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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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ON FREQUENTLY HYPERCYCLIC ABSTRACT HIGHER-ORDER DIFFERENTIAL EQUATIONS *

Belkacem Chaouchi and Marko Kostić

Abstract. In this note, we analyze frequently hypercyclic solutions of abstract higher-order differential equations in separable infinite-dimensional complex Banach spaces. We essentially apply results from the theory of C-regularized semigroups, providing several illustrative examples and possible applications.
Keywords: Higher-order differential equations; regularized semigroups; complex Banach space.

1. Introduction and Preliminaries

As it is well-known, the class of frequently hypercyclic linear continuous operators on separable Fréchet spaces was introduced by F. Bayart and S. Grivaux in 2006 ([1]). Frequent hypercyclicity and various generalizations of this concept are very active fields of research of a great number of mathematicians working in the field of linear topological dynamics (for more details, we may refer e.g. to [2]-[4], [13] and references cited therein).

Frequently hypercyclic properties of abstract first order differential equations have been studied by E. M. Mangino, A. Peris [21] and E. M. Mangino, M. Murillo-Arcila [22], within the framework of theory of strongly continuous semigroups, and the second named author [19], within the theory of integrated and C-regularized semigroups. Frequently hypercyclic abstract second order differential equations have been recently investigated in [20] by using the general notion of C-distribution cosine functions and integrated C-cosine functions. Up to now, we do not have any relevant reference treating the operator theoretical aspects of frequently hypercyclic abstract higher-order differential equations. This fact has strongly influenced us to write this paper.

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The organization, main ideas and novelties of paper are briefly described as follows. Let $(E, \|\cdot\|)$ be a separable infinite-dimensional complex Banach space. We analyze frequently hypercyclic properties of solutions of the abstract Cauchy problem

$$(ACP_n): \begin{cases} u^{(n)}(t) + A_{n-1}u^{(n-1)}(t) + \cdots + A_1u'(t) + A_0u(t) = 0, \ t \geq 0, \\ u^{(k)}(0) = u_k, \ k = 0, \cdots, n-1, \end{cases}$$

where $A_0, \cdots, A_{n-1}$ are closed linear operators on E and $u_0, \cdots, u_{n-1} \in E$; by a strong solution of (ACP_n), we mean any n-times continuously differentiable function $t \mapsto u(t)$, $t \geq 0$ such that the mappings $t \mapsto A_iu^{(i)}(t)$, $t \geq 0$ are continuous for $0 \leq i \leq n-1$ and the initial conditions are satisfied (for more details about the well-posedness of (ACP_n), the reader may consult the monographs [24] by T.-J. Xiao, J. Liang and [16] by the author). In order to investigate frequently hypercyclic properties of solutions to (ACP_n), we convert this problem into corresponding abstract first order differential equation with appropriately chosen operator matrix acting on product space E^n. The proofs of our structural results lean heavily on the use of Lemma 1.1 from [19], where we have recently considered frequent hypercyclicity for C-regularized semigroups following the approach of S. El Mourchid [10, Theorem 2.1] and E. M. Mangino, A. Peris [21, Corollary 2.3]. In contrast to the recent research studies of J. A. Conejero, C. Lizama et al. [5]-[7], where the authors have studied the hypercyclic and chaotic solutions of certain kinds of abstract second and third order differential equations in the spaces of Herzog analytic functions by employing, primarily, the Desch-Schappacher-Webb criterion [9], the operator matrix under our consideration is not bounded and as such does not generate a strongly continuous semigrop on E^n a priori. This is the main reason why we use the theory of C-regularized semigroups in this paper. We construct solutions of (ACP_n) for initial values $(u_0, \cdots, u_{n-1})$ belonging to a certain proper subspace $\tilde{E} \subseteq E^n$ and after that analyze their frequently hypercyclic properties by applying essentially Lemma 1.1, as mentioned above. Motivated by our recent researches [18] and [20], in Definition 1.1 we introduce the notion of a $(W, \tilde{E}, \mathcal{E})$-frequent hypercyclicity. The main goal of Theorem 2.1 is to analyze $(W, \tilde{E}, \mathcal{E})$-frequently hypercyclic solutions of some special classes of problems (ACP_n) in the case that the operator matrix $p(A)$ obtained after the usual convertion generates an entire C-regularized group. After that, we revisit once more the fundamental result [23, Theorem 5] of F. Neubrander. We introduce the notion of $(W, \tilde{E}, \mathcal{E}, (D(A_{n-1}))^n)$-frequent hypercyclicity (Definition 2.1) and consider $(W, \tilde{E}, \mathcal{E}, (D(A_{n-1}))^n)$–frequently hypercyclic solutions of (ACP_n) (Theorem 2.2), provided that the operator $-A_{n-1}$ is the generator of a strongly continuous semigroup on E as well as $D(A_{n-1}) \subseteq D(A_j)$ for $0 \leq j \leq n-2$ (cf. also [16, Theorem 2.10.45] for a generalization of the above-mentioned theorem to abstract time-fractional differential equations). In our approach, we almost always face the situation $\tilde{E} \neq E^n$, which indicates a certain type of subspace frequent hypercyclicity of constructed solutions to (ACP_n) (in [5]-[7], the situation in which $\tilde{E} = E^n$ can really occur). At the end of paper, we provide several examples and applications of our results.

Before explaining the notation used, we would like to note that we will not discuss here frequently hypercyclic properties of systems of evolution equations by using the theory of operator matrices developed by K.-J. Engel and his collaborators (for more details about this subject, we refer the reader to the monograph [12]). By $L(E)$ we denote the space consisting of all continuous linear mappings from E into E. We always assume henceforth that $C \in L(E)$ and C is injective. Let A be a closed linear operator with domain $D(A)$ and range $R(A)$ contained in E, and let $CA \subseteq AC$. Set $D_\infty(A) := \bigcap_{k\in\mathbb{N}} D(A^k)$. The part of A in a linear subspace $\tilde{E}$ of E, $A_{|\tilde{E}}$ shortly, is defined through $A_{|\tilde{E}} := \{(x,y) \in A : x,\ y \in \tilde{E}\}$ (we will identify an operator and its graph henceforth). Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \Big\{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\Big\}.$$

In our framework, the C-resolvent set of A consists of those complex numbers λ for which the operator $\lambda - A$ is injective and $R(C) \subseteq R(\lambda - A)$. The resolvent set of A, denoted by $\rho(A)$, is obtained by plugging $C = I$. For every $\lambda \in \rho(A)$ and $n \in \mathbb{N}$, we have that $(D(A^n), \|\cdot\|_n)$ is a Banach space, where $\|x\|_n := \sum_{i=0}^{n} \|A^i x\|$ $(x \in D(A^n))$. We denote this space simply by $[D(A^n)]$. All operator families considered in this paper will be non-degenerate. Set $\mathbb{N}_n := \{1, \cdots, n\}$ and $\mathbb{N}_n^0 := \mathbb{N}_n \cup \{0\}$ $(n \in \mathbb{N})$.

Suppose that $T \subseteq \mathbb{N}$. The lower density of T, denoted by $\underline{d}(T)$, is defined through:

$$\underline{d}(T) := \liminf_{n\to\infty} \frac{|T \cap [1,n]|}{n}.$$

If $T \subseteq [0,\infty)$, then the lower density of T, denoted by $\underline{d}(T)$, is defined through:

$$\underline{d}_c(T) := \liminf_{t\to\infty} \frac{m(T \cap [0,t])}{t},$$

where $m(\cdot)$ denotes the Lebesgue measure on $[0,\infty)$. A linear operator A on E is said to be frequently hypercyclic iff there exists an element $x \in D_\infty(A)$ (frequently hypercyclic vector of A) such that for each open non-empty subset V of E the set $\{n \in \mathbb{N} : A^n x \in V\}$ has positive lower density.

Motivated by our recent research study of $\mathcal{D}$-hypercyclic and $\mathcal{D}$-topologically mixing properties of abstract degenerate Cauchy problems with Caputo fractional derivatives [18], we introduce the following definition (since we are primarily concerned with applications of C-regularized semigroups, we will consider only non-degenerate differential equations henceforth; the analysis of frequently hypercyclic abstract time-fractional differential equations is far from being trivial and nothing has been said about this theme so far):

Definition 1.1. (cf. also [18, Definition 2]) Suppose that $\emptyset \neq W \subseteq \mathbb{N}_{n-1}^0$, $\tilde{E}$ is a linear subspace of E^n and $\mathcal{E} := (E_i : i \in W)$ is a tuple of linear subspaces of E. Then we say that the abstract Cauchy problem (ACP_n) is $(W, \tilde{E}, \mathcal{E})$-frequently hypercyclic iff there exists a strong solution $t \mapsto u(t)$, $t \geq 0$ of (ACP_n) with the initial values $(u_0, \cdots, u_{n-1}) \in \tilde{E}$ satisfying additionally that, for every tuple of open non-empty subsets V $:= (V_i : i \in W)$ of E, the set $\bigcap_{i\in W}\{t \geq 0 : u^{(i)}(t) \in V_i \cap E_i\}$ has positive lower density.

Introduction of Definition 1.1 is also motivated by some recent results about frequently hypercyclic properties of abstract second order differential equations ([20]). Speaking-matter-of-factly, if the assumptions of [20, Theorem 1] are satisfied, then there exists a closed linear subspace $\tilde{E}$ of E^2 such that the abstract Cauchy problem (ACP_2) with $A_1 \equiv 0$ and $A_0 \equiv -A$ is $(\{0,1\}, \tilde{E}, \mathcal{E})$-frequently hypercyclic, where $\mathcal{E} = (\pi_1(\tilde{E}), \pi_2(\tilde{E}))$ and $\pi_1(\cdot)$, $\pi_2(\cdot)$ denote the first and second projection, respectively. It is also worth noting that the spectral conditions of [20, Theorem 1] are particularly satisfied for a substantially large class of abstract incomplete second order differential equations.

We will use the following definition:

Definition 1.2. Let A be a closed linear operator. If there exists a strongly continuous operator family $(T(t))_{t\geq 0} \subseteq L(E)$ such that:

(i) $T(t)A \subseteq AT(t)$, $t \geq 0$,

(ii) $T(t)C = CT(t)$, $t \geq 0$,

(iii) for all $x \in E$ and $t \geq 0$: $\int_0^t T(s)x\,ds \in D(A)$ and

$$A\int_0^t T(s)x\,ds = T(t)x - Cx,$$

then it is said that A is a subgenerator of a (global) C-regularized semigroup $(T(t))_{t\geq 0}$.

It is well-known that $T(t)T(s) = T(t+s)C$ for all $t,\ s \geq 0$. The integral generator of $(T(t))_{t\geq 0}$ is defined by

$$\hat{A} := \left\{(x,y) \in E \times E : T(t)x - Cx = \int_0^t T(s)y\,ds,\ t \geq 0\right\}.$$

We know that the integral generator of $(T(t))_{t\geq 0}$ is a closed linear operator which is an extension of any subgenerator of $(T(t))_{t\geq 0}$ and satisfies $\hat{A} = C^{-1}AC$ for any subgenerator A of $(T(t))_{t\geq 0}$. If for each fixed element $x \in E$ the mapping $t \mapsto T(t)x$, $t \geq 0$ can be extended to an entire function, then we say that $(T(t))_{t\geq 0}$ is an entire C-regularized group with subgenerator A and integral generator $\hat{A}$ ([8]). Furthermore, it is said that $(T(t))_{t\geq 0}$ is frequently hypercyclic iff there exists an element $x \in E$ (frequently hypercyclic vector of $(T(t))_{t\geq 0}$) such that the mapping $t \mapsto C^{-1}T(t)x$, $t \geq 0$ is well-defined, continuous and that for each open non-empty subset V of E the set $\{t \geq 0 : C^{-1}T(t)x \in V\}$ has positive lower density ([19]).

Throughout the whole paper, we will essentially employ the following result, proved recently in [19]:

Lemma 1.1. *Let $t_0 > 0$ and let A be a subgenerator of a global C-regularized semigroup $(S_0(t))_{t\geq 0}$ on E. Suppose that $R(C)$ is dense in E. Set $T(t)x := C^{-1}S_0(t)x$, $t \geq 0$, $x \in Z_1(A)$. Suppose, further, that there exists a family $(f_j)_{j\in\Gamma}$ of twice continuously differentiable mappings $f_j : I_j \to E$ such that I_j is an interval in $\mathbb{R}$ and $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$. Set $\tilde{E} := \overline{span\{f_j(t) : j \in \Gamma,\ t \in I_j\}}$. Then $A_{|\tilde{E}}$ is a subgenerator of a global $C_{|\tilde{E}}$-regularized semigroup $(S_0(t)_{|\tilde{E}})_{t\geq 0}$ on $\tilde{E}$, $(S_0(t)_{|\tilde{E}})_{t\geq 0}$ is frequently hypercyclic in $\tilde{E}$ and the operator $T(t_0)_{|\tilde{E}}$ is frequently hypercyclic in $\tilde{E}$.*

For more details about C-regularized semigroups and their applications, we refer the reader to the monographs [8] by R. deLaubenfels and [15]-[16] by the author.

2. Formulation and Proof of Main Results

In the formulation of our first structural result, we assume that $N,\ n \in \mathbb{N}$ and $iA_j,\ 1 \leq j \leq N$ are commuting generators of bounded C_0-groups on E. Define $A := (A_1, \cdot\cdot\cdot, A_N)$ and $A^\eta := A_1^{\eta_1} \cdot\cdot\cdot A_N^{\eta_N}$ for any $\eta = (\eta_1, \cdot\cdot\cdot, \eta_N) \in \mathbb{N}_0^N$. If $P(\xi) = [p_{ij}(\xi)]_{n\times n}$ is an arbitrary matrix of complex polynomials in variable $\xi \in \mathbb{R}^N$, then we can write $P(\xi) = \sum_{|\eta|\leq m} P_\eta \xi^\eta$ for a certain integer $m \in \mathbb{N}$ and for certain complex matrices P_η of format $n \times n$. We know that the operator $P(A) := \sum_{|\eta|\leq m} P_\eta A^\eta$ acting with its maximal domain is closable on E^n; moreover, the following holds:

Lemma 2.1. ([8], [16]) *There exists an injective operator $C \in L(E^n)$ with dense range in E^n such that the operator $\overline{P(A)}$ generates an entire C-regularized group $(T(t))_{t\geq 0}$ on E^n such that $T(t)\vec{x} \in D_\infty(P(A))$ for all $\vec{x} \in E^n$.*

Let $\pi_j : E^n \to E$ be the j-th projection $(1 \leq j \leq n)$, let $p_0(\xi), \cdot\cdot\cdot, p_{n-1}(\xi)$ be complex polynomials in variable $\xi \in \mathbb{R}^N$, and let

$$p(A) := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{n-1} \end{bmatrix},$$

where $A_i := p_i(A)$ for $0 \leq i \leq n-1$. Then we have the following:

Theorem 2.1. *Suppose that there exists a family $(F_j)_{j\in\Gamma}$ of twice continuously differentiable mappings $F_j : I_j \to E^n$ such that I_j is an interval in $\mathbb{R}$ and $p(A)F_j(t) = itF_j(t)$ for every $t \in I_j$, $j \in \Gamma$. Set $\tilde{E} := \overline{span\{F_j(t) : j \in \Gamma,\ t \in I_j\}}$. Then the abstract Cauchy problem (ACP_n) is $(\mathbb{N}_{n-1}^0, \tilde{E}, \mathcal{E})$-frequently hypercyclic with $\mathcal{E} := (\pi_1(\tilde{E}), \cdot\cdot\cdot, \pi_n(\tilde{E}))$.*

Proof. By Lemma 2.1, we know that there exists an injective operator $C \in L(E^n)$ with dense range in E^n such that the operator $\overline{P(A)}$ generates an entire C-regularized group $(T(t))_{t\geq 0}$ on E^n such that $T(t)\vec{x} \in D_\infty(P(A))$ for all $\vec{x} \in E^n$. Furthermore, the injective operator C can be chosen such that $C(span\{F_j(t) : j \in \Gamma,\ t \in I_j\}) = span\{F_j(t) : j \in \Gamma,\ t \in I_j\}$ and that $C(\tilde{E})$ is a dense linear subspace of $\tilde{E}$; see [17, Remark 14(ii)]. Due to Lemma 1.1, we know that $(S_0(t)_{|\tilde{E}})_{t\geq 0}$ is frequently hypercyclic in $\tilde{E}$, which implies that there exists a vector $\vec{x} \in \tilde{E}$ such that for each open non-empty subset V in E^n the set $\{t \geq 0 : C^{-1}T(t)\vec{x} \in \tilde{E} \cap V\}$ has positive lower density. Since $C(\tilde{E})$ is a dense linear subspace of $\tilde{E}$, it readily follows that for each open non-empty subset V in E^n the set $\{t \geq 0 : T(t)\vec{x} \in \tilde{E} \cap V\}$ has positive lower density, as well. On the other hand, the function $t \mapsto T(t)\vec{x}$, $t \geq 0$ is a unique solution of the abstract Cauchy problem $\vec{U}'(t) = \overline{p(A)}\vec{U}(t)$, $t \geq 0$; $\vec{U}(0) = C\vec{x}$. Furthermore, $T(t)\vec{x} \in D_\infty(P(A))$ so that the function $t \mapsto T(t)\vec{x}$, $t \geq 0$ is a unique solution of the abstract Cauchy problem $\vec{U}'(t) = p(A)\vec{U}(t)$, $t \geq 0$; $\vec{U}(0) = C\vec{x}$, actually. It is clear that the first, second,..., the n-th component of $T(\cdot)\vec{x}$ is a unique solution of (ACP_n), its first derivative,..., its $(n-1)$-derivative, respectively, with the initial conditions $u_j = \pi_{j+1}(C\vec{x})$, $0 \leq j \leq n-1$. This simply implies the required conclusion. □

Remark 2.1. The most important case for applications is $N = 1$. In this case, let us assume that $f_j : I_j \to E$ is a twice continuously differentiable mapping, $g_j : \{it\,;\, t \in I_j\} \to \mathbb{C} \setminus \{0\}$ is a scalar-valued mapping and $Af_j(t) = g_j(it)f_j(t)$, $t \in I_j$ $(j \in \Gamma)$. If

$$(it)^n + \sum_{l=0}^{n-1}(it)^l P_l\big(g_j(it)\big) = 0, \quad t \in I_j,\ j \in \Gamma, \tag{2.1}$$

then the assumptions of Theorem 2.1 are satisfied with

$$F_j(t) := \big[f_j(t)\ itf_j(t) \cdots\ (it)^{n-1}f_j(t)\big]^T, \quad t \in I_j,\ j \in \Gamma;$$

see e.g. [18, Example 1(ii)].

We continue by observing that Definition 1.1 does not enable one to thoroughly investigate frequently hypercyclic solutions of some important classes of abstract higher-order differential equations already examined in the existing literature. For example, F. Neubrander has analyzed in [23] the well-posedness results for (ACP_n) by reduction this problem into a first order matricial system, employing the matrix

$$\Delta := \begin{bmatrix} -A_{n-1} & I & 0 & \cdots & 0 \\ -A_{n-2} & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -A_1 & 0 & 0 & \cdots & I \\ -A_0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The operator matrix

$$\Psi := \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -A_{n-1} & I & 0 & \cdots & 0 & 0 \\ -A_{n-2} & -A_{n-1} & I & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdots & I & 0 \\ -A_1 & -A_2 & -A_3 & \cdots & -A_{n-1} & I \end{bmatrix}$$

plays an important role in his analysis, as well.

We will use the following notion:

Definition 2.1. Suppose that $\lambda \in \rho(\Delta)$, $\emptyset \neq W \subseteq \mathbb{N}^0_{n-1}$, $\tilde{E}$ is a linear subspace of $(D(A_{n-1}))^n$ and $\mathcal{E} := (E_i : i \in W)$ is a tuple of linear subspaces of E. Then we say that the abstract Cauchy problem (ACP_n) is $(W, \tilde{E}, \mathcal{E}, (D(A_{n-1}))^n)$-frequently hypercyclic iff there exists a strong solution $t \mapsto u(t)$, $t \geq 0$ of (ACP_n) with the initial values $(u_0, \cdots, u_{n-1}) \in \tilde{E}$ satisfying additionally that, for every open non-empty subset V of E^n, the set $\bigcap_{i \in W}\{t \geq 0 : u^{(i)}(t) + \sum_{j=1}^{i} A_{n-i}u^{(i-j)}(t) \in \pi_{i+1}((\lambda - \Delta)^{-n}(\mathrm{V})) \cap E_i\}$ has positive lower density.

This definition is a good one and does not depend on the choice of number $\lambda \in \rho(\Delta)$. This follows from the fact that for each $\lambda \in \rho(\Delta)$ the mapping $\Pi : E^n \to [D(\Delta^n)]$ given by $\Pi\vec{x} := (\lambda - \Delta)^{-n}\vec{x}$, $\vec{x} \in E^n$ is a linear topological isomorphism so that $\{(\lambda - \Delta)^{-n}(\mathrm{V}) : \mathrm{V}$ is an open non-empty subset of $E^n\}$ is equal to the set of all open non-empty subsets of $[D(\Delta^n)]$ and therefore independent of $\lambda \in \rho(\Delta)$.

Our second structural result reads as follows:

Theorem 2.2. *Suppose that the operator $-A_{n-1}$ is the generator of a strongly continuous semigroup on E as well as $D(A_{n-1}) \subseteq D(A_j)$ for $0 \leq j \leq n-2$. Suppose, further, that there exists a family $(F_j)_{j \in \Gamma}$ of twice continuously differentiable mappings $F_j : I_j \to E^n$ such that I_j is an interval in $\mathbb{R}$ and $\Delta F_j(t) = itF_j(t)$ for every $t \in I_j$, $j \in \Gamma$. Set*

$$\tilde{E} := \overline{span\{F_j(t) : j \in \Gamma,\ t \in I_j\}}^{[D(\Delta^n)]}. \tag{2.2}$$

Then the abstract Cauchy problem (ACP_n) is $(\mathbb{N}^0_{n-1}, \Psi^{-1}(\tilde{E}), \mathcal{E})$-frequently hypercyclic with $\mathcal{E} := (\pi_1(\tilde{E}), \cdots, \pi_n(\tilde{E}))$.

Proof. By the proof of [23, Theorem 5], we know the following:

(i) The operator Δ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on E^n and therefore there exists $\lambda \in \rho(\Delta)$.

(ii) The mapping Ψ is a bijection between the spaces $(D(A_{n-1}))^n$ and $D(\Delta^n)$.

(iii) For every $\vec{x} \in D(\Delta^n)$, the mapping $t \mapsto \pi_1(T(t)\vec{x})$, $t \geq 0$ is a strong solution of problem (ACP_n) with the initial value $\vec{y} = \Psi^{-1}\vec{x} \in (D(A_{n-1}))^n$.

From (i), we may deduce that the operator $\Delta_{|D(\Delta^n)}$ generates a strongly continuous semigroup on the space $[D(\Delta^n)]$; see e.g. [11, Chapter II.5]. By Lemma 1.1, it follows that there exists a vector $\vec{x} \in D(\Delta^n) \cap \tilde{E}$ such that for every open non-empty subset V' in $[D(\Delta^n)]$, the set $\{t \geq 0 : T(t)\vec{x} \in \mathrm{V}' \cap \tilde{E}\}$ has positive lower density. Since $\{t \geq 0 : T(t)\vec{x} \in \mathrm{V}' \cap \tilde{E}\} \subseteq \bigcap_{i=1}^{n}\{t \geq 0 : \pi_i(T(t)\vec{x}) \in \pi_i(\mathrm{V}') \cap \pi_i(\tilde{E})\}$, the required assertion follows from a simple analysis involving (i)-(iii) and the fact that the $(i+1)$-projection of $T(\cdot)\vec{x}$ equals $u^{(i)}(\cdot) + \sum_{j=1}^{i} A_{n-i}u^{(i-j)}(\cdot)$ for $0 \leq i \leq n-1$, where $u(\cdot) := \pi_1(T(\cdot)\vec{x})$ is a unique strong solution of problem (ACP_n) with initial value $y = \Psi^{-1}\vec{x}$ (see the equation [23, (1), p. 267]). $\square$

Remark 2.2. Let us assume that $P_0, \cdot \cdot \cdot, P_{n-1}$ are complex polynomials in one variable, $f_j : I_j \to E$ is a twice continuously differentiable mapping, $g_j : \{it\,; t \in I_j\} \to \mathbb{C} \setminus \{0\}$ is a scalar-valued mapping and $Af_j(t) = g_j(it)f_j(t)$, $t \in I_j$ $(j \in \Gamma)$. If (2.1) holds, then the assumptions of Theorem 2.2 are satisfied with $A_s := P_s(A)$ $(0 \leq s \leq n-1)$ and

$$(2.3) \qquad F_j(t) := \big[F_{j1}(t)\ F_{j2}(t) \cdots\ F_{jn}(t)\big]^T, \quad t \in I_j,\ j \in \Gamma,$$

where, for $2 \leq s \leq n$,

$$(2.4) \qquad F_{js}(t) := \sum_{l=0}^{s-2}(it)^l A_{n-s+1+l}f_j(t) + (it)^{s-1}f_j(t), \quad t \in I_1,\ j \in \Gamma.$$

It is worth noting that Theorem 2.1 and Theorem 2.2 provide also sufficient spectral conditions for certain types of (subspace) topologically mixing properties and (subspace) Devaney chaoticity of solutions to (ACP_n); see [18] for more details.

We close the paper by providing some illustrative examples and applications.

Example 2.1. Suppose that $E := L^2(\mathbb{R})$, $c_1 > c > \frac{b}{2} > 0$, the operator $\mathcal{A}_c$ is defined by $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$, $\mathcal{A}_c u := u'' + bxu' + cu$, $u \in D(\mathcal{A}_c)$, $\Omega := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ \lambda \neq c - c_1,\ \mathrm{Re}\,\lambda < c - \frac{b}{2}\}$, $f_1(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}\xi|\xi|^{-(2+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$ and $f_2(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}|\xi|^{-(1+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$ (here, $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on the real line). Consider the equation

$$(2.5) \qquad u'''(t) + \big(c_2 - \mathcal{A}_c\big)u'(t) + c_1 u(t) = 0, \quad t \geq 0,$$

where $c_1 \in \mathbb{C} \setminus \{0\}$ and $c_2 \in \mathbb{C}$. As already observed in [18, Example 1(i)], there exist $t > 0$ and $\epsilon > 0$ such that the equation (2.1) holds with the interval $(i(t-\epsilon), i(t+\epsilon))$ and the obvious choice of polynomials $P_0(\cdot)$, $P_1(\cdot)$ and $P_2(\cdot)$. Since the operator $A_1 := \mathcal{A} - c_2$ generates a strongly continuous semigroup and $D(A_1) \subseteq D(A_0)$, Theorem 2.2 is applicable so that the abstract Cauchy problem (2.5), equipped with initial conditions $u^{(j)}(0) = u_j$ for $0 \leq j \leq 2$, is $(\mathbb{N}_2^0, \Psi^{-1}(\tilde{E}), \mathcal{E})$-frequently hypercyclic, where $F_j(\cdot)$ is given by (2.3)-(2.4) for $j = 1, 2$, $\tilde{E}$ is defined by (2.2) and $\mathcal{E} := (\pi_1(\tilde{E}), \pi_2(\tilde{E}), \pi_3(\tilde{E}))$.

Example 2.2. Suppose that $0 < \gamma \leq 1$, $a > 0$, $p > 2$ and X is a symmetric space of non-compact type and rank one. Then the Laplace-Beltrami operator

$-\Delta^{\natural}_{X,p}$ generates a strongly continuous semigroup on X and we know that $int(P_p) \subseteq \sigma_p(\Delta^{\natural}_{X,p})$, where P_p denotes the parabolic domain defined in [14]. Suppose that (2.1) holds with $\Gamma = \{1\}$, the function $g_1(it) = it$, $t \in I_1$ (I_1 a suitable chosen subinterval of $\mathbb{R}$) and certain complex polynomials $P_0(\cdot), \cdot\cdot\cdot, P_{n-1}(\cdot)$. Then Theorem 2.2 is applicable with operators $A_l := P_l(A)$ $(l \in \mathbb{N}^0_{n-1})$.

Example 2.3. Let us recall that a measurable function $\rho : \mathbb{R} \to (0, \infty)$ is called an admissible weight function iff there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(t) \leq Me^{\omega|t'|}\rho(t + t')$ for all $t,\ t' \in \mathbb{R}$. For such a function $\rho(\cdot)$, we consider the following Banach spaces:

$$L^p_\rho(\mathbb{R}) := \big\{u : \mathbb{R} \to \mathbb{C}\ ; u(\cdot) \text{ is measurable and } ||u||_p < \infty\big\},$$

where $p \in [1, \infty)$ and $||u||_p := (\int_{\mathbb{R}} |u(t)|^p \rho(t)\, dt)^{1/p}$, as well as

$$C_{0,\rho}(\mathbb{R}) := \big\{u : \mathbb{R} \to \mathbb{C}\ ; u(\cdot) \text{ is continuous and } \lim_{t\to\infty} u(t)\rho(t) = 0\big\},$$

with $||u|| := \sup_{t\in\mathbb{R}} |u(t)\rho(t)|$. It is well-known that the operator $A := d/dt$ equipped with domain $D(A) := \{u \in E : u' \in E,\ \text{u}(\cdot) \text{ is absolutely continuous}\}$ generates a strongly continuous translation group on E (see [9, Lemma 4.6]). If we assume that, for every $\lambda \in i\mathbb{R}$, the function $t \mapsto e^{\lambda t}$, $t \in \mathbb{R}$ belongs to the space E and the equation (2.1) holds with $\Gamma = \{1\}$, the function $g_1(it) = it$, $t \in I_1 = \mathbb{R}$ and certain complex polynomials $P_0(\cdot), \cdot\cdot\cdot, P_{n-1}(\cdot)$, then Theorem 2.1 is applicable with operators $A_l := P_l(A)$ $(l \in \mathbb{N}^0_{n-1})$.

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ON ALMOST PARACONTACT ALMOST PARACOMPLEX RIEMANNIAN MANIFOLDS *

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Abstract. Almost paracontact manifolds of odd dimension having an almost paracomplex structure on the paracontact distribution are studied. The components of the fundamental (0,3)-tensor, derived by the covariant derivative of the structure endomorphism and the metric on the considered manifolds in each of the basic classes are obtained. Then, the case of the lowest dimension 3 of these manifolds is considered. An associated tensor of the Nijenhuis tensor is introduced and the studied manifolds are characterized with respect to this pair of tensors. Moreover, a cases of paracontact and para-Sasakian types are commented. A family of examples is given.
Keywords: Paracontact manifold; Riemannian manifold; tensor; metric.

1. Introduction

In 1976, on a differentiable manifold of arbitrary dimension, I. Sato introduced in [10] the concept of (almost) paracontact structure compatible with a Riemannian metric as an analogue of almost contact Riemannian manifold. Then, he studied several properties of the considered manifolds. Later, a lot of geometers develop the differential geometry of these manifolds and in particular of paracontact Riemannian manifolds and para-Sasakian manifolds. In the beginning are the papers [11], [1], [12], [13] and [9] by I. Sato, T. Adati, T. Miyazawa, K. Matsumoto and S. Sasaki.

On an almost paracontact manifold can be considered two kinds of metrics compatible with the almost paracontact structure. If the structure endomorphism induces an isometry on the paracontact distribution of each tangent fibre, then the manifold has an almost paracontact Riemannian structure as in the papers mentioned above. In the case when the induced transformation is antiisometry, then the manifold has a structure of an almost paracontact metric manifold, where

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the metric is semi-Riemannian of type $(n+1, n)$. This case is studied by many geometers, see for example the papers [7], [14] of S. Zamkovoy and G. Nakova.

In 2001, M. Manev and M. Staikova give a classification in [6] of almost paracontact Riemannian manifold of type (n, n) according to the notion given by Sasaki in [9]. These manifolds are of dimension $2n+1$ and the induced almost product structure on the paracontact distribution is traceless, i.e. it is an almost paracomplex structure.

In the present paper, we continue investigations on these manifolds. The paper is organized as follows. In Sect. 2., we recall some facts about the almost paracontact Riemannian manifolds of the considered type and we make some additional comments. In Sect. 3., we reduce the basic classes of the considered manifolds in the case of the lowest dimension 3. In Sect. 4. and Sect. 5., we find the class of paracontact type and the class of normal type of the manifolds studied, respectively, and we obtain some related properties. In Sect. 6., we introduce an associated Nijenhuis tensor and we discuss relevant problems. In Sect. 7., we argue that the classes of the considered manifolds can be determined only by the pair of Nijenhuis tensors. Finally, in Sect. 8., we construct a family of Lie groups as examples of the manifolds of the studied type and we characterize them in relation with the above investigations.

2. Almost paracontact almost paracomplex Riemannian manifolds

Let $(\mathcal{M}, \phi, \xi, \eta)$ be an *almost paracontact manifold*, i.e. $\mathcal{M}$ is an m-dimensional real differentiable manifold with an *almost paracontact structure* (ϕ, ξ, η) if it admits a tensor field ϕ of type $(1,1)$ of the tangent bundle, a vector field ξ and a 1-form η, satisfying the following conditions:

$$\phi\xi = 0, \quad \phi^2 = I - \eta \otimes \xi, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

where I is the identity on the tangent bundle [10].

In [9], it is considered the so-called almost paracontact manifold of type (p, q), where p and q are the numbers of the multiplicity of the ϕ's eigenvalues $+1$ and -1, respectively. Moreover, ϕ has a simple eigenvalue 0. Therefore, we have $\mathrm{tr}\phi = p - q$.

Let us recall that an *almost product structure* P on an differentiable manifold of arbitrary dimension m is an endomorphism on the manifold such that $P^2 = I$. Then a manifold with such a structure is called an *almost product manifold*. In the case when the eigenvalues $+1$ and -1 of P have one and the same multiplicity n, the structure P is called an *almost paracomplex structure* and the manifold is known as an *almost paracomplex manifold* of dimension $2n$ [2]. Then $\mathrm{tr}P = 0$ follows.

Further we consider the case when the dimension of $\mathcal{M}$ is $m = 2n+1$. Then $\mathcal{H} = \ker(\eta)$ is the $2n$-dimensional paracontact distribution of the tangent bundle of $(\mathcal{M}, \phi, \xi, \eta)$, the endomorphism ϕ acts as an almost paracomplex structure on each fiber of $\mathcal{H}$ and the pair $(\mathcal{H}, \phi)$ induces a $2n$-dimensional almost paracomplex manifold. Then we give the following

Definition 2.1. A $(2n+1)$-dimensional differentiable manifold with a structure (ϕ, ξ, η) defined by (2.1) and $\mathrm{tr}\phi = 0$ is called *almost paracontact almost paracomplex manifold*. We denote it by $(\mathcal{M}, \phi, \xi, \eta)$.

Now we can introduce a metric on the considered manifold. It is known from [10] that $\mathcal{M}$ admits a Riemannian metric g which is compatible with the structure of the manifold by the following way:

$$g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y), \quad g(x, \xi) = \eta(x). \tag{2.2}$$

Here and further x, y, z will stand for arbitrary elements of the Lie algebra $\mathfrak{X}(\mathcal{M})$ of tangent vector fields on $\mathcal{M}$ or vectors in the tangent space $T_p\mathcal{M}$ at $p \in \mathcal{M}$.

In [11], an almost paracontact manifold of arbitrary dimension with a Riemannian metric g defined by (2.2) is called an *almost paracontact Riemannian manifold*.

It is easy to conclude that the requirement for a positive definiteness of the metric is not necessary, i.e g can be a pseudo-Riemannian metric. Then, since $g(\xi, \xi) = 1$ follows from (2.1) and (2.2), the signature of g has the form $(2k+1, 2n-2k)$, $k < n$. Since the signature of the metric is not crucial for our considerations, we suppose that g is Riemannian.

Definition 2.2. Let the manifold $(\mathcal{M}, \phi, \xi, \eta)$ be equipped with a Riemannian metric g satisfying (2.2). Then $(\mathcal{M}, \phi, \xi, \eta, g)$ is called an *almost paracontact almost paracomplex Riemannian manifold*.

The decomposition $x = \phi^2 x + \eta(x)\xi$ due to (2.1) generates the projectors h and v on any tangent space of $(\mathcal{M}, \phi, \xi, \eta)$. These projectors are determined by $hx = \phi^2 x$ and $vx = \eta(x)\xi$ and have the properties $h \circ h = h$, $v \circ v = v$, $h \circ v = v \circ h = 0$. Therefore, we have the orthogonal decomposition $T_p\mathcal{M} = h(T_p\mathcal{M}) \oplus v(T_p\mathcal{M})$. Obviously, it generates the corresponding orthogonal decomposition of the space $\mathcal{S}$ of the tensors S of type (0,2) over $(\mathcal{M}, \phi, \xi, \eta)$. This decomposition is invariant with respect to transformations preserving the structures of the manifold. Hereof, we use the following linear operators in $\mathcal{S}$:

$$\begin{aligned} &\ell_1(S)(x, y) = S(hx, hy), \qquad \ell_2(S)(x, y) = S(vx, vy), \\ &\ell_3(S)(x, y) = S(vx, hy) + S(hx, vy). \end{aligned} \tag{2.3}$$

Namely, we have the following decomposition:

$$\mathcal{S} = \ell_1(\mathcal{S}) \oplus \ell_2(\mathcal{S}) \oplus \ell_3(\mathcal{S}), \qquad \ell_i(\mathcal{S}) = \{S \in \mathcal{S} \mid S = \ell_i(S)\}, \quad i = 1, 2, 3.$$

The associated metric $\tilde{g}$ of g on $(\mathcal{M}, \phi, \xi, \eta, g)$ is defined by $\tilde{g}(x, y) = g(x, \phi y) + \eta(x)\eta(y)$. It is shown that $\tilde{g}$ is a compatible metric with $(\mathcal{M}, \phi, \xi, \eta)$ and it is a pseudo-Riemannian metric of signature $(n+1, n)$. Therefore, $(\mathcal{M}, \phi, \xi, \eta, \tilde{g})$ is also an almost paracontact almost paracomplex manifold but with a pseudo-Riemannian metric.

Since the metrics g and $\tilde{g}$ belong to $\mathcal{S}$, then they have corresponding components in the three orthogonal subspaces introduced above and we get them in the following form:

$$\begin{array}{lll} \ell_1(g)=g(\phi\cdot,\phi\cdot)=g-\eta\otimes\eta, & \ell_2(g)=\eta\otimes\eta, & \ell_3(g)=0,\\ \ell_1(\tilde{g})=g(\cdot,\phi\cdot)=\tilde{g}-\eta\otimes\eta, & \ell_2(\tilde{g})=\eta\otimes\eta, & \ell_3(\tilde{g})=0. \end{array}$$

In the final part of the present section we recall the needed notions and results from [6].

