UNIVERSITY OF NIŠ

ISSN 0352-9665 (Print)
ISSN 2406-047X (Online)
COBISS.SR-ID 5881090

FACTA UNIVERSITATIS

Series
MATHEMATICS AND INFORMATICS
Vol. 33, No 4 (2018)

Scientific Journal **FACTA UNIVERSITATIS**
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The cover image taken from http://www.pptbackgrounds.net/binary-code-and-computer-monitors-backgrounds.html.
Publication frequency – one volume, five issues per year.
Published by the University of Niš, Republic of Serbia

This publication was in part supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia
Printed by "UNIGRAF-X-COPY" – Niš, Republic of Serbia

ISSN 0352 – 9665 (Print)
ISSN 2406 – 047X (Online)
COBISS.SR-ID 5881090

FACTA UNIVERSITATIS

SERIES **MATHEMATICS AND INFORMATICS**
Vol. 33, No 4 (2018)

UNIVERSITY OF NIŠ

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[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

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FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 4 (2018), 497–503
https://doi.org/10.22190/FUMI1804497D

ON THE MAPPINGS PRESERVING THE HYPERBOLIC POLYGONS OF TYPE B TOGETHER WITH THEIR HYPERBOLIC AREAS

Oğuzhan Demirel

Abstract. In this paper, we present new characterizations of Möbius transformations and conjugate Möbius transformations by using the mappings preserving the hyperbolic polygons of type B together with their hyperbolic areas.
Keywords. Hyperbolic polygons; Möbius transformations; hyperbolic areas.

1. Introduction

A Möbius transformation $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a mapping of the form $w = \frac{az+b}{cz+d}$ satisfying $ad - bc \neq 0$, where $a, b, c, d \in \mathbb{C}$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The set of all Möbius transformations is a group under composition. Möbius transformations are conformal mappings having many useful properties. For example, a map is Möbius if and only if it preserves cross ratios. As for geometric aspect, circle-preserving is another important characterization of Möbius transformations. There are well-known elementary proofs that if f is a continuous injective map of the extended complex plane $\overline{\mathbb{C}}$ that maps circles into circles, then f is Möbius.

The Möbius invariant property is naturally related to hyperbolic geometry. For instance, see the preservation of triangular domains [6], Lambert and Saccheri quadrilaterals [10], [11], hyperbolic regular polygons [3], hyperbolic regular star polygons [4], polygons of type A [7] and others. The Möbius transformations preserving the open unit disc $B^2 = \{z \in \mathbb{C} : |z| < 1\}$ are precisely those of the form $w = e^{i\theta}\frac{a+z}{1+\overline{a}z}$, where $a, z \in B^2$ and $\theta \in \mathbb{R}$. The Poincaré disc model of hyperbolic geometry is built on B^2, more precisely the points of this model are points of B^2 and the hyperbolic lines of this model are Euclidean semicircular arcs that intersect the boundary of B^2 orthogonally including diameters of B^2. Given two distinct hyperbolic lines which intersect at a point, the measure of the angle between these hyperbolic lines is defined by the Euclidean tangents at the common point.

Received March 12, 2018; accepted April 20, 2018
2010 *Mathematics Subject Classification.* Primary 51M10; Secondary 30F45, 51B10, 51M09, 51M25

Definition 1.1. [1] A Lambert quadrilateral is a hyperbolic quadrilateral with ordered interior angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{2}$ and θ, where $0 < \theta < \frac{\pi}{2}$.

Definition 1.2. [1] A Saccheri quadrilateral is a hyperbolic quadrilateral with ordered interior angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, θ, θ, where $0 < \theta < \frac{\pi}{2}$.

Definition 1.3. [7] A hyperbolic polygon with $n-$sides is called as of type A if it has exactly two interior angles not equal to $\frac{\pi}{2}$.

Definition 1.4. [7] A hyperbolic polygon with $n-$sides is called as of type B if it has exactly a unique interior angle not equal to $\frac{\pi}{2}$.

Saccheri quadrilaterals and Lambert quadrilaterals are convex hyperbolic polygons with 4 sides having type A and type B, respectively.

The transformations defined by $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ satisfying $ad - bc \neq 0$ are known as conjugate Möbius transformations. Clearly a conjugate Möbius transformation is a composition of the complex conjugate function with a Möbius transformation. These transformations, like Möbius transformations, have many beautiful properties. For instance they preserve angle magnitudes of angles, but notice that Möbius transformations preserve the orientation while conjugate Möbius transformations reverse it.

C. Carathéodory [2] proved that every arbitrary one to one correspondence between the points of a circular disc C and a bounded point set C' by which circles lying completely in C are transformed into circles lying in C' must always be either a Möbius transformation or a conjugate Möbius transformation. The following results are well known and they play major roles in our proofs.

Lemma 1.1. *[1] Let $\theta_1, \theta_2, \ldots, \theta_n$ be any ordered $n-$tuple with $0 \leq \theta_j < (n-2)\pi$, $j = 1, \ldots, n$. Then there exists a hyperbolic polygon P with interior angles $\theta_1, \theta_2, \ldots, \theta_n$, occurring in this order around ∂P, if and only if $\theta_1 + \theta_2 + \ldots + \theta_n < (n-2)\pi$.*

Theorem 1.1. *(Gauss-Bonnet theorem for a hyperbolic polygon with n sides) Let P be a hyperbolic convex polygon with $n-$ sides and with interior angles $\theta_1, \theta_2, \ldots, \theta_n$. Then the hyperbolic area $\Delta(P)$ of the polygon P is*

$$\Delta(P) = (n-2)\pi - (\theta_1 + \theta_2 + \ldots + \theta_n) \tag{1.1}$$

Throughout the paper we denote by X' the image of X under f, by $[A_j, A_k]$ the geodesic segment between the points A_j and A_k, by A_jA_k the hyperbolic line passing through the points A_j and A_k, by $A_jA_kA_s$ the hyperbolic triangle with three ordered vertices A_j, A_k and A_s, by $A_1A_2\cdots A_n$ the hyperbolic polygon with $n-$ ordered vertices $A_1, A_2, \cdots A_n$, and by $\angle A_jA_kA_s$ the angle between $[A_j, A_k]$ and $[A_s, A_k]$. We consider the hyperbolic plane $B^2 = \{z \in \mathbb{C} : |z| < 1\}$ with length differential $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$.

2. The Mappings Preserving the Hyperbolic Polygons of Type B Together With Their Hyperbolic Areas

A map $f : B^2 \to B^2$ has the property B, if it preserves $n-$sided hyperbolic polygons having type B, that is if P is a $n-$sided hyperbolic polygon of type B, then $f(P)$ is a $n-$sided hyperbolic polygon of type B, see [7]. J. Liu proved the following result in [7]:

Lemma 2.1. *[7] Let $f : B^2 \to B^2$ be a continuous bijection. If f has Property B for each $n > 3$, then f preserves the vertex where the interior angle is not right.*

Instead of using the continuity condition of functions, we try to obtain a new characterization of Möbius transformations with the condition "$n-$sided hyperbolic polygons preserving property of type B together with their hyperbolic areas " for a fixed $n > 3$. More precisely, when we say f preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas, this means that if P is a $n-$sided hyperbolic polygon of type B with hyperbolic area $\Delta(P) = \sigma$, then $f(P)$ is a $n-$sided hyperbolic polygon of type B with hyperbolic area $\Delta(f(P)) = \sigma$. Area preserving mappings are studied by V. Pambuccian in [8] and by O. Demirel in [5].

Lemma 2.2. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B for a fixed $n > 3$. Then f is injective.*

Proof. Let P and Q be two distinct points in B^2. By *Lemma 2.1*, there exists a hyperbolic polygon, say $A_1A_2\cdots A_n$, satisfying $\angle A_nA_1A_2 = \alpha < \frac{\pi}{2}$, $\angle A_1A_2A_n = \cdots = \angle A_{n-2}A_{n-1}A_n = \angle A_{n-1}A_nA_1 = \frac{\pi}{2}$. There are three cases:

Case 1: Assume $d_H(P,Q) < d_H(A_1,A_2)$, where d_H is hyperbolic distance. $A_1A_2\cdots A_n$ can be carried to the point Q with the help of a hyperbolic isometry g_1 such that $g_1(A_2) = Q$ and $P \in [g_1(A_1), g_1(A_2)]$. Let l be the hyperbolic line passing through P and intersects $g_1(A_{n-1})g_1(A_n)$ perpendicularly. Denote the common point of the hyperbolic lines l and $g_1(A_{n-1})g_1(A_n)$ by S. The existence of the point S is clear since $\angle Pg_1(A_n)g_1(A_{n-1}) < \frac{\pi}{2}$, $\angle Pg_1(A_{n-1})g_1(A_n) < \frac{\pi}{2}$. Hence we construct a hyperbolic polygon $PQg_1(A_3)\cdots g_1(A_{n-1})S$ which is an $n-$sided hyperbolic polygon of type B.

Case 2: Assume $d_H(P,Q) > d_H(A_1,A_2)$. $A_1A_2\cdots A_n$ can be carried to the point Q with the help of a hyperbolic isometry g_2 such that $g_2(A_2) = Q$ and $g_2(A_1) \in [P,Q] = [P, g_2(A_2)]$. Let k be the hyperbolic line passing through P which intersects the hyperbolic line $g_2(A_{n-1})g_2(A_n)$ perpendicularly. Denote the common point of the hyperbolic lines k and $g_2(A_{n-1})g_2(A_n)$ by R. The existence of the point R is clear since $\angle Pg_2(A_n)g_2(A_{n-1}) > \frac{\pi}{2}$. Hence we construct an $n-$sided hyperbolic polygon $RPQg_2(A_3)\cdots g_1(A_{n-2})g_1(A_{n-1})$ of type B.

Case 3: If $d_H(P,Q) = d_H(A_1,A_2)$, then $A_1A_2\cdots A_n$ can be carried to the point Q with the help of a hyperbolic isometry g_3 such that $g_3(A_1) = P$ and $g_3(A_2) = Q$.

Hence we construct an $n-$sided hyperbolic polygon $PQg_3(A_3)\cdots g_3(A_{n-1})g_3(A_n)$ of type B.

As in the cases above, for two arbitrary points P and Q, it is possible to construct an $n-$sided hyperbolic polygon of type B by using these points. Therefore, if $PQB_1B_2\cdots B_n$ is an $n-$sided hyperbolic polygon of type B, then $P'Q'B'_1B'_2\cdots B'_n$ is also an $n-$sided hyperbolic polygon of type B. This ends the proof. $\square$

Lemma 2.3. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B for a fixed $n > 3$. Then f preserves the collinearity and betweenness properties of the points.*

Proof. Let P and Q be two distinct points in B^2 and assume that S be an interior point of $[P, Q]$. By *Lemma 2.2*, one can easily construct an $n-$sided hyperbolic polygon of type B, say $PQA_1\cdots A_{n-2}$. Moreover, there are many more $n-$sided hyperbolic polygons of type B with common side $[P, Q]$ and all of them contain S. Hence the images of all $n-$sided hyperbolic polygons of type B with common side $[P, Q]$ under f are $n-$sided hyperbolic polygons of type B with common side $[P', Q']$ containing S'. Therefore, f preserves the collinearity and betweenness properties of the points. $\square$

Lemma 2.4. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Then f preserves the vertices together with their interior angles.*

Proof. Let $A_1A_2\cdots A_n$ be an $n-$sided hyperbolic polygon of type B (directed counterclockwise) such that $\angle A_nA_1A_2 := \theta \neq \frac{\pi}{2}$. Assume $\angle A'_nA'_1A'_2 = \frac{\pi}{2}$. Clearly, $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$ or $\angle A'_1A'_2A'_3 = \frac{\pi}{2}$. Without loss of generality, we may assume $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$. Now draw a geodesic segment $[A_n, K]$ to the hyperbolic line A_1A_2 where the point K lies on A_1A_2 satisfying $\angle A_nKA_1 = \frac{\pi}{2}$. Notice that if $\theta < \frac{\pi}{2}$, then K lies on $[A_1, A_2]$ and if $\theta > \frac{\pi}{2}$, then A_1 lies on $[K, A_2]$. Since K lies on A_1A_2, by *Lemma 2.3*, the point K' must be lie on $A'_1A'_2$. Hence we construct a new $n-$sided hyperbolic polygon $KA_2\cdots A_n$ of type B. Therefore, $K'A'_2\cdots A'_n$ is also an $n-$sided hyperbolic polygon of type B. Since $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$, we get $\angle A'_{n-1}A'_nK' \neq \frac{\pi}{2}$ which yields $\angle A'_nK'A'_2 = \angle A'_nK'A'_1 = \frac{\pi}{2}$. Obviously, this is a contradiction since the sum of the interior angles of the hyperbolic triangle $A'_nK'A'_1$ is greater then π. Thus we have $\angle A'_nA'_1A'_2 \neq \frac{\pi}{2}$. Because of the fact that f preserves the $n-$sided hyperbolic polygons of type B together with their hyperbolic areas, by *Gauss-Bonnet theorem*, we get $\angle A'_nA'_1A'_2 = \theta$, $\angle A'_{i-1}A'_iA'_{i+1} = \frac{\pi}{2}$ for all $2 \leq i \leq n-1$ and $\angle A'_{n-1}A'_nA'_1 = \frac{\pi}{2}$. $\square$

Lemma 2.5. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Then f preserves hyperbolic distance.*

Proof. Let X, Y and Z be three distinct points in B^2 such that XYZ is a hyperbolic triangle (directed counterclockwise) with $\angle ZXY := \alpha_1, \angle XYZ := \alpha_2$

and $\angle YZX := \alpha_3$. Now, by *Lemma 2.1*, there exists a hyperbolic polygon of type B, say $A_1A_2\ldots A_n$ (directed counterclockwise), such that $\angle A_nA_1A_2 = \alpha_1$. The angle $\angle A_nA_1A_2$ of the hyperbolic polygon $A_1A_2\ldots A_n$ can be moved to the vertex X of the hyperbolic triangle XYZ by an appropriate Möbius transformation g such that the points $g(A_2)$ and $g(A_n)$ lie on the hyperbolic lines XY and XZ, respectively. By the properties of f and g, we immediately get that $g(A_1)'g(A_2)'\ldots g(A_n)'$, that is $X'g(A_2)'\ldots g(A_n)'$, is an $n-$sided hyperbolic polygon of type B. By *Lemma 2.4*, we have $\angle ZXY = \angle A_nA_1A_2 = \angle g(A_n)Xg(A_2) = \angle g(A_n)'X'g(A_2)' = \angle A_n'A_1'A_2' = \angle Z'X'Y' = \alpha_1$. Hence f preserves the interior angle $\angle ZXY$ of the hyperbolic triangle XYZ. Following the same way, one can easily prove that $\angle XYZ = \angle X'Y'Z'$ and $\angle YZX = \angle Y'Z'X'$ hold true. It is well known that, in hyperbolic plane, the lengths of a hyperbolic triangle are determined by its interior angles, see [9]. Therefore, we get that $d_H(X,Y) = d_H(X',Y')$, $d_H(X,Z) = d_H(X',Z')$ and $d_H(Y,Z) = d_H(Y',Z')$. □

Lemma 2.6. *Let $f : B^2 \to B^2$ be a mapping which preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Then f is surjective.*

Proof. To prove that f is surjective, we will show that for any point Y in B^2, there exists a point X in B^2 such that $f(X) = Y$. Let A, B, C be three three distinct points in B^2, each of which is different from Y. Now construct three hyperbolic circles with radius $r_1 = d_H(A',Y)$, $r_2 = d_H(B',Y)$ and $r_3 = d_H(C',Y)$ centered at A', B', C', respectively. These circles meet together only at Y. Because of the fact that f is a distance preserving mapping by *Lemma 2.5*, the pre-images of circles meet together only at a point, say X. Hence, $X' = Y$. □

Theorem 2.1. *The mapping $f : B^2 \to B^2$ is Möbius or conjugate Möbius if, and only if, f preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$.*

Proof. Because of the fact that f is an isometry, the "only if" part is clear. Conversely, we may assume that f preserves $n-$sided hyperbolic polygons of type B together with their hyperbolic areas for a fixed $n > 3$. Without loss of generality we may assume $f(O) = O$ by composing an isometry if necessary. Let x and y be two different points in B^2. Since f preserves the hyperbolic distance by *Lemma 2.5*, one can easily get $d_H(0,x) = d_H(0,x')$ and $d_H(0,y) = d_H(0,y')$, namely $|x| = |x'|$ and $|y| = |y'|$, where $|\cdot|$ is the Euclidean norm. Hence we have $|x-y| = |x'-y'|$, since f preserves the angles by *Lemma 2.4*. Finally, we get

$$2\langle x,y\rangle = |x|^2 + |y|^2 - |x-y|^2 = |x'|^2 + |y'|^2 - |x'-y'|^2 = 2\langle x',y'\rangle. \tag{2.1}$$

Therefore, f preserves the inner product and then is the restriction on B^2 of an orthogonal transformation, that is, f is Möbius transformation or conjugate Möbius transformation by *Carathédory's theorem.* If the orientation of the angles preserved under f, then f is a Möbius transformation, otherwise; f is a conjugate Möbius transformation. □

Corollary 2.1. *The mapping $f : B^2 \to B^2$ is Möbius or conjugate Möbius if, and only if, f preserves the Lambert quadrilaterals together with their hyperbolic areas.*

Naturally, one may wonder whether *Corollary 2.1* is valid for Saccheri quadrilaterals. Now we give the affirmative answer as follows:

Corollary 2.2. *The mapping $f : B^2 \to B^2$ is Möbius or conjugate Möbius if, and only if, f preserves all Saccheri quadrilaterals together with their hyperbolic areas.*

Proof. Because of the fact that f is an isometry, the "only if" part is clear. Conversely, we may assume that f preserves all Saccheri quadrilaterals together with their hyperbolic areas. The injectivity, collinearity and the betweenness properties of f can be easily proved following the ways in the proofs of *Lemma 2.2*, *Lemma 2.3*.

Step 1: We claim that f preserves the right angles of Saccheri quadrilaterals. Let $ABCD$ be a Saccheri quadrilateral with $\angle DAB = \angle ABC = \frac{\pi}{2}$ and $\angle BCD = \angle CDA := \theta < \frac{\pi}{2}$. For each point $X_i \in [A, D]$, there exists a point $Y_i \in [C, B]$ such that X_iABY_i is a Saccheri quadrilateral. Notice that $d_H(A, X_i) = d_H(B, Y_i)$. Assume $\angle Y_iX_iA = \angle BY_iX_i := \theta_i$ for all $i \in I \subset \mathbb{R}$. Since f preserves the Saccheri quadrilaterals together with their hyperbolic areas, we immediately get that $X_i'A'B'Y_i'$ are Saccheri quadrilaterals with $\Delta(X_i'A'B'Y_i') = \Delta(X_iABY_i)$ for all $i \in I$. Notice that, by injectivity property of f, the sets $\{X_i' : i \in I\}$ and $\{Y_i' : i \in I\}$ are consist of collinear points, that is $X_i' \in [A', D']$ and $Y_i' \in [B', C']$ hold true for all $i \in I$. Because of the fact that all the Saccheri quadrilaterals $X_i'A'B'Y_i'$ have common two interior angles $\frac{\pi}{2}, \frac{\pi}{2}$ and have common two vertices A' and B', this implies that $\angle X_i'A'B' = \angle A'B'Y_i' = \frac{\pi}{2}$. Thus f preserves right angles of Saccheri quadrilaterals.

Step 2: By *Step 1*, f preserves the other interior angles of Saccheri quadrilaterals which are not right angles.

Step 3: Let $ABCD$ be a Lambert quadrilateral with $\angle CDA := \theta < \frac{\pi}{2}$ and $\angle DAB = \angle ABC = \angle BCD = \frac{\pi}{2}$. By reflecting $ABCD$ with respect to geodesic BC, we get a Saccheri quadrilateral $AEFD$, where the points E and F are the reflections of the points A and D, respectively. Thus, the quadrilateral $A'E'F'D'$ must be a Saccheri quadrilateral with $\Delta(A'E'F'D') = \Delta(AEFD)$. Since $B \in [A, E]$ and $C \in [D, F]$, we have $B' \in [A', E']$ and $C' \in [D', F']$. Therefore, $A'E'F'D'$ contains two quadrilaterals $A'B'C'D'$ and $B'E'F'C'$. By *Step 1* and *Step 2*, we get $\angle D'A'B' = \angle B'E'F' = \frac{\pi}{2}$ and $\angle C'D'A' = \angle E'F'C' = \theta$. By reflecting $ABCD$ in the geodesic AB, one can easily see that $\angle D'C'B' = \frac{\pi}{2}$ holds true. This implies that C' is the midpoint of D' and F' which implies that $\angle A'B'C' = \frac{\pi}{2}$. Hence the quadrilateral $A'B'C'D'$ must be a Lambert quadrilateral with $\Delta(A'B'C'D') = \Delta(ABCD)$ and this implies that f is a Möbius transformation or a conjugate Möbius transformation.

□

Acknowledgement: The author would like to thank the reviewer for the helpful comments to improve the paper.

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FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 33, No 4 (2018), 505–516
https://doi.org/10.22190/FUMI1804505S

ITERATIONS FOR APPROXIMATING LIMIT REPRESENTATIONS OF GENERALIZED INVERSES

Bilall I. Shaini and Predrag S. Stanimirović

Abstract. Our underlying motivation is the iterative method for the implementation of the limit representation of the Moore-Penrose inverse $\lim_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$ from [Žukovski, Lipcer, On recurent computation of normal solutions of linear algebraic equations, Ž. Vicisl. Mat. i Mat. Fiz. 12 (1972), 843–857] and [Žukovski, Lipcer, On computation pseudoinverse matrices, Ž. Vicisl. Mat. i Mat. Fiz. 15 (1975), 489–492]. The iterative process for the implementation of the general limit formula $\lim_{\alpha\to 0}(\alpha I + R^*S)^{-1}R^*$ was defined in [P.S. Stanimirović, Limit representations of generalized inverses and related methods, Appl. Math. Comput. 103 (1999), 51–68]. In this paper we develop an improvement of this iterative process. The iterative method defined in such a way is able to produce the result in a predefined number of iterative steps. Convergence properties of defined iterations are further investigated.

Keywords. Generalized inverses; Moore-Penrose inverse; Drazin inverse; limit representation; Leverrier-Faddeev algorithm.

1. Introduction

We use the following notation. $\mathbb{C}^{m\times n}$: the set of $m\times n$ complex matrices; $\mathbb{C}_r^{m\times n}$ is the set of rank r: $\mathbb{C}_r^{m\times n} = \{X\in\mathbb{C}^{m\times n} \,:\, \operatorname{rank}(X) = r\}$; $\mathcal{O}$ (resp. $\vec{0}$): the zero matrix of an appropriate order (resp. the zero vector); I_m: identity matrix of the order m; $\mathcal{R}(A)$ and $\mathcal{N}(A)$: the range and the null space of A; $\operatorname{Tr}(A)$: the trace of A.

For any matrix $A\in\mathbb{C}^{m\times n}$ consider the following equations in X:

$$(1)\quad AXA=A,\quad (2)\quad XAX=X,\quad (3)\quad (AX)^*=AX,\quad (4)\quad (XA)^*=XA$$

and if $m=n$, also

$$(5)\quad AX=XA\quad (1^k)\quad A^{k+1}X=A^k.$$

Received December 20, 2018.; Accepted January 22, 2019.
2010 *Mathematics Subject Classification.* Primary 15A09; Secondary 15A23

For a sequence $\mathcal{S}$ of $\{1,2,3,4,5,1^k\}$ the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If X satisfies (1) and (2), it is said to be a reflexive g-inverse of A, whereas $X = A^{\dagger}$ is said to be the Moore-Penrose inverse of A if it satisfies (1)-(4). Also, A_L^{-1} (resp. A_R^{-1}) denote an arbitrary left (resp. right) inverse of A. The group inverse $A^{\#}$ is the unique $\{1,2,5\}$ inverse of A, and exists if and only if $\mathrm{ind}(A) = \min\{k:\ \mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)\} = 1$. A matrix $G = A^D$ is said to be the Drazin inverse of A if (1^k) (for some positive integer k), (2) and (5) are satisfied.

Let there be given invertible matrices M and N of the order m and n, respectively. For any $m \times n$ matrix A, the weighted Moore-Penrose inverse of A is the unique solution $X = A^{\dagger}_{M,N}$ of the matrix equations (1), (2) and the following equations in X:

$$(3M)\qquad (MAX)^* = MAX \qquad\qquad (4N)\qquad (XA)^*N = NXA.$$

the next is valid for a rectangular matrix A [14]:

$$A^{\dagger} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)},\quad A^{\dagger}_{M,N} = A^{(2)}_{\mathcal{R}(A^{\sharp}),\mathcal{N}(A^{\sharp})},\ A^{\sharp} = N^{-1}A^*M, \tag{1.1}$$

where M, N are positive definite matrices. For a given square matrix A the next identities are satisfied:

$$A^{\mathrm{D}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)},\ k \geq \mathrm{ind}(A),\quad A^{\#} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A)}. \tag{1.2}$$

The core inverse of a complex matrix was originated by Baksalary and Trenkler in [1]. A matrix $A^{\textcircled{\#}} \in \mathbb{C}^{n\times n}$ satisfying

$$AA^{\textcircled{\#}} = P_{\mathcal{R}(A)} \text{ and } \mathcal{R}(A^{\textcircled{\#}}) \subseteq \mathcal{R}(A)$$

is called the core inverse of A.

Manjunatha Prasad and Mohana in [10] discovered the core-EP inverse. A matrix X, denoted by $A^{\textcircled{\dagger}}$, is called the core-EP inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies

$$XAX = X,\quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k).$$

The following results can be derived using results from [9]:

$$A^{\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)},\qquad A^{\textcircled{\#}} = A^{(2)}_{\mathcal{R}(A),\mathcal{N}(A^*)}.$$

The remainder of the manuscript is organized as follows. In order to complete the presentation and describe our motivation, limit representations of main generalized inverses are surveyed in Section 2.. Some additional results about the convergence of the iterations proposed in [11] are presented in Section 3.. An efficient method for the improved implementation of defined iterations is considered in Section 4.. An illustrative numerical example is presented in Section 5..

2. Survey of limit representations

Limit representations of main generalized inverses is restated in Proposition 2.1. The inverse of a nonsingular matrix A can be characterized in terms of the limiting process

$$A^{-1} = \lim_{\alpha \to 0} (\alpha I + A)^{-1}, \tag{2.1}$$

wherein it is assumed that $-\alpha \notin \sigma(A)$ and $\sigma(A)$ stands for the set of all eigenvalues of A.

Proposition 2.1. **(a)** [3] *Limit representation of the Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ is equal to*

$$A^{\dagger} = \lim_{\alpha \to 0} \left(\alpha I_n + A^* A\right)^{-1} A^*. \tag{2.2}$$

(b) [7] *Limit representation of the Drazin inverse of a matrix $A \in \mathbb{C}^{n\times n}$ whose index is k can be expressed as the limit*

$$A^{\mathrm{D}} = \lim_{\alpha \to 0} \left(\alpha I_n + A^{l+1}\right)^{-1} A^{l}, \quad l \geqslant k. \tag{2.3}$$

(c) [13] *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose that $G \in \mathbb{C}^{n\times m}$ satisfies $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. In the case of the existence, $A^{(2)}_{T,S}$ is defined by the limit representation*

$$A^{(2)}_{T,S} = \lim_{\alpha \to 0} \left(GA + \alpha I\right)^{-1} G. \tag{2.4}$$

(d)[16] *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = 1$. *Then*

$$A^{\#\!\!\!\bigcirc} = \lim_{\alpha \to 0} AA^*(A^2A^* + \alpha I)^{-1} = \lim_{\alpha \to 0} (AA^*A + \alpha I)^{-1}AA^*. \tag{2.5}$$

$$A^{\#\!\!\!\bigcirc} = \lim_{\alpha \to 0} A(A^2)^*(A^2(A^2)^* + \alpha I)^{-1} = \lim_{\alpha \to 0} (A(A^2)^*A + \alpha I)^{-1}A(A^2)^*. \tag{2.6}$$

(e) [16] *Let $A \in \mathbb{C}^{n\times n}$ and* $\mathrm{ind}(A) = k$. *Then*

$$A^{\dagger\!\!\!\bigcirc} = \lim_{\alpha \to 0} A^k(A^k)^*(A^{k+1}(A^k)^* + \alpha I)^{-1} = \lim_{\alpha \to 0} (A^k(A^k)^*A + \alpha I)^{-1}A^k(A^k)^*. \tag{2.7}$$

The limit representations of the outer inverse in Banach space were investigated in [6].

The following additional notation will be used in this section.

a_t, $\quad t = 1,\dots,m$: tth row of $A \in \mathbb{C}^{m\times n}$; $A_{\underline{t}} = \begin{bmatrix} a_1 \\ \dots \\ a_t \end{bmatrix}$, $t = 1,\dots,m$: the $t \times n$ submatrix which contains the first t rows of $A \in \mathbb{C}^{m\times n}$; $y_{\underline{t}} = A_{\underline{t}}x$, and specially $y_t = a_t x$, $\quad t = 1,\dots,m$;

Our idea in the present paper can be described in three steps. First step is an iterative method for the implementation of the limit representation of the Moore-Penrose inverse $\lim\limits_{\alpha\to 0}(\alpha I + A^*A)^{-1}A^*$ from [17, 18]. Another step is the iterative process for the implementation of the general limit formula $\lim\limits_{\alpha\to 0}(\alpha I + R^*S)^{-1}R^*$, originated in [11]. In this paper we develop an improvement of this iterative process. Detailed description is given in the rest of this section.

In Proposition 2.2 and Proposition 2.3 we restate known iterative methods from [17, 18].

Proposition 2.2. (Žukovski, Lipcer 1972) [17] *For a given $m\times n$ complex matrix A and given $m\times 1$ complex vector y, the solution of the iterative process*

$$
\begin{aligned}
&\gamma_{t+1}^\alpha = \gamma_t^\alpha - \frac{\gamma_t^\alpha a_{t+1}^* a_{t+1}\gamma_t^\alpha}{\alpha + a_{t+1}\gamma_t^\alpha a_{t+1}^*}, \quad \gamma_0^\alpha = I_n, \quad \alpha>0,\\
&x_{t+1}^\alpha = x_t^\alpha + \frac{\gamma_t^\alpha a_{t+1}^*}{\alpha + a_{t+1}\gamma_t^\alpha a_{t+1}^*}\left(y_{t+1} - a_{t+1}x_t^\alpha\right), \quad x_0^\alpha = 0,\\
&\qquad t = 0,\dots,m-1
\end{aligned}
\tag{2.8}
$$

is given by

$$
\begin{aligned}
x_t^\alpha &= \left(\alpha I_n + A_{\underline{t}}{}^* A_{\underline{t}}\right)^{-1} A_{\underline{t}}{}^* y_{\underline{t}}\\
\gamma_t^\alpha &= \left(\alpha I_n + A_{\underline{t}}{}^* A_{\underline{t}}\right)^{-1}\alpha, \quad t=1,\dots,m.
\end{aligned}
$$

Proposition 2.3. (Žukovski, Lipcer 1975) [18] *Let $A\in\mathbb{C}^{m\times n}$. If the rows of the unit matrix I_m are denoted by i_t, $t=1,\dots,m$, then the following iterative method*

$$
\begin{aligned}
&\gamma_{t+1}^\alpha = \gamma_t^\alpha - \frac{\gamma_t^\alpha a_{t+1}^* a_{t+1}\gamma_t^\alpha}{\alpha + a_{t+1}\gamma_t^\alpha a_{t+1}^*}, \quad \gamma_0^\alpha = I_n, \quad \alpha>0,\\
&X_{t+1}^\alpha = X_t^\alpha + \frac{\gamma_t^\alpha a_{t+1}^*}{\alpha + a_{t+1}\gamma_t^\alpha a_{t+1}^*}\left(i_{t+1} - a_{t+1}X_t^\alpha\right), \quad X_0^\alpha = \mathcal{O}\in\mathbb{C}^{n\times m}\\
&\qquad t = 0,\dots,m-1
\end{aligned}
\tag{2.9}
$$

produces the resulting matrices

$$
X_m^\alpha = \left(\alpha I_n + A^*A\right)^{-1}A^*, \quad \gamma_m^\alpha = \left(\alpha I_n + A^*A\right)^{-1}\alpha.
$$

Of special interest are the limits $\lim\limits_{\alpha\to 0} X_m^\alpha = A^\dagger$ [2] and $\lim\limits_{\alpha\to 0}\gamma_m^\alpha = I_n - A^\dagger A$ [17].

An interesting computational scheme was proposed in [17]. This scheme ensures indirect decreasing of the values for α: after computation of the values γ_i^α and X_i^α, $i=1,\dots,m$ by means of (2.9), compute γ_i^α and X_i^α, $i>t$, by means of the rows $a_{m+1}=a_1,\dots,a_{2m}=a_m,\dots$ and the numbers $y_{m+1}=y_1,\dots,y_{2m}=y_m,\dots$ In this case is

$$
X_{mN}^\alpha = X_m^{\alpha/N}, \quad \gamma_{mN}^\alpha = \gamma_m^{\alpha/N}, \quad N=1,2,\dots
\tag{2.10}
$$

where X_{mN}^{α} and γ_{mN}^{α} are defined by

$$X_{mN}^{\alpha} = \left(\alpha I_n + A_{mN}^* A_{mN}\right)^{-1} A_{mN}^*, \quad \gamma_{mN}^{\alpha} = \left(\alpha I_n + A_{mN}^* A_{mN}\right)^{-1} \alpha$$

and A_{mN} is the block matrix which consists of N blocks of $A_m = A$:

$$A_{mN} = \begin{bmatrix} A_m \\ \dots \\ A_m \end{bmatrix} = \begin{bmatrix} A \\ \dots \\ A \end{bmatrix}.$$

The main result in [11] was an approximate method for computing generalized inverses and different matrix expressions involving generalized inverses which are determined by the limit expressions

$$\lim_{\alpha \to 0} \left(\alpha I_q + R^* S\right)^{-1} R^*, \tag{2.11}$$

$$\lim_{\alpha \to 0} \left(\alpha I_q + R^* S\right)^{-1} \alpha, \tag{2.12}$$

where R and S are two arbitrary $p \times q$ complex matrices.

For a given matrix $A \in \mathbb{C}_r^{m \times n}$, in the case $R = S = A$ we obtain the iterative method (2.9) for computing the Moore-Penrose inverse. In the case $m = n$, $R^* = A^l$, $l \geqslant \operatorname{ind}(A)$, $S = A$, we construct an iterative process for implementation of the limit representation (2.3) for computing the Drazin inverse.

The following result from [11] generalizes the iterative process (2.8).

Proposition 2.4. (Stanimirović 1999) [11] *Let given two arbitrary $p \times q$ complex matrices R and S and $p \times 1$ complex vector y. If the rows of the matrices R and S are denoted by r_u and s_u, respectively, $u = 1, \dots, p$, and r_u^* denotes conjugate and transpose of the vector r_u, then the following iterative sequences*

$$\begin{aligned} \gamma_{t+1}^{\alpha} &= \gamma_t^{\alpha} - \frac{\gamma_t^{\alpha} r_{t+1}^* s_{t+1} \gamma_t^{\alpha}}{\alpha + s_{t+1} \gamma_t^{\alpha} r_{t+1}^*}, \quad \gamma_0^{\alpha} = I_q, \quad \alpha > 0, \\ x_{t+1}^{\alpha} &= x_t^{\alpha} + \frac{\gamma_t^{\alpha} r_{t+1}^*}{\alpha + s_{t+1} \gamma_t^{\alpha} r_{t+1}^*} \left(y_{t+1} - s_{t+1} x_t^{\alpha}\right), \quad x_0^{\alpha} = \vec{0}, \\ & t = 0, \dots, p-1 \end{aligned} \tag{2.13}$$

exist if and only if

$$\alpha + s_{t+1} \gamma_t^{\alpha} r_{t+1}^* \neq \vec{0}, \quad t = 0, \dots, p-1. \tag{2.14}$$

In this case, (2.13) *produces the following values:*

$$\begin{aligned} \gamma_t^{\alpha} &= \left(\alpha I_q + R_{\underline{t}}^* S_{\underline{t}}\right)^{-1} \alpha \\ x_t^{\alpha} &= \left(\alpha I_q + R_{\underline{t}}^* S_{\underline{t}}\right)^{-1} R_{\underline{t}}^* y_{\underline{t}}, \quad t = 1, \dots, p, \end{aligned} \tag{2.15}$$

where $R_{\underline{t}}^*$ is $q \times t$ matrix, equal to the conjugate and transpose of the submatrix $R_{\underline{t}}$ of R.

