UNIVERSITY OF NIŠ

ISSN 0352-9665 (Print)
ISSN 2406-047X (Online)
COBISS.SR-ID 5881090

FACTA UNIVERSITATIS

Series
MATHEMATICS AND INFORMATICS
Vol. 33, No 2 (2018)

Scientific Journal **FACTA UNIVERSITATIS**
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The cover image taken from http://www.pptbackgrounds.net/binary-code-and-computer-monitors-backgrounds.html.
Publication frequency – one volume, five issues per year.
Published by the University of Niš, Republic of Serbia

This publication was in part supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia
Printed by "UNIGRAF-X-COPY" – Niš, Republic of Serbia

ISSN 0352 – 9665 (Print)
ISSN 2406 – 047X (Online)
COBISS.SR-ID 5881090

FACTA UNIVERSITATIS

SERIES **MATHEMATICS AND INFORMATICS**
Vol. 33, No 2 (2018)

UNIVERSITY OF NIŠ

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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 2 (2018), 141–152
https://doi.org/10.22190/FUMI1802141H

TOTALLY REAL SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS

Shyamal Kumar Hui[†] **and Tanumoy Pal**

Abstract. The present paper deals with the study of totally real submanifolds and C-totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection. It is proved that the scalar curvatures of C-totally real submanifolds of $(LCS)_n$-manifold with respect to both the said connections are the same.

Keywords: $(LCS)_n$-manifold, totally real submanifold, quarter symmetric metric connection.

1. Introduction

As a generalization of LP-Sasakian manifold, Shaikh [13] recently introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example. Such manifolds have many applications in the general theory of relativity and cosmology ([15], [16]).

The notion of semisymmetric linear connection on a smooth manifold was introduced by Friedmann and Schouten [4]. Then Hayden [6] introduced the idea of metric connection with torsion on a Riemannian manifold. Thereafter Yano [19] studied the semisymmetric metric connection on a Riemannian manifold systematically. As a generalization of the semisymmetric connection, Golab [5] introduced the idea of quarter symmetric linear connection on smooth manifolds. A linear connection $\bar{\nabla}$ in an n-dimensional smooth manifold $\tilde{M}$ is said to be a quarter symmetric connection [5] if its torsion tensor T is of the form

$$\begin{aligned} T(X,Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned} \tag{1.1}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In particular, if $\phi X = X$ then the quarter symmetric connection reduces to a semisymmetric connection. Further,

Received November 13, 2017; accepted March 12, 2018
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25
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if the quarter symmetric connection $\bar{\nabla}$ satisfies the condition $(\bar{\nabla}_X g)(Y, Z) = 0$, for all $X,\ Y,\ Z \in \chi(\tilde{M})$, then $\bar{\nabla}$ is said to be a quarter symmetric metric connection.

Due to important applications in applied mathematics and theoretical physics, the geometry of submanifolds has become a subject of growing interest. Analogous to almost Hermitian manifolds, the invariant and anti-invariant submanifolds [2] are dependent on the behaviour of almost contact metric structure ϕ. A submanifold M of a $(LCS)_n$-manifold manifold $\tilde{M}$ is said to be anti-invariant (or totally real) if for any $X \in T(M)$, $\phi X \in T^{\perp}M$ i.e., $\phi(TM) \subset T^{\perp}M$ at every point of M. A totally real submanifold M of $\tilde{M}$ is a C-totally real submanifold if ξ is normal to M [18]. Consequently, C-totally real submanifolds are anti-invariant. Recently Hui et al. ([1], [7], [8], [9], [17]) studied submanifolds of $(LCS)_n$-manifolds. The present paper deals with the study of totally real submanifolds and C-totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection. It is shown that the scalar curvature of a C-totally real submanifold of $(LCS)_n$-manifold with respect to the Levi-Civita connection and the quarter symmetric metric connection is the same. However, in the case of totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita connection and the quarter symmetric metric connection, they are different. An inequality for the square length of the shape operator in the case of a totally real submanifold of $(LCS)_n$-manifold is derived. The equality case is also considered.

2. Preliminaries

Let $\tilde{M}$ be an n-dimensional Lorentzian manifold [12] admitting a unit time-like concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

satisfies [20]

$$(\tilde{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0, \tag{2.3}$$

$$\tilde{\nabla}_X \xi = \alpha\{X + \eta(X)\xi\}, \quad \alpha \neq 0, \tag{2.4}$$

for $X,\ Y \in \chi(\tilde{M})$, where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function that satisfies

$$\tilde{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. Let us take

$$\phi X = \frac{1}{\alpha}\tilde{\nabla}_X \xi, \tag{2.6}$$

then from (2.4) and (2.6), we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.8}$$

from which it follows that ϕ is a symmetric (1,1) tensor called the structure tensor of the manifold. Thus the Lorentzian manifold $\tilde{M}$ together with the unit time-like concircular vector field ξ, its associated 1-form η and a (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [13]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [11]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold ([13], [14]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.9}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.10}$$

$$\tilde{R}(X, Y)Z = \phi\tilde{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \tag{2.11}$$

for all $X,\ Y,\ Z \in \chi(\tilde{M})$. Using (2.8) in (2.11), we get

$$\begin{aligned}\tilde{R}(X, Y, Z, W) &= \tilde{R}(X, Y, Z, \phi W) + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) \\ &\quad -g(X, Z)\eta(Y)\}\eta(W).\end{aligned} \tag{2.12}$$

Let M be a submanifold of dimension m of a $(LCS)_n$-manifold $\tilde{M}$ ($m < n$) with induced metric g. Also, let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.13}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.14}$$

for all $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are the second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\tilde{M}$ and they are related by [21]

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.15}$$

for any $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. The equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W)+g(h(X,Z),h(Y,W))-g(h(X,W),h(Y,Z)) \tag{2.16}$$

for any vectors $X,\ Y,\ Z,\ W$ tangent to M.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space $\tilde{M}$ such that refracting to M^m, $\{e_1, e_2, \cdots, e_m\}$ is the orthonormal basis to the tangent space T_xM with respect to the induced connection.
We write

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad i, j \in \{1, 2, \cdots, m\} \text{ and } r \in \{m + 1, \cdots, n\}.$$

Then the square length of h is

$$||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j))$$

and the mean curvature H of M associated to ∇ is $H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i)$.

The quarter symmetric metric connection $\bar{\tilde{\nabla}}$ and the Riemannian connection $\tilde{\nabla}$ on a $(LCS)_n$-manifold $\tilde{M}$ are related by [10]

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.17}$$

If $\bar{\tilde{R}}$ and $\tilde{R}$ are the curvature tensors of an $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection $\bar{\tilde{\nabla}}$ and the Riemannian connection $\tilde{\nabla}$, then

$$\begin{aligned} \bar{\tilde{R}}(X,Y,Z,W) = {} & \tilde{R}(X,Y,Z,W) + (2\alpha - 1)[g(\phi X, Z)g(\phi Y, W) \\ & -g(\phi Y, Z)g(\phi X, W)] + \alpha[\eta(Y)g(X, W) \\ & -\eta(X)g(Y, W)]\eta(Z) + \alpha[g(Y, Z)\eta(X) \\ & -g(X, Z)\eta(Y)]\eta(W) \end{aligned} \tag{2.18}$$

for all $X,\ Y,\ Z,\ W \in \chi(\tilde{M})$.
We now recall the following [3]:

Let L be a k-plane section of T_xM and X be a unit vector in L. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_k\}$ of L such that $e_1 = X$. Then the Ricci curvature Ric_L of L at X is defined by [3]

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{2.19}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. Such a curvature is called a k-Ricci curvature.

The scalar curvature τ of the k-plane section L is given by [3]

$$\tau(L) = \sum_{1\le i<j\le k} K_{ij}. \tag{2.20}$$

For each integer k, $2 \leq k \leq n$, the invariant Θ_k on M is defined by [3]

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L.X} Ric_L(X), \qquad x \in M, \tag{2.21}$$

where L runs over all k-plane sections in T_xM and X runs over all unit vectors in L.
The relative null space for M at a point $x \in M$ is defined by [3]

$$\mathcal{N}_x = \{X \in T_xM | h(X, Y) = 0, Y \in T_xM\}. \tag{2.22}$$

3. Theorem-like Environments

This section deals with the study of totally real submanifolds of $(LCS)_n$-manifolds with respect to the Levi-Civita and quarter symmetric metric connection. We prove the following:

Theorem 3.1. *Let M be a totally real submanifold of dimension m $(m < n)$ of a $(LCS)_n$-manifold $\tilde{M}$. Then*

$$m^2||H||^2 = 2\tau + ||h||^2 + (m-1)(\alpha^2 - \rho), \tag{3.1}$$

where τ is the scalar curvature of M.

Proof. Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Now from (2.12) and (2.16), we get

$$\begin{aligned} R(X,Y,Z,W) \;=\; & \tilde{R}(X,Y,Z,\phi W) + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) \\ & -g(X,Z)\eta(Y)\}\eta(W) + g(h(X,W), h(Y,Z)) \\ & -g(h(X,Z), h(Y,W)) \end{aligned} \tag{3.2}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(TM)$.
Since M is a totally real submanifold i.e., anti-invariant, so

$$\tilde{R}(X,Y,Z,\phi W) = g(\tilde{R}(X,Y)Z, \phi W) = 0$$

as $\tilde{R}(X,Y)Z$ is tangent to M and ϕW is normal to M and hence (3.2) yields

$$\begin{aligned} R(X,Y,Z,W) \;=\; & (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(W) \\ & +g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)) \end{aligned} \tag{3.3}$$

for any $X,\ Y,\ Z,\ W \in \Gamma(TM)$. Putting $X = W = e_i$ and $Y = Z = e_j$ in (3.3) and taking summation over $1 \leq i < j \leq m$, we get

$$\begin{aligned} \sum_{1\leq i<j\leq m} R(e_i, e_j, e_j, e_i) \;=\; & (\alpha^2 - \rho) \sum_{1\leq i<j\leq m} [g(e_j, e_j)\eta(e_i)\eta(e_i) - g(e_i, e_j)\eta(e_j)\eta(_j)] \\ & + \sum_{1\leq i<j\leq m} g(h(e_i, e_i), h(e_j, e_j)) \\ & - \sum_{1\leq i<j\leq m} g(h(e_i, e_j), h(e_j, e_i)) \end{aligned}$$

i.e.,

$$2\tau = -(m-1)(\alpha^2 - \rho) + m^2||H||^2 - ||h||^2, \tag{3.4}$$

which implies (3.1). □

Corollary 3.1. *Let M be a C-totally real submanifold of dimension m $(m < n)$ of a $(LCS)_n$-manifold $\tilde{M}$. Then*

$$m^2||H||^2 = 2\tau + ||h||^2.$$

Proof. In a C-totally real submanifold, since $\xi \in \Gamma(T^{\perp}M)$ so, $\eta(X) = 0$ for all $X \in \Gamma(TM)$. Then (3.3) yields

$$R(X,Y,Z,W) \quad = \quad g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),$$

from which, similarly to the above, we can prove that $m^2||H||^2 = 2\tau + ||h||^2$. □

Now let M be a submanifold of dimension m $(m < n)$ of a $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection $\bar{\bar{\nabla}}$ and $\bar{\nabla}$ be the induced connection of M associated to the quarter symmetric metric connection. Also let $\bar{h}$ be the second fundamental form of M with respect to $\bar{\nabla}$. Then the Gauss formula can be written as

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X,Y) \tag{3.5}$$

and hence by virtue of (2.13) and (2.17), we get

$$\bar{\nabla}_X Y + \bar{h}(X,Y) \quad = \quad \nabla_X Y + h(X,Y) + \eta(Y)\phi X - g(\phi X, Y)\xi \tag{3.6}$$

If M is a totally real submanifold of $\tilde{M}$ then $\phi X \in T^{\perp}M$ for any $X \in TM$ and hence $g(\phi X, Y) = 0$ for $X,\ Y \in TM$. So, equating the normal part from (3.6), we get

$$\bar{h}(X,Y) = h(X,Y) + \eta(Y)\phi X. \tag{3.7}$$

Further, if M is C-totally real submanifold of $\tilde{M}$ then $\xi \in T^{\perp}M$ and hence $\eta(Y) = 0$ for all $Y \in TM$. So, (3.7) yields

$$\bar{h}(X,Y) = h(X,Y). \tag{3.8}$$

Let U be a unit tangent vector at $x \in \tilde{M}$ and $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space $\tilde{M}$ such that $e_1 = U$ refracting to M^m, $\{e_1, e_2, \cdots, e_m\}$ is the orthonormal basis to the tangent space T_xM with respect to the induced quarter symmetric metric connection. Then we have the following:

Theorem 3.2. *Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection, then*

$$m^2||H||^2 = 2\bar{\tau} + ||h||^2 + (2m-1)\alpha + m\alpha\eta^2(U), \tag{3.9}$$

where $\bar{\tau}$ is the scalar curvature of M with respect to the induced connection associated to the quarter symmetric metric connection.

Proof. In the case of an $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection, the relation (2.16) becomes

$$\begin{aligned}\bar{\tilde{R}}(X,Y,Z,W) &= \bar{R}(X,Y,Z,W) + g(\bar{h}(X,Z),\bar{h}(Y,W)) \\ &\quad - g(\bar{h}(X,W),\bar{h}(Y,Z)).\end{aligned} \tag{3.10}$$

In view of (2.7) and (2.8), (3.10) yields

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= \tilde{R}(X,Y,Z,\phi W) + (\alpha^2-\rho)\{g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)\}\eta(W) + (2\alpha-1)[g(\phi X,Z)g(\phi Y,W) \\ &\quad - g(\phi Y,Z)g(\phi X,W)] + \alpha[\eta(Y)g(X,W) \\ &\quad - \eta(X)g(Y,W)]\eta(Z) + \alpha[g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)]\eta(W) \\ &\quad + g(\bar{h}(X,W),\bar{h}(Y,Z)) - g(\bar{h}(X,Z),\bar{h}(Y,W)).\end{aligned} \tag{3.11}$$

Since M is totally real, therefore $g(\phi X, Y) = 0$ for all $X,\ Y \in TM$ and (3.7) holds. Thus (3.11) becomes

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= (\alpha^2-\rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\eta(W) \\ &\quad + \alpha[\eta(Y)g(X,W) - \eta(X)g(Y,W)]\eta(Z) \\ &\quad + \alpha[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\eta(W) \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \\ &\quad - \eta(Z)g(h(X,W),\phi Y) - \eta(W)g(\phi X, h(Y,Z)) \\ &\quad + \eta(Z)g(\phi X, h(Y,W)) + \eta(W)g(h(X,Z),\phi Y).\end{aligned} \tag{3.12}$$

Putting $X = W = e_i$ and $Y = Z = e_j$ in (3.12) and taking summation over $1 \le i < j \le m$, we get

$$\begin{aligned}2\bar{\tau} &= -(m-1)(\alpha^2-\rho) - \alpha(1+\eta^2(U))m - \alpha(m-1) \\ &\quad + m^2||H||^2 - ||h||^2,\end{aligned} \tag{3.13}$$

from which (3.9) follows. □

Corollary 3.2. *Let M be a C-totally real submanifold of an $(LCS)_n$-manifold $\tilde{M}$ with respect to the quarter symmetric metric connection. Then*

$$m^2||H||^2 = 2\bar{\tau} + ||h||^2. \tag{3.14}$$

Proof. If M is a C-totally real submanifold then $\eta(Y) = 0$ for all $Y \in TM$ and hence (3.12) implies that

$$\bar{R}(X,Y,Z,W) = g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \tag{3.15}$$

from which, similarly to the above, (3.14) follows. □

From Corollary 3.1 and Corollary 3.2 we get $\tau = \bar{\tau}$ i.e., the scalar curvatures of a C-totally real submanifold of a $(LCS)_n$-manifold with respect to the induced Levi-Civita connection and the induced quarter symmetric metric connection are identical. Thus we can state the following:

Theorem 3.3. *Let M be a C-totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then the scalar curvatures of M with respect to the induced Levi-Civita connection and induced quarter symmetric metric connection are the same.*

Next, we prove the following:

Theorem 3.4. *Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then*
(i) for each unit vector $X \in T_xM$,

$$4Ric(X) \leq m^2||H||^2 + 2(\alpha^2-\rho)(m-2) + 4(m-2)(\alpha^2-\rho)\eta^2(X); \tag{3.16}$$

(ii) in the case of $H(x)$=0, a unit tangent vector X at x satisfies the equality case of (3.16) if and only if X lies in the relative null space $\mathcal{N}_x$ at x.
(iii) the equality case of (3.16) holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point or $m = 2$ and x is a totally umbilical point.

Proof. Let $X \in T_xM$ be a unit tangent vector at x. We choose an orthonormal basis $\{e_1, e_2, \cdots, e_m, e_{m+1}, \cdots, e_n\}$ such that $\{e_1, \cdots, e_m\}$ are tangent to M at x and $e_1 = X$. Then from (3.1), we have

$$\begin{aligned} m^2\|H\|^2 &= 2\tau + \sum_{r=m+1}^{n}\{(h^r_{11})^2 + (h^r_{22} + \cdots + h^r_{mm})^2)\} \\ &\quad -2\sum_{r=m+1}^{n}\sum_{2\leq i<j\leq n} h^r_{ii}h^r_{jj} + (m-1)(\alpha^2-\rho) \\ &= 2\tau + \frac{1}{2}\sum_{r=m+1}^{n}\{(h^r_{11} + \cdots + h^r_{mm})^2 + (h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2\} \qquad (3.17) \\ &\quad +2\sum_{r=m+1}^{n}\sum_{i<j}(h^r_{ij})^2 - 2\sum_{r=m+1}^{n}\sum_{2\leq i<j\leq n} h^r_{ii}h^r_{jj} + (m-1)(\alpha^2-\rho). \end{aligned}$$

From the equation of Gauss, we find

$$K_{ij} = \sum_{r=m+1}^{n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + (\alpha^2 - \rho)\eta^2(e_i),$$

and consequently

$$\sum_{2\le i<j\le m} K_{ij} = \sum_{r=m+1}^{n} \sum_{2\le i<j\le m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + (\alpha^2 - \rho)[m - 2 + \eta^2(X)]. \tag{3.18}$$

Using (3.18) in (3.17), we get

$$\begin{aligned} m^2\|H\|^2 \ \ge\ & 2\tau + \frac{m^2}{2}\|H\|^2 + 2\sum_{r=m+1}^{n}\sum_{j=2}^{m}(h_{1j}^r)^2 - 2\sum_{2\le i<j\le m} K_{ij} \\ & -(m-3)(\alpha^2-\rho) - 2(m-2)(\alpha^2-\rho)\eta^2(X). \end{aligned} \tag{3.19}$$

Therefore,

$$\frac{1}{2}m^2\|H\|^2 \ge 2Ric(X) - (m-3)(\alpha^2-\rho) - 2(m-2)(\alpha^2-\rho)\eta^2(X),$$

from which we get (3.16).

Let us assume that $H(x) = 0$. Then the equality holds in (3.16) if and only if

$$h_{11}^r = h_{22}^r = \cdots = h_{1m}^r = 0 \text{ and } \quad h_{11}^r = h_{22}^r + \cdots + h_{mm}^r, \quad r \in \{m+1, \cdots, n\}.$$

Then $h_{1j}^r = 0$ for every $j \in \{1, \cdots m\}, r \in \{m+1 \cdots n\}$, i.e., $X \in \mathcal{N}_x$.

(iii) The equality case of (3.16) holds for every unit tangent vector at x if and only if

$$h_{ij}^r = 0, i \ne j \text{ and } h_{11}^r + h_{22}^r + \cdots + h_{mm}^r - 2h_{ii}^r = 0.$$

We distinguish two cases:
(a) $m \ne 2$, then x is a totally geodesic point;
(b) $m = 2$, it follows that x is a totally umbilical point.
The converse is trivial. □

Next we obtain the following:

Theorem 3.5. *Let M be a totally real submanifold of a $(LCS)_n$-manifold $\tilde{M}$. Then*

$$\|H\|^2 \ge \frac{2\tau}{m(m-1)} + \frac{1}{m}(\alpha^2 - \rho). \tag{3.20}$$

Proof. We choose an orthonormal basis $\{e_1, \cdots e_m, e_{m+1}, \cdots, e_n\}$ at x such that e_{m+1} is parallel to the mean curvature vector $H(x)$, and $e_1, \cdots, e_m$ diagonalise the shape operator A_{m+1}. Then the shape operator takes the form

$$A_{m+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{3.21}$$

$A_r = (h^r_{ij}), \quad i,j = 1, \cdots, m; r = m+2, \cdots, n, \qquad traceA_r = \sum_{i=1}^{m} h^r_{ii} = 0$

and from (3.1), we get

$$m^2||H||^2 = 2\tau + \sum_{i=1}^{m} a_i^2 + \sum_{r=m+2}^{n} \sum_{i,j=1}^{m} (h^r_{ij})^2 + (m-1)(\alpha^2 - \rho). \tag{3.22}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (m-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j, \tag{3.23}$$

we obtain

$$m^2||H||^2 = \left(\sum_{i=1}^{m} a_i\right)^2 + 2\sum_i a_i^2 - 2\sum_{i<j} a_i a_j \leq m\sum_{i=1}^{m} a_i^2, \tag{3.24}$$

which implies that

$$\sum_i a_i^2 \geq m\|H\|^2. \tag{3.25}$$

In view of (3.25), (3.22) yields

$$m^2\|H\|^2 \geq 2\tau + m\|H\|^2 + (m-1)(\alpha^2 - \rho), \tag{3.26}$$

which implies (3.20). □

Theorem 3.6. *Let M be a totally real submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then for any integer k, $2 \leq k \leq m$ and for any point $x \in M$*

$$\|H\|^2(x) \geq \Theta_k(x) + \frac{1}{m}(\alpha^2 - \rho). \tag{3.27}$$

Proof. Let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1,\cdots,i_k}$ the k-plane section spanned by $e_{i_1}, \cdots, e_{i_k}$. Then, we have [3]

$$\tau(x) \geq \frac{m(m-1)}{2}\Theta_k(x). \tag{3.28}$$

Using (3.28) in (3.20), (3.27) follows. □

Acknowledgement. The authors would like to express their sincere thanks and gratitude to the referees for their valuable suggestions towards the improvement of the paper.

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Ser. Math. Inform. Vol. 33, No 2 (2018), 153–162
https://doi.org/10.22190/FUMI1802153K

ON CERTAIN HESSENBERG MATRICES RELATED WITH LINEAR RECURRENCES

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Abstract. In this paper, we present various results for permanents and determinants of some Hessenberg matrices. Also, some special cases for permanents are given.

Keywords: Hessenberg matrices, permanents, determinants.

1. Introduction

Matrix methods are useful tools deriving some properties of linear recurrences. Some authors obtained many connections between certain sequences and permanents of Hessenberg matrices in the literature [1]-[4],[6],[10]-[12].

The permanent of an $n-$ square matrix $\mathbf{A_n} = [a_{ij}]$ is defined by

$$per\mathbf{A_n} = \sum_{\sigma \in D_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group D_n.

In [9], Minc defined the super-diagonal matrix and showed that the permanent of the matrix equals the order k-Fibonacci number.

In [5], Kılıç derived recurrence relations and generating matrices for the sums of usual tribonacci numbers and 4n subscripted tribonacci sequences $\{T_{4n}\}$, and their sums. Also, the relationships between these sequences and permanents of certain matrices are obtained.

In [6], Kılıç and Taşcı found the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

In [7], Kılıç and Taşcı defined the $n \times n$ tridiagonal Toeplitz $(0, -1, 1)$-matrix $\mathbf{M_n} = [m_{i,j}]$ with $m_{i,i} = -1$ for $1 \leq i \leq n, m_{i,i+1} = m_{i+1,i} = 1$ for $1 \leq i \leq n-1$

Received November 16, 2017; accepted February 23, 2018
2010 *Mathematics Subject Classification.* Primary 11B37; Secondary 05A15

and 0 otherwise, and the $n \times n$ tridiagonal Toeplitz $(0,-1,1)$-matrix $\mathbf{L_n} = [l_{i,j}]$ with $l_{i,i} = -1$ for $2 \le i \le n, l_{i,i+1} = l_{i+1,i} = 1$ for $1 \le i \le n-1$, $l_{1,1} = -\frac{1}{2}$ and 0 otherwise. They showed $per\mathbf{M_n} = F_{-(n+1)}$ and $per\mathbf{L_n} = \frac{L_{-n}}{2}$, where F_n and L_n is the nth Fibonacci and Lucas numbers, respectively.

In [8], Li showed new Fibonacci-Hessenberg matrices and gave another proof of the well-known results relative to the Pell and Perrin numbers.

In [3], Kalman showed that the $(n+k)$-th term of a sequence is defined recursively as a linear combination of the preceding k terms:

$$u_{n+k} = c_0 u_n + c_1 u_{n+1} + ... + c_{k-1} u_{n+k-1} \tag{1.1}$$

in which the initial terms $u_0 = ... = u_{k-2} = 0, u_{k-1} = 1$ and $c_0, c_1, ..., c_{k-1}$ are constants.

In [10], considering the generalized Fibonacci-Narayana sequence $\{G_n(a,c,r)\}$, Ramírez derived some relations between this sequence and a permanent of one type of the upper Hessenberg matrix. For example,

$$per\begin{bmatrix} a & c & c & \cdots & c & & & 0 \\ 1 & a & 0 & 0 & \cdots & c & & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & 1 & a & 0 & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a & 0 & 0 \\ & & & & & 1 & a & 0 \\ & & & & & & 1 & a \end{bmatrix} = G_{n+r-1}(a,c,r),$$

where the generalized Fibonacci-Narayana sequence $\{G_n(a,c,r)\}_{n\in\mathbb{N}}$ is defined as follows:

$$G_n(a,c,r) = aG_{n-1}(a,c,r) + cG_{n-r}(a,c,r),\ 2 \le r \le n,$$

with the initial conditions $G_0(a,c,r) = 0$, $G_i(a,c,r) = 1$, for $i = 1, 2, ..., r-1$.

In [12], Trojovský defined tridiagonal matrices $\mathbf{B_n^\delta} = \left[b_{ij}^\delta\right]$ in the form

$$\begin{cases} 1 & \text{if } i = j \text{ or } i = j-1, \\ (-1)^{j+\delta} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta \in \{0,1\}$ and showed

$$\det \mathbf{B_n^\delta} = \begin{cases} F_{(n+4-6\delta)/2} & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+1)/2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

2. Some results

In this section, we define the sequence $\{R_n(a,b,c,d)\}$ and determine some relationships between the terms of this sequence and permanents of certain upper Hessenberg matrices. A sequence $\{R_n(a,b,c,d)\}$ is defined by for $3 \leq d \leq n$,

$$R_n(a,b,c,d) = aR_{n-1}(a,b,c,d) + bR_{n-2}(a,b,c,d) + cR_{n-d}(a,b,c,d), \tag{2.1}$$

in which $R_0(a,b,c,d) = 0, R_1(a,b,c,d) = R_2(a,b,c,d) = ... = R_{d-2}(a,b,c,d) = 1$ and $R_{d-1}(a,b,c,d) = a$. The sequence $\{R_n(a,b,c,d)\}$ is a generalization of the tribonacci sequence. When $a = b = c = 1$ and $d = 3$, $R_n(1,1,1,3) = T_n$ (the nth tribonacci number). If $c = 0$ and $d = 3$, the generalized Fibonacci sequence $\{U_n(a,b)\}$ is obtained. If $a = b = 1, c = 0$ and $d = 3$, the Fibonacci sequence $\{F_n\}$ is obtained and if $a = c = 1$, $b = 0$ and $d = 3$, the Narayana sequence is obtained.

The generating function $R(z)$ of $R_n(a,b,c,d)$ is given by

$$R(z) = \frac{(a-1+bz)\, z^{d-1} - bz^3 - az^2 + z}{(1-z)\,(1-az-bz^2-cz^d)}.$$

Now we give relationships between terms of the sequence $\{R_n(a,b,c,d)\}$ and the permanents of certain matrices.

For $n \geq 1$, define a $n \times n$ matrix $\mathbf{H_n}(a,b,c,d,k,t) = [h_{i,j}]$ with $h_{i+1,i} = 1$ for $1 \leq i \leq n-2$, $h_{i,i} = a$ for $1 \leq i \leq n-1$, $h_{i,i+1} = b$ for $1 \leq i \leq n-1$, $h_{1,i} = c$ for $3 \leq i \leq d$, $h_{i,d+i-1} = c$ for $2 \leq i \leq n-d+1, h_{n,n-1} = k, h_{n,n} = t$, and 0 otherwise, i.e.,

$$\mathbf{H_n}(a,b,c,d,k,t) = \begin{bmatrix} a & b & c & \cdots & c & 0 & ... & 0 \\ 1 & a & b & 0 & \cdots & c & ... & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & 1 & a & b & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & 1 & a & b \\ 0 & & & & & & k & t \end{bmatrix}. \tag{2.2}$$

Then we give the following Theorem.

Theorem 2.1. *Let $\mathbf{H_n}(a,b,c,d,k,t)$ be the matrix defined in (2.2). Then, for $n \geqslant 1$ and $d \geq 3$,*

$$per\mathbf{H_n}(a,b,c,d,k,t) = kR_{n+d-2}(a,b,c,d) - (ka-t)\, R_{n+d-3}(a,b,c,d), \tag{2.3}$$

where the real numbers k and t.

Proof. (Induction on n) If $n = 1$, then we have

$$per\mathbf{H_1}(a,b,c,d,k,t) = t = kR_{d-1}(a,b,c,d) - (ka-t)\, R_{d-2}(a,b,c,d).$$

Suppose that the equation holds for $n-1$. Then we show that the equation holds for n. Expanding the $per\mathbf{H_n}$ with respect to the last column d times, we write

$$\begin{aligned} & per\mathbf{H_n}(a,b,c,d,k,t) \\ = \; & aper\mathbf{H_{n-1}}(a,b,c,d,k,t) + bper\mathbf{H_{n-2}}(a,b,c,d,k,t) + cper\mathbf{H_{n-d}}(a,b,c,d,k,t). \end{aligned}$$

By our assumption, we have

$$\begin{aligned} per\mathbf{H_n}(a,b,c,d,k,t) &= a\left(kR_{n+d-3}(a,b,c,d) - (ka-t)\,R_{n+d-4}(a,b,c,d)\right) \\ &\quad + b\left(kR_{n+d-4}(a,b,c,d) - (ka-t)\,R_{n+d-5}(a,b,c,d)\right) \\ &\quad + c\left(kR_{n-2}(a,b,c,d) - (ka-t)\,R_{n-3}(a,b,c,d)\right) \\ &= kR_{n+d-2}(a,b,c,d) - (ka-t)\,R_{n+d-3}(a,b,c,d). \end{aligned}$$

Thus, the proof is complete. $\square$

When $t=a$ and $k=1$ in (2.3), we have $per\mathbf{H_n}(a,b,c,d,1,a) = R_{n+d-2}(a,b,c,d)$.

For $n \geq 1$, define a $n \times n$ matrix $\mathbf{E_n}(a,b,c,d,k,t) = [e_{i,j}]$ with $e_{i+1,i} = -1$ for $1 \leq i \leq n-2$, $e_{i,i} = a$ for $1 \leq i \leq n-1$, $e_{i,i+1} = b$ for $1 \leq i \leq n-1$, $e_{1,i} = c$ for $3 \leq i \leq d$, $e_{i,d+i-1} = c$ for $2 \leq i \leq n-d+1$, $e_{n,n-1} = -k$, $e_{n,n} = t$, and 0 otherwise, i.e.,

$$\mathbf{E_n}(a,b,c,d,k,t) = \begin{bmatrix} a & b & c & \cdots & c & 0 & \cdots & 0 \\ -1 & a & b & 0 & \cdots & c & \cdots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & -1 & a & b & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & -1 & a & b \\ 0 & & & & & & -k & t \end{bmatrix}.$$

It is clearly showed from [2] that

$$\det \mathbf{E_n}(a,b,c,d,k,t) = per\mathbf{H_n}(a,b,c,d,k,t).$$

Now, we take the $n \times n$ matrix $\mathbf{H_n}(a,b,c,3,k,t)$ by the following form:

$$\mathbf{H_n}(a,b,c,3,k,t) = \begin{bmatrix} a & b & c & & & & & 0 \\ 1 & a & b & c & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & \\ & & 1 & a & b & c & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots & c \\ & & & & & 1 & a & b \\ 0 & & & & & & k & t \end{bmatrix}.$$

Then, we have

$$per\mathbf{H_n}(a,b,c,3,k,t) = kR_{n+1}(a,b,c,3) - (ka-t)\,R_n(a,b,c,3). \tag{2.4}$$

For example, from [9], for $a = b = c = t = k = 1$ in (2.4), we have that

$$per\mathbf{H_n}(1,1,1,3,1,1) = T_{n+1} = per\mathbf{F}(n,3),$$

where T_n is the nth tribonacci number.

For $n > 1$; we define an $n \times n$ matrix $\mathbf{W_n}(a,b,c,3,k,t)$ as in the compact form, by the definition of $\mathbf{H_n}(a,b,c,3,k,t)$;

$$\mathbf{W_n}(a,b,c,3,k,t) = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ 1 & & & \\ 0 & & & \\ \vdots & & \mathbf{H_{n-1}}(a,b,c,3,k,t) & \\ 0 & & & \end{array}\right]. \tag{2.5}$$

Now, we have the following theorem:

Theorem 2.2. *Let $\mathbf{W_n}(a,b,c,3,k,t)$ be the matrix defined in (2.5). Then, for $n > 2$*

$$per\mathbf{W_n}(a,b,c,3,k,t) = k\sum_{i=1}^{n} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{n} R_{i-1}(a,b,c,3).$$

Proof. (Induction on n) If $n = 3$, we write

$$\begin{aligned} & per\mathbf{W_3}(a,b,c,3,k,t) \\ = \; & k + t + at + bk = k\sum_{i=1}^{3} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{3} R_{i-1}(a,b,c,3). \end{aligned}$$

Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. From the definitions of matrices $\mathbf{H_n}(a,b,c,3,k,t)$ and $\mathbf{W_n}(a,b,c,3,k,t)$, expanding the $per\mathbf{W_{n+1}}(a,b,c,3,k,t)$ with respect to the first column gives us

$$per\mathbf{W_{n+1}}(a,b,c,3,k,t) = per\mathbf{H_n}(a,b,c,3,k,t) + per\mathbf{W_n}(a,b,c,3,k,t).$$

By our assumption and (2.4), we have

$$\begin{aligned} & per\mathbf{W_{n+1}}(a,b,c,3,k,t) \\ = \; & kR_{n+1}(a,b,c,3) - (ka-t)\,R_n(a,b,c,3) \\ & + k\sum_{i=1}^{n} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{n} R_{i-1}(a,b,c,3) \\ = \; & k\sum_{i=1}^{n+1} R_i(a,b,c,3) - (ka-t)\sum_{i=1}^{n+1} R_{i-1}(a,b,c,3). \end{aligned}$$

Thus the proof is obtained. □

When $a=b=c=1$ in (2.1), the sequence $\{R_n\}$, special case of the sequence $\{R_n(a,b,c,d)\}$, is defined by the recurrence

$$R_n = R_{n-1} + R_{n-2} + R_{n-d},\ 3 \le d \le n \tag{2.6}$$

in which $R_0 = 0$, $R_1 = R_2 = ... = R_{d-2} = R_{d-1} = 1$ and especially from (1.1), the sequence $\{S_n\}$ is defined by

$$S_n = S_{n-1} + S_{n-2} + S_{n-d},\ 3 \le d \le n \tag{2.7}$$

in which $S_0 = S_1 = S_2 = ... = S_{d-3} = 0$ and $S_{d-2} = S_{d-1} = 1$. For $d=3$ in (2.6) and (2.7), the sequences $\{R_n\}$ and $\{S_n\}$ coincide as the tribonacci sequence.

For $n>1$, define an $n \times n$ matrix $\mathbf{Z_n} = [z_{i,j}]$ with $z_{i+1,i} = 1$ for $1 \le i \le n-1$, $z_{i,i} = 1$ for $1 \le i \le n$, $z_{1,2} = 0$, $z_{i,i+1} = 1$ for $2 \le i \le n-1$, $z_{1,i} = 1$ for $3 \le i \le d$, $z_{i,d+i-1} = 1$ for $2 \le i \le n-d+1$ and 0 otherwise, i.e.,

$$\mathbf{Z_n} = \begin{bmatrix} 1 & 0 & 1 & \cdots & 1 & 0 & ... & 0 \\ 1 & 1 & 1 & 0 & \cdots & 1 & ... & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ & & 1 & 1 & 1 & 0 & \cdots & 1 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & 1 & 1 & 1 \\ 0 & & & & & & 1 & 1 \end{bmatrix}. \tag{2.8}$$

Then we give the following Theorem.

Theorem 2.3. *Let $\mathbf{Z_n}$ be the matrix defined in (2.8). Then, for $n \ge 5$ and $d \ge 3$,*

$$per\mathbf{Z_n} = R_{n+d-3} + R_{n+d-4} - R_{n+d-5} + S_{n-5}.$$

Proof. We prove this by induction on n. For $n=5$, we write

$$per\mathbf{Z_5} = 2R_{d+1} + 1 = R_{d+2} + R_{d+1} - R_d + S_0.$$

The claim is true for $n=5$. Assume that the claim is true for $n-1$. Thus we show that the claim is true for n. Expanding the $perZ_n$ according to the last column d times, we have

$$per\mathbf{Z_n} = per\mathbf{Z_{n-1}} + per\mathbf{Z_{n-2}} + per\mathbf{Z_{n-d}}.$$

By our assumption, we have

$$\begin{aligned} per\mathbf{Z_n} &= R_{n+d-4} + R_{n+d-5} - R_{n+d-6} + S_{n-6} \\ &\quad + R_{n+d-5} + R_{n+d-6} - R_{n+d-7} + S_{n-7} \\ &\quad + R_{n-3} + R_{n-4} - R_{n-5} + S_{n-d-5}. \end{aligned}$$

From the sequences $\{R_n\}$ and $\{S_n\}$, we write

$$per\mathbf{Z_n} = R_{n+d-3} + R_{n+d-4} - R_{n+d-5} + S_{n-5}.$$

So the proof is complete. $\square$

For $n \geq 1$, define the $n \times n$ matrix $\mathbf{V_n^\delta} = [v_{i,j}]$ with $v_{i+1,i} = 1$ for $1 \leq i \leq n-2$, $v_{i,i} = 1$ for $1 \leq i \leq n-1$, $v_{1,i+1} = v_{i,i+1} = (-1)^{i-\delta}$for $1 \leq i \leq n$ and 0 otherwise, where $\delta \in \{0,1\}$. i.e.,

$$\text{(2.9)} \qquad \mathbf{V_n^\delta} = \begin{bmatrix} 1 & (-1)^{1-\delta} & (-1)^{2-\delta} & \cdots & & (-1)^{n-2-\delta} & (-1)^{n-1-\delta} \\ 1 & 1 & (-1)^{2-\delta} & 0 & \cdots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots \\ & & 1 & 1 & & 0 & \cdots \\ & & & \ddots & \ddots & (-1)^{n-2-\delta} & \ddots \\ & & & & \ddots & 1 & (-1)^{n-1-\delta} \\ 0 & & & & & 1 & 1 \end{bmatrix}.$$

Theorem 2.4. *Let $\mathbf{V_n^\delta}$ be the matrix defined in (2.9). Then, for $n \geq 1$,*

$$\det \mathbf{V_n^\delta} = \begin{cases} (-1)^\delta \left(F_{(n+5+3(-1)^\delta)/2} - 2 - (-1)^\delta\right) & \text{if } n \equiv 0 \pmod 2, \\ (-1)^\delta \left(F_{(n+5)/2} - 2 + (-1)^\delta\right) & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

where F_n is the nth Fibonacci number.

Proof. For $n = 1$, $\det \mathbf{V_1^0} = 1 = (F_3 - 1)$, $\det \mathbf{V_1^1} = 1 = -(F_3 - 3)$ and for $n = 2$, $\det \mathbf{V_2^1} = 0 = -(F_2 - 1)$, $\det \mathbf{V_2^0} = 2 = (F_5 - 3)$.

We show that the claim is true for $n-1$. Using expansion on the first column of $\det V_n^1$, we get as follows

$$\det \mathbf{V_n^1} = \det \mathbf{B_{n-1}^0} - \det \mathbf{V_{n-1}^0}.$$

From (1.2) and the induction hypothesis, we have

$$\begin{aligned} \det \mathbf{V_n^1} &= \begin{cases} F_{n/2} - \left(F_{(n+4)/2} - 1\right) & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+3)/2} - \left(F_{(n+7)/2} - 3\right) & \text{if } n \equiv 1 \pmod 2, \end{cases} \\ &= \begin{cases} -\left(F_{((n+2)/2} - 1\right) & \text{if } n \equiv 0 \pmod 2, \\ -\left(F_{((n+5)/2} - 3\right) & \text{if } n \equiv 1 \pmod 2. \end{cases} \end{aligned}$$

Similarly, using $\det \mathbf{V_n^0} = \det \mathbf{B_{n-1}^1} + \det \mathbf{B_{n-2}^0} + \det \mathbf{V_{n-2}^0}$, the desired result is given. We have the proof. $\square$

Theorem 2.5. *For $n \geq 1$, we have*

$$per\mathbf{V_n^1} = \begin{cases} L_{(n+2)/2} - 1 & \text{if } n \equiv 0 \pmod 2, \\ L_{(n-1)/2} - 1 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

and

$$per\mathbf{V_n^0} = \begin{cases} F_{(n+2)/2} - 1 & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+5)/2} - 1 & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

where F_n is the nth Fibonacci number and L_n is the nth Lucas number.

Proof. Considering $per\mathbf{B_n^\delta} = \begin{cases} F_{(n-2+6\delta)/2} & \text{if } n \equiv 0 \pmod 2, \\ F_{(n+1)/2} & \text{if } n \equiv 1 \pmod 2, \end{cases}$ for $\delta \in \{0, 1\}$ and the equalities

$$per\mathbf{V_n^1} = per\mathbf{B_{n-1}^0} + per\mathbf{V_{n-1}^0} \text{ and } per\mathbf{V_n^0} = per\mathbf{B_{n-1}^1} - per\mathbf{B_{n-2}^0} + per\mathbf{V_{n-2}^0},$$

we have the proof from induction on n. □

3. Some special cases

In this section, we give some special cases of the above theorems:

- For $b = 1$, $c = 0$ and $d = 3$, the generalized Fibonacci sequence $\{U_n(a, 1)\}$,

$$\begin{aligned} & per\mathbf{H_n}(a, 1, 0, 3, k, t) \\ = \; & per \begin{bmatrix} a & 1 & 0 & \cdots & & & \\ 1 & a & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & & 0 \\ & & & & 1 & a & 1 \\ & & & & & k & t \end{bmatrix} = tU_n(a, 1) + kU_{n-1}(a, 1), \end{aligned}$$

$$\begin{aligned} & per\mathbf{W_n}(a, 1, 0, 3, k, t) \\ = \; & per \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 0 & & & H_{n-1} & \\ \vdots & & & & \\ & & & & \end{bmatrix} \\ = \; & \frac{1}{a}\left(tU(_n(a, 1) + (k+t)U(_{n-1}(a, 1) + kU_{n-2}(a, 1) + k(a-1) - t\right), \end{aligned}$$

and then,

$$per\mathbf{W_n}(a, 1, 0, 3, k, t) = per\mathbf{H_n}(a, 1, 0, 3, k, t) + per\mathbf{H_{n-1}}(a, 1, 0, 3, k, t) + k(a-1) + t.$$

- For $a = 1$, $b = 2$, $c = 0$ and $d = 3$ in (2.1), $\{J_n\}$ is the Jacobsthal sequence and for $n \geqslant 2$,

$$\begin{aligned} & per\mathbf{H_n}(1,2,0,3,k,t) \\ = \; & per \begin{bmatrix} 1 & 2 & 0 & & & \\ 1 & 1 & 2 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & 0 \\ & & & 1 & 1 & 2 \\ & & & & k & t \end{bmatrix} = tJ_n + 2kJ_{n-1}. \end{aligned}$$

- For $a = b = k = 1$, $c = 0$ and $d = 3$ in (2.1), $\{F_n\}$ is the Fibonacci sequence,

$$\begin{aligned} & per\mathbf{H_n}(1,1,0,3,1,t) \\ = \; & per \begin{bmatrix} 1 & 1 & 0 & & & \\ 1 & 1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & 0 \\ & & & 1 & 1 & 1 \\ & & & & 1 & t \end{bmatrix} = tF_n + F_{n-1} \end{aligned}$$

and

$$per\mathbf{W_n}(1,1,0,3,1,t) = per \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & & & & \\ 0 & & & H_{n-1} & \\ \vdots & & & & \\ & & & & \end{bmatrix} = t\left(F_{n+1} - 1\right) + F_n.$$

- For $d = 3$ in (2.6) and (2.7), $\{T_n\}$ is the tribonacci sequence and

$$per\mathbf{Z_n} = per \begin{bmatrix} 1 & 0 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \cdots \\ & & & & & 1 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \end{bmatrix} = 2T_{n-1} + T_{n-3} + T_{n-5}.$$

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Ser. Math. Inform. Vol. 33, No 2 (2018), 163–176
https://doi.org/10.22190/FUMI1802163G

ON A GENERALIZATION OF CATALAN POLYNOMIALS

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Abstract. In this paper, we define and study the generalized class of Catalan's polynomials. Thereafter we connect them to the class of Humbert's polynomials and re-found the Humbert recurrence relation [5]. This idea helps us to define a new class of generalized Humbert's polynomials different from those given by H. W. Gould [4] and P. N. Shrivastava [9]. Finally, we establish an explicit formula for a special class of generalized Catalan's polynomials and get two useful combinatorial identities.
Keywords: Catalan's polynomials, Gegenbauer's polynomials,Humbert's polynomials, generating functions.

1. Introduction

We recall that the Catalan numbers C_n are defined for any positive integer n by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and their generating function is

$$C(u) = \frac{1-\sqrt{1-4u}}{2u} = \sum_{n\geq 0} C_n u^n,\ \ |u| < \frac{1}{4}.$$

It is useful here to remember the proof. Writing $C(u) = \frac{1}{2u}\left(1-\sqrt{1-4u}\right)$. Using the fact that for $|u|<1$ and $\alpha \in \mathbb{R}$;

$$(1+u)^{\alpha} = 1 + \sum_{n\geq 1} \begin{bmatrix} \alpha \\ n \end{bmatrix} u^n,\ \text{where}\ \begin{bmatrix} \alpha \\ n \end{bmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))}{n!}.$$

We deduce that

$$C(u) = \frac{1}{2u}\sum_{n\geq 1} \begin{bmatrix} \frac{1}{2} \\ n \end{bmatrix} (-4u)^{n-1}$$

Received December 7, 2017; accepted June 11, 2018
2010 *Mathematics Subject Classification.* Primary 05A15, 11B83; Secondary 05A19

then

$$\begin{aligned} C(u) &= \sum_{n\geq 0} \begin{bmatrix} \frac{1}{2} \\ n \end{bmatrix} (-1)^n 2^{2n+1} u^n \\ &= \sum_{n\geq 0} \frac{1}{n+1} \binom{2n}{n} u^n \end{aligned}$$

For positive integers $a, b \geq 1$, the function $C_n^{a,b}(u) = \frac{1-\sqrt{1-au}}{bu}$ generates numbers $C_n^{a,b}$ of the form $C_n^{a,b} = \frac{a^{n+1}}{2^{2n+1}b} C_n$ and $C_n^{4,2} = C_n$. The idea is to remark that $C_n^{a,b}(u) = \frac{a}{2b} C\left(\frac{au}{4}\right)$. Furthermore $C_{a,b}(u) = \sum_{n\geq 0} \frac{a^{n+1}}{2^{2n+1}b} C_n u^n$.

The class $\{P_n(x)\}_{n\geq 0}$ of Catalan's polynomials [6] is defined by the following linear recurrence relation

$$P_{n+2}(x) = P_{n+1}(x) - xP_n(x) \ \ , n \geq 2 \tag{1.1}$$

and the starting values $P_0(x) = P_1(x) = 1$. The closed form of $P_n(x)$ [6] is

$$P_n(x) = \frac{\left(1+\sqrt{1-4x}\right)^{n+1} - \left(1-\sqrt{1-4x}\right)^{n+1}}{2^{n+1}\sqrt{1-4x}} \tag{1.2}$$

and the bivariate generating function is

$$f(x,t) = \frac{1}{1-t+xt^2} = \sum_{n\geq 0} P_n(x) t^n \tag{1.3}$$

To get the proof, just write

$$xf(x,t) = \sum_{n\geq 0} \left[P_{n+1}(x) - P_{n+2}(x)\right] t^n$$

and

$$xf(x,t) = \frac{1}{t}\left(f(x,t) - 1\right) - \frac{1}{t^2}\left(f(x,t) - 1 - t\right)$$

hence

$$\left(xt^2 - t + 1\right) f(t) = 1.$$

It is well-known that the $(n+1)^{\text{th}}$ Catalan's polynomial $P_n(x)$ is written under the following binomial expression

$$P_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k} (-x)^k, \tag{1.4}$$

where $\lfloor x \rfloor$ is the integer part of x.
Explicitly we get

$$P_{2n}(x) = \sum_{k=0}^{n} \binom{2n-k}{k} (-x)^k$$

and

$$P_{2n+1}(x) = \sum_{k=0}^{n} \binom{2n+1-k}{k} (-x)^k .$$

Furthermore, $P_{2n}(x)$ and $P_{2n+1}(x)$ have the same degree and only the first coefficient corresponding to degree zero is 1.
A new proof of this identity is given in Section 3. using Gegenbauer's polynomials [2] and generalized Catalan's polynomials properties.

2. Generalized class of Catalan's polynomials

Definition 2.1. The generalized class of Catalan's polynomials $\left\{\mathcal{P}_{n,m}^{\lambda,A}(x)\right\}_{n\geq 0}$ is given by the following generating function

$$f_{m,\lambda,A}(x,t) = \frac{1+A(x)t}{(1-mt+xt^m)^{\lambda}} = \sum_{n\geq 0} \mathcal{P}_{n,m}^{\lambda,A}(x)\, t^n, \tag{2.1}$$

where $A(x)$ is any polynomial of $\mathbb{Z}[x]$. With starting values

$$\mathcal{P}_{0,m}^{\lambda,A}(x) = 1 \text{ and } \mathcal{P}_{1,m}^{\lambda,A}(x) = A(x) + \lambda m.$$

To simplify notations let us denote

$$\mathcal{P}_{n,m}^{\lambda,0}(x) = \mathcal{P}_{n,m}^{\lambda}(x) \text{ and } \mathcal{P}_{n,2}^{1,0}(x) = \mathcal{P}_n(x).$$

From the generating function $f_{m,\lambda,A}(x,t)$ we deduce that

$$\mathcal{P}_{n,m}^{\lambda,A}(x) = \mathcal{P}_{n,m}^{\lambda}(x) + A(x)\,\mathcal{P}_{n-1,m}^{\lambda}(x) \tag{2.2}$$

This family generalizes Catalan's polynomials. Using the definition (2.1) we get

$$f_{2,1,0}(x,t) = \frac{1}{(1-2t+xt^2)} = \sum_{n\geq 0} \mathcal{P}_n(x)\, t^n$$

and

$$f_{2,1,0}\left(x,\frac{t}{2}\right) = \frac{1}{\left(1-t+\frac{x}{4}t^2\right)} = f\left(\frac{x}{4},t\right)$$

then

$$2^{-n}\mathcal{P}_n(x) = P_n\left(\frac{x}{4}\right)$$

or

$$\mathcal{P}_n(4x) = 2^n P_n(x)$$

The generalized Catalan's polynomials are related to several polynomial types as Gegenbauer, Humbert-type polynomials. This connection is the subject of Section 3.

The recurrence relation satisfied by the class $\left\{\mathcal{P}_{n,m}^{\lambda,A}(x)\right\}_{n\geq 0}$ according to the positive integers n and m is established in the following theorem

Theorem 2.1. *If* $2 \leq n < m-1$

$$(n+1)\,\mathcal{P}^{\lambda,A}_{n+1,m}(x) = (\lambda+n-2)\,mA(x)\,\mathcal{P}^{\lambda,A}_{n-1,m}(x) - [(n-1)\,A(x) - mn - \lambda m]\,\mathcal{P}^{\lambda,A}_{n,m}(x),$$

if $n \geq m$

$$\begin{aligned}(n+1)\,\mathcal{P}^{\lambda,A}_{n+1,m}(x) &= (1-\lambda m-n+m)\,xA(x)\,\mathcal{P}^{\lambda,A}_{n-m,m}(x) + (n+\lambda-2)\,mA(x)\,\mathcal{P}^{\lambda,A}_{n-1,m}(x)\\ &+ (m-n-\lambda m-1)\,x\mathcal{P}^{\lambda,A}_{n-m+1,m}(x) + [\lambda m-(n-1)\,A(x)+mn]\,\mathcal{P}^{\lambda,A}_{n,m}(x)\end{aligned} \quad (2.3)$$

and for $m \geq 2$

(2.4)

$$\begin{aligned}m\mathcal{P}^{\lambda,A}_{m,m}(x) &= (\lambda+m-3)\,mA(x)\,\mathcal{P}^{\lambda,A}_{m-2,m}(x)\\ &+ \left[\lambda m+m^2-m-(n-2)\,A(x)\right]\mathcal{P}^{\lambda,A}_{m-1,m}(x) - \lambda mx\end{aligned}$$

As a consequence of Theorem 2.1 we get the following corollary.

Corollary 2.1. *If* $2 \leq n < m-1$

$$(n+1)\,\mathcal{P}^{\lambda}_{n+1,m}(x) = m(\lambda+n)\,\mathcal{P}^{\lambda}_{n,m}(x), \quad (2.5)$$

if $n \geq m$

$$(n+1)\,\mathcal{P}^{\lambda}_{n+1,m}(x) - m(n+\lambda)\,\mathcal{P}^{\lambda}_{n,m}(x) + (n-m+1+\lambda m)\,x\mathcal{P}^{\lambda}_{n-m+1,m}(x) = 0$$

and for $m \geq 2$,

$$(\lambda+m-1)\,\mathcal{P}^{\lambda}_{m-1,m}(x) - \mathcal{P}^{\lambda}_{m,m}(x) = \lambda x \quad (2.6)$$

Proof. The relations (2.5), (2.6) and (2.6) of Corollary 2.1 are immediate from the equalities (2.3), (2.3) and (2.4) of Theorem 2.1 by considering $A(x) = 0$. $\square$

2.1. Proof of Theorem 2.1

$$f_{m,\lambda,A}(x,t) = \frac{1+A(x)\,t}{(1-mt+xt^m)^{\lambda}} = \sum_{n\geq 0} \mathcal{P}^{\lambda,A}_{n,m}(x)\,t^n$$

Let $\frac{df_{m,\lambda,A}(x,t)}{dt} = f'_{m,\lambda,A}(x,t)$ then

$$f'_{m,\lambda,A}(x,t) = \sum_{n\geq 1} n\mathcal{P}^{\lambda,A}_{n,m}(x)\,t^{n-1} = \sum_{n\geq 0}(n+1)\,\mathcal{P}^{\lambda,A}_{n+1,m}(x)\,t^n.$$

and

$$\begin{aligned}(1+A(x)\,t)(1-mt+xt^m)\,f'_{m,\lambda,A}(x,t) &= A(x)(1-mt+xt^m)\,f_{m,\lambda,A}(x,t)\\ &- \lambda m\left(xt^{m-1}-1\right)(1+A(x)\,t)\,f_{m,\lambda,A}(x,t)\end{aligned}$$

Taking

$$\Delta=(1-\lambda m)\,xA(x)\,t^{m}-\lambda mxt^{m-1}+(\lambda-1)\,mA(x)\,t+A(x)+\lambda m$$

then

$$\begin{aligned}\Delta f_{m,\lambda,A}(x,t) &= (1-\lambda m)\,xA(x)\sum_{n\geq m}\mathcal{P}_{n-m,m}^{\lambda,A}(x)\,t^{n}-\lambda mx\sum_{n\geq m-1}\mathcal{P}_{n-m+1,m}^{\lambda,A}(x)\,t^{n}\\ &+ (\lambda-1)\,mA(x)\sum_{n\geq 1}\mathcal{P}_{n-1,m}^{\lambda,A}(x)\,t^{n}+(A(x)+\lambda m)\sum_{n\geq 0}\mathcal{P}_{n,m}^{\lambda,A}(x)\,t^{n}\end{aligned}$$

and taking

$$\sigma=(1+A(x)\,t)\,(1-mt+xt^{m})=1+(A(x)-m)\,t-mA(x)\,t^{2}+xt^{m}+xA(x)\,t^{m+1}$$

then

$$\begin{aligned}\sigma f_{m,\lambda,A}'(x,t) &= \sum_{n\geq 0}(n+1)\,\mathcal{P}_{n+1,m}^{\lambda,A}(x)\,t^{n}+(A(x)-m)\sum_{n\geq 1}n\mathcal{P}_{n,m}^{\lambda,A}(x)\,t^{n}\\ &+ x\sum_{n\geq m}(n-m+1)\,\mathcal{P}_{n-m+1,m}^{\lambda,A}(x)\,t^{n}-mA(x)\sum_{n\geq 1}(n-1)\,\mathcal{P}_{n-1,m}^{\lambda,A}(x)\,t^{n}\\ &+ xA(x)\sum_{n\geq m}(n-m)\,\mathcal{P}_{n-m,m}^{\lambda,A}(x)\,t^{n}.\end{aligned}$$

Writing the equality

$$\sigma f_{m,\lambda,A}'(x,t)=\Delta f_{m,\lambda,A}(x,t)$$

in expansion series form and comparing the coefficients of t^n we get the result.

