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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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m-WEAK AMENABILITY OF $(2n)$TH DUALS OF BANACH ALGEBRAS

Mina Ettefagh

Abstract. Let A be a Banach algebra such that its $(2n)$th dual for some $(n \geq 1)$ with first Arens product is $m-$weakly amenable for some $(m > 2n)$. We introduce some conditions by which if m is odd [even], then A is weakly [$2-$weakly] amenable.
Keywords. Banach Algebra; Amenability; normed spaces; bilinear map.

1. Introduction and Preliminaries

Let X be a normed space and $X^{'}$ be the topological dual space of X; the value of $f \in X^{'}$ at $x \in X$ is denoted by $\langle f, x\rangle$. By writing $(X^{'})^{'} = X^{''}$ we regard X as a subspace of $X^{''}$ by means of the natural mapping $i : X \to X^{''}(x \longmapsto \widehat{x})$ where $\langle \widehat{x}, f\rangle = \langle f, x\rangle (f \in X^{'})$. Also we denote the nth dual of X by $X^{(n)}$. The weak topology on X is denoted by $w = \sigma(X, X^{'})$ and weak$^*-$topology on $X^{'}$ is denoted by $w^* = \sigma(X^{'}, X)$.
Now let X, Y and Z be normed spaces and $f : X \times Y \to Z$ be a continuous bilinear map. Arens in [2] offers two extensions f^{***} and f^{t***t} of f from $X^{''} \times Y^{''}$ to $Z^{''}$ as following

$$(1) f^* : Z^{'} \times X \longrightarrow Y^{'} \ \left(\left\langle f^*(z^{'}, x), y\right\rangle = \left\langle z^{'}, f(x, y)\right\rangle\right),$$

$$(2) f^{**} : Y^{''} \times Z^{'} \longrightarrow X^{'} \left(\left\langle f^{**}(y^{''}, z^{'}), x\right\rangle = \left\langle y^{''}, f^*(z^{'}, x)\right\rangle\right),$$

$$(3) f^{***} : X^{''} \times Y^{''} \longrightarrow Z^{''} \left(\left\langle f^{***}(x^{''}, y^{''}), z^{'}\right\rangle = \left\langle x^{''}, f^{**}(y^{''}, z^{'})\right\rangle\right).$$

The mapping f^{***} is the unique extension of f such that $x^{''} \longmapsto f^{***}(x^{''}, y^{''})$ from $X^{''}$ into $Z^{''}$ is $w^* - w^*-$continuous for every $y^{''} \in Y^{''}$. Let now $f^t : Y \times X \to Z$ be the transpose of f defined by $f^t(y, x) = f(x, y)$ for $x \in X$ and $y \in Y$. We can extend f^t as above to f^{t***} and then we have the mapping $f^{t***t} : X^{''} \times Y^{''} \longrightarrow Z^{''}$.

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If $f^{***} = f^{t****t}$ then f is called *Arens regular*. The mapping $y^{''} \longmapsto f^{t****t}(x^{''}, y^{''})$ from $Y^{''}$ into $Z^{''}$ is $w^* - w^*-$continuous for every $x^{''} \in X^{''}$. Arens regularity of f is equivalent to the following equality

$$\lim_i \lim_j \left\langle z^{'}, f(x_i, y_i) \right\rangle = \lim_j \lim_i \left\langle z^{'}, f(x_i, y_i) \right\rangle,$$

whenever both limits exist for any $z^{'} \in Z^{'}$ and all bounded nets (x_i) and (y_j) that w^*-converges to $x^{''} \in X^{''}$ and $y^{''} \in Y^{''}$, respectively.
Throughout this paper A is a Banach algebra. This algebra is called *Arens regular* if its multiplication as a bilinear map $\pi : A \times A \to A(\pi(a, b) = ab)$ is Arens regular. We shall frequently use Goldstine's theorem: for each $a^{''} \in A^{''}$, there is a net (a_i) in A such that $a^{''} = w^* - \lim_i \widehat{a_i}$. Now let $a^{''} = w^* - \lim_i \widehat{a_i}$ and $b^{''} = w^* - \lim_j \widehat{b_j}$ be elements of $A^{''}$. The first and second *Arens products* on $A^{''}$ are denoted by symbols $\Box$ and $\Diamond$ respectively and defined by

$$a^{''} \Box b^{''} = \pi^{***}(a^{''}, b^{''}) \quad , \quad a^{''} \Diamond b^{''} = \pi^{t****t}(a^{''}, b^{''}).$$

It is easy to show that

$$a^{''} \Box b^{''} = w^* - \lim_i w^* - \lim_j \widehat{a_i b_j} \quad , \quad a^{''} \Diamond b^{''} = w^* - \lim_j w^* - \lim_i \widehat{a_i b_j}.$$

On the other hand, we can define above Arens products in stages as follows. Let $a, b \in A$, $f \in A^{'}$ and $F, G \in A^{''}$.

(1) Define $f.a$ in $A^{'}$ by $\langle f.a, b\rangle = \langle f, ab\rangle$, and $a.f$ in $A^{'}$ by $\langle a.f, b\rangle = \langle f, ba\rangle$.

(2) Define $F.f$ in $A^{'}$ by $\langle F.f, a\rangle = \langle F, f.a\rangle$, and $f.F$ in $A^{'}$ by $\langle f.F, a\rangle = \langle F, a.f\rangle$.

(3) Define $F\Box G$ in $A^{''}$ by $\langle F\Box G, f\rangle = \langle F, G.f\rangle$, and $F\Diamond G$ in $A^{''}$ by $\langle F\Diamond G, f\rangle = \langle G, f.F\rangle$.

Then $(A^{''}, \Box)$ and $(A^{''}, \Diamond)$ are Banach algebras, see [2, 7] for further details.

Now let E be a Banach $A-$module, then $E^{'}$ is a Banach $A-$module under actions

$$\langle a.f, x\rangle = \langle f, xa\rangle, \langle f.a, x\rangle = \langle f, ax\rangle \qquad (a \in A, x \in E, f \in E^{'}), \tag{1.1}$$

and $E^{''}$ is a Banach $(A^{''}, \Box)-$module under actions

$$F \bullet \Lambda = w^* - \lim_i w^* - \lim_j \widehat{a_i x_j} \quad , \quad \Lambda \bullet F = w^* - \lim_j w^* - \lim_i \widehat{x_j a_i} \ , \tag{1.2}$$

where $F = w^* - \lim_i \hat{a}_i$ and $\Lambda = w^* - \lim_j \hat{x}_j$ such that (a_i) and (x_j) are bounded nets in A and E, respectively.
For a Banach $A-$module E, the continuous linear map $D : A \to E$ is called a *derivation* if

$$D(ab) = aD(b) + D(a)b \qquad (a, b \in A).$$

For $x \in E$ the derivation $\delta_x : A \to E$ given by $\delta_x(a) = ax - xa$ is called *inner derivation.* The Banach algebra A is called *amenable* if every derivation $D : A \to E'$ is inner, for each Banach A-module E, [12]. If every derivation $D : A \to A'$ $[D : A \to A^{(n)}\ ,\ n \in N]$ is inner, A is called *weakly amenable [n-weakly amenable]*, see also [3, 6] for details.