An approximate method for the implementation of the limit formula (2.11) and its convergence properties were investigated in [11].

Proposition 2.5. (Stanimirović 1999) [11] *Consider $m \times n$ complex matrix A and two $p \times q$ complex matrices R and S, whose rows are denoted by r_t and s_t, $t = 1, \dots, p$, respectively. If the rows of the unit matrix I_p are denoted by i_t, $t = 1, \dots, p$, then the iterations*

$$\begin{aligned} &\gamma_{t+1}^{\alpha} = \gamma_t^{\alpha} - \frac{\gamma_t^{\alpha} r_{t+1}^* s_{t+1} \gamma_t^{\alpha}}{\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^*}, \quad \gamma_0^{\alpha} = I_q, \quad \alpha > 0, \\ &X_{t+1}^{\alpha} = X_t^{\alpha} + \frac{\gamma_t^{\alpha} r_{t+1}^*}{\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^*}\left(i_{t+1} - s_{t+1} X_t^{\alpha}\right), \; X_0^{\alpha} = \mathcal{O} \in \mathbb{C}^{q \times p}, \\ &t = 0, \dots, p-1 \end{aligned} \tag{2.16}$$

converge if and only if

$$\alpha + s_{t+1}\gamma_t^{\alpha} r_{t+1}^* \neq 0, \quad t = 0, \dots, p-1$$

and the limits $\lim\limits_{\alpha \to 0} X_p^{\alpha}$, $\lim\limits_{\alpha \to 0} \gamma_p^{\alpha}$ produce the following results:

$$\lim_{\alpha \to 0} X_p^{\alpha} = \lim_{\alpha \to 0} (\alpha I_q + R^* S)^{-1} R^*, \quad \lim_{\alpha \to 0} \gamma_p^{\alpha} = I_q - \lim_{\alpha \to 0} (\alpha I_q + R^* S)^{-1} R^* S. \tag{2.17}$$

(i) *In the case $p = m$, $q = n$, $R = S = A$ we get*

$$\lim_{\alpha \to 0} X_m^{\alpha} = A^{\dagger}, \quad \lim_{\alpha \to 0} \gamma_m^{\alpha} = I_n - A^{\dagger} A.$$

(ii) *If A is $n \times n$ matrix, selecting the values $p = q = n$, $R^* = A^l$, $l \geqslant \operatorname{ind}(A)$, $S = A$, we obtain*

$$\lim_{\alpha \to 0} X_n^{\alpha} = A^{\mathrm{D}}, \quad \lim_{\alpha \to 0} \gamma_n^{\alpha} = I_n - A^{\mathrm{D}} A.$$

(iii) *In the case $p > q = \operatorname{rank}(S)$, for arbitrary $R \in \mathbb{C}_q^{p \times q}$ such that $R^* S$ is invertible, we get $\lim\limits_{\alpha \to 0} X_p^{\alpha} = S_L^{-1}$.*

(iv) *Consider the case $q > p = \operatorname{rank}(S)$ and an arbitrary matrix $R \in \mathbb{C}^{p \times q}$ such that SR^* is invertible. Then $\lim\limits_{\alpha \to 0} X_p^{\alpha} = S_R^{-1}$.*

(v) *Selection $S = R \in \mathbb{C}^{p \times q}$ in (2.16) implies*

$$\lim_{\alpha \to 0} X_p^{\alpha} = R^{\dagger}, \quad \lim_{\alpha \to 0} \gamma_p^{\alpha} = I_q - R^{\dagger} R.$$

(vi) *For $A \in \mathbb{C}^{n \times n}$, in the case $n = p = q$, $R^* = A^k$, $S = I_n$, the limit value $\lim\limits_{\alpha \to 0} X_n^{\alpha}$ exists if $\operatorname{ind}(A) = k$, in which case $\lim\limits_{\alpha \to 0} X_n^{\alpha} = AA^{\mathrm{D}}$.*

(vii) *If $A \in \mathbb{C}_r^{m \times n}$, $p = q = m = n$, $R^* = \alpha^k I_n$, $S = \alpha^{-k} I_n$, $k = \operatorname{ind}(A) > 0$, then*

$$\lim_{\alpha \to 0} X_n^{\alpha} = (-1)^{k-1}(I - AA^{\mathrm{D}})A^{k-1}.$$

3. Further results on the convergence

Some further results about the convergence with respect to Proposition 3.1.

Theorem 3.1. *Let us observe $m \times n$ complex matrix A and two $p \times q$ complex matrices R and S with rows r_t and s_t, $t = 1, \dots, p$, respectively. If the rows of the unit matrix I_p are denoted by i_t, $t = 1, \dots, p$, then the iterations* (2.16) *converge if and only if*

$$\alpha + s_{t+1}\gamma_t^\alpha r_{t+1}^* \neq 0, \quad t = 0, \dots, p-1.$$

In this case, the limits $\lim\limits_{\alpha \to 0} X_p^\alpha$, $\lim\limits_{\alpha \to 0} \gamma_p^\alpha$ produce the following results:

$$\lim_{\alpha \to 0} X_p^\alpha = \lim_{\alpha \to 0} (\alpha I_q + R^* S)^{-1} R^*, \quad \lim_{\alpha \to 0} \gamma_p^\alpha = I_q - \lim_{\alpha \to 0} (\alpha I_q + R^* S)^{-1} R^* S.$$

(viii) *If A is of rank r, T is a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. If $G \in \mathbb{C}^{n \times m}$ satisfies $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$ and $\operatorname{rank}(GA) = \operatorname{rank}(G)$, then $A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ then in the case $p = m$, $q = n$, $R = G^*$, $S = A$ we get*

$$\lim_{\alpha \to 0} X_m^\alpha = A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)}, \quad \lim_{\alpha \to 0} \gamma_m^\alpha = I_n - A^{(2)}_{\mathcal{R}(G),\mathcal{N}(G)} A.$$

(ix) *If $A \in \mathbb{C}^{n \times n}$ of index $\operatorname{ind}(A) = 1$, the selected values $p = q = n$, $R = A^* A$, $S = A$ initiate*

$$\lim_{\alpha \to 0} X_n^\alpha = A^{\#\!\!\!\bigcirc}, \quad \lim_{\alpha \to 0} \gamma_n^\alpha = I_n - A^{\#\!\!\!\bigcirc} A.$$

(x) *If $A \in \mathbb{C}^{n \times n}$ of index $\operatorname{ind}(A) = k$, the selected values $p = q = n$, $R = (A^k)^* A^k$, $S = A$ initiate*

$$\lim_{\alpha \to 0} X_n^\alpha = A^{\dagger\!\!\!\bigcirc}, \quad \lim_{\alpha \to 0} \gamma_n^\alpha = I_n - A^{\dagger\!\!\!\bigcirc} A.$$

Proof. The proof can be verified using (2.17) in conjunction with (2.4), (2.5) and (2.6). $\square$

4. An improved implementation

In this paper we propose an improvement of the iterative processes (2.13), (2.15) and (2.16). According to the improvement, these iterations can converge in an arbitrary prescribed number of iterations. If b is the required number of iterations , and integers c, d are defined as $c = Quotient[b, p]$, $d = Mod[b, p]$, then the iterations (2.13), (2.15) and (2.16) terminate in $p - 1 + d$ steps, where $c = Quotient[b, p]$, $d = Mod[b, p]$.

Also, an implementation of the introduced approximate methods in the programming package *Mathematica* is developed.

Theorem 4.1. *Let given two arbitrary $p \times q$ complex matrices R and S and $p \times 1$ complex vector y. Let the rows of the matrices R and S be denoted by r_u and s_u, respectively, for each $u = 1, \dots, p$. Also, assume that the rows of the unit matrix I_p are denoted by i_t, $t = 1, \dots, p$. If b is an arbitrary prescribed number of iterations, and integers c, d are defined as $c = Quotient[b, p]$, $d = Mod[b, p]$, then the following iterative sequences:*

$$\begin{aligned}
&\gamma_{t+1}^{\alpha/c} = \gamma_t^{\alpha/c} - \frac{\gamma_t^{\alpha/c} r_{t+1}^* s_{t+1} \gamma_t^{\alpha/c}}{\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^*}, \quad \gamma_0^{\alpha/c} = I_q, \quad \alpha > 0,\\
&x_{t+1}^{\alpha/c} = x_t^{\alpha/c} + \frac{\gamma_t^{\alpha/c} r_{t+1}^*}{\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^*}\left(y_{t+1} - s_{t+1} x_t^{\alpha/c}\right), \quad x_0^{\alpha/c} = \vec{0}, \qquad (4.1)\\
&X_{t+1}^{\alpha/c} = X_t^{\alpha/c} + \frac{\gamma_t^{\alpha/c} r_{t+1}^*}{\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^*}\left(i_{t+1} - s_{t+1} X_t^{\alpha/c}\right), \; X_0^{\alpha/c} = \mathcal{O} \in \mathbb{C}^{q\times p}\\
&\qquad t = 0, \dots, p-1+d
\end{aligned}$$

exist if and only if

$$\alpha/c + s_{t+1}\gamma_t^{\alpha/c} r_{t+1}^* \neq 0, \quad t = 0, \dots, p-1. \qquad (4.2)$$

In the case when (4.2) *holds, the iterations* (4.1) *produce the following values:*

$$\begin{aligned}
\gamma_{p+d-1}^{\alpha/c} &= \gamma_b^{\alpha} = \left(\frac{\alpha}{c} I_q + R_{\underline{d}}{}^* S_{\underline{d}}\right)^{-1} \frac{\alpha}{c}\\
x_{p+d-1}^{\alpha/c} &= x_b^{\alpha} = \left(\frac{\alpha}{c} I_q + R_{\underline{d}}{}^* S_{\underline{d}}\right)^{-1} R_{\underline{d}}^* y_{\underline{d}}, \qquad (4.3)\\
X_{p+d-1}^{\alpha/c} &= X_b^{\alpha} = \left(\frac{\alpha}{c} I_q + R_{\underline{d}}{}^* S_{\underline{d}}\right)^{-1} R_{\underline{d}}^*,
\end{aligned}$$

where $R_{\underline{d}}{}^*$ is $q \times t$ matrix, equal to the conjugate and transpose of the submatrix $R_{\underline{d}}$ of R.

Proof. Utilizing a result from [11], for an arbitrary integer $N \geqslant 1$, we get the following statements for the iterative process:

$$\gamma_{pN}^{\alpha} = \gamma_p^{\alpha/N}, \quad X_{pN}^{\alpha} = X_p^{\alpha/N}, \quad x_{pN}^{\alpha} = x_p^{\alpha/N}, \quad N = 1, 2, \dots. \qquad (4.4)$$

Consequently, after the first $p-1$ iterations we obtain

$$\gamma_p^{\alpha/c} = \gamma_{pc}^{\alpha}, \quad X_p^{\alpha/c} = X_{pc}^{\alpha}, \quad x_p^{\alpha/c} = x_{pc}^{\alpha}.$$

Finally, applying another d iterations we obtain

$$\begin{aligned}
\gamma_{p+d}^{\alpha/c} &= \gamma_{pc+d}^{\alpha} = \gamma_b^{\alpha},\\
X_{p+d}^{\alpha/c} &= X_{pc+d}^{\alpha} = X_b^{\alpha},\\
x_{p+d}^{\alpha/c} &= x_{pc+d}^{\alpha} = x_b^{\alpha}.
\end{aligned}$$

This completes the proof. $\square$

Now we describe implementation of the iterative methods presented in (4.1). Input parameters in the algorithm are:

$r\_$, $s\_$: input matrices R and S;

$it\_$: a prescribed number of iterations;

$alpha\_$: a small real number representing the initial value of the parameter α.

STEP 1. Initial values of used local variables:

```
{m,n}=Dimensions[a];
in=IdentityMatrix[n]; im=IdentityMatrix[m];
g0=in; x0=ConstantArray[0, {n, m}];
```

STEP 2. Implementation of the iterative step. A major problem arising in the implementation of the limit $\lim\limits_{\alpha\to 0} X_p^\alpha$ by means of (4.1) is the increase of dimensions. Namely, according to the property (4.4), decrease of the value α to α/N, $N \geq 1$ requires usage of block matrices γ_{pN}^α, X_{pN}^α, x_{pN}^α. This fact initiates a significant increase of number of arithmetic operations during the iterations. In order to avoid this problem, we use the standard function `Mod` of the programming language *Mathematica*. Further improvement is achieved using the iterations (4.1). Detailed implementation of the iterative rule (4.1) is presented as follows.

```
c=Quotient[it,m]; d=Mod[it,m];
alpha=alpha/c;
i=1;
While[i<=p-1+d,
 j=i;
 If[i>m,j=Mod[i,m];If[j==0,j=m];
 g1=g0-(g0.Transpose[{r[[j]]}].{s[[j]]}.g0)/
       (alpha+({s[[j]]}.g0.Transpose[{r[[j]]}])[[1,1]]);
 x1=x0+g0.Transpose[{r[[j]]}].({in[[j]]}-{s[[j]]}.x0)/
                                (alpha+({s[[j]]}.g0.Transpose[{r[[j]]}])[[1,1]]);
 g0=g1; x0=x1; i=i+1
];
```

STEP 3. Generate the output: `Return[{x1,g1}];`

5. Numerical example

In this section we present a few numerical comparisons between the implementation given in [11] and the implementation introduced in this paper. Assume that R, S are $p\times q$ matrices. Let us denote by b an arbitrary prescribed number of iterations, $c = Quotient[b,p]$ and $d = Mod[b,p]$. Implementation presented in [11] terminates after $b = pc + d$ iterations. On the other hand, modification defined in (4.1) terminates after $p+d-1$ iterations. It is clear that the improved method requires $(p-1)c-1$ iterations less than the original one.

Let us choose the matrices `r=s={{1,2,3},{3,2,1}}`. Using the modified implementation with $\alpha = 0.01$, and the maximal number of steps equal to 20, we perform 301 usual iterations from [11] in only two steps

```
c = 150 d = 1
alpha= 0.0000666667
it=1
x1={{0.07142823129413669,0}, {0.1428564625882733, 0},{0.21428469388241,0}}

it=2
x1={{-0.166663, 0.3333289352552714}, {0.0833331, 0.0833331018524948},
    {0.333329, -0.1666627315502817}}

it=3
x1={{-0.1666627315502817, 0.3333289352552714},
    {0.0833331018524949, 0.0833331018524948},
    {0.3333289352552714, -0.1666627315502817}}
```

Implementation described in [11] gives the following result after 301 iterations:

```
x1={{-0.1666627422808011, 0.3333289429198914},
    {0.0833331072178662,0.083333098020105},
    {0.333328956716534, -0.166662746879682}}
```

6. Conclusion and possible further research

Starting point in our research was the iterative method for the implementation of the limit representation of the Moore-Penrose inverse $A^\dagger = \lim_{\alpha\to 0} (\alpha I + A^*A)^{-1} A^*$ and the matrix expression $I - A^\dagger A$ from [17] and [18]. Further, we used the iterative process for the implementation of the general limit formula $\lim_{\alpha\to 0}(\alpha I + R^*S)^{-1}R^*$ from [11]. In this paper we further investigate convergence of these iterations. Moreover, an improvement of the iterations from from [11] is proposed and investigated. The efficacy of the proposed method is confirmed by its ability to produce the result in a predefined number of iterative steps. Convergence properties of defined iterations and iterations from [11] are further investigated.

An efficient algorithm for the implementation of the iterative processes (2.13), (2.15) and (2.16) from [11] is proposed and described. Firstly, an useful rule for avoiding usage of increasing block matrices during the iterations is proposed. Instead of growing block matrices we propose usage of the function `Mod` on the indices of the input matrices. In addition, according to certain rules, the introduced algorithm can converge in an arbitrary prescribed number of iterations.

Also, an implementation of the introduced approximate methods in the programming package *Mathematica* is developed.

An alternative limit expression of the Drazin inverse of the form

$$A^{\mathrm{D}} = \lim_{\alpha\to 0} (\alpha I_n + A)^{-(l+1)} A^l, \quad l \geqslant k$$

was presented in [5]. One possibility for further research could be development of iterations for the implementation of this alternative limiting formula.

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Ser. Math. Inform. Vol. 33, No 4 (2018), 517–522
https://doi.org/10.22190/FUMI1804517B

MULTIDISKCYCLIC OPERATORS ON BANACH SPACES

Nareen Bamerni

Abstract. In this paper, we define and study multidiskcyclic operators and find some of their properties. Peris (2001) proved that every multihypercyclic operator is hypercyclic. We show the corresponding result for multidiskcyclic operators. In particular, we show that every multidiskcyclic operator is diskcyclic too.
Keywords. Multidiskcyclic operators; Banach space; hypercyclic operator.

1. Introduction

An operator T is called hypercyclic if there is a vector $x \in \mathcal{H}$ such that $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, such a vector x is called hypercyclic for T. The first example of hypercyclic operators in a Banach space was constructed by Rolewicz in 1969 [13]. He proved that if B is a backward shift on the Banach space $\ell^p(\mathbb{N})$ then λB is hypercyclic for any complex number λ; $|\lambda| > 1$. Motivated by Rolewicz's example, supercyclic operators and diskcyclic operators were defined. An operator T is supercyclic if there is a vector $x \in \mathcal{H}$ such that $\mathbb{C}Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where x is called supercyclic vector [9]. An operator T is called diskcyclic if there is a vector $x \in \mathcal{H}$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$ is dense in $\mathcal{H}$, such a vector x is called diskcyclic for T [15]. For more information on these concepts, one may refer to [5, 4, 2].

Recently, these operators were extended to subspaces of Banach spaces, which are called subspace-hypercyclic, subspace-supercyclic and subspace-diskcyclic. For more details on these operators, we refer the reader to [10, 1, 14, 3].

In 1992, Herrero [8] generalized the concepts of hypercyclicity and supercyclicity to multihypercyclicity and multisupercyclicity, respectively as follows:

Received September 30, 2017; accepted December 02, 2018
2010 *Mathematics Subject Classification.* Primary 47A16; Secondary 47B37

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{X})$ is called multihypercyclic (or multisupercyclic), if there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of $\mathcal{X}$ such that $\bigcup_{k=1}^{n} Orb(T, x_k)$ (or $\mathbb{C}\bigcup_{k=1}^{n} Orb(T, x_k)$, respectively) is dense in $\mathcal{X}$.

Herrero [8] posed the following conjecture:

if T is multihypercyclic (or multisupercyclic), then T is hypercyclic (or supercyclic, respectively)

Costakis [7] and Peris [12] independently proved Herrero's conjecture positively. For more information on these concepts, the reader may be refered to [8, 6, 7, 12, 11].

Now, since both multihypercyclic operators and multisupercyclic operators have been defined and studied, then it is natural to define and study multidiskcyclic operators as well. Therefore, the purpose of this section is to define multidiskcyclic operators and find some of their properties which are similar to those of multihypercyclicity and multisupercyclicity. We show that if T is multidiskcyclic, then every positive integer power of T is multidiskcyclic and T^* has at most one eigenvalue; and that one has to have a modulus greater than one. Finally, we show that every multidiskcyclic operator is diskcyclic.

2. Main results

Definition 2.1. Let $L = \{x_1, \cdots, x_m\} \subset \mathcal{X}$, $T \in \mathcal{B}(\mathcal{X})$ and $\mathbb{D}Orb(T, L) = \bigcup_{i=1}^{m} \mathbb{D}Orb(T, x_i)$. If L is minimal such that $\mathbb{D}Orb(T, L)$ is dense, then T is called a multidiskcyclic operator and L is called a diskcyclic set for T.

It is clear form the above definition, that every diskcyclic operator is multidiskcyclic.

The following two results give the common properties between multidiskcyclic operators and diskcyclic operators.

Theorem 2.1. *If T is multidiskcyclic, then T^n is multidiskcyclic for all $n \geqslant 2$.*

Proof. Let L be a diskcyclic set for T, then it is clear that

$$\bigcup_{i=1}^{m} \bigcup_{j=0}^{n-1} \mathbb{D}Orb(T^n, T^j x_i) = \bigcup_{i=1}^{m} \mathbb{D}Orb(T, x_i).$$

It follows that T^n is multidiskcyclic with a multidiskcyclic set $\{T^j x_i : 1 \leqslant i \leqslant m, 0 \leqslant j \leqslant n-1\}$. $\square$

Proposition 2.1. *If T is a multidiskcyclic operator on a Hilbert space $\mathcal{H}$, then T^* has at most one eigenvalue. If $\sigma_p(T^*) = \{\lambda\}$, then λ has a modulus greater than one.*

Proof. Since each multidiskcyclic operator is multisupercyclic, then the adjoint of a multidiskcyclic operator has at most one eigenvalue [11, Theorem 5]. Now, suppose that $\sigma_p(T^*) = \{\lambda\}$. Towards a contradiction assume that $|\lambda| \leqslant 1$.
Let $L = \{x_1, \cdots, x_m\}$ be diskcyclic set for T. Then there exists a unit vector z in which $T^*z = \lambda z$ and

$$\left\{ \bigcup_{i=1}^{m} |\langle \mu T^n x_i, z\rangle| : n \geqslant 0, \mu \in \mathbb{D},\, x_i \in L \right\} \text{ is dense in } \mathbb{R}^+ \cup \{0\}\,. \tag{2.1}$$

Since $|\langle \mu T^n x_i, z\rangle| \leqslant |\mu|\, |\lambda|^n\, \|x_i\|\|z\|$ for all $1 \leqslant i \leqslant m$, and since $|\lambda| \leqslant 1$, then

$$|\langle \mu T^n x_i, z\rangle| \leqslant \|x_i\|\|z\|,$$

that is, $\{\bigcup_{i=1}^{m} |\langle \mu T^n x_i, z\rangle| : n \geqslant 0, \mu \in \mathbb{D},\, x_i \in L\}$ is bounded above, a contradiction to the equation (2.1). □

Miller [11] proved that if T is multihypercyclic (or multisupercyclic) then there exists a vector x such that $Orb(T, x)$ (or $\mathbb{C}Orb(T, x)$, respectively) is somewhere dense. Later on, Bourdon and Feldman [6] showed that the somewhere density of orbit and dense orbit imply to everywhere density of them. It follows that every multihypercyclic (or multisupercyclic) operator is hypercyclic (or supercyclic, respectively). The next theorem shows the analogue of Miller's result for multidiskcyclicity.

Proposition 2.2. *If T is multidiskcyclic, then there exists a vector $x \in \mathcal{X}$ such that the disk orbit of x under T is somewhere dense.*

Proof. Let L be a diskcyclic set for T. Towards a contradiction, suppose that $\mathbb{D}Orb(T, x)$ is nowhere dense for all $x \in \mathcal{X}$. Then, there is $x_k \in L$ such that $\mathbb{D}Orb(T, x_k)$ is nowhere dense. It follows that, $\bigcup_{\substack{i=1 \\ i\neq k}}^{m} \mathbb{D}Orb(T, x_i)$ is dense in $\mathcal{X}$, which is contradiction to the minimality of L. Thus, there exists a vector $x \in \mathcal{X}$ such that $\mathbb{D}Orb(T, x)$ is somewhere dense. □

Since the somewhere density of disk orbit does not imply to everywhere density of it [4, Example 3.14], then we can not apply Bourdon's and Feldman's result [6] to show that every multidiskcyclic is diskcyclic. However, we follow Peris' approach [12] to show that every multidiskcyclic operator is diskcyclic. First, we need the following lemmas.

Lemma 2.1. *[12] If p is a polynomial and α is an eigenvalue of T^*, then $p(T)$ has a dense range if and only if $p(\alpha) \neq 0$.*

The following lemma can be proved by the same way of proving [12, Lemma 3].

Lemma 2.2. *If the interior closure of two disk orbits intersect each other, then they coincide.*

Theorem 2.2. *Let T be a multidiskcyclic operator, then T is diskcyclic.*

Proof. Let n be a positive integer and $L = \{x_1, \cdots, x_n\}$ be a diskcyclic set for T, then

$$\mathcal{X} = \bigcup_{i=1}^{n} \overline{\mathbb{D}Orb(T, x_i)}.$$

Let $n > 1$ (otherwise T is diskcyclic) and $x \in \mathcal{X}$ with $int(\overline{\mathbb{D}Orb(T, x)}) \neq \phi$, then there exists $x_h \in L$ such that $int(\overline{\mathbb{D}Orb(T, x)}) \cap int(\overline{\mathbb{D}Orb(T, x_h)}) \neq \phi$. It follows by Lemma (2.2) that

$$int(\overline{\mathbb{D}Orb(T, x)}) = int(\overline{\mathbb{D}Orb(T, x_h)}).$$

Claim. $Orb(T, x) \subset int(\overline{\mathbb{D}Orb(T, x)})$.
Proof of Claim:
Towards a contradiction, suppose that there exists $T^m x \in Orb(T, x)$ such that $T^m x \notin int(\overline{\mathbb{D}Orb(T, x)})$. It follows that $T^m x \notin int(\overline{\mathbb{D}Orb(T, x_h)})$, thus there exists $1 \leqslant k \leqslant n; k \neq h$ such that $T^m x \in \overline{\mathbb{D}Orb(T, x_k)}$. Since $\overline{\mathbb{D}Orb(T, x_k)}$ is T-invariant, then

$$int(\overline{\mathbb{D}\{T^{m+q}x : q \geqslant 0\}}) \subset int(\overline{\mathbb{D}Orb(T, x_k)}). \tag{2.2}$$

Now, we get

$$int(\overline{\mathbb{D}Orb(T, x_h)}) = int(\overline{\mathbb{D}Orb(T, x)}) = int(\overline{\mathbb{D}\{T^{m+q}x : q \geqslant 0\}}) \subset int(\overline{\mathbb{D}Orb(T, x_k)}).$$

By Lemma 2.2, it follows that

$$int(\overline{\mathbb{D}Orb(T, x_h)}) = int(\overline{\mathbb{D}Orb(T, x_k)}).$$

which is a contradiction. Thus the claim is proved.

To prove the theorem, by applying Proposition 2.1 and Lemma 2.1, we have

$$\mathcal{X} = \overline{P(T)(\mathcal{X})} = \bigcup_{i=1}^{n} \overline{P(T)(\mathbb{D}Orb(T, x_i))}.$$

for every polynomial P with $P(\alpha) \neq 0$ (if $\sigma_p(T^*) = \alpha$). Now, since

$$\overline{P(T)(\mathbb{D}Orb(T, x_i))} = \overline{\mathbb{D}Orb(T, P(T)x_i)},$$

and since L is minimal, then

$$int\left(\overline{\mathbb{D}Orb(T, P(T)x_i)}\right) \neq \phi, \text{ for all } x_i \in L.$$

Thus, for each $i \in \{1, \cdots, n\}$ there exists $j \in \{1, \cdots, n\}$ such that

$$Orb(T, P(T)x_i) \subset int\left(\overline{\mathbb{D}Orb(T, P(T)x_i)}\right) = int\left(\overline{\mathbb{D}Orb(T, x_j)}\right)$$

Let

$$B = \bigcup_{P(\lambda)\neq 0} Orb(T, P(T)x_1) \subset \bigcup_{i=1}^{n} int\left(\overline{\mathbb{D}Orb(T, x_i)}\right)$$

Moreover, $B = \text{span}\,(Orb(T, x_1)) \setminus \overline{(T - \lambda I)\,(\mathcal{X})}$. It follows that B is connected and hence $B \subset int\left(\overline{\mathbb{D}Orb(T, x_1)}\right)$. By [12, Lemma 2], we have B is dense, thus

$$\mathcal{X} = \overline{\mathbb{D}Orb(T, x_1)},$$

which means that, T is diskcyclic. □

Corollary 2.1. *An operator is diskcyclic if and only if it is multidiskcyclic.*

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Ser. Math. Inform. Vol. 33, No 4 (2018), 523–530
https://doi.org/10.22190/FUMI1804523O

SOME GEOMETRIC PROPERTIES OF WEIGHTED LEBESGUE SPACES $L^p_w(G)$

Oğuz Oğur

Abstract. In this paper, we deal with some geometric properties of weighted Lebesgue spaces $L^p_w(G)$, where G is locally compact Abelian group and w is a Beurling weight. Also, we study the uniformly convexity of the space $L^p(G)\cap L^r(G)$ with $1 < p, r < \infty$.
Keywords: weighted Lebesgue space, geometric properties

1. Introduction

Throughout this paper, G is a locally compact Abelian group and dx is a Haar measure on G. If $1 \leq p < \infty$, then $L^p(G)$ will denote the space of functions f such that $|f|^p$ is integrable [2]. A Beurling weight on G is a measurable, locally bounded function w satisfying for each $x, y \in G$ the following two properties: $w(x) \geq 1$ and $w(x + y) \leq w(x).w(y)$. By the definition of $w$it is deduced easily that wdx is a positive measure on G. We denote by $L^p_w(G)$, $1 \leq p < \infty$, the Banach spaces of equivalence classes of real valued measurable functions on G with the system of following norm

$$\|f\|_{p,w} = \left(\int_G |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}} < \infty.$$

The conjugate space of $L^p_w(G)$ is the $L^{p'}_{w'}(G)$, where $w' = w^{1-p'}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. It can be easily seen that $L^p_w(G)$ is a reflexive Banach space [3], [4], [8], [9].

A Banach space X is said to be *strictly convex* if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$, then $|(1-\lambda)x + \lambda y| < 1$ for all $\lambda \in (0, 1)$.

A Banach space X is said to be *uniformly convex if for all* $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that the conditions

$$\|x\| \leq 1,\ \ \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

Received March 03, 2018; accepted July 07, 2018
2010 *Mathematics Subject Classification.* Primary 43A15; Secondary 46E30, 46B20

for all $x, y \in X$.

The number

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1, \;\; \|x-y\| \geq \varepsilon \right\}$$

is called the *modulus of convexity*. Note that if $\varepsilon_1 < \varepsilon_2$, then $\delta(\varepsilon_1) < \delta(\varepsilon_2)$ and $\delta(0) = 0$ since $x = y$ if $\varepsilon = 0$ [1],[7].

We will need some auxiliary lemmas to prove that the spaces $L^p_w(G)$ are uniformly convex whenever $1 < p < \infty$.

Let us first remind that the Minkowski inequality for the space $L^p_w(G)$, $p \geq 1$; If $f, g \in L^p_w(G)$, then

$$\left(\int_G |f(x) + g(x)|^p \, w(x) dx \right)^{\frac{1}{p}} \leq \left(\int_G |f(x)|^p \, w(x) dx \right)^{\frac{1}{p}} + \left(\int_G |g(x)|^p \, w(x) dx \right)^{\frac{1}{p}}.$$

Lemma 1.1. *Let $0 < p < 1$, we have $(a+b)^p \leq a^p + b^p$ for all $a \geq 0$ and $b \geq 0$.*

Lemma 1.2. *If $p \geq 1$, then $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for all positive numbers a and b.*

2. Main Results

Theorem 2.1. *The space $L^p_w(G)$ is convex whenever $0 < p < \infty$.*

Proof. Let $f, g \in L^p_w(G)$. We need to show that $tf + (1-t)g \in L^p_w(G)$ for $0 \leq t \leq 1$. Let us consider this in two cases; $p \geq 1$ and $0 < p < 1$.

Case $p \geq 1$. By lemma 2 and the Minkowski inequality, we have

$$
\begin{aligned}
\int\limits_G |tf(x)+(1-t)g(x)|^p\, w(x)dx &= \int\limits_G \left|(tf(x)+(1-t)g(x))\,(w(x))^{\frac{1}{p}}\right|^p dx \\
&= \left[\left(\int\limits_G \left|(tf(x)+(1-t)g(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right)^{\frac{1}{p}}\right]^p \\
&\leq \left[\left(\int\limits_G \left|(tf(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right)^{\frac{1}{p}}\right. \\
&\quad \left. + \left(\int\limits_G \left|((1-t)g(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right)^{\frac{1}{p}}\right]^p \\
&\leq 2^{p-1}\left[\int\limits_G \left|(tf(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right. \\
&\quad \left. + \int\limits_G \left|((1-t)g(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right] \\
&= 2^{p-1}\left[|t|^p \int\limits_G |f(x)|^p\, w(x)dx\right. \\
&\quad \left. + |1-t|^p \int\limits_G |g(x)|^p\, w(x)dx\right] \\
&= 2^{p-1}\left(|t|^p \|f\|_{p,w}^p + |1-t|^p \|g\|_{p,w}^p\right) \\
&< \infty
\end{aligned}
$$

which shows that $tf+(1-t)g \in L_w^p(G)$ for $p \geq 1$.

Case $0 < p < 1$. Let $f, g \in L_w^p(G)$ and $t \in [0,1]$. By lemma 1, we get

$$
\begin{aligned}
\int\limits_G |tf(x)+(1-t)g(x)|^p\, w(x)dx &= \int\limits_G \left|(tf(x)+(1-t)g(x))\,(w(x))^{\frac{1}{p}}\right|^p dx \\
&\leq \int\limits_G \left|(tf(x))\,(w(x))^{\frac{1}{p}}\right|^p dx + \int\limits_G \left|((1-t)g(x))\,(w(x))^{\frac{1}{p}}\right|^p dx \\
&= |t|^p \|f\|_{p,w}^p + |1-t|^p \|g\|_{p,w}^p \\
&< \infty.
\end{aligned}
$$

This completes the proof. □

Theorem 2.2. *The space $L^p_w(G)$, $1<p<\infty$, is strictly convex.*

Proof. Let $f,g\in L^p_w(G)$ with $f\neq g$, $\|f\|_{p,w}=1, \|g\|_{p,w}=1$ and $0<t<1$. Then, by strictly convexity of $L^p(G)$ we have

$$\begin{aligned}\|(1-t)f+tg\|_{p,w} &= \left(\int_G \left|((1-t)f(x)+tg(x))\,(w(x))^{\frac{1}{p}}\right|^p dx\right)^{\frac{1}{p}} \\ &= \left\|((1-t)f+tg)\,w^{\frac{1}{p}}\right\|_p \\ &< 1.\end{aligned}$$

□

Lemma 2.1. *Let $2\leq p<\infty$ and $a,b\in\mathbb{R}$, then we have*

$$|a+b|^p+|a-b|^p\leq 2^{p-1}\left(|a|^p+|b|^p\right).$$

[5].

Lemma 2.2. *Let $2\leq p<\infty$. For any $f,g\in L_p$, we have*

$$\|f+g\|_p^p+\|f-g\|_p^p\leq 2^{p-1}\left(\|f\|_p^p+\|g\|_p^p\right)$$

[6].

We will also need the following inequality.