3. Generalized class of Humbert's polynomials

The class of Gegenbauer's polynomials [1, 2] $\left\{G_n^{\lambda}(x)\right\}_{n\geq 0}$ is defined by the following generating function

$$G^{\lambda}(x,t)=\frac{1}{(1-2xt+t^{2})^{\lambda}}=\sum_{n\geq 0}G_n^{\lambda}(x)\,t^{n} \tag{3.1}$$

The corresponding recurrence relation is

$$nG_n^{\lambda}(x)=2x\,(n+\lambda-1)\,G_{n-1}^{\lambda}(x)-(n+2\lambda-2)\,G_{n-2}^{\lambda}(x)\,,\quad n\geq 2 \tag{3.2}$$

with starting values $G_0^{\lambda}(x)=1$ and $G_1^{\lambda}(x)=2\lambda x$. Their explicit form is

$$G_n^{\lambda}(x)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^{k}\,\frac{(\lambda)_{n-k}}{k!\,(n-2k)!}\,(2x)^{n-2k} \tag{3.3}$$

where

$$(\lambda)_n = \lambda(\lambda+1)\dots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.$$

and Γ is a gamma function.

Gegenbauer's polynomials are a particular case of Humbert's polynomials $\left\{\Pi_{n,m}^{\lambda}(x)\right\}_{n\geq 0}$ were defined in 1921 by Humbert [5]. Their generating function is

$$\frac{1}{(1-mxt+t^m)^{\lambda}} = \sum_{n\geq 0} \Pi_{n,m}^{\lambda}(x)\, t^n \tag{3.4}$$

Here we define a new generalization of Humbert's polynomials in a way similar to that for Catalan's polynomials different from the class given by H. W. Gould [4]:

$$(C-mxt+yt^m)^p = \sum_{n\geq 0} P_n(m,x,y,p,C)\, t^n$$

and the generalization defined by P. N. Shrivastava [9]:

$$\left(C-axt+bx^l t^m\right)^{-v} = \sum_{n\geq 0} P_n^{(l)}(m,x,a,v,b)\, t^n.$$

Definition 3.1. The generalized Humbert's polynomials of type $\Pi_{n,m}^{\lambda,A}(x)$ are given in means of the function.

$$h_{m,\lambda,A}(x,t) = \frac{1+A(x)\,t}{(1-mxt+t^m)^{\lambda}} = \sum_{n\geq 0} \Pi_{n,m}^{\lambda,A}(x)\, t^n.$$

Then the generalized Gegenbauer's polynomials are defined in means of the generating function

$$\frac{1+A(x)\,t}{(1-2xt+t^2)^{\lambda}} = \sum_{n\geq 0} G_{n,A}^{\lambda}(x)\, t^n.$$

It is obvious that the polynomial $\Pi_{n,m}^{\lambda,A}(x)$ is related to Humbert's polynomial $\Pi_{n,m}^{\lambda}(x)$ by the relation

$$\Pi_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}(x) + A(x)\,\Pi_{n-1,m}^{\lambda}(x) \tag{3.5}$$

and are identical for $A(x)=0$.

Let $A(x)$ and $B(x)$ be two polynomials not forcedly of same degree. Some elementary arithmetic properties of those polynomials are:

$$\Pi_{n,m}^{\lambda,A}(x) + \Pi_{n,m}^{\lambda,-A}(x) = 2\Pi_{n,m}^{\lambda}(x), \tag{3.6}$$

$$\Pi_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda,B}(x) \;=\; [A(x)-B(x)]\,\Pi_{n-1,m}^{\lambda}(x) \tag{3.7}$$

and

$$\Pi_{n,m}^{\lambda,A+B}(x) = \Pi_{n,m}^{\lambda,A}(x) + \Pi_{n,m}^{\lambda,B}(x) - \Pi_{n,m}^{\lambda}(x). \tag{3.8}$$

The recurrence relation of $\Pi_{n,m}^{\lambda,A}(x)$ in means of $\mathcal{P}_{n,m}^{\lambda,A}(x)$, $\mathcal{P}_{n,m}^{\lambda}(x)$ and $\Pi_{n,m}^{\lambda}(x)$ is stated in the following theorem.

Theorem 3.1.

$$\mathcal{P}_{n-1,m}^{\lambda}(x)\left[\Pi_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda}(x)\right] = \Pi_{n-1,m}^{\lambda}(x)\left[\mathcal{P}_{n,m}^{\lambda,A}(x) - \mathcal{P}_{n,m}^{\lambda}(x)\right]. \tag{3.9}$$

This theorem is true for every polynomial $A(x)$ and all positive integers $m, n \geq 2$. At $x = 1$, $\Pi_{n,m}^{\lambda,A}(1)$ is identical to $\mathcal{P}_{n,m}^{\lambda,A}(1)$. The proof of Theorem 3.1 needs the following technical lemma

Lemma 3.1.

$$x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) + x^{-1/m}A(x)\,\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right) \tag{3.10}$$

and for $A(x) = 0$,

$$\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) = x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda}(x) \tag{3.11}$$

Remark 3.1. Taking into account the property (3.11), the relation (3.10) is a reformulation of the equality (2.2) in terms of Humbert's polynomials.

Proof. Writing $f_{m,\lambda,A}(x,t)$ under the following form

$$f_{m,\lambda,A}(x,t) = \frac{1}{(1 - mt + xt^m)^{\lambda}} + \frac{A(x)\,t}{(1 - mt + xt^m)^{\lambda}}.$$

Then

$$f_{m,\lambda,A}(x,t) = \frac{1}{\left(1 - mx^{-1/m}\left(x^{1/m}t\right) + \left(x^{1/m}t\right)^m\right)^{\lambda}} + \frac{A(x)\,t}{\left(1 - mx^{-1/m}\left(x^{1/m}t\right) + \left(x^{1/m}t\right)^m\right)^{\lambda}}$$

and

$$f_{m,\lambda,A}(x,t) = \sum_{n\geq 0}\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m}t^n + A(x)\sum_{n\geq 0}\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m}t^{n+1}.$$

Thus

$$\sum_{n\geq 1}\mathcal{P}_{n,m}^{\lambda,A}(x)\,t^n = \sum_{n\geq 1}\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m}t^n + A(x)\sum_{n\geq 1}\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)x^{n-1/m}t^n.$$

Furthermore

$$\mathcal{P}_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)x^{n/m} + A(x)\,\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)x^{n-1/m}.$$

Finally

$$x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x)=\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)+x^{-1/m}A(x)\,\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right).$$

When $A(x)=0$ the result (3.11) is deduced. □

From the expression (3.11) Lemma 3.1 we deduce that

$$G_n^1\left(x^{-1/2}\right)=\Pi_{n,2}^1\left(x^{-1/2}\right)=x^{-\frac{n}{2}}\mathcal{P}_n(x)$$

and Catalan's polynomials $P_n(x)$ are joined to Gegenbauer's polynomials $G_n^1(x)$ by the following useful relation

$$P_n\left(\frac{x}{4}\right)=2^{-n}x^{\frac{n}{2}}G_n^1\left(x^{-1/2}\right)$$

Each relation leads to

$$P_n(x)=x^{\frac{n}{2}}G_n^1\left((4x)^{-1/2}\right)$$

Taking into account the expression (3.3) of the polynomial $G_n^{\lambda}(x)$ and remarking that $(1)_{n-k}=\Gamma(n-k+1)=(n-k)!$ we deduce that

$$x^{\frac{n}{2}}G_n^1\left((4x)^{-1/2}\right)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{(n-k)!}{k!\,(n-2k)!}x^k$$

Since

$$\frac{(n-k)!}{k!\,(n-2k)!}=\binom{n-k}{k}$$

the binomial sum representation (1.4) of $P_n(x)$ is deduced.

Combining the results in Corollary 2.1 and Lemma 3.1 we get the Humbert recurrence relation [5, 7, 8, 3].

Corollary 3.1. *If* $2\leq n<m-1$

$$(n+1)\,\Pi_{n+1,m}^{\lambda}(x)=m(\lambda+n)\,x\Pi_{n,m}^{\lambda}(x), \tag{3.12}$$

If $n\geq m$

$$(n+1)\,\Pi_{n+1,m}^{\lambda}(x)-mx(n+\lambda)\,\Pi_{n,m}^{\lambda}(x)+(n-m+1+\lambda m)\,\Pi_{n-m+1,m}^{\lambda}(x)=0 \tag{3.13}$$

and

$$(\lambda+m-1)\,x\Pi_{m-1,m}^{\lambda}(x)-\Pi_{m,m}^{\lambda}(x)=\lambda \tag{3.14}$$

The author is thankful to Professor G. V. Milovanović for the information that there is a misprint for the recurrence relation of $\Pi_{n,m}^{\lambda}(x)$ in the works [3], [5] and [7]. The proper one is in the relation 8 [8] rewritten for polynomials $\Pi_{n,m}^{\lambda}(2x/m)$.

Proof. Substituting the value of $\mathcal{P}_{n,m}^{\lambda}(x)$ token from the expression (3.11), Lemma 3.1 in the recurrence formulae (2.5), (2.6) and (2.6) Corollary 2.1, we deduce the recurrence relations (3.12), (3.13) and (3.14) of $\Pi_{n,m}^{\lambda}(x)$. □

For $m = 2$, all the formulae (3.12), (3.13) and (3.14) are reduced to one formula because n is only greater than 1 for $m = 2$. This formula is the well-known recurrence relation [7] of Gegenbauer's polynomials.

$$(n+1)\,G_{n+1}^{\lambda}(x) - 2x\,(n+\lambda)\,G_{n}^{\lambda}(x) + (n-1+2\lambda)\,G_{n-1}^{\lambda}(x) = 0 \ , n \geq 1$$

3.1. Proof of Theorem 3.1

The relation 3.10 states that

$$A(x) = \frac{x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)}{x^{-1/m}\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)}$$

Substitute this value in the equality (3.5) we get

$$\Pi_{n,m}^{\lambda,A}(x) = \Pi_{n,m}^{\lambda}(x) + \frac{x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda,A}(x) - \Pi_{n,m}^{\lambda}\left(x^{-1/m}\right)}{x^{-1/m}\Pi_{n-1,m}^{\lambda}\left(x^{-1/m}\right)}\Pi_{n-1,m}^{\lambda}(x).$$

Using the relation $\Pi_{n,m}^{\lambda}\left(x^{-1/m}\right) = x^{-\frac{n}{m}}\mathcal{P}_{n,m}^{\lambda}(x)$ the result (3.9) of Theorem 3.1 holds.

4. Special class of generalized Catalan's polynomials

In this section we study the special class $M_n(x)$ of generalized Catalan's polynomials defined by the generating function

$$g(x,t) = \frac{1+(1-x)\,t}{1-xt+t^2} = \sum_{n\geq 0} M_n(x)\,t^n$$

and starting values $M_0(x) = M_1(x) = 1$.

The polynomials $M_n(x)$ are an interesting example of generalized Gegenbauer's polynomials. It is enough to note that

$$g(2x,t) = \sum_{n\geq 0} G_{n,A}^{1}(x)\,t^n \text{ with } A(x) = 1-2x$$

and then $M(2x) = G_{n,A}^{1}(x)$.

Proposition 4.1. *The generalized Catalan's polynomials $M_n(x)$ depend on Catalan's polynomials. More precisely, we get the following expression*

$$M_n(x) = x^n P_n\left(x^{-2}\right) + x^{n-1}(1-x)\,P_{n-1}\left(x^{-2}\right) \tag{4.1}$$

We note that for $x = 1$ only $M_n(1) = P_n(1)$ for any positive integer n.

4.1. Proof of the Proposition 4.1

Since

$$f_{2,0,1}(x,t)=\frac{1}{1-2t+xt^2}$$

then

$$f_{2,0,1}\left(x,\frac{t}{2x}\right)=\frac{1}{1-2x^{-1/2}\left(\frac{x^{-1/2}t}{2}\right)+\left(\frac{x^{-1/2}t}{2}\right)^2}$$

and

$$\left(1+\left(\frac{1}{2}-x^{-1/2}\right)x^{-1/2}t\right)f_{2,0,1}\left(x,\frac{t}{2x}\right)=g\left(2x^{-1/2},\frac{x^{-1/2}t}{2}\right).$$

Replacing the variable $x^{-1/2}$ by x we get

$$\left(1+\left(\frac{1}{2}-x\right)xt\right)f_{2,0,1}\left(1/x^2,\frac{x^2}{2}t\right)=g\left(2x,\frac{xt}{2}\right)$$

Thus

$$\sum_{n\geq 0}M_n(2x)\,2^{-n}x^nt^n=\left(1+\left(\frac{1}{2}-x\right)xt\right)\sum_{n\geq 0}\mathcal{P}_n\left(x^{-2}\right)2^{-n}x^{2n}t^n$$

and

$$\sum_{n\geq 0}M_n(2x)\,2^{-n}x^nt^n=\sum_{n\geq 0}\mathcal{P}_n\left(x^{-2}\right)2^{-n}x^{2n}t^n+\left(\frac{1}{2}-x\right)x\sum_{n\geq 1}\mathcal{P}_{n-1}\left(x^{-2}\right)2^{-n+1}x^{2n-2}t^n$$

After getting the series expansion and comparing the coefficients of t^n by using the relationship between $\mathcal{P}_n(x)$ and $P_n(x)$ we conclude that

$$M_n(2x)=(2x)^n\left(P_n\left(\frac{x^{-2}}{4}\right)-P_{n-1}\left(\frac{x^{-2}}{4}\right)\right)+(2x)^{n-1}P_{n-1}\left(\frac{x^{-2}}{4}\right)$$

Replacing $2x$ by x we get

$$M_n(x)=x^n\left[P_n\left(x^{-2}\right)-P_{n-1}\left(x^{-2}\right)\right]+x^{n-1}P_{n-1}\left(x^{-2}\right)$$

and the result 4.1 follows.

4.2. Explicit form of the class $\{M_n)\}_{n\geq 0}$ and application to combinatorics

Applying the formula 4.1 Proposition 4.1 and the recurrence formula 1.1 to $P_n\left(x^{-2}\right)$, we easily found the following recurrence formula of the class $\{M_n(x)\}_{n\geq 0}$.

$$M_{n+2}(x)=xM_{n+1}(x)-M_n(x) \tag{4.2}$$

Table 4.1: First few polynomials

n	$M_n(x)$
0	1
1	1
2	$x-1$
3	x^2-x-1
4	x^3-x^2-2x+1
5	$x^4-x^3-3x^2+2x+1$
6	$x^5-x^4-4x^3+3x^2+3x-1$

In means of this relation the first few polynomials are given in the table 4.1.

Their binomial sum expression is given in the following lemma

Lemma 4.1.

$$M_{2n}(x)=\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+k-1}{n-k-1}x^{2k}-\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+k}{n-k-1}x^{2k+1} \tag{4.3}$$

$$M_{2n+1}(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}x^{2k}+\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n+k}{n-k-1}x^{2k+1}. \tag{4.4}$$

Proof. From the formula (4.1) and the expression (1.4) of $P_n(x)$ we deduce that

$$P_n\left(x^{-2}\right)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n-k}{k}x^{-2k},$$

and

$$\begin{aligned} M_n(x) &= \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}(-1)^k x^{n-1-2k} \\ &+ \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}(-1)^k x^{n-2k}-\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}(-1)^k x^{n-2k} \end{aligned}$$

After simplification, following the parity of n we get the results (4.3) and (4.4) of Lemma 4.1.
□

The coincidence $M_n(1)=P_n(1)$, the explicit formula of $M_n(x)$ found in (4.3) of Lemma 4.1 and the binomial form of $P_n(x)$ include the following two useful combinatorial identities.

Proposition 4.2.

$$\sum_{k=0}^{n-1}(-1)^k\left(n^2+3k^2+2k\right)\frac{(n+k-1)!}{(n-k)!\,(2k+1)!}=(-1)^{n+1} \tag{4.5}$$

$$\sum_{j=k}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n+1}{2j+1}\binom{j}{k}=2^{n-2k}\binom{n-k}{k} \tag{4.6}$$

4.2.1. Proof of Proposition 4.2

The expression 4.3 Lemma 4.1 of the polynomial $M_{2n}(x)$ conducts to

$$M_{2n}(1)=\sum_{k=0}^{n-1}(-1)^{n-k}\left(\binom{n+k-1}{n-k-1}-\binom{n+k}{n-k-1}\right)$$

Then

$$M_{2n}(1)=-(-1)^n\sum_{k=0}^{n-1}(-1)^k\binom{n+k-1}{n-k-2}$$

But $P_{2n}(1)$ can be written in this form

$$P_{2n}(1)=1+(-1)^n\sum_{k=0}^{n-1}(-1)^k\binom{n+k}{n-k}.$$

Since $M_{2n}(1)=P_{2n}(1)$ then

$$1=(-1)^{n+1}\sum_{k=0}^{n-1}(-1)^k\left(\binom{n+k-1}{n-k-2}+\binom{n+k}{n-k}\right).$$

Using the relation

$$\binom{n+k}{n-k}=\frac{(2k+1)(n+k)}{(n-k)(n-k-1)}\binom{n+k-1}{n-k-2}$$

and the fact that

$$\binom{n+k-1}{n-k-2}+\binom{n+k}{n-k}=\left(n^2+3k^2+2k\right)\frac{(n+k-1)!}{(n-k)!\,(2k+1)!}$$

the result (4.5) of Proposition 4.2 holds.

For the second identity let us denote $f(x)=\sqrt{1-4x}$, then from the closed form (1.2) we deduce that

$$f(x)\,P_n(x)=\frac{1}{2^{n+1}}\left[(1+f(x))^{n+1}-(1-f(x))^{n+1}\right].$$

Using the binomial formula we get

$$f(x)\,P_n(x)=\frac{1}{2^{n+1}}\left[\sum_{j=0}^{n+1}\binom{n+1}{j}f^j(x)-\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j f^j(x)\right].$$

Then

$$f(x)\,P_n(x)=\frac{1}{2^{n+1}}\sum_{j=0}^{n+1}\left(1-(-1)^j\right)\binom{n+1}{j}f^j(x).$$

Furthermore

$$P_n(x)=\frac{1}{2^n}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2j+1}f^{2j}(x)$$

and then

$$P_n(x)=\frac{1}{2^n}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2j+1}(1-4x)^j.$$

Using again the binomial formula for the power $(1-4x)^j$ we obtain

$$P_n(x)=\frac{1}{2^n}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{k=0}^{j}\binom{n+1}{2j+1}\binom{j}{k}(-4x)^k.$$

hence

$$P_n(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{1}{2^{n-2k}}\left[\sum_{j=k}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2j+1}\binom{j}{k}\right](-x)^k.$$

After After comparison with the binomial form (1.4) of $P_n(x)$, the result (4.6) of Proposition (4.2) holds.

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Ser. Math. Inform. Vol. 33, No 2 (2018), 177–196
https://doi.org/10.22190/FUMI1802177B

FIXED POINTS OF ALMOST GENERALIZED (α, β)-(ψ, φ)-CONTRACTIVE MAPPINGS IN b−METRIC SPACES

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Abstract. In this paper, we introduce almost generalized (α, β)-(ψ, φ)-contractive maps, and prove some new fixed point results for this class of mappings in b-metric spaces. We provide examples in support of our results. Our results extend/generalize the results of Dutta and Choudhury [8] and Yamaod and Sintunavarat [14].

Keywords: b-metric space, cyclic (α, β)-admissible mapping, almost generalized (α, β)-(ψ, φ)-contractive mappings, fixed point.

1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of metric space. Banach contraction principle is one of the most useful results in fixed point theory. In the direction of generalization of contraction conditions, in 1997, Alber and Guerre-Delabriere [1] introduced weakly contractive maps, which are extensions of contraction maps, and obtained fixed point results in the setting of Hilbert spaces. Rhoades [12] extended this concept to metric spaces. In 2008, Dutta and Choudhury [8] introduced (ψ, φ)- weakly contractive maps and proved the existence of fixed points in complete metric spaces. In continuation to the extensions of contraction maps, Berinde [4] initiated the concept, namely 'weak contractions', which are renamed 'almost contractions', and established fixed point results. For more work on almost contractions, we refer the reader to [3], [5], [8] and [12].

On the other hand, in the direction of generalization of metric spaces, in 1993, Czerwik [7] introduced the concept of b-metric spaces and proved the Banach contraction mapping principle in this setting, where b-metric need not be continuous. Afterwards, many mathematicians studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. In 2014, Alizadeh, Moradlou and

Received April 20, 2017; accepted September 21, 2017
2010 *Mathematics Subject Classification.*

Peyman [2] introduced the notation of cyclic (α,β)-admissible mappings and proved some fixed point results in the setting of complete metric spaces.

The paper is organized as follows. In Section 2, we present preliminaries and earlier papers that we require to develop the main results. In fact, motivated by the work by Alizadeh, Moradlou and Peyman [2], Berinde [4] and Dutta and Choudhury [8], we introduce almost generalized (α,β)-(ψ,φ)-contractive mappings in this section. In Section 3, we prove our main results in which we study the existence of fixed points of almost generalized (α,β)-(ψ,φ)-contractive mappings. In Section 4, we provide examples in support of our results. Our results extend/generalize the results of Dutta and Choudhury [8] and Yamaod and Sintunavarat [14].

2. Preliminaries

Throughout this paper, $\mathbf{R}$ denotes the real line, and $\mathbf{N}$ is the set of all natural numbers.

In this section, we mention some well-known notations, definitions and known results in the literature that we use in the sequel.

Definition 2.1. [10] A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties hold:
(i) ψ is a continuous and nondecreasing function, and
(ii) $\psi(t)=0$ if and only if $t=0$.
We denote the class of all altering distance functions by Ψ

Definition 2.2. [7] Let X be a non-empty set. A function $d:X\times X\to[0,\infty)$ is said to be a b-metric if the following conditions are satisfied;
(i) $0\leq d(x,y)$ for all $x,y\in X$ and $d(x,y)=0$ if and only if $x=y$,
(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$,
(iii) there exists $s\geq 1$ such that $d(x,z)\leq s\big[d(x,y)+d(y,z)\big]$ for all $x,y,z\in X$.
In this case, the pair (X,d) is called a b-metric space with the coefficient s.

Every metric space is a b-metric space with $s=1$. In general, every b-metric space is not a metric space.

Example 2.1. Let $X=\mathbf{R}$, and let the mapping $d:X\times X\to[0,\infty)$ be defined by $d(x,y)=|x-y|^2$ for all $x,y\in X$. Then (X,d) is a b-metric space with coefficient $s=2$, but it is not a metric.

Example 2.2. Let $0<p<1$. We write $l_p(\mathbf{R})=\{\{x_n\}\subseteq\mathbf{R}|\sum_{n=1}^{\infty}|x_n|^p<\infty\}$, and define $d:l_p(\mathbf{R})\times l_p(\mathbf{R})\to[0,\infty)$ by $d(x,y)=(\sum_{n=1}^{\infty}|x_n-y_n|^p)^{\frac{1}{p}}$ for $x=\{x_n\}$, $y=\{y_n\}$ in $l_p(\mathbf{R})$. Then this d is a b-metric with the coefficient $s=2^{\frac{1}{p}}>1$.

Remark 2.1. A b-metric need not be a continuous function. For more details, we refer [9].

Definition 2.3. [6] Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) The b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence in X is b-convergent. In this case, we say that (X, d) is a complete b-metric space. That is, a b-metric space which is b-complete is a complete b-metric space.

Lemma 2.1. [9] Let (X, d) be a b-metric space with $s \geq 1$.

(i) If a sequence $\{x_n\} \subset X$ is a b-convergent sequence, then it admits a unique limit.

(ii) Every b-convergent sequence in X is b-Cauchy.

Definition 2.4. [6] Let (X, d) and (Y, d') be two b-metric spaces. A function $f : X \to Y$ is b-continuous at $x \in X$ if it is b-sequentially continuous at X. That is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.

Definition 2.5. [11] Let A and B be nonempty subsets of X. A mapping $f : A \cup B \to A \cup B$ is said to be cyclic if $f(A) \subset B$ and $f(B) \subset A$.

In the context of the metric space setting, weakly contractive maps are weaker than the contraction maps [[1], [12]].

Theorem 2.1. [8] Let (X, d) be a complete metric space and $f : X \to X$ be a selfmap of X. If there exist ψ, φ in Ψ such that

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \text{ for all } x, y \in X. \tag{2.1}$$

then f has a unique fixed point.

Here we note that if $\psi(t) = t \geq 0$ in (2.1) then we say that f is a weakly contractive map on X, and hence weakly contractive maps are a special case of the maps satisfying the inequality (2.1).

Definition 2.6. [2] Let X be a nonempty set, f be a selfmap on X and $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that f is a cyclic (α, β)-admissible mapping if

(i) for any $x \in X$ with $\alpha(x) \geq 1 \implies \beta(fx) \geq 1$, and

(ii) for any $y \in X$ with $\beta(y) \geq 1 \implies \alpha(fy) \geq 1$.

In the metric space setting, Alizadeh, Moradlou and Peyman [2] defined (α, β)-(ψ, φ)-contractive mappings as follows.

Definition 2.7. [2] Let (X, d) be a metric space and $f : X \to X$ be a cyclic (α, β) - admissible mapping. We say that $f : X \to X$ is an (α, β)-(ψ, φ)- contractive mapping if

$$\begin{aligned} &x, y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \\ &\implies \psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \end{aligned} \tag{2.2}$$

where
$\psi : [0, \infty) \to [0, \infty)$ is a continuous and increasing function and
$\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 2.2. [2] Let (X, d) be a complete metric space, $\alpha,\ \beta : X \to [0, \infty)$ be two mappings and $f : X \to X$ be an $(\alpha, \beta) - (\psi, \varphi)$-contractive mapping. Suppose that the following conditions hold:
(i) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. If either
(ii) f is continuous or
(iii) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$;
then f has a fixed point in X.
Moreover, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all $x, y \in \mathrm{Fix}(f)$, where $\mathrm{Fix}(f)$ is the set of all fixed points of f, then f has a unique fixed point.

Very recently, Yamaod and Sintunavarat [14] introduced $(\alpha, \beta) - (\psi,\ \varphi)$ -contractive mappings in b-metric spaces as follows:

Definition 2.8. [14] Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $\alpha, \beta : X \to [0, \infty)$ be two given mappings. We say that $f : X \to X$ is an $(\alpha, \beta) - (\psi, \varphi)$- contractive mapping if the following condition holds:
for any $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$ implies

$$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) \tag{2.3}$$

where
$M_s(x, y) = max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2s}\}$ and
$\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions.

Theorem 2.3. [14] Let (X, d) be a complete b-metric space with the coefficient $s \geq 1,\ \alpha, \beta : X \to [0, \infty)$ be two mappings and $f : X \to X$ be an (α, β)-(ψ, φ)-contractive mapping. Suppose that
(1) one of the following condition holds;
(1.1) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$
(1.2) There exists $y_0 \in X$ such that $\beta(y_0) \geq 1$
(2) f is continuous
(3) f is cyclic (α, β) -admissible mapping.
Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in the condition (1.1) and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in the condition (1.2) then $\{x_n\}$ and $\{y_n\}$ converge to a fixed point of f.

Remark 2.2. While proving the Cauchy part of Theorem 2.3, the authors Yamaoda and Sintunavarat [14] claimed the following:

"If $\{x_n\}$ is not Cauchy, then there exists an $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ such that $n(k) > m(k) \geq k,\ m(k)$ is even and $n(k)$ is odd, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $n(k)$ is the smallest number such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k)-1}) < \epsilon$."

But '$m(k)$ is even and $n(k)$ is odd' may not be possible due to the following example:

Example 2.3. Let $X = \mathbb{R}$ with the b-metric defined by $d(x, y) = |x - y|^2, x, y \in \mathbb{R}$. We define the sequence $\{x_n\}$ in X by

$$x_n = \begin{cases} 3^n & \text{if } \ n = 1, 3, 5, 7, ... \\ 3^{n+1} & \text{if } \ n = 2, 4, 6, 8, ... \end{cases}$$

Then clearly the sequence $\{x_n\}$ is not b-Cauchy. Let $\epsilon > 0$. If $\{m(k)\}$ and $\{n(k)\}$ are sequences with $m(k)$ is even and $n(k)$ is odd with $n(k) > m(k) \geq k$ and $n(k)$ is the smallest number such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, then we have $n(k) \neq m(k) + 1$, since $d(x_{m(k)}, x_{n(k)}) = d(x_{m(k)}, x_{m(k)+1}) = 0$.

Now, $d(x_{m(k)}, x_{n(k)}) = d(3^{m(k)+1}, 3^{n(k)}) \geq \epsilon$. But $d(x_{m(k)}, x_{n(k)-1}) = d(3^{m(k)+1}, 3^{n(k)-1+1}) = d(3^{m(k)+1}, 3^{n(k)}) \nless \epsilon$. Hence in the negation of the Cauchy part, it is not possible to mention that "$m(k)$ is even and $n(k)$ is odd ".

Thus, in order to get the valid argument, to prove the Cauchy part of the sequence $\{x_n\}$ of Theorem 2.3, we replace the condition (1) of Theorem 2.3 by the following:

(H): " there exists x_0 in X such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$".

Thus the modified version of Theorem 2.3 is the following, and since it follows as a corollary to Theorem 3.1 (we prove Theorem 3.1 in Section 3) and the proof of the Cauchy part of Theorem 3.1 is proved without using the property '$m(k)$ is even and $n(k)$ is odd' (Remark 2.2), we just state this result without proof.

Theorem 2.4. Let (X, d) be a complete b-metric space with the coefficient $s \geq 1,\ \alpha,\ \beta : X \to [0, \infty)$ be two mappings and $f : X \to X$ be an (α, β)-(ψ, φ)-contractive mapping. Suppose that

(1) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$
(2) f is continuous
(3) f is cyclic (α, β)-admissible mapping.

Then f has a fixed point.

Moreover, for $x_0 \in X$ which is as in (1), if the sequence $\{x_n\}$ in X defined by $x_{n+1} = fx_n$ then the sequence $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to a fixed point of f.

Now, we introduce almost generalized (α, β)-(ψ, φ)-contractive mappings in b-metric spaces in the following:

Definition 2.9. Let (X, d) be a b-metric space with the coefficient $s \geq 1$, and let $\alpha, \beta : X \to [0, \infty)$ be two given mappings. Let $f : X \to X$ be a selfmap of X. If there exist $\psi, \varphi \in \Psi$ and $L \geq 0$ such that

$$\text{(2.4)} \quad \begin{aligned} &\text{for all } x, y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \\ &\Longrightarrow \psi(s^3 d(fx, fy)) \leq \psi(M_s(x,y)) - \varphi(M^{'}(x,y)) + LN(x,y), \end{aligned}$$

where
$M_s(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\}$
$M^{'}(x,y) = \max\{d(x,y), d(y,fy)\}$ and
$N(x,y) = \min\{d(x,fx), d(y,fx)\}$,
then we say that f is an almost generalized (α,β)-(ψ,φ) - contractive mapping.

Here we note that if $L = 0$ in (2.4), it becomes:

$$\text{(2.5)} \quad \begin{aligned} &\text{for all } x, y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \\ &\Longrightarrow \psi(s^3 d(fx, fy)) \leq \psi(M_s(x,y)) - \varphi(M^{'}(x,y)). \end{aligned}$$

Further, by the nondecreasing nature of φ, the inequality (2.3) implies (2.5) so that the inequality (2.5) is weaker than (2.3).

Example 2.4. Let $X = [0,1] \cup \{2, 3, ..., \}$. We define $d : X \times X \to [0,\infty)$ by

$$d(n,m) = \begin{cases} 0 & \text{if} \quad n = m \\ |\frac{1}{n} - \frac{1}{m}| & \text{if} \quad n, m \in \{2,4,6,...\} \\ 5 & \text{if} \quad n, m \in \{1,3,5,...\} \\ 2 & \text{otherwise} . \end{cases}$$

Clearly d is a b-metric space with the coefficient $s = \frac{5}{4}$.
Now, we define $f : X \to X$ by

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0,1] \\ 2x - 1 & \text{if} \quad x \in \{1,2,3,...\} \end{cases}$$

and $\alpha, \beta : X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ \frac{x+1}{2} & \text{if } x \in \{1,2,3,...\} \end{cases} \text{ and } \beta(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ \frac{x+2}{3} & \text{if } x \in \{1,2,3,...\}. \end{cases}$$

Now, we show that f is cyclic (α,β)-admissible mapping. Since for any $x \in X$ $\alpha(x) \geq 1 \Leftrightarrow x \in \{1,2,3,...\}$, we have $\beta(fx) = \beta(2x-1) = \frac{2x+1}{3} \geq 1$ for all $x \in \{1,2,3,...\}$. Also, for any $x \in X$ $\beta(x) \geq 1 \Leftrightarrow x \in \{1,2,3,...\}$, we have $\alpha(fx) = \alpha(2x-1) = \frac{2x}{2} = x \geq 1$ for all $x \in \{1,2,3,...\}$.
Therefore, f is a cyclic (α,β)-admissible mapping.

For $x, y \in X$ with $\alpha(x)\beta(x) \geq 1 \iff x, y \in \{1,2,3,...\}$, which implies that $fx = 2x-1$ and $fy = 2y - 1$, therefore fx and fy are odd, and hence $d(fx, fy) = d(2x-1, 2y-1) = 5$ for all $x, y \in \{1,2,3,...\}$.

We choose $\psi(t) = t, \quad \varphi(t) = \frac{3t}{4}, t \geq 0$.
Now, we consider the following cases to show that f is almost generalized (α,β)-(ψ,φ)- contractive mapping with $L = \frac{25}{4}$.

Case (i) : $x, y \in \{1, 3, 5, ...\}$.
In this case,

$$M_s(x,y) = \max\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{d(x,fy)+d(y,fx)}{2(\frac{5}{4})}\}$$
$$= \max\{5,\ 5,\ 5,\ 2(\frac{5+5}{5})\} = \max\{5,\ 4\} = 5,$$

$M'(x,y) = \max\{d(x,y),\ d(y,fy)\} = \max\{5,\ 5\} = 5$, and
$N(x,y) = \min\{d(x,fx),\ d(y,fx)\} = \min\{5,\ 5\} = 5$.
Now, we have

$$\begin{aligned}\psi(s^3 d(fx,fy)) = \ & \psi((\frac{5}{4})^3 d(2x-1, 2y-1)) \ = \ \psi(\frac{625}{64}) \ = \ \frac{625}{64} \\ \leq \ & 5 - (\frac{3}{4})5 \ + \ (\frac{25}{4})5 \\ \leq \ & \psi(M_s(x,y)) \ - \ \varphi(M^{'}(x,y)) \ + \ LN(x,y)\end{aligned}$$

where $L = \frac{25}{4}$.
Case (ii) : $x, y \in \{2,\ 4,\ 6,\ ...\}$.
Here

$$M_s(x,y) = \max\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{d(x,fy)+d(y,fx)}{2(\frac{5}{4})}\}$$
$$= \max\{|\frac{1}{x}-\frac{1}{y}|,\ 2,\ 2,\ 2(\frac{2+2}{5})\} = \max\{|\frac{1}{x}-\frac{1}{y}|,\ 2,\ \frac{8}{5}\} = 2,$$

$M'(x,y) = \max\{d(x,y),\ d(y,fy)\} = \max\{|\frac{1}{x}-\frac{1}{y}|,\ 2\} = 2$, and
$N(x,y) = \min\{d(x,fx),\ d(y,fx)\} = \min\{2,\ 2\} = 2$.
Now, we have

$$\begin{aligned}\psi(s^3 d(fx,fy)) = \ & \psi((\frac{5}{4})^3 d(2x-1, 2y-1)) \ = \ \psi(\frac{625}{64}) \ = \ \frac{625}{64} \leq \ 2 - (\frac{3}{4})2 + (\frac{25}{4})2 \\ \leq \ & \psi(M_s(x,y)) - \varphi(M^{'}(x,y)) + LN(x,y)\end{aligned}$$

where $L = \frac{25}{4}$.
Case (iii) : $x \in \{1,\ 3,\ 5,\ ...\}$, $\quad y \in \{2,\ 4,\ 6,\ ...\}$.
Here

$$M_s(x,y) = \max\{d(x,y),\ d(x,fx),\ d(y,fy),\ \frac{d(x,fy)+d(y,fx)}{2(\frac{5}{4})}\}$$
$$= \max\{2,\ 5,\ 2,\ 2(\frac{5+2}{5})\} = \max\{5,\ 2,\ \frac{14}{5}\} = 5,$$

$M'(x,y) = \max\{d(x,y),\ d(y,fy)\} = \max\{2,\ 2\} = 2$, and
$N(x,y) = \min\{d(x,fx),\ d(y,fx)\} = \min\{5,\ 2\} = 2$.
Now, we have

$$\begin{aligned}\psi(s^3 d(fx,fy)) = \ & \psi((\frac{5}{4})^3 d(2x-1, 2y-1)) \ = \ \psi(\frac{625}{64}) \ = \ \frac{625}{64} \\ \leq \ & 5 - (\frac{3}{4})2 + (\frac{25}{4})2 \\ \leq \ & \psi(M_s(x,y)) - \varphi(M^{'}(x,y)) + LN(x,y)\end{aligned}$$

where $L = \frac{25}{4}$
<u>*Case*</u> (iv) : $y \in \{1,\ 3,\ 5, ...\},\quad x \in \{2,\ 4,\ 6,\ ...\}$.
In this case
$M_s(x,y) = \max\{2,\ 2,\ 5,\ 2\big(\frac{5+2}{5}\big)\} = 5$,
$M'(x,y) = \max\{d(x,y),\ d(y,fy)\} = \max\{2,\ 5\} = 5$, and
$N(x,y) = \min\{d(x,fy),\ d(y,fx)\} = \min\{2,\ 5\} = 2$.
Now, we have

$$\begin{aligned}\psi(s^3 d(fx,fy)) =&\ \psi((\frac{5}{4})^3 d(2x-1, 2y-1)) = \psi(\frac{625}{64})\\ \leq&\ \psi(M_s(x,y)) - \varphi(M^{'}(x,y)) + LN(x,y),\end{aligned}$$

as in the case (iii).
Hence, from all the above cases f is an almost generalized $(\alpha,\beta)-(\psi,\varphi)$-contractive mapping. Here we observe that if $L = 0$ then for any $x, y \in \{1, 3, 5, ...\}$

$$\begin{aligned}\psi(s^3 d(fx,fy)) =&\ \psi((\frac{5}{4})^3 d(2x-1, 2y-1)) \ = \ \psi(\frac{625}{64}) \ \nleq\ \psi(5)\\ \nleq&\ \psi(5) -\ \varphi(5) = \psi(M_s(x,y)) -\ \varphi(M_s(x,y))\end{aligned}$$

for any $\psi, \varphi \in \Psi$, so that f is not an (α,β)-(ψ,φ)- contractive mapping.

Hence the class of 'almost generalized $(\alpha,\beta)-(\psi,\varphi)$- contractive maps' is larger than 'the class of (α,β)-(ψ,φ)- contractive maps'.

Further, we observe that the metric d defined in this example is not a metric in the usual sense for, by choosing $x = 1,\ y = 2$ and $z = 3$, we have $d(x,y) = 5 \nleq 2+2 = d(x,z)+d(z,y)$.

We state the following lemma which is useful to prove our main results.

Lemma 2.2. [13] Suppose (X,d) is a b-metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(m_k, n_k) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon,\ d(x_{m_k}, x_{n_k-1}) < \epsilon$ and

(i) $\epsilon \leq \limsup\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon$

(ii) $\frac{\epsilon}{s} \leq \liminf\limits_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \limsup\limits_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\epsilon$

(iii) $\frac{\epsilon}{s} \leq \liminf\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\epsilon$

(iv) $\frac{\epsilon}{s^2} \leq \liminf\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon$.

3. Main results

Theorem 3.1. Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Let $f : X \to X$ be a selfmapping of X. Assume that there exist two mappings $\alpha, \beta : X \to [0,\infty)$ and $\psi, \varphi \in \Psi$ such that f is an almost generalized $(\alpha,\beta)-(\psi,\varphi)$ - contractive mapping.

Further, suppose that
(1) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
(2) f is continuous,
(3) f cyclic (α, β) - admissible mapping.
Then the sequence $\{x_n\}$ in X defined by $x_{n+1} = fx_n, n = 0, 1, 2, \ldots$, where $x_0 \in X$ is given as in (1) is b-Cauchy and it is b-convergent to z (say) in X, and z is a fixed point of f.

Proof. By (1) we have $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
Now, we define an iterative sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for $n = 0, 1, 2, \ldots$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbf{N} \cup \{0\}$, we have $fx_{n_0} = x_{n_0+1} = x_{n_0}$, so that x_{n_0} is a fixed point of f and we are through.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbf{N} \cup \{0\}$. Since $\alpha(x_0) \geq 1$ and f is cyclic (α, β)-admissible mapping, we have $\beta(x_1) = \beta(fx_0) \geq 1$, and this implies that $\alpha(x_2) = \alpha(fx_1) \geq 1$. By continuing this process, we obtain

$$\alpha(x_{2k}) \geq 1 \text{ and } \beta(x_{2k+1}) \geq 1 \text{ for all } k \in \mathbf{N} \cup \{0\}. \tag{3.1}$$

Since, $\beta(x_0) \geq 1$ and f is a cyclic (α, β)-admissible mapping, we have $\alpha(x_1) = \alpha(fx_0) \geq 1$ and this implies that $\beta(x_2) = \beta(fx_1) \geq 1$. On continuing this process, we obtain

$$\beta(x_{2k}) \geq 1 \text{ and } \alpha(x_{2k+1}) \geq 1 \text{ for all } k \in \mathbf{N} \cup \{0\}. \tag{3.2}$$

Therefore, from (3.1) and (3.2) we have $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbf{N} \cup \{0\}$.
First we claim that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Since $\alpha(x_n)\beta(x_{n+1}) \geq 1$ for all $n \in \mathbf{N} \cup \{0\}$, from (2.4), we have

$$\begin{aligned} \psi(s^3 d(fx_n, fx_{n+1})) \ \leq \ & \psi(M_s(x_n, x_{n+1})) - \varphi(M^{'}(x_n, x_{n+1})) \\ & + LN(x_n, x_{n+1}) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M_s(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s}\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2s}\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}, \end{aligned}$$

$$M'(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, fx_{n+1})\} = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\},$$

and

$$\begin{aligned} N(x_n, x_{n+1}) &= \min\{d(x_n, fx_n), d(x_{n+1}, fx_n)\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1})\} \\ &= \min\{d(x_n, x_{n+1}), 0\} = 0. \end{aligned}$$

Now, if $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$, it follows from (3.3) that

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\
&\leq \psi(s^3 d(fx_n, fx_{n+1})) \\
&\leq \psi(d(x_{n+1}, x_{n+2})) - \varphi(d(x_{n+1}, x_{n+2})) + L(0) \\
&= \psi(d(x_{n+1}, x_{n+2})) - \varphi(d(x_{n+1}, x_{n+2})) \\
&< \psi(d(x_{n+1}, x_{n+2})),
\end{aligned}$$

a contradiction.
Hence
$d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ for all $n \in \mathbf{N} \cup \{0\}$.
Therefore we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\
&\leq \psi(s^3 d(fx_n, fx_{n+1})) \\
&\leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) \\
&< \psi(d(x_n, x_{n+1})).
\end{aligned}$$

Since ψ is nondecreasing, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Thus there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose $r > 0$. Hence we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\
&\leq \psi(s^3 d(fx_n, fx_{n+1})) \\
&\leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) + \theta(0) \\
&= \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})).
\end{aligned} \tag{3.4}$$

On letting $n \to \infty$ and using the continuity of ψ and φ in (3.4), we have
$\psi(r) \leq \psi(s^3 r) \leq \psi(r)) - \varphi(r) < \psi(r)$,
a contradiction.
Hence $r = 0, i.e., \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

We now prove that $\{x_n\}$ is a b-Cauchy sequence. Suppose $\{x_n\}$ is not a b-Cauchy sequence. Then by Lemma 2.2 there exist $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and (i) - (iv) of Lemma 2.2 hold.
Since $\alpha(x_{m_k}) \geq 1$ and $\beta(x_{n_k}) \geq 1$ which implies that $\alpha(x_{m_k})\beta(x_{n_k}) \geq 1$. Now, from (2.4) we have

$$\begin{aligned}
\psi(d(x_{m_k+1}, fx_{n_k+1})) &= \psi(d(fx_{m_k}, fx_{n_k})) \\
&\leq \psi(s^3 d(fx_{m_k}, fx_{n_k})) \\
&\leq \psi(M_s(x_{m_k}, x_{n_k}) - \varphi(M'(x_{m_k}, x_{n_k})) \\
&\qquad + LN(x_{m_k}, x_{n_k}),
\end{aligned} \tag{3.5}$$

where
$M_s(x_{m_k}, x_{n_k}) = \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, fx_{m_k}), d(x_{n_k}, fx_{n_k}), \frac{d(fx_{m_k}, x_{n_k}) + d(x_{m_k}, fx_{n_k})}{2s}\}$,

$M'(x_{m_k}, x_{n_k}) = \max\{d(x_{m_k}, x_{n_k}), d(x_{n_k}, fx_{n_k})\}$, and
$N(x_{m_k}, x_{n_k}) = \min\{d(x_{m_k}, fx_{m_k}), d(x_{n_k}, fx_{m_k})\}$.
On taking Limit supremum as $n \to \infty$, we have

$$\begin{aligned} &\epsilon \leq \limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\} = s\epsilon, \\ &\epsilon \leq \limsup_{k\to\infty} M'(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \} = s\epsilon, \text{ and} \\ &\limsup_{k\to\infty} N(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \} = 0. \end{aligned} \tag{3.6}$$

Also on taking limit infimum as $k \to \infty$, we have
$\epsilon \leq \liminf_{k\to\infty} M'(x_{m_k}, x_{n_k}) \leq \limsup_{k\to\infty} M'(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0\} = s\epsilon.$
Now, using (3.5), we have

$$\begin{aligned} \psi(s\epsilon) = \psi(s^3 \frac{\epsilon}{s^2}) \leq &\ \psi(s^3 \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1})) \\ = &\ \psi(s^3 \limsup_{k\to\infty} d(fx_{m_k}, fx_{n_k}) \\ \leq &\ \psi(\limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k}) - \varphi(\liminf_{k\to\infty} M^{'}(x_{m_k}, x_{n_k})) \\ &\qquad + L \limsup_{k\to\infty} N(x_{m_k}, x_{n_k})) \\ \leq &\ \psi(s\epsilon) - \varphi(\epsilon) \\ < &\ \psi(s\epsilon), \end{aligned}$$

a contradiction. So we conclude that $\{x_n\}$ is a b-Cauchy sequence in (X, d).
Since (X, d) is b-complete, it follows that there exists $z \in X$ such that
$\lim_{n\to\infty} x_n = z.$
Since f is continuous , we have $\lim_{n\to\infty} fx_n = fz$, and
$fz = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = z.$ □

Theorem 3.2. Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Let $f : X \to X$ be a selfmapping of X. Assume that there exist two mappings $\alpha, \beta : X \to [0, \infty)$ and $\psi, \varphi \in \Psi$ such that f is an almost generalized $(\alpha, \beta) - (\psi, \varphi)$ - contractive mapping.

Further, suppose that

(1) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
(2) f cyclic (α, β) - admissible mapping,
(3) If $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\beta(x_n) \geq 1$ for all n, then $\beta(z) \geq 1$.

Then f has a fixed point.

Proof. From the similar arguments as in the proof of Theorem 3.1 we obtain the sequence $\{x_n\}$ is Cauchy and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since (X, d) is

b- complete b- metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. From (3) we have $\beta(z) \geq 1$.
We assume that $fz \neq z$. From the triangular inequality, we have
$d(z, fz) \leq s[d(z, fx_n) + d(fx_n fz)]$.
On taking the limit supremum as $n \to \infty$, we have

$$\frac{1}{s} d(z, fz) \leq \limsup_{n\to\infty} d(fx_n, fz). \tag{3.7}$$

Also we have $d(fx_n, fz) \leq s[d(fx_n, z) + d(zfz)]$.
On taking the limit supremum as $n \to \infty$, we have

$$\limsup_{n\to\infty} d(fx_n, fz) \leq sd(z, fz). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{1}{s} d(z, fz) \ \leq \ \limsup_{n\to\infty} d(fx_n, fz) \leq sd(z, fz). \tag{3.9}$$

From (2.4), we have

$$\begin{aligned} \psi(d(z, fz)) \leq \psi(s^2 d(z, fz)) \ &= \psi(s^3[\frac{1}{s} d(z, fz)]) \\ &\leq \psi(s^3[\limsup_{n\to\infty} d(fx_n, fz)]) \\ &= \limsup_{n\to\infty} \psi(s^3[d(fx_n, fz)]) \\ &\leq \limsup_{n\to\infty} [\psi(M_s(x_n, z)) - \varphi(M^{'}(x_n, z)) \\ &\quad + LN(x_n, z)]. \end{aligned} \tag{3.10}$$

Hence we have

$$\begin{aligned} \psi(d(z, fz)) \leq \psi(s^2 d(z, fz)) \ &\leq \ \limsup_{n\to\infty} \psi(M_s(x_n, z)) \\ &\quad + \ \limsup_{n\to\infty} (-\varphi(M^{'}(x_n, z))) \\ &\quad + \ L \limsup_{n\to\infty} N(x_n, z), \end{aligned} \tag{3.11}$$

where
$d(z, fz) \leq M_s(x_n, z) = \max\{d(x_n, z), d(x_n, fx_n), d(z, fz), \frac{d(x_n, fz) + d(z, fx_n)}{2s}\}$,
$d(z, fz) \leq M'(x_n, z) = \max\{d(x_n, z), d(z, fz)\}$,
$N(x_n, z) = \min\{d(x_n, fx_n), d(z, fx_n)\}$.

On taking the limits of $M_s(x_n, z)$, $M'(x_n, z)$ and $N(x_n, z)$ as $n \to \infty$ and using (3.9), we have

$d(z, fz) \leq \lim_{n\to\infty} M_s(x_n, z) = \max\{0, 0, d(z, fz), \limsup_{n\to\infty} \frac{d(x_n, fz)}{2s}\} = d(z, fz),$

$\lim_{n\to\infty} M'(x_n, z) = \max\{0,\ d(z, fz)\} = d(z, fz),$

$\limsup_{k\to\infty} N(x_n, z) = \{0, 0, \} = 0.$

From (3.11) we have

$$\begin{aligned} \psi(d(z, fz)) \leq&\ \psi(d(z, fz)) - \varphi(d(z, fz)) \\ <&\ \psi(d(z, fz)), \end{aligned} \tag{3.12}$$

a contradiction. Hence $fz = z$. □

Theorem 3.3. In addition to the hypothesis of Theorem 3.1 (Theorem 3.2), if $\alpha(u) \geq 1$ or $\beta(u) \geq 1$ whenever $fu = u$. Then f has a unique fixed point.

Proof. Suppose that u and w be two fixed points of f with $u \neq w$, that is, $fu = u$ and $fw = w$. By the hypothesis we have $\alpha(u) \geq 1$ or $\beta(u) \geq 1$ and $\alpha(w) \geq 1$ or $\beta(w) \geq 1$. Since f is a cyclic (α, β)-admissible mapping, we have
$\alpha(u) \geq 1 \implies \beta(u) = \beta(fu) \geq 1$, and $\beta(u) \geq 1 \implies \alpha(u) = \alpha(fu) \geq 1$,
Therefore we have $\beta(u) \geq 1$ and $\alpha(u) \geq 1$. And also
$\alpha(w) \geq 1 \implies \beta(w) = \beta(fw) \geq 1$, and $\beta(w) \geq 1 \implies \alpha(w) = \alpha(fw) \geq 1$. then we have $\beta(w) \geq 1$ and $\alpha(w) \geq 1$. Hence we have $\alpha(w) \geq 1$, $\alpha(u) \geq 1$, $\beta(w) \geq 1$ and $\beta(u) \geq 1$ this implies $\alpha(u)\beta(w) \geq 1$.
Now, from (2.4) we have

$$\begin{aligned} \psi(d(u, w)) = \psi(d(fu, fw)) &\leq \psi(s^3 d(fu, fw)) \\ &\leq \psi(M_s(u, w)) - \varphi(M'(u, u)) + LN(u, w) \end{aligned} \tag{3.13}$$

Where

$$\begin{aligned} M_s(u, w) &= \max\{d(u, w), d(u, fu), d(w, fw), \frac{d(u, fw) + d(w, fu)}{2s}\} \\ &= \max\{d(u, w), d(u, u), d(w, w), \frac{d(u, w) + d(w, u)}{2s}\} \\ &= \max\{d(u, w), 0, \frac{d(u, w)}{s}\} \\ &= d(u, w) \end{aligned} \tag{3.14}$$

$M'(u, w) = \max\{d(u, w), d(w, fw)\} = \max\{d(u, w), d(w, w)\} = \max\{d(u, w), 0\} = d(u, w),$
$N(u, w) = \min\{d(u, fu), d(w, fu)\} = \min\{d(u, u), d(w, u)\} = \min\{0, d(w, u)\} = 0.$
by using the inequality (3.13), we have

$$\begin{aligned} \psi(d(u, w)) &= \psi(d(fu, fw)) \leq \psi(s^3 d(fu, fw)) \\ &\leq \psi(M_s(u, w)) - \varphi(M'(u, u)) + LN(u, w) \\ &= \psi(d(u, w)) - \varphi(d(u, w)) + L\theta(0) \\ &= \psi(d(u, w)) - \varphi(d(u, w)) \\ &< \psi(d(u, w)), \end{aligned} \tag{3.15}$$

a contradiction.Therefore $u = w$
Hence f has a unique fixed point. □

Definition 3.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$, and A and B be two closed subsets of X such that $A \cap B \neq \emptyset$. Let $f : A \cup B \to A \cup B$ be a mapping. If there exist $\psi, \varphi \in \Psi$ and $L \geq 0$ such that

$$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M^{'}(x, y)) + LN(x, y), \tag{3.16}$$

for all $x \in A$ and $y \in B$. Then we say that f is an almost generalized (A, B)-(ψ, φ)-contractive mapping.

Theorem 3.4. Let A and B be two nonempty closed subsets of a complete b-metric space (X, d) such that $A \cap B \neq \emptyset$, and let $f : A \cup B \to A \cup B$ be a cyclic mapping. If f is an almost generalized (A, B)-(ψ, φ)-contractive mapping, then f has a unique fixed point in $A \cap B$.

Proof. Let us define $\alpha, \beta : A \cup B \to A \cup B$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \qquad \beta(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

For any $x, y \in A \cup B$ with $\alpha(x)\beta(y) \geq 1$, this implies that $x \in A$ and $y \in B$, then from the hypothesis we have
$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M^{'}(x, y)) + LN(x, y)$.
Thus, inequality (3.16) holds. Therefore f is an almost generalized $(\alpha, \beta) - (\psi, \varphi)$-contractive mapping.
Since $A \cap B \neq \emptyset$ there exists $x_0 \in A \cap B$ this implies that $x_0 \in A$ and $x_0 \in B$ hence $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.
Let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $x_n \in B$ for all $n \in \mathbb{N} \cup \{0\}$. Since B is closed we have $x \in B$ hence $\beta(x) \geq 1$.
Therefore all hypotheses of Theorem 3.2 hold.
Hence f has a fixed point. Let u (say) be the fixed point of f. If $u \in A$, then $u = fu \in B$. Similarly, if $u \in B$, then $u = fu \in A$.
Hence $u \in A \cap B$. And also $\alpha(u) \geq 1$ and $\beta(u) \geq 1$. Therefore, by Theorem 3.3, f has a unique fixed point. □

4. Corollaries and examples

Corollary 4.1. Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and $f : X \to X$ be a continuous selfmaping of X. If there exist $\psi, \varphi \in \Psi$ such that
$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M^{'}(x, y)) + LN(x, y)$ for all $x, y \in X$.
Then f has a fixed point.

Proof. By choosing $\alpha(x) = \beta(x) = 1$ in Theorem 3.1, the conclusion of this corollary follows. □

Corollary 4.2. Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Let $f : X \to X$ be a selfmapping of X. Assume that there exist two mappings $\alpha, \beta : X \to [0,\infty)$ and $\psi, \varphi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$
$\implies \psi(d(fx, fy)) \leq \psi(M(x,y)) - \varphi(M'(x,y)) + LN(x,y)$.
Further, suppose that
(1) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
(2) f is continuous,
(3) f cyclic (α,β) - admissible mapping.
Then the sequence $\{x_n\}$ in X defined by $x_{n+1} = fx_n, n = 0,1,2,\dots$, where $x_0 \in X$ is given as in (1) is b-Cauchy and it is b-convergent to z (say) in X, and z is a fixed point of f.

Proof. The result follows from Theorem 3.1 by taking $s = 1$. □

By choosing $s = 1$ and $\alpha(x) = \beta(x) = 1$ in Theorem 3.1, we have the following.

Corollary 4.3. Let (X,d) be a complete metric space and $f : X \to X$ be selfmapping of X. If there exist $\psi, \varphi \in \Psi$ such that
$\psi(d(fx, fy)) \leq \psi(M(x,y)) - \varphi(M'(x,y)) + LN(x,y)$ for all $x, y \in X$.
Then f has a fixed point.

Remark 4.1. Here we observe that Theorem 2.1 is a corollary to Corollary 4.3. For any $x, y \in X$ we have

$$\begin{aligned}\psi(d(fx,fy)) &\leq \psi(d(x,y)) - \varphi(d(x,y)) \leq \psi(M_s(x,y)) - \varphi({M'}_s(x,y)) \\ &\leq \psi(M_s(x,y)) - \varphi(M'(x,y)) + LN(x,y).\end{aligned}$$

Corollary 4.4. (Theorem 2.4) Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Let $f : X \to X$ be a selfmapping of X. Assume that there exist two mappings $\alpha, \beta : X \to [0,\infty)$ and $\psi, \varphi \in \Psi$ such that for all
$x, y \in X$ with $\alpha(x)\beta(y) \geq 1$
$\implies \psi(s^3 d(fx, fy)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y))$
Further, suppose that
(1) there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,
(2) f is continuous,
(3) f cyclic (α,β) - admissible mapping.
Then the sequence $\{x_n\}$ in X defined by $x_{n+1} = fx_n, n = 0,1,2,\dots$, where $x_0 \in X$ is given as in (1) is b-Cauchy and it is b-convergent to z (say) in X, and z is a fixed point of f.