Proposition 1.1. *Let A be a Banach algebra and E be a Banach $A-$module and $D : A \to E$ is a continuous derivation, then $D'' : (A'', \Box) \to E''$ is a continuous derivation, where $E^{''}$ is considered as a Banach $A^{''}-$module in accordance to formula (1.2). ([7], theorem 2.7.17).*

Proposition 1.2. *Let A be a Banach algebra, and let $n \in \mathbb{N}$. If A is $(n+2)-$weakly amenable, then A is n-weakly amenable [6].*

It was shown in [4, 11] that the $n-$weak amenability of $A^{''}$ implies the $n-$weak amenability of A. In [13] it was shown that if the Banach algebra A is complete Arens regular and every derivation $D : A \to A'$ is weakly compact, the weak amenability of $A^{(2n)}$ for some $n \geq 1$ implies of A. The authors in [5, 10] determined the conditions that the $3-$weak amenability of A'' implies the $3-$weak amenability of A, and the $3-$weak amenability of $A^{(2n)}$ for some $(n \geq 1)$ implies the $3-$weak amenability of A.
In this paper we always use the first Arens product $\Box$ on Banach algebra $A^{(2n)}(n \geq 1)$. First, we introduce the following important notation.

If $A^{(3)}$ is considered as a dual space of $A^{''}$, we will use the symbol $A^{(3)} = (A^{''})^{'}$ and the formula (1.1) for $A^{''}-$module actions on $A^{(3)}$. On the other hand, the symbol $A^{(3)} = (A^{'})^{''}$ shows $A^{(3)}$ as the second dual of $A^{'}$, and we will use the formula (1.2) for $A^{''}-$module actions on $A^{(3)}$.

In Section 2 we investigate

- $\triangleright$ two $A^{''}-$module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$,
- $\triangleright$ two $A^{(4)}-$module actions on $A^{(5)} = ((A^{'})^{''})^{''}$ and $A^{(5)} = ((A^{''})^{''})^{'}$,

$\vdots$

- $\triangleright$ two $A^{(2n)}-$module actions on $A^{(2n+1)} = (((A^{'})^{''})^{''}\cdots)^{''}$ and $A^{(2n+1)} = ((((A^{''})^{''})\cdots)^{''})^{'}$,

and also in Section 3 we investigate

- $\triangleright$ two $A^{''}-$module actions on $A^{(4)} = ((A^{'})^{'})^{''}$ and $A^{(4)} = ((A^{''})^{'})^{'}$,
- $\triangleright$ two $A^{(4)}-$module actions on $A^{(6)} = (((A^{'})^{'})^{''})^{''}$ and $A^{(6)} = ((A^{''})^{''})^{'})^{'}$,

$\vdots$

$\triangleright$ two $A^{(2n)}-$module actions on $A^{(2n+2)} = ((((A^{'})^{'})^{''})^{''}\cdots)^{''}$ and $A^{(2n+2)} = (((((A^{''})^{''})\cdots)^{''})^{'})^{'}$.

In these sections we shall frequently use the formulas (1.1) and (1.2), and the induction process. In each case we will find conditions to make two different actions equal. These are generalizations of the methods in [9]. In Section 4 we consider continuous derivations $D : A \to A^{'}$ and $D : A \to A^{''}$. This section is about pulling the inner-ness of $(2n)-$th duals of D down to the inner-ness of D. In our main results in Section 5 we show, using the conditions obtained from previous sections, that $m-$weak amenability of $A^{(2n)}$ for some $n \geq 1$ and $m > 2n$ implies weak or $2-$weak amenability of A.

2. $A^{(2n)}-$Module actions on $A^{(2n+1)}$

First, for $n = 1$, we consider two $A''-$module actions on $A^{(3)}$ when $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$. Let $a^{(3)} = w^* - \lim_\alpha \widehat{a^{'}_\alpha}$, $a^{''} = w^* - \lim_\beta \widehat{a_\beta}$ and $b^{''} = w^* - \lim_i \widehat{b_i}$ in which $(a^{'}_\alpha)$ is a bounded net in $A^{'}$ and (a_β) and (b_i) are bounded nets in A. For the left $A''-$module action on $A^{(3)} = (A^{'})^{''}$ we can write

$$\langle a^{''} \bullet a^{(3)}, b^{''} \rangle = \lim_\beta \lim_\alpha \langle b^{''}, a_\beta . a^{'}_\alpha \rangle = \lim_\beta \lim_\alpha \lim_i \langle a^{'}_\alpha, b_i a_\beta \rangle, \tag{2.1}$$

and for the left $A''-$module action on $A^{(3)} = (A^{''})^{'}$ as dual of $A^{''}$ we can write

$$\langle a^{''} . a^{(3)}, b^{''} \rangle = \langle a^{(3)}, b^{''} \Box a^{''} \rangle = \lim_\alpha \lim_i \lim_\beta \langle a^{'}_\alpha, b_i a_\beta \rangle \tag{2.2}$$

This shows that two left $A''-$module actions on $A^{(3)} = (A^{''})^{'}$ and $A^{(3)} = (A^{'})^{''}$ are not equal. But two right $A''-$module actions $a^{(3)} \bullet a^{''}$ and $a^{(3)} \cdot a^{''}$ are equal, because they are obtained from $\pi^{*(***)}$ and $\pi^{(***)*}$, which obviously are equal.

Proposition 2.1. *Let A be a Banach algebra with the following conditions:*

(i) A is Arens regular,

(ii) the map $A \times A^{'} \to A^{'}$ $\left((a, a^{'}) \longmapsto a.a^{'}\right)$ is Arens regular.

Then two A''-module actions on $A^{(3)} = (A^{'})^{''}$ and $A^{(3)} = (A^{''})^{'}$ coincide.

Proof. It is enough to prove that the left module actions in (2.1) and (2.2) coincide. We begin with the equation (2.1)

$$\begin{aligned}
\langle a'' \bullet a^{(3)}, b'' \rangle &= \lim_\beta \lim_\alpha \langle b'', a_\beta . a'_\alpha \rangle \\
&= \lim_\alpha \lim_\beta \langle b'', a_\beta . a'_\alpha \rangle \qquad \text{(by (ii))} \\
&= \lim_\alpha \lim_\beta \lim_i \langle a_\beta . a'_\alpha, b_i \rangle \\
&= \lim_\alpha \lim_\beta \lim_i \langle a'_\alpha, b_i a_\beta \rangle \\
&= \lim_\alpha \lim_i \lim_\beta \langle a'_\alpha, b_i a_\beta \rangle \qquad \text{(by (i))}.
\end{aligned}$$