Lemma 2.3. *For $2\leq p<\infty$ and any $f,g\in L^p_w(G)$, we have*

$$\|f+g\|_{p,w}^p+\|f-g\|_{p,w}^p\leq 2^{p-1}\left(\|f\|_{p,w}^p+\|g\|_{p,w}^p\right).$$

Proof. Let $f,g\in L^p_w(G)$. Then $fw^{\frac{1}{p}}, gw^{\frac{1}{p}}\in L_p$. By lemma 4, we get

$$\begin{aligned}\|f+g\|_{p,w}^p+\|f-g\|_{p,w}^p &= \left\|fw^{\frac{1}{p}}+gw^{\frac{1}{p}}\right\|_p^p+\left\|fw^{\frac{1}{p}}-gw^{\frac{1}{p}}\right\|_p^p \\ &\leq 2^{p-1}\left(\left\|fw^{\frac{1}{p}}\right\|_p^p+\left\|gw^{\frac{1}{p}}\right\|_p^p\right) \\ &= 2^{p-1}\left(\|f\|_{p,w}^p+\|g\|_{p,w}^p\right).\end{aligned}$$

□

Theorem 2.3. *$L^p_w(G)$ is uniformly convex for $2\leq p<\infty$.*

Proof. Let $f, g \in L_w^p(G)$ with $\|f\|_{p,w} \le 1$, $\|g\|_{p,w} \le 1$ and $\|f-g\|_{p,w} \ge \varepsilon$. Then, we have

$$\|f+g\|_{p,w}^p \le 2^{p-1}\left(\|f\|_{p,w}^p + \|g\|_{p,w}^p\right) - \|f-g\|_{p,w}^p$$

which implies that

$$\begin{aligned} \|f+g\|_{p,w}^p &\le 2^{p-1}.2 - \varepsilon^p \\ &= 2^p\left(1-\left(\frac{\varepsilon}{2}\right)^p\right). \end{aligned}$$

Therefore, we get

$$\left\|\frac{f+g}{2}\right\|_{p,w}^p \le 1 - \left(\frac{\varepsilon}{2}\right)^p.$$

That is, $\delta(\varepsilon) = 1 - \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}$ and this is known to be exact. □

Lemma 2.4. *(The Minkowski inequality for $p \in (0,1)$) Let $0 < p < 1$ and let f and g be positive functions in $L_p(G)$, then $f+g \in L_p(G)$ and*

$$\|f+g\|_p \ge \|f\|_p + \|g\|_p.$$

Lemma 2.5. *If $1 < p < 2$ and $q = \frac{p}{p-1}$, then*

$$|a+b|^q + |a-b|^q \le 2\left(|a|^p + |b|^p\right)^{q-1}$$

for all real numbers a and b [6].

Lemma 2.6. *Let $1 < p \le 2$ and $q = \frac{p}{p-1}$. For any f, $g \in L^p(G)$, we have*

$$\|f+g\|_p^q + \|f-g\|_p^q \le 2\left(\|f\|_p^p + \|g\|_p^p\right)^{q-1}.$$

Theorem 2.4. *Let $1 < p \le 2$ and let $q = \frac{p}{p-1}$. For any f, $g \in L_w^p(G)$, we have*

$$\|f+g\|_{p,w}^q + \|f-g\|_{p,w}^q \le 2\left(\|f\|_{p,w}^p + \|g\|_{p,w}^p\right)^{q-1}.$$

Proof. First notice that

$$\begin{aligned} \|f\|_{p,w}^q &= \left(\left(\int\limits_G |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}}\right)^q \\ &= \left(\int\limits_G |f(x)|^p\, w(x)dx\right)^{\frac{1}{p-1}} \\ &= \left(\int\limits_G |f(x)|^{q(p-1)}\, w(x)dx\right)^{\frac{1}{p-1}} \\ &= \||f|^q\|_{p-1,w}. \end{aligned}$$

Let $f, g \in L^p_w(G)$. By the Minkowski inequality for $0 < r < 1$, we have

$$\left(\int_G |F(x)+G(x)|^r\,dx\right)^{\frac{1}{r}} \geq \left(\int_G |F(x)|^r\,dx\right)^{\frac{1}{r}} + \left(\int_G |G(x)|^r\,dx\right)^{\frac{1}{r}}. \tag{1}$$

Since $1 < p < 2$, we have $0 < \frac{p}{q} < 1$. Let us define $F(x) = \left|(f(x)+g(x))\,w(x)^{\frac{1}{p}}\right|^q$ and $G(x) = \left|(f(x)-g(x))\,w(x)^{\frac{1}{p}}\right|^q$. By lemma 7, we get

$$\begin{aligned}
&\left(\int_G \left|(f(x)+g(x))\,w(x)^{\frac{1}{p}}\right|^p dx\right)^{\frac{q}{p}} + \left(\int_G \left|(f(x)-g(x))\,w(x)^{\frac{1}{p}}\right|^p dx\right)^{\frac{q}{p}} \\
\leq\ & \left[\int_G \left|\left|(f(x)+g(x))\,w(x)^{\frac{1}{p}}\right|^q + \left|(f(x)-g(x))\,w(x)^{\frac{1}{p}}\right|^q\right|^{\frac{p}{q}} dx\right]^{\frac{q}{p}} \\
=\ & \left[\int_G \left|\left|f(x)w(x)^{\frac{1}{p}} + g(x)w(x)^{\frac{1}{p}}\right|^q + \left|f(x)w(x)^{\frac{1}{p}} - g(x)w(x)^{\frac{1}{p}}\right|^q\right|^{\frac{p}{q}} dx\right]^{\frac{q}{p}} \\
\leq\ & \left[\int_G \left(2\left(\left|f(x)w(x)^{\frac{1}{p}}\right|^p + \left|g(x)w(x)^{\frac{1}{p}}\right|^p\right)^{q-1}\right)^{\frac{p}{q}} dx\right]^{\frac{q}{p}} \\
=\ & 2\left[\int_G \left(\left|f(x)w(x)^{\frac{1}{p}}\right|^p + \left|g(x)w(x)^{\frac{1}{p}}\right|^p\right) dx\right]^{\frac{q}{p}} \\
=\ & 2\left[\int_G \left(|f(x)|^p\,w(x) + |g(x)|^p\,w(x)\right) dx\right]^{\frac{q}{p}}
\end{aligned}$$

Thus, we obtain

$$\|f+g\|^q_{p,w} + \|f-g\|^q_{p,w} \leq 2\left(\|f\|^p_{p,w} + \|g\|^p_{p,w}\right)^{q-1}.$$

□

Theorem 2.5. *The space $L^p_w(G)$ is uniformly convex for $1 < p < 2$.*

Proof. Let $f, g \in L^p_w(G)$, $1 < p < 2$, with $\|f\|_{p,w} \leq 1$, $\|g\|_{p,w} \leq 1$ and $\|f-g\|_{p,w} \geq \varepsilon$. Then, by the theorem 4, we have

$$\begin{aligned}
\|f+g\|^q_{p,w} &\leq 2\left(\|f\|^p_{p,w} + \|g\|^p_{p,w}\right)^{q-1} - \|f-g\|^q_{p,w} \\
&\leq 2.2^{q-1} - \varepsilon^q \\
&= 2^q\left(1 - \left(\frac{\varepsilon}{2}\right)^q\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned} \left\| \frac{f+g}{2} \right\|_{p,w} &\leq \left(1-\left(\frac{\varepsilon}{2}\right)^q\right)^{\frac{1}{q}} \\ &\leq 1-\delta \end{aligned}$$

where $\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^q\right)^{\frac{1}{q}}$. $\square$

Let us define $B_{p,r}=L^p(G)\cap L^r(G)$ with $1<p,r<\infty$. It is known that $B_{p,r}$ is a normed space with the norm

$$\|f\|_{p,r}=\max\left\{\|f\|_p\,,\|f\|_r\right\}$$

[7].

Theorem 2.6. *The space $B_{p,r}$ is uniformly convex space for $1<p,r<\infty$.*

Proof. Let $f,g\in B_{p,r}$ with $\|f\|_{p,r}\leq 1$, $\|g\|_{p,r}\leq 1$ and $\|f-g\|_{p,r}\geq\varepsilon$. By definition of the space $B_{p,r}$, we have $f,g\in L^p(G)$ and $f,g\in L^r(G)$. Assume that

$$\|f+g\|_{p,r}=\max\left\{\|f+g\|_p\,,\|f+g\|_r\right\}=\|f+g\|_p\,.$$

By assumption, we have $\|f+g\|_r\leq\|f+g\|_p$.

Let $1<p,r<2$. By lemma 8, we have

$$\|f+g\|_p^q\leq 2\left(\|f\|_p^p+\|g\|_p^p\right)^{q-1}-\|f-g\|_p^q$$

where $q=\frac{p}{p-1}$. Then, we get

$$\begin{aligned} \|f+g\|_{p,r}^q &= \|f+g\|_p^q\leq 2\left(\|f\|_p^p+\|g\|_p^p\right)^{q-1}-\|f-g\|_p^q \\ &\leq 2.2^{q-1}-\varepsilon^q \\ &= 2^q\left(1-\left(\frac{\varepsilon}{2}\right)^q\right) \end{aligned}$$

which gives $\left\|\frac{f+g}{2}\right\|_{p,r}\leq\left(1-\left(\frac{\varepsilon}{2}\right)^q\right)^{\frac{1}{q}}$. By choosing

$\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^q\right)^{\frac{1}{q}}$, the proof is completed for $1<p,r<2$.

If $2\leq p,r<\infty$, then we have, by lemma 4,

$$\begin{aligned} \|f+g\|_{p,r}^p &= \|f+g\|_p^p\leq 2^{p-1}\left(\|f\|_p^p+\|g\|_p^p\right)-\|f-g\|_p^p \\ &\leq 2^p\left(1-\left(\frac{\varepsilon}{2}\right)^p\right) \end{aligned}$$

and we get $\left\|\frac{f+g}{2}\right\|_{p,r}\leq\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}$. If we choose $\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}$, the proof is completed. $\square$

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Ser. Math. Inform. Vol. 33, No 4 (2018), 531–538
https://doi.org/10.22190/FUMI1804531N

SOME CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH q-RUSCHEWEYH DIFFERENTIAL OPERATOR

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Abstract. It is known that the q-analysis (q-calculus) has many applications in mathematics and physics. The notion of the q-derivative D_q of a function f, analytic in the open unit disc, is defined as $D_q f(z) = \frac{f(qz)-f(z)}{(q-1)z}$, $q \in (0,1)$, $(z \neq 0)$ and $D_q f(0) = f'(0)$. Using a q-analogue of the well-known Ruscheweyh differential operator D_q^n of order n, we introduce certain classes $ST_q(n)$ for $n = 0,1,2,...$, and investigate a number of interesting properties such as inclusion and coefficient results. The ideas and techniques in this paper may stimulate further research in this field.

Keywords: Analytic, Starlike functions, q-derivative, Ruscheweyh operator, Subordination

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1.1}$$

The class $S \subset A$ consists of univalent functions. A function $f \in A$ is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ in E if it satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in E).$$

We denote this class by $S^*(\alpha)$. For $\alpha = 0$, we have $S^*(0) = S^*$, is the well-known class of starlike functions. The class $C(\alpha)$, $(0 \leq \alpha < 1)$ consists of convex functions

Received March 21, 2018; accepted: July 07, 2018
2010 *Mathematics Subject Classification.* Primary: 30C45, 30C10; Secondary: 47B38

of order α and can be defined by the relation $f \in C(\alpha)$, if and only if, $zf' \in S^*(\alpha)$.

Let $f_1,\ f_2 \in A$. If there exists a Schwartz function $\phi(z)$ which is analytic in E with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f_1(z) = f_2(\phi(z))$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and write $f_1(z) \prec f_2(z)$, where $\prec$ denote subordination symbol.

For $f \in A$ and given by (1.1), $g : g(z) = z + \sum_{m=2}^{\infty} b_m z^m$, convolution $*$ (Hadamard product) of f and g is defined by

$$(f * g)(z) = \sum_{m=2}^{\infty} a_m b_m z^m.$$

Recently, the use of q-calculus has attracted the attention of many researchers in the field of geometric function theory. Ismail et al. [5] generalized the class S^* with the concept of q-derivative and called this class S_q^* of q-starlike functions. For recent developments, see [10, 11, 12, 13, 14, 17] and the references therein.

We first give some basic definitions and the concept of q-calculus, which we shall use in this paper. For more details, see [3, 8].

A set $B \subset \mathbb{C}$ is called q-geometric if, for $q \in (0,1)$, $qz \in B$, it contains all the sequences $\{zq^m\}_0^\infty$. Jackson [6, 7] defined q-derivative and q-integral of f on the set B as follows:

$$\partial_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \quad (z \neq 0, z \in (0,1)), \tag{1.2}$$

and

$$\int_0^z f(t)\partial_q t = z(1-q) \sum_{m=0}^{\infty} q^m f(zq^m), \quad q \in (0,1),$$

provided that the series converges.

It can easily be seen that, for $m = 1, 2, 3, ...$, and $z \in E$,

$$\partial_q \left\{ \sum_{m=1}^{\infty} a_m z^m \right\} = \sum_{m=1}^{\infty} [m,q] a_m z^{m-1}, \tag{1.3}$$

where

$$[m,q] = 1 + \sum_{i=1}^{m-1} q^i = \frac{1-q^m}{1-q}, \quad [0,q] = 0. \tag{1.4}$$

For any non-negative integer m, the q-number shift factorial is defined by

$$[m,q]! = \begin{cases} 1, & m = 0 \\ [1,q][2,q][3,q]...[m,q], & m = 1,2,3,... \end{cases}$$

Also, the q-generalized Pochhamer symbol for $x > 0$ is given as

$$[m,q]_m = \begin{cases} 1, & m = 0 \\ [x,q][x+1,q]...[x+m\text{-}1,q], & m = 1,2,3,... \end{cases}$$

Throughout this paper, we shall assume $z \in E$, and $q \in (0,1)$, unless stated otherwise.

Using the q-derivative, we define certain new classes of analytic functions given as below.

Definition 1.1. Let $f \in A$. Then f is said to belong to the class ST_q, if

$$\left| \frac{z}{f(z)}(\partial_q f)(z) - \frac{q}{1-q^2} \right| \leq \frac{q}{1-q^2},$$

where $\partial_q f(z)$ is defined by (1.2) on q-geometric set B.

Remark 1.1. We note that, as $q \to 1^-$, the disc $|w(z) - \frac{q}{1-q^2}| \leq \frac{q}{1-q^2}$ becomes the right half plane $\mathrm{Re}\{w(z)\} > \alpha$, $\alpha \in (\frac{1}{2}, 1)$ and the class ST_q reduces to $S^*(\frac{1}{2})$.

Following the argument similar to the one used in [20], it is easily seen that $f \in ST_q$, if and only if,

$$\frac{z\partial_q f(z)}{f(z)} \prec \frac{1}{1-qz}. \tag{1.5}$$

From (1.5) it can be seen that the linear transformation $\frac{1}{1-qz}$ maps $|z| = r$ onto the circle with center $C(r) = \frac{qr^2}{1-q^2r^2}$ and the radius $\sigma(r) = \frac{qr}{1-q^2r^2}$, and we can write

$$\frac{1-qr+qr^2}{(1-qr)(1+qr)} \leq \left\{ \mathrm{Re}\, \frac{z\partial_q f(z)}{f(z)} \right\} \leq \frac{1+qr+qr^2}{(1-qr)(1+qr)}. \tag{1.6}$$

Now, with $\partial_q(\log f(z)) = \frac{\partial_q f(z)}{f(z)}$, $\mathrm{Re}\, \frac{\partial_q f(z)}{f(z)} = r\frac{\partial_q \log|f(z)|}{dr}$ and some computation, we have from (1.6)

$$\frac{1}{r} + \frac{q}{1+qr} \leq \frac{\partial_q}{dr} \log|f(z)| \leq \frac{1}{r} + \frac{q}{1-qr}. \tag{1.7}$$

Taking the q-integral on both sides of (1.7) and simplifying, we get

$$\frac{1}{(1+qr)^{qq_1}} \leq \left| \frac{f(z)}{z} \right| \leq \frac{1}{(1-qr)^{qq_1}}, \quad q_1 = \frac{1-q}{\log q^{-1}}. \tag{1.8}$$

Since $\lim_{q\to 1^-} \frac{1-q}{\log q^{-1}} = 1$, (1.8) gives us a distortion result for $f \in S^*(\frac{1}{2})$ as

$$\frac{r}{(1+r)} \le |f(z)| \le \frac{r}{(1-r)}, \quad \text{see [4]}$$

For $n \in \mathbb{N}_\circ = \{0, 1, 2, 3, ...\}$, the Ruscheweyh derivative D^n of order n, is defined as

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad f \in A.$$

We now proceed to discuss the q-analogue of the Ruscheweyh derivative.

Let the function $F_{q,n+1}$ be defined as

$$F_{q,n+1}(z) = z + \sum_{m=2}^{\infty} \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} z^m, \tag{1.9}$$

where the series converges absolutely in E.

Using (1.9), the q-Ruscheweyh differential operator of order n, $D_q^n : A \to A$ is defined for $f(z)$ given by (1.1) as

$$\begin{aligned} D_q^n f(z) &= F_{q,n+1}(z) * f(z) \\ &= z + \sum_{m=2}^{\infty} \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} a^m z^m, \quad \text{see [9]}. \end{aligned} \tag{1.10}$$

We note that

$$D_q^0 f(z) = f(z) \quad \text{and} \quad D_q' f(z) = z\partial_q f(z).$$

Also (1.10) can be written as

$$D_q^n f(z) = \frac{z\partial_q^n (z^{n-1} f(z))}{[n,q]!}, \quad n \in \mathbb{N}.$$

As $q \to 1^{-1}$, $\lim_{q\to 1^-} F_{q,n+1}(z) = \frac{z}{(1-z)^{n+1}}$, and $\lim_{q\to 1^-} D_q^n f(z) = D^n f(z)$, that is, the q-Ruscheweyh derivative reduces to the Ruscheweyh derivative as $q \to 1^-$. See [18].
The following identity can easily be derived from (1.10).

$$z\partial_q(D_q^n f(z)) = \left(1 + \frac{[n,q]}{q^n}\right) D_q^{n+1} f(z) - \frac{[n,q]}{q^n} D_q^n f(z). \tag{1.11}$$

When $q \to 1^-$, (1.11) reduces to the well-known identity for the Ruscheweyh derivative as

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z).$$

Using the q-operator D_q^n, we define the following.

Definition 1.2. Let $f \in A$ and let the operator $D_q^n : A \to A$ be defined by (1.10). Then $f \in ST_q(n)$, if and only if, $D_q^n f \in ST_q$ in E.

In other words $\frac{z\partial_q(D_q^n f(z))}{D_q^n f(z)} \prec \frac{1}{1-qz}$ implies $f \in ST_q(n)$. We note that, if

$$p(z) \prec \frac{1}{1-qz}, \quad \text{then} \quad \operatorname{Re} p(z) > \frac{1}{1+q}, \quad z \in E.$$

2. Main Results

Theorem 2.1. *For $n \in \mathbb{N}_\circ$, $ST_q(n+1) \subset ST_q(n)$.*

Proof. Let $f \in ST_q(n+1)$. Set

$$\frac{z\partial_q(D_q^n f(z))}{D_q^n f(z)} = p(z). \tag{2.1}$$

We note that $p(z)$ is analytic in E and $p(0) = 1$. We shall show that $p(z) \prec \frac{1}{1-qz}$.

The q-logarithmic differentiation of (2.1) and the use of identity (1.11) yields

$$\frac{z\partial_q(D_q^{n+1} f(z))}{D_q^{n+1} f(z)} = p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}, \quad N_q = \frac{[n,q]}{q^n}. \tag{2.2}$$

Let

$$p(z) = \frac{1}{1 - q\phi(z)}. \tag{2.3}$$

$\phi(z)$ is analytic in E and $\phi(0) = 0$. We shall show that $|\phi(z)| < 1$, for all $z \in E$.

Suppose, on the contrary, that there exists a $z_0 \in E$ such that $|\phi(z_0)| = 1$. Since $f \in ST_q(n+1)$, it follows from (2.2) that

$$\operatorname{Re}\left\{p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}\right\} > \frac{1}{1+q}, \quad N_q = \frac{[n,q]}{q^n}.$$

Using (2.3), we have

$$\operatorname{Re}\left[p(z) + \frac{z\partial_q p(z)}{p(z) + N_q}\right] = \operatorname{Re}\left[\frac{1}{1-q\phi(z_0)} + \frac{qz_0\partial_q\phi(z_0)}{(1-q\phi(z_0))[(1+N_q) - qN_q\phi(z_0)]}\right]. \tag{2.4}$$

Let $\phi(z_0) = e^{i\theta}$. Then

$$\operatorname{Re}\frac{1}{1-q\phi(z_0)} = \frac{1-q\cos\theta}{1-2q\cos\theta+q^2}. \tag{2.5}$$

Also

$$z_0\partial_q\phi(z_0) = k\phi(z_0), \quad k \geq 1, \tag{2.6}$$

by using q-Jacks's Lemma given in [1].
Using (2.5), (2.6), $\phi(z_0) = e^{i\theta}$ in (2.4), we have

$$\operatorname{Re}\left\{\frac{z\partial_q(D_q^{n+1}f(z_0))}{D_q^{n+1}f(z_0)}\right\} = \operatorname{Re}\left\{\frac{1}{1-qe^{i\theta}} + \frac{qke^{i\theta}}{(1-qe^{i\theta})(N_q+1-qN_qe^{i\theta})}\right\}. \tag{2.7}$$

In (2.7), we take $\theta = \pi$, and this gives us

$$\operatorname{Re}\left\{\frac{z\partial_q(D_q^{n+1}f(z))}{D_q^{n+1}f(z)} - \frac{1}{1+q}\right\} < 0, \quad z \in E,$$

which is a contradiction. Thus, $|\phi(z)| < 1$ for all $z \in E$ and this proves $p(z) \prec \frac{1}{1-qz}$. Consequently, $f \in ST_q(n)$ in E. □

Using the identity (1.11) and the definition, the proof of the following result is straightforward.

Theorem 2.2. *Let $f \in ST_q(n)$ and let $I_nf : A \to A$ be defined as*

$$I_nf(z) = \frac{[n+1,q]}{q^nz^n}\int_0^z t^{n-1}f(t)\mathrm{d}_qt, \quad n \in \mathbb{N}_\circ.$$

Then $I_nf(z) \in ST_q(n+1)$.

This operator was introduced by Bernardi [2] for $q \to 1^-$. For $n = 1$, $I_1f(z)$ is the q-analogue of the Libera integral operator, see [15, 16].

In [19], it has been proved that $\cap_{0<q<1}S_q^*(\alpha) = S^*(\alpha)$, $0 \leq \alpha < 1$. From this we can easily deduce that
(i). $\cap_{0<q<1}ST_q = S^*(\frac{1}{2})$.
(ii). $f \in [\cap_{0<q<1}ST_q(n)]$ implies $D^nf \in S^*(\frac{1}{2})$.

We have the following.

Theorem 2.3. $\cap_{n=0}^{\infty}ST_q(n) = \{id\}$ *where id is the identity function.*

Proof. Let $f(z) = z$. Then it follows trivially that $z \in ST_q(n)$, for $n \in \mathbb{N}_\circ$.
On the contrary, assume $f \in \cap_{n=0}^{\infty}ST_q(n)$ with $f(z)$ given by (1.1).
From (1.5) and (1.10), we deduce that $f(z) = z$. □

Theorem 2.4. *Let $f \in ST_q(n)$ and be given by* (1.1). *Then*

$$a_m = O(1)\frac{([n,q]!)([m-1,q]!)}{[m,q]([m+n-1,q]!)}m^{(qq_1+\frac{1}{2})}, \quad q_1 = \frac{1-q}{\log q^{-1}},$$

where $O(1)$ is a constant depending on q.

Proof. Let

$$D_q^n f(z) = z + \sum_{m=2}^{\infty} A_m(n) z^m.$$

Then, from (1.10), we have

$$A_m(n) = \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} a_m.$$

Since $f \in ST_q(n)$, $\quad D_q^n f \in ST_q$, and we can write

$$z\partial_q(D_q^n f(z)) = (D_q^n f(z))(p(z)), \quad p(z) \prec \frac{1}{1-qz}.$$

The Cauchy Theorem, (1.8) and the Schwartz inequality gives us

$$[m,q]|A_m(n)| \leq c_1(q) \frac{1}{(1-r)^{qq_1+\frac{1}{2}}}, \quad q_1 = \frac{1-q}{\log q^{-1}},$$

where $c_1(q)$ is a constant. Taking $r = 1 - \frac{1}{m}$, $(m \to \infty)$, we obtain the desired result. $\square$

As a special case for $n = 0$, $D_q^0 f \in ST_q$ and $a_m = O(1) \cdot m^{(qq_1 - \frac{1}{2})}$, $m \to \infty$. We observe here that, $\lim_{q \to 1^-} ST_q = S^*(\frac{1}{2})$ and $f(z) \prec \frac{z}{1-z}$. Using the Schwartz inequality and subordination, we get

$$a_m = O(1) \cdot \frac{n!(m-1)!}{(m+n-1)!}.$$

Acknowledgements

The author would like to thank the Rector of COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environment.

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Ser. Math. Inform. Vol. 33, No 4 (2018), 539–546
https://doi.org/10.22190/FUMI1804539P

ON A KÄHLER MANIFOLD EQUIPPED WITH LIFT OF QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. The aim of the present paper is to study Kähler manifolds equipped with the lift of a quarter-symmetric non-metric connection. In this paper, a condition on the manifold for being a Kähler manifold with respect to the lift of the quarter-symmetric non-metric connection is obtained. It is further shown that the Nijenhuis tensor with respect to the lift of the quarter-symmetric non-metric connection vanishes. Also, a necessary and sufficient condition for a contravariant almost analytic vector field in a Kähler manifold equipped with the lift of a quarter-symmetric non-metric connection has been found.
Keywords Kähler manifold; non-metric connection; differentiable manifolds.

1. Introduction

In 1975, a linear connection was introduced by S. Golab [5] called quarter-symmetric connection.

A linear connection $\overline{\nabla}$ is said to be a quarter-symmetric connection if the torsion tensor T of $\overline{\nabla}$ has the form

$$T(X,Y) = \omega(Y)\phi X - \omega(X)\phi Y,$$

where ϕ is the tensor field of type (1,1) and X, Y are arbitrary vector fields. A linear connection $\overline{\nabla}$ is said to be a non-metric connection if the covariant derivative of the metric tensor g with respect to $\overline{\nabla}$ does not vanish *i.e.* $\overline{\nabla}g \neq 0$.

The lift function plays an important role in the study of differentiable manifolds. In the last few decades, the theory of lift has been studied by several authors. Furthermore, the study of tangent bundles has been continued by L. S. Das and M. N. I. Khan [3] (2005). They [3] considered a manifold with an almost r-contact structure and obtained an almost complex structure on the tangent bundle. Recently, M. Tekkoyun and S. Civelek [8] (2008) studied and extended the concept of lifts by

Received September 18, 2017; accepted July 24, 2018
2010 *Mathematics Subject Classification.* Primary 53C07; Secondary 53C35, 53C55

considering the structures on complex manifolds. In 2014, the lift was studied with a quarter-symmetric semi-metric connection on tangent bundles by M. N. I. Khan [6]. The same author [7] (2015) also studied the lift equipped with a semi-symmetric non-metric connection on a Kähler manifold. The semi-symmetric non-metric connection has also been considered by B. B. Chaturvedi and P. N. Pandey [2] (2008) in a Kähler manifold. In [2] they showed that the Nijenhuis tensor vanishes in a Kähler manifold equipped with a semi-symmetric non-metric connection. In the same paper, they [2] also proved that if V is a contra-variant almost analytic vector field in a Kähler manifold then V is also a contra-variant almost analytic vector field in a Kähler manifold equipped with a semi-symmetric non-metric connection. Recently, B. B. Chaturvedi and P. Pandey [1] (2015) studied a new type of the metric connection in an almost Hermitian manifold. In that paper, they [1] obtained a condition for a vector field V to be a contravariant almost analytic vector field in an almost Hermitian manifold equipped with a new type of the metric connection.

1.1. Kähler manifold

Let M be an n-(even) dimensional differentiable manifold. If for a tensor field F of type (1,1) and a Riemannian metric g the conditions

$$F^2(X) + X = 0, \quad g(FX, FY) = g(X, Y), \quad (\nabla_X F)Y = 0,$$

hold then M is called Kähler manifold, X, Y are arbitrary vector fields.

1.2. Quarter-symmetric non-metric connection

Let F be a tensor field of type (1,1) then a linear connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y = \nabla_X Y + \omega(Y)FX, \tag{1.1}$$

is called quarter-symmetric non-metric connection, ∇ is the Riemannian connection, ω is 1-form defined by $g(X, \rho) = \omega(X)$ for the associated vector field ρ.
The torsion tensor T and the metric tensor g of $\overline{\nabla}$ are given respectively by

$$T(X,Y) = \omega(Y)FX - \omega(X)FY \quad \textit{and} \quad (\overline{\nabla}_X g)(Y,Z) = \omega(Y)g(X,FZ) + \omega(Z)g(X,FY),$$

for arbitrary vector fields X and Y.

1.3. Tangent Bundle

let M be a differentiable manifold and T_pM denotes the tangent space of M at any point $p \in M$ then the collection of all tangent spaces at $p \in M$ is called the tangent bundle of M and denoted by $T(M) = \cup_{p\in M} T_pM$. Let $\tilde{p} \in T(M)$ then the projection $\pi : T(M) \to M$ defined by $\pi(\tilde{p}) = p$ is called the bundle projection of $T(M)$ over M and the set $\pi^{-1}(p)$ is called the fiber over $p \in M$ and M the base space.

Vertical lift: The composition of two maps $\pi : T(M) \to M$ and $f : M \to \mathbb{R}$ defined by $f^V = fo\pi$ is called the vertical lift of f, where f is a smooth function in M. For $\tilde{p} \in \pi^{-1}(U)$ with induced coordinates (x^h, y^h), the value of $f^V(\tilde{p})$ is constant along each fiber $T_p(M)$ and equal to $f(p)$ *i.e.* $f^V(\tilde{p}) = f^V(x, y) = fo\pi(\tilde{p}) = f(p) = f(x)$.
Complete lift: For a smooth function f in M, a function f^C defined by $f^C = i(df)$ on $T(M)$ is called the complete lift of f. If ∂f is denoted locally by $y^i \partial_i f$ then the complete lift of f is locally denoted by $f^C = y^i \partial_i f = \partial f$.
Let X be a vector field, then for a smooth function f on M, a vector field $X^C \in T(M)$ defined by $X^C f^C = (Xf)^C$ is called the complete lift of X in $T(M)$. If X has the component x^h in M then the component of the complete lift X^C in $T(M)$ is given by $X^C : (x^h, \partial x^h)$ with respect to the induced coordinates in $T(M)$.
For a 1-form ω in M and an arbitrary vector field X, the complete lift of ω is denoted by ω^C and defined by $\omega^C(X^C) = (\omega(X))^C$ [7].

1.4. Induced metric and connection

Let $\tau : S \to M$ be the immersion of an $(n-1)$-dimensional manifold S in M. If we denote the differentiable map $d\tau : T(S) \to T(M)$ of τ by B called the tangent map of τ, $T(S)$ and $T(M)$ being the tangent bundles of S and M, respectively, then the tangent map of B is denoted by $\tilde{B} : T(T(S)) \to T(T(M))$ [7].
Let g be a Riemannian metric in M and the complete lift of g is g^C in $T(M)$. If $\tilde{g}$ denotes the induced metric of g^C on $T(S)$ then we have $\tilde{g}(X, Y) = g^C(\tilde{B}X^C, \tilde{B}Y^C)$, where X, Y are vector fields in S. If ∇ denotes a Riemannian connection on M then ∇^C, the complete lift of ∇, is also a Riemannian connection satisfying $\nabla^C_{X^C} Y^C = (\nabla_X Y)^C$ and $\nabla^C_{X^C} Y^V = (\nabla_X Y)^V$, for the vector fields X, Y in M.
From [7], we know that the lift function has the following properties,

$$
\begin{aligned}
\omega^V(\tilde{B}X^C) &= \omega^V(\tilde{B}X)^{\overline{C}} = \#(\omega^V(X^C)) = \#((\omega(X))^V) \\
&= (\omega(BX))^{\overline{V}}, \omega^C(\tilde{B}X^C) = \omega^C(\tilde{B}X)^{\overline{C}} \\
&= \#(\omega^C(X^C)) = \#((\omega(X))^C) = (\omega(BX))^{\overline{C}}, [X^C, Y^C] \\
&= [X, Y]^C, \quad F^C(X^C) = (F(X))^C, \quad \omega^V(X^C) = (\omega(X))^V, \omega^C(X^C) \\
&= (\omega(X))^C, \quad g^C(X^V, Y^C) = g^C(X^C, Y^V) = (g(X, Y))^V, \qquad (1.2)
\end{aligned}
$$

where X^C, ω^C, F^C, g^C and X^V, ω^V, F^V, g^V are the complete and vertical lifts of X, ω, F, g. $\#$, $\overline{V}$ and $\overline{C}$ denote the operation of restriction, vertical lift and complete lift on $\pi_M^{-1}(\tau(S))$ respectively, X, Y are vector fields in S.

