UNIVERSITY OF NIŠ

ISSN 0352-9665 (Print)
ISSN 2406-047X (Online)
COBISS.SR-ID 5881090

# FACTA UNIVERSITATIS

Series
**MATHEMATICS AND INFORMATICS**
Vol. 33, No 3 (2018)

Scientific Journal **FACTA UNIVERSITATIS**
UNIVERSITY OF NIŠ
Univerzitetski trg 2, 18000 Niš, Republic of Serbia
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Founded in 1986 by **Gradimir V. Milovanović**, Serbian Academy of Sciences and Arts, and Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia
The cover image taken from http://www.pptbackgrounds.net/binary-code-and-computer-monitors-backgrounds.html.
Publication frequency – one volume, five issues per year.
Published by the University of Niš, Republic of Serbia

This publication was in part supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia
Printed by "UNIGRAF-X-COPY" – Niš, Republic of Serbia

ISSN 0352 – 9665 (Print)
ISSN 2406 – 047X (Online)
COBISS.SR-ID 5881090

# FACTA UNIVERSITATIS

*SERIES* **MATHEMATICS AND INFORMATICS**
Vol. 33, No 3 (2018)

UNIVERSITY OF NIŠ

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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 3 (2018), 361-373
https://doi.org/10.22190/FUMI1803361P

# ON m-PROJECTIVE CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS

**Shravan K. Pandey and R.N. Singh**

**Abstract.** The aim of the paper is to characterize generalized Sasakian-space-forms satisfying certain curvature conditions on the m-projective curvature tensor. We study m-projectively semisymmetric, m-projectively flat, $\xi$-m-projectively flat, and m-proje--ctively recurrent generalized Sasakian-space-forms. $W^*.S = 0$ and $W^*.R = 0$ on generalized Sasakian-space-forms are also studied.
**Keywords:** generalized Sasakian-space-forms, m-projectively semisymmetric, m-projectively flat, m-projectively recurrent, $\xi$-m-projectively flat.

## 1. Introduction

Studying the almost Hermitian manifold, Alfred Gray, a well-known geometrician, formulated a principle according to which the so-called curvature identities for the Riemann-Christoffel tensor are key to understanding differential-geometric properties of such manifolds [13]. Many papers are devoted to the study of geometric consequences of these identities and to their analogs for almost contact metric structures. As a continuation of this line of research, we consider some curvature properties of generalized Sasakian-space-forms regarding the m-projective curvature tensor.

A generalized Sasakian-space-form was defined by P. Alegre, D. E. Blair and A. Carriazo in [1] as an almost contact metric manifold ($M^{2n+1}$, $\phi$, $\xi$, $\eta$, g) whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{1.1}$$

where $f_1, f_2, f_3$ are some differential functions on $M^{2n+1}$ and

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$

$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z,$$

Received December 05, 2016; accepted June 10, 2017.
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53D15

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

for any vector field X, Y, Z on $M^{2n+1}$. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well-known Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$. It is known that any three-dimensional $(\alpha,\beta)-$trans-Sasakian manifold with $\alpha, \beta$ depending on $\xi$ is a generalized Sasakian-space-form [2].P.Alegre, A.Carriazo, Y.H.Kim and D.W.Yoon give results in [3] about B.Y.Chen's inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. R. Al-Ghefari, F.R. Al-Solamy and M.H.Shahid analyse in [4] and [5] CR-submanifolds of generalized Sasakian-space-forms. In [9], U.K.Kim studied conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms. U.C.De and A.Sarkar [7] studied generalized Sasakian-space-forms regarding the projective curvature tensor. On the other hand, in 1971, G.P.Pokhariyal and R.S.Mishra [12]defined a tensor field $W^*$ on a Riemannian manifold as

(1.2)
$$\begin{aligned} 'W^*(X,Y,Z,U) ={}& 'R(X,Y,Z,U) - \frac{1}{2(n-1)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)S(X,U) - g(X,Z)S(Y,U)], \end{aligned}$$

where $'W^*(X,Y,Z,U) = g(W^*(X,Y)Z,U)$ and $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. Such a tensor field $W^*$ is known as m-projective curvature tensor. Later, R. H.Ojha [10] defined and studied the properties of the m-projective curvature tensor in Sasakian and Kähler manifolds. He also showed that it bridges the gap between the conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and the H-projective curvature tensor on the other.

Motivated by the above studies, we study here the flatness and symmetry property of generalized Sasakian-space-forms regarding the m-projective curvature tensor. The paper is organized as follows. In section 2, some preliminary results are recalled. In section 3, we study m-projectively semisymmetric generalized Sasakian-space-forms. Section 4 deals with m-projectively flat generalized Sasakian-space-forms. $\xi$-m-projectively flat generalized Sasakian-space-forms are studied in Section 5 and necessary and sufficient condition are obtained for a generalized Sasakian-space-form to be $\xi$-m-projectively flat. In Section 6, m-projectively recurrent generalized Sasakian-space-forms are studied. Section 7 is devoted to the study of generalized Sasakian-space-forms satisfying $W^*.S = 0$. The last section discusses generalized Sasakian-space-forms satisfying $W^*.R = 0$.

## 2. Preliminaries

If on an odd dimensional differentiable manifold $M^{2n+1}$ of the differentiability class $C^{r+1}$ there exists a vector-valued real linear function $\phi$, a 1-form $\eta$, the associated vector field $\xi$ and the Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \phi(\xi) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields $X$ and $Y$, then $(M^{2n+1}, g)$ is said to be an almost contact metric manifold [6] and the structure $(\phi, \xi, \eta, g)$ is called an almost contact metric structure to $M^{2n+1}$. In view of the equations (2.1), (2.2) and (2.3), we have

$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \tag{2.5}$$

Again we know that [1] in a $(2n+1)$-dimensional generalized Sasakian-space-form

(2.6)
$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned}$$

for all vector fields X, Y, Z on $M^{2n+1}$, where R denotes the curvature tensor of $M^{2n+1}$.

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.8}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3). \tag{2.9}$$

For generalized Sasakian-space-forms we also have

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)Y = -R(X,\xi)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)], \tag{2.12}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.13}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.14}$$

where $Q$ is the Ricci operator, i.e. $g(QX, Y) = S(X, Y)$.

A generalized Sasakian space-form is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.15}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on $M^{2n+1}$. For a $(2n+1)$-dimensional $(n > 1)$ almost contact metric manifold the m-projective curvature tensor $W^*$ is given by [12]

(2.16)
$$W^*(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].$$

The m-projective curvature tensor $W^*$ for a generalized Sasakian-space-form is given by

(2.17)
$$W^*(X, Y)\xi = -\frac{(f_1 - f_3)}{(n-1)}[\eta(Y)X - \eta(X)Y] - \frac{1}{2(n-1)}[\eta(Y)QX - \eta(X)QY],$$

$$\eta(W^*(X, Y)\xi) = 0, \tag{2.18}$$

(2.19)
$$W^*(\xi, Y)Z = -\frac{(f_1 - f_3)}{(n-1)}[g(Y, Z)\xi - \eta(Z)Y] - \frac{1}{2(n-1)}[S(Y, Z)\xi - \eta(Z)QY],$$

$$\begin{aligned} \eta(W^*(\xi, Y)Z) = &-\frac{(f_1 - f_3)}{(n-1)}[g(Y, Z) - \eta(Y)\eta(Z)] \\ &- \frac{1}{2(n-1)}[S(Y, Z) - 2n(f_1 - f_3)\eta(Y)\eta(Z)] \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \eta(W^*(X, Y)Z) = &-\frac{(f_1 - f_3)}{(n-1)}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &- \frac{1}{2(n-1)}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \end{aligned} \tag{2.21}$$

## 3. m-Projectively Semisymmetric Generalized Sasakian-Space-Forms

**Definition 3.1.** A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form is said to be m-projectively semisymmetric [7] if it satisfies $(R(X, Y).W^*)(U, V)Z = 0$, where $R(X, Y)$ is to be considered a derivation of the tensor algebra at each point of the manifold for tangent vectors $X, Y$ and $W^*$ is the m-projective curvature tensor of the space-forms.

**Theorem 3.1.** *If a $(2n + 1)-$dimensional $(n > 1)$ generalized Sasakian-space-form is m-projectively semisymmetric then either $f_1 = f_3$ or $M^{2n+1}$ is an Einstein manifold.*

**Proof:** Let us suppose that the generalized Sasakian-space-form is m-projectively semisymmetric. Then we can write

$$(R(\xi,X).W^*)(Y,Z)U=0. \tag{3.1}$$

The above equation can be written as

$$R(\xi,X)W^*(Y,Z)U-W^*(R(\xi,X)Y,Z)U-W^*(Y,R(\xi,X)Z)U-W^*(Y,Z)R(\xi,X)U=0. \tag{3.2}$$

In view of the equation (2.11) the above equation reduces to

$$\begin{aligned}&(f_1-f_3)[g(X,W^*(Y,Z)U)\xi-\eta(W^*(Y,Z)U)X-g(X,Y)W^*(\xi,Z)U\\&+\eta(Y)W^*(X,Z)U-g(X,Z)W^*(Y,\xi)U+\eta(Z)W^*(Y,X)U\\&-g(X,U)W^*(Y,Z)\xi+\eta(U)W^*(Y,Z)X]=0.\end{aligned} \tag{3.3}$$

Now, taking the inner product of the above equation with $\xi$ and using the equation (2.2), we get

$$\begin{aligned}&(f_1-f_3)['W^*(Y,Z,U,X)-\eta(W^*(Y,Z)U)\eta(X)-g(X,Y)\eta(W^*(\xi,Z)U)\\&+\eta(Y)\eta(W^*(X,Z)U)-g(X,Z)\eta(W^*(Y,\xi)U)+\eta(Z)\eta(W^*(Y,X)U)\\&-g(X,U)\eta(W^*(Y,Z)\xi)+\eta(U)\eta(W^*(Y,Z)X)]=0,\end{aligned} \tag{3.4}$$

which on using the equations (2.18), (2.20) and (2.21) gives

$$\begin{aligned}&(f_1-f_3)['R(Y,Z,U,X)-\frac{1}{2(n-1)}\{g(Z,U)S(Y,X)-g(Y,U)S(X,Z)\\&+\{S(X,Z)\eta(Y)-S(X,Y)\eta(Z)\}\eta(U)+2n(f_1-f_3)\{\eta(Z)g(X,Y)\\&-\eta(Y)g(X,Z)\}\eta(U)\}+\frac{(f_1-f_3)}{(n-1)}\{g(Z,U)g(X,Y)-g(Y,U)g(X,Z)\}]\\&=0.\end{aligned} \tag{3.5}$$

Putting $Z=U=e_i$ in the above equation and taking summation over i, $1\leq i\leq 2n+1$, we get

$$(f_1-f_3)[S(X,Y)+(-n)(f_1-f_3)g(X,Y)]=0. \tag{3.6}$$

This gives either $f_1=f_3$ or

$$S(X,Y)=n(f_1-f_3)g(X,Y),$$

which shows that $M^{2n+1}$ is an Einstein manifold. This completes the proof.

## 4. m-Projectively Flat Generalized Sasakian-Space-Forms

**Theorem 4.1.** *A $(2n+1)-$dimensional $(n>1)$ generalized Sasakian-space-form is m-projectively flat if and only if $f_1=\frac{3f_2}{2(1-n)}=f_3$ provided any arbitrary vector field $Z$ is not pointwise collinear with the characteristic vector field $\xi$.*

**Proof:** For a $(2n+1)-$dimensional $(n>1)$ m-projectively flat generalized Sasakian-space-form, we have from the equation (2.16)

$$R(X,Y)Z=\frac{1}{2(n-1)}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]. \tag{4.1}$$

In view of the equations (2.7) and (2.8) the above equation takes the form

$$\begin{aligned}R(X,Y)Z=&\frac{1}{2(n-1)}[2(2nf_1+3f_2-f_3)\{g(Y,Z)X-g(X,Z)Y\}\\&-(3f_2+(2n-1)f_3)\{\eta(Y)X-\eta(X)Y\}\eta(Z)\\&-(3f_2+(2n-1)f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi].\end{aligned} \tag{4.2}$$

By virtue of the equation (2.6) the above equation reduces to

$$\begin{aligned}&f_1\{g(Y,Z)X-g(X,Z)Y\}+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X\\&+2g(X,\phi Y)\phi Z\}+f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi\\&-g(Y,Z)\eta(X)\xi\}=\frac{1}{2(n-1)}[2(2nf_1+3f_2-f_3)\{g(Y,Z)X-g(X,Z)Y\}\\&-(3f_2+(2n-1)f_3)\{\eta(Y)X-\eta(X)Y\}\eta(Z)\\&-(3f_2+(2n-1)f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi].\end{aligned} \tag{4.3}$$

Now, replacing Z by $\phi Z$ in the above equation, we obtain

$$\begin{aligned}&f_1\{g(Y,\phi Z)X-g(X,\phi Z)Y\}+f_2\{-g(X,Z)\phi Y+g(Y,Z)\phi X\\&-2g(X,\phi Y)Z+\eta(X)\eta(Z)\phi Y-\eta(Y)\eta(Z)\phi X+2\eta(Z)g(X,\phi Y)\xi\}\\&+f_3\{g(X,\phi Z)\eta(Y)-g(Y,\phi Z)\eta(X)\}\xi\\&=\frac{1}{2(n-1)}[2(2nf_1+3f_2-f_3)\{g(Y,\phi Z)X-g(X,\phi Z)Y\}\\&-(3f_2+(2n-1)f_3)\{g(Y,\phi Z)\eta(X)-g(X,\phi Z)\eta(Y)\}\xi],\end{aligned} \tag{4.4}$$

which by putting $X=\xi$ takes the form

$$-2(f_1-f_3)g(Y,\phi Z)\xi=0.$$

Then either

$$f_1=f_3 \tag{4.5}$$

or

$$g(Y,\phi Z)=0. \tag{4.6}$$

Suppose $g(Y,\phi Z)=0$. Replacing $Z$ by $\phi Z$ in the equation (4.6) yields

$$g(Y,\phi^2 Z)=0,$$

which implies

$$Z = \eta(Z)\xi.$$

This shows that $Z$ is collinear with $\xi$.
Again replacing X by $\phi X$ in the equation (4.3), we get

$$\begin{aligned}
&f_1\{g(Y,Z)\phi X - g(\phi X,Z)Y\} + f_2\{g(X,Z)\phi Y - \eta(X)\eta(Z)\phi Y \\
&+ g(Y,\phi Z)X - g(Y,\phi Z)\eta(X)\xi + 2g(X,Y)\phi Z - 2\eta(X)\eta(Y)\phi Z\} \\
&+ f_3\{-\eta(Y)\eta(Z)\phi X + g(\phi X,Z)\eta(Y)\xi\} \\
&= \frac{1}{2(n-1)}[2(2nf_1 + 3f_2 - f_3)\{g(Y,Z)\phi X - g(\phi X,Z)Y\} \\
&- (3f_2 + (2n-1)f_3)\eta(Y)\eta(Z)\phi X + (3f_2 + (2n-1)f_3)g(\phi X,Z)\eta(Y)\xi].
\end{aligned} \tag{4.7}$$

Now putting $Y = \xi$ in the above equation, we obtain

$$[(2n+1)f_1 + 3f_2 - 3f_3](\eta(Z)\phi X - g(\phi X,Z)\xi) = 0.$$

Since $\eta(Z)\phi X - g(\phi X,Z)\xi \neq 0$ in general, we obtain

$$(2n+1)f_1 + 3f_2 - 3f_3 = 0. \tag{4.8}$$

From the equations (4.5) and (4.8), we have

$$f_1 = \frac{3f_2}{2(1-n)}. \tag{4.9}$$

Thus, in view of the equations (4.5) and (4.9), we have

$$f_1 = \frac{3f_2}{2(1-n)} = f_3. \tag{4.10}$$

Conversely, suppose $f_1 = \frac{3f_2}{2(1-n)} = f_3$ satisfies a generalized Sasakian-space-form, then we have

$$S(X,Y) = 0, \tag{4.11}$$

$$QX = 0. \tag{4.12}$$

Also, in view of the equation (2.16), we have

$${}'W^*(X,Y,Z,U) = {}'R(X,Y,Z,U), \tag{4.13}$$

where ${}'W^*(X,Y,Z,U) = g(W^*(X,Y)Z,U)$ and ${}'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. Putting $Y = Z = e_i$ in the equation (4.13) and taking summation over $i$, $1 \leq i \leq 2n+1$, we get

$$\sum_{i=1}^{2n+1} {}'W^*(X,e_i,e_i,U) = \sum_{i=1}^{2n+1} {}'R(X,e_i,e_i,U) = S(X,U). \tag{4.14}$$

In view of the equations (4.13) and (2.6), we have

$$
\begin{aligned}
{}'W^*(X,Y,Z,U) &= f_1\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\
&+ f_2\{g(X,\phi Z)g(\phi Y,U) - g(Y,\phi Z)g(\phi X,U) \\
&+ 2g(X,\phi Y)g(\phi Z,U)\} + f_3\{\eta(X)\eta(Z)g(Y,U) \\
&- \eta(Y)\eta(Z)g(X,U) + g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U)\}.
\end{aligned} \tag{4.15}
$$

Now, putting $Y = Z = e_i$ in the above equation and taking summation over $i$, $1 \le i \le 2n+1$, we get

$$
\begin{aligned}
\sum_{i=1}^{2n+1} {}'W^*(X,e_i,e_i,U) &= 2nf_1 g(X,U) + 3f_2 g(\phi X,\phi U) \\
&- f_3\{(2n-1)\eta(X)\eta(U) + g(X,U)\}.
\end{aligned} \tag{4.16}
$$

In view of the equations (4.16), (4.14) and (4.11), we have

$$2nf_1 g(X,U) + 3f_2 g(\phi X,\phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X,U)\} = 0. \tag{4.17}$$

Putting $X = W = e_i$ in the above equation and taking summation over $i$, $1 \le i \le 2n+1$, we get $f_1 = 0$. Then in view of the equation (4.10), $f_2 = f_3 = 0$. Therefore, we obtain from the equation (2.6)

$$R(X,Y)Z = 0. \tag{4.18}$$

Hence in view of the equations (4.18), (4.11) and (4.12), we have $W^*(X,Y)Z = 0$. This completes the proof.

## 5. $\xi$-m-Projectively Flat Generalized Sasakian-Space-Forms

**Definition 5.1.** A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form is said to be $\xi$-m-projectively flat [14], if $W^*(X,Y)\xi = 0$ for all $X, Y \in TM$.

**Theorem 5.1.** *A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form is $\xi$-m-projectively flat if and only if it is an $\eta-$Einstein manifold.*

**Proof:** Let us consider a generalized Sasakian-space-form is $\xi$-m-projectively flat, i.e. $W^*(X,Y)\xi = 0$. Then in view if equation (2.16), we have

$$R(X,Y)\xi = \frac{1}{2(n-1)}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY]. \tag{5.1}$$

By virtue of the equations (2.2), (2.10) and (2.13) the above equation reduces to

$$\eta(Y)QX - \eta(X)QY = -2(f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{5.2}$$

which by putting $Y = \xi$ gives

$$QX = 2(f_1 - f_3)[-X + (n+1)\eta(X)\xi]. \tag{5.3}$$

Now, taking the inner product of the above equation with U, we get

$$S(X,U) = 2(f_1 - f_3)[-g(X,U) + (n+1)\eta(X)\eta(U)], \tag{5.4}$$

which shows that generalized Sasakian-space-form is an $\eta$-Einstein manifold. Conversely, suppose the equation (5.4) is satisfied. Then by virtue of the equations (5.3) and (5.1), we have $W^*(X,Y)\xi = 0$. This completes the proof.

## 6. m-Projectively Recurrent Generalized Sasakian-Space-Forms

**Definition 6.1.** A non-flat Riemannian manifold $M^{2n+1}$ is said to be m-projectively recurrent if its m-projective curvature tensor $W^*$ satisfies the condition

$$\nabla W^* = A \otimes W^*, \tag{6.1}$$

where A is a non-zero 1-form.

**Theorem 6.1.** *If a $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form is m-projectively recurrent, then either $f_1 = f_3$ or it is an Einstein manifold.*

**Proof:** We define a function $f^2 = g(W^*, W^*)$ on $M^{2n+1}$, where the metric g is extended to the inner product between the tensor fields. Then we have

$$f(Yf) = f^2 A(Y).$$

This can be written as

$$Yf = f(A(Y)), (f \neq 0). \tag{6.2}$$

From the above equation, we have

$$X(Yf) - Y(Xf) = \{XA(Y) - YA(X) - A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on $M^{2n+1}$. Then

$$dA(X,Y) = 0, \tag{6.3}$$

i.e. 1-form A is closed.
Now from

$$(\nabla_Y W^*)(Z,U)V = A(Y)W^*(Z,U)V,$$

we have

$$(\nabla_X \nabla_Y W^*)(Z,U)V = \{XA(Y) + A(X)A(Y)\}W^*(Z,U)V. \tag{6.4}$$

In view of the equations (6.3) and (6.4) , we have

$$(R(X,Y).W^*)(Z,U)V = [2dA(X,Y)]W^*(Z,U)V = 0. \tag{6.5}$$

Thus in view of Theorem (3.1), we have either $f_1 = f_3$ or $M^{2n+1}$ is an Einstein manifold.

## 7. Generalized Sasakian-Space-Forms Satisfying $W^*.S = 0$.

**Theorem 7.1.** *A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form satisfying $W^*.S = 0$ is an $\eta$-Einstein manifold.*

**Proof:** Let us consider a generalized Sasakian-space-form $M^{2n+1}$ satisfying $W^*(\xi,X).S = 0$. In this case we can write

$$S(W^*(\xi,X)Y,Z) + S(Y,W^*(\xi,X)Z) = 0. \tag{7.1}$$

In view of the equation (2.19) the above equation reduces to

$$\begin{aligned}&(f_1-f_3)[2n(f_1-f_3)\{g(X,Y)\eta(Z)+g(X,Z)\eta(Y)\}\\&-\{\eta(Y)S(X,Z)+\eta(Z)S(X,Y)\}]+\frac{1}{2}[2n(f_1-f_3)\{S(X,Y)\eta(Z)\\&+S(X,Z)\eta(Y)\}-\{\eta(Y)S(QX,Z)+\eta(Z)S(QX,Y)\}]=0.\end{aligned} \tag{7.2}$$

Now, putting $Z = \xi$ in the above equation, we get

$$S(QX,Y) = 2(f_1-f_3)[(n-1)S(X,Y)+2n(f_1-f_3)g(X,Y)]. \tag{7.3}$$

By virtue of the equation (2.7) the above equation takes the form

$$S(X,Y) = \frac{2n(f_1-f_3)}{K}[2(f_1-f_3)g(X,Y)+(3f_2+(2n-1)f_3)\eta(X)\eta(Y)],$$

where $K = 2nf_1 + 3f_2 + (2n-3)f_3$, which shows that $M^{2n+1}$ is an $\eta$-Einstein manifold. This completes the proof.

## 8. Generalized Sasakian-Space-Forms Satisfying $W^*.R = 0$.

**Theorem 8.1.** *A $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form satisfying $W^*.R = 0$ is an $\eta$-Einstein manifold.*

**Proof:** Suppose $M^{2n+1}$ satisfying $(W^*(\xi,X).R)(Y,Z)U=0$, then it can be written as

$$\begin{aligned}&W^*(\xi,X)R(Y,Z)U-R(W^*(\xi,X)Y,Z)U-R(Y,W^*(\xi,X)Z)U\\&-R(Y,Z)W^*(\xi,X)U=0,\end{aligned}\tag{8.1}$$

which on using the equation (2.19) takes the form

$$\begin{aligned}&\frac{(f_1-f_3)}{(n-1)}[-g(X,R(Y,Z)U)\xi+\eta(R(Y,Z)U)X+g(X,Y)R(\xi,Z)U\\&-\eta(Y)R(X,Z)U+g(X,Z)R(Y,\xi)U-\eta(Z)R(Y,X)U\\&+g(X,U)R(Y,Z)\xi-\eta(U)R(Y,Z)X]-\frac{1}{2(n-1)}[S(X,R(Y,Z)U)\xi\\&-\eta(R(Y,Z)U)QX-S(X,Y)R(\xi,Z)U+\eta(Y)R(QX,Z)U\\&-S(X,Z)R(Y,\xi)U+\eta(Z)R(Y,QX)U-S(X,U)R(Y,Z)\xi\\&+\eta(U)R(Y,Z)QX]=0.\end{aligned}\tag{8.2}$$

Taking the inner product of the above equation with $\xi$, we get

$$\begin{aligned}&\frac{(f_1-f_3)}{(n-1)}[-g(X,R(Y,Z)U)+\eta(R(Y,Z)U)\eta(X)+g(X,Y)\eta(R(\xi,Z)U)\\&-\eta(Y)\eta(R(X,Z)U)+g(X,Z)\eta(R(Y,\xi)U)-\eta(Z)\eta(R(Y,X)U)\\&+g(X,U)\eta(R(Y,Z)\xi)-\eta(U)\eta(R(Y,Z)X)]-\frac{1}{2(n-1)}[S(X,R(Y,Z)U)\\&-\eta(R(Y,Z)U)\eta(QX)-S(X,Y)\eta(R(\xi,Z)U)+\eta(Y)\eta(R(QX,Z)U)\\&-S(X,Z)\eta(R(Y,\xi)U)+\eta(Z)\eta(R(Y,QX)U)-S(X,U)\eta(R(Y,Z)\xi)\\&+\eta(U)\eta(R(Y,Z)QX)]=0.\end{aligned}\tag{8.3}$$

Now using the equations (2.6), (2.11) and (2.12) in the above equation, we get

$$\begin{aligned}&(f_1-f_3)[-f_1\{g(Z,U)g(X,Y)-g(Y,U)g(X,Z)\}\\&-f_2\{g(Y,\phi U)g(\phi Z,X)-g(Z,\phi U)g(\phi Y,X)+2g(Y,\phi Z)g(\phi U,X)\}\\&-f_3\{\eta(Y)\eta(U)g(X,Z)-\eta(Z)\eta(U)g(X,Y)+g(Y,U)\eta(Z)\eta(X)\\&-g(Z,U)\eta(Y)\eta(X)\}+(f_1-f_3)\{g(Z,U)g(X,Y)-g(Y,U)g(X,Z)\}]\\&-\frac{1}{2}[f_1\{g(Z,U)S(X,Y)-g(Y,U)S(X,Z)\}+f_2\{g(Y,\phi U)S(\phi Z,X)\\&-g(Z,\phi U)S(\phi Y,X)+2g(Y,\phi Z)S(\phi U,X)\}+f_3\{\eta(Y)\eta(U)S(X,Z)\\&-\eta(Z)\eta(U)S(X,Y)+2n(f_1-f_3)\eta(X)\eta(Z)g(Y,U)\\&-2n(f_1-f_3)\eta(X)\eta(Y)g(Z,U)\}-(f_1-f_3)\{g(Z,U)S(X,Y)\\&-g(Y,U)S(X,Z)\}]=0.\end{aligned}\tag{8.4}$$

Putting $Z=U=e_i$ in the above equation and summing over i, $1\le i\le 2n+1$, we get

$$S(X,Y)=2(f_1-f_3)[-g(X,Y)+(n+1)\eta(X)\eta(Y)],\tag{8.5}$$

which shows that $M^{2n+1}$ is an $\eta-$Einstein manifold. This completes the proof.

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Ser. Math. Inform. Vol. 33, No 3 (2018), 375-387
https://doi.org/10.22190/FUMI1803375S

# RICCI SOLITONS IN $\alpha$-COSYMPLECTIC MANIFOLDS *

**Jay Prakash Singh and Chawngthu Lalmalsawma**

**Abstract.** The aim of the paper is to study Ricci solitons in $\alpha$-cosymplectic manifolds. Projective, pseudo projective and Weyl conformal curvatures in an $\alpha$-cosymplectic manifolds admitting Ricci solitons have been studied under certain curvature conditions. Also, gradient Ricci solitons in $\alpha$-cosymplectic manifolds have been studied.
**Keywords:** Ricci soliton, gradient Ricci soliton, $\alpha$-cosymplectic manifolds, cosympletic manifolds, $\alpha$-Kenmatsu manifolds

## 1. Introduction

The concept of Ricci soliton was introduced by Hamilton [8] while studying the Ricci flow on surfaces. It is a generalization of an Einstein metric and is defined as a triple $(g, V, \lambda)$ with $g$ a Riemannian metric, $V$ a vector field, and $\lambda$ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $S$ is the Ricci tensor of type $(0, 2)$ and $\pounds$ denotes the Lie derivative operator along the vector field $V$.

The Ricci soliton is said to be shrinking, steady and expanding accordingly as $\lambda$ is negative, zero and positive, respectively [6]. If the vector field $V$ is the gradient of a potential function $-f$, then $g$ is called a gradient Ricci soliton and the equation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

In 2008 Sinha and Sharma [17] started the study of Ricci solitons in contact manifolds. Later Ricci solitons in contact and almost contact manifolds were studied by many authors such as: Ricci solitons in contact metric manifolds by Tripathi

Received August 11, 2017; accepted March 13, 2018
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53D15
*The authors were supported in part by ...

[18], Ricci solitons in manifolds with a quasi-constant curvature by Bejan [2], Ricci solitons in Lorentzian $\alpha$-Sasakian manifolds by Bagewadi [1], Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds by Turan, De and Yildiz [19], Ricci solitons in Kenmotsu manifolds by Nagaraja [12], etc.

The paper is organized as follows: after the introduction and preliminaries, in Section 3 we prove that the Ricci soliton in a Ricci semi-symmetric $\alpha$-cosymplectic manifold of dimension $n$ $(n \geq 2)$, is steady. Section 4 is dedicated to the study of the pseudo-projective semi-symmetric manifold and the projective semi-symmetric manifold. In Section 5 we prove that a Weyl semi-symmetric $\alpha$-Kenmotsu manifold of dimension $n$ $(n \geq 2)$, admitting a Ricci soliton is conformally flat. In Section 6 we study the $\alpha$-cosymplectic manifold satisfying $P(\xi, X) \cdot S = 0$. Finally, we prove that if a gradient Ricci soliton in an $\alpha$-cosymplectic manifold of dimension $n$ $(n \geq 2)$ is expanding, then it is an $\eta$-Einstein manifold.

## 2. Preliminaries

An $n$-dimensional smooth manifold $M$ is said to be an almost contact metric manifold if it admits an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ compatible with $(\phi, \xi, \eta)$ satisfying [3]

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0,$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

On such a manifold, the fundamental form $\Phi$ of $M$ is defined as

$$\Phi(X, Y) = g(\phi X, Y), \qquad X, Y \in \Gamma(TM).$$

In 1967 Blair [4] defined the cosymplectic structure as a quasi-Sasakian structure satisfying $d\eta = 0$. It is to be noted that the notion of cosymplectic manifold introduced by Libermann [11] is different from that of Blair [4]. An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is said to be almost cosymplectic [7] if $d\eta = 0$ and $d\Phi = 0$, where $d$ is the exterior differential operator. The manifold defined by $M = N \times \mathbb{R}$, where $N$ is an almost Kählerian manifold and $\mathbb{R}$ is the real line is the simplest example of the almost cosymplectic manifold [13]. An almost contact manifold $(M, \phi, \xi, \eta)$ is said to be normal if the Nijenhuis torsion

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2(X, Y) + 2d\eta(X, Y)\xi$$

vanishes for any vector fields $X$ and $Y$. A normal almost cosymplectic manifold is a cosymplectic manifold.

An almost contact metric manifold M is said to be almost $\alpha$-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, $\alpha$ being a non-zero real constant.

Kim and Pak [10] combined almost $\alpha$-Kenmotsu and almost cosymplectic manifolds into a new class called almost $\alpha$-cosymplectic manifolds, where $\alpha$ is a scalar. If we join these two classes, we obtain a new notion of an almost $\alpha$-cosymplectic manifold, which is defined by the following formula

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number $\alpha$. A normal almost $\alpha$-cosymplectic manifold is called an $\alpha$-cosymplectic manifold. An $\alpha$-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or $\alpha$-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$.

On such an $\alpha$-cosymplectic manifold, we have

$$(\nabla_X \phi)Y = \alpha\big[g(\phi X, Y)\xi - \eta(Y)\phi X\big] \tag{2.1}$$

and

$$\nabla_X \xi = -\alpha\phi^2 X = \alpha[X - \eta(X)\xi]. \tag{2.2}$$

On an $\alpha$-cosymplectic manifold $M$, the following relations are held ([14], [15])

$$R(\xi, X)Y = \alpha^2\big[\eta(Y)X - g(X, Y)\xi\big], \tag{2.3}$$

$$R(X, Y)\xi = \alpha^2\big[\eta(X)Y - \eta(Y)X\big], \tag{2.4}$$

$$S(\xi, X) = -\alpha^2(n-1)\eta(X), \tag{2.5}$$

$$\eta\big(R(X, Y)Z\big) = \alpha^2\big[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\big]. \tag{2.6}$$

Using (2.2) we have

$$\pounds_\xi g(X, Y) = 2\alpha g(X, Y) - 2\alpha\eta(X)\eta(Y). \tag{2.7}$$

From (1.1) and (2.7) we get

$$S(X, Y) = \alpha\eta(X)\eta(Y) - (\lambda + \alpha)g(X, Y). \tag{2.8}$$

Equation (2.8) yields

$$QX = \alpha\eta(X)\xi - (\lambda + \alpha)X, \tag{2.9}$$

$$S(X, \xi) = -\lambda\eta(X), \tag{2.10}$$

$$r = (1-n)\alpha - \lambda n. \tag{2.11}$$

Comparing (2.5) and (2.10) we get

$$\lambda = \alpha^2(n-1). \tag{2.12}$$

Since $\alpha^2 \geq 0$, for $\alpha \in \mathbb{R}$, from Equation (2.12) we get $\lambda \geq 0$, for all $n \geq 2$. Thus we can state the following:

**Lemma 2.1.** *A Ricci soliton in an $n$-dimensional $\alpha$-cosymplectic manifold, $n \geq 2$, is either steady or expanding.*

We have already stated that an $\alpha$-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or $\alpha$-Kenmotsu under the condition $\alpha \neq 0$, for $\alpha \in \mathbb{R}$. Thus we can state the following lemmas:

**Lemma 2.2.** *A Ricci soliton in an $n$-dimensional $\alpha$-cosymplectic manifold, $n \geq 2$, is steady if and only if it is a cosymplectic manifold.*

**Lemma 2.3.** *A Ricci soliton in an $n$-dimensional $\alpha$-cosymplectic manifold, $n \geq 2$, is expanding if and only if it is an $\alpha$-Kenmotsu manifold.*

## 3. Ricci semi-symmetric $\alpha$-cosymplectic manifold, $n \geq 2$

Consider an $\alpha$-cosymplectic manifold which is Ricci semi-symmetric. Then we have [5]

$$R(X,Y) \cdot S = 0.$$

Now we assume that the condition

$$R(\xi, X) \cdot S(Y,Z) = 0 \tag{3.1}$$

holds in $M$.

From (3.1) it follows that

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{3.2}$$

Using (2.3), (2.8) and (2.10), we get from (3.2)

$$\alpha^2 \Big[ 2\alpha\eta(X)\eta(Y)\eta(Z) - \alpha\eta(Y)g(X,Z) - \alpha\eta(Z)g(X,Y) \Big] = 0,$$

or

$$\alpha^3 \Big[ 2\eta(X)\eta(Y)\eta(Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \Big] = 0. \tag{3.3}$$

Contracting (3.3) over $X$ and $Y$ we get

$$\alpha^3(n-1)\eta(Z) = 0. \tag{3.4}$$

In general, $\eta(Z) \neq 0$. Therefore, $\alpha = 0$. Thus we can state the following:

**Theorem 3.1.** *A Ricci semi-symmetric $\alpha$-cosymplectic manifold, $n \geq 2$, admitting Ricci soliton is a cosymplectic manifold.*

By virtue of Lemma 2.2 we have

**Corollary 3.1.** *A Ricci soliton in a Ricci semi-symmetric $\alpha$-cosymplectic manifold, $n \geq 2$, is steady.*

## 4. Pseudo projective semi-symmetric $\alpha$-cosymplectic manifold, $n \geq 2$

We consider the pseudo projective curvature tensor $P$ of type $(1,3)$ which is defined by [16]

$$\begin{aligned} P(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\big(\frac{a}{n-1} + b\big)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.1}$$

where $R$ is a Riemannian curvature tensor of type $(1,3)$, $r$ is the scalar curvature and $a$ and $b$ are a non-zero constant. From (4.1) we can define a $(0,4)$ type pseudo-projective curvature tensor $\hat{P}$ as follows

$$\begin{aligned} \hat{P}(X,Y,Z,W) &= a\hat{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &- \frac{r}{n}\big(\frac{a}{n-1} + b\big)[g(Y,Z)g(X,W) - g(Y,U)g(Y,W)]. \end{aligned}$$

where $\hat{R}$ is a Riemannian curvature tensor of type $(0,4)$, from which it follows that

$$\sum_{i=1}^{n} \hat{P}(e_i,Y,Z,e_i) = [a + (n-1)b]\big[S(Y,Z) - \frac{r}{n}g(Y,Z)\big]. \tag{4.2}$$

Again from (4.1) we obtain

$$\eta\big(P(X,Y)Z\big) = \big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda+\alpha)b\big] \times \big[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big],$$

or

$$\eta\big(P(X,Y)Z\big) = \beta[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)], \tag{4.3}$$

where $\beta = \big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda+\alpha)b\big]$.