This proves the equality of (2.1) and (2.2). $\square$

Now for $n = 2$ we consider two $A^{(4)}-$module actions on $A^{(5)}$ when $A^{(5)} = ((A')'')''$ and $A^{(5)} = ((A'')'')'$. Let $a^{(5)} = w^* - \lim_\alpha \widehat{a_\alpha^{(3)}}$, $a^{(4)} = w^* - \lim_\beta \widehat{a''_\beta}$ and $b^{(4)} = w^* - \lim_i \widehat{b''_i}$ such that $(a_\alpha^{(3)})$ is a bounded net in $A^{(3)}$ and $(a''_\beta), (b''_i)$ are bounded nets in A''. For the left $A^{(4)}-$module action on $A^{(5)} = ((A')'')''$ we have

$$\langle a^{(4)} \bullet a^{(5)}, b^{(4)} \rangle = \lim_\beta \lim_\alpha \langle b^{(4)}, a''_\beta \bullet a_\alpha^{(3)} \rangle = \lim_\beta \lim_\alpha \lim_i \langle a''_\beta \bullet a_\alpha^{(3)}, b''_i \rangle, \tag{2.3}$$

and for the left $A^{(4)}-$module action on $A^{(5)} = ((A'')'')'$ we have

$$\langle a^{(4)} . a^{(5)}, b^{(4)} \rangle = \langle a^{(5)}, b^{(4)} \Box a^{(4)} \rangle = \lim_\alpha \lim_i \lim_\beta \langle a_\alpha^{(3)}, b''_i \Box a''_\beta \rangle. \tag{2.4}$$

But two right $A^{(4)}-$module actions $a^{(5)} \bullet a^{(4)}$ and $a^{(5)} \cdot a^{(4)}$ are equal. To prove the equality of the left $A^{(4)}-$module actions on $A^{(5)}$, we need the equality of two left $A''-$module actions on $A^{(3)} = (A'')'$ and $A^{(3)} = (A')''$ by the following lemma, whose proof is straightforward.

Lemma 2.1. *Let A be a Banach algebra with the following conditions*

(i) A'' is Arens regular,

(ii) the map $A'' \times A''' \to A'''$ $\Big((a'', a^{(3)}) \longmapsto a''.a^{(3)}\Big)$ is Arens regular.

Then the conditions of the proposition 2.1 hold.

Proposition 2.2. *Let A be a Banach algebra with the conditions of Lemma 2.1, then two $A^{(4)}-$module actions on $A^{(5)} = ((A')'')''$ and $A^{(5)} = ((A'')'')'$ coincide.*

Proof. By Lemma 2.1, two left $A''-$module actions on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$ are equal. We begin with the equality (2.3)

$$\begin{aligned}
\langle a^{(4)} \bullet a^{(5)}, b^{(4)} \rangle &= \lim_\beta \lim_\alpha \langle b^{(4)}, a''_\beta \bullet a_\alpha^{(3)} \rangle \\
&= \lim_\alpha \lim_\beta \langle b^{(4)}, a''_\beta . a_\alpha^{(3)} \rangle \\
&= \lim_\alpha \lim_\beta \lim_i \langle a''_\beta . a_\alpha^{(3)}, b''_i \rangle \\
&= \lim_\alpha \lim_\beta \lim_i \langle a_\alpha^{(3)}, b''_i \Box a''_\beta \rangle \\
&= \lim_\alpha \lim_i \lim_\beta \langle a_\alpha^{(3)}, b''_i \Box a''_\beta \rangle.
\end{aligned}$$

This proves the equality of (2.3) and (2.4). $\square$

We can extend our results to each n, in the following proposition.

Proposition 2.3. *Let A be a Banach algebra with the following conditions for some $n \geq 1$*

(i) A^{2n-2} *is Arens regular,*

(ii) *the map* $A^{(2n-2)} \times A^{(2n-1)} \to A^{(2n-1)}\ ((a,f) \longmapsto a.f)$ *is Arens regular.*

Then two $A^{(2n)}-$*module actions on* $A^{(2n+1)} = ((((A^{''})^{''})\cdots)^{''})^{'}$ *and* $A^{(2n+1)} = ((((A^{'})^{''})\cdots)^{''})^{''}$ *coincide.*

3. $A^{(2n)}-$Module actions on $A^{(2n+2)}$

Our methods in this section are similar to those in Section 2, so we just mention our conclusions very briefly.

Proposition 3.1. *Let A be a Banach algebra with the following conditions*

(i) $A^{''}$ *is Arens regular,*

(ii) *the maps* $A \times A^{'} \to A^{'}\ ((a,f) \longmapsto a.f)$ *and* $A^{'} \times A \to A^{'}\ ((f,a) \longmapsto f \cdot a)$ *are Arens regular.*

Then two A''-module actions on $A^{(4)} = ((A^{'})^{'})^{''}$ *and* $A^{(4)} = ((A^{''})^{'})^{'}$ *coincide.*

To extend our results to $A^{(6)}$ we need the following lemma that is similar to Lemma 2.1.

Lemma 3.1. *Let A be a Banach algebra with the following conditions*

(i) $A^{(4)}$ *is Arens regular,*

(ii) *the maps* $A^{''} \times A^{'''} \to A^{'''}\ ((F,\Lambda) \longmapsto F.\Lambda)$ *and* $A^{'''} \times A^{''} \to A^{'''}\ ((\Lambda,F) \longmapsto \Lambda.F)$ *are Arens regular.*

Then the conditions of the proposition 3.1 hold.

Proposition 3.2. *Let A be a Banach algebra with the conditions of Lemma 3.1, then two* $A^{(4)}$*-module actions on* $A^{(6)} = (((A^{'})^{'})^{''})^{''}$ *and* $A^{(6)} = (((A^{''})^{''})^{'})^{'}$ *coincide.*

Similar to the proposition 2.3 we have the following extension.

Proposition 3.3. *Let A be a Banach algebra with the following conditions for some $n \geq 1$*

(i) $A^{(2n)}$ *is Arens regular,*

(ii) the maps $\big(A^{(2n-2)} \times A^{(2n-1)} \to A^{(2n-1)} (f, \Lambda) \longmapsto f.\Lambda\big)$ *and* $\big(A^{(2n-1)} \times A^{(2n-2)} \to A^{(2n-1)} (\Lambda, f) \longmapsto \Lambda.f\big)$ *are Arens regular.*

Then two $A^{(2n)}-$*module actions on* $A^{(2n+2)} = (((((A^{'})^{'})^{''} \cdots)^{''})^{''}$ *and* $A^{(2n+2)} = (((((A^{''})^{''}) \cdots)^{''})^{'})^{'}$ *coincide.*

Remark 3.1. There are many other module actions in sections 2 and 3 that we do not need to mention. We just introduce the module actions that we will apply in the next sections.

4. Duals of derivations $D : A \to A^{'}$ and $D : A \to A^{''}$

We consider the following duals of the continuous derivation $D : A \to A^{'}$ as in the proposition 1.1

$$
\begin{aligned}
D^{''} &: A^{''} \longrightarrow A^{(3)} = (A^{'})^{''} \\
D^{(4)} &: A^{(4)} = (A^{''})^{''} \longrightarrow A^{(5)} = ((A^{'})^{''})^{''} \\
&\vdots \\
D^{(2n)} &: A^{(2n)} = ((A^{''})^{''} \cdots)^{''} \longrightarrow A^{(2n+1)} = (((A^{'})^{''})^{''} \cdots)^{''},
\end{aligned}
$$

and the following duals of the continuous derivation $D : A \to A^{''} = (A^{'})^{'}$

$$
\begin{aligned}
D^{''} &: A^{''} \longrightarrow A^{(4)} = ((A^{'})^{'})^{''} \\
D^{(4)} &: A^{(4)} = (A^{''})^{''} \longrightarrow A^{(6)} = (((A^{'})^{'})^{''})^{''} \\
&\vdots \\
D^{(2n)} &: A^{(2n)} = ((A^{''})^{''} \cdots)^{''} \longrightarrow A^{(2n+2)} = (((A^{'})^{'})^{''} \cdots)^{''}.
\end{aligned}
$$

We recall that the above $D^{''}, D^{(4)}, \cdots, D^{(2n)}$ are also continuous derivations.