Proof. By hypothesis, we have for all $x, y \in X$ with $\alpha(x)\beta(y) \geq 1$

$$\begin{aligned}\implies \psi(s^3 d(fx,fy)) &\leq \psi(M_s(x,y)) - \varphi(M_s(x,y)) \\ &\leq \psi(M_s(x,y)) - \varphi(M'(x,y)) \\ &\leq \psi(M_s(x,y)) - \varphi(M'(x,y)) + LN(x,y)\end{aligned}$$

Therefore, f satisfies the inequality (2.4) with $L = 0$. Hence by Theorem 3.1, f has a fixed point $\square$

Corollary 4.5. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that $A \cap B \neq \emptyset$, and let $f : A \cup B \to A \cup B$ be a cyclic mapping. If there exist $\psi, \varphi \in \Psi$ and $L \geq 0$ such that
$\psi(d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M^{'}(x, y)) + LN(x, y)$, then f has a unique fixed point in $A \cap B$.

Proof. By choosing $s = 1$ in Theorem 3.4 the conclusion of the corollary follows. $\square$

Corollary 4.6. Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that $A \cap B \neq \emptyset$, and let $f : A \cup B \to A \cup B$ be a cyclic mapping. If there exist $\psi, \varphi \in \Psi$ such that
$\psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M^{'}(x, y))$, then f has a unique fixed point in $A \cap B$.

Proof. follows from Theorem 3.4 by taking $L = 0$. $\square$

Example 4.1. Let $X = [2, 3] \cup \{4, 5, 6, ...\}$, we define $d : X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{x} + \frac{1}{y} & \text{if } x, y \in [2,3] \\ 4 + \frac{1}{x} + \frac{1}{y} & \text{if } x, y \in \{4, 5, 6, ...\} \\ 2 & \text{otherwise.} \end{cases}$$

Clearly, d is a b-metric space with the coefficient $s \geq \frac{89}{80}$.
We define $f : X \to X$ by

$$f(x) = \begin{cases} 3 - \frac{x}{4} & \text{if } x \in [2,3] \\ 2 + \frac{3}{4x} & \text{if } x \in \{3, 4, 5, 6, ...\}. \end{cases}$$

and $\alpha, \beta : X \to [0, \infty)$ by

$$\alpha(x) = \begin{cases} \frac{3}{x} & \text{if } x \in [2,3] \\ 0 & \text{otherwise,} \end{cases} \qquad \beta(x) = \begin{cases} \frac{4}{x} & \text{if } x \in [2,3] \\ 0 & \text{otherwise.} \end{cases}$$

Since for any $x \in X, \alpha(x) \geq 1 \iff x \in [2, 3]$, we have
$\beta(fx) = \frac{3}{fx} = \frac{3}{3-\frac{x}{4}} \geq 1$, and also $x \in X, \beta(x) \geq 1 \iff x \in [2, 3]$, we have
$\alpha(fx) = \frac{4}{fx} = \frac{4}{3-\frac{x}{4}} \geq 1$. Therefore, f is cyclic (α, β) - admissible mapping.
Next, we show that f is an almost generalized $(\alpha, \beta) - (\psi, \varphi)$- contractive mapping.

For $x, y \in X$ with $\alpha(x)\beta(x) \geq 1 \iff x, y \in [2, 3]$. Hence, for $x, y \in [2, 3]$
$fx = 3 - \frac{x}{4}$ and $fy = 3 - \frac{y}{4}$ and for $x \neq y$ we have

$$\begin{aligned} M_s(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\} \\ &= \max\{\frac{1}{x}+\frac{1}{y}, \frac{1}{x}+\frac{1}{fx}, \frac{1}{y}+\frac{1}{fy}, \frac{\frac{1}{x}+\frac{1}{fy}+\frac{1}{y}+\frac{1}{fx}}{2(\frac{89}{80})}\} \geq \frac{2}{3}, \end{aligned}$$

$M'(x,y) \ = \ \max\{d(x,y), d(y,fy) \ = \ \max\{\frac{1}{x}+\frac{1}{y}, \frac{1}{y}+\frac{1}{fy}\} \leq 1$, and
$N(x,y) \ = \ \min\{d(x,fx), d(y,fx)\} \ = \ \min\{\frac{1}{y}+\frac{1}{fx}, \frac{1}{x}+\frac{1}{fx}\} \ \geq \frac{2}{5}$.

We now choose $\psi(t) \ = \ t, \varphi(t) \ = \ \frac{t}{4}, \ t \geq 0$ then, we have

$$
\begin{aligned}
\psi(s^3 d(fx,fy)) &= d((\frac{89}{80})^3 d(3-\frac{x}{4}, 3-\frac{y}{4})) \\
&= (\frac{89}{80})^3(\frac{1}{3-\frac{x}{4}} + \frac{1}{3-\frac{y}{4}}) \\
&\leq (\frac{89}{80})^3(\frac{8}{9}) \qquad (4.1)\\
&\leq \frac{2}{3} - \frac{1}{4}(1) + 3(\frac{2}{5}) \\
&\leq \psi(M_s(x,y) \ - \ \varphi(M'(x,y)) \ + \ LN(x,y).
\end{aligned}
$$

Hence f is an almost generalized $(\alpha,\beta)-(\psi,\varphi)$- contractive mapping with $L=3$. Therefore, f satisfies all the hypotheses of Theorem 3.1 and $x=\frac{12}{5}$ is a fixed point of f.

Here we observe that when $L=0$, the inequality (2.4) fails to hold for $x=2$ and $y=3$, we have

$$
\begin{aligned}
\psi(s^3 d(fx,fy)) &= d((\frac{89}{80})^3 d(f2,f3)) \ = \ (\frac{89}{80})^3(\frac{38}{45}) \\
&\nleq \psi(\frac{9}{10}) - \varphi(\frac{5}{6}) \\
&= \psi(M_s(2,3) \ - \ \varphi(M'(2,3)) \ = \ \psi(M_s(x,y) \ - \ \varphi(M'(x,y)).
\end{aligned}
$$

for any ψ and φ. Hence f is not an (α,β)-(ψ,φ)-contractive mapping. Therefore Theorem 2.4 is not applicable.

Further, this example shows the importance of L in the inequality (2.4) of the almost generalized (α,β)-(ψ,φ)- contractive mapping.

Remark 4.2. From Example 4.1 and Corollary 4.4 we observe that Theorem 3.1(Theorem 3.2) is a generalization of Theorem 2.4.

Example 4.2. Let $X=\{0,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}\cup[1,2]$. We define $d: X\times X \to [0,\infty)$ by

$$
d(x,y) = \begin{cases} 0 & \text{if } \ x=y \\ |x-y| & \text{if } \ x,y \in \{0,\frac{1}{2},\frac{1}{4},\frac{1}{6},...\} \\ 6 & \text{if } \ x,y \in \{\frac{1}{3},\frac{1}{5},\frac{1}{7},...\} \text{ and } x \neq y \\ 2 & \text{otherwise.} \end{cases}
$$

Then it is easy to see that (X,d) is a b-metric space with the coefficient $s=\frac{3}{2}$. Now, we define $f: X \to X$ by

$$
fx = \begin{cases} 2-x & \text{if } \ x \in \{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\} \\ \frac{x}{2}+\frac{1}{2} & \text{if } \ x \in [1,2]. \end{cases}
$$

and $\alpha,\beta: X \to [0,\infty)$ by

$$
\alpha(x) = \beta(x) = \begin{cases} x & \text{if } \ x \in [1,2] \\ 0 & \text{otherwise.} \end{cases} .
$$

Since for any $x \in X, \alpha(x) \geq 1 \iff x \in [1,2]$, we have
$\beta(fx) = \beta(\frac{x}{2}+\frac{1}{2}) = \frac{x}{2}+\frac{1}{2} \geq 1$. Since $\alpha(x) = \beta(x)$, clearly f is a cyclic (α,β)-admissible mapping.
We choose $\psi(t) = t, \varphi(t) = \frac{t}{4}$ for $t \geq 0$.
Now, we show that f is almost generalized $(\alpha,\beta)-(\psi,\varphi)$- contractive mapping with $L = \frac{22}{8}$.
Since $\alpha(x)\beta(y) \geq 1 \iff x, y \in [1,2]$, for $x, y \in [1,2]$, we have
$M_s(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2\left(\frac{3}{2}\right)}\} = \{2, \frac{2+2}{2(\frac{3}{2})}\} = 2$,
$M'(x,y) = \max\{d(x,y), d(y,fy)\} = \{2,2\} = 2$ and
$N(x,y) = \text{Min}\{d(x,fx), d(y,fx)\} = \{2,2\} = 2$.
Hence, for $x, y \in [1,2]$, we have

$$\begin{aligned}
\psi(s^3 d(fx,fy)) = \ & \psi((\frac{3}{2})^3 d(\frac{x}{2}+\frac{1}{2}, \frac{y}{2}+\frac{1}{2})) = \psi(\frac{27}{8}2) = \frac{27}{4} \\
\leq \ & 2 - \frac{2}{4} + (\frac{22}{8})2 \\
\leq \ & \psi(M_s(x,y)) - \varphi(M'(x,y)) + LN(x,y).
\end{aligned}$$

Hence f is an almost generalized (α,β)-(ψ,φ) - contractive mapping with $L = \frac{22}{8}$ and the condition (3) of Theorem 3.2 holds trivially. Therefore f satisfies all the hypotheses of Theorem 3.2 and $x = 1$ is a fixed point of f.

Here we observe that when $L = 0$, the inequality (2.4) fails to hold for $x = 1$ and $y = 2$, we have

$$\begin{aligned}
\psi(s^3 d(fx,fy)) &= \psi((\frac{3}{2})^3 d(f1,f2)) = \psi((\frac{3}{2})^3(2)) = \psi(\frac{27}{8}(2)) \\
&\not\leq \psi(2) - \varphi(2) = \psi(M_s(x,y)) - \varphi(M_s(x,y))
\end{aligned}$$

for any ψ and φ, so that f is not an (α,β)-(ψ,φ)-contractive mapping.

Further, we observe that the metric d defined in this example is not a metric in the usual sense, for $x = \frac{1}{3},\ y = \frac{1}{5}$ and $z = \frac{1}{2}$ then
$d(x,y) = 6 \not\leq 2+2 = d(x,z) + d(z,y)$.

Example 4.3. Let $X = [0.5,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ \frac{1}{x}+\frac{1}{y} & \text{if} \quad x,y \in [0.5,1] \\ 4+\frac{1}{x}+\frac{1}{y} & \text{if} \quad x,y \in (1,\infty) \\ 2 & \text{otherwise} \end{cases}$$

Clearly, d is a b-metric space with the coefficient $s \geq \frac{3}{2}$.
Let $A = [0.5,1]$ and $B = [1,\infty)$, we define $f : A \cup B \to A \cup B$ by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \in A \\ \frac{1}{2}+\frac{1}{2x} & \text{if} \quad x \in B. \end{cases}$$

Now, we have for $x \in A, fx = \frac{1}{x} \in [1,\infty) = B$ which implies that $fA \subset B$ and also for $x \in B, fx = \frac{1}{2}+\frac{1}{2x} \in [0.5,1] = A$ which implies that $fB \subset A$. Hence f is cyclic.

We choose $\psi(t)=t, \varphi(t)=\frac{t}{2}\ \ t\geq 0$.
For $x\in A$ and $y\in B$ we have

$$
\begin{aligned}
M_s(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2s}\} \\
&= \max\{2,2,2, \frac{\frac{1}{x}+\frac{1}{\frac{1}{2}+\frac{1}{2x}}+4+\frac{1}{y}}{2(\frac{3}{2})}\} \geq 2,
\end{aligned}
$$

$M'_s(x,y) =\ \max\{d(x,y), d(y,fy)\} = \max\{2,2\} = 2$,
and $N(x,y) =\ \min\{d(x,fx), d(y,fx\} = \max\{2, 4+\frac{1}{y}+x\} = 2$.
Now, we have

$$
\begin{aligned}
\psi(s^3 d(fx,fy)) &= \psi(\frac{3}{2}d(\frac{1}{x}, \frac{1}{2}+\frac{1}{2x}) \\
&= \psi((\frac{3}{2})^3(2) \\
&\leq 2-\frac{2}{2}+6(2) \\
&\leq \psi(M_s(x,y)) - \varphi(M_s(x,y)) + LN(x,y).
\end{aligned} \tag{4.2}
$$

Therefore f is an almost generalized (A,B)-(ψ,φ)-contractive mapping with $L=6$.

Hence f satisfies all the hypotheses of Theorem 3.4 and $x=1$ is a unique fixed point of f.

Further, we observe the following:
if we define

$$
\alpha(x) = \begin{cases} 1 & \text{if} \quad x\in A \\ 0 & \text{otherwise,} \end{cases} \qquad \beta(x) = \begin{cases} 1 & \text{if} \quad x\in B \\ 0 & \text{otherwise,} \end{cases}
$$

then f is a cyclic (α,β)-admissible mapping and by (4.2), we have f is an almost generalized (α,β)-(ψ,φ)-contractive mapping. Hence f satisfies all the hypotheses of Theorem 3.1, along with the hypothesis of Theorem 3.3, and f has a unique fixed point 1.

But for $x=\frac{67}{100}$ and $y=1$, we have

$$
\begin{aligned}
\psi(d(fx,fy)) = \ \psi(d(f\frac{67}{100}, f1)) \ &= \ \psi(d(|\frac{100}{67}-1|) \ = \ \psi(\frac{33}{67}) \\
\nleq \ \psi(\frac{33}{100}) - \varphi(\frac{33}{100}) \ &= \ \psi(d(x,y)) \ - \ \varphi(d(x,y)),
\end{aligned}
$$

for any $\psi,\varphi\in\Psi$ so that the inequality (2.1) fails to hold. Hence Theorem 2.1 is not applicable.

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Ser. Math. Inform. Vol. 33, No 2 (2018), 197–202
https://doi.org/10.22190/FUMI1802197A

GERAGHTY EXTENSION TO k-DIMENSION

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Abstract. In this paper, we extend the Geraghty result [7] to k-dimension.

Keywords: Fixed point, Geraghty extension, k-dimension, metric space.

1. Introduction and Preliminaries

It is known that the Banach contraction principle is considered as one of the most important theorems in the classical functional analysis. There are many generalizations of this theorem. The following generalization is due to M. Geraghty [7].

Theorem 1.1. *[7] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If T satisfies the following inequality:*

$$d(Tx, Ty) \leq \beta(d(x, y))\, d(x, y)$$

for all $x, y \in X$, where $\beta : [0, \infty) \to [0, 1)$ is a function which satisfies the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \textit{implies} \quad \lim_{n\to\infty} t_n = 0.$$

Then T has a unique fixed point $u \in X$ and $\{T^n x\}$ converges to u for each $x \in X$.

The above result has been generalized by many authors. For details, see [1, 2, 3, 4, 5, 6, 8, 9].

$\mathbb{N}$ (resp. $\mathbb{N}_0$) denotes a set of positive (nonnegative) integers. We denote by $\mathcal{F}$ a set of functions β given in Theorem 1.1. The aim of this paper is to generalize and extend Theorem 1.1 to k-dimension. To be more clear, we will consider non-self mappings $T : X^k \to X$ involving a Geraghty type contraction in the class of metric spaces. Note that in the given contraction (it corresponds later to (2.1)),

Received January 30, 2018; accepted March 20, 2018
2010 *Mathematics Subject Classification.* 47H10; 54H25; 46J10

we consider two k-uplets of the form $(u_1, u_2, \ldots, u_k)$ and $(u_2, u_3, \ldots, u_{k+1})$, that is, there is a repetition of $(k-1)$-components, which are u_2, u_3, …, u_k. This fact is different from all known multidimensional fixed point results where the two considered k-uplets are not generally dependent, i.e., of the form $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$.

2. Main results

Our main result is

Theorem 2.1. *Let (X, d) be a complete metric space and $k \in \mathbb{N}$. Let $T : X^k \to X$ be such that*

(2.1)
$$d(T(u_1, u_2, \ldots, u_k), T(u_2, u_3, \ldots, u_{k+1}))$$
$$\leq \beta\left(M((u_1, u_2, \ldots, u_k), (u_2, u_3, \ldots, u_{k+1}))\right) M((u_1, x_2, \ldots, u_k), (u_2, u_3, \ldots, u_{k+1})),$$

for all $u_1, u_2, \ldots, u_k, u_{k+1}$ in X, where $\beta \in \mathcal{F}$ and $M : X^k \times X^k \to [0, \infty)$ is as

$$M((u_1, u_2, \ldots, u_k), (u_2, u_3, \ldots, u_{k+1}))$$
$$= \max\{d(u_k, u_{k+1}), d(u_k, T(u_1, u_2, \ldots, u_k)), d(u_{k+1}, T(u_2, u_3, \ldots, u_{k+1}))\}.$$

Then there is a point u in X such that $T(u, u, \ldots, u) = u$.

Proof. We split the proof into several steps.
Step 1: Let $k \in \mathbb{N}$ be fixed. Consider as the initial point the k-uplet point $(x_1, x_2, \ldots, x_k) \in X^k$. Let

$$x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \quad \text{for all } n \in \mathbb{N}.$$

In view of (2.1),

(2.2)
$$d(x_{n+k+1}, x_{n+k+2}) = d(T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}), T(x_{n+2}, x_{n+3}, \ldots, x_{n+k+1}))$$
$$\leq \beta\left(M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1}))\right)$$
$$M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})).$$

Now,

$$M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1}))$$
$$= \max\{d(x_{n+k}, x_{n+k+1}), d(T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}), x_{n+k}), d(T(x_{n+2}, x_{n+3}, \ldots, x_{n+k+1}), x_{n+k+1}$$
$$= \max\{d(x_{n+k}, x_{n+k+1}), d(x_{n+k+2}, x_{n+k+1})\}.$$

The case that $M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})) = d(x_{n+k+2}, x_{n+k+1})$ for some n, is impossible. Indeed, by (2.2) and the fact that $\beta \in \mathcal{F}$,

$$d(x_{n+k+1}, x_{n+k+2}) \leq \beta\left(d(x_{n+k+2}, x_{n+k+1})\right) d(x_{n+k+2}, x_{n+k+1}) < d(x_{n+k+2}, x_{n+k+1}),$$

which is a contradiction. Hence $M((x_{n+1}, x_{n+2}, \ldots, x_{n+k}), (x_{n+2}, x_{n+3}, \ldots, x_{n+k+1})) = d(x_{n+k}, x_{n+k+1})$ for all $n \geq 0$. Again by (2.2),

(2.3)
$$d(x_{n+k+1}, x_{n+k+2}) \leq \beta\left(d(x_{n+k}, x_{n+k+1})\right) d(x_{n+k}, x_{n+k+1}) < d(x_{n+k}, x_{n+k+1})$$
for all $n \in \mathbb{N}_0$.

So the sequence $\{d(x_{n+k}, x_{n+k+1})\}$ is non-negative and non-increasing. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. So for a large n, $d(x_{n+k}, x_{n+k+1}) > 0$. (2.3) implies that

$$\frac{d(x_{n+k+1}, x_{n+k+2})}{d(x_{n+k}, x_{n+k+1})} \leq \beta(d(x_{n+k}, x_{n+k+1})) < 1.$$

Taking the limit as $n \to \infty$, we get that

$$\lim_{n\to\infty} \beta\left(d(x_{n+k}, x_{n+k+1})\right) = 1.$$

Since $\beta \in \mathcal{F}$,

$$\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = 0. \tag{2.4}$$

Step 2: We shall prove that $\{x_{n+k}\}$ is a Cauchy sequence. We argue by contradiction. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(p)+k}\}$ and $\{x_{n(p)+k}\}$ of $\{x_{n+k}\}$ with $m(p) > n(p) > p$ such that for every p

$$d(x_{m(p)+k}, x_{n(p)+k}) \geq \varepsilon. \tag{2.5}$$

Moreover, corresponding to $n(p)$ we can choose $m(p)$ in such a way that it is the smallest integer with $m(p) > n(p)$ and satisfying (2.5). Then

$$d(x_{m(p)+k-1}, x_{n(p)+k}) < \varepsilon. \tag{2.6}$$

By the triangle inequality, (2.5) and (2.6), we get

$$\begin{aligned} d(x_{n(p)+k-1}, x_{m(p)+k-1}) &\leq d(x_{n(p)+k-1}, x_{n(p)+k}) + d(x_{n(p)+k}, x_{m(p)+k-1}) \\ &< \varepsilon + d(x_{n(p)+k-1}, x_{n(p)+k}), \end{aligned} \tag{2.7}$$

and

(2.8)
$$\begin{aligned} \varepsilon &\leq d(x_{n(p)+k}, x_{m(p)+k}) \\ &\leq d(x_{n(p)+k}, x_{n(p)+k-1}) + d(x_{n(p)+k-1}, x_{m(p)+k-1}) + d(x_{m(p)+k-1}, x_{m(p)+k}). \end{aligned}$$

Using (2.4) in (2.7) and (2.8), we obtain

$$\lim_{p\to\infty} d(x_{n(p)+k-1}, x_{m(p)+k-1}) = \varepsilon. \tag{2.9}$$

On the other hand,

$$\begin{aligned}
&M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))\\
&=\max\{d(x_{n(p)+k-1},x_{m(p)+k-1}),d(T(x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),x_{n(p)+k-1}),\\
&d(T(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}),x_{m(p)+k-1})\}\\
&=\max\{d(x_{n(p)+k-1},x_{m(p)+k-1}),d(x_{n(p)+k},x_{n(p)+k-1}),d(x_{m(p)+k},x_{m(p)+k-1})\}.
\end{aligned}$$

In view of (2.4) and (2.9),
(2.10)

$$\lim_{p\to\infty} M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))=\varepsilon.$$

By (2.1) and (2.5),

(2.11)

$$\begin{aligned}
&\varepsilon\le d(x_{n(p)+k},x_{m(p)+k})\\
&=d(T(x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),T(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))\\
&\le\beta\left(M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))\right)\\
&M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))\\
&<M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))\\
&=\max\{d(x_{n(p)+k-1},x_{m(p)+k-1}),d(x_{n(p)+k},x_{n(p)+k-1}),d(x_{m(p)+k},x_{m(p)+k-1})\}.
\end{aligned}$$

Using (2.10), we deduce from (2.11)

$$\lim_{p\to\infty}\beta\left(M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))\right)=1.$$

Since $\beta\in\mathcal{F}$, we have

$$\lim_{p\to\infty} M((x_{n(p)},x_{n(p)+1},\dots,x_{n(p)+k-1}),(x_{m(p)},x_{m(p)+1},\dots,x_{m(p)+k-1}))=0,$$

which is a contradiction with respect to (2.10). Thus $\{x_{n+k}\}$ is Cauchy in (X,d).
Step 3: Now, by using the completeness property of X, there exists a point u in X such that

$$\lim_{n\to\infty} x_{n+k}=u. \tag{2.12}$$

Assume that $u\ne T(u,u,\dots,u)$. We have

$$\begin{aligned}
&M((x_n,x_{n+1},\dots,x_{n+k-1}),(u,u,\dots,u))\\
&\max\{d(x_{n+k-1},u),d(x_{n+k-1},T(x_n,x_{n+1},\dots,x_{n+k-1}),d(u,T(u,u,\dots,u))\}\\
&=d(u,x_{n+k})+\max\{d(x_{n+k-1},u),d(x_{n+k-1},x_{n+k}),d(u,T(u,u,\dots,u))\}
\end{aligned}$$

From (2.4) and (2.12),

$$\lim_{n\to\infty} M((x_n,x_{n+1},\dots,x_{n+k-1}),(u,u,\dots,u))=d(u,T(u,u,\dots,u)). \tag{2.13}$$

On the other hand, by (2.1)

$$\begin{aligned} d(u,T(u,u,\ldots,u)) &\le d(u,x_{n+k})+d(x_{n+k},T(u,u,\ldots,u))\\ &= d(u,x_{n+k})+d(T(x_n,x_{n+1},\ldots,x_{n+k-1}),T(u,u,\ldots,u))\\ &\le d(u,x_{n+k})+\beta\left(M((x_n,x_{n+1},\ldots,x_{n+k-1}),(u,u,\ldots,u))\right)\\ &.M((x_n,x_{n+1},\ldots,x_{n+k-1}),(u,u,\ldots,u))\\ &< d(u,x_{n+k})+M((x_n,x_{n+1},\ldots,x_{n+k-1}),(u,u,\ldots,u)). \end{aligned} \tag{2.14}$$

Using (2.13) in (2.14), we obtain

$$\lim_{n\to\infty}\beta\left(M((x_n,x_{n+1},\ldots,x_{n+k-1}),(u,u,\ldots,u))\right)=1,$$

that is,

$$\lim_{n\to\infty}M((x_n,x_{n+1},\ldots,x_{n+k-1}),(u,u,\ldots,u))=0.$$

It is a contradiction with respect to (2.13). Thus, $d(u,T(u,u,\ldots,u))=0$. This completes the proof. □

Remark 2.1. Taking $k=1$ in Theorem 2.1, we get a generalization of Theorem 1.1. Our main result is then a generalization and an extension of the Geraghty theorem to k-dimension.

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FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 2 (2018), 203–215
https://doi.org/10.22190/FUMI1802203M

ON NUMERICAL EVALUATION OF THE PACKET-ERROR RATE FOR BINARY PHASE-MODULATED SIGNALS RECEPTION OVER GENERALIZED–K FADING CHANNELS *

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Abstract. We present a numerical evaluation of the packet error rate (PER) for digital binary phase modulations over wireless communication channels. The analysis is valid for a quasistatic fading communication channel, where multipath fading and shadowing appear simultaneously. The approach is based on a numerical evaluation of the signal-to-noise ratio threshold that is further used in PER computation. We analyze the threshold and PER dependence on signal power, multipath fading and shadowing severity, as well as packet length.

Keywords: bit error rate, packet error rate, wireless communication channel.

1. Introduction

The quality of service in communication systems is usually described by bit error rate (BER) and packet error rate (PER). The BER is a probability that a transmitted bit over a channel will be wrongly detected in the receiver due to the noise and interferences over the channel. The PER is a probability that a packet of bits (or symbols) will be wrongly detected. The packet is detected wrongly if at least one bit (or symbol) is wrongly detected. Both metrics are associated to the physical layer of communication systems. However, PER is a very important metric in designing across multiple protocol layers of wireless networks [11], [12].

It is very hard to calculate the exact value of PER, especially when encoding and decoding algorithms are implemented. Because of that, many efforts have been made in order to analytically or numerically approximate PER. Chatzigeorgiou at al. [3], [4] developed a threshold-based method for approximating PER over quasistatic fading channels. They examined both single-input single-output and

Received July 19, 2017; accepted October 19, 2017
2010 *Mathematics Subject Classification.* Primary 94A05; Secondary 94A40, 94A14, 60G35
*The authors were supported in part by the Ministry of Science of the Republic of Serbia under grant III44006, and by the Norwegian Ministry of Foreign Affairs under the NORBAS project (grant 2011/1383).

multiple-input multiple-output channels. They observed a situation when a direct propagation component does not exist in the channel, i.e., fading is described by the Rayleigh probability density function (PDF). Xi at al. [14] proposed a novel analytical approach for evaluating the signal-to-noise (SNR) ratio threshold required for computation of PER. Their analysis is valid basically for the Rayleigh fading channel, but it was also extended for a more general case when besides scattering propagation components there is also a direct signal propagation component. In other words, their analysis is valid for the Nakagami-m fading channel, too. Wang et al. [11], [12] suggested an accurate approximation of the PER of diversity receivers over the Rayleigh fading channel when different error correction coding schemes are implemented.

All previously mentioned works were applicable in the situation when only multipath fading exists in the channel. However, very often, besides multipath fading, shadowing appears simultaneously during signal transmission [5, 9]. In this case, signal variations at the receiver input can be accurately described by the Gamma-shadowed Nakagami-m PDF. This PDF is also known as the generalized-K PDF [10, 2, 9]. The aim of this paper is to provide a numerical method for evaluating PER for binary phase shift keying over the composite fading channel. We give an approach for evaluating the SNR threshold and after that use this threshold for estimating PER. We examine the effect of signal power, packet length, multipath fading severity and shadowing sharpness on the numerical value of the SNR threshold, and consequently on PER. Approximate PER values are expressed in terms of Meijer's G functions [13] with appropriate arguments.

The paper is organized as follows. The system model is described in more detail in Section 2. Approximation for PER is discussed in Section 3, with preliminary numerical results indicating validity of the approximation. Section 4 examines a procedure for obtaining the threshold level value required for approximating PER, and proposes a simple method for its computation. An example is given for realistic system parameters. In Section 5, we present numerical results of the system analysis, and further validate the approximations made in the previous sections. Some concluding remarks are presented in the final section.

2. System model

BER represents the time-average of the ratio of wrongly decoded bits over a total number of transfered bits. If the process of the receiver operation is considered ergodic, as is the case for the most processes relevant in telecommunications, then the time-average is equal to the ensemble-average. Therefore, BER is equal to error probability P_e. The most important parameter on which BER depends is SNR. The most common model of noise treats it as having a zero-mean Gaussian probability density function, with the effective noise level being equal to variance of the distribution. In general, higher SNR values lead to lower BER, so BER is a strictly decreasing function of SNR.

In cases where there are other random influences on the signal level, an important parameter for the receiver BER is signal level statistics. In general, BER is an average of its instantaneous value for a fixed signal level, over the signal-to-noise statistics, or BER = E{BER(SNR)}. The situation of signal level varying significantly is almost synonymous with modern wireless communications, be it mobile, cellular or Wi-Fi. It is almost universally recognized from the user's experience point that sometimes it is enough to move a few centimeters while talking over your mobile to suddenly lose the signal or encounter 'poor' signal levels. The effect is attributed to signal fading, which is a propagation effect of quasi-randomly interfering signal copies producing unpredictable signal levels. It is also accompanied by signal shadowing, which is a random process of the signal being attenuated through the obstacles in the propagation path. These two effects combined can significantly degrade the user experience with wireless technologies. The combined influence of multipath fading and signal shadowing can be described by the generalized-K fading model, which is the previously discussed relevant signal level statistics. Its probability density function is given by [7]:

$$
\begin{aligned}
p(m_m, m_s, \rho_0, \rho) \quad = \quad & 2\frac{\rho^{\frac{m_m+m_s-2}{2}}}{\Gamma(m_m)\Gamma(m_s)}\left(\frac{m_m m_s}{\rho_0}\right)^{\frac{m_m+m_s}{2}} \\
& \mathrm{K}_{m_m-m_s}\left(2\sqrt{\rho\frac{m_m m_s}{\rho_0}}\right),
\end{aligned}
\tag{2.1}
$$

where $\mathrm{K}_\nu(\cdot)$ is a modified Bessel function of the first kind, of order ν [6, (8.432)], and $\Gamma(\cdot)$ is a Gamma function. PDF is defined for positive values of ρ, and for $\rho_0 \geqslant 0, m_m > 0, m_s > 0$. It is suitable as a channel model when $m_m \geqslant 1/2$.

On the other hand, these types of data transfers are usually centered around the group transfer of bit packets, and are considered packet-radio communications. Therefore, in contrast to BER, the more important performance measure for these types of telecommunication systems is the packet-error-rate.

If individual bits in the packet are not mutually correlated, than PER can be expressed as $\mathrm{PER} = \mathrm{E}\{1 - (1 - \mathrm{BER})^{l_p}\}$:

$$
\mathrm{PER}(m_m, m_s, \rho_0) = \int_0^{+\infty} \left[1 - \big(1 - \mathrm{BER}(\rho)\big)^{l_p}\right] p(m_m, m_s, \rho_0, \rho)\, \mathrm{d}\rho,
\tag{2.2}
$$

which is the ensemble-average of probability that the whole packet of l_p bits is received correctly without any errors. In the previous equation, the parameter m_m represents the multipath fading parameter, while m_s is the shadowing parameter, as discussed in the previous paragraph. The parameter ρ_0 represents the average value of SNR, over the signal level varying statistics.

Individual bits have BER that depends on the modulation format and demodulation operation of the receiver, and for the phase modulation formats of interest it can be expressed as:

$$\mathrm{BER}(\rho) = \begin{cases} \dfrac{1}{2}\mathrm{erfc}(\sqrt{\rho}), & \text{for BPSK modulation,} \\ \dfrac{1}{2}e^{-\rho/2}, & \text{for DBPSK modulation.} \end{cases} \tag{2.3}$$

Under the assumption that the signal level is constant, and the noise at the receiver is additive white Gaussian, the instantaneous SNR in the previous equation is designated as ρ.

As is obvious from the previous equations, one cannot express the receiver PER in a closed form by combining (2.2, 2.2, 2.3). Therefore, the performance analysis is limited to numerical computation of (2.2), or its approximations.

3. Packet-error rate approximation

One of the adopted PER approximations concentrates on the following form:

$$\mathrm{PER} = \int_0^{\gamma_\omega} p(x)\,\mathrm{d}x, \tag{3.1}$$

where $\mathrm{PER} = \mathrm{PER}(m_m, m_s, \rho_0)$, and $p(x) = p(m_m, m_s, \rho_0, x)$. The threshold value γ_ω that satisfies the equation is to be determined. The integral on the right-hand side is by definition the cumulative distribution function (CDF) corresponding to PDF $p(x)$, which can be expressed in the form of Meijer's G function [1] as:

$$\mathrm{CDF}(m_m, m_s, \rho_0, \rho) = \frac{1}{\Gamma(m_m)\Gamma(m_s)} G_{1,3}^{2,1}\left(\rho\frac{m_m m_s}{\rho_0} \,\middle|\, \begin{matrix} 1 \\ m_s, m_m, 0 \end{matrix}\right), \tag{3.2}$$

where $G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a1, \cdots, a_p \\ b1, \cdots, b_q \end{matrix}\right)$ denotes Meijer's G function defined by [6, (9.301)] [8, (16.17.1)].

In order to obtain the value of the threshold γ_ω, we have to solve the equation:

$$\mathrm{PER}(m_m, m_s, \rho_0) = \mathrm{CDF}(m_m, m_s, \rho_0, \gamma_\omega) \tag{3.3}$$

According to available literature, the motivation behind this approach is that the threshold γ_ω will not depend significantly on the values of m_m, m_s, and ρ_0, thus enabling one threshold to be used across the range of values of interest. As a consequence, PER is expressed in the aforementioned form of Meijer's G function, which can be used for further analytic or numerical manipulation.

In order to validate this assumption, we have computed numerical values of the threshold γ_ω for a realistic range of values $0.5 \leqslant m_m \leqslant 6$, $0.3 \leqslant m_s \leqslant 12$, and $-5 \leqslant \rho_0 \leqslant 25$, and the results are shown in Fig. 3.1. The results are computed

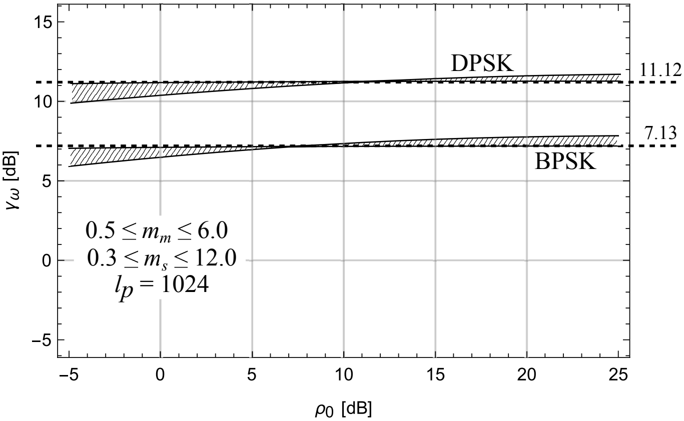

FIG. 3.1: Threshold values for a range of fading and shadowing propagation conditions. Patterned areas represent the range of threshold values as the propagation parameters are swept across their ranges.

for a fixed packet length of 1024 bits. Numerical values are obtained by using *Mathematica*'s *FindRoot* implementation of Newton's method for solving non-linear equations, and care has been taken to ensure the results are computed with enough working precision to justify the accuracy of the solutions. The figure shows that the threshold somewhat depends on a particular set of values, but it remains in a fairly narrow band of possible values. By averaging the threshold values over uniformly distributed points in m_m, m_s and ρ_0 that we computed in the simulation, we get an average value of $\gamma_\omega \approx 7.13$ dB for BPSK modulation, and $\gamma_\omega \approx 11.12$ dB for DBPSK.

4. Approximation for the threshold level

Formally, the solution to (3.3) can be written in the form of an inverse function:

$$\gamma_\omega = \mathrm{CDF}^{-1}(\mathrm{PER}), \tag{4.1}$$

since CDF is a monotonically increasing function.

From the previous experience in the evaluation of telecommunication systems performance, we are confident that, at least for a significant range of parameter values, the cumulative distribution function exhibits log–log behavior, i.e. it can be approximated to some extent by a straight line in the log–log scale. Therefore, we proceed by imposing the log–log scale for the cumulative distribution function. We further assume that the function can be expanded to power series:

$$\log\left[\mathrm{CDF}\left(e^x\right)\right] = \sum_{i=0}^{N} a_i x^i + R_N(x). \tag{4.2}$$

In the previous equation we used $\mathrm{CDF}(x) = \mathrm{CDF}(m_m, m_s, \rho_0, x)$ notation in order to avoid a lengthy and non-essential list of function arguments. Coefficients a_n can be determined, for example, as the coefficients in the Maclaurin series of the function:

$$a_n = \frac{1}{n!}\frac{\mathrm{d^n}}{\mathrm{d}x^n} \log\left[\mathrm{CDF}(e^x)\right]\Big|_{x=0}. \tag{4.3}$$

The first four a_n coefficients are given in Table 4.1. Meijer's G functions exhibit a closure in the sense that if the function argument is a constant multiple of the constant power of the argument, the derivatives and antiderivatives with respect to the argument are also expressible as G-functions. This is clearly reflected in the coefficients shown in Table 4.1, where we have expressed a_n in terms of $a_{n-1}, a_{n-2}, \cdots, a_1$. It should be possible to formulate a general expression for a_n, but we have not done so here. Instead we have focused on the cases $N \leqslant 4$, which enable a relatively simple symbolic representation of the approximate inverse function CDF^{-1}. In order to solve (3.3) using a series representation of (4.2), we write a nonlinear equation:

$$\sum_{i=0}^{N} a_i x^i - \log(\mathrm{PER}) = 0. \tag{4.4}$$

Let us assume that at least one real solution to this polynomial equation exists and can be expressed in a symbolic form as:

$$x = f(a_1, a_2, \cdots, a_N, \mathrm{PER}). \tag{4.5}$$

Then, the approximate inverse function CDF^{-1} can be expressed as:

$$\mathrm{CDF}^{-1}(x) \approx e^{f(a_1, a_2, \cdots, a_N, \mathrm{PER})}. \tag{4.6}$$

Let us examine a simple linear approximation, i.e. $N = 1$. We can directly write the function f:

$$f(a_0, a_1, \mathrm{PER}) = \frac{\log(\mathrm{PER}) - a_0}{a_1}. \tag{4.7}$$

From this solution and (4.6), it follows directly that the approximate threshold γ_{ω_1} is:

$$\gamma_{\omega_1} = \left(\frac{\mathrm{PER}}{\mathrm{CDF}(1)}\right)^{1/a_1}, \tag{4.8}$$

where a_1 is given in Table 4.1.

A numerical example for illustrative telecommunication system parameters can be the following: $m_m = 6$, $m_s = 12$, $\rho_0 = 3000$, modulation format - DBPSK.

Table 4.1: The first four coefficients a_n for the Maclaurin series of (4.2)

n	a_n
0	$\log\left[\mathrm{CDF}(1)\right]$
1	$\frac{\Gamma(m_m)\Gamma(m_s)}{\mathrm{CDF}(1)} \frac{m_m m_s}{\rho_0} G_{0,2}^{2,0}\left(\frac{m_m m_s}{\rho_0} \,\middle\vert\, \begin{matrix} - \\ m_m-1, m_s-1 \end{matrix}\right)$
2	$\frac{a_1}{2}(1-a_1)+ \frac{\Gamma(m_m)\Gamma(m_s)}{2\,\mathrm{CDF}(1)} \left(\frac{m_m m_s}{\rho_0}\right)^2 G_{1,3}^{2,1}\left(\frac{m_m m_s}{\rho_0} \,\middle\vert\, \begin{matrix} -1 \\ m_m-2, m_s-2, 0 \end{matrix}\right)$
3	$a_1\left(\frac{a_1^2}{6}+\frac{a_1}{2}+a_2+\frac{1}{3}\right)-a_2+ \frac{\Gamma(m_m)\Gamma(m_s)}{6\,\mathrm{CDF}(1)} \left(\frac{m_m m_s}{\rho_0}\right)^3 G_{1,3}^{2,1}\left(\frac{m_m m_s}{\rho_0} \,\middle\vert\, \begin{matrix} -2 \\ m_m-3, m_s-3, 0 \end{matrix}\right)$
4	$\frac{a_1}{4}\left(\frac{a_1^3}{6}-a_1^2+\frac{11}{6}a_1-1\right)+\frac{a_2}{2}\left(a_1^2+a_2+a_1+\frac{11}{6}\right)+a_3\left(a_1-\frac{3}{2}\right)+ \frac{\Gamma(m_m)\Gamma(m_s)}{24\,\mathrm{CDF}(1)} \left(\frac{m_m m_s}{\rho_0}\right)^4 G_{1,3}^{2,1}\left(\frac{m_m m_s}{\rho_0} \,\middle\vert\, \begin{matrix} -3 \\ m_m-4, m_s-4, 0 \end{matrix}\right)$

The cumulative distribution function value at argument value 1 for given system parameters is:

$$\mathrm{CDF}(1) = 7.94648 \times 10^{-19}.$$

The coefficient a_1 has the value of:

$$a_1 = 5.99589.$$

The numerical value of PER, obtained by a numerical evaluation of the integral (2.2) is:

$$\mathrm{PER} = 8.64275 \times 10^{-12}.$$

From the equation (4.8), we get the threshold value:

$$\gamma_{\omega_1} = 14.91247.$$

Once the threshold value is computed given the initial system parameters, one can approximate the PER values for different system parameters as:

$$\mathrm{PER}(m_m, m_s, \rho_0) \approx \mathrm{CDF}(m_m, m_s, \rho_0, \gamma_{\omega_1}). \tag{4.9}$$

5. Numerical results and discussion

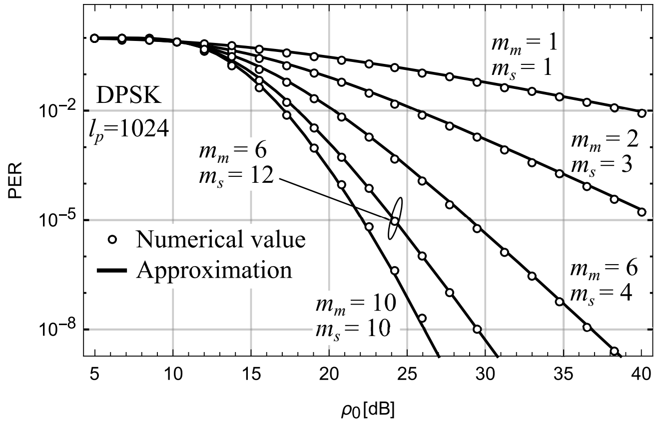

FIG. 5.1: Numerical values of DBPSK PER compared to approximate values obtained in the example for $N = 1$.

In order to further validate the assumption about a universal usage of the threshold value for different parameters, we have calculated numerical values for PER and compared them to the approximate values obtained using (4.9) for different system parameters. The results are shown in Fig. 5.1. The threshold value is the same as the one determined in the example in the previous section, and it is used in all obtained approximate PER values. For a wide range of the multipath fading parameter values m_m, the shadowing parameter m_s, and the average signal-to-noise ratio ρ_0, we can see very good agreement between the precisely computed numerical values and the approximate values of PER. It is not unexpected that the best match is achieved for the values m_m and m_s, for which we have calculated the threshold γ_ω. The practical values of PER that are of interest in telecommunications range from 10^{-1} to 10^{-9}, and Fig. 5.1 is shown in a logarithmic scale to better illustrate this magnitude range of PER values.

Fig. 5.2 shows the results of the approximation when applied to the BPSK modulation format. Linear approximation, i.e. $N = 1$, results in a threshold value of $\gamma_\omega = 6.112566$ when the modulation format is BPSK, and this value is used for all the curves shown in Fig. 5.2. Numerically obtained PER values are shown with circle marks, and the overall figure is similar to that for DBPSK. The exception is that, in general, the system performs better when using BPSK, compared to the case when the system uses DBPSK. The same value of PER is achieved in BPSK when SNR is lower, i.e. a lower SNR is required to make the system perform as well as in the DBPSK case. This can be viewed as increased receiver sensitivity. In the reverse sense, when we look at BPSK as a reference and than compare DBPSK with it, we usually say that DBPSK incurs power penalty for its lower complexity.

After reviewing Figs. 5.1 and 5.2, we conclude that the assumptions made in writing PER approximation (4.9) are not unfounded. We further investigate numerically the influence of using better than linear approximations in obtaining threshold

Table 5.1: Threshold values obtained by N-th order approximation, for $m_m = 6, m_s = 12, \rho_0 = 30$ dB, and packet size $l_p = 1024$

N	γ_{ω_N}[DBPSK]	γ_{ω_N}[BPSK]
1	14.912471	6.112566
2	14.949862	6.119436
3	14.983723	6.123588
4	15.006688	6.125465
7	15.026646	6.126391

values γ_ω. If we try to obtain the threshold for $N = 2$, and for the same system parameters as in the linear example, we get the following closed form:

$$\gamma_{\omega_2} = \exp\left[-\frac{a_1}{2a_2} - \sqrt{\left(\frac{a_1}{2a_2}\right)^2 + \frac{1}{a_2}\log\frac{\mathrm{PER}}{\mathrm{CDF}(1)}}\,\right], \tag{5.1}$$

which evaluates to: $\gamma_{\omega_2} = 14.949862$.

Approximations of higher order are also possible and obviously expressible in a closed form for $N = 3$ and $N = 4$, but we have not developed the expressions due to their complexity. After the fourth order, the polynomial equation is solvable numerically in the general case, and the results are not of great interest to wireless engineers. The results shown in Table 5.1 summarize the results we have obtained for both modulation formats and for the same system parameters. We clearly see that the approximation order has only secondary influence on the numerically evaluated values of γ_ω, which is not particularly significant when used in the approximate expression for PER.

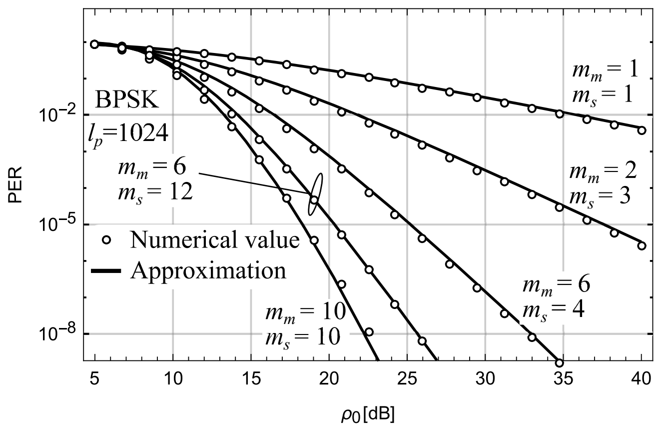

FIG. 5.2: Numerical values of BPSK PER compared to the approximate values obtained in the example for $N = 1$.

Having in mind good agreement of approximation to performance under different system parameters, we conclude that the small variations of the threshold γ_ω hardly justify the use of higher order approximations, especially when used in a wide range of system parameters.

Dependence of the threshold value on fading and shadowing parameters is shown in Fig. 5.4. The figure clearly indicates that the threshold depends on the propagation parameters. However, in the parameter range of interest, this variation of threshold has relatively low influence on wireless system performance. On the other hand, if one of the parameters, either m_m or m_s, is larger than the other one, the threshold value 'saturates' and its variation is significantly lower. This indicates that the system performance may be limited mainly by the lower of the two parameters, $\min(m_m, m_s)$. Fig. 5.4 also shows approximate threshold values obtained via approximations of order 1 and 2. Linear approximation slightly underestimates the threshold in cases where the propagation parameter values are close to each other, and in such cases it would be a better choice to use the second-order approximation whose results are closer to the numerical results.

Fig. 5.5 shows the dependence of SNR threshold on the packet length l_p. This dependence is stronger than the dependencies on the propagation parameters and SNR, and this behavior is expected. In general, as the performance of DBPSK is somewhat poorer than the performance of BPSK, from (3.1) it follows that the corresponding DBPSK threshold is always larger. On the other hand, when the packet length increases, so does the probability of packet errors, which is again reflected in the corresponding threshold increase, as shown in Fig. 5.5.

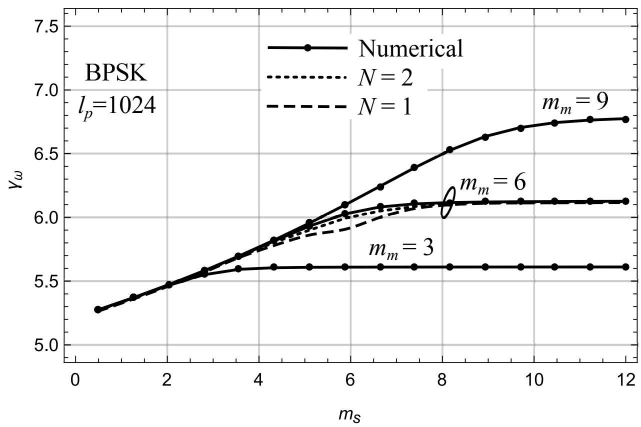

FIG. 5.3: Dependence of threshold values on the multipath fading parameter m_m and the shadowing parameter m_s, while SNR is fixed at 30 dB. Full line curves are obtained numerically, the long-dash is the first-order, while the short-dash curve is the second-order approximation.

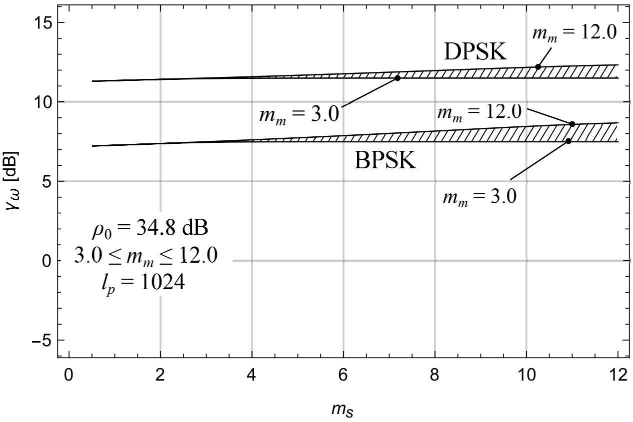

FIG. 5.4: Dependance of threshold γ_ω on the propagation parameters, with same decibel scale as in Fig. 3.1, for comparison.

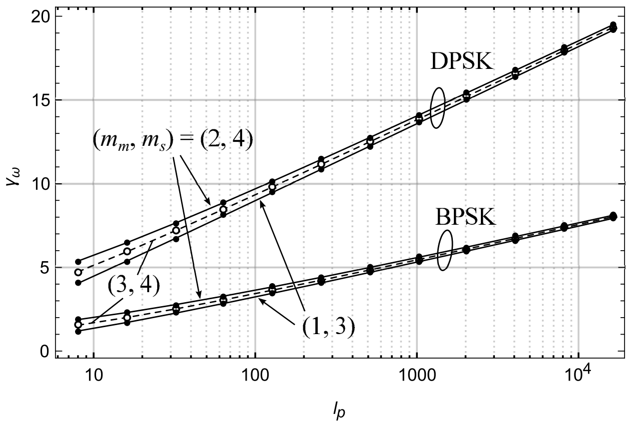

FIG. 5.5: Threshold value γ_ω versus packet length for both the modulation formats and the practical values of fading and shadowing parameters.

6. Conclusion

In this paper, we have analyzed PER in detecting BPSK and DBPSK signals transmitted over the quasistatic Gamma-shadowed Nakagami-m wireless channel. A numerical approach has been proposed for determining SNR threshold required for approximate PER evaluation. The resulting method provides means for determining the threshold level in a symbolic form that is suitable for analysis of different system parameter influence. The results illustrate that the threshold strongly depends on the packet length and much less so on the propagation parameters and SNR. The numerical values of PER indicate that a single threshold value may be used for PER calculation in a wide range of system parameters. Even better results can be obtained when the threshold is calculated separately for specific fading severity and shadowing sharpness.

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FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 2 (2018), 217–230
https://doi.org/10.22190/FUMI1802217S

SOME TYPES OF η-RICCI SOLITONS ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. In this paper, we study some types of η-Ricci solitons on Lorentzian para-Sasakian manifolds and we give an example of η-Ricci solitons on a 3-dimensional Lorentzian para-Sasakian manifold. We obtain the conditions for η-Ricci solitons on φ-conformally flat, φ-conharmonically flat and φ-projectively flat Lorentzian para-Sasakian manifolds. The existence of η-Ricci solitons implies that (M, g) is an η-Einstein manifold. In these cases there is no Ricci soliton on M with the potential vector field ξ.
Keywords: η-Ricci solitons, Lorentzian para-Sasakian structure, conformal curvature, conharmonic curvature and projective curvature.

1. Introduction

In 1982, Hamilton [12] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij.}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0,$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively. Ricci solitons have

Received July 11, 2017; accepted December 05, 2017
2010 *Mathematics Subject Classification.* Primary 53B30, 53C44; Secondary 53C15, 53C50.

been studied in many contexts: on Kähler manifolds [10], on contact and Lorentzian manifolds [1, 7, 15, 21], on Sasakian [14], α-Sasakian [15], on Kenmotsu [2] etc. In paracontact geometry, Ricci solitons firstly appeared in the paper of G. Calvaruso and D. Perrone [8]. Ricci solitons on 3-dimensional normal paracontact manifolds were studied by C. L. Bejan and M. Crasmareanu [3].

A more general notion is that of η-Ricci soliton introduced by J. T. Cho and M. Kimura [9], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [7]. Recently, η-Ricci solitons on para-Kenmotsu manifolds were studied by A. M. Blaga [4] and η-Ricci solitons on Lorentzian para-Sasakian manifolds were also studied by A. M. Blaga [5].

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞ and ∇ be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor R (see [20]), the Weyl conformal curvature tensor C (see [23]), the conharmonic curvature tensor H (see [16]) and the projective curvature tensor P (see [23]) of (M, g) are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.2}$$

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(Y,Z)QX - g(X,Z)QY], \tag{1.4}$$

respectively, where Q is the Ricci operator, defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, $r = tr(S)$ is the scalar curvature and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

This paper is organized as follows: Section 2 consists of the basic definitions of the Lorentzian para-Sasakian manifold. In Section 3, we define Ricci and η-Ricci soliton on $(M, \varphi, \xi, \eta, g)$ and also give an example of η-Ricci solitons on a 3-dimensional Lorentzian para-Sasakian manifold. In Section 4, we obtain the conditions for η-Ricci solitons on φ-conformally flat, φ-conharmonically flat and φ-projectively flat Lorentzian para-Sasakian manifolds. The existence of η-Ricci solitons implies that (M, g) is an η-Einstein manifold. In these cases there is no Ricci soliton on M with the potential vector field ξ.

2. Lorentzian para-Sasakian manifolds

The notion of a Lorentzian para-Sasakian manifold was introduced by K. Matsumoto [17].

An n-dimensional differential manifold M^n is a Lorentzian para-Sasakian (LP-Sasakian) manifold if it admits a $(1,1)$-tensor field φ, contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g, which satisfy

$$\varphi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which imply

$$(a)\ \ \varphi\xi = 0, \quad (b)\ \ \eta(\varphi X) = 0, \quad (c)\quad rank(\varphi) = n-1, \tag{2.2}$$

Then M^n admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

and M^n is said to admit a Lorentzian almost paracontact structure (φ,ξ,η,g). In this case, we have

$$(a)\quad g(X,\xi) = \eta(X), \qquad (b)\quad \nabla_X\xi = \varphi X, \tag{2.4}$$

$$(\nabla_X\varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

If we put

$$\Omega(X,Y) = g(X,\varphi Y) = g(\varphi X, Y) = \Omega(Y,X), \tag{2.6}$$

for any vector fields X and Y, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$-tensor field.

Also, since the vector field is closed in an LP-Sasakian manifold, we have

$$(\nabla_X\eta)(Y) = \Omega(X,Y) = g(\varphi X,Y) = (\nabla_Y\eta)(X), \qquad \nabla_\xi\eta = 0, \tag{2.7}$$

for any vector fields X and Y.

Also, in an LP-Sasakian manifold (M^n,φ,ξ,η,g), for any $X,Y,Z \in \chi(M^n)$, the following relations hold:

$$\eta(\nabla_X\xi) = 0, \quad \nabla_\xi\xi = 0, \tag{2.8}$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.9}$$

$$\eta(R(X,Y)\xi) = 0, \tag{2.10}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.11}$$

$$L_\xi\varphi = 0, \quad L_\xi\eta = 0, \quad L_\xi g = 2g(\varphi\cdot,\cdot), \tag{2.12}$$

where R is the Riemann curvature tensor field, L is the Lie derivatives and ∇ is the Levi-Civita connection associated to g.

3. Ricci and η-Ricci Solitons on (M,φ,ξ,η,g)

Let (M,φ,ξ,η,g) be paracontact metric manifolds. Consider the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = 0, \tag{3.1}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we have:

$$2S(X,Y) = -g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any $X,Y\in\chi(M)$, or equivalent:

$$S(X,Y) = -g(\varphi X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y), \tag{3.3}$$

for any $X,Y\in\chi(M)$. The data (g,ξ,λ,μ) satisfying the equation (3.1) is said to be an η-Ricci soliton on M [9]; in particular, if $\mu=0$, (g,ξ,λ) is a Ricci soliton [13] and it is called shrinking, steady or expanding accordingly as λ is negative, zero or positive, respectively [11]. In [18] and [19] the the authors proved that on a Lorentzian para-Sasakian manifold (M,φ,ξ,η,g), the Ricci tensor field satisfies

$$S(X,\xi) = (dim(M)-1)\eta(X), \tag{3.4}$$

$$S(\varphi X,\varphi Y) = S(X,Y) + (dim(M)-1)\eta(X)\eta(Y). \tag{3.5}$$

Again putting $X=\varphi X$ and $Y=\varphi Y$ in the equation (3.3), we get

$$S(\varphi X,\varphi Y) = -g(X,\varphi Y) - \lambda g(\varphi X,\varphi Y), \tag{3.6}$$

for any $X,Y\in\chi(M)$. From (3.3) and (3.4), we obtain

$$\mu - \lambda = n-1. \tag{3.7}$$

Putting $X = Y = e_i$ in (3.3) and summing over $i = 1, 2, ..., n$, we have

$$r = \sum_{i=1}^{n} S(e_i, e_i) = -\psi - \lambda n - \mu, \tag{3.8}$$

where $\psi = tr\varphi$.