In the cited paper, the manifolds under study are called almost paracontact Riemannian manifolds of type (n,n). The structure group of $(\mathcal{M},\phi,\xi,\eta,g)$ is $\mathcal{O}(n)\times\mathcal{O}(n)\times 1$, where $\mathcal{O}(n)$ is the group of the orthogonal matrices of size n.

The tensor F of type (0,3) plays a fundamental role in differential geometry of the considered manifolds. It is defined by:

$$F(x,y,z)=g\big((\nabla_x\phi)\,y,z\big), \tag{2.4}$$

where ∇ is the Levi-Civita connection of g. The basic properties of F with respect to the structure are the following:

$$\begin{aligned} F(x,y,z)&=F(x,z,y)\\ &=-F(x,\phi y,\phi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi). \end{aligned} \tag{2.5}$$

The relations of $\nabla\xi$ and $\nabla\eta$ with F are:

$$(\nabla_x\eta)y=g\left(\nabla_x\xi,y\right)=-F(x,\phi y,\xi). \tag{2.6}$$

If $\{\xi;e_i\}$ $(i=1,2,\dots,2n)$ is a basis of the tangent space $T_p\mathcal{M}$ at an arbitrary point $p\in\mathcal{M}$ and $\big(g^{ij}\big)$ is the inverse matrix of the matrix (g_{ij}) of g, then the following 1-forms are associated with F:

$$\theta(z)=g^{ij}F(e_i,e_j,z),\qquad \theta^*(z)=g^{ij}F(e_i,\phi e_j,z),\qquad \omega(z)=F(\xi,\xi,z). \tag{2.7}$$

These 1-forms are known also as the Lee forms of the considered manifolds. Obviously, the identities $\omega(\xi)=0$ and $\theta^*\circ\phi=-\theta\circ\phi^2$ are always valid.

There, it is made a classification of the almost paracontact almost paracomplex Riemannian manifolds with respect to F. The vector space $\mathbb{F}$ of all tensors F with the properties (2.5) is decomposed into 11 subspaces $\mathbb{F}_i$ $(i=1,2,\dots,11)$, which are orthogonal and invariant with respect to the structure group of the considered manifolds. This decomposition induces a classification of the manifolds under study. An almost paracontact almost paracomplex Riemannian manifold is said to be in the class $\mathcal{F}_i$ $(i=1,2,\dots,11)$, or briefly an $\mathcal{F}_i$-manifold, if the tensor F belongs to the subspace $\mathbb{F}_i$. Such a way, it is obtained that this classification consists of 11 basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, …, $\mathcal{F}_{11}$. The intersection of the basic classes is the special class $\mathcal{F}_0$ determined by the condition $F(x,y,z)=0$. Hence $\mathcal{F}_0$ is the class of the considered manifolds with ∇-parallel structures, i.e. $\nabla\phi=\nabla\xi=\nabla\eta=\nabla g=\nabla\tilde{g}=0$.

Moreover, it is given the conditions for F determining the basic classes $\mathcal{F}_i$ of $(\mathcal{M}, \phi, \xi, \eta, g)$ and the components of F corresponding to $\mathcal{F}_i$. It is said that $(\mathcal{M}, \phi, \xi, \eta, g)$ belongs to the class $\mathcal{F}_i$ $(i = 1, 2, \dots, 11)$ if and only if the equality $F = F_i$ is valid. In the last expression, F_i are the components of F in the subspaces $\mathbb{F}_i$ and they are given by the following equalities:

$$
\begin{aligned}
F_1(x,y,z) &= \tfrac{1}{2n}\big\{g(\phi x,\phi y)\theta(\phi^2 z) + g(\phi x,\phi z)\theta(\phi^2 y) \\
&\qquad - g(x,\phi y)\theta(\phi z) - g(x,\phi z)\theta(\phi y)\big\},\\
F_2(x,y,z) &= \tfrac{1}{4}\big\{2F(\phi^2 x,\phi^2 y,\phi^2 z) + F(\phi^2 y,\phi^2 z,\phi^2 x) + F(\phi^2 z,\phi^2 x,\phi^2 y)\\
&\qquad - F(\phi y,\phi z,\phi^2 x) - F(\phi z,\phi y,\phi^2 x)\big\}\\
&\quad - \tfrac{1}{2n}\big\{g(\phi x,\phi y)\theta(\phi^2 z) + g(\phi x,\phi z)\theta(\phi^2 y)\\
&\qquad - g(x,\phi y)\theta(\phi z) - g(x,\phi z)\theta(\phi y)\big\},\\
F_3(x,y,z) &= \tfrac{1}{4}\big\{2F(\phi^2 x,\phi^2 y,\phi^2 z) - F(\phi^2 y,\phi^2 z,\phi^2 x) - F(\phi^2 z,\phi^2 x,\phi^2 y)\\
&\qquad + F(\phi y,\phi z,\phi^2 x) + F(\phi z,\phi y,\phi^2 x)\big\},\\
F_4(x,y,z) &= \tfrac{\theta(\xi)}{2n}\big\{g(\phi x,\phi y)\eta(z) + g(\phi x,\phi z)\eta(y)\big\},\\
F_5(x,y,z) &= \tfrac{\theta^*(\xi)}{2n}\big\{g(x,\phi y)\eta(z) + g(x,\phi z)\eta(y)\big\},\\
F_6(x,y,z) &= \tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi) + F(\phi^2 y,\phi^2 x,\xi) + F(\phi x,\phi y,\xi)\\
&\qquad + F(\phi y,\phi x,\xi)]\eta(z)\\
&\qquad + [F(\phi^2 x,\phi^2 z,\xi) + F(\phi^2 z,\phi^2 x,\xi) + F(\phi x,\phi z,\xi)\\
&\qquad + F(\phi z,\phi x,\xi)]\eta(y)\big\}\\
&\quad - \tfrac{\theta(\xi)}{2n}\big\{g(\phi x,\phi y)\eta(z) + g(\phi x,\phi z)\eta(y)\big\}\\
&\quad - \tfrac{\theta^*(\xi)}{2n}\big\{g(x,\phi y)\eta(z) + g(x,\phi z)\eta(y)\big\},\\
F_7(x,y,z) &= \tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi) - F(\phi^2 y,\phi^2 x,\xi) + F(\phi x,\phi y,\xi)\\
&\qquad - F(\phi y,\phi x,\xi)]\eta(z)\\
&\qquad + [F(\phi^2 x,\phi^2 z,\xi) - F(\phi^2 z,\phi^2 x,\xi) + F(\phi x,\phi z,\xi)\\
&\qquad - F(\phi z,\phi x,\xi)]\eta(y)\big\},\\
F_8(x,y,z) &= \tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi) + F(\phi^2 y,\phi^2 x,\xi) - F(\phi x,\phi y,\xi)\\
&\qquad - F(\phi y,\phi x,\xi)]\eta(z)\\
&\qquad + [F(\phi^2 x,\phi^2 z,\xi) + F(\phi^2 z,\phi^2 x,\xi) - F(\phi x,\phi z,\xi)\\
&\qquad - F(\phi z,\phi x,\xi)]\eta(y)\big\},\\
F_9(x,y,z) &= \tfrac{1}{4}\big\{[F(\phi^2 x,\phi^2 y,\xi) - F(\phi^2 y,\phi^2 x,\xi) - F(\phi x,\phi y,\xi)\\
&\qquad + F(\phi y,\phi x,\xi)]\eta(z)\\
&\qquad + [F(\phi^2 x,\phi^2 z,\xi) - F(\phi^2 z,\phi^2 x,\xi) - F(\phi x,\phi z,\xi)\\
&\qquad + F(\phi z,\phi x,\xi)]\eta(y)\big\},\\
F_{10}(x,y,z) &= \eta(x)F(\xi,\phi^2 y,\phi^2 z),\\
F_{11}(x,y,z) &= \eta(x)\big\{\eta(y)\omega(z) + \eta(z)\omega(y)\big\}.
\end{aligned}
\tag{2.8}
$$

It is easy to conclude that a manifold of the considered type belongs to a direct sum of two or more basic classes, i.e. $(\mathcal{M}, \phi, \xi, \eta, g) \in \mathcal{F}_i \oplus \mathcal{F}_j \oplus \cdots$, if and only if the fundamental tensor F on $(\mathcal{M}, \phi, \xi, \eta, g)$ is the sum of the corresponding components F_i, F_j, $\dots$ of F, i.e. the following condition is satisfied $F = F_i + F_j + \cdots$.

Finally in this section, we obtain immediately

Proposition 2.1. *The dimensions of the subspaces $\mathbb{F}_i$ $(i = 1, 2, \dots, 11)$ in the decomposition of the space $\mathbb{F}$ of the tensors F on $(\mathcal{M}, \phi, \xi, \eta, g)$ are the following:*

$$\begin{array}{lll}
\dim \mathbb{F}_1 = 2n, & \dim \mathbb{F}_2 = n(n-1)(n+2), & \dim \mathbb{F}_3 = n^2(n-1), \\
\dim \mathbb{F}_4 = 1, & \dim \mathbb{F}_5 = 1, & \dim \mathbb{F}_6 = (n-1)(n+2), \\
\dim \mathbb{F}_7 = n(n-1), & \dim \mathbb{F}_8 = n^2, & \dim \mathbb{F}_9 = n^2, \\
\dim \mathbb{F}_{10} = n^2, & \dim \mathbb{F}_{11} = 2n. &
\end{array}$$

Proof. Using the characteristic symmetries of the fundamental tensor F and the form of its components from (2.8) in each of $\mathcal{F}_i$ $(i = 1, 2, \dots, 11)$, we get the equalities in the statement. □

3. The components of the fundamental tensor for dimension 3

Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be the manifold under study with the lowest dimension, i.e. $\dim \mathcal{M} = 3$ (or $n = 1$) and let the system of three vectors $\{e_0 = \xi, e_1 = e, e_2 = \phi e\}$ be a ϕ-basis which satisfies the following conditions:

$$\begin{array}{c}
g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = 1, \\
g(e_0, e_1) = g(e_1, e_2) = g(e_0, e_2) = 0.
\end{array} \tag{3.1}$$

We denote the components of the tensors F, θ, θ^* and ω with respect to the ϕ-basis $\{e_0, e_1, e_2\}$ as follows $F_{ijk} = F(e_i, e_j, e_k)$, $\theta_k = \theta(e_k)$, $\theta^*_k = \theta^*(e_k)$ and $\omega_k = \omega(e_k)$. The properties (2.5) and (3.1) imply the equalities $F_{i12} = F_{i21} = 0$ and $F_{i11} = -F_{i22}$ for any i. Then, bearing in mind (2.7), we obtain for the Lee forms the following:

$$\begin{array}{lll}
\theta_0 = F_{110} + F_{220}, & \theta_1 = F_{111} = -F_{122} = -\theta^*_2, & \omega_1 = F_{001}, \\
\theta^*_0 = F_{120} + F_{210}, & \theta_2 = F_{222} = -F_{211} = -\theta^*_1, & \omega_2 = F_{002}, \\
 & & \omega_0 = 0.
\end{array} \tag{3.2}$$

The arbitrary vectors x, y, z in $T_p\mathcal{M}$, $p \in \mathcal{M}$, have the expression $x = x^i e_i$, $y = y^i e_i$, $z = z^i e_i$ with respect to $\{e_0, e_1, e_2\}$.

Proposition 3.1. *The components F_i $(i = 1, 2, \dots, 11)$ of the fundamental tensor F for a 3-dimensional almost paracontact almost paracomplex Riemannian manifold are the following:*

$$\begin{array}{l}
F_1(x, y, z) = \left(x^1\theta_1 - x^2\theta_2\right)\left(y^1z^1 - y^2z^2\right), \\
F_2(x, y, z) = F_3(x, y, z) = 0, \\
F_4(x, y, z) = \frac{\theta_0}{2}\left\{x^1\left(y^0z^1 + y^1z^0\right) + x^2\left(y^0z^2 + y^2z^0\right)\right\}, \\
F_5(x, y, z) = \frac{\theta^*_0}{2}\left\{x^1\left(y^0z^2 + y^2z^0\right) + x^2\left(y^0z^1 + y^1z^0\right)\right\}, \\
F_6(x, y, z) = F_7(x, y, z) = 0, \\
F_8(x, y, z) = \lambda\left\{x^1\left(y^0z^1 + y^1z^0\right) - x^2\left(y^0z^2 + y^2z^0\right)\right\}, \\
F_9(x, y, z) = \mu\left\{x^1\left(y^0z^2 + y^2z^0\right) - x^2\left(y^0z^1 + y^1z^0\right)\right\}, \\
F_{10}(x, y, z) = \nu\, x^0\left(y^1z^1 - y^2z^2\right), \\
F_{11}(x, y, z) = x^0\left\{\omega_1\left(y^0z^1 + y^1z^0\right) + \omega_2\left(y^0z^2 + y^2z^0\right)\right\},
\end{array} \tag{3.3}$$

where

$$\lambda = F_{110} = -F_{220}, \qquad \mu = F_{120} = -F_{210}, \qquad \nu = F_{011} = -F_{022}$$

and the components of the Lee forms are given in (3.2).

Proof. Using the expressions (2.8) of F_i for the corresponding classes $\mathcal{F}_i$ ($i = 1, \dots, 11$), the equalities (2.5), (3.1) and (3.2), we obtain the corresponding form of F_i for the lowest dimension of the considered manifold. □

As a result of Proposition 3.1, we establish the truthfulness of the following

Theorem 3.1. *The 3-dimensional almost paracontact almost paracomplex Riemannian manifolds belong to the basic classes $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_8$, $\mathcal{F}_9$, $\mathcal{F}_{10}$, $\mathcal{F}_{11}$ and to their direct sums.*

Let us remark that for the considered manifolds of dimension 3, the basic classes $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, $\mathcal{F}_7$ are restricted to the special class $\mathcal{F}_0$.

4. Paracontact almost paracomplex Riemannian manifolds

Let $(\mathcal{M}, \phi, \xi, \eta, g)$, $\dim \mathcal{M} = 2n+1$, be an almost paracontact almost paracomplex Riemannian manifold such that the following condition is satisfied:

$$2g(x, \phi y) = (\mathfrak{L}_\xi g)(x, y), \tag{4.1}$$

where the Lie derivative $\mathfrak{L}$ of g along ξ has the following form in terms of $\nabla\eta$:

$$(\mathfrak{L}_\xi g)(x, y) = (\nabla_x \eta)y + (\nabla_y \eta)x. \tag{4.2}$$

Bearing in mind (2.6) and (4.2), $\mathfrak{L}_\xi g$ is expressed by F as follows:

$$(\mathfrak{L}_\xi g)(x, y) = -F(x, \phi y, \xi) - F(y, \phi x, \xi). \tag{4.3}$$

In [11], it is said that an m-dimensional almost paracontact Riemannian manifold endowed with the property $2g(x, \phi y) = (\nabla_x \eta)y + (\nabla_y \eta)x$ is a *paracontact Riemannian manifold*.

Definition 4.1. An almost paracontact almost paracomplex Riemannian manifold satisfied (4.1) is called *paracontact almost paracomplex Riemannian manifold*.

Now we determine the class of paracontact almost paracomplex Riemannian manifolds with respect to the basic classes $\mathcal{F}_i$. Firstly, we compute $\mathfrak{L}_\xi g$ on each $\mathcal{F}_i$-manifold using (4.3) and (2.8). Then we obtain

Proposition 4.1. *Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) $(\mathfrak{L}_\xi g)(x,y) = 0$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$*;*

b) $(\mathfrak{L}_\xi g)(x,y) = -\frac{1}{n}\theta(\xi)g(x,\phi y)$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_4$*;*

c) $(\mathfrak{L}_\xi g)(x,y) = -\frac{1}{n}\theta^*(\xi)g(\phi x,\phi y)$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_5$*;*

d) $(\mathfrak{L}_\xi g)(x,y) = 2\big(\nabla_x\eta\big)y$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_6 \oplus \mathcal{F}_9$*;*

e) $(\mathfrak{L}_\xi g)(x,y) = -\eta(x)\omega(\phi y) - \eta(y)\omega(\phi x)$ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_{11}$*.*

It is known that ξ is a Killing vector field when $\mathfrak{L}_\xi g = 0$. Therefore, the latter proposition implies

Corollary 4.1. *An almost paracontact almost paracomplex Riemannian manifold* $(\mathcal{M},\phi,\xi,\eta,g)$ *has a Killing vector field* ξ *if and only if* $(\mathcal{M},\phi,\xi,\eta,g)$ *belongs to* $\mathcal{F}_i$ $(i = 1,2,3,7,8,10)$ *or to their direct sums.*

We denote by ${\mathcal{F}_4}'$ the subclass of $\mathcal{F}_4$ determined by $\theta(\xi) = -2n$, i.e.

$$ {\mathcal{F}_4}' = \big\{\mathcal{F}_4 \,|\, \theta(\xi) = -2n\big\}. \tag{4.4} $$

Then, the component ${F_4}'$ of F corresponding to the subclass ${\mathcal{F}_4}'$ is

$$ {F_4}'(x,y,z) = -g(\phi x,\phi y)\eta(z) - g(\phi x,\phi z)\eta(y). \tag{4.5} $$

Theorem 4.1. *Paracontact almost paracomplex Riemannian manifolds belong to* ${\mathcal{F}_4}'$ *or to its direct sums with* $\mathcal{F}_1$*,* $\mathcal{F}_2$*,* $\mathcal{F}_3$*,* $\mathcal{F}_7$*,* $\mathcal{F}_8$ *and* $\mathcal{F}_{10}$*.*

Proof. Let us consider an arbitrary almost paracontact almost paracomplex Riemannian manifold, i.e. $F = F_1 + \ldots + F_{11}$. Using the expressions (2.8) of F_i for the corresponding classes $\mathcal{F}_i$ $(i = 1,\ldots,11)$ and the condition (4.1), we obtain

$$ F = F_1 + F_2 + F_3 + {F_4}' + F_7 + F_8 + F_{10}, \tag{4.6} $$

where ${F_4}'$ is determined by (4.5).

Vice versa, if (4.6) holds true, then it implies (4.1) by (4.3), i.e. $(\mathcal{M},\phi,\xi,\eta,g)$ is a paracontact almost paracomplex Riemannian manifold. Supposing that $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to some of $\mathcal{F}_i$ $(i = 1,2,3,7,8,10)$ or their direct sum, it follows that g is degenerate. Therefore, the component ${F_4}'$ is indispensable and we get the statement. □

Let us remark that ${\mathcal{F}_4}'$ and $\mathcal{F}_0$ are subclasses of $\mathcal{F}_4$ without common elements.

Moreover, bearing in mind Corollary 4.1 and Theorem 4.1, we conclude that paracontact almost paracomplex Riemannian manifolds with a Killing vector field ξ do not exist, i.e. for the manifolds studied, there is no analogue of a K-contact manifold.

In [13], it is introduced the notion of a para-Sasakian Riemannian manifold of an arbitrary dimension by the condition $\phi x = \nabla_x \xi$. The same condition determines a special kind of paracontact almost paracomplex Riemannian manifolds. These manifolds we call para-Sasakian paracomplex Riemannian manifolds. Then, using (4.4), we obtain the truthfulness of the following

Theorem 4.2. *The class of the para-Sasakian paracomplex Riemannian manifolds is* $\mathcal{F}_4{}'$.

5. The Nijenhuis tensor

5.1. Introduction of the Nijenhuis tensor

Let us consider the product manifold $\check{\mathcal{M}}$ of an almost paracontact almost paracomplex manifold $(\mathcal{M}, \phi, \xi, \eta)$ and the real line $\mathbb{R}$, i.e. $\check{\mathcal{M}} = \mathcal{M} \times \mathbb{R}$. We denote a vector field on $\check{\mathcal{M}}$ by $\left(x, a\frac{\mathrm{d}}{\mathrm{d}r}\right)$, where x is tangent to $\check{\mathcal{M}}$, r is the coordinate on $\mathbb{R}$ and a is a function on $\mathcal{M} \times \mathbb{R}$. Further, we use the denotation $\partial_r = \frac{\mathrm{d}}{\mathrm{d}r}$ for brevity. Following [10], we define an almost paracomplex structure $\check{P}$ on $\check{\mathcal{M}}$ by:

$$\check{P}\,(x,\, a\partial_r) = \left(\phi x + \frac{a}{r}\xi,\, r\eta(x)\partial_r\right) \tag{5.1}$$

that implies

$$\check{P}x = \phi x, \qquad \check{P}\xi = r\partial_r, \qquad \check{P}\partial_r = \frac{1}{r}\xi.$$

Further, we use the setting $\zeta = r\partial_r$. It easy to check that $\check{P}^2 = I$ and $\mathrm{tr}\check{P} = 0$. In the case when $\check{P}$ is integrable, it is said that the almost paracontact structure (ϕ, ξ, η) is *normal*.

It is known, the vanishing of the Nijenhuis torsion $\left[\check{P}, \check{P}\right]$ of $\check{P}$ is a necessary and sufficient condition for integrability of $\check{P}$. According to [10], the condition of normality is equivalent to vanishing of the following four tensors:

$$\begin{aligned} &N^{(1)}(x,y) = [\phi,\phi](x,y) - \mathrm{d}\eta(x,y)\xi,\\ &N^{(2)}(x,y) = (\mathfrak{L}_{\phi x}\eta)(y) - (\mathfrak{L}_{\phi y}\eta)(x),\\ &N^{(3)}(x) = (\mathfrak{L}_{\xi}\phi)(x),\\ &N^{(4)}(x) = (\mathfrak{L}_{\xi}\eta)(x), \end{aligned} \tag{5.2}$$

where the Nijenhuis torsion of ϕ is determined by:

$$[\phi,\phi](x,y) = [\phi x, \phi y] + \phi^2[x,y] - \phi[\phi x, y] - \phi[x, \phi y] \tag{5.3}$$

and $\mathrm{d}\eta$ is the exterior derivative of η given by:

$$\mathrm{d}\eta(x,y) = (\nabla_x\eta)y - (\nabla_y\eta)x. \tag{5.4}$$

According to (2.6) and (5.4), $\mathrm{d}\eta$ is expressed by F as follows:

$$\mathrm{d}\eta(x,y) = -F(x,\phi y,\xi) + F(y,\phi x,\xi). \tag{5.5}$$

Let $(\mathcal{M},\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional almost paracontact almost paracomplex Riemannian manifold.

In [10], it is proved that the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$, $N^{(4)}$. Then $N^{(1)}$ is denoted simply by N, i.e.

$$N(x,y)=[\phi,\phi](x,y)-\mathrm{d}\eta(x,y)\xi, \tag{5.6}$$

and it is called the *Nijenhuis tensor of the structure* (ϕ,ξ,η). Therefore, an almost paracontact structure (ϕ,ξ,η) is normal if and only if its Nijenhuis tensor is zero.

Obviously, N is an antisymmetric tensor, i.e. $N(x,y)=-N(y,x)$. According to (5.3), (5.4) and (5.6), the tensor N has the following form in terms of the covariant derivatives of ϕ and η with respect to ∇:

$$N(x,y)=(\nabla_{\phi x}\phi)y-(\nabla_{\phi y}\phi)x-\phi(\nabla_x\phi)y+\phi(\nabla_y\phi)x-(\nabla_x\eta)y\,\xi+(\nabla_y\eta)x\,\xi.$$

The corresponding tensor of type (0,3) of the Nijenhuis tensor on $(\mathcal{M},\phi,\xi,\eta,g)$ is defined by equality $N(x,y,z)=g\left(N(x,y),z\right)$. Then, using (2.4) and (2.6), we express N in terms of the fundamental tensor F as follows:

$$\begin{aligned}N(x,y,z)=F(\phi x,y,z)-F(\phi y,x,z)-F(x,y,\phi z)+F(y,x,\phi z)\\+\eta(z)\left\{F(x,\phi y,\xi)-F(y,\phi x,\xi)\right\}.\end{aligned} \tag{5.7}$$

Proposition 5.1. *The Nijenhuis tensor on an almost paracontact almost paracomplex Riemannian manifold has the following properties:*

$$\begin{array}{ll}N(\phi^2x,\phi y,\phi z)=-N(\phi^2x,\phi^2y,\phi^2z), & N(\phi^2x,\phi^2y,\phi^2z)=N(\phi x,\phi y,\phi^2z),\\ N(x,\phi^2y,\phi^2z)=-N(x,\phi y,\phi z), & N(\phi^2x,\phi^2y,z)=N(\phi x,\phi y,z),\\ N(\xi,\phi y,\phi z)=-N(\xi,\phi^2y,\phi^2z), & N(\phi x,\phi y,\xi)=N(\phi^2x,\phi^2y,\xi).\end{array}$$

Proof. The equalities from the above follow by direct computations from the properties (2.5) and the expression (5.7). □

In [10], there are given the following relations between the tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$:

$$\begin{aligned}&N^{(2)}(x,y)=-\eta\left(N^{(1)}(x,\phi y)\right)-\eta\left(N^{(1)}(\phi x,\xi)\right)\eta(y),\\&N^{(3)}(x)=-N^{(1)}(\phi x,\xi),\\&N^{(4)}(x)=-N^{(2)}(\phi x,\xi),\qquad N^{(4)}(x)=-\eta\left(N^{(3)}(\phi x)\right).\end{aligned} \tag{5.8}$$

Applying the expression (5.7) to equalities (5.8), we obtain the form of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ in terms of the fundamental tensor F:

$$\begin{aligned}&N^{(2)}(x,y)=-F(x,y,\xi)+F(y,x,\xi)-F(\phi x,\phi y,\xi)+F(\phi y,\phi x,\xi),\\&N^{(3)}(x,y)=F(\xi,x,y)-F(x,y,\xi)+F(\phi x,\phi y,\xi),\\&N^{(4)}(x)=-F(\xi,\xi,\phi x),\end{aligned} \tag{5.9}$$

where it is used the denotation $N^{(3)}(x,y)=g\left(N^{(3)}(x),y\right)$.

Table 5.1: Nijenhuis tensors

	$N^{(1)}(x,y,z)$	$N^{(2)}(x,y)$	$N^{(3)}(x,y)$	$N^{(4)}(x)$
$\mathcal{F}_1$	0	0	0	0
$\mathcal{F}_2$	0	0	0	0
$\mathcal{F}_3$	$-2\Big\{F(\phi x,\phi y,\phi z)+F(\phi^2 x,\phi^2 y,\phi z)\Big\}$	0	0	0
$\mathcal{F}_4$	0	0	0	0
$\mathcal{F}_5$	0	0	0	0
$\mathcal{F}_6$	0	0	0	0
$\mathcal{F}_7$	$4F(x,\phi y,\xi)\eta(z)$	$-4F(x,y,\xi)$	0	0
$\mathcal{F}_8$	$2\Big\{\eta(x)F(y,\phi z,\xi)-\eta(y)F(x,\phi z,\xi)\Big\}$	0	$-2F(x,y,\xi)$	0
$\mathcal{F}_9$	$2\Big\{\eta(x)F(y,\phi z,\xi)-\eta(y)F(x,\phi z,\xi)\Big\}$	0	$-2F(x,y,\xi)$	0
$\mathcal{F}_{10}$	$-\eta(x)F(\xi,y,\phi z)+\eta(y)F(\xi,x,\phi z)$	0	$F(\xi,x,y)$	0
$\mathcal{F}_{11}$	$\eta(z)\Big\{\eta(x)\omega(\phi y)-\eta(y)\omega(\phi x)\Big\}$	$\eta(y)\omega(x)-\eta(x)\omega(y)$	$\eta(y)\omega(x)$	$-\omega(\phi x)$

Proposition 5.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an $\mathcal{F}_i$-manifold $(i=1,2,\dots,11)$. Then the four tensors $N^{(k)}$ $(k=1,2,3,4)$ on this manifold have the form in the respective cases, given in Table 5.1.*

Proof. We apply direct computations, using (2.8), (5.7) and (5.9). □

By virtue Proposition 5.2, we have the following

Theorem 5.1. *An almost paracontact almost paracomplex Riemannian manifold $(\mathcal{M},\phi,\xi,\eta,g)$ has:*

a) vanishing $N^{(1)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$ or to their direct sums;

b) vanishing $N^{(2)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_6$, $\mathcal{F}_8$, $\mathcal{F}_9$, $\mathcal{F}_{10}$ or to their direct sums;

c) vanishing $N^{(3)}$ if and only if it belongs to the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_7$ or to some of their direct sums;

d) vanishing $N^{(4)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_{10}$ or to their direct sums.

Bearing in mind Theorem 5.1, we conclude the following

Corollary 5.1. *The class of normal almost paracontact almost paracomplex Riemannian manifolds is $\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6$.*

5.2. The exterior derivative of the structure 1-form

According to (2.3), the 2-form $\mathrm{d}\eta$ on $(\mathcal{M},\phi,\xi,\eta,g)$ can be decomposed as follows:

$$\mathrm{d}\eta=\ell_1(\mathrm{d}\eta)+\ell_3(\mathrm{d}\eta),$$

$$\begin{aligned}\ell_1(\mathrm{d}\eta)(x,y) &= \mathrm{d}\eta(hx,hy), \qquad \ell_2(\mathrm{d}\eta)(x,y)=0,\\ \ell_3(\mathrm{d}\eta)(x,y) &= \mathrm{d}\eta(vx,hy)+\mathrm{d}\eta(hx,vy).\end{aligned} \tag{5.10}$$

The next proposition gives geometric conditions for vanishing the components of $\mathrm{d}\eta$.

Proposition 5.3. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) the paracontact distribution $\mathcal{H}$ of $(\mathcal{M},\phi,\xi,\eta,g)$ is involutive if and only if $\ell_1(\mathrm{d}\eta)=0$;

b) the integral curves of ξ are geodesics on $(\mathcal{M},\phi,\xi,\eta,g)$ if and only if $\ell_3(\mathrm{d}\eta)=0$.

Proof. It is said that $\mathcal{H}$ is an involutive distribution when $[x,y]$ belongs to $\mathcal{H}$ for $x,y\in\mathcal{H}$, i.e. $\eta([hx,hy])=0$ holds for arbitrary x and y. By virtue of the identity $\eta([hx,hy])=-\mathrm{d}\eta(hx,hy)$ and (5.10), we have the equality $\eta([hx,hy])=-\ell_1(\mathrm{d}\eta)(x,y)$. This accomplishes the proof of a).

As it is known, the integral curves of ξ are geodesics on $(\mathcal{M},\phi,\xi,\eta,g)$ if and only if $\nabla_\xi\xi$ vanishes. The equality (5.10) implies that $\ell_3(\mathrm{d}\eta)=0$ is valid if and only if $\mathrm{d}\eta(x,\xi)=0$ holds. Applying (5.4) and (2.6), we obtain the equality $\mathrm{d}\eta(x,\xi)=-g(\nabla_\xi\xi,x)$. Then, it is clear that b) holds true. □

Next, we compute $\mathrm{d}\eta$ on the considered manifold belonging to each of the basic classes and obtain the following

Proposition 5.4. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) $\mathrm{d}\eta(x,y)=0$ if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_i$ $(i=1,\dots,6,9,10)$ or to their direct sums;

b) $\mathrm{d}\eta(x,y)=\ell_1(\mathrm{d}\eta)(x,y)=2\big(\nabla_x\eta\big)y$ if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_7$, $\mathcal{F}_8$ or $\mathcal{F}_7\oplus\mathcal{F}_8$;

c) $\mathrm{d}\eta(x,y)=\ell_3(\mathrm{d}\eta)(x,y)=-\eta(x)\omega(\phi y)+\eta(y)\omega(\phi x)$ if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_{11}$.

By Proposition 5.3 and Proposition 5.4, we get the following theorem, which gives a geometric characteristic of the manifolds of some classes with respect to the form $\mathrm{d}\eta$.

Theorem 5.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then we have:*

a) the structure 1-form η is closed if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_i$ $(i=1,\dots,6,9,10)$ or to their direct sums;

b) the paracontact distribution $\mathcal{H}$ of $(\mathcal{M},\phi,\xi,\eta,g)$ is involutive if and only if $(\mathcal{M},\phi,\xi,\eta,g)$ belongs to $\mathcal{F}_i$ $(i=1,\dots,6,9,10,11)$ or to their direct sums;

c) the integral curves of the structure vector field ξ *are geodesics on* $(\mathcal{M}, \phi, \xi, \eta, g)$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_i$ $(i = 1, \dots, 10)$ *or to their direct sums.*

5.3. The Nijenhuis torsion of the structure endomorphism of the paracontact distribution

Proposition 5.5. *Let* $(\mathcal{M}, \phi, \xi, \eta, g)$ *be an almost paracontact almost paracomplex Riemannian manifold. Then for the Nijenhuis torsion of* ϕ *we have:*

a) $[\phi, \phi](x, y) = 0$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_i$ $(i = 1, 2, 4, 5, 6, 11)$ *or to their direct sums;*

b) $[\phi, \phi](x, y) = -2\left\{\phi\left(\nabla_{\phi x}\phi\right)\phi y + \phi\left(\nabla_{\phi^2 x}\phi\right)\phi^2 y\right\}$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_3$*;*

c) $[\phi, \phi](x, y) = -2\left(\nabla_x \eta\right)(y)\,\xi$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_7$*;*

d) $[\phi, \phi](x, y) = -2\left\{\eta(x)\nabla_y \xi - \eta(y)\nabla_x \xi - \left(\nabla_x \eta\right)(y)\,\xi\right\}$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_8$*;*

e) $[\phi, \phi](x, y) = -2\left\{\eta(x)\nabla_y \xi - \eta(y)\nabla_x \xi\right\}$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_9$*;*

f) $[\phi, \phi](x, y) = -\eta(x)\,\phi\left(\nabla_\xi \phi\right) y + \eta(y)\,\phi\left(\nabla_\xi \phi\right) x$ *if and only if* $(\mathcal{M}, \phi, \xi, \eta, g)$ *belongs to* $\mathcal{F}_{10}$*.*

Proof. Using (5.6) and the forms of the Nijenhuis tensor N and the 2-form $\mathrm{d}\eta$, given in Proposition 5.2 and Proposition 5.4, respectively, we get the statements from the above by direct computations. □

Now, we specialize the form of $[\phi, \phi]$ for the class of paracontact almost paracomplex Riemannian manifolds and we find its subclasses of manifolds whose almost paracomplex structure ϕ on $\mathcal{H}$ is integrable.

Theorem 5.3. *Let* $(\mathcal{M}, \phi, \xi, \eta, g)$ *be a paracontact almost paracomplex Riemannian manifold. Then it has:*

a) an integrable almost paracomplex structure ϕ*, i.e.* $[\phi, \phi] = 0$*, if and only if the manifold belongs to* $\mathcal{F}_4{}'$ *or to its direct sums with* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*;*

b) an nonintegrable almost paracomplex structure ϕ*, i.e.* $[\phi, \phi] \neq 0$ *if and only if the manifold belongs to the rest of the classes, given in Theorem 4.1.*

Proof. We establish the truthfulness of the statements using Theorem 4.1 and Proposition 5.5. □

Bearing in mind Theorem 5.3 a), the manifolds from the classes $\mathcal{F}_4{}'$, $\mathcal{F}_1 \oplus \mathcal{F}_4{}'$, $\mathcal{F}_2 \oplus \mathcal{F}_4{}'$ and $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4{}'$ we call *paracontact paracomplex Riemannian manifolds.* In the other case, the manifolds from the rest of the classes, given in Theorem 4.1, we call *paracontact almost paracomplex Riemannian manifolds.*

6. The associated Nijenhuis tensor

By analogy with the skew-symmetric Lie bracket (the commutator), determined by $[x,y]=\nabla_x y-\nabla_y x$, let us consider the symmetric braces (the anticommutator), defined by $\{x,y\}=\nabla_x y+\nabla_y x$ as in [5]. Bearing in mind the definition of the Nijenhuis torsion $\left[\check{P},\check{P}\right]$ of an almost paracomplex structure $\check{P}$ on $\check{\mathcal{M}}$, we give a definition of a tensor $\left\{\check{P},\check{P}\right\}$ of type $(1,2)$ as follows:

$$\left\{\check{P},\check{P}\right\}(\check{x},\check{y})=\{\check{x},\check{y}\}+\left\{\check{P}\check{x},\check{P}\check{y}\right\}-\check{P}\left\{\check{P}\check{x},\check{y}\right\}-\check{P}\left\{\check{x},\check{P}\check{y}\right\},$$

where the action of $\check{P}$ is given in (5.1) and the anticommutator on the tangent bundle of $\check{\mathcal{M}}$ is determined by:

$$\left\{(x,a\partial_r),(y,b\partial_r)\right\}=\big(\{x,y\},(x(b)+y(a))\,\partial_r\big).$$

We call $\left\{\check{P},\check{P}\right\}$ an *associated Nijenhuis tensor of the almost paracomplex manifold* $\big(\check{\mathcal{M}},\check{P}\big)$. Obviously, this tensor is symmetric with respect to its arguments, i.e. $\left\{\check{P},\check{P}\right\}(\check{x},\check{y})=\left\{\check{P},\check{P}\right\}(\check{y},\check{x})$.

Since the almost paracomplex manifold $\big(\check{\mathcal{M}},\check{P}\big)$ is generated from the almost paracontact almost paracomplex manifold $(\mathcal{M},\phi,\xi,\eta)$, we seek to express the associated Nijenhuis tensor $\left\{\check{P},\check{P}\right\}$ by tensors for the structure (ϕ,ξ,η). Since $\left\{\check{P},\check{P}\right\}$ is a tensor field of type (1,2) on $\check{\mathcal{M}}$, it suffices to compute the following two expressions:

$$\begin{aligned}\left\{\check{P},\check{P}\right\}\big((x,0),(y,0)\big)&=\big\{(x,0),(y,0)\big\}+\big\{\check{P}(x,0),\check{P}(y,0)\big\}-\check{P}\big\{\check{P}(x,0),(y,0)\big\}\\&\qquad-\check{P}\big\{(x,0),\check{P}(y,0)\big\}\\&=\big(\{x,y\},0\big)+\big\{(\phi x,\eta(x)\zeta),(\phi y,\eta(y)\zeta)\big\}\\&\quad-\check{P}\big\{(\phi x,\eta(x)\zeta),(y,0)\big\}-\check{P}\big\{(x,0),(\phi y,\eta(y)\zeta)\big\}\\&=\big(\{\phi,\phi\}(x,y)-(\mathfrak{L}_\xi g)(x,y)\xi,\\&\quad((\mathfrak{L}_\xi g)(\phi x,y)+(\mathfrak{L}_\xi g)(x,\phi y))\,\zeta\big)\end{aligned}$$

and

$$\begin{aligned}\left\{\check{P},\check{P}\right\}\big((x,0),(0,\zeta)\big)&=\big\{(x,0),(0,\zeta)\big\}+\big\{\check{P}(x,0),\check{P}(0,\zeta)\big\}-\check{P}\big\{\check{P}(x,0),(0,\zeta)\big\}\\&\qquad-\check{P}\big\{(x,0),\check{P}(0,\zeta)\big\}\\&=\big\{(\phi x,\eta(x)\zeta),(\xi,0)\big\}-\check{P}\big\{(\phi x,\eta(x)\zeta),(0,\zeta)\big\}\\&\qquad-\check{P}\big\{(x,0),(\xi,0)\big\}\\&=(\{\phi x,\xi\}-\phi\{x,\xi\},(\mathfrak{L}_\xi g)(x,\xi)\zeta)\,.\end{aligned}$$

In the latter expressions, we use the Lie derivative $(\mathfrak{L}_\xi g)(x,y)$, determined by (4.2), of the Riemannian metric g of $(\mathcal{M},\phi,\xi,\eta,g)$.