Lemma 4.1. *Let A be a Banach algebra with the hypothesis of the proposition 2.1. If the second dual $D^{''}$ of the continuous derivation $D : A \to A^{'}$ is inner, then D is inner.*

Proof. Since $D^{''} : A^{''} \longrightarrow A^{(3)} = (A^{'})^{''}$ is inner, there is $a^{(3)} \in A^{(3)}$ such that for every $a^{''} \in A^{''}$ we have

$$D^{''}(a^{''}) = a^{''} \bullet a^{(3)} - a^{(3)} \bullet a^{''},$$

Now let $a' =: i^*(a^{(3)})$, where $i : A \longrightarrow A''$ is the natural map and so $i^* : (A'')' = A^{(3)} \longrightarrow A'$. Then for each $a, b \in A$ we can write

$$\begin{aligned}
\langle D(a), b\rangle &= \langle D''(\widehat{a}), \widehat{b}\rangle \\
&= \langle \widehat{a} \bullet a^{(3)} - a^{(3)} \bullet \widehat{a}, \widehat{b}\rangle \\
&= \langle a^{(3)}, \widehat{b}\Box\widehat{a} - \widehat{a}\Box\widehat{b}\rangle \qquad \text{(by proposition 2.1)} \\
&= \langle a^{(3)}, \widehat{ba - ab}\rangle \\
&= \langle i^*(a^{(3)}), ba - ab\rangle \\
&= \langle a', ba - ab\rangle \\
&= \langle a.a' - a'.a, b\rangle,
\end{aligned}$$

hence $D(a) = a.a' - a'.a$. $\square$

Using the reasoning similar to that in the proof of the previous lemma we have the next lemmas.

Lemma 4.2. *Let A be a Banach algebra with hypothesis of the proposition 2.3. If $(2n)-th$ dual $D^{(2n)}$ of the continuous derivation $D : A \to A'$ is inner for some $n \geq 1$, then $D^{(2n-2)}, \cdots, D''$ and D are inner.*

Lemma 4.3. *Let A be a Banach algebra with the hypothesis of the proposition 3.3. If $(2n)-th$ dual $D^{(2n)}$ of the continuous derivation $D : A \to A''$ is inner for some $n \geq 1$, then $D^{(2n-2)}, \cdots, D''$ and D are inner.*

5. Main results

The results of this section are immediate consequences of the previous sections, and so the proofs will be very short.

Proposition 5.1. *Let A be a Banach algebra with the hypothesis of the proposition 2.1. If A'' is weakly amenable, then A is weakly amenable.*

Proof. Suppose that $D : A \to A'$ is a continuous derivation. Then $D'' : A'' \longrightarrow A^{(3)} = (A')''$ is a continuous derivation by the proposition 1.1. But two $A''-$module actions on $A^{(3)} = (A')''$ and $A^{(3)} = (A'')'$ are equal by the proposition 2.1, hence $D'' : A'' \longrightarrow A^{(3)} = (A'')'$ is also a continuous derivation in which $A^{(3)} = (A'')'$ is considered a dual of A''. Since A'' is weakly amenable, then D'' is inner. Therefore D is inner by Lemma 4.1. This completes the proof. $\square$

Using the same reasoning as in the proof of the previous proposition we have the next results.

Proposition 5.2. *Let A be a Banach algebra with the conditions in the proposition 2.3 for some $n \geq 1$. If $A^{(2n)}$ is weakly amenable, then A is weakly amenable.*

Proof. This is a consequence of Lemma 4.2. $\square$

Proposition 5.3. *Let A be a Banach algebra with the conditions of the proposition 3.3 for some $n \geq 1$. If $A^{(2n)}$ is 2-weakly amenable, then A is 2-weakly amenable.*

Proof. This is a consequence of Lemma 4.3. $\square$

Finally we obtain the following general results.

Corollary 5.1. *Let $n \geq 1, m > 2n$ and suppose that A is a Banach algebra such that the conditions of the preposition 2.3 hold for n. If $A^{(2n)}$ is $m-$weakly amenable and m is odd, then A is weakly amenable.*

Proof. $A^{(2n)}$ is weakly amenable by the proposition 1.2, and hence A is weakly amenable by the proposition 5.2. $\square$

Corollary 5.2. *Let $n \geq 1, m > 2n$ and suppose that A be a Banach algebra such that the conditions of the preposition 3.3 hold for n. If $A^{(2n)}$ is $m-$weakly amenable and m is even, then A is 2-weakly amenable.*

Proof. $A^{(2n)}$ is $2-$weakly amenable by the proposition 1.2, and hence A is 2-weakly amenable by the proposition 5.3. $\square$

Example 5.1. Take a non-reflexive complex Banach space A and a bounded linear map $\varphi : A \longrightarrow \mathbb{C}$. One can define a multiplication on A by

$$ab =: \langle \varphi, b \rangle a \ , (a, b \in A).$$

This makes A a Banach algebra which is called *ideally factored algebra associated to φ* and sometimes it is denoted by A_φ, [1]. One can write for $a, b \in A$

$$\varphi(ab) = \varphi(\langle \varphi, b \rangle a) = \langle \varphi, a \rangle \langle \varphi, b \rangle = \varphi(ba),$$

this shows that φ is multiplicative. It is easy to conclude the following equations

$$\begin{aligned}
a^{'}.a &= \langle a^{'}, a \rangle \varphi \\
a.a^{'} &= \langle \varphi, a \rangle a^{'} \\
a^{''} \Box b^{''} &= a^{''} \Diamond b^{''} = \langle b^{''}, \varphi \rangle a^{''} \\
a^{'''}.a^{''} &= \langle a^{'''}, a^{''} \rangle \widehat{\varphi} \\
a^{''}.a^{'''} &= \langle a^{''}, \varphi \rangle a^{'''},
\end{aligned}$$

whenever $a \in A, a^{'} \in A^{'}, a^{''}, b^{''} \in A^{''}$ and $a^{'''} \in A^{'''}$. Now for bounded nets (a_i) and $(a_j^{'})$ in A and $A^{'}$, respectively, we have

$$\begin{aligned}
w^* - \lim_j w^* - \lim_i \widehat{a_i a_j^{'}} &= w^* - \lim_j w^* - \lim_i \langle \varphi, a_i \rangle \widehat{a_j^{'}} \\
&= \lim_i \langle \varphi, a_i \rangle w^* - \lim_j \widehat{a_j^{'}}.
\end{aligned}$$