Now we assume that the condition

$$R(\xi,X) \cdot P(Y,Z)W = 0 \tag{4.4}$$

holds in $M$.

From (4.4) it follows that

$$\begin{aligned} R(\xi,X)P(Y,Z)W - P(R(\xi,X)Y,Z)W - P(Y,R(\xi,X)Z)W \\ -P(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{4.5}$$

Using (2.3) in (4.5) we find

$$\begin{aligned} \alpha^2\Big[\eta\big(P(Y,Z)W\big)X - \hat{P}(Y,Z,W,X)\xi - \eta(Y)P(X,Z)W \\ +g(X,Y)P(\xi,Z)W - \eta(Z)P(Y,X)W + g(X,Z)P(Y,\xi)W \\ -\eta(W)P(Y,Z)X + g(X,W)P(Y,Z)\xi\Big] = 0, \end{aligned} \tag{4.6}$$

where $\hat{P}(Y,Z,W,X) = g\big(X, P(Y,Z)W\big)$.

Taking the inner product of (4.5) with $\xi$ we get

$$\begin{aligned} \alpha^2\Big[\eta\big(P(Y,Z)W\big)\eta(X) - \hat{P}(Y,Z,W,X) - \eta(Y)\eta\big(P(X,Z)W\big) \\ +g(X,Y)\eta\big(P(\xi,Z)W\big) - \eta(Z)\eta\big(P(Y,X)W\big) + g(X,Z)\eta\big(P(Y,\xi)W\big) \\ -\eta(W)\eta\big(P(Y,Z)X\big) + g(X,W)\eta\big(P(Y,Z)\xi\big)\Big] = 0. \end{aligned} \tag{4.7}$$

By virtue of (4.3), (4.7) yields

$$\alpha^2\Big[\hat{P}(Y,Z,W,X) + \beta\big\{g(X,Y)g(Z,W) - g(X,Z)g(Y,W)\big\}\Big] = 0. \tag{4.8}$$

Contracting (4.8) over $X$ and $Y$ and using (4.2) we get

$$\alpha^2\Big[[a+(n-1)b]\big\{S(Z,W) - \frac{r}{n}g(Z,W)\big\} + \beta(n-1)g(Z,W)\Big] = 0. \tag{4.9}$$

We suppose that the $\alpha$-cosymplectic manifold is an $\alpha$-Kenmotsu manifold i.e., $\alpha \neq 0$. Thus (4.9) can be written as

$$S(Z,W) = \Big[\frac{r}{n} - \frac{\beta(n-1)}{a+(n-1)b}\Big]g(Z,W),$$

or

$$S(Z,W) = \rho g(Z,W), \tag{4.10}$$

where $\rho = \Big[\frac{r}{n} - \frac{\beta(n-1)}{a+(n-1)b}\Big]$.

Hence we have the following theorem:

**Theorem 4.1.** *A pseudo-projective semi-symmetric $\alpha$-Kenmotsu manifold, $n \geq 2$, admitting a Ricci soliton is an Einstein manifold.*

Again, contracting (4.9) over $Z$ and $W$, we get

$$n(n-1)\alpha^2\beta = 0. \tag{4.11}$$

From (4.11) it follows that

$$\alpha^2\beta = 0,$$

or

$$\alpha^2\big[a\alpha^2 + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + (\lambda+\alpha)b\big] = 0. \tag{4.12}$$

If we put $a = 1$ and $b = -\frac{1}{(n-1)}$ then (4.1) takes the form

$$\begin{aligned} P(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y] \\ &= \tilde{P}(X,Y)Z, \end{aligned} \tag{4.13}$$

where $\tilde{P}(X,Y)Z$ is the projective curvature tensor and is a particular case of $P$.

Now putting $a = 1$ and $b = -\frac{1}{(n-1)}$ in (4.12) and making use of (2.12) we get

$$\alpha^3 = 0,$$

or

$$\alpha = 0. \tag{4.14}$$

Thus we can state the following:

**Theorem 4.2.** *A projective semi-symmetric $\alpha$-cosymplectic manifold, $n \geq 2$, admitting a Ricci soliton is a cosymplectic manifold.*

By virtue of Lemma 2.2 we have

**Corollary 4.1.** *A Ricci soliton in a projective semi-symmetric $\alpha$-cosymplectic manifold, $n \geq 2$, is steady.*

## 5. Weyl semi-symmetric $\alpha$-cosymplectic manifold, $n > 2$

We consider the Weyl conformal curvature tensor $C$ of type $(1,3)$ which is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\big[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ -S(X,Z)Y\big] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{5.1}$$

where $R$ is a Riemannian curvature tensor of type $(1,3)$. From (4.1) we can define a $(0,4)$ type Weyl conformal curvature tensor $\hat{C}$ as follows:

$$\begin{aligned} \hat{C}(X,Y,Z,W) &= \hat{R}(X,Y,Z,W) - \frac{1}{n-2}\big[g(Y,Z)S(X,W) \\ &- g(X,Z)S(Y,W) + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\big] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned}$$

where $\hat{R}$ is a Riemannian curvature tensor of type $(0,4)$. From which it follows that

$$\sum_{i=1}^{n} \hat{C}(e_i,Y,Z,e_i) = 0. \tag{5.2}$$

Again, from (5.1) we obtain

$$\eta\big(C(X,Y)Z\big) = 0. \tag{5.3}$$

Now we assume that the condition

$$R(\xi,X)\cdot C(Y,Z)W = 0 \tag{5.4}$$

holds in $M$.

From (5.4) it follows that

$$\begin{aligned} R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W - C(Y,R(\xi,X)Z)W \\ -C(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{5.5}$$

Using (2.3) in (5.5) we find

$$\begin{aligned} \alpha^2\Big[\eta\big(C(Y,Z)W\big)X - \hat{C}(Y,Z,W,X)\xi - \eta(Y)C(X,Z)W \\ +g(X,Y)C(\xi,Z)W - \eta(Z)C(Y,X)W + g(X,Z)C(Y,\xi)W \\ -\eta(W)C(Y,Z)X + g(X,W)C(Y,Z)\xi\Big] = 0, \end{aligned} \tag{5.6}$$

where $\hat{C}(Y,Z,W,X) = g\big(X,C(Y,Z)W\big)$.

Taking the inner product of (5.6) with $\xi$ we get

$$\begin{aligned} \alpha^2\Big[\eta\big(C(Y,Z)W\big)\eta(X) - \hat{C}(Y,Z,W,X) - \eta(Y)\eta\big(C(X,Z)W\big) \\ +g(X,Y)\eta\big(C(\xi,Z)W\big) - \eta(Z)\eta\big(C(Y,X)W\big) + g(X,Z)\eta\big(C(Y,\xi)W\big) \\ -\eta(W)\eta\big(C(Y,Z)X\big) + g(X,W)\eta\big(C(Y,Z)\xi\big)\Big] = 0. \end{aligned} \tag{5.7}$$

By virtue of Equation (5.3), (5.7) yields

$$\alpha^2\hat{C}(Y,Z,W,X) = 0. \tag{5.8}$$

We suppose that the $\alpha$-cosymplectic manifold is an $\alpha$-Kenmotsu manifold i.e., $\alpha \neq 0$. Then we have

$$\hat{C}(Y,Z,W,X) = 0. \tag{5.9}$$

Thus we can state the following:

**Theorem 5.1.** *A Weyl semi-symmetric $\alpha$-Kenmotsu manifold, $n > 2$, admitting a Ricci soliton is conformally flat.*

### 6. $\alpha$-cosymplectic manifold, $n \geq 2$ satisfying $P(\xi, X) \cdot S = 0$

Making use of (2.3), (2.8) and (2.10) in (4.1) we get

$$P(\xi,Y)Z = \big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + \lambda b\big][\eta(Z)Y - g(Y,Z)\xi] \\ +\alpha b[\eta(Y)\eta(Z)\xi - g(Y,Z)\xi],$$

or

$$P(\xi,Y)Z = \beta[\eta(Z)Y - g(Y,Z)\xi] + \gamma[\eta(Y)\eta(Z)\xi - g(Y,Z)\xi], \tag{6.1}$$

where $\beta = \big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big) + \lambda b\big]$ and $\gamma = \alpha b$.

Now we consider that a given manifold satisfies

$$P(\xi, X) \cdot S(Y, Z) = 0,$$

from which it follows that

$$S(P\big(\xi, X)Y, Z\big) + S\big(Y, P(\xi, X)Z\big) = 0. \tag{6.2}$$

Using (6.1) in (6.2) yields

$$\beta\eta(Y)S(X,Z) - \beta g(X,Y)S(\xi,Z) + \gamma\eta(X)\eta(Y)S(\xi,Z) \\ -\gamma g(X,Y)S(\xi,Z) + \beta\eta(Z)S(X,Y) - \beta g(X,Z)S(\xi,Y) \\ +\gamma\eta(X)\eta(Z)S(\xi,Y) - \gamma g(X,Z)S(\xi,Y) = 0. \tag{6.3}$$

Making use of (2.8) and (2.10) in (6.3) we get

$$\big(\alpha\beta - \lambda\gamma\big)[2\eta(X)\eta(Y)\eta(Z) - g(X,Z)\eta(Y) \\ -g(X,Y)\eta(Z)] = 0. \tag{6.4}$$

Contracting (6.4) over $X$ and $Y$ we get

$$\big(\alpha\beta - \lambda\gamma\big)(1-n)\eta(Z) = 0. \tag{6.5}$$

Putting $Z = \xi$ in (6.5) yields

$$\big(\alpha\beta - \lambda\gamma\big)(1-n) = 0, \tag{6.6}$$

from which it follows that

$$\big(\alpha\beta - \lambda\gamma\big) = 0,$$

or

$$\alpha\big[\alpha^2 a + \frac{r}{n}\big(\frac{a}{n-1} + b\big)\big] = 0. \tag{6.7}$$

We suppose that the $\alpha$-cosymplectic manifold is an $\alpha$-Kenmotsu manifold i.e., $\alpha \neq 0$. Then (6.7) yields

$$\big[\alpha^2 a + \frac{r}{n}(\frac{a}{n-1} + b)\big] = 0,$$

or

$$\alpha^2 = -\frac{r}{n}(\frac{1}{n-1} + \frac{b}{a}). \tag{6.8}$$

Thus we can state the following:

**Theorem 6.1.** *If an $\alpha$-cosymplectic manifold, $n \geq 2$, admitting a Ricci soliton and satisfying $P(\xi, X) \cdot S = 0$ is an $\alpha$-Kenmotsu manifold, then it satisfies $\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big)$.*

By virtue of Lemma 2.3 we have

**Corollary 6.1.** *If a Ricci soliton in an $\alpha$-cosymplectic manifold, $n \geq 2$, satisfying $P(\xi, X) \cdot S = 0$ is expanding, then $\alpha^2 = -\frac{r}{n}\big(\frac{1}{n-1} + \frac{b}{a}\big)$.*

For $a = 1$ and $b = -\frac{1}{(n-1)}$, from (6.6)

$$\alpha^3 = 0,$$

or

$$\alpha = 0. \tag{6.9}$$

Thus we can state the following:

**Theorem 6.2.** *An $\alpha$-cosymplectic manifold, $n \geq 2$, admitting a Ricci soliton and satisfying $\tilde{P}(\xi, X) \cdot S = 0$ is a cosymplectic manifold.*

By virtue of Lemma 2.3 we have

**Corollary 6.2.** *A Ricci solitons in an $\alpha$-cosymplectic manifold, $n \geq 2$, satisfying $\tilde{P}(\xi, X) \cdot S = 0$ is steady.*

## 7. Gradient Ricci soliton in $\alpha$-cosymplectic manifolds

From Equation (1.2) we have

$$\nabla\nabla f = S + \lambda g. \tag{7.1}$$

This can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{7.2}$$

where $D$ is the gradient operator of $g$. Using (7.2) we can obtain

$$R(X,Y)Df = (\nabla_X Q)Y + (\nabla_Y Q)X. \tag{7.3}$$

Taking the inner product of (7.3) with $\xi$ we get

$$g\big(R(X,Y)Df,\xi\big) = g\big((\nabla_X Q)Y,\xi\big) + g\big((\nabla_Y Q)X,\xi\big). \tag{7.4}$$

Using (2.2) and (2.9) we have

$$g\big((\nabla_\xi Q)Y,\xi\big) = 0, \tag{7.5}$$

and

$$g\big((\nabla_Y Q)\xi,\xi\big) = 0. \tag{7.6}$$

By virtue of (7.5) and (7.6), Equation (7.4) yields

$$g\big(R(\xi,Y)Df,\xi\big) = 0. \tag{7.7}$$

Again, using (2.3) in (7.7) we get

$$g\big(R(\xi,Y)Df,\xi\big) = \alpha^2\big[\eta(Y)\eta(Df) - g(Y,Df)\big]. \tag{7.8}$$

From (7.7) and (7.8) we have

$$\alpha^2\big[\eta(Y)\eta(Df) - g(Y,Df)\big] = 0. \tag{7.9}$$

Now we suppose that $\alpha \neq 0$, i.e., the given manifold is an $\alpha$-Kenmotsu manifold. Equation (7.9) yields

$$\eta(Y)\eta(Df) = g(Y,Df). \tag{7.10}$$

From (7.10) we obtain

$$Df = (\xi f)\xi. \tag{7.11}$$

Using (7.11) in (7.2)

$$Y(\xi f)\xi + \alpha(\xi f)\big[Y - \eta(Y)\xi\big] = QY + \lambda Y. \tag{7.12}$$

Taking the inner product of (7.12) with $X$, we obtain

$$Y(\xi f)\eta(X) + \alpha(\xi f)\big[g(X,Y) - \eta(X)\eta(Y)\big] = S(X,Y) + \lambda g(X,Y). \tag{7.13}$$

Putting $X = \xi$ and using (2.10) in (7.13) we get

$$Y(\xi f) = S(\xi,Y) + \lambda\eta(Y) = 0. \tag{7.14}$$

From (7.14) it is clear that $\xi f$ is constant. Thus (7.13) in (7.14) yields

$$\alpha(\xi f)\big[g(X,Y) - \eta(X)\eta(Y)\big] = S(X,Y) + \lambda g(X,Y),$$

or

$$S(X,Y) = \big[\alpha(\xi f) - \lambda\big]g(X,Y) - \alpha(\xi f)\eta(X)\eta(Y). \tag{7.15}$$

Hence we can state the following:

**Theorem 7.1.** *If an $\alpha$-cosymplectic manifold, $n \geq 2$, admitting a gradient Ricci soliton is an $\alpha$-Ketmotsu manifold, then it is an $\eta$-Einstein manifold.*

By virtue of Lemma 2.2 we have

**Corollary 7.1.** *If a gradient Ricci soliton in an $\alpha$-cosymplectic manifold, $n \geq 2$, is expanding, then it is an $\eta$-Einstein manifold.*

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Ser. Math. Inform. Vol. 33, No 3 (2018), 389-399
https://doi.org/10.22190/FUMI1803389Y

# PROJECTIVE CHANGE BETWEEN RANDERS METRIC AND EXPONENTIAL $(\alpha, \beta)$-METRIC

**Ganga Prasad Yadav and Paras Nath Pandey**

**Abstract.** In this paper, we find conditions to characterize the projective change between two $(\alpha, \beta)$-metrics, such as exponential $(\alpha, \beta)$-metric, $L = \alpha e^{\frac{\beta}{\alpha}}$ and Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ on a manifold with dim $n > 2$, where $\alpha$ and $\overline{\alpha}$ are two Riemannian metrics, $\beta$ and $\overline{\beta}$ are two non-zero 1-forms. We also discuss special curvature properties of two classes of $(\alpha, \beta)$-metrics.

**Keywords**: Finsler space, $(\alpha, \beta)$-metric, projective change, Randers metric, Berwlad, Riemannian metric.

## 1. Introduction

M. Matsumoto [10] introduced the concept of $(\alpha, \beta)$-metric on a differentiable manifold with local coordinates $x^i$, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M^n$. M. Hashiguchi and Y. Ichijyo [6] studied some special $(\alpha, \beta)$-metrics and obtained interesting results. In the projective Finsler geometry, there is a remarkable theorem called Rapcsak [14] theorem, which plays an important role in the projective geometry of Finsler spaces. In fact, this theorem gives the necessary and sufficient condition for a Finsler space to be projective to another Finsler space.

The projective change between two Finsler spaces has been studied by many authors ([2], [5], [8], [11], [12], [16]). In 1994, S. Bacso and M. Matsumoto [2] studied the projective change between Finsler spaces with $(\alpha, \beta)$-metric. In 2008, H. S. Park and Y. Lee [11] studied the projective changes between a Finsler space with $(\alpha, \beta)$-metric and the associated Riemannian metric. The authors Z. Shen and Civi Yildirim [16] studied a class of projectively flat metrics with a constant flag curvature. In 2009, Ningwei Cui and Yi-Bing Shen [5] studied projective change between two classes of $(\alpha, \beta)$-metrics. Also the author N. Cui [4] studied the $S$-curvature of some $(\alpha, \beta)$-metrics. In this paper, we find conditions to characterize

Received November 17, 2017; accepted April 19, 2018
2010 *Mathematics Subject Classification.* Primary 53B40; Secondary 53C60

the projective change between two $(\alpha,\beta)$-metrics, such as the exponential $(\alpha,\beta)$-metric, $L=\alpha e^{\frac{\beta}{\alpha}}$ and Randers metric $\overline{L}=\overline{\alpha}+\overline{\beta}$ on a manifold with dim $n>2$, where $\alpha$ and $\overline{\alpha}$ are two Riemannian metrics, $\beta$ and $\overline{\beta}$ are two non-zero 1-forms. In addition, we discuss special curvature properties of two classes of $(\alpha,\beta)$-metrics.

## 2. Preliminaries

The terminology and notation are referred to ([15], [9], [1]). Let $M^n$ be a real smooth manifold of dimension $n$ and let $F^n=(M^n,L)$ be a Finsler space on the differentiable manifold $M^n$ endowed with the fundamental function $L(x,y)$. We use the following notation:

$$(2.1)\qquad \begin{cases} g_{ij}=\frac{1}{2}\dot\partial_i\dot\partial_j L^2, \\ C_{ijk}=\frac{1}{2}\dot\partial_k g_{ij}, \\ h_{ij}=g_{ij}-l_il_j, \\ \gamma^i_{jk}=\frac{1}{2}g^{ir}(\partial_j g_{rk}+\partial_k g_{rj}-\partial_r g_{jk}), \\ G^i=\frac{1}{2}\gamma^i_{jk}y^jy^k,\ G^i_j=\dot\partial_j G^i, \\ G^i_{jk}=\dot\partial_k G^i_j,\ G^i_{jkl}=\dot\partial_l G^i_{jk}, \end{cases}$$

where $\dot\partial_i\equiv\frac{\partial}{\partial y^i}$.

**Definition 2.1.** A change $L\to\overline{L}$ of a Finsler metric on the same underlying manifold $M$ is called projective change if any geodesic in $(M,L)$ remains to be geodesic in $(M,\overline{L})$ and vice versa.

A Finsler metric is projectively related to another metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics $\alpha$ and $\overline{\alpha}$ are projectively related if and only if their spray coefficients have the relation [5]

$$(2.2)\qquad G^i_\alpha=G^i_{\overline{\alpha}}+\lambda_{x^k}y^ky^i,$$

where $\lambda=\lambda(x)$ is a scalar function on the based manifold.

Two Finsler metric $F$ and $\overline{F}$ are projectively related if and only if their spray coefficients have the relation [5]

$$(2.3)\qquad G^i=\overline{G}^i+P(y)y^i,$$

where $P(y)$ is a scalar function and homogeneous of degree one in $y^i$.

**Definition 2.2.** A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

For a given Finsler metric $L=L(x,y)$, the geodesic of $L$ is given by

$$(2.4)\qquad \frac{d^2x^i}{dt^2}+2G^i(x,\frac{dx}{dt})=0,$$

where $G^i = G^i(x,y)$ are called geodesic coefficients, which are given by

$$G^i = \frac{g^{il}}{4}\Big\{[L^2]_{x^m y^l} y^m - [L^2]_{x^l}\Big\}. \tag{2.5}$$

Let $\phi = \phi(s)$, $|s| < b_0$, be a positive $C^\infty$ satisfying the following

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \ (|s| \le b < b_0). \tag{2.6}$$

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, $\beta = b_iy^i$ is a 1-form satisfying $\|\beta_x\|_\alpha < b_0$ for all $x \in M$, then $L = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, is called an (regular) $(\alpha,\beta)$-metric. In this case, the fundamental form of the metric tensor induced by $L$ is positive definite. Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be the covariant derivative of $\beta$ with respect to $\alpha$.
Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}) \ \ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

$\beta$ is closed if and only if $s_{ij} = 0$ [17]. Let $s_j = b^is_{ij}$, $s^i_j = a^{il}s_{lj}$, $s_0 = s_iy^i$, $s^i_0 = s^i_jy^j$ and $r_{00} = r_{ij}y^iy^j$.

The relation between the geodesic coefficient $G^i$ of $L$ and the geodesic coefficient $G^i_\alpha$ of $\alpha$ is given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{r_{00} - 2Q\alpha s_0\}\{\psi b^i + \Theta\alpha^{-1}y^i\}, \tag{2.7}$$

where

$$\begin{aligned}
\Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},\\
Q &= \frac{\phi'}{\phi - s\phi'},\\
\psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}$$

**Definition 2.3.** [5] Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right), \tag{2.8}$$

where $G^i$ is the spray coefficient of $L$. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called a Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant [13]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes (2.8). This shows that the Douglas tensor is a non-Riemannian quantity. In what follows, we use quantities with a bar to denote the corresponding quantities of metric $\overline{L}$. We compute the Douglas tensor of a general $(\alpha, \beta)$-metric. Let

$$\widehat{G}^i = G^i_\alpha + \alpha Q s^i_0 + \psi\{r_{00} - 2Q\alpha s_0\}b^i. \tag{2.9}$$

Using (2.9) in (2.7), we have

$$G^i = \widehat{G}^i + \Theta\{r_{00} - 2Q\alpha s_0\}\alpha^{-1}y^i. \tag{2.10}$$

Clearly, $G^i$ and $\widehat{G}^i$ are projective equivalents according to (2.3). They have the same Douglas tensor. Let

$$T^i = \alpha Q s^i_0 + \psi\{r_{00} - 2Q\alpha s_0\}b^i. \tag{2.11}$$

Then $\widehat{G}^i = G^i_\alpha + T^i$, thus

$$\begin{aligned} D^i_{jkl} &= \widehat{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \end{aligned} \tag{2.12}$$

To simplify (2.12), we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k, \;\; s_{y^k} = \alpha^{-2}(b_k\alpha - s y_k),$$

where $y_i = a_{il}y^l$, $\alpha_{y^k} = \frac{\partial \alpha}{\partial y^k}$. Then

$$\begin{aligned} [\alpha Q s^m_0]_{y^m} &= \alpha^{-1}y_m Q s^m_0 + \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s^m_0 \\ &= Q's_0 \end{aligned}$$

and

$$\begin{aligned} \psi(r_{00} - 2Q\alpha s_0)b^m]_{y^m} &= \psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &+ 2\psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0], \end{aligned}$$

where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.11), we get

$$\begin{aligned} T^m_{y^m} &= Q's_0 + \psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &+ 2\psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \end{aligned} \tag{2.13}$$

We assume that the $(\alpha, \beta)$-metrics $L$ and $\overline{L}$ have the same Douglas tensor, i.e., $D^i_{jkl} = \widehat{D}^i_{jkl}$. Thus from (2.8) and (2.12), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \overline{T}^i - \frac{1}{n+1}(T^m_{y^m} - \overline{T}^m_{y^m})y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \overline{T}^i - \frac{1}{n+1}(T^m_{y^m} - \overline{T}^m_{y^m})y^i, \tag{2.14}$$

where $H^i_{00} = H^i_{jk}y^j y^k$.

**Theorem 2.1.** *[4] For the special form of $(\alpha, \beta)$-metric, $L = \alpha + \epsilon\beta + k\left(\frac{\beta^2}{\alpha}\right)$, where $\epsilon$, $k$ are non-zero constant, the following are equivalent:*

- *$L$ has an isotropic $S$-curvature, i.e., $S = (n+1)c(x)L$ for some scalar function $c(x)$ on $M$.*
- *$L$ has an isotropic mean Berwald curvature.*
- *$\beta$ is a killing one form of constant length with respect to $\alpha$. This is equivalent to $r_{00} = s_0 = 0$.*
- *$L$ has a vanished $S$-curvature, i.e., $S = 0$.*
- *$L$ is a weak Berwald metric, i.e., $E = 0$.*

## 3. Projective Change between Randers Metric and Exponential $(\alpha, \beta)$-metric

In this section, we find the projective relation between two $(\alpha, \beta)$-metrics on the same underlying manifold $M$ of dimension $n > 2$. For $(\alpha, \beta)$-metric $L = \alpha e^{\frac{\beta}{\alpha}}$, one can prove by (2.6) that $L$ is a regular Finsler metric if and only if 1-form $\beta$ satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficient are given by (2.7) with

$$\begin{cases} \Theta = \frac{1-2s}{2(1+b^2-s-s^2)}, \\ Q = \frac{1}{1-s}, \\ \psi = \frac{1}{2(1+b^2-s-s^2)}. \end{cases} \tag{3.1}$$

Using (3.1) in (2.7), we get

$$\begin{aligned} G^i &= G^i_\alpha + \frac{\alpha^2}{\alpha - \beta}s^i_0 + \frac{1}{2(\alpha^2 - \beta^2 + \alpha^2 b^2 - \alpha\beta)}\left[r_{00} - \frac{2\alpha^2}{\alpha - \beta}s_0\right] \\ &\times [\alpha^2 b^i + (\alpha - 2\beta)y^i]. \end{aligned} \tag{3.2}$$

For the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$, one can also prove by (2.6) that $\overline{L}$ is a regular Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficients are given by (2.7) with

$$\overline{\Theta} = \frac{1}{2(1+s)}, \quad \overline{Q} = 1, \quad \overline{\psi} = 0. \tag{3.3}$$

First, we prove the following lemma:

**Lemma 3.1.** *Let $L = \alpha e^{\frac{\beta}{\alpha}}$ and $\overline{L} = \overline{\alpha} + \overline{\beta}$ be two $(\alpha, \beta)$-metrics on a manifold $M$ with dimension $n > 2$. Then they have the same Douglas tensor if and only if both metrics are Douglas metrics.*

*Proof.* First, we prove the sufficient condition. Let $L$ and $\overline{L}$ be Douglas metrics and the corresponding Douglas tensor $D^i_{jkl}$ and $\widehat{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\widehat{D}^i_{jkl} = 0$, that is, both metrics have the same Douglas tensor. Next, we prove the necessary condition. If $L$ and $\overline{L}$ have the same Douglas tensor, then (2.14) holds.
Using (2.13), (3.1) and (3.3) in (2.14), we have

$$H^i_{00} = \frac{A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2}{K\alpha^6 + U\alpha^5 + M\alpha^4 + N\alpha^3 + V\alpha^2 + R} - \overline{\alpha}\,\overline{s}^i_0, \tag{3.4}$$

where

$$\begin{aligned}
A^i &= (1+b^2)[2s^i_0(1+b^2) - 2s_0],\\
B^i &= (1+b^2)[r_{00}b^i - 2\beta(3+b^2)s^i_0 + 2\beta s_0 b^i\\
&- 2\lambda s s_0(1+s)y^i - 2\lambda r_0 y^i] - 2\lambda s_0 y^i,\\
C^i &= \beta(3+2b^2)(2\lambda r_0 y^i - r_{00}b^i) - 2\lambda\beta s s_0(2+b^2)y^i\\
&+ 4\lambda\beta s_0(1+b^2)y^i + 2\beta^2(1-2b^2)s^i_0 - \lambda b^2 r_{00}y^i,\\
D^i &= 2\beta^3(3+2b^2)s^i_0 + r_{00}\beta^2(2+b^2)b^i\\
&+ 2\lambda\beta(\beta s_0 + 2\beta s^2 s_0 - \beta b^2 r_0 - 2\beta r_0 - s^2 r_{00})y^i,\\
E^i &= \beta^2 r_{00}[\beta b^i + \lambda(3b^2 - 4s^2 - 2\beta b^2)y^i]\\
&+ 2\lambda\beta^3(s s_0 - r_0 - 2s_0)y^i - 2\beta^4 s^i_0,\\
F^i &= 2\lambda\beta^3(\beta r_0 - \beta s_0 + s^2 r_{00})y^i\\
&- 2\beta^5 s^i_0 - \beta^4 r_{00}b^i,\\
\lambda &= \frac{1}{n+1}
\end{aligned}$$

and

$$\begin{aligned}
K &= 2(1+b^2)^2,\ \ U = 4\beta(b^4 - 3b^2 - 2),\ \ M = 2\beta^2(b^2+2)^2,\\
N &= 4\beta^3(1+b^2),\ \ V = -4\beta^4(2+b^2),\ \ R = 2\beta^6.
\end{aligned}$$

Then (3.4) is equivalent to

$$\begin{aligned}
&A^i\alpha^7 + B^i\alpha^6 + C^i\alpha^5 + D^i\alpha^4 + E^i\alpha^3 + F^i\alpha^2\\
&= (K\alpha^6 + U\alpha^5 + M\alpha^4 + N\alpha^3 + V\alpha^2 + R)(H^i_{00} + \overline{\alpha}\,\overline{s}^i_0).
\end{aligned} \tag{3.5}$$

Replacing $y^i$ in (3.5) by $-y^i$, we have

$$\begin{aligned}
&- A^i\alpha^7 + B^i\alpha^6 - C^i\alpha^5 + D^i\alpha^4 - E^i\alpha^3 + F^i\alpha^2\\
&= (K\alpha^6 - U\alpha^5 + M\alpha^4 - N\alpha^3 + V\alpha^2 + R)(H^i_{00} - \overline{\alpha}\,\overline{s}^i_0).
\end{aligned} \tag{3.6}$$

Subtracting (3.6) from (3.5), we get

$$\begin{aligned} A^i\alpha^7 + C^i\alpha^5 + E^i\alpha^3 &= (U\alpha^5 + N\alpha^3)H^i_{00} \\ &+ (K\alpha^6 + M\alpha^4 + V\alpha^2 + R)\overline{\alpha}\ \overline{s}^i_0. \end{aligned} \tag{3.7}$$

From (3.7), we have

$$\begin{aligned} \alpha^2[A^i\alpha^5 + C^i\alpha^3 + E^i\alpha - (U\alpha^3 + N\alpha)H^i_{00} \\ -\overline{\alpha}\ \overline{s}^i_0(K\alpha^4 + M\alpha^2 + V)] = R\overline{\alpha}\ \overline{s}^i_0. \end{aligned} \tag{3.8}$$

From (3.8), $R\overline{\alpha}\ \overline{s}^i_0$ has the factor $\alpha^2$, i.e., the term $R\overline{\alpha}\ \overline{s}^i_0 = 2\beta^6\,\overline{\alpha}\ \overline{s}^i_0$ has the factor $\alpha^2$. We can study two cases for Riemannian metric.

**Case (i):** If $\overline{\alpha} \neq \mu(x)\alpha$, then $R\overline{s}^i_0 = 2\beta^6\overline{s}^i_0$ has the factor $\alpha^2$. Note that $\beta^2$ has no factor $\alpha^2$. Then the only possibility is that $\beta\overline{s}^i_0$ has the factor $\alpha^2$. Then for each i there exists a scalar function $\eta^i = \eta^i(x)$ such that $\beta\overline{s}^i_0 = \eta^i\alpha^2$ which is equivalent to $b_j\overline{s}^i_k + b_k\overline{s}^i_j = 2\eta^i\alpha_{jk}$. When $n > 2$ and we assume that $\eta^i \neq 0$, then

$$\begin{aligned} 2 &\geq rank(b_j\overline{s}^i_k) + rank(b_k\overline{s}^i_j) \\ &> rank(b_j\overline{s}^i_k + b_k\overline{s}^i_j) \\ &= rank(2\eta^i\alpha_{jk}) > 2, \end{aligned}$$

which is impossible unless $\eta^i = 0$. Then $\beta\overline{s}^i_0 = 0$. Since $\beta \neq 0$, we have $\overline{s}^i_0 = 0$, which says that $\overline{\beta}$ is closed.

**Case (ii):** If $\overline{\alpha} = \mu(x)\alpha$, then (3.7), becomes

$$\begin{aligned} R\mu(x)\overline{s}^i_0 &= \alpha^2[A^i\alpha^4 + C^i\alpha^2 + E^i - (U\alpha^2 + N)H^i_{00} \\ &- \mu(x)\overline{s}^i_0(K\alpha^4 + M\alpha^2 + V)]. \end{aligned} \tag{3.9}$$

From (3.9), we can see that $\mu(x)R\overline{s}^i_0$ has the factor $\alpha^2$. i.e., $\mu(x)R\overline{s}^i_0 = 2\mu(x)\overline{s}^i_0\beta^6$ has the factor $\alpha^2$. Note that $\mu(x) \neq 0$ for all $x \in M$ and $\beta^2$ has no factor $\alpha^2$. The only possibility is that $\beta\overline{s}^i_0$ has the factor $\alpha^2$. As the similar reason in case $(i)$, we have $\overline{s}^i_0 = 0$, when $n > 2$, which says that $\overline{\beta}$ is closed.

M. Hashiguchi [7] proved that the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ is a Douglas metric if and only if $\overline{\beta}$ is closed. Thus $\overline{L}$ is a Douglas metric. Since $L$ is projectively related to $\overline{L}$, then both $L$ and $\overline{L}$ are Douglas metrics. $\square$

**Theorem 3.1.** *The Finsler metric $L = \alpha e^{\frac{\beta}{\alpha}}$ is projectively related to $\overline{L} = \overline{\alpha} + \overline{\beta}$ if and only if the following conditions are satisfied*

$$\begin{cases} G^i_\alpha = G^i_{\overline{\alpha}} + \theta y^i - \tau\xi\alpha^2 b^i, \\ b_{i|j} = \tau[(1 + 2b^2)a_{ij} - 3b_ib_j], \\ d\overline{\beta} = 0, \end{cases} \tag{3.10}$$

*where $b^i = a^{ij}b_j$, $b = \|\beta\|_\alpha$, $b_{i|j}$ denotes the coefficient of the covariant derivatives of $\beta$ with respect to $\alpha$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on a manifold $M$ with dimension $n > 2$.*

*Proof.* First, we prove the necessary condition. Since the Douglas tensor is invariant under projective changes between two Finsler metrics, if $L$ is projectively related to $\overline{L}$, then they have the same Douglas tensor. According to Lemma 3.1, we obtain that both $L$ and $\overline{L}$ are Douglas metrics.

We know that the Randers metric $\overline{L} = \overline{\alpha} + \overline{\beta}$ is a Douglas metric if and only if $\overline{\beta}$ is closed, i.e., $d\overline{\beta} = 0$.
The Finsler metric $L = \alpha e^{\frac{\beta}{\alpha}}$ is a Douglas metric if and only if

$$b_{i|j} = \tau[(1 + 2b^2) - 3b_i b_j], \tag{3.11}$$

for some scalar function $\tau = \tau(x)$ [3], where $b_{i|j}$ denotes the coefficient of the covariant derivatives of $\beta = b_i y^i$ with respect to $\alpha$. In this case, $\beta$ is closed. Since $\beta$ is closed, $s_{ij} = 0 \Rightarrow b_{i|j} = b_{j|i}$. Thus $s^i_0 = 0$ and $s_0 = 0$.
By using (3.11), we have $r_{00} = \tau[(1 + 2b^2)\alpha^2 - 3\beta^2]$. Substituting all these in (3.2), we get

$$G^i = G^i_\alpha + \tau \frac{[(1 + 2b^2)(\alpha^3 - 2\alpha^2\beta) - 3\alpha\beta^2 + 6\beta^3]}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)} y^i + \tau\xi\alpha^2 b^i, \tag{3.12}$$

where $\xi = \frac{\tau[(1+2b^2)\alpha^2 - 3\beta^2]b^i}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}$.
Since $L$ is projectively related to $\overline{L}$, this is a Randers change between $L$ and $\overline{\alpha}$. Noticing that $\overline{\beta}$ is closed, then $L$ is projectively related to $\overline{\alpha}$. Thus, there is a scalar function $P = P(y)$ on $TM - \{0\}$ such that

$$G^i = G^i_{\overline{\alpha}} + P y^i. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\left[P + \frac{3\alpha\beta^2 - 6\beta^3 - (1 + 2b^2)(\alpha^3 - 3\alpha^2\beta)}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right] y^i = G^i_\alpha - G^i_{\overline{\alpha}} + \tau\xi\alpha^2 b^i. \tag{3.14}$$

Note that the RHS of the above equation is a quadratic form. Then there must be one form $\theta = \theta_i y^i$ on $M$, such that

$$P + \frac{3\alpha\beta^2 - 6\beta^3 - (1 + 2b^2)()\alpha^3 - 3\alpha^2\beta}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)} = \theta.$$

Thus (3.14) becomes

$$G^i_\alpha = G^i_{\overline{\alpha}} + \theta y^i - \tau\xi\alpha^2 b^i. \tag{3.15}$$

Equations (3.11) and (3.12) together with (3.15) complete the proof of the necessity. Since $\overline{\beta}$ is closed, it suffices to prove that $L$ is projectively related to $\overline{\alpha}$. From (3.12) and (3.15), we have

$$G^i = G^i_{\overline{\alpha}} + \left[\theta + \frac{\tau[(1 + 2b^2)(\alpha^3 - 3\alpha^2\beta) - 3\alpha\beta^2 + 6\beta^3]}{2(\alpha^2 - \beta^2 + b^2\alpha^2 - \alpha\beta)}\right] y^i,$$

that is, $L$ is projectively related to $\overline{\alpha}$ $\square$

From the above theorem, we get the following corollaries.

