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ALMOST CONFORMAL RICCI SOLITONS ON LP-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to classify almost conformal Ricci solitons on Lorentzian para-Sasakian manifolds. In this paper, we prove that such manifolds with infinitesimal contact vector field V is η-Einstein and the scalar curvature of the manifold is constant, where V is potential vector field. Moreover, we show that an almost conformal Ricci soliton on Lorentzian para-Sasakian manifold becomes a con-formal Ricci soliton and it is shrinking, steady or expanding according as the dimension of the manifold is greater than 3 or equal to 3 or less than 3. Also we prove that V is strictly infinitesimal contact vector field.
Keywords: Ricci solitons, Lorentzian para-Sasakian manifolds, η-Einstein manifold.

1. Introduction

In 1982, R. S. Hamilton [13] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1.1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1.1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$,

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where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [3]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.2) are interesting and useful in physics and are often referred as quasi-Einstein ([4],[5]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [10] who discusses some aspects of it. Recently, the notion of almost Ricci soliton have introduced [21] by Piagoli, Riogoli, Rimoldi and Setti.

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([6], [7], [11], [14], [15], [16], [25], [24]) and many others.

In [9], during 2003-2004, Fischer developed the notion of conformal Ricci flow which is a generalization of the classical Ricci flow. The conformal Ricci flow on a $2n+1$-dimensional smooth closed connected oriented manifold M is defined by the following equation:

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{2n+1}) = -pg \tag{1.3}$$

and $r(g) = -1$, where p is a scalar non-dynamical field which depends on time, $r(g)$ is the scalar curvature of the manifold.

In 2015, Basu and Bhattacharyya [1] introduced the concept of conformal Ricci soliton by the equation

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.4}$$

where λ is constant. Conformal Ricci soliton is the generalization of Ricci soliton.

Pigola et al. first introduced [21] the notion of almost Ricci soliton in 2010. In 2014, Sharma has also studied [22] the almost Ricci soliton and has also done some gloriuos research works. Recently, in 2018, Ghosh and Patra also have studied [12] the almost Ricci solitons on contact geometry. In Riemannian manifold (M, g), almost Ricci soliton is defiend by the equation

$$\pounds_V g + 2S = 2\lambda g, \tag{1.5}$$

where λ is a smooth function on M. The almost Ricci soliton is said to be shrinking, steady or expanding according as λ is positive, zero or negative.

Recently in [8], Dutta, Basu and Bhattacharyya have introduced the notion of almost conformal Ricci soliton by

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.6}$$

where λ is a smooth function on M. The almost conformal Ricci soliton is said to be shrinking, steady or expanding according as λ is positive, zero or negative.

In the present paper, after introduction, we study Lorentzian para-Sasakian manifolds which is stated as LP-Sasakian manifolds afterwards. In section 3, we characterize almost conformal Ricci solitons on LP-Sasakian manifolds and we prove several important results.

2. LP-Sasakian manifolds

In 1989, Matsumoto [18] introduced the notion of LP-Sasakian manifolds or in short LP-Sasakian manifolds. An example of a five dimensional LP-Sasakian manifold was given by Matsumoto, Mihai and Rosaca [19].

Let M be an n-dimensional differential manifold endowed with a $(1,1)$ tensor field φ, a vector field ξ, a 1-form η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor g_p: $T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,...,+)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\varphi^2(X) = X + \eta(X)\xi,\ \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y. Then, such a structure (φ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold with the structure (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold. In the Lorentzian almost paracontact manifold M, the following relations hold [18]:

$$\varphi\xi = 0,\ \eta(\varphi X) = 0, \tag{2.3}$$

$$\Phi(X,Y) = \Phi(Y,X), \tag{2.4}$$

where $\Phi(X,Y) = g(X, \varphi Y)$. A Lorentzian almost paracontact manifold M equipped with the structure (φ, ξ, η, g) is called an LP-Sasakian manifold if

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi^2 X, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In an LP-Sasakian manifold M with the structure (φ, ξ, η, g) it is easily seen that

$$\nabla_X \xi = \varphi X, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = g(\varphi X, Y) = (\nabla_Y \eta)(X), \tag{2.7}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.10}$$

for all vector fields X, Y on M. LP-Sasakian manifolds have been studied by several authors such as ([2], [17], [20], [23]) and many others.

Definition 2.1. ([12]) A vector field V on a contact manifold is said to be an infinitesimal contact vector field if it preserve the contact form η, that is

$$\pounds_V \eta = \psi \eta, \tag{2.11}$$

for some smooth function ψ on M. When $\psi = 0$ on M, the vector field V is called a strict infinitesimal contact vector field.