This proves Arens regularity of the map $A \times A^{'} \to A^{'} \left((a, a^{'}) \longmapsto a.a^{'}\right)$. Since A is not reflexive, there exist bounded nets (a_i) and $(a_j^{'})$ in A and $A^{'}$, respectively such that $\lim_i \lim_j \langle a_j^{'}, a_i\rangle \neq \lim_j \lim_i \langle a_j^{'}, a_i\rangle$, and hence the map $A^{'} \times A \to A^{'} \left((a^{'}, a) \longmapsto a^{'}.a\right)$ is not Arens regular, because

$$\begin{aligned} w^* - \lim_j w^* - \lim_i \widehat{a_j^{'} a_i} &= w^* - \lim_j w^* - \lim_i \langle a_j^{'}, a_i\rangle \varphi \\ &\neq w^* - \lim_i w^* - \lim_j \langle a_j^{'}, a_i\rangle \varphi. \end{aligned}$$

By using a similar reasoning we conclude that the map $A^{''} \times A^{'''} \to A^{'''} \left((a^{''}, a^{'''}) \longmapsto a^{''}.a^{'''}\right)$ is Arens regular, but the map $A^{'''} \times A^{''} \to A^{'''} \left((a^{'''}, a^{''}) \longmapsto a^{'''}.a^{''}\right)$ is not Arens regular. It is obvious that the algebras A and $A^{(2n)}$ for all $n \geq 1$ are Arens regular. In fact we have $(A_\varphi)^{''} = (A^{''})_\varphi$. Finally, all the conditions of propositions in section 2 hold , but the conditions of section 3 hold in commutative case.

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INSERTION OF A CONTRA-CONTINUOUS FUNCTION BETWEEN TWO COMPARABLE CONTRA-$\alpha-$CONTINUOUS (CONTRA-$C-$CONTINUOUS) FUNCTIONS *

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Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on topological spaces on which the kernel of sets is open.
Keywords: Insertion, Strong binary relation, $C-$open set, Semi-preopen set, $\alpha-$open set, Contra-continuous function, Lower cut set.

1. Introduction

The concept of a $C-$open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in [12]. The authors define a set S to be a $C-$*open* set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a $C-$*closed* set if its complement (denoted by S^c) is a $C-$open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha-$open set and a $C-$open set or equivalently a subset of a topological space is closed if and only if it is an $\alpha-$closed set and a $C-$closed set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha-$continuous and $C-$continuous or equivalently a function is contra-continuous if and only if it is contra-$\alpha-$continuous and contra-$C-$continuous.
Recall that a subset A of a topological space (X, τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called $\alpha-$*closed* if its complement is $\alpha-$open or equivalently if A is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called *semi-preopen or* $\beta-$*open*. A set is *semi-preclosed or* $\beta-$*closed* if its complement is

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semi-preopen or $\beta-$open.
In [7] it was shown that a set A is $\beta-$open if and only if $A \subseteq Cl(Int(Cl(A)))$. A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].
Recall that a real-valued function f defined on a topological space X is called $A-$continuous [23] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [4, 11]. In the recent literature many topologists have focused their research in the direction of investigating different types of generalized continuity.
J. Dontchev in [5] introduced a new class of mappings called contra-continuity.S. Jafari and T. Noiri in [13, 14] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha-$continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 22].
Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-$C-$continuous* , *contra-$\alpha-$continuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. $C-$closed , $\alpha-$closed) in X[5].
Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [19].
If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.
The following definitions are modifications of conditions considered in [17].
A property P, defined relative to a real-valued function on a topological space, is a *$cc-$property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are $cc-$properties, the following terminology is used:(i) A space X has the *weak $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$.(ii) A space X has the *$cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g < h < f$.(iii) A space X has the *strong $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.(iv) A space X has the *weakly $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 , f has property P_2 and $f - g$ has property P_2, then there exists a

contra-continuous function h such that $g < h < f$.
In this paper, for a topological space whose $\Lambda-$sets or kernel of sets are open, is given a sufficient condition for the weak $cc-$insertion property. Also for a space with the weak $cc-$insertion property, we give a necessary and sufficient condition for the space to have the $cc-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for the insertability of a contra-continuous function, the necessary definitions and terminology are stated.
Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^{Λ} and A^{V} as follows:
$A^{\Lambda} = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^{V} = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
In [6, 18, 21], A^{Λ} is called the *kernel* of A.
The family of all $\alpha-$open, $\alpha-$closed, $C-$open and $C-$closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.
We define the subsets $\alpha(A^{\Lambda}), \alpha(A^{V}), C(A^{\Lambda})$ and $C(A^{V})$ as follows:
$\alpha(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in \alpha O(X, \tau)\}$,
$\alpha(A^{V}) = \cup\{F : F \subseteq A, F \in \alpha C(X, \tau)\}$,
$C(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in CO(X, \tau)\}$ and
$C(A^{V}) = \cup\{F : F \subseteq A, F \in CC(X, \tau)\}$.
$\alpha(A^{\Lambda})$ (resp. $C(A^{\Lambda})$) is called the $\alpha - kernel$ (resp. $C - kernel$) of A.
The following first two definitions are modifications of conditions considered in [15, 16].
Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \; \bar{\rho} \; y$ if and only if $y \; \rho \; v$ implies $x \; \rho \; v$ and $u \; \rho \; x$ implies $u \; \rho \; y$ for any u and v in S.
Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:
1) If $A_i \; \rho \; B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \; \rho \; C$ and $C \; \rho \; B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.
2) If $A \subseteq B$, then $A \; \bar{\rho} \; B$.
3) If $A \; \rho \; B$, then $A^{\Lambda} \subseteq B$ and $A \subseteq B^{V}$.
The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:
Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.
We now give the following main result:
Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on

the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [16] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X. ■

The above proof used the technique of theorem 1 in [15].

Theorem 2.2. Let P_1 and P_2 be $cc-$property and X be a space that satisfies the weak $cc-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2.The space X has the $cc-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by contra-continuous functions.

Proof. Theorem 2.1 of [20].■

3. Applications

The abbreviations $c\alpha c$ and cCc are used for contra-$\alpha-$continuous and contra-$C-$continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2, we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case

$\alpha(A^\Lambda) \subseteq \alpha(B^V)$ (resp. $C(A^\Lambda) \subseteq C(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha-$open (resp. $C-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $\alpha(A(f,t_1)^\Lambda) \subseteq \alpha(A(g,t_2)^V)$ (resp. $C(A(f,t_1)^\Lambda) \subseteq C(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-$\alpha-$continuous (resp. contra-$C-$continuous) function is contra-continuous.

Proof. Let f be a real-valued contra-$\alpha-$continuous (resp. contra-$C-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$.■

Corollary 3.3. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the strong $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$. Set $h = (f+g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function.■

Corollary 3.4. If for each pair of disjoint subsets G_1, G_2 of X , such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$.

Proof. Let g and f be real-valued functions defined on X, such that g is $c\alpha c$ (resp. cCc) and f is cCc (resp. $c\alpha c$), with $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $C(A^\Lambda) \subseteq \alpha(B^V)$ (resp. $\alpha(A^\Lambda) \subseteq C(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $C-$open (resp. $\alpha-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $C(A(f,t_1)^\Lambda) \subseteq \alpha(A(g,t_2)^V)$ (resp. $\alpha(A(f,t_1)^\Lambda) \subseteq C(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets G_1, G_2 of X, such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1, F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$.