Then, we define the following four tensors $\widehat{N}^{(k)}$ $(k=1,2,3,4)$ of type (1,2), (0,2), (1,1), (0,1), respectively:

$$\begin{aligned}&\widehat{N}^{(1)}(x,y)=\{\phi,\phi\}(x,y)-(\mathfrak{L}_\xi g)(x,y)\xi,\\&\widehat{N}^{(2)}(x,y)=(\mathfrak{L}_\xi g)(\phi x,y)+(\mathfrak{L}_\xi g)(x,\phi y),\\&\widehat{N}^{(3)}(x)=\{\phi x,\xi\}-\phi\{x,\xi\},\\&\widehat{N}^{(4)}(x)=(\mathfrak{L}_\xi g)(x,\xi),\end{aligned}\tag{6.1}$$

where $\{\phi, \phi\}$ is the symmetric tensor of type (1,2) determined by:

$$\{\phi, \phi\}(x, y) = \{\phi x, \phi y\} + \phi^2\{x, y\} - \phi\{\phi x, y\} - \phi\{x, \phi y\}. \tag{6.2}$$

By direct converting their definitions, we find relations between the four tensors $\widehat{N}^{(k)}$ as follows:

$$\begin{aligned}
&\widehat{N}^{(2)}(x, y) = -\eta\left(\widehat{N}^{(1)}(x, \phi y)\right) - \eta\left(\widehat{N}^{(1)}(\phi x, \xi)\right)\eta(y),\\
&\widehat{N}^{(3)}(x) = \widehat{N}^{(1)}(\phi x, \xi) - \eta(x)\phi\widehat{N}^{(1)}(\xi, \xi)\\
&\widehat{N}^{(4)}(x) = -\eta\left(\widehat{N}^{(1)}(x, \xi)\right) = \tfrac{1}{2}g\left(\widehat{N}^{(1)}(\xi, \xi), x\right),\\
&\widehat{N}^{(4)}(x) = \widehat{N}^{(2)}(\phi x, \xi), \qquad \widehat{N}^{(4)}(x) = -\eta\left(\widehat{N}^{(3)}(\phi x)\right).
\end{aligned} \tag{6.3}$$

Theorem 6.1. *For an almost paracontact almost paracomplex Riemannian manifold we have:*

a) if $\widehat{N}^{(1)}$ vanishes, then all the other tensors $\widehat{N}^{(2)}$, $\widehat{N}^{(3)}$ and $\widehat{N}^{(4)}$ vanish;

b) if any one of $\widehat{N}^{(2)}$ and $\widehat{N}^{(3)}$ vanishes, then $\widehat{N}^{(4)}$ vanishes.

Proof. The statements above are consequences of the relations (6.3) between $\widehat{N}^{(k)}$ $(k = 1, 2, 3, 4)$. □

Therefore, $\widehat{N}^{(1)}$ plays a main role between them and we denote it simply by $\widehat{N}$, i.e.

$$\widehat{N}(x, y) = \{\phi, \phi\}(x, y) - (\mathfrak{L}_\xi g)(x, y)\xi \tag{6.4}$$

and we call it an *associated Nijenhuis tensor of the structure* (ϕ, ξ, η, g). Obviously, $\widehat{N}$ is symmetric, i.e. $\widehat{N}(x, y) = \widehat{N}(y, x)$. Applying the expressions (6.2), (4.2) and (6.4), the associated Nijenhuis tensor has the following form in terms of $\nabla\phi$ and $\nabla\eta$:

$$\widehat{N}(x, y) = (\nabla_{\phi x}\phi)y + (\nabla_{\phi y}\phi)x - \phi(\nabla_x\phi)y - \phi(\nabla_y\phi)x - (\nabla_x\eta)y\,\xi - (\nabla_y\eta)x\,\xi.$$

The corresponding tensor of type (0,3) is defined by $\widehat{N}(x, y, z) = g\left(\widehat{N}(x, y), z\right)$. According to (2.4) and (2.6), we express $\widehat{N}$ in terms of the fundamental tensor F as follows:

$$\begin{aligned}
\widehat{N}(x, y, z) = &F(\phi x, y, z) + F(\phi y, x, z) - F(x, y, \phi z) - F(y, x, \phi z)\\
&+ \eta(z)\left\{F(x, \phi y, \xi) + F(y, \phi x, \xi)\right\}.
\end{aligned} \tag{6.5}$$

Proposition 6.1. *The associated Nijenhuis tensor on an almost paracontact almost paracomplex Riemannian manifold has the following properties:*

$$\begin{aligned}
&\widehat{N}(\phi^2 x, \phi y, \phi z) = -\widehat{N}(\phi^2 x, \phi^2 y, \phi^2 z), \qquad &&\widehat{N}(\phi^2 x, \phi^2 y, \phi^2 z) = \widehat{N}(\phi x, \phi y, \phi^2 z),\\
&\widehat{N}(x, \phi^2 y, \phi^2 z) = -\widehat{N}(x, \phi y, \phi z), &&\widehat{N}(\phi^2 x, \phi^2 y, z) = \widehat{N}(\phi x, \phi y, z),\\
&\widehat{N}(\xi, \phi y, \phi z) = -\widehat{N}(\xi, \phi^2 y, \phi^2 z), &&\widehat{N}(\phi x, \phi y, \xi) = \widehat{N}(\phi^2 x, \phi^2 y, \xi).
\end{aligned}$$

Table 6.1: Associated Nijenhuis tensors

	$\widehat{N}^{(1)}(x,y,z)$	$\widehat{N}^{(2)}(x,y)$	$\widehat{N}^{(3)}(x,y)$	$\widehat{N}^{(4)}(x)$
$\mathcal{F}_1$	$\frac{2}{n}\Big\{g(x,\phi y)\theta(\phi^2 z)-g(\phi x,\phi y)\theta(\phi z)\Big\}$	0	0	0
$\mathcal{F}_2$	$-2\Big\{F(\phi x,\phi y,\phi z)+F(\phi^2 x,\phi^2 y,\phi z)\Big\}$	0	0	0
$\mathcal{F}_3$	0	0	0	0
$\mathcal{F}_4$	$\frac{2}{n}\theta(\xi)g(x,\phi y)\eta(z)$	$-\frac{2}{n}\theta(\xi)g(\phi x,\phi y)$	0	0
$\mathcal{F}_5$	$\frac{2}{n}\theta^*(\xi)g(\phi x,\phi y)\eta(z)$	$-\frac{2}{n}\theta^*(\xi)g(x,\phi y)$	0	0
$\mathcal{F}_6$	$4F(x,\phi y,\xi)\eta(z)$	$-4F(x,y,\xi)$	0	0
$\mathcal{F}_7$	0	0	0	0
$\mathcal{F}_8$	$-2\Big\{\eta(x)F(y,\phi z,\xi)+\eta(y)F(x,\phi z,\xi)\Big\}$	0	$2F(x,y,\xi)$	0
$\mathcal{F}_9$	$-2\Big\{\eta(x)F(y,\phi z,\xi)+\eta(y)F(x,\phi z,\xi)\Big\}$	0	$2F(x,y,\xi)$	0
$\mathcal{F}_{10}$	$-\eta(x)F(\xi,y,\phi z)-\eta(y)F(\xi,x,\phi z)$	0	$F(\xi,x,y)$	0
$\mathcal{F}_{11}$	$\eta(z)\Big\{\eta(x)\omega(\phi y)+\eta(y)\omega(\phi x)\Big\}$ $-2\eta(x)\eta(y)\omega(\phi z)$	$-\eta(x)\omega(y)-\eta(y)\omega(x)$	$\omega(x)\eta(y)+2\eta(x)\omega(y)$	$-\omega(\phi x)$

Proof. The results follow form the properties (2.5) of F and the expression (6.5). □

Applying (6.5) to (6.3), we give the form of $\widehat{N}^{(2)}$, $\widehat{N}^{(3)}$ and $\widehat{N}^{(4)}$ in terms of F:

$$\begin{aligned}
&\widehat{N}^{(2)}(x,y)=-F(x,y,\xi)-F(y,x,\xi)-F(\phi x,\phi y,\xi)-F(\phi y,\phi x,\xi),\\
&\widehat{N}^{(3)}(x,y)=F(\xi,x,y)+F(x,y,\xi)-F(\phi x,\phi y,\xi),\\
&\widehat{N}^{(4)}(x)=-F(\xi,\phi x,\xi),
\end{aligned}\tag{6.6}$$

where we use the denotation $\widehat{N}^{(3)}(x,y)=g\left(\widehat{N}^{(3)}(x),y\right)$.

Proposition 6.2. *Let $(\mathcal{M},\phi,\xi,\eta,g)$ be an $\mathcal{F}_i$-manifold $(i=1,2,\dots,11)$. Then the four tensors $\widehat{N}^{(k)}$ $(k=1,2,3,4)$ on this manifold have the form in the respective cases, given in Table 6.1.*

Proof. The calculations are made, using (6.5), (6.6) and the expression (2.8) of each of F_i for the corresponding class $\mathcal{F}_i$. □

As a result of Proposition 6.2, we establish the truthfulness of the following

Theorem 6.2. *An almost paracontact almost paracomplex Riemannian manifold $(\mathcal{M},\phi,\xi,\eta,g)$ has:*

a) vanishing $\widehat{N}^{(1)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_3$, $\mathcal{F}_7$ or to their direct sum;

b) vanishing $\widehat{N}^{(2)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_7$, …, $\mathcal{F}_{10}$ or to their direct sums;

c) vanishing $\widehat{N}^{(3)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1,\dots,\mathcal{F}_7$ or to their direct sums;

d) vanishing $\widehat{N}^{(4)}$ if and only if it belongs to some of the basic classes $\mathcal{F}_1, \dots, \mathcal{F}_{10}$ or to their direct sums.

By virtue of Theorem 6.2, we obtain the following

Corollary 6.1. *The class of almost paracontact almost paracomplex Riemannian manifolds with a vanishing associated Nijenhuis tensor $\widehat{N}$ is $\mathcal{F}_3 \oplus \mathcal{F}_7$.*

7. The pair of Nijenhuis tensors and the classification of the considered manifolds

In the previous two sections, by (5.7) and (6.5), we give the expressions of the Nijenhuis tensor N and its associated $\widehat{N}$ by the tensor F, respectively. Here, we find how the fundamental tensor F is determined by the pair of Nijenhuis tensors. Since F is used for classifying the manifolds studied, we can expressed the classes $\mathcal{F}_i$ $i = (1, 2, \dots, 11)$ only by the pair $(N, \widehat{N})$.

Theorem 7.1. *Let $(\mathcal{M}, \phi, \xi, \eta, g)$ be an almost paracontact almost paracomplex Riemannian manifold. Then its fundamental tensor is expressed by N and $\widehat{N}$ by the formula:*

$$\begin{aligned} F(x,y,z) = &\tfrac{1}{4}\big[N(\phi x,y,z) + N(\phi x,z,y) + \widehat{N}(\phi x,y,z) + \widehat{N}(\phi x,z,y)\big] \\ &- \tfrac{1}{2}\eta(x)\big[N(\xi,y,\phi z) + \widehat{N}(\xi,y,\phi z) + \eta(z)\widehat{N}(\xi,\xi,\phi y)\big]. \end{aligned} \tag{7.1}$$

Proof. Taking the sum of (5.7) and (6.5), we obtain:

$$F(\phi x,y,z) - F(x,y,\phi z) = \frac{1}{2}\big[N(x,y,z) + \widehat{N}(x,y,z)\big] - \eta(z)F(x,\phi y,\xi). \tag{7.2}$$

The identities (2.5) together with (2.1) imply:

$$F(x,y,\phi z) + F(x,z,\phi y) = \eta(z)F(x,\phi y,\xi) + \eta(y)F(x,\phi z,\xi). \tag{7.3}$$

A suitable combination of (7.2) and (7.3) yields:

$$F(\phi x,y,z) = \frac{1}{4}\big[N(x,y,z) + N(x,z,y) + \widehat{N}(x,y,z) + \widehat{N}(x,z,y)\big]. \tag{7.4}$$

Applying (2.1), we obtain from (7.4) the following:

$$\begin{aligned} F(x,y,z) = &\tfrac{1}{4}\big[N(\phi x,y,z) + N(\phi x,z,y) + \widehat{N}(\phi x,y,z) + \widehat{N}(\phi x,z,y)\big] \\ &+ \eta(x)F(\xi,y,z). \end{aligned} \tag{7.5}$$

Set $x = \xi$ and $z \to \phi z$ into (7.2) and use (2.1) to get:

$$F(\xi,y,z) = -\frac{1}{2}\big[N(\xi,y,\phi z) + \widehat{N}(\xi,y,\phi z)\big] + \eta(z)\omega(y). \tag{7.6}$$

Finally, using (6.5) and the general identities $\omega(\xi) = 0$, we obtain:

$$\omega(z) = -\frac{1}{2}\widehat{N}(\xi,\xi,\phi z). \tag{7.7}$$

Substitute (7.7)into (7.6) and the obtained identity insert into (7.5) to get (7.1). □

Corollary 7.1. *The class of almost paracontact almost paracomplex Riemannian manifolds with vanishing tensors N and $\widehat{N}$ is the special class $\mathcal{F}_0$.*

8. A family of Lie groups as manifolds of the studied type

Let $\mathcal{L}$ be a $(2n+1)$-dimensional real connected Lie group and let its associated Lie algebra with a global basis $\{E_0, E_1, \dots, E_{2n}\}$ of left invariant vector fields on $\mathcal{L}$ be defined by:

$$[E_0, E_i] = -a_i E_i - a_{n+i} E_{n+i}, \qquad [E_0, E_{n+i}] = -a_{n+i} E_i + a_i E_{n+i}, \tag{8.1}$$

where $a_1, \dots, a_{2n}$ are real constants and $[E_j, E_k] = 0$ in other cases.

Let (ϕ, ξ, η) be an almost paracontact almost paracomplex structure determined for any $i \in \{1, \dots, n\}$ by:

$$\begin{array}{lll} \phi E_0 = 0, & \phi E_i = E_{n+i}, & \phi E_{n+i} = E_i, \\ \xi = E_0, & \eta(E_0) = 1, & \eta(E_i) = \eta(E_{n+i}) = 0. \end{array} \tag{8.2}$$

Let g be a Riemannian metric defined by:

$$\begin{array}{l} g(E_0, E_0) = g(E_i, E_i) = g(E_{n+i}, E_{n+i}) = 1, \\ g(E_0, E_j) = g(E_j, E_k) = 0, \end{array} \tag{8.3}$$

where $i \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, 2n\}$, $j \neq k$. Thus, since (2.1) is satisfied, the induced $(2n+1)$-dimensional manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ is an almost paracontact almost paracomplex Riemannian manifold.

Let us remark that in [8] the same Lie group is considered with an appropriate almost contact structure and a compatible Riemannian metric. Then, the generated almost cosymplectic manifold is studied. On the other hand, in [3], the same Lie group is equipped with an almost contact structure and B-metric. Then, the obtained manifold is characterized. Moreover, in [4], the case of the lowest dimension is considered and properties of the constructed manifold are determined.

Let us consider the constructed almost paracontact almost paracomplex Riemannian manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ of dimension 3, i.e. for $n = 1$.

According to (8.1) and (8.3) for $n = 1$, by the Koszul equality

$$2g\left(\nabla_{E_i} E_j, E_k\right) = g\left([E_i, E_j], E_k\right) + g\left([E_k, E_i], E_j\right) + g\left([E_k, E_j], E_i\right)$$

for the Levi-Civita connection ∇ of g, we obtain:

$$\begin{array}{ll} \nabla_{E_1} E_0 = a_1 E_1 + a_2 E_2, & \nabla_{E_2} E_0 = a_2 E_1 - a_1 E_2, \\ \nabla_{E_1} E_1 = -\nabla_{E_2} E_2 = -a_1 E_0, & \nabla_{E_1} E_2 = \nabla_{E_2} E_1 = -a_2 E_0, \end{array} \tag{8.4}$$

and the others $\nabla_{E_i} E_j$ are zero.

Then, using (8.4), (8.2), (2.4) and (3.3), we get the following components $F_{ijk} = F(E_i, E_j, E_k)$ of the fundamental tensor:

$$F_{101} = F_{110} = F_{202} = F_{220} = -a_2, \qquad F_{102} = F_{120} = -F_{201} = -F_{210} = -a_1,$$

and the other components of F are zero. Thus, we have the expression of F for arbitrary vectors $x = x^i E_i$, $y = y^i E_i$, $z = z^i E_i$ as follows:

$$\begin{aligned} F(x,y,z) = &-a_2 \left\{x^1 \left(y^0 z^1 + y^1 z^0\right) + x^2 \left(y^0 z^2 + y^2 z^0\right)\right\} \\ &-a_1 \left\{x^1 \left(y^0 z^2 + y^2 z^0\right) - x^2 \left(y^0 z^1 + y^1 z^0\right)\right\}. \end{aligned} \tag{8.5}$$

Bearing in mind the latter equality, we obtain that F has the following form:

$$F(x,y,z) = F_4(x,y,z) + F_9(x,y,z),$$

by virtue of (3.3) for $\mu = -a_1$, $\theta_0 = -2a_2$. Therefore, we have proved the following

Proposition 8.1. *The constructed 3-dimensional almost paracontact almost paracomplex Riemannian manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ belongs to:*

a) $\mathcal{F}_4 \oplus \mathcal{F}_9$ if and only if $a_1 \neq 0$, $a_2 \neq 0$;

b) $\mathcal{F}_4$ if and only if $a_1 = 0$, $a_2 \neq 0$;

c) $\mathcal{F}_9$ if and only if $a_1 \neq 0$, $a_2 = 0$;

d) $\mathcal{F}_0$ if and only if $a_1 = 0$, $a_2 = 0$.

Finally, we get the following

Proposition 8.2. *The constructed 3-dimensional almost paracontact almost paracomplex Riemannian manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ has the following properties:*

a) It has vanishing $N^{(4)}$ and $\widehat{N}^{(4)}$;

b) It is a normal almost paracontact almost paracomplex Riemannian manifold with vanishing $\widehat{N}^{(3)}$ if and only if $a_1 = 0$ and arbitrary a_2;

c) It is a para-Sasakian paracomplex Riemannian manifold if and only if $a_1 = 0$, $a_2 = 1$;

d) It has vanishing $\widehat{N}^{(2)}$ if and only if $a_2 = 0$ and arbitrary a_1.

Proof. According to (8.5), (3.3) and Proposition 5.2, we find the following form of the Nijenhuis tensor of $(\mathcal{L}, \phi, \xi, \eta, g)$:

$$N(x,y,z) = -2a_1 \left\{\left(x^1 y^2 - x^2 y^1\right) z^0 + \left(x^0 y^1 - x^1 y^0\right) z^1 - \left(x^0 y^2 - x^2 y^0\right) z^2\right\}.$$

From the latter equality and (5.8) (or alternatively from (8.5) and (5.9)), we have:

$$N^{(2)}(x,y) = 2a_1(x^1 y^1 - x^2 y^2), \quad N^{(3)}(x,y) = 2a_1(x^1 y^2 - x^2 y^1), \quad N^{(4)}(x) = 0.$$

Similarly, for the associated Nijenhuis tensor of $(\mathcal{L}, \phi, \xi, \eta, g)$ we obtain:

$$\widehat{N}(x,y,z) = -4a_2 \left(x^1 y^2 + x^2 y^1\right) z^0 + 2a_1 \left\{\left(x^0 y^1 + x^1 y^0\right) z^1 - \left(x^0 y^2 + x^2 y^0\right) z^2\right\}.$$

By virtue of (6.3) and the equality from above (or in other way by (8.5) and (6.6)), we get:

$$\widehat{N}^{(2)}(x,y) = 4a_2\left(x^1y^1 + x^2y^2\right), \quad \widehat{N}^{(3)}(x,y) = -2a_1\left(x^1y^2 - x^2y^1\right), \quad \widehat{N}^{(4)}(x) = 0.$$

As a conclusion, the obtained results imply the propositions in a), b) and d). Moreover, the case of the $\mathcal{F}_4{}'$-manifold, i.e. the proposition in c), follows from Proposition 8.1 b). □

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ON GENERALIZED ϕ-RECURRENT AND GENERALIZED CONCIRCULARLY ϕ-RECURRENT $N(\kappa)$-PARACONTACT METRIC MANIFOLDS

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Abstract. The purpose of the present paper is to study generalized ϕ-recurrent, generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifolds and generalized ϕ-recurrent paracontact metric manifolds of constant curvature.
Keywords:generalized ϕ-recurrent, generalized concirculary ϕ -recurrent, $N(\kappa)$-paracontact metric manifold.

1. Introduction

Almost paracontact metric structures are the natural odd-dimensional analogue to almost paraHermitian structures, just like almost contact metric structures correspond to the almost Hermitian ones. The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [6] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy, [10]. An important class among paracontact metric manifolds is that of the κ-spaces, which satisfy the nullity condition [2]. This class includes the para-Sasakian manifolds [6, 10], the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all X, Y vector fields on the manifold [11], etc.

Let M be an $2n + 1$-dimensional connected semi-Riemannian manifold with semi-Riemannian metric g and Levi-Civita connection ∇. M is called *locally symmetric* if its curvature tensor is parallel with respect to ∇. The notion of locally symmetric manifold has been weakend such as *recurrent manifold* by Walker [9], in 1977 Takahashi [8] introduced the notion of *local ϕ -symmetry* on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry, De et al. [3] introduced and studied the notion of *ϕ-recurrent* Sasakian manifold. Then in [4] and [7], De and Gazi and Peyghan et al. studied ϕ-recurrent $N(\kappa)$-contact metric manifolds. Dubey [5] introduced the notion of *generalized recurrent* manifold.

Motivated by these considerations, the author make the first contribution to study generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifolds (which includes both the

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notion of local ϕ-symmetry and also ϕ-recurrence) and generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifolds.

The paper is organized as follows:

Section 2 is preliminary section, where we recall basic facts which we will need throughout the paper. In Section 3, we prove that a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold for $\kappa \neq -1, 0$. We show that in a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold, the characteristic vector field ξ and the vector field $\rho_1\kappa + \rho_2$ associated to the 1-form $A\kappa + B$ are co-directional. We find the relation between associated 1-forms A and B for a three dimensional generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold. In Section 4, we mainly give the relation between associated 1-forms A and B in a generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold (M^{2n+1}, g) of constant curvature $c \neq 0$. In Section 5, we prove that a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold for $\kappa \neq -1, 0$. We give the relation between associated 1-forms A and B for a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold and we show that in a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold, the characteristic vector field ξ and the vector field $\rho_1 c + \rho_2$ associated to the 1-form $Ac + B$ are co-directional. Finally, we show that for a three dimensional generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold, r is not necessarily be a constant.

2. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold and ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a one-form on M. Then (ϕ, ξ, η) is called an *almost paracontact structure* on M if

(i) $\phi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1,$

(ii) the tensor field ϕ induces an almost paracomplex structure on the distribution $D = \ker \eta$, that is the eigendistributions $D^{\pm}$, corresponding to the eigenvalues ± 1, have equal dimensions, $dim\, D^+ = dim\, D^- = n$.

The manifold M is said to be an *almost paracontact manifold* if it is endowed with an almost paracontact structure [10].

Let M be an almost paracontact manifold. M will be called an *almost paracontact metric manifold* if it is additionally endowed with a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.1}$$

For such manifold, we have

$$\eta(X) = g(X, \xi), \phi(\xi) = 0, \eta \circ \phi = 0. \tag{2.2}$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.3}$$

usually called *fundamental form*.

For an almost paracontact manifold, there exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a *ϕ-basis.*

On an almost paracontact manifold, one defines the $(1,2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X,Y) = [\phi,\phi](X,Y) - 2d\eta(X,Y)\xi, \tag{2.4}$$

where $[\phi,\phi]$ is the *Nijenhuis torsion* of ϕ

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be *normal* [10]. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$

$$J(X, \lambda\frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt}),$$

is integrable.

If $d\eta(X,Y) = g(X, \phi Y)$, then (M, ϕ, ξ, η, g) is said to be *paracontact metric manifold.* In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [10] that h anti-commutes with ϕ and satisfies

$$i) h\xi = 0, \quad ii) trh = trh\phi = 0, \quad iii) \nabla\xi = -\phi + \phi h, \tag{2.5}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).

Moreover $h = 0$ if and only if ξ is a Killing vector field. In this case (M, ϕ, ξ, η, g) is said to be a *K-paracontact manifold.* Similarly as in the class of almost contact metric manifolds [1], a normal almost paracontact metric manifold will be called *para-Sasakian* if $\Phi = d\eta$.

On an almost paracontact metric manifold M, if the Ricci operator satisfies

$$Q = \alpha id + \beta\eta \otimes \xi,$$

where both α and β are smooth functions, then the manifold is said to be an *η-Einstein manifold.* An η-Einstein manifold with β vanishing and α a constant is obviously an Einstein manifold.

The *κ-nullity distribution* $N(\kappa)$ of a semi-Riemannian manifold M is defined by

$$N(\kappa) : p \to N_p(\kappa) = \{Z \in T_pM \mid R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)\}, \tag{2.6}$$

for some real constant κ. If the characteristic vector field ξ belongs to $N(\kappa)$, then we call a paracontact metric manifold an *$N(\kappa)$-paracontact metric manifold.* For a $N(\kappa)$-paracontact metric manifold [2] we have,

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y), \tag{2.7}$$
$$S(X,\xi) = 2n\kappa\eta(X), \tag{2.8}$$
$$h^2 = (1+\kappa)\phi^2. \tag{2.9}$$

for all X, Y vector fields on M, where κ is constant and S is the Ricci tensor.

Lemma 2.1. *[2]In any $(2n+1)$-dimensional paracontact (κ,μ)-manifold (M,ϕ,ξ,η,g) such that $\kappa \neq -1$, the Ricci operator Q is given by*

$$Q = (2(1-n)+n\mu)I + (2(n-1)+\mu)h + (2(n-1)+n(2\kappa-\mu))\eta\otimes\xi. \tag{2.10}$$

Using (2.10), we have

$$S(\phi X,\phi Y) = S(X,Y) - 4(1-n)g(X,Y) + (4(1-n)-2n\kappa)\eta(X)\eta(Y). \tag{2.11}$$

Definition 2.1. A $N(\kappa)$-paracontact metric manifold is said to be a generalized ϕ-recurrent if its curvature tensor R satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)(g(Y,Z)X - g(X,Z)Y), \tag{2.12}$$

where A and B are two 1-forms, B is non zero and they are defined by

$$A(X) = g(X,\rho_1), \quad B(X) = g(X,\rho_2), \tag{2.13}$$

where ρ_1 and ρ_2 are vector fields associated with 1-forms A, B respectively.

Definition 2.2. A (2n+1)-dimensional $N(\kappa)$-paracontact metric manifold is called a generalized concircular ϕ-recurrent if its concircular curvature tensor C

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.14}$$

satisfies the condition

$$\phi^2((\nabla_W C)(X,Y)Z) = A(W)C(X,Y)Z + B(W)(g(Y,Z)X - g(X,Z)Y), \tag{2.15}$$

where A and B are defined as (2.13) and $r = tr(S)$ is the scalar curvature.

In the above definitions, X, Y, Z, W are arbitrary vector fields and not necessarily orthogonal to ξ.

Remark 2.1. A flat manifold satisfies $R = 0$ and $\nabla R = 0$, so flat manifolds are trivial examples of generalized ϕ-recurrent paracontact metric manifolds.

3. Generalized ϕ-recurrent $N(\kappa)$ -paracontact metric manifolds

Theorem 3.1. *For $\kappa \neq -1, 0$, a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold.*

Proof. In view of (2.12), we get

$$\begin{aligned}(\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi \\ = A(W)R(X,Y)Z + B(W)(g(Y,Z)X - g(X,Z)Y).\end{aligned} \tag{3.1}$$

Taking the inner product on both sides of (3.1) with U, we obtain

$$\begin{aligned} g((\nabla_W R)(X,Y)Z,U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) &= A(W)g(R(X,Y)Z,U) \\ &\quad +B(W)(g(Y,Z)g(X,U) \\ &\quad -g(X,Z)g(Y,U)). \end{aligned} \tag{3.2}$$

Let e_i, $1 \leq i \leq 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and getting the summation over i, one can get

$$\begin{aligned} &(\nabla_W S)(Y,Z) - \sum_{i=1}^{2n+1} \varepsilon_i \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ &= A(W)S(Y,Z) + 2nB(W)g(Y,Z). \end{aligned} \tag{3.3}$$

Now, let calculate the second term of the left hand side of the above equation by replacing Z by ξ. Using (2.6) and the fact that $(\nabla_W g) = 0$, we get

$$\varepsilon_i g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{3.4}$$

Putting $Z = \xi$ in (3.3) and using (2.8) and (3.4), we obtain

$$(\nabla_W S)(Y,\xi) = 2n\eta(Y)(\kappa A(W) + B(W)). \tag{3.5}$$

Using the property (iii) of (2.5) and (2.8) in $(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi)$, we have

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= 2n\kappa(\nabla_W \eta)(Y) + S(Y,\phi W - \phi h W) \\ &= 2n\kappa g(-\phi W + \phi h W, Y) + S(Y,\phi W - \phi h W). \end{aligned} \tag{3.6}$$

Comparing equations (3.5) and (3.6), we get

$$2n\eta(Y)(\kappa A(W) + B(W)) = 2n\kappa g(-\phi W + \phi h W, Y) + S(Y,\phi W - \phi h W). \tag{3.7}$$

Replacing Y by ϕY in the last equation and using (2.1) and (2.11), we obtain

$$\begin{aligned} 0 &= (2n\kappa - 4(1-n))g(W,Y) + (-2n\kappa + 4(1-n))g(W,hY) \\ &\quad +(-2n\kappa + 4(1-n) - 2n\kappa)\eta(Y)\eta(W) + S(Y,W) - S(Y,hW). \end{aligned} \tag{3.8}$$

Employing (2.9) and (2.10) in (3.8),we get

$$\begin{aligned} S(Y,W) &= 2(-n-\kappa+1)g(W,Y) + 2(n\kappa+n-1)g(hW,Y) \\ &\quad +2(n(\kappa+1)+\kappa-1)\eta(Y)\eta(W). \end{aligned} \tag{3.9}$$

Putting $W = hW$ in (3.9) and using again (2.9) and (2.10), we have

$$2\kappa g(hW,Y) = 2n\kappa(\kappa+1)g(W,Y) - 2n\kappa(\kappa+1)\eta(Y)\eta(W).$$

By the assumption of $\kappa \neq 0$, the last equations returns to

$$g(hW,Y) = n(\kappa+1)(g(W,Y) - \eta(Y)\eta(W)). \tag{3.10}$$

Using (3.10) in (3.9), we get

$$S(Y,W) = \alpha g(W,Y) + \beta\eta(Y)\eta(W),$$

where $\alpha = 2[(-n-\kappa+1)+n(\kappa+1)(n\kappa+n-1)]$, $\beta = 2[n(\kappa+1)+(\kappa-1)-n(\kappa+1)(n\kappa+n-1)]$. Hence, we can conclude that the manifold is η-Einstein manifold. $\square$

Theorem 3.2. *For a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g), the characteristic vector field ξ and the vector field $\rho_1\kappa+\rho_2$ associated to the 1-form $A\kappa+B$ are co-directional.*

Proof. Two vector fields P and Q are said to be co-directional if $P = fQ$, where f is a non-zero scalar, that is $g(P, X) = fg(Q, X)$ for all X.

Taking inner product of (3.1) with ξ, we have

$$A(W)g(R(X,Y)Z,\xi) + B(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = 0. \tag{3.11}$$

Then by the use of second Bianchi identity, we can write

$$\begin{aligned} & A(W)g(R(X,Y)Z,\xi) + B(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \\ & +A(Y)g(R(W,X)Z,\xi) + B(Y)(g(X,Z)\eta(W) - g(W,Z)\eta(X)) \\ & +A(X)g(R(Y,W)Z,\xi) + B(X)(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)) \\ = {} & 0. \end{aligned} \tag{3.12}$$

From (2.6), it follows that

$$g(R(X,Y)Z,\xi) = \kappa(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z)). \tag{3.13}$$

Using (3.13) in (3.12), we get

$$\begin{aligned} & \kappa\left\{\begin{array}{c} A(W)[(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z))] \\ +A(Y)[(-\eta(X)g(W,Z) + \eta(W)g(X,Z))] \\ A(X)[(-\eta(W)g(Y,Z) + \eta(Y)g(W,Z))] \end{array}\right\} \\ & +B(W)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \\ & +B(Y)(g(X,Z)\eta(W) - g(W,Z)\eta(X)) \\ & +B(X)(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)) \\ = {} & 0. \end{aligned} \tag{3.14}$$

Replacing $Y = Z$ by e_i in (3.14) and taking summation over i, $1 \le i \le 2n+1$, we obtain

$$(2n-1)\left[\kappa(A(W)\eta(X) - A(X)\eta(W)) + B(W)\eta(X) - B(X)\eta(W)\right] = 0. \tag{3.15}$$

Putting $X = \xi$ in the last equation, we have

$$\begin{aligned} \kappa(A(W) - \eta(W)\eta(\rho_1)) &= -(B(W) - \eta(W)\eta(\rho_2)) \\ \eta(W)(\kappa\eta(\rho_1) + \eta(\rho_2)) &= \kappa A(W) + B(W). \end{aligned} \tag{3.16}$$

where $\eta(\rho_1) = g(\xi, \rho_1) = A(\xi)$ and $\eta(\rho_2) = g(\xi, \rho_2) = B(\xi)$. From (3.16), we complete the proof of the theorem. □

Theorem 3.3. *Let (M^3, g) be a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Then $B(W) = -\kappa A(W)$.*

Proof. We recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \tag{3.17}$$

where Q is the Ricci-operator, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. Let (M^3, g) be a generalized ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Replacing Z by ξ in (3.17) and using (2.8), we have

$$R(X,Y)\xi = (2\kappa - \frac{r}{2})(\eta(Y)X - \eta(X)Y) + \eta(Y)QX - \eta(X)QY. \tag{3.18}$$

Comparing (2.7) with (3.18), we get

$$(\kappa - \frac{r}{2})(\eta(Y)X - \eta(X)Y) = \eta(X)QY - \eta(Y)QX. \tag{3.19}$$

Putting $Y = \xi$ in (3.19) and using (2.8), we obtain

$$QX = (\frac{r}{2} - \kappa)X + (3\kappa - \frac{r}{2})\eta(X)\xi, \tag{3.20}$$

which gives

$$S(X,Y) = (\frac{r}{2} - \kappa)g(X,Y) + (3\kappa - \frac{r}{2})\eta(X)\eta(Y). \tag{3.21}$$

By taking account of (3.20) and (3.21) in (3.17), one can get

$$\begin{aligned} R(X,Y)Z &= (3\kappa - \frac{r}{2})(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ &\quad +(\frac{r}{2} - 2\kappa)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{3.22}$$

Taking the covariant derivative of the last equation according to W, we deduce that

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= -\frac{dr(W)}{2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ &\quad +\frac{dr(W)}{2}(g(Y,Z)X - g(X,Z)Y) \\ &\quad +(3\kappa - \frac{r}{2})\begin{pmatrix} (g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\nabla_W\xi \\ +(g(Y,Z)\xi - \eta(Z)Y)(\nabla_W\eta)(X) \\ -(g(X,Z)\xi - \eta(Z)X)(\nabla_W\eta)(Y) \\ +(\eta(Y)X - \eta(X)Y)(\nabla_W\eta)(Z). \end{pmatrix}. \end{aligned} \tag{3.23}$$

Now, let Y be a non-zero vector field orthogonal to ξ and $X = Z = \xi$. Using (2.5), (3.23) follows that

$$(\nabla_W R)(\xi,Y)\xi = -2(3\kappa - \frac{r}{2})(\nabla_W\eta)(\xi)Y = 0. \tag{3.24}$$

By virtue of (2.12) and (3.24), we obtain

$$A(W)R(\xi,Y)\xi - B(W)Y = 0. \tag{3.25}$$

From (2.7), we have

$$R(\xi,Y)\xi = -\kappa Y. \tag{3.26}$$

If we use (3.26) in (3.25), it follows that the requested relation holds. This completes the proof of the theorem. $\square$

4. Generalized ϕ-recurrent paracontact metric manifolds of constant curvature

Theorem 4.1. *[10]If a paracontact manifold M^{2n+1} is of constant sectional curvature c and dimension $2n+1 \geqslant 5$, then $c=-1$ and $|h|^2=0$.*

Theorem 4.2. *If a generalized ϕ-recurrent paracontact metric manifold (M^{2n+1}, g) is of constant curvature and $(2n+1) \geqslant 5$, then $A(W) = B(W)$.*

Proof. Let (M^{2n+1}, g) be a generalized ϕ-recurrent paracontact metric manifold of constant curvature c and $(2n+1) \geqslant 5$. From Theorem 4.1, we have $c=-1$. So, we can write

$$R(X,Y)Z = -(g(Y,Z)X - g(X,Z)Y). \tag{4.1}$$

Taking the covariant derivative of the last equation according to W, we deduce that

$$(\nabla_W R)(X,Y)Z = 0. \tag{4.2}$$

Now, let Y be a non-zero vector field orthogonal to ξ and $X = Z = \xi$. From (4.1), we have

$$R(\xi, Y)\xi = Y. \tag{4.3}$$

By using (2.12), (4.2) and (4.3), we have

$$0 = A(W) - B(W)$$

which completes the proof. □

Theorem 4.3. *If a generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold (M^{2n+1}, g) is of constant curvature $c \neq 0$, then $B(W) = -\kappa A(W)$.*

Proof. Let us consider a $(2n+1)$-dimensional generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold which has constant curvature c. So, we have

$$R(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y). \tag{4.4}$$

Replacing Z by ξ in (4.4), we get

$$R(X,Y)\xi = c(\eta(Y)X - \eta(X)Y). \tag{4.5}$$

From (2.7) and (4.5), we obtain

$$c(\eta(Y)X - \eta(X)Y) = \kappa(\eta(Y)X - \eta(X)Y). \tag{4.6}$$

Now, let Y be a non-zero vector field orthogonal to ξ and $X = \xi$. So, (4.6) returns to $c = \kappa \neq 0$. Because of the manifold is $N(\kappa)$-paracontact metric manifold, we have

$$R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y). \tag{4.7}$$

Taking the covariant derivative of the last equation according to W, we deduce that

$$(\nabla_W R)(X,Y)Z = -\kappa((\nabla_W g)(X,Z)Y - ((\nabla_W g)(Y,Z)X)) = 0. \tag{4.8}$$

Putting $Y = Z = \xi$ in (2.12), and taking account of (4.7) and (4.8), we obtain

$$0 = (X - \eta(X)\xi)(A(W)\kappa + B(W)). \tag{4.9}$$

If X is a non-zero vector field orthogonal to ξ, from (4.9), we get

$$0 = A(W)\kappa + B(W).$$

□

Remark 4.1. If a generalized ϕ-recurrent $N(\kappa \neq 0)$-paracontact metric manifold (M^{2n+1}, g) is of constant curvature $c \neq 0$, and $(2n+1) \geqslant 5$, then $\kappa = -1$.

5. Generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifolds

Theorem 5.1. *For $\kappa \neq -1, 0$, a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1}, g) is an η-Einstein manifold.*

Proof. Let us consider a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold. From (2.15), we have

$$(\nabla_W C)(X,Y)Z - \eta((\nabla_W C)(X,Y)Z)\xi = A(W)C(X,Y)Z + B(W)(g(Y,Z)X - g(X,Z)Y). \tag{5.1}$$

Taking the inner product on both sides of (5.1) with U, we obtain

$$\begin{aligned} g((\nabla_W C)(X,Y)Z,U) - \eta((\nabla_W C)(X,Y)Z)\eta(U) &= A(W)g(C(X,Y)Z,U) \\ &\quad + B(W)(g(Y,Z)g(X,U) \\ &\quad - g(X,Z)g(Y,U)). \end{aligned} \tag{5.2}$$

Let e_i, $1 \leq i \leq 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (5.2) and taking summation over i, we thus get

$$\begin{aligned} (\nabla_W S)(Y,Z) &= \frac{dr(W)}{2n+1}g(Y,Z) - \frac{dr(W)}{(2n+1)2n}(g(Y,Z) - \eta(Y)\eta(Z)) \\ &= A(W)(S(Y,Z) - \frac{r}{2n+1}g(Y,Z)) + B(W)2ng(Y,Z). \end{aligned} \tag{5.3}$$

If we make use of the property (iii) of (2.5) and (2.8) in (5.3), we obtain

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= \frac{dr(W)}{2n+1}\eta(Y) \\ &\quad + A(W)\eta(Y)\left(2n\kappa - \frac{r}{2n+1}\right) + B(W)2n\eta(Y). \end{aligned} \tag{5.4}$$

On the other hand, using again the property (iii) of (2.5) and (2.8), we can evaulate $(\nabla_W S)(Y,\xi)$ as

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) \\ &= -2n\kappa g(Y,\phi W - \phi hW) + S(Y,\phi W - \phi hW). \end{aligned} \tag{5.5}$$

Comparing (5.4) to (5.5), we have

$$\begin{aligned} S(Y,\phi W-\phi hW) &= 2n\kappa g(Y,\phi W-\phi hW)+\frac{dr(W)}{2n+1}\eta(Y) \\ &\quad +A(W)\eta(Y)\left(2n\kappa-\frac{r}{2n+1}\right)+B(W)2n\eta(Y). \end{aligned} \tag{5.6}$$

If we use (2.9), (2.10) and (2.11) after putting ϕY instead of Y in (5.6), we get

$$\begin{aligned} S(Y,W) &= 2(-n-\kappa+1)g(Y,W)+2(n-1+n\kappa)g(Y,hW) \\ &\quad +2((n-1)+\kappa(n+1))\eta(Y)\eta(W). \end{aligned} \tag{5.7}$$

If we replace W by hW in the last equation, we can immediately observe that

$$g(Y,hW)=n(1+\kappa)(g(Y,W)-\eta(Y)\eta(W)). \tag{5.8}$$

Using (5.8) in (5.7), we have

$$S(Y,W)=\alpha g(W,Y)+\beta\eta(Y)\eta(W),$$

where $\alpha=2((-n-\kappa+1)+(n-1+n\kappa)n(1+\kappa))$, $\beta=2((n-1)+\kappa(n+1)-(n-1+n\kappa)n(1+\kappa))$. Namely, manifold is η-Einstein manifold. □

Theorem 5.2. *Let (M^{2n+1},g) be a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Then $(\frac{r}{(2n+1)2n}-\kappa)A(W)=B(W)$.*

Proof. Putting $Y=Z=e_i$ in (5.2) and taking summation over i, one can get

$$\begin{aligned} &(\nabla_W S)(X,U)-\frac{dr(W)}{2n+1}g(X,U)-(\nabla_W S)(X,\xi)\eta(U)+\frac{dr(W)}{2n+1}\eta(X)\eta(U) \\ &= A(W)(S(X,U)-\frac{r}{2n+1}g(X,U))+B(W)2ng(X,U). \end{aligned} \tag{5.9}$$

Putting $U=\xi$ in (5.9) and using (2.8), we have

$$A(W)(2n\kappa-\frac{r}{2n+1})\eta(X)+2nB(W)\eta(X)=0. \tag{5.10}$$

Setting $X=\xi$ in the last equation, we get the requested relation which completes the proof of the theorem. □

Theorem 5.3. *For a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold (M^{2n+1},g), the characteristic vector field ξ and the vector field $\rho_1\gamma+\rho_2$ associated to the 1-form $A\gamma+B$ are co-directional.*

Proof. Two vector fields P and Q are said to be co-directional if $P=fQ$, where f is a non-zero scalar, that is $g(P,X)=fg(Q,X)$ for all X.

Taking inner product of (5.1) with ξ, we have

$$A(W)g(C(X,Y)Z,\xi)+B(W)(g(Y,Z)\eta(X)-g(X,Z)\eta(Y))=0. \tag{5.11}$$

In virtue of (2.14) and (5.11), we get

$$A(W)g(R(X,Y)Z,\xi) = (A(W)\frac{r}{(2n+1)2n} - B(W))(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{5.12}$$

Then by the use of second Bianchi identity, we obtain

$$\begin{aligned} & A(W)g(R(X,Y)Z,\xi) + A(Y)g(R(W,X)Z,\xi) + A(X)g(R(Y,W)Z,\xi) \\ = & (A(W)\frac{r}{(2n+1)2n} - B(W))(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) + \\ & (A(Y)\frac{r}{(2n+1)2n} - B(Y))(g(X,Z)\eta(W) - g(W,Z)\eta(X)) + \\ & (A(X)\frac{r}{(2n+1)2n} - B(X))(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)). \end{aligned} \tag{5.13}$$

From (2.6), it follows that

$$g(R(X,Y)Z,\xi) = \kappa(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z)).$$

Using the last equation in (5.13), we get

$$\begin{aligned} & A(W)[\kappa(-\eta(Y)g(X,Z) + \eta(X)g(Y,Z))] + \\ & A(Y)[\kappa(-\eta(X)g(W,Z) + \eta(W)g(X,Z))] + \\ & A(X)[\kappa(-\eta(W)g(Y,Z) + \eta(Y)g(W,Z))] \\ = & (A(W)\frac{r}{(2n+1)2n} - B(W))(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) + \\ & (A(Y)\frac{r}{(2n+1)2n} - B(Y))(g(X,Z)\eta(W) - g(W,Z)\eta(X)) + \\ & (A(X)\frac{r}{(2n+1)2n} - B(X))(g(W,Z)\eta(Y) - g(Y,Z)\eta(W)). \end{aligned} \tag{5.14}$$

Replacing $Y = Z$ by e_i in (5.14) and taking summation over i, $1 \leq i \leq 2n+1$, we obtain

$$(1-2n)\begin{pmatrix} (\kappa - \frac{r}{(2n+1)2n})(A(X)\eta(W) - A(W)\eta(X)) \\ +B(X)\eta(W) - B(W)\eta(X) \end{pmatrix} = 0. \tag{5.15}$$

Putting $X = \xi$ in the last equation, we have

$$\eta(W)(\eta(\rho_2) + \gamma\eta(\rho_1)) = A(W)\gamma + B(W), \tag{5.16}$$

where $\gamma = (\kappa - \frac{r}{(2n+1)2n})$, $\eta(\rho_1) = g(\xi,\rho_1) = A(\xi)$ and $\eta(\rho_2) = g(\xi,\rho_2) = B(\xi)$. From (5.16), we complete the proof of the theorem. □

Theorem 5.4. *Let (M^3, g) be a generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold. Then $B(W) = -\frac{dr(W)}{6} + (\frac{r}{6} - \kappa)A(W)$.*

Proof. Using (3.22) in (2.14), we get

$$\begin{aligned} C(X,Y)Z \; = \; & (3\kappa - \frac{r}{2})(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\ & +(\frac{r}{3} - 2\kappa)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{5.17}$$

It is readily taken that the covariant derivative of the above expression

$$
\begin{aligned}
(\nabla_W C)(X,Y)Z &= -\frac{dr(W)}{2}(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) \\
&\quad + \frac{dr(W)}{3}(g(Y,Z)X - g(X,Z)Y) \\
&\quad + (3\kappa - \frac{r}{2})\begin{pmatrix} (g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\nabla_W\xi \\ +(g(Y,Z)\xi - \eta(Z)Y)(\nabla_W\eta)(X) \\ -(g(X,Z)\xi - \eta(Z)X)(\nabla_W\eta)(Y) \\ +(\eta(Y)X - \eta(X)Y)(\nabla_W\eta)(Z). \end{pmatrix}. \qquad (5.18)
\end{aligned}
$$

Let us assume that Y is a non-zero vector field orthogonal to ξ and $X = Z = \xi$. Using the property (iii) of (2.5) and (5.18), we have

$$(\nabla_W C)(\xi, Y)\xi = \frac{dr(W)}{6}Y. \qquad (5.19)$$

It follows (2.12) and (5.19) from that

$$A(W)C(\xi, Y)\xi - B(W)Y = \frac{dr(W)}{6}Y. \qquad (5.20)$$

From (2.7) and (2.14), we have

$$C(\xi, Y)\xi = (-\kappa + \frac{r}{6})Y. \qquad (5.21)$$

If we employ (5.21) in (5.20), we immediately see that one is able to get the requested equation. □

Remark 5.1. In a three dimensional generalized concirculary ϕ-recurrent $N(\kappa)$-paracontact metric manifold, r is not necessarily be a constant.

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EFFICIENT ENCODINGS TO HYPERELLIPTIC CURVES OVER FINITE FIELDS *

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Abstract. Many cryptosystems are based on the difficulty of the discrete logarithm problem in finite groups. In this case elliptic and hyperelliptic cryptosystems are more noticed because they provide good security with smaller size keys. Since these systems were used for cryptography, it has been an important issue to transform a random value in finite field into a random point on an elliptic or hyperelliptic curve in a deterministic and efficient method. In this paper we propose a deterministic encoding to hyperelliptic curves over finite field. For cryptographic desires it is important to have an injective encoding. In finite fields with characteristic three we obtain an injective encoding for genus two hyperelliptic curves.
Keywords: Cryptosystem; hyperelliptic curves; injective encoding; finite field.

1. Introduction

We first recall that a hyperelliptic curve H of gunes g is a curve by the equation $y^2 = f(x)$, where f a squarefree, monic polynomial of degree 2g + 1. Every hyperelliptic curve of genus 1 is called an elliptic curve. In fact an elliptic curve over the finite field $\mathbb{F}_q$ is the set $E(\mathbb{F}_q)$ which includes all of the points (x, y) such that

$$y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{F}_q$ with an additional point that is called infinity. The points on $E(\mathbb{F}_q)$ with ∞ form an additive abelian group but for $g \geq 2$ there is not a group law on the points of a hyperelliptic curve. However the divisor group of H is denoted by $Div(H)$ is a free abelian group. A divisor $D \in Div(H)$ is a formal sum $D = \sum m_P P$ where $m_P \in \mathbb{Z}$ and $m_P = 0$ for all but finitely many $P \in H$. Then the degree of D is defined by $degD = \sum m_P$. The divisors of degree 0 form a subgroup of $Div(H)$ which is denoted by $Div^0(H)$. For every $f \in \bar{\mathbb{F}_q}(H)$ the divisor of f is defined

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by $div(f) = \sum ord_P(f)P$ where $ord_P(f)$ is the order of vanishing of f at P. A divisor $D \in Div(H)$ is called a principal divisor if it has the form $D = div(f)$ for some $f \in \bar{\mathbb{F}_q}(H)$. Two divisors D_1, D_2 are called linearly equivalent if $D_1 - D_2$ is principal. The group of principal divisors of H is denoted by $Princ(H)$. Since every principal divisor has degree 0, $Princ(H)$ is a subgroup of $Div^0(H)$. The jacobian of H over $\mathbb{F}_q$ is defined by $J(H) = Div^0(H)/Princ(H)$. Since in many cryptosystems we need to a group we use the group $J(H)$ rather than the set of points on H. We have:

$$(\sqrt{q} - 1)^{2g} \leq \#J(H) \leq (\sqrt{q} + 1)^{2g}$$

Therefore $\#J(H) \approx q^g$.

2. Background

Encoding from finite fields element into the points of a given elliptic or hyperelliptic curve is a more challenging problem and requires to be studied more carefully. Before 2006 the usual method was Try and Increment. It was to take $x \in \mathbb{F}_q$ and check whether this value corresponds to a valid abscissa of a point on the elliptic curve. If not, try another abscissa until one of them works. One defect of this algorithm is that the number of operation is not constant. namely the number of steps depends on the input x.

Algorithm 1 Try and Increment Method

Require: : u an integer.
Ensure: : Q, a point of $E(\mathbb{F}_q)$.
  for $i = 0$ **to** $k - 1$ **do**
    (a) set $x = u + i$
    (b) If x^3+ax+b is a quadratic residue in $\mathbb{F}_q$, then return $Q = (x, (x^3+ax+b)^{\frac{1}{2}})$
  end for
  return $\perp$

The twisted curves method was to apply curve and its twist as suggested in [6]. If E is defined by $y^2=x^3+ax+b$ over $\mathbb{F}_q$, the twist of E is a curve E^d defined by

$$dy^2=x^3+ax+b,$$

where d is a quadratic non-residue in $\mathbb{F}_q$. Then for every x there exists y such that the point (x, y) belongs to E or E^d. This method was little noticed since it requires calculation on two curves and this doubles the running time.

When $q\equiv 2 \ (mod\ 3)$ the map $x \rightarrow x^3$ is a bijection from $\mathbb{F}_q^*$ to itself. If E is defined by the equation $y^2 = x^3 + b$, the map

$$f : u \longrightarrow ((u^2 - b)^{\frac{1}{3}}, u)$$

gives a bijection from $\mathbb{F}_q$ to affine points on the curve E. Therefore these curves are supersingular for every b. The MOV attack gives an efficient computable method

which enables to reduce the DLP on a supersingular elliptic curve to DLP on a finite field [15]. Therefore in order to avoid this attack, much larger parameters must be used.
In 2006 the first algorithm for encoding to elliptic curves in deterministic polynomial time was proposed by Shallue and Woestijne [16]. The algorithm is based on the Skalba equality which says that there exist four maps $X_1(t), X_2(t), X_3(t), X_4(t)$ such that

$$f(X_1(t))f(X_2(t))f(X_3(t)) = (X_4(t))^2,$$

where $f(x) = X^3 + aX + b$. Then in a finite field for a fixed parameter t, there exists $1 \leqslant j \leqslant 3$ such that $f(X_j(t))$ is a quadratic residue. This implies that $(X_j(t), \sqrt{f(X_j(t))})$ is a point on $E : y^2 = f(x)$. For $q \equiv 3(mod\ 4)$ computing the square root $\sqrt{f(X_j(t))}$ is simply an exponentiation but for $q \equiv 1(mod\ 4)$, no deterministic algorithm has been found for computing the square root. If we have a non quadratic residue in $\mathbb{F}_q$ we can apply Tonelli Shanks algorithm to compute the square root. Using Skalba equality the authors of [16] show that a modification of Tonelli-Shanks algorithm can compute square roots deterministicaly in time $\mathrm{O}(\log^4 q)$. Shallue-Woestijne method runs in time $\mathrm{O}(\log^4 q)$ for any field size $q = p^n$ and in time $\mathrm{O}(\log^3 q)$ when $q \equiv 3\,(mod\ 4)$. The maps were simplified and generalized to hyperelliptic curves of the forms $y^2 = x^n + ax + b$ and $y^2 = x^n + ax^2 + bx$ by Ulas in 2007 [18]. We recall these maps for elliptic curves in the following result.

Lemma 2.1. *Let $f(x) = x^3 + ax + b$ and*

$$\begin{aligned} X_1(t,u) &= u \\ X_2(t,u) &= \frac{-b}{a}\left(1 + \frac{1}{t^4 f(u)^2 + t^2 f(u)}\right) \\ X_3(t,u) &= t^2 f(u) X_2(t,u) \\ U(t,u) &= t^3 f(u)^2 f(X_2(t,u)). \end{aligned}$$

Then

$$U(t,u)^2 = f(X_1(t,u)).f(X_2(t,u)).f(X_3(t,u))$$

In 2009 Icart proposed another method for encoding to elliptic curves [13]. If $q \equiv 2\,(mod\ 3)$ the map $x \to x^3$ is a bijection in $\mathbb{F}_q$ and cube roots are uniquely defined with $x^{\frac{1}{3}} = x^{\frac{2q-1}{3}}$.Icart defined an encoding as follows:

$$f_{a,b}\colon \mathbb{F}_{p^n} \longrightarrow E_{a,b}$$

$$u \longrightarrow (x, y)\,,$$

where

$$x = \left(v^2 - b - \frac{u^6}{27}\right)^{\frac{1}{3}} + \frac{u^2}{3} \qquad y = ux + v \qquad v = \frac{3a - u^4}{6u}$$

He fixed $f_{a,b}(0) = O$, the neutral element of the elliptic curve. Icart proved that for all $p \in E_{a,b}$ the set $f_{a,b}^{-1}(p)$ is computable in polynomial time and

$\left|f_{a,b}^{-1}(p)\right| \leqslant 4$, namely a point has at most 4 preimages. He also proved that his algorithm works with complexity $O\left(\log^3 q\right)$ and conjectured that the image of $f_{a,b}$ contains $\frac{5}{8}.\#E_{a,b}(\mathbb{F}_q)+O(q^{\frac{1}{2}})$. Icart's conjecture was proved by Farashahi, Shparlinski and Voloch[9].

Brier et al [5] proposed a further simplification of the Shallue-Woestijne-Ulas algorithm for elliptic curves over finite field $\mathbb{F}_q$ with $q \equiv 3(mod\ 4)$. They showed every point $p=(x,y)$ has at most 8 preimages.

For cryptographic purposes it is important to have an injective encoding into an elliptic curve. In 2011 Farashahi [8] described an injective encoding to Hessian curves with a point of order 3 over $\mathbb{F}_q$ where $q \equiv 2(mod\ 3)$.
Fouque, Jeux and Tibouchi [10] proposed an injective encoding to elliptic curves of the form

$$E_c^{\delta} : y^2 = x^3 - 4\delta x^2 + \delta(c+\delta/c)^2 x,$$

where $c \in \mathbb{F}_q \backslash \{-1,0,1\}, \delta = \pm 1$.
Bernstein, Hamburg, Krosnova and Lange [3] proposed an injective encoding for elliptic curves of the form

$$E_{a,b} : y^2 = x(x^2+ax+b)$$

with $a,b \in \mathbb{F}_q$.

Foque and Tibouchi [11] proposed a deterministic encoding in to hyperelliptic curves of the form

$$y^2 = x^{2g+1} + a_1 x^{2g-1} + \cdots + a_g x,$$

where g is the genus of the curve.
We need to take some security considerations for choosing a hyperelliptic curves. In this context, we have two important sequences:
1. If g is large there exists a subexponential algorithm for solving the discrete logarithm problems in $J(\mathbb{F}_q)$.[1]
2. If g is small such that $g \geqslant 5$ the attack by gaudry can solve discrete logarithm problem in $J(\mathbb{F}_q)$.[12]
Therefore for cryptographic desires we must consider the hyperelliptic curves of genus 2,3,4.

3. Main result

In this section we first propose an algorithm for encoding to hyperelliptic curves of the form $y^2 = x^n + ax^{n-1} + bx$ over finite field $\mathbb{F}_q$. Then we show our proposed method defines an injective encding where $n=5$ (genus is 2) and q is a power of 3.

Lemma 3.1. *Let $g(x) = x^n + ax^{n-1} + bx$. If λ is a quadratic non-residue such that for some $x \in \mathbb{F}_q$ we have*

$$g(\lambda.x) = \lambda g(x) \tag{3.1}$$

then either x or $\lambda.x$ is the abscissa of a point on the $y^2 = g(x)$. Moreover for each λ the value

$$x = \frac{a(1-\lambda^{n-2})}{(\lambda^{n-1}-1)} \tag{3.2}$$

satisfies (3.1).

Proof. Since λ is not a quadratic residue, if x satisfies (3.1) then either $g(\lambda.x)$ or $g(x)$ must be a square in $\mathbb{F}_q$. Therefor either x or $\lambda.x$ must be abscissa of a point on the curve $y^2 = g(x)$. Moreover we have:

$$\begin{aligned} g(\lambda x) &= \lambda g(x) \\ (\lambda x)^n + a(\lambda x)^{n-1} + b(\lambda x) &= \lambda(x^n + ax^{n-1} + bx) \\ \lambda^{n-1}x^n + a\lambda^{n-2}x^{n-1} + bx &= x^n + ax^{n-1} + bx \\ \lambda^{n-1}x + a\lambda^{n-2} &= x + a \\ x &= \frac{a(1-\lambda^{n-2})}{(\lambda^{n-1}-1)}. \end{aligned}$$

□

Theorem 3.1. *Let $q \equiv 3(mod\ 4)$ and for any $t \in \mathbb{F}_q$*

$$\begin{aligned} X_1(t) &= \frac{a(1-(-t)^{2n-4})}{((-t)^{2n-2}-1)} \\ X_2(t) &= -t^2 X_1(t) \\ U(t) &= tg(X_1(t)) \end{aligned}$$

Then

$$(U(t))^2 = -g(X_1(t))g(X_2(t))$$

Proof. since $q \equiv 3(mod\ 4)$, -1 is a quadratic non-residue and we can take $\lambda = -t^2$ in previous lemma. Therefore $X_1(t) = x$ and $X_2(t) = \lambda x$ and we have:

$$\begin{aligned} g(X_1(t))g(X_2(t)) &= g(x)g(\lambda.x) = \lambda g(x)^2 = -t^2 g(x)^2 \\ &= -(tg(x))^2 = -(U(t))^2 \end{aligned}$$

□

Remark 3.1. Let $P = (X_P, Y_P)$ be a point generated by this method. We solve the equations $X_1(t) = X_P$ and $X_2(t) = X_P$ to compute the pre-images of P. Since $degX_1(t) = 2n-2$ and $degX_2(t) = 2n-2$ each equation has at most $2n-2$ solutions. The minus sign in the final step of the algorithm makes that set of points obtained of form $(X_1, g_1^{\frac{q+1}{4}})$ and set of points obtained of form $(X_2, -g_2^{\frac{q+1}{4}})$ are separated. Hence a point has at most $2n-2$ pre-images.

Algorithm 2 Encoding Algorithm

Require: : $\mathbb{F}_q$ such that $q \equiv 3(mod\ 4)$, parameters $a, t \in \mathbb{F}_q$.
Ensure: : $(x, y) \in H_{n,a,b}(\mathbb{F}_q)$ where $H_{n,a,b} : y^2 = x^n + ax^{n-1} + bx$.
If $t = 0$ then return $(0, 0)$
If $t = \pm 1$ then return O
$\lambda \longleftarrow -t^2$
$X_1 \longleftarrow \frac{a(1-\lambda^{n-2})}{(\lambda^{n-1}-1)}$
$X_2 \longleftarrow \lambda.X_1$
$g_1 = X_1^n + aX_1^{n-1} + bX_1; g_2 = X_2^n + aX_2^{n-1} + bX_2$
If g_1 is a square, return $(X_1, g_1^{\frac{q+1}{4}})$, otherwise return $(X_2, -g_2^{\frac{q+1}{4}})$

3.1. Injective encoding

In this section we express an injective encoding for hyperelliptic curves of the form $H_{2,a,b} : y^2 = x^5 + ax^4 + bx$. If we want to use our proposed algorithm for $n = 5$ we have:

$$X_1(\lambda) = \frac{a(1-\lambda^3)}{(\lambda^4-1)} \qquad X_2(\lambda) = \frac{a(\lambda-\lambda^4)}{(\lambda^4-1)}$$

for every quadratic non-residue λ.
If $X_1(\lambda_1) = X_1(\lambda_2)$ we have:

$$\lambda_1^4 - \lambda_1^4\lambda_2^3 + \lambda_2^3 - \lambda_2^4 + \lambda_1^3\lambda_2^4 - \lambda_1^3 = 0.$$

We divide the sides of this equation by $1 - \lambda_1$ and $1 - \lambda_2$ and $\lambda_1 - \lambda_2$. Then we have:

$$(\lambda_2^2 + \lambda_2 + 1)\lambda_1^2 + (\lambda_2^2 + \lambda_2)\lambda_1 + \lambda_2^2 = 0. \tag{3.3}$$

The discremnant of equation 3.3 is $\Delta_1 = \lambda_2^2(-3\lambda_2^2 - 2\lambda_2 - 3)$. Therefore if Δ_1 is a quadratic non-residue, this equation has no solution.
It also follows from $X_2(\lambda_1) = X_2(\lambda_2)$ that:

$$\lambda_1^4 - \lambda_2^4 + \lambda_1\lambda_2^4 - \lambda_1^4\lambda_2 - \lambda_1 + \lambda_2 = 0.$$

Similarly if we divide the sides of this equation by $1 - \lambda_1$ and $1 - \lambda_2$ and $\lambda_1 - \lambda_2$, we have:

$$\lambda_1^2 + \lambda_1(\lambda_2 + 1) + \lambda_2^2 + \lambda_2 + 1 = 0 \tag{3.4}$$

The discremnant of equation 3.4 is $\Delta_2 = -3\lambda_2^2 - 2\lambda_2 - 3$. Therefore if Δ_2 is a quadratic non-residue, this equation has no solution. By looking at equations $\Delta_1 = \lambda_2^2(-3\lambda_2^2 - 2\lambda_2 - 3)$ and $\Delta_2 = -3\lambda_2^2 - 2\lambda_2 - 3$, it can be concluded that they are quadratic non-residues if for any λ as quadratic non-residue the value $\Delta = -3\lambda^2 - 2\lambda - 3$ is a quadratic non-residue.

Definition 3.1. Let p be a prime number and $q = p^n$ for $n \in \mathbb{N}$. If $\{\beta_1, \ldots, \beta_{n-1}\}$ be a basis for $\mathbb{F}_q$ over $\mathbb{F}_p$, for every element $a \in \mathbb{F}_q$ we have:

$$a = a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1} \qquad a_i \in \mathbb{F}_p$$

We define set S which consists of half the field elements as follows:

$$S = S_0 \cup S_1 \cup \cdots \cup S_{n-1}$$

Such that

$$\begin{aligned}
S_0 &= \{(a_0, a_1, \ldots, a_n) : 0 < a_0 \leq \frac{p-1}{2}, \forall 1 \leq i \leq n \quad a_i = 0\} \\
S_1 &= \{(a_0, a_1, \ldots, a_n) : 0 < a_1 \leq \frac{p-1}{2}, \forall 2 \leq i \leq n \quad a_i = 0\} \\
&\vdots \\
S_{n-1} &= \{(a_0, a_1, \ldots, a_n) : 0 < a_n \leq \frac{p-1}{2}\}.
\end{aligned}$$

It is easy to see that S has cardinality $\frac{p^n-1}{2}$ and for each $x \in \mathbb{F}_q$ exactly one of x or $-x$ is in $\mathbb{F}_q$.

Corollary 3.1. *If we consider $H_{2,a,b}$ over finite fields of characteristic 3, the algorithm 2 defines an injective encoding from S into points $H_{2,a,b}$.*

Proof. Since $Char(\mathbb{F}_q) = 3$ we have $\Delta = \lambda$. Therefore the Δ value is always a quadratic non residue. Since each of λ comes by two values $\pm t$, every point (x, y) in the outpot of this algorithm has exactly 2 preimages in $\mathbb{F}_q$. Therefore for the elements of S we have an injective encoding. $\square$

Remark 3.2. We know that the set of points on $H_{n,a,b}$ is not a group. Therefore if for cryptographic purposes we need to be in a group, we can map $H_{n,a,b}$ to the jacobian J of $H_{n,a,b}$ which is an abelian group. If we use the jacobian of a hyperelliptic curve instead of an elliptic curve over a finite field $\mathbb{F}_q$ we can reduce key size by having the same level of security. In our case by using a hyperelliptic curve of genus 2 over a finite field $q \simeq 3^{80}$ we have the same level of security when we use an elliptic curve group where $q \simeq 3^{160}$.

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COMPARISON OF THE INFLUENCE OF DIFFERENT NORMALIZATION METHODS ON TWEET SENTIMENT ANALYSIS IN THE SERBIAN LANGUAGE

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Abstract. Given the growing need to quickly process texts and extract information from the data for various purposes, correct normalization that will contribute to better and faster processing is of great importance. The paper presents the comparison of different methods of short text (tweet) normalization. The comparison is illustrated by the example of text sentiment analysis. The results of an application of different normalizations are presented, taking into account time complexity and sentiment algorithm classification accuracy. It has been shown that using cutting to n-gram normalization, better or similar results are obtained compared to language-dependent normalizations. Including the time complexity, it is concluded that the application of this language-independent normalization gives optimal results in the classification of short informal texts.

1. Introduction

Normalization is an important step in text preparation for any type of machine processing. Normalization can be language-independent and language-dependent. Language-dependent normalization better preserves text properties and reduces the word to morphologically correct form. The problems of language-dependent normalization are unavailability and robustness of lexical resources and complexity of the normalization algorithms. Language-independent normalization reduces words to forms that do not necessarily have to be morphologically correct. On the other hand, specific lexical resources are not required for the use of language-independent normalizations. Cutting the word to the character n-grams of a certain length, as a way of language independent normalization, can have its advantages in particular text processing. This normalization is much faster than linguistic normalization and is preferred if it achieves satisfactory processing precision. This paper presents the effect of text normalization on the classification of short texts (tweets) based

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on the sentiment. Three types of normalization have been processed, two linguistic (stemming and lemmatization) and one language-independent (cutting words to character n-grams). The rest of the paper is organized as follows. The paper begins with a description of related work on different normalization methods for sentiment analysis in Section 2. Section 3. contains information about the dataset. Section 4. describes different normalization methods. Section 5. shows the results of the sentiment lexicon normalization. The experimental results and time complexity are included in Section 6. Finally, we conclude and present suggestions for future work in Section 7.

2. Related work

The sentiment analysis has been an ongoing topic of research recently. It seeks to determine the attitude expressed in the text. The sentiment can be analyzed at the level of the whole text, sentences or one aspect of the text [16]. The sentiment can be expressed discreetly (positive, negative and neutral) or on a scale from positive to negative. There are corpus-based and lexicon-based approaches to determine the polarity of the corresponding text [19]. The corpus-based approach (supervised approach) uses the methods of machine learning over a marked set of data. The lexicon-based approach (unsupervised approach) determines the polarity based on the sentiment lexicon. The sentiment lexicon contains words that can have discrete values (-1, 0, 1 or positive, neutral, negative) or values on a scale (eg. from -10 to +10). Sentiment lexicons with discrete values are Bing Liu's Opinion Lexicon [4] and MPQA Subjectivity Lexicon [21]. Sentiment lexicon containing the value of the polarity that has a specific scale is SentiWordNet [18]. When it comes to methods that determine the sentiment, most researchers use supervised learning methods [3], although a considerable number of approaches provide analysis by methods of unsupervised lexicon-based [19] and [15] or combined semi-supervised learning [2]. Although a machine-based approach gives better classification, a lexicon-based approach takes precedence in situations where a set of marked data is not available, and when a classifier training time is crucial. In the sentiment analysis, there are challenges such as the treatment of phenomena of negation, sarcasm, irony, and others. The sentiment analysis is closely related to the language. A sentiment analysis in the Serbian language was made for a set of newspaper articles [14], film reviews [1] and a set of tweets [8]. Normalization of text is a part of every kind of text processing and sentiment analysis. The effect of normalization on various problems of text processing is different. The normalization results vary depending on the type of text to which they are applied and the language in which the text is written. There are a small number of papers concerning the normalization of short texts in the Serbian language and its related languages (Bosnian, Croatian and Montenegrin). Linguistic normalization of the texts in the Serbian language was performed by D. Vitas et al. [20], who described tools and resources for the processing of texts in the Serbian language. In addition to linguistic normalization, normalization by stemming can be done using stemmers. The authors of the paper [1] dealt with the impact of morphological, stemming and word embedding normalization on the sentiment classification of text in the Serbian language. They used a movie review corpus and found that stemmer gives better results compared to lemmatizer and that adding bag-of-words attributes increases the accuracy of classification methods. The application of the word embedding method (which requires lemmatization) and the string kernel method (which does not require any normalization) in the sentiment analysis of informal short texts in the Croatian language is shown by L. Rotim et al. [17]. Their results show that word embedding outperforms string kernels, which in turn outperform word and n-gram bag-of-words baselines. Alternative methods of normalization are known, such as cutting off the same length and n-gram analysis [11].

3. Dataset

The increasing use of social networks and the text availability make them popular for research. In this paper, the experimental dataset consists of tweets in the Serbian language. Tweets are short, informal texts that contain a lot of incorrectly written words, use of slang, irony, and sarcasm. If we take all these into consideration, we can conclude that normalization and classification of such informally written texts is a very demanding task. Liu et al., [7] created special systems for normalizing such texts. Tweets were collected using the Twitter Streaming API in the period from 30 November 2016 to 30 June 2017. The dataset was manually labeled by three people, two men, and one woman. Background of annotators are the following: a doctor of medicine, an electrical engineer and a student of the Serbian language and literature. In case of disagreement on tweet marking between any two or all three annotators, the tweet is thrown out of the set. The final dataset consists of 7663 tweets, 4193 of which are marked negative, 2625 neutral and 818 positives.

4. Normalization

Normalization is considered the process consisting of two phases. In the first phase, a tokenization that is linguistically independent and specific to the type of data is performed. The tokenizer deals with words that appear in tweets but do not carry the meaning (such as retweet, via, etc.), spaces in the text, numbers, dates, and punctuation characters in such a way that output tokens are only those that affect the meaning of the text. The second phase of text normalization is partially or completely linguistically dependent. In this paper, it involves removing the stop words specific for the Serbian language and reducing different forms of the words to their base. Reduction of the number of different types of words appearing in the dataset is done in three ways: stemming (ST), normalization by using morphological lexicon (MN) and cutting to the character of 4-grams, 5-grams, 6-grams and 7-grams (4G, 5G, 6G, 7G).dgsd

4.1. Stemming

Stemming belongs partly to linguistic, and partly to the heuristic approach of reducing different word forms to their root. Stemming removes the word suffixes by cutting them to the root of the word. There are publicly available stemming algorithms in the Serbian language. The paper used a stemming algorithm for the Serbian language described by N. Miloević [13]. This stemmer consists of a list of irregular verbs: moći/can, hteti/want, jesam/I am, and biti/to be. For each form of these verbs (a total of 68 infections), the word is reduced to its morphological root. Stemmer also contains a list of 289 suffixes and substitutions that are added if the suffix is taken from the word. Stemmer modifies the first words containing letters with diacritic characters, so the letters with a diacritic sign are replaced with two letters: "š" is modified to "sx", "č" to "cx", "ć" to "cy", "đ" to "dx" and "ž" to "zx". Further, if the word belongs to the list of one of the 4 irregular verbs, then it is shifted to the root; otherwise, the longest suffix contained in the word is found, and the word is modified according to the rule for that suffix. If the word does not contain any of the suffixes, then the stemmer returns the original word. This stemmer is set to stem words longer than 4 characters and those with more than 3 characters after the suffix is taken away. Stemmers created by Kešelj and Šipka [5] are also based on rules with suffixes. The algorithm for stemming of Ljubešić [10] is rule-based and achieves F1 97.64

4.2. Normalization with the morphological lexicon

Lemmatization is the process of reducing different forms of a single word to its linguistic root - lemma. Different forms of one and the same word occur when this word appears in different grammatical cases, grammatical gender, grammatical number, and grammatical tense or grammatical person. Lemmatization is used in the morphological analysis of the text; the morphological lexicon is used for the process of word reduction. The morphological lexicon contains all the forms of the word and word lemma. The lemma is derived from the form of words and other labels needed to uniquely map to its corresponding lemma. The additional tags of the word include the aforementioned information about the grammatical case, grammatical gender, grammatical number, grammatical tense or grammatical person and other characteristics - depending on the type of the word that is reduced to the lemma. For the application of lemmatization, a corpus with labeled word POS tag is required. Due to the absence of such corpus, normalization is applied in this paper by using the morphological lexicon that takes the first lemma for the corresponding word form, without taking into account the characteristics of the word. This normalization has defects in relation to real lemmatization. Since lemmatization is performed by using a large number of rules (each word is considered separately), it is accordingly more complex and time-consuming in comparison to stemming. The morphological lexicon of Krstev et al. is used for lemmatization [6]. It consists of 3,630,613 entries for 85,721 lemmas covering 11 PoS: 646

867 nouns, 2 315 640 adjectives, 654 159 verbs, 3233 adverbs, 4 794 numerals, 83 conjunctions, 218 interjections, 169 prepositions, 5 321 pronouns, 103 particles and 26 abbreviations.

4.3. Cutting to n-gram

An alternative way of normalization, which does not require any lexical resources is normalization by cutting the word into the first n characters. For n, the values of 4, 5, 6, and 7 are taken. This way of text normalization certainly results in information loss, and it may happen that different words with different sentiments are reduced to the same n-gram (ambicija/ambition, besplatno/free (positive words) and ambis/ambis, bespomoćno/helpless (negative words) are reduced to the same 4-grams - ambi, besp). However, the advantages of reduction of a large number of word forms with the same first n characters may bring greater benefits in comparison to losses (e.g.ljubav/love, ljubavni/love, ljubavisati/love, ljubavi/love are reduced to the same 4 grams ljuba/love). The gains and losses obtained by this normalization were experimentally shown (in Section 5.). In the example of the Twitter sentiment classification, this normalization was experimentally shown to positively affect the accuracy of classification.

