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A NOTE FOR A GENERALIZATION OF THE DIFFERENTIAL EQUATION OF SPHERICAL CURVES

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Abstract. The differential equation characterizing a spherical curve in $\mathbb{R}^3$ expresses the radius of curvature of the curve in terms of its torsion. In this paper, we have given a generalization of this equation for a curve lying in an arbitrary surface in $\mathbb{R}^3$. Moreover, we have established the analogue of the Frenet equations for a curve lying in a surface of $\mathbb{R}^3$. We have also revisited some formulas for the geodesic torsion of a curve lying in a surface of $\mathbb{R}^3$.
Keywords: spherical curves, differential geometry, Frenet equations.

1. Introduction

The curves to be considered here are curves in the Euclidean space $\mathbb{R}^3$ of the form $\alpha = \alpha(s)$, $s \in [0, L]$, where s is the arc length which is of class C^3. For such a curve, the following facts are well known.

There exists two functions κ, τ defined on $[0, L]$ that determine completely the shape of the curve in $\mathbb{R}^3$. The functions κ and τ are respectively the curvature and the torsion of the curve. Such a curve $\alpha : [0, L] \longrightarrow \mathbb{R}^3$ have a Frenet frame (T, N, B) which is a map on $[0, L]$, $s \longmapsto (T(s), N(s), B(s))$ that satisfies the Frenet

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equations

$$\left\{\begin{array}{lll} T' & = & \kappa N \\ N' & = & -\kappa T - \tau B \\ B' & = & \tau N \end{array}\right. , \tag{1.1}$$

where the prime ($'$) denotes the differentiation with respect to arc length. For more information see [1, 3].

The condition for a curve to be a spherical curve, (i.e) it lies on a sphere, is usually given in form

$$\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' + \frac{\tau}{\kappa} = 0. \tag{1.2}$$

One can ask what the analogous of the equation (1.2) is when the curve is assumed to be in an arbitrary surface in $\mathbb{R}^3$. One of the aims is to give an answer to this question.

When a curve such as the above mentioned is assumed to lie in a given surface $\Sigma \subset \mathbb{R}^3$, then there exists two other invariants κ_n and τ_g defined on $[0, L]$ which are unique except for the sign (depending on the orientation of Σ). The functions κ_n and τ_g defined on $[0, L]$ are the normal curvature and the geodesic curvature of the curve.

Let Σ be a surface on $\mathbb{R}^3$. We will assume that Σ is oriented by choice of a unit normal field

$$\xi : \Sigma \longrightarrow S^2. \tag{1.3}$$

For a curve $\alpha : [0, L] \longrightarrow \mathbb{R}^3$ given as above, and lying in Σ, there are two naturel frames along α (see [1]). The first is Frenet frame (T, N, B) given above. For the second, let denoted by $\xi = \xi(s)$ be the restriction of ξ on α; and we consider the second frame $(T, \xi \times T, \xi)$ where $\times$ is the vector product in $\mathbb{R}^3$. These two frames (T, N, B) and $(T, \xi \times T, \xi)$ are the positively oriented in $\mathbb{R}^3$ as we will see later.

In [2] it is shown that the differential equation characterizing a spherical curve can be solved explicitly to express the radius of curvature of the curve in terms of its torsion. The author of [6] gives a necessary condition for a curve to be a spherical curve. In Minkowski space the characterization of curve lying on pseudohyperbolical space and Lorentzian hypersphere are stated both depending on curvature functions and character of Serret-Frenet frame of the curve, respectively. For detail see [4, 5, 7]. The main results of this paper is to prove the following results.

Theorem 1.1. *Under the assumptions and notations above, we have the following*

i) the trihedron $(T, \xi, T \times \xi)$ and the functions κ, τ, κ_n and τ_g satisfy the following equation

$$\left\{\begin{array}{lll} T' & = & \kappa_n \xi + \sqrt{\kappa^2 - \kappa_n^2}(\xi \times T) \\ \xi' & = & -\kappa_n T + \tau_g(\xi \times T) \\ (T \times \xi)' & = & -\sqrt{\kappa^2 - \kappa_n^2} T - \tau_g(\xi \times T) \end{array}\right. , \tag{1.4}$$

ii)

$$\left(\frac{\kappa_n}{\kappa}\right)' = -(\tau - \tau_g)\sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2} \tag{1.5}$$

iii)

$$\tau_g^2 = -(K - 2H\kappa_n + \kappa_n^2) \tag{1.6}$$

where K and H are respectively the restriction of mean curvature and the Gauss curvature of Σ to α.