(ii) If G is a $C-$open (resp. $\alpha-$open) subset of X which is contained in an $\alpha-$closed (resp. $C-$closed) subset F of X, then there exists a closed subset H of X such that $G \subseteq H \subseteq H^{\Lambda} \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are $C-$open (resp. $\alpha-$open) and $\alpha-$closed (resp. $C-$closed) subsets of X, respectively. Hence, F^c is an $\alpha-$open (resp. $C-$open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^{\Lambda} \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^{\Lambda} \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is $\alpha-$open and G_2 is $C-$open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is an $\alpha-$closed subset of X. Hence by (ii) there exists a closed set H such that $G_2 \subseteq H \subseteq H^{\Lambda} \subseteq G_1^c$.

But

$$H \subseteq H^{\Lambda} \Rightarrow H \cap (H^{\Lambda})^c = \varnothing$$

and

$$H^{\Lambda} \subseteq G_1^c \Rightarrow G_1 \subseteq (H^{\Lambda})^c.$$

Furthermore, $(H^{\Lambda})^c$ is a closed subset of X. Hence $G_2 \subseteq H, G_1 \subseteq (H^{\Lambda})^c$ and $H \cap (H^{\Lambda})^c = \varnothing$. This means that condition (i) holds.■

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is $\alpha-$open and G_2 is $C-$open, can be separated by closed subsets of X then there exists a contra-continuous function $h : X \to [0, 1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is $\alpha-$open and G_2 is $C-$open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is an $\alpha-$closed subset of X containing the $C-$open subset G_2 of X,by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq G_1^c.$$

Note that $H_{1/2}$ is also an $\alpha-$closed subset of X and contains G_2, and G_1^c is an $\alpha-$closed subset of X and contains the $C-$open subset $H_{1/2}^{\Lambda}$ of X. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^{\Lambda} \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq H_{3/4} \subseteq H_{3/4}^{\Lambda} \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$. Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^\Lambda)^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently h is a contra-continuous function. ■

Lemma 3.3. Suppose that X is a topological space such that every two disjoint $C-$open and $\alpha-$open subsets of X can be separated by closed subsets of X. The following conditions are equivalent:

(i) Every countable convering of $C-$closed (resp. $\alpha-$closed) subsets of X has a refinement consisting of $\alpha-$closed (resp. $C-$closed) subsets of X such that for every $x \in X$, there exists a closed subset of X containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{G_n\}$ of $C-$open (resp. $\alpha-$open) subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of $\alpha-$closed (resp. $C-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of $C-$open (resp. $\alpha-$open) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of $C-$closed (resp. $\alpha-$closed) subsets of X . By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a closed subset of X and $V_n^\Lambda \subseteq G_n^c$. By setting $F_n = (V_n^\Lambda)^c$, we obtain a decreasing sequence of closed subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $C-$closed (resp. $\alpha-$closed) subsets of X, we set for $n \in \mathbb{N}, G_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{G_n\}$ is a decreasing sequence of $C-$open (resp. $\alpha-$open) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of $\alpha-$closed (resp. $C-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.Now we define the subsets W_n of X in the following manner:

W_1 is a closed subset of X such that $F_1^c \subseteq W_1$ and $W_1^\Lambda \cap G_1 = \varnothing$.

W_2 is a closed subset of X such that $W_1^\Lambda \cup F_2^c \subseteq W_2$ and $W_2^\Lambda \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of closed sets. Moreover, we have

(i) $W_n^\Lambda \subseteq W_{n+1}$

(ii) $F_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus W_{n-1}^\Lambda$.

Then since $W_{n-1}^\Lambda \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of closed sets and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$$\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \cdots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}$$

These sets are closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are closed sets, and for every point in X we can find a closed set containing the point that intersects only finitely many elements of that refinement.■

Corollary 3.5. If every two disjoint $C-$open and $\alpha-$open subsets of X can be separated by closed subsets of X and, in addition, every countable covering of $C-$closed (resp. $\alpha-$closed) subsets of X has a refinement that consists of $\alpha-$closed (resp. $C-$closed) subsets of X such that for every point of X we can find a closed subset containing that point such that it intersects only a finite number of refining members then X has the weakly $cc-$insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$).

Proof. Since every two disjoint $C-$open and $\alpha-$open sets can be separated by closed subsets of X, therefore by Corollary 3.4, X has the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$. Now suppose that f and g are real-valued functions on X with $g < f$, such that g is $c\alpha c$ (resp. cCc), f is cCc (resp. $c\alpha c$) and $f-g$ is cCc (resp. $c\alpha c$). For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$

Since $f-g$ is cCc (resp. $c\alpha c$), hence $A(f-g, 3^{-n+1})$ is a $C-$open (resp. $\alpha-$open) subset of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $C-$open (resp. $\alpha-$open) subsets of X and furthermore since $0 < f-g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of $\alpha-$closed (resp. $C-$closed) subsets of X such that $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.2, the pair $A(f-g, 3^{-n+1})$ and $X \setminus D_n$ of $C-$open (resp. $\alpha-$open) and $\alpha-$open (resp. $C-$open) subsets of X can be completely separated by contra-continuous functions. Hence by Theorem 2.2, there exists a contra-continuous function h defined on X such that $g < h < f$, i.e., X has the weakly $cc-$insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$).■.

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KENMOTSU MANIFOLDS ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. The objective of the present paper is to study the Kenmotsu manifold admitting the Schouten-van Kampen connection. We study the Kenmotsu manifold admitting the Schouten-van Kampen connection satisfying certain curvature conditions. Also, we prove the equivalent conditions for the Ricci soliton in a Kenmotsu manifold to be steady with respect to the Schouten-van Kampen connection.
Keywords: Ricci solitons, Kenmotsu manifolds, Schouten-van Kampen connection, concircular curvature tensor, projective curvature tensor, conharmonic curvature tensor, shrinking.

1. Introduction

The Schouten-van Kampen connection has been introduced for studying non-holomorphic manifolds. It preserves - by parallelism - a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2] [9] [17]. Then, Olszak studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure [14]. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and established certain curvature properties with respect to this connection. Recently, Gopal Ghosh [7] and Yildiz [24] studied the Schouten-van Kampen connection in Sasakian manifolds and f-Kenmotsu manifolds, respectively. Kenmotsu manifolds introduced by Kenmotsu in 1971[10] have been extensively studied by many authors [20] [15] [16]. In 1982, Hamilton [8] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Since then the Ricci flow has become a powerful tool for the study of Riemannian manifolds. The Ricci soliton, considered to be a self-similar solution to the Ricci flow is a Riemannian metric g on a manifold M, together with a vector field V such that

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{1.1}$$

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where L_V denotes the Lie derivative along V, and S and λ are respectively the Ricci tensor and a constant. A Ricci soliton is said to be shrinking or steady or expanding depending on whether λ is negative, zero or positive. A Ricci soliton is said to be a gradient Ricci soliton if the vector field V is the gradient of some smooth function f on M. In [18], Sharma started the study of Ricci solitons in the K-contact geometry. In 2016, the authors in [21] explained the nature of Ricci solitons in f-Kenmotsu manifolds with a semi-symmetric non-metric connection. Ramesh Sharma et al. [18] [19], De et al. [4][1], and Nagaraja et al. [12] [11] [13] extensively studied Ricci solitons in contact metric manifolds in many different ways.

