FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 2 (2022), 417-437
https://doi.org/10.22190/FUMI220321029V
Original Scientific Paper

ADDITIVE PROPERTIES OF THE DRAZIN INVERSE FOR MATRICES AND BLOCK REPRESENTATIONS: A SURVEY

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Abstract. In this paper, a review of a development of the Drazin inverse for the sum of two matrices has been given. Since this topic is closely related to the problem of finding the Drazin inverse of a 2×2 block matrix, the paper also offers a survey of this subject.
Keywords:Drazin inverse, block matrix, additive properties.

1. Introduction

The concept of the Drazin inverse was introduced in 1958, by Michael P. Drazin, in his celebrated paper [1]. Drazin defined this generalized inverse in a class which is wider than a class of matrices – in associative rings, and named it "pseudo–inverse". The definition which was given by Drazin is as follows.

Definition 1.1. [1] Let $\mathcal{R}$ be an associative ring and $x \in \mathcal{R}$. If there exists an element $c \in \mathcal{R}$, which satisfies the following relations

(i) $cx = xc$,

(ii) $x^m = x^{m+1}c$, for some $m \in \mathbb{N}$,

(iii) $c = c^2x$,

Received March 21, 2022. accepted April 08, 2022.
Communicated by Dragana Cvetković-Ilić
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2010 *Mathematics Subject Classification.* Primary 15A09; Secondary xxxxx, xxxxx

then x is pseudo–invertible in $\mathcal{R}$ and c is the *pseudo–inverse* of x, denoted by $c = x'$.

Furthermore, in [1, Theorem 1], Drazin proved that the pseudo–inverse x' is unique, when x is pseudo–invertible by Definition 1.1. In addition, he opened the problem of additivity of pseudo–inverse and obtained the following result.

Theorem 1.1. [1, Corollary 1] *If $x_1, \ldots, x_j$ ($j \in \mathbb{N}$, $j \geq 2$), are pseudo–invertible elements of associative ring $\mathcal{R}$, such that $x_s x_t = 0$ ($s, t \in \{1, ..., j\}$, $s \neq t$), then $x_1 + ... + x_j$ is also pseudo–invertible element in $\mathcal{R}$, with $(x_1 + ... + x_j)' = {x_1}' + ... + {x_j}'$.*

Further, in the mentioned paper [1], Drazin remarked that the results that he obtained are also applicable to matrices.

After the publication [1] of Drazin, in the late sixties of the previous century, this "pseudo–inverse" was studied more detailed in the matrix concept and it was named "the Drazin inverse" [2, 3, 4, 5, 6, 7].

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices and let $A \in \mathbb{C}^{n\times n}$. We denote by $\mathcal{R}(A), \mathcal{N}(A)$ and $\text{rank}(A)$, the range, the null space and the rank of matrix A, respectively. The smallest nonnegative integer k, such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$, is called the index of matrix A, denoted by $\text{ind}(A)$. For every matrix $A \in \mathbb{C}^{n\times n}$, such that $\text{ind}(A) = k$, there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies the following relations:

$$A^{k+1}A^d = A^k, \;\; A^d A A^d = A^d, \;\; AA^d = A^d A.$$

The matrix A^d is called the Drazin inverse of A [8, 9]. Clearly, $\text{ind}(A) = 0$ if and only if A is nonsingular, and in that case A^d reduces to A^{-1}. Through this paper we use notation $A^\pi = I - AA^d$ to denote the projection on $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$. Also, we agree that $A^0 = I$ and $\sum_{i=1}^{k-j} A_i = 0$, for $k \leq j$, where $i, j, k \in \mathbb{N}$.

The Drazin inverse of square matrices has applications in various areas. Some of the elementary applications of the Drazin inverse of a square matrix were given by S.L. Campbel [10, 11]. In the following we have the application of the Drazin inverse in solving the singular system of differential equations [10, 11].

Let $E, F \in \mathbb{C}^{n\times n}$, where E is singular. Assume that there exists a scalar μ such that $\mu E + F$ is regular matrix. Then the general solution of the singular system of differential equations:

$$Ex'(t) + Fx(t) = 0, \quad t \geq t_0$$

is given by

$$x(t) = e^{-\hat{E}^d \hat{F}(t-t_0)} \hat{E}^d \hat{E} q,$$

where

$$\hat{E} = (\mu E + F)^{-1}E, \quad \hat{F} = (\mu E + F)^{-1}F,$$

and q is an arbitrary vector of dimension n.

2. The Drazin Inverse for the Sum of Two Matrices

Let $P, Q \in \mathbb{C}^{n\times n}$. As we have already noticed in Introduction, the problem of finding the explicit formula for $(P+Q)^d$ was posed by Drazin in 1958 [1]. In the matrix concept, the result which was obtained by Drazin is: when $PQ = 0$ and $QP = 0$, then $(P+Q)^d = P^d + Q^d$. This problem remained unnoticed until 2001. Namely, in 2001, R. E. Hartwig, G. Wang and Y. Wei studied the problem of additivity of the Drazin inverse of complex matrices (see [12]). Using Cline's formula [13], $(AB)^d = A((BA)^d)^2B$, and the representation of $(P+Q)^d$:

$$(2.1)\ (P+Q)^d = \left(\begin{bmatrix} I & Q \end{bmatrix}\begin{bmatrix} P \\ I \end{bmatrix}\right)^d = \begin{bmatrix} I & Q \end{bmatrix}\left(\begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}^d\right)^2\begin{bmatrix} P \\ I \end{bmatrix},$$

they obtained the result which generalize Drazin's result, which is presented in the following theorem.

Theorem 2.1. [12, Theorem 2.1] *Let $P, Q \in \mathbb{C}^{n\times n}$ and let* $\operatorname{ind}(P) = r$, $\operatorname{ind}(Q) = s$. *If $PQ =$, then*

$$(P+Q)^d = Y_1 + Y_2,$$

where

$$(2.2)\qquad Y_1 = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1},\ Y_2 = \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi.$$

After mentioned publication [12], this topic attracted a great attention and a plenty of papers on this subject were published. In 2010, H. Yang and X. Liu [14], also using the representation (2.1), derived the formula for $(P+Q)^d$ under conditions $PQP = 0$, $PQ^2 = 0$ and thereby generalized the result from Theorem 2.1. In the following theorem we have the mentioned formula for $(P+Q)^d$.