5. Normalized sentiment lexicons

The sentiment lexicons contain words that are marked positive or negative. They serve in the sentiment analysis in the lexicon-based method. The sentiment lexicon used in this paper as the starting lexicon consisted of 5632 words (reduced to the morphological root), 4058 of which were negative and 1574 positive words [12]. The three described normalizations were used, and three resulting lexicons were obtained and used in a dataset, normalized by one of the three normalizations (stemming, lemmatization or cutting to n-grams).

5.1. Normalization of sentiment lexicon

The application of normalization to sentiment lexicons affects the total number of words in the lexicon as well as their quality. By using linguistic dependent normalizers (stemmer and lemmatizer), the words with the same or similar meaning are reduced to their common root. Using n-gram analysis, words from a lexicon are cut into n-grams, without taking into account the meaning of the word. Due to the characteristics of the corresponding normalizers, the resulting lexicons have a significantly different number of words [11]. The small numbers of words with different sentiments are transformed to the same root, due to which they become contradictory. Normalizations based on language rules produce a lower number of such words. Contradictory data will be excluded from the sentiment lexicons. Table 5.1 shows the results after normalization of words in lexicons and removal of contradictory words. The number of words in the lexicons is displayed as well as the total number of different roots obtained after normalization. For better comparison, the results are presented in cases when no normalization is applied (NN) and the results of the application of different normalizations: stemming (ST), morphological

Table 5.1: Number of words in normalized lexicons after the removal of contradictory words and number of different roots to which they are reduced by normalizations

Type of normalization	NN	ST	NM	4G	5G	6G	7G
Number of different words	5632	5596	5632	4139	5116	5481	5576
Number of different bases	5632	5218	5632	2271	3506	4283	4803

vocabulary normalization (NM), 4-grams (4G), 5-grams (5G), 6-grams (6G) and 7-grams (7G).

The number of different words in nominalized lexicons decreases due to the exclusion of contradictory words, which is best expressed in normalization by cutting to 4-grams. Stemming is also part of the word ejected. The number of different normalized roots to which the words from the basic lexicon are reduced is the smallest in cutting to 4 grams, which increases with the length n. The number of occurring sentiment words in the data set is calculated for lexicons. The total number of sentiment words is: for stems 10777; for lemmas 12920; for 4-grams 22466; for 5-grams 15284; for 6-grams 10697; for 7-grams 7874. The distribution of the number of occurrences of the sentiment in the corpus of polarity is shown in Figure 5.1. The number of occurrences of terms from normalized lexicons in tweets from the corpus is the largest for 4-grams and 5-grams and for the lemmas. This distribution indicates that words cut to 4-grams are best mapped in tweets, which is expected due to the number of different word forms, beginning with the same 4-gram. What is visible from the chart and Table 5.1 is that the effect of normalization is reduced with the increase of n length, so that by cutting to 7-grams, we obtain those that are inclined towards results without normalization.

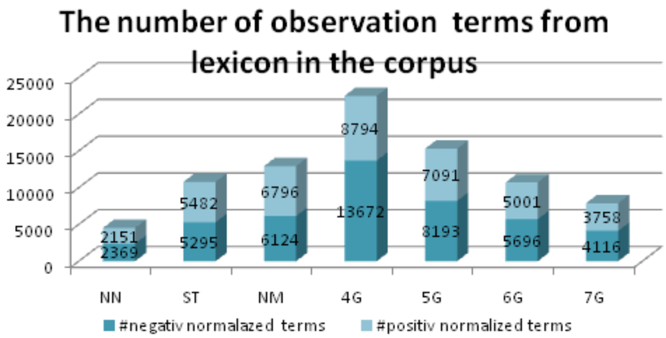

Fig. 5.1: The number of observation terms from lexicon in the corpus

In the next chapter, the quality of collected lexicon is verified by examining whether these words from normalized lexicon appear in tweets with the corresponding polarity, and how such normalized lexicon affects the classification by sentiment.

6. Results and discussion

The results of normalization are presented in two directions. In the first direction, the analysis of sentiment lexicons is done in the way that provides validation over the marked set of tweets, so the appearance of specific words in tweets by class is calculated. The second direction of results is based on the sentiment analysis of a normalized dataset using normalized sentiment lexicons.

6.1. Validation of the sentiment lexicon over a tweet dataset

The validity of lexicon is based on a set of tweets that are normalized by corresponding normalization. For each normalized root of sentiment word (stem, lemma or n-gram), the calculation (the number of occurrences of that word in tweets with the positive and negative sentiment) is done. If the sentiment word occurs within the negation scope, its appearance is counted as having appeared in a tweet with the opposite polarity. Normalization of the score is done by dividing the number of occurrences with the number of tweets from that class (the data set is unbalanced). The score is calculated by the formula (6.1) with n as a number of affirmative occurrences in positive tweets, and nn number of occurrences with negation in positive tweets; m is a number of affirmative occurrences in negative tweets, and nm is the number of occurrences with negation in negative tweets.

$$score = (n - nn)/num\_positive\_tw - (m - nm)/num\_negativ\_tw. \tag{6.1}$$

By classifying the sentiment words based on whether they appear more in positive or in negative tweets, the effect of normalization on sentiment analysis is tested. The sentiment is assigned to the word in the following way:
positive- if they appear more in positive than in negative tweets, (score>0).
negative- if they appear more in negative than in positive tweets, (score<0).
Sentiment word need not be classified (when score=0) for two reasons. The first reason is that sentiment word does not appear in the corpus, and second is that sentiment word appears equally in positive and negative tweets; this latter case is rare. Table 6.1 shows the classification results of positive sentiment words, negative sentiment words and all sentiment words from lexicons when different types of normalization are applied. Formula (6.2) presents precision (Pre), as a the number of correctly classified sentiment words($num\_corectly\_classified$) divided by the total number of sentiment words classified as belonging to the corresponding sentiment class ($total\_num\_classified$). Formula (6.3) presents recall (Rcall) as the number of correctly classified sentiment word($num\_corectly\_classified$) divided by the total number of sentiment words that actually belong to the corresponding sentiment class($total\_num$). Measure F1 uses a combination of Precision and Recall presented in formula (6.4), giving more relevant results with an unbalanced dataset.

$$Pre = num\_corectly\_classified/total\_num\_classified \tag{6.2}$$

Table 6.1: The obtained precision and recall for different types of used normalizations. The corresponding maximums are labeled red.

		NN	**ST**	**NM**	**4G**	**5G**	**6G**	**7G**
ACC negative words	Pre	82%	81%	80%	84%	82%	81%	80%
	Rcall	16%	27%	25%	60%	45%	35%	28%
	F1	26%	41%	38%	70%	58%	49%	41%
ACC positive words	Pre	79%	67%	66%	52%	60%	64%	69%
	Rcall	13%	24%	19%	47%	37%	28%	22%
	F1	22%	35%	30%	49%	46%	39%	34%
ACC all words	Pre	81%	77%	77%	75%	76%	76%	77%
	Rcall	15%	26%	23%	57%	43%	33%	26%
	F1	25%	39%	36%	65%	55%	46%	39%

$$Rcall = num\_corectly\_classified/total\_num \tag{6.3}$$

$$F1 = 2 * Pre * Rcall/(Pre + Rcall) \tag{6.4}$$

From the obtained results, it can be concluded that n-grams (n < 7) are well classified by sentiment (the classification has the best F1 score). The reason is that a larger number of n-grams were found in the set of tweets compared to stemms and lemmas. Being informal texts, tweets often contain misspelled words that are rarer at length up to 6 letters. On the other hand, the Serbian language, being morphologically rich is difficult to process, and a large number of words are found in forms that are not adequately processed by stemmer and lemmatizer, hence such sentimental words cannot be found in the sentiment lexicon. Testing the improvement of classification of sentiment words by cutting them to n-grams versus stemmer and lemmatizer is done using Mc Nemar's test. We made a correlation matrix for classification by using n-gram analysis and lemmatization and 4-gram analysis and stemming. In both cases, the value of p <0.0001 was found, i.e. cutting on n-grams had statistically significant influence on the improvement of sentiment word classification.

6.2. Application of normalized lexicon to tweet sentiment analysis

The influence of the three normalization methods was tested on Twitter sentiment analysis. Normalized lexicons and normalized data set were used to determine the sentiment in two experiments. The first experiment classified tweets based only on the words found in the sentiment lexicon. In another experiment, the methodology for learning Multinomial Logistic Regression (MLR) was used for classification by sentiment. Figure 6.1 shows the system architecture from collecting data, through normalization to sentiment analysis.

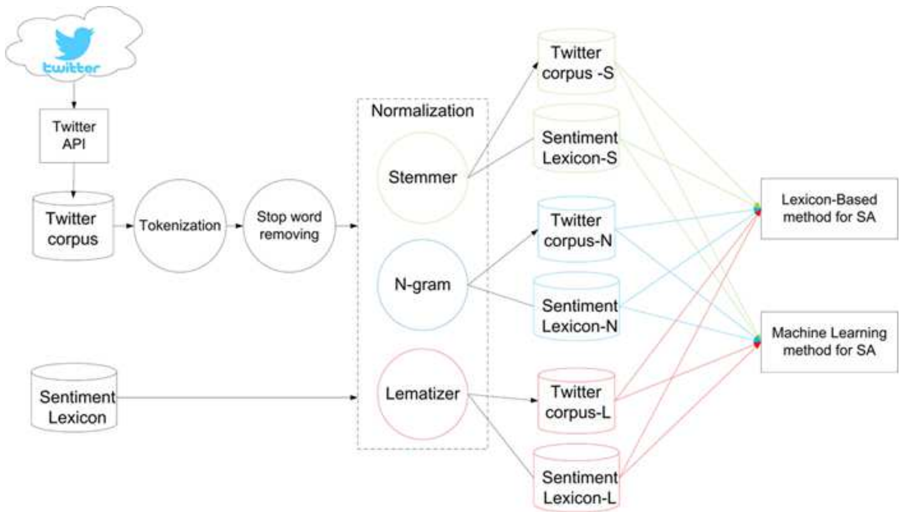

FIG. 6.1: System architecture from collecting tweets, through the normalization process to the sentiment analysis

In the first experiment, the quality of the prediction for three normalization methods was performed by a lexicon-based method. The advantage of this classification method is that it does not require training and is independent of the dataset. Although the results are worse than in the machine learning approach, this algorithm gives us a better insight into the impact of different normalizations of the sentiment analysis. The sentiment is calculated according to formula 6.5. The sum of numbers of positive sentiment terms and negative sentiment terms within the negation scope that appear in tweets is given in the sumPos attribute. The sum of numbers of negative sentiment terms and positive sentiment terms in the negation scope from that appear in tweets is given in the sumNeg attribute [9].

$$D_{it} = \begin{cases} \textit{positive} & \text{if sumPos>sumNeg} \\ \textit{neutral} & \text{if sumPos=sumNeg} \\ \textit{negative} & \text{if sumPos<sumNeg} \end{cases} \tag{6.5}$$

The results obtained by this method are given in Table 6.2. The table contains results when no normalization, stemming, morphologic dictionary or cutting to n-grams (n = 4,5,6 and 7) are applied. The results of the 3-class (3K) classification of

Table 6.2: Correctly Classified tweets using Lexicon-Based method depending on normalization

Normalization	NN	ST	NM	4G	5G	6G	7G
Lexicon-Based 3K	47.09%	50.51%	48.98%	49.25%	49.10%	50.13%	49.15%
Lexicon-Based 2K	33.45%	52.17%	50.73%	59.69%	53.54%	49.51%	43.76%

Table 6.3: Correctly Classified tweets using machine lerning method depending on normalization

Normalization	NN	ST	NM	4G	5G	6G	7G
MLR-3K	59.86%	64.17%	62.45%	59.23%	60.61%	62.49%	61.80%
MLR-2K	84.71%	85.27%	84.25%	83.96%	84.45%	84.47%	84.71%

tweets in positive, neutral and negative are presented, as well as the classification for 2-class (2K) positive and negative. If we only look at the classification of positive and negative tweets, we get 4-gram normalization giving the best accuracy. However, neutral tweets distort the classification quite a lot, so when classifying a group of tweets with three classes, normalization using stemmer gives the best results. In this case, cutting to n-gram finds a large number of n-grams with the sentiment in tweets, even in neutral, which classifies them into positive or negative. Neutral tweets often carry a part of the sentiment that is not clearly defined, and this can be solved only by introducing the classification of tweets into several sentiment groups. On the other hand, by cutting the word on n-grams, a set of sentiment words are lost if, after normalization, they are infiltrated into a group of contradictory ones. The omission of these sentiment words distorts the sentiment analysis classification. The classification quality is shown using a percentage of accurately classified tweets using formula (6.6).

$$Correct\_classif = num\_corect\_classif\_tweets/total\_num\_of\_tweets \tag{6.6}$$

In the second experiment, supervised machine learning was performed using MLR in 10-fold cross-validation (Table 6.3). The attributes used for this method are the following: sumNeg, sumPoz, the number of words in negation scope, the number of words in the tweet. Here we see a significant increase in the results obtained for all three normalization methods, where stemming achieved the best result. The normalization by cutting to the n-grams, in this case, is the best for 6-grams and is more accurate than the normalization using the morphological lexicon.

6.3. Complexity of algorithms

Large amounts of data available for processing require techniques to quickly achieve results. In order to measure the complexity of sentiment analysis algorithms, the time complexity of determining the sentiment for one tweet has been shown, by using all three types of normalization. Differences in complexity of the sentiment algorithm in different modes of normalization are reflected in the size of lexical resources or the number of rules, used to normalize tweet and determine sentiment by formula (6.7). The first part of normalization does not depend on the way words are reduced and it will not be considered. Only the part that is specific for each normalization is considered. Sentiment analysis algorithm does not directly depend on normalization, but indirectly through the size of sentiment lexicon which is obtained by normalization. How the complexity of sentiment analysis algorithm depends on normalization is presented through the complexity of sentiment lexicon used (6.8).

$$iT = tweet\_normalization + determ\_senti \tag{6.7}$$

$$determ\_senti = num\_of\_words\_in\_tweet * num\_of\_words\_in\_senti\_lex \tag{6.8}$$

1. The normalization with a stemmer, as a linguistic dependent normalization, depends on the number of rules used in stemming and the size of the stemmed sentiment lexicon. Based on previously presented values, it is obtained as described in detail in Subsection 5.1. and Subsection 4.1. that:
tweet_normalization = number_of_words_in_tweet * number_od_rules_in_stemmer= m * (68 + 289)
determining_sentiment = m * 5218
iT = m * 5575

2. The use of morphological lexicon is the most expensive process due to robust lexical resources it uses. The size of morphological lexicon determines the complexity of normalization. For the normalization of morphological lexicon based on formulas (6.7) and (6.8), the following complexity is obtained:
tweet_normalization = number_of_words_in_tweet *number_od_rules_in_lemmatizer= m * 3,630,613
determining_sentiment = m * 5632
iT = m * 3636245

3. Cutting to n-grams requires the fewest resources, as shown in Table 6.4. The complexity of normalization is reduced to the number of words in the tweet. The sentiment lexicon normalized by cutting into n-grams is also smaller than the stemmed and lemmatized lexicon. Depending on length n, the complexity by cutting to n-grams is:
tweet_normalization = number_of_words_in_tweet * number_of_ngram_rules= m * 1

The obtained results indicate that normalization by cutting to n-grams is the least required and the fastest normalization algorithm and gives better results than lemmatization. If we compare it with stemming, the results are also satisfactory,

Table 6.4: Complexity of sentiment analyses in case of cutting to n-gram normalization

n	determining_sentiment	iT
4	m*2271	m*2272
5	m*3506	m*3507
6	m*4283	m*4284
7	m*4803	m*4804

since the classification into two classes of sentiments is always satisfactory. The problem occurs more due to the nature of sentiment analysis, i.e. unclearly defined type of neutral tweets that mainly carry both sentiments words by nature.

7. Conclusion

Cutting to n-grams maps a great number of words in tweets, so the number of accurately classified tweets is large. The problem arises with words that are thrown out of the sentiment lexicon because they are reduced to words with the opposite sentiment, therefore they do not participate in the sentiment analysis. Another problem is that neutral tweets contain the sentiment word that makes them difficult for classification. The results show that cutting off sentiment words into n-grams gives good results in classifying sentiment words in tweets, especially due to the informal form of tweet writing. Taking into account the accuracy of classification, the minimum of lexical resources, and the simplicity of application, cutting to n-grams is a method that has the advantage over linguistic dependent normalization in the Twitter sentiment analysis. In linguistic dependent normalizers, the use of stemmers takes precedence over the normalization with the morphological lexicon, both due to low complexity of the algorithm and the best result in the tweet sentiment classification in the 3-class dataset. In order to improve results, the sentiment analysis algorithm itself should be improved. Improving the result is possible using domain sentiment lexicon with sentiments that are also validated on the appropriate corpus. The introduction of several degrees in the sentiment analysis would significantly solve the problem of neutral tweet classification by sentiment.

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NEW TYPE INEQUALITIES FOR $\mathbb{B}^{-1}$-CONVEX FUNCTIONS INVOLVING HADAMARD FRACTIONAL INTEGRAL

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Abstract. Abstract convexity is an important area of mathematics in recent years and it has very significant applications areas like inequality theory. The Hermite-Hadamard Inequality is one of these applications. In this article, we studied Hermite-Hadamard Inequalities for $\mathbb{B}^{-1}$-convex functions via Hadamard fractional integral.
Keywords: Hermite-Hadamard Inequality; Hadamard fractional integral; convexity.

1. Introduction

Abstract convexity has an important area in convexity theory and it becomes different convexity types. These abstract convexity types have significant applications variety fields like mathematical economy, operation research, inequality theory and optimization theory. Additionally, $\mathbb{B}^{-1}$-convexity is one of these abstract convexity types ([1, 4]). It has applications to mathematical economy and inequality theory ([4, 19]). There are many articles about $\mathbb{B}^{-1}$-convexity ([2, 3, 10, 16, 22]).

Hermite-Hadamard inequalities can be given as an application on inequality theory for abstract convexity types ([6, 7, 8, 9, 13, 14, 17, 20]). We proved this inequality for $\mathbb{B}^{-1}$-convex functions ([19]). Hermite-Hadamard inequality is an integral inequality and it has be given with classic integral operator up to now. But, recently, the studies are made with fractional integral operators that are more general([5, 11, 12, 15, 18, 21]). Therefore, we study Hermite-Hadamard inequalities involving Hadamard fractional integral operator for $\mathbb{B}^{-1}$-convexity.

The outline of paper as follows: Second section is given on two parts that are required definitions and theorems of $\mathbb{B}^{-1}$-convexity and Hermite-Hadamard inequality for $\mathbb{B}^{-1}$-convex functions, respectively. In third section, we give Hermite-Hadamard type inequalities involving Hadamard fractional integral.

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2. Preliminaries

In this section, we give some required definition and theorems.

Lets recall the following definition of fractional integral type. Along the paper, let $f : [a, b] \to \mathbb{R}$ be a given function, where $0 \leq a < b < +\infty$ and $f \in L_1[a, b]$. Also, $\Gamma(\alpha)$ is the Gamma function.

Definition 2.1. [11] The left-sided Hadamard fractional integral $\mathbf{J}_{a+}^{\alpha}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \qquad x > a \tag{2.1}$$

provided that the integral exists. The right-sided Hadamard fractional integral $\mathbf{J}_{b-}^{\alpha}$ of order $\alpha > 0$ of f is defined by

$$\mathbf{J}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \qquad x < b \tag{2.2}$$

provided that the integral exists.

2.1. $\mathbb{B}^{-1}$-convexity

For $r \in \mathbb{Z}^-$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$ to itself; $\boldsymbol{x} = (x_1, x_2, ..., x_n) \to \Phi_r(\boldsymbol{x}) = (\varphi_r(x_1), \varphi_r(x_2), ..., \varphi_r(x_n))$ is homeomorphism from $\mathbb{R}_*^n$ to itself.

For a finite nonempty set $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)}\right\} \subset \mathbb{R}_*^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{i=1}^m t_i \Phi_r(\boldsymbol{x}^{(i)})\right) : t_i \geq 0, \sum_{i=1}^m t_i = 1\right\}.$$

We denote by $\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)} \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}} \boldsymbol{x}^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m)}\right\}, ..., \min\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $\boldsymbol{x}^{(i)}$.

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.2. [1] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r \in \mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of $\mathbb{R}_*^n$, is called $\mathbb{B}^{-1}$-polytope of A.

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}^n_{++}$.

Theorem 2.1. [1] For all nonempty finite subsets $A = \left\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, ..., \boldsymbol{x}^{(m)}\right\} \subset \mathbb{R}^n_{++}$ we have

$$Co^{-\infty}(A) = \lim_{r\to-\infty} Co^r(A) = \left\{ \overset{m}{\underset{i=1}{\wedge}} t_i \boldsymbol{x}^{(i)} : t_i \geq 1, \min_{1\leq i\leq m} t_i = 1 \right\} .$$

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.3. [1] A subset U of $\mathbb{R}^n_*$ is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.

By Theorem 2.1, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$:

Theorem 2.2. [1] A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if for all $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in U$ and all $\lambda \in [1, \infty)$ one has $\lambda \boldsymbol{x}^{(1)} \wedge \boldsymbol{x}^{(2)} \in U$.

Remark 2.1. As a result of Theorem 2.2, we can say that $\mathbb{B}^{-1}$-convex sets in $\mathbb{R}_{++}$ are positive intervals.

Definition 2.4. [10] For $U \subset \mathbb{R}^n_*$, a function $f : U \to \mathbb{R}_*$ is called a $\mathbb{B}^{-1}$-convex function if $epi^*(f) = \{(\boldsymbol{x}, \mu) \,|\boldsymbol{x} \in U, \mu \in \mathbb{R}_*, \mu \geq f(\boldsymbol{x})\}$ is a $\mathbb{B}^{-1}$-convex set.

In $\mathbb{R}^n_{++}$, we can give the following fundamental theorem which provides a sufficient and necessary condition for $\mathbb{B}^{-1}$-convex functions [10].

Theorem 2.3. Let $U \subset \mathbb{R}^n_{++}$ and $f : U \to \mathbb{R}_{++}$. The function f is $\mathbb{B}^{-1}$-convex if and only if the set U is $\mathbb{B}^{-1}$-convex and one has the inequality

$$f(\lambda \boldsymbol{x} \wedge \boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) \wedge f(\boldsymbol{y}) \tag{2.3}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in U$ and all $\lambda \in [1, +\infty)$.

2.2. Hermite-Hadamard Inequality for $\mathbb{B}^{-1}$-convex Functions

We proved the following theorem that gives the Hermite-Hadamard inequality involving classic integral for $\mathbb{B}^{-1}$-convex functions in [19].

Theorem 2.4. Suppose $f : [a, b] \subset \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is a $\mathbb{B}^{-1}$-convex function. Then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \begin{cases} \frac{f(a)(a+b)}{2a}, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{2bf(a)f(b)-a\left[(f(a))^2+(f(b))^2\right]}{2(b-a)f(a)}, & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases} . \tag{2.4}$$

3. Hermite-Hadamard Type Inequalities Involving Hadamard Fractional Integral

Hadamard fractional integral is one of the important fractional integral types. So, we introduce Hermite-Hadamard type inequalities including Hadamard fractional integral in this section.

Theorem 3.1. *Let $\alpha > 0$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then*

$$\boldsymbol{J}_{a+}^{\alpha} f(b) \leq \begin{cases} \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \left(\ln \frac{b}{\lambda a}\right)^{\alpha-1} d\lambda, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \left(\ln \frac{b}{\lambda a}\right)^{\alpha-1} d\lambda + \frac{f(b)\left(\ln \frac{bf(a)}{af(b)}\right)^{\alpha}}{\Gamma(\alpha+1)}, & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}. \tag{3.1}$$

Proof. The inequality (2.3) is satisfy for f, because of its $\mathbb{B}^{-1}$-convexity of f. Lets' multiply both sides of this inequality by $\frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b-\ln \lambda a]^{1-\alpha}}$ then integrate with respect to λ over $[1,+\infty)$. For the left sided of inequality, we have that

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda + \int_{\frac{b}{a}}^{+\infty} \frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} f(\min\{\lambda a,b\})\, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{1}{\lambda\left(\ln \frac{b}{\lambda a}\right)^{1-\alpha}} f(\lambda a)\, d\lambda \\
&= \int_a^b \left[\ln \frac{b}{t}\right]^{\alpha-1} \frac{f(t)}{t} dt = \Gamma(\alpha)\, \mathbf{J}_{a+}^{\alpha} f(b).
\end{aligned}$$

For the right sided of the inequality, we have to examine two cases of $\frac{f(b)}{f(a)}$. One of these cases is $\frac{b}{a} \leq \frac{f(b)}{f(a)}$ and in this situation the equation is

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda + \\
&+ \int_{\frac{b}{a}}^{+\infty} \frac{(\ln(\min\{\lambda a,b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\}\, d\lambda \\
&= f(a) \int_1^{\frac{b}{a}} \left(\ln \frac{b}{\lambda a}\right)^{\alpha-1} d\lambda.
\end{aligned}$$

Thus, with these calculations the first part of requested inequality is

$$\mathbf{J}_{a^+}^{\alpha} f(b) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} \left(\ln \frac{b}{\lambda a}\right)^{\alpha-1} d\lambda \,. \tag{3.2}$$

The other case is $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$, thence we obtain the followings:

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\} \, d\lambda \\
&= \int_1^{\frac{f(b)}{f(a)}} \frac{1}{\lambda\left(\ln \frac{b}{\lambda a}\right)^{1-\alpha}} \lambda f(a) \, d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{1}{\lambda\left(\ln \frac{b}{\lambda a}\right)^{1-\alpha}} \lambda f(a) \, d\lambda \\
&= f(a) \int_1^{\frac{f(b)}{f(a)}} \left(\ln \frac{b}{\lambda a}\right)^{\alpha-1} d\lambda + \frac{f(b)\left(\ln \frac{bf(a)}{af(b)}\right)^{\alpha}}{\alpha}.
\end{aligned}$$

Hence, we attain that

$$\mathbf{J}_{a^+}^{\alpha} f(b) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} \left(\ln \frac{b}{\lambda a}\right)^{\alpha-1} d\lambda + \frac{f(b)\left(\ln \frac{bf(a)}{af(b)}\right)^{\alpha}}{\Gamma(\alpha+1)} \,. \tag{3.3}$$

Finally, the inequality (3.1) can be get from (3.2) and (3.3). $\square$

Theorem 3.2. *Let $\alpha > 0$. If f is a $\mathbb{B}^{-1}$-convex function on $[a,b]$, then*

(3.4)

$$\boldsymbol{J}_{b^-}^{\alpha} f(a) \leq \begin{cases} \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} (\ln\lambda)^{\alpha-1} d\lambda, & \frac{b}{a} \leq \frac{f(b)}{f(a)} \\ \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} (\ln\lambda)^{\alpha-1} d\lambda + \frac{f(b)}{\Gamma(\alpha+1)} \left[\left(\ln\frac{b}{a}\right)^{\alpha} - \left(\ln\frac{f(b)}{f(a)}\right)^{\alpha}\right], & 1 \leq \frac{f(b)}{f(a)} < \frac{b}{a} \end{cases}.$$

Proof. Let the function f be $\mathbb{B}^{-1}$-convex. Thus, the inequality (2.3) holds. If we multiply by $\frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a,b\})-\ln a]^{1-\alpha}}$ and integrate with respect to λ over $[1,+\infty)$ to this inequality, then we have

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} f(\min\{\lambda a, b\}) \, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} f(\min\{\lambda a, b\}) \, d\lambda + \\
&+ \int_{\frac{b}{a}}^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} f(\min\{\lambda a, b\}) \, d\lambda \\
&= \int_a^b \left(\ln\frac{t}{a}\right)^{\alpha-1} \frac{f(t)}{t} dt = \Gamma(\alpha) \mathbf{J}_{b^-}^{\alpha} f(a) \,.
\end{aligned}$$

There are two following situations that have to been examined for the right sided of inequality. The first is $\frac{b}{a} \leq \frac{f(b)}{f(a)}$. Hence,

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} \min\{\lambda f(a), f(b)\} \, d\lambda \\
&= \int_1^{\frac{b}{a}} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} \min\{\lambda f(a), f(b)\} \, d\lambda + \\
&+ \int_{\frac{b}{a}}^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln(\min\{\lambda a, b\}) - \ln a]^{1-\alpha}} \min\{\lambda f(a), f(b)\} \, d\lambda \\
&= f(a) \int_1^{\frac{b}{a}} (\ln \lambda)^{\alpha-1} \, d\lambda.
\end{aligned}$$

Therefore, the inequality is obtained

$$\mathbf{J}_{b^-}^{\alpha} f(a) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{b}{a}} (\ln \lambda)^{\alpha-1} \, d\lambda \ . \tag{3.5}$$

The second case is $1 \leq \frac{f(b)}{f(a)} < \frac{b}{a}$. At this stage, we get that

$$\begin{aligned}
&\int_1^{+\infty} \frac{(\ln(\min\{\lambda a, b\}))'}{[\ln b - \ln \lambda a]^{1-\alpha}} \min\{\lambda f(a), f(b)\} \, d\lambda \\
&= \int_1^{\frac{f(b)}{f(a)}} \frac{1}{(\ln \lambda)^{1-\alpha}} f(a) \, d\lambda + \int_{\frac{f(b)}{f(a)}}^{\frac{b}{a}} \frac{1}{\lambda (\ln \lambda)^{1-\alpha}} f(b) \, d\lambda \\
&= f(a) \int_1^{\frac{f(b)}{f(a)}} (\ln \lambda)^{\alpha-1} \, d\lambda + \frac{f(b)}{\alpha} \left[\left(\ln \frac{b}{a} \right)^{\alpha} - \left(\ln \frac{f(b)}{f(a)} \right)^{\alpha} \right].
\end{aligned}$$

So, the inequality is in that form:

$$\mathbf{J}_{b^-}^{\alpha} f(a) \leq \frac{f(a)}{\Gamma(\alpha)} \int_1^{\frac{f(b)}{f(a)}} (\ln \lambda)^{\alpha-1} \, d\lambda + \frac{f(b)}{\Gamma(\alpha+1)} \left[\left(\ln \frac{b}{a} \right)^{\alpha} - \left(\ln \frac{f(b)}{f(a)} \right)^{\alpha} \right] \ . \tag{3.6}$$

Consequently, we have proven the inequality (3.4) by using the inequalities (3.5) and (3.6). □

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STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS AND SOME PROPERTIES IN 2-NORMED SPACES

Sevim Yegül and Erdinç Dündar

Abstract. In this study, we introduced the concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Also, were studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces.
Keywords: Uniform convergence, Statistical Convergence, Double sequences of Functions, Statistical Cauchy sequence, 2-normed Spaces.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [16] and Schoenberg [35]. Gökhan et al. [21] introduced the concepts of pointwise statistical convergence and statistical Cauchy sequence of real-valued functions. Balcerzak et al. [5] studied statistical convergence and ideal convergence for sequence of functions. Duman and Orhan [7] studied μ-statistically convergent function sequences. Gökhan et al. [22] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. Dündar and Altay [8,9] studied the concepts of pointwise and uniformly $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of double sequences of functions and investigated some properties about them. Also, a lot of development have been made about double sequences of functions (see [4,14,20]).

The concept of 2-normed spaces was initially introduced by Gähler [18,19] in the 1960's. Gürdal and Pehlivan [25] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Sharma and Kumar [32] introduced statistical convergence, statistical Cauchy sequence, statistical limit points and statistical cluster points in probabilistic 2-normed space. Statistical convergence and statistical Cauchy sequence

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of functions in 2-normed space were studied by Yegül and Dündar [37]. Sarabadan and Talebi [31] presented various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathcal{I}$-equistatistically convergence and study $\mathcal{I}$-equistatistically convergence of sequences of functions. Futhermore, a lot of development have been made in this area (see [1–3, 6, 15, 23, 24, 26–29, 33, 34]).

2. Definitions and Notations

Now, we recall the concepts of double sequences, density, statistical convergence, 2-normed space and some fundamental definitions and notations (See [5, 10–13, 17, 19–21, 23–25, 30–32, 36]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$.

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1|; \quad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

Let $(X, \|.,.\|)$ be a finite dimensional 2-normed space and $u = \{u_1, \cdots, u_d\}$ be a basis of X. We can define the norm $\|.\|_\infty$ on X by $\|x\|_\infty = \max\{\|x, u_i\| : i = 1, ..., d\}$.

Associated to the derived norm $\|.\|_\infty$, we can define the (closed) balls $B_u(x, \varepsilon)$ centered at x having radius ε by $B_u(x, \varepsilon) = \{y : \|x - y\|_\infty \leq \varepsilon\}$, where $\|x - y\|_\infty = \max\{\|x - y, u_j\|, j = 1, ..., d\}$.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be convergent to f if $f_n(x) \to f(x)(\|.,.\|_Y)$ for each $x \in X$. We write $f_n \to f(\|.,.\|_Y)$. This can be expressed by the formula $(\forall y \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), y\| < \varepsilon$.

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $\delta(K) = \lim\limits_{n\to\infty} \frac{1}{n}|K_n|$, if it exists.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be (pointwise) statistical convergent to f, if for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} \frac{1}{n}\big|\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\}\big| = 0$, for each $x \in X$ and each nonzero $z \in Y$. It means that for each $x \in X$ and each nonzero $z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$, a.a. (almost all) n. In this case, we write

$$st - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \quad or \quad f_n \to_{st} f(\|.,.\|_Y).$$

The sequence of functions $\{f_n\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exists a number $k = k(\varepsilon, z)$ such that $\delta(\{n \in \mathbb{N} : \|f_n(x) - f_k(x), z\| \geq \varepsilon\}) = 0$, for each $x \in X$, i.e., $\|f_n(x) - f_k(x), z\| < \varepsilon$, a.a. n.

Let X be a 2-normed space. A double sequence (x_{mn}) in X is said to be convergent to $L \in X$, if for every $z \in X$, $\lim\limits_{m,n\to\infty} \|x_{mn} - L, z\| = 0$. In this case, we write $\lim\limits_{n,m\to\infty} x_{mn} = L$ and call L the limit of (x_{mn}).

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \leq m$, $k \leq n$. That is, $K_{mn} = |\{(j,k) : j \leq m, k \leq n\}|$, where $|A|$ denotes the number of elements in A. If the double sequence $\left\{\frac{K_{mn}}{mn}\right\}$ has a limit then we say that K has double natural density and is denoted by $d_2(K) = \lim\limits_{m,n\to\infty} \frac{K_{mn}}{mn}$.

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

Let $\{x_{mn}\}$ be a double sequence in 2-normed space $(X, \|.,.\|)$. The double sequence (x_{mn}) is said to be statistically convergent to L, if for every $\varepsilon > 0$, the set $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L, z\| \geq \varepsilon\}$ has natural density zero for each nonzero z in X, in other words (x_{mn}) statistically converges to L in 2-normed space $(X, \|.,.\|)$ if $\lim\limits_{m,n\to\infty} \frac{1}{mn}\big|\{(m,n) : \|x_{mn} - L, z\| \geq \varepsilon\}\big| = 0$, for each nonzero z in X. It means that for each $z \in X$, $\|x_{mn} - L, z\| < \varepsilon$, $a.a.$ (m,n). In this case, we write $st - \lim\limits_{m,n\to\infty} \|x_{mn}, z\| = \|L, z\|$.

A double sequence (x_{mn}) in 2-normed space $(X, \|.,.\|)$ is said to be statistically Cauchy sequence in X, if for every $\varepsilon > 0$ and every nonzero $z \in X$ there exist two number $M = M(\varepsilon, z)$ and $N = N(\varepsilon, z)$ such that $d_2\big(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{MN}, z\| \geq \varepsilon\}\big) = 0$, i.e., for each nonzero $z \in X$, $\|x_{mn} - x_{MN}, z\| < \varepsilon$, a.a. (m,n).

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that $|f_{mn}(x) - f(x)| < \varepsilon$, for all $m, n > N$. In this case we write $\lim\limits_{m,n\to\infty} f_{mn}(x) = f(x)$ *or* $f_{mn} \to f$, on S.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly convergent to f on a set $S \subset \mathbb{R}$, if for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that for all $m, n > N$ implies $|f_{mn}(x) - f(x)| < \varepsilon$, for all $x \in S$. In this case we write $f_{mn} \rightrightarrows f$, on S.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i \ \ and \ \ n \leq j : |f_{mn}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon,\ \ a.a.\ (m,n)$. In this case, we write $st - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \ or \ f_{mn} \to_{st} f$, on S.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i \ \ and \ \ n \leq j : |f_{mn}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for all $x \in S$, i.e., for all $x \in S$, $|f_{mn}(x) - f(x)| < \varepsilon,\ \ a.a.\ (m,n)$. In this case we write $f_{mn} \rightrightarrows f$, on S.

Let $\{f_{mn}\}$ be a double sequence of functions defined on a set S. A double sequence $\{f_{mn}\}$ is said to be statistically Cauchy if for every $\varepsilon > 0$, there exist $N(= N(\varepsilon))$ and $M(= M(\varepsilon))$ such that $|f_{mn}(x) - f_{MN}(x)| < \varepsilon$ a.a. (m,n) and for each (fixed) $x \in S$, i.e.,

$$\lim_{i,j\to\infty} \frac{1}{ij} |\{(m,n), m \leq i \ \ and \ \ n \leq j : |f_{mn}(x) - f_{MN}(x)| \geq \varepsilon\}| = 0$$

for each (fixed)$x \in S$

Lemma 2.1. *[9] Let f and f_{mn}, $m,n = 1,2,...$, be continuous functions on $D = [a,b] \subset \mathbb{R}$. Then $f_{mn} \rightrightarrows f$ on D if and only if $\lim_{m,n\to\infty} c_{mn} = 0$, where $c_{mn} = \max_{x\in D} |f_{mn}(x) - f(x)|$.*

3. Main Results

In this paper, we study concepts of convergence, statistical convergence and statistical Cauchy sequence of double sequences of functions and investigate some properties and relationships between them in 2-normed spaces.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be two double sequences of functions, f and g be two functions from X to Y.

Definition 3.1. A double sequence $\{f_{mn}\}$ is said to be pointwise convergent to f if, for each point $x \in X$ and for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x,\varepsilon)$ such that for all $m,n \geq k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \to f(\|.,.\|_Y)$.

Definition 3.2. A double sequence $\{f_{mn}\}$ is said to be uniformly convergent to f, if for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that for all $m, n > k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for all $x \in X$ and for every $z \in Y$. In this case, we write $f_{mn} \rightrightarrows f(\|.,.\|_Y)$.