Example 3.1. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent frame field on M given by [22]

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

where a is a non-zero constant such that $a \neq 1$. Let g be the Lorentzian metric defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,$$

$$g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1.$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$, for any $U \in \chi(M)$ and φ be the $(1, 1)$-tensor field defined by

$$\varphi E_1 = -E_1, \quad \varphi E_2 = -E_2 \quad and \quad \varphi E_3 = 0.$$

Then, using the linearity of φ and g, we have $\eta(E_3) = -1, \varphi^2 U = U + \eta(U)E_3$ and $g(\varphi U, \varphi W) = g(U, W) + \eta(U)\eta(W)$, for any $U, W \in \chi(M)$. Thus for $E_3 = \xi, (\varphi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad -g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1,$$

$$\nabla_{E_2} E_1 = ae^z E_2, \quad \nabla_{E_2} E_2 = -ae^z E_1 - E_3, \quad \nabla_{E_2} E_3 = -E_2,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

It can be easily seen that for $E_3 = \xi, (\varphi, \xi, \eta, g)$ is a Lorentzian para-Sasakian structure on M. Consequently, $(M, \varphi, \xi, \eta, g)$ is a Lorentzian para-Sasakian manifold.

Also, the Riemannian curvature tensor R is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$R(E_1,E_2)E_2 = (1-a^2e^{2z})E_1,\ R(E_1,E_3)E_3 = -E_1,\ R(E_2,E_1)E_1 = (1-a^2e^{2z})E_2,$$

$$R(E_2,E_3)E_3 = -E_2,\ R(E_3,E_1)E_1 = E_3,\ R(E_3,E_2)E_2 = E_3 + ae^z E_1.$$

Then, the Ricci tensor S is given by

$$S(E_1,E_1) = S(E_2,E_2) = -a^2e^{2z}, \quad S(E_3,E_3) = -2.$$

From (3.3), we obtain $S(E_1,E_1) = 1-\lambda$ and $S(E_3,E_3) = \lambda-\mu$, therefore $\lambda = 1+a^2e^{2z}$, and $\mu = 3+a^2e^{2z}$. The data (g,ξ,λ,μ) for $\lambda = 1+a^2e^{2z}$, and $\mu = 3+a^2e^{2z}$ defines an η-Ricci soliton on the Lorentzian para-Sasakian manifold M.

4. Main results

In this section, we consider an η-Ricci soliton on φ-conformally flat, φ-conharmonically flat and φ-projectively flat Lorentzian para-Sasakian manifolds.

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \varphi(T_p(M^n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n)) \oplus L(\xi_p).$$

Let us consider the following particular cases:

(1) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to L(\xi_p)$, i.e., the projection of the image of C in $\varphi(T_p(M^n))$ is zero.

(2) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n))$, i.e., the projection of the image of C in $L(\xi_p)$ is zero.

$$C(X,Y)\xi = 0. \tag{4.1}$$

(3) $C : \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \to L(\xi_p)$, i.e., when C is restricted to $\varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n))$, the projection of the image of C in $\varphi(T_p(M^n))$ is zero. This condition is equivalent to

$$\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0, \tag{4.2}$$

(see[6]).

Here the cases (1), (2) and (3) are conformally symmetric, ξ-conformally flat and φ-conformally flat, respectively. The cases (1) and (2) were considered in [24] and [25], respectively. The case (3) was considered in [6] for M a K-contact manifold.

Now we will study the condition (4.2) for η-Ricci solitons on Lorentzian para-Sasakian manifolds.

Definition 4.1. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition (4.2) is called φ-conformally flat.

Suppose that $(M^n, g), n > 3$, is a φ-conformally flat Lorentzian para-Sasakian manifold. It is easy to see that $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(C(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. So by the use of (1.2), φ-conformally flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &\quad -g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) \\ &\quad -g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)] - \frac{r}{(n-1)(n-2)} \\ &\quad [g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{4.3}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n; then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.3) and summing over $i = 1, ..., n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) &= \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &\quad -g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) \\ &\quad -g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)] - \frac{r}{(n-1)(n-2)} \\ &\quad \sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)]. \end{aligned} \tag{4.4}$$

It can be easy to verify that

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{4.5}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + n - 1, \tag{4.6}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \tag{4.7}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n+1, \tag{4.8}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z). \tag{4.9}$$

So applying (4.5) – (4.9) into (4.4), we obtain

$$S(\varphi Y, \varphi Z) = \left(\frac{r}{n-1} - 1\right) g(\varphi Y, \varphi Z). \tag{4.10}$$

Using (3.6) and (3.8) in (4.10), we get

$$(n-1) g(Y, \varphi Z) = (\psi + \mu + \lambda + n - 1) g(\varphi Y, \varphi Z), \tag{4.11}$$

for any $Y, Z \in \chi(M^n)$ and for $Y \mapsto \varphi Y$, we get

$$(n-1) g(\varphi Y, \varphi Z) = (\psi + \mu + \lambda + n - 1) g(Y, \varphi Z). \tag{4.12}$$

Adding the previous two equations, we have

$$(\psi + \mu + \lambda + 2n - 2)[g(Y, \varphi Z) - g(\varphi Y, \varphi Z)] = 0, \tag{4.13}$$

for any $Y, Z \in \chi(M^n)$ and follows

$$\psi + \mu + \lambda + 2n - 2 = 0. \tag{4.14}$$

Now using (3.7) in (4.24), we get

$$\lambda = \frac{3 - \psi - 3n}{2} \quad \textit{and} \quad \mu = \frac{1 - \psi - n}{2}. \tag{4.15}$$

Hence, we can state the following:

Theorem 4.1. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conformally flat, then*

$$\lambda = \frac{3 - \psi - 3n}{2} \quad \textit{and} \quad \mu = \frac{1 - \psi - n}{2}.$$

Corollary 4.1. *If (φ, ξ, η, g) is a φ-conformally flat Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, then there is no Ricci soliton with a potential vector field ξ.*

From (3.3), (3.7) and (4.11), we obtain

$$\begin{aligned} S(X,Y) &= -\left(\frac{\psi + n\lambda + \mu + n - 1}{n-1}\right) g(X,Y) \\ &\quad - \left(\frac{\psi + \lambda + \mu n + n - 1}{n-1}\right) \eta(X)\eta(Y). \end{aligned} \tag{4.16}$$

Hence, we can state the following:

Proposition 4.1. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conformally flat, then (M^n, g) is an η-Einstein manifold.*

Let H be the conharmonic curvature tensor of M^n.

Definition 4.2. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\varphi^2 H(\varphi X, \varphi Y)\varphi Z = 0,$$

is called φ-conharmonically flat.

Now our aim is to find the characterization of η-Ricci solitons on Lorentzian para-Sasakian manifolds satisfying the above condition.

Assume that $(M^n, g), n > 3$, is a φ-conharmonically flat Lorentzian para-Sasakian manifold. It can be easily seen that $\varphi^2 H(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(H(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.3), φ-conharmonically flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &\quad - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) \\ &\quad - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)]. \end{aligned} \tag{4.17}$$

In a manner similar to the method in the proof of Theorem (4.1), choosing $\{e_1, e_2, ..., e_{n-1}, \xi\}$ the local orthonormal basis of vector fields in M^n, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.17) and summing over $i = 1, ..., n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) &= \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &\quad - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) \\ &\quad - g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)]. \end{aligned} \tag{4.18}$$

So applying (4.5) – (4.9) into (4.18), we get

$$S(\varphi Y, \varphi Z) = -(r+1)g(\varphi Y, \varphi Z). \tag{4.19}$$

Using (3.6) and (3.8) in the above equation, we get

$$g(Y, \varphi Z) = (-\psi - \lambda n - \lambda - \mu + 1)g(\varphi Y, \varphi Z), \tag{4.20}$$

for any $Y, Z \in \chi(M^n)$ and for $Y \mapsto \varphi Y$, we get

$$g(\varphi Y, \varphi Z) = (-\psi - \lambda n - \lambda - \mu + 1)g(Y, \varphi Z). \tag{4.21}$$

Adding the previous two equations, we have

$$(-\psi - \lambda n - \lambda - \mu + 2)[g(Y, \varphi Z) - g(\varphi Y, \varphi Z)] = 0, \tag{4.22}$$

for any $Y, Z \in \chi(M^n)$ and follows

$$[\psi + \lambda(n+1) + \mu - 2] = 0. \tag{4.23}$$

In view of (3.7) and (4.23), we obtain

$$\lambda = \frac{-(\psi + n - 3)}{(n+2)} \quad and \quad \mu = \frac{-\psi + n^2 + 1}{(n+2)}. \tag{4.24}$$

Hence, we can state the following:

Theorem 4.2. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conharmonically flat, then*

$$\lambda = \frac{-(\psi + n - 3)}{(n+2)} \quad and \quad \mu = \frac{-\psi + n^2 + 1}{(n+2)}.$$

Corollary 4.2. *If (φ, ξ, η, g) is a φ-conharmonically flat Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, then there is no Ricci soliton with the potential vector field ξ.*

From (3.3), (3.7) and (4.20), we obtain

$$\begin{aligned} S(X,Y) &= (\psi + n\lambda + \mu - 1)g(X,Y) \\ &\quad + (\psi + n\mu + \lambda - 1)\eta(X)\eta(Y). \end{aligned} \tag{4.25}$$

Hence, we can state the following:

Proposition 4.2. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-conharmonically flat, then (M^n, g) is η-Einstein manifold.*

Let P be the projective curvature tensor of M^n.

Definition 4.3. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0,$$

is called φ-projectively flat.

Assume that $(M^n, g), n > 3$, is a φ-projectively flat Lorentzian para-Sasakian manifold. It can be easily seen that $\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1.4), φ-projectively flat means

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-1}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) \\ &\quad -g(\varphi X, \varphi Z)S(\varphi Y, \varphi W). \end{aligned} \tag{4.26}$$

Similar to the proof of Theorem (4.1), we can suppose that $\{e_1, e_2, ..., e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M^n, then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.26) and summing over $i = 1, ..., n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) &= \frac{1}{n-1}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) \\ &\quad -g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \end{aligned} \tag{4.27}$$

So applying (4.5) – (4.9) into (4.27), we get

$$nS(\varphi Y, \varphi Z) = rg(\varphi Y, \varphi Z), \tag{4.28}$$

In view of $(3.6), (3.8)$ and (4.28), we obtain

$$ng(Y, \varphi Z) = (\psi + \mu)g(\varphi Y, \varphi Z), \tag{4.29}$$

for any $Y, Z \in \chi(M^n)$ and for $Y \mapsto \varphi Y$, we get

$$ng(\varphi Y, \varphi Z) = (\psi + \mu)g(Y, \varphi Z). \tag{4.30}$$

Adding the previous two equations, we have

$$(\psi + \mu + n)[g(Y, \varphi Z) - g(\varphi Y, \varphi Z)] = 0, \tag{4.31}$$

for any $Y, Z \in \chi(M^n)$ and follows

$$\psi + \mu + n = 0. \tag{4.32}$$

In view of (3.7) and (4.32), we obtain

$$\lambda = -\psi - 2n + 1 \quad \textit{and} \quad \mu = -(\psi + n). \tag{4.33}$$

Hence, we can state the following:

Theorem 4.3. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-projectively flat, then*

$$\lambda = -\psi - 2n + 1 \quad \textit{and} \quad \mu = -(\psi + n).$$

Corollary 4.3. *If (φ, ξ, η, g) is a φ-projectively flat Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, then there is no Ricci soliton with the potential vector field ξ.*

From (3.3), (3.7) and (4.29), we obtain

$$\begin{aligned} S(X, Y) \quad = \quad & \left(\frac{\psi + \mu - n\lambda}{n}\right) g(X, Y) \\ & + \left(\frac{\psi + \mu - \mu n}{n}\right) \eta(X)\eta(Y). \end{aligned} \tag{4.34}$$

Hence, we can state the following:

Proposition 4.3. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the n-dimensional manifold M^n, (g, ξ, λ, μ) is an η-Ricci soliton on M^n and M^n is φ-projectively flat, then (M^n, g) is an η-Einstein manifold.*

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Ser. Math. Inform. Vol. 33, No 2 (2018), 231–254
https://doi.org/10.22190/FUMI1802231P

VECTOR BUNDLES AND PARACONTACT FINSLER STRUCTURES

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Abstract. Almost paracontact and normal almost paracontact Finsler structures on a vector bundle are defined. Finding some conditions, integrability of these structures is studied. Moreover, we define paracontact metric, para- Sasakian and K-paracontact Finsler structures and study some properties of these structures. For a K-paracontact Finsler structure, we find vertical and horizontal flag curvatures. Then, defining the vertical φ-flag curvature, we prove that every locally symmetric para-Sasakian Finsler structure has a negative vertical φ-flag curvature. Finally, we define the horizontal and vertical Ricci tensors of a para-Sasakian Finsler structure and study some curvature properties of them.
Keywords: Finsler structure, paracontact structure, Sasakian structure, symmetry, vector bundle.

1. Introduction

Contact geometry has a very close relationship with physical concepts. This geometry was introduced by Sophus Lie in his works on PDEs. Contact theory is in contrast with foliation theory. In contact theory, the investigators try to study a distribution which is no longer integrable (even locally). This does not occur for any one-dimensional distribution, but in upper-dimensional distributions we can find such structures whose vector fields are not tangent to any submanifold of the main manifold.

If a notion can be investigated in the case of contact structures, it can be studied for paracontact structures as well. These structures were first introduced by Sato [11]. Then Sasaki focused on some interesting concepts of these structures when he studied as for contact structures [9, 10]. Recently, many mathematician such as Bejan, Calvaruso, Druţă-Romaniuc, Ivanov, Kaneyuki, Cappelletti-Montano and Zamakovoy studied interesting properties of these structures [1, 2, 3, 4, 7, 8, 17, 18].

Received October 31, 2017; accepted December 12, 2017
2010 *Mathematics Subject Classification.* 53D15, 53C05

The notion of vector bundle is one of many important geometric objects that have interesting applications in physics [15, 16]. On vector bundles, Sinha, Prasad and Yadav defined some structures similar to the almost contact (paracontact) structures [12, 13, 14]. But the definitions presented by them are not well-defined (see 4.1) and cannot be realized in practical situations. Recently, Yalınıs and Çalışkan introduced and studied some concepts about the contact structure on vector bundles based on the same definitions [19]. These incorrect definitions led to some bugs in their study (see 4.1, 4.3, 4.2). After studying and modifying the definitions, we submitted this paper to arxiv.org (see arXiv:1302.0647) in 2013. But in 2014, Kazan and Karadağ (without considering our paper) submitted and published a paper with similar results on paracontact structures on vector bundles. Also, their study was based on incorrect definitions (see [5]). Moreover, their study led to some pitfalls in numerous results and discussions. We mention these mistakes as remarks in the current text.

In this paper, we define almost paracontact Finsler structures and normal almost paracontact Finsler structures on a vector bundle E and introduce some conditions for the integrability (normality) of these structures. We provide some equivalent conditions for the normality of an almost paracontact Finsler structure. Then, using a pseudo-metric G on E, similarly to [17], we consider the following compatibility condition for this structure:

$$G(\phi X, \phi Y) = -G(X, Y) + \eta(X)\eta(Y).$$

We also define the paracontact metric Finsler structure, para-Sasakian Finsler structure and K-paracontact Finsler structure. We find some conditions under which a paracontact metric Finsler structure is a K-paracontact structure. Then we get conditions under which a paracontact metric Finsler structure on a vector bundle E reduces to a K-paracontact Finsler structure. For a K-paracontact Finsler structure on a vector bundle E, we find vertical and horizontal flag curvatures. We define the vertical ϕ-flag curvature and prove that every locally symmetric para-Sasakian Finsler structure has a vertical ϕ-flag curvature $-\frac{1}{4}$.

Finally, we define horizontal and vertical Ricci tensors of a para-Sasakian Finsler manifold and study some of their curvature properties.

2. Preliminaries

Let $E(M) = (E, \pi, M)$ be a vector bundle with an $(n+m)$-dimensional total space E, n-dimensional base space M and the projection map π, such that $\pi : E \to M$, $u \in E \to \pi(u) = x \in M$ where $u = (x, y)$ and $y = \pi^{-1}(x)$ is the fibre of $E(M)$ over x. We denote by V_uE the local fibre of the vertical bundle VE at $u \in E$ and by H_uE the complementary space of V_uE in the tangent space T_uE at u to the total space E. Thus we have

$$T_uE = H_uE \oplus V_uE. \tag{2.1}$$

A nonlinear connection N on the total space E of $E(M)$ is a differentiable distribution $H : E \to T_uE$, $u \in E \to H_u \subset T_uE$ with the property (2.1) (see [6]).

We denote by (x^i, y^a), $i = 1, \ldots, n$, $a = 1, \ldots, m$, the canonical coordinates of a point $u \in E$. Then $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\}$ is the natural basis and $\{dx^i, dy^a\}$ is its dual basis on E. It is easy to see that $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\}$ is the basis on E adapted to decomposition (2.1) and $\{dx^i, \delta y^a\}$ is its basis (co-basis), where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \quad \delta y^a = dy^a + N_i^a dx^i,$$

and N_i^a are the coefficients of a nonlinear connection N. Now, we consider the horizontal and the vertical projectors h and v of the nonlinear connection, which are determined by the direct decomposition (2.1). These projectors can be expressed with respect to the adapted basis as follows:

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^a} \otimes \delta y^a.$$

Using the above projectors, any vector field X on E can be uniquely written as $X = hX + vX$. In the following, we adopt the notations

$$hX = X^H, \quad vX = X^V$$

and we say X^H and X^V are horizontal and vertical components of X. Thus, any vector field X on E can be uniquely written in the form

$$X = X^H + X^V.$$

In the adapted basis, we have $X = X^i(x, y)\frac{\delta}{\delta x^i} + \bar{X}^a(x, y)\frac{\partial}{\partial y^a}$ and

$$X^H = X^i(x, y)\frac{\delta}{\delta x^i}, \qquad X^V = \bar{X}^a(x, y)\frac{\partial}{\partial y^a}. \tag{2.2}$$

Now, let ω be a 1-form on E. Then it can be uniquely written as $\omega = \omega^H + \omega^V$. In the adapted basis, we have $\omega = \omega_i(x, y)dx^i + \bar{\omega}_a(x, y)\delta y^a$ and

$$\omega^H = \omega_i(x, y)dx^i, \quad \omega^V = \bar{\omega}_a(x, y)\delta y^a. \tag{2.3}$$

A tensor field T on the vector bundle E is called a distinguished tensor field (briefly, a d-tensor) of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ if it has the following property

$$\begin{aligned} &T(\omega_{i_1}, \ldots, \omega_{i_p}, \omega_{a_1}, \ldots, \omega_{a_r}, X_{j_1}, \ldots, X_{j_q}, X_{b_1}, \ldots, X_{b_s}) \\ &\quad = T(\omega^H_{i_1}, \ldots, \omega^H_{i_p}, \omega^V_{a_1}, \ldots, \omega^V_{a_r}, X^H_{j_1}, \ldots, X^H_{j_q}, X^V_{b_1}, \ldots, X^V_{b_s}), \end{aligned}$$

where ω_{i_k}, ω_{a_l}, $(k = 1, \ldots, p,\ l = 1, \ldots, r)$ are 1-forms on E and X_{j_v}, X_{b_w}, $(v = 1, \ldots, q,\ w = 1, \ldots, s)$ are vector fields on E. For instance, the components X^H and

X^V from (2.2) of a vector field X are d-vector fields. Also the components ω^H and ω^V of an 1-form ω, from (2.3) are d-1-form fields. In the adapted basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\}$ and adapted co-basis $\{dx^i, \delta y^a\}$, T is expressed by

$$\begin{aligned} T = T^{i_1,\ldots,i_p,a_1,\ldots,a_r}_{j_1,\ldots,j_q,b_1,\ldots,b_s} \frac{\delta}{\delta x^{i_1}} \otimes \ldots \otimes \frac{\delta}{\delta x^{i_p}} \otimes \frac{\partial}{\partial y^{a_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{a_r}} \\ \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_q} \otimes \delta y^{b_1} \otimes \ldots \otimes \delta y^{b_s}. \end{aligned}$$

A linear connection D on E is called a distinguished connection (briefly, d-connection) if it preserves by parallelism the horizontal distribution, that is $Dh = 0$. Since $Id = h + v$, then $Dh = 0$ implies that $Dv = 0$. Thus a d-connection preserves by parallelism the vertical distribution. Therefore, we can write

$$\begin{aligned} D_X Y &= (D_X Y^H)^H + (D_X Y^V)^V, \\ D_X \omega &= (D_X \omega^H)^H + (D_X \omega^V)^V, \end{aligned}$$

where X, Y are vector fields on E and ω is a 1-form on E.

A d-connection with respect to the adapted basis has the following form

$$\begin{cases} D_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = F^k_{ij} \frac{\delta}{\delta x^k}, & D_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^b} = \bar{F}^c_{ib} \frac{\partial}{\partial y^c}, \\ D_{\frac{\partial}{\partial y^a}} \frac{\delta}{\delta x^j} = C^k_{aj} \frac{\delta}{\delta x^k}, & D_{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial y^b} = \bar{C}^c_{ab} \frac{\partial}{\partial y^c}. \end{cases}$$

For this connection, there is an associated pair of operators in the algebra of d-tensor fields. For any vector field X on E, set

$$D^H_X Y = D_{X^H} Y, \quad D^V_X Y = D_{X^V} Y \quad D^H_X f = X^H(f), \quad D^V_X f = X^V(f),$$

where Y is a vector field and f is a smooth function on E. We call D^H (D^V) the operator of h-covariant (v-covariant) derivation. If ω is a 1-form on E, we define

$$\begin{aligned} (D^H_X \omega)Y &= X^H(\omega(Y)) - \omega(D^H_X Y), \\ (D^V_X \omega)Y &= X^V(\omega(Y)) - \omega(D^V_X Y), \end{aligned}$$

for any vector fields X, Y on E.

Now, we consider the pseudo-metric structure G on E which is symmetric and non-degenerate as $G = G^H + G^V$, where $G^H(X, Y) = G(X^H, Y^H)$ is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$, symmetric and non-degenerate on $H_u E$ and $G^V(X, Y) = G(X^V, Y^V)$ is of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, symmetric and non-degenerate on $V_u E$. In the adapted basis, we can write

$$G = g_{ij}(x, y) dx^i \otimes dx^j + h_{ab}(x, y) \delta y^i \otimes \delta y^j.$$

A d-connection D on E is called a *metrical d-connection* with respect to G if $D_X G = 0$ holds for every vector field X on E.

For a d-connection D, we consider the torsion T defined by

$$T(X,Y) = D_X Y - D_Y X - [X,Y], \quad \forall X,Y \in \chi(E),$$

where $\chi(E)$ is the set of all vector fields on E. The torsion of a d-connection D on E is completely determined by the following five tensor fields:

$$\begin{aligned} T^H(X^H,Y^H) &= D^H_X Y^H - D^H_Y X^H - [X^H,Y^H]^H, \\ T^V(X^H,Y^H) &= -[X^H,Y^H]^V, \\ T^H(X^H,Y^V) &= -D^V_Y X^H - [X^H,Y^V]^H, \\ T^V(X^H,Y^V) &= D^H_X Y^V - [X^H,Y^V]^V, \\ T^V(X^V,Y^V) &= D^V_X Y^V - D^V_Y X^V - [X^V,Y^V]^V, \end{aligned}$$

which are called $(h)h$-torsion, $(v)h$-torsion, $(h)hv$-torsion, $(v)hv$-torsion and $(v)v$-torsion, respectively. A d-connection D is said to be *symmetric* if the $(h)h$-torsion and $(v)v$-torsion vanish. In this paper, we use the symmetric metrical d-connection and we call it *Finsler connection*. It is easy to see that the following relations hold for the Finsler connection

$$\begin{aligned} 2G(D^H_X Y^H, Z^H) &= X^H G(Y^H,Z^H) + Y^H G(X^H,Z^H) - Z^H G(X^H,Y^H) \\ &\quad + G([X^H,Y^H],Z^H) - G([X^H,Z^H],Y^H) - G([Y^H,Z^H],X^H), \end{aligned} \tag{2.4}$$

$$\begin{aligned} 2G(D^V_X Y^V, Z^V) &= X^V G(Y^V,Z^V) + Y^V G(X^V,Z^V) - Z^V G(X^V,Y^V) \\ &\quad + G([X^V,Y^V],Z^V) - G([X^V,Z^V],Y^V) - G([Y^V,Z^V],X^V). \end{aligned} \tag{2.5}$$

Finally, we consider the curvature of a Finsler connection D as follows

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \quad \forall X,Y,Z \in \chi(E).$$

As D preserves by parallelism the horizontal and vertical distributions, from the above equation, we see that the operator $R(X,Y)$ carries horizontal vector fields into horizontal vector fields and vertical vector fields into verticals. Consequently, we have the following

$$R(X,Y)Z = (R(X,Y)Z^H)^H + (R(X,Y)Z^V)^V \quad \forall X,Y,Z \in \chi(E).$$

Since $R(X,Y)$ is skew symmetric with respect to X and Y, then the curvature of a Finsler connection D on E is completely determined by the following six tensor fields

$$\left\{ \begin{aligned} R(X^H,Y^H)Z^H &= D^H_X D^H_Y Z^H - D^H_Y D^H_X Z^H - D_{[X^H,Y^H]}Z^H, \\ R(X^H,Y^H)Z^V &= D^H_X D^H_Y Z^V - D^H_Y D^H_X Z^V - D_{[X^H,Y^H]}Z^V, \\ R(X^V,Y^H)Z^H &= D^V_X D^H_Y Z^H - D^H_Y D^V_X Z^H - D_{[X^V,Y^H]}Z^H, \\ R(X^V,Y^H)Z^V &= D^V_X D^H_Y Z^V - D^H_Y D^V_X Z^V - D_{[X^V,Y^H]}Z^V, \\ R(X^V,Y^V)Z^H &= D^V_X D^V_Y Z^H - D^V_Y D^V_X Z^H - D_{[X^V,Y^V]}Z^H, \\ R(X^V,Y^V)Z^V &= D^V_X D^V_Y Z^V - D^V_Y D^V_X Z^V - D_{[X^V,Y^V]}Z^V. \end{aligned} \right. \tag{2.6}$$

In the sequel, the restriction of the tensor field R to the horizontal (respectively vertical) distribution will be called horizontal (respectively vertical) curvature of D.

3. Almost Paracontact Finsler Structure

We consider a tensor field ϕ, a 1-form η and a vector field ξ on E, given by:

$$\phi = \phi^i_j(x,y)\frac{\delta}{\delta x^i}\otimes dx^j + \bar{\phi}^a_b(x,y)\frac{\partial}{\partial y^a}\otimes \delta y^b, \tag{3.1}$$

$$\eta = \eta_i(x,y)dx^i + \bar{\eta}_a(x,y)\delta y^a, \quad \xi = \xi^i(x,y)\frac{\delta}{\delta x^i} + \bar{\xi}^a(x,y)\frac{\partial}{\partial y^a}. \tag{3.2}$$

Definition 3.1. *Suppose that ϕ, η and ξ are given by (3.1) and (3.2) on E such that*

$$\phi^2 = I - \eta^H\otimes\xi^H - \eta^V\otimes\xi^V, \quad \eta^H(\xi^H) = \eta^V(\xi^V) = 1, \tag{3.3}$$

where

$$\eta^H = \eta_i(x,y)dx^i, \quad \eta^V = \bar{\eta}_a(x,y)\delta y^a, \quad \xi^H = \xi^i(x,y)\frac{\delta}{\delta x^i}, \quad \xi^V = \bar{\xi}^a(x,y)\frac{\partial}{\partial y^a}.$$

Then (ϕ,η,ξ) is called an almost paracontact Finsler structure on E and E is called an almost paracontact Finsler vector bundle.

Now, we are going to consider some properties of an almost paracontact Finsler structure. First, we prove the following.

Theorem 3.1. *Suppose that E has an almost paracontact Finsler structure, then the following holds*

$$\phi(\xi^H) = \phi(\xi^V) = 0, \quad \eta^H\circ\phi = \eta^V\circ\phi = 0.$$

Proof. By (3.3) and $\eta^V(\xi^H) = 0$, we have

$$\phi^2(\xi^H) = \xi^H - \eta^H(\xi^H)\xi^H = 0.$$

Then $\phi(\xi^H) = 0$ or $\phi(\xi^H)$ is a nontrivial eigenvector of ϕ corresponding to the eigenvalue 0. Since $\phi(\xi^H)\in HE$, then $\eta^V(\phi(\xi^H)) = 0$. Using (3.3), we obtain

$$0 = \phi^2(\phi(\xi^H)) = \phi(\xi^H) - \eta^H(\phi(\xi^H))\xi^H \quad \text{or} \quad \phi(\xi^H) = \eta^H(\phi(\xi^H))\xi^H.$$

Now, if $\phi(\xi^H)$ is nontrivial eigenvector of the eigenvalue 0, then $\eta^H(\phi(\xi^H)) \neq 0$. Thus we have

$$0 = \phi^2(\xi^H) = \eta^H(\phi(\xi^H))\phi(\xi^H) = (\eta^H(\phi(\xi^H)))^2\xi^H \neq 0,$$

which is a contradiction. Therefore $\phi(\xi^H) = 0$. Similarly, we get $\phi(\xi^V) = 0$.

On the other hand, since $\phi(\xi^H) = 0$ then we get

$$\begin{aligned}\eta^H(\phi(X))\xi^H &= \eta^H(\phi(X^H))\xi^H = \phi(X^H) - \phi^3(X^H)\\ &= \phi(X^H) - \phi(X^H) + \phi(\eta^H(X^H)\xi^H) = 0,\end{aligned}$$

for any $X\in\chi(E)$. Hence $\eta^H\circ\phi = 0$. Similarly, we have $\eta^V\circ\phi = 0$. $\square$

Remark 3.1. *Let us put*

$$\phi^H = \phi^i_j(x,y)\frac{\delta}{\delta x^i}\otimes dx^j \quad and \quad \phi^V = \bar{\phi}^a_b\frac{\partial}{\partial y^a}\otimes \delta y^b.$$

Then by Theorem 3.1, we deduce that (ϕ^H, η^H, ξ^H) *and* (ϕ^V, η^V, ξ^V) *are almost paracontact structures on sub-bundles* HE *and* VE*, respectively.*

Proposition 3.1. *Let* E *be endowed with an almost paracontact Finsler structure* (ϕ, η, ξ)*. Then rank*$\phi = (\dim E) - 2$*.*

Proof. It is sufficient to show that $\ker\phi =<\xi^H> \oplus <\xi^V>$. Since $\phi\xi^H = \phi\xi^V = 0$, then we have $<\xi^H> \oplus <\xi^V> \subseteq \ker\phi$. Now, let $\bar{\xi} \in \ker\phi$. Then $\phi\bar{\xi} = 0$ and (3.3) give us

$$\bar{\xi} = \eta^H(\bar{\xi})\xi^H + \eta^V(\bar{\xi})\eta^V \in <\xi^H> \oplus <\xi^V>,$$

i.e., $\ker\phi \subseteq <\xi^H> \oplus <\xi^V>$. Thus $\ker\phi =<\xi^H> \oplus <\xi^V>$. □

We say that an almost paracontact Finsler structure (ϕ, η, ξ) on the vector bundle E is *normal*, if the following holds

$$N^{(1)}(X,Y) = N_\phi(X,Y) - d\eta^H(X,Y)\xi^H - d\eta^V(X,Y)\xi^V = 0,$$

where X, Y are vector fields on E.

Now, we are going to give some equivalent conditions for the normality of structure (ϕ, η, ξ). For this reason, we introduce three tensors $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ and show that the vanishing of $N^{(1)}$ implies the vanishing of these tensors. First, we define the tensor $N^{(2)}$ on T_uE as follows

$$\begin{aligned} N^{(2)}(X^H, Y^H) &= (\pounds^H_{\phi X}\eta^H)(Y^H) - (\pounds^H_{\phi Y}\eta^H)(X^H), \\ N^{(2)}(X^V, Y^V) &= (\pounds^V_{\phi X}\eta^V)(Y^V) - (\pounds^V_{\phi Y}\eta^V)(X^V), \\ N^{(2)}(X^V, Y^H) &= (\pounds^V_{\phi X}\eta^H)(Y^H) + (\pounds^V_{\phi X}\eta^V)(Y^H) \\ &\quad -(\pounds^H_{\phi Y}\eta^H)(X^V) - (\pounds^H_{\phi Y}\eta^V)(X^V). \end{aligned}$$

To define $N^{(3)}$ and $N^{(4)}$, we consider the following cases:

Case 1: For $X^H, \xi^H \in H_uE$, we define

$$N^{(3)}(X^H) = (\pounds^H_\xi\phi)(X^H), \qquad N^{(4)}(X^H) = (\pounds^H_\xi\eta^H)(X^H).$$

Case 2: For $X^V, \xi^V \in V_uE$, we define

$$N^{(3)}(X^V) = (\pounds^V_\xi\phi)(X^V), \qquad N^{(4)}(X^V) = (\pounds^V_\xi\eta^V)(X^V).$$

Case 3: For $X^H \in H_uE$ and $\xi^V \in V_uE$, we define

$$N^{(3)}(X^H) = (\pounds^V_\xi\phi)(X^H), \qquad N^{(4)}(X^H) = (\pounds^V_\xi\eta^H)(X^H).$$

Case 4: For $X^V \in V_uE$ and $\xi^H \in H_uE$, we define

$$N^{(3)}(X^V) = (\pounds^H_\xi\phi)(X^V), \qquad N^{(4)}(X^V) = (\pounds^H_\xi\eta^V)(X^V).$$

Theorem 3.2. *For any almost paracontact Finsler structure (ϕ, η, ξ) the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$.*

Proof. If $N^{(1)} = 0$, then for X^H and ξ^H we have

$$\begin{aligned}0 &= N^{(1)}(X^H, \xi^H)\\ &= \phi^2[X^H, \xi^H] + [\phi X^H, \phi\xi^H] - \phi[\phi X^H, \xi^H] - \phi[X^H, \phi\xi^H]\\ &- d\eta^H(X^H, \xi^H)\xi^H - d\eta^V(X^H, \xi^H)\xi^V\\ &= \phi^2[X^H, \xi^H] - \phi[\phi X^H, \xi^H] - d\eta^H(X^H, \xi^H)\xi^H - d\eta^V(X^H, \xi^H)\xi^V.\end{aligned}$$

Applying η^H to (3.4), we obtain

$$d\eta^H(X^H, \xi^H) = 0,$$

which gives

$$\begin{aligned}N^{(4)}(X^H) = (\pounds^H_\xi \eta^H)(X^H) &= \xi^H\big(\eta^H(X^H)\big) - \eta^H[\xi^H, X^H]\\ &= -d\eta^H(X^H, \xi^H) = 0.\end{aligned}$$

Since $d\eta^H(X^H, \xi^H) = 0$, then by (3.4) we have

$$0 = \phi^2[X^H, \xi^H] - \phi[\phi X^H, \xi^H] = \phi\big((\pounds^H_\xi \phi)X^H\big). \tag{3.4}$$

Similarly to (3.4), we obtain

$$\begin{aligned}0 &= \eta^H\Big(N^{(1)}(\phi X^H, \xi^H)\Big) = d\eta^H(\xi^H, \phi X^H),\\ 0 &= \eta^V\Big(N^{(1)}(\phi X^H, \xi^H)\Big) = d\eta^V(\xi^H, \phi X^H),\end{aligned}$$

which imply that

$$\eta^H([\xi^H, \phi X^H]) = 0, \quad \eta^V([\xi^H, \phi X^H]) = 0.$$

Applying ϕ to (3.4) and using the above equation, we have $(\pounds^H_\xi \phi)X^H = 0$, i.e., $N^{(3)}(X^H) = 0$. Applying η^H to the following

$$\begin{aligned}0 = N^{(1)}(\phi X^H, Y^H) &= [X^H, \phi Y^H] - \eta^H(X^H)[\xi^H, \phi Y^H] + \phi Y^H\big(\eta^H(X^H)\big)\xi^H\\ &-\phi[X^H, Y^H] - \phi[\phi X^H, \phi Y^H] + [\phi X^H, Y^H] - \phi X^H\big(\eta^H(Y^H)\big)\xi^H\\ &+\eta^H(X^H)\phi[\xi^H, Y^H] + \eta^V[\phi X^H, Y^H]\xi^V,\end{aligned}$$

and using $\eta^H([\xi^H, \phi X^H]) = 0$, we get

$$\begin{aligned}0 &= -\eta^H[\phi Y^H, X^H] + \phi Y^H(\eta^H(X^H)) + \eta^H[\phi X^H, Y^H] - \phi X^H(\eta^H(Y^H))\\ &= (\pounds^H_{\phi Y}\eta^H)X^H - (\pounds^H_{\phi X}\eta^H)Y^H.\end{aligned}$$

Thus $N^{(2)}(X^H,Y^H)=0$. In a similar way, we can conclude the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ from the vanishing of $N^{(1)}$, when X^V and Y^V belong to V_uE. Now we prove the result when one of them belongs to V_uE and the other belongs to H_uE.

Similarly to (3.4), the vanishing of $N^{(1)}$ implies that

$$\begin{aligned} 0 &= N^{(1)}(X^V,\xi^H) \\ &= \phi^2[X^V,\xi^H]-\phi[\phi X^V,\xi^H]-d\eta^H(X^V,\xi^H)\xi^H-d\eta^V(X^V,\xi^H)\xi^V. \end{aligned} \tag{3.5}$$

Applying η^V and η^H to (3.5), we get

$$d\eta^V(X^V,\xi^H)=0, \quad d\eta^H(X^V,\xi^H)=0. \tag{3.6}$$

But we have

$$N^{(4)}(X^V)=(\pounds^H_\xi\eta^V)(X^V)=\xi^H(\eta^V(X^V))-\eta^V[\xi^H,X^V]=-d\eta^V(X^V,\xi^H).$$

Therefore the first part of (3.6) gives us $N^{(4)}(X^V)=0$. Using (3.5) and (3.6), we obtain

$$\begin{aligned} 0=\phi(N^{(1)}(X^V,\xi^H)) &= \phi[X^V,\xi^H]-[\phi X^V,\xi^H] \\ &= (\pounds^H_\xi\phi)(X^V) \\ &= N^{(3)}(X^V). \end{aligned}$$

Therefore $N^{(3)}(X^V)=0$. In a similar way to (3.5), we obtain

$$\begin{aligned} 0 &= \eta^H\Big(N^{(1)}(\xi^V,\phi Y^H)\Big)=-d\eta^H(\xi^V,\phi Y^H), \\ 0 &= \eta^V\Big(N^{(1)}(\xi^V,\phi Y^H)\Big)=-d\eta^V(\xi^V,\phi Y^H) \end{aligned}$$

which gives us

$$\eta^H[\xi^V,\phi Y^H]=0, \quad \eta^V[\xi^V,\phi Y^H]=0. \tag{3.7}$$

Using (3.7), we get

$$\begin{aligned} 0 &= \eta\Big(N^{(1)}(\phi X^V,Y^H)\Big) \\ &= \eta^H([X^V,\phi Y^H])+\eta^V([X^V,\phi Y^H])+\phi Y^H(\eta^V(X^V))+\eta^V([\phi X^V,Y^H]) \\ &\qquad -\phi X^V(\eta^H(Y^H))+\eta^H([\phi X^V,Y^H]) \\ &= -N^{(2)}(X^V,Y^H), \end{aligned}$$

i.e., $N^{(2)}(X^V,Y^H)=0$. $\square$

4. Paracontact Finsler Structures

A pseudo-metric structure G on E satisfying the conditions

$$G^H(\phi X, \phi Y) = -G^H(X, Y) + \eta^H(X)\eta^H(Y), \tag{4.1}$$

$$G^V(\phi X, \phi Y) = -G^V(X, Y) + \eta^V(X)\eta^V(Y), \tag{4.2}$$

is said to be compatible with the structure (ϕ, η, ξ). In this case, the quadruplet (ϕ, η, ξ, G) is called an almost paracontact metric Finsler structure and E is called an almost paracontact metric Finsler vector bundle. From (4.1) and (4.2) we deduce

$$G(\phi X, \phi Y) = -G(X, Y) + \eta^H(X)\eta^H(Y) + \eta^V(X)\eta^V(Y).$$

By (4.1) and (4.2) we have

$$G^H(X, \xi) = \eta^H(X), \quad G^V(X, \xi) = \eta^V(X), \tag{4.3}$$

which gives us $G(X, \xi) = \eta(X)$. Using (4.1)-(4.3), one can also obtain

$$G(X^H, \phi Y^H) = -G(\phi X^H, Y^H), \;\; G(X^V, \phi Y^V) = -G(\phi X^V, Y^V).$$

Now, we define the fundamental 2-form Φ by

$$\Phi(X, Y) = G(X, \phi Y), \quad \forall X, Y \in \chi(E),$$

which gives

$$\begin{aligned} &\Phi(X^H, Y^H) = G^H(X, \phi Y), \quad \Phi(X^V, Y^V) = G^V(X, \phi Y), \\ &\Phi(X^V, Y^H) = -\Phi(Y^H, X^V) = G(X^V, \phi Y^H) = 0. \end{aligned} \tag{4.4}$$

Remark 4.1. *In [5, 12, 13, 14, 19], to define contact and paracontact Finsler structures the authors considered a tensor field ϕ of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. According to (2.4), ϕ has the following local expression*

$$\phi = \phi^{ia}_{jb} \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial y^a} \otimes dx^j \otimes \delta y^b.$$

Thus for $X = X^k(x, y)\frac{\delta}{\delta x^k} + \bar{X}^c(x, y)\frac{\partial}{\partial y^c}$ we have

$$\phi(X) = \phi^{ia}_{kb} X^k \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial y^a} \otimes \delta y^b + \phi^{ia}_{jc} \bar{X}^c \frac{\delta}{\delta x^i} \otimes dx^j \otimes \frac{\partial}{\partial y^a}.$$

This shows that $\phi(X)$ is not a vector field on E and so $G(\phi X, \phi Y)$ is not well-defined. Therefore, ϕ can not be a tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Also, in the definition of contact and paracontact Finsler structures they considered the condition

$$\eta^H(\xi^H) + \eta^V(\xi^V) = 1, \tag{4.5}$$

and using it, they deduced $G^H(X,\xi) = \eta^H(X)$ *from* $G^H(\phi X, \phi Y) = -G^H(X,Y) + \eta^H(X)\eta^H(Y)$ *or* $G^H(\phi X, \phi Y) = G^H(X,Y) - \eta^H(X)\eta^H(Y)$ *(see (3.9) of [5] and (2.8) of [19]). But it is easy to see that this result is not true unless* $\eta^H(\xi^H) = 1$*. In a similar way, we can deduce that the condition* $\eta^V(\xi^V) = 1$ *is necessary. According to these reasons, definitions of contact and paracontact Finsler structures in [5, 12, 13, 14, 19] are not true mathematically. Moreover, the condition (4.5) breaks down the idea of inheritance properties by vertical and horizontal slices of a paracontact structure on a vector bundle (obviously this idea needs the condition* $\eta^H(\xi^H) = \eta^V(\xi^V) = 1$ *to be different from the (4.5) one).*

Definition 4.1. *An almost paracontact metric Finsler structure* (ϕ, η, ξ, G) *is called a paracontact metric Finsler structure if*

$$d\eta^H(X,Y) = \Phi(X^H, Y^H), \quad d\eta^V(X,Y) = \Phi(X^V, Y^V). \tag{4.6}$$

By (4.4) and (4.6), it follows that $d\eta(X,Y) = G(X, \phi Y)$. Then we get the following

$$d\eta(X^H, Y^H) = G(X^H, \phi Y^H) = G^H(X, \phi Y) = d\eta^H(X,Y).$$

Similarly, we obtain

$$d\eta(X^V, Y^V) = d\eta^V(X,Y) \quad \text{and} \quad d\eta(X^V, Y^H) = d\eta(X^H, Y^V) = 0.$$

Thus we deduce that (ϕ, η, ξ, G) is a paracontact metric Finsler structure if and only if the following holds

$$\begin{aligned} d\eta(X^H, Y^H) &= d\eta^H(X,Y) = G^H(X, \phi Y), \\ d\eta(X^V, Y^V) &= d\eta^V(X,Y) = G^V(X, \phi Y), \\ d\eta(X^H, Y^V) &= d\eta(X^V, Y^H) = 0. \end{aligned}$$

Moreover, if this structure is normal then it is called *para-Sasakian Finsler structure.*

Let (ϕ, η, ξ, G) be a paracontact metric Finsler structure on E. If ξ^H and ξ^V are Killing vector fields with respect to G^H and G^V, respectively, then (ϕ, η, ξ, G) is called a *K-paracontact Finsler structure* on E and E is called a *K-paracontact Finsler vector bundle.*

Theorem 4.1. *Let* (ϕ, η, ξ, G) *be a paracontact metric Finsler structure on* E*. Then* $N^{(2)} = N^{(4)} = 0$*. Moreover,* $N^{(3)} = 0$ *if and only if* ξ^H *and* ξ^V *are Killing vector fields with respect to* G^H *and* G^V*, respectively.*

Proof. Since (ϕ, η, ξ, G) is a paracontact metric Finsler structure on E, then we have

$$0 = G^H(\xi^H, \phi X^H) = d\eta^H(\xi^H, X^H) = (\pounds^H_\xi \eta^H)(X^H) = N^{(4)}(X^H).$$

We also have

$$d\eta^H(\phi X^H, Y^H) = G^H(\phi X^H, \phi Y^H) = -G^H(X^H, \phi^2 Y^H) = -d\eta^H(X^H, \phi Y^H),$$

which gives us $N^{(2)}(X^H, Y^H) = 0$. Similarly, we obtain $N^{(2)}(X^V, Y^V) = 0$. Using (4.1) and (4.6), we get

$$d\eta^H(\phi X^V, Y^H) = d\eta^H(\phi Y^H, X^V) = d\eta^V(\phi X^V, Y^H) = d\eta^V(\phi Y^H, X^V) = 0.$$

The above equations gives us $N^{(2)}(X^V, Y^H) = 0$.

Now, we prove the second part of the Theorem. According to

$$\pounds_\xi^H d\eta^H = i_\xi^H(d^2\eta^H) + d \circ i_\xi^H d\eta^H = d \circ i_\xi^H d\eta^H,$$

Since $N^{(4)} = 0$, then we obtain

$$(i_{\xi^H} d\eta^H)(X^H) = d\eta^H(\xi^H, X^H) = N^{(4)}(X^H) = 0. \tag{4.7}$$

By assumption, we have

$$d\eta^H(\xi^H, X^V) = G^H(\xi^H, \phi X^V) = 0. \tag{4.8}$$

By (4.8), it follows that

$$(i_{\xi^H} d\eta^H)(X^V) = d\eta^H(\xi^H, X^V) = 0. \tag{4.9}$$

Then (4.7) and (4.9) imply that $i_{\xi^H} d\eta^H = 0$ and consequently $\pounds_\xi^H d\eta^H = 0$. Similarly, we obtain $\pounds_\xi^V d\eta^V = 0$. Therefore, we get

$$0 = (\pounds_\xi^H d\eta^H)(X, Y^H) = (\pounds_\xi^H G^H)(X, \phi Y^H) + G^H(X, (\pounds_\xi^H \phi)(Y^H)), \tag{4.10}$$

$$0 = (\pounds_\xi^H d\eta^H)(X, Y^V) = (\pounds_\xi^H G^H)(X, \phi Y^V) + G^H(X, (\pounds_\xi^H \phi)(Y^V)), \tag{4.11}$$

$$0 = (\pounds_\xi^V d\eta^V)(X, Y^H) = (\pounds_\xi^V G^V)(X, \phi Y^H) + G^V(X, (\pounds_\xi^V \phi)(Y^H)), \tag{4.12}$$

$$0 = (\pounds_\xi^V d\eta^V)(X, Y^V) = (\pounds_\xi^V G^V)(X, \phi Y^V) + G^V(X, (\pounds_\xi^V \phi)(Y^V)). \tag{4.13}$$

By these equations, we conclude that if $\pounds_\xi^H G^H = \pounds_\xi^V G^V = 0$, then $N^{(3)} = 0$.

Conversely, let $N^{(3)} = 0$. Then from (4.10)-(4.13) we get

$$(i)\ (\pounds_\xi^H G^H)(X, \phi Y) = 0, \quad (ii)\ (\pounds_\xi^V G^V)(X, \phi Y) = 0. \tag{4.14}$$

Now, we show that $(\pounds_\xi^H G^H)(X,Y)=0$. It is easy to see that

$$(\pounds_\xi^H G^H)(X^V,Y^V)=0.$$

Using part (i) of (4.14), we obtain

$$\begin{aligned}(\pounds_\xi^H G^H)(X^H,Y^H)&=(\pounds_\xi^H G^H)(X^H,\phi^2Y^H)+\eta^H(Y^H)(\pounds_\xi^H G^H)(X^H,\xi^H)\\&=\eta^H(Y^H)(\pounds_\xi^H G^H)(X^H,\xi^H).\end{aligned}$$

Since $N^{(4)}=0$, then we have

$$(\pounds_\xi^H G^H)(X^H,\xi^H)=(\pounds_\xi^H \eta^H)(X^H)=0. \tag{4.15}$$

The relations (4.15) and (4.15) give us

$$(\pounds_\xi^H G^H)(X^H,Y^H)=0.$$

By part (i) of (4.14), we get

$$\begin{aligned}(\pounds_\xi^H G^H)(X^H,Y^V)&=(\pounds_\xi^H G^H)(X^H,\phi^2Y^V)+\eta^V(Y^V)(\pounds_\xi^H G^H)(X^H,\xi^V)\\&=-\eta^V(Y^V)G^H(X^H,[\xi^H,\xi^V]).\end{aligned}$$

Again, using part (i) of (4.14), it follows that

$$\begin{aligned}0=(\pounds_\xi^H G^H)(\xi^V,\phi^2Y^H)&=-G^H([\xi^H,\xi^V],\phi^2Y^H)\\&=-G^H([\xi^H,\xi^V],Y^H)+\eta^H(Y^H)G^H([\xi^H,\xi^V],\xi^H)\\&=-G^H([\xi^H,\xi^V],Y^H)+\eta^H(Y^H)\eta^H([\xi^H,\xi^V]).\end{aligned}$$

Since $N^{(4)}=0$, then we have

$$0=(\pounds_{\xi^V}\eta^H)(\xi^H)=-\eta^H([\xi^V,\xi^H]). \tag{4.16}$$

Plugging (4.16) in (4.16) implies that

$$G^H([\xi^H,\xi^V],Y^H)=0.$$

Then (4.16) reduces to the following

$$(\pounds_\xi^H G^H)(X^H,Y^V)=0.$$

It follows that $(\pounds_\xi^H G^H)(X,Y)=0$, where $X,Y\in\chi(E)$. Similarly, we can obtain $(\pounds_\xi^V G^V)(X,Y)=0$. This completes the proof. $\square$

Remark 4.2. *In [5], the authors used the equivalence between the Killing property of ξ and the Killing properties of ξ^H and ξ^V several times (see Lemma 5.1 and Corollary 5.2 of [5]). But it is not true. Indeed, if ξ is Killing then ξ^H and ξ^V are not Killing, necessarily.*

In the next proposition, we explain an important relation as a big widget for our next purposes.

Proposition 4.1. *Let (ϕ,η,ξ,G) be an almost paracontact metric Finsler structure on E. Then the following hold*

$$
\begin{aligned}
2G((D_{X^H}\phi)Y^H,Z^H) &= -d\Phi(X^H,\phi Y^H,\phi Z^H) - d\Phi(X^H,Y^H,Z^H)\\
&\quad - G\big(N^{(1)}(Y^H,Z^H),\phi X^H\big) + N^{(2)}(Y^H,Z^H)\eta(X^H)\\
&\quad + d\eta^H(\phi Y^H,X^H)\eta(Z^H) - d\eta^H(\phi Z^H,X^H)\eta(Y^H),\\
2G((D_{X^V}\phi)Y^V,Z^V) &= -d\Phi(X^V,\phi Y^V,\phi Z^V) - d\Phi(X^V,Y^V,Z^V)\\
&\quad - G(N^{(1)}(Y^V,Z^V),\phi X^V) + N^{(2)}(Y^V,Z^V)\eta(X^V)\\
&\quad + d\eta^V(\phi Y^V,X^V)\eta(Z^V) - d\eta^V(\phi Z^V,X^V)\eta(Y^V).
\end{aligned}
$$

Proof. By a simple calculation, we get

$$
\begin{aligned}
d\Phi(X^H,\phi Y^H,\phi Z^H) = &- X^H(\Phi(Y^H,Z^H)) - \phi Y^H(g(Z^H,X^H))\\
&+ \phi Y^H(\eta(Z^H)\eta(X^H)) + \phi Z^H(G(X^H,Y^H))\\
&- \phi Z^H(\eta(X^H)\eta(Y^H)) - G([X^H,\phi Y^H],Z^H)\\
&+ \eta^H([X^H,\phi Y^H])\eta(Z^H) - G([\phi Z^H,X^H],Y^H)\\
&+ \eta^H([\phi Z^H,X^H]\eta(Y^H) - \Phi([\phi Y^H,\phi Z^H],X^H).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
G(N^{(1)}(Y^H,Z^H),\phi X^H) = \Phi([Y^H,Z^H],X^H) &+ \Phi([\phi Y^H,\phi Z^H],X^H)\\
&+ G([\phi Y^H,Z^H],X^H) - \eta^H([\phi Y^H,Z^H])\eta(X^H)\\
&+ G([Y^H,\phi Z^H],X^H) - \eta^H([Y^H,\phi Z^H])\eta(X^H).
\end{aligned}
$$

Moreover, the following holds

$$
\begin{aligned}
d\eta^H(\phi Y^H,X^H)\eta(Z^H) &= \phi Y^H(\eta(X^H))\eta(Z^H) - \eta^H([\phi Y^H,X^H])\eta(Z^H),\\
d\eta^H(\phi Z^H,X^H)\eta(Y^H) &= \phi Z^H(\eta(X^H))\eta(Y^H) - \eta^H([\phi Z^H,X^H])\eta(Y^H).
\end{aligned}
$$

If we denote the right-hand side of (4.17) by I, then using the above equations we can obtain the following

$$
\begin{aligned}
I = \quad & \phi Y^H(G(Z^H,X^H)) - \phi Z^H(G(X^H,Y^H)) + G([X^H,\phi Y^H],Z^H)\\
&+ G([\phi Z^H,X^H],Y^H) - Y^H(\Phi(Z^H,X^H)) - Z^H(\Phi(X^H,Y^H))\\
&+ \Phi([X^H,Y^H],Z^H) + \Phi([Z^H,X^H],Y^H) - G([\phi Y^H,Z^H],X^H)\\
&- G([Y^H,\phi Z^H],X^H).
\end{aligned}
\tag{4.17}
$$

Since D is a Finsler connection, then it is G-compatible and its $(h)h$-torsion vanishes. Thus (4.17) reduces to following

$$I = G((\nabla_{X^H}\phi)Y^H, Z^H) - G(\nabla_{X^H}Z^H, \phi Y^H) - G(\nabla_{X^H}\phi Z^H, Y^H). \tag{4.18}$$

On the other hand, we have

$$X^H G(Z^H, \phi Y^H) = G(\nabla_{X^H}Z^H, \phi Y^H) + G(Z^H, \nabla_{X^H}\phi Y^H), \tag{4.19}$$

$$X^H G(\phi Z^H, Y^H) = G(\nabla_{X^H}\phi Z^H, Y^H) + G(\phi Z^H, \nabla_{X^H}Y^H). \tag{4.20}$$

Since $G(Z^H, \phi Y^H) = G(\phi Z^H, Y^H)$, then by (4.19) and (4.20) we get

$$G(\nabla_{X^H}Z^H, \phi Y^H) + G(\nabla_{X^H}\phi Z^H, Y^H) = -G((\nabla_{X^H}\phi)Y^H, Z^H). \tag{4.21}$$

Plugging (4.21) in (4.18) give us (4.17). Similarly, we can obtain (4.17). $\square$

Proposition 4.2. *Let (ϕ, η, ξ, G) be a paracontact metric Finsler structure on E. Then the following holds*

$$\begin{aligned} 2G((D_{X^H}\phi)Y^H, Z^H) &= -G(N^{(1)}(Y^H, Z^H), \phi X^H) + d\eta^H(\phi Y^H, X^H)\eta(Z^H) \\ &\quad - d\eta^H(\phi Z^H, X^H)\eta(Y^H), \\ 2G((D_{X^V}\phi)Y^V, Z^V) &= -G(N^{(1)}(Y^V, Z^V), \phi X^V) + d\eta^V(\phi Y^V, X^V)\eta(Z^V) \\ &\quad - d\eta^V(\phi Z^V, X^V)\eta(Y^V). \end{aligned} \tag{4.22}$$

Moreover, we get $D_\xi \phi = 0$.

Proof. By Proposition 4.1, we can get (4.22), (4.22). Thus we prove $D_\xi\phi = 0$. By $N^{(2)} = 0$, we obtain $d\eta^H(\phi X^H, \xi^H) = 0$. So plugging $X = \xi^H$ in (4.22) we get the following

$$G((D_{\xi^H}\phi)Y^H, Z^H) = 0,$$

which gives us

$$G^H((D_{\xi^H}\phi)Y^H, Z) = 0.$$

We also have $G^H((D_{\xi^H}\phi)Y^V, Z) = 0$. Therefore, we obtain

$$G^H((D_{\xi^H}\phi)Y, Z) = 0.$$

It means that $D_{\xi^H}\phi = 0$. Similarly, we get $D_{\xi^V}\phi = 0$. Therefore, $D_\xi\phi = 0$. $\square$

Using Theorem 4.1, we conclude the following.