Theorem 2.2. [14, Theorem 2.1] *Let $P, Q \in \mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(P) = r$, $\operatorname{ind}(Q) = s$. *If $PQP = 0$ and $PQ^2 = 0$, then*

$$(P+Q)^d = Y_1 + Y_2 + \left(Y_1\left(P^d\right)^2 + \left(Q^d\right)^2 Y_2 - Q^d\left(P^d\right)^2 - \left(Q^d\right)^2 P^d\right)PQ,$$

where Y_1 and Y_2 are defined as in (2.2).

Using the representation (2.1) and induction by $k \in \mathbb{N}$, in 2011, J. Višnjić and D. S. Cvetković–Ilić [15], generalized the results from Theorem 2.1 and Theorem 2.2. This result is given in the next theorem.

Theorem 2.3. [15, Theorem 2.1] *Let us define for $j \in \mathbb{N}$, the set*

$$U_j = \{(p_1, q_1, p_2, q_2, ..., p_j, q_j) : \sum_{i=1}^{j} p_i + \sum_{i=1}^{j} q_i = j-1,\ p_i, q_i \in \{0, 1, ..., j-1\},\ i = \overline{1, j}\}.$$

Let $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$ *and* $k \in \mathbb{N}$. *If*

$$PQ\prod_{i=1}^{k}(P^{p_i}Q^{q_i}) = 0, \tag{2.3}$$

for every $(p_1, q_1, p_2, q_2, ..., p_k, q_k) \in U_k$, *then*

$$(P+Q)^d = Y_1 + Y_2 + \sum_{i=1}^{k-1}\Big(Y_1(P^d)^{i+1} + (Q^d)^{i+1}Y_2 - \sum_{j=1}^{i+1}(Q^d)^j(P^d)^{i+2-j}\Big)PQ(P+Q)^{i-1},$$

where Y_1 *and* Y_2 *are defined as in (2.2).*

Depending on $k \in \mathbb{N}$, we have special cases of Theorem 2.3. For $k = 1$, the condition (2.3) is $PQ = 0$, hence we get Theorem 2.1 as a corollary of Theorem 2.3. In the case $k = 2$, the condition (2.3) is: $PQP = 0$ and $PQ^2 = 0$. Therefore, we have Theorem 2.2 as a direct corollary of Theorem 2.3. If we consider the case $k = 3$, we have the following corollary:

Corollary 2.1. [15, Corollary 2.3] *Let* $P, Q \in \mathbb{C}^{n\times n}$ *be such that* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$. *If* $PQP^2 = 0$, $PQPQ = 0$, $PQ^2P = 0$ *and* $PQ^3 = 0$ *then*

$$\begin{aligned}(P+Q)^d &= Y_1 + Y_2 + \left(Y_1(P^d)^2 + (Q^d)^2Y_2 - \sum_{i=1}^{2}(Q^d)^i(P^d)^{3-i}\right)PQ \\ &\quad + \left(Y_1(P^d)^3 + (Q^d)^3Y_2 - \sum_{i=1}^{3}(Q^d)^i(P^d)^{4-i}\right)(PQP + PQ^2),\end{aligned}$$

where Y_1 *and* Y_2 *are defined by (2.2).*

We remark that every of these results can also be expressed by its symmetrical formulation (by using the conjugate transpose of matrices P and Q). For example, symmetrical conditions of the conditions of Corollary 2.1 are: $Q^2PQ = 0$, $PQPQ = 0$, $QP^2Q = 0$ and $P^3Q = 0$. Hence, the additive formula given under conditions $QPQ = 0$, $QP^2Q = 0$, $P^3Q = 0$ [16, Theorem 3.2], is also a special case of Theorem 2.3. Also, we remark that additive result from Theorem 2.3 is extended to a Banach algebra (see [17]).

Many other authors also studied the problem of additivity of the Drazin inverse and offered formulas for $(P+Q)^d$, with some side conditions for matrices P and Q. In the following list, we give some conditions for matrices P and Q, under which were obtained formulas for $(P+Q)^d$. Furthermore, a list of authors who derived mentioned formulas are given and the year of publication of these results.

(1) N. Castro–González, 2005:
$P^dQ = 0$, $PQ^d = 0$ and $Q^\pi PQP^\pi = 0$ [18, Theorem 2.5];

(2) M. F Martínez–Serrano and N. Castro-González, 2009:
$P^2Q = 0$ and $Q^2 = 0$ [19, Theorem 2.2];

(3) N. Castro–González, E. Dopazo and M. F. Martínez–Serrano, 2009:
$P^2Q = 0$ and $PQ^2 = 0$ [20, Theorem 2.3];

(4) C. Bu, C. Feng and S. Bai, 2012:
$P^2Q = 0$ and $Q^2P = 0$ [16, Theorem 3.1];

(5) C. Bu and C. Zhang, 2013:
$S_{2i-1}PS_{2i-1}P = 0$, $S_{2i-1}PS_{2i-1}QS_{2(i-1)} = 0$ and $QS_{2(i-1)}PQS_{2(i-1)}Q^2 = 0$, where $S_i = (P + Q)^i$ and $i \geq 1$ [21, Theorem 3.1];

(6) J. Višnjić, 2015:
$P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$ [22, Theorem 2.1];

(7) L. Sun et al. 2016:
(7.1) $P^2QP = 0$, $PQ^2 = 0$ and $(QP)^2 = 0$ [23, Theorem 3.1],
(7.2) $P^2QP = 0$, $P^3Q = 0$ and $Q^2 = 0$ [23, Theorem 3.3];

(8) E. Dopazo, M.F. Martínez–Serrano, J. Robles, 2016:
(8.1) $PQ^2 = 0$, $P^2QP^\pi = 0$ and $P^dQ = 0$ [24, Theorem 4.1 (i)],
(8.2) $PQ^2 = 0$, $PQPP^\pi = 0$ and $P^dQP = 0$ [24, Theorem 4.1 (ii)];

(9) X. Yang, X. Liu and F. Chen, 2017:
$P^2QP = 0$, $Q^2PQ = 0$, $Q^2P^2 = 0$, $PQ^2P = 0$, $P^3Q = 0$ and $Q^3P = 0$ [25, Theorem 3.1];

(10) M. Dana and R. Yousefi, 2018:
$PQP = 0$, $QPQ = 0$, $P^2Q^2 = 0$ and PQ^3 [26, Theorem 4];

(11) R. Yousefi and M. Dana, 2018:
(11.1) $P^2QP = 0$, $P^2Q^2 = 0$ and $QPQ = 0$ [27, Theorem 2.1],
(11.2) $PQP^2 = 0$, $PQ^2 = 0$ and $QP^3 = 0$ [27, Theorem 3.1];

(12) L. Guo, J. Chen and H. Zou, 2019:
$PQ^iP = 0$, for $i = 1, 2, \ldots, n$ [28, Theorem 2.1].