Theorem 3.1. *Let D be a compact subset of X and f and f_{mn}, $(m, n = 1, 2, ...)$, be continuous functions on D. Then,*

$$f_{mn} \rightrightarrows f(\|.,.\|_Y)$$

on D if and only if

$$\lim_{m,n\to\infty} c_{mn} = 0,$$

where $c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\|$.

Proof. Suppose that $f_{mn} \rightrightarrows f(\|.,.\|_Y)$ on D. Since f and f_{mn} are continuous functions on D, so $(f_{mn}(x) - f(x))$ is continuous on D, for each $(m, n) \in \mathbb{N} \times \mathbb{N}$. Since $f_{mn} \rightrightarrows f(\|.,.\|_Y)$ on D then, for each $\varepsilon > 0$, there is a positive integer $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $m, n > k_0$ implies

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}$$

for all $x \in D$ and every $z \in Y$. Thus, when $m, n > k_0$ we have

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2} < \varepsilon.$$

This implies

$$\lim_{m,n\to\infty} c_{mn} = 0.$$

Now, suppose that

$$\lim_{m,n\to\infty} c_{mn} = 0.$$

Then, for each $\varepsilon > 0$, there is a positive integer $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$0 \le c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| < \varepsilon,$$

for $m, n > k_0$ and every $z \in Y$. This implies that $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for all $x \in D$, every $z \in Y$ and $m, n > k_0$. Hence, we have

$$f_{mn} \rightrightarrows f(\|.,.\|_Y),$$

for all $x \in D$ and every $z \in Y$. $\square$

Definition 3.3. A double sequence $\{f_{mn}\}$ is said to be (pointwise) statistical convergent to f, if for every $\varepsilon > 0$,

$$\lim_{i,j\to\infty} \frac{1}{ij} \left|\{(m, n), m \le i, n \le j : \|f_{mn}(x) - f(x), z\| \ge \varepsilon\}\right| = 0,$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$. It means that for each (fixed) $x \in X$ and each nonzero $z \in Y$,

$$\|f_{mn}(x) - f(x), z\| < \varepsilon, \quad a.a. \ (m,n).$$

In this case, we write

$$st - \lim_{m,n\to\infty} \|f_{mn}(x) - z\| = \|f(x), z\| \quad or \quad f_{mn} \longrightarrow_{st} f(\|.,.\|_Y).$$

Remark 3.1. $\{f_{mn}\}$ is any double sequence of functions and f is any function from X to Y, then set

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon, \ \ for \ each \ x \in X \ and \ each \ z \in Y\} = \emptyset,$$

since if $z = \overrightarrow{0}$ (0 vektor), $\|f_{mn}(x) - f(x), z\| = 0 \ngeq \varepsilon$ so the above set is empty.

Theorem 3.2. *If for each $x \in X$ and each nonzero $z \in Y$,*

$$st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad and \quad st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|g(x), z\|$$

then, for each $x \in X$ and each nonzero $z \in Y$

$$\|f_{mn}(x), z\| = \|g_{mn}(x), z\|$$

(i.e., $f = g$).

Proof. Assume $f \neq g$. Then, $f - g \neq \overrightarrow{0}$, so there exists a $z \in Y$ such that f, g and z are linearly independent (such a z exists since $d \geq 2$). Therefore, for each $x \in X$ and each nonzero $z \in Y$,

$$\|f(x) - g(x), z\| = 2\varepsilon, \quad with \quad \varepsilon > 0.$$

Now, for each $x \in X$ and each nonzero $z \in Y$, we get

$$\begin{aligned} 2\varepsilon = \|f(x) - g(x), z\| &= \|(f(x) - f_{mn}(x)) + (f_{mn}(x) - g(x)), z\| \\ &\leq \|f_{mn}(x) - g(x), z\| + \|f_{mn}(x) - f(x), z\| \end{aligned}$$

and so

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - g(x), z\| < \varepsilon\} \subseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}.$$

But, for each $x \in X$ and each nonzero $z \in Y$,

$$d_2\left(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - g(x), z\| < \varepsilon\}\right) = 0,$$

then contradicting the fact that $f_{mn} \longrightarrow_{st} g(\|.,.\|_Y)$. □

Theorem 3.3. *If $\{g_{mn}\}$ is a convergent sequence of double sequences of functions such that $f_{mn} = g_{mn}$, a.a. (m,n) then, $\{f_{mn}\}$ is statistically convergent.*

Proof. Suppose that for each $x \in X$ and each nonzero $z \in Y$,

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\}) = 0 \;\; \textit{and} \;\; \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|,$$

then for every $\varepsilon > 0$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}$$

$$\subseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}$$

$$\cup \{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\}.$$

Therefore,

$$\begin{aligned} d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) \\ \leq d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon) \\ + d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}\}). \end{aligned} \tag{3.1}$$

Since $\lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$, the set $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}$ contains finite number of integers and so

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0.$$

Using inequality (3.1) we get for every $\varepsilon > 0$

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,$$

for each $x \in X$ and each nonzero $z \in Y$ and so consequently

$$st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

□

Theorem 3.4. *If $st - \lim \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$, then $\{f_{mn}\}$ has a subsequence of function $\{f_{m_i n_i}\}$ such that*

$$\lim_{i\to\infty} \|f_{m_i n_i}(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$.

Proof. Proof of this Theorem is as an immediate consequence of Theorem 3.3. □

Theorem 3.5. *Let $\alpha \in \mathbb{R}$. If for each $x \in X$ and each nonzero $z \in Y$,*

$$st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \;\; \textit{and} \;\; st - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

then

(i) $st - \lim_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|$ *and*

(ii) $st - \lim_{m,n\to\infty} \|\alpha f_{mn}(x), z\| = \|\alpha f(x), z\|$.

Proof. **(i)** Suppose that

$$st-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\| \ \ and \ \ st-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|g(x),z\|$$

for each $x\in X$ and each nonzero $z\in Y$. Then, $\delta(K_1)=0$ and $\delta(K_2)=0$ where

$$K_1=K_1(\varepsilon,z):\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\frac{\varepsilon}{2}\right\}$$

and

$$K_2=K_2(\varepsilon,z):\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-g(x),z\|\geq\frac{\varepsilon}{2}\right\}$$

for every $\varepsilon>0$, each $x\in X$ and each nonzero $z\in Y$. Let

$$K=K(\varepsilon,z)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|(f_{mn}(x)+g_{mn}(x))-(f(x)+g(x)),z\|\geq\varepsilon\}.$$

To prove that $\delta(K)=0$, it suffices to show that $K\subset K_1\cup K_2$. Let $(m_0,n_0)\in K$ then, for each $x\in X$ and each nonzero $z\in Y$,

$$\|(f_{m_0n_0}(x)+g_{m_0n_0}(x))-(f(x)+g(x)),z\|\geq\varepsilon. \tag{3.2}$$

Suppose to the contrary, that $(m_0,n_0)\notin K_1\cup K_2$. Then, $(m_0,n_0)\notin K_1$ and $(m_0,n_0)\notin K_2$. If $(m_0,n_0)\notin K_1$ and $(m_0,n_0)\notin K_2$ then, for each $x\in X$ and each nonzero $z\in Y$,

$$\|f_{m_0n_0}(x)-f(x),z\|<\frac{\varepsilon}{2} \ \ and \ \ \|g_{m_0n_0}(x)-g(x),z\|<\frac{\varepsilon}{2}.$$

Then, we get

$$\begin{aligned}\|(f_{m_0n_0}(x)&+g_{m_0n_0}(x))-(f(x)+g(x)),z\|\\ &\leq\|f_{m_0n_0}(x)-f(x),z\|+\|g_{m_0n_0}(x)-g(x),z\|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &=\varepsilon,\end{aligned}$$

for each $x\in X$ and each nonzero $z\in Y$, which contradicts (3.2). Hence, $(m_0,n_0)\in K_1\cup K_2$ and so $K\subset K_1\cup K_2$.

(ii) Let $\alpha\in\mathbb{R}$ $(\alpha\neq0)$ and for each $x\in X$ and each nonzero $z\in Y$,

$$st-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|.$$

Then, we get

$$d_2\left(\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\frac{\varepsilon}{|\alpha|}\right\}\right)=0.$$

Therefore, for each $x\in X$ and each nonzero $z\in Y$, we have

$$\begin{aligned}\{(m,n)\in\mathbb{N}\times\mathbb{N}&:\|\alpha f_{mn}(x)-\alpha f(x),z\|\geq\varepsilon\}\\ &=\{(m,n)\in\mathbb{N}\times\mathbb{N}:|\alpha|\|f_{mn}(x)-f(x),z\|\geq\varepsilon\}\\ &=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|\geq\frac{\varepsilon}{|\alpha|}\right\}.\end{aligned}$$

Hence, density of the right hand side of above equality equals 0. Therefore, for each $x \in X$ and each nonzero $z \in Y$, we have

$$st - \lim_{m,n\to\infty} \|\alpha f_{mn}(x), z\| = \|\alpha f(x), z\|.$$

□

Theorem 3.6. *A double sequence of functions $\{f_{mn}\}$ is pointwise statistically convergent to a function f if and only if there exists a subset $K_x = \{(m,n)\} \subseteq \mathbb{N}\times\mathbb{N}$, $m,n = 1,2,...$ for each (fixed) $x \in X$ $d_2(K_x) = 1$ and $\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each (fixed) $x \in X$ and each nonzero $z \in Y$.*

Proof. Let $st_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$. For $r = 1,2,...$ put

$$K_{r,x} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x), z\| \geq \frac{1}{r}\}$$

and

$$M_{r,x} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x), z\| < \frac{1}{r}\}$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$. Then, $d_2(K_{r,x}) = 0$ and

$$M_{1,x} \supset M_{2,x} \supset ... \supset M_{i,x} \supset M_{i+1,x} \supset ... \tag{3.3}$$

and

$$d_2(M_{r,x}) = 1, \qquad r = 1,2,... \tag{3.4}$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$.

Now, we have to show that for $(m,n) \in M_{r,x}$, $\{f_{mn}\}$ is convergent to f. Suppose that $\{f_{mn}\}$ is not convergent to f. Therefore, there is $\varepsilon > 0$ such that

$$\|f_{mn}(x), z\| = \|f(x), z\| \geq \varepsilon$$

for infinitely many terms and some $x \in X$ and each nonzero $z \in Y$. Let

$$M_{\varepsilon,x} = \{(m,n) : \|f_{mn}(x) - f(x), z\| < \varepsilon\}$$

and $\varepsilon > \frac{1}{r}$ $(r = 1,2,...)$. Then, $d_2(M_{\varepsilon,x}) = 0$ and by (3.3) $M_{r,x} \subset (M_{\varepsilon,x})$. Hence, $d_2(M_{r,x}) = 0$ which contradicts (3.4). Therefore, $\{f_{mn}\}$ is convergent to f.

Conversely, suppose that there exists a subset $K_x = \{(m,n)\} \subseteq \mathbb{N}\times\mathbb{N}$ for each (fixed) $x \in X$ and each nonzero $z \in Y$ such that $d_2(K_x) = 1$ and $\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, i.e., there exist an $N(x,\varepsilon)$ such that for each (fixed) $x \in X$, each nonzero $z \in Y$ and each $\varepsilon > 0$, $m,n \geq N$ implies $\|f_{mn}(x), z\| = \|f(x), z\| < \varepsilon$. Now,

$$K_{\varepsilon,x} = \{(m,n) : \|f_{mn}(x), z\| \geq \varepsilon\} \subseteq \mathbb{N}\times\mathbb{N} - \{(m_{N+1}, n_{N+1}), (m_{N+2}, n_{N+2}), ...\}$$

for each (fixed) $x \in X$ and each nonzero $z \in Y$. Therefore, $d_2(K_{\varepsilon,x}) \leq 1 - 1 = 0$ for each (fixed) $x \in X$ and each nonzero $z \in Y$. Hence, $\{f_{mn}\}$ is pointwise statistically convergent to f. □

Definition 3.4. A double sequence of functions $\{f_{mn}\}$ is said to uniformly statistically convergent to f, if for every $\varepsilon > 0$ and for each nonzero $z \in Y$,

$$\lim_{i,j\to\infty} \frac{1}{ij}|\{(m,n), m \leq i, n \leq j : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}| = 0,$$

for all $x \in X$. That is, for all $x \in X$ and for each nonzero $z \in Y$

$$\|f_{mn}(x) - f(x), z\| < \varepsilon, \quad a.a \ (m,n). \tag{3.5}$$

In this case, we write $f_{mn} \rightrightarrows_{st} f(\|.,.\|_Y)$.

Theorem 3.7. *Let D be a compact subset of X and f and $\{f_{mn}\}$, $m,n = 1,2,...$ be continuous functions on D. Then,*

$$f_{mn} \rightrightarrows_{st} f(\|.,.\|_Y)$$

on D if and only if

$$st_2 - \lim_{m,n\to\infty} \|c_{mn}(x), z\| = 0,$$

where $c_{mn} = \max_{x\in S} \|f_{mn}(x) - f(x), z\|$.

Proof. Suppose that $\{f_{mn}\}$ uniformly statistically convergent to f on D. Since f and $\{f_{mn}\}$ are continuous functions on D, so $(f_{mn}(x) - f(x))$ is continuous on D, for each $m,n \in \mathbb{N}$. By statistically convergence for $\varepsilon > 0$

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,$$

for each $x \in D$ and for each nonzero $z \in Y$. Hence, for $\varepsilon > 0$ it is clear that

$$c_{mn} = \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}$$

for each $x \in D$ and for each nonzero $z \in Y$. Thus we have

$$st - \lim_{m,n\to\infty} c_{mn} = 0.$$

Now, suppose that $st - \lim_{m,n\to\infty} c_{mn} = 0$. We let following set

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \varepsilon\},$$

for $\varepsilon > 0$ and for each nonzero $z \in Y$. Then, by hypothesis we have $d_2(A(\varepsilon)) = 0$. Since for $\varepsilon > 0$

$$\max_{x\in D} \|f_{mn}(x) - f(x), z\| \geq \|f_{mn}(x) - f(x), z\| \geq \varepsilon$$

we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \subset A(\varepsilon)$$

and so

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}) = 0,$$

for each $x \in D$ and for each nonzero $z \in Y$. This proves the theorem. $\square$

Now, we can give the relations between well-known convergence models and our studied models as the following result.

Corollary 3.1. ***(i)*** $f_{mn} \rightrightarrows f(\|.,.\|_Y) \Rightarrow f_{mn} \longrightarrow f(\|.,.\|_Y) \Rightarrow f_{mn} \longrightarrow_{st} f(\|.,.\|_Y)$.
(ii)$f_{mn} \rightrightarrows f(\|.,.\|_Y) \Rightarrow f_{mn} \rightrightarrows_{st} f(\|.,.\|_Y) \Rightarrow f_{mn} \longrightarrow_{st} f(\|.,.\|_Y)$.

Now, we give the concept of statistical Cauchy sequence and investigate relationships between statistical Cauchy sequence and statistical convergence of double sequences of functions in 2-normed space.

Definition 3.5. The double sequences of functions $\{f_{mn}\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exist two numbers $k = k(\varepsilon, z)$, $t = t(\varepsilon, z)$ such that

$$d_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{kt}(x), z\| \geq \varepsilon\}) = 0, \text{ for each (fixed) } x \in X,$$

i.e., for each nonzero $z \in Y$,

$$\|f_{nm}(x) - f_{kt}(x), z\| < \varepsilon, \quad a.a. \ (m,n).$$

Theorem 3.8. *Let $\{f_{mn}\}$ be a statistically Cauchy sequence of double sequence of functions in a finite dimensional 2-normed space $(X, \|.,.\|)$. Then, there exists a convergent sequence of double sequences of functions $\{g_{mn}\}$ in $(X, \|.,.\|)$ such that $f_{mn} = g_{mn}$, for a.a. (m,n).*

Proof. First note that $\{f_{mn}\}$ is a statistically Cauchy sequence of functions in $(X, \|.\|_\infty)$. Choose a natural number $k(1)$ and $j(1)$ such that the closed ball $B_u^1 = B_u(f_{k(1)j(1)}(x), 1)$ contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$. Then, choose a natural number $k(2)$ and $j(2)$ such that the closed ball $B_2 = B_u(f_{k(2)j(2)}(x), \frac{1}{2})$ contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$. Note that $B_u^2 = B_u^1 \cap B_2$ also contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$. Thus, by continuing of this process, we can obtain a sequence $\{B_u^r\}_{r \geq 1}$ of nested closed balls such that diam $(B_u^r) \leq \frac{1}{2^r}$. Therefore,

$$\bigcap_{r=1}^{\infty} B_u^r = \{h(x)\},$$

where h is a function from X to Y. Since each B_u^r contains $f_{mn}(x)$ for a.a. (m,n) and for each $x \in X$, we can choose a sequence of strictly increasing natural numbers $\{S_r\}_{r \geq 1}$ such that for each $x \in X$,

$$\frac{1}{mn}|\{(m,n) \in \mathbb{N} \times \mathbb{N} : f_{mn}(x) \notin B_u^r\}| < \frac{1}{r}, \ \ if \ \ m,n > S_r.$$

Put $T_r = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m,n > S_r, \ \ f_{mn}(x) \notin B_u^r\}$ for each $x \in X$, for all $r \geq 1$ and $R = \bigcup_{r=1}^{\infty} R_r$. Now, for each $x \in X$, define the sequence of functions $\{g_{mn}\}$ as following

$$g_{mn}(x) = \begin{cases} h(x) & , \quad \text{if} \quad (m,n) \in R \times R \\ f_{mn}(x) & , \quad \text{otherwise.} \end{cases}$$

Note that, $\lim_{m,n\to\infty} g_{mn}(x) = h(x)$, for each $x \in X$. In fact, for each $\varepsilon > 0$ and for each $x \in X$, choose a natural number m such that $\varepsilon > \frac{1}{r} > 0$. Then, for each $m, n > S_r$ and for each $x \in X$, $g_{mn}(x) = h(x)$ or $g_{mn}(x) = f_{mn}(x) \in B_u^r$ and so in each case

$$\|g_{mn}(x) - h(x)\|_\infty \leq diam(B_u^r) \leq \frac{1}{2^{r-1}}.$$

Since, for each $x \in X$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : g_{mn}(x) \neq f_n(x)\} \subseteq \{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \notin B_u^r\},$$

we have

$$\begin{aligned}
&\frac{1}{mn}|\{(m,n) \in \mathbb{N}\times\mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\}| \\
&\qquad\qquad \leq \frac{1}{mn}|\{(n,m) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \notin B_u^r\}| \\
&\qquad\qquad < \frac{1}{r},
\end{aligned}$$

and so

$$d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : g_{mn}(x) \neq f_{mn}(x)\}) = 0.$$

Thus, $g_{mn}(x) = f_{mn}(x)$ for a.a. m,n and for each $x \in X$ in $(X, \|.\|_\infty)$. Suppose that $\{u_1, ..., u_d\}$ is a basis for $(X, \|.,.\|)$. Since, for each $x \in X$,

$$\lim_{m,n\to\infty} \|g_{mn}(x) - h(x)\|_\infty = 0 \;\; and \;\; \|g_{mn}(x) - h(x), u_i\| \leq \|g_{mn}(x) - h(x)\|_\infty$$

for all $1 \leq i \leq d$, then we have

$$\lim_{m,n\to\infty} \|g_{mn}(x) - h(x), z\|_\infty = 0,$$

for each $x \in X$ and each nonzero $z \in X$. It completes the proof. □

Theorem 3.9. *The sequence $\{f_{mn}\}$ is statistically convergent if and only if $\{f_{mn}\}$ is a statistically Cauchy sequence of double sequence of functions.*

Proof. Assume that f be function from X to Y and $st - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$ and $\varepsilon > 0$. Then, for each $x \in X$ and each nonzero $z \in Y$, we have

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}, \quad a.a. \;\; (m,n).$$

If $k = k(\varepsilon, z)$ and $t = t(\varepsilon, z)$ are chosen so that for each $x \in X$ and each nonzero $z \in Y$,

$$\|f_{kt}(x) - f(x), z\| < \frac{\varepsilon}{2},$$

and so we have

$$\begin{aligned} \|f_{mn}(x) - f_{kt}(x), z\| &\leq \|f_{mn}(x) - f(x), z\| + \|f(x) - f_{kt}(x), z\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad a.a. \ (m,n). \end{aligned}$$

Hence, $\{f_{mn}\}$ is statistically Cauchy sequence of double sequence of functions.

Now, assume that $\{f_{mn}\}$ is statistically Cauchy sequence of double sequence of function. By Theorem 3.8, there exists a convergent sequence $\{g_{mn}\}$ from X to Y such that $f_{mn} = g_{mn}$ for a.a. (m,n). By Theorem 3.3, we have

$$st - \lim \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. □

Theorem 3.10. *Let $\{f_{mn}\}$ be a double sequence of functions. The following statements are equivalent*

(i) $\{f_{mn}\}$ is (pointwise) statistically convergent to $f(x)$,

(ii) $\{f_{mn}\}$ is statistically Cauchy,

(iii) There exisits a subsequence $\{g_{mn}\}$ of $\{f_{mn}\}$ such that $\lim\limits_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$.

Proof. Proof of this Theorem is as an immediate consequence of Theorem 3.6 and Theorem 3.9. □

Definition 3.6. Let D be a compact subset of X and $\{f_{mn}\}$ be a double sequence of functions on D. $\{f_{mn}\}$ is said to be statistically uniform Cauchy if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exists $k = k(\varepsilon, z)$, $t = t(\varepsilon, z)$ such that

$$d_2(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{kt}(x), z\| \geq \varepsilon\}) = 0$$

for all $x \in X$.

Theorem 3.11. *Let D be a compact subset of X and $\{f_{mn}\}$, be a sequence of bounded functions on D. Then, $\{f_{mn}\}$ is uniformly statistically convergent if and only if it is uniformly statistically Cauchy on D.*

Proof. Proof of this theorem is similar the Theorem 3.9. So, we omit it. □

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SEQUENCE SPACES OVER n-NORMED SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION OF ORDER (α, β)

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Abstract. In the present paper, we introduce sequence spaces over n-normed spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ of order (α, β). We examine some topological properties and prove some inclusion relations between the resulting sequence spaces.

Keywords: Musielak-Orlicz function; lacunary sequence; n-normed spaces; statistical convergence; paranorm space

1. Introduction and preliminaries

Mursaleen and Noman [29] introduced the notion of λ-convergent and λ-bounded sequences as follows : Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [29] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

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This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

The concept of 2-normed spaces was initially developed by Gähler [14] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [22]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, if we take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then, the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} given by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Let w be the space of all real or complex sequences $x = (x_k)$. Lindenstrauss and Tzafriri [20] used the idea of Orlicz function to define the following sequence space

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\},$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [20] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [33]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

For some other recent works related to sequence spaces, we refer the interested reader to [4, 9, 16, 17, 18, 19, 21, 23, 24, 27, 30, 31, 32, 34, 35, 44] and reference therein.

The notion of statistical convergence was introduced by Fast [10]. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory (see [1, 2, 3, 5, 8, 12, 15, 26, 28, 36, 37]). In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. In the recent past, Çolak [6] introduced the concept of statistical convergence order α (also see [7, 38]).

By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience.

$$N_\theta = \Big\{x \in w : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - l| = 0, \text{ for some } \ l\Big\}.$$

The notion of lacunary statistically convergent sequences of order (α, β) was first defined by Şengül [40] and then studied in [41, 42, 43, 25]. Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \le \beta \le 1$ be given. We say that the sequence $x = (x_k)$ is $S_\alpha^\beta(\theta)$-statistically convergent(or lacunary statistically convergent sequences of order (α, β)) if there is a real number L such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \ge \epsilon\}|^\beta = 0,$$

where $I_r = (k_{r-1}, k_r]$ and h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, \cdots, h_r^\alpha, \cdots)$ and $|\{k \le n : k \subset E\}|^\beta$ denotes the βth power of number of elements of E not exceeding n. In the present case this convergence is indicated by $S_\alpha^\beta(\theta) - \lim x_k = L$. $S_\alpha^\beta(\theta)$ will denote the set of all $S_\alpha^\beta(\theta)$-statistically convergent sequences. If $\theta = (2^r)$, then we will write S_α^β (see [39]). If $\alpha = \beta = 1$ and $\theta = (2^r)$, then we obtain the notion of statistical convergence. The choice of $\beta = 1$ and $\theta = (2^r)$ gives the notion of statistical convergence of order α due to Çolak [6]. Further, if we take $\alpha = \beta = 1$, then we obtain the notion of lacunary statistical convergence given by Fridy and Orhan [13].

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. In the present paper, we define the following sequence spaces:

$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 = \Big\{x = (x_k) \in w :$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta = 0, \ \rho > 0, \ s \ge 0\Big\},$$

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)=\Big\{x=(x_k)\in w:$$
$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,\ldots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta=0,\text{for some } L,\ \rho>0,\ s\geq 0\Big\}$$

and
$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty=\Big\{x=(x_k)\in w:$$
$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta<\infty,\ \ \rho>0,\ s\geq 0\Big\}.$$

If we take $\mathcal{M}(x)=x$, we get

$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0=\Big\{x=(x_k)\in w:$$
$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)^{p_k}\Big]^\beta=0,\ \ \rho>0,\ s\geq 0\Big\},$$

$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)=\Big\{x=(x_k)\in w:$$
$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}\|\Big)^{p_k}\Big]^\beta=0,\ \ \text{for some } L,\ \ \rho>0,\ s\geq 0\Big\}$$

and
$$w_\alpha^\beta(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_\infty=\Big\{x=(x_k)\in w:$$
$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)^{p_k}\Big]^\beta<\infty,\ \ \rho>0,\ s\geq 0\Big\}.$$

If we take $p=(p_k)=1$ for all $k\in\mathbb{N}$, we have

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)_0=\Big\{x=(x_k)\in w:$$
$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta=0,\ \ \rho>0,\ s\geq 0\Big\},$$

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)=\Big\{x=(x_k)\in w:$$
$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^\beta=0,\ \ \text{for some } L,\ \rho>0,\ s\geq 0\Big\}$$

and
$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)_\infty=\Big\{x=(x_k)\in w:$$
$$\sup_r\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta<\infty,\ \ \rho>0,\ s\geq 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

2. Main results

In this section, we study some topological properties of sequence spaces over n-normed spaces defined by a Musielak-Orlicz function of order (α, β) and prove some inclusion relations between the resulting spaces.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers the spaces $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$, $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$ and $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k), y = (y_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(\alpha x + \beta y)}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta = 0.$$

Since $x = (x_k), y = (y_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$, there exist positive numbers $\rho_1, \rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing, convex function and so by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(\alpha x+\beta y)}{\rho_3}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta \\
&\le \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\alpha\Lambda_k(x)}{\rho_3}, z_1, \cdots, z_{n-1}\| + \|\frac{\beta\Lambda_k(y)}{\rho_3}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&\le K\frac{1}{h_r^\alpha}\sum_{k\in I_r} \frac{1}{2^{p_k}} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&+ K\frac{1}{h_r^\alpha}\sum_{k\in I_r} \frac{1}{2^{p_k}} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&\le K\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&+ K\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&\to 0 \text{ as } r\to\infty.
\end{aligned}$$

Thus we have $\alpha x + \beta y \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Hence $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ is a linear space. Similarly, we can prove that $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$ and $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers. Then $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ is a topological linear spaces paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \le 1\Big\},$$

where $H = \max(1, \sup_k p_k) < \infty$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Since $M_k(0) = 0$ we get $g(0) = 0$. Again if $g(x) = 0$ then

$$\inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \le 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$ there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \le 1.$$

Thus

$$\begin{aligned}
&\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \\
&\qquad \le \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}}.
\end{aligned}$$

Suppose $(x_k) \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Lambda_k(x) \neq 0$ for each $k \in \mathbb{N}$. Let $\epsilon \to 0$ then

$$\|\frac{\Lambda_k(x)}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\| \to \infty.$$

It follows that

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\epsilon}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore $\Lambda_k(x) = 0$ for each k and thus $(x_k) = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x+y)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \\
&= \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)+\Lambda_k(y)}{\rho_1+\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big[\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big] \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big[\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big]\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \\
&\leq 1.
\end{aligned}$$

Since ρ, ρ_1 and ρ_2 are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1\Big\} \\
&\leq \inf\Big\{(\rho_1)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1\Big\} \\
&+ \inf\Big\{(\rho_2)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1\Big\}.
\end{aligned}$$

Therefore $g(x+y) \leq g(x)+g(y)$. Finally we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition

$$g(\mu x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(\mu x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Thus

$$g(\mu x) = \inf\Big\{(|\mu|t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{t}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu|^{\sup p_r})$, we have

$$g(\mu x) \leq \max(1, |\mu|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{t}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

So the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subseteq w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.*

Proof. Let $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$, then there exists a positive number ρ_1 such that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta = 0.$$

Define $\rho = 2\rho_1$. Since (M_k) is non-decreasing, convex and so by using inequality (1.1), we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&= \sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)+L-L}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta \\
&\leq K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\frac{1}{2^{p_k}}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta \\
&+ K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\frac{1}{2^{p_k}}\Big[\Big[M_k\Big(\|\frac{L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&\leq K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&+ K\sup_r \frac{1}{h_r^\alpha}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{L}{\rho_1}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \\
&< \infty.
\end{aligned}$$

Hence $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$. □

Theorem 2.4. *Let* $0 < \inf p_k = h \le p_k \le \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k), \mathcal{M}' = (M'_k)$ *be Musielak-Orlicz functions satisfying* Δ_2*-condition, then we have*

(i) $w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$;

(ii) $w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$;

(iii) $w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.

Proof. Let $x = (x_k) \in w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ then we have

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s}\Big[\Big[M'_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta = 0.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \le t \le \delta$. Let $(y_k) = M'_k\Big[\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]$ for all $k \in \mathbb{N}$. We can write

$$\frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s}\Big[\Big(M_k[y_k]\Big)^{p_k}\Big]^\beta = \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\le\delta}} k^{-s}\Big[\Big(M_k[y_k]\Big)^{p_k}\Big]^\beta + \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\ge\delta}} k^{-s}\Big[\Big(M_k[y_k]\Big)^{p_k}\Big]^\beta.$$

So we have

$$\frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\le\delta}} k^{-s}\Big[\big(M_k[y_k]\big)^{p_k}\Big]^\beta \le [M_k(1)]^H \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\le\delta}} k^{-s}\Big[\big(M_k[y_k]\big)^{p_k}\Big]^\beta$$

$$\le [M_k(2)]^H \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\le\delta}} k^{-s}\Big[\big(M_k[y_k]\big)^{p_k}\Big]^\beta \tag{2.1}$$

for $y_k > \delta, y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Since $(M_k)'s$ are non-decreasing and convex, it follows that

$$M_k(y_k) < M_k(1 + \frac{y_k}{\delta}) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k(\frac{2y_k}{\delta}).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$M_k(y_k) < \frac{1}{2}T\frac{y_k}{\delta}M_k(2) + \frac{1}{2}T\frac{y_k}{\delta}M_k(2) = T\frac{y_k}{\delta}M_k(2).$$

Hence,

$$\frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\ge\delta}} k^{-s}\big(M_k[y_k]^{p_k}\big)^\beta \le \max\Big(1, (T\frac{M_k(2)}{\delta})^H\Big) \frac{1}{h_r^\alpha} \sum_{\substack{k\in I_r\\ y_k\le\delta}} k^{-s}\big([y_k]^{p_k}\big)^\beta \tag{2.2}$$

From equation (2.1) and (2.2), we have $x = (x_k) \in w_\alpha^\beta(\mathcal{M}\circ\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. This completes the proof of (i). Similarly we can prove that

$$w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|).$$

and

$$w_\alpha^\beta(\mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M} \circ \mathcal{M}', \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty.$$

□

Theorem 2.5. *Let $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. Then for a Musielak-Orlicz function $\mathcal{M} = (M_k)$ which satisfies Δ_2-condition, we have*
(i) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$;
(ii) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$;
(iii) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.

Proof. The proof is on similar lines. We omit the details. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $0 < h = \inf p_k$. Then $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} k^{-s}\big(\big(M_k(t)\big)^{p_k}\big)^\beta = \infty \tag{2.3}$$

for some $t > 0$.

Proof. Let $w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Suppose that (2.3) does not hold. Therefore there are subinterval $I_{r(j)}$ of the set of interval I_r and a number $t_0 > 0$, where

$$t_0 = \|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\| \text{ for all } k,$$

such that

$$\frac{1}{h_{r(j)}^\alpha} = \sum_{k\in I_{r(j)}} k^{-s}\big(\big(M_k(t_0)\big)^{p_k}\big)^\beta \le K < \infty, m = 1, 2, 3, \cdots \tag{2.4}$$

Let us define $x = (x_k)$ as follows :

$$\Lambda_k(x) = \begin{cases} \rho t_0, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases}.$$

Thus, by (2.4), $x \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$. But $x \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Hence (2.3) must hold.

Conversely, suppose that (2.3) holds and let $x \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$. Then for each r,

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \leq K < \infty. \tag{2.5}$$

Suppose that $x \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Then for some number $\epsilon > 0$, there is a number k_0 such that for a subinterval $I_{r(j)}$, of the set of interval I_r,

$$\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\| > \epsilon \text{ for } k \geq k_0.$$

From properties of sequence of Orlicz functions, we obtain

$$\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta \geq M_k(\epsilon)^{p_k},$$

which contradicts (2.3), by using (2.5). Hence we get

$$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0.$$

This completes the proof. $\square$

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*
(i) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$;
(ii) $w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$;
(iii) $\sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s}\big((M_k(t))^{p_k}\big)^\beta < \infty$ *for all* $t > 0$.

Proof. (i) $\Rightarrow$ (ii). Let (i) holds. To verify (ii), it is enough to prove

$$w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0 \subset w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty.$$

Let $x = (x_k) \in w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Then for $\epsilon > 0$ there exists $r \geq 0$, such that

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta < \epsilon.$$

Hence there exists $K > 0$ such that

$$\sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[\|\frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1}\|\Big]^{p_k}\Big]^\beta < K.$$

So we get $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$.
(ii) $\Rightarrow$ (iii). Let (ii) holds. Suppose (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s}\big[(M_k(t))^{p_k}\big]^\beta = \infty$$

and therefore we can find a subinterval $I_{r(j)}$, of the set of interval I_r such that

$$\frac{1}{h^{\alpha}_{r(j)}}\sum_{k\in I_{r(j)}} k^{-s}\Big[\Big(M_k(\frac{1}{j})\Big)^{p_k}\Big]^{\beta} > j,\ j=1,2,3,\cdots \tag{2.6}$$

Let us define $x=(x_k)$ as follows :

$$\Lambda_k(x)=\begin{cases}\frac{\rho}{j}, & k\in I_{r(j)}\\ 0, & k\notin I_{r(j)}\end{cases}.$$

Then $x=(x_k)\in w^{\beta}_{\alpha}(\Lambda,p,s,\|\cdot,\cdots,\cdot\|)_0$. But by (2.6), $x\notin w^{\beta}_{\alpha}(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_{\infty}$, which contradicts (ii). Hence (iii) must holds.
(iii) $\Rightarrow$ (i). Let (iii) holds and suppose $x=(x_k)\in w^{\beta}_{\alpha}(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_{\infty}$. Suppose that $x=(x_k)\notin w^{\beta}_{\alpha}(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_{\infty}$, then

$$\sup_r\frac{1}{h^{\alpha}_r}\sum_{k\in I_r} k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^{\beta}=\infty. \tag{2.7}$$

Let $t=\|\frac{\Lambda_k(x)}{\rho},z_1,z_2,\cdots,z_{n-1}\|$ for each k, then by (2.7)

$$\sup_r\frac{1}{h^{\alpha}_r}\sum_{k\in I_r} k^{-s}\big[\big(M_k(t)\big)^{p_k}\big]^{\beta}=\infty$$

which contradicts (iii). Hence (i) must holds. □

Theorem 2.8. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*
(i) $w^{\beta}_{\alpha}(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0\subset w^{\beta}_{\alpha}(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0$*;*
(ii) $w^{\beta}_{\alpha}(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_0\subset w^{\beta}_{\alpha}(\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)_{\infty}$*;*
(iii) $\inf_r\frac{1}{h^{\alpha}_r}\sum_{k\in I_r} k^{-s}\big[\big(M_k(t)\big)^{p_k}\big]^{\beta}>0$ *for all* $t>0$.

Proof. (i) $\Rightarrow$ (ii). It is obvious.
(ii) $\Rightarrow$ (iii). Let the inclusion in (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r\frac{1}{h^{\alpha}_r}\sum_{k\in I_r} k^{-s}\big[\big(M_k(t)\big)^{p_k}\big]^{\beta}=0 \text{ for some } t>0,$$

and we can find a subinterval $I_{r(j)}$, of the set of interval I_r such that

$$\frac{1}{h^{\alpha}_{r(j)}}\sum_{k\in I_{r(j)}} k^{-s}\big[\big(M_k(j)\big)^{p_k}\big]^{\beta}<\frac{1}{j},\ j=1,2,3,\cdots \tag{2.8}$$

Let us define $x=(x_k)$ as follows :

$$\Lambda_k(x)=\begin{cases}\rho j, & k\in I_{r(j)}\\ 0, & k\notin I_{r(j)}\end{cases}.$$

Thus by (2.8), $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ but $x = (x_k) \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_\infty$, which contradicts (ii). Hence (iii) must hold.
(iii) $\Rightarrow$ (i). Let (iii) holds. Suppose that $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$. Then

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta \to 0 \text{ as } r \to \infty. \tag{2.9}$$

Again suppose that $x = (x_k) \notin w_\alpha^\beta(\Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)_0$ for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$, of the set of interval I_r, we have

$$\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \geq \epsilon \text{ for all } k.$$

Then from properties of the Orlicz function, we can write

$$\Big[\Big[M_k \Big(\| \frac{\Lambda_k(x)}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta \geq \big[\big(M_k(\epsilon) \big)^{p_k} \big]^\beta.$$

Consequently, by (2.9), we have

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \big[\big(M_k(\epsilon) \big)^{p_k} \big]^\beta = 0,$$

which contradicts (iii). Hence (i) must hold. $\square$

Theorem 2.9. *(i) If* $0 < \inf p_k \leq p_k \leq 1$ *for all* $k \in \mathbb{N}$, *then*

$$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|) \subseteq w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, s, \|\cdot, \cdots, \cdot\|).$$

(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, *for all* $k \in \mathbb{N}$, *then*

$$w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, s, \|\cdot, \cdots, \cdot\|) \subseteq w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|).$$

Proof. (i) Let $x = (x_k) \in w_\alpha^\beta(\mathcal{M}, \Lambda, \theta, p, s, \|\cdot, \cdots, \cdot\|)$, then

$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that
$$\lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[M_k \Big(\| \frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^\beta$$

$$\leq \lim_{r \to \infty} \frac{1}{h_r^\alpha} \sum_{k \in I_r} k^{-s} \Big[\Big[M_k \Big(\| \frac{\Lambda_k(x) - L}{\rho}, z_1, z_2, \cdots, z_{n-1} \| \Big) \Big]^{p_k} \Big]^\beta,$$

therefore,

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta=0.$$

Hence

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)\subseteq w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|).$$

(ii) Let $p_k\geq 1$ for each k and $\sup p_k<\infty$. Let $x=(x_k)\in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)$, then for each $\rho>0$, we have

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta=0<1.$$

Since $1\leq p_k\leq\sup p_k<\infty$, we have

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\Big]^\beta\\
&\quad\leq\lim_{r\to\infty}\frac{1}{h_r^\alpha}\sum_{k\in I_r}k^{-s}\Big[M_k\Big(\|\frac{\Lambda_k(x)-L}{\rho},z_1,z_2,\cdots,z_{n-1}\|\Big)\Big]^\beta\\
&\quad=0\\
&\quad<1.
\end{aligned}$$

Therefore $x=(x_k)\in w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)$, for each $\rho>0$. Hence

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|)\subseteq w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|).$$

This completes the proof of the theorem. □

Theorem 2.10. *If* $0<\inf p_k\leq p_k\leq\sup p_k=H<\infty$, *for all* $k\in\mathbb{N}$, *then*

$$w_\alpha^\beta(\mathcal{M},\Lambda,\theta,p,s,\|\cdot,\cdots,\cdot\|)=w_\alpha^\beta(\mathcal{M},\Lambda,\theta,s,\|\cdot,\cdots,\cdot\|).$$

Proof. The proof is on similar lines, we omit the details. □

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SOME EQUIVALENT QUASINORMS ON $L_{\phi,E}$

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Abstract. In this paper we define a new operator ideal $L_{\phi,E}$ by using block sequence spaces and symmetric norming function. Also we define different quasi-norms on this class and deal with equivalence of these quasi-norms.