3. Almost conformal Ricci solitons on LP-Sasakian manifolds

This section is devoted to study almost conformal Ricci solitons on LP-Sasakian manifolds with the potential vector field V is an infinitesimal contact vector field. Then we obtain

$$\begin{aligned}
(\pounds_V d\eta)(X, Y) &= \pounds_V d\eta(X, Y) - d\eta(\pounds_V X, Y) - d\eta(X, \pounds_V Y) \\
&= \pounds_V g(X, \phi Y) - g(\pounds_V X, \phi Y) - g(X, \phi \pounds_V Y) \\
&= \pounds_V g(X, \phi Y) - g(\pounds_V X, \phi Y) - g(X, \pounds_V \phi Y - (\pounds_V \phi)Y) \\
&= \pounds_V g(X, \phi Y) - g(\pounds_V X, \phi Y) - g(X, \pounds_V \phi Y) + g(X, (\pounds_V \phi)Y) \\
&= (\pounds_V g)(X, \phi Y) + g(X, (\pounds_V \phi)Y), \qquad (3.1)
\end{aligned}$$

for any vector fields X and Y on M.
Then using (1.6) in (3.1) we get

$$\begin{aligned}
(\pounds_V d\eta)(X, Y) &= -2S(X, \phi Y) + [2\lambda - (p + \frac{2}{2n+1})]g(X, \phi Y) \\
&\quad + g(X, (\pounds_V \phi)Y), \qquad (3.2)
\end{aligned}$$

for any smooth vector fields X and Y on M.
As V is an infinitesimal contact vector field, from (2.11) we have

$$\pounds_V d\eta = d\pounds_V \eta = (d\psi) \wedge \eta + \psi(d\eta), \tag{3.3}$$

from which it follows that

$$(\pounds_V d\eta)(X,Y) = \frac{1}{2}\{d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\} + \psi g(X,\phi Y). \tag{3.4}$$

for any vector fields X and Y on M.
In view of (3.2) and (3.4) we infer

$$\begin{aligned} &-2S(X,\phi Y) + [2\lambda - (p + \frac{2}{2n+1})]g(X,\phi Y) + g(X,(\pounds_V\phi)Y) \\ &= \frac{1}{2}\{d\psi(X)\eta(Y) - d\psi(Y)\eta(X)\} + \psi g(X,\phi Y), \end{aligned} \tag{3.5}$$

and hence we get

$$\begin{aligned} 2(\pounds_V\phi)Y &= 4Q\phi Y + 2[\psi - 2\lambda + (p + \frac{2}{2n+1})]\phi Y \\ &\quad + \eta(Y)D\psi - d\psi(Y)\xi, \end{aligned} \tag{3.6}$$

for any vector field Y on M.
Substituting $Y = \xi$ in (3.6) yields

$$2(\pounds_V\phi)\xi = D\psi - (\xi\psi)\xi. \tag{3.7}$$

The equation (1.6) can be exhibited as

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X,Y), \tag{3.8}$$

for any vector fields X and Y on M. Tracing the above equation gives

$$2divV = -2r + (2n+1)[2\lambda - (p + \frac{2}{2n+1})]. \tag{3.9}$$

Let Ω be the volume form of M, that is,

$$\Omega = \eta \wedge (d\eta)^n \neq 0. \tag{3.10}$$

Taking Lie derivative of the foregoing equation along the vector field V and using (2.11) and (3.3) yields we have

$$\begin{aligned} \pounds_V\Omega &= \psi\Omega + n[(\eta\wedge\eta)(d\eta)^{n-1}d\psi + \psi\eta\wedge(d\eta)^n] \\ &= \psi\Omega + n\psi\Omega \\ &= (n+1)\psi\Omega. \end{aligned} \tag{3.11}$$

Applying the formula [12] $\pounds_V \Omega = (\mathrm{divV})\Omega$ on (3.11) we get

$$(divV)\Omega = (n+1)\psi\Omega, \tag{3.12}$$

and hence

$$divV = (n+1)\psi. \tag{3.13}$$

With help of (3.9), from (3.13) it follows that

$$r = -(n+1)\psi + \frac{2n+1}{2}[\lambda - (p + \frac{2}{2n+1})]. \tag{3.14}$$

The equation (1.6) can be expressed as

$$(\pounds_V g)(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X,Y). \tag{3.15}$$