**Corollary 3.1.** *The Finsler metric $L=\alpha e^{\frac{\beta}{\alpha}}$ is projectively related to $\overline{L}=\overline{\alpha}+\overline{\beta}$ if and only if they are Douglas metrics and the spray coefficients of $\alpha$ and $\overline{\alpha}$ have the following relation*

$$G^i_\alpha = G^i_{\overline{\alpha}} + \theta y^i - \tau\xi\alpha^2 b^i,$$

*where $b^i=a^{ij}b_j$, $\tau=\tau(x)$ is a scalar function and $\theta=\theta_i y^i$ is one form on a manifold $M$ with dimension $n\geq 2$.*

Further, we assume that the Randers metric $\overline{L}=\overline{\alpha}+\overline{\beta}$ is locally Minkowskian, where $\overline{\alpha}$ is a Euclidean metric and $\overline{\beta}=\overline{b}_i y^i$ is one form with $\overline{b}_i$ =constant. Then (3.10) can be written as

$$\begin{cases} G^i_\alpha = \theta y^i - \tau\xi\alpha^2 b^i, \\ b_{i|j} = \tau[(1+2b^2)a_{ij} - 3b_i b_j]. \end{cases} \tag{3.16}$$

Thus, we state

**Corollary 3.2.** *The Finsler metric $\alpha e^{\frac{\beta}{\alpha}}$ is projectively related to $\overline{L}=\overline{\alpha}+\overline{\beta}$ if and only if $L$ is projectively flat, that is, $L$ is projectively flat if and only if* (3.16) *holds.*

## 4. Special Curvature Properties of two $(\alpha,\beta)$-metrics

We know that the Berwald curvature tensor of a Finsler metric $L$ is defined by [9]

$$G = G^i_{jkl}dx^j \otimes \partial_i \otimes dx^k \otimes dx^l, \tag{4.1}$$

where $G^i_{jkl}=[G^i]_{y^j y^k y^l}$ and $G^i$ are the spray coefficients of $L$. The mean Berwald curvature tensor is defined by

$$E = E_{ij}dx^i \otimes dx^j, \tag{4.2}$$

where $E_{ij}=\frac{1}{2}G^m_{mij}$.
A Finsler space is said to be of the isotropic mean Berwald curvature if

$$E_{ij} = \frac{n+1}{2}c(x)L_{y^i y^j}, \tag{4.3}$$

where $c(x)$ is scalar function on $M$.

In this section, we assume that $(\alpha,\beta)$-metric $L=\alpha e^{\frac{\beta}{\alpha}}$ has some special curvature properties.

**Theorem 4.1.** *The Finsler metric $L=\alpha e^{\frac{\beta}{\alpha}}$ having an isotropic $S$-curvature or isotropic mean Berwald curvature is projectively related to $\overline{L}=\overline{\alpha}+\overline{\beta}$ if and only if the following conditions hold:*

- *$\alpha$ is projectively related to $\overline{\alpha}$,*
- *$\beta$ is parallel with respect to $\alpha$, i.e., $b_{i|j} = 0$,*
- *$\overline{\beta}$ is closed, i.e., $d\overline{\beta} = 0$,*

*where $b_{i|j}$ denotes the coefficient of the covariant derivative of $\beta$ with respect to $\alpha$.*

*Proof.* The sufficiency is obvious from Theorem (3.2). For the necessary condition, from Theorem 3.1, if $L$ is projectively related to $\overline{L}$, then

$$b_{i|j} = \tau[(1+2b^2)a_{ij} - 3b_i b_j],$$

where $\tau = \tau(x)$ is scalar function. Transvecting the above equation with $y^i$ and $y^j$, we have

$$r_{00} = \tau[(1+2b^2)\alpha^2 - 3\beta^2]. \tag{4.4}$$

From Theorem 2.4, if $L$ has an isotropic $S$-curvature or an equivalently isotropic mean Berwald curvature, then $r_{00} = 0$. If $\tau \neq 0$, then (4.4) gives

$$(1+2b^2)\alpha^2 - 3\beta^2 = 0, \tag{4.5}$$

which is equivalent to

$$(1+2b^2)a_{ij} - 3b_i b_j = 0. \tag{4.6}$$

Transvecting (4.6) with $a^{il}$, we get

$$(1+2b^2)\delta^l_j - 3b^l b_j = 0. \tag{4.7}$$

Contracting $l$ and $j$ in (4.7), we have $n + (2n-3)b^2 = 0$, which is impossible. Thus $\tau = 0$. Substituting in Theorem 3.2, we complete the proof. $\square$

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Ser. Math. Inform. Vol. 33, No 3 (2018), 401–408
https://doi.org/10.22190/FUMI1803401M

# SOME RESULTS ON GENERALIZED $(k, \mu)$-PARACONTACT METRIC MANIFOLDS

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**Abstract.** The aim of this paper is to study the Codazzi type of the Ricci tensor in generalized $(k, \mu)$-paracontact metric manifolds. We also study the cyclic parallel Ricci tensor in generalized $(k, \mu)$-paracontact metric manifolds. Further, we characterize generalized $(k, \mu)$-paracontact metric manifolds whose structure tensor $\phi$ is $\eta$-parallel. Finally, we investigate locally $\phi$-Ricci symmetric generalized $(k, \mu)$-paracontact metric manifolds.

**Keywords**: Generalized $(k, \mu)$-paracontact metric manifold, Codazzi type of tensor, cyclic parallel Ricci tensor, $\eta$-parallel $\phi$-tensor, locally $\phi$-Ricci symmetric.

## 1. Introduction

In 1985, Kaneyuki and Williams [8] introduced the idea of paracontact geometry. A systematic investigation on paracontact metric manifolds was done by Zamkovoy [12]. Recently, Cappelletti-Montano et al [5] introduced a new type of paracontact geometry, the so-called paracontact metric $(k, \mu)$ space, where $k$ and $\mu$ are constants. This is known [2] about the contact case $k \leq 1$, but in the paracontact case there is no restriction of $k$. Recently, three-dimensional generalized $(k, \mu)$-paracontact metric manifolds were studied by Kupeli Erken et al [9, 10].
Zamkovoy [12] studied paracontact metric manifolds and some remarkable subclasses named para-Sasakian manifolds. In particular, in recent years, many authors have pointed to the importance of paracontact geometry and, in particular, para-Sasakian geometry. Several papers have established relationships with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. A normal paracontact metric manifold is a para-Sasakian manifold. An almost paracontact metric manifold is a para-sasakian manifold if and only if [12]

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X. \tag{1.1}$$

Received December 07, 2017; accepted March 22, 2018
2010 *Mathematics Subject Classification.* Primary 53C05; Secondary 53D15, 53C15

A. Gray [7] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. The Ricci tensor S of type (0,2) is said to be cyclic parallel if it is non-zero and satisfies the condition

$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0. \tag{1.2}$$

Again, a Riemannian or a pseudo-Riemannian manifold is said to be of Codazzi type if its Ricci tensors of type (0,2) is non-zero and satisfy the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{1.3}$$

for all vector fields $X$, $Y$, $Z$. On a contact metric manifold there is an associated CR-structure which is integrable if and only if the structure tensor $\phi$ is $\eta$-parallel, that is,

$$g((\nabla_X \phi)Y, Z) = 0,$$

for all vector fields $X, Y, Z$ in the contact distribution $D(\eta = 0)$. In 2005, Boeckx and Cho [3] considered a milder condition that $h$ is $\eta$-parallel, that is,

$$g((\nabla_X h)Y, Z) = 0,$$

for all vector fields $X$, $Y$, $Z$ in the contact distribution $D$.

The paper is organized in the following way:
In Section 2, we discuss some basic results of paracontact metric manifolds. Further, we characterize the Codazzi type of the Ricci tensor in generalized $(k,\mu)$-paracontact metric manifolds. In Section 4, we investigate the cyclic parallel Ricci tensor in generalized $(k,\mu)$-paracontact metric manifolds. In the next section we study $\eta$-parallel $\phi$-tensor in a generalized $(k,\mu)$-paracontact metric manifold. Finally, we investigate locally $\phi$-Ricci symmetric generalized $(k,\mu)$-paracontact metric manifolds.

## 2. Preliminaries

An odd dimensional smooth manifold $M^n (n > 1)$ is said to be an almost paracontact manifold [8] if it carries a $(1,1)$-tensor $\phi$, a vector field $\xi$ and a 1-form $\eta$ satisfying :
(i) $\phi^2 X = X - \eta(X)\xi$, for all $X \in \chi(M)$,
(ii)$\eta(\xi) = 1$, $\phi(\xi) = 0$, $\eta \circ \phi = 0$,
(iii) the tensor field $\phi$ induces an almost paracomplex structure on each fiber of $D = ker(\eta)$, that is, the eigen distributions $D^+_\phi$ and $D^-_\phi$ of $\phi$ corresponding to the eigenvalues 1 and $-1$, respectively, have an equal dimension $n$.

An almost paracontact structure is said to be normal [8] if and only if the $(1,2)$ type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. A para-Sasakian manifold is a normal paraconatact metric manifold. If an almost paracontact manifold admits a pseudo-Riemannian metric $g$ such that

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.1}$$

for $X$, $Y \in \chi(M)$, then we say that $(M, \phi, \xi, \eta, g)$ is an almost paracontact metric manifold. Any such pseudo-Riemannian metric is of signature $(n+1, n)$. An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ [12]. In a paracontact metric manifold we define $(1,1)$-type tensor fields $h$ by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds_\xi \phi$ is the Lie derivative of $\phi$ along the vector field $\xi$. Then we observe that $h$ is symmetric and anti-commutes with $\phi$. Also $h$ satisfies the following conditions [12]:

$$h\xi = 0, \; tr(h) = tr(\phi h) = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.3}$$

for all $X \in \chi(M)$, where $\nabla$ denotes the Levi-Civita connection of the pseudo-Riemannian manifold.
Moreover, $h$ vanishes identically if and only if $\xi$ is a Killing vector field. In this case, $(M, \phi, \xi, \eta, g)$ is said to be a $K$-paracontact manifold [11].

Generalized $(k, \mu)$-paracontact metric manifolds were studied by Erken et al. [10] and Erken [9]. A generalized $(k, \mu)$-paracontact metric manifold means a three-dimensional paracontact metric manifold which satisfies the curvature condition

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.4}$$

where $k$ and $\mu$ are smooth functions.

In a generalized $(k \neq -1, \mu)$-paracontact manifold the following results hold [4, 5, 9, 10]

$$h^2 = (1 + k)\phi^2, \tag{2.5}$$

$$\xi(k) = 0, \tag{2.6}$$

$$Q\xi = 2k\xi, \tag{2.7}$$

$$(\nabla_\xi h)(Y) = \mu h(\phi Y), \tag{2.8}$$

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X \;\; &= \;\; -(1 + k)[2g(X, \phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X] \\ &\quad\; +(1 - \mu)(\eta(X)\phi hY - \eta(Y)\phi hX), \end{aligned} \tag{2.9}$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \text{ for } \; k \neq -1 \tag{2.10}$$

$$h \; grad\mu = gradk, \tag{2.11}$$

$$(\nabla_X \eta)(Y) = -g(\phi X, Y) + g(\phi h X, Y), \tag{2.12}$$

$$QX = (\frac{r}{2} - k)X + (-\frac{r}{2} + 3k)\eta(X)\xi + \mu hX, k \neq -1, \tag{2.13}$$

where $X$ is any vector fields on $M$, $Q$ is the Ricci operator of $M$, $r$ denotes the scalar curvature of $M$.
From ( 2.13), we have

$$S(X,Y) = (\frac{r}{2} - k)g(X,Y) + (-\frac{r}{2} + 3k)\eta(X)\eta(Y) + \mu g(hX,Y), k \neq -1. \tag{2.14}$$

## 3. The Codazzi type of the Ricci tensor in generalized $(k, \mu)$-paracontact metric manifolds

In this section we characterize generalized $(k, \mu)$-paracontact metric manifolds whose Ricci tensor is of Codazzi type.
Then we have

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z), \tag{3.1}$$

which implies $r$ =constant.

Now from (2.14) we have

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \{\frac{(Xr)}{2} - (Xk)\}g(Y,Z) + \{-\frac{(Xr)}{2} + 3(Xk)\}\eta(Y)\eta(Z)\\ &+\{-\frac{r}{2} + 3k\}\{(\nabla_X\eta)(Y)\eta(Z) + \eta(Y)(\nabla_X\eta)(Z)\} + (X\mu)g(hY,Z)\\ &+\mu g((\nabla_X h)(Y), Z)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(\nabla_Y S)(X,Z) &= \{\frac{(Yr)}{2} - (Yk)\}g(X,Z) + \{-\frac{(Yr)}{2} + 3(Yk)\}\eta(X)\eta(Z)\\ &+\{-\frac{r}{2} + 3k\}\{(\nabla_Y\eta)(X)\eta(Z) + \eta(X)(\nabla_Y\eta)(Z)\} + (Y\mu)g(hX,Z)\\ &+\mu g((\nabla_Y h)(X), Z).\end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) in (3.1) yields

$$\begin{aligned}&\{\frac{(Xr)}{2} - (Xk)\}g(Y,Z) + \{-\frac{(Xr)}{2} + 3(Xk)\}\eta(Y)\eta(Z)\\ &+\{-\frac{r}{2} + 3k\}\{(\nabla_X\eta)(Y)\eta(Z) + \eta(Y)(\nabla_X\eta)(Z)\} + (X\mu)g(hY,Z)\\ &+\mu g((\nabla_X h)(Y), Z) = \{\frac{Yr}{2} - Yk\}g(X,Z)\\ &+\{-\frac{(Yr)}{2} + 3(Yk)\}\eta(X)\eta(Z) + \{-\frac{r}{2} + 3k\}\{(\nabla_Y\eta)(X)\eta(Z)\\ &+\eta(X)(\nabla_Y\eta))(Z)\} + (Y\mu)g(hX,Z) + \mu g((\nabla_Y h)(X), Z).\end{aligned} \tag{3.4}$$

Substituting $Z=\xi$ in (3.4) gives

$$\begin{aligned}
&\{\frac{(Xr)}{2}-(Xk)\}\eta(Y)+\{-\frac{(Xr)}{2}+3(Xk)\}\eta(Y)+\{-\frac{r}{2}+3k\}\{(\nabla_X\eta)(Y)\\
&+\eta(Y)(\nabla_X\eta))(\xi)\}+\mu\eta((\nabla_Xh)(Y))=\{\frac{Yr}{2}-Yk\}\eta(X)\\
&+\{-\frac{(Yr)}{2}+3(Yk)\}\eta(X)+\{-\frac{r}{2}+3k\}\{(\nabla_Y\eta)(X)+\eta(X)(\nabla_Y\eta))(\xi)\}\\
&+\mu\eta((\nabla_Yh)(X)). \qquad (3.5)
\end{aligned}$$

Putting $X=\xi$ in (3.5) and using $r$ =constant, we obtain

$$\mu\eta((\nabla_\xi h)(Y))=2(Yk)+\mu\eta((\nabla_Yh)(\xi))=0. \qquad (3.6)$$

Applying (2.8) in (3.6), we have $(Yk)=0$, which implies $k$ =constant. Hence from (2.11), we get either $h=0$ or $\mu$ =constant. Thus, we can state the following

**Theorem 3.1.** *If in a generalized $(k,\mu)$-paracontact metric manifold with $k\neq-1$ the Ricci tensor is of Codazzi type, then the manifold is either a $(k,\mu)$-paracontact metric manifold or a $K$-paracontact manifold.*

## 4. The cyclic parallel Ricci tensor in generalized $(k,\mu)$-paracontact metric manifolds

This section is devoted to the study of the cyclic parallel Ricci tensor in generalized $(k,\mu)$-paracontact metric manifolds
If the Ricci tensors is cyclic parallel, then we have

$$(\nabla_ZS)(X,Y)+(\nabla_XS)(Y,Z)+(\nabla_YS)(Z,X)=0, \qquad (4.1)$$

which implies $r=constant$.

Now from the equation (2.14), we obtain

$$\begin{aligned}
&\{\frac{(Zr)}{2}-(Zk)\}g(X,Y)+\{-\frac{(Zr)}{2}+3(Zk)\}\eta(X)\eta(Y)\\
&+\{-\frac{r}{2}+3k\}\{(\nabla_Z\eta)(X)\eta(Y)+\eta(X)(\nabla_Z\eta)(Y)\}+(Z\mu)g(hX,Y)\\
&+\mu g((\nabla_Zh)(X)Y)+\{\frac{Xr}{2}-Xk\}g(Y,Z)+\{-\frac{(Xr)}{2}+3(Xk)\}\eta(Y)\eta(Z)\\
&+\{-\frac{r}{2}+3k\}\{(\nabla_X\eta)(Y)\eta(Z)+\eta(Y)(\nabla_X\eta)(Z)\}+(X\mu)g(hY,Z)\\
&+\mu g((\nabla_Xh)(Y),Z)+\{(Yr)-(Yk)\}g(Z,X)+\{-\frac{(Yr)}{2}+3(Yk)\}\eta(Z)\eta(X)\\
&+\{-\frac{r}{2}+3k\}\{(\nabla_Y\eta)(Z)\eta(X)+\eta(Z)(\nabla_Y\eta)(X)\}+(Y\mu)g(hZ,X)\\
&+\mu g((\nabla_Yh)(Z),X)=0. \qquad (4.2)
\end{aligned}$$

Substituting $X = Y = \xi$ and applying (2.8) in (4.2) yields

$$2(Zk) + (-\frac{r}{2} + 3k)(\nabla_\xi \eta)(Z) + (-\frac{r}{2} + 3k)(\nabla_\xi \eta)(Z) = 0. \tag{4.3}$$

Now using (2.12) in (4.3), we have

$$(Zk) = 0. \tag{4.4}$$

Therefore, $k$ =constant. Hence from (2.11), we have either $h = 0$ or $\mu$ =constant. This leads to the following:

**Theorem 4.1.** *If in a generalized $(k, \mu)$-paracontact metric manifold with $k \neq -1$ the Ricci tensor is cyclic parallel, then the manifold is either a $(k, \mu)$-paracontact metric manifold or a $K$-paracontact manifold.*

## 5. The $\eta$-parallel $\phi$-tensor in generalized $(k, \mu)$-paracontact metric manifolds

In this section we study the $\eta$-parallel $\phi$-tensor in generalized $(k, \mu)$-paracontact metric manifolds
If the $(1, 1)$ tensor $\phi$ is $\eta$-parallel, then we have [1]

$$g((\nabla_X \phi)Y, Z) = 0. \tag{5.1}$$

From (2.10) and (5.1), we get

$$-g(X, Y)\eta(Z) + g(hX, Y)\eta(Z) + g(X, Z)\eta(Y) - g(hX, Z)\eta(Y) = 0. \tag{5.2}$$

Putting $Z = \xi$ in (5.2) yields

$$-g(X, Y) + g(hX, Y) + \eta(X)\eta(Y) = 0. \tag{5.3}$$

Substituting $X = hX$ in (5.3), we have

$$-g(hX, Y) - (k + 1)g(X, Y) + (k + 1)\eta(X)\eta(Y) = 0. \tag{5.4}$$

Adding (5.3) and (5.4), we obtain

$$(k + 2)\{g(X, Y) - \eta(X)\eta(Y)\} = 0. \tag{5.5}$$

Thus we have $k = -2$, that is, $k$ = constant. Using (2.11) we have $h\ grad\mu = 0$. Therefore, either $h = 0$ or $\mu$ =constant.
Thus we can state the following:

**Theorem 5.1.** *If in a generalized $(k, \mu)$-paracontact metric manifold with $k \neq -1$, the tensor $\phi$ is $\eta$-parallel, then the manifold is either a $(k, \mu)$-paracontact metric manifold or a $K$-paracontact manifold.*

## 6. Locally $\phi$-Ricci symmetric generalized $(k,\mu)$-paracontact manifolds

A paracontact metric manifold is said to be locally $\phi$-Ricci symmetric [6] if it satisfies

$$\phi^2(\nabla_X Q)(Y) = 0, \tag{6.1}$$

for all vector fields $X$, $Y$ orthogonal to $\xi$, where $Q$ is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Taking the covariant derivative of (2.13) with respect to $Y$ and applying $\phi^2$ we get

$$-\{\frac{(Yr)}{2} - Yk\}X - (Y\mu)hX + \mu\phi^2((\nabla_Y h)X) = 0. \tag{6.2}$$

Interchanging $X$ and $Y$ in (6.2), we have

$$-\{\frac{(Xr)}{2} - Xk\}Y - (X\mu)hY + \mu\phi^2((\nabla_X h)Y) = 0. \tag{6.3}$$

Subtracting (6.3) from (6.2), we obtain

$$\begin{aligned}&\{\frac{(Yr)}{2} - Yk\}X - \{\frac{(Xr)}{2} - Xk\}Y + (Y\mu)hX - (X\mu)hY \\ &+\mu\phi^2((\nabla_X hY) - (\nabla_Y hX)) = 0.\end{aligned} \tag{6.4}$$

Applying (2.9) in (6.4), we get

$$\{\frac{(Yr)}{2} - Yk\}X - \{\frac{(Xr)}{2} - Xk\}Y + (Y\mu)hX - (X\mu)hY = 0. \tag{6.5}$$

Substituting $X = \xi$ in (6.5) yields

$$-\frac{1}{2}(\xi r)Y - (\xi\mu)hY + \{\frac{Yr}{2} - Yk\}\xi = 0. \tag{6.6}$$

Taking the inner product with $Z$ from (6.6), we have

$$-\frac{1}{2}(\xi r)g(Y,Z) - (\xi\mu)g(hY,Z) = 0. \tag{6.7}$$

Let $\{e_i\}$, $i = 1,2,3$ be a local orthonormal basis in the tangent space $T_PM$ at each point p $\in M$. Substituting $Y = Z = e_i$ in (6.7) and summing over $i = 1$ to 3, we infer that $\xi r = 0$, since $k \neq -1$.
This leads to the following:

**Theorem 6.1.** *If a generalized $(k,\mu)$-paracontact metric manifold with $k \neq -1$, is locally $\phi$-Ricci symmetric, then the characteristic vector field $\xi$ leaves the scalar curvature invariant.*

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Ser. Math. Inform. Vol. 33, No 3 (2018), 409– 416
https://doi.org/10.22190/FUMI1803409B

# ON TZITZEICA CURVES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

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**Abstract.** In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. We characterize such curves according to their curvatures. We show that there is no Tz-curve with constant curvatures (W-curves). We consider Salkowski (TC-curve) and anti-Salkowski curves.
**Keywords**: Tz-curves, W-curves, TC-curves

## 1. Introduction

Gheorgha Tzitzeica, a Romanian mathematician (1872-1939), introduced a class of curves, nowadays called Tzitzeica curves, and a class of surfaces of the Euclidean 3-space called Tzitzeica surfaces. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x = x(s)$ for which the ratio of its torsion $\kappa_2$ and the square of the distance $d_{osc}$ from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e.,

$$\frac{\kappa_2}{d_{osc}^2} = a \tag{1.1}$$

where $d_{osc} = \langle N_2, x\rangle$ and $a \neq 0$ is a real constant, $N_2$ is the binormal vector of $x$.

In [3] the authors gave the connections between the Tzitzeica curve and the Tzitzeica surface in a Minkowski 3-space and the original ones from the Euclidean 3-space. In [7] the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in a Euclidean space. In [12], the elliptic cylindrical curves verifying Tzitzeica condition were adapted to the Minkowski 3-space. In [2], the authors gave the necessary and sufficient condition for a space curve to become a Tzitzeica curve. The new classes of symmetry reductions for the Tzitzeica curve equation were determined. In [1], the authors were interested in the curves of Tzitzeica type and they investigated the conditions for non-null general helices, pseudo-spherical curves and pseudo-spherical general helices to become of Tzitzeica type in a Minkowski space $\mathbb{E}_1^3$.

Received December 19, 2017; accepted June 20, 2018
2010 *Mathematics Subject Classification.* Primary 53A04; Secondary 53A05

A Tzitzeica surface in $\mathbb{E}^3$ is a spatial surface $M$ given with the parametrization $X(u,v)$ for which the ratio of its Gaussian curvature $K$ and the distance $d_{\tan}$ from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e.,

$$\frac{K}{d_{\tan}^4} = a_1 \tag{1.2}$$

for a constant $a_1$. The orthogonal distance from the origin to the tangent plane is defined by

$$d_{\tan} = \left\langle X, \overrightarrow{U} \right\rangle \tag{1.3}$$

where $X$ is the position vector of the surface and $\overrightarrow{U}$ is a unit normal vector of the surface.

The asymptotic lines of a Tzitzeica surface with a negative Gausssian curvature are Tzitzeica curves [7]. In [18], the authors gave the necessary and sufficient condition for the Cobb-Douglas production hypersurface to be a Tzitzeica hypersurface. In addition, a new Tzitzeica hypersurface was obtained in parametric, implicit and explicit forms in [8]

In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. Furthermore, we investigate a Tzitzeica curve in a Euclidean 3-space $\mathbb{E}^3$ whose position vector $x = x(s)$ satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s), \tag{1.4}$$

for some differentiable functions, $m_i(s)$, $0 \leq i \leq 2$, where $\{T, N_1, N_2\}$ is the Frenet frame of $x$. We characterize such curves according to their curvatures. We show that there is no Tzitzeica curve in $\mathbb{E}^3$ with constant curvatures (W-curves). We give the relations between the curvatures of the Tz-Salkowski curve (TC-curve) and the Tz-anti-Salkowski curve.

## 2. Basic Notations

Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in a Euclidean 3-space $\mathbb{E}^3$. Let us denote $T(s) = x'(s)$ and call $T(s)$ a unit tangent vector of $x$ at $s$. We denote the curvature of $x$ by $\kappa_1(s) = \|x''(s)\|$. If $\kappa_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve $x$ at $s$ is given by $x^{''}(s) = \kappa_1(s)N_1(s)$. The unit vector $N_2(s) = T(s) \times N_1(s)$ is called the unit binormal vector of $x$ at $s$. Then we have the Serret-Frenet formulae:

$$\begin{aligned} T'(s) &= \kappa_1(s)N_1(s), \\ N_1'(s) &= -\kappa_1(s)T(s) + \kappa_2(s)N_2(s), \\ N_2'(s) &= -\kappa_2(s)N_1(s), \end{aligned} \tag{2.1}$$

where $\kappa_2(s)$ is the torsion of the curve $x$ at $s$ (see, [10]).

If the Frenet curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$ of $x$ are constant functions then $x$ is called a screw line or a helix [9]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations then F. Klein and S. Lie called them *W-curves* [14]. It is known that a curve $x$ in $\mathbb{E}^3$ is called a *general helix* if the ratio $\kappa_2(s)/\kappa_1(s)$ is a nonzero constant [16]. Salkowski (resp. anti-Salkowski) curves in a Euclidean space $\mathbb{E}^3$ are generally known as the family of curves with A constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization [15, 17] (for T.C-curve see also [13]).

For a space curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$, the planes at each point of $x(s)$ spanned by $\{T, N_1\}$, $\{T, N_2\}$ and $\{N_1, N_2\}$ are known as the *osculating plane*, the *rectifying plane* and *normal plane*, respectively. If the position vector $x$ lies on its rectifying plane, then $x(s)$ is called *rectifying curve* [5]. Similarly, the curve for which the position vector $x$ always lies in its osculating plane is called *osculating curve.* Finally, $x$ is called *normal curve* if its position vector $x$ lies in its normal plane.

Rectifying curves characterized by the simple equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s), \tag{2.2}$$

where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $N_2(s)$ are tangent and binormal vector fields of $x$, respectively [5, 6].

For a regular curve $x(s)$, the position vector $x$ can be decomposed into its tangential and normal components at each point:

$$x = x^T + x^N. \tag{2.3}$$

A curve in $\mathbb{E}^3$ is called *N-constant* if the normal component $x^N$ of its position vector $x$ is of constant length [4, 11]. It is known that a curve in $\mathbb{E}^3$ is congruent to an $N$-constant curve if and only if the ratio $\frac{\kappa_2}{\kappa_1}$ is a non-constant linear function of an arc-length function $s$, i.e., $\frac{\kappa_2}{\kappa_1}(s) = c_1 s + c_2$ for some constants $c_1$ and $c_2$ with $c_1 \neq 0$ [4]. Further, an $N$-constant curve $x$ is called first kind if $\left\|x^N\right\| = 0$, otherwise second kind [11].

## 3. Tzitzeica Curves in $\mathbb{E}^3$

In the present section we characterize Tzitzeica curves in $\mathbb{E}^3$ in terms of their curvatures.

**Definition 3.1.** Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s) \neq 0$. If the torsion of $x$ satisfies the condition

$$\kappa_2(s) = a.d_{osc}^2, \tag{3.1}$$

for some real constant $a$ then $x$ is called Tzitzeica curve (Tz-curve), where

$$d_{osc} = \langle N_2, x\rangle \tag{3.2}$$

is the orthogonal distance from the origin to the osculating plane of $x$.

We have the following result.

**Proposition 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. If $x$ is a Tz-curve, then the equation*

$$\kappa_2' \langle x, N_2 \rangle + 2\kappa_2^2 \langle x, N_1 \rangle = 0 \tag{3.3}$$

*holds.*

*Proof.* Let $x$ be a unit speed curve in $\mathbb{E}^3$, then by the use of the equations (3.1) and (3.2) we get

$$\frac{\kappa_2(s)}{\langle N_2, x \rangle^2} = a \neq 0. \tag{3.4}$$

Further, differentiating the equation (3.4), we obtain the result. □

**Definition 3.2.** Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s) \neq 0$. Then $x$ is a spherical curve if and only if

$$\frac{\kappa_2(s)}{\kappa_1(s)} = \left( \frac{\kappa_1'(s)}{\kappa_2(s)\kappa_1^2(s)} \right)' \tag{3.5}$$

holds [9].

**Theorem 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed spherical curve in $\mathbb{E}^3$. If $x$ is a Tz-curve then the equation*

$$\frac{\kappa_2'(s)}{2\kappa_2^3(s)} = \frac{\kappa_1(s)}{\kappa_1'(s)} \tag{3.6}$$

*holds between the curvatures of $x$.*

*Proof.* Let $x$ be a unit speed spherical curve in $\mathbb{E}^3$. Then we have

$$\|x\| = r \tag{3.7}$$

where $r$ is the radius of the sphere. Differentiating the equation (3.7) with respect to $s$, we get

$$\langle x, T \rangle = 0. \tag{3.8}$$

Further, differentiating the equation (3.8), we have

$$\langle x, N_1 \rangle = -\frac{1}{\kappa_1}. \tag{3.9}$$

By differentiating the equation (3.9), we obtain

$$\langle x, N_2 \rangle = \frac{\kappa_1'}{\kappa_1^2 \kappa_2}. \tag{3.10}$$

Finally, substituting (3.9) and (3.10) into (3.3), we get the result. □

**Corollary 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed spherical Tz-curve in $\mathbb{E}^3$. Then the torsion of $x$ satisfies the equation*

$$\kappa_2 = \sqrt{\frac{\kappa_1''\kappa_1 - 2\left(\kappa_1'\right)^2}{3\kappa_1^2}}. \tag{3.11}$$

*Proof.* Substituting (3.6) into (3.5), we get the result. □

**Corollary 3.2.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski spherical Tz-curve in $\mathbb{E}^3$. Then the curvature of $x$ is given by*

$$\kappa_1 = \frac{\sqrt{3}\kappa_2}{c_1 \sin\left(\sqrt{3}\kappa_2 s\right) - c_2 \cos\left(\sqrt{3}\kappa_2 s\right)} \tag{3.12}$$

*where $c_1$, $c_2$ are integral constants and $\kappa_2$ is the constant torsion of $x$.*

*Proof.* Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski spherical Tz-curve in $\mathbb{E}^3$. Then from (3.11), we obtain the differential equation

$$\kappa_1''\kappa_1 - 2\left(\kappa_1'\right)^2 - 3\kappa_1^2\kappa_2^2 = 0 \tag{3.13}$$

which has the solution (3.12). □

**Lemma 3.1.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$ whose position vector satisfies the parametric equation*

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) \tag{3.14}$$

*for some differentiable functions, $m_i(s)$, $0 \le i \le 2$. If $x$ is a Tz-curve then we get*

$$\begin{aligned} m_0' - \kappa_1 m_1 &= 1, \\ m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\ m_2' + \kappa_2 m_1 &= 0, \\ \kappa_2' m_2 + 2\kappa_2^2 m_1 &= 0. \end{aligned} \tag{3.15}$$

*Proof.* Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in $\mathbb{E}^3$. Then, by taking the derivative of (3.14) with respect to the parameter $s$ and using the Frenet formulae, we obtain

$$\begin{aligned} x'(s) = {} & (m_0'(s) - \kappa_1(s)m_1(s))T(s) \\ & +(m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) \\ & +(m_2'(s) + \kappa_2(s)m_1(s))N_2(s). \end{aligned} \tag{3.16}$$

Further, using the equations (3.3) and (3.16), we get (3.15). □

**Theorem 3.2.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski Tz-curve in $\mathbb{E}^3$ (with the curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$) given with the parametrization (3.14). Then $x$ is congruent to a rectifying curve with the parametrization*

$$x(s) = (s + c_1)\,T(s) + c_2 N_2(s) \tag{3.17}$$

*where $c_1$ and $c_2$ are integral constants.*

*Proof.* Let $x$ be a unit speed anti-Salkowski Tz-curve in $\mathbb{E}^3$. Then, the torsion $\kappa_2$ of $x$ is constant. From the equation (3.15), we get

$$\begin{aligned} m_0 &= s + c_1 \\ m_1 &= 0 \\ m_2 &= c_2 \end{aligned} \tag{3.18}$$

where $c_1$ and $c_2$ are integral constants. Finally, substituting (3.18) into (3.14), we get the result. □

**Corollary 3.3.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed anti-Salkowski Tz-curve in $\mathbb{E}^3$ (with curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$) given with the parametrization (3.14). Then $x$ is congruent to $N$-constant curve of second kind.*

**Corollary 3.4.** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed Salkowski Tz-curve in $\mathbb{E}^3$ (with the curvatures $\kappa_1 > 0$ and $\kappa_2 \neq 0$) given with the parametrization (3.14). Then we have*

$$m_1'' + \left(\kappa_1^2 + 3\kappa_2^2\right) m_1 + \kappa_1 = 0 \tag{3.19}$$

*where the curvature $\kappa_1$ of $x$ is a real constant.*

*Proof.* Let $x$ be a unit speed Salkowski Tz-curve in $\mathbb{E}^3$. Hence, the curvature $\kappa_1$ of $x$ is constant, from the equation (3.15), we get the result. □

**Corollary 3.5.** *There is no Tz-curve with a constant curvature and a constant torsion. (i.e. Tz-W-curve)*

*Proof.* Let $x$ be a unit speed Tz-curve in $\mathbb{E}^3$ with a constant curvature and a constant torsion. (i. e. Tz-W-curve). Then, using (3.15), we obtain

$$\frac{\kappa_1(s)}{\kappa_2(s)} = \frac{c_2}{s + c_1} \tag{3.20}$$

which is a contradiction. □

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Ser. Math. Inform. Vol. 33, No 3 (2018), 417–449
https://doi.org/10.22190/FUMI1803417A

# A NEW LOG-LOCATION REGRESSION MODEL WITH INFLUENCE DIAGNOSTICS AND RESIDUAL ANALYSIS

**Emrah Altun, Haitham M. Yousof and G. G. Hamedani**

**Abstract.** A new four-parameter lifetime model called Odd Log-Logistic Burr XII distribution is defined and investigated. Some of its mathematical properties are derived. Some useful characterization results based on the ratio of two truncated moments, based on the hazard function, as well as on the conditional expectation of certain functions of a random variable, are presented. The maximum likelihood method is used to estimate the model parameters by means of a graphical Monte Carlo simulation study. Moreover, we introduce a new log-location regression model based on the proposed distribution. The Jackknife estimation method as an alternative method is used to estimate the unknown parameters of a new regression model. The generalized cook distance and likelihood distance measures are used to detect possible influential observations. Martingale and modified deviance residuals are defined to detect outliers and evaluate the model assumptions. The potentiality of the new regression model is illustrated by means of a real data set.

**Keywords**: Regression Model; Burr XII Distribution; Residual Analysis; Influential Diagnostics; Simulation; Jackknife Estimation Method.