Keywords: Operator ideal; sequence spaces; norming function; quasi-norm.

1. Introduction

The operator ideal theory has a special importance in functional analysis. One of the most important methods of constructing operator ideals is using $s-$ numbers. Pietsch defined the approximation numbers of a bounded linear operator in Banach spaces, in 1963 [14]. Later on, the other examples of $s-$numbers, namely Kolmogorov numbers, Weyl numbers, etc. are introduced to the Banach space setting.

In this paper, we denote the set of all natural numbers and nonnegative real numbers by $\mathbb{N}$ and $\mathbb{R}^+$, respectively.

A finite rank operator is defined as a bounded linear operator whose dimension of the range space is finite [10].

Let ω be the set of all real valued sequences. A sequence space is any vector subspace of ω.

Maddox defined the linear space $l(p)$ as follows in [8]:

$$l(p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} |x_n|^{p_n} < \infty \right\},$$

where (p_n) is a bounded sequence of strictly positive real numbers.

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The set of all sequences whose generalized weighted mean transforms are in the space $l(p)$ is the sequence space $l(u,v;p)$ which is is introduced by Altay and Başar in [1] as follows:

$$l(u,v;p)=\left\{x\in\omega:\sum_{n=1}^{\infty}\left|u_n\sum_{k=1}^{n}v_kx_k\right|^{p_n}<\infty\right\},$$

where $u_n,\ v_k\neq 0$ for all $n,\ k\in\mathbb{N}$.

If $p_n=p$ for all $n\in\mathbb{N}$, $l(u,v;p)=Z(u,v;l_p)$ which is defined by Malkowsky and Savaş [12] as follows:

$$Z(u,v;l_p)=\left\{x\in\omega:\sum_{n=1}^{\infty}\left|u_n\sum_{k=1}^{n}v_kx_k\right|^{p}<\infty\right\},$$

where $1<p<\infty$.

The Cesaro sequence space ces_p is defined as

$$ces_p=\left\{x=(x_k)\in\omega:\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^{p}<\infty\right\},$$

where $1<p<\infty$ ([18], [21], [22]). Afterwards, Mursaleen and Khan defined the Cesaro vector-valued sequence space by

$$Ces(X,p,q)=\left\{x=(x_k):\sum_{k=1}^{\infty}\left(\frac{1}{Q_k}\sum_{n=1}^{k}|x_n|\right)^{p_k}<\infty\right\},$$

where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers and $Q_n=\sum\limits_{k=0}^{n}q_k$, $(n\in\mathbb{N})$ [13]. Here if $q_k=1$ for each k then $Ces(X,p,q)$ is reduced to ces_p.

Let E and F be real or complex Banach spaces. $\mathcal{L}(E,F)$ and $\mathcal{L}$ denotes the space of all bounded linear operators from E to F and the space of all bounded linear operators between any two arbitrary Banach spaces, respectively.

A map $s=(s_n):\mathcal{L}\to\mathbb{R}^+$ assigning to every operator $T\in\mathcal{L}$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an $s-$number sequence if the following conditions are satisfied for all Banach spaces E,F,E_0 and F_0:

$(S1)$ $\|T\|=s_1(T)\geq s_2(T)\geq\ldots\geq 0$ for every $T\in\mathcal{L}(E,F)$,

$(S2)$ $s_{m+n-1}(S+T)\leq s_m(S)+s_n(T)$ for every $S,T\in\mathcal{L}(E,F)$ and $m,n\in\mathbb{N}$,

$(S3)$ $s_n(RST)\leq\|R\|\,s_n(S)\,\|T\|$ for some $R\in\mathcal{L}(F,F_0)$, $S\in\mathcal{L}(E,F)$ and $T\in\mathcal{L}(E_0,E)$, where E_0,F_0 are arbitrary Banach spaces,

$(S4)$ If $rank(T)\leq n$, then $s_n(T)=0$,

$(S5)$ $s_n(I:l_2^n\to l_2^n)=1$, where I denotes the identity operator on the $n-$dimensional Hilbert space l_2^n, where $s_n(T)$ denotes the $n-th$ $s-$number of the operator T [2].

One of the example of s-number sequence is the approximation number, which is defined by Pietsch. The $n-$th approximation number, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf\{\|T - A\| : A \in \mathcal{L}(E,F), rank(A) < n\},$$

where $T \in \mathcal{L}(E,F)$ and $n \in \mathbb{N}$ [14].

Pietsch [14] defined an operator $T \in \mathcal{L}(E,F)$ to be l_p type operator if $\sum\limits_{n=1}^{\infty}(a_n(T))^p < \infty$ for $0 < p < \infty$. Then, in [3] Constantin defined the class of $ces-p$ type operators by using the Cesaro sequence spaces, where an operator $T \in \mathcal{L}(E,F)$ is called $ces-p$ type if $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{k=1}^{n}a_k(T)\right)^p < \infty,\ \ 1 < p < \infty$. Afterwards Tita in [24] proved that the class of l_p type operators and the class of $ces-p$ type operators are coincides.

As a generalization of l_p type operators, $A-p$ type operators were examined in [5]. Also in [9], [10], [11] Maji and Srivastava studied the class $A^{(s)}-p$ of $s-$type ces_p operators using $s-$number sequence and Cesaro sequence spaces and they introduced a new class $A^{(s)}_{p,q}$ of $s-$type $ces(p,q)$ operators by using weighted Cesaro sequence space for $1 < p < \infty$. Recently, the class of $s-$type $Z(u,v;l_p)$ operators have been defined and studied on some properties of this class in [4].

The idea of quasi-normed operator ideals is developed by the fact that, some important operator ideals which do not possess a natural norm should also be covered. There exists a lot of different quasi-norms on every operator ideal. In addition to this, the nice quasi-norms are determined by the completeness of the corresponding topology [15].

Now give the definitions of operator ideal and quasi-norm:

Let E' be the dual of E, which is composed of continuous linear functionals on E. Let $x' \in E'$ and $y \in F$, then the map $x' \otimes y : E \to F$ is defined by

$$(x' \otimes y)(x) = x'(x)\,y,\ \ x \in E.$$

A subcollection $\mathfrak{F}$ of $\mathcal{L}$ is called an operator ideal if each component $\mathfrak{F}(E,F) = \mathfrak{F} \cap \mathcal{L}(E,F)$ satisfies the following conditions:

$(OI-1)$ if $x' \in E'$, $y \in F$, then $x' \otimes y \in \mathfrak{F}(E,F)$,

$(OI-2)$ if $S,T \in \mathfrak{F}(E,F)$, then $S+T \in \mathfrak{F}(E,F)$,

$(OI-3)$ if $S \in \mathfrak{F}(E,F)$, $T \in \mathcal{L}(E_0,E)$ and $R \in \mathcal{L}(F,F_0)$, then $RST \in \mathfrak{F}(E_0,F_0)$[15].

A function $\alpha : \mathfrak{F} \to \mathbb{R}^+$ is said to be a quasi-norm on the operator ideal $\mathfrak{F}$ if the following conditions hold:

$(QN-1)$ If $x' \in E'$, $y \in F$, then $\alpha(x' \otimes y) = \|x'\|\,\|y\|$;

$(QN-2)$ there exists a constant $C \geq 1$ such that $\alpha(S+T) \leq C[\alpha(S) + \alpha(T)]$;

$(QN-3)$ if $S \in \mathfrak{F}(E,F), T \in \mathcal{L}(E_0,E)$ and $R \in \mathcal{L}(F,F_0)$, then $\alpha(RST) \leq \|R\|\,\alpha(S)\,\|T\|$[15].

In particular if $C = 1$ then α becomes a norm on the operator ideal $\mathfrak{F}$.

An ideal $\mathfrak{F}$ with a quasi-norm α,which is denoted by $[\mathfrak{F}, \alpha]$, is said to be a quasi-Banach operator ideal if each component $\mathfrak{F}(E, F)$ is complete under the quasi-norm α.

Let ℓ_∞ be the space of all bounded real sequences and $K \subset \ell_\infty$ be the set of all sequences x such that $card\{i \in \mathbb{N}, x_i \neq 0\} < n$ and $x_1 \geq x_2 \geq \ldots \geq 0$.

A function $\phi : K \to \mathbb{R}$ is called symmetric norming function, if the following conditions satisfied

(ϕ1) $\phi(x) > 0$ for every $x \in K$,

(ϕ2) $\phi(\alpha x) = \alpha\phi(x)$ for every $x \in K$ and $\alpha \geq 0$,

(ϕ3) $\phi(x + y) \leq \phi(x) + \phi(y)$ for every $x, y \in K$

(ϕ4) $\phi(1, 0, 0, \ldots) = 1$

(ϕ5) if the inequality $\sum\limits_1^k x_i \leq \sum\limits_1^k y_i$ holds for $k = 1, 2, \ldots,$ then $\phi(x) \leq \phi(y)$ holds [28].

It's given that ([27], [19]) for all symmetric norming functions ϕ, the function $\phi_{(p)}$ defined as

$$\phi_{(p)} : (x_i) \in K \to \left(\phi\left(\{x_i^p\}\right)\right)^{\frac{1}{p}}, 1 \leq p \leq \infty$$

is also a symmetric norming function. For more details on symmetric norming functions we refer to ([7], [20], [23], [25]-[27], [30], [31]).

By using the properties of symmetric norming function and the sequence $(a_n(T))$, the class $\mathcal{L}_\phi(E, F)$ is defined in [25] and [29] as follows

$$\mathcal{L}_\phi(E, F) = \{T \in \mathcal{L}(E, F) : \phi(\{a_n(T)\}) < \infty\}.$$

Let $E = (E_n)$ be a partition of finite subsets of the positive integer such that

$$\max E_n < \min E_{n+1}$$

for $n = 1, 2, \ldots$. In [6] Foroutannia defined the sequence space $l_p(E)$ as

$$l_p(E) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\}, (1 \leq p < \infty)$$

with the seminorm $|||\cdot|||_{p,E}$, which is defined in the following way:

$$|||x|||_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{\frac{1}{p}}.$$

For example, if $E_n = \{2n - 1, 2n\}$ for all n, then $x = (x_n) \in l_p(E)$ if and only if $\sum\limits_{n=1}^{\infty} |x_{2n-1} + x_{2n}|^p < \infty$. It is obvious that $|||\cdot|||_{p,E}$ cannot be a norm, since we have

$|||x|||_{p,E} = 0$ while $x = (1, -1, 0, 0, \ldots) \neq \theta$ and $E_n = \{2n-1, 2n\}$ for all n. In the special case $E_n = \{n\}$ for $n = 1, 2, \ldots$, we have $l_p(E) = l_p$ and $|||x|||_{p,E} = \|x\|_p$.

For more information about block sequence spaces, we refer to [16],[17].

In [32], the class of $l_p(E)$ type operators, which is denoted by $L_{p,E}(E,F)$, is given and it is shown that this class is a quasi-Banach operator ideal by the quasinorm

$$\|T\|_{p,E} = \left(\sum_{n=1}^{\infty} \left(\sum_{j \in E_n} s_j(T) \right)^p \right)^{\frac{1}{p}}.$$

Also a new class of operators $L_{\phi_{(p)},E}$ is defined. Further it is proved that by quasinorm

$$\|T\|_{\phi_{(p)},E} = \phi_{(p)} \left(\left\{ \sum_{j \in E_i} s_j(T) \right\} \right)$$

this class is a quasi-Banach operator ideal.

2. Main Results

Now we define a new class $L_{\phi,E}(E,F)$ including the class $L_\phi(E,F)$ as

$$L_{\phi,E}(E,F) = \left\{ T \in \mathcal{L}(E,F) : \phi\left(\left\{ \sum_{j \in E_n} s_j(T) \right\} \right) < \infty \right\}.$$

For example if we take $E_n = \{n\}$ for $n = 1, 2, \ldots$, we have $L_{\phi,E}(E,F) = L_\phi(E,F)$. Also if we take $E_n = \{2n-1, 2n\}$ for all n, we get $\phi(\{s_{2n-1}(T) + s_{2n}(T)\}) < \infty$.

In this section we show some equivalent quasinorms on operator ideal $L_{\phi,E}(E,F)$.

Theorem 2.1. $\|T\|_{\phi,E} = \phi\left(\left\{ \sum_{j \in E_n} s_j(T) \right\} \right)$ *is a quasinorm on operator ideal* $L_{\phi,E}(E,F)$.

Proof. If $x' \in E$ and $y \in F$, then the equality

$$\phi\left(\left\{ \sum_{j \in E_n} s_j(x' \otimes y) \right\} \right) = \phi(\{s_1(x' \otimes y)\}) = \|x' \otimes y\| = \|x'\| \, \|y\| < \infty$$

holds since $x' \otimes y$ is a rank one operator, $s_n(x' \otimes y) = 0$ for $n \geq 2$. Therefore $\|x' \otimes y\|_{\phi,E} = \|x'\| \, \|y\|$ and $x' \otimes y \in L_{\phi,E}$.

Let $S, T \in L_{\phi,E}$. Then we have that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{j\in E_n}s_j\left(S+T\right)\right) &\leq \sum_{n=1}^{\infty}\left(\sum_{j\in E_n}s_{2j-1}\left(S+T\right)+\sum_{j\in E_n}s_{2j}\left(S+T\right)\right)\\ &\leq \sum_{n=1}^{\infty}\left(2\sum_{j\in E_n}s_{2j-1}\left(S+T\right)\right)\\ &\leq \sum_{n=1}^{\infty}\left(2\sum_{j\in E_n}s_j\left(S\right)+s_j\left(T\right)\right).\end{aligned}$$

By using ($\phi5$) we can get

$$\begin{aligned}\phi\left(\left\{\sum_{j\in E_n}s_j\left(S+T\right)\right\}\right) &\leq \phi\left(\left\{2\left(\sum_{j\in E_n}s_j\left(S\right)+\sum_{j\in E_n}s_j\left(T\right)\right)\right\}\right)\\ &\leq 2\left[\phi\left(\left\{\left(\sum_{j\in E_n}s_j\left(S\right)\right)\right\}\right)+\phi\left(\left\{\left(\sum_{j\in E_n}s_j\left(T\right)\right)\right\}\right)\right]\\ &< \infty.\end{aligned}$$

It follows that

$$\left\|S+T\right\|_{\phi,E}\leq 2\left(\left\|S\right\|_{\phi,E}+\left\|T\right\|_{\phi,E}\right)$$

and also $S+T\in L_{\phi,E}$.

We have that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{j\in E_n}s_j\left(RST\right)\right) &\leq \sum_{n=1}^{\infty}\left(\sum_{j\in E_n}\left\|R\right\|\left\|T\right\|s_j\left(S\right)\right)\\ &\leq \left\|R\right\|\left\|T\right\|\sum_{n=1}^{\infty}\left(\sum_{j\in E_n}s_j\left(S\right)\right).\end{aligned}$$

By using the properties of ϕ function, we obtain

$$\phi\left(\left\{\sum_{j\in E_n}s_j\left(RST\right)\right\}\right)\leq\left\|R\right\|\left\|T\right\|\phi\left(\left\{\sum_{j\in E_n}s_j\left(S\right)\right\}\right)<\infty$$

and also the inequality

$$\left\|RST\right\|_{\phi,E}\leq\left\|R\right\|\left\|T\right\|\left\|S\right\|_{\phi,E}$$

holds.

Hence, $\left\|T\right\|_{\phi,E}=\phi\left(\left\{\sum_{j\in E_n}s_j\left(T\right)\right\}\right)$ is a quasinorm on operator ideal $L_{\phi,E}\left(E,F\right)$. □

Proposition 2.1. *The quasinorm* $\|T\|^{+}_{\phi,E} = \phi\left(\left\{\sum\limits_{j\in E_n} s_{2j-1}(T)\right\}\right)$ *is equivalent with* $\|T\|_{\phi,E}$.

Proof. The equivalence can be easily seen from the fact that

$$\sum_{n=1}^{k}\sum_{j\in E_n} s_{2j-1}(T) \le \sum_{n=1}^{k}\sum_{j\in E_n} s_j(T) \le 2\sum_{n=1}^{k}\sum_{j\in E_n} s_{2j-1}(T).$$

□

Remark 2.1. For the particular case if $E_n = \{n\}$ for $n = 1, 2, \dots,$ we get Proposition 1.1 in [28].

Proposition 2.2. *The quasinorm* $\|T\|_{\phi_{(p)},E}$ *is equivalent with*

$$\|T\|^{\nabla}_{\phi_{(p)},E} = \phi_{(p)}\left(\left\{\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in E_i} s_j(T)\right\}\right), 1 < p < \infty$$

where

$$E_n = \{nN - N + 1, nN - N + 2, \dots, nN\}\, for all\ n.$$

Proof. This is a consequence of Hardy's inequality.

$$\sum_{n=1}^{k}\left(\sum_{j\in E_n} s_j(T)\right)^p \le \sum_{n=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in E_i} s_j(T)\right)^p \le \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{k}\left(\sum_{j\in E_n} s_j(T)\right)^p.$$

□

Remark 2.2. In particular case if we take $N = 1$ we get Proposition 1.2 in [28].

Theorem 2.2. *If* (α_n) *is a nonincreasing positive sequence and* $\lim \alpha_{Nn} \neq 0$, *then the quasinorm* $\|T\|_{\phi_{(p)},E}$ *is equivalent with the quasinorm*

$$\|T\|^{\circ}_{\phi_{(p)},E} = \phi_{(p)}\left(\left\{\frac{1}{\alpha_1 + \ldots + \alpha_n}\sum_{i=1}^{n}\sum_{j\in E_i} s_j(T)\right\}\right), 1 < p < \infty \text{ where}$$

$$E_n = \{nN - N + 1, nN - N + 2, \dots, nN\}\, for\ all\ n\ .$$

Proof. We know that the sequences (α_n) and $(s_n(T))$ are decreasing, so we can write that

$$\frac{1}{n\alpha_1} n\alpha_{Nn}\sum_{j\in E_i} s_j(T) = \frac{\alpha_{Nn}}{\alpha_1}\sum_{j\in E_i} s_j(T) \le \frac{1}{\alpha_1 + \ldots + \alpha_n}\sum_{i=1}^{n}\sum_{j\in E_i} \alpha_j s_j(T)$$
$$\le \frac{1}{n\alpha_{Nn}}\alpha_1\sum_{i=1}^{n}\sum_{j\in E_i} s_j(T).$$

If $\lim \alpha_{Nn} = \alpha \neq 0$, we get

$$\frac{\alpha}{\alpha_1}\sum_{j\in E_i} s_j\left(T\right) \leq \frac{1}{\alpha_1+\ldots+\alpha_n}\sum_{i=1}^{n}\sum_{j\in E_i}\alpha_j s_j\left(T\right) \leq \frac{\alpha_1}{\alpha}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in E_i} s_j\left(T\right)\right).$$

By using Hardy's inequality we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{\alpha}{\alpha_1}\sum_{j\in E_i} s_j\left(T\right)\right)^p &\leq \sum_{i=1}^{n}\left(\frac{1}{\alpha_1+\ldots+\alpha_n}\sum_{i=1}^{n}\sum_{j\in E_i}\alpha_j s_j\left(T\right)\right)^p \\ &\leq \sum_{i=1}^{n}\left(\frac{\alpha_1}{\alpha}\right)^p\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in E_i} s_j\left(T\right)\right)^p \\ &\leq \left(\frac{\alpha_1}{\alpha}\right)^p\left(\frac{p}{p-1}\right)^p\sum_{i=1}^{n}\sum_{j\in E_i}\left(s_j\left(T\right)\right)^p,\ 1<p<\infty.\end{aligned}$$

By the property $(\phi 5)$ it results

$$\frac{\alpha_1}{\alpha}\left\|T\right\|_{\phi_{(p)},E} \leq \left\|T\right\|^{\circ}_{\phi_{(p)},E} \leq \frac{\alpha_1}{\alpha}\frac{p}{p-1}\left\|T\right\|_{\phi_{(p)},E}.$$

Hence $\left\|T\right\|_{\phi_{(p)},E}$ is equivalent with $\left\|T\right\|^{\circ}_{\phi_{(p)},E}$. □

Remark 2.3. In particular case if we take $N = 1$ then we get Theorem 1.4 in [28].

Theorem 2.3. *Let (u_n) and (w_n) are sequences of non-negative real numbers such that $u_1 \geq u_2 \geq ... \geq u_n \geq \ldots$ and $w_1 \leq w_2 \leq ... \leq w_n \leq ...$ and $w_n \leq n \leq \frac{w_n}{u_n}$. Let $\lim\limits_{n\to\infty} u_{Nn} \neq 0$, then the quasinorm $\left\|T\right\|_{\phi_{(p)},E}$ is equivalent to*

$$\left\|T\right\|^{\gamma}_{\phi_{(p)},E} = \phi_{(p)}\left(\left\{\frac{1}{w_n}\sum_{i=1}^{n}\sum_{j\in E_i} u_j s_j\left(T\right)\right\}\right) \text{ for } 1 \leq p < \infty.$$

where

$$E_n = \{nN-N+1, nN-N+2, \ldots, nN\} \text{ for all } n.$$

Proof. Since the sequences (u_n) and $(a_n(T))$ are decreasing, we can write

$$\frac{1}{n} n u_{Nn}\sum_{j\in E_i} s_j\left(T\right) \leq \frac{1}{w_n}\sum_{i=1}^{n}\sum_{j\in E_i} u_j s_j\left(T\right) \leq \frac{1}{n u_{Nn}} u_1\sum_{i=1}^{n}\sum_{j\in E_i} s_j\left(T\right).$$

Summing from $n=1$ to k, we get

$$\begin{aligned}\sum_{n=1}^{k}\left(u_{Nn}\sum_{j\in E_i}s_j\left(T\right)\right)^p &\leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}\sum_{j\in E_i}u_js_j\left(T\right)\right)^p\\ &\leq \sum_{n=1}^{k}\left(\frac{u_1}{nu_{Nn}}\sum_{i=1}^{n}\sum_{j\in E_i}s_j\left(T\right)\right)^p.\end{aligned}$$

If $\lim\limits_{n\to\infty}u_{Nn}=u\neq 0$, then we obtain

$$\begin{aligned}u^p\sum_{n=1}^{k}\left(\sum_{j\in E_i}s_j\left(T\right)\right)^p &\leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}\sum_{j\in E_i}u_js_j\left(T\right)\right)^p\\ &\leq \left(\frac{u_1}{u}\right)^p\sum_{n=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{j\in E_i}s_j\left(T\right)\right)^p\end{aligned}$$

for every $k\in\mathbb{N}$. By using Hardy's inequality, we get

$$\begin{aligned}u^p\sum_{n=1}^{k}\left(\sum_{j\in E_i}s_j\left(T\right)\right)^p &\leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}\sum_{j\in E_i}u_js_j\left(T\right)\right)^p\\ &\leq \left(\frac{u_1}{u}\right)^p\left(\frac{p}{p-1}\right)^p\sum_{i=1}^{n}\left(\sum_{j\in E_i}s_j\left(T\right)\right)^p\end{aligned}$$

for every $k\in\mathbb{N}$. From the properties of the function ϕ, we obtain that

$$u\left\|T\right\|_{\phi_{(p)},E}\leq\left\|T\right\|_{\phi_{(p)},E}^{\gamma}\leq\left(\frac{u_1}{u}\right)\left(\frac{p}{p-1}\right)\left\|T\right\|_{\phi_{(p)},E}.$$

□

Remark 2.4. For the particular case, if we choose $N=1$, we get Theorem 2.2 in [30]. And also if we take $u_i=\alpha_i$ and $w_n=\alpha_1+\alpha_2+...+\alpha_n$ in Theorem 3, where $N=1$ then we obtain Theorem 1.4 in [28], where $\alpha_1\leq 1$. If we take $u_i=1$ and $w_n=n$ in Theorem 3, then we obtain Proposition 1.2 in [28].

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NONLINEAR INVARIANTS OF PLANAR POINT CLOUDS TRANSFORMED BY MATRICES

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Abstract. The goal of this paper is to present invariants of planar point clouds, that is functions which take the same value before and after a linear transformation of a planar point cloud via a 2×2 invertible matrix. In the approach we adopt here, these invariants are functions of two variables derived from the least squares straight line of the planar point cloud under consideration. A linear transformation of a point cloud induces a nonlinear transformation of these variables. The said invariants are solutions to certain Partial Differential Equations, which are obtained by employing Lie theory. We find cloud invariants in the general case of a four−parameter transformation matrix, as well as, cloud invariants of various one−parameter sets of transformations which can be practically implemented. Case studies and simulations which verify our findings are also provided.

1. Introduction

Analysing level shapes is the key problem in many computer science areas, as image analysis, geometric computing, optical character recognition e.t.c. [2, 7]. Usually, by means of the modern sensing technology, we make detailed scans of complex plane objects by generating point cloud data, consisting from thousands or millions of points. Then we study the underlying properties, either by creating appropriate models or by discovering properties which remain constant under sets of transformations or under the influence of noise distortions.

In particular, when we deal with planar set of points, a basic approach, which is widely used, is that of determining quantities which can characterize collectively the behaviour of the whole set, as well as its change, when a transformation is applied to it. In other words, we determine quantities which can represent the planar set of points under consideration, as a whole.

One approach along this line is the classical work of Ming−Kuei Hu [6], who introduced the moment invariants methodology, followed in the course of time by

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many others [3, 4, 9, 10], to mention but a few. The key element of their approach was to introduce the so−called moments of planar figures, in order to identify a planar geometrical figure as a whole, and then to study their invariants under translation, similitude and orthogonal transformations.

In the present paper, we consider planar set of points, called henceforth point cloud or cloud of points. We advocate a different approach, and in order to characterize collectively the behaviour of the whole cloud of points we introduce two variables M and H. These variables stem from the least squares line assigned to these points. In fact M is the slope of this line, and H is a variation of the $y-$intercept of this line.

Any transformation of the cloud of points, by means of a 2×2 matrix, induces a nonlinear transformation, to be precise a rational one, of the quantities M and H. We assume throughout this paper that any 2×2 transformation matrix of a cloud of points is invertible.

The purpose of this paper is to find cloud invariants, or shortly invariants, expressed in terms of the variables M and H. By the term "invariants" [8] we mean functions which take the same value at the original and at the transformed values of M and H, when a cloud of points undergoes a transformation with a 2×2 matrix. In order to solve this problem we use Lie Theory [5, 1]. As a result, the problem is reduced to solving certain Partial Differential Equations. Any solution to these PDEs, provides us with a cloud invariant.

Since the problem, in its general form, does not have a solution which can be practically implemented, we are examining the special case of an one−parameter set of transformations. This is the case when the entries of a transformation matrix are functions of one parameter only. In this case a general solution is found by using Lie theory implemented with symbolic computation.

At first sight it might seem that restricting ourselves to a one−parameter set of transformations, useful as it may be, cannot be of great use. However, this is not the case, because as we point out in section 5.2.1. *any* given matrix belongs to one such one−parameter set of transformations. As a result we find a family of cloud invariants for *any* given matrix and this certainly lends itself to practical implementation.

By practical implementation we mean that these invariants can be used as a tool for studying changes of planar figures and for creating proper software which monitors and displays these changes in real time. This would have many applications in optical character recognition, as well as, in image analysis and computer graphics techniques; icons created by the same "source" will be readily identified.

This potential application of our results suggests a direction for future research. The cloud of points may come from an icon which has a parabolic− like shape. In this case it is natural to look for cloud invariants which are expressed in terms of variables which appear as coefficients, or variations thereof, of the parabola which is the best fit for the cloud points we consider. Comparison with already existing methodologies, which address the same questions, via simulations and computational experiments, will be also the subject of future research.

In section 2. we introduce the variables M and H, and we find the transformation of these variables which is induced from a transformation with a 2×2 matrix of the cloud of points under consideration. In section 3. we define the notion of an invariant function, and we prove in the current case, Lie's theory fundamental result that the nonlinear invariant condition is equivalent to a linear condition provided that the invariant function is properly analytic. Moreover, by using this linear condition, we find the cloud invariants in the general case of a four−parameter transformation matrix of the cloud of points under consideration. In section 4. we find a family of cloud invariants for a general one−parameter set of transformations. In section 5. we find families of cloud invariants for various sets of transformations. We also find a family of cloud invariants for a "linear" one−parameter set of transformations and we point out that *any* given matrix belongs to such a set. In section 6. we verify our results with simulations and computational experiments in a cloud of 10.000 points. In section 7. we close the paper with some concluding remarks.

2. The Basic Quantities and their Transformations

In this section we present two quantities M and H which characterize collectively a cloud of points and serve as the independent variables of the invariant functions we are going to construct. They originate from the least squares straight line fitted to the cloud of points under consideration.

Let (x_i, y_i), $i = 1, ..., N$, be a cloud of points on the plane. We define the quantities:

$$M = \frac{N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{N\sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2}, \quad \text{and,} \tag{2.1}$$

$$H = \frac{N\sum_{i=1}^{N} y_i^2 - (\sum_{i=1}^{N} y_i)^2}{N\sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2}. \tag{2.2}$$

M is the slope of the least squares straight line and H is suggested by the calculations. We call them the **linear coefficients** of the cloud. Sometimes M is referred as the **slope of the cloud** and H as **the constant term** of the cloud. A transformation of the cloud of points under the action of a 2×2 matrix induces a transformation to M and H. This last transformation is of prime importance to our construction of invariant functions and so we proceed to find it. Firstly, we need a definition.

Definition 2.1. *Let (x_i, y_i), $i = 1, \ldots, N$, be a cloud of points and $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha, \beta, \gamma, \delta \in \mathbf{R}$, a given 2×2 matrix. Let us suppose that every point (x_i, y_i), $i = 1, \ldots, N$, of the cloud undergoes a transformation $T_A : \begin{pmatrix} x_i \\ y_i \end{pmatrix} \to \begin{pmatrix} \hat{x}_i \\ \hat{y}_i \end{pmatrix}$ according to the rule $\begin{pmatrix} \hat{x}_i \\ \hat{y}_i \end{pmatrix} = A \begin{pmatrix} x_i \\ y_i \end{pmatrix}$.*

We say that the cloud is transformed under the matrix A, and in particular we say that the cloud $(\hat{x}_i, \hat{y}_i)$, $i = 1, \ldots, N$, is the transformation of the cloud (x_i, y_i), $i = 1, \ldots, N$, under the matrix A.

The transformation of M and H induced by a transformation of the cloud of points via a matrix A is given in the following Theorem.

Theorem 2.1. *Let (x_i, y_i), $i = 1, 2, \ldots, N$, be a cloud of points with linear coefficients M and H. Let $(\hat{x}_i, \hat{y}_i)$, $i = 1, 2, \ldots, N$, be the transformation of the cloud (x_i, y_i), $i = 1, \ldots, N$, under a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha, \beta, \gamma, \delta \in \mathbf{R}$. Let $\hat{M}$ and $\hat{H}$ be the linear coefficients of the cloud $(\hat{x}_i, \hat{y}_i)$, $i = 1, 2, \ldots, N$. Then the following relations hold*

$$\hat{M} = \frac{(\alpha\delta + \beta\gamma)M + \beta\delta H + \alpha\gamma}{2\alpha\beta M + \beta^2 H + \alpha^2}, \tag{2.3}$$

$$\hat{H} = \frac{2\gamma\delta M + \delta^2 H + \gamma^2}{2\alpha\beta M + \beta^2 H + \alpha^2}. \tag{2.4}$$

Proof: Let (x_i, y_i), $i = 1, 2, \ldots, N$, be a cloud of points with linear coefficients M and H. It is convenient to define the quantities M_n, H_n, and D, as follows

$$M_n = N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i, \tag{2.5}$$

$$H_n = N\sum_i y_i^2 - (\sum_i y_i)^2, \tag{2.6}$$

$$D = N\sum_i x_i^2 - (\sum_i x_i)^2. \tag{2.7}$$

The relations (2.1) and (2.2) which define the linear coefficients M and H of the cloud of points under consideration can now be written in the following shorter form:

$$M = \frac{M_n}{D}, \quad H = \frac{H_n}{D}. \tag{2.8}$$

A transformation of the cloud of points under a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha, \beta, \gamma, \delta \in \mathbf{R}$, induces a transformation to the quantities M_n, H_n, D. The induced transformed values $\hat{M}_n$, $\hat{H}_n$, $\hat{D}$, which are assigned to the cloud $(\hat{x}_i, \hat{y}_i)$, $i = 1, 2, \ldots, N$, are calculated as follows:

$$\hat{M}_n = N\sum_i \hat{x}_i \hat{y}_i - \sum_i \hat{x}_i \sum_i \hat{y}_i =$$

$$= N\sum_i (\alpha x_i + \beta y_i)(\gamma x_i + \delta y_i) - \sum_i (\alpha x_i + \beta y_i) \sum_i (\gamma x_i + \delta y_i) =$$

$$= \alpha\gamma \left[N \sum x_i^2 - \left(\sum_i x_i \right)^2 \right] + (\alpha\delta + \beta\gamma) \left[N \sum_i x_i y_i - \sum_i x_i \sum_i y_i \right] +$$

$$+\beta\delta \left[N \sum_i y_i^2 - \left(\sum_i y_i^2 \right) \right] = \alpha\gamma D + (\alpha\delta + \beta\gamma) M_n + \beta\delta H_n, \tag{2.9}$$

$$\hat{H}_n = N \sum_i \hat{y}_i^2 - \left(\sum_i \hat{y}_i \right)^2 = N \sum_i (\gamma x_i + \delta y_i) - \left[\sum_i (\gamma x_i + \delta y_i) \right]^2 =$$

$$= \gamma^2 \left[N \sum_i x_i^2 - \left(\sum_i x_i \right)^2 \right] + \delta^2 \left[N \sum_i y_i^2 - \left(\sum_i y_i^2 \right) \right] + \tag{2.10}$$

$$+2\gamma\delta \left[N \sum_i x_i y_i - \sum_i x_i \sum_i y_i \right] = \gamma^2 D + \delta^2 H_n + 2\gamma\delta M_n,$$

and,

$$\hat{D} = N \sum_i \hat{x}_i^2 - (\hat{x}_i)^2 = N \sum_i (\alpha x_i + \beta y_i)^2 - \left[\sum_i (\alpha x_i + \beta y_i) \right]^2 =$$

$$= \alpha^2 \left[N \sum_i x_i^2 - \left(\sum_i x_i \right)^2 \right] + \beta^2 \left[N \sum_i y_i^2 - \left(\sum_i y_i^2 \right) \right] + \tag{2.11}$$

$$+2\alpha\beta \left[N \sum_i x_i y_i - \sum_i x_i \sum_i y_i \right] = \alpha^2 D + \beta^2 H_n + 2\alpha\beta M_n.$$

From relations (2.8), (2.9), (2.10), and (2.11), we conclude that the linear coefficients $\hat{M}$ and $\hat{H}$ of the cloud $(\hat{x}_i, \hat{y}_i)$, $i = 1, 2, \ldots, N$, are given by the relations (2.3), (2.4) and the theorem has been proved. $\square$

We denote the set of transformations (2.3) and (2.4) by $\mathcal{T}(A)$. These are transformations of the form

$$\hat{M} = \mathcal{M}(M, H, \alpha, \beta, \gamma, \delta), \qquad \hat{H} = \mathcal{H}(M, H, \alpha, \beta, \gamma, \delta). \tag{2.12}$$

If A is a matrix with entries $(\alpha, \beta; \gamma, \delta)$, then we can associate to it an element of the set $\mathcal{T}(A)$, namely the transformation given by (2.3) and (2.4). We denote this transformation by $\mathcal{T}(A)_{(\alpha,\beta,\gamma,\delta)}$. The following remarks are in order regarding this association:

- The set of transformations $\mathcal{T}(A)$ form a Lie group, under the usual composition of transformations, if and only if the set of matrices A form also a Lie group, under the usual multiplication of matrices, namely the group $GL(2)$, i.e., the group of 2×2 invertible matrices.

- This association is not one−to−one. Indeed, one can easily check that

$$\mathcal{T}(A)_{(\alpha,\beta,\gamma,\delta)} = \mathcal{T}(A)_{(\kappa\alpha,\kappa\beta,\kappa\gamma,\kappa\delta)},$$

  $\kappa \in R,\ \kappa \neq 0$. Therefore *all* matrices κA are associated to the same element $\mathcal{T}(A)_{(\alpha,\beta,\gamma,\delta)}$ of $\mathcal{T}(A)$.

- We can make the association between the sets A and $\mathcal{T}(A)$ one−to−one by assigning arbitrarily a fixed non−zero value to any of the entries $(\alpha, \beta; \gamma, \delta)$ of the matrices of the set A.

- In this paper we prefer not to make this association one−to−one because this may give the false impression that restrictions are imposed to the set of transformations A which act on the cloud of points under consideration.