Additive properties of the Drazin inverse were investigated not only in the matrix concept, but also in the class of operators, in rings, in Banach algebra (for example, see [17, 20, 29, 30, 31, 32, 33, 34, 35]).

3. Representations for the Drazin Inverse of 2×2 Block Matrices

Let M be a complex block matrix of a form:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.1}$$

where A and D are square matrices, not necessarily of the same size. In 1979, S. L. Campbell and C. D. Meyer [36] posed the problem of finding the Drazin inverse of matrix M, in terms of A^d and D^d, with arbitrary blocks A, B, C and D. Since then, many authors have studied this problem and offered some formulas for M^d, with some restrictions upon the blocks of matrix M. In the present, there is no general expression of M^d, with no side conditions on blocks of matrix M, so this problem is still an open one. However, it is shown that the Drazin inverse of 2×2 block matrix has applications in several areas, such as differential and difference equations and perturbation theory of the Drazin inverse (see [10, 11, 12, 36, 37, 38]), so this topic is still of the great significance. We will present one of known examples of application of the Drazin inverse of a 2×2 block matrix (see [38]).

Consider the second-order system:

$$Ex''(t) + Fx'(t) + G = 0, \tag{3.2}$$

where $E, F, G \in \mathbb{C}^{n\times n}$ and G is nonsingular. Then there is nonzero λ, such that $\lambda^2 E + \lambda F + G$ is invertible. For $x(t) = e^{\lambda t} y(t)$, (3.2) is equivalent to:

$$(\lambda^2 E + \lambda F + G)^{-1} E y''(t) + (\lambda^2 E + \lambda F + G)^{-1}(F + 2\lambda E) y'(t) + y(t) = 0.$$

For $w(t) = y'(t)$, the above system is equivalent to the first order system:

$$\begin{bmatrix} 0 & -I \\ \tilde{E} & \tilde{F} \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix}' + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.3}$$

where $\tilde{E} = (\lambda^2 E + \lambda F + G)^{-1} E$ and $\tilde{F} = (\lambda^2 E + \lambda F + G)^{-1}(F + 2\lambda E)$. For sufficiently small μ, we have that $\mu \begin{bmatrix} 0 & -I \\ \tilde{E} & \tilde{F} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ is invertible. In order to express solutions of (3.3), in terms of $\tilde{E}$ and $\tilde{F}$, we need to find the Drazin inverse of 2×2 block matrix:

$$\hat{E} = \begin{bmatrix} I & -\mu I \\ \mu\tilde{E} & \mu\tilde{F} + I \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\mu I \\ \mu\tilde{E} & \mu\tilde{F} \end{bmatrix}.$$

Primarily due to its applications, the topic of finding the Drazin inverse of 2×2 block matrix M is still studied and new formulas for M^d, with less restrictive conditions for blocks of M, are developed. We remark that the literature on this subject is truly sizeable and here we will present some of the results, which we found interesting.

3.1. The connection between M^d and $(P+Q)^d$

The problem of finding the Drazin inverse of the sum of two matrices is closely related to the problem of finding the Drazin inverse of 2×2 complex block matrix. We have that

$$(P+Q)^d = \left(\begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \right)^d = \begin{bmatrix} P & Q \end{bmatrix} \left(\begin{bmatrix} P & Q \\ P & Q \end{bmatrix}^d \right)^2 \begin{bmatrix} I \\ I \end{bmatrix},$$

therefore we can see the connection between the Drazin inverse of the sum of two matrices and the Drazin inverse of 2×2 block matrix (see [27, Theorem 3.1]). Also, as it is presented in (2.1), the Drazin inverse of the sum $P+Q$ can be expressed using the Drazin inverse of 2×2 block matrix $\begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}$. Hence, one can use this approach and obtain a formula for $(P+Q)^d$, applying formulas for M^d (for example, see [24]).

Conversely, matrix M can be presented as a sum of two matrices, so we can use formulas for $(P+Q)^d$ to derive representations for M^d (for example, see [22]). The last approach is widely used in the current literature and there are many ways to split matrix M as a sum of two matrices. For example, matrix M can be presented as a sum:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}, \quad \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix},$$

$$\begin{bmatrix} A & B \\ CAA^d & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ CA^\pi & 0 \end{bmatrix}, \quad \begin{bmatrix} A^2A^d & AA^dB \\ C & D \end{bmatrix} + \begin{bmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{bmatrix}, \text{ etc.}$$

3.2. The Drazin inverse of anti–triangular block matrices

As we have noticed in the previous subsection, matrix M, given by (3.1), can be presented as a sum of two matrices, where one of them can be triangular ($B=0$ or $C=0$ in (3.1)) or anti–triangular ($A=0$ or $D=0$ in (3.1)). Hence, triangular and anti–triangular matrices are widely used for deriving formulas for M^d, and consequently for $(P+Q)^d$. In 1977, two groups of authors, independently of each other, derived a general expression of the Drazin inverse of lower and upper triangular matrices. Actually, R. E. Hartwig and J. M. Shoaf [39], obtained mentioned formula in the ring concept, and C. D. Meyer and N. D. Rose [40] in the matrix concept. This formula is known as Hartwig–Meyer–Rose formula and it is presented in the following theorem.