This paper is structured as follows. After a brief review of Kenmotsu manifolds in Section 2, in Section 3 we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature with respect to the Schouten-van Kampen connection, study the curvature properties of the Kemotsu manifold admitting the Schouten-van Kampen connection, and prove the conditions for the Kenmotsu manifold admitting the Schouten-van Kampen connection to be isomorphic to the hyperbolic space. In the last section we prove the equivalent conditions for the Ricci soliton in a Kenmotsu manifold admitting the Schouten-van Kampen connection to be steady.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ g(X,\xi) = \eta(X),\ \eta(\xi) = 1, \eta \circ \phi = 0, \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [3] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold the following relations hold [6].

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{2.5}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.7}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{2.8}$$

for any vector fields X, Y, Z on M, where R denote the curvature tensor of type $(1,3)$ on M.

3. Kenmotsu manifolds admitting Schouten-van Kampen connection

Throughout this paper we associate $*$ with the quantities with respect to the Schouten-van Kampen connection. The Schouten-van Kampen connection ∇^* associated to the Levi-Civita connection ∇ is given by [14]

$$\nabla^*_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{3.1}$$

for any vector fields X, Y on M.
Using (2.4) and (2.5), the above equation yields,

$$\nabla^*_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X. \tag{3.2}$$

By taking $Y = \xi$ in (3.2) and using (2.4) we obtain

$$\nabla^*_X \xi = 0. \tag{3.3}$$

We now calculate the Riemann curvature tensor R^* using (3.2) as follows:

$$R^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y. \tag{3.4}$$

Using (2.6) and taking $Z = \xi$ in (3.4), we get

$$R^*(X,Y)\xi = 0. \tag{3.5}$$

On contracting (3.4), we obtain the Ricci tensor S^* of a Kenmotsu manifold with respect to the Schouten-van Kampen connection ∇^* as

$$S^*(Y,Z) = S(Y,Z) + 2ng(Y,Z). \tag{3.6}$$

This gives

$$Q^*Y = QY + 2nY. \tag{3.7}$$

Contracting with respect to Y and Z in (3.6), we get

$$r^* = r + 2n(2n+1), \tag{3.8}$$

where r^* and r are the scalar curvatures with respect to the Schouten-van Kampen connection ∇^* and the Levi-Civita connection ∇, respectively.

From the above discussions we state the following:

Theorem 3.1. *The curvature tensor R^*, the Ricci tensor S^* and the scalar curvature r^* of a Kenmotsu manifold M with respect to the Schouten-van Kampen connection ∇^* are given by (3.4), (3.6) and (3.8), respectively. Further, the curvature tensor R^* of ∇^* satisfies*
i) $R^*(X,Y)Z = -R^*(Y,X)Z$,
ii) $R^*(X,Y,Z,W) + R^*(Y,X,Z,W) = 0$,
iii) $R^*(X,Y,Z,W) + R^*(X,Y,W,Z) = 0$,
iv) $R^*(X,Y)Z + R^*(Y,Z)X + R^*(Z,X)Y = 0$,
v) S^* *is symmetric.*

From (3.6), it follows that

Theorem 3.2. *A Kenmotsu manifold M admitting the Schouten-van Kampen connection is Ricci flat with respect to the Schouten-van Kampen connection if and only if M is an Einstein manifold with respect to Levi-Civita connection.*

Now, if $R^*(X,Y)Z = 0$, then by virtue of (3.4), we get

$$R(X,Y,Z,U) = g(X,Z)g(Y,U) - g(Y,Z)g(X,U). \tag{3.9}$$

Thus, we state that

Theorem 3.3. *Let M be a Kenmotsu manifold admitting the Schouten-van Kampen connection. The curvature tensor of M with respect to the Schouten-van Kampen connection vanishes if and only if M with respect to the Levi-Civita connection is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.*

An interesting invariant of the concircular transformation is concircular curvature tensor. The concircular curvature tensor [22] C^* with respect to the Schouten-van Kampen connection ∇^* is defined by

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.10}$$

for all vector fields X, Y, Z on M.
If C^* vanishes, the conditions in theorem (3.1) are satisfied.

Definition 3.1. A Kenmotsu manifold with respect to the Schouten-van Kampen connection ∇^* is said to be ξ- concircularly flat if $C^*(X,Y)\xi = 0$.

In view of (3.4) and (3.8) in (3.10), we get

$$\begin{aligned} C^*(X,Y)Z &= R(X,Y)Z + g(Y,Z)X - g(X,Z)Y \\ &- \frac{r + 2n(2n+1)}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.11}$$

By taking $Z = \xi$ in (3.11) and then using (2.1) and (2.6), we find

$$C^*(X,Y)\xi = \frac{r + 2n(2n+1)}{2n(2n+1)}R(X,Y)\xi. \tag{3.12}$$

Thus, from (3.4), (3.8), (3.11) and (3.12), we have the following theorem:

Theorem 3.4. *Let M be a Kenmotsu manifold admitting the Schouten-van Kampen connection. In M, the following three conditions are equivalent:*
i) M is ξ- concircularly flat,
ii) $r = -2n(2n+1)$,
iii) $r^* = 0$.

Definition 3.2. A Kenmotsu manifold is said to be ϕ-concircularly flat with respect to the Schouten-van Kampen connection ∇^* if

$$g(C^*(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{3.13}$$

for any vector fields X, Y, Z on M.

Using (3.10) in (3.13), we have

$$\begin{aligned} g(R^*(\phi X, \phi Y)\phi Z, \phi W) &= \frac{r^*}{2n(2n+1)}\{g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{3.14}$$

Let $\{e_1, e_2, e_3, ...e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1, \phi e_2, \phi e_3, ...\phi e_{2n+1}\}$ is also a local orthonormal basis. If we put $X = W = e_i$ in (3.14) and summing up with respect to $i, 1 \leqslant i \leqslant 2n+1$, we obtain

$$\begin{aligned} \sum_{i=1}^{2n} g(R^*(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{r^*}{2n(2n+1)}\sum_{i=1}^{2n}\{g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\}. \end{aligned} \tag{3.15}$$

From (3.15), it follows that

$$S^*(\phi Y, \phi Z) = \frac{r^*(2n-1)}{2n(2n+1)}g(\phi Y, \phi Z). \tag{3.16}$$

Using (2.1), (3.6) and (3.8) in (3.16), we get

$$S(\phi Y, \phi Z) + 2ng(\phi Y, \phi Z) = \frac{(r + 2n(2n+1))(2n-1)}{2n(2n+1)}g(\phi Y, \phi Z). \tag{3.17}$$

By using (2.2) and (2.8) in (3.17), we obtain

$$S(Y, Z) + 2n\eta(Y)\eta(Z) + \{2n - \frac{(r + 2n(2n+1))(2n-1)}{2n(2n+1)}\}g(\phi Y, \phi Z) = 0. \tag{3.18}$$