- Needless to say that the results are identical regardless of whether we make or we do not make the association between the sets A and $\mathcal{T}(A)$ one−to−one.

We note that in the search for invariants we do not need to restrict to the case where A, and therefore $\mathcal{T}(A)$, is a group. As it will become evident from the proof in the next section, and as it will be demonstrated in the example given in subsection 5.2.1., what it is really necessary is that the set A, and subsequently the set $\mathcal{T}(A)$, must contain the identity element. The identity elements of both A and $\mathcal{T}(A)$ are obtained when $\alpha = 1, \beta = 0, \gamma = 0,$ and, $\delta = 1$. For brevity we write $e = (1, 0, 0, 1)$ and we denote by e_i, $i = 1, 2, 3, 4$, its components.

3. Invariants

A main objective in cloud of points theory is that of finding invariants. These are quantities which remain unchanged whenever a cloud of points is transformed under the action of a 2×2 matrix. Invariant quantities enable us to recognize clouds of points arising from the same "source".

In our approach the entities which identify a cloud of points are M and H. Therefore, we are looking for invariants which are functions of these two quantities. This is formalized in the following definition:

Definition 3.1. *Let (x_i, y_i), $i = 1, ..., N$, be a cloud of points on the plane with linear coefficients M and H. Let $(\hat{x}_i, \hat{y}_i)$, $i = 1, \ldots, N$, be the transformation of the cloud (x_i, y_i), $i = 1, \ldots, N$, under a matrix A. Let $\hat{M}$, $\hat{H}$ be the linear coefficients of the cloud $(\hat{x}_i, \hat{y}_i)$, $i = 1, \ldots, N$. $\hat{M}$ and $\hat{H}$ are the transformed values of M and H under the induced set of transformations $\mathcal{T}(A)$. We say that a function $I : \mathbf{R}^2 \to \mathbf{R}$ is a cloud invariant if and only if*

$$I(\hat{M}, \hat{H}) = I(M, H). \tag{3.1}$$

The next theorem is the key result in our study because it provides us with a mechanism for finding cloud invariants. Its proof, is the proof in our case, of Lie's theory fundamental result [1] that the nonlinear condition (3.1) is equivalent to a linear condition provided that the invariant function is properly analytic. The details are as follows:

Theorem 3.1. *Let (x_i, y_i), $i = 1, ..., N$, be a cloud of points on the plane with linear coefficients M and H, and let $(\hat{x}_i, \hat{y}_i)$, $i = 1, ..., N$, be the transformation of the cloud (x_i, y_i), $i = 1, ..., N$, under a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. A function $I : \mathbf{R}^2 \to \mathbf{R}$, analytic in the parameters $\alpha, \beta, \gamma, \delta$, is a cloud invariant if and only if the following equations hold simultaneously*

$$\xi^1_\alpha \frac{\partial I}{\partial M} + \xi^2_\alpha \frac{\partial I}{\partial H} = 0, \tag{3.2}$$

$$\xi^1_\beta \frac{\partial I}{\partial M} + \xi^2_\beta \frac{\partial I}{\partial H} = 0, \tag{3.3}$$

$$\xi^1_\gamma \frac{\partial I}{\partial M} + \xi^2_\gamma \frac{\partial I}{\partial H} = 0, \tag{3.4}$$

$$\xi^1_\delta \frac{\partial I}{\partial M} + \xi^2_\delta \frac{\partial I}{\partial H} = 0, \tag{3.5}$$

where,

$$\xi^1_Q = \left(\frac{\partial \hat{M}}{\partial Q}\right)_e = \frac{\partial \mathcal{M}(M, H, 1, 0, 0, 1)}{\partial Q}, \xi^2_Q = \left(\frac{\partial \hat{H}}{\partial Q}\right)_e = \frac{\partial \mathcal{H}(M, H, 1, 0, 0, 1)}{\partial Q},$$

$Q = \alpha, \beta, \gamma, \delta.$

Proof: Let (x_i, y_i), $i = 1, ..., N$, be a cloud of points on the plane with linear coefficients M and H. Let $(\hat{x}_i, \hat{y}_i)$, $i = 1, \ldots, N$, be the transformation of the cloud (x_i, y_i), $i = 1, \ldots, N$, under a matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Let $\hat{M}$, $\hat{H}$ be the linear coefficients of the cloud $(\hat{x}_i, \hat{y}_i)$, $i = 1, \ldots, N$. $\hat{M}$ and $\hat{H}$ are the transformed values of M and H under the induced set of transformations $\mathcal{T}(A)$, given by equations (2.3) and (2.4). Let $I(\hat{M}, \hat{H})$ be a real−valued function analytic in the parameters $\alpha, \beta, \gamma,$ and δ. The Taylor expansion of $I(\hat{M}, \hat{H})$, with center e, reads:

$$\begin{aligned} I(\hat{M}, \hat{H}) &= I(M, H) + (\alpha - 1)\left(\frac{\partial I(\hat{M}, \hat{H})}{\partial \alpha}\right)_e + \beta\left(\frac{\partial I(\hat{M}, \hat{H})}{\partial \beta}\right)_e + \\ &\quad \gamma\left(\frac{\partial I(\hat{M}, \hat{H})}{\partial \gamma}\right)_e + (\delta - 1)\left(\frac{\partial I(\hat{M}, \hat{H})}{\partial \delta}\right)_e + + \frac{1}{2!}\left((\alpha - 1)^2\right. \\ &\quad \left(\frac{\partial^2 I(\hat{M}, \hat{H})}{\partial \alpha^2}\right)_e + \beta^2\left(\frac{\partial^2 I(\hat{M}, \hat{H})}{\partial \beta^2}\right)_e + \gamma^2\left(\frac{\partial^2 I(\hat{M}, \hat{H})}{\partial \gamma^2}\right)_e + \end{aligned}$$

$$(\delta-1)^2\left(\frac{\partial^2 I(\hat{M},\hat{H})}{\partial\delta^2}\right)_e + 2(\alpha-1)\beta\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e + \cdots$$

$$2\gamma(\delta-1)\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\gamma}\right)_e\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\delta}\right)_e\Bigg) + \cdots\,. \tag{3.6}$$

The form of the functional dependence of $I(\hat{M},\hat{H})$ on the parameters $\alpha,\beta,\gamma,\delta$ allows to simplify (3.6) in a way which suggests the conclusion of the theorem. To illustrate the point at hand we use the chain rule to evaluate the derivative $\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e$:

$$\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e = \left(\frac{\partial I(\hat{M},\hat{H})}{\partial\hat{M}}\frac{\partial\hat{M}}{\partial\alpha} + \frac{\partial I(\hat{M},\hat{H})}{\partial\hat{H}}\frac{\partial\hat{H}}{\partial\alpha}\right)_e \tag{3.7}$$

$$\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e = \left(\frac{\partial I(\hat{M},\hat{H})}{\partial\hat{M}}\right)_e\left(\frac{\partial\hat{M}}{\partial\alpha}\right)_e + \left(\frac{\partial I(\hat{M},\hat{H})}{\partial\hat{H}}\right)_e\left(\frac{\partial\hat{H}}{\partial\alpha}\right)_e \tag{3.8}$$

By introducing the quantities:

$$\xi^1_Q = \left(\frac{\partial\hat{M}}{\partial Q}\right)_e,\ \xi^2_Q = \left(\frac{\partial\hat{H}}{\partial Q}\right)_e,\ Q = \alpha,\beta,\gamma,\delta, \tag{3.9}$$

and by noting

$$\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\hat{M}}\right)_e = \frac{\partial I(M,H)}{\partial M},\ \text{and,}\ \left(\frac{\partial I(\hat{M},\hat{H})}{\partial\hat{\mathrm{H}}}\right)_e = \frac{\partial I(M,H)}{\partial H}, \tag{3.10}$$

we can rewrite equation (3.8) in the following shorter form

$$\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e = \xi^1_\alpha\frac{\partial I(M,H)}{\partial M} + \xi^2_\alpha\frac{\partial I(M,H)}{\partial H} \tag{3.11}$$

By introducing the operator

$$X_\alpha = \xi^1_\alpha\frac{\partial}{\partial M} + \xi^2_\alpha\frac{\partial}{\partial H}, \tag{3.12}$$

we rewrite equation (3.11) as

$$\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e = X_\alpha I, \tag{3.13}$$

where for short we wrote I instead of $I(M,H)$. For the second order derivative $\left(\frac{\partial^2 I(\hat{M},\hat{H})}{\partial\alpha^2}\right)_e$ we have:

$$\begin{aligned}\left(\frac{\partial^2 I(\hat{M},\hat{H})}{\partial\alpha^2}\right)_e &= \left(\frac{\partial\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)}{\partial\alpha}\right)_e = \xi^1_\alpha \frac{\partial\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e}{\partial M} + \xi^2_\alpha \frac{\partial\left(\frac{\partial I(\hat{M},\hat{H})}{\partial\alpha}\right)_e}{\partial H} \\ &= X_\alpha\left(X_\alpha I\right) \qquad (3.14)\end{aligned}$$

A similar analysis applies to the derivatives of all orders in the Taylor expansion (3.6). As a result the Taylor expansion (3.6) reads:

$$\begin{aligned}I(\hat{M},\hat{H}) &= I(M,H) + \sum_{i=1}^{4}(\mathcal{Q}_i - e_i)\left(X_{\mathcal{Q}_i} I\right) + \\ &\frac{1}{2!}\sum_{i,j=1}^{4}(\mathcal{Q}_i - e_i)(\mathcal{Q}_j - e_j)X_{\mathcal{Q}_i}\left(X_{\mathcal{Q}_j} I\right) + \\ &\frac{1}{3!}\sum_{i,j,k=1}^{4}(\mathcal{Q}_i - e_i)(\mathcal{Q}_j - e_j)(\mathcal{Q}_k - e_k)X_{\mathcal{Q}_i}\left(X_{\mathcal{Q}_j}\left(X_{\mathcal{Q}_k} I\right)\right) + \cdots \qquad (3.15)\end{aligned}$$

For convenience, by $\mathcal{Q}$ we denote the vector $(\alpha,\beta,\gamma,\delta)$, and by $\mathcal{Q}_i$, $i = 1,2,3,4$, its components.

From equation (3.15) we conclude that when the *linear infinitesimal* conditions

$$X_{\mathcal{Q}_i} I = 0, \quad \mathcal{Q}_i = \alpha,\beta,\gamma,\delta, \qquad (3.16)$$

are satisfied then $I(\hat{M},\hat{H}) = I(M,H)$. Therefore I is a cloud invariant.

Conversely, when I is a cloud invariant, then $I(\hat{M},\hat{H}) = I(M,H)$, and equation (3.15) gives:

$$\begin{aligned}&\sum_{i=1}^{4}(\mathcal{Q}_i - e_i)\left(X_{\mathcal{Q}_i} I\right) + \frac{1}{2!}\sum_{i,j=1}^{4}(\mathcal{Q}_i - e_i)(\mathcal{Q}_j - e_j)X_{\mathcal{Q}_i}\left(X_{\mathcal{Q}_j} I\right) + \\ &\frac{1}{3!}\sum_{i,j,k=1}^{4}(\mathcal{Q}_i - e_i)(\mathcal{Q}_j - e_j)(\mathcal{Q}_k - e_k)X_{\mathcal{Q}_i}\left(X_{\mathcal{Q}_j}\left(X_{\mathcal{Q}_k} I\right)\right) + \cdots = 0 \qquad (3.17)\end{aligned}$$

For every pair of values M and H equation (3.17) becomes a polynomial in the variables $\alpha,\beta,\gamma,\delta$. Consequently equation (3.17) can only hold if for every pair of values M and H the coefficients of the polynomial vanish, i.e., if the following relations hold

$$\begin{aligned}X_{\mathcal{Q}_i} I &= X_{\mathcal{Q}_i}\left(X_{\mathcal{Q}_j} I\right) = \\ &X_{\mathcal{Q}_i}\left(X_{\mathcal{Q}_j}\left(X_{\mathcal{Q}_k} I\right)\right) = \cdots = 0, \quad i,j,k \in \{1,2,3,4\}, \qquad (3.18)\end{aligned}$$

for every pair of values M and H. If $X_{\mathcal{Q}_i} I = 0, \quad \mathcal{Q}_i = \alpha, \beta, \gamma, \delta$, the rest of the relations (3.18) follow. Equations (3.16), $X_{\mathcal{Q}_i} I = 0, \quad \mathcal{Q}_i = \alpha, \beta, \gamma, \delta$, are nothing but equations (3.2), (3.3), (3.4), and (3.5), respectively. This completes the proof. □

Sophus Lie's great advance was to replace the complicated, nonlinear finite invariance condition (3.1) by the vastly more useful linear infinitesimal condition (3.16) and to recognize that if a function satisfies the infinitesimal condition then it also satisfies the finite condition, and vice versa, provided that the function is analytic in the parameters α, β, γ, and δ.

It is to be noted that in the proof of Lie's main Theorem (3.1) we used the following:

1. The assumption that cloud invariant I is a function analytic in the parameters α, β, γ, and δ.
2. The assumption that the set of transformations $\mathcal{T}(A)$, and subsequently the set of transformations A, contain the identity element, which is obtained when $\alpha = 1,\ \beta = 0,\ \gamma = 0$, and $\delta = 1$.
3. The chain rule for the differentiation of composite functions.

Nowhere in the proof of Lie's main Theorem (3.1) is the assumption made that the set of transformations $\mathcal{T}(A)$ is closed under the usual composition of transformations, or equivalently, that the associated set of matrices A is closed under the usual matrix multiplication. This will become evident and exemplified in subsection 5.2.1. where we find cloud invariants I under a set of transformations $\mathcal{T}(A)$ which are such that the associated set of matrices A are not closed under the usual multiplication of matrices.

3.1. Cloud invariants in the general case

As a first application of Theorem (3.1) we find the cloud invariants under a general matrix A. This is the content of the next Corollary.

Corollary 3.1. *If a cloud of points is transformed via a matrix* $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, *then the only cloud invariants are:*

1. *The constant functions* $I(M, H) = c$, $c \in R$.
2. *The level curve* $I(M, H) = 0$ *of the function* $I(M, H) = \dfrac{H}{M^2} - 1$.

Proof: According to Theorem (3.1) a function $I(\hat{M}, \hat{H})$, analytic in the parameters $\alpha, \beta, \gamma, \delta$, is a cloud invariant if and only if it satisfies the system of PDEs (3.2), (3.3), (3.4), and (3.5), which read:

$$\xi^1_\alpha(\mathbf{x})\frac{\partial I}{\partial M}+\xi^2_\alpha(\mathbf{x})\frac{\partial I}{\partial H} = -M\frac{\partial I}{\partial M}-2H\frac{\partial I}{\partial H}=0, \tag{3.19}$$

$$\xi^1_\beta(\mathbf{x})\frac{\partial I}{\partial M}+\xi^2_\beta(\mathbf{x})\frac{\partial I}{\partial H} = (H-2M^2)\frac{\partial I}{\partial M}-2HM\frac{\partial I}{\partial H}=0, \tag{3.20}$$

$$\xi^1_\gamma(\mathbf{x})\frac{\partial I}{\partial M}+\xi^2_\gamma(\mathbf{x})\frac{\partial I}{\partial H} = \frac{\partial I}{\partial M}-2M\frac{\partial I}{\partial H}=0, \tag{3.21}$$

$$\xi^1_\delta(\mathbf{x})\frac{\partial I}{\partial M}+\xi^2_\delta(\mathbf{x})\frac{\partial I}{\partial H} = M\frac{\partial I}{\partial M}+2H\frac{\partial I}{\partial H}=0. \tag{3.22}$$

We easily find that the only solutions to the last system of equations are:

1. The constant functions $I(M,H)=c$, $c\in R$.
2. The level curve $I(M,H)=0$ of the function $I(M,H)=\dfrac{H}{M^2}-1$.

This completes the proof.□

The second invariant implies in particular that when the values of M and H are such that $H=M^2$, then their transformed values $\hat{H}$ and $\hat{M}$ are such that $\hat{H}=\hat{M}^2$.

4. The One−Parameter Case

The cloud invariants under a general matrix A, given in Corollary 3.1, do not lend themselves to practical implementation. This leads us to examining particular cases of A. We start by considering the one−parameter case, i.e. the case where the entries of the matrix A are analytic functions of a single parameter φ. Interestingly enough it turns out that in this case we can find cloud invariants in closed form which can be practically implemented. The first step to prove this assertion is the next Corollary.

Corollary 4.1. *Let (x_i,y_i), $i=1,...,N$, be a cloud of points on the plane with linear coefficients M and H, and let $(\hat{x}_i,\hat{y}_i)$, $i=1,...,N$, be the transformation of the cloud (x_i,y_i), $i=1,...,N$, under a matrix $A(\varphi)=\begin{pmatrix}\alpha(\varphi) & \beta(\varphi)\\ \gamma(\varphi) & \delta(\varphi)\end{pmatrix}$, where $\alpha(\varphi)$, $\beta(\varphi)$, $\gamma(\varphi)$, and $\delta(\varphi)$, are real analytic functions of a parameter $\varphi\in\mathbf{R}$. We assume that there exists a value of φ, denoted by φ^*, such that $A(\varphi^*)=\mathbf{I}$, $\mathbf{I}$ the 2×2 identity matrix. An analytic function $I:\mathbf{R}^2\to\mathbf{R}$ is a cloud invariant if and only if the next equation holds:*

$$[(H-2M^2)\beta'(\varphi^*)-\delta M+\gamma'(\varphi^*)]\frac{\partial I}{\partial M}+2[\gamma'(\varphi^*)M-\beta'(\varphi^*)HM-\delta H]\frac{\partial I}{\partial H}=0, \tag{4.1}$$

where, $\delta=\alpha'(\varphi^*)-\delta'(\varphi^*)$.

Proof: In order to find the cloud invariants we apply Theorem 3.1. The key point is that in the case under consideration *all* the entries of the matrix $A(\varphi)$ are functions of a single parameter φ. This implies in particular that equations (3.2), (3.3), (3.4), and (3.5), whose solution space are the cloud invariants, reduce to one equation:

$$\xi^1_\varphi \frac{\partial I}{\partial M} + \xi^2_\varphi \frac{\partial I}{\partial H} = 0, \tag{4.2}$$

$$\xi^1_\varphi = \left(\frac{\partial \hat{M}}{\partial \varphi}\right)_e = \frac{\partial \mathcal{M}(M,H,1,0,0,1)}{\partial \varphi}, \xi^2_\varphi = \left(\frac{\partial \hat{H}}{\partial \varphi}\right)_e = \frac{\partial \mathcal{H}(M,H,1,0,0,1)}{\partial \varphi}.$$

Differentiation is now with respect to φ, that is $Q = \varphi$. Consequently cloud invariants in the one−parameter case are solutions to equation (4.2) which reads:

$$[(H-2M^2)\beta'(\varphi^*) - \delta M + \gamma'(\varphi^*)]\frac{\partial I}{\partial M} + 2[\gamma'(\varphi^*)M - \beta'(\varphi^*)HM - \delta H]\frac{\partial I}{\partial H} = 0,$$

where, $\delta = \alpha'(\varphi^*) - \delta'(\varphi^*)$. This completes the proof. □

It is difficult to obtain in closed form the whole set of solutions of equation (4.1). However we can find, in closed form, a wide subclass of solutions of equation (4.1). This is the content of the following Theorem.

Theorem 4.1. *A class of solutions of equation (4.1), and hence a family of invariants of a cloud of points* (x_i, y_i), $i = 1, ..., N$, *when it is transformed under a matrix* $A(\varphi) = \begin{pmatrix} \alpha(\varphi) & \beta(\varphi) \\ \gamma(\varphi) & \delta(\varphi) \end{pmatrix}$, *is given by:*

$$I(M,H) = F\left(\frac{M^2 - H}{(H\beta'(\varphi^*) - \gamma'(\varphi^*) + \delta M)^2}\right), \tag{4.3}$$

where $F(.)$, *is an arbitrary real valued function and* $\delta = \alpha'(\varphi^*) - \delta'(\varphi^*)$. *We assume that* $\alpha(\varphi)$, $\beta(\varphi)$, $\gamma(\varphi)$, *and* $\delta(\varphi)$, *are real analytic functions of a single parameter* $\varphi \in \mathbf{R}$. *We also assume that there exists a value of* φ, *denoted by* φ^*, *such that* $A(\varphi^*) = \mathbf{I}$, $\mathbf{I}$ *being the* 2×2 *identity matrix.*

Proof: In order to find solutions of equation (4.1) we use the undetermined coefficients method. This method consists in seeking for solutions $I(M,H)$ of the form:

$$F\left(\frac{\sum_{i=1}^n \sum_{j=1}^m a_{ij} M^i H^j}{\sum_{i=1}^n \sum_{j=1}^m b_{ij} M^i H^j}\right), \tag{4.4}$$

where a_{ij} and b_{ij} are unknown coefficients to be determined. By substituting this particular form of the solution into equation (4.1) we obtain that a polynomial in the two variables M and H is equal to zero. The resulting condition, the coefficients of the polynomial are equal to zero, gives solution (4.3). This completes the proof. □

A Corollary of the previous theorem is that we can find cloud invariants, when the cloud is transformed under a matrix, provided the matrix is an element of the one−parameter set of transformations $A(\varphi)$, considered in this theorem.

Corollary 4.2. *Let* (x_i, y_i), $i = 1, ..., N$, *be a cloud of points and let this cloud be transformed under a matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $aij \in \mathbf{R}$. *If there exist real valued, analytic, functions* $\alpha(\varphi), \beta(\varphi), \gamma(\varphi), \delta(\varphi)$ *and values* φ^*, φ_1, *such that:*

1. $\alpha(\varphi^*) = 1, \beta(\varphi^*) = 0, \gamma(\varphi^*) = 0, \delta(\varphi^*) = 1$
2. $\alpha(\varphi_1) = a_{11}, \beta(\varphi_1) = a_{12}, \gamma(\varphi_1) = a_{21}, \delta(\varphi_1) = a_{22}$

then, the quantity

$$I(M, H) = F\left(\frac{M^2 - H}{(H\beta'(\varphi^*) - \gamma'(\varphi^*) + \delta M)^2}\right),$$

where $F(.)$ *is an arbitrary real valued function and* $\delta = \alpha'(\varphi^*) - \delta'(\varphi^*)$, *is a cloud invariant.*

Proof: This is an immediate consequence of Theorem 4.1. □

5. Examples of Invariants

In this section, by using Theorem theorem 4.1, we find cloud invariants when a cloud is transformed under various sets of transformations $A(\varphi)$. As we pointed out in section 2. it is not necessary the set $A(\varphi)$ to form a group under the usual multiplication of matrices. Firstly we consider sets of transformations $A(\varphi)$ which do form a group and then we consider a set $A(\varphi)$ which does not form a group. Finally, in the last subsection, by using the previous findings, we find cloud invariants for any given matrix.

5.1. Sets $A(\varphi)$ which form a group

We start with simpler sets of transformations $A(\varphi)$ and gradually proceed to more general cases.

5.1.1. $A(\varphi)$ is a diagonal matrix

We start by assuming that $A(\varphi)$ is diagonal and has the form:

$$A(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & \varphi \end{pmatrix}, \tag{5.1}$$

where $\varphi \in \mathbf{R}$.

In this case we can easily see that $\varphi^* = 1$ and that $\beta'(\varphi^*) = 0$, $\gamma'(\varphi^*) = 0$, $\delta = \alpha'(\varphi^*) - \delta'(\varphi^*) = -1$. Therefore, according to Theorem 4.1, a family of cloud invariants is:

$$F\left(\frac{M^2 - H}{(-M)^2}\right) = h\left(\frac{H}{M^2}\right), \tag{5.2}$$

where $F(\cdot)$ and $h(\cdot)$ are arbitrary real valued functions.

5.1.2. $A(\varphi)$ is an upper triangular matrix

We assume that $A(\varphi)$ is upper triangular and has the form:

$$A(\varphi) = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}, \tag{5.3}$$

$\varphi \in \mathbf{R}$.

In this case we easily verify that $\varphi^* = 0$ and that $\beta'(\varphi^*) = 1$, $\gamma'(\varphi^*) = 0$, $\delta = \alpha'(\varphi^*) - \delta'(\varphi^*) = 0$. Consequently, according to Theorem 4.1, a family of cloud invariants is:

$$F\left(\frac{M^2 - H}{(H \cdot 1)^2}\right) = F\left(\frac{M^2 - H}{H^2}\right), \tag{5.4}$$

where $F(\cdot)$ is an arbitrary real valued function.

5.1.3. $A(\varphi)$ is a lower triangular matrix

We assume that $A(\varphi)$ is lower triangular and has the form:

$$A(\varphi) = \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}, \tag{5.5}$$

$\varphi \in \mathbf{R}$.

In this case we easily find that $\varphi^* = 0$ and that $\beta'(\varphi^*) = 0$, $\gamma'(\varphi^*) = 1$, $\delta = \alpha'(\varphi^*) - \delta'(\varphi^*) = 0$. Consequently, according to Theorem 4.1, a family of cloud invariants is:

$$F\left(\frac{M^2 - H}{(-1)^2}\right) = h(H - M^2), \tag{5.6}$$

where $F(\cdot)$ and $h(\cdot)$ are arbitrary real valued functions.

5.1.4. $A(\varphi)$ is a rotation matrix

Finally, we assume that $A(\varphi)$ is a rotation matrix and has the form:

$$A(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \tag{5.7}$$

$\varphi \in \mathbf{R}$.

In this case we easily obtain that $\varphi^* = 0$ and that $\beta'(\varphi^*) = 1,\ \gamma'(\varphi^*) = -1,\ \delta = \alpha'(\varphi^*) - \delta'(\varphi^*) = 0$. Consequently, according to Theorem 4.1, a family of cloud invariants is:

$$F\left(\frac{M^2 - H}{(H+1)^2}\right), \tag{5.8}$$

where $F(\cdot)$ is an arbitrary real valued function.

5.2. A set $A(\varphi)$ which does not form a group

5.2.1. A "linear" matrix

A set of transformations $A(\varphi)$ which subsumes the sets of transformations considered in subsections 5.1.1., 5.1.2., and 5.1.3. is the set of "linear" matrices

$$A(\varphi) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \varphi \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11}\varphi & a_{12} + b_{12}\varphi \\ a_{21} + b_{21}\varphi & a_{22} + b_{22}\varphi \end{pmatrix}, \tag{5.9}$$

where $a_{ij}, b_{ij} \in \mathbf{R}$, and φ is a real free parameter. The set of matrices $A(\varphi^*)$ does not, in general, form a group under matrix multiplication. However, we assume that there exists a value of φ, denoted by φ^*, such that $A(\varphi^*) = \mathbf{I}$, $\mathbf{I}$ being the 2×2 identity matrix. One can easily check that such a value φ^* exists if and only if the entries a_{ij}, b_{ij} satisfy one of the following conditions:

$$b_{22} \neq 0 \wedge a_{21} = \frac{(a_{22}-1)\,b_{21}}{b_{22}} \wedge a_{12} = \frac{(a_{22}-1)\,b_{12}}{b_{22}} \wedge a_{11} = \frac{a_{22}b_{11} - b_{11} + b_{22}}{b_{22}}, \tag{5.10}$$

or

$$b_{22} = 0 \wedge a_{22} = 1 \wedge b_{21} \neq 0 \wedge a_{12} = \frac{a_{21}b_{12}}{b_{21}} \wedge a_{11} = \frac{a_{21}b_{11} + b_{21}}{b_{21}}, \tag{5.11}$$

or

$$b_{22} = 0 \wedge b_{21} = 0 \wedge a_{22} = 1 \wedge a_{21} = 0 \wedge b_{12} \neq 0 \wedge a_{11} = \frac{a_{12}b_{11} + b_{12}}{b_{12}}, \tag{5.12}$$

or

$$b_{22} = 0 \wedge b_{21} = 0 \wedge b_{12} = 0 \wedge a_{22} = 1 \wedge a_{21} = 0 \wedge a_{12} = 0 \wedge b_{11} \neq 0. \tag{5.13}$$

In this case we easily find that $\beta'(\varphi^*) = b_{12},\ \gamma'(\varphi^*) = b_{21},\ \delta = \alpha'(\varphi^*) - \delta'(\varphi^*) = b_{11} - b_{22}$. According to Theorem 4.1, a family of cloud invariants is:

$$F\left(\frac{M^2 - H}{(Hb_{12} - b_{21} + (b_{11} - b_{22})M)^2}\right), \tag{5.14}$$

where $F(\cdot)$ is an arbitrary real valued function.

5.2.2. Cloud invariants for an arbitrary matrix

For *any* given matrix there always exists a set of transformations $A(\varphi)$, of the form (5.9), which contains this matrix. In fact one can easily prove that there exists a two parameter family of such sets $A(\varphi)$. According to Corollary 4.2, a family of cloud invariants, when a cloud of points is transformed under this matrix, is given by relation (5.14). As a case study we consider the matrix:

$$\mathcal{M} = \begin{pmatrix} 0.4 & -0.4 \\ -0.05 & 0.9 \end{pmatrix}. \tag{5.15}$$

Let $A(\varphi)$ be the set of transformations:

$$A(\varphi) = \begin{pmatrix} -2 & -2 \\ -1/4 & 1/2 \end{pmatrix} + \varphi \begin{pmatrix} 12 & 8 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2+12\varphi & -2+8\varphi \\ -1/4+\varphi & 1/2+2\varphi \end{pmatrix}. \tag{5.16}$$

One can easily check that $A(0.2) = \mathcal{M}$. We have $b_{12} = 8,\ b_{21} = 1,\ b_{11} - b_{22} = 10$. According to Corollary 4.2, a family of cloud invariants is:

$$F\left(\frac{M^2 - H}{(8H - 1 + 10M)^2}\right), \tag{5.17}$$

where $F(\cdot)$ is an arbitrary real valued function. Since there exists a two parameter family of sets $A(\varphi)$, of the form (5.9), which contain $\mathcal{M}$, there exists a two parameter family of cloud invariants of the form (5.17), when a cloud is transformed via $\mathcal{M}$. However, the explicit form of this two parameter family of cloud invariants is not needed here.

6. Simulations

To see how the above theory works in practise, we consider a cloud of 10.000 points forming the scheme of Figure 6.1. Using relations (2.1) and (2.2), we calculate the linear coefficients M and H of the cloud. We find $M = 1.52244$ and $H = 2.46998$. We transform now this cloud by using various matrices.

Firstly we let the the diagonal matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ to act on the cloud. Both the initial and the transformed schemes are depicted in Figure 6.2. Using relations (2.3) and (2.4) we calculate the linear coefficients of the new cloud and we obtain $\hat{M} = 3.04488$ and $\hat{H} = 9.87991$. According to our findings in subsection 5.1.1., a family of cloud invariants, when a cloud is transformed with the diagonal matrix A, is $I(M,H) = F(H/M^2)$, where $F(\cdot)$ is an arbitrary real valued function. Indeed, for the initial and the transformed linear coefficients, we find $H/M^2 = \hat{H}/\hat{M}^2 = 1.06564$. It follows that we have $I(M,H) = I(\hat{M},\hat{H})$ for *any* real valued function $F(\cdot)$.

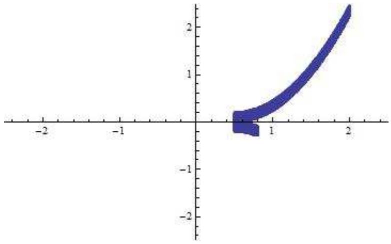

FIG. 6.1: The Original Scheme

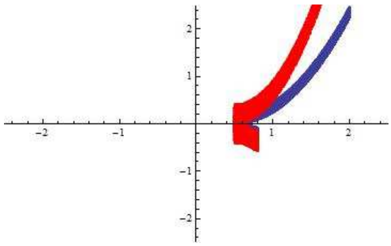

FIG. 6.2: A Diagonal Transformation

As a second example of transformation, we let the upper triangular matrix $B = \begin{pmatrix} 1 & 0.7 \\ 0 & 1 \end{pmatrix}$ to act on the cloud. The result of this transformation is given in Figure 6.3. The transformation of the cloud we consider under B has linear coefficients $\hat{M} = 0.748882$ and $\hat{H} = 0.568896$. As we found in subsection 5.1.2., a family of cloud invariants, when a cloud is transformed with the upper triangular matrix B, is $I(M, H) = F((M^2 - H)/H^2)$, where $F(\cdot)$ is an arbitrary real valued function. Indeed, we have $(M^2 - H)/H^2 = (\hat{M}^2 - \hat{H})/\hat{H}^2 = -0.0249396$. Consequently we have $I(M, H) = I(\hat{M}, \hat{H})$ for *any* real valued function $F(\cdot)$.

As a third example of transformation, we act on the cloud of points with a rota-

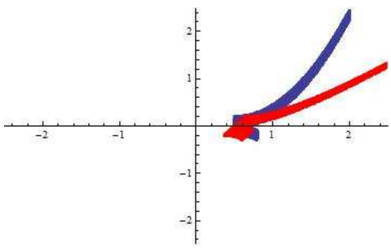

FIG. 6.3: An Upper Triangular Transformation

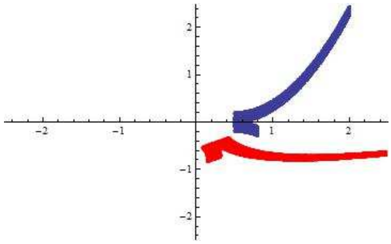

FIG. 6.4: A Rotation

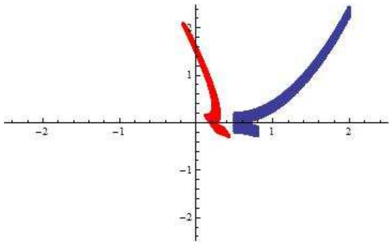

FIG. 6.5: An Arbitrary Matrix

tion matrix $C = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$. The initial and the rotated clouds are shown in Figure 6.4. The linear coefficients of the rotated cloud are $\hat{M} = -0.0364518$ and $\hat{H} = 0.0143303$. We found in subsection 5.1.4., that a family of cloud invariants, when a cloud is transformed with the rotation matrix C, is $I(M,H) = F\left(\frac{M^2 - H}{(H+1)^2}\right)$, where $F(\cdot)$ is an arbitrary real valued function. Indeed, we have $\frac{M^2 - H}{(H+1)^2} = \frac{\hat{M}^2 - \hat{H}}{(\hat{H}+1)^2} = -0.0126368$. Henceforth we have $I(M,H) = I(\hat{M},\hat{H})$, for *any* real valued function $F(\cdot)$.

Finally, we act on the cloud of points with the matrix $D = \begin{pmatrix} 0.4 & -0.4 \\ -0.05 & 0.9 \end{pmatrix}$. The result of this transformation is shown in Figure 6.5. The linear coefficients of the transformed cloud are $\hat{M} = -4.86159$ and $\hat{H} = 27.4371$. We found in subsection 5.2.1., that a family of cloud invariants, when a cloud is transformed with the matrix D, is $I(M,H) = F\left(\frac{M^2 - H}{(8H - 1 + 10M)^2}\right)$, where $F(\cdot)$ is an arbitrary real valued function. Indeed, we have $\frac{M^2 - H}{(8H - 1 + 10M)^2} = \frac{\hat{M}^2 - \hat{H}}{(8\hat{H} - 1 + 10\hat{M})^2} = -0.000131743$. Consequently we obtain $I(M,H) = I(\hat{M},\hat{H})$, for *any* real valued function $F(\cdot)$.

We note that the results we obtained by considering the aforementioned cloud of points verify our findings in section 5.

7. Concluding Remarks

We have studied transformations, with 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha, \beta, \gamma, \delta \in \mathbf{R}$, of planar set of points, called clouds of points for convenience. Our aim in this paper is to find cloud invariants, i.e. functions which take the same value when they are evaluated for the initial and for the transformed cloud of points. It is natural the cloud invariants to be functions of variables which carry information for the whole cloud. The cloud invariants we find are functions of two such variables M and H.

M and H are functions of the coordinates of the points of the cloud. As a result we find that any transformation of a cloud of points by a 2×2 matrix induces a nonlinear transformation $(M, H) \to (\hat{M}, \hat{H})$, $\hat{M} = \mathcal{M}(M, H, \alpha, \beta, \gamma, \delta)$, $\hat{H} = \mathcal{H}(M, H, \alpha, \beta, \gamma, \delta)$, given explicitly by equations (2.3) and (2.4), of the variables M and H.

M and H originate from the best fitting straight line through the cloud of points under consideration. This straight line is determined by the least squares fitting technique. Henceforth by definition a cloud invariant is any function $I(M, H)$ which satisfies the relation (3.1), $I(M, H) = I(\hat{M}, \hat{H})$, where $\hat{M}$ and $\hat{H}$ are the values of the variables M and H for the transformed cloud.

We find cloud invariants by using Lie theory. Lie theory replaces the complicated, nonlinear finite invariance condition (3.1) by the more useful and tractable linear infinitesimal condition (3.16) provided that the function $I(\hat{M}, \hat{H})$ is analytic in the parameters α, β, γ, and δ. Linear condition (3.16) is a set of linear PDEs. Any solution to this system of PDEs gives a cloud invariant.

Cloud invariants can be practically implemented in various fields, e.g. in optical character recognition, in image analysis and computer graphics techniques, by providing the necessary tools in order to identify icons created by the same "source". The cloud invariants we find for the general four−parameter case, when a cloud is transformed with a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, cannot be practically implemented.

However, the cloud invariants we find for various one−parameter groups of transformations can be practically implemented. In particular we find cloud invariants for a group consisting of diagonal matrices, for a group consisting of upper triangular matrices, for a group consisting of lower triangular matrices, and for the group of rotations $SO(2)$.

More importantly, for the practical implementation of our findings, we find cloud invariants for *any* given matrix. We find these cloud invariants by noticing that *any* given matrix belongs to a one−parameter "linear" set of transformations of the form $\mathcal{A} + \mathcal{B}\varphi$, $\mathcal{A}$ and $\mathcal{B}$ are given 2×2 matrices, and $\varphi \in \mathbf{R}$. Our findings are verified by examples and simulations in a cloud of 10.000 points.

We expressed the cloud invariants in terms of the variables M and H. M and H are essentially the coefficients of the straight line which is the best fit for the cloud of points under consideration. This provides a natural guide for future research. With a view to apply our results in fields such as character recognition, the next logical step is to consider the case where the cloud of points originates from an icon which has a parabolic−like shape.

In this case we will look for cloud invariants which are expressed in terms of variables which appear as coefficients, or variations thereof, of the parabola which is the best fit for the cloud of points under consideration. For similar reasons, subsequently, we will look for new cloud invariants expressed in terms of variables which appear as coefficients in third or higher degree curves. We will compare our findings, with simulations and computational experiments, with those acquired by other approaches.

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