Theorem 3.1. [39, 40] *Let M_1 and M_2 be block matrices of a form:*

$$M_1 = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}, \quad M_2 = \begin{bmatrix} B & C \\ 0 & A \end{bmatrix},$$

where A and B are square matrices, with $\operatorname{ind}(A)=k$, $\operatorname{ind}(B)=l$. *Then* $\max\{k,l\} \le \operatorname{ind}(M_i) \le k+l$, *for $i\in\{1,2\}$, and*

$$M_1^d = \begin{bmatrix} A^d & 0 \\ X & B^d \end{bmatrix}, \quad M_2^d = \begin{bmatrix} B^d & X \\ 0 & A^d \end{bmatrix},$$

where

$$X = X(B,C,A) = \sum_{i=0}^{l-1} (B^d)^{i+2} C A^i A^\pi + \sum_{i=0}^{k-1} B^\pi B^i C (A^d)^{i+2} - B^d C A^d.$$

However, until today there is no general expression for the Drazin inverse of anti–triangular block matrix. This problem was opened in 1983 by S. L. Campbel [11], as an application in solving second order differential equations. These matrices are also used in some applications like reducing the sizes of the matrices involved in numerical analysis, saddle-point problems, optimization problems and graph theory [10, 41, 42, 43, 44]. Primarily due to its applications, many scientists have investigated special cases of this problem and offered some formulas, which are valid under certain conditions.

Let N be an anti–triangular block matrix:

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{3.4}$$

In 2005, N. Castro–González and E. Dopazo [45] studied the open problem of finding a formula for N^d. They noticed that a matrix of a form (3.4), where $A = B = I$, is involved in calculating the Drazin inverse of bordered matrices of the form $\begin{bmatrix} I & P^t \\ Q & UV^t \end{bmatrix}$, where U, V, P and Q are $n \times k$ matrices [46]. Therefore, they derived the formula for matrix N, when $A = B = I$. This formula is presented in the next theorem.

Theorem 3.2. [45, Theorem 3.3] *Let* $F = \begin{bmatrix} I & I \\ E & 0 \end{bmatrix}$, *where* $E \in \mathbb{C}^{l\times l}$ *and* $\operatorname{ind}(E) = r$. *Then*

$$F^d = \begin{bmatrix} Z_1 E^\pi & E^d + Z_2 E^\pi \\ E^d E + Z_2 E E^\pi & -E^d + (Z_1 - Z_2)E^\pi \end{bmatrix},$$

where

$$Z_1 = \sum_{j=0}^{r-1} (-1)^j C(2,j) E^j, \ \ Z_2 = \sum_{j=0}^{r-1} (-1)^j C(2j+1,j) E^j \quad \text{and} \quad C(n,k) = \begin{pmatrix} n \\ k \end{pmatrix}.$$

Using the result from Theorem 3.2, N. Castro–González and E. Dopazo derived the representation for N^d, under assumptions $CA^dA = C$ and $BCA^d = A^dBC$:

Theorem 3.3. [45, Theorem 4.1] *Let N be matrix of a form (3.4). If $CA^dA = C$ and $BCA^d = A^dBC$, then*

$$N^d = \begin{bmatrix} \left(T_1 + A^dBCT_2\right)(BC)^\pi A & \left((BC)^d + T_1(BC)^\pi\right) B \\ C\left((BC)^d + T_1(BC)^\pi\right) & C\left(-A\left((BC)^d\right)^2 + T_2(BC)^\pi\right) B \end{bmatrix},$$

where

$$T_1 = \sum_{j=0}^{r-1} C(2j,j)(A^d)^{2j+2}(BC)^j \ \text{and} \ T_2 = \sum_{j=0}^{r-1} C(2j+1,j)(A^d)^{2j+3}(BC)^j,$$

with $\operatorname{ind}((A^d)^2BC) = r$ *and* $C(n,k) = \begin{pmatrix} n \\ k \end{pmatrix}$.

In 2009, M. F. Martínez–Serrano and N. Castro-González [19] investigated the Drazin inverse for 2×2 block matrix M. As corollaries of their results, they derived formulas for the Drazin inverse of anti–triangular block matrix N, defined as in (3.4), under condition $ABC = 0$ and also under condition $BCA = 0$.

Theorem 3.4. [19, Corollary 3.9, Corollary 3.10] *Let N be a matrix of a form (3.4), with $\operatorname{ind}(BC) = r$ and $\operatorname{ind}(A^2) = s$.*

(i) If $ABC = 0$, then

$$N^d = \begin{bmatrix} (\Psi_1 A^d + \Psi_2)A & (\Psi_1 A^d + \Psi_2)B \\ C(\Psi_1 A^d + \Psi_2) & C\left(\Psi_1 (A^d)^2 + (BC)^d \Psi_2 A - (BC)^d A^d\right) B \end{bmatrix},$$

where

$$\Psi_1 = \sum_{i=0}^{r-1} (BC)^\pi (BC)^i (A^d)^{2i+1} \text{ and } \Psi_2 = \sum_{i=0}^{s-1} ((BC)^d)^{i+1} (BC)^i A^{2i} A^\pi.$$

(ii) If $BCA = 0$, then

$$N^d = \begin{bmatrix} A(A^d \Phi_1 + \Phi_2) & (A^d \Phi_1 + \Phi_2)B \\ C(A^d \Phi_1 + \Phi_2) & C\left((A^d)^2 \Phi_1 + A\Phi_2 (BC)^d - A^d (BC)^d\right) B \end{bmatrix},$$

where

$$\Phi_1 = \sum_{i=0}^{r-1} (A^d)^{2i+1} (BC)^i (BC)^\pi \text{ and } \Phi_2 = \sum_{i=0}^{s-1} A^{2i} A^\pi ((BC)^d)^{i+1}.$$

Also in 2009, C. Deng and Y. Wei [43] investigated the Drazin inverse of a 2×2 anti–triangular operator matrix of a form (3.4). They obtained formulas for N^d, which are valid when the following conditions are satisfied:

$$\text{(3.5)} \qquad \begin{cases} \text{(i) } ABC = 0 \text{ [43, Theorem 3.3];} \\ \text{(ii) } BCA^\pi = 0 \text{ and } AA^d BC = 0 \text{ [43, Theorem 3.6];} \\ \text{(iii) } A^\pi AB = 0 \text{ and } BCAA^d = 0 \text{ [43, Theorem 3.8].} \end{cases}$$