2. Lift of quarter-symmetric non-metric connection on a Kähler manifold

Taking the complete lift of the equation (1.1), we get

$$(\overline{\nabla}_{BX} BY)^{\overline{C}} = (\nabla_{BX} BY)^{\overline{C}} + (\omega(BY)B(FX))^{\overline{C}}. \qquad (2.1)$$

Simplifying (2.1), we have

$$\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C = \nabla^C_{\tilde{B}X^C}\tilde{B}Y^C + \omega^C(\tilde{B}Y^C)\tilde{B}(FX)^V + \omega^V(\tilde{B}Y^C)\tilde{B}(FX)^C. \tag{2.2}$$

A connection $\overline{\nabla}^C$ defined by (2.2) is called the lift of a quarter-symmetric non-metric connection $\overline{\nabla}$.
Replacing Y by FY, the equation (2.2) gives

$$\begin{aligned}\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}(FY)^C &= \nabla^C_{\tilde{B}X^C}\tilde{B}(FY)^C + \omega^C(\tilde{B}(FY)^C)\tilde{B}(FX)^V\\ &\quad +\omega^V(\tilde{B}(FY)^C)(\tilde{B}(FX))^C.\end{aligned} \tag{2.3}$$

Also, operating F^C on the equation (2.2), we get

$$F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C) = F^C(\nabla^C_{\tilde{B}X^C}\tilde{B}Y^C) - \omega^C(\tilde{B}Y^C)\tilde{B}X^V - \omega^V(\tilde{B}Y^C)\tilde{B}X^C. \tag{2.4}$$

Subtracting (2.4) from (2.3), we have

$$\begin{aligned}(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}F^C)(\tilde{B}Y^C) &= \omega^C(\tilde{B}(FY)^C)\tilde{B}(FX)^V + \omega^V(\tilde{B}(FY)^C)\tilde{B}(FX)^C\\ &+ \omega^C(\tilde{B}Y^C)\tilde{B}X^V + \omega^V(\tilde{B}Y^C)\tilde{B}X^C.\end{aligned} \tag{2.5}$$

Thus, we can state

Theorem 2.1. *Let M be a Kähler manifold equipped with the lift of a quarter-symmetric non-metric connection $\overline{\nabla}^C$ then M is a Kähler manifold with respect to $\overline{\nabla}^C$ if and only if*

$$\begin{aligned}&\omega^C(\tilde{B}(FY)^C)\tilde{B}(FX)^V + \omega^V(\tilde{B}(FY)^C)\tilde{B}(FX)^C\\ &\quad+\omega^C(\tilde{B}Y^C)\tilde{B}X^V + \omega^V(\tilde{B}Y^C)\tilde{B}X^C = 0.\end{aligned} \tag{2.6}$$

Now, if we denote

$$\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C) = \omega^C(\tilde{B}Y^C)\tilde{B}(FX)^V + \omega^V(\tilde{B}Y^C)\tilde{B}(FX)^C. \tag{2.7}$$

and define a tensor $'\overline{H}^C$ of type (0,3) by

$$'\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C, \tilde{B}Z^C) = g^C(\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C), \tilde{B}Z^C), \tag{2.8}$$

then, the equations (2.7) and (2.8) together give

$$\begin{aligned}'\overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C, \tilde{B}Z^C) &= \omega^C(\tilde{B}Y^C)g^C(\tilde{B}(FX)^V, \tilde{B}Z^C)\\ &\quad+\omega^V(\tilde{B}Y^C)g^C(\tilde{B}(FX)^C, \tilde{B}Z^C)\end{aligned} \tag{2.9}$$

Replacing Y and Z by FY and FZ in (2.9), respectively, we get

$$\begin{aligned}'\overline{H}^C(\tilde{B}X^C, \tilde{B}(FY)^C, \tilde{B}(FZ)^C) &= \omega^C(\tilde{B}(FY)^C)g^C(\tilde{B}(FX)^V, \tilde{B}(FZ)^C)\\ &+ \omega^V(\tilde{B}(FY)^C)g^C(\tilde{B}(FX)^C, \tilde{B}(FZ)^C)\end{aligned} \tag{2.10}$$

Subtracting (2.9) from (2.10), we find

$$
\begin{aligned}
&{}'\overline{H}^C(\tilde{B}X^C, \tilde{B}(FY)^C, \tilde{B}(FZ)^C) -{}' \overline{H}^C(\tilde{B}X^C, \tilde{B}Y^C, \tilde{B}Z^C) \\
&= \omega^C(\tilde{B}(FY)^C)g^C(\tilde{B}(FX)^V, \tilde{B}(FZ)^C) \\
&+\omega^V(\tilde{B}(FY)^C)g^C(\tilde{B}(FX)^C, \tilde{B}(FZ)^C) \\
&-\omega^C(\tilde{B}Y^C)g^C(\tilde{B}(FX)^V, \tilde{B}Z^C) - \omega^V(\tilde{B}Y^C)g^C(\tilde{B}(FX)^C, \tilde{B}Z^C),
\end{aligned} \tag{2.11}
$$

which shows that ${}'\overline{H}^C$ is a hybrid in the last two slots if and only if the right hand side of (2.11) vanishes.
We also know that a necessary and sufficient condition to be a Kähler manifold with respect to the connection D defined by $D_XY = \nabla_XY + H(X,Y)$ is that ${}'H$ defined by ${}'H(X,Y,Z) = g(H(X,Y),Z)$ is a hybrid in the last two slots [4].
Hence from the above discussions, we conclude the following

Theorem 2.2. *Let M be a Kähler manifold equipped with the lift of a quarter-symmetric non-metric connection $\overline{\nabla}^C$ then a necessary and sufficient condition for M to be a Kähler manifold with respect to the connection $\overline{\nabla}^C$ is that*

$$
\begin{aligned}
&\omega^C(\tilde{B}(FY)^C)g^C(\tilde{B}(FX)^V, \tilde{B}(FZ)^C) + \omega^V(\tilde{B}(FY)^C)g^C(\tilde{B}(FX)^C, \tilde{B}(FZ)^C) \\
&-\omega^C(\tilde{B}Y^C)g^C(\tilde{B}(FX)^V, \tilde{B}Z^C) - \omega^V(\tilde{B}Y^C)g^C(\tilde{B}(FX)^C, \tilde{B}Z^C) = 0.
\end{aligned} \tag{2.12}
$$

Corollary 2.1. *Also, replacing X by FX in (2.12), we have*

$$
\begin{aligned}
&\omega^C(\tilde{B}(FY)^C)\tilde{B}(FX)^V + \omega^V(\tilde{B}(FY)^C)\tilde{B}(FX)^C \\
&+\omega^C(\tilde{B}Y^C)\tilde{B}X^V + \omega^V(\tilde{B}Y^C)\tilde{B}X^C = 0,
\end{aligned} \tag{2.13}
$$

which verifies the condition of the Kähler manifold obtained in (2.6). □

Let ${}'F$ denotes the 2-form of the Riemannian metric g defined by ${}'F(Y,Z) = g(FY,Z)$ then the complete lift of ${}'F$ is denoted and defined by

$$
{}'F^C(\tilde{B}Y^C, \tilde{B}Z^C) = g^C(\tilde{B}(FY)^C, \tilde{B}Z^C). \tag{2.14}
$$

Taking the covariant differentiation of (2.14), we get

Corollary 2.2.

$$
\begin{aligned}
&(\overline{\nabla}^C_{\tilde{B}X^C}{}'F^C)(\tilde{B}Y^C, \tilde{B}Z^C) = (\nabla^C_{\tilde{B}X^C}{}'F^C)(\tilde{B}Y^C, \tilde{B}Z^C) \\
&+\omega^C(\tilde{B}Y^C)g^C(\tilde{B}X^V, \tilde{B}Z^C) + \omega^V(\tilde{B}Y^C)g^C(\tilde{B}X^C, \tilde{B}Z^C) \\
&-\omega^C(\tilde{B}Z^C)g^C(\tilde{B}Y^C, \tilde{B}X^V) - \omega^V(\tilde{B}Z^C)g^C(\tilde{B}Y^C, \tilde{B}X^C).
\end{aligned}
$$

By taking the cyclic sum over X, Y, Z of the equation (2.15), we obtain

$$
(\overline{\nabla}^C_{\tilde{B}X^C}{}'F^C)(\tilde{B}Y^C, \tilde{B}Z^C) + (\overline{\nabla}^C_{\tilde{B}Y^C}{}'F^C)(\tilde{B}Z^C, \tilde{B}X^C)
$$

$$
\begin{aligned}
&+(\overline{\nabla}^C_{\tilde{B}Z^C}{}'F^C)(\tilde{B}X^C, \tilde{B}Y^C)\\
&= \omega^C(\tilde{B}Y^C)g^C(\tilde{B}X^V, \tilde{B}Z^C) + \omega^V(\tilde{B}Y^C)g^C(\tilde{B}X^C, \tilde{B}Z^C)\\
&+\omega^C(\tilde{B}Z^C)g^C(\tilde{B}Y^V, \tilde{B}X^C) + \omega^V(\tilde{B}Z^C)g^C(\tilde{B}Y^C, \tilde{B}X^C)\\
&+\omega^C(\tilde{B}X^C)g^C(\tilde{B}Z^V, \tilde{B}Y^C) + \omega^V(\tilde{B}X^C)g^C(\tilde{B}Z^C, \tilde{B}Y^C)\\
&-\omega^C(\tilde{B}Z^C)g^C(\tilde{B}Y^C, \tilde{B}X^V) - \omega^V(\tilde{B}Z^C)g^C(\tilde{B}Y^C, \tilde{B}X^C)\\
&-\omega^C(\tilde{B}X^C)g^C(\tilde{B}Z^C, \tilde{B}Y^V) - \omega^V(\tilde{B}X^C)g^C(\tilde{B}Z^C, \tilde{B}Y^C)\\
&-\omega^C(\tilde{B}Y^C)g^C(\tilde{B}X^C, \tilde{B}Z^V) - \omega^V(\tilde{B}Y^C)g^C(\tilde{B}X^C, \tilde{B}Z^C). \qquad (2.15)
\end{aligned}
$$

Thus, we can state the following

Theorem 2.3. *Let M be a Kähler manifold equipped with the lift of a quarter-symmetric non-metric connection $\overline{\nabla}^C$ then the relation (2.16) holds.*

Also, it is well known that the Nijenhuis tensor N with respect to the Riemannian connection ∇ is given by

$$
\begin{aligned}
N(X,Y) &= [FX, FY] - [X,Y] - F[FX,Y] - F[X,FY] && (2.16)\\
&= \nabla_{FX}FY - \nabla_{FY}FX - \nabla_X Y + \nabla_Y X && (2.17)\\
&-F\nabla_{FX}Y + F\nabla_Y FX - F\nabla_X FY + F\nabla_{FY}X. && (2.18)
\end{aligned}
$$

If N^C denotes the complete lift of the Nijenhuis tensor N then the equation (2.17) gives the Nijenhuis tensor $\overline{N}^C$ with respect to the connection $\overline{\nabla}^C$ as follows

$$
\begin{aligned}
\overline{N}^C(\tilde{B}X^C, \tilde{B}Y^C) &= \overline{\nabla}^C_{\tilde{B}(FX)^C}\tilde{B}(FY)^C - \overline{\nabla}^C_{\tilde{B}(FY)^C}\tilde{B}(FX)^C\\
&-\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}Y^C + \overline{\nabla}^C_{\tilde{B}Y^C}\tilde{B}X^C - F^C(\overline{\nabla}^C_{\tilde{B}(FX)^C}\tilde{B}Y^C)\\
&+F^C(\overline{\nabla}^C_{\tilde{B}Y^C}\tilde{B}(FX)^C) - F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}(FY)^C) && (2.19)\\
&+F^C(\overline{\nabla}^C_{\tilde{B}(FY)^C}\tilde{B}X^C). && (2.20)
\end{aligned}
$$

By help of (2.2), the equation (2.18) reduces to

$$\overline{N}^C(\tilde{B}X^C, \tilde{B}Y^C) = 0. \qquad (2.21)$$

Hence, we have

Theorem 2.4. *Let M be a Kähler manifold equipped with the lift of a quarter-symmetric non-metric connection $\overline{\nabla}^C$ then the Nijenhuis tensor $\overline{N}^C$ with respect to the connection $\overline{\nabla}^C$ vanishes.*

3. Contravariant almost analytic vector field on a Kähler manifold

We know that in an almost Hermitian manifold, a necessary and sufficient condition for a vector field W to be a contravariant almost analytic vector field is that

$$\nabla_{FX}W = (\nabla_W F)X + F(\nabla_X W). \qquad (3.1)$$

For a Kähler manifold the equation (3.1) reduces to

$$\nabla_{FX}W - F(\nabla_X W) = 0. \tag{3.2}$$

Now, replacing X by FX and Y by W in (2.2) we have

$$\overline{\nabla}^C_{\tilde{B}(FX)^C}\tilde{B}W^C = \nabla^C_{\tilde{B}(FX)^C}\tilde{B}W^C - \omega^C(\tilde{B}W^C)\tilde{B}X^V - \omega^V(\tilde{B}W^C)\tilde{B}X^C. \tag{3.3}$$

Again, replacing Y by W and then taking F^C in (2.2), we get

$$F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}W^C) = F^C(\nabla^C_{\tilde{B}X^C}\tilde{B}W^C) - \omega^C(\tilde{B}W^C)\tilde{B}X^V - \omega^V(\tilde{B}W^C)\tilde{B}X^C. \tag{3.4}$$

Subtracting (3.4) from (3.3), we obtain

$$\overline{\nabla}^C_{\tilde{B}(FX)^C}\tilde{B}W^C - F^C(\overline{\nabla}^C_{\tilde{B}X^C}\tilde{B}W^C) = \nabla^C_{\tilde{B}(FX)^C}\tilde{B}W^C - F^C(\nabla^C_{\tilde{B}X^C}\tilde{B}W^C). \tag{3.5}$$

Thus, we have the following theorem

Theorem 3.1. *Let M be a Kähler manifold equipped with the lift of a quarter-symmetric non-metric connection $\overline{\nabla}^C$ then a necessary and sufficient condition for a vector field W to be a contravariant almost analytic vector field with respect to the connection $\overline{\nabla}^C$ is that it is a contravariant almost analytic vector field with respect to the connection ∇^C.* □

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Ser. Math. Inform. Vol. 33, No 4 (2018), 547–559
https://doi.org/10.22190/FUMI1804547K

SOME NOTES CONCERNING TACHIBANA AND VISHNEVSKII OPERATORS IN THE TANGENT BUNDLE

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Abstract. The main aim of the present paper is to study Tachibana and Vishnevskii operators for the Lorentzian almost r-para-contact structure in the tangent bundle.
Keywords: Tangent bundle, Vertical lift, Complete lift, Lie derivative, Tachibana operator, Vishnevskii operator.

1. Introduction

The study of differential geometry of the tangent bundle is a very fruitful domain of differential geometry because the theory provides many new problems in modern differential geometry. The study of differential geometry of the tangent bundle started promptly in 1960s by Davis, Sasaki, Yano and Davis, Tachibana and many others. Yano and Ishihara have studied vertical, complete and horizontal lifts of tensors and connection. The first author studied lifts of a hypersurface with connections to tangent bundles and a Kähler manifold in 2014 [7] and 2016 [8]. Also, different structures on tanent bundles have been studied by several authors such as Das and the first author (2005) [4], Tekkoyun(2006) [6], the first author(2017) [15] and many others.

I. Sato [17] introduced the notion of almost contact structure on differential geometry. An almost paracontact Riemannian manifold and an almost product Riemannian manifold were studied by Adati [19] while the almost r-contact structure was introduced by Vanzura [10]. In [13], Motsumoto initiated the study of Lorentzian paracontact manifolds. The Lorentzian almost r-paracontact structure in the tangent bundle was studied by Khan and Jun [14].

The paper is organized as follows: In Section 2, we recall some basic definitions of vertical, complete and horizontal lifts and the Lie derivative. Section 3 deals with Tachibana and Vishnevskii operators associated with the Lorentzian almost r-para-contact structure in the tangent bundle.

Received November 26, 2017; accepted April 10, 2018
2010 *Mathematics Subject Classification.* Primary 53D15; Secondary 58A30

2. Preliminaries

Let M be an n-dimensional differentiable manifold and let $T(M) = \bigcup_{p\in M} T_p(M)$ be its tangent bundle. Then $T(M)$ is also a differentiable manifold [1]. Let $X = \sum_{i=1}^n x^i(\frac{\partial}{\partial x^i})$ and $\eta = \sum_{i=1}^n \eta^i dx^i$ be the expressions in local coordinates for the vector field X and the 1-form η in M. Let (x^i, y^i) be local coordinates of point in $T(M)$ induced naturally from the coordinate chart $U(x^i)$ in M.

2.1. Vertical lifts

If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \longrightarrow M$ and $f : M \longrightarrow R$, so that

$$f^V = fo\pi \tag{2.1}$$

Thus, if a point $\tilde{p}\epsilon\pi^{-1}(U)$ has induced the coordinates (x^h, y^h) then

$$f^V(\tilde{p}) = f^V(x,y) = fo\pi(\tilde{p}) = f(p) = f(x) \tag{2.2}$$

Thus the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f . Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\tau(M)$ into the tensor algebra $\tau(T(M))$ with respect to constant coefficients by the conditions

$$(P\otimes Q)^V = P^V \otimes Q^V, (P+R)^V = P^V + R^V \tag{2.3}$$

P, Q and R being arbitrary elements of $\tau(M)$ [3].

Furthermore, the vertical lifts of tensor fields obey the general properties [1, 2]:

$$(f.g)^V = f^V g^V, (f+g)^V = f^V + g^V \tag{a}$$

$$(X+Y)^V = X^V + Y^V, (f.X)^V = f^V X^V, X^V f^V = 0, [X^V, Y^V] = 0 \tag{b}$$

$$(f.\eta)^V = f^V \eta^V, \eta^V(X^V) = 0, X^V(Y^V) = 0, \tag{c}$$

$\forall f, g \in \tau_0^0(M), X, Y \in \tau_0^1(M), \phi \in \tau_1^1(M)$.

2.2. Complete lifts

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$f^C = i(df)$$

and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \tau_0^1(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$.

The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\eta \in \tau_0^1(M)$ Then a 1-form η^C in $T(M)$ defined by

$$\eta^C(X^C) = (\eta(X))^C$$

X being an arbitrary vector field in M. We call η^C the complete lift of η.

The complete lifts to a unique algebra isomorphism of the tensor algebra $\tau(M)$ into the tensor algebra $\tau(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, (P + R)^C = P^C + R^C$$

P, Q and R being arbitrary elements of $\tau(M)$.

Moreover, the complete lifts of tensor fields obey the general properties [1, 2]:

(a) $$(fX)^C = f^C X^V + f^V X^C = (Xf)^C, X^C f^V = (Xf)^V, X^V f^C = (Xf)^V,$$

(b) $$\phi^V XC = (\phi X)^V, \phi^C X^V = (\phi X)^V, (\phi X)^C = \phi^C X^C;$$

(c) $$\eta^V X^C = (\eta(X))^C, \eta^C X^V = (\eta(X))^V$$

(d) $$[X^V, Y^C] = [X, Y]^C, I^C = I, I^V I^C = X^V, [X^C, Y^C] = [X, Y]^C$$

$\forall f, g \in \tau_0^0(M), X, Y \in \tau_0^1(M), \phi \in \tau_1^1(M)$.

2.3. Horizontal lifts

Let (x^h, y^h) be a local coordinate system in an open set $\pi^{-1}(U) \subset T(M)$ where U is an arbitrary coordinate neighborhood in M. Suppose that a tensor field S in M by

$$S = S^{i,\ldots,h}_{l,k,\ldots,j} \frac{\partial}{\partial x^i} \otimes \ldots\ldots \otimes \frac{\partial}{\partial x^h} \otimes dx^l \otimes dx^k \otimes \ldots\ldots \otimes dx^j$$

and a tensor field $\gamma_x S$ in $\pi^{-1}(U)$ by

$$\gamma_x S = \left(X^l S^{i,\dots,h}_{l,k,\dots,j}\right) \frac{\partial}{\partial y^i} \otimes \dots\dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots\dots \otimes dx^j$$

and a tensor field γS in $\pi^{-1}(U)$ by

$$\gamma S = \left(y^l S^{i,\dots,h}_{l,k,\dots,j}\right) \frac{\partial}{\partial y^i} \otimes \dots\dots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \dots\dots \otimes dx^j.$$

The tensor fields $\gamma_x S$ and γS defined in each $\pi^{-1}(U)$ determine respectively global tensor fields in $T(M)$.

Let ∇ be an affine connection in M. If f is a function in M then the gradient of f dented by ∇f in M.

Apply the operation γ to γf and get $\gamma(\nabla f)$.

Put

$$\nabla_\gamma f = \gamma(\nabla f).$$

The horizontal lift f^H of $f \in \tau_0^0(M)$ to the tangent bundle $T(M)$ by

$$(f)^H = f^C - \nabla_\gamma f \tag{2.4}$$

Let $X \in \tau_0^1(M)$. Then the horizontal lift X^H of X defined by

$$X^H = X^C - \nabla_\gamma X \tag{2.5}$$

in $T(M)$, where

$$\nabla_\gamma X = \gamma(\nabla X)$$

The horizontal lift X^H of X has the components

$$\begin{bmatrix} x^h \\ -\Gamma_i^h x^i \end{bmatrix} \tag{2.6}$$

with respect to the induced coordinates in $T(M)$, where $\Gamma_i^h = y^j \Gamma_{ji}^h$.

Suppose that $\eta \in \tau_1^0(M)$. Then the 1-form η^C in $T(M)$ defined by the horizontal lift S^H of the tensor field S of an arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S \tag{2.7}$$

for all $P, Q \in \tau(M)$. We have

$$\nabla_\gamma(P \otimes Q) = (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q)$$

or

$$(P \otimes Q)^H = P^H \otimes Q^V + P^V \otimes Q^H. \tag{2.8}$$

In addition, the horizontal lifts of tensor fields obey the general properties [1, 2]:

$$\begin{array}{ll} \text{(a)} & X^H f^V = (Xf)^V, F^V X^H = (FX)^V, F^C X^H = (FX)^H + (\nabla_\gamma F) X^H \\ \text{(b)} & \eta^V (X^H) = (\eta(X))^H, \eta^C (X^H) = (\eta(X))^C - \gamma(\eta \circ (\nabla X), ; \\ \text{(c)} & \eta^H (X^C) = \eta^H (\nabla_\gamma X), \eta^H (X^H) = 0 \end{array}$$

$\forall f, g \in \tau_0^0(M), X, Y \in \tau_0^1(M), \eta \in \tau_1^0(M), F \in \tau_1^1(M).$

Let$X$ be a vector field in an n-dimensional differentiable manifold M. The differential transformation L_X is called the Lie derivative with respect to X if

$$\begin{array}{ll} \text{(a)} & L_X f = Xf, \forall f \in \tau_0^0(M) \\ \text{(b)} & L_X Y = [X, Y]. \end{array}$$

The Lie derivative $L_X F$ of a tensor field F of type (1, 1) with respect to a vector field X is defined by [1]

$$(L_X F) = [X, FY] - F[X, Y] \tag{2.9}$$

where $[,]$ is the Lie bracket.

Let M be an n-dimensional differentiable manifold. Differential transformation of algebra $T(M)$ defined by

$$D = \nabla_X : T(M) \to T(M), X \in \tau_0^1(M), \tag{2.10}$$

is called covariant derivation with respect to a vector field X if

$$\begin{array}{ll} \text{(a)} & \nabla_{fX+gY} t = f \nabla_X t + g \nabla_Y t, \\ \text{(b)} & \nabla_X f = Xf, \forall f, g \in \tau_0^0(M), \forall X, Y \in \tau_0^1(M), \forall t \in \tau(M). \end{array}$$

and a transformation defined by

$$\nabla : \tau_0^1(M) \times \tau_0^1(M) \to \tau_0^1(M) \tag{2.11}$$

is called affine connection [1].

Proposition 2.1. *For any $X, Y \in \tau_0^1(M)$[1]*

$$\begin{array}{ll} \textit{(a)} & [X^V, Y^H] = [X, Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_X Y)^V \\ \textit{(b)} & [X^C, Y^H] = [X, Y]^H - \gamma(L_X Y), \\ \textit{(c)} & [X^H, Y^V] = [X, Y]^V + (\nabla_Y X)^V, \\ \textit{(d)} & [X^C, Y^H] = [X, Y]^H - \gamma \hat{R}(X, Y) \end{array}$$

where $\hat{\nabla}$ is an affine connection in M defined by

$$\hat{\nabla}_X Y = \nabla_Y X + [X, Y]$$

and $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$.

Proposition 2.2. *For any $X, Y \in \tau_0^1(M), f \in \tau_0^0(M)$ and ∇^H is the horizontal lift of the affine connection ∇ to $T(M)$ [1]*

$$\begin{array}{ll} (a) & \nabla^H_{X^V} Y^V = 0, \\ (b) & \nabla^H_{X^V} Y^H = 0, \\ (c) & \nabla^H_{X^H} Y^V = (\nabla_X Y)^V, \\ (d) & \nabla^H_{X^H} Y^H = (\nabla_X Y)^H. \end{array}$$

3. Tachibana and Vishnevskii operators associated with the Lorentzian almost r-para-contact structure in the tangent bundle

Let M be a differentiable manifold of C^∞ class and $T(M)$ denotes the tangent bundle of M. Suppose that there are a tensor field ϕ of type (1, 1), a vector field ξ_p and a 1-form $\eta_p, p = 1, 2, ...r$ satisfying [5, 6, 11]

$$\begin{array}{ll} \text{(a)} & \phi^2 = I - \sum_{p=1}^r \xi_p \otimes \eta_p \\ \text{(b)} & \phi\xi_p = 0 \\ \text{(c)} & \eta_p \circ \phi = 0 \\ \text{(d)} & \eta_p(\xi_q) = \delta_{pq} \end{array} \tag{3.1}$$

where $p = 1, 2, ...r$ and δ_{pq} denote the Kronecker delta. Thus the manifold M satisfying conditions (3.1) will be said to possess a Lorentzian almost r-para-contact structure [13, 14].

Let us suppose that the base space M admits the Lorentzian almost r-para-contact structure. Then there exists a tensor field ϕ of type (1, 1), $r(C^\infty)$ vector fields $\xi_1, \xi_2, \ldots \xi_p$, and $r(C^\infty)$ 1-forms $\eta_1, \eta_2, \ldots \eta_p$, such that equation (3.1) are satisfied. Taking the complete lifts of the equation (3.1) we obtain the following:

$$\begin{array}{ll} \text{(a)} & (\phi^H)^2 = I + \sum_{p=1}^r \{\xi_p^V \otimes \eta_p^H - \xi_p^H \otimes \eta_p^V\} \\ \text{(b)} & \phi^H \xi_p^V = 0, \phi^H \xi_p^C = 0 \\ \text{(c)} & \eta_p^V \circ \phi^H = 0, \eta_p^H \circ \phi^V = 0, \eta_p^H \circ \phi^H = 0, \eta_p^V \circ \phi^V = 0 \\ \text{(d)} & \eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0, \eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = 1 \end{array} \tag{3.2}$$

Let us define the element $\tilde{J}$ of $J_0^1 T(M)$ by

$$\tilde{J} = \phi^H + \sum_{p=1}^{r} \left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right) \tag{3.3}$$

then in the view of the equation (3.2), it is easily shown that

$$\tilde{J}^2 X^V = X^V, \tilde{J}^2 X^H = X^H$$

which means that $\tilde{J}$ is an almost product structure in $T(M)$ [3, 14]. Now in view of the equation (3.3), we have

$$\begin{array}{lll} & \text{(a)} & \tilde{J}X^H = (\phi X)^H + \sum_{p=1}^{r} \left\{(\eta_p(X))^V \xi_p^V\right\} \\ (3.4) & \text{(b)} & \tilde{J}X^V = (\phi X)^V - \sum_{p=1}^{r} \left\{(\eta_p(X))^V \xi_p^H\right\} \end{array}$$

for all $X \in \tau_0^1(M)$.

3.1. Tachibana Operator

Let $\phi \in \tau_1^1(M)$ and $\tau(M) = \sum_{r,s=0}^{\infty} \tau_r^s(M)$ be a tensor algebra over R. A map $\Phi_\phi|_{r+s>0}$ is called the Tachibana operator or Φ_ϕ operator on M if [9]

$$\begin{array}{ll} \text{(a)} & \Phi_\phi \textit{ is linear with respect to constant coefficient }, \\ \text{(b)} & \Phi_\phi : \tau^*(M) \to \tau_{s+1}^r(M) \textit{ for all } r \textit{ and } s \\ \text{(c)} & \Phi_\phi(K \otimes^C L) = (\Phi_\phi K) \otimes L + K \otimes \Phi_\phi L \textit{ for all } K, L \in \tau^*(M), \\ \text{(d)} & \Phi_{\phi X} Y = -(L_Y \phi) X forall X, Y \in \tau_0^1(M) \\ & \textit{where } L_Y \textit{ is Lie derivation with respect to } Y, \\ \text{(e)} & (\Phi_{\phi\eta})Y = (d(\tau_Y \eta(\Phi X) - (d(\tau_Y(\eta \circ \Phi)X + \eta((L_Y \phi)X) \\ (3.5) & \quad = (\Phi X(\tau_Y \eta))(\Phi X) - X(\tau_{\phi X} \eta) + \eta((L_Y \phi)X) \end{array}$$

for all $\eta \in \tau_1^0(M)$ and $X, Y \in \tau_0^1$, where $\tau_Y \eta = \eta(X) = \eta \otimes^C Y, \tau_r^{*s}(M)$ the module of the pure tensor field of type (r, s) on M with respect to the affinor field φ.

Theorem 3.1. *For the Tachibana operator on M, L_X the operator Lie derivation with respect to $X, \tilde{J} \in \tau_1^1(T(M))$ defined by $\tilde{J} = \phi^H + \sum_{p=1}^{r} \left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)$ and $\eta(Y) = 0$, we have*

$$
\begin{aligned}
&(a) && \Phi_{\tilde{J}Y^V} X^H = -((\hat{\nabla}_X \phi)Y)^V + \textstyle\sum_{p=1}^{r}((\hat{\nabla}_X \eta_p)Y)^V \xi_p^H \\
&(b) && \Phi_{\tilde{J}Y^H} X^H = -((L_X \phi)Y)^H + \gamma \hat{R}(X, \phi Y) - \textstyle\sum_{p=1}^{r}((L_X \eta_p)Y)^V \xi_p^V - \tilde{J}\gamma \hat{R}(X, Y) \\
&(c) && \Phi_{\tilde{J}Y^V} X^V = 0 \\
&(d) && \Phi_{\tilde{J}Y^H} X^V = -((L_X \phi)Y)^V + ((\nabla_X \phi)Y)^V + \textstyle\sum_{p=1}^{r}((L_X \eta_p)Y)^V \xi_p^H \\
&(3.6) && - \textstyle\sum_{p=1}^{r}((\nabla_X \eta_p)Y)^V \xi_p^H
\end{aligned}
$$

where $X, Y \in \tau_0^1(M)$*, a tensor field* $\phi \in \tau_1^1(M)$*, a vector field* ξ *and a 1-form* $\eta \in \tau_1^0(M)$*.*

Proof.

$$
\begin{aligned}
(a)\Phi_{\tilde{J}Y^V} X^H &= -(L_{X^H} \tilde{J})Y^V = -(L_{X^H} \tilde{J}Y^V - \tilde{J} L_{X^H} Y^V), \quad \textit{since} \;\; L_X Y = [X, Y] \\
&= -[X^H, \tilde{J}Y^V] + \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)[X^H, Y^V] \\
&= -[X^H.(\phi Y)^V] + \phi^H([X, Y]^V + (\nabla_X Y)^V) + \sum_{p=1}^{r} \eta_p^V([X, Y]^V + (\nabla_Y X)^V)\xi_p^V \\
&\quad - \sum_{p=1}^{r} \eta_p^H([X, Y]^V + (\nabla_Y X)^V)\xi_p^H \\
&= -[X^H, (\phi Y)^V](\nabla_{\phi Y} X)^V + \phi^H([X, Y]^V + (\nabla_Y X)^V) + \sum_{p=1}^{r} \eta_p^V([X, Y]^V \\
&\quad + (\nabla_Y X)^V)\xi_p^V - \sum_{p=1}^{r} \eta_p^H([X, Y]^V + (\nabla_Y X)^V)\xi_p^H \\
&= -((\hat{\nabla}_X \phi)Y)^V - (\phi \hat{\nabla}_X Y)^V - \sum_{p=1}^{r}((L_X \eta_p)Y)^V \xi_p^H + \sum_{p=1}^{r}((\hat{\nabla}_X \eta_p)Y)^V \xi_p^H \\
&\quad - \sum_{p=1}^{r}(\eta_p(L_X Y))^V \xi_p^H \\
&\qquad \textit{as} \;\; \eta(L_X Y) = -(L_X \eta_p)Y \\
&= -((\hat{\nabla}_X \phi)Y)^V + \sum_{p=1}^{r}((\hat{\nabla}_X \eta_p)Y)^V \xi_p^H.
\end{aligned}
$$

(3.7)

$$
(b)\Phi_{\tilde{J}Y^H} X^H = -(L_{X^H} \tilde{J})Y^H = -(L_{X^H} \tilde{J}Y^H - \tilde{J} L_{X^H} Y^H) \quad \textit{since} \;\; L_X Y = [X, Y]
$$

$$= -[X^H, \tilde{J}Y^H] + \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)[X^H, Y^H]$$

$$= -[X^H, (\phi Y)^H] + \phi^H[X^H, Y^H] + \sum_{p=1}^{r}\eta_p^V[X^H, Y^H]\xi_p^V - \sum_{p=1}^{r}\eta_p^H[X^H, Y^H]\xi_p^H$$

$$since [X^H, Y^H] = [X, Y]^H - \gamma\hat{R}(X, Y),$$

$$= -((L_X\phi)Y)^H + \gamma\hat{R}(X, \phi Y) - \phi^H\gamma\hat{R}(X, Y)) - \sum_{p=1}^{r}((L_X\eta_p)Y)^V\xi_p^V$$

$$-\sum_{p=1}^{r}((\eta_p^V\gamma\hat{R}(X, Y)\xi_p^V + \sum_{p=1}^{r}((\eta_p^H\gamma\hat{R}(X, Y)\xi_p^H$$

$$= -((L_X\phi)Y)^H + \gamma\hat{R}(X, \phi Y) - \sum_{p=1}^{r}((L_X\eta_p)Y)^V\xi_p^V - \tilde{J}\gamma\hat{R}(X, Y). \tag{3.8}$$

$$\text{(c)}\Phi_{\tilde{J}Y^V}X^V = -(L_{X^V}\tilde{J})Y^V = -(L_{X^V}\tilde{J}Y^V - \tilde{J}L_{X^V}Y^V) \quad since \;\; L_XY = [X, Y]$$

$$= -[X^V, \tilde{J}Y^V] + \tilde{J}[X^V, Y^V], \qquad [X^V, Y^V] = 0$$

$$= -[X^V, \phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)Y^V]$$

$$as \;\; (\eta_p(Y)\xi_p)^H = 0$$

$$= -[X^V, (\phi Y)^V] + \sum_{p=1}^{r}[X^V, (\eta_p(Y)\xi_p)^H] = 0 \tag{3.9}$$

$$\text{(d)}\Phi_{\tilde{J}Y^H}X^V = -(L_{X^V}\tilde{J})Y^H = -L_{X^V}\tilde{J}Y^H + \tilde{J}L_{X^V}Y^H), \quad since \;\; L_XY = [X, Y]$$

$$= -[X^V, \tilde{J}Y^H] + \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)[X^V, Y^H]$$

$$= -[X, \phi Y]^V + (\nabla_X\phi Y)^V + \phi^H([X, Y]^V - (\nabla_XY)^V) + \sum_{p=1}^{r}\eta_p^V([X, Y]^V - (\nabla_XY)^V)\xi_p^V$$

$$-\sum_{p=1}^{r}\eta_p^H([X, Y]^V - (\nabla_XY)^V)\xi_p^H$$

$$since \quad \eta_pL_XY = L_X\eta_p(Y) - (L_X\eta_p)Y, \;\; \eta_p\nabla_XY = \nabla_X\eta_p(Y) - (\nabla_X\eta_p)Y$$

$$= -((L_X\phi)Y)^V + ((\nabla_X\phi)Y)^V + \sum_{p=1}^{r}((L_X\eta_p)Y)^V\xi_p^H - \sum_{p=1}^{r}((\nabla_X\eta_p)Y)^V\xi_p^H. \tag{3.10}$$

□

Corollary 3.1. *If we put $Y = \xi_p$ i.e. $\eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0$, $\eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = 1$, then we have*

$$\text{(a)}\qquad \Phi_{\tilde{J}\xi_p^V} X^H = \textstyle\sum_{p=1}^r (L_{\xi_p}X)^H - \gamma\hat{R}(X,\xi_P) - (\hat{\nabla}_X\phi)^V - (\hat{\nabla}_X\eta_p)\xi_p^V\xi_p^H$$

$$\begin{aligned}\text{(b)}\qquad \Phi_{\tilde{J}\xi_p^H} X^H = {} & (\hat{\nabla}_X\xi_p)^V - ((L_X\phi)\xi_p)^H + \phi^H\gamma\hat{R}(X,\xi_p) - \textstyle\sum_{p=1}^r((L_X\eta_p)\xi_p)^V\xi_p^V \\ & - \textstyle\sum_{p=1}^r \eta_p^V\gamma\hat{R}(X,\xi_p)\xi_p^V + \sum_{p=1}^r \eta_p^H\gamma\hat{R}(X,\xi_p)\xi_p^H\end{aligned}$$

$$\text{(c)}\qquad \Phi_{\tilde{J}\xi_p^V} X^V = (\hat{\nabla}_\xi)_p X^V$$

$$\text{(d)}\qquad \Phi_{\tilde{J}\xi_p^H} X^V = -((L_X\phi)\xi_p)^V + \textstyle\sum_{p=1}^r((L_X\eta_p)\xi_p)^V\xi_p^H - \sum_{p=1}^r((\nabla_X\eta_p)\xi_p)^V\xi_p^H.$$

3.2. Vishnevskii Operator

Let ∇ be a linear connection and ϕ be a tensor field of type (1,1) on M. If the condition (d) of the Tachibana operator is replaced by

$$\text{(D)}\qquad \Psi_{\phi X}Y = \nabla_{\phi X}Y - \phi\nabla_X Y \tag{3.11}$$

for any $X, Y \in \tau_0^1(M)$. A map $\Psi_\phi : \tau^*(M) \to \tau(M)$, which satisfies conditions (a), (b), (c), (e) of the Tachibana operator and the condition (D), is called the Vishnevskii operator on M[9, 11].