## 1. Introduction

The Pearson system of distributions was originally introduced as an effort for modeling visibly skewed observations. It was well known at the time how to adjust a theoretical model to fit the first two cumulants or moments of observed data. In his original paper and analogously to the Pearson system of densities, Burr [4] proposed another system of distributions that includes twelve types of cdfs (cumulative distribution function) which yield a variety of density shapes. This system is obtained by considering cdfs satisfying a differential equation whose solution is given by

$$G(t) = \frac{1}{\exp\left[-\int \psi(t)\, dt\right] + 1},$$

Received December 21, 2017; accepted January 15, 2018
2010 *Mathematics Subject Classification.* Primary 60E05; Secondary 62J05

where $\psi(t)$ is chosen such that $G(t)$ is a cdf on the real line. Twelve choices for $\psi(t)$ made by Burr resulted in twelve distributions which might be useful for modeling Data. The principal aim in choosing one of these forms of distributions is to facilitate the mathematical analysis to which it is subjected, while attaining a reasonable approximation. Burr ([4], [5], [6]) and others (see Burr and Cislak [7], Hatke [18], Rodriguez [24]) devoted special attention to one of these forms, denoted by type XII, whose distribution function $G(x)$ is

$$G(t;\alpha,\beta,\lambda)=\left\{1-\left[1+\left(\frac{t}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\},x\geq 0. \tag{1.1}$$

Both $\alpha$ and $\beta$ are shape parameters, $\lambda>0$ is a scale parameter. A location parameter can easily be introduced to make (1.1) a four parameter model. The corresponding probability density function (pdf) of (1.1) is

$$g(t;\alpha,\beta,\lambda)=\alpha\beta\lambda^{-\alpha}t^{\alpha-1}\left[1+\left(\frac{t}{\lambda}\right)^{\alpha}\right]^{-\beta-1},x>0. \tag{1.2}$$

The Burr XII (BXII) model has many applications in different areas including acceptance sampling plans, reliability and failure time modeling. Tadikamalla [28] studied the BXII model and its related models, namely: Pareto type II (Lomax), log-logistic, compound Weibull gamma and Weibull exponential distributions. Zimmer et al. [31] proposed a new three-parameter Burr XII distribution. This distribution, having Weibull and logistic as sub-models, is a very popular distribution for modeling lifetime data and phenomena with monotone failure rates. Shao [29] studied the maximum likelihood estimation for the three-parameter BXII model. Soliman [27] studied the estimation of parameters from progressively censored data using the Burr-XII model. Recently, Silva et al. [25] proposed a new location-scale regression model based on the BXII model; Silva et al. [26] proposed a residual for the log-BXII regression distribution whose empirical model is close to normality; Afify et al. [2] studied the Weibull BXII distribution; Cordeiro et al. [11] proposed a double BXII model with forty special cases; Yousof et al. [30] proposed and studied the Topp Leone generated Burr XII distribution, among others.

Gleaton and Lynch [14] defined the cdf of the so-called odd log-logistic-G (OLL-G) family (for $x>0$) by

$$F(x;\theta,\xi)=\frac{G(x,\boldsymbol{\xi})^{\theta}}{G(x,\boldsymbol{\xi})^{\theta}+\bar{G}(x,\boldsymbol{\xi})^{\theta}}\ . \tag{1.3}$$

The OLL-G density function is

$$f(x;\theta,\xi)=\frac{\theta g(x,\boldsymbol{\xi})\left[G(x,\boldsymbol{\xi})\bar{G}(x,\boldsymbol{\xi})\right]^{\theta-1}}{\left[G(x,\boldsymbol{\xi})^{\theta}+\bar{G}(x,\boldsymbol{\xi})^{\theta}\right]^{2}}, \tag{1.4}$$

where $\theta > 0$ is the shape parameter and $\xi = \xi_k = (\ \xi_1, \xi_2, ...)$ is a parameters vector. A random variable (rv) $X$ with pdf (1.4) is denoted by $X \sim$OLL-G $(\theta, \xi)$. In the last decade, researchers have showed a great interest in introducing a new family of distributions by adding parameter(s) to OLL-G family. Recent extensions of the OLL-G family can be cited as follows: the Zografos-Balakrishnan odd log-logistic family of distributions by Cordeiro et al. [8], the generalized odd log-logistic family by Cordeiro et al. [9], the beta dd log-logistic generalized family of distributions by Cordeiro et al. [10], and a new generalized odd log-logistic family of distributions by Haghbin et al. [16].

Here, a new extension of the BXII distribution is proposed by means of the OLL-G family. Inserting (1.1) and (1.2) in (1.3) and (1.4), the cdf and pdf of the odd log-logistic BXII (OLLBXII) distribution are defined as

$$F(x) = F(x;\theta,\alpha,\beta,\lambda) = \frac{\overbrace{\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}}^{A_i}}{\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta} + \underbrace{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\theta\beta}}_{B_i}}, \quad x \geq 0, \tag{1.5}$$

and

$$\begin{aligned} f(x) = f(x;\theta,\alpha,\beta,\lambda) &= \theta\alpha\beta\lambda^{-\alpha}x^{\alpha-1}\left[1+\left(\tfrac{x}{\lambda}\right)^{\alpha}\right]^{-\beta-1} \\ &\times \frac{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right)^{\theta-1}}{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}+\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\theta\beta}\right)^{2}}, \quad x > 0, \end{aligned} \tag{1.6}$$

respectively.

The paper is organized as follows: The graphical presentation and motivation for the new model are presented in Section 2. In Section 3, we derive some mathematical properties of the new model. In Section 4, some useful characterization results based on the ratio of two truncated moments, based on the hazard function, and based on the conditional expectation of certain functions of a random variable are presented. In Section 5, the maximum likelihood method is discussed to estimate the model parameters by means of a Monte Carlo simulation study. A new log-location regression model and its estimation via maximum likelihood method and Jackknife estimation method, sensitivity analysis, and residual analysis are presented and displayed in Section 6. In Section 7, two applications to real data sets are performed to demonstrate the empirically importance of the new model. Finally, some conclusions and future work are given in Section 8.

## 2. Graphical presentation and motivation

The importance of pdf (1.6) can be summarized as follows: the OLLBXII model contains some well-known models as its sub-models. More clearly, the BXII model is a special sub-model when $\theta = 1$. For $\theta = \lambda = \alpha = 1$ and $\theta = \lambda = \beta = 1$, we have the standard Lomax and standard log-logistic distributions, respectively. For $\lambda = \alpha = 1$ we have the OLL-Lomax distribution. For $\lambda = \beta = 1$ we have the OLL-LL distribution. For $\beta \to 1$ we have the OLL-Weibull distribution.

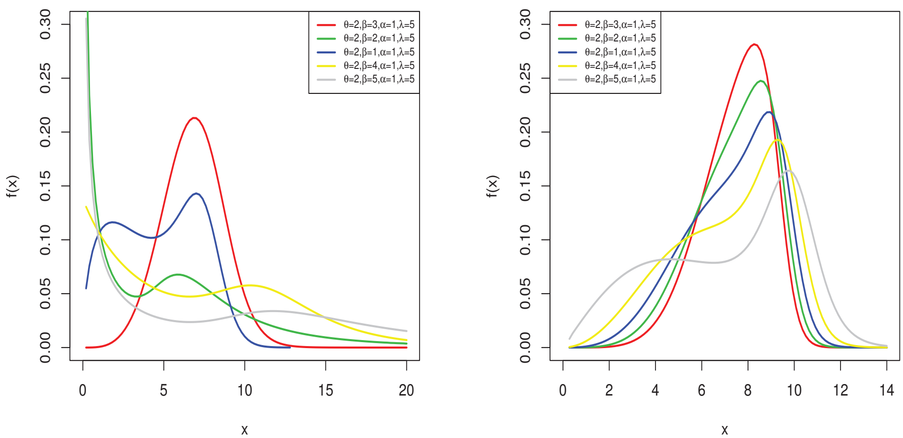


FIG. 2.1: The pdf plots of OLLBXII distribution for several parameter values.

We are motivated to introduce OLLBXII distribution because it contains a number of the aforementioned known lifetime models as illustrated above. The hrf of OLLBXII distribution exhibits decreasing, upside-down, and bathtub hazard rates as illustrated in Figure 2.2. It is shown in Section 3 that OLLBXII distribution can be viewed as a mixture of the two-parameter BXII distribution. It can be viewed as a suitable model for fitting the left-skewed, right-skewed, symmetric and bimodal data sets as illustrated in Figure 2.1.

Moreover, Figure 2.3 displays the hrf regions of OLLBXII distribution for fixed $\alpha = 4$, $\lambda = 0.1$ parameters. The developed computational codes are provided in Appendix. As seen from Figure 2.3, when the parameter $\theta < 0.255$, the hrf of OLLBXII distribution is decreasing, otherwise, it is upside-down. Similar results can be obtained for different parameter combinations by using the computational codes given in Appendix.

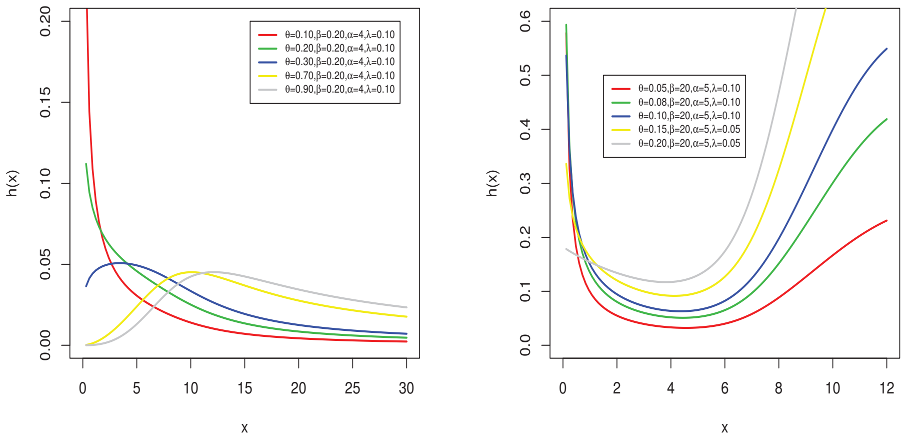


FIG. 2.2: The hrf plots of OLLBXII distribution for several parameter values.

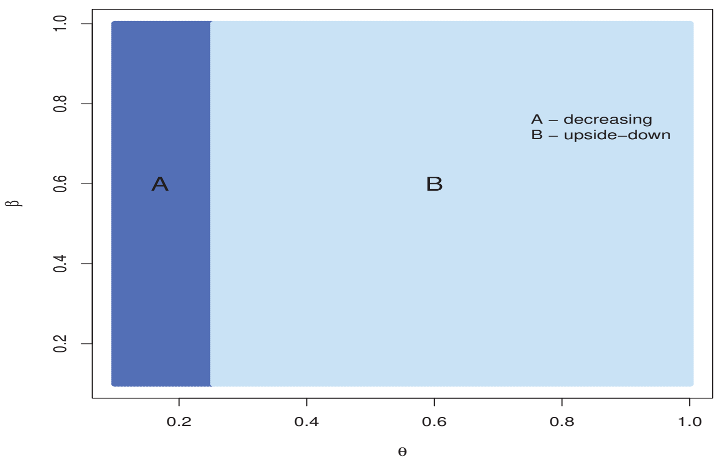


FIG. 2.3: The hrf regions of OLLBXII distribution for fixed $\alpha = 4$, $\lambda = 0.1$ parameters.

## 3. Mathematical Properties

### 3.1. Quantile function

Let $U$ have a uniform $U(0,1)$ distribution, the quantile function (qf) of OLLBXII distribution is defined by

$$Q\left(u\right)=\lambda\left\{\left[\frac{\left(1-u\right)^{\frac{1}{\theta}}}{u^{\frac{1}{\theta}}+\left(1-u\right)^{\frac{1}{\theta}}}\right]^{-\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}, \tag{3.1}$$

follows the density function (1.6). The following algorithm can be used to generate random variables from density (1.6).

**Algorithm 3.1. Algorithm**

1. Generate $U \sim U(0,1)$
2. Set $X=\lambda\left[\left\{\frac{(1-U)^{1/\theta}}{U^{1/\theta}+(1-U)^{1/\theta}}\right\}^{-1/\beta}\right]^{1/\alpha}$

The effects of the shape parameters of the new model can be measured by the skewness and kurtosis using the qf (3.1). These measures, called Bowley's skewness and Moors's kurtosis, are given respectively by

$$Skewness=\frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}$$

and

$$Kurtosis=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

The plots of Bowley's skewness and Moors's kurtosis of the BOLL-GHN distribution are displayed in Figure 3.1. Figures 3.1(a) and (b) display the effects of the parameters $\beta$ and $\theta$ on skewness and kurtosis measures for fixed $\alpha=10, \lambda=0.5$. Figures 3.1(c) and (d) display the effects of the parameters $\alpha$ and $\theta$ on skewness and kurtosis measures for fixed $\beta=10, \lambda=0.5$. As seen in Figure 3.1; when the parameters $\alpha, \beta$ and $\theta$ increase, skewness and kurtosis decrease.

### 3.2. Mixture representation

We provide a very useful linear representation for the OLLBXII cdf. First, we use a power series for the quantity $A_i$ ($\theta > 0$ real) given by

$$A_i=\sum_{k=0}^{\infty}a_k\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{k}, \tag{3.2}$$

where $a_k = \sum\limits_{j=k}^{\infty} (-1)^{k+j} \binom{\theta}{j}\binom{j}{k}$. For any real $\theta > 0$, we consider the generalized binomial expansion

$$B_i = \sum_{k=0}^{\infty} (-1)^k \binom{\theta}{k} \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k . \tag{3.3}$$

Inserting (3.2) and (3.3) in equation (1.5), we obtain

$$F(x) = \frac{\sum\limits_{k=0}^{\infty} a_k \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k}{\sum\limits_{k=0}^{\infty} b_k \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k},$$

where $b_k = a_k + (-1)^k \binom{\theta}{k}$. The ratio of the two power series can be expressed as

$$F(x) = \sum_{k=0}^{\infty} c_k \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^k = \sum_{k=0}^{\infty} c_k \Pi_k\left(x; \alpha, \beta, \lambda\right), \tag{3.4}$$

where $\Pi_k\left(x; \alpha, \beta, \lambda\right) = \left[G(x, \alpha, \beta, \lambda)\right]^k$ is the exponentiated BXII cdf with power parameter $k$, and the coefficients $c_k$'s (for $k \geq 0$) are determined from the recurrence equation

$$c_k = b_0^{-1} \left(a_k + b_0^{-1} \sum_{w=0}^{\infty} b_w c_{k-w}\right).$$

By differentiating (3.4), the pdf of $X$ can be expressed as

$$f(x) = \sum_{k=0}^{\infty} c_{1+k} \pi_{1+k}\left(x; \alpha, \beta, \lambda\right) = \sum_{r=0}^{\infty} v_r g(x; \alpha, (1+r)\,\beta, \lambda), \tag{3.5}$$

where $\pi_{1+k}\left(x; \alpha, \beta, \lambda\right)$ is the exponentiated BXII density with power parameter $k+1$, $g(x; \alpha, (1+r)\,\beta, \lambda)$ is the BXII density with parameters $\alpha, (1+r)\,\beta$ and $\lambda$ and

$$v_r = \sum_{k=0}^{\infty} (-1)^r \frac{(1+k)}{(1+r)} c_{k+1} \binom{k}{r}$$

### 3.3. Moments and cumulants

Let $W$ be a random variable having BXII distribution (1.2) with parameters $\alpha$ and $\beta$ and $\lambda$. For $n < \alpha\beta \Leftrightarrow \frac{n}{\alpha} < \beta$, the $n^{th}$ ordinary and incomplete moments of $W$ are given respectively, by

$$\mu'_n = \beta \lambda^n \, \mathrm{B}\left(\beta - n\alpha^{-1}, 1 + n\alpha^{-1}\right)$$

and

$$\varphi_n(z) = \beta\, \lambda^n \mathrm{B}\left(z^{\alpha}; \beta - n\alpha^{-1}, 1 + n\alpha^{-1}\right),$$

where

$$\mathrm{B}(a,b)=\int_0^\infty t^{a-1}\,(1+t)^{-(a+b)}dt$$

and

$$\mathrm{B}(z;a,b)=\int_0^z t^{a-1}\,(1+t)^{-(a+b)}dt$$

are beta and incomplete beta functions of the second type, respectively. So, several structural properties of the OLLBXII model can be obtained from (3.4) and those properties of BXII distribution.

The $n^{th}$ ordinary moment of $X$ is given by

$$\mu'_n=E(X^n)=\sum_{r=0}^{\infty}v_r\int_{-\infty}^{\infty}x^n\,g(x;\alpha,(1+r)\,\beta,\lambda)dx,$$

and (for $0<(1+r)\,\beta-n\alpha^{-1}$)

$$\mu'_n=\sum_{r=0}^{\infty}v_r\,(1+r)\,\beta\lambda^n\,\mathrm{B}\left((1+r)\,\beta-n\alpha^{-1},1+n\alpha^{-1}\right). \tag{3.6}$$

By setting $n=1$ in (3.6), we have the mean of $X$. The last integration can be computed numerically for most parent distributions. The $n^{th}$ central moment of $X$, say $\mu_n$, is given by

$$\mu_n=E\,(X-\mu'_1)^n=\sum_{m=0}^{n}\binom{n}{m}(-\mu'_1)^{n-m}\,\mu'_{n-m}.$$

The cumulants ($\kappa_s$) of $X$ are determined from the ordinary moments as (for $s\geq 2$)

$$\kappa_s=\mu'_s-\sum_{k=1}^{s-1}\binom{s-1}{k-1}\kappa_k\,\mu'_{s-k},$$

where $\kappa_1=\mu'_1$. The skewness ($\gamma_1=\kappa_3/\kappa_2^{3/2}$) and kurtosis ($\gamma_2=\kappa_4/\kappa_2^2$) of $X$ are just the third and fourth standardized cumulants. They are important to derive Edgeworth expansions for the cdf and pdf of the standardized sum and mean of independent and identically distributed random variables with OLLBXII distribution.

### 3.4. Moment generating function

Let $X$ have OLLBXII$(\theta,\alpha,\beta,\lambda)$ distribution. The mgf of $X$, say $M\,(t)$, using the Maclaurin series expansion of an exponential function (Abramowitz and Stegun [3]), can be written as

$$M\,(t)=E\left[\exp\,(tX)\right]=\sum_{m=0}^{\infty}(-1)^m\,E(X^m)/m!.$$

Another representation for M(t) can be obtained from (3.4) as an infinite weighted sum

$$M\left(t\right)=\sum_{r=0}^{\infty}v_{r}M_{1+r}\left(t\right),$$

where $M_{1+r}\left(t\right)$ is the mgf of the BXII density with parameters $\alpha,\left(1+r\right)\beta$ and $\lambda$. Paranaíba et al. [20] introduced a simple exemplification for the mgf of the three-parameter BXII distribution. In a similar manner, we provide another exemplification for the mgf, say $M_{1+r}(t)$, of the BXII$(\alpha,\left(1+r\right)\beta,\lambda)$ model. For $0>t$, we can write

$$M\left(t\right)=\alpha\beta\left(1+r\right)\lambda^{-\alpha}\int_{0}^{\infty}\exp\left(yt\right)y^{\alpha-1}\left[1+\left(\frac{y}{\lambda}\right)^{\alpha}\right]^{-\beta(1+r)-1}dy.$$

Next, we require the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x|\begin{array}{c}a_{1},...,a_{p}\\ b_{1},...,b_{q}\end{array}\right)=\left(2\pi i\right)^{-1}\int_{L}\frac{\prod_{j=1}^{m}\Gamma\left(b_{j}+t\right)\prod_{j=1}^{n}\Gamma\left(1-a_{j}-t\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_{j}+t\right)\prod_{j=m+1}^{p}\Gamma\left(1-b_{j}-t\right)}x^{-t}dt,$$

where $i=\sqrt{-1}$ is the complex unit and $L$ denotes an integration path (Gradshteyn and Ryzhik [15], Section 9.3). The Meijer G-function contains, as particular cases, many integrals with elementary and special functions (see Prudnikov et al. [21]). We now assume that $\alpha=m\beta^{-1}$, where $m$ and $\beta$ are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. We have the following result, which holds for $m$ and $\beta$ positive integers, $-1<\mu$ and $0>p$ (Prudnikov et al. [22], p. 21),

$$I\left(p,\mu,m\beta^{-1},v\right)=\int_{0}^{\infty}\exp\left(-px\right)x^{\mu}\left(1+x^{m\beta^{-1}}\right)^{v}dx,$$

or

$$I\left(p,\mu,m\beta^{-1},v\right)=VG_{\beta+m,\beta}^{\beta,\beta+m}\left(\left(mp^{-1}\right)^{m}|\begin{array}{c}\triangle\left(m,-\mu\right),\triangle\left(\beta,v+1\right)\\ \triangle\left(\beta,0\right)\end{array}\right),$$

where

$$V=\beta^{-v}\left[\Gamma\left(-v\right)\right]^{-1}m^{v+\frac{1}{2}}p^{-(\mu+1)}\left(2\pi\right)^{-\frac{m-1}{2}}$$

and

$$\triangle\left(\beta,a\right)=a\beta^{-1},\left(a+1\right)\beta^{-1},...,\left(a+\beta\right)\beta^{-1}.$$

The mgf of of the BXII$(\alpha,\beta,\lambda)$ can be written as

$$M\left(t\right)=mI\left(-\lambda t,m\beta^{-1}-1,m\beta^{-1},-\beta-1\right),t<0.$$

Hence, the mgf of of the OLLBXII$(\theta,\alpha,\left(1+r\right)\beta,\lambda)$ can be expressed as

$$M_{X}\left(t\right)=m\sum_{r=0}^{\infty}v_{r}I\left(-\lambda t,m\left[\beta\left(r+1\right)\right]^{-1}-1,m\left[\beta\left(r+1\right)\right]^{-1},-\left[\beta\left(r+1\right)+1\right]\right).$$

### 3.5. Incomplete moment

The $s^{th}$ incomplete moment, say $\varphi_s(t)$, of OLLBXII distribution is given by $\varphi_s(t) = \int_0^t x^s\, f(x)dx$. From the equation (3.4),

$$\varphi_s(t) = \sum_{r=0}^{\infty} v_r \int_0^t x^s\, g(x;\alpha,\beta\,(r+1)\,,\lambda)dx,$$

and using the lower incomplete gamma function, we have (for $s < \alpha\beta$)

$$\varphi_s(t) = \sum_{r=0}^{\infty} v_r\, \beta\,(r+1)\,\lambda^s\, \mathrm{B}\left(t^{\alpha};\beta\,(r+1)\, - s\alpha^{-1}, 1+s\alpha^{-1}\right).$$

The $1^{st}$ incomplete moment of $X$, denoted by $\varphi_1\,(t)$, is simply determined from $\varphi_s(t)$ by taking $s = 1$. The $1^{st}$ incomplete moment has important applications related to the residual life, the mean waiting time and Bonferroni and Lorenz curves.

### 3.6. Moments of reversed residual life and mean waiting time

The $s^{th}$ moment of the reversed residual life, say $R_s(t) = E\left[(t-X)^s \mid X \le t\right]$ for $t > 0$ and $s = 1, 2, \ldots$, uniquely determines $F(x)$. Then, $R_s(t)$ is defined by

$$R_s(t) = \frac{1}{F(t)} \int_0^t (t-x)^s dF(x).$$

The $s^{th}$ moment of the reversed residual life of $X$ is

$$R_s(t) = \frac{1}{F(t)} \sum_{i=0}^{n} \sum_{r=0}^{\infty} \frac{(-1)^i s!}{i!(s-i)!} v_r\, \beta\,(r+1)\,\lambda^s\;\mathrm{B}\left(t^{\alpha};\beta\,(r+1)\, - s\alpha^{-1}, 1+s\alpha^{-1}\right).$$

The mean waiting time (MWT) or the mean inactivity time (MIT), also named the mean reversed residual life function, $R_1(t) = E[(t-X) \mid X \le t]$, represents the waiting time elapsed since the failure of a component on condition that this failure has occurred in $(0,x)$. The MIT of $X$ can be obtained by setting $s = 1$ in the above equation.

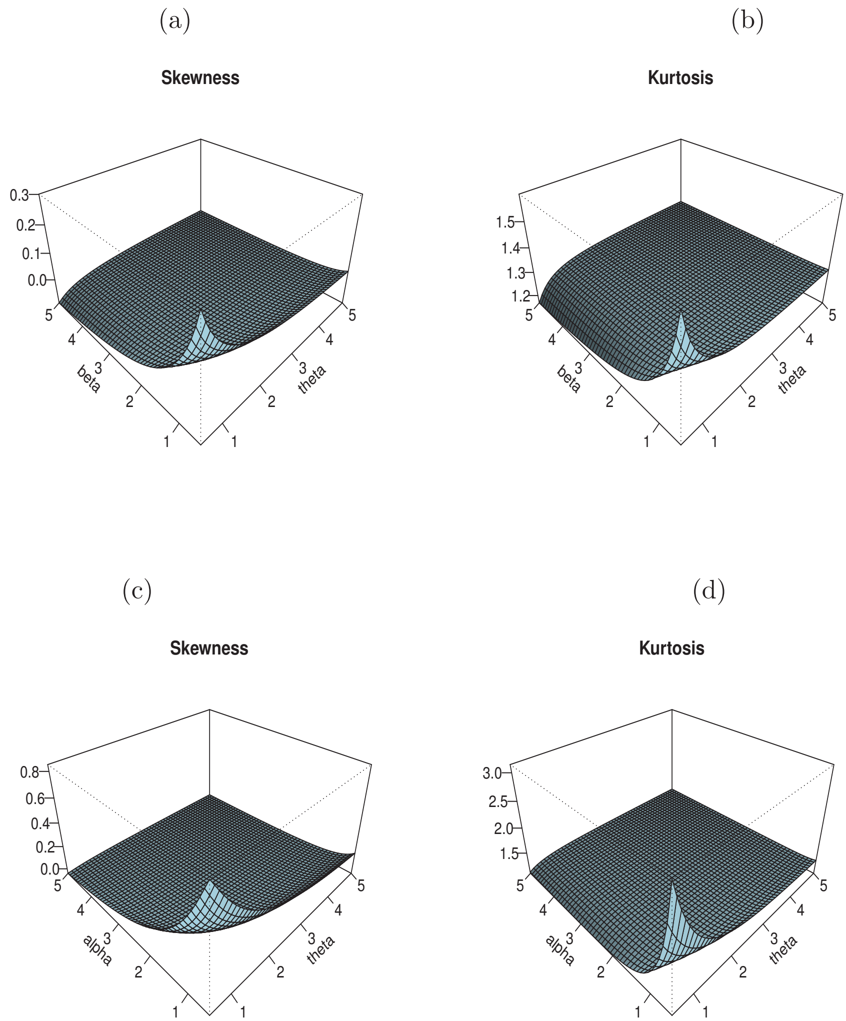


Fig. 3.1: The skewness and kurtosis plots of OLLBXII distribution for several parameter values.

## 4. Characterizations

In this section we present certain characterizations of OLLBXII distribution. These characterizations are in terms of: $(i)$ the ratio of two truncated moments; $(ii)$ the hazard function and $(iii)$ conditional expectations of functions of the random variable. One of the advantages of characterization $(i)$ is that the cdf is not required to have a closed form. We present our characterizations $(i)-(iii)$ in three subsections.

### 4.1. Characterizations based on the ratio of two truncated moments

In this subsection we present characterizations of OLLBXII distribution in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to Glänzel [12], see Theorem 1 of Appendix A. Note that the result also holds when the interval $H$ is not closed. Moreover, as mentioned above, it could also be applied when the cdf $F$ does not have a closed form. As shown in Glänzel [13], this characterization is stable in the sense of weak convergence.

**Proposition 4.1.** Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)=\frac{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}^{\theta}+\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta\theta}\right)^{2}}{\left(\left\{1-\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right\}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\right)^{\theta-1}}$ and $q_2(x)=q_1(x)\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}$ for $x>0$. The random variable $X$ has pdf (1.6) if and only if the function $\eta$ defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta},\qquad x>0.$$

Proof. Let $X$ be a random variable with pdf (1.6), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\theta\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta},\qquad x>0,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\frac{\theta}{2}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-2\beta},\qquad x>0,$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{1}{2}q_1(x)\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}<0\quad for\ \ x>0.$$

Conversely, if $\eta$ is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha\beta\lambda^{-\alpha}x^{\alpha-1}}{1+\left(\frac{x}{\lambda}\right)^{\alpha}}\qquad x>0,$$

and hence

$$s(x) = \log\left\{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{\beta}\right\}, \quad x>0.$$

Now, in view of Theorem 1, $X$ has density $(1.6)$.

**Corollary 4.1.** Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. The pdf of $X$ is (6) if and only if there exist functions $q_2$ and $\eta$ defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha\beta\lambda^{-\alpha}x^{\alpha-1}}{1+\left(\frac{x}{\lambda}\right)^{\alpha}} \quad x>0.$$

The general solution to the differential equation in Corollary 4.1 is

$$\eta(x) = \left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{\beta}\left[-\int \alpha\beta\lambda^{-\alpha}x^{\alpha-1}\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-\beta}\left(q_1(x)\right)^{-1} q_2(x) + D\right],$$

where $D$ is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with $D=0$. However, it should be also noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 1.

### 4.2. Characterization based on hazard function

It is known that the hazard function, $h_F$, of a twice differentiable distribution function, $F$, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of OLLBXII distribution, for $\theta=1$, which is not of the above trivial form.

**Proposition 4.2.** Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The pdf of $X$ is (1.6), for $\theta=1$, if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \frac{\alpha-1}{x}h_F(x) = -\frac{\alpha^2\beta\lambda^{-2\alpha}x^{2(\alpha-1)}}{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^2}, \quad x>0.$$

Proof. If $X$ has pdf (1.6), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{x^{-(\alpha-1)}h_F(x)\right\} = \alpha\beta\lambda^{-\alpha}\frac{d}{dx}\left\{\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-1}\right\}, \quad x>0,$$

from which, we obtain

$$h_F(x) = \frac{\alpha\beta\lambda^{-\alpha}x^{\alpha-1}}{1+\left(\frac{x}{\lambda}\right)^{\alpha}}, \quad x>0,$$

which is the hazard function of OLLBXII distribution.

### 4.3. Characterization based on the conditional expectation of certain functions of the random variable

In this subsection we employ a single function $\psi$ of $X$ and characterize the distribution of $X$ in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work [17], so we will just state it here as a proposition, which can be used to characterize OLLBXII distribution.

**Proposition 4.3.** Let $X:\Omega\to(d,e)$ be a continuous random variable with *cdf* $F$. Let $\psi(x)$ be a differentiable function on $(d,e)$ with $\lim_{x\to e^-}\psi(x)=1$. Then for $\delta\neq 1$,

$$E[\psi(X) \mid X\geq x] = \delta\psi(x), \qquad x\in(d,e)$$

if and only if

$$\psi(x) = (1-F(x))^{\frac{1}{\delta}-1}, \qquad x\in(d,e).$$

**Remark 4.3.** $(A)$ For $(d,e)=(0,\infty)$, $\psi(x)=\left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]^{-1}$ and $\delta=\frac{\beta}{\beta+1}$, Proposition 4.3 provides a characterization of OLLBXII distribution.

## 5. Estimation

If $X$ follows the OLLBXII distribution with vector of parameters $\mathbf{\Phi}=(\theta,\alpha,\beta,\lambda)^T$, the log-likelihood for $\mathbf{\Phi}$ from a single observation x of $X$ is given by

$$\begin{aligned}\ell(\mathbf{\Phi}) &= \log\theta+\log\alpha+\log\beta-\alpha\log\lambda-(\beta+1)\log s \\ &\quad +(\theta-1)\log\left[\left(1-s^{-\beta}\right)s^{-\beta}\right]-2\log\left[\left(1-s^{-\beta}\right)^{\theta}+s^{-\theta\beta}\right],\end{aligned}$$

where $s = \left[1+\left(\frac{x}{\lambda}\right)^{\alpha}\right]$. The components of the unit score vector $U = U(\mathbf{\Phi}) = \left(\frac{\partial\theta}{\ell(\mathbf{\Phi})}, \frac{\partial\alpha}{\ell(\mathbf{\Phi})}, \frac{\partial\beta}{\ell(\mathbf{\Phi})}, \frac{\partial\lambda}{\ell(\mathbf{\Phi})}\right)^{T} = \left(U\left(\theta\right), U\left(\alpha\right), U\left(\beta\right), U\left(\lambda\right)\right)^{T}$ are given by

$$U\left(\theta\right) = \theta^{-1} + \log\left[\left(1-s^{-\beta}\right)s^{-\beta}\right] - 2\frac{\left(1-s^{-\beta}\right)^{\theta}\log\left(1-s^{-\beta}\right) - \beta s^{-\theta\beta}\log s}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}},$$

$$\begin{aligned} U\left(\alpha\right) &= \alpha^{-1} - \log\lambda - \frac{\left(\beta+1\right)p}{s} \\ &\quad + \left(\theta-1\right)\frac{\beta p s^{-2\beta-1} - \beta p\left(1-s^{-\beta}\right)s^{-\beta-1}}{\left(1-s^{-\beta}\right)s^{-\beta}} \\ &\quad -2\frac{\theta\beta p s^{-\beta-1}\left(1-s^{-\beta}\right)^{\theta-1} - \theta\beta p s^{-\theta\beta-1}}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}}, \end{aligned}$$

$$\begin{aligned} U\left(\beta\right) &= \beta^{-1} - \log s + \left(\theta-1\right)\frac{s^{-2\beta}\log s - s^{-\beta}\left(1-s^{-\beta}\right)\log s}{\left(1-s^{-\beta}\right)s^{-\beta}} \\ &\quad -2\frac{\theta s^{-\beta}\left(1-s^{-\beta}\right)^{\theta-1}\log s - \theta s^{-\theta\beta}\log s}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}} \end{aligned}$$

and

$$\begin{aligned} U\left(\lambda\right) &= -\alpha\lambda^{-1} - \frac{\left(\beta+1\right)q}{s} + \left(\theta-1\right)\frac{\beta q s^{-2\beta-1} - \beta q\left(1-s^{-\beta}\right)s^{-\beta-1}}{\left(1-s^{-\beta}\right)s^{-\beta}} \\ &\quad -2\frac{\theta q s^{-\beta-1}\left(1-s^{-\beta}\right)^{\theta-1} - \theta\beta q s^{-\theta\beta-1}}{\left(1-s^{-\beta}\right)^{\theta} + s^{-\theta\beta}}, \end{aligned}$$

where $p = \left(\frac{x}{\lambda}\right)^{\alpha}\log\left(\frac{x}{\lambda}\right)$ and $q = \alpha x^{\alpha}\lambda^{-\alpha-1}$. For a random sample $x = \left(x_1, ..., x_n\right)^{T}$ of size $n$ from $X$, the total log-likelihood is $\ell_n(\mathbf{\Phi}) = \sum_{i=0}^{n}\ell^{(i)}(\mathbf{\Phi})$, where $\ell^{(i)}(\mathbf{\Phi})$ is the log-likelihood for the i$^{th}$ observation. The total score function is $U_n = \sum_{i=0}^{n}U^{(i)}$, where $U^{(i)}$ has the form given before. Maximization of $\ell(\mathbf{\Phi})$ (or $\ell_n(\mathbf{\Phi})$) can be easely performed using well-established routines such as the nlm or optimize in the R statistical package. Setting these equations to zero, $U(\mathbf{\Phi}) = 0$, and solving them simultaneously gives the MLE $\widehat{\mathbf{\Phi}}$ b of $\mathbf{\Phi}$. These equations cannot be solved analytically and statistical software can be used to evaluate them numerically using iterative techniques such as the Newton-Raphson algorithm.

The parameter estimation procedure of the OLLBXII model can be summarized as follows:

- The **optim** function of R software is used to minimize the minus log-likelihood function of the BXII model by means of the Nelder-Mead (NM) optimization method. There is no need to provide the derivatives of the objective function for the NM method.

- The estimated parameters of BXII distribution is used as the initial values of the OLLBXII model. The initial value of the additional parameter $\theta$ is chosen as 1. Then, the parameter estimations of the OLLBXII model are obtained with the **optim** function as given in first step.
- The inverse of the estimated Hessian matrix is used to obtain the corresponding standard errors.

### 5.1. Simulation Study

In this section, the parameter estimation efficiency of the MLE method is evaluated for the parameters of OLLBXII distribution by means of the Monte Carlo simulation. The following simulation procedure is implemented:

1. Set the sample size $n$ and the vector of parameters $\boldsymbol{\theta} = (\theta, \beta, \alpha, \lambda)$
2. Generate random observations of size $n$ from OLLBXII$(\theta, \beta, \alpha, \lambda)$ distribution
3. Using the generated random observations in Step 2, estimate $\hat{\boldsymbol{\theta}}$ by means of MLE method
4. Repeat steps 2 and 3, $N$ times
5. Using $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$ compute the mean relative estimates (MREs) and mean square errors (MSEs) via the following equations:

$$MRE = \sum_{j=1}^{N} \frac{\hat{\boldsymbol{\theta}}_{i,j} \big/ \boldsymbol{\theta}_{\boldsymbol{i}}}{N} \text{ and } MSE = \sum_{j=1}^{N} \frac{\left(\hat{\boldsymbol{\theta}}_{\boldsymbol{i,j}} - \boldsymbol{\theta}_{\boldsymbol{i}}\right)^2}{N}, \ i = 1, 2, 3, 4.$$

The statistical software R is used to obtain simulation results. The chosen parameter values for simulation study are $\boldsymbol{\theta} = (0.5, 5, 5, 0.5)$, $N = 10,000$ and $n = (50, 55, 60, ..., 500)$. We expect that MREs are closer to one when the MSEs are near zero. Figures 4 and 5 display the estimated biases, MSEs and MREs. As seen from these figures, the estimated MSEs for all parameters tend to zero for large sample sizes and the values of MREs tend to one. The biases for the parameters $\theta, \beta$ and $\alpha$ are positive whereas the biases for the parameter $\lambda$ is negative. The biases for all the parameters tend to zero for large sample sizes. It is clear that the estimates of parameters are asymptotically unbiased. Therefore, the MLE is an appropriate method for estimating parameters of the OLLBXII distribution. Similar results can be obtained for different parameter vectors.