Results mentioned above were generalized by E. Dopazo, M. F. Martínez–Serrano and J. Robles [24], in 2016. Namely, these authors obtained explicit representations for N^d in the following cases:

$$\text{(3.6)} \begin{cases} \text{(a) } ABCA^\pi = 0 \text{ and } AA^d BC = 0 \text{ [24, Theorem 2.1];} \\ \text{(b) } A^\pi BCA = 0 \text{ and } BCAA^d = 0 \text{ [24, Theorem 2.3];} \\ \text{(c) } BCA^\pi A = 0, BCA^\pi B = 0 \text{ and } A^d BCA = 0 \text{ [24, Theorem 2.4];} \\ \text{(d) } AA^\pi BC = 0, CA^\pi BC = 0 \text{ and } ABCA^d = 0 \text{ [24, Theorem 2.6].} \end{cases}$$

Furthermore, in 2021, D. Zhang, Y. Yin and D. Mosić [47] studied the Drazin inverse of anti–triangular block matrix N of a form (3.4), for $C = I$ and also for arbitrary C. Actually, they derived the representation for the Drazin inverse of the matrix $\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$, when $AA^dBAA^d = 0$, $BA^\pi B = 0$ and $BA^\pi A = 0$ [47, Theorem 3.2]. Further, they used a factorization of N:

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

and obtained the formula for N^d under conditions $AA^dBCAA^d = 0$, $BCA^\pi BC = 0$ and $BCA^\pi A = 0$ [47, Theorem 3.3]. Mentioned formula is generalization of the case (c) from the list (3.6) and the case (ii) from the list (3.5).

Recently, J. Višnjić and I. Ilić [48] offered representations for N^d when the following conditions are satisfied:

(1) $AA^\pi BCA = 0, CA^\pi BCA = 0$ and $A^dBCA^d = 0$ [48, Theorem 3.1];

(2) $ABCA^\pi A = 0, ABCA^\pi B = 0$ and $A^dBCA^d = 0$ [48, Theorem 3.2].

We remark that the case (1) from the list above is a generalization of the cases (b) and (d) from the list (3.6), and also of the case (ii) from Theorem 3.4. In addition, the case (2) from the previous list is an extension of the cases (a) and (c) from the list (3.6), and also of the cases (i) and (ii) from the list (3.5).

Lately, D. Zhang, Y. Yin and D. Mosić [49] generalized the case $BCA = 0$ from Theorem 3.4, for operator matrix. Actually, these authors derived the formula for N^d when $BCA^3 = 0$, $BCABC = 0$ and $BCA^2BC =$ is valid [49, Theorem 3.2].

3.3. The Drazin inverse of block matrices with zero or nonsingular generalized Schur complement

Let M be a complex block matrix of a form (3.1). In the case when A is nonsingular, the Schur complement of A in M is defined by $Z = D - CA^{-1}B$ [8]. It is well known that, if A is regular matrix, then the invertibility of a 2×2 block matrix M is equivalent to the invertibility of the Schur complement of A in M [50], and in that case:

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BZ^{-1}CA^{-1} & -A^{-1}BZ^{-1} \\ -Z^{-1}CA^{-1} & Z^{-1} \end{bmatrix}. \tag{3.7}$$

If A is a singular matrix, we define the generalized Schur complement, based on the Drazin inverse of A [8], in the following way:

$$S = D - CA^dB. \tag{3.8}$$

Through the rest of this paper, we will assume that generalized Schur complement S is defined as in (3.8).

In 1989, J. Miao [51] studied the problem of finding the Drazin inverse of a complex block matrix M, with some restrictions on the generalized Scuhr complement. In the case when the generalized Schur complement is nonsingular and some certain conditions hold, this author obtained the following representation for M^d.

Theorem 3.5. [51] *Let M be a complex block matrix of a form (3.1). If $CA^\pi = 0$, $A^\pi B = 0$ and the generalized Schur complement $S = D - CA^dB$ is nonsingular, then:*

$$M^d = \begin{bmatrix} A^d + A^dBS^{-1}CA^d & -A^dBS^{-1} \\ -S^{-1}CA^d & S^{-1} \end{bmatrix}.$$

In 1998, Y. Wei [52] also considered representations for the Drazin inverse of complex block matrices and extended the result from Theorem 3.5. Namely, this author proved that when condition of invertibility of the generalized Schur complement is replaced with some weaker conditions, then the Drazin inverse of a 2×2 block matrix M can be expressed analogously to the expression (3.7) of the ordinary inverse of nonsingular block matrix.

Theorem 3.6. [52, Theorem 1] *Let M be a complex block matrix of a form (3.1). If $CA^\pi = 0$, $A^\pi B = 0$, $BS^\pi = 0$, $S^\pi C = 0$ and $DS^\pi = 0$, then*

$$M^d = \begin{bmatrix} A^d + A^dBS^dCA^d & -A^dBS^d \\ -S^dCA^d & S^d \end{bmatrix}. \tag{3.9}$$

Many other authors also investigated when the Drazin inverse of a 2×2 block matrix M adopts the form (3.9) and offered some results on this topic (for example, see [53, 54, 55, 56]).

Together with the case when the generalized Schur complement is nonsingular, the case when the generalized Schur complement is equal to zero is also studied. In the mentioned paper [51] of J. Miao, the author derived the representation for M^d, in the case when the generalized Schur complement is equal to zero and conditions $CA^\pi = 0$ and $A^\pi B = 0$ are satisfied. This well–known result is presented in the following theorem.

Theorem 3.7. [51] *Let M be a complex block matrix of a form (3.1), such that $S = 0$. If $CA^\pi = 0$ and $A^\pi B = 0$ then:*

$$M^d = \begin{bmatrix} I \\ CA^d \end{bmatrix} \left((AW)^d\right)^2 A \begin{bmatrix} I & A^dB \end{bmatrix},$$

where $W = AA^d + A^dBCA^d$.