Theorem 4.2. *Let (ϕ, η, ξ, G) is a paracontact metric Finsler structure on E. Then this structure is a K-paracontact structure if and only if $N^{(3)} = 0$.*

Since a para-Sasakian Finsler structure is normal, then we have $N^{(3)} = 0$. Thus from the above proposition we deduce the following.

Corollary 4.1. *Any para-Sasakian structure on E is a K-paracontact structure.*

Now, we are going to find some conditions under which a paracontact metric Finsler structure on a vector bundle E reduces to a K-paracontact Finsler structure. More precisely, we prove the following theorem.

Theorem 4.3. *Let (ϕ, η, ξ, G) be a paracontact metric Finsler structure on E. Then this structure is a K-paracontact Finsler structure if and only if*

$$\left\{ \begin{array}{ll} (i)\ D^H_X \xi^H = -\frac{1}{2}\phi X^H, & (ii)\ G^H([\xi^H, X^V]^H, Y^H) = 0, \\ (iii)\ D^V_X \xi^V = -\frac{1}{2}\phi X^V, & (iv)\ G^V([\xi^V, X^H]^V, Y^V) = 0. \end{array} \right. \tag{4.23}$$

Proof. Let (ϕ, η, ξ, G) be a K-paracontact Finsler structure. Then the following holds

$$\pounds^H_\xi G^H = \pounds^V_\xi G^V = 0.$$

We have

$$0 = (\pounds^H_\xi G^H)(X^V, Y^H) = -G^H([\xi^H, X^V]^H, Y^H),$$
$$0 = (\pounds^V_\xi G^V)(X^H, Y^V) = -G^V([\xi^V, X^H]^V, Y^V),$$

which gives us (ii) and (iv) of (4.23).

It is easy to see that, the following holds

$$(\pounds^H_\xi G)(X^H, Y^H) = (\pounds^H_\xi G^H)(X^H, Y^H).$$

Therefore

$$\begin{aligned} 0 = (\pounds^H_\xi G)(X^H, Y^H) &= \pounds^H_\xi G(X^H, Y^H) - G(\pounds^H_\xi X^H, Y^H) - G(X^H, \pounds^H_\xi Y^H) \\ &= \pounds^H_\xi G(X^H, Y^H) - G([\xi^H, X^H]^H, Y^H) - G(X^H, [\xi^H, Y^H]^H). \end{aligned}$$

Since D is symmetric, then we have

$$[\xi^H, X^H]^H = D^H_\xi X^H - D^H_X \xi^H. \tag{4.24}$$

Plugging (4.24) in (4.24) yields

$$0 = (D^H_\xi G)(X^H, Y^H) + G(D^H_X \xi^H, Y^H) + G(X^H, D^H_Y \xi^H).$$

Since D is G-compatible, then $D^H_{\xi} G = 0$. Thus

$$G(D^H_X \xi^H, Y^H) = -G(X^H, D^H_Y \xi^H). \tag{4.25}$$

Similarly, we get

$$G(D^V_X \xi^V, Y^V) = -G(X^V, D^V_Y \xi^V). \tag{4.26}$$

Using (2.4), we obtain

$$2G(D^H_X \xi^H, Y^H) - 2G(X^H, D^H_Y \xi^H) = 2d\eta(X^H, Y^H). \tag{4.27}$$

By (4.25) and (4.27) we have

$$2G(D^H_X \xi^H, Y^H) - 2G(X^H, D^H_Y \xi^H) = 4G(D^H_X \xi^H, Y^H). \tag{4.28}$$

(4.27) and (4.28) give us

$$2G(D^H_X \xi^H, Y^H) = d\eta(X^H, Y^H) = G(X^H, \phi Y^H) = -G(\phi X^H, Y^H).$$

Hence

$$D^H_X \xi^H = -\frac{1}{2}\phi X^H.$$

Similarly, using (4.26) we can deduce that $D^V_X \xi^V = -\dfrac{1}{2}\phi X^V$.

Conversely, suppose that (4.23) holds. Then from part (i) of (4.23) we have

$$\begin{aligned} 0 &= (\pounds^H_{\xi} G^H)(X^H, Y^H) \\ &= G(D^H_X \xi^H, Y^H) + G(X^H, D^H_Y \xi^H) \\ &= -\frac{1}{2}[G(\phi X^H, Y^H) + G(X^H, \phi Y^H)] = 0. \end{aligned}$$

Also (ii) gives us

$$(\pounds^H_{\xi} G^H)(X^V, Y^H) = 0.$$

Therefore, considering

$$(\pounds^H_{\xi} G^H)(X^V, Y^V) = 0,$$

we deduce $\pounds^H_{\xi} G^H = 0$. By a similar method, we can obtain $\pounds^V_{\xi} G^V = 0$. This completes the proof. $\square$

Lemma 4.1. *Let (ϕ, η, ξ, G) be a K-paracontact Finsler structure on a vector bundle E. Then the following holds*

$$R(X^V, \xi^V)\xi^V = -\frac{1}{4}(X^V - \eta^V(X^V)\xi^V), \tag{4.29}$$

$$R(X^H, \xi^H)\xi^H = -\frac{1}{4}(X^H - \eta^H(X^H)\xi^H) - D_{[X^H, \xi^H]^V}\xi^H. \tag{4.30}$$

Proof. Using $[X^V, \xi^V]^H = 0$, $D_\xi \phi = 0$ and (2.6), we obtain

$$\begin{aligned} R(X^V, \xi^V)\xi^V &= \frac{1}{2}\phi(D_{\xi^V} X^V + [X^V, \xi^V]^V) \\ &= -\frac{1}{4}\phi^2(X^V) \\ &= -\frac{1}{4}\big[X^V - \eta^V(X^V)\xi^V\big]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} R(X^H, \xi^H)\xi^H &= \frac{1}{2}\phi(D_{\xi^H} X^H + [X^H, \xi^H]^H) - D_{[X^H,\xi^H]^V}\xi^H \\ &= -\frac{1}{4}\phi^2(X^H) - D_{[X^H,\xi^H]^V}\xi^H \\ &= -\frac{1}{4}(X^H - \eta^H(X^H)\xi^H) - D_{[X^H,\xi^H]^V}\xi^H. \end{aligned}$$

This completes the proof. □

Theorem 4.4. *Let (ϕ, η, ξ, G) be a K-paracontact Finsler structure on E. Then the following holds*

(i) *the vertical flag curvature of all plane sections containing ξ^V is equal to $-\frac{1}{4}$;*

(ii) *the horizontal flag curvature of all plane sections containing ξ^H is equal to $-\frac{1}{4}$ if and only if $G(D^V_{[X^H,\xi^H]}\xi^H, X^H) = 0$.*

Proof. Let X^V be a unit vector field orthogonal to ξ^V. Then

$$\eta^V(X^V) = 0.$$

Consequently, (4.29) gives us

$$R(X^V, \xi^V)\xi^V = -\frac{1}{4}X^V.$$

Therefore, we get

$$K(X^V, \xi^V) = G^V(R(X^V, \xi^V)\xi^V, X^V) = -\frac{1}{4}G(X^V, X^V) = -\frac{1}{4}.$$

Similarly, if X^H is a unit vector field orthogonal to ξ^H, then from (4.30) we get

$$\begin{aligned} K(X^H, \xi^H) &= G^H(R(X^H, \xi^H)\xi^H, X^H) \\ &= -\frac{1}{4}G(X^H, X^H) - G(D_{[X^H,\xi^H]^V}\xi^H, X^H) \\ &= -\frac{1}{4} - G(D_{[X^H,\xi^H]^V}\xi^H, X^H). \end{aligned}$$

Therefore $K(X^H, \xi^H) = -\frac{1}{4}$ holds if and only if $G(D_{[X^H,\xi^H]^V}\xi^H, X^H) = 0$. □

Remark 4.3. *In Theorem 6.1 of [5] and Theorem 4.2 from [19], the authors consider only* $G^H(R(X^H,\xi^H)\xi^H,X^H)$ *and* $G^V(R(X^V,\xi^V)\xi^V,X^V)$ *to compute the flag curvature of a plane which contains* ξ*. But they forgot some terms such as* $G^H(R(X^V,\xi^H)\xi^H,X^H)$ *and* $G^H(R(X^V,\xi^V)\xi^H,X^H)$ *in computing the flag curvature. Indeed, they computed only vertical and horizontal flag curvatures.*

Now, we are going to study some properties of the para-Sasakian Finsler structure on a vector bundle. First, we prove the following.

Theorem 4.5. *Let* (ϕ,η,ξ,G) *be a para-Sasakian Finsler structure on a vector bundle* E*. Then the following relations hold*

$$(D^H_X\phi)Y^H=\frac{1}{2}\{\eta^H(Y^H)X^H-G^H(X^H,Y^H)\xi^H\}, \tag{4.31}$$

$$(D^V_X\phi)Y^V=\frac{1}{2}\{\eta^V(Y^V)X^V-G^V(X^V,Y^V)\xi^V\}. \tag{4.32}$$

Moreover, the Riemannian curvature satisfies the following

$$R(X^V,Y^V)\xi^V=\frac{1}{4}\{\eta^V(X^V)Y^V-\eta^V(Y^V)X^V\}, \tag{4.33}$$

$$R(X^H,Y^H)\xi^H=\frac{1}{4}\{\eta^H(X^H)Y^H-\eta^H(Y^H)X^H\}-D^V_{[X^H,Y^H]}\xi^H. \tag{4.34}$$

Proof. Since (ϕ,η,ξ,G) is a para-Sasakian Finsler structure, then $\Phi=d\eta$ and $N^{(1)}=N^{(2)}=0$. Thus by (4.17), we obtain

$$\begin{aligned}2G((D^H_X\phi)Y^H,Z^H)&=d\eta^H(\phi Y^H,X^H)\eta(Z^H)-d\eta^H(\phi Z^H,X^H)\eta(Y^H)\\&=G(\phi Y^H,\phi X^H)\eta(Z^H)-G(\phi Z^H,\phi X^H)\eta(Y^H)\\&=-G(X^H,Y^H)\eta(Z^H)+G(X^H,Z^H)G(\xi^H,Y^H)\\&=G(\eta(Y^H)X^H-G(X^H,Y^H)\xi^H,Z^H).\end{aligned}$$

This implies (4.31). With similar computations, one can obtain (4.32).

Using (2.6), Theorem 4.2 and Corollary 4.1, we have

$$\begin{aligned}R(X^V,Y^V)\xi^V&=D^V_XD^V_Y\xi^V-D^V_YD^V_X\xi^V-D^V_{[X^V,Y^V]}\xi^V\\&=D^V_X(-\frac{1}{2}\phi Y^V)-D^V_Y(-\frac{1}{2}\phi X^V)+\frac{1}{2}\phi[X^V,Y^V]^V\\&=-\frac{1}{2}(D^V_X\phi)Y^V+\frac{1}{2}(D^V_Y\phi)X^V.\end{aligned} \tag{4.35}$$

By (4.32) and (4.35) we get

$$R(X^V,Y^V)\xi^V=\frac{1}{4}\{\eta^V(X^V)Y^V-\eta^V(Y^V)X^V\}.$$

Similarly, using (4.31) we obtain

$$\begin{aligned} R(X^H, Y^H)\xi^H &= D^H_X D^H_Y \xi^H - D^H_Y D^H_X \xi^H - D^H_{[X^H,Y^H]}\xi^H - D^V_{[X^H,Y^H]}\xi^H \\ &= D^H_{X}(-\frac{1}{2}\phi Y^H) - D^H_{Y}(-\frac{1}{2}\phi X^H) + \frac{1}{2}\phi[X^H, Y^H]^H - D^V_{[X^H,Y^H]}\xi^H \\ &= -\frac{1}{2}(D^H_X\phi)Y^H + \frac{1}{2}(D^H_Y\phi)X^H - D^V_{[X^H,Y^H]}\xi^H \\ &= \frac{1}{4}\{\eta^H(X^H)Y^H - \eta^H(Y^H)X^H\} - D^V_{[X^H,Y^H]}\xi^H. \end{aligned}$$

This completes the proof. □

A plane section in V_uE is called a vertical ϕ-section if there exists a unit vector X^V in V_uE orthogonal to ξ^V such that $\{X^V, \phi X^V\}$ span the section. The vertical flag curvature $K(X^V, \phi X^V)$ is called *vertical ϕ-flag curvature.*

Proposition 4.3. *Let (ϕ, η, ξ, G) be a para-Sasakian Finsler structure on E. Suppose that E is locally symmetric. Then it has a vertical ϕ-flag curvature $-\frac{1}{4}$.*

Proof. Let $X^V \neq 0$ be a vector field on E orthogonal to ξ^V. Then we have $\eta^V(X^V) = G^V(X^V, \eta^V) = 0$. By direct conclusion we obtain

$$\begin{aligned} (D_{\phi X^V}R)(X^V, \phi X^V)\xi^V = \frac{1}{2}\Big[\ &\phi R(X^V, \phi X^V)\phi X^V - \frac{1}{4}G^V(X^V, \phi X^V)\phi^2 X^V \\ &+ \frac{1}{4}G^V(\phi X^V, \phi X^V)\phi X^V\Big]. \end{aligned} \tag{4.36}$$

Considering $G(X^V, \phi X^V) = -G(\phi X^V, X^V)$, we have $G(X^V, \phi X^V) = 0$. Using this equation and noting that E is locally symmetric (4.36) gives us

$$\phi R(X^V, \phi X^V)\phi X^V + \frac{1}{4}G^V(\phi X^V, \phi X^V)\phi X^V = 0. \tag{4.37}$$

By (4.37), we get

$$G(\phi R(X^V, \phi X^V)\phi X^V, \phi X^V) + \frac{1}{4}G(\phi X^V, \phi X^V)G(\phi X^V, \phi X^V) = 0. \tag{4.38}$$

Since $\eta^V(X^V) = 0$, then (4.38) gives us

$$G(R(X^V, \phi X^V)\phi X^V, X^V) = \frac{1}{4}G^2(\phi X^V, \phi X^V).$$

Therefore, we obtain

$$K(X^V, \phi X^V) = \frac{G(R(X^V, \phi X^V)\phi X^V, X^V)}{G(X^V, X^V)G(\phi X^V, \phi X^V)} = -\frac{1}{4}.$$

It means that E has a vertical ϕ-flag curvature $-\frac{1}{4}$. □

4.1. Horizontal and Vertical Ricci Tensors

The *horizontal Ricci tensor* S^H of an $(n+m)$-dimensional para-Sasakian Finsler manifold E is given by

$$\begin{aligned} S^H(X^H, Y^H) &= \sum_{i=1}^{n-1} G(R(X^H, E_i^H)E_i^H, Y^H) + G(R(X^H, \xi^H)\xi^H, Y^H) \\ &= \sum_{i=1}^{n-1} G(R(E_i^H, X^H)Y^H, E_i^H) + G(R(\xi^H, X^H)Y^H, \xi^H), \end{aligned}$$

where $\{E_1^H, E_2^H, \dots, E_{n-1}^H, \xi^H\}$ is a local orthonormal frame of H_uE. Similarly, the *vertical Ricci tensor* of an $(n+m)$-dimensional para-Sasakian Finsler manifold E is given by

$$\begin{aligned} S^V(X^V, Y^V) &= \sum_{i=1}^{m-1} G(R(X^V, E_i^V)E_i^V, Y^V) + G(R(X^V, \xi^V)\xi^V, Y^V) \\ &= \sum_{i=1}^{m-1} G(R(E_i^V, X^V)Y^V, E_i^V) + G(R(\xi^V, X^V)Y^V, \xi^V), \end{aligned}$$

where $\{E_1^V, E_2^V, \dots, E_{m-1}^V, \xi^V\}$ is a local orthonormal frame of V_uE.

Proposition 4.4. *The horizontal and vertical Ricci tensors S^H and S^V of a $(n+m)$-dimensional para-Sasakian Finsler manifold satisfy the following equations:*

$$(4.39)\qquad \begin{cases} (i)\ S^H(X^H, \xi^H) = \frac{1-n}{4}\eta^H(X^H) - \sum_{i=1}^{n-1} G(D^V_{[E_i^H, X^H]}\xi^H, E_i^H), \\ (ii)\ S^V(X^V, \xi^V) = \frac{1-m}{4}\eta^V(X^V), \\ (iii)\ S^H(\xi^H, \xi^H) = \frac{1-n}{4} - \sum_{i=1}^{n-1} G(D^V_{[E_i^H, \xi^H]}\xi^H, E_i^H), \\ (iv)\ S^V(\xi^V, \xi^V) = \frac{1-m}{4}. \end{cases}$$

Proof. Using (4.34) and (4.39), one can obtain the following:

$$\begin{aligned} S^H(X^H, \ \xi^H) &= \sum_{i=1}^{n-1} G\big(R(E_i^H, X^H)\xi^H, \quad E_i^H\big) \\ &= \sum_{i=1}^{n-1} G\Big(\frac{1}{4}\eta^H(E_i^H)X^H - \frac{1}{4}\eta^H(X^H)E_i^H - D^V_{[E_i^H, X^H]}\xi^H, \ E_i^H\Big). \end{aligned} \tag{4.40}$$

Since E_i^H is orthogonal to ξ^H, then we have $\eta^H(E_i^H) = G(E_i^H, \xi^H) = 0$. By (4.40) and $G(E_i^H, E_i^H) = 1$, we get part (i) of (4.39). Plugging $X^H = \xi^H$ in (i) and using

$\eta^H(X^H) = 1$ implies (iii). Similarly, (4.33) and (4.39) give us

$$\begin{aligned} S^V(X^V, \xi^V) &= \sum_{i=1}^{m-1} G\big(R(E_i^V, X^V)\xi^V,\ E_i^V\big) \\ &= \frac{1}{4}\sum_{i=1}^{m-1} G\Big(\eta^V(E_i^V)X^V - \eta^V(X^V)E_i^V,\ E_i^V\Big) \\ &= \frac{1-m}{4}\eta^V(X^V). \end{aligned}$$

By setting $X^V = \xi^V$ in (4.41), we get (iv). □

According to parts (i) and (iii) of (4.39), one can deduce the following easily.

Corollary 4.2. *For an $(n+m)$-dimensional para-Sasakian Finsler manifold, the following holds*

i) $S^H(X^H, \xi^H) = \frac{1-n}{4}\eta^H(X^H)$ *is equivalent to vanishing of*

$$\sum_{i=1}^{n-1} G(D^V_{[E_i^H, X^H]}\xi^H, E_i^H);$$

ii) $S^H(\xi^H, \xi^H) = \frac{1-n}{4}$ *is equivalent to vanishing of*

$$\sum_{i=1}^{n-1} G(D^V_{[E_i^H, \xi^H]}\xi^H, E_i^H).$$

Using Lemma 4.1, we have the following proposition.

Proposition 4.5. *The horizontal and vertical Ricci tensors S^H and S^V of a $(n+m)$-dimensional K-paracontact Finsler vector bundle satisfy the following equations:*

$$S^H(\xi^H, \xi^H) = \frac{1-n}{4} - \sum_{i=1}^{n-1} G(D^V_{[E_i^H, \xi^H]}\xi^H, E_i^H), \quad S^V(\xi^V, \xi^V) = \frac{1-m}{4}.$$

Proposition 4.5 have an easy consequence as follows.

Corollary 4.3. *For a $(n+m)$-dimensional K-paracontact Finsler vector bundle E, $S^H(\xi^H, \xi^H) = \frac{1-n}{4}$ is equivalent to the vanishing of*

$$\sum_{i=1}^{n-1} G(D^V_{[E_i^H, \xi^H]}\xi^H, E_i^H).$$

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Ser. Math. Inform. Vol. 33, No 2 (2018), 255–268
https://doi.org/10.22190/FUMI1802255D

ON THE QUASI-CONFORMAL CURVATURE TENSOR OF AN ALMOST KENMOTSU MANIFOLD WITH NULLITY DISTRIBUTIONS

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Abstract. The objective of the present paper is to characterize quasi-conformally flat and ξ-quasi-conformally flat almost Kenmotsu manifolds with (k,μ)-nullity and $(k,\mu)'$-nullity distributions, respectively. Also we characterize almost Kenmotsu manifolds with vanishing extended quasi-conformal curvature tensor and extended ξ-quasi-conformally flat almost Kenmotsu manifolds such that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution.
Keywords: Almost Kenmotsu manifold, Einstein manifold, Weyl conformal curvature tensor, Quasi-conformal curvature tensor, Extended quasi-conformal curvature tensor.

1. Introduction

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g and let $T(M)$ be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of (M,g) is defined by

$$g(QX,Y) = S(X,Y), \tag{1.1}$$

where S denotes the Ricci tensor of type $(0,2)$ on M and $X,Y \in T(M)$. The Weyl conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.2}$$

for $X,Y,Z \in T(M)$, where R and r denote the Riemannian curvature tensor and scalar curvature of M, respectively.

Received February 08, 2018; accepted May 05, 2018
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C35

For a $(2n+1)$-dimensional Riemannian manifold, the quasi-conformal curvature tensor $\tilde{C}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.3}$$

where a and b are two scalars. The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [21]. If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor.

A $(2n+1)$-dimensional Riemannian manifold will be called a manifold of the quasi-constant curvature if the Riemannian curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad +g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)],\end{aligned} \tag{1.4}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, p, q are scalars and there exists a unit vector field ρ satisfying $g(X,\rho) = T(X)$. The notion of the quasi-constant curvature for Riemannian manfiolds was introduced by Chen and Yano [4].

At present, the study of nullity distributions is a very interesting topic on almost contact metric manifolds. The notion of k-nullity distribution was introduced by Gray [10] and Tanno [15] in the study of Riemannian manifolds (M,g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.5}$$

for any X, $Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Blair, Koufogiorgos and Papantonio [1] introduced the generalized notion of k-nullity distribution, named (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1}$, ϕ, ξ, η, $g)$, which is defined for any $p \in M$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned}N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\mu[g(Y,Z)hX - g(X,Z)hY]\},\end{aligned} \tag{1.6}$$

where $h = \frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation.

In [7] Dileo and Pastore introduce the notion of $(k,\mu)'$-nullity distribution, another generalized notion of k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p \in M^{2n+1}$ and $k,\mu \in \mathbb{R}$ as follows:

$$\begin{aligned}N_p(k,\mu)' = \{Z \in T_pM : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\mu[g(Y,Z)h'X - g(X,Z)h'Y]\},\end{aligned} \tag{1.7}$$

where $h' = h \circ \phi$.

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ, ξ, η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2],[3]),

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{1.8}$$

where I denotes the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$ hold; both can be derived from (1.8) easily.
If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y of T_pM^{2n+1}, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \Phi Y)$ for any X, Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [2]. Recently in ([7],[8],[9],[13],[14]), an almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also, Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X,Y. It is well known [11] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

At each point $p \in M$,we have

$$T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\}$$

where $\{\xi_p\}$ is 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p. Then the Weyl conformal curvature tensor C is a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi\}.$$

Three particular cases can be considered as follows:
(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, the projection of the image of C in $\phi(T_p(M))$ is zero .
(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M))$, that is, the projection of the image of C in $\{\xi\}$ is zero.
(3) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, when C is restricted to $\phi(T_p(M)) \times \phi(T_p(M))$, the projection of the image of C in $\phi(T_p(M))$ is zero, which is equivalent to $\phi^2 C(\phi X, \phi Y)\phi Z = 0$.

Definition 1.1. [22] A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be ξ-conformally flat if the linear operator $C(X, Y)$ is an endomorphism of $\phi(T(M))$, that is, if

$$C(X, Y)\phi(T(M)) \subset \phi(T(M)).$$

Then it follows immediately that

Proposition 1.1. *[22] On a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following conditions are equivalent.*

(a) M^{2n+1} *is ξ-conformally flat,*

(b) $\eta(C(X, Y)Z) = 0,$

(c) $\phi^2 C(X, Y)Z = -C(X, Y)Z,$

(d) $C(X, Y)\xi = 0,$

where $X, Y, Z \in T(M)$.

Almost Kenmotsu manifolds have been studied by several authors such as Dileo and Pastore ([7]-[9]), Wang and Liu ([16]-[20]), De and Mandal([5], [6], [12]) and many others. In the present paper we like to study quasi-conformal curvature tensor of almost Kenmotsu manifolds with (k, μ) and $(k, \mu)'$-nullity distributions, respectively. Also, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended ξ-quasi-conformally flat almost Kenmotsu manifolds with (k, μ)-nullity distribution.
The paper is organized as follows:
In Section 2, we give a brief account on almost Kenmotsu manifolds with ξ belonging to the (k, μ)-nullity distribution and ξ belonging to the $(k, \mu)'$-nullity distribution. Section 3 deals with quasi-conformally flat and ξ-quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the (k, μ)-nullity distribution. As a consequence of the main result, we obtain several corollaries. Section 4 is devoted to the study of quasi-conformally flat almost Kenmotsu manifolds with the characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distribution. In the final section, we discuss vanishing extended quasi-conformal curvature tensor in an almost Kenmotsu manifold and extended ξ-quasi-conformally flat almost Kenmotsu manifolds with (k, μ)-nullity distribution.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [13]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.1}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.2}$$

(2.3) $$\phi l\phi - l = 2(h^2 - \phi^2),$$

(2.4) $$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y,$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([7], [18])

(2.5) $$h = 0 \Leftrightarrow h' = 0, \;\; h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2).$$

3. Quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution

In this section we study quasi-conformally flat and ξ-quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution.
From (1.6) we obtain

(3.1) $$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

where $k, \mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 3.1. *[7] Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n+1)$. Suppose that the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a wrapped product of an open interval and an almost Kähler manifold.*

In view of Lemma 3.1 it follows from the equation (3.1),

(3.2) $$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

(3.3) $$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X,$$

(3.4) $$S(X,\xi) = -2n\eta(X),$$

(3.5) $$Q\xi = -2n\xi,$$

for any vector fields X, Y on M^{2n+1}.

Theorem 3.1. *An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (k,μ)-nullity distribution is quasi-conformally flat if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof: Let us first consider the manifold M^{2n+1} which is quasi-conformally flat, that is,

(3.6) $$\tilde{C}(X,Y)Z = 0,$$

for any vector fields X, Y, Z on M^{2n+1}.
From (1.3) we have

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ &\quad + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &\quad + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.7}$$

Putting $Z = \xi$ in the above equation and using (3.2) and (3.4) we get

$$\begin{aligned} \eta(X)g(Y,W) - \eta(Y)g(X,W) &= \frac{b}{a}[-2n\eta(X)g(Y,W) + 2n\eta(Y)g(X,W) \\ &\quad + S(Y,W)\eta(X) - S(X,W)\eta(Y)] \\ &\quad + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b][g(X,W)\eta(Y) \\ &\quad - g(Y,W)\eta(X)]. \end{aligned} \tag{3.8}$$

Putting $Y = \xi$ in the above equation we obtain after simplification

$$S(X,W) = \alpha g(X,W) + \beta\eta(X)\eta(W), \tag{3.9}$$

where $\alpha = \frac{a}{b}[\frac{2bn}{a} + \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b] + 1]$ and $\beta = \frac{a}{b}[-\frac{4bn}{a} - \frac{r}{a(2n+1)}[\frac{a}{2n} + 2b] - 1]$.
Therefore, we have $\alpha + \beta = -2n$.
Now using the above relation, (3.9) implies

$$r = 2n(\alpha - 1). \tag{3.10}$$

In [7], Dileo and Pastore proved that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the sectional curvature $K(X,\xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$. Using this value of r we obtain from (3.10), $\alpha = -2n$. This implies $\beta = 0$.
Hence (3.9) reduces to

$$S(X,W) = -2ng(X,W). \tag{3.11}$$

From (3.7) we obtain

$$\begin{aligned} aR(X,Y)Z &= -b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad + \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.12}$$

Using the value of r and (3.11) in (3.12) yields

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y], \tag{3.13}$$

which implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.
Conversely, suppose that the manifold is locally isometric to the hyperbolic space

$\mathbb{H}^{2n+1}(-1)$. That is, (3.13) holds.
Contracting X in (3.13) yields

$$S(Y,Z) = -2ng(Y,Z). \tag{3.14}$$

Hence (3.13) and (3.14) together implies $\tilde{C}(X,Y)Z = 0$. That is, the manifold is quasi-conformally flat.
Hence the theorem is proved.

Now, if $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor. Hence we can state the following:

Corollary 3.1. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is conformally flat if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

The above corollary has been proved by De and Mandal [5].

Theorem 3.2. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)- nullity distribution is ξ-quasi-conformally flat if and only if the manifold is an Einstein manifold.*

Proof: Let us consider a manifold that is ξ-quasi-conformally flat. That is,

$$\tilde{C}(X,Y)\xi = 0,$$

which implies

$$\begin{aligned} aR(X,Y)\xi &= -b[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] \\ &\quad + \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,\xi)X - g(X,\xi)Y]. \end{aligned} \tag{3.15}$$

Using (3.2) and (3.4) and $r = -2n(2n+1)$ we get from the above equation

$$\eta(Y)QX - \eta(X)QY = -2n[\eta(Y)X - \eta(X)Y], \tag{3.16}$$

Putting $Y = \xi$ in the above equation we obtain

$$QX = -2nX, \tag{3.17}$$

which implies $S(X,Y) = -2ng(X,Y)$. That is, the manifold is Einstein.
Conversely, assume that the manifold is Einstein. Then there exists a scalar λ such that

$$S(X,Y) = \lambda g(X,Y). \tag{3.18}$$

In an almost Kenmotsu manifold with (k,μ)-nullity distribution, the scalar curvature $r = -2n(2n+1)$. This implies $\lambda = -2n$. Now

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.19}$$

Using (3.18) we get

$$\tilde{C}(X,Y)Z = a[R(X,Y)Z + (g(Y,Z)X - g(X,Z)Y)]. \tag{3.20}$$

Putting $Z = \xi$ in the above equation and using (3.2) we obtain

$$\tilde{C}(X,Y)\xi = 0,$$

which implies that the manifold is ξ-quasi-conformally flat.

If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor.
Thus we are in a position to state the following:

Corollary 3.2. *An almost Kenmotsu manifold with (k,μ)-nullity distribution is ξ-conformally flat if and only if it is Einstein.*

4. Quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution

In this section we study ξ-quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.5) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. Before presenting our main theorem we recall some results:

Lemma 4.1. *(Prop. 4.1 and Prop. 4.3 of [7]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as a simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvatures are given by the following:*

(a) $K(X,\xi) = k - 2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k + 2\lambda$ *if* $X \in [-\lambda]'$,

(b) $K(X,Y) = k - 2\lambda$ *if* $X, Y \in [\lambda]'$;
$K(X,Y) = k + 2\lambda$ *if* $X, Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} *has a constant negative scalar curvature* $r = 2n(k-2n)$.

Lemma 4.2. *(Lemma 3 of [16]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution . If $h' \neq 0$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{4.1}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

From (1.7) we have,

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{4.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (4.2)

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{4.3}$$

Contracting X in (4.2), we have

$$S(Y,\xi) = 2nk\eta(Y). \tag{4.4}$$

Moreover, in an almost Kenmotsu manifold with $(k,\mu)'$-nullity distribution

$$\nabla_X \xi = X - \eta(X)\xi + h'X \tag{4.5}$$

and

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y) \tag{4.6}$$

holds.

Theorem 4.1. *A $(2n+1)$-dimensional$(n > 1)$ quasi-conformally flat almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution is either conformally flat or of a quasi-constant curvature.*

Proof: Let us assume that the manifold M^{2n+1} is quasi-conformally flat, that is,

$$\tilde{C}(X,Y)Z = 0, \tag{4.7}$$

for any vector fields X, Y, Z on M^{2n+1}.
From (1.3) we have

$$\begin{aligned} a\tilde{R}(X,Y,Z,W) &= b[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ &\quad +S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &\quad +\frac{r}{(2n+1)}[\frac{a}{2n} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4.8}$$

Putting $Z = \xi$ in the above equation and using (4.2) and (4.4) we have

$$\begin{aligned} &ak[\eta(Y)g(X,W) - \eta(X)g(Y,W)] + a\mu[\eta(Y)g(h'X,W) - \eta(X)g(h'Y,W)] \\ &= b[2nk\eta(X)g(Y,W) - 2nk\eta(Y)g(X,W) - \eta(Y)S(X,W) + \eta(X)S(Y,W)] \\ &\quad +\frac{r}{(2n+1)}[\frac{a}{2n} + 2b][\eta(Y)g(X,W) - \eta(X)g(Y,W)]. \end{aligned} \tag{4.9}$$

Putting $Y=\xi$ in the above equation and using (4.4) we get after simplifying

$$\begin{aligned} S(X,W) &= [-2nk+\frac{r}{b(2n+1)}[\frac{a}{2n}+2b]-\frac{ak}{b}]g(X,W) \\ &+[4nk-\frac{r}{b(2n+1)}[\frac{a}{2n}+2b]+\frac{ak}{b}]\eta(X)\eta(W)-\frac{a\mu}{b}g(h'X,W). \end{aligned} \tag{4.10}$$

Let us denote

$$A=-2nk+\frac{r}{b(2n+1)}[\frac{a}{2n}+2b]-\frac{ak}{b} \tag{4.11}$$

and

$$B=4nk-\frac{r}{b(2n+1)}[\frac{a}{2n}+2b]+\frac{ak}{b}. \tag{4.12}$$

Then, we see that

$$A+B=2nk. \tag{4.13}$$

Putting $X=W=e_i$ in (4.10), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $i=1,2,3....,(2n+1)$, we get

$$r=A(2n+1)+B. \tag{4.14}$$

From (4.13) and (4.14) we get

$$A=\frac{r}{2n}-k. \tag{4.15}$$

From (4.11) and (4.15), it follows that

$$-2nk+\frac{r}{b(2n+1)}[\frac{a}{2n}+2b]-\frac{ak}{b}=\frac{r}{2n}-k.$$

The above relation gives

$$(a+2nb-b)(r-2nk(2n+1))=0. \tag{4.16}$$

Hence, either $a+2nb-b=0$ or $r=2nk(2n+1)$.

Let us suppose that $a+2nb-b=0$. Then we see that $b=-\frac{a}{2n-1}$. Hence, from (1.3), it follows that $\tilde{C}(X,Y)Z=aC(X,Y)Z$, where $C(X,Y)Z$ is the Weyl conformal curvature tensor. So, in this case, the quasi-conformally flat manifold is conformally flat.

Now, if $r = 2nk(2n+1)$, then from (4.10) we obtain

$$S(X,W) = 2nkg(X,W) - \frac{a\mu}{b}g(h'X,W). \tag{4.17}$$

Using (4.17) in (4.8) yields

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= k[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad -\mu[g(h'X,Z)g(Y,W) - g(h'Y,Z)g(X,W) \\ &\quad +g(h'Y,W)g(X,Z) - g(h'X,W)g(Y,Z)].\end{aligned} \tag{4.18}$$

From (4.1) and (4.17), it follows that

$$g(h'X,W) = l[g(X,W) - \eta(X)\eta(W)], \tag{4.19}$$

where $l = \frac{2nb(k+1)}{a\mu-2nb} = -\frac{nb(k+1)}{a+nb}$, by Lemma 4.1.
Using (4.19) in (4.18) we get

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W) \\ &\quad +g(Y,Z)\eta(X)\eta(W) - g(Y,W)\eta(X)\eta(Z)],\end{aligned} \tag{4.20}$$

where $p = k - 4l$ and $q = 2l$.
This completes the proof.

5. Extended quasi-conformal curvature tensor of an almost Kenmotsu manifold with (k,μ)-nullity distribution

In this section we study vanishing extended quasi-conformal curvature tensor and extended ξ-quasi-conformally flat almost Kenmotsu manifolds with ξ belonging to (k,μ)-nullity distribution.
The extended form of quasi-conformal curvature tensor can be written as

$$\begin{aligned}\tilde{C}_e(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y] \\ &\quad -\eta(X)\tilde{C}(\xi,Y)Z - \eta(Y)\tilde{C}(X,\xi)Z - \eta(Z)\tilde{C}(X,Y)\xi.\end{aligned} \tag{5.1}$$

Theorem 5.1. *In an almost Kenmotsu manifold with ξ belonging to (k,μ)-nullity distribution, the extended quasi-conformal curvature tensor vanishes if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof: Putting $Y = Z = \xi$ and supposing that the extended quasi-conformal tensor vanishes, we get from (5.1)

$$\begin{aligned}&aR(X,\xi)\xi + b[S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi] + (a+4nb)(X - \eta(X)\xi) \\ &\quad -\eta(X)\tilde{C}(\xi,\xi)\xi - \tilde{C}(X,\xi)\xi - \tilde{C}(X,\xi)\xi = 0.\end{aligned} \tag{5.2}$$

Now, using (3.4) and (3.5) the above equation reduces to

$$bQX = -2nbX + 2\tilde{C}(X,\xi)\xi. \tag{5.3}$$

Now, Using (3.2), (3.4) and (3.5) we obtain

$$\tilde{C}(X,\xi)\xi = 2nbX + bQX. \tag{5.4}$$

Putting the value of $\tilde{C}(X,\xi)\xi$ in (5.3) we get

$$QX = -2nX, \tag{5.5}$$

which implies

$$S(X,Y) = -2ng(X,Y). \tag{5.6}$$

This shows that the manifold is Einstein. Since, the extended quasi-conformal curvature tensor vanishes, we have from (5.1)

$$\begin{aligned} aR(X,Y)Z &= -b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -(a+4nb)[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\eta(X)\tilde{C}(\xi,Y)Z + \eta(Y)\tilde{C}(X,\xi)Z + \eta(Z)\tilde{C}(X,Y)\xi. \end{aligned} \tag{5.7}$$

Now, making use of (3.3), (3.4), (3.5) and (5.5) we obtain

$$\tilde{C}(\xi,Y)Z = 0,\ \tilde{C}(X,\xi)Z = 0.$$

Again since the manifold is Einstein, we have from Theorem 3.2

$$\tilde{C}(X,Y)\xi = 0.$$

Putting these values in (5.7) and using (5.6) we get

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]. \tag{5.8}$$

This implies that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

Conversely, suppose that the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. That is, (5.8) holds.
Contracting X in (5.8) yields

$$S(Y,Z) = -2ng(Y,Z). \tag{5.9}$$

Now, as shown earlier in this theorem

$$\tilde{C}(\xi,Y)Z = \tilde{C}(X,\xi)Z = \tilde{C}(X,Y)\xi = 0.$$

Then, making use of (5.8), (5.9) and the above values, we obtain from (5.1) that

$$\tilde{C}_e(X,Y)Z = 0.$$

Hence the theorem is proved.

Theorem 5.2. *An almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution is extended ξ-quasi-conformally flat if and only if the manifold is Einstein.*

Proof: Suppose $\tilde{C}_e(X,Y)\xi = 0$ and putting $Y = \xi$, we get from (5.1)

$$\begin{aligned} aR(X,\xi)\xi + b[S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi] + (a+4nb)(X-\eta(X)\xi) \\ -\eta(X)\tilde{C}(\xi,\xi)\xi - \tilde{C}(X,\xi)\xi - \tilde{C}(X,\xi)\xi = 0. \end{aligned} \tag{5.10}$$

Now, using (3.4) and (3.5) the above equation reduces to

$$bQX = -2nbX + 2\tilde{C}(X,\xi)\xi. \tag{5.11}$$

Now, Using (3.2), (3.4) and (3.5) we obtain

$$\tilde{C}(X,\xi)\xi = 2nbX + bQX. \tag{5.12}$$

Putting the value of $\tilde{C}(X,\xi)\xi$ in (5.11) we get

$$QX = -2nX, \tag{5.13}$$

which implies that the manifold is Einstein.

Conversely, if the manifold is Einstein then obviously $\tilde{C}_e(X,Y)\xi = 0$. Hence the theorem is established.

Acknowledgement: The authors are thankful to the referees for their valuable suggestions towards the improvement of this paper. The author Dibakar Dey is supported in part by CSIR, India (File no: 09/028(1010)/2017-EMR-1).

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Ser. Math. Inform. Vol. 33, No 2 (2018), 269–277
https://doi.org/10.22190/FUMI1802269B

REMARKS ON METALLIC WARPED PRODUCT MANIFOLDS

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Abstract. We characterize the metallic structure on the product of two metallic manifolds in terms of metallic maps and provide a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic. We discuss a particular case of the product manifolds and we construct an example of the metallic warped product Riemannian manifold.
Keywords: Riemannian manifold, metallic warped product, projection mapping.

1. Introduction

Starting from a polynomial structure, which was generally defined by S. I. Goldberg, K. Yano and N. C. Petridis in ([8],[9]), we consider a polynomial structure on an m-dimensional Riemannian manifold (M, g), called by us a *metallic structure* ([6],[11],[7],[12]), determined by a $(1,1)$-tensor field J which satisfies the equation:

$$J^2 = pJ + qI, \tag{1.1}$$

where I is the identity operator on the Lie algebra of vector fields on M identified with the set of smooth sections $\Gamma(T(M))$ (and we will simply denote $X \in T(M)$) with p and q are non-zero natural numbers). From the definition, we easily get the recurrence relation:

$$J^{n+1} = g_{n+1} \cdot J + g_n \cdot I, \tag{1.2}$$

where $(\{g_n\}_{n\in\mathbb{N}^*})$ is the generalized secondary Fibonacci sequence defined by $g_{n+1} = pg_n + qg_{n-1}$, $n \geq 1$ with $g_0 = 0$, $g_1 = 1$ and p, $q \in \mathbb{N}^*$.

If (M, g) is a Riemannian manifold endowed with a metallic structure J such that the Riemannian metric g is J-compatible (i.e. $g(JX, Y) = g(X, JY)$, for any $X, Y \in T(M)$), then (M, g, J) is called a *metallic Riemannian manifold*. In this case:

$$g(JX, JY) = pg(X, JY) + qg(X, Y), \tag{1.3}$$

Received March 30, 2018; accepted May 24, 2018
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

for any $X, Y \in T(M)$.

It is known ([13]) that an almost product structure F on M induces two metallic structures:

$$J_{\pm} = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I \tag{1.4}$$

and, conversely, every metallic structure J on M induces two almost product structures:

$$F_{\pm} = \pm\frac{2}{2\sigma_{p,q} - p}J - \frac{p}{2\sigma_{p,q} - p}I, \tag{1.5}$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number, which is a positive solution of the equation $x^2 - px - q = 0$, for p and q non-zero natural numbers.

In particular, if the almost product structure F is compatible with the Riemannian metric, then J_+ and J_- are metallic Riemannian structures.

On a metallic manifold (M, J) there exist two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators l and m ([13]) given by:

$$l = -\frac{1}{2\sigma_{p,q} - p}J + \frac{\sigma_{p,q}}{2\sigma_{p,q} - p}I, \quad m = \frac{1}{2\sigma_{p,q} - p}J + \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p}I. \tag{1.6}$$

The analogue concept of a locally product manifold is considered in the context of metallic geometry. Precisely, we say that the metallic Riemannian manifold (M, g, J) is *locally metallic* if J is parallel with respect to the Levi-Civita connection associated to g.

2. Metallic warped product Riemannian manifolds

2.1. Warped product manifolds

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions n and m, respectively. Denote by p_1 and p_2 the projection maps from the product manifold $M_1 \times M_2$ onto M_1 and M_2 and by $\widetilde{\varphi} := \varphi \circ p_1$ the lift to $M_1 \times M_2$ of a smooth function φ on M_1. In this case, we call M_1 *the base* and M_2 *the fiber* of $M_1 \times M_2$. The unique element $\widetilde{X}$ of $T(M_1 \times M_2)$ that is p_1-related to $X \in T(M_1)$ and to the zero vector field on M_2 will be called the *horizontal lift of* X and the unique element $\widetilde{V}$ of $T(M_1 \times M_2)$ that is p_2-related to $V \in T(M_2)$ and to the zero vector field on M_1 will be called the *vertical lift of* V. Also denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

For $f > 0$ a smooth function on M_1, consider the Riemannian metric on $M_1 \times M_2$:

$$\widetilde{g} := p_1^* g_1 + (f \circ p_1)^2 p_2^* g_2. \tag{2.1}$$

Definition 2.1. ([4]) The product manifold of M_1 and M_2 together with the Riemannian metric $\widetilde{g}$ defined by (2.1) is called *the warped product* of M_1 and M_2 by the warping function f [and it is denoted by $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$].

Note that if f is constant (equal to 1), the warped product becomes the usual product of the Riemannian manifolds.

For $(x,y) \in \widetilde{M}$, we shall identify $X \in T(M_1)$ with $(X_x, 0_y) \in T_{(x,y)}(\widetilde{M})$ and $Y \in T(M_2)$ with $(0_x, Y_y) \in T_{(x,y)}(\widetilde{M})$ ([3]).

The projection mappings of $T(M_1 \times M_2)$ onto $T(M_1)$ and $T(M_2)$, respectively, denoted by $\pi_1 =: Tp_1$ and $\pi_2 =: Tp_2$ verify:

$$\pi_1 + \pi_2 = I, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0. \tag{2.2}$$

The Riemannian metric of the warped product manifold $\widetilde{M} = M_1 \times_f M_2$ equals to:

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2), \tag{2.3}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$ and we notice that the leaves $M_1 \times \{y\}$, for $y \in M_2$, are totally geodesic submanifolds of $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g})$.

If we denote by $\widetilde{\nabla}$, ${}^{M_1}\nabla$, ${}^{M_2}\nabla$ the Levi-Civita connections on $\widetilde{M}$, M_1 and M_2, we know that for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$ ([14]):

$$\widetilde{\nabla}_{(X_1,X_2)}(Y_1, Y_2) = ({}^{M_1}\nabla_{X_1} Y_1 - \frac{1}{2} g_2(X_2, Y_2) \cdot grad(f^2),$$

$${}^{M_2}\nabla_{X_2} Y_2 + \frac{1}{2f^2} X_1(f^2) Y_2 + \frac{1}{2f^2} Y_1(f^2) X_2). \tag{2.4}$$

In particular:

$$\widetilde{\nabla}_{(X,0)}(0, Y) = \widetilde{\nabla}_{(0,Y)}(X, 0) = (0, X(\ln(f))Y).$$

Let R, R_{M_1}, R_{M_2} be the Riemannian curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{R_{M_1}}$, $\widetilde{R_{M_2}}$ the lift on $\widetilde{M}$ of R_{M_1} and R_{M_2}. Then:

Lemma 2.1. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, Y, $Z \in \mathcal{L}(M_1)$ and any U, V, $W \in \mathcal{L}(M_2)$, we have:*

1. $R(X,Y)Z = \widetilde{R_{M_1}}(X,Y)Z$;
2. $R(U,X)Y = \frac{1}{f} H^f(X,Y)U$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
3. $R(X,Y)U = R(U,V)X = 0$;
4. $R(U,V)W = \widetilde{R_{M_2}}(U,V)W - \frac{|grad(f)|^2}{f^2}[g(U,W)V - g(V,W)U]$;
5. $R(X,U)V = \frac{1}{f} g(U,V) \widetilde{\nabla}_X grad(f)$.

Let S, S_{M_1}, S_{M_2} be the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}. Then:

Lemma 2.2. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, $Y \in \mathcal{L}(M_1)$ and any V, $W \in \mathcal{L}(M_2)$, we have:*

1. $S(X,Y) = \widetilde{S_{M_1}}(X,Y) - \frac{m}{f} H^f(X,Y)$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
2. $S(X,V) = 0$;
3. $S(V,W) = \widetilde{S_{M_2}}(V,W) - \left[\frac{\Delta(f)}{f} + (m-1)\frac{|grad(f)|^2}{f^2}\right] g(V,W)$.

Remark 2.1. For the case of product Riemannian manifolds:

i) the Riemannian curvature tensors verify ([2]):

$$R(\widetilde{X},\widetilde{Y})\widetilde{Z} = (R_1(X_1,Y_1)Z_1, R_2(X_2,Y_2)Z_2), \tag{2.5}$$

for any $\widetilde{X} = (X_1,X_2), \widetilde{Y} = (Y_1,Y_2), \widetilde{Z} = (Z_1,Z_2) \in T(M_1 \times M_2)$, where R, R_1 and R_2 are respectively the Riemannian curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2);

ii) the Ricci curvature tensors verify ([2]):

$$S(\widetilde{X},\widetilde{Y}) = S_1(X_1,Y_1) + S_2(X_2,Y_2), \tag{2.6}$$

for any $\widetilde{X} = (X_1,X_2), \widetilde{Y} = (Y_1,Y_2) \in T(M_1 \times M_2)$, where S, S_1 and S_2 are respectively the Ricci curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2).

Note that the Riemannian curvature tensor of a locally metallic Riemannian manifold has the following properties:

Proposition 2.1. *If (M,g,J) is a locally metallic Riemannian manifold, then for any $X,Y,Z \in T(M)$:*

$$R(X,Y)JZ = J(R(X,Y)Z), \tag{2.7}$$

$$R(JX,Y) = R(X,JY), \tag{2.8}$$

$$R(JX,JY) = qR(JX,Y) + pR(X,Y), \tag{2.9}$$

$$R(J^{n+1}X,Y) = g_{n+1} \cdot R(JX,Y) + g_n \cdot R(X,Y), \tag{2.10}$$

where $(\{g_n\}_{n\in\mathbb{N}^*})$ is the generalized secondary Fibonacci sequence defined by $g_{n+1} = pg_n + qg_{n-1}$, $n \geq 1$ with $g_0 = 0$, $g_1 = 1$ and p, $q \in \mathbb{N}^*$.

Proof. The locally metallic condition $\nabla J = 0$ is equivalent to $\nabla_X JY = J(\nabla_X Y)$, for any $X,Y \in T(M)$ and (2.7) follows from the definition of R. The relations (2.8), (2.9) and (2.10) follow from the symmetries of R and from the recurrence relation $J^{n+1} = g_{n+1} \cdot J + g_n \cdot I$. □

Theorem 2.1. *If* $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is a locally metallic Riemannian warped product manifold, then* M_2 *is* $\widetilde{J}$*-invariant submanifold of* $\widetilde{M}$*.*

Proof. Applying (2.8) from Proposition 2.1 and Lemma 2.1, we obtain $H^f(X,Y)\widetilde{J}U = H^f(\widetilde{J}X,Y)U$, for any X, $Y \in \mathcal{L}(M_1)$ and any $U \in \mathcal{L}(M_2)$, where H^f is the lift on $\widetilde{M}$ of $Hess(f)$. $\square$

2.2. Metallic warped product Riemannian manifolds

2.2.1. Metallic Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by the projection operators

The endomorphism

$$F := \pi_1 - \pi_2 \tag{2.11}$$

verifies $F^2 = I$ and $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$, thus F is an almost product structure on $M_1 \times M_2$.

By using relations (1.4) we can construct on $M_1 \times M_2$ two metallic structures, given by:

$$\widetilde{J}_{\pm} = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I. \tag{2.12}$$

Also from $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$ follows $\widetilde{g}(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.2. *There exist two metallic Riemannian structures* $\widetilde{J}_{\pm}$ *on* $(\widetilde{M}, \widetilde{g})$ *given by:*

$$\widetilde{J}_{\pm} = \pm\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I, \tag{2.13}$$

where $\widetilde{M} = M_1 \times_f M_2$ *and* $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$*, for any* $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$*.*

Note that for $\widetilde{J}_{+} = \frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$, the projection operators are $\pi_1 = m$, $\pi_2 = l$ and for $\widetilde{J}_{-} = -\frac{2\sigma_{p,q} - p}{2}F + \frac{p}{2}I$ we have $\pi_1 = l$, $\pi_2 = m$, where m and l are given by (1.6).

Remark 2.2. If we denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{g}$, we obtain that $\widetilde{\nabla}F = 0$ [hence $\widetilde{\nabla}\widetilde{J}_{\pm} = 0$ and so $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J}_{\pm})$ is a locally metallic Riemannian manifold].

For the case of a product Riemannian manifold $(\widetilde{M} = M_1 \times M_2, \widetilde{g})$ with $\widetilde{g}$ given by (2.1) for $f = 1$ and $\widetilde{J}_{\pm}$ defined by (2.13), we deduce that the Riemann curvature of $\widetilde{\nabla}$ verifies (2.7), (2.8), (2.9), (2.10).

2.2.2. Metallic Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by two metallic structures on M_1 and M_2

For any vector field $\widetilde{X} = (X, Y) \in T(M_1 \times M_2)$ we define a linear map $\widetilde{J}$ of tangent space $T(M_1 \times M_2)$ into itself by:

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.14}$$

where J_1 and J_2 are two metallic structures defined on M_1 and M_2, respectively, with $J_i^2 = pJ_i + qI$, $i \in \{1, 2\}$ and p, q non zero natural numbers. It follows that:

$$\widetilde{J}^2\widetilde{X} = \widetilde{J}(J_1 X, J_2 Y) = (J_1^2 X, J_2^2 Y) = p(J_1 X, J_2 Y) + q(X, Y). \tag{2.15}$$

Also from $g_i(J_i X_i, Y_i) = g_i(X_i, J_i Y_i)$, $i \in \{1, 2\}$, we get $\widetilde{g}(\widetilde{J}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.3. *If (M_1, g_1, J_1) and (M_2, g_2, J_2) are metallic Riemannian manifolds with $J_i^2 = pJ_i + qI$, $i \in \{1, 2\}$ and p, q non-zero natural numbers, then there exists a metallic Riemannian structure $\widetilde{J}$ on $(\widetilde{M}, \widetilde{g})$ given by:*

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.16}$$

for any $\widetilde{X} = (X, Y) \in T(\widetilde{M})$, where $\widetilde{M} = M_1 \times_f M_2$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

For the case of a product Riemannian manifold $(\widetilde{M} = M_1 \times M_2, \widetilde{g})$ with $\widetilde{g}$ given by (2.1) for $f = 1$ and $\widetilde{J}_{\pm}$ defined by (2.13), we deduce that the Riemann curvature of $\widetilde{\nabla}$ verifies (2.7), (2.8), (2.9), (2.10).

Now we shall obtain a characterization of the metallic structure on the product of two metallic manifolds (M_1, J_1) and (M_2, J_2) in terms of *metallic maps*, that are smooth maps $\Phi : M_1 \to M_2$ satisfying:

$$T\Phi \circ J_1 = J_2 \circ T\Phi.$$

In a way similar to the case of Golden manifolds ([5]), we have:

Proposition 2.2. *The metallic structure $\widetilde{J} := (J_1, J_2)$ given by (2.16) is the only metallic structure on the product manifold $\widetilde{M} = M_1 \times M_2$ such that the projections p_1 and p_2 on the two factors M_1 and M_2 are metallic maps.*

A necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic will be further provided:

Theorem 2.4. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by (2.1) and* $\widetilde{J}$ *given by (2.16)) be the warped product of the locally metallic Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is locally metallic if and only if:*

$$\begin{cases} (df^2 \circ J_1) \otimes I = df^2 \otimes J_2 \\ g_2(J_1 \cdot, \cdot) \cdot grad(f^2) = g_2(\cdot, \cdot) \cdot J_1(grad(f^2)) \end{cases}.$$

Proof. Replacing the expression of $\widetilde{\nabla}$ from (2.4), under the assumptions ${}^{M_1}\nabla J_1 = 0$ and ${}^{M_2}\nabla J_2 = 0$ we obtain the conclusion. □

Theorem 2.5. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by (2.1) and* $\widetilde{J}$ *(2.16)) be the warped product of the metallic Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. If* M_1 *and* M_2 *have* J_1*- and* J_2*-invariant Ricci tensors, respectively (i.e.* $Q_{M_i} \circ J_i = J_i \circ Q_{M_i}$*,* $i \in \{1, 2\}$*), then* $\widetilde{M}$ *has* $\widetilde{J}$*-invariant Ricci tensor if and only if*

$$Hess(f)(J_1 \cdot, \cdot) - Hess(f)(\cdot, J_1 \cdot) \in \{0\} \times T(M_2).$$

Proof. If we denote by S, S_{M_1}, S_{M_2} the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}, by using Lemma 2.2, for any X, $Y \in \mathcal{L}(M_1)$, we have:

$$S(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(\widetilde{J}X, Y) - \frac{m}{f} H^f(\widetilde{J}X, Y) = \widetilde{S_{M_1}}(X, \widetilde{J}Y) - \frac{m}{f} H^f(\widetilde{J}X, Y) =$$

$$= S(X, \widetilde{J}Y) + \frac{m}{f} H^f(X, \widetilde{J}Y) - \frac{m}{f} H^f(\widetilde{J}X, Y),$$

where H^f is the lift on $\widetilde{M}$ of $Hess(f)$. Also, for any V, $W \in \mathcal{L}(M_2)$, we obtain:

$$S(\widetilde{J}V, W) = \widetilde{S_{M_2}}(\widetilde{J}V, W) - [f\Delta(f) + (m-1)|grad(f)|^2] g_2(J_2 V, W) =$$

$$= \widetilde{S_{M_2}}(V, \widetilde{J}W) - [f\Delta(f) + (m-1)|grad(f)|^2] g_2(V, J_2 W) = S(V, \widetilde{J}W).$$

□

Example 2.1. Consider $M := \{(u, \alpha_1, \alpha_2, ..., \alpha_n), u > 0, \alpha_i \in [0, \frac{\pi}{2}], i \in \{1, ..., n\}\}$ and let $f : M \to \mathbb{R}^{2n}$ be the immersion given by:

$$f(u, \alpha_1, ..., \alpha_n) := (u \cos \alpha_1, u \sin \alpha_1, ..., u \cos \alpha_n, u \sin \alpha_n). \tag{2.17}$$

We can find a local orthonormal frame of the submanifold M in $\mathbb{R}^{2n}$, spanned by the vectors:

$$Z_0 = \sum_{i=1}^{n} \left(\cos \alpha_i \frac{\partial}{\partial x_i} + \sin \alpha_i \frac{\partial}{\partial y_i} \right), \quad Z_i = -u \sin \alpha_i \frac{\partial}{\partial x_i} + u \cos \alpha_i \frac{\partial}{\partial y_i}, \tag{2.18}$$

for any $i \in \{1, ..., n\}$.

We remark that $\|Z_0\|^2 = n$, $\|Z_i\|^2 = u^2$, $Z_0 \perp Z_i$, for any $i \in \{1, ..., n\}$ and $Z_i \perp Z_j$, for $i, j \in \{1, ..., n\}$ with $i \neq j$.

In the next considerations, we shall denote by:

$$(X^1, Y^1, ..., X^k, Y^k, X^{k+1}, Y^{k+1}, ..., X^n, Y^n) =: (X^i, Y^i, X^j, Y^j),$$

for any $k \in \{2, ..., n-1\}$, $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$.

Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the $(1,1)$-tensor field defined by:

$$J(X^i, Y^i, X^j, Y^j) := (\sigma X^i, \sigma Y^i, \overline{\sigma} X^j, \overline{\sigma} Y^j), \tag{2.19}$$

for any $k \in \{2, ..., n-1\}$, $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$, where $\sigma := \sigma_{p,q}$ is the metallic number and $\overline{\sigma} = 1 - \sigma$. It is easy to verify that J is a metallic structure on $\mathbb{R}^{2n}$ (i.e. $J^2 = pJ + qI$).

Moreover, the metric $\overline{g}$, given by the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$, is J-compatible and $(\mathbb{R}^{2n}, \overline{g}, J)$ is a metallic Riemannian manifold.

From (2.18) we get:

$$JZ_0 = \sigma \sum_{i=1}^{k} \left(\cos\alpha_i \frac{\partial}{\partial x_i} + \sin\alpha_i \frac{\partial}{\partial y_i} \right) + \overline{\sigma} \sum_{j=k+1}^{n} \left(\cos\alpha_j \frac{\partial}{\partial x_j} + \sin\alpha_j \frac{\partial}{\partial y_j} \right)$$

and, for any $k \in \{2, ..., n-1\}, i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$ we get:

$$JZ_i = \sigma Z_i, \quad JZ_j = \overline{\sigma} Z_j.$$

We can verify that JZ_0 is orthogonal to $span\{Z_1, ..., Z_n\}$ and

$$\cos(\widehat{JZ_0, Z_0}) = \frac{k\sigma + (n-k)\bar{\sigma}}{\sqrt{n(k\sigma^2 + (n-k)\bar{\sigma}^2)}}. \tag{2.20}$$

Consider the manifolds M_1 and M_2 with $TM_1 = span\{Z_0\}$ and $TM_2 = span\{Z_1, ..., Z_n\}$. Then $M := M_1 \times_u M_2$ with the Riemannian metric tensor $g = ndu^2 + u^2 \sum_{i=1}^{n} d\alpha_i^2$ is a warped product (semi-slant) submanifold of the metallic Riemannian manifold $(\mathbb{R}^{2n}, \langle \cdot, \cdot \rangle, J)$.

Acknowledgements. The authors thank the referee for his/her valuable suggestions that definitely improved the paper.

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Ser. Math. Inform. Vol. 33, No 2 (2018), 279–293
https://doi.org/10.22190/FUMI1802279O

BILATERAL AND BILINEAR GENERATING FUNCTIONS FOR THE MODIFIED GENERALIZED SYLVESTER POLYNOMIALS

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Abstract. The present study deals with some new properties for the modified generalized Sylvester polynomials. The results obtained here include various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the modified generalized Sylvester polynomials and the generalized Lauricella functions. Finally, we get several interesting results of this theorem.
Keywords: Sylvester polynomial, generating function, Lauricella function.

1. Introduction

Generalized functions occupy pride of place in literature on special functions. Their importance, which is mounting everyday, stems from the fact that they generalize the well-known one variable special functions, namely, Hermite polynomials, Laguerre polynomials, Legendre polynomials, Gegenbauer polynomials, Jacobi polynomials, Rice polynomials, Generalized Sylvester polynomials, etc. All these polynomials are closely associated with problems of applied nature. For example, Gegenbauer polynomials are deeply connected with axially symmetric potentials in dimensions and contain the Legendre and Chebyshev polynomials as special cases. The hypergeometric functions of which the Jacobi polynomials is a special case are important in many cases of mathematics analysis and its applications.

We define the modified generalized Sylvester polynomials $f_n(x;a,b)$ as follows (see [10]):

$$f_n(x;a,b) = \frac{(bx)^n}{n!}\, {}_2F_0\left[-n, ax; -; (-bx)^{-1}\right]. \tag{1.1}$$

where $b \neq 0$ is an arbitrary constant.

Received November 16, 2017; accepted January 17, 2018
2010 *Mathematics Subject Classification.* Primary 33C45, 33C47, 33C65

When $a = 1$ and $b = 1$ then (1.1) becomes

$$f_n(x;1,1) = \phi_n(x). \tag{1.2}$$

We call the polynomials $f_n(x;a,b)$ modified generalized Sylvester polynomials in view of the relations (1.2). For $a = 1$ and b by (1.1) becomes A.K. Agarwal and H.L. Manocha [8] generalization of Sylvester polynomials.

The following generating relations hold for (1.1) (see, [10]):

$$\sum_{n=0}^{\infty} f_n(x;a,b)t^n = (1-t)^{-ax}e^{bxt} \quad (|t| < 1) \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} (\lambda)_n\, f_n(x;a,b)t^n = (1-bxt)^{-\lambda}\ {}_2F_0\left[\lambda, ax; -; \left(\frac{t}{1-bxt}\right)\right], \tag{1.4}$$

where ${}_2F_0$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters $(p,q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned} {}_pF_q\left[\begin{array}{ll} \alpha_1,...,\alpha_p; & \\ \beta_1,...,\beta q; & z \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n}\frac{z^n}{n!} \\ &= {}_pF_q\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta q; z\right) \end{aligned}$$

and $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 = 1 \text{ and } (\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}\,(\lambda \in \mathbb{C})$$

in terms of the familiar Gamma function.

Lemma 1.1. *The following generating function holds true [2]:*

$$\sum_{n=0}^{\infty} \binom{n+k}{n} f_{n+k}(x;a,b)t^n = (1-t)^{-ax-k}e^{bxt}f_k(x;a,b(1-t)). \tag{1.5}$$

Proof. If we write $t+u$ instead of t in (1.3) , we get

$$\begin{aligned} \sum_{n=0}^{\infty} f_n(x;a,b)(t+u)^n &= (1-t-u)^{-ax}e^{bx(t+u)} \\ \sum_{n=0}^{\infty} f_n(x;a,b)\sum_{k=0}^{n}\binom{n}{k}t^{n-k}u^k &= (1-t)^{-ax}\left(1-\frac{u}{1-t}\right)^{-ax}e^{bxt}e^{bxu}. \end{aligned}$$

Replacing n by $n+k$ in the last relation, we may write that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{n} f_{n+k}(x;a,b)t^n u^m = (1-t)^{-ax}e^{bxt}\sum_{k=0}^{\infty}(1-t)^{-k} f_k(x;a,b(1-t))u^k$$

From the coefficients of u^k on both sides of the equality, one can get the desired result. $\square$

Lemma 1.2. *The following addition formula holds for the modified generalized Sylvester polynomials* $f_n(x;a,b)$:

$$f_n(x_1+x_2;a,b) = \sum_{m=0}^{n} f_{n-m}(x_1;a,b) f_m(x_2;a,b). \tag{1.6}$$

Proof. Replacing x by x_1+x_2 in (1.3), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n(x_1+x_2;a,b)t^n &= (1-t)^{-ax_1-ax_2}e^{b(x_1+x_2)t} \\
&= (1-t)^{-ax_1}e^{bx_1t}(1-t)^{-ax_2}e^{bx_2t} \\
&= \sum_{n=0}^{\infty} f_n(x_1;a,b)t^n \sum_{m=0}^{\infty} f_m(x_2;a,b)t^m \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_n(x_1;a,b) f_m(x_2;a,b)t^{n+m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} f_{n-m}(x_1;a,b) f_m(x_2;a,b)t^{n}.
\end{aligned}$$

From the coefficients of t^n on both sides of the last equality, one can get the desired result. $\square$

The main objective of this paper is to study different properties of the modified generalized Sylvester polynomials. Various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are given. In addition, we derive a theorem giving certain families of bilateral generating functions for the modified generalized Sylvester polynomials and the generalized Lauricella functions.

2. Bilinear and Bilateral Generating Functions

This section presents several families of bilinear and bilateral generating functions for the modified generalized Sylvester polynomials $f_n(x;a,b)$ given by (1.1) without using Lie algebraic techniques but with the help of a similar method as considered in [4], [5], [6].

We begin by stating the following theorem.

Theorem 2.1. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* μ,ψ*, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k \qquad (a_k\neq 0)$$

and

$$\Theta_{n,p}^{\mu,\psi}(x;a,b;y_1,...,y_r;\xi):=\sum_{k=0}^{[n/p]}a_kf_{n-pk}(x;a,b)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $p\in\mathbb{N}$*, we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x;a,b;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n=(1-t)^{-ax}e^{bxt}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{2.1}$$

provided that each member of (2.1) exists.

Proof. For convenience, let S denote the first member of the assertion (2.1) of Theorem 2.1. Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kf_{n-pk}(x;a,b)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^kt^{n-pk}.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,f_n(x;a,b)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^kt^n\\&=\sum_{n=0}^{\infty}f_n(x;a,b)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\\&=(1-t)^{-ax}e^{bxt}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

Theorem 2.2. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathbb{N})$ *and of complex order* μ,ψ*, let*

$$\Lambda_{\mu,\psi}^{n,p}(x_1+x_2;a,b;y_1,...,y_r;t):=\sum_{k=0}^{[n/p]}a_kf_{n-pk}(x_1+x_2;a,b)\Omega_{\mu+\psi k}(y_1,...,y_r)t^k,$$

where $a_k\neq 0$, $n,p\in\mathbb{N}$ *and the notation* $[n/p]$ *means the greatest integer less than or equal to* n/p.

Then, for $p \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} \sum_{l=0}^{[k/p]} a_l f_{n-k}(x_1; a, b) f_{k-pl}(x_2; a, b) \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l$$

$$= \Lambda_{\mu,\psi}^{n,p}(x_1 + x_2; a, b; y_1, ..., y_r; t) \tag{2.2}$$

provided that each member of (2.2) exists.

Proof. *For convenience, let T denote the first member of the assertion (2.2) of Theorem 2.2. Then, upon substituting for the polynomials $f_n(x_1 + x_2; a, b)$ from the (1.6) into the left-hand side of (2.2), we obtain*

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]} \sum_{k=0}^{n-pl} a_l f_{n-k-pl}(x_1; a, b) f_k(x_2; a, b) \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l \\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl} f_{n-k-pl}(x_1; a, b) f_k(x_2; a, b) \right) \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l \\ &= \sum_{l=0}^{[n/p]} a_l f_{n-pl}(x_1 + x_2; a, b) \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l \\ &= \Lambda_{\mu,\psi}^{n,p}(x_1 + x_2; a, b; y_1, ..., y_r; t). \end{aligned}$$

□

Theorem 2.3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,p,q}(x; a, b; y_1, ..., y_r; t) := \sum_{n=0}^{\infty} a_n f_{m+qn}(x; a, b) \Omega_{\mu+pn}(y_1, ..., y_r) t^n$$

where $a_n \neq 0$ and

$$\theta_{n,p,q}(y_1, ..., y_r; z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1, ..., y_r) z^k.$$

Then, for $p, q \in \mathbb{N}$; we have

$$\sum_{n=0}^{\infty} f_{m+n}(x; a, b) \theta_{n,p,q}(y_1, ..., y_r; z) t^n$$

$$= (1-t)^{-ax-m} e^{bxt} \Lambda_{\mu,p,q}\left(x; a, b(1-t); y_1, ..., y_r; z(\frac{t}{1-t})^q\right) \tag{2.3}$$

provided that each member of (2.3) exists.

Proof. For convenience, let T denote the first member of the assertion (2.3) of Theorem 2.3. Then,

$$T=\sum_{n=0}^{\infty} f_{m+n}(x;c)\sum_{k=0}^{[n/q]}\begin{pmatrix} m+n \\ n-qk\end{pmatrix} a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k t^n.$$

Replacing n by $n+qk$ and then using (1.5), we may write that

$$\begin{aligned}
T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix} m+n+qk \\ n\end{pmatrix} f_{m+n+qk}(x;c)a_k\,\Omega_{\mu+pk}(y_1,...,y_r)z^k t^{n+qk}\\
&= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\begin{pmatrix} m+n+qk \\ n\end{pmatrix} f_{m+n+qk}(x;a,b)t^n\right) a_k\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k\\
&= \sum_{k=0}^{\infty} a_k(1-t)^{-ax-m-qk}e^{bxt}f_{m+qk}\left(x;a,b(1-t)\right)\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k\\
&= (1-t)^{-ax-m}e^{bxt}\sum_{k=0}^{\infty} a_k(1-t)^{-qk}f_{m+qk}\left(x;a,b(1-t)\right)\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k\\
&= (1-t)^{-ax-m}e^{bxt}\Lambda_{\mu,p,q}\left(x;a,b(1-t);y_1,...,y_r;z(\frac{t}{1-t})^q\right)
\end{aligned}$$

which completes the proof. □

3. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=\Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r)$$

in Theorem 2.1, where the multivariable polynomials $\Phi^{(\alpha)}_{\mu+\psi k}(x_1,...,x_r)$ [4], generated by

$$(1-x_1t)^{-\alpha}e^{(x_2+...+x_r)\ t}=\sum_{n=0}^{\infty}\Phi^{(\alpha)}_{n}\left(x_1,...,x_r\right)t^n. \qquad (3.1)$$
$$\left(\alpha\in\mathbb{C}\ ;\ \ |t|<\left\{|x_1|^{-1}\right\}\right).$$

We are thus led to the following result which provides a class of bilateral generating functions for the multivariable polynomials $\Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r)$ and the modified generalized Sylvester polynomials.

Corollary 3.1. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r)\zeta^k \quad (a_k\neq 0\,,\ \ \mu,\psi\in\mathbb{C})\,,$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k f_{n-pk}(x;a,b)\Phi^{(\alpha)}_{\mu+\psi k}(y_1,...,y_r)\frac{\zeta^k}{t^{pk}}t^n = (1-t)^{-ax}e^{bxt}\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) \tag{3.2}$$

provided that each member of (3.2) exists.

Remark 3.1. Using the generating relation (3.1) for the multivariable polynomials $\Phi_n^{(\alpha)}(x_1,...,x_r)$ and getting $a_k=1$, $\mu=0$, $\psi=1$ in Corollary 3.1, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} f_{n-pk}(x;a,b)\Phi^{(\alpha)}_{k}(y_1,...,y_r)\zeta^k t^{n-pk} = (1-t)^{-ax}e^{bxt}(1-y_1\zeta)^{-\alpha}e^{(y_2+...+y_r)\zeta}.$$

$$\left(|\zeta|<\left\{|y_1|^{-1}\right\},\quad |t|<1\right)$$

If we set $r=1$, $y_1=x_3$ and

$$\Omega_{\mu+\psi k}(x_3)=f_{\mu+\psi k}(x_3;a,b)$$

in Theorem 2.2, we have the following bilinear generating functions for the modified generalized Sylvester polynomials.

Corollary 3.2. *If*

$$\Lambda^{n,p}_{\mu,\psi}(x_1+x_2;a,b;x_3;a,b;t) := \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x_1+x_2;a,b)f_{\mu+\psi k}(x_3;a,b)t^k$$

$$(a_k \neq 0,\ \mu,\psi\in\mathbb{C})$$

then, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l f_{n-k}(x_1;a,b)f_{k-pl}(x_2;a,b)f_{\mu+\psi l}(x_3;a,b)t^l = \Lambda^{n,p}_{\mu,\psi}(x_1+x_2;a,b;x_3;a,b;t) \tag{3.3}$$

provided that each member of (3.3) exists.

Remark 3.2. Taking $a_l=1$, $\mu=0$, $\psi=1$, $p=1$, $t=1$ in Corollary 3.2, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{k} f_{n-l}(x_1+x_2;a,b)f_l(x_3;a,b)=f_n(x_1+x_2+x_3;a,b).$$

If we set $s=r$ and

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=u_{\mu+\psi k}^{(\alpha_1,\ldots,\alpha_r)}(y_1,...,y_r)$$

in Theorem 2.3, where the Erkus-Srivastava polynomials $u_n^{(\alpha_1,\ldots,\alpha_r)}(y_1,...,y_r)$ is generated by [7],

$$\prod_{j=1}^{r}\left\{(1-x_jt^{m_j})^{-\alpha_j}\right\}=\sum_{n=0}^{\infty}u_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,...,x_r)\,t^n$$

$$\left(\alpha_j\in\mathbb{C}\ (j=1,...,r)\ ;\ |t|<\min\left\{|x_1|^{-1/m_1},...,|x_r|^{-1/m_r}\right\}\right.$$

we get a family of the bilateral generating functions for the Erkus-Srivastava polynomials and the modified generalized Sylvester polynomials as follows:

Corollary 3.3. *If*

$$\Lambda_{\mu,p,q}(x;a,b;y_1,...,y_r;t)\ :=\sum_{n=0}^{\infty}a_nf_{m+qn}(x;a,b)u_{\mu+pn}^{(\alpha_1,\ldots,\alpha_r)}(y_1,...,y_r)t^n$$
$$(a_n\neq 0,\ m\in\mathbb{N}_0,\ \mu,\psi\in\mathbb{C})$$

and

$$\theta_{n,p,q}(y_1,...,y_r;z):=\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_ku_{\mu+pk}^{(\alpha_1,\ldots,\alpha_r)}(y_1,...,y_r)z^k$$

where $n,p\in\mathbb{N}$, then we have

$$\sum_{n=0}^{\infty}f_{m+n}(x;a,b)\theta_{n,p,q}(y_1,...,y_r;z)t^n \tag{3.4}$$
$$=\ (1-t)^{-ax-m}e^{bxt}\Lambda_{\mu,p,q}\left(x;a,b(1-t);y_1,...,y_r;z\left(\frac{t}{1-t}\right)^q\right)$$

provided that each member of (3.4) exists.

Furthermore, for every suitable choice of the coefficients a_k ($k\in\mathbb{N}_0$), if the multivariable functions $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.1, 2.2, 2.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the modified generalized Sylvester polynomials given explicitly by (1.1).

4. Miscellaneous Properties

In this section we give some properties for the modified generalized Sylvester polynomials $f_n(x;a,b)$ given by (1.1).

Theorem 4.1. *The modified generalized Sylvester polynomials $f_n(x;a,b)$ have the following integral representation:*

$$f_n(x;a,b) = \frac{1}{n!\Gamma(ax)} \int_0^\infty e^{-u} u^{ax-1} (bx+u)^n du$$

where, $Re(ax) > 0$.

Proof. If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt, \qquad (Re(v) > 0)$$

on the left-hand side of the generating function (1.3), we have

$$\begin{aligned}
\sum_{n=0}^\infty f_n(x;a,b)t^n &= \frac{1}{\Gamma(ax)} \int_0^\infty e^{-(1-t)u} u^{ax-1} e^{bxt} du \\
&= \frac{1}{\Gamma(ax)} \int_0^\infty e^{-u} u^{ax-1} e^{(bx+u)t} du \\
&= \frac{1}{\Gamma(ax)} \int_0^\infty e^{-u} u^{ax-1} \sum_{n=0}^\infty (bx+u)^n \frac{t^n}{n!} du \\
&= \sum_{n=0}^\infty \left(\frac{1}{n!\Gamma(ax)} \int_0^\infty e^{-u} u^{ax-1} (bx+u)^n du \right) t^n
\end{aligned}$$

From the coefficients of t^n on both sides of the last equality, one can get the desired result. $\square$

We now discuss some miscellaneous recurrence relations of the modified generalized Sylvester polynomials. By differentiating each member of the generating function relation (1.3) with respect to x and using

$$\sum_{n=0}^\infty \sum_{k=0}^\infty A(k,n) = \sum_{n=0}^\infty \sum_{k=0}^n A(k,n-k),$$

we have

$$\frac{d}{dx}f_n(x;a,b) = bf_{n-1}(x;a,b) + a\sum_{m=0}^{n-1}\frac{1}{(m+1)}f_{n-m-1}(x;a,b).$$

Besides, by differentiating each member of the generating function relation (1.3) with respect to t, we have the following recurrence relation for these polynomials:

$$(n+1)f_{n+1}(x;a,b) = x\left(bf_n(x;a,b) + a\sum_{m=0}^{n}f_{n-m}(x;a,b)\right).$$

5. The Generalized Lauricella Functions

In the present section, we derive various families of bilateral generating functions for the modified generalized Sylvester polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions. The four Appell functions of two variables, denoted by F_1, F_2, F_3 and F_4 were generalized by Lauricella functions of n variables which are denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ [2] and

$$F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \; F_C^{(2)} = F_4, \quad F_D^{(2)} = F_1.$$

A further generalization of the familiar Kampé de Fériet hypergeometric function in two variables is due to Srivastava and Daoust who defined the generalized Lauricella (or the Srivastava-Daoust) function as follows [3]:

$$F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}\left(\begin{array}{ccccc} \left[(a):\theta^{(1)},...,\theta^{(n)}\right]: & \left[(b^{(1)}):\phi^{(1)}\right]; & ...; & \left[(b^{(n)}):\phi^{(n)}\right]; & \\ & & & & z_1,...,z_n \\ \left[(c):\psi^{(1)},...,\psi^{(n)}\right]: & \left[(d^{(1)}):\delta^{(1)}\right]; & ...; & \left[(d^{(n)}):\delta^{(n)}\right]; & \end{array}\right)$$

$$= \sum_{m_1,...,m_n=0}^{\infty}\Omega(m_1,...,m_n)\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},$$

where, for convenience,

$$\Omega(m_1,...,m_n) := \frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+...+m_n\psi_j^{(n)}}}\frac{\prod\limits_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod\limits_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}\cdots\frac{\prod\limits_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod\limits_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\theta_j^{(k)} \; (j=1,...,A;\; k=1,...,n) \;\text{ and }\; \phi_j^{(k)} \; (j=1,...,B^{(k)};\; k=1,...,n),$$

$$\psi_j^{(k)} \; (j=1,...,C;\; k=1,...,n) \;\text{ and }\; \delta_j^{(k)} \; (j=1,...,D^{(k)};\; k=1,...,n)$$

are real constants and $\left(b^{(k)}_{B^{(k)}}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b^{(k)}_j \ (j = 1, ..., B^{(k)};\ k = 1, ..., n)$$

with similar interpretations for other sets of parameters [1]. Here, as usual, $(\lambda)_v$ denotes the Pochhammer symbol.

For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1, m_2, ..., m_s)\}_{m_1,...,m_s\in\mathbb{N}_0}$ of real or complex parameters, let $\varphi_n(u_1; u_2, ..., u_s)$ of s (real or complex) variables $u_1; u_2, ..., u_s$ defined by [1]

$$\begin{aligned}\varphi_n(u_1; u_2, ..., u_s) \quad : &= \sum_{m_1=0}^{n} \sum_{m_2,...,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\ &\times\Omega\left(f(m_1, ..., m_s), m_2, ..., m_s\right) \frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!} \end{aligned} \tag{5.1}$$

where, for convenience,

$$((b))_{m_1\phi} = \prod_{j=1}^{B}(b_j)_{m_1\phi_j} \text{ and } ((d))_{m_1\delta} = \prod_{j=1}^{D}(d_j)_{m_1\delta_j}.$$

Theorem 5.1. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} f_n(x; a, b)\varphi_n(u_1; u_2, ..., u_s)t^n \\ = \quad &(1-t)^{-ax}e^{bxt} \sum_{m_1,k,m_2,...,m_s=0}^{\infty} \frac{((b))_{(m_1+k)\phi}(ax)_k}{((d))_{(m_1+k)\delta}} \\ &\times\Omega(f((m_1+k), ..., m_s), m_2, ..., m_s)\frac{(-u_1bxt)^{m_1}}{m_1!}\frac{\left(\frac{u_1t}{t-1}\right)^k}{k!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!},\end{aligned}$$

where $\varphi_n(u_1; u_2, ..., u_s)$ is given by (5.1).

Proof. By using the relationship (1.5), it is easily observed that

$$\begin{aligned}&\sum_{n=0}^{\infty} f_n(x; a, b)\varphi_n(u_1; u_2, ..., u_s)t^n \\ = \quad &\sum_{n=0}^{\infty} f_n(x; a, b) \sum_{m_1=0}^{n} \sum_{m_2,...,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\ &\times \Omega\left(f(m_1, ..., m_s), m_2, ..., m_s\right) \frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}t^n \\ = \quad &\sum_{m_1,m_2,...,m_s=0}^{\infty} \frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1, ..., m_s), m_2, ..., m_s\right)\end{aligned}$$

$$\times\,(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\;(1-t)^{-ax-m_1}e^{bxt}f_{m_1}(x;a,b(1-t))$$

$$= \quad (1-t)^{-ax}e^{bxt}\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,\ldots,m_s),m_2,\ldots,m_s\right)$$

$$\times\,(-\frac{u_1t}{1-t})^{m_1}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\frac{(bx(1-t))^{m_1}}{m_1!}\sum_{k=0}^{m_1}(-m_1)_k(ax)_k\frac{(-bx(1-t))^{-k}}{k!}$$

$$= \quad (1-t)^{-ax}e^{bxt}$$

$$\times\sum_{m_1,k,m_2,\ldots,m_s=0}^{\infty}\frac{((b))_{(m_1+k)\phi}}{((d))_{(m_1+k)\delta}}\Omega\left(f((m_1+k),m_2,\ldots,m_s),m_2,\ldots,m_s\right)(ax)_k$$

$$\frac{(-u_1bxt)^{m_1}}{m_1!}\frac{(\frac{u_1t}{t-1})^k}{k!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}.$$

□

By appropriately choosing the multiple sequence $\Omega(m_1,m_2,\ldots,m_s)$ in Theorem 5.1, we obtain several interesting results as follows which give bilateral generating functions for the generalized Sylvester polynomials and the generalized Lauricella (or the Srivastava-Daoust) functions.

I.By letting

$$\Omega\left(f(m_1,\ldots,m_s),m_2,\ldots,m_s\right)$$
$$= \frac{\prod_{j=1}^{A}(a_{1j})_{m_1\theta_j^{(1)}+\ldots+m_s\theta_j^{(s)}}}{\prod_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+\ldots+m_s\psi_j^{(s)}}}\frac{\prod_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}\cdots\frac{\prod_{j=1}^{B^{(s)}}(b_j^{(s)})_{m_s\phi_j^{(s)}}}{\prod_{j=1}^{D^{(s)}}(d_j^{(s)})_{m_s\delta_j^{(s)}}}$$

in Theorem 5.1, we obtain the following result.

Corollary 5.1. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty}f_n(x;a,b)F_{E:D;D^{(2)};\ldots;D^{(s)}}^{A:B+1;B^{(2)};\ldots;B^{(s)}}$$
$$\left(\begin{array}{llllll}\left[(a_1):\theta^{(1)},\ldots,\theta^{(s)}\right]: & [-n:1], & [(b):\phi]; & [(b^{(2)}):\phi^{(2)}]; & \ldots; & [(b^{(s)}):\phi^{(s)}];\\ \left[(c):\psi^{(1)},\ldots,\psi^{(s)}\right]: & & [(d):\delta]; & [(d^{(2)}):\delta^{(2)}]; & \ldots; & [(d^{(s)}):\delta^{(s)}];\end{array}\; u_1,u_2,\ldots,u_s\right)t^n$$

$$= \quad (1-t)^{-ax}e^{bxt}F_{E+D:0;0;D^{(2)};\ldots;D^{(s)}}^{A+B:0;1;B^{(2)};\ldots;B^{(s)}}$$
$$\left(\begin{array}{llllll}\left[(e):\varphi^{(1)},\ldots,\varphi^{(s+1)}\right]: & -, & [ax:1]; & [(b^{(2)}):\phi^{(2)}]; & \ldots; & [(b^{(s)}):\phi^{(s)}];\\ \left[(f):\xi^{(1)},\ldots,\xi^{(s+1)}\right]: & - & -; & [(d^{(2)}):\delta^{(2)}]; & \ldots; & [(d^{(s)}):\delta^{(s)}];\end{array}\right.$$

$$\left. (-u_1bxt), (\tfrac{u_1t}{t-1}), u_2, ..., u_s \right)$$

where the coefficients e_j, f_j, $\varphi_j^{(s)}$ *and* $\xi_j^{(s)}$ *are given by*

$$e_j = \begin{cases} a_{1j}, & (1 \le j \le A) \\ b_{j-A}, & (A < j \le A+B) \end{cases}$$

$$f_j = \begin{cases} c_j, & (1 \le j \le E) \\ d_{j-E}, & (E < j \le E+D) \end{cases}$$

$$\varphi_j^{(r)} = \begin{cases} \theta_j^{(1)} & (1 \le j \le A;\ 1 \le r \le 2) \\ \theta_j^{(r-1)} & (1 \le j \le A;\ 2 < r \le s+1) \\ \phi_{j-A} & (A < j \le A+B;\ 1 \le r \le 2) \\ 0 & (A < j \le A+B;\ 2 < r \le s+1) \end{cases}$$

and

$$\xi_j^{(r)} = \begin{cases} \psi_j^{(1)} & (1 \le j \le E;\ 1 \le r \le 2) \\ \psi_j^{(r-1)} & (1 \le j \le E;\ 2 < r \le s+1) \\ \delta_{j-E} & (E < j \le E+D;\ 1 \le r \le 2) \\ 0 & (E < j \le E+D;\ 2 < r \le s+1) \end{cases}$$

respectively.

II. *Upon setting*

$$\Omega\left(f(m_1, ..., m_s), m_2, ..., m_s\right) = \frac{(a_1)_{m_1+...+m_s}(b_2)_{m_2}...(b_s)_{m_s}}{(c_1)_{m_1}...(c_s)_{m_s}}$$

and

$$\phi = \delta = 0 \quad (\textit{that is, } \phi_1 = ... = \phi_B = \delta_1 = ... = \delta_D = 0)$$

in Theorem 5.1, we obtain the following result.

Corollary 5.2. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} f_n(x; a, b) F_A^{(s)}\left[a_1, -n, b_2, ..., b_s; c_1, ..., c_s; u_1, u_2, ..., u_s\right] t^n$$

$$= (1-t)^{-ax} e^{bxt} F^{1:0;1;1;...;1}_{1:0;0;1;...;1}$$

$$\left(\begin{array}{llllll} [(a_1) : 1, ..., 1] : & -; & [ax : 1]; & [b_2 : 1]; & ...; & [b_s : 1]; \\ \left[(c_1) : \psi^{(1)}, ..., \psi^{(s+1)}\right] : & -; & -; & [c_2 : 1]; & ...; & [c_s : 1]; \end{array} \right.$$

$$\left. (-u_1bxt), (\tfrac{u_1t}{t-1}), u_2, ..., u_s \right),$$

where the coefficients $\psi^{(\eta)}$ *are given by*

$$\psi^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq s+1) \end{cases}$$

and $F_A^{(s)}$ *is the first kind of Lauricella functions.*

III. If we put

$$\Omega\left(f(m_1, ..., m_s), m_2, ..., m_s\right) = \frac{(a_1^{(1)})_{m_2} ... (a_1^{(s-1)})_{m_s} (a_2^{(1)})_{m_2} ... (a_2^{(s-1)})_{m_s}}{(c)_{m_1 + \cdots + m_s}}$$

and

$$B = 1,\ \phi_1 = 1 and \delta = 0$$

in Theorem 5.1, we obtain the following result.

Corollary 5.3. *The following bilateral generating function holds true:*

$$\begin{aligned} &\sum_{n=0}^{\infty} f_n(x; a, b) F_B^{(s)} \left[-n, a_1^{(1)}, ..., a_1^{(s-1)}, b_1, a_2^{(1)}, ..., a_2^{(s-1)}; c; u_1, u_2, ..., u_s\right] t^n \\ = \ &(1-t)^{-ax} e^{bxt} F_{1:0;0;0;...;0}^{1:0;1;2;...;2} \\ &\left(\begin{array}{lllllll} \left[(b_1) : \theta^{(1)}, ..., \theta^{(s+1)}\right] : & -; & [ax : 1]; & \left[a^{(1)} : 1\right]; & ...; & \left[a^{(s-1)} : 1\right]; \\ [(c) : 1, ..., 1] : & -; & -; & -; & ...; & -; \end{array} \right. \\ &\left. \qquad\qquad\qquad\qquad (-u_1 bxt), (\tfrac{u_1 t}{t-1}), u_2, ..., u_s \right), \end{aligned}$$

where the coefficients $\theta^{(\eta)}$ *are given by*

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq k+1) \end{cases}$$

and $F_B^{(s)}$ *is the second kind of Lauricella functions.*

IV.By letting

$$\Omega\left(f(m_1, ..., m_s), m_2, ..., m_s\right) = \frac{(a_1)_{m_1 + ... + m_s} (b_2)_{m_2} ... (b_s)_{m_s}}{(c)_{m_1 + ... + m_s}}$$

and

$$\phi = \delta = 0,$$

in Theorem 5.1, we obtain the following result.

Corollary 5.4. *The following bilateral generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty} f_n(x;a,b)F_D^{(s)}\left[a_1,-n,b_2,...,b_s;c;u_1,u_2,...,u_s\right]t^n\\ = \quad &(1-t)^{-ax}e^{bxt}F_D^{(s+1)}\left[a_1,0,ax,b_2,...,b_s;c;(-u_1bxt),(\frac{u_1t}{t-1}),u_2,...,u_s\right]\end{aligned}$$

and $F_D^{(s)}$ *is the forth kind of Lauricella functions.*

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Ser. Math. Inform. Vol. 33, No 2 (2018), 295–305
https://doi.org/10.22190/FUMI1802295A

NUMERICAL RECKONING FIXED POINTS FOR BERINDE MAPPINGS VIA A FASTER ITERATION PROCESS

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Abstract. In this paper we prove that the M-iteration process converges strongly faster than S-iteration and Picard-S iteration processes. Moreover, the $M-$ iteration process is faster than the S_n iteration process with a sufficient condition for weak contractive mapping defined on a normed linear space. We also give two numerical reckoning examples to support our main theorem. For approximating fixed points, all codes were written in MAPLE
Keywords: Iteration process, fixed point, weak contractive mapping, normed linear space.

1. Introduction and Preliminaries

Let K be a non-empty convex subset of a Banach space X and let $T : K \to K$ be a mapping. A point p is called the fixed point of a mapping t if $Tp = p$ and $F(T)$ represents the set of all fixed points of the mapping T.

It is well known that any linear or non-linear equation including differential equations and integral equations, can be transferred into a fixed point problem. For example, non-linear equations $x^2 - \sin x = 0$, and $x^3 lnx + e^x = 0$ cannot be solved easily. But, after transferring them into fixed point problems such as $Tx = x$, we can approximate the fixed point or the fixed points of the mapping of T with the help of iteration schemes. Thus, it is clear that any fixed point of T is also a solution of the corresponding equations.

One of the main conclusions which guarantees the existence of a fixed point was given by S. Banach in 1922 which is also called the Banach contraction principle and given as follows:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists a $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y)$$

Received March 19, 2018; accepted April 12, 2018
2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10

for all $x, y \in X$, then T has a unique fixed point.

Then since the Banach contraction principle was defined, many researchers have studied fixed point theory on different classes of mapping, on different types of spaces, and on different iteration processes.

In this paper, we give some useful results about some iteration schemes for finding fixed points of T. Firstly, we give some well-known iteration processes. Let (X, d) be a metric space and let $T : X \to X$ be a mapping and let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0, 1]$. For $x_0 \in X$,

- Picard iteration (1890) [12]

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots \tag{1.1}$$

- Mann iteration (1953) [9]

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \dots \tag{1.2}$$

- Ishikawa iteration (1974) [7]

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \quad n = 0, 1, 2, \dots \end{cases} \tag{1.3}$$

- S-iteration (2007) [2]

$$\begin{cases} \xi_{n+1} &= (1 - \alpha_n)T\xi_n + \alpha_n T\mu_n \\ \mu_n &= (1 - \beta_n)\xi_n + \beta_n T\xi_n, \quad n = 0, 1, 2, \dots \end{cases} \tag{1.4}$$

- Picard-S iteration (2014) [6]

$$\begin{cases} p_{n+1} &= Tq_n \\ q_n &= (1 - \alpha_n)Tp_n + \alpha_n Tr_n \\ r_n &= (1 - \beta_n)p_n + \beta_n Tp_n \end{cases} \tag{1.5}$$

- S_n iteration (2016) [14]

$$\begin{cases} u_{n+1} &= (1 - \alpha_n)Tw_n + \alpha_n Tv_n \\ v_n &= (1 - \beta_n)u_n + \beta_n v_n \\ w_n &= (1 - \gamma_n)u_n + \gamma_n Tu_n, \quad n = 0, 1, 2, \dots \end{cases} \tag{1.6}$$

- M-iteration (2018) [16]

$$\begin{cases} x_{n+1} &= Ty_n \\ y_n &= Tz_n \\ z_n &= (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \dots \end{cases} \tag{1.7}$$

In 2004, Berinde [4] introduced the concept of contractive mappings on metric space (X, d) as follows.

Definition 1.1. Let T be a mapping on a metric space (X, d). Then T is called a Berinde mapping if there exists $\delta \in [0, 1)$ and $L \in [0, \infty)$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx) \quad \forall x, y \in X \tag{1.8}$$

for all $x, y \in X$.

He also proved that any Zamfirescu mapping satisfies the weak contractive condition. Thus, the class of weak contractive mappings is wider than the class of Zamfirescu mapping. We refer the readers to [17, 3] to learn more about Zamfirescu and Berinde mapping.

In order to compare convergence rates between two iteration processes, we use the following useful definitions.

Definition 1.2. [4] Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers that converge to x and y, respectively and suppose that there exists

$$L : \lim_{n \to \infty} \frac{|x_n - x|}{|y_n - y|}$$

1. If $L = 0$, then $\{x_n\}$ converges faster to x than $\{y_n\}$ to y.
2. If $0 < L < \infty$, then $\{x_n\}$ and $\{y_n\}$ have the same rate of convergence.

Definition 1.3. [1] Let $(X, \|\cdot\|)$ be a normed linear space and $\{u_n\}$ and $\{v_n\}$ converging to the same point $p \in X$ and following the error estimates

$$\begin{aligned} \|u_n - p\| &\leq a_n \quad \forall n \in \mathbb{N}; \\ \|v_n - p\| &\leq b_n \quad \forall n \in \mathbb{N}; \end{aligned}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers. If $\{a_n\}$ converges faster than $\{b_n\}$ then $\{u_n\}$ converges faster than $\{v_n\}$ to p.

2. Approximation Results

Recently, Gursoy and Karakaya [6] proved that the Picard-S iteration process converges faster than all Picard [12], Mann [9], Ishikawa [7], Noor [10], SP [11], CR [5], S [2], S* [8], Abbas [1], Normal-S [13] and two-step Mann [15] iteration processes for contraction mappings.

In 2016, Sintunavarat and Pitea [14] defined a new three step iteration which is called S_n iteration. They also showed that their iteration converges faster than Mann, Ishikawa and S-iteration processes for mappings satisfying Berinde contractive condition.

In 2018, Ullah and Arshad [16] defined a new three step iteration process, called M-iteration process for finding fixed points of mappings and they get some convergence results for Suzuki generalized nonexpansive mappings in uniformly convex Banach spaces. They also showed that the M-iteration process converges faster than the Picard-S iteration and the S-iteration process for Suzuki generalized nonexpansive mappings.

Our purpose in this paper is to prove that the M-iteration process converges faster than the S_n iteration process with a sufficient condition and faster than the S-iteration and Picard-S iteration processes for weak contractive mappings. We support our result with two numerical examples.

Theorem 2.1. *Let K be a non-empty closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : K \to K$ be a mapping satisfying the weak contractive condition (1.8) with a fixed point p. Suppose that the sequences $\{x_n\}$, $\{u_n\}$, $\{p_n\}$ and $\{\xi_n\}$ are defined by the iteration processes M, S_n, Picard-S and S-iteration processes, respectively. Also the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\alpha, 1-\alpha]$, $[\beta, 1-\beta]$, and $[\gamma, 1-\gamma]$, respectively, $\alpha, \beta, \gamma \in (0, \frac{1}{2})$. Then the M-iteration process converges faster than the S and Picard-S iteration processes. Moreover, if $\alpha(2-\gamma) < \gamma$ then the M-iteration process is also faster than the S_n-iteration process.*

Proof. By using the M-iteration, we can get the following result

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\le \delta \|y_n - p\| \\ &= \delta \|Tz_n - p\| \\ &\le \delta^2 \|z_n - p\| \\ &= \delta^2 \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \\ &\le \delta^2 [(1-\alpha_n)\|x_n - p\| + \alpha_n \delta \|x_n - p\|] \\ &= (1-(1-\delta)\alpha_n)\delta^2 \|x_n - p\| \end{aligned} \tag{2.1}$$

for all $n \in \mathbb{N}$. Therefore,

$$\|x_n - p\| \le \{(1-(1-\delta)\alpha)\delta^2\}^n \|x_0 - p\|$$

for all $n \in \mathbb{N}$. Choose

$$a_n := \{(1-(1-\delta)\alpha)\delta^2\}^n \|x_0 - p\|.$$

By using the Picard-S iteration, we get

$$\begin{aligned} \|r_n - p\| &\le \|(1-\beta_n)p_n + \beta_n T p_n - p\| \\ &\le (1-\beta_n)\|p_n - p\| + \beta_n \|Tp_n - p\| \\ &\le (1-\beta_n)\|p_n - p\| + \beta_n \delta \|p_n - p\| \\ &= (1-(1-\delta)\beta_n)\|p_n - p\|. \end{aligned} \tag{2.2}$$

Using the Picard-S again and from (2.2), we have

$$\begin{aligned}\|q_n - p\| &\leq \|(1-\alpha_n)Tp_n + \alpha_n Tr_n - p\| \\ &\leq (1-\alpha_n)\|Tp_n - p\| + \alpha_n\|Tr_n - p\| \\ &\leq (1-\alpha_n)\delta\|p_n - p\| + \alpha_n\delta\|r_n - p\| \\ &\leq (1-\alpha_n)\delta\|p_n - p\| + \alpha_n\delta(1-(1-\delta)\beta_n)\|p_n - p\| \\ &= (1-(1-\delta)\alpha_n\beta_n)\delta.\|p_n - p\|. \qquad (2.3)\end{aligned}$$

From (2.3), we get

$$\begin{aligned}\|p_{n+1} - p\| &= \|Tq_n - p\| \\ &\leq \delta\|q_n - p\| \\ &\leq (1-(1-\delta)\alpha_n\beta_n)\delta^2\|p_n - p\|. \qquad (2.4)\end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$\|p_n - p\| \leq \{(1-(1-\delta)\alpha\beta)\delta^2\}^n\|p_0 - p\|$$

for all $n \in \mathbb{N}$. Let

$$b_n := \{(1-(1-\delta)\alpha\beta)\delta^2\}^n\|p_0 - p\|.$$

As proved in Theorem 2.1 of [14], we have

$$\|u_n - p\| \leq \{1-(1-\delta)\beta[\gamma_n - \alpha_n + \alpha_n\gamma_n]\}^n\|u_0 - p\| \qquad (2.5)$$

and

$$\|\xi_n - p\| \leq [1-(1-\delta)\alpha\beta]^n\|\xi_0 - p\| \qquad (2.6)$$

for all $n \in \mathbb{N}$. Choose

$$c_n := \{1-(1-\delta)\beta[\gamma - \alpha + \alpha.\gamma]\}^n\|u_0 - p\|.$$

and

$$d_n := [1-(1-\delta)\alpha\beta]^n\|\xi_0 - p\|.$$

Since $\alpha(2-\gamma) < \gamma$, we obtain

$$1-(1-\delta)\beta(\gamma - \alpha + \alpha.\gamma) < 1-(1-\delta)\alpha\beta < 1.$$

Now using the definition (1.2) and the definition (1.3) we get the following results.

$$\begin{aligned}\lim_{n\to\infty}\|x_n - p\| &\leq \lim_{n\to\infty}[(1-(1-\delta)\alpha)\delta^2]^n\|x_0 - p\| = 0, \\ \lim_{n\to\infty}\|p_n - p\| &\leq \lim_{n\to\infty}[(1-(1-\delta)\alpha\beta)\delta^2]^n\|p_0 - p\| = 0, \\ \lim_{n\to\infty}\|u_n - p\| &\leq \lim_{n\to\infty}[1-(1-\delta)\beta[\gamma - \alpha + \alpha.\gamma]]^n\|u_0 - p\| = 0, \\ \lim_{n\to\infty}\|\xi_n - p\| &\leq \lim_{n\to\infty}[1-(1-\delta)\alpha\beta]^n\|\xi_0 - p\| = 0.\end{aligned}$$

Now we give convergence rates of the above iterations as follows;

- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{(1-(1-\delta)\alpha)^n \delta^{2n}}{(1-(1-\delta)\alpha\beta)^n \delta^{2n}} . \dfrac{\|x_0 - p\|}{\|p_0 - p\|} = 0,$
- $\lim_{n\to\infty} \dfrac{a_n}{c_n} = \dfrac{(1-(1-\delta)\alpha)^n \delta^{2n}}{[1-(1-\delta)\beta[\gamma - \alpha + \alpha.\gamma]]^n} . \dfrac{\|x_0 - p\|}{\|u_0 - p\|} = 0,$
- $\lim_{n\to\infty} \dfrac{a_n}{d_n} = \dfrac{(1-(1-\delta)\alpha)^n \delta^{2n}}{[1-(1-\delta)\alpha\beta]^n} . \dfrac{\|x_0 - p\|}{\|\xi_0 - p\|} = 0.$

Therefore, the conclusion follows. □

3. Numerical Results

Now we give numerical examples to support our theorem. In both examples, we choose functions satisfying the weak contraction condition. It can be understood easily with the help of the mean value theorem.

Example 3.1. Let $D = [-10, 10]$ be a subset of a usual normed space $\mathbb{R}$ and let $T : D \to D$ be a mapping such that $Tx = \sin(\cos x)$ for all $x \in D$. Choose $\alpha = \beta = 0.12$ and $\gamma = 0.24$ and $\alpha_n = \beta_n = \gamma_n = 0.25$ for all $n \in \mathbb{N}$. It is obvious that T has a unique fixed point $p = 0.69481969073079 \in D$. Moreover, the sequences of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ and the parameters α, β and γ satisfy the condition of Theorem 2.1.

For an arbitrary initial point $x_0 = 2$, the values of the iterations of S, Picard-S, S_n and M are given in Table 1. Thus, it is obvious that the M-iteration process converges faster than all other iterations. Now we give the graphs of these iterations to show their convergence behaviours in Figure 1.

In the next example, we use an exponential function which also satisfies the weak contractive condition.

Example 3.2. Let $D = [0, 5]$ be a subset of a usual normed space $\mathbb{R}$ and let $T : D \to D$ be a mapping such that $Tx = e^{\frac{4}{4+x^2}}$ for all $x \in D$. Choose $\alpha = \beta = 0.2$,$\gamma = 0.45$ and $\alpha_n = \beta_n = \gamma_n = 0.50$ for all $n \in \mathbb{N}$. It is obvious that T has a unique fixed point $p = 1.7579448713504 \in D$. Moreover, the sequences of $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ and the parameters α, β and γ satisfy the conditions of Theorem 2.1.

For an arbitrary initial point $x_0 = 5$, the values of the iterations of S, Picard-S, S_n and M are given in Table 2. Thus, it is obvious that the M-iteration process converges faster than all other iterations. Now we give the graphs of these iterations in Figure 2 to see their convergence behaviours.

Step	Siteration	Picard-S	S_n iteration	M iteration
1	2.00000000000000	2.00000000000000	2.000000000000000	2.00000000000000
2	−0.2606345908683	0.82274676877756	0.059666857688088	0.83357849776340
3	0.82742078111148	0.72063145818922	0.79298563965090	0.71374670426113
4	0.63322481776167	0.69984773469901	0.66834020139151	0.69737415165409
5	0.71929615165795	0.69579056568185	0.70061313646443	0.69516366956216
6	0.68437145798799	0.69500682583460	0.69335657926482	0.69486599536890
7	0.69915397516439	0.69485574834868	0.69515086974096	0.69482592374820
8	0.69299964736283	0.69482663793094	0.69473790460681	0.69482052974562
9	0.69558010394304	0.69482102922656	0.69483851648450	0.69481980366892
10	0.69450131531509	0.69481994861403	0.69481510357798	0.69481970593317
11	0.69495287230429	0.69481974041622	0.69482075802148	0.69481969277715
12	0.69476395803525	0.69481970030350	0.69481943293436	0.69481969100624
13	0.69484300965349	0.69481969257512	0.69481975113705	0.69481969076786
14	0.69480993330735	0.69481969108613	0.69481967622364	0.69481969073577
15	0.69482377345466	0.69481969079925	0.69481969414590	0.69481969073146
16	0.69481798240850	0.69481969074398	0.69481968991371	0.69481969073088
17	0.69482040553572	0.69481969073333	0.69481969092373	0.69481969073080
18	0.69481939163787	0.69481969073127	0.69481969068473	0.69481969073079
19	0.69481981587892	0.69481969073088	0.69481969074169	0.69481969073079
20	0.69481963836559	0.69481969073080	0.69481969072819	0.69481969073079
21	0.69481971264172	0.69481969073079	0.69481969073140	0.69481969073079
22	0.69481968156270	0.69481969073079	0.69481969073065	0.69481969073079
23	0.69481969456696	0.69481969073079	0.69481969073082	0.69481969073079
24	0.69481968912563	0.69481969073079	0.69481969073078	0.69481969073079
25	0.69481969140243	0.69481969073079	0.69481969073079	0.69481969073079
26	0.69481969044976	0.69481969073079	0.69481969073079	0.69481969073079
27	0.69481969084838	0.69481969073079	0.69481969073079	0.69481969073079
28	0.69481969068158	0.69481969073079	0.69481969073079	0.69481969073079
29	0.69481969075137	0.69481969073079	0.69481969073079	0.69481969073079
30	0.69481969072217	0.69481969073079	0.69481969073079	0.69481969073079
⋮	⋮	⋮	⋮	⋮
39	0.69481969073079	0.69481969073079	0.69481969073079	0.69481969073079
40	0.69481969073079	0.69481969073079	0.69481969073079	0.69481969073079

Table 3.1: Comparative results of Example 3.1

Step	S iteration	Picard-S	S_n iteration	M iteration
1	5.00000000000000	5.00000000000000	5.00000000000000	5.00000000000000
2	1.42941764430671	1.93846644431469	1.40485245885178	1.87003729972110
3	1.84419392945226	1.77947624628915	1.75019355009770	1.76199242097177
4	1.73187212214594	1.76120146917096	1.75603091341028	1.75819390853271
5	1.76700050742008	1.75849723073996	1.75778491609830	1.75796491928449
6	1.75460689343929	1.75804523322895	1.75790258123863	1.75794676700153
7	1.75923776754111	1.75796396996576	1.75794020608824	1.75794507130585
8	1.75742781464024	1.75794862884936	1.75794356116350	1.75794489420287
9	1.75815703047109	1.75794562943728	1.75794471266250	1.75794487412885
10	1.75785610751453	1.75794502733560	1.75794482305018	1.75794487170534
11	1.75798260400559	1.75794490395775	1.75794486551196	1.75794487139762
12	1.75792862247490	1.75794487825579	1.75794486935670	1.75794487135688
13	1.75795194515923	1.75794487282874	1.75794487112933	1.75794487135130
14	1.75794176333736	1.75794487166981	1.75794487126058	1.75794487135052
15	1.75794624777708	1.75794487141994	1.75794487134225	1.75794487135040
16	1.75794425757424	1.75794487136563	1.75794487134601	1.75794487135038
17	1.75794514670100	1.75794487135375	1.75794487135012	1.75794487135038
18	1.75794474716322	1.75794487135112	1.75794487135015	1.75794487135038
19	1.75794492762700	1.75794487135055	1.75794487135039	1.75794487135038
20	1.75794484573944	1.75794487135042	1.75794487135036	1.75794487135038
21	1.75794488305041	1.75794487135039	1.75794487135039	1.75794487135038
22	1.75794486598680	1.75794487135038	1.75794487135038	1.75794487135038
23	1.75794487381696	1.75794487135038	1.75794487135038	1.75794487135038
24	1.75794487021279	1.75794487135038	1.75794487135038	1.75794487135038
25	1.75794487187643	1.75794487135038	1.75794487135038	1.75794487135038
26	1.75794487110653	1.75794487135038	1.75794487135038	1.75794487135038
27	1.75794487146368	1.75794487135038	1.75794487135038	1.75794487135038
28	1.75794487129764	1.75794487135038	1.75794487135038	1.75794487135038
29	1.75794487137498	1.75794487135038	1.75794487135038	1.75794487135038
30	1.75794487133888	1.75794487135038	1.75794487135038	1.75794487135038
⋮	⋮	⋮	⋮	⋮
39	1.75794487135039	1.75794487135038	1.75794487135038	1.75794487135038
40	1.75794487135038	1.75794487135038	1.75794487135038	1.75794487135038

Table 3.2: Comparative results of Example 3.2

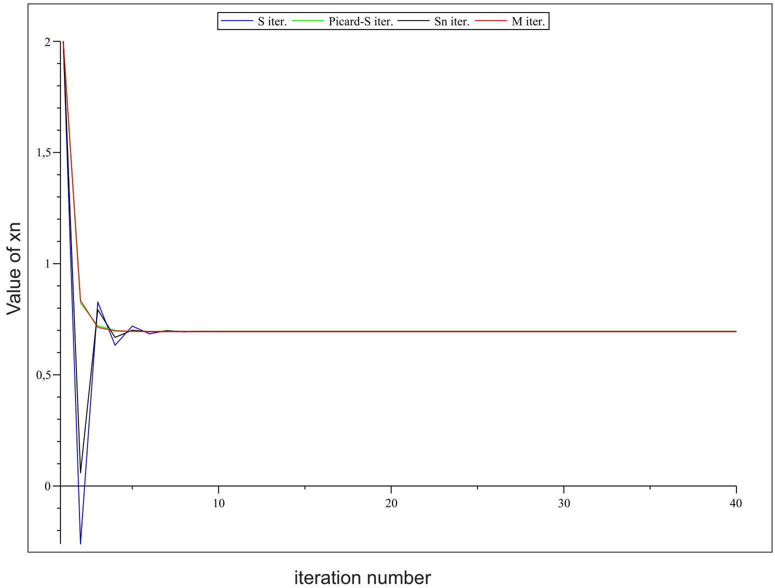

Fig. 3.1: Behaviour of the iterations given in Example 1

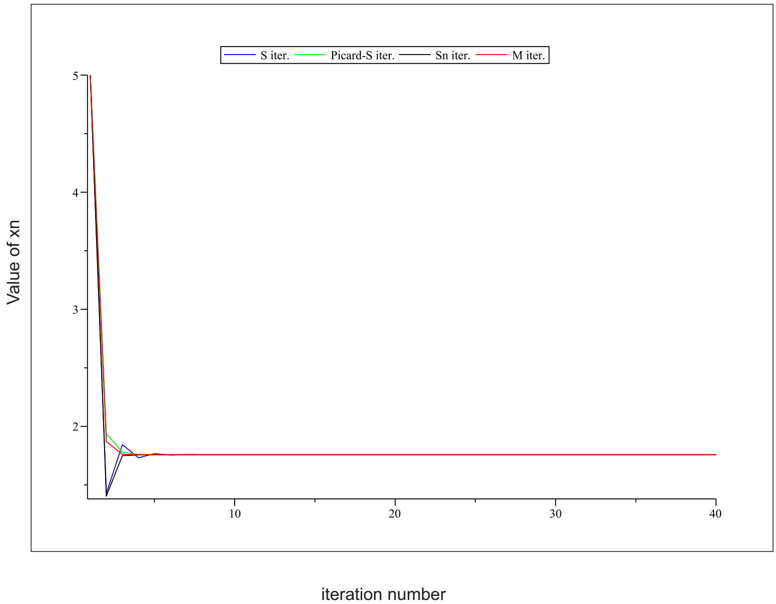

Fig. 3.2: Behaviour of the iterations given in Example 2

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Ser. Math. Inform. Vol. 33, No 2 (2018), 307–324
https://doi.org/10.22190/FUMI1802307A

MULTIGENERATOR GABOR FRAMES ON LOCAL FIELDS

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Abstract. The main objective of this paper is to provide a complete characterization of multigenerator Gabor frames on a periodic set Ω in K. In particular, we provide some necessary and sufficient conditions for the multigenerator Gabor system to be a frame for $L^2(\Omega)$. Furthermore, we establish a complete characterization of multigenerator Parseval Gabor frames.
Keywords: Multigenerator Gabor frames, periodic set, signal processing.

1. Introduction

The concept of frames in a Hilbert space was originally introduced by Duffin and Schaeffer [3] in the context of non-harmonic Fourier series. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. Frames did not seem to generate much interest until the ground-breaking work of Daubechies *et al.* [4]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. Since then the theory of frames began to be more widely investigated, and now it is found to be useful in signal processing, image processing, harmonic analysis, sampling theory, data transmission with erasures, quantum computing and medicine. Recently, more applications of the theory of frames are found in diverse areas including optics, filter banks, signal detection and in the study of Bosev spaces and Banach spaces. We refer the reader to [1], [5] for an introduction to frame theory and its applications.

The most important concrete realization of frame is Gabor frame. Gabor systems are collections of functions

$$\mathcal{G}(a,b,\psi) = \Big\{ M_{mb}T_{na}\psi(x) =: e^{2\pi imax}\psi(x-na) : m,n \in \mathbb{Z} \Big\} \tag{1.1}$$

which are built by the combined action of modulations and translations of a single function and hence can be viewed as the set of time-frequency shifts of $\psi(x) \in L^2(\mathbb{R})$

Received July 19, 2017; accepted October 19, 2017
2010 *Mathematics Subject Classification.* Primary 42C15; Secondary 42C40, 43A70

along the lattice $a\mathbb{Z}\times b\mathbb{Z}$ in $\mathbb{R}^2$. Such systems, also called *Weyl-Heisenberg systems*, were introduced by Gabor [2] with the aim of constructing efficient, time-frequency localized expansions of signals as an infinite linear combinations of elements in [1.1]. The system $\mathcal{G}(a,b,\psi)$ given by [1.1] is called a *Gabor frame* if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leqslant \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \left|\langle f, M_{mb}T_{na}\psi\rangle\right|^2 \leqslant B\|f\|_2^2, \tag{1.2}$$

holds for every $f \in L^2(\mathbb{R})$, and we call the optimal constants A and B the *lower frame bound* and the *upper frame bound*, respectively. A *tight Gabor frame* refers to the case when $A = B$, and a *normalized tight frame* refers to the case when $A = B = 1$. Gabor systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. One of the most important problem in practice is therefore to determine conditions for Gabor systems to be frames. In practice, once the window function has been chosen, the first question to investigate for Gabor analysis is to find the values of the time-frequency parameters a, b such that $\mathcal{G}(a,b,\psi)$ is a frame. Therefore, the product ab will decide whether the system $\mathcal{G}(a,b,\psi)$ constitutes a frame or is even complete for $L^2(\mathbb{R})$ or not. In this context, a useful tool is the Ron and Shen [8] criterion. By using this criterion, Gröchenig et al.[6] have proved that the system $\mathcal{G}(a,b,\psi)$ cannot be a frame for $L^2(\mathbb{R})$ if $|ab| > 1$ and have also shown that the system $\mathcal{G}(a,b,\psi)$ will form an orthonormal basis for $L^2(\mathbb{R})$ if $|ab| = 1$.