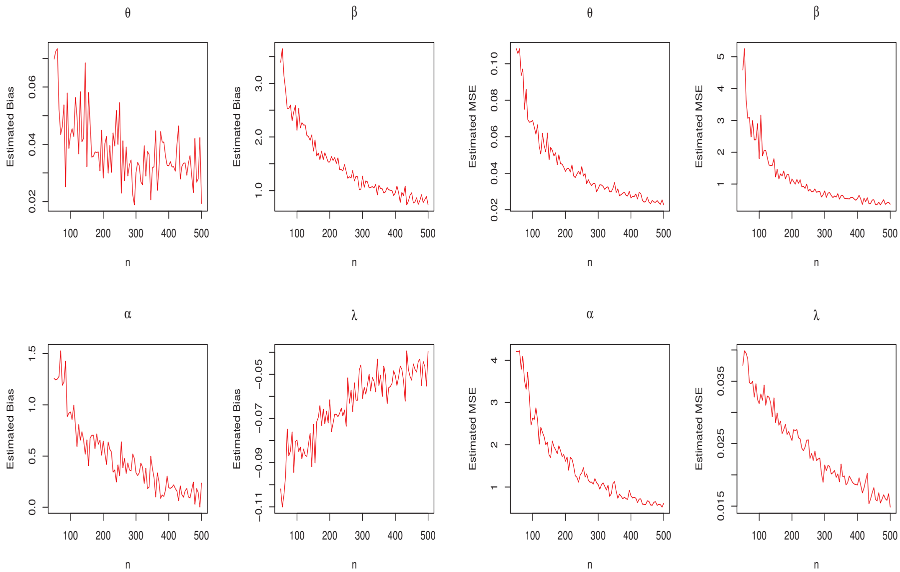


FIG. 5.1: Estimated biases and MSEs for the chosen parameter values.

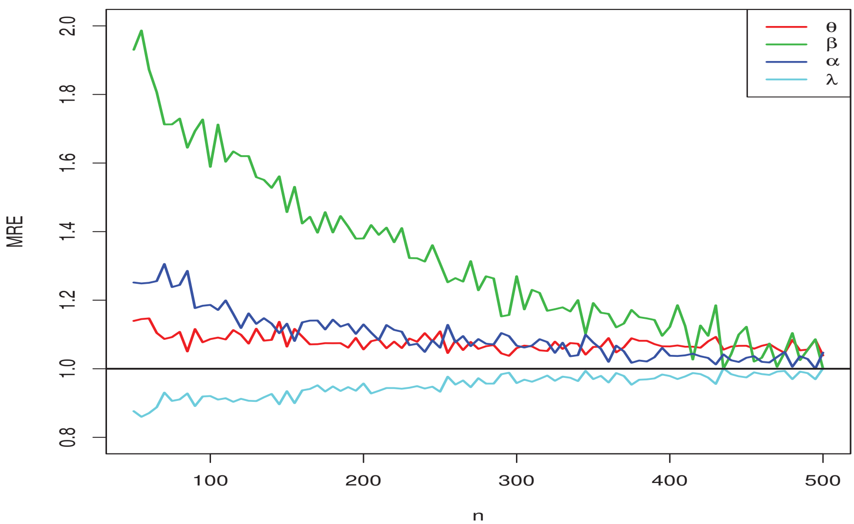


FIG. 5.2: Estimated MREs for the chosen parameter values.

## 6. Log-OLLBXII regression model

Let $X$ denotes a random variable with OLLBXII distribution and let $Y = \log(X)$. The density function of $Y$ (for $y \in \text{Re}$) for $\alpha = 1/\sigma$ and $\lambda = \exp(\mu)$, can be expressed as

(6.1)
$$f(y)=\frac{\frac{\theta\beta}{\sigma}\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-(\beta+1)}\exp\left(\frac{y-\mu}{\sigma}\right)\left[\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta-1}}{\left[\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta}+\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta\theta}\right]^{2}},$$

where $\mu \in \text{Re}$ is the location parameter, $\sigma > 0$ is the scale parameter, $\theta > 0$ and $\beta > 0$ are the shape parameters. We refer to equation (6.1) as the Log-OLLBXII (LOLLBXII) distribution, say $Y \sim \text{LOLLBXII}(\theta, \beta, \sigma, \mu)$. The plots in Figure 6.1 show shapes of density function (6.1) for selected parameter values. They reveal that this distribution is a good candidate to model left and right skewed and symmetric lifetime data sets. The survival function corresponding to (6.1) is given by

$$S(y) = 1 - \frac{\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta}}{\left[1-\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta}\right]^{\theta}+\left(1+\exp\left(\frac{y-\mu}{\sigma}\right)\right)^{-\beta\theta}} \tag{6.2}$$

and the hrf is simply $h(y) = f(y)/S(y)$. The standardized random variable $Z = (Y - \mu)/\sigma$ has density function

(6.3)
$$f(z)=\frac{\theta\beta(1+\exp(z))^{-(\beta+1)}\exp(z)\left[\left[1-(1+\exp(z))^{-\beta}\right](1+\exp(z))^{-\beta}\right]^{\theta-1}}{\left[\left[1-(1+\exp(z))^{-\beta}\right]^{\theta}+(1+\exp(z))^{-\beta\theta}\right]^{2}}.$$

### 6.1. Estimation

#### 6.1.1. Maximum Likelihood Estimation

Based on the LOLLBXII density, we propose a linear location-scale regression model linking the response variable $y_i$ and the explanatory variable vector $\mathbf{v}_i^{\intercal} = (v_{i1}, \ldots, v_{ip})$ given by

$$y_i = \mathbf{v}_i^{\intercal}\boldsymbol{\beta} + \sigma z_i, \; i = 1, \ldots, n, \tag{6.4}$$

where the random error $z_i$ has density function (6.3), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^{\intercal}$, and $\sigma > 0$, $\theta > 0$ and $\beta > 0$ are unknown parameters. The parameter $\mu_i = \mathbf{v}_i^{\intercal}\boldsymbol{\beta}$ is the location of $y_i$. The location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^{\intercal}$ is represented by a linear model $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\beta}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^{\intercal}$ is a known model matrix. The LOLLBXII

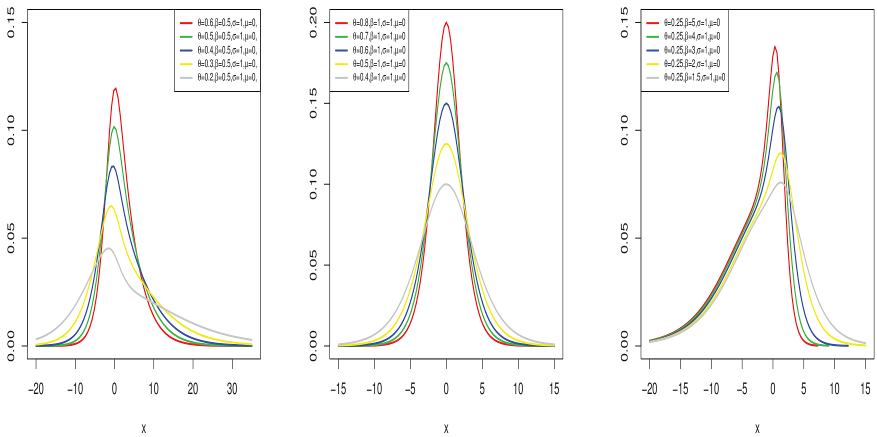


FIG. 6.1: Plots of the LOLLBXII density function for some parameter values.

model (6.4) provides new avenues for modeling several types of data sets. Note that when the parameter $\theta = 1$, the LOLLBXII regression model reduces to the Log-BXII (LBXII) regression model introduced by Silva et al. [25].

Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of $n$ independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$ where $x_i$ and $c_i$ are lifetime and censoring times, respectively. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let $F$ and $C$ be the sets of individuals for which $y_i$ is the log-lifetime or log-censoring, respectively. The log-likelihood function for the vector of parameters $\boldsymbol{\tau} = (\theta, \beta, \sigma, \boldsymbol{\beta}^\intercal)^\intercal$ from model (6.4) has the form $l(\boldsymbol{\tau}) = \sum\limits_{i \in F} l_i(\boldsymbol{\tau}) + \sum\limits_{i \in C} l_i^{(c)}(\boldsymbol{\tau})$, where $l_i(\boldsymbol{\tau}) = \log[f(y_i)]$, $l_i^{(c)}(\boldsymbol{\tau}) = \log[S(y_i)]$, $f(y_i)$ is the density (6.1) and $S(y_i)$ is the survival function (6.2) of $Y_i$. The total log-likelihood function for $\boldsymbol{\tau}$ is given by

$$
\begin{aligned}
\ell(\tau) = {}& r \log\left(\frac{\theta\beta}{\sigma}\right) - (\beta+1) \sum_{i \in F} \log\left(1 + \exp(z_i)\right) + \sum_{i \in F} z_i \\
&+ (\theta - 1) \sum_{i \in F} \log\left[\left[1 - (1+\exp(z_i))^{-\beta}\right](1+\exp(z_i))^{-\beta}\right] \\
&-2 \sum_{i \in F} \log\left[\left[1 - (1+\exp(z_i))^{-\beta}\right]^{\theta} + (1+\exp(z_i))^{-\beta\theta}\right] \\
&+ \sum_{i \in C} \log\left[1 - \frac{\left[1-(1+\exp(z_i))^{-\beta}\right]^{\theta}}{\left[1-(1+\exp(z_i))^{-\beta}\right]^{\theta} + (1+\exp(z_i))^{-\beta\theta}}\right],
\end{aligned} \tag{6.5}
$$

where $z_i = (y_i - \mathbf{v}_i^\intercal \boldsymbol{\beta})/\sigma$ and $r$ is the number of uncensored observations (failures). The MLE $\widehat{\boldsymbol{\tau}}$ of the vector of unknown parameters can be evaluated by maximizing the log-likelihood function (6.5). The optim function of R software is used to estimate $\widehat{\boldsymbol{\tau}}$. Under the standard regularity conditions, the asymptotic distribution of $(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau})$ is multivariate normal $N_{p+3}(0, K(\boldsymbol{\tau})^{-1})$, where $K(\boldsymbol{\tau})$ is the expected information matrix. The asymptotic covariance matrix $K(\boldsymbol{\tau})^{-1}$ of $\widehat{\boldsymbol{\tau}}$ can be approximated by the inverse of the $(p+3)\times(p+3)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\tau})$, whose elements are evaluated numerically. The approximate multivariate normal distribution $N_{p+3}(0, -\ddot{\mathbf{L}}(\boldsymbol{\tau})^{-1})$ for $\widehat{\boldsymbol{\tau}}$ can be used, in the classical way, to construct approximate confidence intervals for the parameters in $\boldsymbol{\tau}$.

The likelihood ratio (LR) statistic can be used for comparing the sub-model of the LOLLBXII regression model. For example, the LR statistic can be used to discriminate between the LOLLBXII and LBXII regression models since they are nested models, or equivalently to test $H_0 : \theta = 1$. The LR statistic reduces to $w = 2[\ell(\hat{\theta}, \hat{\beta}, \sigma, \hat{\boldsymbol{\beta}}) - \ell(1, \tilde{\beta}, \tilde{\sigma}, \tilde{\boldsymbol{\beta}})]$, where $(\hat{\theta}, \hat{\beta}, \hat{\sigma}, \hat{\boldsymbol{\beta}})$ are the unrestricted MLEs and $(1, \tilde{\beta}, \tilde{\sigma}, \tilde{\boldsymbol{\beta}})$ are the restricted estimates under $H_0$. The statistic $w$ is asymptotically (as $n \to \infty$) distributed as $\chi^2_k$, where $k$ is difference of two parameter vectors of nested models. For example, $k = 1$ for the above hypothesis test.

### 6.1.2. Jackknife Estimation Method

We used the Jackknife estimation method as an alternative method to estimate the unknown parameters of LOLLBXII regression model. This method is based on "leave one out" procedure. Let $\hat{\boldsymbol{\tau}}$ be the parameter estimation of whole sample and $\hat{\boldsymbol{\tau}}_{-i}$ be the parameter estimation when the $i_{th}$ observation is dropped from the sample. The pseudo-value of $i_{th}$ observation is given by

$$\tilde{\boldsymbol{\tau}}_i = n\hat{\boldsymbol{\tau}} - (n-1)\,\hat{\boldsymbol{\tau}}_{-i}. \tag{6.6}$$

Then, Jackknife estimation of $\boldsymbol{\tau}$, is given by

$$\hat{\boldsymbol{\tau}}_{jack} = \frac{1}{n}\sum_{i=1}^{n} \tilde{\boldsymbol{\tau}}_i. \tag{6.7}$$

It is clear that Jackknife estimation of $\boldsymbol{\tau}$ is the average of pseudo-values. Confidence intervals of Jackknife estimates are

$$\hat{\boldsymbol{\tau}}_{jack} \pm t_{\alpha/2,(n-1)}\frac{s}{\sqrt{n}}, \tag{6.8}$$

where $t_{\alpha/2,(n-1)}$ is the value that is exceeded with probability $\alpha/2$ for the t distribution with $n-1$ degrees of freedom. The parameter estimation of the LOLLBXII regression model can be obtained by means of the theory described above.

### 6.2. Sensitivity analysis

A first tool to perform the sensitivity analysis, as stated before, is by means of global influence starting from case deletion. Case deletion is a popular method to investigate the influence of taking out the $i_{th}$ case from the data on the parameters estimates. This method compares the $\hat{\boldsymbol{\tau}}$ with $\hat{\boldsymbol{\tau}}_{-i}$ where $\hat{\boldsymbol{\tau}}_{-i}$ is the estimated parameters when the $i_{th}$ case is dropped from the data. If there is a big differences between $\hat{\boldsymbol{\tau}}_{-i}$ and $\hat{\boldsymbol{\tau}}$, the dropped observation could be considered as an influential observation.

Here, generalized cook distance and likelihood distance measures are used to detect the possible influential observations. These measures are described below.

#### 6.2.1. Generalized cook distance

Generalized Cook distance (GD) is given by

$$GD_i(\boldsymbol{\tau}) = (\hat{\boldsymbol{\tau}}_{-i} - \hat{\boldsymbol{\tau}})^T \left[ -\ddot{L}(\hat{\boldsymbol{\tau}}) \right] (\hat{\boldsymbol{\tau}}_{-i} - \hat{\boldsymbol{\tau}}), \tag{6.9}$$

where $-\ddot{L}(\hat{\boldsymbol{\tau}})$ is the observed information matrix.

#### 6.2.2. Likelihood Distance

The Likelihood Distance (LD) is given by

$$LD_i(\hat{\boldsymbol{\tau}}) = 2\left\{\ell(\hat{\boldsymbol{\tau}}) - \ell(\hat{\boldsymbol{\tau}}_{-i})\right\}, \tag{6.10}$$

where $\ell(\hat{\boldsymbol{\tau}})$ is the estimated log likelihood value of whole data set and $\ell(\hat{\boldsymbol{\tau}}_{-i})$ is the estimated log likelihood value when the $i_{th}$ observations is dropped.

### 6.3. Residual analysis

Residual analysis has critical role in checking the adequacy of the fitted model. In order to analyse departures from error assumption, two types of residuals are considered: martingale and modified deviance residuals.

#### 6.3.1. Martingale residual

The martingale residuals are defined in the counting process and takes the values between $+1$ and$-\infty$ (see, Fleming and Harrington(1994) for details). The martingale residuals for the LOLLBXII model are,

$$r_{M_i} = \begin{cases} 1 + \log\left(1 - \frac{\left[1-(1+\exp(z_i))^{-\beta}\right]^\theta}{\left[1-(1+\exp(z_i))^{-\beta}\right]^\theta + (1+\exp(z_i))^{-\beta\theta}}\right) & \text{if} i \in F, \\ \log\left(1 - \frac{\left[1-(1+\exp(z_i))^{-\beta}\right]^\theta}{\left[1-(1+\exp(z_i))^{-\beta}\right]^\theta + (1+\exp(z_i))^{-\beta\theta}}\right) & \text{if} i \in C, \end{cases} \tag{6.11}$$

where $z_i = (y_i - \mu)/\sigma$.

### 6.3.2. Modified deviance residual

The main drawback of the martingale residual is that when the fitted model is correct, it is not symmetrically distributed about zero. To overcome this problem, a modified deviance residual was proposed by Therneau et al. (1990). The modified deviance residual is given by

$$r_{D_i} = \begin{cases} sign\,(r_{M_i})\,\{\,-2\,[r_{M_i} + \log\,(1 - r_{M_i})]\}^{1/2}, \ \text{if} i \in F \\ sign\,(r_{M_i})\,\{\,-2r_{M_i}\}^{1/2}, \ \text{if} i \in C, \end{cases} \tag{6.12}$$

where $\hat{r}_{M_i}$ is the martingale residual.

## 7. Applications

In this section, we provide two applications to real data sets to illustrate the flexibility of the OLLBXII distribution and the LOLLBXII regression model. The statistical software R is used for all numerical computations. The following goodness-of-fit measures are used to compare the OLLBXII model with the BXII model: Cramer von Mises (W*), Anderson Darling (A*), estimated $-\ell$. In general, the smaller the values of these statistics, the better the fit to the data. Moreover, LR test is also used to compare the models.

### 7.1. Turbocharger data set

We compare the fitting performance of the OLLBXII model with its sub-model. The first data set comes from Xu et al. [32] and it represents the time to failure (103 h) of turbocharger of one type of engine. The data are as follows: 1.6 3.5 4.8 5.4 6.0 6.5 7.0 7.3 7.7 8.0 8.4 2.0 3.9 5.0 5.6 6.1 6.5 7.1 7.3 7.8 8.1 8.4 2.6 4.5 5.1 5.8 6.3 6.7 7.3 7.7 7.9 8.3 8.5 3.0 4.6 5.3 6.0 8.7 8.8 9.0.

The total-time-test (TTT) plot, introduced by Aarset [1], is used to obtain the empirical behavior of the hazard rate of used data set. When the shape of TTT plot has a straight diagonal line, the hazard rate is constant. When the shapes of TTT plot have a convex or concave, the hazard rates are monotonically increasing or decreasing, respectively. Figure 8 displays the TTT plot of the used data set. Based on Figure 8, it is clear that the empirical hazard rate of the used data set is monotonically increasing.

Table 7.1 gives W* and A* statistics and log-likelihood values. Based on Tabl 7.1, it is clear that OLLBXII distribution provides superior fit and therefore could be chosen as a more adequate model than BXII for used data set.

Moreover, the profile log-likelihood functions of OLLBXII distribution are displayed in Figure 7.2. Figure 7.2 reveals that the likelihood equations of OLLBXII distribution have solutions that are maximizers.

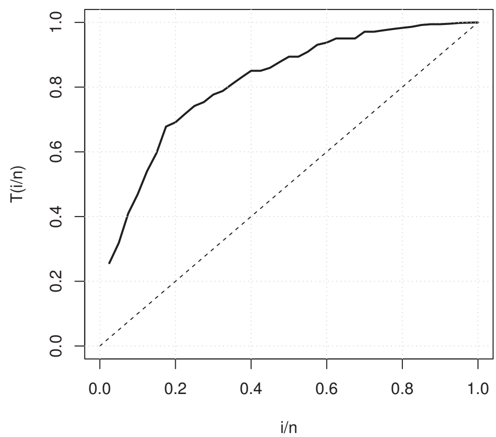


FIG. 7.1: The TTT plot of used data set.

Table 7.1: Fitting summary of the models: MLE estimates and their standard errors, $A^{\star}$, $W^{\star}$ and estimated $-\ell$

| Models | $\theta$ | $\alpha$ | $\beta$ | $\lambda$ | $A^{\star}$ | $W^{\star}$ | $-\ell$ |
|---|---|---|---|---|---|---|---|
| BXII | | 118.7304 | 3.879613 | 0.042166 | 0.582 | 0.0783 | 82.53635 |
| | | 181.7223 | 0.5222 | 0.0181 | | | |
| OLLBXII | 0.3051 | 118.058 | 10.2619 | 0.09023 | 0.1365 | 0.02005 | 78.34025 |
| | 0.1053 | 368.047 | 3.04463 | 0.03201 | | | |

Table 7.2 shows the LR statistics and the corresponding $p$-values. From Table 7.2, the computed $p$-value is smaller than 0.05, so the null hypotheses are rejected. We conclude that the OLLBXII model fits the first data better than the its sub-model according to the LR test results.

More information can be provided in Figure 7.3 by a histogram of the data with fitted lines of the pdfs for all distributions. Figure 7.3(a) suggests that the OLLBXII fits left-skewed data very well. Then, we present the plots of the fitted density, cumulative and survival functions with the probability-probability (P-P) plot for the OLLBXII model in Figure 7.3(b). They reveal a good adjustment for the data of the estimated density, cumulative and survival functions of OLLBXII distribution.

Table 7.2: The LR test results for third data set.

| | Hypotheses | LR | $p$-value |
|---|---|---|---|
| OLLBXII versus BXII | $H_0 : \theta = 1$ | 8.392 | 0.004 |

### 7.2. HIV data set

The hypothetical dataset contains 100 observations on HIV+ subjects belonging to an Health Maintenance Organization (HMO). The HMO wants to evaluate the survival time of these subjects. In this hypothetical data set, subjects were enrolled from January 1, 1989 until December 31, 1991. Study follow up then ended on December 31, 1995. This data set is reported in Hosmer and Lemeshow [19] and can also be found in R package *Bolstad2*. We adopt the LOLLBXII regression model to analyze this dataset. The variables involved in the study are: $y_i$ - observed survival time (in months); $cens_i$ - censoring indicator (0= alive at study end or lost to follow-up,1=death due to AIDS or AIDS related factors), $x_{i1}(1 = yes, 0 = no)$ represents the history of drug use and $x_{i2}$ represents the ages of patients.

We consider the following regression model

$$y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1} x_{i1} + \boldsymbol{\beta_2} x_{i2} + \sigma z_i,$$

where $y_i$ has the LOLLBXII density, for $i = 1, \dots, 100$.

#### 7.2.1. Maximum Likelihood Estimation

The MLE method is used to estimate unknown parameters of the LOLLBXII and LBXII regression models. Table 7.3 lists the MLEs of the model parameters of the LBXII and LOLLBXII regression models fitted to the current data and the log-likelihood and AIC values. These results indicate that the LOLLBXII regression model has the lowest values of these statistics, and so the LOLLW-W model provides better fitting than LBXII model for current data. For the fitted regression models, note that $\beta_0$, $\beta_1$ and $\beta_2$ is marginally significant at the 5% level.

LR test is used to compare the LOLLBXII and LBXII regression models. Table 7.4 shows the LR statistic and the corresponding $p$-value for the used data set. Based on the figures in Table 7.4, the computed $p$-value is smaller than 0.05, so the null hypotheses are rejected. We conclude that the LOLLBXII regression model provides better fits than its sub-model according to the LR test results.

#### 7.2.2. Jackknife Estimation Method

Here, the Jackknife estimation method is used to estimate the unknown parameters of LOLLBXII regression model. In Table 7.5, the jackknife estimates for the parameters of the LOLLBXII regression model are reported. From Table 7.5, we conclude

Table 7.3: MLEs of the parameters, their standard errors and $p$-values, the estimated $-\ell$ and AIC statistic.

| | LOLL-BXII | | | LBXII | | |
|---|---|---|---|---|---|---|
| Parameters | Estimates | Std. Errors | p-values | Estimates | Std. Errors | p-values |
| $\theta$ | 0.977 | 1.356 | - | - | - | - |
| $\beta$ | 2.940 | 6.389 | - | 0.867 | 0.361 | - |
| $\sigma$ | 0.705 | 1.112 | - | 0.566 | 0.080 | - |
| $\boldsymbol{\beta_0}$ | 6.675 | 3.227 | 0.039 | 4.755 | 0.804 | <0.001 |
| $\boldsymbol{\beta_1}$ | -0.090 | 0.014 | <0.001 | -0.070 | 0.017 | <0.001 |
| $\boldsymbol{\beta_2}$ | -0.974 | 0.210 | <0.001 | -0.902 | 0.220 | <0.001 |
| $-\ell$ | 127.942 | | | 130.152 | | |
| AIC | 267.885 | | | 270.304 | | |

Table 7.4: The LR test results for HIV+ data set.

| | Hypotheses | LR | $p$-value |
|---|---|---|---|
| LOLLBXII versus LBXII | $H_0 : \alpha = 1$ | 4.4198 | 0.035 |

that the parameters $\beta_0$ and $\beta_1$ are significant when the jackknife estimation method is used.

### 7.2.3. Sensitivity Analysis

Here, possible influential observations are analysed with measures described in Section 6.2.. Figure 7.4 displays the results of the generalized Cook distance,$GD_i\left(\boldsymbol{\tau}\right)$. Based on Figure 7.4, cases 41, 48 and 92 can be considered as possible influential observations.

Moreover, the effects of $i_{th}$ observation on parameters of LOLLBXII regression model is analysed and displayed in Figure 7.5. Based on this figure, it is clear that the most influential observations are 41 and 48.

Table 7.5: Jackknife estimates for the parameters of LOLLBXII regression model

| Parameters | Estimates | Std. Errors | 95% confidence intervals |
|---|---|---|---|
| $\theta$ | 0.933 | 0.147 | [0.641; 1.224] |
| $\beta$ | 2.862 | 0.060 | [2.743; 2.980] |
| $\sigma$ | 0.659 | 0.165 | [0.331; 0.987] |
| $\boldsymbol{\beta_0}$ | 6.647 | 0.203 | [ 6.243; 7.050] |
| $\boldsymbol{\beta_1}$ | -0.092 | 0.015 | [-0.121; -0.063] |
| $\boldsymbol{\beta_2}$ | -0.926 | 0.538 | [-1.994; 0.142] |

### 7.2.4. Residual Analysis

Figure 7.6 displays the index plot of the modified deviance residuals and its Q-Q plot against to $N(0,1)$ quantiles for Stanford heart transplant data set. Based on Figure 7.6 we conclude that none of observed values appears as possible outliers. Therefore, the fitted model is appropriate for these data set.

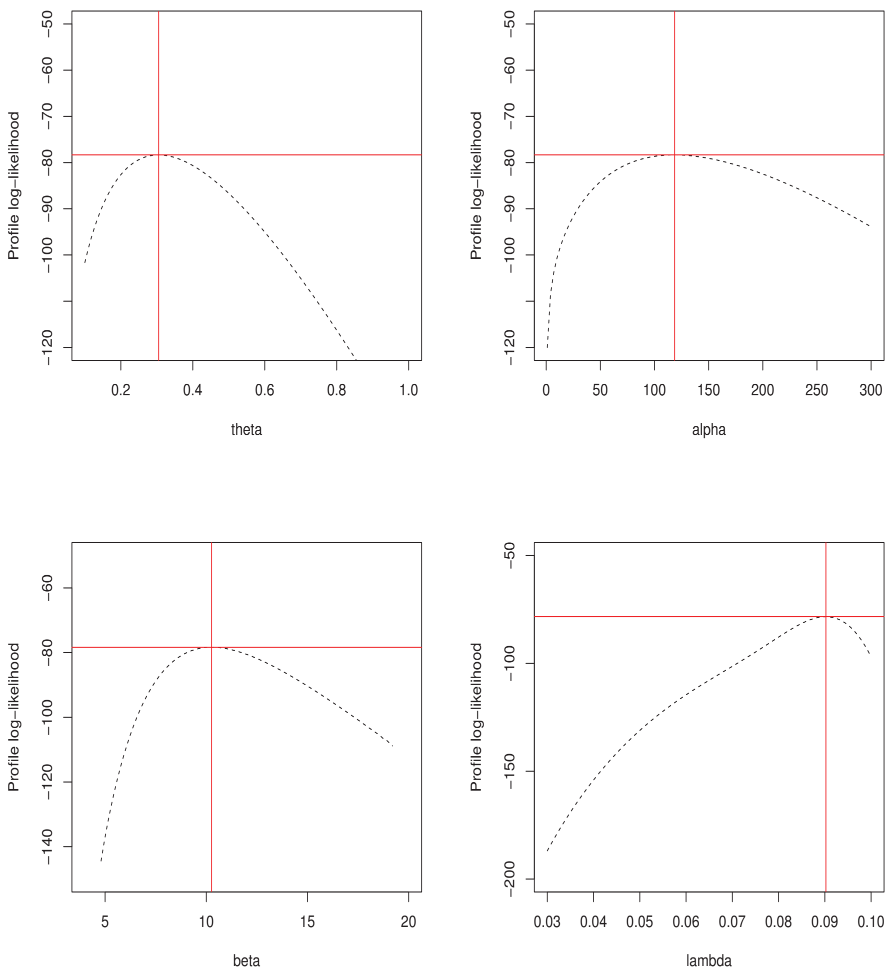


Fig. 7.2: Profile log-likelihood plots of OLLBXII distribution

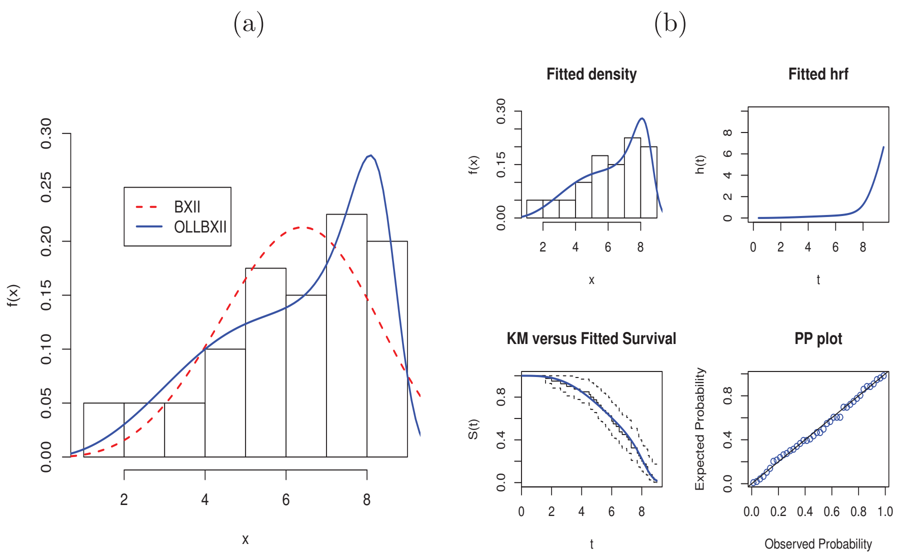


FIG. 7.3: Fitted densities of distributions for the first data set

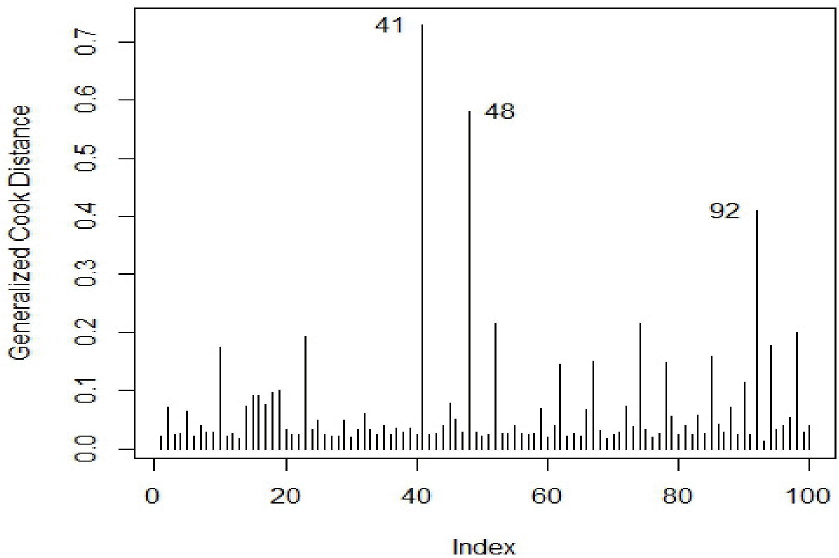


FIG. 7.4: Index plot of generalized cook distance.

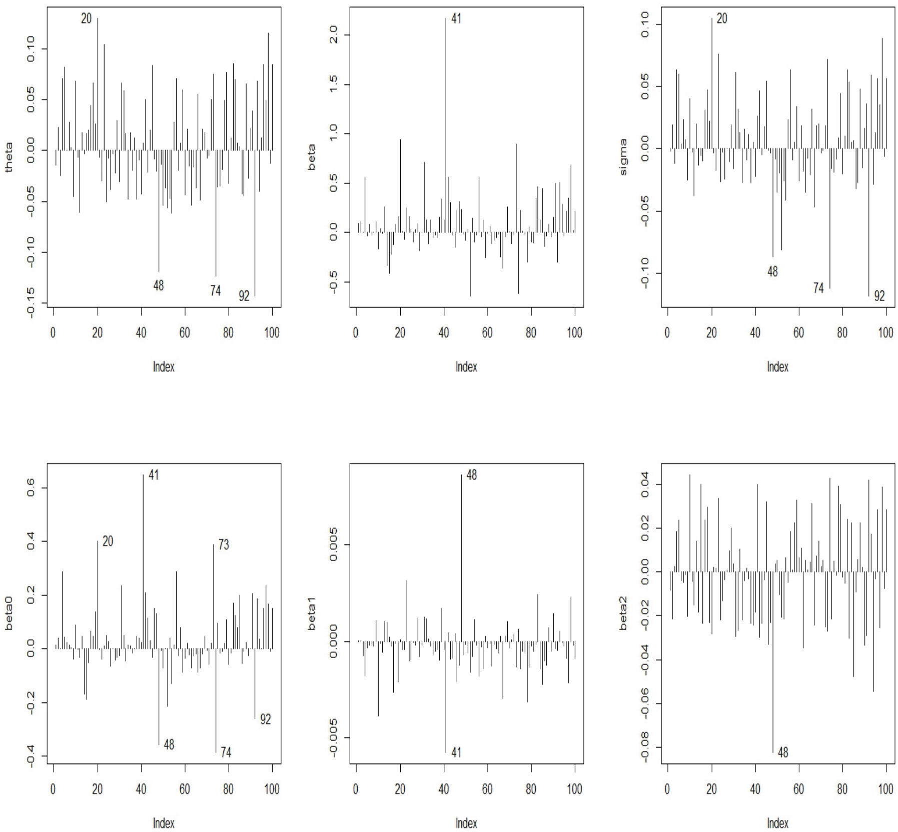


FIG. 7.5: The effects of observations on parameter values.

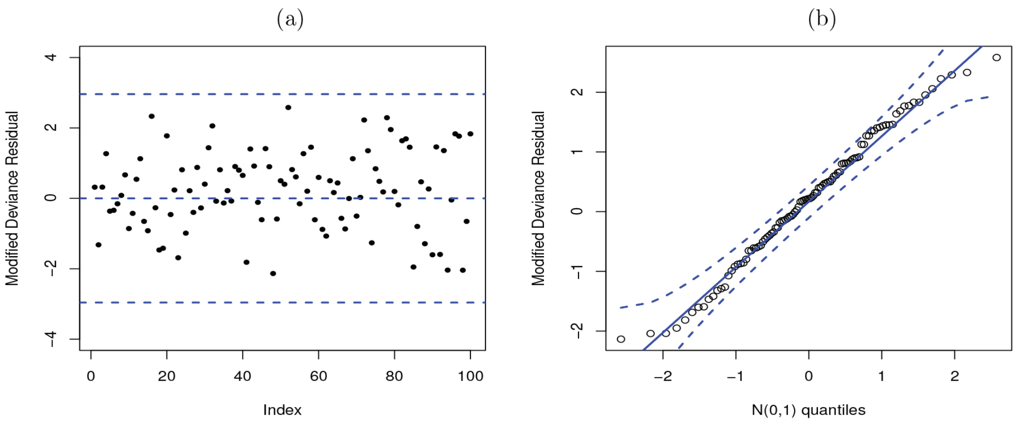


FIG. 7.6: (a) Index plot of the modified deviance residual and (b) Q-Q plot for modified deviance residual.

## 8. Concluding remarks

We propose a new lifetime model called Odd Log-logistic Burr XII distribution. Some of its mathematical properties are derived. Some useful characterization results based on the ratio of two truncated moments, based on the hazard function, and based on the conditional expectation of certain functions of the random variable are presented. The maximum likelihood method is used to estimate the model parameters by means of a graphical Monte Carlo simulation study. Moreover, we introduce a new log-location regression model based on the proposed distribution. The Jackknife estimation method is employed as an alternative method to estimate the unknown parameters of the new regression model. The generalized cook distance and likelihood distance measures are used to detect possible influential observations. Martingale and modified deviance residuals are defined to detect outliers and evaluate the model assumptions. The potentiality of the new regression model is illustrated by means of real data sets. Additionally, the index plot of the generalized cook distance and the plots for the effects of observations on the model parameters are presented.

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## Appendix A

**Theorem 1.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \rightarrow H$ be a continuous random variable with the distribution function $F$ and let $q_1$ and $q_2$ be two real functions defined on $H$ such that

$$\mathbf{E}\left[q_2\left(X\right) \mid X \geq x\right] = \mathbf{E}\left[q_1\left(X\right) \mid X \geq x\right] \eta\left(x\right), \quad x \in H,$$

is defined with some real function $\eta$. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and $F$ is twice continuously differentiable and strictly monotone function on the set $H$. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of $H$. Then $F$ is uniquely determined by the functions $q_1, q_2$ and $\eta$ , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u) q_1(u) - q_2(u)} \right| \exp(-s(u)) \; du \ ,$$

where the function $s$ is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and $C$ is the normalization constant, such that $\int_H dF = 1$.