In 2006, the paper [38] of R. E. Hartwig, X. Li and Y. Wei was published, where representations for the Drazin inverse of 2×2 block matrices were studied in the cases when the generalized Schur complement is either nonsingular or zero. In the mentioned paper, authors obtained representations for M^d, which generalize the representations from Theorem 3.5 and 3.7. These representations are presented in the next two theorems.

Theorem 3.8. [38, Theorem 3.1] *Let M be a complex block matrix of a form (3.1), such that the generalized Schur complement S is nonsingular. If $CA^\pi B = 0$ and $AA^\pi B = 0$, then*

$$M^d = \left(I + \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix} R\right) R \left(I + \sum_{i=0}^{r-1} R^{i+1} \begin{bmatrix} 0 & 0 \\ CA^\pi A^i & 0 \end{bmatrix}\right),$$

where $\operatorname{ind}(A) = r$ *and*

$$R = \begin{bmatrix} A^d + A^d BS^{-1}CA^d & -A^d BS^{-1} \\ -S^{-1}CA^d & S^{-1} \end{bmatrix}.$$

As a corollary of the result above, R. E. Hartwig, X. Li and Y. Wei also obtained a representation for M^d, when $CA^\pi B = 0$, $CA^\pi A = 0$ and S is nonsingular (see [38, Corollary 3.2]). In the case when $S = 0$, these authors derived the following result.

Theorem 3.9. [38, Theorem 4.1] *Let M be a complex block matrix of a form (3.1), such that $S = 0$. If $CA^\pi B = 0$ and $AA^\pi B = 0$, then*

$$\operatorname{ind}(M) \leq \operatorname{ind}(AW) + \operatorname{ind}(A) + 2$$

and

$$M^d = \left(I + \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix} R_1\right) R_1 \left(I + \sum_{i=0}^{r-1} R_1^{i+1} \begin{bmatrix} 0 & 0 \\ CA^\pi A^i & 0 \end{bmatrix}\right),$$

where $\operatorname{ind}(A) = r$*,*

$$R_1 = \begin{bmatrix} I \\ CA^d \end{bmatrix} \left((AW)^d\right)^2 A \begin{bmatrix} I & A^d B \end{bmatrix}$$

and $W = AA^d + A^d BCA^d$.

Furthermore, in [38, Corollary 4.2], a representation for M^d is derived when $CA^\pi B = 0$, $CA^\pi A = 0$ and $S = 0$, as a corollary of Theorem 3.9.

In the years that came after the publication [38], many authors have been considering this topic. In the case when S is nonsingular, there were few generalizations of known results. Namely, in 2009, M. F. Martínez–Serrano and N. Castro-González [19] extended the result from Theorem 3.8 and of its corollary [38, Corollary 3.2]. These representations are given under the following conditions:

(i) $A^2 A^\pi B = 0$, $CAA^\pi B = 0$, $CA^\pi B = 0$ and S is nonsingular [19, Theorem 3.11];

(ii) $CA^2 A^\pi = 0$, $CAA^\pi B = 0$, $CA^\pi B = 0$ and S is nonsingular [19, Corollary 3.12].

In 2012, C. Bu, C. Feng and P. Dong [57] also studied representations for M^d in the case when S is nonsingular. In order to extend the results from Theorem 3.8 and of its corollary [38, Corollary 3.2], these authors offered the representations for M^d when the following conditions are satisfied:

(a) $AA^\pi BC = 0$, $AA^\pi BD = 0$, $CA^\pi BC = 0$, $CA^\pi BD = 0$ and $S = 0$ [57, Theorem 3.2];

(b) $BCA^\pi A = 0$, $DCA^\pi A = 0$, $BCA^\pi B = 0$, $DCA^\pi B = 0$ and $S = 0$ [57, Theorem 3.3].

It turned out that the conditions (a) from the list above are equivalent to the conditions of Theorem 3.8 and that the conditions (b) are equivalent to the conditions of [38, Corollary 3.2] (see [58, Remark 3.5]).

L. Xia and B. Deng [59] also studied the case when S is nonsingular. Actually, in 2017, a paper of these authors was published, where a representations for M^d are derived, when the following conditions are satisfied:

(I) $A^\pi BC = 0$, $CA^\pi B = 0$, $BD + AB = 0$ and S is nonsingular [59, Theorem 3.5];

(II) $BCA^\pi = 0$, $CA^\pi B = 0$, $CA + DC = 0$ and S is nonsingular [59, Theorem 3.6].

Contrary to the case when the generalized Schur complement is nonsingular, many authors offered representations for M^d in the case when the generalized Schur complement is equal to zero, which extend representations from Theorem 3.9 and [38, Corollary 4.2]. In the following list, we give some of the conditions, under which are derived representations for M^d in the case when $S = 0$.

(1) M. F Martínez–Serrano and N. Castro-González, 2009:
  (1.1) $A^2A^\pi B = 0$, $CAA^\pi B = 0$, $BCA^\pi B = 0$ and $S = 0$ [19, Theorem 3.1],
  (1.2) $CA^2A^\pi = 0$, $CAA^\pi B = 0$, $CA^\pi BC = 0$ and $S = 0$ [19, Theorem 3.2],
  (1.3) $ABCA^\pi = 0$, BCA^π is nilpotent and $S = 0$ [19, Theorem 3.3],
  (1.4) $A^\pi BCA = 0$, $A^\pi BC$ is nilpotent and $S = 0$ [19, Corollary 3.4],
  (1.5) $ABC = 0$ and $S = 0$ [19, Theorem 3.6],
  (1.6) $BCA = 0$ and $S = 0$ [19, Corollary 3.7];

(2) H. Yang and X. Liu, 2010:
  (2.1) $AA^\pi BC = 0$, $CA^\pi BC = 0$ and $S = 0$ [14, Theorem 3.3],
  (2.2) $BCA^\pi A = 0$, $BCA^\pi B = 0$ and $S = 0$ [14, Theorem 3.3];

(3) C. Bu, C. Feng and S. Bai, 2012:
  (3.1) $ABCA^\pi = 0$, $A^\pi ABC = 0$ and $S = 0$ [16, Theorem 4.1],
  (3.2) $ABCA^\pi = 0$, $CBCA^\pi = 0$ and $S = 0$ [16, Theorem 4.3],
  (3.3) $CA^\pi BC = 0$, $A^2A^\pi BC$=0, $CAA^\pi BC = 0$ and $S = 0$ [16, Theorem 4.4];