Theorem 3.2. *For Ψ_ϕ the Vishnevskii operator on M and ∇^H the horizontal lift of an affine connection ∇ in M to $T(M)$, $\tilde{J} \in \tau_1^1(T(M))$ defined by (3.3), we have*

$$\begin{aligned}
&\text{(a)}\quad \Psi_{\tilde{J}X^V}Y^H = -\textstyle\sum_{p=1}^r((\eta_p(X)\nabla_\xi)_p Y^H \\
&\text{(b)}\quad \Psi_{\tilde{J}X^H}Y^V = ((\hat{\nabla}_Y\phi)X)^V - ((L_X\phi)X)^V + \textstyle\sum_{p=1}^r(\eta_p\hat{\nabla}_Y X)^V\xi_p^H \\
&\qquad\qquad - \textstyle\sum_{p=1}^r(\eta_p L_Y X)^V\xi_p^H \\
&\text{(c)}\quad \Psi_{\tilde{J}X^V}Y^V = -\textstyle\sum_{p=1}^r(\eta_p(X))^V)\nabla_{\xi_p^H}^H Y^V \\
&\text{(d)}\quad \Psi_{\tilde{J}X^H}Y^H = ((\hat{\nabla}_Y\phi)X)^H - ((L_X\phi)X)^H - \textstyle\sum_{p=1}^r(\eta_p\hat{\nabla}_Y X)^V\xi_p^V \\
&\qquad\qquad + \textstyle\sum_{p=1}^r(\eta_p L_Y X)^V\xi_p^V
\end{aligned} \tag{3.12}$$

where $X, Y \in \tau_0^1(M)$, a tensor field $\phi \in \tau_1^1(T(M))$, vector fields ξ_p and a 1-form $\eta_p \in \tau_1^0, p = 1 \ldots r$.

Proof.

$$\text{(a)}\Psi_{\tilde{J}X^V}Y^H = \nabla_{\tilde{J}X^V}^H Y^H - \tilde{J}\nabla_{X^V}^H Y^H$$

$$= \nabla^H_{(\phi^H + \sum_{p=1}^r(\xi_p^V\otimes\eta_p^V - \xi_p^H\otimes\eta_p^H))X^V} Y^H - \left(\phi^H + \sum_{p=1}^r\left(\xi_p^V\otimes\eta_p^V - \xi_p^H\otimes\eta_p^H\right)\right)\nabla_{X^V}^H Y^H$$

$$= \nabla_{(\phi X)^V - \sum_{p=1}^{r}(\eta_p X)^V \sigma_p^H} Y^H \quad as \nabla^H_{X^V} Y^H = 0$$

$$= -\sum_{p=1}^{r}(\eta_p X)^V (\nabla_{\xi_p} Y)^H \quad as \nabla^H_{(\phi X)^V} Y^H = 0$$

$$= -\sum_{p=1}^{r}(\eta_p(X)\nabla_{\xi_p} Y)^H. \tag{3.13}$$

$$(b)\Psi_{\tilde{J}X^H} Y^V = \nabla^H_{\tilde{J}X^H} Y^H - \tilde{J}\nabla^H_{X^H} Y^V$$

$$= \nabla^H_{\left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)X^H} Y^V - \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^H} Y^V$$

$$= \nabla^H_{(\phi X)^H} Y^V - \phi^H (\nabla_X Y)^V + \sum p = 1^r \eta_p^H (\nabla_X Y)^V \xi_p^H$$

$$= (\hat{\nabla}_Y \phi X)^V + [\phi X, Y]^V - \phi^H((\hat{\nabla}_Y X)^V + [X,Y]^V) + \sum_{p=1}^{r}\eta_p^H((\hat{\nabla}_Y X)^V + [X,Y]^V)\xi_p^H$$

$$= ((\hat{\nabla}_Y \phi)X)^V - ((L_Y\phi)X)^V + \sum_{p=1}^{r}(\eta_p \hat{\nabla}_Y X)^V \xi_p^H - \sum_{p=1}^{r}(\eta_p \hat{L_Y} X)^V \xi_p^H \tag{3.14}$$

$$(c)\Psi_{\tilde{J}X^V} Y^V = \nabla^H_{\tilde{J}X^V} Y^V - \tilde{J}\nabla^H_{X^V} Y^V$$

$$= \nabla^H_{\left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)X^V} Y^V - \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^V} Y^V$$

$$= \nabla^H_{(\phi X)^V} Y^V - \sum_{p=1}^{r}\eta_p(X))^V)\nabla^H_{\xi_p^H} Y^V$$

$$= -\sum p = 1^r \eta_p(X))^V)\nabla^H_{\xi_p^H} Y^V \quad as \nabla^H_{(\phi X)^V} Y^V = 0. \tag{3.15}$$

$$(d)\Psi_{\tilde{J}X^H} Y^H = \nabla^H_{\tilde{J}X^H} Y^H - \tilde{J}\nabla^H_{X^H} Y^H$$

$$= \nabla^H_{\left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)X^H} Y^H - \left(\phi^H + \sum_{p=1}^{r}\left(\xi_p^V \otimes \eta_p^V - \xi_p^H \otimes \eta_p^H\right)\right)\nabla^H_{X^H} Y^H$$

$$= \nabla^H_{(\phi X)^H} Y^H - \phi^H (\nabla_X Y)^H - \sum_{p=1}^{r}\eta_p^V (\nabla_X Y)^H \xi_p^V$$

$$= (\hat{\nabla}_Y \phi X)^H + [\phi X, Y]^H - \phi^H((\hat{\nabla}_Y X)^H + [X,Y]^H) - \sum_{p=1}^{r}\eta_p^V((\hat{\nabla}_Y X)^H + [X,Y]^H)\xi_p^H$$

$$= ((\hat{\nabla}_Y \phi)X)^H - ((L_Y\phi)X)^H - \sum_{p=1}^{r}(\eta_p \hat{\nabla}_Y X)^V \xi_p^V - \sum_{p=1}^{r}(\eta_p L_Y X)^V \xi_p^V \tag{3.16}$$

□

Corollary 3.2. *If we put $Y = \xi_p$ i.e. $\eta_p^H(\xi_p^H) = \eta_p^V(\xi_p^V) = 0$, $\eta_p^H(\xi_p^V) = \eta_p^V(\xi_p^H) = 1$, then we have*

(a) $$\Psi_{\tilde{J}\xi_p^V} Y^H = -(\nabla_\xi)_p Y^H$$

(b) $$\Psi_{\tilde{J}\xi_p^H} Y^V = -\phi^H(\hat{\nabla}_Y \xi_p)^V + (\phi L_Y \xi_p)^V + \sum_{p=1}^r (\eta_p(\hat{\nabla}_Y \xi_p)^V \xi_p^H - \sum_{p=1}^r (\eta_p(L_Y \xi_p)^V)\xi_p^H$$

(c) $$\Psi_{\tilde{J}\xi_p^V} Y^V = -(\nabla_{\xi_p} Y)^V$$

(d) $$\Psi_{\tilde{J}\xi_p^H} Y^H = ((\hat{\nabla}_Y \phi)\xi_p)^H + (\phi[Y, \xi_p])^H + \sum_{p=1}^r ((\hat{\nabla}_Y \eta_p)\xi_p)^V \xi_p^V - \sum_{p=1}^r ((L_Y \eta_p)\xi_p)^V \xi_p^V.$$

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Ser. Math. Inform. Vol. 33, No 4 (2018), 561–575
https://doi.org/10.22190/FUMI1804561B

CONHARMONIC CURVATURE TENSOR OF A QUARTER-SYMMETRIC METRIC CONNECTION IN A KENMOTSU MANIFOLD

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Abstract. The aim of the present paper is to study Kenmotsu manifolds admitting a quarter-symmetric metric connection whose conharmonic curvature tensor satisfies certain curvature conditions.
Keywords. Kenmotsu manifolds; curvature; conharmonic curvature tensor.

1. Introduction

Manifolds known as Kenmotsu manifolds were studied by K. Kenmotsu in 1972 [17]. They set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension [26]. Consider an almost contact metric manifold M^{2n+1}, with the structure (ϕ, ξ, η, g) given by a tensor field ϕ of type $(1, 1)$, a vector field ξ, a 1-form η satisfying $\phi^2 = -I+\eta\otimes\xi$, $\eta(\xi) = 1$, and a Riemannian metric g such that $g(\phi X, \phi Y) = g(X, Y)-\eta(X)\eta(Y)$ for any vector field X and Y. The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y. The normality of an almost contact metric manifold is expressed by the vanishing of the tensor field $N = [\phi, \phi]+2d\eta\otimes\xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [6]. For more details we refer to Blair's books ([6],[7]). A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta\wedge\Phi$. It is well known that Kenmotsu manifolds can be characterized through their Levi-Civita connection, by $(\nabla_X\phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y, Z. Moreover, Kenmotsu proved that such a manifold M^{2n+1} is locally a warped product $]-\varepsilon, \varepsilon[\times_f N^{2n}$, N^{2n} being a Kähler manifold and $f^2 = ce^{2t}$, c is a positive constant.

More recently in ([18],[21]) and [12], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu.

Received December 23, 2017; accepted July 14, 2018
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Kenmotsu manifolds have been studied by Barman and De ([4], [5]) Barman [2], Kim and Pak [18] and many others.

In 1924, Friedmann and Schouten [13] introduced the idea of a semi-symmetric connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor $\tilde{T}$ of the connection $\tilde{\nabla}$ satisfies

$$\tilde{T}(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form and ρ is a vector field defined by

$$u(X) = g(X,\rho), \tag{1.2}$$

for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1932, Hayden [14] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M,g). A semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric metric connection if $\tilde{\nabla} g = 0$.

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M,g) was given by Yano [27]: $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)\rho$, where $u(X) = g(X,\rho)$.

In 1975, Golab [15] defined and studied the quarter-symmetric connection in differentiable manifolds with affine connections. A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is called a quarter-symmetric connection [15] if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.3}$$

where η is a 1-form and ϕ is a (1,1) tensor field.

In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [13]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If, moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0, \tag{1.4}$$

for all $X, Y, Z \in \chi(M)$, then the quarter-symmetric connection $\bar{\nabla}$ is said to be a quarter-symmetric metric connection.

After Golab [15] and Rastogi ([23], [24]) the systematic study of quarter-symmetric metric connection have been continued by Mishra and Pandey [19], Yano and Imai [28], Mukhopadhyay, Roy and Barua [20], De and Biswas [10], Taleshian and Parakasha [25], Barman [3] and many others.

Let M be a Riemannian manifold of dimension n equipped with two metric tensors g and $\grave{g}$. If a transformation of M does not change the angle between two tangent vectors at a point with respect to g and $\grave{g}$, then such a transformation is said to be a conformal transformation of the metrics on the Riemannian manifold. Under conformal transformation, the length of the curves are changed but the angles made by the curves remain the same.

Let us consider a Riemannian manifold M with two metric tensors g and $\grave{g}$ such that they are related by

$$\grave{g}(X,Y) = e^{2\sigma} g(X,Y), \tag{1.5}$$

where σ is a real function on M.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general, a harmonic function is not transformed into a harmonic function. The condition under which a harmonic function remains invariant has been studied by Ishii [16] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.5) satisfying the condition

$$\sigma,^i_i + \sigma,_i \sigma,^i = 0, \tag{1.6}$$

where the comma denotes the covariant differentiation with respect to the metric g.

Let C denote the conharmonic curvature tensor of type $(1,3)$ with respect to the Levi-Civita connection which is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[&g(Y,Z)QX - g(X,Z)QY \\ &+S(Y,Z)X - S(X,Z)Y], \end{aligned} \tag{1.7}$$

where $S(Y,Z) = g(QY,Z)$.

Taking the inner product of (1.7) with W, we have

$$\begin{aligned} 'C(X,Y,Z,W) =' R(X,Y,Z,W) - \frac{1}{2n-1}[&g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+S(Y,Z)g(X,W) - S(X,Z)g(Y,W)], \end{aligned} \tag{1.8}$$

where $'C(X,Y,Z,W) = g(C(X,Y)Z,W)$, $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$, R and S are the curvature tensor and the Ricci tensor with respect to the Levi-Civita connection, respectively.

A manifold is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = a'g(X,Y)$, where a' is a constant on the manifold.

A manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on the manifold.

In this paper we study the conharmonic curvature tensor on Kenmotsu manifolds with respect to the quarter-symmetric metric connection. The paper is organized as follows. After the introduction in Section 2, we give a brief account of the Kenmotsu manifolds. In section 3, we express the quarter-symmetric metric connection on Kenmotsu manifolds. Section 4 is devoted to the study of the semi ϕ-conharmonically flat on Kenmotsu manifolds admitting the quarter-symmetric metric connection and we prove that the manifold is an Einstein manifold with respect to the Levi-Civita connection. Section 5 deals with the ξ-conharmonically flat on Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Section 6 contains the ϕ-conharmonically flat on Kenmotsu manifolds admitting the quarter-symmetric metric connection. We get the manifold to be an η- Einstein manifold with respect to the Levi-Civita connection. Finally, we construct an example of a 3-dimensional Kenmotsu manifold admitting the quarter-symmetric metric connection to support the results obtained in Section 5.

2. Kenmotsu Manifolds

Let M be an $(2n+1)$-dimensional almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and the Riemannian metric g on M satisfying [6]

$$\eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta(\phi(X)) = 0,\ g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X ,Y on $\chi(M)$. A manifold with an almost contact metric structure (ϕ,ξ,η,g) is an almost Kenmotsu manifold if the following conditions are satisfied

$$d\eta = 0; \quad d\Omega = 2\eta\wedge\Omega,$$

where Ω being the 2-form defined by $\Omega(X,Y) = g(X,\phi Y)$. Any normal almost Kenmotsu manifold is a Kenmotsu manifold. An almost contact metric structure (ϕ,ξ,η,g) is a Kenmotsu manifold [17] if and only if

$$(\nabla_X\phi)(Y) = g(\phi X,Y)\xi - \eta(Y)\phi X. \tag{2.4}$$

Here we denote the Kenmotsu manifold of dimension $(2n+1)$ by M. From the above relations, it follows that

$$g(X,\phi Y) = -g(\phi X,Y), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.9}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.10}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.11}$$

where R and S denote the curvature tensor and the Ricci tensor of M,, respectively, with respect to the Levi-Civita connection.

Let M be a Kenmotsu manifold. M is said to be a *η-Einstein manifold* if there exist real valued functions λ_1, λ_2 such that

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y).$$

For $\lambda_2 = 0$, the manifold M is an Einstein manifold.

Now we state the following:

Lemma 2.1. *[17] Let M be an η-Einstein Kenmotsu manifold of the form $S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y)$. If $\lambda_2 = constant$ $(or, \lambda_1 = constant)$, then M is an Einstein one.*

3. Quarter-symmetric metric connection on Kenmotsu manifolds

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M,g) has been obtained by Sular, *Özgür* and De [11] which is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.1}$$

Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇ and the curvature tensor $\bar{R}$ of M with respect to the quarter-symmetric metric connection $\bar{\nabla}$ [11] given by

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z + \eta(X)g(\phi Y, Z)\xi - \eta(Y)g(\phi X, Z)\xi - \\ \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X \end{aligned} \tag{3.2}$$

and

$$\bar{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X, \tag{3.3}$$

where $X, Y, Z \in \chi(M)$, the set of all differentiable vector fields on M.

The above equation (3.2) yields

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z.$$

Taking the inner product of (3.2) with W [11], we have

$$\begin{aligned} {}'\bar{R}(X,Y,Z,W) ={}' R(X,Y,Z,W) + \eta(X)\eta(W)g(\phi Y,Z) \\ -\eta(Y)\eta(W)g(\phi X,Z) - \eta(X)\eta(Z)g(\phi Y,W) \\ +\eta(Y)\eta(Z)g(\phi X,W), \end{aligned} \tag{3.4}$$

where ${}'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$, ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

A relation between the Ricci tensor $\bar{S}$ of $\bar{\nabla}$ and the Ricci tensors S of ∇ on (M,g) has been obtained by S [11] which is obtained by

$$\bar{S}(Y,Z) = S(Y,Z) + g(\phi Y,Z) \tag{3.5}$$

and also

$$\bar{S}(Y,\xi) = -2n\eta(Y). \tag{3.6}$$

In view of (3.5) yields

$$\bar{Q}Y = QY + \phi Y, \tag{3.7}$$

where $\bar{S}(Y,Z) = g(\bar{Q}Y,Z)$.

Again the scalar curvature tensor $\bar{r}$ of the quarter-symmetric metric connection $\bar{\nabla}$ and the scalar curvature tensor r of the Levi-Civita connection ∇ on (M,g) is defined by [11], so we get

$$\bar{r} = r, \tag{3.8}$$

From (2.9), it is implied that

$$r = -2n(2n+1).$$

4. Semi ϕ-conharmonically flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Let ${}'\bar{C}$ denote the conharmonic curvature tensor of type $(0,4)$ with respect to the quarter-symmetric metric connection which is defined by

$$\begin{aligned} {}'\bar{C}(X,Y,Z,W) ={}' \bar{R}(X,Y,Z,W) - \frac{1}{2n-1}[g(Y,Z)\bar{S}(X,W) - g(X,Z)\bar{S}(Y,W) \\ +\bar{S}(Y,Z)g(X,W) - \bar{S}(X,Z)g(Y,W)], \end{aligned} \tag{4.1}$$

where$'\bar{C}(X,Y,Z,W) = g(\bar{C}(X,Y)Z,W)$, $'\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$ and $X,Y,Z \in \chi(M)$, the set of all differentiable vector fields on M.

Let $\mathbf{C}$ be the Weyl conformal curvature tensor of a $(2n+1)$-dimensional manifold M. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into a direct sum $\chi_p(M) = \phi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $\chi_p(M)$ generated by ξ_p. Then we have a map:

$\mathbf{C} : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M)) \oplus L(\xi_p)$.

It may be natural to consider the following particular cases:

(1)$\mathbf{C} : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow L(\xi_p)$, i.e, the projection of the image of $\mathbf{C}$ in $\phi(\chi_p(M))$ is zero.

(2)$\mathbf{C} : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \longrightarrow \phi(\chi_p(M))$, i.e, the projection of the image of $\mathbf{C}$ in $L(\xi_p)$ is zero.

$$\mathbf{C}(X,Y)\xi = 0.$$

(3)$\mathbf{C} : \phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M)) \longrightarrow L(\xi_p)$, i.e, when $\mathbf{C}$ is restricted to $\phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M))$, the projection of the image of $\mathbf{C}$ in $\phi(\chi_p(M))$ is zero. This condition is equivalent to

$$\phi^2\mathbf{C}(\phi X, \phi Y)\phi Z = 0.$$

Here the cases 1, 2 and 3 are conformally symmetric, ξ-conformally flat and ϕ-conformally flat, respectively. The cases (1) and (2) were considered in [9] and [29], respectively. The case (3) was considered in [8] for the case M is a K-contact manifold. Furthermore, in [1], the authors studied contact metric manifolds satisfying (3). Analogous to the definition of ξ-conformally flat and ϕ-conformally flat, we give the following definitions:

Definition 4.1. A Kenmotsu manifold is said to be semi-ϕ-conharmonically flat with respect to the quarter-symmetric metric connection if

$$g(\bar{C}(\phi X, Y)Z, \phi W) = 0. \tag{4.2}$$

Definition 4.2. A Kenmotsu manifold is said to be an Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X,Y) = a'g(X,Y),$$

where a' is a constant on the manifold.

Putting $X = \phi X$ and $W = \phi W$ in (4.1), we get

$$
\begin{aligned}
{}'\bar{C}(\phi X, Y, Z, \phi W) ={}' \bar{R}(\phi X, Y, Z, \phi W) - \frac{1}{2n-1}[g(Y, Z)\bar{S}(\phi X, \phi W) -\\
g(\phi X, Z)\bar{S}(Y, \phi W) + \bar{S}(Y, Z)g(\phi X, \phi W) -\\
\bar{S}(\phi X, Z)g(Y, \phi W)].
\end{aligned}
\tag{4.3}
$$

In view of (2.1), (2.2), (3.4) and (4.3) yields

$$
\begin{aligned}
{}'\bar{C}(\phi X, Y, Z, \phi W) ={}' R(\phi X, Y, Z, \phi W) - \eta(Y)\eta(Z)g(X, \phi W)\\
-\frac{1}{2n-1}[g(Y, Z)\bar{S}(\phi X, \phi W) - g(\phi X, Z)\bar{S}(Y, \phi W)\\
+\bar{S}(Y, Z)g(\phi X, \phi W) - \bar{S}(\phi X, Z)g(Y, \phi W)].
\end{aligned}
\tag{4.4}
$$

Applying (2.1), (2.2), (2.3), (2.5) and (3.5) in (4.4), it follows that

$$
\begin{aligned}
{}'\bar{C}(\phi X, Y, Z, \phi W) ={}' R(\phi X, Y, Z, \phi W) - \eta(Y)\eta(Z)g(X, \phi W)\\
-\frac{1}{2n-1}[g(Y, Z)S(X, W) + 2n\eta(X)\eta(W)g(Y, Z) - g(Y, Z)g(X, \phi W)\\
-g(\phi X, Z)S(Y, \phi W) + g(Y, W)g(X, \phi Z) - \eta(Y)\eta(W)g(X, \phi Z)\\
+g(X, W)S(Y, Z) - \eta(X)\eta(W)S(Y, Z) + g(X, W)g(\phi Y, Z)\\
-\eta(X)\eta(W)g(\phi Y, Z) - g(Y, \phi W)S(\phi X, Z) - g(X, Z)g(\phi Y, W)\\
+\eta(X)\eta(Z)g(\phi Y, W)].
\end{aligned}
\tag{4.5}
$$

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.5) and summing over $i = 1$ to $2n$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{2n} g(\bar{C}(\phi e_i, Y)Z, \phi e_i) = \sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) - \sum_{i=1}^{2n} \eta(Y)\eta(Z)g(e_i, \phi e_i)\\
-\frac{1}{2n-1}\sum_{i=1}^{2n}[g(Y, Z)S(e_i, e_i) + 2n\eta(e_i)\eta(e_i)g(Y, Z) - g(Y, Z)g(e_i, \phi e_i)\\
-g(\phi e_i, Z)S(Y, \phi e_i) + g(Y, e_i)g(e_i, \phi Z) - \eta(Y)\eta(e_i)g(e_i, \phi Z)\\
+g(e_i, e_i)S(Y, Z) - \eta(e_i)\eta(e_i)S(Y, Z) + g(e_i, e_i)g(\phi Y, Z)\\
-\eta(e_i)\eta(e_i)g(\phi Y, Z) - g(Y, \phi e_i)S(\phi e_i, Z) - g(e_i, Z)g(\phi Y, e_i)\\
+\eta(e_i)\eta(Z)g(\phi Y, e_i)].
\end{aligned}
\tag{4.6}
$$

Again using (2.1), (2.2), (2.5) and (4.2) in (4.6), we see that

$$S(Y,Z) = -2n^2 g(Y,Z) - (n - \frac{3}{2}) g(Y, \phi Z). \tag{4.7}$$

Interchanging Y with Z in (4.7), implies that

$$S(Y,Z) = -2n^2 g(Y,Z) - (n - \frac{3}{2}) g(Z, \phi Y). \tag{4.8}$$

By adding (4.7) and (4.8) and using (2.5), we have

$$S(Y,Z) = -2n^2 g(Y,Z).$$

Therefore, $S(Y,Z) = a' g(Y,Z)$,
where $a' = -2n^2$.

This means that the manifold is an Einstein manifold with respect to the Levi-Civita connection.

Summing up we can state the following:

Theorem 4.1. *If a Kenmotsu manifold is semi-ϕ-conharmonically flat with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold.*

5. ξ -conharmonically flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Let $\bar{C}$ denote the conharmonic curvature tensor of type $(1,3)$ with respect to the quarter-symmetric metric connection which is defined by

$$\begin{aligned} \bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[&g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y \\ &+ \bar{S}(Y,Z)X - \bar{S}(X,Z)Y], \end{aligned} \tag{5.1}$$

where $\bar{S}(Y,Z) = g(\bar{Q}Y,Z)$ and $X,Y,Z \in \chi(M)$, the set of all differentiable vector fields on M.

Definition 5.1. A Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be ξ-conharmonically flat if $\bar{C}(X,Y)\xi = 0$.

Putting $Z = \xi$ in (5.1), it follows that

$$\begin{aligned} \bar{C}(X,Y)\xi = \bar{R}(X,Y)\xi - \frac{1}{2n-1}[&g(Y,\xi)\bar{Q}X - g(X,\xi)\bar{Q}Y \\ &+ \bar{S}(Y,\xi)X - \bar{S}(X,\xi)Y]. \end{aligned} \tag{5.2}$$

Using (2.1), (2.2), (3.3), (3.6) and (3.7) in (5.2), we get

$$\bar{C}(X,Y)\xi = C(X,Y)\xi + \frac{2(n-1)}{2n-1}[\eta(Y)\phi X - \eta(X)\phi Y]. \tag{5.3}$$

If $n = 1$, then the above equation (5.3) implies that

$$\bar{C}(X,Y)\xi = C(X,Y)\xi.$$

Now, we are in a position to state the following:

Theorem 5.1. *A three-dimensional Kenmotsu manifold is ξ -conharmonically flat with respect to the quarter-symmetric metric connection if the manifold is also ξ -conharmonically flat with respect to the Levi-Civita connection.*

6. ϕ -conharmonically flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Definition 6.1. A Kenmotsu manifold is said to be ϕ-conharmonically flat with respect to the quarter-symmetric metric connection if

$$g(\bar{C}(\phi X,\phi Y)\phi Z,\phi W) = 0, \tag{6.1}$$

where $X, Y, Z, W \in \chi(M)$, the set of all differentiable vector fields on M.

Definition 6.2. A Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold .

Putting $Y = \phi Y$ and $Z = \phi Z$ in (4.3), we get

$$\begin{aligned} {}'\bar{C}(\phi X,\phi Y,\phi Z,\phi W) &= {}'\bar{R}(\phi X,\phi Y,\phi Z,\phi W) \\ -\frac{1}{2n-1}[g(\phi Y,\phi Z)\bar{S}(\phi X,\phi W) &- g(\phi X,\phi Z)\bar{S}(\phi Y,\phi W) \\ +\bar{S}(\phi Y,\phi Z)g(\phi X,\phi W) &- \bar{S}(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{6.2}$$

Using (2.1), (2.2) and (3.4) in (6.2), we have

$$\begin{aligned} {}'\bar{C}(\phi X,\phi Y,\phi Z,\phi W) &= {}'R(\phi X,\phi Y,\phi Z,\phi W) \\ -\frac{1}{2n-1}[g(\phi Y,\phi Z)\bar{S}(\phi X,\phi W) &- g(\phi X,\phi Z)\bar{S}(\phi Y,\phi W) \\ +\bar{S}(\phi Y,\phi Z)g(\phi X,\phi W) &- \bar{S}(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{6.3}$$

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (6.3) and summing over $i = 1$ to $2n$, we obtain

$$\begin{aligned}\sum_{i=1}^{2n} g(\bar{C}(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \sum_{i=1}^{2n} g(R(\phi e_i, \phi Y,)\phi Z, \phi e_i)\\ -\frac{1}{2n-1}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)\bar{S}(\phi e_i, \phi e_i) &- g(\phi e_i, \phi Z)\bar{S}(\phi Y, \phi e_i)\\ +\bar{S}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) &- \bar{S}(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].\end{aligned}\tag{6.4}$$

In view of (3.8), (3.), (6.1) and (6.4), we take the form

$$\begin{aligned}S(\phi Y, \phi Z) - \frac{1}{2n-1}[-2n(2n+1)g(\phi Y, \phi Z)\\ +2(n-1)\bar{S}(\phi Y, \phi Z)] = 0.\end{aligned}\tag{6.5}$$

Applying (2.1), (2.2) and (3.5) in (6.5), it is implied that

$$S(\phi Y, \phi Z) = -2n(2n+1)g(\phi Y, \phi Z) - 2(n-1)g(Y, \phi Z).\tag{6.6}$$

Interchanging Y with Z in (6.6), we get

$$S(\phi Y, \phi Z) = -2n(2n+1)g(\phi Y, \phi Z) - 2(n-1)g(Z, \phi Y).\tag{6.7}$$

By adding (6.6) and (6.7) and using (2.5), we have

$$S(\phi Y, \phi Z) = -2n(2n+1)g(\phi Y, \phi Z).\tag{6.8}$$

By virtue of (2.3) and (6.8) we yield

$$S(Y, Z) = -2n(2n+1)g(Y, Z) + 4n^2\eta(Y)\eta(Z).$$

Therefore, $S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z)$,
where $a = -2n(2n+1)$ and $b = 4n^2$.
From which it follows that the manifold is an η-Einstein manifold.

This leads us to state the following:

Theorem 6.1. *If a Kenmotsu manifold is ϕ -conharmonically flat with respect to the quarter-symmetric metric connection, then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

Since a and b are both constant, in view of Lemma 2.1, we conclude the following:

Corollary 6.1. *If a Kenmotsu manifold is ϕ -conharmonically flat with respect to the quarter-symmetric metric connection, then the manifold is an Einstein manifold one.*

7. Example

In this section we construct an example on a Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ which verify the result in Section 3 and Section 5 of $\bar{\nabla}$.

We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. We choose the vector fields

$$e_1 = z\frac{\partial}{\partial x}, \;\; e_2 = z\frac{\partial}{\partial y}, \;\; e_3 = -z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be a 1-form defined by

$$\eta(Z) = g(Z, e_3)$$

for any $Z \in \chi(M)$.

Let ϕ be a $(1, 1)$-tensor field defined by

$$\phi e_1 = -e_2, \;\; \phi e_2 = e_1, \;\; \phi e_3 = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = 1$$

$$\phi^2(Z) = -Z + \eta(Z)e_3$$

and

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $U, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. The 1-form η is closed. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold and is also normal. So, it is a Kenmotsu manifold.

Then we have

$$[e_1, e_2] = 0, \ [e_1, e_3] = e_1, \ [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by [22]

$$\begin{aligned} 2g(\nabla_X Y, W) &= Xg(Y, W) + Yg(X, W) - Wg(X, Y) - g(X, [Y, W]) \\ &\quad - g(Y, [X, W]) + g(W, [X, Y]). \end{aligned} \tag{7.1}$$

Using Koszul's formula we get the following

$$\nabla_{e_1} e_1 = -e_3, \ \nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \ \nabla_{e_2} e_2 = -e_3, \ \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_3 = 0.$$

Using (3.1) in the above equation, we obtain

$$\bar{\nabla}_{e_1} e_1 = -e_3, \ \bar{\nabla}_{e_1} e_2 = 0, \ \bar{\nabla}_{e_1} e_3 = e_1,$$

$$\bar{\nabla}_{e_2} e_1 = 0, \ \bar{\nabla}_{e_2} e_2 = -e_3, \ \bar{\nabla}_{e_2} e_3 = e_2,$$

$$\bar{\nabla}_{e_3} e_1 = e_2, \ \bar{\nabla}_{e_3} e_2 = -e_1, \ \bar{\nabla}_{e_3} e_3 = 0.$$

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1, e_2)e_2 = -e_1, \ R(e_1, e_3)e_3 = -e_1, \ R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \ R(e_3, e_1)e_1 = -e_3, \ R(e_3, e_2)e_2 = -e_3,$$

and

$$\bar{R}(e_1, e_2)e_2 = -e_1, \ \bar{R}(e_1, e_3)e_3 = -e_2 - e_1, \ \bar{R}(e_2, e_1)e_1 = -e_2,$$

$$\bar{R}(e_2, e_3)e_3 = e_1 - e_2, \ \bar{R}(e_3, e_1)e_1 = -e_3, \ \bar{R}(e_3, e_2)e_2 = -e_3.$$

With the help of the above results we can express the Ricci tensor as follows:

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2$$

and

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = \bar{S}(e_3, e_3) = -2.$$

From the above expressions we can easily verify the equation (3.5). Also, it follows that the scalar curvature with respect to the Levi-Civita connection and quarter-symmetric metric connection is equal to -6.

Let X and Y be any two vector fields given by
$X = a_1 e_1 + a_2 e_2 + a_3 e_3$ and $Y = b_1 e_1 + b_2 e_2 + b_3 e_3$ where a_i, b_i, for all $i = 1, 2, 3$

are all non-zero real numbers.

Using the above curvature tensors and the Ricci tensors of the Levi-Civita connection and quarter-symmetric metric connection, respectively, we obtain

$$\bar{C}(X,Y)\xi = 3(a_1b_3 - a_3b_1)e_1 + 3(a_2b_3 - a_3b_2)e_2 = C(X,Y)\xi.$$

Hence, the manifold under consideration satisfies the Theorem 5.1 of Section 5.

Acknowledgements. The author wishes to express his sincere thanks and gratitude to the referee for his valuable suggestions towards the improvement of the paper.

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Ser. Math. Inform. Vol. 33, No 4 (2018), 577–586
https://doi.org/10.22190/FUMI1804577M

SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD ADMITTING A TYPE OF RICCI QUARTER-SYMMETRIC METRIC CONNECTION *

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Abstract. The aim of the present paper is to study submanifolds of a Riemannian manifold admitting a type of Ricci quater-symmetric metric connection. We have proved that the induced connection is also a Ricci quarter-symmetric metric connection. We have also considered the mean curvature and the shape operator of the submanifold with respect to the Ricci quarter-symmetric metric connection. We have obtained the Gauss, Codazzi and Ricci equations with respect to the Ricci quarter-symmetric metric connection. Finally, we have considered the totally geodesicness and obtained the relation between the sectional curvatures of the manifold and its submanifold with respect to the Ricci quarter-symmetric metric connection.
Keywords. Riemannian manifold; submanifolds; metric connection; curvature.

1. Introduction

Let ∇ be a linear connection in an $n-$dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is called non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is called non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita

Received February 13, 2018; accepted April 05, 2018.
2010 *Mathematics Subject Classification.* Primary 53B05; Secondary 53C40.
*The author is thankful to the University Grants Commission(UGC) of India because the work was supported by a grant from UGC-Minor Research Project.

connection. In 1975, S. Golab[8] introduced the notion of a quarter-symmetric linear connection in a differentiable manifold. In 1980, R. S. Mishra and S. N. Pandey [11] deduced some properties of the Riemannian, Kaehlerian and Sasakian manifolds that admits quarter-symmetric metric connection. In 1972, T. Imai[10] found some properties of a Riemannian manifold and hypersurfaces of a Riemannian manifold with a semisymmetric metric connection. In 1976, Z. Nakao[13] studied submanifolds of a Riemannian manifold with semisymmetric metric connection. Also in 1994, Agashe and Chafle[1] studied submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection. In 2010, C. Ozgur[14], studied on submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection. Also Zhao etal([9],[16],[17])studied on Riemannian manifolds with Quarter-symmetric metric connection. De etal ([7],[5],[6],[3],[12], [15]) studied Quarter-symmetric and Ricci Quarter-symmetric metric connections in Riemannian and Contact manifolds. Later in 2000, S. Ali and R. Nivas[2] studied on submanifolds immersed in a manifold with quarter-symmetric metric connection.

A linear connection is said to be a Ricci quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \pi(Y)QX - \pi(X)QY,$$

where π is a $1-$form and Q is the Ricci tensor operator defined by

$$g(QX,Y) = S(X,Y),$$

where S is the Ricci tensor of type $(0,2)$.