## Appendix B

```
library(numDeriv)
rm(list=ls(all=TRUE))


f=function(x,theta,beta,alpha,lambda,a,b)
{


f=G(x,beta,alpha,lambda,a,b)**theta/(G(x,beta,alpha,lambda,a,b)
**theta+(1-G(x,beta,alpha,lambda,a,b))**theta)
ff=theta*g(x,beta,alpha,lambda,a,b)*(G(x,beta,alpha,lambda,a,b)
*(1-G(x,beta,alpha,lambda,a,b)))**(theta-1)/
((G(x,beta,alpha,lambda,a,b)
**theta+(1-G(x,beta,alpha,lambda,a,b))**theta))**2


fff=ff/(1-f)
return(fff)
}


g=function(x,beta,alpha,lambda,a,b){dburr(x,beta,alpha,lambda)}
G=function(x,beta,alpha,lambda,a,b){pburr(x,beta,alpha,lambda)}


pdf=function(x){f(x[1],theta,beta,alpha,lambda)}
pdf2=function(y,theta,beta,alpha,lambda){f(y,theta,beta,alpha,
    lambda)}

troca=function(){
  y=seq(0.1,15,0.1); mod=c(); deriv=c()
  ate=pdf2(y,theta,beta,alpha,lambda)
```

```
  ate=ate[ate!=Inf] ;  n=length(ate)
  for(i in 1:n){deriv=c(deriv,grad(func=pdf,x=c(y[i])))}
  sinal=sign(deriv)
  change=c()
  for(j in 1:n-1){
    change1=ifelse(sinal[j]==sinal[j+1],0,1); change=c(change,
       change1)}
  position=which(change %in% c(1))

  if (sum(change)==0) mod<-ifelse(sinal[1]>0,"+","-")
  if (sum(change)>0) mod<-ifelse(sinal[position]>sinal[position
     +1],"+-","-+")

  if (identical(mod,c("+"))) mod<<-"crescente"
  if (identical(mod,c("+-"))) mod<<-"modal"
  if (identical(mod,c("+-","-+"))) mod<<-"n"
  if (identical(mod,c("+-","-+","+-"))) mod<<-"m"
  if (identical(mod,c("-"))) mod<<-"decrescente"
  if (identical(mod,c("-+"))) mod<<-"banheira"
  if (identical(mod,c("-+","+-"))) mod<<-"inv(n)"
  if (identical(mod,c("-+","+-","-+"))) mod<<-"w"
  return(c(sum(change)))}

#fixing parameters
alpha=4;lambda=0.1; alphax=c(); betax=c(); a2=c(); a3=c()

for(theta in seq(0.1,1,0.005)){
  for(beta in seq(0.1,1,0.005)){
  alphax=c(alphax,theta);betax=c(betax,beta);a=troca();a2=c(a2,
     a); a3=c(a3,mod)}}

ff=factor(a3,labels=1:2)
ff1=as.numeric(ff)
ff1[ff1==1]='royalblue1'       #decres
ff1[ff1==2]='slategray1'     # inv (n)
ff1[ff1==3]='darkslategray3' #m bimo
ff1[ff1==4]='slategray1'     #mod
plot(alphax,betax,col=ff1,pch=16,cex=1,ylab =expression(beta)
,xlab=expression(theta))


text(0.17,0.6,'A',col=1,cex=1.5)
text(0.6,0.6,'B',col=1,cex=1.5)
legend(0.7,0.8,c("A - decreasing","B - upside-down"
                    ),bty="n",cex=1)
```

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Ser. Math. Inform. Vol. 33, No 3 (2018), 451-465
https://doi.org/10.22190/FUMI1803451T

# CONNECTEDNESS IN SOFT m-STRUCTURE *

**Samajh Singh Thakur and Alpa Singh Rajput**

**Abstract.** In the present paper, we introduce the concept of soft connectedness in a soft m-structure and study some of its properties and characterizations.

**Keywords**: Soft m-structure, Soft m-connectedness and Soft m-connectedness between soft sets.

## 1. Introduction

The concept of soft set is fundamentally important in almost every scientific field. Soft set theory is a new mathematical tool dealing with uncertainty and has been applied in several directions since its introduction by Molodtsov [19] in 1999. The operations on soft sets and soft structures have been studied in [1, 16, 23]. Maji et. al [15] gave the first practical application of soft sets in decision theory. In 2011 Shabir and Naz [22] initiated a study of soft topological spaces. In recent years, many soft topological concepts such as soft connectedness and their strong forms [8, 11, 17, 20, 24],soft separation axioms [14, 20, 22], weak and strong forms of soft open sets and soft continuity [17, 2, 3, 4, 5, 6, 9, 10, 12, 13, 25] have been introduced and studied. Recently, the authors of this paper [21] initiated a study of soft m-structures. In the present paper we introduce the concept of soft connectedness in soft m-structures and we study some of its properties and characterizations.

## 2. Preliminaries

Let U be an initial universe set, E be a set of parameters, P(U) denote the power set of U and A $\subseteq$ E.

**Definition 2.1.** [19] A pair (F, A) is called a soft set over U, where F is a mapping given by F: A $\rightarrow$ P(U). In other words, a soft set over U is a parameterized family of subsets of the universe U. For all e $\in$ A, F(e) may be considered a set of e-approximate elements of the soft set (F, A).

Received December 07, 2016; accepted June 11, 2018.
2010 *Mathematics Subject Classification.* Primary 54A40, 54D10; Secondary 06D72
*The authors were supported in part by . . .

**Definition 2.2.** [16] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), denoted by (F, A) $\subseteq$ (G, B), if

(a) A $\subseteq$ B and

(b) F (e) $\subseteq$ G (e) for all e $\in$ E.

**Definition 2.3.** [16] Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal denoted by (F, A) = (G, B) if (F, A) $\subseteq$ (G, B) and (G, B) $\subseteq$ (F, A).

**Definition 2.4.** [7] The complement of a soft set (F, A), denoted by $(F, A)^c$, is defined by $(F, A)^c$ = ($F^c$, A), where $F^c$ : A $\to$ P(U) is a mapping given by $F^c$(e) = U $-$ F(e), for all e $\in$ E.

**Definition 2.5.** [16] Let a soft set (F, A) over U.

(a) A null soft set denoted by $\phi$ if for all e $\in$ A, F (e) = $\phi$.

(b) An absolute soft set denoted by $\widetilde{U}$, if for each e $\in$ A, F(e) = U.

Clearly, $\widetilde{U}^c = \phi$ and $\phi^c = \widetilde{U}$.

**Definition 2.6.** [7] The union of two sets (F, A) and (G, B) over a common universe U is a soft set (H, C), where C = A $\cup$ B and for all e $\in$ C,

$$H(e) = \begin{cases} F(e), & if e \in A - B \\ G(e), & if e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

**Definition 2.7.** [7] The intersection of two soft sets (F, A) and (G, B) over a common universe U is a soft set (H, C) where C = A $\cap$ B and H(e) = F(e) $\cap$ G(e) for each e $\in$ E.

Let X and Y be initial universe sets and E and K be non-empty sets of the parameters, S(X, E) denotes the family of all soft sets over X, and S(Y, K) denotes the family of all soft sets over Y.

**Definition 2.8.** [12] Let S(X,E) and S(Y,K) be families of soft sets. Let u: X $\to$ Y and p: E $\to$ K be mappings. Then a mapping $f_{pu}$: S(X, E) $\to$ S(Y, K) is defined as:

(i)Let (F, A) be a soft set in S(X, E). The image of (F, A) under $f_{pu}$, written as $f_{pu}$ (F, A) = ( $f_{pu}$(F), p(A)), is a soft set in S(Y,K) such that

$$f_{pu}(F)(k) = \begin{cases} \bigcup_{e \in p^{-1}(k) \bigcap A} u(F(e)), & p^-1(k) \bigcap A \neq \phi \\ \phi, & p^{-1}(k) \bigcap A = \phi \end{cases}$$

For all k ∈ K.
(ii) Let (G , B) be a soft set in S(Y , K). The inverse image of (G, B) under $f_{pu}$, written as $f_{pu}^{-1}$(G,B) = ($f_{pu}^{-1}$(G),$p^{-1}$(B))), is a soft set in S(X,E) such that

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}G(p(e)), & p(e)\in B \\ \phi, & p(e)\notin B \end{cases}$$

For all e ∈ E.

**Definition 2.9.** [25]Let $f_{pu}$ : S(X, E) → S(Y, K) be a mapping and u : X → Y and p : E → K be mappings. Then $f_{pu}$ is soft onto, if u : X → Y and p : E → K are onto and $f_{pu}$ is soft one-one, if u : X → Y and p : E → K are one-one.

**Definition 2.10.** [22] A subfamily $\tau$ of S(X , E) is called a soft topology over X if:

1. $\widetilde{\phi}$ , $\widetilde{X}$ belong to $\tau$ .
2. The union of any number of soft sets in $\tau$ belongs to $\tau$.
3. The intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triplet (X, $\tau$, E) is called a soft topological space over X. The members of $\tau$ are called soft open sets in X and their complements are called soft closed sets in X.

**Definition 2.11.** If (X ,$\tau$ , E) is a soft topological space and a soft set (F, E) over X.

(a) The soft closure of (F, E) is denoted by Cl(F,E), and defined as the intersection of all soft closed super sets of (F,E) [22].

(b) The soft interior of (F, E) is denoted by Int(F,E), and defined as the soft union of all soft open subsets of (F, E) [25].

**Definition 2.12.** [25] The soft set (F,E) ∈ S(X,E) is called a soft point if there exist x ∈ X and e ∈ E such that F(e) = {x} and F(e') = $\phi$ for each e' ∈ E − {e}, and the soft point (F,E) is denoted by $x_e$.

**Definition 2.13.** A soft set (A, E) of a soft topological space(X,$\tau$,E) is called :

**(a)** Soft regular open (A, E) = Int(Cl(A, E)) [6];

**(b)** Soft $\alpha$-open if (A, E) ⊂ Int(Cl(Int(A, E))) [3] ;

**(c)** Soft semi-open if (A, E) ⊂ Cl(Int(A, E)) [17] ;

**(d)** Soft preopen if (A, E) ⊂ Int(Cl(A, E)) [2] ;

**(e)** Soft b-open if (A, E) ⊂ Int(Cl(A, E)) ∪ Cl(Int (A, E)) [5].

**(f)** Soft $\beta$-open if (A, E) $\subset$ Cl(Int(Cl(A, E))) [4]

The family of all soft regular open (resp. soft $\alpha$-open, soft semi-open, soft preopen, soft $\beta$-open, soft b-open) sets of X will be denoted by SRO(X,E) (resp. S$\alpha$O(X,E), SSO(X,E), SPO(X,E), S$\beta$O(X, E), SbO(X, E)).

**Definition 2.14.** Let (A,E ) be a soft subset of a soft topological space (X,$\tau$,E). Then:

**(a)** The intersection of all soft semi-open sets containing (A, E) is called semi-closure of (A,E ). It is denoted by sCl(A,E) [17].

**(b)** The intersection of all soft preopen sets containing (A, E) is called preclosure of (A,E). It is denoted by pCl(A,E)[2].

**(c)** The intersection of all soft $\alpha$ open sets containing (A,E) is called $\alpha$-closure of (A,E). It is denoted by $\alpha$Cl(A,E )[3].

**(d)** The intersection of all soft b-open sets containing (A,E) is called b-closure of (A,E). It is denoted by bCl(A,E)[5].

**(e)** The intersection of all soft $\beta$-open sets containing (A,E) is called $\beta$-closure of (A,E). It is denoted by $\beta$Cl(A,E)[4].

**Definition 2.15.** A soft mapping $f_{pu}$ : (X,$\tau$,E)$\rightarrow$ (X,$\sigma$,K) is said to be :

**(a)** Soft continuous if $f_{pu}^{-1}$ (U, K) $\in \tau$ for every soft set (U, K) $\in \sigma$ [25] .

**(b)** Soft $\alpha$-continuous if $f_{pu}^{-1}$ (U, K) $\in$ S$\alpha$O(X, E) for every soft set (U, K) $\in \sigma$ [3].

**(c)** Soft semi-continuous if $f_{pu}^{-1}$ (U, K) $\in$ SSO(X, E) for every soft set (U, K) $\in \sigma$ [17].

**(d)** Soft precontinuous if $f_{pu}^{-1}$ (U, K) $\in$ SPO(X, E) for every soft set (U, K) $\in \sigma$ [2].

**(e)** Soft b-continuous if $f_{pu}^{-1}$ (U, K) $\in$ SbO(X, E) for every soft set (U, K) $\in \sigma$ [5].

**(f)** Soft $\beta$-continuous if $f_{pu}^{-1}$ (U, K) $\in$ S$\beta$O(X, E) for every soft set (U, K) $\in \sigma$ [4].

**Definition 2.16.** A soft mapping $f_{pu}$ : (X,$\tau$,E)$\rightarrow$ (X,$\sigma$,K) is said to be :

**(a)** Soft open if $f_{pu}$(U, E) $\in \sigma$ for every soft set (U, E) $\in \tau$ [26].

**(b)** Soft $\alpha$-open if $f_{pu}$ (U, E) $\in$ S$\alpha$O(Y, K) for every soft set (U, E) $\in \tau$ [3].

**(c)** Soft semi-open if $f_{pu}$(U, E) $\in$ SSO(Y, K) for every soft set (U, E) $\in \tau$ [17].

**(d)** Soft preopen if $f_{pu}$(U, E) $\in$ SPO(Y, K) for every soft set (U, E) $\in \tau$ [2].

**(e)** Soft b-open if $f_{pu}$(U, E) $\in$ SbO(Y, K) for every soft set (U, E) $\in \tau$ [5].

**(f)** Soft $\beta$-open if $f_{pu}$(U, E) $\in$ S$\beta$O(Y, K) for every soft set (U, E) $\in \tau$ [4].

**Definition 2.17.** [14] Let (X,$\tau$,E) be a soft topological space, and (A,E),(B,E) be two soft sets over X. The soft sets (A,E) and (B,E) are said to be soft-separated, if (A,E) $\cap$ Cl(B,E) = $\phi$ and Cl(A,E) $\cap$ (B,E) = $\phi$.

**Definition 2.18.** [14] Let (X,$\tau$,E) be a soft topological space and if there exist two non-empty soft separated sets (A,E),(B,E) such that (A,E) $\cup$ (B,E) = $\tilde{X}$, then (A,E) and (B,E) are said to be a soft disconnection for a soft topological space (X,$\tau$,E).(X,$\tau$,E) is said to be soft-disconnected if (X,$\tau$,E) has a soft disconnection. Otherwise, (X,$\tau$,E) is said to be soft-connected.

**Definition 2.19.** [17] Let (X,$\tau$ ,E) be a soft topological space. The nonempty soft sets (F,A) and (F,B) in S(X,E) are called soft semi-separated iff sCl(F,A) $\cap$ (F,B) = (F,A) $\cap$ sCl(F,B) = $\phi$.

**Definition 2.20.** [17] Let (X,$\tau$ ,E)be a soft topological space. If there does not exist a soft semi-separation of X, then it is said to be soft s-connected.

**Definition 2.21.** [24] Let (X,$\tau$ ,E) be a soft topological space. The nonempty soft sets (F,A) and (F,B) in S(X,E) are called soft preseparated iff pCl(F,A) $\cap$ (F,B) = (F,A) $\cap$ pCl(F,B) = $\phi$.

**Definition 2.22.** [24] Let (X,$\tau$ ,E)be a soft topological space. If there does not exist a soft preseparation of X, then it is said to be soft P-connected.

**Definition 2.23.** [21] A subfamily $m_{(X,E)}$ of S(X,E) is called a soft minimal structure (briefly soft m-structure) over X if $\phi \in m_{(X,E)}$ and $\tilde{X} \in m_{(X,E)}$.

(X,$m_{(X,E)}$) is called a soft space with a soft minimal structure $m_{(X,E)}$ or simply a soft m-space. Each member of $m_{(X,E)}$ is called a soft m-open set and the complement of a soft m-open set is called a soft m-closed set.

**Remark 2.1.** [21] Let (X,$\tau$,E) be a soft topological space. Then the families $\tau$, SSO(X,E), SPO(X,E), S$\alpha$O(X,E), S$\beta$O(X,E), S$b$O(X,E), SRO(X,E) are all soft m-structures over X.

**Definition 2.24.** [21] Let X be a nonempty set, E be a set of parameters and $m_{(X,E)}$ be a soft m-structure over X. The soft $m_{(X,E)}$-closure and the soft $m_{(X,E)}$-interior of the soft set (A,E) over X are defined as follows:

(1) $m_{(X,E)}$-Cl(A,E) = $\cap$ {(F,E) : (A,E) $\subset$ (F,E) ,$(F,E)^c \in m_{(X,E)}$ }.

(2) $m_{(X,E)}$-Int(A,E) = $\cup$ {(F,E) : (F,E) $\subset$ (A,E) ,(F,E) $\in m_{(X,E)}$ }.

**Remark 2.2.** [21] Let (X,$\tau$,E) be a soft topological space and (A,E) be a soft set over X. If $m_{(X,E)} = \tau$ (respectively SO(X,E), SPO(X,E), S$\alpha$O(X,E), S$\beta$O(X,E), S$b$O(X,E)), then we have:

(1) $m_{(X,E)}$-Cl(A,E) = Cl(A,E) (resp. sCl(A,E) ,pCl(A,E),$\alpha$Cl(A,E),$\beta$ Cl(A,E), bCl(A,E)).

(2) $m_{(X,E)}$-Int(A,E)= Int(A,E) (resp. sInt(A,E) ,pInt(A,E),$\alpha$Int(A,E),$\beta$Int(A,E), bInt(A,E)).

**Theorem 2.1.** *[21] Let S(X,E) be a family of soft sets and $m_{(X,E)}$ a soft minimal structure over X.*

*For soft sets (A,E) and (B,E) of X, the following holds:*

**(a)** *(i): $m_{(X,E)}$-Int$(A,E)^c = (m_{(X,E)} - Cl(A,E))^c$ and (ii) : $m_{(X,E)}$-Cl$(A,E)^c = (m_{(X,E)} - Int(A,E))^c$.*

**(b)** *If $(A,E)^c \in m_{(X,E)}$, then $m_{(X,E)}$-Cl(A,E) = (A,E) and if (A,E) $\in m_{(X,E)}$ ,then $m_{(X,E)}$-Int(A,E) = (A,E).*

**(c)** *$m_{(X,E)}$-Cl($\phi$) = $\phi$ ,$m_{(X,E)}$-Cl($\tilde{X}$) = $\tilde{X}$ , $m_{(X,E)}$-Int($\phi$) = $\phi$ ,$m_{(X,E)}$-Int($\tilde{X}$) = $\tilde{X}$.*

**(d)** *If (A,E) $\subset$ (B,E), then $m_{(X,E)}$-Cl(A,E) $\subset$ $m_{(X,E)}$-Cl(B,E), $m_{(X,E)}$-Int(A,E) $\subset$ $m_{(X,E)}$-Int(B,E).*

**(e)** *(A,E) $\subset$ $m_{(X,E)}$-Cl(A,E) and $m_{(X,E)}$-Int(A,E) $\subset$ (A,E).*

**(f)** *$m_{(X,E)}$-Cl($m_{(X,E)}$-Cl(A,E)) = $m_{(X,E)}$-Cl(A,E) and $m_{(X,E)}$-Int($m_{(X,E)}$-Int(A,E)) = $m_{(X,E)}$-Int(A,E).*

**Definition 2.25.** [21] A soft mapping $f_{pu}$ : (X,$m_{(X,E)}$) $\rightarrow$ (Y,$m_{(Y,K)}$), where the minimal soft structure $m_{(X,E)}$ and $m_{(Y,K)}$ over X and Y, respectively, is said to be soft M-continuous if for each $x_e \in$ S(X,E) and each (V,K) $\in m_{(Y,K)}$ containing $f_{pu}$ ($x_e$), there exists (U,E) $\in m_{(X,E)}$ containing $x_e$ such that $f_{pu}$(U,E) $\subset$ (V,K).

Throughout this paper soft clopen means soft closed and open.

## 3. Connectedness in soft m-structure

**Definition 3.1.** [21] A soft minimal structure $m_{(X,E)}$ over X is said to have the property **B** if the union of any family of subsets belongs to $m_{(X,E)}$ belongs to $m_{(X,E)}$.

**Definition 3.2.** Let X be a nonempty set, E be a set of parameters and $m_{(X,E)}$ be a soft m-structure over X with property **B**. In (X,$m_{(X,E)}$) two nonempty soft sets (A,E) and (B,E) over X are called soft m-separated iff $m_{(X,E)}$-Cl(A,E) $\cap$ (B,E) = (A,E) $\cap$ $m_{(X,E)}$-Cl(B,E) = $\phi$.

**Remark 3.1.** Let (X,$\tau$,E) be a soft topological space over X. If $m_{(X,E)} = \tau$ (resp. SSO(X,E),SPO(X,E),SbO(X,E)) and $m_{(X,E)}$-Cl(A,E) = Cl(A,E) (resp. sCl(A,E), pCl(A,E), bCl(A,E)) we get definitions of soft separated( resp. soft semi-separated, soft preseparated, soft b-separated) sets.

**Definition 3.3.** Let $m_{(X,E)}$ be a soft m-structure over X with the property **B**. Then (X,$m_{(X,E)}$) is said to be soft m-connected if there does not exist two nonempty soft m-separated sets (A,E) and (B,E) over X, such that (A,E) $\cup$ (B,E) = $\tilde{X}$. Otherwise it is soft m-disconnected. In this case, the pair (A,E) and (B,E) is called soft m-disconnection over X.

**Remark 3.2.** Let (X,$\tau$,E) be a soft topological space over X. If we replace soft m-separated by soft separated (resp. soft semi-separated, soft preseparated, soft b-separated) sets we get a definition for soft connectedness (resp. soft semi-connectedness, soft preconnectedness, soft b-connectedness).

**Theorem 3.1.** *Let (X,$m_{(X,E)}$) be a soft m-space with the property* **B**. *Then the following conditions are equivalent:*

*(1) (X,$m_{(X,E)}$) has a soft m-disconnection.*

*(2) There exist two disjoint soft m-closed sets (A,E) ,(B,E) $\in m_{(X,E)}$ such that (A,E) $\cup$ (B,E) = $\tilde{X}$.*

*(3) There exist two disjoint soft m-open sets (A,E) ,(B,E) $\in m_{(X,E)}$ such that (A,E) $\cup$ (B,E) = $\tilde{X}$.*

*(4) (X,$m_{(X,E)}$) has a proper soft m-open and soft m-closed set over X.*

Proof: (1) $\rightarrow$ (2) : Let (X,$m_{(X,E)}$) have a soft m-disconnection (A,E) and (B,E). Then (A,E) $\cap$ (B,E) = $\phi$ and

$m_{(X,E)}$-Cl(A,E) = $m_{(X,E)}$-Cl(A,E) $\cap$ ((A,E) $\cup$ (B,E)) = ($m_{(X,E)}$-Cl(A,E) $\cap$ (A,E)) $\cup$ ($m_{(X,E)}$-Cl(A,E) $\cap$ (B,E)) = (A,E).

Therefore, (A,E) is a soft m-closed set over X. Similarly, we can see that (B,E) is also a soft m-closed set over X.

(2) $\rightarrow$ (3) : Let (X,$m_{(X,E)}$) has a soft m-disconnection (A,E) and (B,E) such that (A,E) and (B,E) are soft m-closed. Then $(A, E)^c$ and $(B, E)^c$ are soft m-open sets in $m_{(X,E)}$. Then it is easy to see $(A, E)^c \cap (B, E)^c = \phi$ and $(A, E)^c \cup (B, E)^c = \tilde{X}$.

(3) $\rightarrow$ (4) : Let (X,$m_{(X,E)}$) have a soft m-disconnection (A,E) and (B,E) such that (A,E) and (B,E) are soft m-open over X. Then (A,E) and (B,E are also soft closed in (X,$m_{(X,E)}$).

(4) $\rightarrow$ (1) : Let (X,$m_{(X,E)}$) has a proper soft m-open and soft m-closed set (F,E) over X. Put (H,E) = $(F, E)^c$. Then (H,E) and (F,E) are non-empty soft m-closed sets in (X,$m_{(X,E)}$). (H,E) $\cap$ (F,E) = $\phi$ and (H,E) $\cup$ (F,E) = $\tilde{X}$. Therefore, (H,E) and (F,E) is a soft m-disconnection of (X,$m_{(X,E)}$).

**Remark 3.3.** Let (X,$\tau$,E) be a soft topological space over X ,if $m_{(X,E)} = \tau$ (resp. SSO(X,E),SPO(X,E),SbO(X,E)) Then the following conditions are equivalent:

(1) (X,$\tau$,E) has a soft disconnection (resp. soft semi-disconnection, soft pre disconnection, soft b-disconnection).

(2) There exist two disjoint soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) sets (A,E) ,(B,E) such that (A,E) $\cup$ (B,E) = $\tilde{X}$.

(3) There exist two disjoint soft open (resp. soft semi-open, soft pre-open, soft b-open) sets (A,E) ,(B,E) such that (A,E) $\cup$ (B,E) = $\tilde{X}$.

(4) (X,$\tau$,E) has a proper soft open(resp. soft semi-open, soft pre-open, soft b-open) and soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) set over X.

**Corollary 3.1.** *Let (X,$m_{(X,E)}$) be a soft m-space with the property* **B**. *Then the following conditions are equivalent: (1) (X,$m_{(X,E)}$) is a soft m-connected.*

*(2) There does not exist two disjoint soft m-closed sets (A,E), (B,E) $\in m_{(X,E)}$ such that (A,E) $\cup$ (B,E) = $\tilde{X}$.*

*(3) There does not exist two disjoint soft m-open sets (A,E), (B,E) $\in m_{(X,E)}$ such that (A,E) $\cup$ (B,E) = $\tilde{X}$.*

*(4) (X,$m_{(X,E)}$) at most has two soft m-closed and soft m-open sets over X, that is, $\phi$ and $\tilde{X}$.*

**Remark 3.4.** Let (X,$\tau$,E) be a soft topological space over X ,if $m_{(X,E)} = \tau$ (resp. SSO(X,E),SPO(X,E),SbO(X,E)). Then the following conditions are equivalent:

(1) (X,$\tau$,E) is a soft connected (resp. soft semi-connected, soft preconnected ,soft b-connected).

(2) There does not exist two disjoint soft closed (resp. soft semi-closed, soft preclosed, soft b-closed) sets (A,E) ,(B,E) such that (A,E) $\cup$ (B,E) = $\tilde{X}$.

(3) There does not exist two disjoint soft open (resp. soft semi-open, soft pre-open, soft b-open) sets (A,E), (B,E) such that (A,E) $\cup$ (B,E) = $\tilde{X}$.

(4) (X,$\tau$,E) has a proper soft open(resp. soft semi-open, soft pre-open, soft b-open) and soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed)set over X.

**Definition 3.4.** Let (X,$m_{(X,E)}$) be a soft m-space with the property **B**, Y $\subset$ X in (X,$m_{(X,E)}$). The soft space (Y,$m_{(Y,E)}$) is called a soft m-subspace of (X,$m_{(X,E)}$) if $m_{(Y,E)} = \{$(A,E) $\cap \tilde{Y}$ : (A,E) $\in m_{(X,E)}\}$.

**Lemma 3.1.** *Let (X,$m_{(X,E)}$) be a soft m-space with the property* **B**, *(Y,$m_{(Y,E)}$) be a soft m-subspace of (X,$m_{(X,E)}$). If (A,E) $\subset \tilde{Y} \subset \tilde{X}$. Then $m_{(Y,E)}$-Cl(A,E) = $m_{(X,E)}$-Cl(A,E) $\cap \tilde{Y}$.*

Proof: We have $m_{(Y,E)}$-Cl(A,E) = $\cap$ {(F,E): (A,E) $\subset$ (F,E), $\tilde{Y}$ -(F,E) $\in m_{(Y,E)}$)}= $\cap$ {(F,E) $\cap \tilde{Y}$: (A,E) $\subset$ (F,E)$\cap \tilde{Y}$, $\tilde{X}$ - (F,E) $\in m_{(X,E)}$)}= $\cap$ {(F,E) $\cap \tilde{Y}$: (A,E) $\subset$ (F,E), $\tilde{X}$ - (F,E) $\in m_{(X,E)}$)} = $\cap$ { (F,E) : (A,E) $\subset$ (F,E) ,$\tilde{X}$ - (F,E) $\in m_{(X,E)}$} $\cap$ $\tilde{Y}$ = $m_{(X,E)}$-Cl(A,E) $\cap \tilde{Y}$.

Therefore, the lemma holds.

**Lemma 3.2.** *Let (X,$m_{(X,E)}$) be a soft m-space with the property* **B**, *(Y,$m_{(Y,E)}$) be a soft m-subspace of (X,$m_{(X,E)}$). If (A,E) and (B,E) are soft sets in (Y,$m_{(Y,E)}$), then (A,E) and (B,E) are soft m-separated in (Y,$m_{(Y,E)}$) if and only if (A,E) and (B,E) are soft m-separated in (X,$m_{(X,E)}$).*

Proof: We have $m_{(Y,E)}$-Cl(A,E) $\cap$ (B,E) $=$ ($m_{(X,E)}$-Cl(A,E) $\cap$ $\tilde{Y}$) $\cap$ (B,E) $=$ $m_{(X,E)}$-Cl(A,E) $\cap$ (B,E) by lemma 3.1.

Similarly, we have

$m_{(Y,E)}$-Cl(B,E) $\cap$ (A,E) $=$ $m_{(X,E)}$-Cl(B,E) $\cap$ (A,E).

Therefore, the lemma holds.

**Lemma 3.3.** *Let (X,$m_{(X,E)}$) be a soft m-space with the property* **B**, *$\tilde{Y} \subset \tilde{X}$. (Y,$m_{(Y,E)}$) be a soft m-subspace of (X,$m_{(X,E)}$). (Y,$m_{(Y,E)}$) is soft m-connected. If (A,E) and (B,E) are soft m-separated in (X,$m_{(X,E)}$), such that $\tilde{Y} \subset$ (A,E) $\cup$ (B,E), then $\tilde{Y} \subset$ (A,E) or $\tilde{Y} \subset$ (B,E).*

*Proof: We have $\tilde{Y} \subset$ (A,E) $\cup$ (B,E),we have $\tilde{Y}$ =( $\tilde{Y} \cap$ (A,E) )$\cup$ ($\tilde{Y} \cap$ (B,E)). By lemma 3.2 ,$\tilde{Y} \cap$ (A,E) and $\tilde{Y} \cap$ (B,E) are soft m-separated in (Y,$m_{(Y,E)}$). Since (Y,$m_{(Y,E)}$) is soft m-connected, we have $\tilde{Y} \cap$ (A,E) $= \phi$ or $\tilde{Y} \cap$ (B,E) $= \phi$. Therefore, $\tilde{Y} \subset$ (A,E) or $\tilde{Y} \subset$ (B,E).*

**Lemma 3.4.** *Let $\{(X_\alpha, m_{(X_\alpha,E)}$: $\alpha \in J$ $\}$ be a soft family non-empty soft m-connected subspaces of (X,$m_{(X,E)}$). If $\bigcap_{\alpha\in J} X_\alpha \neq \phi$, then $(\cup_{\alpha\in J}X_\alpha, \cup_{\alpha\in J}m_{(X_\alpha,E)}$ is a soft m-connected subspace of (X,$m_{(X,E)}$).*

*Proof: Let $Y = (\bigcup_{\alpha\in J} X_\alpha)$. Choose a soft point $x_e \in \tilde{Y}$. Let (C,E) and (D,E) be a soft m-disconnection of $(\cup_{\alpha\in J}X_\alpha, \cup_{\alpha\in J}m_{(X_\alpha,E)}$. Then, $x_e \in$ (C,E) and $x_e \in$ (D,E), we assume that $x_e \in$ (C,E).For each $\alpha \in J$. Since $\{(X_\alpha, m_{(X_\alpha,E)}$ is soft m-connected, it follows from lemma 3.3 that $\tilde{(X_\alpha)} \subset$ (C,E) or $\tilde{(X_\alpha)} \subset$ (D,E). Therefore, we have $\tilde{Y} \subset$ (C,E) since $x_e \in$ (C,E) and then (D,E) $= \phi$, which is a contradiction. Thus $(\cup_{\alpha\in J}X_\alpha, \cup_{\alpha\in J}m_{(X_\alpha,E})$ is a soft m-connected subspace of (X,$m_{(X,E)}$).*

**Theorem 3.2.** *Let $\{(X_\alpha, m_{(X_\alpha,E)})$: $\alpha \in J$ $\}$ be a soft family non-empty soft m-connected subspaces of (X,$m_{(X,E)}$).If $X_\alpha \cap X_\beta \neq \phi$ for $\alpha, \beta \in J$ ,then $(\cup_{\alpha\in J}X_\alpha, m_{(\cup_{\alpha\in J}X_\alpha,E)})$ is a soft m-connected subspace of (X,$m_{(X,E)}$).*

Proof : Let $\alpha_o \in$ J. For $\beta \in$ J,Put $A_\beta = X_{\alpha_o} \cup X_\beta$ By lemma 3.4 , $\{(A_\beta, m_{(X_\beta,E)}$ is soft m-connected. Then, $\{\{(A_\beta, m_{(X_\beta,E)} : \beta \in J\}$ is a family soft m-connected subspace of (X,$m_{(X,E)}$) and $\bigcap_{\beta\in J} A_\beta = X_{\alpha_o} \neq \phi$. Obviously, $(\bigcup_{\alpha\in J} X_\alpha = (\bigcup_{\beta\in J} A_\beta$. It follows from lemma 3.4 that $(\cup_{\alpha\in J}X_\alpha, \cup_{\alpha\in J}m_{(X_\alpha,E)}$ is a soft m-connected subspace of (X,$m_{(X,E)}$).

**Theorem 3.3.** *Let (X,$m_{(X,E)}$) be a soft m-space with the property* **B**, *$\tilde{Y} \subset \tilde{X}$. (Y,$m_{(Y,E)}$) be a soft m-subspace of (X,$m_{(X,E)}$). If $\tilde{Y} \subset \tilde{A} \subset m_{(X,E)}$-Cl(F,E), then*

*(A,$m_{(A,E)}$) is a soft connected m-subspace of (X,$m_{(X,E)}$). In particular, $m_{(X,E)}$-Cl(F,E) is a soft connected m-subspace of (X,$m_{(X,E)}$).*

*Proof: Let (C,E) and (D,E) be a soft m-disconnection of (A,$m_{(A,E)}$). By lemma 3.3, we have $\tilde{A} \subset$ (C,E) or $\tilde{A} \subset$ (D,E). We assume that $\tilde{A} \subset$ (C,E). By lemma 3.2, we have $m_{(X,E)}$-Cl(C,E) $\cap$ (D,E) $= \phi$ and, hence, $\tilde{A} \cap$ (D,E) $= \phi$, which is a contradiction.*

**Theorem 3.4.** *Let $f_{pu}$ : (X,$m_{(X,E)}$) $\to$ (Y,$m_{(Y,K)}$) be a soft M-continuous mapping, where $m_{(X,E)}$ and $m_{(Y,K)}$ are soft minimal structures over X and Y, respectively. If (X,$m_{(X,E)}$) is soft m-connected, then the soft image of (X,$m_{(X,E)}$) is also soft m-connected.*

Proof: Let $f_{pu}$ : (X,$m_{(X,E)}$) $\to$ (Y,$m_{(Y,K)}$) be a soft continuous mapping. Conversely, suppose that (Y,$m_{(Y,K)}$) is soft m-disconnected and the pair (A,K) and (B,K) is a soft m-disconnection of (Y,$m_{(Y,K)}$). Since $f_{pu}$ : (X,$m_{(X,E)}$) $\to$ (Y,$m_{(Y,K)}$) is soft continuous, then $f_{pu}^{-1}$(A,K) $\in m_{(X,E)}$, $f_{pu}^{-1}$(B,K) $\in m_{(X,E)}$. Clearly, the pair $f_{pu}^{-1}$(A,K) and $f_{pu}^{-1}$(B,K) is a soft m-disconnection of (X,$m_{(X,E)}$), which is a contradiction. Hence, (Y,$m_{(Y,K)}$) is soft m-connected. This completes the proof.

**Remark 3.5.** Let (X,$\tau$,E) and (Y,$\vartheta$,K) be two soft topological spaces over X and Y, respectively. If $m_{(X,E)} = \tau$ ,$m_{(Y,K)} = \vartheta$. $f_{pu}$ : (X ,$\tau$,E) $\to$ (Y,$\vartheta$,K) is a soft continuous mapping. If (X ,$\tau$,E) is soft connected (resp. soft semi-connected, soft pre connected, soft b-connected) then the soft image of (X ,$\tau$,E) is also soft connected (resp. soft semi-connected, soft preconnected, soft b-connected).

**Definition 3.5.** Let $m_{(X,E)}$ be a soft m-structure over X. A soft set (F,E) in (X,$m_{(X,E)}$ ) is soft m-connected if it is soft m-connected as a soft m-subspace.