(4) J. Višnjić, 2015:
(4.1) $ABCA^\pi A = 0$, $ABCA^\pi B = 0$ and $S = 0$ [22, Theorem 3.1],
(4.2) $AA^\pi BCA = 0$, $CA^\pi BCA = 0$ and $S = 0$ [22, Theorem 3.2];

(5) L. Sun et al, 2016:
(5.1) $A^2A^\pi BC = 0$, $BCA^\pi BC = 0$, $CAA^\pi BC = 0$ and $S = 0$ [23, Theorem 4.1],
(5.2) $BCA^2A^\pi = 0$, $BCAA^\pi B = 0$, $BCA^\pi BC = 0$ and $S = 0$ [23, Theorem 4.2],
(5.3) $A^2BC = 0$, $ABCA = 0$, $ABCB = 0$ and $S = 0$ [23, Theorem 4.3],
(5.4) $BCA^2 = 0$, $ABCA = 0$, $CBCA = 0$ and $S = 0$ [23, Theorem 4.4];

(6) X. Yang, X. Liu and F. Chen, 2017:
$A^2BCA^\pi A = 0$, $A^2BCA^\pi B = 0$, $A^\pi ABC = 0$ and $S = 0$ [25, Theorem 4.1];

(7) M. Dana and R. Yousefi, 2018:
$CAA^\pi B = 0$, $A^2A^\pi BC = 0$, $A^\pi BCA^2 = 0$, $A^\pi BCB = 0$ and $S = 0$ [26, Theorem 8];

(8) R. Yousefi and M. Dana, 2018:
(8.1) $AA^\pi BC = 0$, $BCA^\pi B = 0$ and $S = 0$ [27, Theorem 2.3],
(8.2) $CA^\pi BC = 0$, $BCAA^\pi = 0$ and $S = 0$ [27, Theorem 2.4],
(8.3) $BCA^d = 0$, $BCAA^\pi B = 0$, $CA^2A^\pi = 0$ and $S = 0$ [27, Theorem 3.3],
(8.4) $A^dBC = 0$, $CAA^\pi BC = 0$, $A^2A^\pi B = 0$ and $S = 0$ [27, Theorem 3.4].

3.4. The Drazin inverse of block matrices with no restrictions on generalized Schur complement

Let M be a 2×2 block matrix of the form (3.1), with arbitrary blocks A, B, C and D. As it was already mentioned, the problem of finding the formula for M^d was posed in 1979 [36] and it remains open. However, it was studied by many authors and some special cases of this problem are solved.

In 2001, D. S. Djordjević and P. S. Stanimirović [37] studied the problem of finding the representation for M^d, under some specific conditions, in the concept of operators. The result obtained by these authors is presented in the following theorem, in the matrix concept.

Theorem 3.10. [37, Theorem 5.3] *Let M be a matrix of the form (3.1). If $BC = 0$, $DC = 0$ and $BD = 0$, then*

$$M^d = \begin{bmatrix} A^d & (A^d)^2B \\ C(A^d)^2 & D^d + C(A^d)^3B \end{bmatrix}.$$

In the paper [38], published in 2006, R. E. Hartwig, X. Li and Y. Wei derived the following two representations for M^d:

Theorem 3.11. [38, Lemma 2.2] *Let M be a matrix of the form (3.1). If $BC = 0$, $DC = 0$ and D is nilpotent, then*

$$M^d = \begin{bmatrix} I \\ CA^d \end{bmatrix} A^d \begin{bmatrix} I & \sum_{i=0}^{j-1}(A^d)^{i+1}BD^i \end{bmatrix},$$

where $j = \mathrm{ind}(D)$. Furthermore, $\mathrm{ind}(M) \leq \mathrm{ind}(A) + \mathrm{ind}(D) + 1$.

Corollary 3.1. [38, Corollary 2.3] *Let M be a matrix of the form (3.1). If $BC = 0$, $BD = 0$ and D is nilpotent, then*

$$M^d = \begin{bmatrix} I \\ \sum_{i=0}^{j-1} D^iC(A^d)^{i+1} \end{bmatrix} A^d \begin{bmatrix} I & A^dB \end{bmatrix},$$

where $j = \mathrm{ind}(D)$. In addition, $\mathrm{ind}(M) \leq \mathrm{ind}(A) + \mathrm{ind}(D) + 1$.

In 2008, D. S. Cvetković–Ilić [60] obtained the representation for M^d when $BC = 0$ and $DC = 0$, without additional condition $BD = 0$ from Theorem 3.10, or D is nilpotent matrix from Theorem 3.11. This representation for M^d is given in the next theorem.

Theorem 3.12. [60, Corollary 2.1] *Let M be a matrix of the form (3.1) and* $\mathrm{ind}(A) = r$, $\mathrm{ind}(D) = s$. *If $BC = 0$ and $DC = 0$, then*

$$M^d = \begin{bmatrix} A^d & X \\ C(A^d)^2 & D^d + CXD^d + CA^dX \end{bmatrix},$$

where

$$X = X(A, B, D) = \sum_{i=0}^{s-1}(A^d)^{i+2}BD^iD^\pi + \sum_{i=0}^{r-1} A^\pi A^iB(D^d)^{i+2} - A^dBD^d \tag{3.10}$$

In the mentioned paper [60], the author also obtained the formula for M^d, when $AB = 0$ and $CB = 0$. This representation for M^d is given in the following theorem.

Theorem 3.13. [60, Corollary 2.2] *Let M be a matrix of the form (3.1) and* $\mathrm{ind}(A) = r$, $\mathrm{ind}(D) = s$. *If $AB = 0$ and $CB = 0$, then*

$$M^d = \begin{bmatrix} A^d + BXA^d + BD^dX & B(D^d)^2 \\ X & D^d + CXD^d \end{bmatrix},$$

where $X = X(D, C, A)$ is defined by (3.10).

In 2009, D. S. Cvetković–Ilić [61] derived another representation for M^d, when the blocks of matrix M satisfy conditions $CA = 0$ and $CB = 0$. This representation is presented in the next theorem.