Motivated by these studies, in this paper we study submanifolds of a Riemannian manifold admitting a type of Ricci quarter-symmetric metric connection.

The present paper is organized as follows: After the preliminaries, in Section 3, we consider submanifold of a Riemannian manifold endowed with a Ricci quater-symmetric metric connection and show that the induced connection on a submanifold of a Riemannian manifold with a Ricci quarter-symmetric metric connection is also a Ricci quarter-symmetric metric connection. We also show that the mean curvature vector of the Riemannian manifold with respect to the Levi-Civita connection and Ricci quarter-symmetric metric connection coincide if and only if the scalar curvature vanishes. In the last part of this section we prove that the shape operators with respect to the Levi-Civita connection are simultaneously diagonalizable if and only if the shape operators with respect to the Ricci quarter-symmetric metric connection are simultaneously diagonalizable. Finally, we study the Gauss and Codazzi equation with respect to the Ricci quarter-symmetric metric connection and prove that the normal connection $\nabla^{\perp}$ is flat if and only if all second fundamental tensors with respect to the Ricci quarter-symmetric and the Levi-Civita connection are simultaneously diagonalizable and also if the submanifold is totally geodesic with respect to the Ricci quarter-symmetric metric connection, then the sectional curvature of the manifold and its submanifold are identical.

2. Preliminaries

Let $\bar{M}$ be an $m-$dimensional manifold with a Riemannian metric g and $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$[18]. We define a linear connection ∇^* on $\bar{M}$ by

$$\nabla^*_X Y = \bar{\nabla}_X Y + \pi(Y)QX - S(X,Y)\rho \tag{2.1}$$

for arbitrary vector fields X, Y of $\bar{M}$, where ρ is the vector field defined by $g(X,\rho) = \pi(X)$ and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$,S is the Ricci tensor of $(0,2)-$type.

Using (2.1), the torsion tensor T^* with respect to the connection ∇^* is given by

$$T^*(X,Y) = \pi(Y)QX - \pi(X)QY. \tag{2.2}$$

A linear connection ∇^* satisfying the condition (2.2) is called a Ricci quarter-symmertic connection. Also using (2.1), we have

$$(\nabla^*_Z g)(X,Y) = (\bar{\nabla}_Z g)(X,Y) = 0.$$

Hence the connection is a metric connection.

We denote by R^* the curvature tensor of $\bar{M}$ with respect to the Ricci quarter-symmetric metric connection ∇^*. So we have

$$\begin{aligned}
R^*(X,Y)Z &= \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z \\
&= \bar{R}(X,Y)Z + (\bar{\nabla}_X\pi)(Z)QY - (\bar{\nabla}_Y\pi)(Z)QX \\
&\quad + \pi(Z)\{(\bar{\nabla}_X Q)Y - (\bar{\nabla}_Y Q)X\} + \{(\bar{\nabla}_Y S)(X,Z) \\
&\quad - (\bar{\nabla}_X S)(Y,Z)\}\rho + S(X,Z)\bar{\nabla}_Y\rho - S(Y,Z)\bar{\nabla}_X\rho \\
&\quad + \pi(Z)\{\pi(QY)QX - \pi(QX)QY + S(Y,QX)X \\
&\quad - S(X,QY)Y\} + \pi(\rho)\{S(X,Z)QY - S(Y,Z)QX\} \\
&\quad + \{S(Y,Z)S(X,\rho) - S(X,Z)S(Y,\rho)\}\rho,
\end{aligned} \tag{2.3}$$

where $\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z$ is the curvature tensor of the manifold with respect to the Levi-Civita connection $\bar{\nabla}$. The Riemannian curvature tensors of the connections ∇^* and $\bar{\nabla}$ are defined by

$$R^*(X,Y,Z,W) = g(R^*(X,Y,Z),W)$$

and

$$\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y,Z),W)$$

respectively.

3. submanifolds of a Riemannian manifold with a Ricci quarter-symmetric metric connection

Let M be an $n-$dimensional submanifold of a Riemannian manifold $\bar{M}$ with a Ricci quarter-symmetric metric connection. Decomposing the vector field ρ on M uniquely into their tangent and normal components ρ^T and $\rho^\perp$ respectively we have

$$\rho = \rho^T + \rho^\perp.$$

The Gauss formula for a submanifold M of a Riemannian manifold $\bar{M}$ with respect to the Riemannian connection $\bar{\nabla}$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y), \tag{3.1}$$

where X and Y are vector fields tangent to M and m is the second fundamental form of M in $\bar{M}$. If $m = 0$, then M is called totally geodesic with respect to the Riemannian connection. $H = \frac{1}{n} trace\, m$ is called the mean curvature vector of the submanifold. If $H = 0$, then M is called minimal. For the second fundamental form m, the covariant derivative of m is defined by

$$(\bar{\nabla}_X m)(Y, Z) = \nabla^\perp_X m(Y, Z) - m(\nabla_X Y, Z) - m(Y, \nabla_X Z).$$

for any vector field X tangent to M. $\bar{\nabla}$ is called the Van der Waerden-Bortolotti connection of M, that is, $\bar{\nabla}$ is the connection in $TM \oplus T^\perp M$ built with ∇ and $\nabla^\perp$[4].

Let $\dot{\nabla}$ be the induced connection from the Ricci quarter-symmetric metric connection. We define

$$\nabla^*_X Y = \dot{\nabla}_X Y + \dot{m}(X, Y), \tag{3.2}$$

where $\dot{m}$ is the induced second fundamental form.

The equation (3.2) is the Gauss equation with respect to the Ricci quarter-symmetric metric connection ∇^*

Using (2.1), from (3.1) and (3.2) we have

$$\begin{aligned} \dot{\nabla}_X Y + \dot{m}(X, Y) &= \nabla_X Y + m(X, Y) + \pi(Y) QX \\ &\quad - S(X, Y)\rho^T - S(X, Y)\rho^\perp. \end{aligned} \tag{3.3}$$

Now taking the tangential and normal parts we have

$$\dot{\nabla}_X Y = \nabla_X Y + \pi(Y) QX - S(X, Y)\rho^T \tag{3.4}$$

and

$$\dot{m}(X, Y) = m(X, Y) - S(X, Y)\rho^\perp. \tag{3.5}$$

If $\dot{m} = 0$, then M is called totally geodesic with respect to the Ricci quarter-symmetric connection.

From (3.4), we have

$$\dot{T}(X,Y) = \dot{\nabla}_X Y - \dot{\nabla}_Y X - [X,Y] = \pi(Y)QX - \pi(X)QY, \tag{3.6}$$

where $\dot{T}$ is the torsion tensor of M with respect to $\dot{\nabla}$ and X, Y are vector fields tangent to M.

Moreover using (3.4) we have

$$(\dot{\nabla}_X g)(Y,Z) = (\nabla_X g)(Y,Z). \tag{3.7}$$

In view of (2.1), (3.4), (3.6) and (3.7) we can state the following:

Theorem 3.1. *The induced connection on a submanifold of a Riemannian manifold with a Ricci quarter-symmetric metric connection is also a Ricci quarter-symmetric metric connection.*

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space of M. We define the mean curvature vector $\dot{H}$ of M with respect to the Ricci quarter-symmetric metric connection $\dot{\nabla}$ by

$$\dot{H} = \frac{1}{n}\sum_{i=1}^{n} \dot{m}(e_i, e_i).$$

So from (3.5), we find

$$\dot{H} = H - \frac{r}{n}\rho^{\perp}.$$

If $H = 0$, then M is called minimal with respect to the Ricci quarter-symmetric metric connection.

So we have the following result:

Theorem 3.2. *If M be an $n-$dimensional submanifold of an $m-$dimensional Riemannian manifold $\bar{M}$, then the mean curvature vector of M with respect to the Levi-Civita connection and the Ricci quarter-symmetric metric connection coincide if and only if the scalar curvature vanishes.*

Let N be a normal vector field on M. From (2.1), we have

$$\nabla^*_X N = \bar{\nabla}_X N + \pi(N)QX. \tag{3.8}$$

The usual Weingarten formula is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad N \in T^{\perp}(M) \tag{3.9}$$

where $-A_N X$ and $\nabla^{\perp}_X N$ are the tangential and normal parts of $\bar{\nabla}_X N$. From (3.8) and (3.9) we get

$$\nabla^*_X N = -\dot{A}_N X + \nabla^{\perp}_X N, \quad \textit{where } \dot{A}_N = (A_N - \pi(N)Q)I, \tag{3.10}$$

which is the Weingarten formulae for a submanifold of a Riemannian manifold with respect to the Ricci quarter-symmetric metric connection.

Since A_N is symmetric, it is easy to see that

$$g(\dot{A}_N X, Y) = g(X, \dot{A}_N Y)$$

and

$$g([\dot{A}_N, \dot{A}_L]X, Y) = g(X, [A_N, A_L]Y), \tag{3.11}$$

where $[\dot{A}_N, \dot{A}_L] = \dot{A}_N \dot{A}_L - \dot{A}_L \dot{A}_N$, $[A_N, A_L] = A_N A_L - A_L A_N$, N and L are unit normal vector fields on M.

Theorem 3.3. *If M be an $n-$dimensional submanifold of an $m-$dimensional Riemannian manifold $\bar{M}$ admitting the Ricci quarter-symmetric metric connection, then the shape operators with respect to Levi-Civita connection are simultaneously diagonalizable if and only if the shape operators with respect to then Ricci quarter-symmetric metric connection are simultaneously diagonalizable.*

4. Gauss and codazzi equation with respect to the Ricci quarter-symmetric metric connection

We denote the curvature tensor of $\bar{M}$ with respect to the Ricci quarter-symmetric metric connection ∇^* by

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$$

and that of M with respect to the induced Ricci quarter-symmetric metric connection $\dot{\nabla}$ by

$$\dot{R}(X,Y)Z = \dot{\nabla}_X \dot{\nabla}_Y Z - \dot{\nabla}_Y \dot{\nabla}_X Z - \dot{\nabla}_{[X,Y]} Z.$$

We shall now find the equation of Gauss-Codazzi with respect to the Ricci quarter-symmetric metric connection.

$$\begin{aligned} R^*(X,Y)Z &= \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z \\ &= \dot{\nabla}_X \dot{\nabla}_Y Z + \dot{m}(X, \dot{\nabla}_Y Z)) - \dot{A}_{\dot{m}(Y,Z)} X + \nabla^{\perp}_X \dot{m}(Y,Z) \\ &\quad - \dot{\nabla}_Y \dot{\nabla}_X Z - \dot{m}(Y, \dot{\nabla}_X Z)) + \dot{A}_{\dot{m}(X,Z)} Y - \nabla^{\perp}_Y \dot{m}(X,Z) \\ &\quad - \dot{\nabla}_{[X,Y]} Z - \dot{m}([X,Y],Z) \\ &= \dot{R}(X,Y)Z - \dot{A}_{\dot{m}(Y,Z)} X + \dot{A}_{\dot{m}(X,Z)} Y \\ &\quad + \dot{m}(X, \dot{\nabla}_Y Z) - \dot{m}(Y, \dot{\nabla}_X Z) - \dot{m}([X,Y],Z) \\ &\quad + \nabla^{\perp}_X \dot{m}(Y,Z) - \nabla^{\perp}_Y \dot{m}(X,Z). \end{aligned} \tag{4.1}$$

Taking account of (3.10), we have

$$\begin{aligned} R^*(X,Y)Z &= \dot{R}(X,Y)Z - A_{\dot{m}(Y,Z)}X - \pi(\dot{m}(Y,Z))X + A_{\dot{m}(X,Z)}Y \\ &\quad +\pi(\dot{m}(X,Z))Y + \dot{m}(X,\dot{\nabla}_Y Z) - \dot{m}(Y,\dot{\nabla}_X Z) \\ &\quad -\dot{m}([X,Y],Z) + \nabla^{\perp}_X \dot{m}(Y,Z) - \nabla^{\perp}_Y \dot{m}(X,Z). \end{aligned} \tag{4.2}$$

Since $g(A_N X, Y) = g(h(X,Y),N)$, using (3.5) we obtain

$$\begin{aligned} R^*(X,Y,Z,W) &= \dot{R}(X,Y,Z,W) - g(A_{\dot{m}(Y,Z)}X,W) + g(A_{\dot{m}(X,Z)}Y,W) \\ &\quad +\pi(\dot{m}(Y,Z))g(QX,W) - \pi(\dot{m}(X,Z))g(QY,W) \\ &= \dot{R}(X,Y,Z,W) - g(m(Y,Z),m(X,W)) \\ &\quad +g(m(X,Z),m(Y,W)) + S(Y,Z)\pi(m(X,W)) \\ &\quad -S(X,Z)\pi(m(Y,W)) + S(X,W)\pi(m(Y,Z)) \\ &\quad -S(Y,W)\pi(m(X,Z)) + \pi(\rho^{\perp})[S(X,Z)S(Y,W) \\ &\quad -S(Y,Z)S(X,W)], \end{aligned} \tag{4.3}$$

where W is a tangent vector field on M.

From (4.2), the normal component of $R^*(X,Y)Z$ is given by

$$\begin{aligned} (R^*(X,Y)Z)^{\perp} &= \dot{m}(X,\dot{\nabla}_Y Z) - \dot{m}(Y,\dot{\nabla}_X Z) - \dot{m}([X,Y],Z) \\ &\quad +\nabla^{\perp}_X \dot{m}(Y,Z) - \nabla^{\perp}_Y \dot{m}(X,Z) \\ &= (\nabla^{\perp}_X \dot{m})(Y,Z) - (\nabla^{\perp}_Y \dot{m})(X,Z) \\ &\quad +\pi(Y)\dot{m}(QX,Z) - \pi(X)\dot{m}(QY,Z), \end{aligned} \tag{4.4}$$

where $(\nabla^{\perp}_X \dot{m})(Y,Z) = \nabla^{\perp}_X \dot{m}(Y,Z) - \dot{m}(\dot{\nabla}_X Y, Z) - \dot{m}(\dot{\nabla}_Y X, Z)$.

It is called the van der Waerden-Bortolotti connection with respect to the Ricci quarter-symmetric metric connection.

Also the equation (4.4) is the equation of Codazzi with respect to the Ricci quarter-symmetric metric connection.

From (3.2) and (3.10), we get

$$\begin{aligned} \nabla^*_X \nabla^*_Y N_1 &= -\dot{A}_{\nabla^{\perp}_Y N_1} X - \dot{m}(\dot{A}_{N_1} Y, X) \\ &\quad +\nabla^{\perp}_X \nabla^{\perp}_Y N_1 - \dot{\nabla}_X \dot{A}_{N_1} Y, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \nabla^*_Y \nabla^*_X N_1 &= -\dot{A}_{\nabla^{\perp}_X N_1} Y - \dot{m}(\dot{A}_{N_1} X, Y) \\ &\quad +\nabla^{\perp}_Y \nabla^{\perp}_X N_1 - \dot{\nabla}_Y \dot{A}_{N_1} X \end{aligned} \tag{4.6}$$

and

$$\nabla^*_{[X,Y]} N_1 = -\dot{A}_{N_1}[X,Y] + \nabla^{\perp}_{[X,Y]} N_1. \tag{4.7}$$

So using (4.5)-(4.7), we have

$$\begin{aligned} R^*(X,Y,N_1,N_2) &= R^{\perp}(X,Y,N_1,N_2) - g(\dot{m}(\dot{A}_{N_1}Y,X),N_2) \\ &\quad +g(\dot{m}(\dot{A}_{N_1}X,Y),N_2), \end{aligned} \tag{4.8}$$

where N_1 and N_2 are normal vector fields on M. Hence in view of (3.5) and (3.10) the equation (4.8) turns into

$$\begin{aligned} R^*(X,Y,N_1,N_2) &= R^{\perp}(X,Y,N_1,N_2) - g(m(A_{N_1}Y,X),N_2) \\ &\quad +S(A_{N_1}Y,X)g(\rho^{\perp},N_2) + g(m(A_{N_1}X,Y),N_2) \\ &\quad -S(A_{N_1}X,Y)g(\rho^{\perp},N_2) \\ &= R^{\perp}(X,Y,N_1,N_2) + g([N_1,N_2]X,Y). \end{aligned} \tag{4.9}$$

The equation (4.9) is the equation of Ricci with respect to the Ricci quarter-symmetric metric connection.

If $\bar{M}$ is a space of constant curvature c with respect to the connection $\bar{\nabla}$, then the equation (2.2) reduce to

$$\begin{aligned} R^*(X,Y)Z &= c\{g(y,Z)X - g(X,Z)Y\} + (\bar{\nabla}_X\pi)(Z)QY - (\bar{\nabla}_Y\pi)(Z)QX \\ &\quad +\pi(Z)\{(\bar{\nabla}_XQ)Y - (\bar{\nabla}_YQ)X\} + \{(\bar{\nabla}_YS)(X,Z) \\ &\quad -(\bar{\nabla}_XS)(Y,Z)\}\rho + S(X,Z)\bar{\nabla}_Y\rho - S(Y,Z)\bar{\nabla}_X\rho \\ &\quad +\pi(Z)\{\pi(QY)QX - \pi(QX)QY + S(Y,QX)X \\ &\quad -S(X,QY)\} + \pi\{(S(X,Z)QY - S(Y,Z)QX\} \\ &\quad +\{S(Y,Z)S(X,\rho) - S(X,Z)S(Y,\rho)\}\rho. \end{aligned} \tag{4.10}$$

From (4.10)we have $R^*(X,Y,N_1,N_2) = 0$. Therefore using (3.11) and (4.9) we obtain

$$R^{\perp}(X,Y,N_1,N_2) = g([N_2,N_1]X,Y) = g([N_2^*,N_1^*]X,Y).$$

Hence we can state the following theorem:

Theorem 4.1. *If M be an $n-$dimensional submanifold of an $m-$dimensional space of constant curvature $\bar{M}(c)$ admitting Ricci quarter-symmetric metric connection, then the normal connection $\nabla^{\perp}$ is flat if and only if all second fundamental tensors with respect to the Ricci quarter-symmetric and the Levi-Civita connection are simultaneously diagonalizable.*

From the equation (4.3) we have

$$R^*(X,Y,Y,X) = \dot{R}(X,Y,Y,X) - g(m(X,X),m(Y,Y))$$

$$
\begin{aligned}
&+g(m(X,Y),m(Y,X))+S(Y,Y)\pi(m(X,X))\\
&-S(X,X)\pi(m(Y,Y))-S(X,Y)\pi(m(X,Y))\\
&-S(Y,X)\pi(m(X,Y))+\pi(\rho^{\perp})[S(X,Y)S(Y,X)\\
&-S(Y,Y)S(X,X)]. \qquad (4.11)
\end{aligned}
$$

Now if the sectional curvature of $\bar{M}$ and M at a point $p \in \bar{M}$ with respect to the Ricci quarter-symmetric metric connection is denoted by κ^* and $\dot{\kappa}$ respectively, then the equation (4.11) reduce to

$$
\begin{aligned}
\kappa^* \quad = \quad & \dot{\kappa}-g(m(X,X),m(Y,Y))+g(m(X,Y),m(Y,X))\\
&+S(Y,Y)\pi(m(X,X))-S(X,X)\pi(m(Y,Y))\\
&-S(X,Y)\pi(m(X,Y))-S(Y,X)\pi(m(X,Y))\\
&+\pi(\rho^{\perp})[S(X,Y)S(Y,X)-S(Y,Y)S(X,X)]. \qquad (4.12)
\end{aligned}
$$

If we consider M is totally geodesic with respect to the Ricci quarter-symmetric metric connection, then from (3.5) we get $m(X,Y) = S(X,Y)\rho^{\perp}$ and using this result, the equation (4.12) becomes

$$\kappa^* = \dot{\kappa}.$$

Hence we have the following theorem:

Theorem 4.2. *Let M be an $n-$dimensional submanifold of an $m-$dimensional Riemannian manifold $\bar{M}$ admitting a Ricci quarter-symmetric metric connection. If M is totally geodesic with respect to the Ricci quarter-symmetric metric connection, then the sectional curvatures κ^* and $\dot{\kappa}$ of $\bar{M}$ and M (resp.) are identical.*

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Ser. Math. Inform. Vol. 33, No 4 (2018), 587–597
https://doi.org/10.22190/FUMI1804587N

SOME RESULTS ON $(k, \mu)'$-ALMOST KENMOTSU MANIFOLDS *

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Abstract. In this paper, we study the quasi-conformal curvature tensor $\tilde{C}$ and projective curvature tensor P on a $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1} of dimension greater than 3. We obtain that if M^{2n+1} is non-Kenmotsu and satisfies $R \cdot \tilde{C} = 0$ or $P \cdot P = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Keywords: Almost Kenmotsu manifold, $(k, \mu)'$-nullity condition, quasi-conformal curvature tensor, projective curvature tensor.

1. Introduction

In 1972, K. Kenmotsu introduced a new class of almost contact metric manifolds, nowadays known as Kenmotsu manifolds [8]. The concept of almost Kenmotsu manifolds, regarded as a generalization of Kenmotsu manifolds, was studied by Janssens and Vanhecke (see [4]). In 2007, Pitiş [7] published a book containing many systematic studies related to Kenmotsu manifolds. Some geometric properties and fundamental formulas of almost Kenmotsu manifolds were obtained by Kim and Pak [11] and Pastore et al. [5, 6]. Several authors studied almost Kenmotsu manifolds considering some curvature conditions (see [12, 13, 14]). Recently, some curvature properties of some types of almost Kenmotsu manifolds were obtained by Wang and Liu in [15, 16, 17, 18].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat (see [2]). For $n \geq 1$, M is locally projectively flat if and

Received March 12, 2018; accepted May 15, 2018
2010 *Mathematics Subject Classification.* Primary 53D15; Secondary 53C25
*The authors were supported in part by the National Natural Science Foundation of China (No. 11431009)

only if the projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)U = R(X,Y)U - \frac{1}{2n}[S(Y,U)X - S(X,U)Y] \tag{1.1}$$

for any vector fields $X, Y, U \in \mathfrak{X}(M)$, where S is the Ricci tensor of M.

The Weyl conformal curvature tensor C on a $(2n+1)$-dimensional manifold M is defined by [20]

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y] \\ &- \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \end{aligned} \tag{1.2}$$

for any vector fields X, Y, Z on M, where S, Q and r denote the Ricci curvature tensor, the Ricci operator with respect to the metric g and the scalar curvature, respectively. Note that the Weyl conformal curvature tensor on any three dimension Riemannian manifold vanishes.

For a $(2n+1)$-dimensional manifold M, the quasi-conformal curvature tensor $\tilde{C}$ is defined by [21]

$$\begin{aligned} \tilde{C}(X,Y)Z =& aR(X,Y)Z - \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y,Z)X - g(X,Z)Y] \\ &+ b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.3}$$

where a and b are two constants. If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to the Weyl conformal curvature tensor.

In this paper, we aim to extend some known results regarding the projective and quasi-conformal curvature tensor on Kenmotsu manifolds (see [1, 2, 9, 10]) to a class of almost Kenmotsu manifolds. In Section 2, we recall some basic formulas and properties of almost Kenmotsu manifolds and the notion of $(k,\mu)'$-almost Kenmotsu manifolds. In Section 3, we introduce some properties of such manifolds used to prove our main results. In Section 4 and 5, we classify almost Kenmotsu manifolds satisfying $R \cdot \tilde{C} = 0$ and $P \cdot P = 0$, respectively.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost contact metric manifold of dimension $2n+1$, equipped with an almost contact metric structure (ϕ, ξ, η, g) (see [3]) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi) \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(M)$, where ϕ, ξ, η, g and $\mathfrak{X}(M)$ denote a $(1,1)$-tensor field, a vector field, a 1-form, the Riemannian metric and the Lie algebra of all differentiable vector fields on M^{2n+1}, respectively.

The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any fields $X, Y \in \mathfrak{X}(M)$. M^{2n+1} is called an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The almost contact metric manifold is said to be normal if the Nijenhuis tensor of ϕ is given by $[\phi,\phi] = -2d\eta \otimes \xi$, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold [4].

On an almost Kenmotsu manifold M^{2n+1}, the two $(1,1)$-type tensor fields $l = R(\cdot,\xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi\phi$ are symmetric, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. Then we get

$$h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0. \tag{2.3}$$

We also have the following formulas presented in [5, 6]:

$$\nabla_X\xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi\xi = 0), \tag{2.4}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$trl = S(\xi,\xi) = g(Q\xi,\xi) = -2n - trh^2, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X \tag{2.7}$$

for any $X, Y \in \mathfrak{X}(M)$, where $h' = h \circ \phi$ and S, Q, ∇, $\mathfrak{X}(M)$ denote the Ricci tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the Lie algebra of all vector fields on M^{2n+1}, respectively.

3. Some properties of $(k,\mu)'$-almost Kenmotsu manifolds

If the characteristic vector field ξ of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfies the $(k,\mu)'$-nullity condition (see [6]), then it is called a $(k,\mu)'$-almost Kenmotsu manifold. The $(k,\mu)'$-nullity condition is defined as follows:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y] \tag{3.1}$$

for any vector fields X, Y, where both k and μ are constant on M^{2n+1}. M^{2n+1} is said to be a (k,μ)-almost manifold Kenmotsu manifold if there holds $R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$ for any vector fields X, Y and $k, \mu \in \mathbb{R}$. A (k,μ)-almost Kenmotsu manifold satisfies $k = -1$ and $h = 0$ (see [6]). A (k,μ)-almost Kenmotsu manifold is a special case of $(k,\mu)'$-almost Kenmotsu manifolds. Following [6], on any $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1}, we have

$$h'^2 X = -(k+1)X + (k+1)\eta(X)\xi \tag{3.2}$$

for any vector field $X \in \mathfrak{X}(M)$ and $\mu = -2$. From (3.2), we know that $h' = 0$ is equivalent to $k = -1$ and $h' \neq 0$ everywhere if and only if $k < -1$. Furthermore, by (3.1) and the symmetry of the Riemannian curvature tensor R, it is easy to see that

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] - 2[g(h'X,Y)\xi - \eta(Y)h'X] \tag{3.3}$$

for any $X, Y \in \mathfrak{X}(M)$. In case of $k < -1$, we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces of h' corresponding two eigenvalues $\lambda > 0$ and $-\lambda$, respectively. Obviously, by (3.2), we have

$$\lambda = \sqrt{-k-1} > 0. \tag{3.4}$$

Before presenting one of our main results, we give the following two lemmas.

Lemma 3.1. [6, Proposition 4.2] *Let M^{2n+1} be a $(k,\mu)'$-almost Kenmotsu manifold such that $h' = 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies*

$$\begin{aligned}
&(3.5) & R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0,\\
&(3.6) & R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0,\\
&(3.7) & R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},\\
&(3.8) & R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,\\
&(3.9) & R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\\
&(3.10) & R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

Lemma 3.2. [18, Lemma 3.2] *Let M^{2n+1} be a $(k,\mu)'$-almost Kenmotsu manifold such that $h' \neq 0$. Then the Ricci operator of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.11}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Proof. See the proof of [19, Lemma 3.2]. □

4. $(k,\mu)'$-almost Kenmotsu manifolds satisfying $R(X,Y)\cdot\tilde{C} = 0$

In this section, we consider a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} satisfying the condition

$$R(X,Y)\cdot\tilde{C} = 0, \tag{4.1}$$

or equivalently

$$\begin{aligned}
(R(X,Y)\cdot\tilde{C})(U,V)W &= R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U,V)W\\
&\quad - \tilde{C}(U,R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W\\
&= 0
\end{aligned} \tag{4.2}$$

for any $X, Y, U, V, W \in \mathfrak{X}(M)$.

From the definition of $\tilde{C}$ (see (1.3)), we have

$$
\begin{aligned}
\tilde{C}(\xi,Y)Z = {} & [ak - \frac{r}{2n+1}(\frac{a}{2n}+2b) + 2nkb - 2nb]g(Y,Z)\xi \\
& - [ak - \frac{r}{2n+1}(\frac{a}{2n}+2b) + 2nkb - 2nb]\eta(Z)Y \\
& - (-a\mu + 2nb)g(h'Y,Z)\xi + (-a\mu+2nb)\eta(Z)h'Y,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\tilde{C}(\xi,Y)\xi = {} & [ak - \frac{r}{2n+1}(\frac{a}{2n}+2b) + 2nkb - 2nb]\eta(Y)\xi \\
& - [ak - \frac{r}{2n+1}(\frac{a}{2n}+2b) + 2nkb - 2nb]Y \\
& + (-a\mu + 2nb)h'Y,
\end{aligned} \tag{4.4}
$$

where r, a and b denote the scalar curvature and two constants, respectively. Let us denote by $A = [ak - \frac{r}{2n+1}(\frac{a}{2n}+2b) + 2nkb - 2nb]$, $B = -A$, $D = (-a\mu + 2nb)$ and $E = -D$.

Substituting $X = U = \xi$ in (4.2) we have

$$
\begin{aligned}
(R(\xi,Y)\cdot\tilde{C})(\xi,V)W &= R(\xi,Y)\tilde{C}(\xi,V)W - \tilde{C}(R(\xi,Y)\xi,V)W \\
& - \tilde{C}(\xi,R(\xi,Y)V)W - \tilde{C}(\xi,V)R(\xi,Y)W \\
&= 0
\end{aligned} \tag{4.5}
$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Making use of (3.3), (4.3) and (4.4) we calculate every term in equation (4.5) straightly. Then we have

$$
\begin{aligned}
& R(\xi,Y)\tilde{C}(\xi,V)W \\
= & k[g(Y,\tilde{C}(\xi,V)W)\xi - \eta(\tilde{C}(\xi,V)W))Y] \\
& + \mu[g(h'Y,\tilde{C}(\xi,V)W)\xi - \eta(\tilde{C}(\xi,V)W))h'Y] \\
= & k\{A[\eta(Y)g(V,W)\xi - \eta(W)g(Y,V)\xi] \\
& + E[\eta(Y)g(h'V,W)\xi - \eta(W)g(Y,h'V)\xi]\} \\
& - k\{A[g(V,W)Y - \eta(W)\eta(V)Y] + Eg(h'V,W)Y\} \\
& + \mu\{-A\eta(W)g(h'Y,V)\xi - E\eta(W)g(h'Y,h'V)\xi\} \\
& - \mu\{A[g(V,W)h'Y - \eta(W)\eta(V)h'Y] + Eg(h'V,W)h'Y\}.
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
& \tilde{C}(R(\xi,Y)\xi,V)W \\
= & k\eta(Y)\tilde{C}(\xi,V)W - k\tilde{C}(Y,V)W - \mu\tilde{C}(h'Y,V)W \\
= & k\{A[\eta(Y)g(V,W)\xi - \eta(W)\eta(Y)V] + E[\eta(Y)g(h'V,W)\xi \\
& - \eta(W)\eta(Y)h'V]\} - k\tilde{C}(Y,V)W - \mu\tilde{C}(h'Y,V)W.
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
&\tilde{C}(\xi, R(\xi, Y)V)W \\
=&kg(Y,V)\tilde{C}(\xi,\xi)W - k\eta(V)\tilde{C}(\xi,Y)W \\
&+ \mu g(h'Y,V)\tilde{C}(\xi,\xi)W - \mu\eta(V)\tilde{C}(\xi,h'Y)W \\
=&-k\{A[\eta(V)g(Y,W)\xi - \eta(W)\eta(V)Y] \\
&+ E[\eta(V)g(h'Y,W)\xi - \eta(W)\eta(V)h'Y]\} \\
&- \mu\{A[\eta(V)g(h'Y,W)\xi - \eta(W)\eta(V)h'Y]\} \\
&+ E[\eta(V)g(h'^2Y,W)\xi - \eta(W)\eta(V)h'^2Y]\}.
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&\tilde{C}(\xi, V)R(\xi, Y)W \\
=&kg(Y,W)\tilde{C}(\xi,V)\xi - k\eta(W)\tilde{C}(\xi,V)Y \\
&+ \mu g(h'Y,W)\tilde{C}(\xi,V)\xi - \mu\eta(W)\tilde{C}(\xi,V)h'Y \\
=&k\{A[g(Y,W)\eta(V)\xi - g(Y,W)V] + Dg(Y,W)h'V\} \\
&- k\{A[\eta(W)g(V,Y)\xi - \eta(Y)\eta(W)V] \\
&+ E[\eta(W)g(h'V,Y)\xi - \eta(Y)\eta(W)h'V]\} \\
&+ \mu\{A[g(h'Y,W)\eta(V)\xi - g(h'Y,W)V] + Dg(h'Y,W)h'V\} \\
&- \mu\{A\eta(W)g(V,h'Y)\xi + E\eta(W)g(h'V,h'Y)\xi\}
\end{aligned}
\tag{4.9}
$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Substituting (4.6)-(4.9) into (4.5) and using (3.2) gives

$$
\begin{aligned}
&k\tilde{C}(Y,V)W + \mu\tilde{C}(h'Y,V)W - kAg(V,W)Y \\
&-kEg(h'V,W)Y - \mu Ag(V,W)h'Y - \mu Eg(h'V,W)h'Y \\
&+kE\eta(V)g(h'Y,W)\xi - kE\eta(W)\eta(V)h'Y - \mu E(k+1)\eta(V)g(Y,W)\xi \\
&+\mu E(k+1)\eta(V)\eta(W)Y + kAg(Y,W)V + kEg(Y,W)h'V \\
&+\mu Ag(h'Y,W)V + \mu Eg(h'Y,W)h'V = 0
\end{aligned}
\tag{4.10}
$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Substituting $Y = h'Y$ in (4.10) and using (3.2) we obtain

$$
\begin{aligned}
&k\tilde{C}(h'Y,V)W - \mu(k+1)\tilde{C}(Y,V)W - kAg(V,W)h'Y \\
&-kEg(h'V,W)h'Y + \mu A(k+1)g(V,W)Y + \mu E(k+1)g(h'V,W)Y \\
&-kE(k+1)\eta(V)g(Y,W)\xi + kE(k+1)\eta(V)\eta(W)Y \\
&-\mu E(k+1)\eta(V)g(h'Y,W)\xi + \mu E(k+1)\eta(V)\eta(W)h'Y + kAg(h'Y,W)V \\
&+kEg(h'Y,W)h'V - \mu A(k+1)g(Y,W)V - \mu E(k+1)g(Y,W)h'V = 0
\end{aligned}
\tag{4.11}
$$

for any $Y, V, W \in \mathfrak{X}(M)$. Subtracting μ multiple of (4.11) from k multiple of (4.10) and using $\mu = -2$ implies

$$
\begin{aligned}
&(k+2)^2\tilde{C}(Y,V)W - (k+2)^2\{Ag(V,W)Y + Eg(h'V,W)Y \\
&-E\eta(V)g(h'Y,W)\xi + E\eta(V)\eta(W)h'Y - Ag(Y,W)V - Eg(Y,W)h'V\} = 0
\end{aligned}
\tag{4.12}
$$

for any $Y, V, W \in \mathfrak{X}(M)$. Next, we assume that $Y = V = W \in [-\lambda]'$ in (1.3), where $[-\lambda]'$ is eigenspace of h' corresponding eigenvalue $-\lambda$. Thus, by applying Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned} &\tilde{C}(Y,V)W \\ &=[a(k+2\lambda) - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 4nb(\lambda - 1)][g(V,W)Y - g(Y,W)V] \end{aligned} \tag{4.13}$$

for any $Y, V, W \in \mathfrak{X}(M)$.