Corollary 1.1. *If the curve α lying in a sphere with τ and κ' are nowhere zero in $[0, L]$, then equation* (1.5) *implies* (1.2).

The paper is organized as follows: in Section 2, we recall some results and definitions which we use for the proof of our main results. In Section 3, we prove the main results of this paper.

2. Preliminaries

Let $\alpha = \alpha(s)$, $s \in [0, L]$ be a regular curve of classe C^3 lying on an oriented surface Σ in $\mathbb{R}^3$. An orientation of Σ is determined by a choice of a unit normal $\xi : \Sigma \longrightarrow S^2$.

If $p \in \Sigma$, a basis (u, v) of $T_p\Sigma$ is positively oriented if $(u, v, \xi(p))$ is a positive basis of $\mathbb{R}^3$. A basis of $\mathbb{R}^3$ of the form $(u, v, u \times v)$ is positively oriented. So the Frenet frame $(T(s), N(s), B(s))$ on α is positively oriented at every $s \in [0, L]$. The second frame $(T(s), \xi(s) \times T(s), \xi(s))$, $s \in [0, L]$ considered above have the same orientation that the basis $(\xi(s), T(s), \xi(s) \times T(s))$, $s \in [0, T]$. Therefore, on α the "trihedron" (T, N, B) and $(T, \xi \times T, \xi)$ are positively oriented.

For each $s \in [0, L]$, we define the angle $\theta = \theta(s)$ between $N(s)$ and $\xi(s)$ by

$$\langle N(s), \xi(s)\rangle = \cos\theta(s). \tag{2.1}$$

And we have the following relation

$$N(s) = \cos\theta(s)\xi(s) + \sin\theta(s)(\xi(s) \times T(s)), \quad s \in [0, T]. \tag{2.2}$$

Now let us recall some basic facts for a curve $\alpha = \alpha(s)$ given as above and lying on a surface $\Sigma \subset \mathbb{R}^3$.

If p is a point of Σ, the Gauss map $\xi : \Sigma \longrightarrow S^2$ is a differential map and its differential $d_p\xi$ at p is a self-adjoint endomorphism of $T_p\Sigma$. The fact that $d_p\xi : T_p\Sigma \longrightarrow T_p\Sigma$ is a self-adjoint map allows to associate a quadratic form Π_p in T_pS. The quadratic form Π_p is defined on $T_p\Sigma$ by

$$\Pi_p(v) = -\langle d_p\xi(v), v\rangle \tag{2.3}$$

is called the second fundamental form of Σ at p.

Definition 2.1. A curve α in Σ passing through p, κ the curvature of α at p and $\cos\theta = \langle N, \xi\rangle$, where N is the normal vector of α at p; the number

$$\kappa_n = \kappa \cos\theta \tag{2.4}$$

is called the normal curvature of $\alpha \in \Sigma$ at p.

If $p = p(s) \in \Sigma$, the following interpretation of Π_p is well known:

$$\begin{aligned} \Pi_p\Big(\alpha'(s)\Big) &= -\langle d_p\xi\Big(\alpha'(s)\Big), \alpha'(s)\rangle \\ &= -\langle \xi'(s), \alpha'(s)\rangle \\ &= \langle N(s), \alpha''(s)\rangle \qquad (2.5)\\ &= \langle N(s), \kappa N\rangle(p) = \kappa_n(p) \qquad (2.6) \end{aligned}$$

In the other words, the value of the second fundamental form Π_p at a unit vector $v \in T_p\Sigma$ is equal to the normal curvature of a regular curve passing through p and tangent to v.

Now let us come back to the linear map $d_p\xi$. It is known that for each $p \in \Sigma$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p\Sigma$ such that $d_p\xi(e_1) = -k_1e_1$, $d_p\xi(e_2) = -k_2e_2$. Moreover, k_1 and k_2 ($k_1 \geq k_2$) are the maximum and the minimum of the second fundamental form Π_p restricted to the unit circle of $T_p\Sigma$. That is, they are the extreme values of the normal curvature at p.

The point $p \in \Sigma$ is called an umbilic point if $k_1(p) = k_2(p)$.