Theorem 3.14. [61, Theorem 2.1] *Let M be a matrix of the form (3.1) and* $\mathrm{ind}(A) = r$, $\mathrm{ind}(D) = s$. *If $AB = 0$ and $CB = 0$, then*

$$M^d = \begin{bmatrix} A^d + X_2 C & X_1 \\ (D^d)^2 C & D^d \end{bmatrix},$$

where

$$X_i = \sum_{j=0}^{s-1} (A^d)^{i+j+1} B D^j D^\pi + \sum_{j=0}^{r-1} A^\pi A^j B (D^d)^{i+j+1} - \sum_{j=0}^{i-1} (A^d)^{j+1} B (D^d)^{i-j},$$

for $i \in \{1, 2\}$.

M. Catral, D. D. Olesky and P. Van Den Driessche, in the paper [44] published in 2009, obtained a general expression of the Drazin inverse of a bipartite matrix $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. This formula is given in the following theorem.

Theorem 3.15. [44, Theorem 2.1] *Let M be matrix of a form (3.1), such that $A = 0$ and $D = 0$. Then*

$$M^d = \begin{bmatrix} 0 & (BC)^d B \\ C(BC)^d & 0 \end{bmatrix}.$$

Furthermore, if $\mathrm{ind}(BC) = s$, *then* $\mathrm{ind}(A) \leq 2s + 1$.

All of the previous mentioned results were generalized a lot of times. In the contiuation, we have provided a list of some of the conditions under which were obtained mentioned generalizations:

(1) N. Castro–González, E. Dopazo, M. F. Martínez–Serrano, 2009:
(1.1) $BCA = 0$, $BD = 0$ and BC is nilpotent [20, Theorem 4.2],
(1.2) $BCA = 0$, $DC = 0$ and $BD = 0$ [20, Theorem 4.4],
(1.3) $BCA = 0$, $DC = 0$ and D is nilpotent [20, Theorem 4.5];

(2) E. Dopazo and M. F. Martínez–Serrano, 2010:
(2.1) $BC = 0$, $BDC = 0$ and $BD^2 = 0$ [62, Theorem 2.2],
(2.2) $BD^\pi C = 0$, $BDD^d = 0$ and $DD^\pi C = 0$ [62, Theorem 2.5],
(2.3) $BD = 0$, $D^\pi CA = 0$ and $D^\pi CB = 0$ [62, Theorem 2.7];

(3) A. S. Cvetković and G. V. Milovanović, 2010: (3.1) $ABC = 0$, $DC = 0$ and $BD = 0$ [63, Theorem 1],
(3.2) $ABC = 0$, $DC = 0$ and BC is nilpotent [63, Theorem 2],
(3.3) $ABC = 0$, $DC = 0$ and D is nilpotent [63, Theorem 3];

(4) C. Bu and K. Zhang, 2010:
(4.1) $ABC = 0$ and $DC = 0$ [64, Theorem 2.2],
(4.2) $ABC = 0$ and $BD = 0$ [64, Theorem 2.3];

(5) D. S. Cvetković–Ilić, 2011:
(5.1) $BCA = 0$ and $BD = 0$ [17, Theorem 1.5],
(5.2) $ABC = 0$ and $DC = 0$ [17, Theorem 1.6],

(6) H. Yang and X. Liu, 2011:
(6.1) $BCA = 0$, $BCB = 0$, $DCA = 0$ and $DCB = 0$ [14, Theorem 3.1],
(6.2) $ABC = 0$, $ABD = 0$, $CBC = 0$ and $CBD = 0$ [14, Theorem 3.2];

(7) J. Ljubisavljević and D. S. Cvetković–Ilić, 2011:
(7.1) $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$ [15, Theorem 3.1],
(7.2) $BCA = 0$, $DCA = 0$, $CBC = 0$ and $CBD = 0$ [15, Theorem 3.2],
(7.3) $BD^{\pi}C = 0$, $BDD^d = 0$, $DD^{\pi}CA = 0$ and $DD^{\pi}CB = 0$ [15, Theorem 3.3],
(7.4) $BD = 0$, $D^{\pi}CA^2 = 0$, $D^{\pi}CAB = 0$ and $D^{\pi}CBC = 0$ [15, Theorem 3.4];

(8) J. Ljubisavljević and D. S. Cvetković–Ilić, 2013:
(8.1) $BCA = 0$, $DCA = 0$ and $DCB = 0$ [65, Theorem 3.1],
(8.2) $BCA = 0$, $ABD = 0$ and $CBD = 0$ [65, Theorem 3.2],
(8.3) $BCA = 0$, $DCA = 0$ and $CBD = 0$ [65, Theorem 3.3],
(8.4) $ABC = 0$, $ABD = 0$ and $CBD = 0$ [65, Theorem 3.4],
(8.5) $ABC = 0$, $ABD = 0$ and $DCB = 0$ [65, Theorem 3.5],
(8.6) $ABC = 0$, $DCA = 0$ and $DCB = 0$ [65, Theorem 3.6];

(9) D. Zhang, Y. Yin and D. Mosić, 2022:
(9.1) $BCA^3 = 0$, $BCA^2BC = 0$, $BCABC = 0$, $ABD = 0$ and $CBD = 0$ [49, Theorem 4.1],
(9.2) $BCA^3 = 0$, $BCA^2BC = 0$, $BCABC = 0$, $DCA = 0$ and $DCB = 0$ [49, Theorem 4.2],
(9.3) $BCA^3 = 0$, $BCA^2BC = 0$, $BCABC = 0$, $BDC = 0$ and $D^2C = 0$ [49, Theorem 4.3].

4. Conclusion

At the moment, there is no formula for $(P+Q)^d$ without any side conditions for matrices P and Q. Hence, this problem remains open. Since the problem of finding the explicit representation for M^d is closely connected to the problem of finding the Drazin inverse of a sum of two matrices, this problem also remains open. However, since the Drazin inverse of matrices has many applications, it is important to find as many formulas as we can for computing the Drazin inverse of a sum of two matrices (and for the Drazin inverse of a 2×2 block matrix).

Ackonwledgement

The author was supported in part by Grant No. 451-03-68/2022-14/200113 of the Ministry of Education, Science and Technological Development, Republic of Serbia.

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