With the help of (4.13) and assuming $Y = V = W \in [-\lambda]'$, from (4.12) we get

$$2nb(k+2)^2(\lambda - 1 - k)[g(V,W)Y - g(Y,W)V] = 0. \tag{4.14}$$

Putting (3.4) into (4.14) we have

$$\lambda(\lambda - 1)^2(\lambda + 1)^3 = 0. \tag{4.15}$$

In view of the fact $\lambda > 0$, we obtain $\lambda = 1$ and hence $k = -2$. From [6, Corollary 4.2] and [5, Theorem 6], we know that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Therefore we have the following:

Theorem 4.1. *If a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} of dimension greater than 3 satisfies $R \cdot \tilde{C} = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Since quasi-conformally symmetric manifold ($\nabla \tilde{C} = 0$) implies $R \cdot \tilde{C} = 0$, therefore from Theorem 4.1 we state the following:

Corollary 4.1. *A quasi-conformally symmetric non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold $M^{2n+1}(n > 1)$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Since $R \cdot R$ implies $R \cdot \tilde{C} = 0$, we get the following:

Corollary 4.2. *A semisymmetric non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold $M^{2n+1}(n > 1)$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

The above corollary has been proved by Wang and Liu [15].

5. $(k,\mu)'$-almost Kenmotsu manifolds satisfying $P(X,Y) \cdot P = 0$

In this section, we consider a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifolds M^{2n+1} satisfying the condition

$$P(X,Y) \cdot P = 0, \tag{5.1}$$

which implies

$$(5.2)\quad \begin{aligned}&(P(X,Y)\cdot P)(U,V)W\\ =&P(X,Y)P(U,V)W-P(P(X,Y)U,V)W\\ &-P(U,P(X,Y)V)W-P(U,V)P(X,Y)W\\ =&0\end{aligned}$$

for any $X,Y,U,V,W\in\mathfrak{X}(M)$.

Making use of (1.1), we get

$$(5.3)\quad \begin{aligned}&P(X,Y)P(U,V)W\\ &=R(X,Y)R(U,V)W-\frac{1}{2n}S(V,W)R(X,Y)U+\frac{1}{2n}S(U,W)R(X,Y)V\\ &-\frac{1}{2n}\{S(Y,R(U,V)W)X-\frac{1}{2n}S(V,W)S(Y,U)X+\frac{1}{2n}S(U,W)S(Y,V)X\}\\ &+\frac{1}{2n}\{S(X,R(U,V)W)Y-\frac{1}{2n}S(V,W)S(X,U)Y+\frac{1}{2n}S(U,W)S(X,V)Y\},\end{aligned}$$

$$(5.4)\quad \begin{aligned}&P(P(X,Y)U,V)W\\ &=R(R(X,Y)U,V)W-\frac{1}{2n}S(Y,U)R(X,V)W+\frac{1}{2n}S(X,U)R(Y,V)W\\ &-\frac{1}{2n}\{S(V,W)R(X,Y)U-\frac{1}{2n}S(V,W)S(Y,U)X+\frac{1}{2n}S(V,W)S(X,U)Y\}\\ &+\frac{1}{2n}\{S(R(X,Y)U,W)V-\frac{1}{2n}S(Y,U)S(X,W)V+\frac{1}{2n}S(X,U)S(Y,W)V\},\end{aligned}$$

$$(5.5)\quad \begin{aligned}&P(U,P(X,Y)V)W\\ &=R(U,R(X,Y)V)W-\frac{1}{2n}S(Y,V)R(U,X)W+\frac{1}{2n}S(X,V)R(U,Y)W\\ &-\frac{1}{2n}\{S(R(X,Y)V,W)U-\frac{1}{2n}S(Y,V)S(X,W)U+\frac{1}{2n}S(X,V)S(Y,W)U\}\\ &+\frac{1}{2n}\{S(U,W)R(X,Y)V-\frac{1}{2n}S(U,W)S(Y,V)X+\frac{1}{2n}S(U,W)S(X,V)Y\},\end{aligned}$$

$$(5.6)\quad \begin{aligned}&P(U,V)P(X,Y)W\\ &=R(U,V)R(X,Y)W-\frac{1}{2n}S(Y,W)R(U,V)X+\frac{1}{2n}S(X,W)R(U,V)Y\\ &-\frac{1}{2n}\{S(V,R(X,Y)W)U-\frac{1}{2n}S(Y,W)S(V,X)U+\frac{1}{2n}S(X,W)S(V,Y)U\}\\ &+\frac{1}{2n}\{S(U,R(X,Y)W)V-\frac{1}{2n}S(Y,W)S(U,X)V+\frac{1}{2n}S(X,W)S(U,Y)V\}.\end{aligned}$$

Substituting (5.3)-(5.6) into (5.2), we have

$$\begin{aligned}
&(R(X,Y)\cdot R)(U,V)W-\frac{1}{2n}\{S(Y,R(U,V)W)X-S(X,R(U,V)W)Y\}\\
&+\frac{1}{2n}\{S(Y,U)R(X,V)W-S(X,U)R(Y,V)W-S(R(X,Y)U,W)V\}\\
&+\frac{1}{2n}\{S(Y,V)R(U,X)W-S(X,V)R(U,Y)W+S(R(X,Y)V,W)U\}\\
&+\frac{1}{2n}\{S(Y,W)R(U,V)X-S(X,W)R(U,V)Y+S(V,R(X,Y)W)U\\
&-S(U,R(X,Y)W)V\}=0
\end{aligned}\tag{5.7}$$

for any vector fields $X,Y,U,V,W\in\mathfrak{X}(M)$. If (5.1) holds, putting $Y=U=\xi$ into (5.7), we obtain

$$\begin{aligned}
&(R(X,\xi)\cdot R)(\xi,V)W-\frac{1}{2n}\{S(\xi,R(\xi,V)W)X-S(X,R(\xi,V)W)\xi\}\\
&+\frac{1}{2n}\{S(\xi,\xi)R(X,V)W-S(X,\xi)R(\xi,V)W-S(R(X,\xi)\xi,W)V\}\\
&+\frac{1}{2n}\{S(\xi,V)R(\xi,X)W-S(X,V)R(\xi,\xi)W+S(R(X,\xi)V,W)\xi\}\\
&+\frac{1}{2n}\{S(\xi,W)R(\xi,V)X-S(X,W)R(\xi,V)\xi+S(V,R(X,\xi)W)\xi\\
&-S(\xi,R(X,\xi)W)V\}=0
\end{aligned}\tag{5.8}$$

for any vector fields $X,V,W\in\mathfrak{X}(M)$. In Section 4, we know that $S(\xi,V)=2nk\eta(V)$, using the equation and (3.1), we have

$$\begin{aligned}
&S(R(\xi,X)Y,Z)\\
=&2n\{\eta(Z)[k^2g(X,Y)-2kg(h'X,Y)]\\
&+\eta(Y)[kg(X,Z)-k(k+1)\eta(Z)\eta(X)\\
&+kg(X,h'Z)-2g(h'X,Z)-2g(h'X,h'Z)]\}
\end{aligned}\tag{5.9}$$

for any vector fields $X,Y,Z\in\mathfrak{X}(M)$. Combining (5.9) with (5.8) and assuming that $X\in[\lambda]$ and $V=W\in[-\lambda]$ in (5.8) are eigenvector fields of h' corresponding two eigenvalues λ and $-\lambda$, respectively. Thus, by applying Lemma 3.1, we obtian

$$(R(X,\xi)\cdot R)(\xi,V)W=[k^2+2k\lambda+k(k+2)]g(V,W)X.\tag{5.10}$$

On the other hand, by a straightforward computation and applying Lemma 3.1, Wang and Liu [15, Theorem 1.1] obtained the following relation (one can check it by a direct calculation).

$$\begin{aligned}
&(R(X,\xi)\cdot R)(\xi,V)W\\
=&R(X,\xi)R(\xi,V)W-R(R(X,\xi)\xi,V)W\\
&-R(\xi,R(X,\xi)V)W-R(\xi,V)R(X,\xi)W\\
=&[(k-2\lambda)(k+2)-k^2+4\lambda^2]g(V,W)X.
\end{aligned}\tag{5.11}$$

From (5.10) and (5.11), we get $\lambda^2(\lambda-1)=0$. In view of the fact $\lambda>0$, we obtain $\lambda=1$ and hence $k=-2$. From [6, Corollary 4.2] and [5, Theorem 6] we can know that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

Consequently, we have the following theorem:

Theorem 5.1. *If a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} satisfies $P\cdot P=0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Acknowledgment. The authors would like to thank the referee for his or her valuable suggestions and comments that led to the improvement of this paper.

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Ser. Math. Inform. Vol. 33, No 4 (2018), 599–611
https://doi.org/10.22190/FUMI1804599P

ON A SUBSPACE OF A SPECIAL FINSLER SPACE

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Abstract. The present paper deals with the properties of a Finsler space F_n^* whose metric is obtained from the metric of another Finsler space F_n defined over the same manifold, with the help of a contravariant vector $v^i(x^j)$ satisfying the condition $LC_{jkr}v^r = \rho h_{jk}$, where L, h_{jk} and C_{jkr} are metric function, angular metric tensor and Cartan tensor of F_n, respectively, and ρ is a scalar function of positional coordinates x^i. Apart from obtaining expressions for different geometric objects of F_n^*, a subspace of F_n^* is studied. Apart from other results for the subspace of F_n^*, certain conditions for a subspace of F_n^* to be totally geodesic and projectively flat have been obtained.

Keywords: Finsler space; subspace; projective change; totally geodesic subspace; projectively flat space.

1. Introduction

In 1952, S. Kikuchi [11] studied the theory of a subspace of a Finsler space. H. Rund [3] in 1959, H. Yasuda [4] in 1987, T. Sakaguchi [12] in 1988 and many others mathematicians contributed significantly to the theory of Finsler subspaces and obtained many important and interesting results. In 1980 during the study of conformally flat Finsler spaces, H. Izumi [2] introduced a vector b_i which is $v-$covariant constant ($b_i|_j = 0$) and satisfies the condition $LC^r_{jk}b_r = \rho h_{jk}$, where ρ is a scalar independent of directional arguments y^i. He called such vector b_i as $h-$ vector. In 1990, B. N. Prasad [1] studied a Finsler space with a special metric $ds = (g_{ij}(dx)dx^i dx^j)^{1/2} + b_i(x, y)dx^i$, where b_i is an $h-$vector, and obtained the Cartan connections. In 2008, M. K. Gupta and P. N. Pandey [6] worked on subspaces of a Finsler space with a special metric by taking this $h-$ vector.

Let $F_n = (M_n, L)$ be a Finsler space and $F_n^* = (M_n, L^*)$ be another Finsler space over the same manifold M_n, whose metric L^* is obtained from the metric L of F_n by

(1.1) $$L^*(x, y) = L(x, y) + v_i(x, y)y^i,$$

Received March 27, 2018; accepted April 10, 2018
2010 *Mathematics Subject Classification.* 53B40

where $v_i = g_{ij}v^j$, g_{ij} is the metric tensor of F_n and $v^i(x^j)$ is a contravariant vector satisfying

$$LC_{jkr}v^r = \rho h_{jk}, \tag{1.2}$$

where ρ is a scalar function of positional coordinates x^i.
We call such a Finsler space $F_n^* = (M_n, L^*)$ as a special Finsler space. This special Finsler space F_n^* is a generalization of the Finsler spaces considered by the authors ([1], [6]). The aim of the present paper is to obtain the Cartan connections and to study a subspace of the Finsler space $F_n^* = (M_n, L^*)$.

2. Preliminaries

Let the Cartan connection of an $n-$dimensional Finsler space $F_n = (M_n, L)$ is given by the triad $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, where $G^i_j = F^i_{jk}y^k$ and C^i_{jk} is the associated Cartan tensor. If $X_i(x,y)$ be a covariant vector field then its $h-$ and $v-$ covariant derivatives with respect to the Cartan connection $C\Gamma$ are given by

$$X_{i|k} = \partial_k X_i - (\dot{\partial}_r X_i)G^i_k - X_r F^r_{ik} \tag{2.1}$$

and

$$X_i|_k = \dot{\partial}_k X_i - X_r C^r_{ik} \tag{2.2}$$

respectively. Here ∂_k and $\dot{\partial}_k$ denote the partial derivatives with respect to x^k and y^k respectively, and ∂_k and $\dot{\partial}_k$ stand for $\partial/\partial x^k$ and $\partial/\partial y^k$ respectively.

The components of the metric tensor g_{ij} and the angular metric tensor h_{ij} of the Finsler space $F_n = (M_n, L)$ are defined respectively by

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2 \tag{2.3}$$

and

$$h_{ij} = L\dot{\partial}_i\dot{\partial}_j L. \tag{2.4}$$

Differentiating (2.3) partially with respect to y^k, we obtain a tensor C_{ijk} of type $(0,3)$ defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}. \tag{2.5}$$

This tensor is called the Cartan tensor and its degree of homogeneity in y^i is -1.
The normalized supporting element $l_i = y_i/L$ satisfies $l_i = \dot{\partial}_i L$. From the equations (2.3) and (2.4), we obtain the relation

$$g_{ij} = h_{ij} + l_i l_j \tag{2.6}$$

among the metric tensor g_{ij}, the angular metric tensor h_{ij} and the normalized supporting element l_i. The $h-$covariant derivatives and $v-$ covariant derivatives of g_{ij}, h_{ij} and l_i satisfy [9]

$$\begin{array}{llllll} \text{(a)} & g_{ij|k} = 0 & \text{(b)} & h_{ij|k} = 0 & \text{(c)} & L_{|i} = 0 \\ \text{(d)} & l^i_{|j} = 0 & \text{(e)} & l_i|_j = \frac{1}{L} h_{ij} & \text{(f)} & L|_i = l_i. \end{array} \tag{2.7}$$

Let $M_m(1 < m < n)$ be an $m-$dimensional subspace of the $n-$ dimensional manifold M_n represented parametrically by the equations

$$x^i = x^i(u^\alpha) \qquad i = 1, 2, ...n; \quad \alpha = 1, 2, ...m, \tag{2.8}$$

where u^α denote the Gaussian cordinates on the subspace M_m.

Let $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ be the projection factors [3] and the matrix $||B^i_\alpha||$ of this projection factors be supposed to be of rank m. If y^i, the supporting element, is assumed to be tangential to the subspace M_m then it can be written in terms of the projection factors as

$$y^i = B^i_\alpha(u) w^\alpha, \qquad \alpha = 1, 2, ...m. \tag{2.9}$$

Here $w = (w^\alpha)$ is assumed to be the supporting element at the point (u^α) of the subspace M_m. The metric $L(x, y)$ of the Finsler space $F_n = (M_n, L)$ induces the metric

$$\bar{L}(u, w) = L(x(u), y(u, w)) \tag{2.10}$$

on the subspace M_m. Thus, we obtain an $m-$dimensional Finsler subspace $F_m = (M_m, \bar{L}(u, w))$ of the space $F_n = (M_n, L)$.
Let $g_{\alpha\beta}(u, w)$ defined by

$$g_{\alpha\beta}(u, w) = \frac{1}{2} \frac{\partial^2 \bar{L}^2}{\partial w^\alpha \partial w^\beta}, \tag{2.11}$$

be the metric tensor of the subspace F_m. Successive partial differentiations of (2.10) with respect to w^α and w^β give

$$g_{\alpha\beta}(u, w) = g_{ij}(x, y) B^i_\alpha B^j_\beta. \tag{2.12}$$

A covariant vector Y_i which satisfies the condition

$$Y_i B^i_\alpha(u) = 0 \tag{2.13}$$

is called normal to the subspace F_m. Clearly, these are m equations for determinations of n functions Y_i. So, there exist $(n - m)$ linearly independent and mutually orthogonal unit vectors $Y^i_{(a)}$, (say), satisfying the following conditions

$$g_{ij} B^i_\alpha Y^j_{(a)} = 0 \tag{2.14}$$

and

$$Y_i^{(a)} = g_{ij} Y_{(a)}^j, \tag{2.15}$$

where $(a) = m+1, m+2, ...n$. Further, (2.14) and (2.15) imply that

$$g_{ij} Y_{(a)}^i Y_{(b)}^j = \delta_{(a)(b)}, \qquad \{(a),(b) = m+1, m+2, ...n\}. \tag{2.16}$$

If $B_i^\alpha(u,w)$ is the reciprocal of the projection factors B_α^i defined by

$$B_i^\alpha(u,w) = g^{\alpha\beta} B_\beta^j \, g_{ij}, \tag{2.17}$$

then, in view of (2.12), we have

$$B_\alpha^i B_i^\beta = \delta_\alpha^\beta. \tag{2.18}$$

From (2.14), (2.15), (2.16), (2.17) and (2.18), we have

$$\begin{array}{llll} \text{(a)} & B_\alpha^i Y_i^{(a)} = 0 & \text{(b)} & Y_{(a)}^i B_i^\alpha = 0 \\ \text{(c)} & Y_{(a)}^i Y_i^{(b)} = \delta_{(b)}^{(a)} & \text{(d)} & B_\alpha^i B_j^\alpha + Y_{(a)}^i Y_j^{(a)} = \delta_j^i. \end{array} \tag{2.19}$$

If the triad $IC\Gamma = (F_{\beta\gamma}^\alpha, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$, where $G_\beta^\alpha = F_{\beta\gamma}^\alpha y^\gamma$, is the induced Cartan connection of the Finsler subspace F_m then the second fundamental tensor $H_{\alpha\beta}^{(a)}$ and the normal curvature vector $H_\alpha^{(a)}$ with respect to induced Cartan connection $IC\Gamma$ can be expressed in the direction of the normal vector $Y_{(a)}^i$ by

$$H_{\alpha\beta}^{(a)} = Y_i^{(a)}(B_{\alpha\beta}^i + F_{jk}^i B_\alpha^j B_\beta^k) + M_{(b)\alpha}^{(a)} H_\beta^{(b)} \tag{2.20}$$

and

$$H_\alpha^{(a)} = Y_i^{(a)}(B_{0\alpha}^i + F_{0j}^i B_\alpha^j) \tag{2.21}$$

respectively, where

$$M_{(b)\alpha}^{(a)} = C_{jk}^i Y_i^{(a)} Y_{(b)}^j B_\alpha^k, \tag{2.22}$$

$$B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \qquad B_{0\alpha}^i = v^\beta B_{\beta\alpha}^i. \tag{2.23}$$

The contraction of (2.20) by v^α gives us

$$H_{0\beta}^{(a)} = v^\alpha H_{\alpha\beta}^{(a)} = H_\beta^{(a)}. \tag{2.24}$$

3. The Finsler space $F_n^* = (M_n, L^*)$

Let $v^i = v^i(x^j)$ be a contravariant vector field in a Finsler space $F_n = (M_n, L)$ satisfying the condition (1.2).
Differentiating $v_i = g_{ir}(x,y)v^r$ partially with respect to y^j and using the condition (1.2), we obtain

$$L(\dot{\partial}_j v_i) = 2\rho h_{ij}. \tag{3.1}$$

Consider an $n-$ dimensional Finsler space $F_n^* = (M_n, L^*)$ whose metric function $L^*(x,y)$ is obtained from the metric of the space F_n by the transformation (1.1). Throughout the paper, the geometric objects related to F_n^* will be asterisked $*$.

Differentiating (1.1) partially with respect to y^k and using (3.1), we get

$$L_k^* = L_k + v_k, \tag{3.2}$$

where $L_k^* = \dot{\partial}_k L^*$.
The normalized supporting element l_i^* of F_n^* can be written as

$$l_k^* = l_k + v_k. \tag{3.3}$$

Differentiating (3.2) partially with respect to y^j and using (3.1), we obtain

$$L_{jk}^* = L_{jk} + 2\rho h_{jk}/L, \tag{3.4}$$

where $L_{jk} = \dot{\partial}_j \dot{\partial}_k L$ and $L_{jk}^* = \dot{\partial}_j \dot{\partial}_k L^*$.
Using (2.4) in (3.4), we get

$$L_{jk}^* = (1+2\rho)L_{jk}. \tag{3.5}$$

Partial Differentiation of (3.5) with respect to y^i gives

$$L_{ijk}^* = (1+2\rho)L_{ijk}, \tag{3.6}$$

where $L_{ijk} = \dot{\partial}_k L_{ij}$ and $L_{ijk}^* = \dot{\partial}_k L_{ij}^*$.
In view of (2.4), the angular metric tensor h_{ij}^* of the Finsler space F_n^* is given as

$$h_{ij}^* = \tau(1+2\rho)h_{ij}, \tag{3.7}$$

where $\tau = \frac{L^*}{L}$.
From (3.3), (3.7) and (2.6), the fundamental metric tensor g_{ij}^* of the Finsler space F_n^* is given by

$$g_{ij}^* = \tau(1+2\rho)g_{ij} + v_i v_j + l_i v_j + v_i l_j + (1-\tau(1+2\rho))l_i l_j. \tag{3.8}$$

Keeping $g^{*ij}g_{jk}^* = \delta_k^i$ in view, the inverse metric tensor g^{*ij} is given by

$$g^{*ij} = \frac{1}{\tau(1+2\rho)}g^{ij} - \frac{1}{\tau^2(1+2\rho)}(l^i v^j + v^i l^j) + \frac{\tau(1+2\rho)+v^2-1}{\tau^3(1+2\rho)}l^i l^j \tag{3.9}$$

in the Finsler space F^*_n.
Differentiating (3.8) partially with respect to y^k, we obtain the Cartan tensor C^*_{ijk} of F^*_n as

$$C^*_{ijk} = \tau(1+2\rho)C_{ijk} + \frac{(1+2\rho)}{2L}(h_{jk}c_i + h_{ki}c_j + h_{ij}c_k). \tag{3.10}$$

Here $c_i = v_i - (\tau - 1)l_i$. Thus, we have

Theorem 3.1. *The components of the metric tensor g^*_{ij}, the inverse metric tensor g^{*ij}, the angular metric tensor h^*_{ij} and the Cartan tensor C^*_{ijk} of the Finsler space F^*_n whose metric L^* is obtained from the metric L of the Finsler space F_n by (1.1), are given by (3.8), (3.9), (3.7) and (3.10) respectively.*

4. The Cartan connection of the Finsler space $F^*_n = (M_n, L^*)$

In this section, we find the Cartan connection of the Finsler space $F^*_n = (M_n, L^*)$. Since L_{ij} is $h-$covariant constant with respect to the Cartan connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$, i.e. $L_{ij|k} = 0$, (2.1) gives

$$\partial_k L_{ij} = L_{ijr}F^r_{0k} + L_{rj}F^r_{ik} + L_{ir}F^r_{jk}, \tag{4.1}$$

where $L_{ijk} = \dot{\partial}_k L_{ij}$ and $F^r_{0k} = F^r_{ik}y^i = G^r_k$.
Differentiating (3.5) covariantly with respect to x^i, we get

$$\partial_i L^*_{jk} = (1+2\rho)\partial_i L_{jk} + 2\rho_i L_{jk}, \tag{4.2}$$

where $\partial_i \rho = \rho_i$.
In view of (4.1), (3.5) and (3.6), (4.2) can be written as

$$(1+2\rho)\{L_{jkr}(F^{*r}_{0i} - F^r_{0i}) + L_{kr}(F^{*r}_{ji} - F^r_{ji}) + L_{jr}(F^{*r}_{ki} - F^r_{ki})\} = 2\rho_i L_{jk}. \tag{4.3}$$

Let D^i_{jk} be the difference of the connections F^{*i}_{jk} and F^i_{jk}, i.e.

$$D^i_{jk} = F^{*i}_{jk} - F^i_{jk}. \tag{4.4}$$

In view of (4.4), (4.3) reduces to

$$(1+2\rho)(L_{jkr}D^r_{0i} + L_{rk}D^r_{ji} + L_{jr}D^r_{ki}) = 2\rho_i L_{jk}. \tag{4.5}$$

Cyclic rotation of the indices i, j and k gives

$$(1+2\rho)(L_{kir}D^r_{0j} + L_{ri}D^r_{jk} + L_{kr}D^r_{ij}) = 2\rho_j L_{ki}, \tag{4.6}$$

and

$$(1+2\rho)(L_{ijr}D^r_{0k} + L_{rj}D^r_{ki} + L_{ir}D^r_{jk}) = 2\rho_k L_{ij}. \tag{4.7}$$

Using $L_{k|j} = 0$ in (2.1), we have

$$\partial_j L_k = L_{kr} F^r_{0j} + L_r F^r_{jk}. \tag{4.8}$$

Differentiating (3.2) partially with respect to x^j and then using (4.8) and (2.1), we obtain

$$L^*_{kr} F^*_{0j} + L^* F^*_{kj} = (1+2\rho) L_{kr} F^r_{0j} + (L_r + v_r) F^r_{kj} + v_{k|j}. \tag{4.9}$$

In view of (3.2), (3.3), (3.5), and (4.4), (4.9) reduces to

$$(1+2\rho) L_{kr} D^r_{0j} + (l_r + v_r) D^r_{kj} = v_{k|j}. \tag{4.10}$$

Here subscript '0'denotes the contraction by the supporting element y^k.
Now, we propose

Theorem 4.1. *If $F_n = (M_n, L)$ and $F^*_n = (M_n, L^*)$ are two Finsler spaces over the same manifold M_n and $L^*(x,y)$ is given by (1.1), then the Cartan connection of F^*_n is completely determined by (4.5) and (4.10).*

To prove Theorem 4.1, first we have to prove the following lemma

Lemma 4.1. *The system of equations*

$$\begin{aligned} &(a)\quad (1+2\rho) L_{jk} A^k = B_j \\ &(b)\quad (l_k + v_k) A^k = B \end{aligned} \tag{4.11}$$

has a unique solution

$$A^k = (1+2\rho)^{-1} L B^k + \tau^{-1}(B - (1+2\rho) L B_v) l^k, \tag{4.12}$$

where $\tau = (L^*/L)$, $B_v = B_i v^i$ and $B^i = g^{ij} B_j$ for given B_j and B such that $B_j l^j = 0$.

Proof. From $h_{jk} = L L_{jk}$ and (2.6), (4.11(a)) can be written as

$$g_{jk} A^k = (1+2\rho)^{-1} L B_j + l_j (l_k A^k). \tag{4.13}$$

Transvecting (4.13) with v^j, we get

$$v_k A^k = (1+2\rho)^{-1} L B_v + (\tau - 1) l_k A^k, \tag{4.14}$$

where $B_v = B_i v^i$.
In view of (4.11(b)), (4.14) implies

$$l_k A^k = \tau^{-1}(B - (1+2\rho)^{-1} L B_v). \tag{4.15}$$

Thus, from (4.13) and (4.15), we have

$$g_{jk} A^k = (1+2\rho)^{-1} L B_j + \tau^{-1}(B - (1+2\rho)^{-1} L B_v) l_j. \tag{4.16}$$

Contraction of (4.16) by g^{ij} gives the solution

$$A^i = (1+2\rho)^{-1} L B^i + \tau^{-1}(B - (1+2\rho)^{-1} L B_v) l^i \tag{4.17}$$

of the given system, where $B^i = g^{ij} B_j$ and $\tau = \frac{L^*}{L}$. □

Thus, we are in a position to prove Theorem 4.1. We complete the proof of Theorem 4.1 if we find the value of D^i_{jk}.
We will find the value of D^i_{jk} in three steps. In the first step, we will find the value of D^i_{00}, in the second step we will find D^i_{j0} and in the last step we will find D^i_{jk}.

In view of (4.10), we have

$$(1+2\rho)L_{jr}D^r_{0k} + (l_r+v_r)D^r_{jk} = v_{j|k}. \tag{4.18}$$

Simultaneously adding and subtracting (4.18) and (4.10), we get

$$(1+2\rho)(L_{jr}D^r_{0k} + L_{kr}D^r_{0j}) + 2(l_r+v_r)D^r_{jk} = v_{j|k} + v_{k|j} \tag{4.19}$$

and

$$(1+2\rho)(L_{jr}D^r_{0k} - L_{kr}D^r_{0j}) = v_{j|k} - v_{k|j}. \tag{4.20}$$

If we take

$$(a)\quad v_{j|k} + v_{k|j} = 2s_{jk} \qquad (b)\quad v_{j|k} - v_{k|j} = 2t_{jk}, \tag{4.21}$$

then (4.19) and (4.20) become

$$(1+2\rho)(L_{jr}D^r_{0k} + L_{kr}D^r_{0j}) + 2(l_r+v_r)D^r_{jk} = 2s_{jk}. \tag{4.22}$$

$$(1+2\rho)(L_{jr}D^r_{0k} - L_{kr}D^r_{0j}) = 2t_{jk}. \tag{4.23}$$

Subtracting (4.7) from the addition of (4.5) and (4.6), we have

$$2L_{kr}D^r_{ij} + (1+2\rho)(L_{jkr}D^r_{0i} + L_{kir}D^r_{0j} - L_{ijr}D^r_{0k}) = 2(\rho_i L_{jk} + \rho_j L_{ki} - \rho_k L_{ij}). \tag{4.24}$$

Transvection of (4.22), (4.23) and (4.24) with y^k and utilization of $L_{ij}y^j = 0$ give us

$$(1+2\rho)L_{jr}D^r_{00} + 2(l_r+v_r)D^r_{0j} = 2s_{j0}, \tag{4.25}$$

$$(1+2\rho)L_{jr}D^r_{00} = 2t_{j0}, \tag{4.26}$$

$$(1+2\rho)(L_{jr}D^r_{0i} + L_{ir}D^r_{0j} + L_{ijr}D^r_{00}) = 2\rho_0 L_{ij}. \tag{4.27}$$

Transvecting (4.25) with y^j, we find

$$(l_r+v_r)D^r_{00} = s_{00}. \tag{4.28}$$

Applying Lemma 4.1 in (4.26) and (4.28), we get

$$D^r_{00} = \frac{L}{L^*(1+2\rho)}\{2L^* t^r_0 + l^r((1+2\rho)s_{00} - 2Lt_{v0})\}. \tag{4.29}$$

Here $t_0^r = g^{ir} t_{i0}$ and $t_{v0} = t_{i0} v^i$.
Putting k in palace of i in (4.27) and then adding with (4.23), we find

$$L_{jr} D_{0k}^r = \frac{1}{2(1+2\rho)}(2t_{jk} + 2\rho_0 L_{jk} - \frac{1}{2}(1+2\rho) L_{jkr} D_{00}^r). \tag{4.30}$$

If we take

$$\frac{1}{2(1+2\rho)}(2t_{jk} + 2\rho_0 L_{jk} - \frac{1}{2}(1+2\rho) L_{jkr} D_{00}^r) = A_{jk}, \tag{4.31}$$

then (4.30) reduces to

$$L_{jr} D_{0k}^r = A_{jk}. \tag{4.32}$$

From (4.29) and (4.31), we have

$$A_{jk} = \frac{1}{2L^*(1+2\rho)}\{2L^*(t_{jk} - LL_{jkr} t_0^r) + L_{jk}((1+2\rho)s_{00} - 2Lt_{v0} + 2L^*\rho_0)\}. \tag{4.33}$$

This shows that A_{jk} is known.
If we write

$$s_{k0} - \frac{1}{2}(1+2\rho) L_{kr} D_{00}^r = A_k, \tag{4.34}$$

the equation (4.25) assumes the form

$$(l_r + v_r) D_{0k}^r = A_k. \tag{4.35}$$

Putting the value of D_{00}^r from (4.29) in (4.34), we get

$$A_k = s_{k0} - LL_{kr} t_o^r. \tag{4.36}$$

In view of Lemma 4.1, the system of equations (4.32) and (4.35) give

$$D_{0k}^r = \frac{L}{L^*}(L^* A_k^r + l^r(A_k - LA_{vk})), \tag{4.37}$$

where $A_{vk} = A_{jk} v^j$ and $A_k^r = g^{ri} A_{ik}$.
Now we can express (4.22) in the form

$$(l_r + v_r) D_{jk}^r = B_{jk}, \tag{4.38}$$

where

$$B_{jk} = s_{jk} - \frac{1}{2}(1+2\rho)(L_{jr} D_{0k}^r + L_{kr} D_{0j}^r). \tag{4.39}$$

The equation (4.24) may be written as

$$L_{ir} D_{jk}^r = B_{ijk}, \tag{4.40}$$

where

$$B_{ijk} = (\rho_j L_{ki} + \rho_k L_{ij} - \rho_i L_{jk}) - \frac{1}{2}(1+2\rho)(L_{kir}D^r_{0j} + L_{ijr}D^r_{0k} - L_{jkr}D^r_{0i}). \tag{4.41}$$

Putting the value of D^r_{0i} from (4.37), we see that B_{jk} and B_{ijk} are known quantities. Applying the Lemma 4.1, for the system of equations (4.38) and (4.40), we obtain

$$D^r_{jk} = \frac{L}{L^*}\{L^* B^r_{jk} + l^r(B_{jk} - LB_{vjk})\}, \tag{4.42}$$

where $B^r_{jk} = g^{ir}B_{ijk}$ and $B_{vjk} = B_{ijk}v^i$. The quantities B_{jk} and B_{ijk} are given by respectively (4.39) and (4.41) together with (4.37).
Thus, the proof is completed.