Definition 2.2. In terms of the principal curvatures k_1, k_2, the Gauss curvature K and the mean curvature H are given by:

$$K = k_1k_2 \qquad H = \frac{k_1 + k_2}{2}. \tag{2.7}$$

3. Proof of the main results

3.1. Proof of the theorem

For three vectors $u, v, w \in \mathbb{R}^3$, the following formulas will be used:

$$u \times (v \times w) = \langle u, w\rangle v - \langle u, v\rangle w. \tag{3.1}$$

And for an orthonormal positive oriented basis (u, v, w) in $\mathbb{R}^3$, the following relations

$$u \times v = w, \qquad w \times u = v, \tag{3.2}$$

will be also used.

Now assume that for $s \in [0, L]$, $\alpha(s)$ lies in a surface Σ. For the geodesic torsion τ_g of α at $p = \alpha(s)$, $s \in]0, L[$ we have the well known two formulas:

$$\tau_g(s) = \tau - \frac{d\theta}{dt} = \cos\phi \sin\phi (k_1 - k_2), \tag{3.3}$$

where τ is the torsion of α, θ is the angle between $\xi(s)$ and $N(s)$, ϕ is the angle that T makes with the principal direction e_1 and k_1, k_2 are principal curvatures associated with the orthonormal basis $\{e_1, e_2\}$ (assumed to be positively oriented in $T_p\Sigma$).

Here we will use another formulas for τ_g with is given in the lemma below.

Lemma 3.1. *In the notations given above, we have*

$$\tau_g(s) = \langle \xi'(s), \xi \times T \rangle, \quad s \in]0, L[. \tag{3.4}$$

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of $T_p\Sigma$ such that

$$d_p\xi(e_1) = -k_1 e_1, \quad d_p\xi(e_2) = -k_2 e_2.$$

where $p = \alpha(s)$. We can assume that $e_1 \times e_2 = \xi(s)$; thus $(e_1, e_2, \xi(s))$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. We put $T = \cos\varphi e_1 + \sin\varphi e_2$ and we have

$$\begin{aligned}
\langle \xi'(s), \xi \times T \rangle &= \langle d_p\xi(T), \xi \times T \rangle \\
&= \langle -\cos\varphi k_1 e_1 - \sin\varphi k_2 e_2, \xi \times (\cos\varphi e_1 + \sin\varphi e_2) \rangle \\
&= \langle -\cos\varphi k_1 e_1 - \sin\varphi k_2 e_2, -\sin\varphi e_1 + \cos\varphi e_2) \rangle \\
&= \cos\varphi \sin\varphi (k_1 - k_2).
\end{aligned}$$

This show (3.4) by (3.3). □

Let us show (i) in Theorem 1.1.
For convenience, we will drop the point $p = \alpha(s) \in \Sigma$ in the formulas.
- From θ defined by $\cos\theta = \langle \xi, N \rangle$ the normal N which is normal to T becomes

$$N = \cos\theta\xi + \sin\theta T \times \xi,$$

and

$$\begin{aligned}
T' &= \kappa N \\
&= \kappa \cos\theta\xi + \kappa \sin\theta\xi \times T \\
&= \kappa_n \xi + \kappa\sqrt{1 - \cos^2\theta}\xi \times T \\
&= \kappa_n \xi + \sqrt{\kappa^2 - \kappa_n^2}\xi \times T.
\end{aligned}$$

- Since $\langle \xi, \xi \rangle = 1$, then $\xi' = aT + bT \times \xi$ for some numbers a and b.
We have

$$\begin{aligned}
a &= \langle \xi', T \rangle \\
&= \langle \xi, T \rangle' - \langle \xi, T' \rangle \\
&= -\kappa \langle \xi, N \rangle \\
&= -\kappa \cos\theta \\
&= -\kappa_n
\end{aligned}$$

and by (3.4) we get

$$b = \langle \xi', \xi \times T \rangle = \tau_g.$$

Thus we get $\xi' = -\kappa_n T + \tau_g \xi \times T$.
- We have $(\xi \times T)' = cT + d\xi$ for some constants c and d. We get

$$\begin{aligned} c &= \langle (\xi \times T)', T \rangle \\ &= \langle \xi \times T, T \rangle' - \langle \xi \times T, T' \rangle \\ &= -\kappa \langle \xi \times T, N \rangle \\ &= -\kappa \langle \xi \times T, \cos\theta \xi + \sin\theta T \times \xi \rangle \\ &= -\kappa \sin\theta \\ &= -\sqrt{\kappa^2 - \kappa_n^2}. \end{aligned}$$

and by (3.4), we get

$$d = \langle (\xi \times T)', \xi \rangle = \langle (\xi \times T), \xi \rangle' - \langle \xi \times T, \xi' \rangle = -\tau_g.$$

Thus $(\xi \times T)' = -\sqrt{\kappa^2 - \kappa_n^2} T - \tau_g \xi$.
This show the (i) of the theorem.