Putting $X = Y = \xi$ in (3.15) we infer

$$g(\pounds_V \xi, \xi) = \lambda - n - (p - 1 + \frac{2}{2n+1}). \tag{3.16}$$

Replacing Y by ξ and using (2.10) and (2.11) we find

$$g(\pounds_V \xi, X) = [\psi + 2(n-1) - 2\lambda + (p + \frac{2}{2n+1})]\eta(X), \tag{3.17}$$

from which it follows that

$$\pounds_V \xi = [\psi + 2(n-1) - 2\lambda + (p + \frac{2}{2n+1})]\xi. \tag{3.18}$$

Operating ϕ on the last equation we find that

$$\phi(\pounds_V \xi) = 0. \tag{3.19}$$

With the help of (3.19), from (3.7) we have

$$D\psi = (\xi\psi)\xi, \tag{3.20}$$

which implies that

$$d\psi = (\xi\psi)\eta. \tag{3.21}$$

Taking exterior derivative of (3.21) we get

$$d(\xi\psi) \wedge \eta + (\xi\psi)d\eta = 0. \tag{3.22}$$

Taking wedge product of the above with η we deduce that

$$d\psi(\xi) = 0. \tag{3.23}$$

Making use of (3.23) in (3.21) we get

$$d\psi = 0, \tag{3.24}$$

from which it follows that ψ is constant.
From the hypothesis ψ is constant, so integrating (3.13) and applying divergence theorem [12] we obtain

$$\psi = 0, \tag{3.25}$$

which implies that V is a stictly infinitesimal contact vector field. Hence we can state the following:

Theorem 3.1. *If an LP-Sasakian manifold admits almost conformal Ricci solitons with the potential vector field V is an infinitesimal contact vector field, then V becomes a strictly infinitesimal contact vector field.*

Using (3.18) in (3.16) and then using (3.25) we infer

$$\lambda = n - 1. \tag{3.26}$$

Therefore, we are in a position to state the following:

Theorem 3.2. *An LP-Sasakian manifold admitting almost conformal Ricci solitons with the potential vector field is an infinitesimal contact vector field are shrinking, steady or expanding according as the dimension of the manifolds is greater than* 3 *or equal to* 3 *or less than* 3.

Also from (3.26) it is clear that λ is constant and consequently the almost conformal Ricci soliton becomes conformal Ricci soliton and hence we have the following:

Theorem 3.3. *An LP-Sasakian manifold admitting an almost conformal Ricci soliton with the potential vector field is an infinitesimal contact vector field reduces to a conformal Ricci soliton.*

Using (3.25) in (3.6) we get

$$(\pounds_V \phi)Y = 2Q\phi Y + \delta\phi Y, \tag{3.27}$$

where $\delta = p + \frac{2}{2n+1} - 2\lambda$.
Also, using (3.25) in (2.11) we get

$$(\pounds_V \eta) = 0. \tag{3.28}$$

Now,

$$(\pounds_V \phi)Y = \pounds_V \phi Y - \phi(\pounds_V Y). \tag{3.29}$$

Substituting $Y = \phi Y$ in the preceding equation we deduce that

$$(\pounds_V \phi)\phi Y = \pounds_V Y - \phi(\pounds_V \phi Y). \tag{3.30}$$

Operating ϕ on (3.29) we have

$$\phi(\pounds_V \phi)Y = \phi(\pounds_V \phi Y) - \pounds_V Y - \eta(\pounds_V Y)\xi. \tag{3.31}$$

Adding the equations (3.30) and (3.31) we obtain

$$\phi(\pounds_V \phi)Y + (\pounds_V \phi)\phi Y = -\eta(\pounds_V Y)\xi. \tag{3.32}$$

Let us assume that $Q\phi = \phi Q$. With the help of the assumption, first term of (2.1) and (3.27), from (3.32) we find that

$$S(X,Y) = -\delta g(X,Y) - (\delta + n - 1)\eta(X)\eta(Y) - \eta(\pounds_V Y)\eta(X), \tag{3.33}$$

where $\delta = p + \frac{2}{2n+1} - 2\lambda$.
Putting $X = \phi^2 X$ and $Y = \phi^2 Y$ in the last equation and the using the second term of (2.2), second term of (2.3) and (2.10) we have

$$S(X,Y) = -\delta g(X,Y) - (\delta + n - 1)\eta(X)\eta(Y), \tag{3.34}$$

where $\delta = p + \frac{2}{2n+1} - 2\lambda$ and we can can state the following:

Theorem 3.4. *If an LP-Sasakian manifold admits almost conformal Ricci solitons with the potential vector field V is an infinitesimal contact vector field and the Ricci operator commutes with the structure tensor ϕ, then M is an η-Einstein manifold.*

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