5. Subspace of the Finsler space $F^*_n = (M_n, L^*)$

Suppose F_m and F^*_m are the subspaces of the Finsler spaces F_n and F^*_n respectively.
Contracting (2.14) by w^α and using (2.9) and $y^i g_{ij} = y_j$, we obtain

$$y_j Y^j_{(a)} = 0. \tag{5.1}$$

Again contracting (3.8) with $Y^i_{(a)}Y^j_{(b)}$ and using (2.16), (5.1) and $\tau = (L^*/L)$, we have

$$g^*_{ij}Y^i_{(a)}Y^j_{(b)} = \frac{L^*}{L}(1+2\rho)\delta_{(a)(b)} + v_i Y^i_{(a)} v_k Y^k_{(b)}. \tag{5.2}$$

Fixing the index (a) and taking $(a) = (b)$ in (5.2), we get

$$g^*_{ij}Y^i_{(a)}Y^j_{(a)} = \frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2. \tag{5.3}$$

Hence

$$g^*_{ij}\left(\frac{Y^i_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2}}\right)\left(\frac{Y^j_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho) + (v_r Y^r_{(a)})^2}}\right) = 1. \tag{5.4}$$

From (5.4), it is clear that $\left(\frac{Y^i_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho)+(v_r Y^r_{(a)})^2}}\right)$ is a unit vector.
Contracting (3.8) by $B^i_\alpha Y^j_{(a)}$ and using (2.14) & (5.1), we obtain

$$g^*_{ij}B^i_\alpha Y^j_{(a)} = (v_j Y^j_{(a)})(v_i + l_i)B^i_\alpha. \tag{5.5}$$

From (5.5), we can say that $Y^j_{(a)}$ is normal to the subspace F^*_m if and only if the condition

$$(v_j Y^j_{(a)})(v_i + l_i)B^i_\alpha = 0 \tag{5.6}$$

holds. This implies at least one of the conditions $v_jY^j_{(a)} = 0$ and $(v_i + l_i)B^i_\alpha = 0$. Suppose $(v_i + l_i)B^i_\alpha = 0$. Contracting this condition by w^α and using $B^i_\alpha w^\alpha = y^i$, we get $L + v_iy^i = L^* = 0$ which is not possible. Hence, we have the first condition, i.e.

$$v_jY^j_{(a)} = 0. \tag{5.7}$$

Thus, the vector $Y^j_{(a)}$ is normal to the subspace F^*_m if and only if the vector v_j is tangent to the subspace F_m. From (5.4), (5.5) and (5.7), we find that $\left(\frac{Y^j_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho)}}\right)$ is a unit normal vector of the subspace F^*_m. In view of (2.14), (2.15) and (2.16), we obtain

$$Y^{*j}_{(a)} = \frac{Y^j_{(a)}}{\sqrt{\frac{L^*}{L}(1+2\rho)}}. \tag{5.8}$$

Contracting (3.8) by $Y^{*i}_{(a)}$ and using (2.15), we obtain

$$Y^{*(a)}_j = \sqrt{\frac{L^*}{L}(1+2\rho)}Y^{(a)}_i. \tag{5.9}$$

Thus, we have

Theorem 5.1. *Let $F^*_n = (M_n, L^*)$ be a Finsler space whose metric function L^* is obtained from the metric function L of the Finsler space $F_n = (M_n, L)$ by the transformation (1.1). If F^*_m and F_m are $m-$ dimensional subspaces of F^*_n and F_n respectively, then the vector $v_i(x, y)$ satisfying the condition (1.2) is tangent to F_m if and only if any vector $Y^i_{(a)}$ normal to F_m is also normal to F^*_m.*

Let us assume that the transformation given by (1.1) is projective. Then, we have

$$G^{*i}_j = G^i_j + p_jy^i + p\delta^i_j, \tag{5.10}$$

where p is a function of directional argument and of homogeneity one in y^i and $\dot\partial_jp = p_j$.
Contracting (4.4) by y^k and using $F^i_{jk}y^k = G^i_j$, we get

$$G^{*i}_j = G^i_j + D^i_{0j}. \tag{5.11}$$

Thus, from (5.10) and (5.11), we have

$$D^i_{0j} = p_jy^i + p\delta^i_j. \tag{5.12}$$

Contracting (5.12) by $B^j_\alpha Y^{(a)}_i$, using (2.15) and (2.19(a)), we obtain

$$Y^{(a)}_i D^i_{0j} B^j_\alpha = 0. \tag{5.13}$$

If every geodesic in the subspace F_m with respect to the induced metric is also a geodesic in the enveloping space F_n, the subspace F_m is called totally geodesic subspace and this type of space is characterized by

$$H^{(a)}_{\alpha} = 0, \tag{5.14}$$

i.e. its normal curvature vector vanishes identically.
In view of (2.21), the normal curvature vector $H^{*(a)}_{\alpha}$ of the subspace F^*_m in the direction $Y^{(a)}_i$ is given by

$$H^{*(a)}_{\alpha} = Y^{*(a)}_i (B^i_{0\alpha} + G^{*i}_j B^j_{\alpha}). \tag{5.15}$$

Using (5.9) and (5.11) in (5.15), we get

$$H^{*(a)}_{\alpha} = \sqrt{\frac{L^*}{L}(1+2\rho)}\; H^{(a)}_{\alpha} + \sqrt{\frac{L^*}{L}(1+2\rho)}\; Y^{(a)}_i D^i_{0j} B^j_{\alpha} = 0. \tag{5.16}$$

In view of (5.13), (5.16) reduces to

$$H^{*(a)}_{\alpha} = \sqrt{\frac{L^*}{L}(1+2\rho)}\; H^{(a)}_{\alpha}. \tag{5.17}$$

$\sqrt{\frac{L^*}{L}(1+2\rho)} \neq 0$ for $\sqrt{\frac{L^*}{L}(1+2\rho)} = 0$ implies $\rho = -(1/2)$, a contradiction to the fact that ρ is a function of x^i. Hence, we conclude from (5.17) that $H^{(a)}_{\alpha}$ vanishes if and only if $H^{*(a)}_{\alpha}$ vanishes as $\sqrt{\frac{L^*}{L}(1+2\rho)} \neq 0$. Therefore, we have

Theorem 5.2. *If a contravariant vector field v^i satisfying the condition (1.2) is tangent to a subspace F_m of the space F_n then F_m is totally geodesic if and only if the subspace F^*_m of F^*_n is totally geodesic.*

If there exists a projective change between the Finsler spaces $F_n = (M_n, L)$ and $F^*_n = (M_n, L^*)$ over the underlying manifold M_n such that the later space is locally Minkowskian then the space F_n is said to be projectively flat.

In 2005, M. Kitayama [7] showed that a totally geodesic subspace of a projectively flat Finsler space is also projective.

Makato Matsumoto [8] proved that a Finsler space $F^n (n > 3)$ is projectively flat if the Weyl torsion tensor W^i_{jk} and the Douglas tensor D^i_{jkh} vanish, i.e.

$$(a) \quad W^i_{jk} = 0, \qquad (b) \quad D^i_{jkh} = 0 \tag{5.18}$$

and the converse part is also true.
Under a projective change $W^{*i}_{jk} = W^i_{jk}$ and $D^{*i}_{jkh} = D^i_{jkh}$, i.e. both tensors are invariant [10]. Thus, we conclude following theorem, from Theorem 5.2 in view of (5.18)

Theorem 5.3. *Let the metric function L^* of a Finsler space $F^*_n = (M_n, L^*)$ be obtained from the metric function L of a projective flat Finsler space $F_n = (M_n, L), n > 3$, by the transformation (1.1). If a subspace F_m of F_n is totally geodesic and the vector field v^i satisfying (1.2) is tangential to it, then the corresponding subspace F^*_m of F^*_n is projectively flat.*

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Ser. Math. Inform. Vol. 33, No 4 (2018), 613–626
https://doi.org/10.22190/FUMI1804613T

QUASI STATISTICAL CONVERGENCE IN CONE METRIC SPACES

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Abstract. The main purpose of this paper is to define a new type of statistical convergence of sequences in a cone metric space and investigate the relations of these sequences with some other sequences.

Keywords: Cone metric, statistical convergence, statistical boundedness.

1. Introduction and Preliminaries

The study of statistical convergence apparently goes back to Steinhaus [19] and Fast [7]. This concept has been studied under different names in spaces such as topological spaces, cone metric spaces etc. (see, for example [5],[8],[9],[12],[13],[14],[18]). Long-Guang and Xian [11] suggested the idea of a cone metric space. The main difference with a metric is that a cone metric is valued in an ordering Banach space. Later, several authors studied cone metric spaces and applied different names. This concept takes a vital role in computer science, statistics and some other research areas as well as general topology (see, for example [2],[2],[7],[11],[16]). The definition of statistical convergence and statistical boundedness of a sequence in a cone metric space was studied by Kedian, Shou and Ying [13]. In [10], the authors defined the concept of a quasi-statistical filter. Also it is known that statistical convergence is related to Cesaro summability and strong-Cesaro summability (see, for example [4],[3],[18]). Recently, Sakaoğlu and Yurdakadim [15] defined the notions of quasi-statistical convergence and strongly-Cesaro summability by relying on [4], [3], [10] and [18], and they found some inclusion theorems between these concepts. In the present paper, we introduce the quasi-statistical convergence and quasi-statistical boundedness of a sequence on a cone metric space, and obtain some theorems related to quasi-statistically convergent sequences. Later, we give the definition of strongly-quasi summable sequences in a cone metric space and we also investigate some theorems related to quasi-statistically convergent sequences and

Received June 21, 2016; accepted August 06, 2016
2010 *Mathematics Subject Classification.* Primary 40A35; Secondary 40G15, 40F05

strongly–quasi summable sequences. Finally, we present some results related to these theorems.

Throughout this paper, by $\mathbb{N}$ and $\mathbb{R}$ we denote the set of all positive integers and the set of all real numbers, respectively. For a subset S of $\mathbb{N}$, $|S|$ stands for the cardinality of S.

Definition 1.1. ([7]) Let $S \subset \mathbb{N}$ and $S(m) = \{i \in S : i \leq m\}$ for each $m \in \mathbb{N}$. If the following limit exists, then

$$\delta(S) = \lim_{m\to\infty} \frac{|S(m)|}{m}$$

is called the asymptotic (or natural) density of S. It is clear that $\delta(S) \in [0,1]$. Also, if $\delta(S) = 1$, then S is said to be statistically dense. It can be easily obtained that $\delta(\mathbb{N} - S) = 1 - \delta(S)$ for each $S \subset \mathbb{N}$.

Definition 1.2. ([8]) A sequence (x_m) in $\mathbb{R}$ is said to be statistically convergent to a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \{i \leq m : |x_i - x| \geq \varepsilon\} = 0$$

or equivalently

$$\lim_{m\to\infty} \frac{1}{m} \{i \leq m : |x_i - x| < \varepsilon\} = 1.$$

Definition 1.3. ([1]) Let E be a real Banach space. A subset P of E is called a cone if it satisfies the following conditions:

(1) $P \neq \emptyset$, $P \neq \{0\}$ and P is closed.

(2) $ax + by \in P$ for all $x, y \in P$ and $a, b \in \mathbb{R}$ with $a, b \geq 0$.

(3) If $x \in P$ and $-x \in P$, then $x = 0$ for all $x, y \in P$.

A partial ordering " $\preceq$ " with respect to P is defined by $x \preceq y \Leftrightarrow y - x \in P$. Also, we mean $x \prec y \Leftrightarrow x \preceq y$, $x \neq y$ and $x \prec\prec y \Leftrightarrow y - x \in E^+$, where E^+ denotes the interior of P; that is $E^+ = \{c \in E : 0 \prec\prec c\}$. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying this inequality is called the normal constant of P.

In this study, we always suppose that E is a Banach space, P is a cone in E with $E^+ \neq \emptyset$ and " $\preceq$ " is a partial ordering with respect to P.

Definition 1.4. ([17]) Let X be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $0 \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

2. $d(x,y) = d(y,x)$ for all $x, y \in X$,

3. $d(x,y) \preceq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, (X,d) is called a cone metric space.

Definition 1.5. ([11]) A sequence (x_n) in a cone metric space (X,d) is said to be convergent to $x \in X$ if for every $c \in E^+$ there exists a natural number N such that $d(x_n, x) \prec\prec c$ for all $n > N$.

Definition 1.6. ([13]) A sequence (x_n) in a cone metric space (X,d) is said to be statistically convergent to $x \in X$ if for every $c \in E^+$

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : d(x_k, x) \prec\prec c\}\right| = 1.$$

It is denoted by st-$\lim\limits_{n\to\infty} x_n = x$.

Definition 1.7. ([13]) A sequence (x_n) in a cone metric space (X,d) is said to be statistically bounded if there exist $\alpha \in X$ and $c \in E^+$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : d(x_k, \alpha) \preceq c\}\right| = 1.$$

Definition 1.8. ([15]) Let $s = (s_n)$ be a sequence of positive real numbers such that

$$\lim_n s_n = \infty \quad \text{and} \quad \limsup_n \frac{s_n}{n} < \infty. \tag{1.1}$$

The quasi density of a subset $K \subset \mathbb{N}$ with respect to the sequence $s = (s_n)$ is defined by

$$\delta_s(K) = \lim_{n\to\infty} \frac{1}{s_n} \left|\{k \leq n : k \in K\}\right|.$$

A sequence (x_n) in $\mathbb{R}$ is called quasi-statistical convergent to x provided that for every $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\}$ has quasi-density zero. It is denoted by st_q-$\lim_{n\to\infty} x_n = x$.

Throughout the study, we assume that $s = (s_n)$ and $t = (t_n)$ are sequences of positive real numbers satisfying the conditions in (1.1).

Definition 1.9. ([15]) A sequence (x_n) in $\mathbb{R}$ is said to be strongly quasi-summable to $x \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} |x_k - x| = 0.$$

2. Main Results

In this section, we first define the quasi-statistical convergence of a sequence in a cone metric space. Later, we give some results related to this concept.

Definition 2.1. A sequence (x_n) in a cone metric space (X,d) is said to be quasi-statistical convergent to a point $x \in X$ if for every $c \in E^+$ we have

$$\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : d(x_k, x) \prec\prec c\}\right| = 1$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : c \preceq d(x_k, x)\}\right| = 0.$$

We denote it by $st_q - \lim\limits_{n\to\infty} x_n = x$. If we take $(s_n) = (n)$, then we obtain that (x_n) is statistical convergent.

Theorem 2.1. *Let (x_n) be a sequence in a cone metric space (X,d). If (x_n) is convergent to $x \in X$, then it is quasi-statistical convergent to x.*

Proof. Let $\lim\limits_{n\to\infty} x_n = x$. Then, for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec\prec c$ for every $n > n_0$. It follows that

$$\frac{1}{s_n} \left|\{k \le n : c \preceq d(x_k, x)\}\right| \le \frac{n_0}{s_n}$$

which means $\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : c \preceq d(x_k, x)\}\right| = 0$. Hence, (x_n) is quasi-statistical convergent to x. $\square$.

The converse of the previous theorem does not hold which can be seen from the following example.

Example 2.1. Let $E = \mathbb{R}$, $P = [0,\infty)$ and $X = \mathbb{R}$. Consider X with usual metric $d(x,y) = |x-y|$. Let $s_n = n^{3/4}$. Define a sequence (x_n) as follows:

$$x_n = \begin{cases} 0, & n \ne m^2 \text{ for all } m \in \mathbb{N} \\ n, & n = m^2 \text{ for some } m \in \mathbb{N} \end{cases}$$

It is obvious that (x_n) is not convergent. On the other hand, it is quasi-statistical convergent to 0. Indeed, given any $c \in E^+$, we obtain the inclusion

$$\{n : c \preceq d(x_n, 0)\} \subset \{n : n = m^2, m \in \mathbb{N}\}.$$

Hence we conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : c \preceq d(x_k, 0)\}\right| &\le \lim_{n\to\infty} \frac{1}{s_n} \left|\left\{k \le n : k = m^2, m \in \mathbb{N}\right\}\right| \\ &= \lim_{n\to\infty} \frac{1}{s_n} \left[\left|\sqrt{n}\right|\right] = 0. \end{aligned}$$

Theorem 2.2. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is quasi-statistical convergent to $x \in X$, then it is statistical convergent to x.*

Proof. Let $st_q\text{-}\lim_{n\to\infty} x_n = x$ and $M = \sup_n \frac{s_n}{n}$. Then, for every $c \in E^+$, we have $\lim_{n\to\infty} \frac{1}{s_n} |\{k \le n : c \preceq d(x_k, x)\}| = 0$. The statistical convergence of the sequence (x_n) follows from the following inequality

$$\frac{1}{n} |\{k \le n : c \preceq d(x_k, x)\}| \le \frac{M}{s_n} |\{k \le n : c \preceq d(x_k, x)\}| .$$

□.

The converse of the previous theorem does not hold which can be seen from the following example.

Example 2.2. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \ge 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ be the cone metric defined by $d(x, y) = (|x - y|, \alpha |x - y|)$, where $\alpha > 0$ is a constant. Assume that the sequence (s_n) satisfies $\lim_n \frac{\sqrt{n}}{s_n} = \infty$. We can choose a subsequence (s_{n_p}) such that $s_{n_p} > 1$ for each $p \in \mathbb{N}$. Consider the sequence (x_n) defined by

$$x_n = \begin{cases} s_n, & n = m^2 \text{and } s_n \in \{s_{n_p} : p \in \mathbb{N}\} \\ 1, & n = m^2 \text{ and } s_n \notin \{s_{n_p} : p \in \mathbb{N}\} \quad (m \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$d(x_n, 0) = \begin{cases} (s_n, \alpha s_n), & n = m^2 \text{ and } s_n \in \{s_{n_p} : p \in \mathbb{N}\} \\ (1, \alpha), & n = m^2 \text{ and } s_n \notin \{s_{n_p} : p \in \mathbb{N}\} \quad (m \in \mathbb{N}) \\ (0, 0), & \text{otherwise.} \end{cases}$$

It is easy to see that (x_n) is statistical convergent to zero. Now, we show that (x_n) is not quasi-statistical convergent to zero; that is,

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \le n : c \preceq d(x_k, 0)\}| \ne 0.$$

For $c = (1, \alpha) \in E^+$ and $n \in \mathbb{N}$, we have

$$|\{k \le n : c \preceq d(x_k, 0)\}| = \left|\{k \le n : k = m^2, m \in \mathbb{N}\}\right|$$

and

$$\frac{1}{s_n} |\{k \le n : c \preceq d(x_k, 0)\}| = \frac{1}{s_n} \left(\sqrt{n} - r_n\right), \tag{2.1}$$

where $0 \le r_n < 1$. If we take the limit of (2.1) as $n \to \infty$, we conclude that (x_n) is not quasi-statistical convergent to zero.

Consequently, we have the following diagram:

convergent $\Rightarrow$ quasi-statistical convergent $\Rightarrow$ statistical convergent

Theorem 2.3. *Assume that*

$$h = \inf_n \frac{s_n}{n} > 0. \tag{2.2}$$

If a sequence (x_n) in a cone metric space (X, d) is statistical convergent to $x \in X$, then it is quasi-statistical convergent to x.

Proof. The proof follows from the inequality

$$\frac{1}{n} |\{k \leq n : c \preceq d(x_k, L)\}| \geq h \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, L)\}| .$$

□.

Corollary 2.1. *Assume that the sequence (s_n) satisfies (2.2). Then, (x_n) is statistical convergent to x if and only if (x_n) is quasi-statistical convergent to x.*

Theorem 2.4. *If (x_n) is quasi-statistical convergent to x in a cone metric space (X, d), then there is a sequence (y_n) which is convergent to x and quasi-statistical null sequence (z_n) such that $x_n = y_n + z_n$ for all $n \in \mathbb{N}$.*

Proof. Let $st_q - \lim_{n\to\infty} x_n = x$. If the terms of the sequence (x_n) is constant after a certain stage, then the proof is trivial. Otherwise given any $c \in E^+$, we can find an increasing sequence of positive integers (N_j) such that $N_0 = 0$ and $\frac{1}{s_n}\left|\left\{k \leq n : \frac{e}{j} \preceq d(x_k, x)\right\}\right| < \frac{1}{j}$ for all $n > N_j$ $(j = 1, 2, ...)$. Let us define (y_k) and (z_k) as follows:

$$\begin{array}{llll} z_k = 0 & \text{and} \quad y_k = x_k; & \text{if } N_0 < k \leq N_1, & \\ z_k = 0 & \text{and} \quad y_k = x_k; & \text{if } d(x_k, x) \prec\prec \frac{e}{j}, & N_j < k \leq N_{j+1}, \\ z_k = x_k - x & \text{and} \quad y_k = x; & \text{if } \frac{e}{j} \preceq d(x_k, x), & N_j < k \leq N_{j+1}. \end{array}$$

It is easy to see that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$. Now, we show that (y_k) is convergent to x. Given any $c \in E^+$, choose $j \in \mathbb{N}$ such that $\frac{e}{j} \prec\prec c$.

$$\text{If } \frac{e}{j} \preceq d(x_k, x) \text{ for } k > N_j, \text{ then } d(y_k, x) = d(x, x) = 0.$$

If $d(x_k, x) \prec\prec \frac{e}{j}$ for $k > N_j$, then $d(y_k, x) = d(x_k, x) \prec\prec \frac{e}{j} \prec\prec c$. Hence, it follows that $\lim_{k\to\infty} y_k = x$.

To show that (z_k) is quasi-statistical null sequence; it is enough to prove that

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : z_k \neq 0\}| = 0.$$

For $c \in E^+$, it is clear that the inclusion

$$\{k \leq n : c \preceq d(z_k, 0)\} \subseteq \{k \leq n : z_k \neq 0\}$$

holds for all $n \in \mathbb{N}$. Thus, we have

$$|\{k \leq n : c \preceq d(z_k, 0)\}| \leq |\{k \leq n : z_k \neq 0\}|.$$

Given any $\delta > 0$ there is a $j \in \mathbb{N}$ such that $\frac{1}{j} < \delta$. If $N_j < k \leq N_{j+1}$, we have

$$|\{k \leq n : z_k \neq 0\}| = \left|\left\{k \leq n : \frac{e}{j} \preceq d(x_k, x)\right\}\right|.$$

Thus, we have

$$\frac{1}{s_n}|\{k \leq n : z_k \neq 0\}| \leq \frac{1}{s_n}\left|\left\{k \leq n : \frac{e}{v} \preceq d(x_k, x)\right\}\right| < \frac{1}{v} < \frac{1}{j} < \delta$$

for $N_v < k \leq N_{v+1}$ and $v > j$ which concludes the proof. □.

The following result is an immediate consequence of the previous theorem.

Corollary 2.2. *If (x_n) is quasi-statistical convergent to x, then it has a subsequence (y_n) which is convergent to x.*

Definition 2.2. A sequence (x_n) in a cone metric space (X, d) is said to be quasi-statistical Cauchy if for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{s_n}|\{k \leq n : d(x_k, x_{n_0}) \prec\prec c\}| = 1$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{s_n}|\{k \leq n : c \preceq d(x_k, x_{n_0})\}| = 0.$$

Theorem 2.5. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is a Cauchy sequence, then it is a quasi-statistical Cauchy sequence.*

Proof. Let (x_n) be a Cauchy sequence. Then, for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \prec\prec c$ for every $n, m \geq n_0$. It follows that

$$\frac{1}{s_n}|\{k \leq n : c \preceq d(x_k, x_{n_0})\}| \leq \frac{n_0}{s_n}$$

which means $\lim_{n\to\infty} \frac{1}{s_n}|\{k \leq n : c \preceq d(x_k, x_{n_0})\}| = 0$. Hence, (x_n) is quasi-statistical Cauchy. □.

The sequence given in Example 2.1 is also a quasi-statistical Cauchy sequence which is not a Cauchy sequence.

Theorem 2.6. *Let (x_n) be a sequence in a cone metric space (X, d). If (x_n) is a quasi-statistical Cauchy sequence, then it is a statistical Cauchy sequence.*

Proof. Let (x_n) be a quasi-statistical Cauchy sequence. Then, for every $c \in E^+$ there exists $n_0 \in \mathbb{N}$ such that $\lim\limits_{n\to\infty} \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}| = 0$. Thus we have

$$\begin{aligned}\frac{1}{n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}| &= \frac{s_n}{n} \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}| \\ &\leq K \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, x_{n_0})\}|,\end{aligned}$$

where $K = \sup_n \frac{s_n}{n}$. This implies that (x_n) is a statistical Cauchy sequence in X. □.

Consequently, we have the following diagram:

Cauchy $\Rightarrow$ quasi-statistical Cauchy $\Rightarrow$ statistical Cauchy

Definition 2.3. A sequence (x_n) in a cone metric space (X, d) is said to be quasi-statistical bounded if there exist $\alpha \in X$ and $c \in E^+$ such that

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : c \preceq d(x_k, \alpha)\}| = 0.$$

Theorem 2.7. *If (x_n) is quasi-statistical bounded sequence in a cone metric space (X, d), then it is statistical bounded.*

Proof. Let (x_n) be a quasi-statistical bounded sequence, $\alpha \in X$ and $H = \sup\limits_n \frac{s_n}{n}$. Since the inequality

$$\frac{1}{n} |\{k \leq n : c \preceq d(x_k, \alpha)\}| \leq \frac{H}{s_n} |\{k \leq n : c \preceq d(x_k, \alpha)\}|$$

holds, the proof follows immediately. □.

Lemma 2.1. *Let P be a normal cone with normal constant K. The following statements hold for sequences (x_n) and (y_n) in a cone metric space (X, d).*

1. $st_q - \lim\limits_{n\to\infty} x_n = x \Leftrightarrow st_q - \lim\limits_{n\to\infty} d(x_n, x) = 0$
2. *If* $st_q - \lim\limits_{n\to\infty} x_n = x$ *and* $st_q - \lim\limits_{n\to\infty} y_n = y$, *then* $st_q - \lim\limits_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Proof. (1) Suppose that $st_q - \lim\limits_{n\to\infty} x_n = x$. Then, for every $c \in E^+$, we have

$$\lim_{n\to\infty} \frac{1}{s_n} |\{k \leq n : d(x_k, x) \prec\prec c\}| = 1.$$

Given any $\varepsilon > 0$, choose $c \in E^+$ such that $K \|c\| < \varepsilon$. Suppose that $k \in \mathbb{N}$ satisfies $d(x_k, x) \prec\prec c$. Since P is a normal cone with normal constant K, we can write

$$\|d(x_k, x)\| \leq K \|c\| < \varepsilon.$$

Consequently, we obtain

$$\frac{1}{s_n}\left|\left\{k \le n : d\left(x_k, x\right) \prec\prec c\right\}\right| \le \frac{1}{s_n}\left|\left\{k \le n : \left\|d\left(x_k, x\right)\right\| < \varepsilon\right\}\right|.$$

Hence, we conclude that

$$\lim_{n\to\infty}\frac{1}{s_n}\left|\left\{k \le n : \left\|d\left(x_k, x\right)\right\| < \varepsilon\right\}\right| = 1$$

which means $st_q - \lim\limits_{n\to\infty} d\left(x_n, x\right) = 0$.

Conversely, suppose that $st_q - \lim\limits_{n\to\infty} d\left(x_n, x\right) = 0$. Then for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty}\frac{1}{s_n}\left|\left\{k \le n : \left\|d\left(x_k, x\right)\right\| < \varepsilon\right\}\right| = 1.$$

Given any $c \in E^+$, we can find an $\varepsilon > 0$ such that $c - a \in E^+$ for all $a \in E$ with $\|a\| < \varepsilon$. Hence, if we choose $k \in \mathbb{N}$ such that $\|d(x_k, x)\| < \varepsilon$, then we obtain $d(x_k, x) \prec\prec c$ which implies that the inclusion $\{k : \|d(x_k, x)\| < \varepsilon\} \subset \{k : d(x_k, x) \prec\prec c\}$ holds. It follows that

$$\frac{1}{s_n}\left|\left\{k \le n : \left\|d\left(x_k, x\right)\right\| < \varepsilon\right\}\right| \le \frac{1}{s_n}\left|\left\{k \le n : d\left(x_k, x\right) \prec\prec c\right\}\right|.$$

Thus, we conclude that $\lim\limits_{n\to\infty}\frac{1}{s_n}\left|\left\{k \le n : d\left(x_k, x\right) \prec\prec c\right\}\right| = 1$ and so $st_q - \lim\limits_{n\to\infty} x_n = x$.

(2) Suppose $st_q - \lim\limits_{n\to\infty} x_n = x$ and $st_q - \lim\limits_{n\to\infty} y_n = y$. Given any $\varepsilon > 0$, choose $c \in E^+$ such that $\|c\| < \frac{\varepsilon}{4K+2}$. For $k \in \mathbb{N}$ with $d(x_k, x) \prec\prec c$ and $d(y_k, y) \prec\prec c$, we have $\|d(x_k, y_k) - d(x, y)\| < \varepsilon$ from the proof of Lemma 5 in [11]. Hence, the inclusion

$$\{k : \|d(x_k, y_k) - d(x, y)\| \ge \varepsilon\} \subset \{k : c \preceq d(x_k, x)\} \cup \{k : c \preceq d(y_k, y)\}$$

holds. It follows that

$$\lim_{n\to\infty}\frac{1}{s_n}\left|\left\{k \le n : \|d(x_k, y_k) - d(x, y)\| \ge \varepsilon\right\}\right| = 0$$

which means that $st_q - \lim\limits_{n\to\infty} d\left(x_n, y_n\right) = d\left(x, y\right)$. $\square$.

Remark 2.1. Note that P does not need to be a normal cone to prove the sufficiency condition in 1 of Lemma 2.1. That is; if $st_q - \lim\limits_{n\to\infty} d\left(x_n, x\right) = 0$ in a cone metric space (X, d), then we have $st_q - \lim\limits_{n\to\infty} x_n = x$.

Theorem 2.8. *Let (x_n) and (y_n) be two sequences in a cone metric space (X, d). If $st_q - \lim\limits_{n\to\infty} y_n = y$ and $d\left(x_n, y\right) \preceq d\left(y_n, y\right)$ for every $n \in \mathbb{N}$, then $st_q - \lim\limits_{n\to\infty} x_n = y$.*

Proof. Suppose that $st_q - \lim_{n\to\infty} y_n = y$ and $d(x_n, y) \preceq d(y_n, y)$ for every $n \in \mathbb{N}$. The proof follows from the fact that

$$\frac{1}{s_n}\left|\{k \le n : d(y_k, y) \preceq c\}\right| \le \frac{1}{s_n}\left|\{k \le n : d(x_k, y) \preceq c\}\right|.$$

$\square$.

Definition 2.4. A sequence (x_n) in a cone metric space (X, d) is said to be strongly quasi-summable to x, if

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} \|d(x_k, x)\| = 0$$

holds.

We will use N_q^s and S_q^s for the set of all strongly quasi-summable sequences and all quasi-statistical convergent sequences, respectively. That is,

$$N_q^s = \left\{(x_n) : \lim_{n\to\infty} \frac{1}{s_n} \sum_{k=1}^{n} \|d(x_k, x)\| = 0 \text{ for some } x\right\}$$

and

$$S_q^s = \left\{(x_n) : \lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : c \preceq d(x_k, x)\}\right| = 0 \text{ for some } x \in \mathbb{R} \text{ and for all } c \in E^+\right\}$$

If we take $t = (t_n)$ instead of $s = (s_n)$, we will write N_q^t and S_q^t instead of N_q^s and S_q^s, respectively.

Theorem 2.9. *Let $s_n \le t_n$ for every $n \in \mathbb{N}$. If a sequence (x_n) in a cone metric space (X, d) is quasi-statistical convergent to x with respect to $s = (s_n)$, then (x_n) sequence is quasi-statistical convergent to x with respect to $t = (t_n)$.*

Proof. Suppose that for every $c \in E^+$ we have $\lim_{n\to\infty} \frac{1}{s_n} \left|\{k \le n : c \preceq d(x_k, x)\}\right| = 0$. Since $s_n \le t_n$ holds for every $n \in \mathbb{N}$, we have the inequality

$$\frac{1}{s_n}\left|\{k \le n : c \preceq d(x_k, x)\}\right| \ge \frac{1}{t_n}\left|\{k \le n : c \preceq d(x_k, x)\}\right|.$$

Letting $n \to \infty$ in both sides of the above inequality, we obtain that the sequence (x_n) is quasi-statistical convergent to x with respect to $t = (t_n)$. $\square$.

Now, we consider the sequence (x_n) in Example 2.2 and if we take $t_n = n$ and $s_n = n^{1/4}$, then we observe that the sequence (x_n) is quasi-statistical convergent to zero with respect to the sequence $t = (t_n)$ but the sequence (x_n) is not quasi-statistical convergent to zero with respect to the sequence $s = (s_n)$. Thus, the following result can given as a consequence of this theorem.

Corollary 2.3. *Let $s_n \leq t_n$ for every $n \in \mathbb{N}$. Then, the inclusion $S_q^s \subset S_q^t$ strictly holds.*

Theorem 2.10. *Let $s_n \leq t_n$ for every $n \in \mathbb{N}$. If a sequence (x_n) in a cone metric space (X, d) is strongly quasi-summable to x with respect to $s = (s_n)$, then the sequence (x_n) is quasi-statistical convergent to x with respect to $t = (t_n)$.*

Proof. Let $\lim\limits_{n\to\infty} \frac{1}{s_n} \sum\limits_{k=1} \|d(x_k, x)\| = 0$. By using the fact that

$$\sum_{k=1}^{n} \|d(x_k, x)\| = \sum_{\substack{k=1 \\ \|d(x_k,x)\| \geq \varepsilon}}^{n} \|d(x_k, x)\| + \sum_{\substack{k=1 \\ \|d(x_k,x)\| < \varepsilon}}^{n} \|d(x_k, x)\| \geq \varepsilon \left|\{k \leq n : \|d(x_k, x)\| \geq \varepsilon\}\right|$$

and $s_n \leq t_n$ for every $n \in \mathbb{N}$, we obtain

$$\frac{1}{\varepsilon} \frac{1}{s_n} \sum_{k=1}^{n} \|d(x_k, x)\| \geq \frac{1}{t_n} \left|\{k \leq n : \|d(x_k, x)\| \geq \varepsilon\}\right|.$$

Since the limit of the left side equals to zero, we have $st_q - \lim\limits_{n\to\infty} d(x_n, x) = 0$ with respect to $t = (t_n)$. From Remark 2.1, we conclude that $st_q - \lim\limits_{n\to\infty} x_n = x$ with respect to $t = (t_n)$. $\square$.

The converse of this theorem is not always true.

Example 2.3. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ be the cone metric defined by $d(x, y) = (|x - y|, |x - y|)$. Consider the sequence (x_n) defined by

$$x_n = \begin{cases} 1, & n = m^2 \\ 0, & n \neq m^2 \end{cases} \quad m \in \mathbb{N}$$

Let $(s_n) = \left(n^{\frac{1}{4}}\right)$ and $(t_n) = (n)$. We have

$$d(x_n, 0) = \begin{cases} (1, 1), & n = m^2 \\ (0, 0), & n \neq m^2 \end{cases} \quad m \in \mathbb{N}$$

Hence, given any $c \in E^+$ and $n \in \mathbb{N}$, we obtain

$$\frac{1}{t_n} \left|\{k \leq n : d(x_k, 0) \prec\prec c\}\right| \geq \frac{1}{t_n} \left|\left\{k \leq n : n \neq m^2\right\}\right|.$$

Since the limit of the right side equals 1, we conclude that the sequence (x_n) is quasi-statistical convergent to zero with respect to $t = (t_n)$.

Now, we will show that the sequence (x_n) is not strongly quasi-summable to zero with respect to $s = (s_n)$. It is clear that

$$\|d(x_k, 0)\| = \begin{cases} \sqrt{2}, & k = m^2 \\ 0, & k \neq m^2 \end{cases} \quad m \in \mathbb{N}$$

Then, we obtain that

$$\begin{aligned}\sum_{k=1}^{n}\|d(x_k,0)\| &= 0\left|\{k\le n: k\neq m^2 \text{ for all } m\in\mathbb{N}\}\right| \\ &+ \sqrt{2}\left|\{k\le n: k=m^2 \text{ for some } m\in\mathbb{N}\}\right| \\ &= 0.\left(n-[|\sqrt{n}|]\right)+\sqrt{2}\left([|\sqrt{n}|]\right).\end{aligned}$$

and so

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\|d(x_k,0)\| = \lim_{n\to\infty}\frac{1}{s_n}\sqrt{2}\left([|\sqrt{n}|]\right)=\infty.$$

Consequently, we find that

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\|d(x_k,0)\|\neq 0.$$

which means the sequence (x_n) is not strongly quasi-summable to zero with respect to $s=(s_n)$.

Corollary 2.4. *Let $s_n\le t_n$ for every $n\in\mathbb{N}$. The inclusion $N_q^s\subset S_q^t$, strictly holds.*

Theorem 2.11. *Let $s_n\le t_n$ for every $n\in\mathbb{N}$. If a sequence (x_n) in a cone metric space (X,d) is strongly quasi-summable to x with respect to $s=(s_n)$, then the sequence (x_n) is strongly quasi-summable sequence to x with respect to $t=(t_n)$.*

Proof. Suppose that the sequence (x_n) is strongly quasi-summable to x with respect to $s=(s_n)$. Then, we have

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\|d(x_k,x)\|=0.$$

From the fact that $s_n\le t_n$ for every $n\in\mathbb{N}$, we have the following inequality

$$\frac{1}{s_n}\sum_{k=1}^{n}\|d(x_k,x)\|\ge\frac{1}{t_n}\sum_{k=1}^{n}\|d(x_k,x)\|.$$

Hence, we conclude that $\lim\limits_{n\to\infty}\frac{1}{t_n}\sum\limits_{k=1}^{n}\|d(x_k,x)\|=0.$ $\square$.

But the converse of this theorem is not always true. To observe this, consider the sequences (x_n), $s=(s_n)$ and $t=(t_n)$ defined in Example 2.3. It can be shown that $(x_n)\in N_q^t$ and $(x_n)\notin N_q^s$. Thus, the following corollary can be given as a result of this theorem.

Corollary 2.5. *Let $s_n\le t_n$ for every $n\in\mathbb{N}$. The inclusion $N_q^s\subset N_q^t$, strictly holds.*

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FACTA Universitatis. Series, Mathematics and informatics / editor-in-chief Predrag S. Stanimirović. - 1986, N° 1- . - Niš : University of Niš, 1986- (Niš : Unigraf-X-Copy). - 24 cm

Tekst na engl. jeziku. - Drugo izdanje na drugom medijumu: Facta Universitatis. Series: Mathematics and Informatics (Online) = ISSN 2406-047X
ISSN 0352-9665 = Facta Universitatis. Series: Mathematics and informatics
COBISS.SR-ID 5881090

FACTA UNIVERSITATIS

Series
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Vol. 33, No 4 (2018)

Contents