**Remark 3.6.** Let (X,$\tau$,E) be a soft topological space over X. A soft set (F,E) in (X,$\tau$,E) is soft connected (resp. soft semi-connected, soft preconnected and soft b-connected) if it is soft connected (resp. soft semi-connected, soft preconnected and soft b-connected) as a soft subspace.

**Theorem 3.5.** *Let $m_{(X,E)}$ be a soft m-structure over X ,(G,E) be a soft m-connected set in (X,$m_{(X,E)}$) and (F,E) be a soft set over X such that (G,E) $\subset$ (F,E) $\subset m_{(X,E)}$-Cl(G,E). Then(F,E) is soft m-connected.*

Proof: It is sufficient that $m_{(X,E)}$-Cl(G,E) is soft m-connected. On the contrary, suppose that $m_{(X,E)}$-Cl(G,E) is soft m-disconnected. Then there exists a soft m-disconnection ((H,E),(K,E) ) of $m_{(X,E)}$-Cl(G,E). That is, there are ((H,E)$\cap$ (G,E)),((K,E)$\cap$ (G,E)) soft sets in (G,E) such that ((H,E)$\cap$ (G,E))$\cap$ ((K,E)$\cap$ (G,E)) = ((H,E) $\cap$ (K,E))$\cap$ (G,E) = $\phi$, and ((H,E)$\cap$ (G,E))$\cup$ ((K,E)$\cap$(G,E)) = ((H,E)$\cup$ (K,E))$\cap$ (G,E) = (G,E). This yields that the pair ((H,E) $\cap$ (G,E)) and ((K,E)$\cap$ (G,E)) is a soft m-disconnection of (G,E), which is a contradiction. This proves that $m_{(X,E)}$-Cl(G,E) is soft m-connected. Hence, the proof is complete.

**Lemma 3.5.** *Let $m_{(X,E)}$ be a soft m-structure over X with the property* **B***, and let (A,E) and (B,E) be two soft sets over X. In (X,$m_{(X,E)}$) the following statements are equivalent:*

*(1) $\phi$, $\tilde{X}$ are only soft m-open and soft m-closed set in $m_{(X,E)}$.*

*(2) (X,$m_{(X,E)}$) is not a soft union of two disjoint soft sets (A,E) and (B,E) $\in m_{(X,E)}$.*

*(3)(X,$m_{(X,E)}$) is not a soft union of two disjoint soft sets $(A,E)^c$ and $(B,E)^c$ $\in m_{(X,E)}$.*

*(4)(X,$m_{(X,E)}$) is not a soft union of two nonempty soft m-separated sets.*

**Remark 3.7.** Let (X, $\tau$ ,E) be a soft topological space over X, so we put $m_{(X,E)} = \tau$ (resp. SSO(X,E),SPO(X,E),SbO(X,E)). Also, let (A,E) and (B,E) be two soft sets over X. In (X, $\tau$ ,E) the following statements are equivalent:

(1) $\phi$ and $\tilde{X}$ are only soft clopen (resp. soft semi-clopen, soft preclopen, soft b-clopen) sets in (X, $\tau$ ,E).

(2) (X, $\tau$ ,E) is not a soft union of two soft disjoint soft open(resp. soft semi-open ,soft pre open, soft b-open) sets .

(3) (X, $\tau$ ,E) is not a soft union of two soft disjoint soft closed (resp. soft semi-closed, soft preclosed, soft b-closed) sets.

(4) (X, $\tau$ ,E) is not a soft union of two nonempty soft separated(soft semi separated, soft preseparated, soft b-separated) sets.

**Theorem 3.6.** *Let $m_{(X,E)}$ be a soft m-structure over X with the property* **B***. In (X,$m_{(X,E)}$) the following statements are equivalent:*

*(1) (X, $m_{(X,E)}$) is a soft m-connected space.*

*(2)(X, $m_{(X,E)}$) is not a soft union of any two soft m-separated sets.*

Proof : (1) $\rightarrow$ (2) : Assume (1). Suppose (2) is false, then let (A,E) and (B,E) be two soft m-separated sets such that $\tilde{X}$ = (A,E) $\cup$ (B,E). Since (X, $m_{(X,E)}$) is soft m-connected $m_{(X,E)}$-Cl(A,E) $\cap$ (B,E)=(A,E) $\cap$ $m_{(X,E)}$-Cl(B,E) = $\phi$. Since (A,E) $\subset$ $m_{(X,E)}$-Cl(A,E) and (B,E) $\subset$ $m_{(X,E)}$-Cl(B,E),then (A,E) $\cup$ (B,E) = $\phi$. Now $m_{(X,E)}$-Cl(A,E) $\subset$ $(B,E)^c$ =(A,E). Hence, $m_{(X,E)}$-Cl(A,E) = (A,E). Therefore, $(A,E)^c \in m_{(X,E)}$.By the same way we show that $(B,E)^c \in m_{(X,E)}$ which is a contradiction with remark 3.5. This shows that (2) is true. Therefore (1) $\rightarrow$ (2).

(2) $\rightarrow$ (1) : Assume that (2) is not true. Let $(A,E)^c$ and $(B,E)^c$ be two soft m-disjoint nonempty and $(A,E)^c$ and $(B,E)^c \in m_{(X,E)}$ such that $\tilde{X} = (A,E)^c \cup (B,E)^c$. Then, $m_{(X,E)}$-Cl$(A,E)^c$ $\cap$ (B,E)=(A,E) $\cap$ $m_{(X,E)}$-Cl$(B,E)^c$ = $(A,E)^c \cap (B,E)^c = \phi$. This contradicts the hypothesis in (2). This show that (1) is true. Therefore, (2) $\rightarrow$ (1).

**Remark 3.8.** Let (X, $\tau$ ,E) be a soft topological space over X, so we put $m_{(X,E)} = \tau$. Then, the following statements are equivalent:

(1) (X, $\tau$ ,E) is a soft connected (soft semi-connected, soft preconnected, soft b-connected) space.

(2) (X, $\tau$ ,E) is not the soft union of any two soft separated (soft semi separated, soft preseparated, soft b-separated) sets.

**Remark 3.9.** (1) Let $m_{(X,E)}$ be a soft m-structure over X with the property **B**, and let (A,E) be a soft set over X. If $\phi \neq$ (A,E) $\subset$ (X,$m_{(X,E)}$) then (A,E) is a soft m-connected set in $m_{(X,E)}$ whenever (X,$m_{(X,E)}$) is a soft m-connected space.

(2) Let (X, $\tau$ ,E) be a soft topological space over X, so we put $m_{(X,E)} = \tau$. If $\phi \neq$ (A,E) $\subset$ (X, $\tau$ ,E) then (A,E) is a soft connected (soft semi-connected, soft preconnected, soft b-connected) set over X whenever (X, $\tau$ ,E) is a soft connected (soft semi-connected, soft preconnected, soft b-connected) space.

**Theorem 3.7.** *Let $m_{(X,E)}$ be a soft m-structure over X with the property* **B**. *In (X,$m_{(X,E)}$), let the soft set (A,E) be a soft m-connected set. Let (B,E) and (C,E) be soft m-separated sets. If (A,E) $\subset$ (B,E) $\cup$ (C,E). Then, either (A,E) $\subset$ (B,E) or (A,E) $\subset$ (C,E).*

Proof: Suppose (A,E) is a soft m-connected set and (B,E),(C,E) are soft m-separated sets such that (A,E) $\subset$ (B,E) $\cup$ (C,E). Let (A,E) notsubset (B,E) and (A,E) is not a subset of (C,E). Suppose ($A_1$,E) = (B,E) $\cap$ (A,E) $\neq \phi$ and ($A_2$,E) = (C,E) $\cap$ (A,E) $\neq \phi$. Then, (A,E) = ($A_1$,E) $\cup$ ($A_2$,E). Since ($A_1$,E) $\subset$ (B,E). Hence, $m_{(X,E)}$-Cl($A_1$,E) $\subset$ $m_{(X,E)}$-Cl(B,E). Since $m_{(X,E)}$-Cl(B,E)$\cap$ (C,E) = $\phi$ then $m_{(X,E)}$-Cl ($A_1$,E) $\cap$ ($A_2$,E) = $\phi$. Since ($A_2$,E) $\subset$ (C,E). Hence, $m_{(X,E)}$-Cl($A_2$,E) $\subset$ $m_{(X,E)}$-Cl(C,E). Since $m_{(X,E)}$-Cl(C,E)$\cap$ (B,E) = $\phi$. Then $m_{(X,E)}$-Cl ($A_2$,E) $\cap$ ($A_1$,E) = $\phi$. But (A,E) = ($A_1$,E)$\cup$ ($A_2$,E). Therefore, (A,E) is not a soft m-connected space. This is a contradiction. Then either (A,E) $\subset$ (B,E) or (A,E) $\subset$ (C,E).

**Remark 3.10.** Let (X, $\tau$ ,E) be a soft topological space over X, so we put $m_{(X,E)} = \tau$. Also, let (A,E) be a soft connected (resp. soft semi-connected, soft preconnected, soft b-connected) set. Let (B,E) and (C,E) be soft separated (resp. soft semi-separated, soft preseparated, soft b-separated) sets. If (A,E) $\subset$ (B,E) $\cup$ (C,E) then either (A,E) $\subset$ (B,E) or (A,E) $\subset$ (C,E).

Let $m_{(X,E)}$ be a soft m-structure over X with the property **B**. In (X,$m_{(X,E)}$), let the soft set (A,E) be a soft m-connected set, then $m_{(X,E)}$-Cl(A,E) is soft m-connected.

Proof: Suppose the soft set (A,E) is a soft m-connected set and $m_{(X,E)}$-Cl(A,E) is not. Then there exist two soft m-separated sets (B,E) and (C,E) such that $m_{(X,E)}$-Cl(A,E) = (B,E) $\cup$ (C,E). But (A,E) $\subset$ $m_{(X,E)}$-Cl(A,E),then (A,E) = (B,E) $\cup$ (C,E) and since (A,E) is a soft m-connected set, then by Theorem 3.7 either (A,E) $\subset$ (B,E) or (A,E) $\subset$ (C,E).

(i) If (A,E) $\subset$ (B,E) then $m_{(X,E)}$-Cl(A,E) $\subset$ $m_{(X,E)}$-Cl(B,E). But $m_{(X,E)}$-Cl(B,E) $\cap$ (C,E) = $\phi$. Hence, $m_{(X,E)}$-Cl(A,E)$\cap$ (C,E) = $\phi$. Since (C,E) $\subset$ $m_{(X,E)}$-Cl(A,E), then (C,E) = $\phi$ this is a contradiction.

(ii) If (A,E) $\subset$ (C,E) then in the same way we can prove that (B,E) = $\phi$, which is a contradiction. Therefore, $m_{(X,E)}$-Cl(A,E) is soft m-connected.

**Remark 3.11.** Let (X, $\tau$, E) be soft topological space over X, we put $m_{(X,E)} = \tau$ let soft set (A,E) be a soft connected (resp. soft semi connected,soft pre connected, soft b-connected)set then $m_{(X,E)}$-Cl(A,E) is soft connected(resp. soft semi connected,soft pre connected, soft b-connected).

**Theorem 3.8.** *Let $m_{(X,E)}$ be a soft m-structure over X with the property* **B**. *In (X,$m_{(X,E)}$), let the soft set (A,E) be a soft m-connected set and (A,E) $\subset$ (B,E) $\subset$ $m_{(X,E)}$-Cl(A,E) then (B,E) is soft m-connected.*

Proof: If (B,E) is not soft m-connected, then there exist two soft sets (C,E) and (D,E) such that $m_{(X,E)}$-Cl(C,E) $\cap$ (D,E) = (C,E) $\cap$ $m_{(X,E)}$-Cl(D,E)= $\phi$ and (B,E) = (C,E) $\cup$ (D,E). Since (A,E) $\subset$ (B,E), thus either (A,E) $\subset$ (C,E) or (A,E) $\subset$ (D,E). Suppose (A,E) $\subset$ (C,E) then $m_{(X,E)}$-Cl(A,E) $\subset$ $m_{(X,E)}$-Cl(C,E), thus $m_{(X,E)}$-Cl(A,E) $\subset$ (D,E) = $m_{(X,E)}$-Cl(C,E) $\subset$ (D,E) = $\phi$. But (D,E) $\subset$ (B,E) $\subset$ $m_{(X,E)}$-Cl(A,E), thus $m_{(X,E)}$-Cl(A,E) $\cap$ (D,E) =(D,E). Therefore, (D,E) =$\phi$ which is a contradiction. Thus, (B,E) is a soft m-connected set.

If (A,E) $\subset$ (B,E), then we can prove that (C,E) = $\phi$. This is a contradiction. Then (B,E) is soft m-connected.

**Remark 3.12.** Let (X, $\tau$ ,E) be a soft topological space over X, so we put $m_{(X,E)}$ = $\tau$. Also, let the soft set (A,E) be a soft connected (resp. soft semi-connected, soft preconnected, soft b-connected) set and (A,E) $\subset$ (B,E) $\subset$ $m_{(X,E)}$-Cl(A,E), then (B,E) is soft connected (resp. soft semi-connected, soft preconnected, soft b-connected).

**Remark 3.13.** Let (X,$\tau$,E) be a soft topological space over X, and (F,E) be a soft set over X. (X,$\tau$,E) is soft connected (soft semi-connected, soft preconnected, soft b-connected) if and only if there does not exist nonempty soft set (F,E) over X which is both soft open (resp. soft semi-open, soft preopen, soft b-open) and soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) set over X.

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Ser. Math. Inform. Vol. 33, No 3 (2018), 467-479
https://doi.org/10.22190/FUMI1803467B

# ON SOME NEW GENERALIZATIONS OF CERTAIN GAMIDOV INTEGRAL INEQUALITIES IN TWO INDEPENDENT VARIABLES AND THEIR APPLICATIONS

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**Abstract.** The aim of this paper is to establish some new nonlinear Gamidov integral inequalities in two independent variables which can give the explicit bounds on unknown functions. To show the feasibility of the obtained inequalities, some illustrative examples are also introduced.

**Keywords**: Integral equation, mean value Theorem, Gamidov integral inequality.

## 1. Introduction

The integral inequalities which provide explicit bounds on unknown functions play an important role in the development of the theory of differential and integral equations. For instance, see $[1-19]$ and the references given therein. During the past few years, an enormous amount of effort has been devoted to the discovery of new types of inequalities and their applications in various branches of ordinary and partial differential and integral equations.

In [8] , Sh.G.Gamidov, while studying the boundary value problem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms

$$u(t) \leq c + \int_a^t f(s)u(s)ds + \int_a^b g(s)u(s)ds, \tag{1.1}$$

for $t \in [a, b]$, under some suitable conditions on the functions involved in (1.1).

Pachpatte obtained the following interesting explicit bounds on certain integral inequalities which appear in [14] :

Received June 07, 2017; accepted July 21, 2018
2010 *Mathematics Subject Classification.* 26D15, 26D20, 39A12

(1.2) $$u(t) \leq a(t) + \int_a^t f(t,s)u(s)ds + \int_a^b g(s)u(s)ds.$$

Very recently, K. Cheng, C. Guo in [5] discussed the following general version in two independent variables:

(1.3)
$$u(x,y) \leq a(x,y)+b(x,y)\int_0^x \int_0^y f(s,t)u(s,t)dsdt+c(x,y)\int_0^M \int_0^N g(s,t)u(s,t)dsdt,$$

for $(x,y) \in [0,M] \times [0,N]$.

Motivated by the results above and the inequalities obtained in [5,8,10,14], we give a generalization of nonlinear Gamidov integral inequalities in two independent variables which can be used as a tool to study the boundedness of solutions of integral equations. Some applications are also given to illustrate the usefulness of some of our results.

Before establishing our main results, we need the following lemmas.

**Lemma 1.1.** [5] *Assume $u(x,y)$, $a(x,y)$, $c(x,y)$, $g(x,y) \in C([0,M]\times[0,N], [0,\infty))$ and*
$$u(x,y) \leq a(x,y) + c(x,y)\int_0^M \int_0^N u(s,t)g(s,t)dsdt,$$

*for $(x,y) \in [0,M] \times [0,N]$. If $\int_0^M \int_0^N c(s,t)g(s,t)dsdt < 1$, then the following explicit estimate*

$$u(x,y) \leq a(x,y) + \frac{c(x,y)\int_0^M \int_0^N a(s,t)g(s,t)dsdt}{1-\int_0^M \int_0^N c(s,t)g(s,t)dsdt},$$

*holds for $(x,y) \in [0,M] \times [0,N]$.*

**Lemma 1.2.** [9] *Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

(1.4) $$a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}},$$

*for any $K > 0$.*

## 2. Main Result

In what follows, $\mathbb{R}$ denotes the set of real numbers $\mathbb{R}_+ = [0,\infty)$, $I_1 = [0,M]$, and $I_2 = [0,N]$ are given subsets of $\mathbb{R}$. Let $\Delta = I_1 \times I_2$, $C(U,V)$ denotes the collection of continuous functions from $U$ to $V$. Now let us give the main results of this paper.

**Lemma 2.1.** *Asssume that* $u(x,y)$, $a(x,y)$, $c(x,y)$, $g(x,y) \in C(\Delta, \mathbb{R}_+)$ *and* $n : \mathbb{R}_+ \to \mathbb{R}_+$ *a differentiable increasing function on* $]0,+\infty[$ *with the continuous non-increasing first derivative* $n^{'}$ *on* $]0,+\infty[$. *If*

$$u(x,y) \leq a(x,y) + c(x,y) \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt, \tag{2.1}$$

*then the following explicit estimate*

$$u(x,y) \leq a(x,y) + \frac{c(x,y) \int_0^M \int_0^N g(s,t)n(a(s,t))dsdt}{1 - \int_0^M \int_0^N c(s,t)g(s,t)n\acute{}(a(s,t))dsdt}, \tag{2.2}$$

*holds for* $(x,y) \in \Delta$, *provided that*

$$\int_0^M \int_0^N c(s,t)g(s,t)n'(a(s,t))dsdt < 1. \tag{2.3}$$

*Proof.* Obviously, $\int_0^M \int_0^N g(s,t)n(u(s,t))dsdt$ is a constant.

Letting

$$\Omega = \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt, \tag{2.4}$$

from (2.1), we have

$$u(x,y) \leq a(x,y) + c(x,y)\Omega. \tag{2.5}$$

Since $n$ is increasing on $]0,+\infty[$, then

$$n(u(x,y)) \leq n(a(x,y) + c(x,y)\Omega). \tag{2.6}$$

Applying the mean value Theorem for the function $n$, then for every $x_1 \geq y_1 > 0$ there exists $c \in ]y_1, x_1[$ such that

$$n(x_1) - n(y_1) = n\acute{}(c)(x_1 - y_1) \leq n'(y_1)(x_1 - y_1). \tag{2.7}$$

Which gives

$$n(u(x,y)) \leq n'(a(x,y))c(x,y)\Omega + n(a(x,y)), \tag{2.8}$$

taking into account that $g(x,y)$ is positive, then

$$g(x,y)n(u(x,y)) \leq g(x,y)n'(a(x,y))c(x,y)\Omega + g(x,y)n(a(x,y)). \tag{2.9}$$

Integrating both sides of (2.9) on $\Delta$, we obtain

$$\begin{aligned} \Omega &= \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt \\ &\leq \Omega \int_0^M \int_0^N c(s,t)g(s,t)n'(a(s,t))dsdt \\ &+ \int_0^M \int_0^N g(s,t)n(a(s,t))dsdt. \end{aligned} \tag{2.10}$$

It follows from (2.10) that

$$\Omega \leq \frac{\int_0^M \int_0^N g(s,t)n(a(s,t))dsdt}{1 - \int_0^M \int_0^N c(s,t)g(s,t)n'(a(s,t))dsdt}.$$

Substituting the inequality above into (2.5), we get the explicit estimate (2.2) for $u(x,y)$. $\square$

**Remark 2.1.** By taking $n(x) = x$, the inequality given in Lemma 2.1 reduces to the inequality given in Lemma 1.1.

**Corollary 2.1.** *Suppose that the conditions of Lemma 2.1 hold. Then*

$$u(x,y) \leq a(x,y) + c(x,y) \int_0^M \int_0^N g(s,t) \arctan(u(s,t))dsdt.$$

*Implies*

$$u(x,y) \leq a(x,y) + \frac{c(x,y) \int_0^M \int_0^N g(s,t) \arctan(a(s,t))dsdt}{1 - \int_0^M \int_0^N \dfrac{c(s,t)g(s,t)}{1 + a^2(s,t)}dsdt}, \quad . \tag{2.11}$$

*for $(x,y) \in \Delta$, provided that*

$$\int_0^M \int_0^N \frac{c(s,t)g(s,t)}{1 + a^2(s,t)}dsdt < 1,$$

*and if*

$$u(x,y) \leq a(x,y) + c(x,y) \int_0^M \int_0^N g(s,t) \ln(u(s,t) + 1)dsdt,$$

*then*

$$u(x,y) \le a(x,y) + \frac{c(x,y)\int_0^M \int_0^N g(s,t)\ln(a(s,t)+1)dsdt}{1-\int_0^M \int_0^N \frac{c(s,t)g(s,t)}{1+a(s,t)}dsdt},$$

*for* $(x,y) \in \Delta$, *provided that*

$$\int_0^M \int_0^N \frac{c(s,t)g(s,t)}{1+a(s,t)}dsdt < 1.$$

**Theorem 2.1.** *Assume that* $a(x,y)$, $b(x,y)$, $c(x,y)$, $f(x,y)$, $g(x,y) \in C(\Delta,\mathbb{R}_+)$ *and* $a(x,y)$, $b(x,y)$, $c(x,y)$ *are nondecreasing in* $x$ *and* $y$. *Let* $n : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $]0,+\infty[$ *with continuous non-increasing first derivative* $n'$ *on* $]0,+\infty[$. *If* $u(x,y) \in C(\Delta,\mathbb{R}_+)$ *satisfies*

$$u(x,y) \le a(x,y) + b(x,y)\int_0^x \int_0^y f(s,t)u(s,t)dsdt + c(x,y)\int_0^M \int_0^N g(s,t)n(u(s,t))dsdt, \tag{2.12}$$

*then, we have*

$$u(x,y) \le A^*(x,y) + C^*(x,y) \times \frac{\int_0^M \int_0^N g(s,t)n(A^*(s,t))dsdt}{1-\int_0^M \int_0^N C^*(s,t)g(s,t)n'(A^*(s,t))dsdt}, \tag{2.13}$$

*for* $(x,y) \in \Delta$, *provided that*

$$\int_0^M \int_0^N C^*(s,t)g(s,t)n'(A^*(s,t))dsdt < 1,$$

*where*

$$\begin{aligned} A^*(x,y) &= a(x,y)\exp\left\{b(x,y)\int_0^x \int_0^y f(s,t)dsdt\right\}, \\ C^*(x,y) &= c(x,y)\exp\left\{b(x,y)\int_0^x \int_0^y f(s,t)dsdt\right\}. \end{aligned} \tag{2.14}$$

*Proof.* Fixing any arbitrary $(X,Y) \in \Delta$ , then for $(x,y) \in \Delta_1 = [0,X] \times [0,Y]$, from (2.12), we have

$$u(x,y) \le a(X,Y) + b(X,Y)\int_0^x \int_0^y f(s,t)u(s,t)dsdt + c(X,Y)\int_0^M \int_0^N g(s,t)n(u(s,t))dsdt, \tag{2.15}$$

where we apply that $a(x,y)$, $b(x,y)$, and $c(x,y)$ are nondecreasing in $x$ and $y$.

Define a function $v(x,y), (x,y) \in \Delta_1$ by the right side of (2.15). Then, $v(x,y)$ is positive and nondecreasing in $x$ and $y$ and

$$u(x,y) \leq v(x,y). \tag{2.16}$$

Furthermore, we have

$$v(0,y) = a(X,Y) + c(X,Y) \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt, \tag{2.17}$$

$$\begin{aligned} \frac{\partial}{\partial x} v(x,y) &= b(X,Y) \int_0^y f(x,t)u(x,t)dt \\ &\leq b(X,Y) \int_0^y f(x,t)v(x,t)dt \\ &\leq (b(X,Y) \int_0^y f(x,t)dt)v(x,y). \end{aligned} \tag{2.18}$$

Since $v(x,y)$ is nondecreasing in $y$, from (2.18), one gets

$$\frac{(\partial/\partial x)\, v(x,y)}{v(x,y)} \leq b(X,Y) \int_0^y f(x,t)dt. \tag{2.19}$$

Now, keeping $y$ fixed in (2.19), setting $x = s$, and integrating the last inequality with respect to $s$ from 0 to $x$, we get

$$v(x,y) \leq \ v(0,y) \exp\left\{ b(X,Y) \int_0^x \int_0^y f(s,t)dsdt \right\}. \tag{2.20}$$

It follows from (2.16) and (2.17) that

$$\begin{aligned} u(x,y) &\leq \left[ a(X,Y) + c(X,Y) \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt \right] \\ &\quad \times \exp\left\{ b(X,Y) \int_0^x \int_0^y f(s,t)dsdt \right\} \\ &= a(X,Y) \exp\left\{ b(X,Y) \int_0^x \int_0^y f(s,t)dsdt \right\} \\ &\quad + c(X,Y) \exp\left\{ b(X,Y) \int_0^x \int_0^y f(s,t)dsdt \right\} \\ &\quad \times \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt \\ &= A_1(x,y,X,Y) + C_1(x,y,X,Y) \int_0^M \int_0^N g(s,t)n(u(s,t))dsdt, \end{aligned} \tag{2.21}$$

where

$$(2.22)\qquad \begin{aligned} A_1(x,y,X,Y) &= a(X,Y)\exp\left\{b(X,Y)\int_0^x\int_0^y f(s,t)dsdt\right\}, \\ C_1(x,y,X,Y) &= c(X,Y)\exp\left\{b(X,Y)\int_0^x\int_0^y f(s,t)dsdt\right\}. \end{aligned}$$

Using Lemma 2.1, from (2.21), we easily obtain

$$(2.23)\quad \begin{aligned} u(x,y) \le\ & A_1(x,y,X,Y)+C_1(x,y,X,Y) \\ & \times\frac{\int_0^M\int_0^N g(s,t)n(A_1(s,t,X,Y))dsdt}{1-\int_0^M\int_0^N C_1(s,t,X,Y)g(s,t)n'(A_1(s,t,X,Y))dsdt}, \end{aligned}$$

since the inequality (2.23) holds for all $(x,y)\in\Delta_1$ , taking $x=X$ and $y=Y$, we have

$$(2.24)\quad \begin{aligned} u(X,Y) &\le A_1(X,Y,X,Y)+C_1(X,Y,X,Y)\times \\ &\quad \frac{\int_0^M\int_0^N g(s,t)n(A_1(s,t,X,Y))dsdt}{1-\int_0^M\int_0^N C_1(s,t,X,Y)g(s,t)n'(A_1(s,t,X,Y))dsdt} \\ &= A^*(X,Y)+C^*(X,Y)\times \\ &\quad \frac{\int_0^M\int_0^N g(s,t)n(A^*(s,t))dsdt}{1-\int_0^M\int_0^N C^*(s,t)g(s,t)n'(A^*(s,t))dsdt}, \end{aligned}$$

for $(X,Y)\in\Delta$, where $A^*(X,Y)$ and $C^*(X,Y)$ are defined as in (2.14).

Taking into account that $X$ and $Y$ are arbitrary, we replace $X$ and $Y$ by $x$ and $y$, respectively, and we get the required inequality in (2.13). $\square$

**Remark 2.2.** If we take $n(x)=x$, then Theorem 2.1 reduces to Theorem 2 in [5].

**Theorem 2.2.** *Let $a(x,y), b(x,y),\ c(x,y), f(x,y)$ and $g(x,y)$ be as in Theorem 2.1. If $u(x,y)\in C(\Delta,\mathbb{R}_+)$ satisfies*

(2.25)

$$u^p(x,y)\le a(x,y)+b(x,y)\int_0^x\int_0^y f(s,t)u^q(s,t)dsdt+c(x,y)\int_0^M\int_0^N g(s,t)n(u(s,t))dsdt,$$

*where $p\ge q\ge 0,\ p\ge 1$ are constants, then*

$$(2.26)\qquad \begin{aligned} u(x,y) \le\ & A^*(x,y)+C^*(x,y)\times \\ & \frac{\int_0^M\int_0^N G^*(s,t)n(A^*(s,t))dsdt}{1-\int_0^M\int_0^N C^*(s,t)G^*(s,t)n'(A^*(s,t))dsdt}, \end{aligned}$$

*for* $(x,y)\in\Delta$, *provided that*

$$\int_0^M\int_0^N C^*(s,t)G^*(s,t)n'(A^*(s,t))dsdt<1,$$

*where*

$$\begin{aligned}(2.27)\qquad A^*(x,y) &= A_1(x,y)\exp\left\{B_1(x,y)\int_0^x\int_0^y F^*(s,t)dsdt\right\},\\ C^*(x,y) &= C_1(x,y)\exp\left\{B_1(x,y)\int_0^x\int_0^y F^*(s,t)dsdt\right\},\end{aligned}$$

*and*

$$\begin{aligned}A_1(x,y) &= \frac{1}{p}K^{\frac{1-p}{p}}b(x,y)\int_0^x\int_0^y f(s,t)\left[\frac{q}{p}K^{(q-p)/p}a(s,t)+\frac{p-q}{p}\ K^{q/p}\right]dsdt\\ &\quad+\frac{1}{p}K^{\frac{1-p}{p}}a(x,y)+\frac{p-1}{p}K^{\frac{1}{p}},\end{aligned}$$

$$B_1(x,y)=\frac{q}{p}K^{(q-p)/p}b(x,y),C_1(x,y)=\frac{1}{p}K^{\frac{1-p}{p}}c(x,y),$$

$$\begin{aligned}(2.28)\qquad F^*(x,y)&=f(x,y),\\ G^*(x,y)&=g(x,y),\end{aligned}$$

*Proof.* Define a function $w(x,y)$ by

$$\begin{aligned}(2.29)\qquad w(x,y) &= b(x,y)\int_0^x\int_0^y f(s,t)u^q(s,t)dsdt\\ &\quad+c(x,y)\int_0^M\int_0^N g(s,t)n(u(s,t))dsdt,\end{aligned}$$

for $(x,y)\in\Delta$. Then, from (2.29) , we have

$$(2.30)\qquad u^p(x,y)\le a(x,y)+w(x,y).$$

Applying Lemma 1.2, we get

(2.31)

$$u(x,y)\le(a(x,y)+w(x,y))^{1/p}\ \le\frac{1}{p}K^{\frac{1-p}{p}}(a(x,y)+w(x,y))+\frac{p-1}{p}K^{\frac{1}{p}}=v(x,y).$$

$$u^q(x,y)\le(a(x,y)+w(x,y))^{q/p}\ \le\frac{q}{p}K^{(q-p)/p}(a(x,y)+w(x,y))\ +\frac{p-q}{p}\ K^{q/p}.$$

It follows from (2.29),(2.30) and (2.31) that

$$
\begin{aligned}
(2.32) \qquad w(x,y) \quad \leq \quad & b(x,y)\int_0^x\int_0^y f(s,t) \\
& \times\left[\frac{q}{p}K^{(q-p)/p}\left(a(s,t)+w(s,t)\right) \ + \frac{p-q}{p}\ K^{q/p}\right]dsdt \\
& + c(x,y)\int_0^M\int_0^N g(s,t)n(v(s,t))dsdt,
\end{aligned}
$$

taking into-account that $\frac{1}{p}K^{\frac{1-p}{p}}w(x,y)\leq v(x,y)$, we have

$$
\begin{aligned}
w(x,y) \quad \leq \quad & b(x,y)\int_0^x\int_0^y f(s,t)\left[\frac{q}{p}K^{(q-p)/p}a(s,t)+\frac{p-q}{p}\ K^{q/p}\right]dsdt \\
& +qK^{(q-1)/p}b(x,y)\int_0^x\int_0^y f(s,t)v(s,t)dsdt \\
(2.33) \qquad & +c(x,y)\ \int_0^M\int_0^N g(s,t)\ n(v(s,t))dsdt.
\end{aligned}
$$

Multiplying both sides of (2.33) by $\frac{1}{p}K^{\frac{1-p}{p}}$ and adding $\frac{1}{p}K^{\frac{1-p}{p}}a(x,y)+\frac{p-1}{p}K^{\frac{1}{p}}$ to both sides of the resultant inequality, we obtain

$$
\begin{aligned}
(2.34) \qquad v(x,y) \quad \leq \quad & A_1(x,y)+B_1(x,y)\int_0^x\int_0^y F^*(s,t)v(s,t)dsdt \\
& +C_1(x,y)\int_0^M\int_0^N G^*(s,t)n(v(s,t))dsdt,
\end{aligned}
$$

where $A_1(x,y), B_1(x,y), C_1(x,y), F^*(x,y)$ and $G^*(x,y)$ are defined as in (2.28).

Note that $A_1(x,y), B_1(x,y)$ and $C_1(x,y)$ are nonnegative, continuous, and non-decreasing for $(x,y)\in\Delta$.A suitable application of Theorem 2.1 to (2.34) gives

$$
\begin{aligned}
(2.35) \qquad u(x,y) \quad \leq \quad & v(x,y)\leq A^*(x,y)+C^*(x,y)\times \\
& \frac{\int_0^M\int_0^N G^*(s,t)n(A^*(s,t))dsdt}{1-\int_0^M\int_0^N C^*(s,t)G^*(s,t)n'(A^*(s,t))dsdt},
\end{aligned}
$$

where $A^*(x,y)$ and $C^*(x,y)$ are defined as in (2.27). □

**Remark 2.3.** If we take $n(x)=x$, then Theorem 2.2 reduces to Theorem 6 in [5].

## 3. Applications

In this section, we shall illustrate how our main results can be applied to study the boundedness and uniqueness of the solution to certain integral equations in two independent variables.

**Example 3.1.** Consider the following integral equation:

$$z(x,y)=a(x,y)+b(x,y)\int_0^x\int_0^y F(s,t,z)dsdt+c(x,y)\int_0^M\int_0^N G(s,t,z)dsdt, \tag{3.1}$$

for $(x,y)\in\Delta$, where $z(x,y)\in C(\Delta,\mathbb{R}), a(x,y), b(x,y), c(x,y)\in C(\Delta,\mathbb{R}_+)$ are nondecreasing in $x$ and $y$, $F(x,y,z), G(x,y,z)\in C(\Delta\times\mathbb{R},\mathbb{R})$.