Gabor analysis is a pervasive signal processing method for decomposing and reconstructing signals from their time frequency projections and also in the context of speech processing, texture segmentation, pattern and object recognition. In order to analyze the dynamic time frequency samples of the signals that contain a wide range of spatial and frequency components, the resolution of which is normally very poor, the single windowed Gabor expansion is not suitable. To address this issue, one of the best choices is the multigenerator Gabor system with a set of multiple windows of various time frequency localizations in frame system. The representation of signals of multiple and time-varying frequencies would have their corresponding windowing templates and resolutions relate to. The concept of multigenerator Gabor system is introduced by Zibulski and Zeevi [12] and they [13] discussed the frame operator associated with the multigenerator Gabor frame by invoking the concept of piecewise Zak transform. They pointed out that the Ballian-Low theorem for the multigenerator Gabor frame is more generalized to the consideration of a scheme of multigenerator which makes it possible to overcome in a way the constraint imposed by the single window in the original theorem. Since then a lot of research [13]-[18] has been carried out in both theory and application aspects of the multigenerator Gabor frame as they can increase the degree of freedom by incorporating windows of various types and widths. For more information on this topic, we refer the reader to [1], [5].

For modeling a signal that appears periodically but intermittently, $a\mathbb{Z}$-periodic set in $\mathbb{R}$ can be used. In this direction, some authors considered the Gabor analysis in $L^2(\mathbb{S})$, where $\mathbb{S}$ is an $a\mathbb{Z}$-periodic set in $\mathbb{R}$. Although the classical Gabor analysis tools in $L^2(\mathbb{S})$ can be adjusted to treat such a scenario by padding with zeros outside the set $\mathbb{S}$, Gabor systems that fit exactly such a scenario might have been more efficient. Gabardo and Li [19] obtained density results for Gabor systems associated with periodic subsets of the real line. Lian and Li [20] studied the Gabor frame sets for subspaces. They pointed out that only a periodic $\mathbb{S}$ in $\mathbb{R}$ is suitable for Gabor analysis.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact Abelian groups. For example, any field endowed with a discrete topology is a local field. For this reason we consider only non-discrete fields. Local fields are essentially of two types (excluding connected local fields $\mathbb{R}$ and $\mathbb{C}$). Local fields of the characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics. For more details we refer the reader to the book by Taibleson [11].

The local field K is a natural model for the structure of Gabor frame systems, as well as a domain upon which one can construct Gabor basis functions. Recently, there has been a substantial body of work concerned with the construction of Gabor frames on K or, more generally, on local fields of positive characteristic. Jiang et al.[7] constructed Gabor frames on local fields of positive characteristic using basic concepts of operator theory and have established a necessary and sufficient conditions for the system $\{M_{u(m)b}T_{u(n)a}\psi =: \chi_m(bx)\psi\big(x-u(n)a\big)\}_{m,n\in\mathbb{N}_0}$ to be a frame for $L^2(K)$. Shah [9] established a complete characterization of Gabor frames on local fields by virtue of two basic equations in the frequency domain and provides the algorithm for constructing an orthonormal Gabor basis for $L^2(K)$. Recent results related to Gabor frames on local fields of positive characteristic can be found in [9],[10], and the references therein.

Motivated and inspired by the above work, our aim is to investigate multigenerator Gabor systems on a periodic set in local field and provide complete characterizations for such systems to be frameS. Moreover, necessary and sufficient condition for such a system to be a Parseval Gabor frame. Our results also hold for the Cantor dyadic group and the Vilenkin groups as they are local fields of positive characteristic.

The rest of this paper is organized as follows. In Section 2., we discuss some preliminary facts about Fourier analysis on local fields of positive characteristic and also some results to be used throughout the paper. In Section 3., we establish necessary and sufficient conditions for the multigenerator Gabor system to be a frame for $L^2(\Omega)$. In Section 4., we obtain a complete characterization of multigenerator Parseval Gabor frames.

2. Preliminaries on local fields and basic facts about frames

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote additive and multiplicative groups of K, respectively. If K is any field and is endowed with a discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field we mean a field K which is locally compact, non-discrete and totally disconnected. p-adic fields are examples of local fields. More details can be found in [11,13]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leqslant \max\{|x|, |y|\}$ for all $x, y \in K$. The property (c) is called the *ultrametric inequality*. The set $\mathfrak{D} = \{x \in K : |x| \leqslant 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as a result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, there exists an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called the *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [11], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \backslash \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For the proof of this fact we refer the reader to [11]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \left\{x \in K : |x| < q^{-k}\right\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of the distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [13], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx. \tag{2.2}$$

Furthermore, the properties of the Fourier transform on a local field K are much similar to those on the real line. In particular, the Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q,\ n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1},\ \ 0 \le a_k < p,\ \ \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.3}$$

Also, for

$$n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \dots, s,$$

we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.4}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leqslant s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geqslant 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$ be as above. We define the character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.5}$$

We also denote the test function space on K by $\mathcal{S}$, i.e., each function f in $\mathcal{S}$ is a finite linear combination of functions of the form $\mathbf{1}_k(x-h), h \in K, k \in \mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that $\mathcal{S}$ is dense in $L^p(K), 1 \leqslant p < \infty$, and each function in $\mathcal{S}$ is of compact support and so is its Fourier transform.

A measurable set Ω in a local field K is said to be $a-$ periodic if $\Omega + u(n)a = \Omega$, for every $n \in \mathbb{N}_0$. Let Ω be an a-periodic subset of K. Then it is clear that Ω is av-periodic for every $v \in \mathbb{N}$. Denote $\Omega^0 = G_a \cap \Omega$ and

$$L^2(\Omega) = \left\{f \in L^2(K) : supp(f) \subset \Omega\right\} \tag{2.6}$$

where $G_a = \{x \in \Omega : |x| \leqslant |a|\}$. Clearly, it is a Hilbert space with the inner product in $L^2(K)$.

Definition 2.1. Let a and b be any two fixed elements in K. For a fixed positive integer L, let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subseteq L^2(\Omega)$, define the multi-generator Gabor system

(2.7)
$$\mathcal{G}(a,b,\Psi) := \Big\{ M_{u(m)b_\ell} T_{u(n)a} \psi^\ell =: \chi_m(b_\ell x)\psi^\ell\big(x - u(n)a\big) : n, m \in \mathbb{N}_0, 1 \leqslant \ell \leqslant L \Big\},$$

where $M_{u(m)b_\ell} f(x) = \chi_m(b_\ell x) f(x)$ and $T_{u(n)a} f(x) = f\big(x - u(n)a\big)$ are the modulation and translation operators defined on $L^2(K)$, respectively. We call the Gabor system $\mathcal{G}(a,b,\Psi)$ a *Gabor frame* for $L^2(\Omega)$, if there exist constants C and D, $0 < C \leqslant D < \infty$ such that

$$C\big\|f\big\|_2^2 \leqslant \sum_{\ell=1}^{L} \sum_{m\in\mathbb{N}_0} \sum_{n\in\mathbb{N}_0} \big|\big\langle f, M_{u(m)b_\ell} T_{u(n)a_\ell} \psi^\ell \big\rangle\big|^2 \leqslant D\big\|f\big\|_2^2 \tag{2.8}$$

The following Lemma follows from the frames associated with shift invariant spaces(see [11] or [1]).

Lemma 2.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a family of elements in $L^2(K)$ and suppose that for $b > 0$,*

$$B = \frac{1}{|b|} \sup_{x\in K} \sum_{k\in\mathbb{N}_0} \left| \sum_{n\in\mathbb{N}_0} f_n(x)\overline{f_n(x - b^{-1}u(k))} \right| < \infty, \tag{2.9}$$

then $\big\{M_{u(m)b} f_n : m, n \in \mathbb{N}_0\big\}$ is Bessel sequences with the upper bound B for $L^2(K)$. Furthermore, if

$$A = \frac{1}{|b|} \inf_{x\in K} \left\{ \sum_{n\in\mathbb{N}_0} |f_n(x)|^2 - \sum_{k\in\mathbb{N}} \left| \sum_{n\in\mathbb{N}_0} f_n(x)\overline{f_n(x - b^{-1}u(k))} \right| \right\} > 0, \tag{2.10}$$

then $\big\{M_{u(m)b} f_n : m, n \in \mathbb{N}_0\big\}$ is a frame with bounds A and B.

3. Necessary and Sufficient Conditions for Multigenerator Gabor System to be frame for $L^2(\Omega)$

In this section, we establish some necessary and sufficient conditions for the multi-generator Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) to be a frame for $L^2(\Omega)$. Before we proceed to the main results, we first provide the relationship between the Gabor system in $L^2(K)$ and its subspace $L^2(\Omega)$.

Theorem 3.1. *Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subseteq L^2(\Omega)$ and $a, b > 0$, then the following results hold.*

(a) If the Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a frame for $L^2(K)$, then it is a frame for $L^2(\Omega)$.

(b) If the Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a Bessel sequence for $L^2(\Omega)$ with the upper bound B, then it is a Bessel sequence for $L^2(K)$ with the same upper bound.

Proof. The part (a) clearly follows from the fact that $L^2(\Omega) \subset L^2(K)$. Now we proceed to prove part(b). Suppose that the multigenerator Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a Bessel sequence for $L^2(\Omega)$. Then there exists a constant $B>0$ such that

$$\sum_{\ell=1}^{L}\sum_{n\in\mathbb{N}_0}\left|\left\langle f, M_{u(m)b}T_{u(n)a}\psi^{\ell}\right\rangle\right|^2 \leqslant B\|f\|^{,}.\ \ \forall f\in L^2(\Omega). \tag{3.1}$$

Further we observe that

$$\begin{aligned}\left\langle f, M_{u(m)b}T_{u(n)a}\psi^{\ell}\right\rangle &= \int_K f(x)\overline{\psi^{\ell}(x-u(n)a)}\chi_m(bx)\,dx\\ &= \int_\Omega f(x)\overline{\psi^{\ell}(x-u(n)a)}\chi_m(bx)\,dx,\end{aligned} \tag{3.2}$$

as $f\overline{\psi^{\ell}} \in L^2(\Omega), 1\leqslant \ell \leqslant L$, for all $f\in L^2(K)$. Therefore, it follows that

$$\left\langle f, M_{u(m)b}T_{u(n)a}\psi^{\ell}\right\rangle = \left\langle f\mathbf{1}_\Omega, M_{u(m)b}T_{u(n)a}\psi^{\ell}\right\rangle \tag{3.3}$$

Thus for all $f\in L^2(K)$, we have

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{n\in\mathbb{N}_0}\left|\left\langle f, M_{u(m)b}T_{u(n)a}\psi^{\ell}\right\rangle\right|^2 &= \sum_{\ell=1}^{L}\sum_{n\in\mathbb{N}_0}\left|\left\langle f\mathbf{1}_\Omega, M_{u(m)b}T_{u(n)a}\psi^{\ell}\right\rangle\right|^2\\ &\leqslant B\|f\mathbf{1}_\Omega\|^2\\ &\leqslant B\|f\|^2.\end{aligned} \tag{3.4}$$

This clearly implies that the multigenerator Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a Bessel sequence for $L^2(K)$ with the same upper bound B. $\square$

Now we state the sufficient condition for the multigenerator Gabor system $\mathcal{G}(a,b\Psi)$ given by (2.7) to be a frame for $L^2(\Omega)$.

Theorem 3.2. *Let $\Psi=\{\psi_1,\psi_2,\dots,\psi_L\}\subseteq L^2(K)$ and $a,b>0$ suppose that*

$$B=\frac{1}{|b|}\sup_{x\in G_{b^{-1}}}\sum_{k\in\mathbb{N}_0}\left|\sum_{\ell=1}^{L}\sum_{n\in\mathbb{N}_0}T_{(n)a}\psi^{\ell}(x)\overline{T_{u(n)a}\psi^{\ell}(x-b^{-1}u(k))}\right|<\infty, \tag{3.5}$$

then the multigenerator Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a Bessel sequences with the upper bound B for $L^2(\Omega)$. Furthermore, if

$$A=\frac{1}{|b|}\inf_{x\in G_{b^{-1}}}\left\{\sum_{n\in\mathbb{N}_0}\left|\sum_{\ell=1}^{L}T_{u(n)a}\psi^{\ell}(x)\right|^2-\sum_{k\in\mathbb{N}}\left|\sum_{\ell=1}^{L}\sum_{n\in\mathbb{N}_0}T_{(n)a}\psi^{\ell}(x)\overline{T_{u(n)a}\psi^{\ell}(x-b^{-1}u(k))}\right|\right\}>0, \tag{3.6}$$

then the multigenerator Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a frame for $L^2(\Omega)$ with bounds A and B.

Proof. Define

$$H_1(x) = \sum_{k\in\mathbb{N}_0} \left| \sum_{\ell=1}^{L} \sum_{n\in\mathbb{N}_0} T_{(n)a}\psi^\ell(x)\overline{T_{u(n)a}\psi^\ell(x-b^{-1}u(k))} \right|. \tag{3.7}$$

$$H_2(x) = \sum_{n\in\mathbb{N}_0} \left| \sum_{\ell=1}^{L} T_{u(n)a}\psi^\ell(x) \right|^2 - \sum_{k\in\mathbb{N}} \left| \sum_{\ell=1}^{L} \sum_{n\in\mathbb{N}_0} T_{(n)a}\psi^\ell(x)\overline{T_{u(n)a}\psi^\ell(x-b^{-1}u(k))} \right| \tag{3.8}$$

Clearly H_1 and H_2 are $b^{-1}-$periodic functions. Thus

$$B = \frac{1}{|b|}\sup_{x\in K} \sum_{k\in\mathbb{N}_0} \left| \sum_{\ell=1}^{L} \sum_{n\in\mathbb{N}_0} T_{(n)a}\psi^\ell(x)\overline{T_{u(n)a}\psi^\ell(x-b^{-1}u(k))} \right| < \infty, \tag{3.9}$$

$$A = \frac{1}{|b|}\inf_{x\in K} \left\{ \sum_{n\in\mathbb{N}_0} \left| \sum_{\ell=1}^{L} T_{u(n)a}\psi^\ell(x) \right|^2 - \sum_{k\in\mathbb{N}} \left| \sum_{\ell=1}^{L} \sum_{n\in\mathbb{N}_0} T_{(n)a}\psi^\ell(x)\overline{T_{u(n)a}\psi^\ell(x-b^{-1}u(k))} \right| \right\} > 0, \tag{3.10}$$

Define

$$f_n(x) = T_{u(k)a}\psi^\ell(x), \tag{3.11}$$

where $n = \ell + su(k)$, $1 \leqslant \ell \leqslant L$. Then, one obtains from (3.9) and (3.10) that

$$B = \frac{1}{|b|}\sup_{x\in K} \sum_{k\in\mathbb{N}_0} \left| \sum_{n\in\mathbb{N}_0} f_n(x)\overline{f_n(x-b^{-1}u(k))} \right| < \infty, \tag{3.12}$$

$$A = \frac{1}{|b|}\inf_{x\in K} \left\{ \sum_{n\in\mathbb{N}_0} |f_n(x)|^2 - \sum_{k\in\mathbb{N}} \left| \sum_{n\in\mathbb{N}_0} f_n(x)\overline{f_n(x-b^{-1}u(k))} \right| \right\} > 0, \tag{3.13}$$

respectively. By invoking Lemma 2.1, and the fact $L^2(\Omega) \subset L^2(K)$, the result follows. □

Now we prove the necessary condition for the multigenerator Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) to be a frame for $L^2(\Omega)$, which depends on the interplay among the functions $\psi^1, \psi^2, \ldots, \psi^L$ and the parameters $a, b_1, \ldots, b_L$ and the periodic set Ω.

Theorem 3.3. *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subseteq L^2(\Omega)$, and $a, b_1, b_2, \ldots, b_L > 0$. Suppose that the Gabor system $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a multigenerator Gabor frame for $L^2(\Omega)$ with bounds A and B, then*

$$A\mathbf{1}_{\mathbf{\Omega}}(x) \leqslant \sum_{\ell=1}^{L} \left\{ \frac{1}{|b_\ell|} \sum_{n\in\mathbb{N}_0} |\psi^\ell(x-u(n)a)|^2 \right\} \leqslant B\mathbf{1}_{\mathbf{\Omega}}(x), \quad a.e. \;\; K. \tag{3.14}$$

Proof. We first note that Ω is an a-periodic subset of K. Therefore, $\psi^\ell(\cdot - u(n)a) \in L^2(\Omega)$ for all $n \in \mathbb{N}, 1 \leqslant \ell \leqslant L$. Thus

$$\sum_{\ell=1}^{L} \left\{ \frac{1}{|b_\ell|} \sum_{n\in\mathbb{N}_0} |\psi^\ell(x - u(n)a)|^2 \right\} = 0, \quad a.e.\ K \setminus \Omega. \tag{3.15}$$

We establish the proof by contradiction. Assume that the upper bound condition in (3.14) is not true on Ω. Then there exists a measurable set $\Xi \subset \Omega$ with positive measure such that

$$\sum_{\ell=1}^{L} \left\{ \frac{1}{|b_\ell|} \sum_{n\in\mathbb{N}_0} |\psi^\ell(x - u(n)a)|^2 \right\} > B \quad a.e. \quad on \quad \Xi. \tag{3.16}$$

We can assume that Ξ is contained in a ball Υ with the diameter of $|b|^{-1}$. Setting

$$\Xi_0 = \left\{ x \in \Xi : \frac{1}{|b_\ell|} \sum_{n\in\mathbb{N}_0} |\psi^\ell(x - u(n)a)|^2 \geqslant \frac{1}{|b_\ell|} + B \right\}$$

and

$$\Xi_k = \left\{ x \in \Xi : \frac{1}{|b_\ell|(k+1)} + B \leqslant \frac{1}{|b_\ell|} \sum_{n\in\mathbb{N}_0} |\psi^\ell(x - u(n)a)|^2 < \frac{1}{|b_\ell| k} + B \right\}, \quad k \in \mathbb{N}.$$

Thus we obtain a partition of Ξ into disjoint measurable sets among which at least one say, Ξ_s, has positive measure.

Now consider the function $f = \mathbf{1}_{\Xi_s}$, the characteristic function on Ξ_s and note that $\|f\|^2 = |\Xi_s|$. Clearly, for any $n \in \mathbb{N}_0$, the function $f\overline{T_{u(n)a}\psi^\ell}$ has support in Ξ_s. Since Ξ_s is contained in a ball Υ with the diameter of $|b|^{-1}$ and the functions $\left\{ \sqrt{|b_\ell|}\chi_m(b_\ell x) : m \in \mathbb{N}_0, 1 \leqslant \ell \leqslant L \right\}$ constitutes an orthonormal basis for $L^2(\Upsilon)$ for every ball Υ of the diameter $|b|^{-1}$, we have

$$\begin{aligned} \sum_{m\in\mathbb{N}_0} \left| \left\langle f, M_{u(m)b_\ell} T_{u(n)a_\ell} \psi^\ell \right\rangle \right|^2 &= \sum_{m\in\mathbb{N}_0} \left| \left\langle f\overline{T_{u(n)a_\ell}\psi^\ell}, M_{u(m)b_\ell} \right\rangle \right|^2 \\ &= \tfrac{1}{|b_\ell|} \int_K |f(x)|^2\, |\psi^\ell(x - u(n)a)|^2 dx. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{m\in\mathbb{N}_0} \left| \left\langle f, M_{u(m)b_\ell} T_{u(n)a_\ell} \psi^\ell \right\rangle \right|^2 &= \tfrac{1}{|b_\ell|} \int_{\Xi_s} |f(x)|^2 \sum_{n\in\mathbb{N}_0} |\psi^\ell(x - u(n)a)|^2\, dx \\ &\geqslant \left\{ B + \tfrac{1}{|b_\ell|(s+1)} \right\} \|f\|^2. \end{aligned}$$

This is a contradiction to the assumption that B is the upper frame bound for the multigenerator Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.7). In a similar vein, we can show that if the lower bound condition in (3.14) is violated, then A cannot be the lower bound for the multigenerator Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.7). □

4. Characterizations of Parseval Multigenerator Gabor Frame

In this section, we will provide the characterization of Parseval multigenerator Gabor frames. The following Lemma is very useful in this section.

Lemma 4.1. *Let f be a bounded measurable function with compact support and let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subseteq L^2(\Omega)$, then for $a, b > 0$ and $1 \leqslant \ell \leqslant L$, we have*

$$\begin{aligned}\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} & \left|\langle f, M_{u(m)b}T_{u(n)a}\psi^\ell\rangle\right|^2 \\ & = \frac{1}{|b|}\int_K |f(x)|^2 \big|\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x-u(n)a\big)\big|^2\,dx \\ & \qquad + \sum_{k\in\mathbb{N}_0}\frac{1}{|b|}\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big) \\ & \qquad\qquad \times \sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}dx\end{aligned}$$

Proof. We have

$$\begin{aligned}
\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} & \left|\langle f, M_{u(m)b}T_{u(n)a}\psi^\ell\rangle\right|^2 \\
&= \sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\int_K \left| f(x)\overline{\psi^\ell\big(x-u(n)a\big)}\,\overline{\chi_m(bx)}\,dx\right|^2 \\
&= \sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\int_K \left| f(x+u(n)a)\overline{\psi^\ell(\xi)}\,\overline{\chi_m(bx)}\,dx\right|^2 \\
&= \sum_{n\in\mathbb{N}_0}\frac{1}{|b|}\int_K \overline{f\big(x+u(n)a\big)}\,\psi^\ell(\xi) \\
&\qquad\qquad \times \sum_{\ell\in\mathbb{N}_0} f\big(x+b^{-1}u(\ell)+u(n)a\big)\overline{\psi^\ell\big(x+b^{-1}u(\ell)\big)}, dx \\
&= \sum_{n\in\mathbb{N}_0}\frac{1}{|b|}\int_K |f\big(x+u(n)a)|^2|\psi^\ell(x)|^2\,dx \\
&+ \sum_{n\in\mathbb{N}_0}\frac{1}{|b|}\int_K \overline{f\big(x+u(n)a\big)}\,\psi^\ell(x) \\
&\qquad\qquad \times \sum_{k\in\mathbb{N}} f\big(x+b^{-1}u(k)+u(n)a\big)\overline{\psi^\ell\big(x+b^{-1}u(k)\big)}dx \\
&= \sum_{n\in\mathbb{N}_0}\frac{1}{|b|}\int_K |f(x)|^2|\psi^\ell\big(x-u(n)a)\big)|^2\,dx \\
&+ \sum_{n\in\mathbb{N}_0}\frac{1}{|b|}\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big) \\
&\qquad\qquad \times \sum_{k\in\mathbb{N}} f\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\,dx.
\end{aligned}$$

□

Theorem 4.1. *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subseteq L^2(\Omega)$, and $a, b_1, b_2, \ldots, b_L > 0$. Suppose that the Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.7) is a tight frame for $L^2(\Omega)$ with $A = 1$,then*

$$\sum_{\ell=1}^{L}\frac{1}{|b_\ell|}\left\{\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x-u(n)a)\big)|^2\right\} = \mathbf{1}_\Omega, \quad a.e. \quad K. \tag{4.1}$$

Furthermore, if $b_1 = b_2 = \cdots = b_L = b(say)$, then for $k \in \mathbb{N}$, we have

$$\sum_{\ell=1}^{L}\left\{\sum_{n\in\mathbb{N}_0}\psi^{\ell}\big(x-u(n)a)\big)|^2\right\} = |b|\mathbf{1}_{\Omega}, \quad a.e. \quad K. \tag{4.2}$$

$$\sum_{\ell=1}^{L}\left\{\sum_{k\in\mathbb{N}}\psi^{\ell}\big(x+b^{-1}u(m)\big)\overline{\psi^{\ell}\big(x+b^{-1}u(k)-u(n)a\big)}\right\} = 0, \tag{4.3}$$

hold a.e in K.

Proof. Define

$$\nu^1 = \min\left\{\frac{1}{|b_\ell|} : 1 \leqslant \ell \leqslant L\right\} \tag{4.4}$$

Consider

$$\mathcal{D} = \big\{f : f \in L^2(\Omega) \quad and\ suppf \subset (G_{\nu^1}\cap\Omega)\big\} \tag{4.5}$$

Since $\mathcal{G}(a,b,\Psi)$ given by (2.7) is a tight frame for $L^2(\Omega)$ with $A = 1$. Then

$$\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\big|\langle f, M_{u(m)b}T_{u(n)a}\psi^{\ell}\rangle\big|^2 \leqslant \|f\|^2, \quad \forall f\in\mathcal{D}. \tag{4.6}$$

By Invoking Lemma4.1, for all $f \in \mathcal{D}$ and fixed ℓ, n, we have

(4.7)
$$\begin{aligned}\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0}\big|\langle f, M_{u(m)b}T_{u(n)a}\psi^{\ell}\rangle\big|^2 &= \sum_{m\in\mathbb{N}_0}\left|\int_K f(x)\overline{\psi^{\ell}(x-u(n)a)\chi_m(b_\ell x)}\,dx\right|^2\\ &= \tfrac{1}{|b_\ell|}\int_K |f(x)\overline{\psi^{\ell}(x-u(n)a)}|^2\,dx\\ &= \frac{1}{|b_\ell|}\int_{G_{\nu^1}} |f(x)\psi^{\ell}(x-u(n)a)|^2\,dx\end{aligned}$$

Thus for any $f \in \mathcal{D}$, we have

$$\int_{G_{\nu^1}}|f(x)|^2\,dx = \int_{G_{\nu^1}}|f(x)|^2\sum_{\ell=1}^{L}\left\{\sum_{n\in\mathbb{N}_0}\psi^{\ell}\big(x-u(n)a)\big)|^2\right\}dx \tag{4.8}$$

implies that

$$\sum_{\ell=1}^{L}\left\{\sum_{n\in\mathbb{N}_0}\psi^{\ell}\big(x-u(n)a)\big)|^2\right\} = 1 \quad a.e. G_{\nu^1}\cap\Omega. \tag{4.9}$$

which gives the desired result (4.1) and its particular case (4.2).

Next we proceed to prove (4.3). For fixed $\ell, 1 \leqslant \ell \leqslant L$, by using Lemma 4.1, we have

(4.10)
$$\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b}T_{u(n)a}\psi^\ell\rangle\right|^2$$
$$= \frac{1}{|b|}\int_K |f(x)|^2 |\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x-u(n)a)\big)|^2\,dx$$
$$+\sum_{k\in\mathbb{N}_0}\frac{1}{|b|}\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big)$$
$$\times\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}dx$$

Then,

(4.11)
$$\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b}T_{u(n)a}\psi^\ell\rangle\right|^2$$
$$= \int_K |f(x)|^2 |\sum_{\ell=1}^{L}\frac{1}{|b|}\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x-u(n)a)\big)|^2\,dx$$
$$+\sum_{k\in\mathbb{N}_0}\frac{1}{|b|}\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big)$$
$$\times\sum_{\ell=1}^{L}\left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\right)dx$$

By combining (4.11) with (4.2), it follows that

(4.12)
$$\sum_{k\in\mathbb{N}_0}\frac{1}{|b|}\quad\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big)$$
$$\times\sum_{\ell=1}^{L}\left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\right)dx = 0.$$

By using the change $u(k) \to -u(k)$, it can be seen that the contribution in the above sum for any value of $u(k)$ is a complex conjugate of the contribution from the value $-u(k)$. Therefore, we have

$$\sum_{k\in\mathbb{N}} Re\left\{\frac{1}{|b|}\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big)\Theta_k(x)dx\right\}=0. \tag{4.13}$$

where

$$\Theta_k(x)=\sum_{\ell=1}^{L}\left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\right)=0. \tag{4.14}$$

To establish the required result, we consider three cases. First we consider the case when $x\in\Omega$. Since Ω is an $a-$ periodic set, then $x-u(n)a\in\Omega$ for all $n\in\mathbb{N}_0$. Therefore

$$\psi^\ell(x-u(n)a)=0,\quad \forall n\in\mathbb{N}_0 \tag{4.15}$$

Thus

$$\sum_{\ell=1}^{L}\left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\right)=0,\quad \forall k\in\mathbb{N}. \tag{4.16}$$

The second case is when $x-b^{-1}u(k)\notin\Omega$ for fixed $k\in\mathbb{N}$. Then $x-u(n)a-b^{-1}u(k)\notin\Omega$ for all $n\in\mathbb{N}_0$. Therefore

$$\sum_{\ell=1}^{L}\left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\right)=0. \tag{4.17}$$

The third case is when $x\in\Omega$ and $x-b^{-1}u(k)\in\Omega$ for fixed $k\in\mathbb{N}$. Let Γ be any ball of the radius at most b^{-1} and denote $\Gamma\cap\Omega$ by Γ^0 and $(\Gamma-b^{-1}u(k))\cap(\Omega+b^{-1}u(k)$ by $\Gamma^{'}$. If the measure of $\Gamma\cap\Gamma^{'}$ is zero, then $x\notin\Gamma^0$ a.e. or $x\notin\Gamma-b^{-1}u(k)$ a.e, thus

$$\sum_{\ell=1}^{L}\left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b^{-1}u(k)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(n)a\big)}\right)=0. \tag{4.18}$$

If the measure of $\Gamma^0\cap\Gamma^{'}$ is positive. We define a function $f\in L^2(\Omega)$ by

$$f(x)=\begin{cases}\exp\{-arg\Theta_{k_0}(x)\}, & x\in\Gamma^0\cap\Gamma^{'}\\[2ex] \qquad 1\quad, & x\in\Gamma^0\cap\Gamma^{'}-b^{-1}u(k),\\[2ex] \qquad 0\quad, & \text{otherwise.}\end{cases} \tag{4.19}$$

Then, by (4.13) we obtain

$$\int_{\Gamma^0\cap\Gamma'} |\Theta_{k_0}(x)|\, dx = 0.$$

It follows that $\Theta_{k_0(x)} = 0$, a.e, on $\Gamma\cap\Gamma^{'}$. Since Γ is an arbitrary ball of the radius at most $\frac{1}{b}$, we conclude that $\Theta_{k_0}(x) = 0$, a.e, in Ω. A simple computation shows that

$$\Theta_{-k_0}(x) = \Theta_{k_0}(x + b^{-1}u(k_0))$$

Thus the desired result follows. □

To proceed further, we first define some notations. For $b_1, b_2, \cdots, b_L > 0$, we define

$$\nu^1 = \min_{1\leqslant \ell\leqslant L}\left\{\frac{1}{|b_\ell|}\right\}$$

and for $j \geqslant 2$,

$$\nu^j = \min_{1\leqslant \ell\leqslant L}\left\{\frac{1}{|b_\ell|} : |b_\ell| < \frac{1}{\nu^{j-1}}\right\}. \tag{4.20}$$

Also, we define

$$\mathcal{I}_j = \left\{\ell : |b_\ell| = \frac{1}{\nu^j}, 1 \leqslant \ell \leqslant L\right\}.$$

Then there exists a unique $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\mathcal{I} \neq \phi \ \ \textit{for} \ \ 1 \leqslant j \leqslant j_0, \\ &\mathcal{I}_{j_1}\cap\mathcal{I}_{j_2} = \phi, \ \ \textit{for} \ \ j_1 \neq j_2, \\ &\textstyle\bigcup_{j=1}^{j_0}\mathcal{I}_j = \{1, 2, \cdots, L\}. \end{aligned} \tag{4.21}$$

Theorem 4.2. *Let j_0 be a unique positive integer satisfying (4.21). Suppose that $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subseteq L^2(\Omega)$, and $a, b_1, b_2, \ldots, b_L > 0$ satisfy*

$$\sum_{\ell=1}^{L}\frac{1}{|b_\ell|}\left\{\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x - u(n)a)\big)|^2\right\} = \mathbf{1}_\Omega, \tag{4.22}$$

(4.23)

$$\sum_{\ell\in\mathcal{I}_j}\left\{\sum_{m\in\mathbb{N}_0}\psi^\ell\big(x + b^{-1}u(m)\big)\overline{\psi^\ell\big(x + b^{-1}u(k) - u(m)a\big)}\right\} = 0, \ \ \textit{for} \ \ k \in \mathbb{N}, \ 1 \leqslant j \leqslant j_0$$

a.e in K. Then the Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.7) is a tight frame for $L^2(\Omega)$ with $A = 1$.

Proof. For fixed $\ell = 1, 2, \ldots, L$, using Lemma 4.1, we obtain

$$\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b_\ell}T_{u(n)a}\psi^\ell\rangle\right|^2 = \frac{1}{|b_\ell|}\int_K |f(x)|^2 \Big|\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x-u(n)a)\big)\Big|^2\,dx + \sum_{k\in\mathbb{N}_0}\frac{1}{|b_\ell|}\int_K \overline{f(x)}\,\psi^\ell\big(x-u(n)a\big) \times \left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b_\ell^{-1}u(k)\big)\overline{\psi^\ell\big(x+b_\ell^{-1}u(k)-u(n)a\big)}\right)dx \tag{4.24}$$

which implies that,

$$\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b_\ell}T_{u(n)a}\psi^\ell\rangle\right|^2 = \int_K\sum_{\ell=1}^{L}\frac{1}{|b_\ell|}|f(x)|^2\Big|\sum_{n\in\mathbb{N}_0}\psi^\ell\big(x-u(n)a)\big)\Big|^2\,dx + (\blacksquare), \tag{4.25}$$

where

$$(\blacksquare) = \int_K\sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\frac{1}{|b_\ell|}\quad\overline{f(x)}\,\psi^\ell\big(x-u(n)a\big) \left(\sum_{n\in\mathbb{N}}\psi^\ell\big(x+b_\ell^{-1}u(k)\big)\overline{\psi^\ell\big(x+b_\ell^{-1}u(k)-u(n)a\big)}\right)dx. \tag{4.26}$$

On combining (4.26) with (4.22), it follows that

$$\sum_{\ell=1}^{L}\sum_{m\in\mathbb{N}_0}\sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b_\ell}T_{u(n)a}\psi^\ell\rangle\right|^2 = \int_K |f(x)|^2\,dx + (\blacksquare). \tag{4.27}$$

Define

$$\Theta_k^j(x) = \sum_{\ell\in\mathcal{I}_j}\left\{\sum_{m\in\mathbb{N}_0}\psi^\ell\big(x+b^{-1}u(m)\big)\overline{\psi^\ell\big(x+b^{-1}u(k)-u(m)a\big)}\right\}\quad \textit{for}\ \ 1\leqslant j\leqslant j_0 \tag{4.28}$$

Then, from (4.23) we obtain

$$(\blacksquare) = \sum_{k\in\mathbb{N}} \left\{ \int_K \sum_{j=1}^{j_0} \frac{1}{|b_j|} \overline{f(x)} f(x - b_j^{-1} u(k) \Theta_k^j(x) \right\} = 0. \tag{4.29}$$

From this together with (4.27), it follows that

$$\sum_{\ell=1}^{L} \sum_{m\in\mathbb{N}_0} \sum_{n\in\mathbb{N}_0} \left|\langle f, M_{u(m)b_\ell} T_{u(n)a} \psi^\ell \rangle\right|^2 = \int_K |f(x)|^2\, dx = \|f\|, \quad \forall f \in L^2(\Omega).$$

Therefore, the Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.7) is a tight frame for $L^2(\Omega)$ with $A = 1$. $\square$

From the above two theorems, we obtain the following theorem, which is a necessary and sufficient condition for the multigenerator Parseval Gabor frame.

Theorem 4.3. *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \subseteq L^2(\Omega)$, and $a, b > 0$. Then the Gabor system $\mathcal{G}(a, b, \Psi)$ given by (2.7) is a tight frame for $L^2(\Omega)$ with $A = 1$ if and only if*

$$\sum_{\ell=1}^{L} \frac{1}{|b|} \left\{ \sum_{n\in\mathbb{N}_0} \psi^\ell\big(x - u(n)a)\big)|^2 \right\} = \mathbf{1}_\Omega,$$

$$\sum_{\ell=1}^{L} \left\{ \sum_{n\in\mathbb{N}} \psi^\ell\big(x + b^{-1}u(n)\big) \overline{\psi^\ell\big(x + b^{-1}u(k) - u(n)a\big)} \right\} = 0,$$

hold a.e in K.

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Ser. Math. Inform. Vol. 33, No 2 (2018), 325–336
https://doi.org/10.22190/FUMI1802325K

ROUGHLY GEODESIC $B - r-$PREINVEX FUNCTIONS ON CARTAN HADAMARDMANIFOLDS *

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Abstract. In this paper, we introduce a new class of functions called roughly geodesic $B - r-$ preinvex function on a Hadamard manifold and establish some properties of roughly geodesic $B - r-$ preinvex functions on Hadamard manifolds. It is observed that a local minimum point for a scalar optimization problem is also a global minimum point under roughly geodesic B-r- preinvexity on the Hadamard manifolds. The results presented in this paper extend to and generalize the results in the literature.
Keywords: Hadamard manifold, preinvex function, minimum point.

1. Introduction

In mathematics, the concept of convexity is well known and it contributes a fundamental character to engineering, mathematical economics, optimization theory, management science and Riemannian manifolds. One of the most significant applications of the convex function is that any local minimum is also a global minimum. However, convexity does not give accurate results in real world mathematical problems and economic models. For this reason various authors have introduced concepts of generalized convex functions. Initially in 1981 Hanson [11] presented a significant generalization of the convex function which was later known as an invex function. Further, the convex set and the convex function were generalized by Ben-Israel and Mond [10], and called invex set and preinvex function, correspondingly. The characterization of preinvex functions and its applications in optimization theory have been discussed in [14, 27]. Noor [19, 23] studied the equilibrium problems and variational inequalities under these functions. Many articles have appeared in the literature on preinvex functions (see, [1, 2, 3, 5, 9, 12, 16, 20, 28]).

Received December 19, 2017; accepted May 15, 2018
2010 *Mathematics Subject Classification.* Primary 58E17; Secondary 90C26.
*This work is supported by the Dean ship of Scietific Resaerch, University of Tabuk under the project number: 0114-1438-S. . .

Few results related to optimization theory and nonlinear analysis have been enhanced on Riemannian manifolds from the Euclidean space. Geodesic convexity proposed by Rapcsak [25] and Udriste [26], which is a natural generalization of convexity in which linear space is exchanged by Riemannian manifolds. Furthermore, on a Riemannian manifold the concept of invexity was introduced by Pini [24] and its generalization was explored by Mititelu [18]. On a Riemannian manifold, the geodesic invex set, geodesic η-preinvex function and geodesic η-invex function have been explained by Barani and Pouryayevali [9] and they have also discussed the relationships between these functions. Moreover, the geodesic α-preinvex function is a generalization of the notion of geodesic η-preinvexity introduced by R.P. Agrawal et al. [1]. Recently, the notions of B-invex set and B-invex function were studied on Riemannian manifolds by Zhou and Huang [28].

Analyzing the discussion of Barani and Pouryayevali [9] and Zhou and Huang [28], we attempt to deliberate the notions of geodesic $B-r$-invex set and geodesic $B-r$-preinvex function on a Riemannian manifold. These functions are a generalization of the preinvex function defined in [9, 5, 28]. Barani [8] presented the convexity and monotonicity of set valued mapping on Hadamard manifolds and presented the mean value theorem. Zou et al. [32] introduced the classical Penot generalized directional derivative and Clarke's generalized gradient and used these to discuss the first and second order necessary and sufficient conditions for a minimum point of the nonlinear programming problem. Recently, Jana and Nahak [13] obtained the optimality conditions for the nonlinear optimization problem under generalized invexities on a Riemannian manifold.

The paper is divided sectionally in the following way. In Section 2, we recall specific preliminaries, definitions and results, which are applied to demonstrate the work of this paper. We derive a new class function, namely, geodesic $B-r$-preinvex and geodesic $B-r$-invex function in Section 3. Some properties and relations between the geodesic log-preinvex and the geodesic $B-r$-invex function on a Riemannian manifold are studied in Section 4. Moreover, in Section 5, we discuss the results based on a lower semi-continuous log-preinvexity function with a proximal sub-differential and observe that for a mathematical optimization problem with log-preinvexity on a Riemannian manifold, its local minimum point is also a global minimum point. Finally, we obtain the mean value inequality on Hadamard manifolds in Section 6 and discuss the conclusions of the paper in Section 7.

2. Preliminaries

The present paper is based on the concept of generalized convexity on Cartan Hadamard manifolds. The objective of this paper is to present a new notion of roughly geodesic $B-r-$ invex and roughly geodesic $B-r-$ preinvex functions, and some properties and relationships between these functions are discussed. First of all, we apply the smoothness condition on a roughly geodesic $B-r-$ preinvex function with lower semi-continuity and try to obtain the existence condition for a global minimum. Finally, we obtain the mean value inequality for $B-r-$ preinvex function on Cartan Hadamard manifolds.

To recall some basic definitions of and results for Riemannian manifolds for further study, we refer the reader to ([9], [17], [28], [24]) and references therein.

Definition 2.1. A simply connected complete Riemannian manifold with a non-positive sectional curvature is called a Hadamard manifold. On a Hadamard manifold $\bar{M}$, there exists an exponential mapping $exp_p : T_p\bar{M} \to \bar{M}$ such that $exp_p v = \gamma_v(1)$, where γ_v is a geodesic defined by its position p and velocity γ at p.

Lemma 2.1.(Cartan-Hadamard theorem) Suppose X be a connected complete metric space which is locally convex. Then, with respect to the induced length metric d, the universal cover of X is a geodesic convex space. Let $(\bar{M}, \langle .,. \rangle)$ be a Hadamard manifold with a Riemannian metric $\langle .,. \rangle$. For a subset $U \subset \bar{M}$, a mapping $\eta \times \eta \to T\bar{M}$ is a function such that for every $u, v \in U$, $\eta(u,v) \in T\bar{M}$.

Definition 2.2. The geodesic distance $d(u,v)$ is the length of a minimal geodesic segment between any two points u, v on a manifold.

Definition 2.3. For a mapping $\psi : U \to R$, if the following limit

$$limit_{\lambda \to 0} \frac{\psi(exp_v \lambda\eta(u,v)) - \psi(v)}{\lambda\|\eta(u,v)\|},$$

exists, then ψ is said to be a $\eta(u,v)-$differentiable mapping at $v \in \bar{M}$,

Moreover, the $\eta(u,v)-$differential of ψ at v is given by

$$d_{\eta(u,v)}\psi(v) = limit_{\lambda \to 0} \frac{\psi(exp_v \lambda\eta(u,v)) - \psi(v)}{\lambda\|\eta(u,v)\|}.$$

Definition 2.4[9]. On a Riemannian manifold $\bar{M}$ for the function $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ such that $\eta(u,v) \in T_v\bar{M}$, for every $u, v \in \bar{M}$. A non-empty subset U of $\bar{M}$ is said to be a geodesic invex set with respect to η if for every $u, v \in U$, there exists a unique geodesic $\gamma_{u,v} : [0,1] \to \bar{M}$ such that

$$\gamma_{u,v}(0) = v, \quad \gamma'_{u,v}(0) = \eta(u,v), \quad \gamma_{u,v}(s) \in U, \quad for\ all\ s \in [0,1].$$

3. Results and Discussion

The present section is divided into three subsections.

3.1. Geodesic B-invex sets and roughly geodesic $B-r-$preinvex functions

Convexity and its generalizations play an important role in the development of optimality conditions and duality theory. Various generalizations of convexity have appeared in literature. In [10], Ben-Israel and Mond introduced a new generalization of the convex function. Craven [29] named the invex function. Antczak [5] presented a generalization of the V-invex function [15] and the $r-$invex function [4], called a V-r-invex function. Bector and Singh [30] and Suneja et al. [31] introduced B-vex and B-preinvex functions. Recently, Zhou and Huang [28] defined the geodesic $B-$invex set as follows:

Definition 3.1 [28]. The set U is said to be a geodesic $B-$invex set on a Hadamard manifold with respect to η and $b(u,v,\lambda) : U \times U \times [0,1] \to R_+$, if for all $u, v \in U$ and $\lambda \in [0,1]$ such that $exp_v \lambda b\eta(u,v) \in U$.

U is said to be a geodesic $B-$invex set with respect to η on a Hadamard manifold, if U is $B-$invex for all $u, v \in U$ on a Hadamard manifold with respect to η.

Definition 3.2 [28]. Let U be a geodesic $B-$invex set. Then a mapping $\psi : U \to T\bar{M}$ is said to be a roughly geodesic B-preinvex function with respect to η with a roughness degree ρ at $v \in U$, if there exists $b(u,v,\lambda) : U \times U \times [0,1] \to R$ such that

$$f(exp_v \lambda b\eta(u,v)) \leq \lambda b\psi(u) + (1-\lambda b)\psi(v),$$

for all $u \in U$ and $\lambda \in [0,1]$ with $d(u,v) \geq \rho$, ψ is said to be a roughly geodesic $B-$preinvex function on U with respect to η, if it is a roughly geodesic B-preinvex function at any $v \in U$ with respect to the same η on U.

Now we introduce a roughly geodesic $B-r-$preinvex function on $\bar{M}$.

Definition 3.3. For a geodesic $B-$invex set U, the mapping $\psi : U \to T\bar{M}$ is said to be a roughly geodesic $B-r-$preinvex function with respect to η with a roughness degree ρ at $v \in U$, if there exists $b(u,v,\lambda) : U \times U \times [0,1] \to R_+$ such that

$$\psi(exp_v \lambda b\eta(u,v)) \leq \begin{cases} log(\lambda b e^{r\psi(u)} + (1-\lambda b)e^{r\psi(v)})^{\frac{1}{r}} & \text{if } r \neq 0, \\ \lambda b\psi(u) + (1-\lambda b)\psi(v) & \text{if } r = 0. \end{cases}$$

for any $u \in S$ and $\lambda \in [0,1]$ with $d(u,v) \geq \rho$. ψ is said to be a roughly geodesic $B-r-$preinvex function on U with respect to η and b, if it is a roughly geodesic $B-r-$preinvex function at any $v \in U$ with respect to η on U.

The function ψ is called a strictly roughly geodesic $B-r-$ preinvex function, if the above inequality holds as a strict inequality.

Remark 3.1. Every roughly geodesic $B-$preinvex function and geodesic $\eta-$preinvex function are a roughly geodesic $B-r-$preinvex function for $r=0$ and $b=1$, respectively. However, the converse does not hold in general.

We provide the following non-trivial example for the geodesic $B-r-$preinvex function but not a geodesic $B-$ preinvex function.

Example 3.1. Let $\bar{M}=\{e^{i\theta}:0<\theta<1\}$ and $\psi:\bar{M}\to R$ defined by $\psi(e^{i\theta})=\cos\theta$ with $u,v\in\bar{M}$, $u=e^{i\alpha}$ and $v=e^{i\beta}$. If $exp_v\lambda b\eta(u,v))=e^{i((1-\lambda b)\beta+\lambda b\alpha)}$, then ψ is a geodesic $B-r-$preinvex function but not a geodesic $B-$preinvex function at $\alpha=\frac{\pi}{2}$, $\beta=\frac{\pi}{4}, b=2$, since $\cos[\frac{\pi}{4}+\frac{\pi}{4}2\lambda]>\frac{1-2\lambda}{\sqrt{2}}$ for $\lambda=\frac{3}{4}$.

Proposition 3.1. If $\psi:U\to R$ is a roughly $B-r-$preinvex function with respect to $\eta:U\times U\to T\bar{M}$ and $v\in U$, then for any real number $k\in R$, the level set $U_k=\{u|u\in U,\psi(u)\leq k\}$ is a geodesic $B-$invex set.

Proof. For any $u,v\in U_k$, we have $\psi(u)\leq k$, $\psi(v)\leq k$. Since ψ is a roughly geodesic $B-r-$preinvex function, then we have

$$\psi(exp_v\lambda b\eta(u,v))\leq log(\lambda be^{r\psi(u)}+(1-\lambda b)e^{r\psi(v)})^{\frac{1}{r}},$$

or

$$\begin{aligned} e^{r\psi(exp_v\lambda b\eta(u,v))} &\leq \lambda be^{r\psi(u)}+(1-\lambda b)e^{r\psi(v)} \\ &\leq \lambda be^{rk}+(1-\lambda b)e^{rk}. \end{aligned}$$

Equivalently,

$$e^{r\psi(exp_v\lambda b\eta(u,v))}\leq e^{rk},$$

or

$$\psi(exp_v\lambda b\eta(u,v)\leq k.$$

Therefore, $exp_v\lambda b\eta(u,v)\in U_k$ for all $\lambda\in[0,1]$, and the result is proved.

Theorem 3.1. Let U be a geodesic $B-$invex set and let $\psi:U\to R$ be a roughly geodesic $B-r-$preinvex with respect to $\eta:U\times U\to T\bar{M}$ with a roughness degree ρ on U. Then $epi(\psi)$ is a $B-$invex set on $U\times R$.

Proof. Let ψ be a roughly geodesic $B-r-$preinvex function with respect to $\eta:U\times U\to T\bar{M}$ with a roughness degree ρ on U. Then there exists $b(u,v,\lambda):U\times U\times[0,1]\to R_+$ such that

$$\psi(exp_v\lambda b\eta(u,v))\leq log(\lambda be^{r\psi(u)}+(1-\lambda b)e^{r\psi(v)})^{\frac{1}{r}},$$

where $exp_v\lambda b\eta(u,v)\in U$ and $d(u,v)\geq\rho$. Assume that $(u,\alpha),(v,\beta)\in epi(\psi)$. Then it is easy to see that $\psi(u)\leq\alpha,\psi(v)\leq\beta$, from these observations we have

$$\begin{aligned} \psi(exp_v\lambda b\eta(u,v)) &\leq log(\lambda be^{r\psi(\alpha)}+(1-\lambda b)e^{r\psi(\beta)})^{\frac{1}{r}} \\ &= log(e^{r\beta}+(e^{r\alpha}-e^{r\beta})\lambda b)^{\frac{1}{r}}. \end{aligned}$$

Therefore,

$$(exp_v\lambda b\eta(u,v), log(e^{r\beta} + (e^{r\alpha} - e^{r\beta})\lambda b)^{\frac{1}{r}}) \in epi(\psi),$$

which implies that $epi(\psi)$ is a $B-$invex set on $U \times R$.

Theorem 3.2. If $\phi_i : U \to R, i = 1, 2, \ldots, m$ are roughly geodesic $B - r-$preinvex functions with respect to the same $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ with a roughness degree ρ on U, then the set defined by $\bar{M} = \{u \in U : \phi_i(u) \leq 0, i = 1, 2, \ldots, m\}$ is a geodesic $B-$ invex set with respect to η.

Proof. Since $\phi_i(u), i = 1, 2, \ldots, m$, are roughly geodesic $B - r-$preinvex functions, then there exists $b(u, v, \lambda) : \bar{M} \times \bar{M} \times [0, 1] \to R_+$, such that

$$\phi_i(exp_v\lambda b\eta(u,v)) \leq log(\lambda b e^{r\phi_i(u)} + (1-\lambda b)e^{r\phi_i(v)})^{\frac{1}{r}} \leq log(\lambda b e^0 + (1-\lambda b)e^0)^{\frac{1}{r}},$$

or equivalently,

$$\phi_i(exp_v\lambda b\eta(u,v)) \leq 0, \quad i = 1, 2, \ldots, m,$$

and so $exp_v\lambda b\eta(u,v) \in \bar{M}$. Thus, $\bar{M}$ is a geodesic $B-$ invex set.

Theorem 3.3. Let U be a geodesic B-invex set . If $\psi : U \to R$ is an $\eta-$differentiable roughly geodesic $B - r-$preinvex function with respect to $\eta : U \times U \to R$ with a roughness degree ρ at $v \in U$. Then there exists a function $\bar{b}(u,v) : U \times U \to R_+$ such that

$$\|\eta(u,v)\|d_{\eta(u,v)}\psi(v) \leq \frac{\bar{b}(u,v)}{r}e^{-r\psi(v)}(e^{r\psi(u)} - e^{r\psi(v)}),$$

for each $u \in U$ with $d(u,v) \geq \rho$ and $\bar{b}(u,v) = lim_{\lambda\to 0}b(u,v,\lambda)$.

Proof. If ψ is a roughly geodesic $B - r-$preinvex function at v, then there exists $b(u,v,\lambda) : U \times U \times [0,1] \to R_+$ such that

$$\psi(exp_v\lambda b\eta(u,v)) \leq log[\lambda b e^{r\psi(u)} + (1-\lambda b)e^{r\psi(v)}]^{\frac{1}{r}},$$

for each $u \in U$ and $\lambda \in [0,1]$ with $d(u,v) \geq \rho$. Since ψ is $\eta-$ differentiable at v, we have

$$d_{\eta(u,v)}\psi(v) = lim_{\lambda\to 0}\frac{\psi(exp_v\lambda b\eta(u,v)) - \psi(v)}{\lambda\|\eta(u,v)\|},$$

and so

$\psi(v) + d_{\eta(u,v)}\psi(v)\lambda\|\eta(u,v)\| + O^2(\lambda b) = \psi(exp_v\lambda b\eta(u,v)) \leq log[\lambda b e^{r\psi(u)} + (1 - \lambda b)e^{r\psi(v)}]^{\frac{1}{r}}$,

or

$$e^{r\psi(v)+rd_{\eta(u,v)}\psi(v)\lambda\|\eta(u,v)\|+rO^2(\lambda b)} - e^{r\psi(v)} \leq \lambda b(e^{r\psi(u)} - e^{r\psi(v)}).$$

Dividing by λ and taking the limit $\lambda \to 0$, we get

$$\|\eta(u,v)\| d_{\eta(u,v)}\psi(v) \le \frac{\bar{b}(u,v)}{r} e^{-r\psi(v)}(e^{r\psi(u)} - e^{r\psi(v)}).$$

Remark 3.2. If $r = 0$, then the result in the above Theorem is similar to the result of Theorem 4.4 [28].

3.2. Roughly Geodesic $B - r-$ Preinvexity and semi continuity

Now we discuss geodesic $B-r-$preinvexity on a Cartan Hadamard manifold under a proximal subdifferential of a lower semi-continuous function. First, we recall the definition of the proximal subdifferentiable of a function defined on a Riemannian manifold [9].

Definition 3. 4. Let $\bar{M}$ be a Riemannian manifold and let $\psi : \bar{M} \to (-\infty, \infty]$ be a lower semi-continuous function. A point $\xi \in T_v\bar{M}$ is said to be the proximal subgradient of ψ at $v \in dom(\psi)$, if there exist positive numbers δ and σ such that

$$\psi(u) \ge \psi(v) + < \xi, exp_v^{-1}u >_v - \sigma d^2(u,v),$$

for all $u \in B(v, \delta)$, where $dom\psi = \{u \in \bar{M} : \psi(u) < \infty\}$. The set of all proximal subgradients of $v \in \bar{M}$ is denoted by $\partial_p\psi(v)$.

Theorem 3.4. Let $\bar{M}$ be a Hadamard manifold and U be a geodesic $B-$invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ and $b(u,v,\lambda) : U \times U \times [0,1] \to R$. Let $\psi : U \to R$ be a roughly geodesic $B - r-$ preinvex function. If $\bar{u} \in U$ is a local minimum of the problem

$$\text{(P)} \qquad \text{Minimize } \psi(u), \quad \text{subject to } u \in U,$$

then $\bar{u}$ is a global minimum of (P).

Proof. If $\bar{u} \in U$ is a local minimum, then there exists a neighbourhood $N_\epsilon(\bar{u})$ such that

$$\psi(\bar{u}) \le \psi(u), \tag{3.1}$$

for all $u \in U \cap N_\epsilon(\bar{u})$.

If $\bar{u}$ is not a global minimum of ψ, then there exists a point $u^* \in U$ such that

$$\psi(u^*) < \psi(\bar{u}),$$

or

$$e^{r\psi(u^*)} < e^{r\psi(\bar{u})}.$$

As U is a geodesic $B-$invex set with respect to η and b, there exists a unique geodesic $\gamma(\lambda b) = exp_v(\lambda b\eta(u^*, \bar{u}))$ such that $\gamma(0) = \bar{u}$, $\gamma'(0) = \bar{b}(u^*, \bar{u})\eta(u^*, \bar{u})$ where $lim_{\lambda\to 0}b(u^*, \bar{u}, \lambda) = \bar{b}(u^*, \bar{u})$, $exp_v(\lambda b\eta(u^*, \bar{u})) \in U$, for all $\lambda \in [0, 1]$. If we choose $\epsilon > 0$ such that $d(\gamma(\lambda b), \bar{u}) < \epsilon$, then $\gamma(\lambda b) \in N_\epsilon(\bar{x})$. From the roughly geodesic $B - r-$ preinvexity of ψ, we have

$$\psi(exp_v\lambda b\eta(u^*, \bar{u})) \leq log(\lambda be^{r\psi(u^*)} + (1 - \lambda b)e^{r\psi(\bar{u})})^{\frac{1}{r}}.$$

Equivalently, we have

$$e^{r\psi(exp_v\lambda b\eta(u^*,\bar{u}))} \leq \lambda be^{r\psi(u^*)} + (1 - \lambda b)e^{r\psi(\bar{u})} < \lambda be^{r\psi(\bar{u})} + (1 - \lambda b)e^{r\psi(\bar{u})} = e^{r\psi(\bar{u})},$$

or

$$\psi(exp_v\lambda b\eta(u^*, \bar{u})) < \psi(\bar{u}), \quad for\ all\ \lambda \in (0, 1].$$

Therefore, for each $exp_v\lambda b\eta(u^*, \bar{u})) \in U \cap N_\epsilon(\bar{u})$, $\psi(exp_v\lambda b\eta(u^*, \bar{u}))) < \psi(\bar{u})$, which is a contradiction of (3.1). Hence the result.