Let us show (ii) in Theorem 1.1.

We have $\frac{\kappa_n}{\kappa} = \cos\theta$. Differentiating this relation, we get

$$\begin{aligned} \left(\frac{\kappa_n}{\kappa}\right)' &= -\frac{d\theta}{dt} \sin\theta \\ &= -(\tau - \tau_g)\sqrt{1 - \cos^2\theta} \\ &= -(\tau - \tau_g)\sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2}. \end{aligned}$$

This show (ii).

Let us show (iii) in Theorem 1.1.

Let $\{e_1, e_2\}$ be the unit orthonormal basis of $T_p\Sigma$ such that $d_p\xi(e_1) = -k_1 e_1$ and $d_p\xi(e_2) = -k_2 e_2$ as the recalls in section 2. And let φ be defined by $\cos\varphi = \langle e_1, T \rangle$; and then we can write $T = \cos\varphi e_1 + \sin\varphi e_2$, under the assumption that $e_1 \times e_2 = \xi$, i.e (e_1, e_2, ξ) is a positive oriented basis of $\mathbb{R}^3 = T_p\mathbb{R}^3$.
We have $\xi' = -\kappa \cos\theta T + \tau_g \xi \times T$ by (i). Also we have

$$\begin{aligned} \xi' &= d_p\xi(T) \\ &= -\cos\varphi k_1 e_1 - \sin\varphi k_2 e_2. \end{aligned} \tag{3.5}$$

Thus

$$\begin{aligned} \xi' &= -\kappa\cos\theta T + \tau_g T \times \xi \\ &= -\kappa\cos\theta(\cos\varphi e_1 + \sin\varphi e_2) + \tau_g(-\cos\varphi e_2 + \sin\varphi e_1) \\ &= (-\kappa\cos\theta\cos\varphi + \sin\varphi\tau_g)e_1 + (\kappa\cos\theta\sin\varphi - \tau_g\cos\varphi)e_2. \end{aligned} \tag{3.6}$$

By the computation given in (3.5) and (3.6) above one gets easily that

$$\begin{cases} (k_1 - \kappa\cos\theta)\cos\varphi + \tau_g \sin\varphi = 0 \\ (k_2 - \kappa\cos\theta)\sin\varphi + \tau_g \cos\varphi = 0 \end{cases}.$$

By writing the last relation in matrix form:

$$\begin{pmatrix} k_1 - \kappa\cos\theta & -\tau_g \\ \tau_g & k_2 - \kappa\cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

one gets the determinant

$$\begin{vmatrix} k_1 - \kappa\cos\theta & -\tau_g \\ \tau_g & k_2 - \kappa\cos\theta \end{vmatrix} = 0$$

$\Rightarrow k_1k_2 - \kappa\cos\theta(k_1 + k_2) + \kappa^2\cos^2\theta + \tau_g^2 = 0$
$\Rightarrow K - 2\kappa_n H + \kappa_n^2 + \tau_g^2 = 0.$
Thus we have

$$\tau_g^2 = -(K - 2H\kappa_n + \kappa_n^2).$$

This shows (iii). So the theorem is proved.

3.2. Proof of the corollary

We assume that α lies in a sphere in $\mathbb{R}^3$ of radius R. We consider the equation (ii):

$$\left(\frac{\kappa_n}{\kappa}\right)' = -(\tau - \tau_g)\sqrt{1 - \left(\frac{\kappa_n}{\kappa}\right)^2}.$$

It is well known that, on a sphere every point is an umbilic point. This fact is important in the proof that on the sphere the second fundamental form is a constant (see [8]). That is, for any unit tangent vector v at $p = \alpha(s)$ belong to this sphere we have $\Pi_p(v) = \pm\frac{1}{R}$ and the Gauss curvature K and mean curvature H are constants $(K = \frac{1}{R^2}, H = \pm\frac{1}{R})$. This shows that the geodesic curvature τ_g of α is zero.
Thus the equation (ii) becomes

$$\pm\frac{1}{R}\left(\frac{1}{\kappa}\right)' = -\tau\sqrt{1 - \frac{1}{R^2\kappa^2}},$$

that implies

$$\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 + \left(\frac{1}{\kappa}\right)^2 = R^2.$$

By differentiating this equation and by using $\kappa' \neq 0$, one gets easily (ii). This shows the corollary.

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