**Theorem 3.1.** *Assume that the functions $F$ and $G$ in (3.1) satisfy the conditions*

$$\begin{aligned} |F(s,t,z)| &\leq f(s,t)\,|z|\,, \\ |G(s,t,z)| &\leq g(s,t)n(|z|), \end{aligned} \tag{3.2}$$

*where $f(s,t)$, $g(s,t)$ and $n$ are defined as in Theorem 2.1.*

*If $z(x,y)$ is the unique solution of (3.1), then*

$$\begin{aligned} |z(x,y)| \leq\ & A^*(x,y)+C^*(x,y)\times \\ & \frac{\int_0^M\int_0^N g(s,t)n(A^*(s,t))dsdt}{1-\int_0^M\int_0^N C^*(s,t)g(s,t)n'(A^*(s,t))dsdt}, \end{aligned} \tag{3.3}$$

*for $(x,y)\in\Delta$, provided that*

$$\int_0^M\int_0^N C^*(s,t)g(s,t)n'(A^*(s,t))dsdt<1, \tag{3.4}$$

*where $A^*(x,y), C^*(x,y)$ are defined in (2.14).*

*Proof.* Assume that $z(x,y)$ is the unique solution of (3.1), from (3.2) we have

$$\begin{aligned} |z(x,y)| \leq\ & a(x,y)+b(x,y)\int_0^x\int_0^y f(s,t)\,|z(s,t)|\,dsdt \\ & +c(x,y)\int_0^M\int_0^N g(s,t)n(|z(s,t)|)dsdt. \end{aligned} \tag{3.5}$$

Now an application of Theorem 2.1 to (3.5), yields the required inequality in (3.3). $\square$

**Corollary 3.1.** *If we take in (3.2), $n(z)=\arctan(z)$, then the unique solution of (3.1) can be expressed as*

$$\begin{aligned} |z(x,y)| \leq\ & A^*(x,y)+C^*(x,y)\times \\ & \frac{\int_0^M\int_0^N g(s,t)\arctan(A^*(s,t))dsdt}{1-\int_0^M\int_0^N \frac{C^*(s,t)g(s,t)dsdt}{1+A^{*2}(s,t)}}, \end{aligned}$$

*provided that*

$$\int_0^M \int_0^N \frac{C^*(s,t)g(s,t)dsdt}{1+A^{*2}(s,t)} < 1.$$

*If we take* $n(z) = \ln(z+1)$, *then the unique solution of* (3.1) *can be expressed as*

$$\begin{aligned} |z(x,y)| &\leq A^*(x,y) + C^*(x,y) \times \\ &\quad \frac{\int_0^M \int_0^N g(s,t)\ln(A^*(s,t)+1)dsdt}{1 - \int_0^M \int_0^N \frac{C^*(s,t)g(s,t)dsdt}{1+A^*(s,t)}}, \end{aligned}$$

*provided that*

$$\int_0^M \int_0^N \frac{C^*(s,t)g(s,t)dsdt}{1+A^*(s,t)} < 1.$$

**Proposition 3.1.** *Assume that the functions $F$ and $G$ in* (3.1) *satisfy the conditions*

$$\begin{aligned} |F(s,t,z)| - F(s,t,\overline{z}) &\leq f(s,t)\,|z-\overline{z}|\,, \\ |G(s,t,z)| - G(s,t,\overline{z}) &\leq g(s,t)n(|z-\overline{z}|), \end{aligned} \tag{3.6}$$

*where $f(s,t)$, $g(s,t)$ and $n$ are defined as in Theorem* 2.1 *with* $n(0)=0$. *If*

$$\int_0^M \int_0^N C^*(s,t)g(s,t)n'(A^*(s,t))dsdt < 1,$$

*where $A^*$ and $C^*$ are defined as in Theorem 2.1, and $z(x,y)$ is a solution of* (3.1), *then* (3.1) *has at most one solution.*

*Proof.* Let $z(x,y)$ and $\overline{z}(x,y)$ be two solutions of (3.1), then

$$\begin{aligned} \overline{z}(x,y) &= a(x,y) + b(x,y)\int_0^x \int_0^y F(s,t,\overline{z})dsdt \\ &\quad + c(x,y)\int_0^M \int_0^N G(s,t,\overline{z})dsdt, \\ z(x,y) &= a(x,y) + b(x,y)\int_0^x \int_0^y F(s,t,z)dsdt \\ &\quad + c(x,y)\int_0^M \int_0^N G(s,t,z)dsdt. \end{aligned} \tag{3.7}$$

From (3.7), we have

$$
\begin{aligned}
(3.8)\quad |z(x,y)-\overline{z}(x,y)| &\le b(x,y)\int_0^x\int_0^y |F(s,t,z)-F(s,t,\overline{z})|\,dsdt \\
&\quad +c(x,y)\int_0^M\int_0^N |G(s,t,z)-G(s,t,\overline{z})|\,dsdt \\
&\le b(x,y)\int_0^x\int_0^y f(s,t)\,|z-\overline{z}|\,dsdt \\
&+ c(x,y)\int_0^M\int_0^N g(s,t)n(|z-\overline{z}|)dsdt.
\end{aligned}
$$

According to Theorem 2.1, we obtain that $|z(x,y)-\overline{z}(x,y)|\le 0$, which implies $z(x,y)=\overline{z}(x,y)$ for $(x,y)\in\Delta$. $\square$

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Ser. Math. Inform. Vol. 33, No 3 (2018), 481-496
https://doi.org/10.22190/FUMI1803481A

# ON APPROXIMATION OF FIXED POINTS OF MEAN NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

**Ali Abkar and Mojtaba Rastgoo**

**Abstract.** A new iterative algorithm for approximating fixed points of mean nonexpansive mappings in CAT(0) spaces is introduced. As a result, a $\triangle$-convergence theorem is established. The result we obtain improves and extends several recent results in the literature. Finally, some numerical examples are presented to illustrate the main result and to compare the new algorithm with some existing ones.
**Keywords:** Iterative algorithm; CAT(0) space; weak convergence; $\triangle$-convergence; mean nonexpansive mapping

## 1. Introduction

Fixed point theory of metric spaces was initiated by the celebrated *Banach contraction principle* which states that every contraction on a complete metric space has a unique fixed point; moreover, the fixed point can be approximated by Picard's iterates. Perhaps the most influential fixed point theorem in metric fixed point theory is the theorem due to F. E. Browder and D. Gohde; in 1965, F. E. Browder [10] and D. Gohde [9] independently proved that every nonexpansive self-mapping of a closed, convex, and bounded subset of a uniformly convex Banach space has a fixed point. Fixed point theory in Cartan-Alexandrov-Toponogov spaces, or briefly in CAT(0) spaces, was first studied by W. A. Kirk (see [30, 29]. Among other things, he proved that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space has a fixed point. Since then the fixed point theorems for various mappings in a CAT(0) space have been developed rapidly and numerous papers have appeared (see for example [1, 2, 31, 15, 6, 17, 18] and the references therein).

As a generalization of nonexpansive mappings, in 1975, Zhang [26] introduced the concept of a *mean nonexpansive mapping* in Banach spaces and proved the existence and uniqueness of fixed points for this type of mappings in Banach spaces with the normal structure. The mean nonexpansive mappings were extensively studied

Received October 03, 2017; accepted March 12, 2018
2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10, 47J25

by Wu and Zhang [7], and by Yang and Cui [32]. In 2010, Nakprasit [13] provided an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive and showed that increasing mean nonexpansiveness implies Suzuki-generalized nonexpansiveness. In 2012, Ouahab [3] proved a fixed point theorem for strong semigroups of mean nonexpansive mappings in uniformly convex Banach spaces. In this paper, we shall study mean nonexpansive mappings in the context of CAT$(0)$ spaces.

Let $(X, d)$ be a metric space and $x, y$ be two fixed elements in $X$ such that $d(x, y) = l$. A geodesic path from $x$ to $y$ is an isometry $c : [0, l] \to c([0, 1]) \subset X$ such that $c(0) = x$, $c(l) = y$. The image of a geodesic path between two points is called a geodesic segment. A metric space $(X, d)$ is called a geodesic space if every two points of $X$ are joined by a geodesic segment. A geodesic triangle represented by $\triangle(x, y, z)$ in a geodesic space consists of three points $x, y, z$ and the three segments joining each pair of the points. A comparison triangle of a geodesic triangle $\triangle(x, y, z)$, denoted by $\overline{\triangle}(x, y, z)$ or $\triangle(\overline{x}, \overline{y}, \overline{z})$, is a triangle in the Euclidean space $\mathbb{R}^2$ such that $d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y})$, $d(x, z) = d_{\mathbb{R}^2}(\overline{x}, \overline{z})$, and $d(y, z) = d_{\mathbb{R}^2}(\overline{y}, \overline{z})$. This is obtainable by using the triangle inequality, and it is unique up to isometry on $\mathbb{R}^2$. Bridson and Haefliger [16] have shown that such a triangle always exists. A geodesic segment joining two points $x, y$ in a geodesic space $X$ is represented by $[x, y]$. Every point $z$ in the segment is represented by $\alpha x \oplus (1 - \alpha)y$, where $\alpha \in [0, 1]$, that is, $[x, y] := \{\alpha x \oplus (1 - \alpha)y : \ \alpha \in [0, 1]\}$. A subset $\mathcal{C}$ of a metric space $X$ is called convex if for all $x, y \in \mathcal{C}$, $[x, y] \subset \mathcal{C}$. A geodesic space is called a CAT$(0)$ space if for every geodesic triangle $\triangle$ and its comparison $\overline{\triangle}$, the following inequality is satisfied: $d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$ for all $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$. Complete CAT$(0)$ spaces are often called Hadamard spaces (see [28, 24, 25]. Examples of CAT$(0)$ spaces include the $\mathbb{R}$-tree, Hadamard manifolds, and the Hilbert ball equipped with the hyperbolic metric. For more details on these spaces, see for example [19, 14, 8]. A geodesic space $(X, d)$ is called hyperbolic (see [12, 23]) if, for any $x, y, z \in X$,

$$d(\frac{1}{2}z \oplus \frac{1}{2}x, \frac{1}{2}z \oplus \frac{1}{2}y) \leq \frac{1}{2}d(x, y).$$

The class of hyperbolic spaces include the normed spaces, CAT$(0)$ spaces, and some others. Bashir Ali in [4] presented an example of a hyperbolic space that is not a normed space. Therefore, the class of hyperbolic spaces is more general than the class of normed spaces.

Let $\mathcal{C}$ be a nonempty subset of a CAT$(0)$ spaces $(X, d)$. A self-mapping $T : \mathcal{C} \to \mathcal{C}$ is called *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in \mathcal{C}$. The mapping $T$ is called *quasi-nonexpansive* if $Fix(T) = \{x \in \mathcal{C} : Tx = x\} \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in \mathcal{C}$ and $p \in Fix(T)$.

In 2015, Zhou and Cui in [11] introduced an iterative algorithm to approximate fixed points of mean nonexpansive mappings in CAT$(0)$ spaces; this algorithm is defined in the following way:

$$\begin{cases} x_1 \in \mathcal{C}, \\ x_{n+1} = (1-t_n)x_n \oplus t_n T(y_n), \\ y_n = (1-s_n)x_n \oplus s_n T(x_n), \quad n \geq 1, \end{cases}$$

where $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are some sequences in $(0,1)$.

In this paper, we introduce a new iterative algorithm for approximating fixed points of mean nonexpansive mappings in CAT(0) spaces. Under suitable conditions, we prove the $\triangle$-convergence theorem for our algorithm. The results we obtain improve and extend several recent results in the literature; they also complement many known existing results. We then provide some numerical examples to illustrate our main result. In this way, we display the efficiency of our proposed algorithm.

## 2. Preliminaries

Throughout this article, $(X,d)$ will stand for a metric space. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to $x$, and $x_n \to x$ to indicate that the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x$.
We start by recalling some basic definitions.

**Definition 2.1.** Let $\mathcal{C}$ be a nonempty subset of $(X,d)$. A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if

$$d(Tx,Ty) \leq d(x,y), \quad \forall x,y \in \mathcal{C}.$$

**Definition 2.2.** Let $\mathcal{C}$ be a nonempty subset of $(X,d)$. A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be mean nonexpansive if

$$d(Tx,Ty) \leq a\, d(x,y) + b\, d(x,Ty), \quad \forall x,y \in \mathcal{C},$$

where $a$ and $b$ are two nonnegative real numbers such that $a+b \leq 1$.

Obviously, every nonexpansive mapping is a mean nonexpansive mapping (with $a=1$ and $b=0$). Note that a mean nonexpansive mapping is not necessarily continuous as the following example shows, so that mean nonexpansive mappings are not necessarily nonexpansive.

**Example 2.1.** Suppose that $T : [0,1] \to [0,1]$ is a mapping defined by

$$Tx = \begin{cases} \dfrac{x}{5} + \dfrac{5}{12} & x \in [0, \frac{1}{2}); \\ \frac{x}{6} + \frac{5}{12} & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then $T$ is mean nonexpansive with $a = \frac{1}{3}, b = \frac{2}{3}$, but not continuous at $x = \frac{1}{2}$. Thus, $T$ is not a nonexpansive mapping.

**Example 2.2.** Suppose that $T : [0,1] \to [0,1]$ is a mapping defined by

$$Tx = \begin{cases} \dfrac{1-x}{3} & x \in [0,1] \text{ is rational;} \\ \frac{1+x}{5} & x \in [0,1] \text{ is irrational.} \end{cases}$$

Then $T$ is mean nonexpansive with $a = \frac{1}{3}, b = \frac{2}{3}$, but not continuous at any point in $[0,1]$ except $x = \frac{1}{4}$, the fixed point of $T$.

In 2008, Suzuki [27] introduced Suzuki-generalized nonexpansive mappings in Banach spaces.

**Definition 2.3.** Let $\mathcal{C}$ be a nonempty subset of $(X,d)$. A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be Suzuki-generalized nonexpansive if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq d(x,y)$$

for all $x, y \in \mathcal{C}$.

In [13], Nakprasit provided an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive and showed that increasing mean nonexpansive mappings are Suzuki-generalized nonexpansive.
We now turn to some known facts regarding CAT(0) spaces.

**Lemma 2.1.** *([20], Lemma 2.5) Let $(X,d)$ be a CAT(0) space. Then*

$$d((1-\alpha)x \oplus \alpha y, z)^2 \leq (1-\alpha)d(x,z)^2 + \alpha d(y,z)^2 - \alpha(1-\alpha)d(x,y)^2$$

*for all $\alpha \in [0,1]$ and $x, y, z \in X$.*

**Lemma 2.2.** *([5], Lemma 4.5) Let $x$ be a given point in a CAT(0) space $(X,d)$ and $\{t_n\}$ be a sequence in a closed interval $[a,b]$ with $0 < a \leq b < 1$ and $0 < a(1-b) \leq \dfrac{1}{2}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in $X$ such that*

1. $\limsup_{n\to\infty} d(x_n, x) \leq r$,
2. $\limsup_{n\to\infty} d(y_n, x) \leq r$,
3. $\limsup_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$

*for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

**Theorem 2.1.** *([11], Theorem 3.1) Let $\mathcal{C}$ be a nonempty bounded closed convex subset of a complete CAT(0) space $(X, d)$ and $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping with $b < 1$. Then $T$ has a fixed point.*

**Theorem 2.2.** *([11], Theorem 3.2) Let $(X, d)$ be a complete CAT(0) space and $\mathcal{C}$ be a nonempty bounded closed convex subset of $X$. Let $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping with $b < 1$, and let $\{x_n\} \subset \mathcal{C}$ be an approximate fixed point sequence (i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$) and $\{x_n\} \rightharpoonup \omega$. Then $T(\omega) = \omega$.*

**Definition 2.4.** Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $(X, d)$.

1. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by
$$r(\{x_n\}) := \inf_{x\in X}\{r(x, \{x_n\})\},$$
where $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x_n, x)$.
2. The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set
$$A(\{x_n\}) := \{x \in X : \ r(x, \{x_n\}) = r(\{x_n\})\}.$$

In 2006, Dhompongsa et al proved that $A(\{x_n\})$ consists of exactly one point for each bounded sequence $\{x_n\}$ in a CAT(0) space (see Proposition 7 in [22]). We recall that a bounded sequence $\{x_n\}$ in $X$ is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has a regular subsequence. It is now time to give the concept of $\triangle$-convergence in a CAT(0) space.

**Definition 2.5.** [31] Let $(X, d)$ be a CAT(0) space. A sequence $\{x_n\}$ in $X$ is said to $\triangle$-converge to $x \in X$ if and only if $x$ is the unique asymptotic center of all subsequences of $\{x_n\}$. In this case, we write $\triangle - \lim_{n\to\infty} x_n = x$ and $x$ is called the $\triangle$-limit of $\{x_n\}$.

**Proposition 2.1.** *([5], Proposition 3.12). Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $(X, d)$ and let $\mathcal{C} \subset X$ be a closed convex subset which contains $\{x_n\}$. Then,*

- $\triangle - \lim_{n\to\infty} x_n = x$ *implies* $\{x_n\} \rightharpoonup x$;
- *if* $\{x_n\}$ *is regular, then* $\{x_n\} \rightharpoonup x$ *implies* $\triangle - \lim_{n\to\infty} x_n = x$.

**Lemma 2.3.** *The following assertions in a CAT(0) space hold:*

- *[20] Every bounded sequence in a complete $CAT(0)$ space has a $\triangle$-convergent subsequence.*

- *[21] If $\{x_n\}$ is a bounded sequence in a closed convex subset $\mathcal{C}$ of a complete CAT(0) space $(X, d)$, then the asymptotic center of $\{x_n\}$ is in $\mathcal{C}$.*

- *[20] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space $(X, d)$ with $A(\{x_n\}) = \{p\}$, $\{\nu_n\}$ is a subsequence of $\{x_n\}$ with $A(\{\nu_n\}) = \{\nu\}$, and the sequence $\{d(x_n, \nu)\}$ converges, then $p = \nu$.*

**Lemma 2.4.** *([11], Lemma 4.4) Let $\mathcal{C}$ be a nonempty closed convex subset of a complete CAT(0) space $(X, d)$ and $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping. If $\{x_n\}$ is a sequence in $\mathcal{C}$ such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $\triangle-\lim_{n\to\infty} x_n = p$, then $T(p) = p$.*

**Remark 2.1.** By Lemma 2.4 and Proposition 2.1 (ii), if $\{x_n\}$ in Theorem 2.2 is regular, then the condition $b < 1$ in Theorem 2.2 can be removed.

## 3. Weak Convergence Theorem

We begin this section by proving a $\triangle$-convergence theorem for mean nonexpansive mappings in CAT(0) spaces. Here we introduce a new iterative algorithm to approximate the fixed point of our mapping. We shall then compare our algorithm with that of Zhou and Cui [11].

**Theorem 3.1.** *Let $(X, d)$ be a complete CAT(0) space, $\mathcal{C}$ be a nonempty, bounded closed convex subset of $(X, d)$ and $T : \mathcal{C} \to \mathcal{C}$ be a mean nonexpansive mapping with $b < 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$, also $\{\alpha_n\}$ be a sequence in a closed interval $[r, s]$ with $0 < r \leq s < 1$ and $0 < r(1 - s) \leq \frac{1}{2}$. Then $\{x_n\}_{n=1}^{\infty}$ which is defined by*

$$(3.1) \qquad \begin{cases} x_1 \in \mathcal{C}, \\ z_n = T((1-\alpha_n)x_n \oplus \alpha_n T(x_n)), \\ y_n = T((1-\beta_n)z_n \oplus \beta_n T(z_n)), \\ x_{n+1} = T((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), \end{cases}$$

*is $\triangle$-convergent to some point $p \in Fix(T)$.*

*Proof.* By using Theorem 2.1, we get $Fix(T) \neq \emptyset$. Next, we will divide the proof into three steps.

**Step 1**. First, we will prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in Fix(T)$, where $\{x_n\}$ is defined by (3.3). For this purpose, let $p \in Fix(T)$, using the fact that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ we obtain

$$
\begin{aligned}
d(z_n, p) &= d(T((1-\alpha_n)x_n \oplus \alpha_n T(x_n)), p) \\
&\leq a\left[d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p)\right] \\
&\quad +b\left[d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p)\right] \\
&\leq d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(T(x_n), p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n a d(x_n, p) + \alpha_n b d(x_n, p) \\
&\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\
&\leq d(x_n, p) \qquad (3.2)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Also, we have

$$
\begin{aligned}
d(y_n, p) &= d(T((1-\beta_n)z_n \oplus \beta_n T(z_n)), p) \\
&\leq a\left[d((1-\beta_n)z_n \oplus \beta_n T(z_n), p)\right] \\
&\quad +b\left[d((1-\beta_n)z_n \oplus \beta_n T(z_n), p)\right] \\
&\leq d((1-\beta_n)z_n \oplus \beta_n T(z_n), p) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n d(T(z_n), p) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n a d(z_n, p) + \beta_n b d(z_n, p) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\
&\leq d(z_n, p) \\
&\leq d(x_n, p) \qquad (3.3)
\end{aligned}
$$

for all $n \in \mathbb{N}$. From (3.3), (3.4) and (3.5) and using the fact that $\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$, we conclude that

$$\begin{aligned}
d(x_{n+1}, p) &= d(T((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), p) \\
&\leq a\Bigg[d((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), p)\Bigg] \\
&\qquad +b\Bigg[d((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), p)\Bigg] \\
&\leq d((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n), p) \\
&\leq (1-\gamma_n)d(T(z_n), p) + \gamma_n d(T(y_n), p) \\
&\leq (1-\gamma_n)ad(z_n, p) + (1-\gamma_n)bd(z_n, p) + \gamma_n ad(y_n, p) + \gamma_n bd(y_n, p) \\
&\leq (1-\gamma_n)d(z_n, p) + \gamma_n d(y_n, p) \\
&\leq d(x_n, p) \qquad (3.4)
\end{aligned}$$

Consequently, we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $n \geq 1$. This implies that $\{x_n\}$ is bounded and decreasing. Hence, $\lim_{n\to\infty} d(x_n, p)$ exists. Thus, $\{x_n\}$ is bounded.

**Step 2**. In this step, we will prove that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Without loss of generality, we may assume that

$$\mathbf{r} := \lim_{n\to\infty} d(x_n, p). \qquad (3.5)$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} d(T(x_n), p) &\leq \limsup_{n\to\infty} \Bigg[ad(x_n, p) + bd(x_n, p)\Bigg] \\
&\leq \limsup_{n\to\infty} d(x_n, p) \\
&\leq \mathbf{r} \qquad (3.6)
\end{aligned}$$

from (3.2), we conclude that

$$\limsup_{n\to\infty} d(z_n, p) \leq \mathbf{r} \qquad (3.7)$$

now, we can write

$$
\begin{aligned}
\mathbf{r} = \limsup_{n\to\infty} d(x_{n+1}, p) &= \limsup_{n\to\infty} d(T((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), p) \\
&\leq a\Big[d((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), p)\Big] \\
&\quad +b\Big[d((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n)), p)\Big] \\
&\leq d((1-\gamma_n)T(z_n) \oplus \gamma_n T(y_n), p) \\
&\leq (1-\gamma_n)d(T(z_n), p) + \gamma_n d(T(y_n), p) \\
&\leq (1-\gamma_n)ad(z_n, p) \\
&\quad +(1-\gamma_n)bd(z_n, p) + \gamma_n ad(y_n, p) + \gamma_n bd(y_n, p) \\
&\leq (1-\gamma_n)d(z_n, p) + \gamma_n d(y_n, p) \\
&\leq (1-\gamma_n)d(z_n, p) + \gamma_n d(z_n, p) \\
&\leq d(z_n, p),
\end{aligned}
$$

which implies that

$$
\mathbf{r} \leq \limsup_{n\to\infty} d(z_n, p). \tag{3.8}
$$

From (3.7) and (3.8), we have

$$
\begin{aligned}
\mathbf{r} &= \limsup_{n\to\infty} d(z_n, p) \\
&= \limsup_{n\to\infty} d(T((1-\alpha_n)x_n \oplus \alpha_n T(x_n)), p). \\
&\leq a \limsup_{n\to\infty} \Big[d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p)\Big] \\
&\quad +b \limsup_{n\to\infty} \Big[d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p)\Big] \\
&\leq \limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) \qquad (3.9)
\end{aligned}
$$

Also

$$
\begin{aligned}
\limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) &\le \limsup_{n\to\infty} \Big[(1-\alpha_n)d(x_n,p) + \alpha_n d(T(x_n),p)\Big] \\
&\le \limsup_{n\to\infty} \Big[(1-\alpha_n)d(x_n,p) \qquad (3.10) \\
&\qquad + \alpha_n a d((x_n),p) + \alpha_n b d((x_n),p)\Big] \\
&\le \limsup_{n\to\infty} \Big[(1-\alpha_n)d(x_n,p) + \alpha_n d((x_n),p)\Big] \\
&\le \limsup_{n\to\infty} d(x_n,p) = \mathbf{r} \qquad (3.11)
\end{aligned}
$$

From (3.9) and (3.11), we have

$$\limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T(x_n), p) = \mathbf{r} \tag{3.12}$$

By using Lemma 2.2 with (3.5), (3.6) and (3.12), we have

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \tag{3.13}$$

Therefore, Step 2 is proved.

**Step 3**. Define

$$\Omega_\triangle(x_n) := \bigcup_{\{\nu_n\}\subseteq\{x_n\}} A(\{\nu_n\}) \subseteq Fix(T).$$

We claim that the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of $T$ and $\Omega_\triangle(x_n)$ consists of exactly one point. Assume that $\nu \in \Omega_\triangle(x_n)$. From the definition of $\Omega_\triangle(x_n)$, there is a subsequence $\{\nu_n\}$ of $\{x_n\}$ such that $A(\{\nu_n\}) = \{\nu\}$. From assertion $(\mathcal{A}_1)$ in Lemma 2.3, there exists a subsequence $\{\rho_n\}$ of $\{\nu_n\}$ such that $\triangle - \lim_{n\to\infty} \rho_n = \rho \in \mathcal{C}$. Using Lemma 2.4, we conclude that $\rho \in Fix(T)$. Since $\{d(\nu_n, \rho)\}$ converges, by assertion $(\mathcal{A}_2)$ in Lemma 2.3, we obtain $\nu = \rho$. Therefore, $\Omega_\triangle(x_n) \subseteq Fix(T)$. Finally, we show that $\Omega_\triangle(x_n)$ consists of exactly one point. Let $\{\nu_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{\nu_n\}) = \{\nu\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $\nu = \rho \in Fix(T)$. Since $\{d(x_n, \rho)\}$ converges, by assertion $(\mathcal{A}_3)$ in Lemma 2.3, we have $x = \rho \in Fix(T)$, that is, $\Omega_\triangle(x_n) = x$. This completes the proof. □

## 4. Numerical Experiments and Comparison

In this section, we supply a numerical example of a mean nonexpansive mapping satisfying the conditions of Theorem 3.1, and some numerical experiment results to explain the conclusion of our algorithm.

**Example 4.1.** Consider $X = \mathbb{R}$ with its usual metric, so $X$ is also a complete CAT(0) space. Let $\mathcal{C} = [-1, 1]$ which clearly is a bounded closed convex subset of $X$. Define the mapping $T : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$Tx = \begin{cases} \dfrac{x}{5} + \dfrac{5}{12} & x \in [-1, \frac{1}{2}); \\ \frac{x}{6} + \frac{5}{12} & x \in [\frac{1}{2}, 1]. \end{cases}$$

$T$ is discontinuous at $x = 0.5$; consequently, $T$ is neither nonexpansive nor contractive. Now, we prove that $T$ is mean nonexpansive.

Case 1: $x, y \in [-1, \frac{1}{2})$. By the definition of $T$,

$$\begin{aligned} d(T(x), T(y)) &= \frac{1}{4} d(\frac{4}{5}x, \frac{4}{5}y) \\ &= \frac{1}{4} d(x - \frac{y}{5} + \frac{y}{5} - \frac{x}{5}, y - x + x - \frac{y}{5}) \\ &\leq d(x, y) + \frac{1}{4} d(-\frac{y}{5}, -x) + \frac{1}{4} d(\frac{y}{5}, x) + \frac{1}{4} d(-\frac{x}{5}, -\frac{y}{5}) \\ &\leq \frac{1}{4} d(x, y) + \frac{1}{2} d(x, T(y)) + \frac{1}{4} d(T(x), T(y)). \end{aligned}$$

This implies that $d(T(x), T(y)) \leq \frac{1}{3} d(x, y) + \frac{2}{3} d(x, T(y))$.

Case 2: $x \in [-1, \frac{1}{2}), y \in [\frac{1}{2}, 1]$. In this case, we have

$$\begin{aligned} d(T(x), T(y)) &= d(\frac{x}{5}, \frac{y}{5}) \\ &= d(\frac{x}{5} + \frac{T(y)}{5} - \frac{T(y)}{5}, \frac{y}{5} + \frac{T(x)}{5} - \frac{T(x)}{5}) \\ &\leq \frac{1}{5} d(x, T(x)) + \frac{1}{5} d(T(x), T(y)) + \frac{1}{5} d(y, T(y)) \\ &\leq \frac{1}{5} d(x, T(y)) + \frac{1}{5} d(T(x), T(y)) + \frac{1}{5} d(T(x), T(y)) \\ &\quad + \frac{1}{5} d(x, y) + \frac{1}{5} d(x, T(y)) \\ &= \frac{2}{5} d(x, T(y)) + \frac{2}{5} d(T(x), T(y)) + \frac{1}{5} d(x, y). \end{aligned}$$

This implies that $d(T(x), T(y)) \leq \frac{1}{3} d(x, y) + \frac{2}{3} d(x, T(y))$.

Case 3: $y \in [-1, \frac{1}{2}), x \in [\frac{1}{2}, 1]$. The argument is similar to the one in Case 2.

Case 4: $x, y \in [\frac{1}{2}, 1]$. The proof is the same as in Case 1.

Hence, $T$ is mean nonexpansive by taking $a = \frac{1}{3}, b = \frac{2}{3}$.

Clearly, 0.5 is the only fixed point of the mapping $T$. Put $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{n + 100}$. By using MATHEMATICA, we computed the iterates of the algorithm for two different initial points $x_1 = -0.9 \in [-1, 1]$ and $x_1 = 0.9 \in [-1, 1]$. Finally, using numerical experiments we compared the Zhou and Cui iteration process with our algorithm (see Table 4.1). Moreover, the convergence behavior of these algorithms is shown in Figure 4.1. We conclude that $x_n$ converges to 0.5.

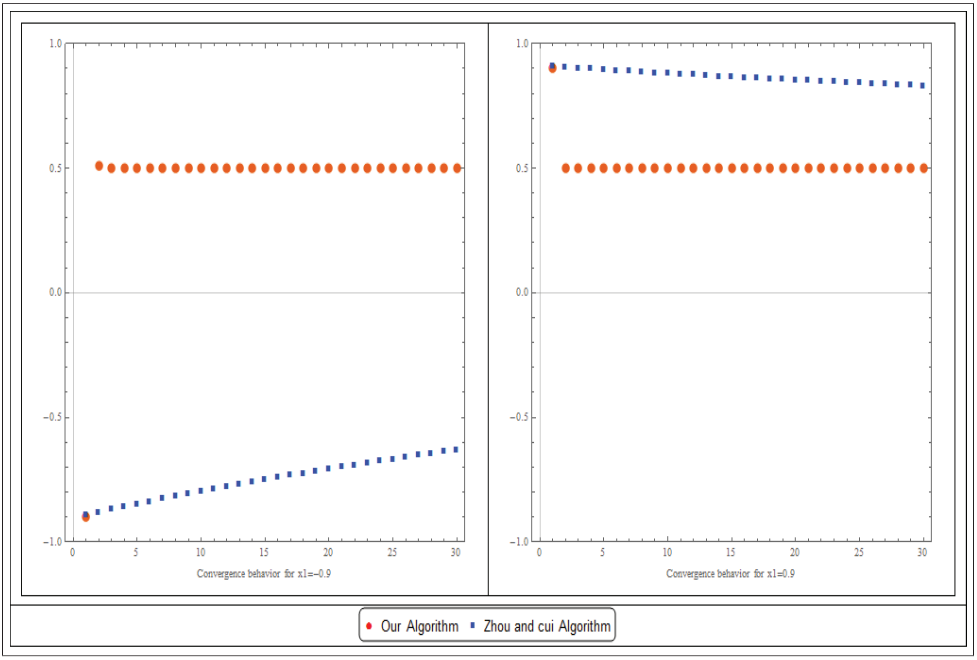


Figure 4.1: Convergence behaviors corresponding to $x_1 = -0.9$ and $x_1 = 0.9$ for 30 steps.

**Example 4.2.** Consider $X = \mathbb{R}^2$ equipped with the Euclidean norm. Let $x = (x_1, x_2) \in \mathbb{R}^2$, then the squared distance of $x$ from the origin, $O$, is

$$\|x\|^2 = x_1^2 + x_2^2.$$

Consider $\mathcal{C} = [-1, 1] \times [-1, 1]$ which is a bounded, closed, and convex subset of $X$. We define the mapping $\mathrm{K} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\mathrm{K}(x_1, x_2) := (\frac{1}{3}x_1, \frac{1}{3}x_2)$$

K is a nonexpansive mapping. This means that K is a mean nonexpansive mapping with $a = 1$ and $b = 0$. Clearly, zero is the only fixed point of the mapping K. In this case, our algorithm is the following:

$$(4.1)\qquad \begin{cases} x_{(1)} = (x_{(1)_1}, x_{(1)_2}) \in \mathcal{C}, \\ (z_{(n)_1}, z_{(n)_2}) = K((1-\alpha_n)(x_{(n)_1}, x_{(n)_2}) + \alpha_n K(x_{(n)_1}, x_{(n)_2})), \\ (y_{(n)_1}, y_{(n)_2}) = K((1-\beta_n)(z_{(n)_1}, z_{(n)_2}) + \beta_n K(z_{(n)_1}, z_{(n)_2})), \\ (x_{(n+1)_1}, x_{(n+1)_2}) = K((1-\gamma_n)K(z_{(n)_1}, z_{(n)_2}) + \gamma_n K(y_{(n)_2})). \end{cases}$$

Put $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{n+100}$. By using MATHEMATICA, we computed the iterates of the algorithm (4.1) for $x_{(1)} = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{C}$ for 500 steps. Finally, using numerical experiments we compared the Zhou and Cui iteration process with our algorithm (4.1). The convergence behavior of these algorithms is shown in Figure 4.2. The conclusion is that $x_n$ converges to $(0, 0)$.

Table 4.1: Numerical results corresponding to $x_1 = -0.9$ and $x_1 = 0.9$ for 30 steps.

| Step | Our Algorithm for $x_1 = -0.9$ | Zhou and Cui Algorithm for $x_1 = -0.9$ | Our Algorithm for $x_1 = 0.9$ | Zhou and Cui Algorithm for $x_1 = 0.9$ |
|---|---|---|---|---|
| 1 | $-0.9$ | $-0.9$ | 0.9 | 0.9 |
| 2 | 0.509646 | $-0.888724$ | 0.501821 | 0.896694 |
| 3 | 0.500044 | $-0.877647$ | 0.500008 | 0.893448 |
| 4 | 0.5 | $-0.866763$ | 0.5 | 0.89026 |
| 5 | 0.5 | $-0.856069$ | 0.5 | 0.887127 |
| 6 | 0.5 | $-0.845558$ | 0.5 | 0.88405 |
| 7 | 0.5 | $-0.835227$ | 0.5 | 0.881026 |
| 8 | 0.5 | $-0.483408$ | 0.5 | 0.878054 |
| 9 | 0.5 | $-0.815081$ | 0.5 | 0.875132 |
| 10 | 0.5 | $-0.805258$ | 0.5 | 0.87226 |
| 11 | 0.5 | $-0.795596$ | 0.5 | 0.869436 |
| 12 | 0.5 | $-0.786091$ | 0.5 | 0.866658 |
| 13 | 0.5 | $-0.776739$ | 0.5 | 0.863926 |
| 14 | 0.5 | $-0.767537$ | 0.5 | 0.861238 |
| 15 | 0.5 | $-0.75848$ | 0.5 | 0.858594 |
| 16 | 0.5 | $-0.749565$ | 0.5 | 0.855991 |
| 17 | 0.5 | $-0.740788$ | 0.5 | 0.85343 |
| 18 | 0.5 | $-0.732147$ | 0.5 | 0.850909 |
| 19 | 0.5 | $-0.723638$ | 0.5 | 0.848428 |
| 20 | 0.5 | $-0.715258$ | 0.5 | 0.845984 |
| 21 | 0.5 | $-0.707003$ | 0.5 | 0.843578 |
| 22 | 0.5 | $-0.698872$ | 0.5 | 0.841209 |
| 23 | 0.5 | $-0.690861$ | 0.5 | 0.838875 |
| 24 | 0.5 | $-0.682967$ | 0.5 | 0.836576 |
| 25 | 0.5 | $-0.675188$ | 0.5 | 0.834311 |
| 26 | 0.5 | $-0.667521$ | 0.5 | 0.832079 |
| 27 | 0.5 | $-0.659964$ | 0.5 | 0.82988 |
| 28 | 0.5 | $-0.652514$ | 0.5 | 0.827713 |
| 29 | 0.5 | $-0.645169$ | 0.5 | 0.825576 |
| 30 | 0.5 | $-0.637927$ | 0.5 | 0.82347 |

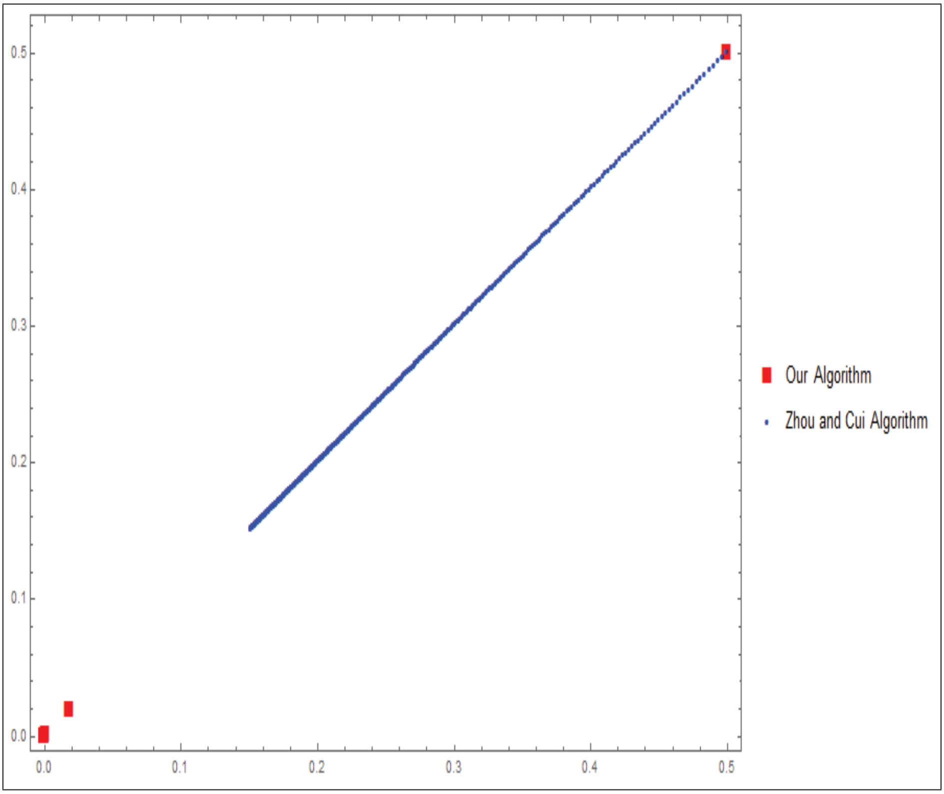


Figure 4.2: Convergence behaviors corresponding to $x_1 = (\frac{1}{2}, \frac{1}{2})$ for 500 steps.

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**FACTA Universitatis**. Series, Mathematics and informatics / editor-in-chief Predrag S. Stanimirović. - 1986, N° 1- . - Niš : University of Niš, 1986- (Niš : Unigraf-X-Copy). - 24 cm

Tekst na engl. jeziku. - Drugo izdanje na drugom medijumu: Facta Universitatis. Series: Mathematics and Informatics (Online) = ISSN 2406-047X
ISSN 0352-9665 = Facta Universitatis. Series: Mathematics and informatics
COBISS.SR-ID 5881090

**FACTA UNIVERSITATIS**

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Vol. 33, No 3 (2018)

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