Theorem 3.5. Let $\bar{M}$ be a Cartan-Hadamard manifold and U be a geodesic $B-$invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ with $\eta(u, v) \neq 0$ for all $u \neq v$. Assume that $\psi : U \to (-\infty, \infty]$ is lower semi-continuous roughly geodesic $B - r-$ preinvex function and $v \in dom(\psi)$, $\xi \in \partial_p\psi(v)$. Then there exists a positive number δ such that

$$e^{r\psi(u)} - e^{r\psi(v)} \geq e^{r\psi(v)} < \xi, \eta(u, v) >_v, \quad for\ all\ u \in U \cap B(v, \delta).$$

Proof. From the definition of $\partial_p\psi(v)$, there are positive numbers δ and σ such that

$$\psi(u) \geq \psi(v) + < \xi, exp_v^{-1}u >_v -\sigma d^2(u, v), \quad for\ all\ u \in B(v, \delta). \tag{3.2}$$

Now, fix $u \in U \cap B(v, \delta)$. Since U is a geodesic $B-$invex set with respect to η, there exists a unique geodesic $\gamma_{u,v}(\lambda b) = exp_v(\lambda b\eta(u, v)) : [0, 1] \to \bar{M}$ such that $\gamma_{u,v}(0) = v$, $\gamma'_{u,v}(0) = b\eta(u, v)$, $\gamma_{u,v}(\lambda b) \in U$, for all $\lambda \in [0, 1]$. If we choose $\lambda_0 = \frac{\delta}{\|\eta(u,v)\|_v}$, then $exp_v(\lambda b\eta(u, v)) \in U \cap B(v, \delta)$ for all $\lambda \in [0, \lambda_0)$.
From the roughly geodesic $B - r-$preinvexity of ψ, we get

$$\psi(exp_v(\lambda b(\eta(u, v)) \leq log(\lambda be^{r\psi(u)} + (1 - \lambda b)e^{r\psi(v)})^{\frac{1}{r}},$$

or

$$e^{r\psi(exp_v(\lambda b(\eta(u,v))} \leq \lambda be^{r\psi(u)} + (1 - \lambda b)e^{r\psi(v)}. \tag{3.3}$$

Using (3.1) for each $\lambda \in (0, \lambda_0)$, we get

$$\begin{aligned}\psi(exp_v(\lambda b(\eta(u, v)) &\geq \psi(v) + < \xi, exp_v^{-1}exp_v(\lambda b\eta(u, v)) >_v -\sigma d^2(exp_v(\lambda b\eta(u, v), v)\\ &= \psi(v) + \langle\xi, \lambda b\eta(u, v)\rangle_v - \sigma d^2(exp_v(\lambda b(\eta(u, v)), v).\end{aligned}$$

Since $\bar{M}$ is a Cartan-Hadamard manifold for each $\lambda \in (0, \lambda_0)$, we have

$$d^2(exp_v(\lambda b\eta(u,v), v) = \|\lambda b\eta(u,v)\|_v^2 = \lambda^2 b^2 \|\eta(u,v)\|_v^2.$$

Thus, we have

$$\psi(exp_v(\lambda b(\eta(u,v)) \geq \psi(v) + \langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2 \|b\eta(u,v)\|_v^2,$$

or

$$e^{r\psi(exp_v(\lambda(\eta(u,v))} \geq e^{r\psi(v)} e^{\langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2 \|b\eta(u,v)\|_v^2}. \tag{3.4}$$

Inequalities (3.3) and (3.4) give

$$\lambda b e^{r\psi(u)} + (1-\lambda b)e^{r\psi(v)} \geq e^{r\psi(v)} e^{\langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2 \|b\eta(u,v)\|_v^2}.$$

By further calculation we arrive at

$$b(e^{r\psi(u)} - e^{r\psi(v)}) \geq e^{r\psi(v)} \frac{1}{\lambda}[e^{\langle \xi, \lambda b\eta(u,v)\rangle_v - \sigma\lambda^2 \|b\eta(u,v)\|_v^2} - 1],$$

taking the limit $\lambda \to 0$

$$e^{r\psi(u)} - e^{r\psi(v)} \geq e^{r\psi(v)} \langle \xi, \eta(u,v)\rangle_v.$$

Which proves the theorem completely.

3.3. Mean value inequality

In the present sub-section, we obtain the mean value inequality for $B-r-$preinvex function defined on a Cartan Hadamard manifold.

In the continuation of Definition 2.3, we have the following definition.

A set P_{xy} is said to be a closed $\eta-$path joining the points x and $y = \gamma(1)$, if

$$P_{xy} = \{v : v = \gamma(s), s \in [0,1]\}.$$

An open $n-$path connecting the points x and y is a set of the type

$$P_{xy}^0 = \{v : v = \gamma(s), s \in (0,1)\}.$$

If $x = y$, then $P_{xy}^0 = \phi$.

Theorem 3.6. Let $\bar{M}$ be a Cartan Hadamard manifold and U be a geodesic $B-$invex set with respect to $\eta : \bar{M} \times \bar{M} \to T\bar{M}$ such that $\eta(x,y) \neq 0$ for all

$x, y \in U, x \neq y$. Let $\gamma_{y,x}(s) = exp_x(sb\eta(y,x))$ for all $x, y \in U$, $s \in [0,1]$ and $z = \gamma_{x,y}(1)$. Then the function $\psi : U \to R$ is to be a geodesic $B - r-$preinvex if and only if the following inequality

$$e^{r\psi(u)} \leq e^{r\psi(x)} + \frac{e^{r\psi(y)} - e^{r\psi(x)}}{\langle \eta(y,x), \eta(y,x)\rangle_x} \langle exp_x^{-1} u, \eta(y,x)\rangle_x. \tag{3.5}$$

holds, for all $u \in P_{zx}$.

Proof. Let $\psi : U \to R$ be a $B - r-$preinvex function. If $u = x$ or $u = z$. Then the inequality (3.5) is true trivially. If $u \in P_{zx}$ then $u = exp(sb\eta(y,x))$, for $s \in [0,1]$. Since U is a $B-$invex set, then for $u \in U$, we have

$$s = \frac{\langle exp_x^{-1} u, \eta(y,x)\rangle_x}{b\langle \eta(y,x), \eta(y,x)\rangle_x}.$$

By the $B - r-$invexity of ψ, we have

$$\psi(u) = \psi(exp_x(sb\eta(y,x))) \leq log(sbe^{r\psi(y)} - (1 - sb)e^{r\psi(x)})^{\frac{1}{r}},$$

or

$$\begin{aligned} e^{r\psi(u)} &\leq sbe^{r\psi(y)} + (1 - sb)e^{r\psi(x)} \\ &= e^{r\psi(x)} + sb(e^{r\psi(y)} - e^{r\psi(x)}). \end{aligned}$$

Utilizing the value of s we get the required inequality.

Conversely, suppose the inequality (3.5) is true. Let $x, y \in U$ and $u = exp_x(sb\eta(y,x))$, for some $s \in [0,1]$. Then for $u \in U$, we have $\psi(u) = \psi(exp_x(sb\eta(y,x)))$. From (**??**)

$$\begin{aligned} e^{r\psi(u)} &\leq e^{r\psi(x)} + \frac{e^{r\psi(y)} - e^{r\psi(x)}}{\langle \eta(y,x), \eta(y,x)\rangle_x} \langle exp_x^{-1} u, \eta(y,x)\rangle_x \\ &= e^{r\psi(x)} + \frac{e^{r\psi(y)} - e^{r\psi(x)}}{\langle \eta(y,x), \eta(y,x)\rangle_x} \langle exp_x^{-1} exp_x(sb\eta(y,x), \eta(y,x)\rangle_x \\ &= sbe^{r\psi(y)} + (1 - sb)e^{r\psi(x)}, \end{aligned}$$

or

$$\psi(u) \leq log(sbe^{r\psi(y)} + (1 - sb)e^{r\psi(x)})^{\frac{1}{r}}.$$

Equivalently,

$$\psi(exp_x(sb\eta(y,x)) \leq log(sbe^{r\psi(y)} + (1 - sb)e^{r\psi(x)})^{\frac{1}{r}},$$

which shows that ψ is a geodesic $B - r-$preinvex function on U.

Remark 3.2. If $r = 0$ and $b = 1$, then the mean value inequality becomes the inequality proved in [9].

4. Conclusion

In the present paper, we have defined the concept of the roughly geodesic $B-r-$ preinvex function on a Riemannian manifold. This function generalizes the preinvex functions defined in ([1], [2], [3], [4], [9], [16], [24]). Further, we have proved that a local minimum point is also a global minimum point for a scalar optimization problem under the aforesaid function. Finally, the mean value inequality is also proved involving a geodesic $B-r$ preinvex function. This inequality generalizes the inequality obtained in ([1], [9]). As a future work, the findings of this paper can be utilized for multiobjective mathematical problems on Riemannian manifolds.

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Ser. Math. Inform. Vol. 33, No 2 (2018), 337–360
https://doi.org/10.22190/FUMI1802337M

A BRANCH-AND-BOUND ALGORITHM FOR A PSEUDO-BOOLEAN OPTIMIZATION PROBLEM WITH BLACK-BOX FUNCTIONS *

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Abstract. We consider a conditional pseudo-Boolean optimization problem with both the objective function and all constraint functions given algorithmically (black-box functions) and defined on $\{0,1\}^n$ only. We suppose that these functions have certain properties, for example, unimodality and monotonicity. To solve problems of this type, we propose an optimization algorithm based on finding boundary points of the feasible region and the branch-and-bound method. The developed algorithm is aimed at the reception of an exact solution of an optimization problem. In addition, this algorithm can be used as an improvement of approximate algorithms such as the greedy heuristic and the random search algorithms for finding boundary points. Even after a small number of iterations (branchings), a significant improvement of the found feasible solution is achieved.
Keywords: Pseudo-Boolean optimization problem, branch-and-bound method, Constrained pseudo-Boolean optimization problem.

1. Introduction

In the optimization model construction, many problems are naturally formalized as pseudo-Boolean optimization problems. A typical formulation of a pseudo-Boolean optimization problem is as follows. Let $X = (x_1, \ldots, x_n)$ be a set of n independent binary variables and $f(X)$ be a real-valued function to be optimized: $f : S \to \mathbb{R}$, where $S \subset \{0,1\}^n$ is a subregion of Boolean variables space defined by a given system of constraints imposed on the values of variables X.

If $S = \{0,1\}^n$, that is, no constraints are imposed on the choice of variables $x_1, \ldots, x_n$ then such a problem is called an unconstrained pseudo-Boolean optimization problem. For its solution, in [2], exact algorithms based on the detection of the optimized function behavioural features in binary variables space are worked

Received March 06, 2018; accepted June 11, 2018
2010 *Mathematics Subject Classification.* Primary 90C57; Secondary 90C27, 90C09
*Results were obtained in the framework of the state task 2.5527.2017/8.9 of the Ministry of Education and Science of the Russian Federation

out. These features were used to construct and justify effective exact algorithms. In particular, an exact optimization algorithm demanding $n + 1$ calculations of a function was developed for a strictly monotone pseudo-Boolean function.

A special feature of these algorithms is that they do not require algebraic definition of an objective function. It may be so called a black-box function. We can calculate the value of the function in points $\{0, 1\}^n$ only. The issue of constructing algorithms for solving pseudo-Boolean optimization problems with black-box functions are considered in this paper.

The most "primitive" way to find the exact solution of a pseudo-Boolean optimization problem is to search all possible combinations of values of binary variables. The number of such combinations is equal to 2^n. For a lot of real problems it is unacceptable. To decrease the number of calculations, unpromising combinations of values of variables (such combinations form the subregions of the original space of binary variables) should not be considered. But to reveal them it is necessary to know the properties of the function, that is, the behaviour of the function on the points (combinations of variables). Such approaches as the dynamic programming method [6] and the branch-and-bound method [17] are based on the exclusion of sets of unpromising alternatives.

Many practical problems of choice are formalized as pseudo-Boolean optimization problems with constraints on choosing a combination of the variables; in this case in the behaviour of the objective function and constraints there are peculiarities which allow one to construct acceptable algorithms to find the exact solution. It is the problem of construction of such algorithms for a widespread class of problems that is considered in this paper.

In this paper, we propose an algorithm for solving problems of conditional pseudo-Boolean optimization based on the branch-and-bound scheme and using properties of functions of the optimization model for estimating upper bounds and eliminating unpromising solutions. The branch-and-bound method was originally developed to solve integer linear programming problems [17]. Then, based on this scheme, algorithms were developed to solve special classes of problems, such as nonlinear programming problems [1, 15, 25, 26], the traveling salesman problem [11, 21, 22], facility location [10, 20, 24], network design [8, 14, 16].

In addition, various modifications of the original branch-and-bound algorithm have been developed, combining the branch-and-bound principles with other techniques, such as cutting planes [11, 19, 21, 26], column generation [5, 9, 12], genetic and evolutionary algorithms [7, 13, 23].

2. Problem statement and basic notions

2.1. Constrained pseudo-Boolean optimization problem

Let us consider the problem of the following form:

$$C(X) \to \max_{X \in B_2^n}, \tag{2.1}$$

$$A_j(X) \leqslant H_j, j = 1, \dots, m, \tag{2.2}$$

where $B_2^n = \{0,1\}^n$ is a space of binary variables, $C(X)$ and $A_j(X)$ are pseudo-Boolean functions (real-valued functions of binary variables) which are generally defined implicitly (algorithmically).

To describe the proximity of the vectors (points in space B_2^n), we shall apply the notion of neighborhood [2]. Two points $X_1, X_2 \in B_2^n$ are *k-neighboring* if they differ in values of k coordinates. Let the set of all points k-neighbouring to point X be called the level of some point X and denoted as $O_k(X)$. The level $O_1(X)$ can be presented as the neighborhood of point X.

Point $X^* \in B_2^n$ is called the *local minimum* of the pseudo-Boolean function f, if $f(X^*) < f(X)$ for all $X \in O_1(X^*)$. The notion of a local maximum is introduced similarly. If a function has the only point of a local minimum (maximum), it is often called *unimodal*.

In many works devoted to pseudo-Boolean functions special classes of functions with definite properties are considered. In this paper we shall consider a class of monotone functions which are rather often met in practical problems.

The unimodal function f is called *monotone* on B_2^n if for each $X^k \in O_k(X^*)$ $(k = 1, \dots, n)$ the following condition is met: $f(X^{k-1}) \leqslant f(X^k)$ for all $X^{k-1} \in O_{k-1}(X^*) \cap O_1(X^k)$, where X^* is a local minimum of the function. That is, a function is a monotone one if it does not decrease while moving away from the point of minimum. If a sign of inequality is strict, then the function is strictly monotone.

It is easy to show that if the function is strictly monotone then the only points of local minimum and maximum differ in the value of n coordinates.

Let us take two points $Y, Z \in B_2^n$ the values of some coordinates in which coincide: $y_i = z_i, i \in A \subset \{1, \dots, n\}$; $y_j \neq z_j, j \notin A$. Let a set of all points X the values of whose variables with $i \in A$ index are fixed and equal to $x_i = y_i = z_i$ and the values of all the rest variables can take any values, be called a *subcube* $K(Y, Z)$ (Figure 2.1). In [2] subcube $K(Y, Z)$ is introduced as the union of the shortest paths from Y to Z.

2.2. Properties of a set of feasible solutions

Let us introduce some notions for points placed in a binary space in a particular way [4].

- Point $Y \in A$ is a *boundary point* of set A if $\exists X \in O_1(Y)$, such that $X \notin A$.
- Point $Y \in O_i(X^0) \cap A$ is called a *limiting point* of set A with reference point $X^0 \in A$ if $X \notin A$ for any $X \in O_1(Y) \cap O_{i+1}(X^0)$ (Figure 2.2).

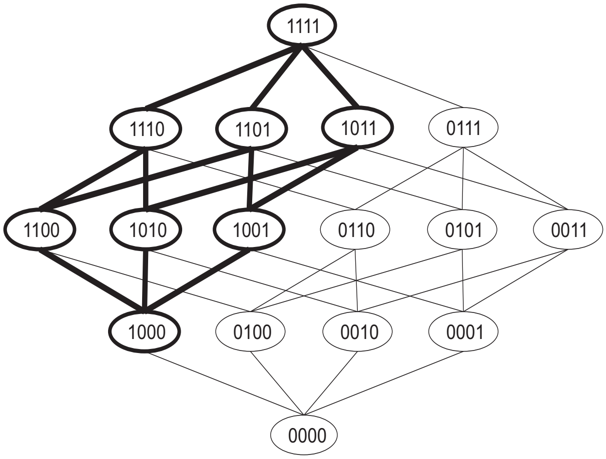

FIG. 2.1: An example of a subcube in the binary space

- Let the constraint which determines the subregion of Boolean variables space be called *active* if the optimal solution of the constrained optimization problem does not coincide with the optimal solution of a corresponding optimization problem without regard to the constraint. In other words, a constraint is active if the optimal solution of an unconstrained problem is unfeasible for a problem with the constraint.

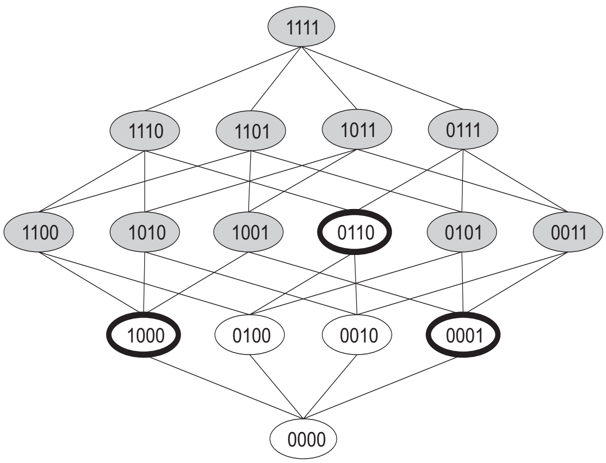

FIG. 2.2: An example of limiting points

One of the properties of a feasible set of solutions looks like this:

Let us consider the problem (2.1)-(2.2). If the objective function is a monotone unimodal function and the constraint is active, then the optimal solution of the problem will be the point belonging to the subset of limiting points of the set S of feasible solutions with reference point X^0 in which the objective function possesses the minimum value:

$$C(X^0) = \min_{X \in B_2^n} C(X).$$

Also it is not difficult to show that if the constraint function (2.2) is an unimodal pseudo-Boolean function then the set of feasible solutions S of the problem is a connected set.

3. Class of monotone pseudo-Boolean functions

Let us consider a class of problems of the following form

$$C(X) \to \max_{X \in B_2^n},$$
$$A(X) \leqslant H,$$

where the objective function $C(X)$ and the function $A(X)$ determining the system of constraints belong to the class of monotone pseudo-Boolean functions.

Let us note some properties of classes of unimodal and monotone pseudo-Boolean functions which form the considered class of problems. In optimization algorithms construction it is necessary to take these properties into account.

First of all, let us consider a following property that will be used later on. On the basis of the definitions of a subcube and monotonicity of a pseudo-Boolean function it can be argued that if a function f increases steadily from $X^0 \in B_2^n$ then for any point $Y \in B_2^n$ is fulfilled:

a) $f(X) \leqslant f(Y)$ for all $X \in K(X^0, Y)$;

b) $f(X) \geqslant f(Y)$ for all $X \in K(Y, X^1)$, where $X^1 = (1 - x_1^0, \ldots, 1 - x_n^0) \in O_n(X^0)$.

3.1. Properties of constraint functions

Unimodal constraint function

Let us consider a constraint function $A(X)$ which has the unique minimum in the point $X^0 \in B_2^n$. Let us denote $X^1 \equiv X \in O_n(X^0)$.

As it was noted above, a set of feasible points in this case is a connected set.

The main property which follows from the definitions introduced above is:

If the function $A(X)$ is a unimodal one (it has the unique local minimum in the point X^0) and on a level $O_k(X^0)$ all points are unfeasible or limiting, then on a level $O_l(X^0)$ where $l > k$ there are no feasible points. It can be illustrated with the following picture (Figure 3.1).

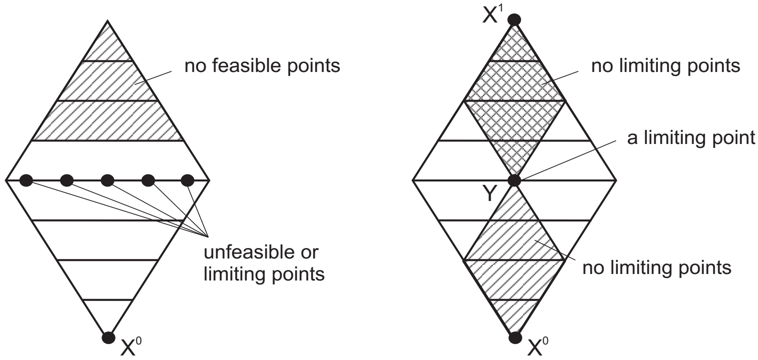

FIG. 3.1: Case of an unimodal constraint function (left) and case of a monotone constraint function (right)

2. Monotone constraint function

Let us consider a constraint function which increases steadily from the point $X^0 \in B_2^n$. On the basis of notions of a subcube and monotonicity we can deduce the following properties:

a) If the function $A(X)$ is monotone and a point $Y \in B_2^n$ is feasible (satisfies the constraint $A(Y) \leqslant H$) then any point $X \in K(X^0, Y)$ is also feasible.

b) If the function $A(X)$ is monotone and a point $Y \in B_2^n$ is unfeasible (doesn't satisfy the constraint $A(Y) \leqslant H$) then any point $X \in K(Y, X^1)$ is also unfeasible.

Generalizing these properties and the notion of a limiting point one can conclude that if the function $A(X)$ is monotone and a point $Y \in B_2^n$ is limiting then any point $X \in K(X^0, Y) \setminus Y$ is not limiting, and any point $X \in K(Y, X^1) \setminus Y$ is not limiting either (that is, while looking for all the other limiting points the subcubes $K(X^0, Y)$ and $K(Y, X^1)$ can be excluded from consideration.

3.2. Properties of objective functions

1. Unimodal objective function

If f is an unimodal function on B_2^n with the local minimum point X^0 then

$$\min_{X_j^k \in O_k(X^0)} f(X_j^k) \leqslant \min_{X_j^{k+1} \in O_{k+1}(X^0)} f(X_j^{k+1}).$$

This implies that if the function $C(X)$ is unimodal (it has the unique local maximum in the point X^1) and the solution giving the maximum value of the function $C(X)$ on a level $O_k(X^1)$ is feasible then on a level $O_l(X^1)$ where $l > k$ there is no the optimal solution (Figure 3.2).

2. Monotone objective function

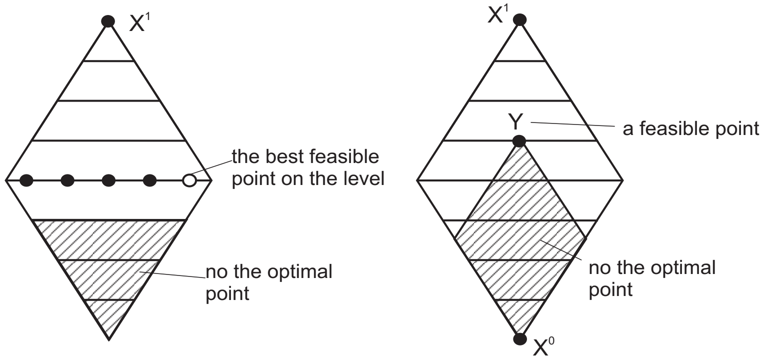

FIG. 3.2: Case of an unimodal objective function (left) and case of a monotone objective function (right)

If the objective function $C(X)$ increases steadily from the point $X^0 \in B_2^n$ then the optimal solution belongs to the subset of limiting points. From the property considered at the beginning of this unit we have the following.

If the function $C(X)$ is monotone and the solution $Y \in B_2^n$ is feasible (satisfies the constraint $A(Y) \leqslant H$) then in the subcube $K(X^0, Y)$ there is no the optimal solution.

4. A scheme of the branch-and-bounds method for a problem with black-box functions

The basis of the branch-and-bounds method is the idea of sequential partition of a set of feasible solutions into subsets. At each step of the method the elements of partition are checked to find out whether the given subset contains an optimal solution. The check is carried out by means of calculating the upper bound for an objective function on a given subset. If the upper bound is not better than the *record* – the best of the found solutions – then the subset can be discarded. A checked subset can be also discarded if the best solution was found in it. If the value of the objective function on a found solution is better than the record then the record is changed. On finishing the algorithm work the record is the result of its work.

If one manages to discard all elements of partition then the record is the optimal solution of the problem. Otherwise the most promising subset (for example, with the greatest value of the upper bound) is chosen from those which were discarded, and it is partitioned. New subsets are checked again, and so on.

It is obvious that the use of specific structural peculiarities of the problem allows one to construct a workable branch and bound algorithm.

Let us consider the application of the scheme for the solution of optimization problem in which all variables are binary, and the objective function and the constraint are unimodal and monotone.

The most widely used variant of application of the branch-and-bounds scheme for the solution of pseudo-Boolean optimization problems is the following. The problem of continuous optimization which is relaxation of the original problem is being solved (for example, with a simplex algorithm). As a result we have solution X^*, which will not be binary in general. Then the problem is divided into two subproblems and two mutually exclusive constraints exhausting all possibilities are added. For example, let component x_i' in X^* be not binary. Then constraints $x_i' = 0$ and $x_i' = 1$ appear in corresponding subproblems. Further branching occurs similarly.

Such an approach is suitable for problems in which the objective function and the constraints are defined explicitly (in the form of algebraic expressions). But the problem under consideration consists of functions defined algorithmically (black-box functions), that is, it is impossible to calculate the value of the function in the point which is not binary. Therefore there appeared the necessity to investigate other variants of application of the scheme.

Here we shall consider the question of application of the branch-and-bounds scheme for optimization problems in which the objective function and the constraints are defined algorithmically. Namely, for the problem (2.1)-(2.2), in which the functions $C(X)$ and $A(X)$ increase monotonically from the point $X^0 = (x_1^0, \ldots, x_n^0)$.

The simplest algorithm of the branch-and-bounds method based on the properties of the considered class of problems will look like this. In the first stage of branching set B_2^n is partitioned into two equicardinal subsets: $S_1^0 = \{X \in B_2^n : x_1 = 0\}$ and $S_1^1 = \{X \in B_2^n : x_1 = 1\}$ (let's call it branching of the first order). Each of these subsets is a subcube of dimension $n-1$, the cardinality of the subsets is 2^{n-1}. Partition of elements of the space B_2^n for $n = 4$ into two subcubes is shown on Figure 4.1. In the next stages of branching each of subcubes is partitioned into two subcubes and so on. For example, S_1^0 is partitioned into $S_2^{00} = \{X \in B_2^n : x_1 = 0, x_2 = 0\}$ and $S_2^{01} = \{X \in B_2^n : x_1 = 0, x_2 = 1\}$ (Figure 4.2). So, after branching of the k-th order there appear subcubes consisting of 2^{n-k} elements (vectors).

In the subset got after branching of the k-th order k coordinates are fixed for any binary vector from this subset. Let the vector in which variable coordinates are equal to corresponding coordinates of initial vector X^0 be called *"lower" point* $\underline{X}$ and the vector in which all variable coordinates are opposite to corresponding coordinates of X^0 be called *"upper" point* $\overline{X}$:

$$\overline{X} = (x_1, \ldots, x_k, 1 - x_{k+1}^0, 1 - x_{k+2}^0, \ldots, 1 - x_n^0),$$
$$\underline{X} = (x_1, \ldots, x_k, x_{k+1}^0, x_{k+2}^0, \ldots, x_n^0).$$

The objective function and the constraint function increase monotonically on B_2^n with chosen initial point $X^0 = (x_1^0, \ldots, x_n^0)$ from where it follows that in the

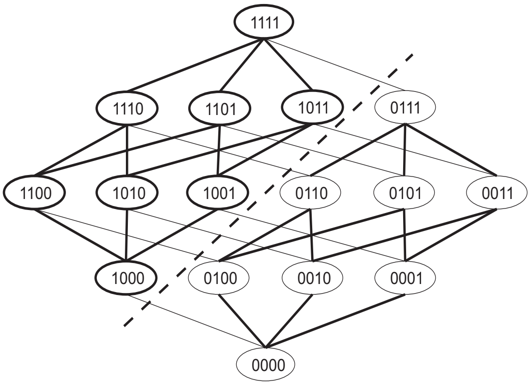

FIG. 4.1: Partition B_2^4 into two subcubes

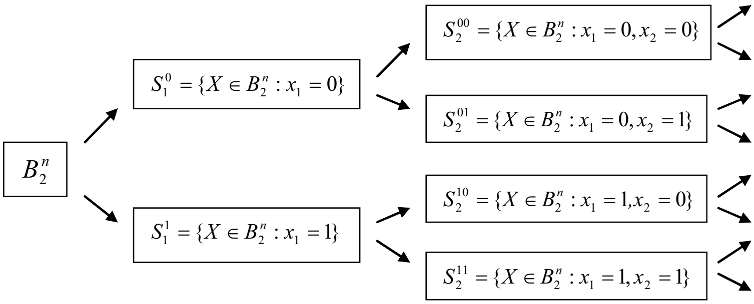

FIG. 4.2: The scheme of branching

“upper” point of the subcube they take the greatest value, and in the “lower” point the smallest one.

The subset (subcube) is excluded from consideration in three cases:

1. In point $\underline{X}$ the constraint is not performed; in this case all solutions in the subset are unfeasible.

2. In point $\overline{X}$ the constraint is performed; then this solution is the best one in the subset, and it is compared with the record.

3. In point $\overline{X}$ the constraint is not performed, but the objective function in it takes the value which is smaller than the record.

Otherwise further branching of this subset takes place.

At the first stage the value of the objective function in any feasible point of the space B_2^n can be taken as the record. If in the considering subset $K(\underline{X}, \overline{X})$ the point $\overline{X}$ is feasible and the value of the criterial function in it is greater than the record then the record is changed.

It is easy to show that the received solution will be exact. The constraint $A(X) \leqslant H$ partitions the set B_2^n into two subsets one of which satisfies the constraint and the other does not. From the condition of monotonicity of functions C (X) and A (X) it follows that the solution of the problem will be the point belonging to the subset of limiting points.

In case 1 in subcube $K(\underline{X}, \overline{X})$ there are no limiting points. In case 2 only point $\overline{X}$ in the subcube can be limiting. In case 3 there are limiting points in the subcube, but they are worse than those found before. So the scheme provides exact solution of the problem.

The considered approach allows one to easily calculate the lower bound which is equal to the value of the objective function in the upper point of the subcube.

Though the approach described above offers considerable reduction of the number of points to be searched in the process of finding the optimal solution nevertheless this process is labour intensive as it may require a great deal of branching.

The next part of the paper describes the optimization algorithm combining the schemes of the branch-and-bounds method and the rule of subcubes truncation considered in the previous part.

5. The optimization algorithm

Let us consider the problem (2.1)-(2.2) in which functions $C(X)$ and $A(X)$ monotonically increase from point $X^0 = (x_1^0, \dots, x_n^0)$. Let's denote $X^1 = O_n(X^0)$, $X^1 = (x_1^1, \dots, x_n^1)$. All set of points of the space B_2^n can be presented as the subcube $K(X^0, X^1)$.

5.1. Branching

Let us suppose that some limiting point $X' \in B_2^n$ is found. Then subcubes $K(X', X^0)$ and $K(X', X^1)$ can be excluded from further consideration.

Let us introduce an auxiliary variable

$$z_i = \begin{cases} x_i, & \text{if } x_i^0 = 0, \\ \bar{x}_i, & \text{if } x_i^0 = 1. \end{cases}$$

Then subcube $K(X', X^0)$ can be represented as a set of points for which the following boolean expression is true:

$$T^0 = \bigwedge_{i: x_i' = x_i^0} \bar{z}_i.$$

And subcube $K(X', X^1)$ can be described as follows

$$T^1 = \bigwedge_{i: x_i' = x_i^1} z_i.$$

For the sake of convenience let's denote a set of indexes for which $x'_i = x^1_i$ is fulfilled as $A(X') = \{i_1, \ldots, i_k\}$ and a set of indexes for which $x'_i = x^0_i$ is fulfilled as $B(X') = \{i_1, \ldots, i_{n-k}\}$. It is obvious that $|A(X')| = k$ and $|A(X')| = n - k$ where k is the number of the level on which point $K(X') \in O_k(X^0)$ is located. Then we may write:

$$T^0 = \bigwedge_{i \in B(X')} \bar{z}_i, \quad T^1 = \bigwedge_{i \in A(X')} z_i.$$

Let us partition subcube $K(X^0, X^1)$ into two parts:

$$\begin{tikzcd}[column sep=small] & K(X^0, X^1) \arrow[dl] \arrow[dr] & \\ (T^0 = 1) \vee (T^1 = 1) & & (T^0 = 0) \wedge (T^1 = 0) \end{tikzcd}$$

The left part, as it was stated above, is excluded from further consideration. The right part $(T^0 = 0) \wedge (T^1 = 0)$ can be represented as a set of subcubes.

Let us consider the condition $(T^1 = 0)$. It is fulfilled if $z_i = 0$ for at least one $i \in A(X')$. If $|A(X')| > 1$ then a set of points fulfilling the condition $(T^1 = 0)$ can be represented only as a number of subcubes, but not as one subcube. The most evident way to partition this set of points into k subcubes is to fix alternately the value of variable $z_i = 0$ for $i \in A(X')$. In this case we receive k subcubes of dimension $n - 1$. The disadvantage of this method is that the received subcubes substantially intersect each other.

To avoid this we shall use the following approach. We shall get the first subcube K^1_1 having fixed one variable $z_{i_1} = 0$. For the second K^1_2 we shall fix two variables : $z_{i_1} = 1$ and $z_{i_2} = 0$. For the third K^1_3 - three variables: $z_{i_1} = 1$, $z_{i_2} = 1$ and $z_{i_3} = 0$. And so on. For the k-th subcube K^1_k: $z_{i_s} = 1, s = 1, \ldots, k-1$, $z_{i_k} = 0$.

As a result we get k subcubes of different dimensions. Such an approach guarantees that the received subcubes don't intersect.

The same procedure is offered for condition $(T^0 = 0)$. The corresponding set of points should be partitioned into $(n - k)$ subcubes by fixing variables $j \in B(X')$. For the first subcube K^0_1 we shall fix one variable $z_{j_1} = 1$. For the second subcube K^0_2 we shall fix two variables: $z_{j_1} = 0$ and $z_{j_2} = 1$. For the third K^0_3 - three variables: $z_{j_1} = 0$, $z_{j_2} = 0$ and $z_{j_3} = 1$. For $(n-k)$-th subcube K^0_{n-k}: $z_{j_s} = 0, s = 1, \ldots, n-k-1$, $z_{j_{n-k}} = 0$.

As a result we get two sets of subcubes: $K^1_1, \ldots, K^1_k$ and $K^0_1, \ldots, K^0_{n-k}$. A set of points fulfilling the condition $(T^0 = 0) \wedge (T^1 = 0)$ corresponds to the union of all possible intersections of pairs of subcubes taken from these two sets:

$$\bigcup_{\substack{i \in A(X') \\ j \in B(X')}} (K^1_i \cap K^0_j).$$

So, having found in subcube $K(X^0, X^1)$ some limiting point $X' \in O_k(X^0)$ we partition this subcube into two parts one of which is discarded (subcubes $K(X', X^0)$

and $K(X', X^1)$), and from the other part $k \cdot (n-k)$ new branches are formed. Each of these branches is a subcube which can be subjected to the same procedure of branching as described above.

5.2. Upper bound

Let us denote the upper and the lower points of some subcube as $\overline{X}$ and $\underline{X}$ respectively. $z_i^{\underline{X}} = 0$ is fulfilled in point $\underline{X}$ for all free (unfixed) variables, and $z_i^{\overline{X}} = 1$ is fulfilled in point $\overline{X}$ for all free variables. For fixed variables naturally $z_i^{\underline{X}} = z_i^{\overline{X}}$.

Subcube $K(\underline{X}, \overline{X})$ can contain an optimal solution only if the following conditions are fulfilled:

1. There are feasible solutions in the subcube.
2. The upper bound of the corresponding branch is above the best found solution.

As the constraint function $A(X)$ increases monotonically from point X^0, then within subcube $K(\underline{X}, \overline{X})$ function $A(X)$ increases monotonically from point $\underline{X}$ possessing its minimum value in this point. Therefore if point $\underline{X}$ is unfeasible then all points of this subcube are unfeasible. The objective function $C(X)$ within the subcube also increases monotonically from point $\underline{X}$ possessing its maximum value in point $\overline{X}$. Point $\overline{X}$ itself can be unfeasible, but value $C(\overline{X})$ can be used as the upper bound of the branch corresponding to the subcube.

Also, if point $\overline{X}$ is feasible then all other points of this subcube are a fortiori not better; besides, in this case there are no limiting points in the subcube with the possible exception of $\overline{X}$.

So, subcube $K(\underline{X}, \overline{X})$ is excluded from further search if at least one of the following conditions is fulfilled:

- Point $\underline{X}$ is unfeasible.
- Point $\overline{X}$ is feasible.
- The value of upper bound $C(\overline{X})$ does not exceed the already found best feasible value of the objective function.

Such a check including calculation of the upper bound requires the scanning of only two points of the subcube.

5.3. Search for limiting points

To carry out branching in a way described above it is necessary to find some limiting point belonging to the considered subcube $K(\underline{X}, \overline{X})$. This solution should not necessarily be the best one in the given subcube. However, a good solution can exceed the record (the best already found feasible solution) and it also can increase the chances to discard new branches got in the process of further branching (if their upper bound will turn out to be lower).

The simplest stochastic algorithm for search of limiting points is as follows. The search begins from $\underline{X}$. At each step the algorithm chooses a feasible neighbouring point on the following level moving along the way of increasing of the objective function to the bound of a feasible area. In case of necessity the procedure is repeated several times and the best point is chosen from the found limiting points.

Algorithm "Random search"

1. Suppose $l = 1$.
2. Suppose $X_1 = \underline{X}$, $i = 1$.
3. Randomly choose a point $X_{i+1} \in O_1(X_i) \cap O_i(\underline{X}) \cap \{X \in K(\underline{X}, \overline{X}) : A(X) \leqslant H\}$, $i = i + 1$. If there are no such points go to step 4, otherwise the cycle is repeated.
4. $Y_l = X_i$. If $l < L$ then $l = l + 1$ and go to step 2.
5. Define X^* from the condition

$$C(X^*) = \max_{l=1,\dots,L} C(Y_l).$$

Defined number L is a number of limiting points which is planned to find. As $card\{O_1(X_k) \cap O_{k+1}(\underline{X})\} = n_K - k$, where $X_k \in O_k(X)$, n_K is the dimension of subcube $K(\underline{X}, \overline{X})$ (the number of free variables) then from current search point X_k the algorithm looks through not more than $n_K - k$ following points. So the computational complexity of the algorithm can be calculated as follows

$$T \leqslant L \cdot \sum_{i=0}^{n-1} (n_K - i) = L \cdot \frac{n_K(n_K + 1)}{2}.$$

A regular algorithm using greedy heuristics is an alternative to random search of limiting points.

Greedy algorithms are natural heuristics in which at each step the most effective at the given moment decision is made without considering what happens at the following steps of search.

For the problem being considered a greedy algorithm can have the following form.

Algorithm "Greedy"

1. Suppose $X_1 = \underline{X}$, $i = 1$.
2. Calculate $C(X_j)$ and $A(X_j)$ for $X_j \in O_1(X) \cap O_i(\underline{X})$, $j = 1, \dots, n_K - i + 1$.

3. If there is no X_j for which $A(X_j) \leqslant H$, then $X^* = X$ is the solution of the problem.

4. From those X_j for which $A(X_j) \leqslant H$ find $X = \arg\max_{X_j} \lambda(X_j)$.

5. $i = i + 1$, go to step 2.

Here $\lambda(X_j) = C(X_j)/A(X_j)$ or $\lambda(X_j) = C(X_j)$.

In more detail these and other algorithms of search of limiting points have been considered in [3].

5.4. The algorithmic scheme

The procedures described above are the main elements of which an algorithm of search of an optimal solution consists. Now we shall consider the algorithm itself.

The first step is the choice of a branch for branching. In the first cycle there is only one branch which corresponds to the binary space of n dimension. In the following cycles, when there are several open branches, the branch with the maximum upper-bound estimate is chosen. If there are no open branches then the algorithm finishes its work.

At the second step the search for an approximate solution representing some limiting point in the corresponding subcube in a chosen branch is carried out. If the value of the objective function in this point is better than the record (the best found solution) then the change of the record occurs. The search of an limiting point can be carried out with the help of, for example, a random-search algorithm or a greedy algorithm described above.

At the third step the procedure of branching of a chosen branch according to the found limiting point is performed. The check of received branches is carried out. If there are feasible solutions in a branch and the upper bound is greater than the record then this branch is added to the list of open branches.

After completing some number of such cycles one should interrupt in order to sort the branches by the value of the upper bound and close the branches the upper bound of which is less than the record.

The search is stopped if there are no open branches left. In this case it can be argued that the exact solution of the problem (global constrained maximum) is found.

While solving the problems of great dimensions it can be inaccessible due to excessively large amount of search time. The achievement of a number of formed branches or a number of branching of some defined value can serve as a stopping criterion.

The algorithm is shown schematically in Figure 5.1.

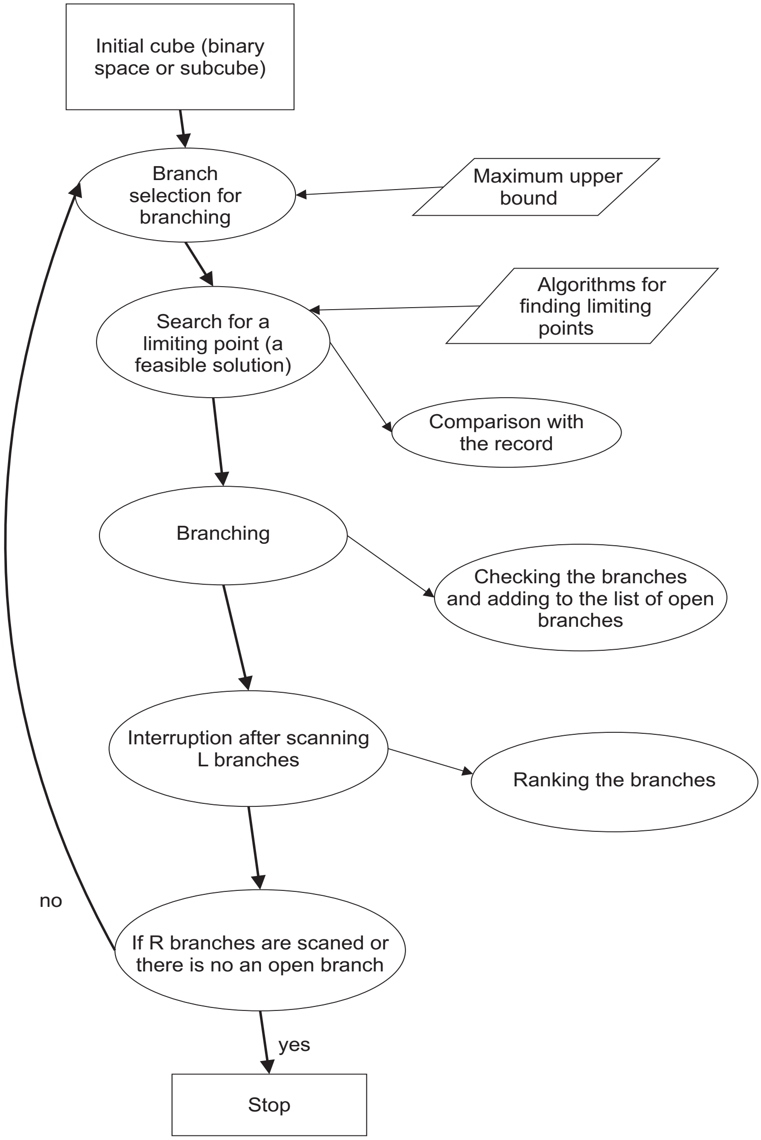

FIG. 5.1: The algorithmic scheme

6. Experimental investigation

This paper gives the results of the experimental investigation of the described algorithm work on the constrained pseudo-Boolean optimization problems generated randomly. Objective functions and constraints have the following form:

$$C(X) = \sum_{i=1}^{n} c_1^i x_i + \sum_{i=1}^{n-1} c_2^i x_i x_{i+1} + \sum_{i=1}^{n-2} c_3^i x_i x_{i+1} x_{i+2} \to \max,$$
$$A(X) = \sum_{i=1}^{n} a_1^i x_i + \sum_{i=1}^{n-1} a_2^i x_i x_{i+1} + \sum_{i=1}^{n-2} a_3^i x_i x_{i+1} x_{i+2} \leqslant b,$$
$$X \in \{0,1\}^n,$$

where coefficients $c_1^i, c_2^i, c_3^i, a_1^i, a_2^i, a_3^i$ are random numbers taken from the range $[0, 20]$; $b = A(X^r)$, where $X^r = (x_1^r, \ldots, x_n^r)$ is a randomly chosen point: $x_i^r = 1$ with probability 1/4 and $x_i^r = 0$ with probability 3/4 (this bias is made due to some real-world problems, in this case the set of feasible points is less than the set of infeasible points). As all coefficients are non-negative numbers then functions $C(X)$ and $A(X)$ are monotone ones with the minimum in point $(0, \ldots, 0)$ and unconstrained maximum in point $(1, \ldots, 1)$.

Efficiency of the algorithm will be characterized by the search time and achieved value of the objective function (if the maximum of the objective function is defined inexact or there is not proof that the solution is exact). By the search time (or the time complexity) we will mean the number of computing values of the objective function (and/or the constraint function) that the algorithm has made (the number of points that the algorithm has scanned).

At first let us investigate how fast the optimization algorithm finds an exact solution. For this purpose series of tests were conducted on the problems of small dimension: $n = 10, 15, 20$. 500 tasks were solved for each dimension value. A simple algorithm "random search" with repetition number $L = 1$ was used to find boundary points.

The distribution of values of time complexity and the number of branches that occur as a result of complete solution of the problem are shown in the graphs (Figure 6.1 and Figure 6.2). In the experiments it is guaranteed that the exact solution of the problem has been found (i.e. there are not open branches remained). For comparison, time complexity of the exhaustive search for the dimension $n = 10$ is $2^{10} = 1024$, for $n = 15$ complexity is $2^{15} = 32768$, for $n = 20$ complexity is $2^{20} = 1048576$.

Further the graphs (Figure 6.3 and Figure 6.4) present the distribution of the time complexity and the number of branches when the exact solution has been found, but absence of a better solution has not been guaranteed yet (i.e.there are open branches remained).

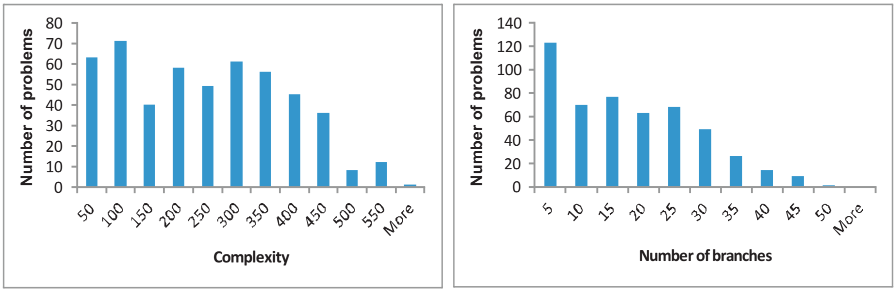

FIG. 6.1: Time complexity and number of branches for $n = 10$ (the exact solution is justified)

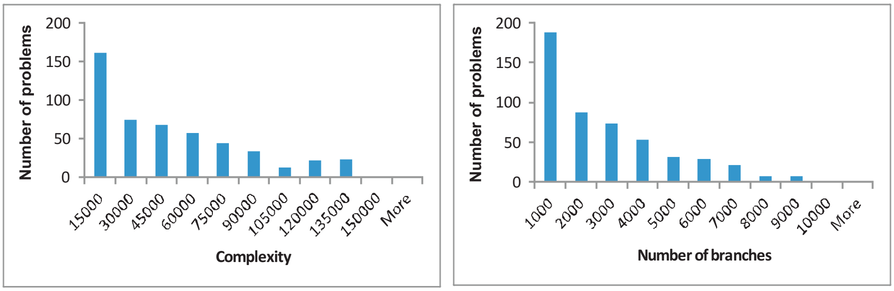

FIG. 6.2: Time complexity and number of branches for $n = 20$ (the exact solution is justified)

As can be seen from the graphs, the exact solution is found usually well before completion of the full search.

It should be noted that the results of the solutions of problems generated in such a way differ greatly from problem to problem therefore it doesn't make sense to give the mean values of efficiency indexes. Instead the results of solution of separate problems are given here what in this case is more demonstrative.

During the search the number of open branches is being changed significantly. The first few cycles it is increased rapidly and towards the end of the search it is gradually reduced, approaching zero. Absence of open branches upon completion of the search means that the solution is exact, that is there is no a better feasible solution under the given conditions.

The pictures (Figure 6.5, Figure 6.6 and Figure 6.7) show examples for the dynamics of change in the number of open branches and the value of the record in the process of search. The number of branching is shown on X-axis, the amount of open branches on the left of Y-axis, the value of the record on the right of Y-axis.

Also here was examined the question of how the proposed optimization algo-

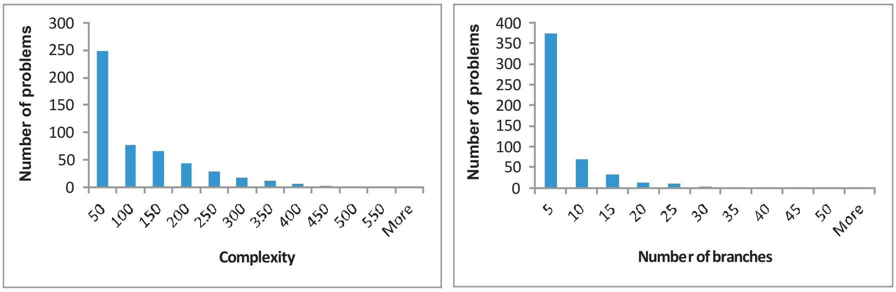

FIG. 6.3: Time complexity and number of branches for $n = 10$ (the exact solution is not justified)

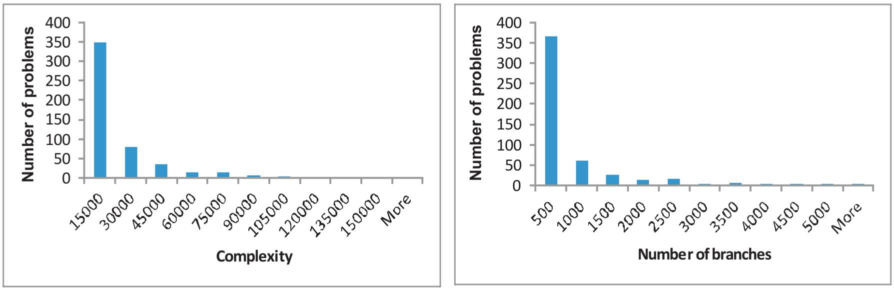

FIG. 6.4: Time complexity and number of branches for $n = 20$ (the exact solution is not justified)

rithm can improve the solution obtained individually by the algorithm of search for boundary points (the greedy heuristic or random search of boundary points).

To find the first approximate solution the greedy optimization algorithm described above was used. The obtained solution was used as a branching point in accordance with the procedure described above for branching the optimization space. To find solutions in the formed branches also the greedy algorithm was used. Values of the best solutions found during the search by the branch and bound algorithm presented in the tables.

Number of branching	Found solution	Time complexity
Greedy algorithm	47	34
1	47	82
2	56	141
4	58	202
7	73	281

Table 6.1: The greedy algorithm and the branch and bound algorithm, $n = 10$

Number of branching	Found solution	Time complexity
Greedy algorithm	31	90
1	52	216
2	53	341
4	66	492
8	67	862
19	68	1703
32	74	2645
88	78	5403
153	79	8840

Table 6.2: The greedy algorithm and the branch and bound algorithm, $n = 20$

Number of branching	Found solution	Time complexity
Greedy algorithm	74	165
1	74	407
2	76	707
5	89	1529
12	106	2850
34	108	6161
139	109	19937
183	111	24662
541	118	54243
589	119	58145
1515	123	126962

Table 6.3: The greedy algorithm and the branch and bound algorithm, $n = 30$

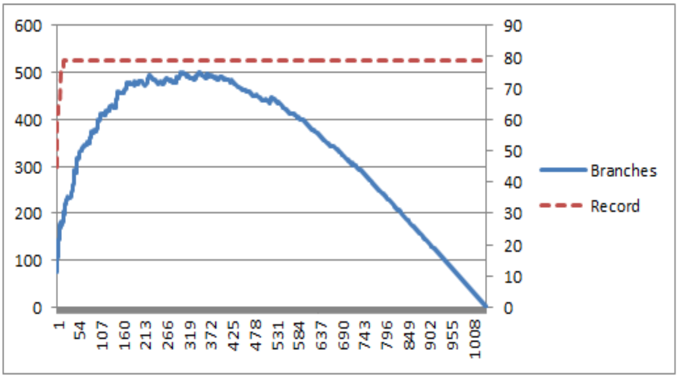

FIG. 6.5: Amount of open branches and value of the record for $n = 20$

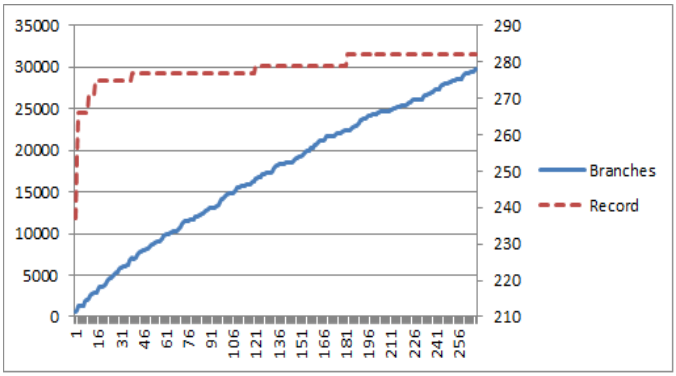

FIG. 6.6: Amount of open branches and value of the record for $n = 50$

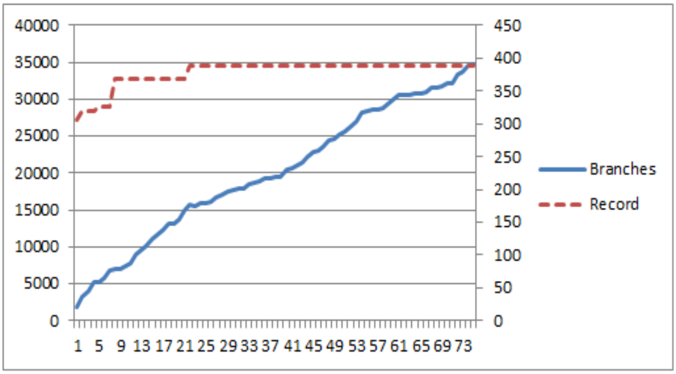

FIG. 6.7: Amount of open branches and value of the record for $n = 100$

Number of branching	Found solution	Time complexity
Greedy algorithm	342	1810
1	342	4874
3	371	12318
8	374	26684
35	388	88616
67	401	158873
70	406	163781

Table 6.4: The greedy algorithm and the branch and bound algorithm, $n = 100$

Number of branching	Found solution	Time complexity
Greedy algorithm	708	7380
1	743	20153
2	784	22181
4	798	35944
7	800	70062
11	803	112484
12	840	114011

Table 6.5: The greedy algorithm and the branch and bound algorithm, $n = 200$

7. Conclusions

The main peculiarity of the considered class of problems is that an objective function and constraint functions are supposed to be defined implicitly, that is, calculations of the functions in points are possible, but their algebraic notation is not known. On the one hand, such problems are often met in practice, for example, when it is necessary to turn to a data array to calculate a function. On the other hand, even for problems for which algebraic notation of functions is possible, these functions can be considered as algorithmically defined, which significantly simplifies the work with an available optimization model.

Such a class of models restricts the number of optimization algorithms available for application. Of course it is always possible to apply the local search algorithm or the algorithms of genetic type, but they do not guarantee finding of the exact solution, and one cannot say how close the found solution is to the optimal one.

At the same time in many practical problems objective functions and constraints have the same properties, such as unimodality and monotonicity. And these properties are not taken into account in application of universal algorithms.

The approach presented in this paper is aimed at the reception of an exact solution of an optimization problem. The realized way of branching divides a branch which represents a subcube of binary variables space into a great number of branches a significant part of which is at once subjected to exclusion. It offers quick reduction of the area in which an optimal solution can be found.

The developed algorithm can be also applied for the problems of high dimensionality. For all that, of course, it will not be proved that the found solution is an optimal one, if there are still unconsidered open branches. In this case such an algorithm can be considered as improvement of approximate algorithms of boundary points search, such as a greedy algorithm and random search of boundary points. Such improvement even on a small number of iterations (branchings) offers significant improvement of the found feasible solution.

From now on it is planned to investigate the work of the algorithm on practical problems: for example, on the problem of capacity planning, the problem of searching for rules in data in logical algorithms of classification. It is interesting to compare this algorithm with popular search algorithms such as local search with multi-start and algorithms of genetic type.

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FACTA Universitatis. Series, Mathematics and informatics / editor-in-chief Predrag S. Stanimirović. - 1986, N° 1- . - Niš : University of Niš, 1986- (Niš : Unigraf-X-Copy). - 24 cm

Tekst na engl. jeziku. - Drugo izdanje na drugom medijumu: Facta Universitatis. Series: Mathematics and Informatics (Online) = ISSN 2406-047X
ISSN 0352-9665 = Facta Universitatis. Series: Mathematics and informatics
COBISS.SR-ID 5881090

FACTA UNIVERSITATIS

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Mathematics and Informatics

Vol. 33, No 2 (2018)

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