Hence by contracting (3.18), we get

$$r = -2n. \tag{3.19}$$

By substituting the equation (3.19) in (3.10), we get

$$C^*(X, Y)Z = R(X, Y)Z + \frac{1}{2n+1}\{g(Y, Z)X - g(X, Z)Y\}. \tag{3.20}$$

This leads to the following:

Theorem 3.5. *Let the Kenmotsu manifold M admitting the Schouten-van Kampen connection be ϕ-concircularly flat. Then M is of constant sectional curvature $-\frac{1}{2n+1}$ if and only if the concircular curvature tensor C^* vanishes.*

We consider

$$C^*.S^* = S^*(C^*(X,Y)Z,U) + S^*(Z, C^*(X,Y)U). \tag{3.21}$$

By making use of (3.10) and (3.6) in (3.21), we obtain

$$\begin{aligned} C^*.S^* &= S(R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}, U) \\ &+ S(Z, R(X,Y)U - \frac{r}{2n(2n+1)}\{g(Y,U)X - g(X,U)Y\}). \end{aligned} \tag{3.22}$$

Suppose $C^*.S^* = 0$. Then we have

$$S^*(C^*(X,Y)Z,U) + S^*(Z, C^*(X,Y)U) = 0. \tag{3.23}$$

Taking $U = \xi$ in (3.23) and using (3.6), it follows that

$$S^*(Z, C^*(X,Y)\xi) = 0. \tag{3.24}$$

Making use of (2.1), (2.6) and (3.11) in (3.24), we get

$$\frac{r + 2n(2n+1)}{2n(2n+1)} S^*(Z, \eta(X)Y - \eta(Y)X) = 0. \tag{3.25}$$

Replacing X by ξ in (3.25) and using (2.1) and (3.6), we see that

$$\frac{r + 2n(2n+1)}{2n(2n+1)} \{S(Z,Y) + 2ng(Z,Y)\} = 0. \tag{3.26}$$

Contracting (3.26) with respect to Y and Z, we get

$$r = -2n(2n+1). \tag{3.27}$$

From (3.22) and (3.27), we obtain

$$S(Y,Z) = -2ng(Y,Z). \tag{3.28}$$

Thus M is an Einstein manifold.
Again, by substituting (3.27) in (3.11), we obtain

$$C^*(X,Y)Z = R(X,Y)Z + \{g(Y,Z)X - g(X,Z)Y\}. \tag{3.29}$$

Thus, from the above discussion and using (3.4), (3.8) and (3.12), we state the following:

Theorem 3.6. *Let M be a Kenmotsu manifold admitting the Schouten-van Kampen connection. Then $C^*.S^* = 0$ if and only if $S(Y,Z) = -2ng(Y,Z)$.*
Further if $C^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

Theorem 3.7. *If in a Kenmotsu manifold M admitting the Schouten-van Kampen connection, $C^*.S^* = 0$ holds, then the following three conditions are equivalent:*
i) M is ξ- concircularly flat,
ii) $r = -2n(2n+1)$,
iii) $r^* = 0$.

The projective curvature tensor [23] P^* with respect to the Schouten-van Kampen connection ∇^* is defined by

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n}\{S^*(Y,Z)X - S^*(X,Z)Y\}. \tag{3.30}$$

If the projective curvature tensor P^* with respect to the Schouten-van Kampen connection ∇^* vanishes, then from (3.30), we have

$$R^*(X,Y)Z = \frac{1}{2n}\{S^*(Y,Z)X - S^*(X,Z)Y\}. \tag{3.31}$$

Now in view of (3.4) and (3.6), (3.31) takes the form

$$\begin{aligned} &g(R(X,Y)Z,W) + g(Y,Z)g(X,W) - g(X,Z)g(Y,W) = \\ &\frac{1}{2n}[\{S(Y,Z) + 2ng(Y,Z)\}g(X,W) - \{S(X,Z) + 2ng(X,Z)\}g(Y,W)]. \end{aligned} \tag{3.32}$$

Now taking $W = \xi$ in (3.32), we obtain

$$S(Y,Z)\eta(X) - S(X,Z)\eta(Y) = 2n\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}. \tag{3.33}$$

Again, setting $X = \xi$ in (3.33), we get

$$S(Y,Z) = -2ng(Y,Z). \tag{3.34}$$

Contracting the above equation (3.34), we get

$$r = -2n(2n+1). \tag{3.35}$$

Using (3.34) in (3.31), we have $R^* = 0$.
Thus we state the following:

Theorem 3.8. *Let M be a Kenmotsu manifold admitting the Schouten-van Kampen connection. In M, the vanishing of the projective curvature tensor with respect to the Schouten-van Kampen connection leads to the vanishing of the curvature tensor with respect to the Schouten-van Kampen connection.*

By making use of (3.4) and (3.6) in (3.30), we get

$$P^*(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}. \tag{3.36}$$

Suppose $(P^*(X,Y).S^*)(Z,U) = 0$ holds in a Kenmotsu manifold M. Then we have

$$S^*(P^*(X,Y)Z,U) + S^*(Z,P^*(X,Y)U) = 0. \tag{3.37}$$

Taking $X = \xi$ in the equation (3.37), we get

$$S^*(P^*(\xi,Y)Z,U) + S^*(Z,P^*(\xi,Y)U) = 0. \tag{3.38}$$

By using (3.36), equation (3.38) turns into

$$S^*(Y,Z)\eta(U) + S^*(Y,U)\eta(Z) = 0. \tag{3.39}$$

In view of the equation (3.6), (3.39) becomes

$$S(Y,Z)\eta(U) + S(Y,U)\eta(Z) + 2n\{g(Y,Z)\eta(U) + g(Y,U)\eta(Z)\} = 0. \tag{3.40}$$

In (3.40), taking $U = \xi$ and contracting with respect to Y and Z, we get

$$S(Y,Z) = -2ng(Y,Z). \tag{3.41}$$

and

$$r = -2n(2n+1). \tag{3.42}$$

Again, by substituting (3.42) in (3.30), we obtain

$$P^*(X,Y)Z = R(X,Y)Z + \{g(Y,Z)X - g(X,Z)Y\}. \tag{3.43}$$

Thus we can state that

Theorem 3.9. *In a Kenmotsu manifold M admitting the Schouten-van Kampen connection, $P^*.S^* = 0$ if and only if $S(Y,Z) = -2ng(Y,Z)$.*
Further, if $P^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.

The conharmonic curvature tensor [5] K^* with respect to the Schouten-van Kampen connection ∇^* is defined by

$$\begin{aligned} K^*(X,Y)Z &= R^*(X,Y)Z - \frac{1}{2n-1}\{S^*(Y,Z)X - S^*(X,Z)Y \\ &+ g(Y,Z)Q^*X - g(X,Z)Q^*Y\}. \end{aligned} \tag{3.44}$$

If the conharmonic curvature tensor K^* with respect to the Schouten-van Kampen connection ∇^* vanishes, then from (3.44), we have

$$\begin{aligned} R^*(X,Y)Z &= \frac{1}{2n-1}\{S^*(Y,Z)X - S^*(X,Z)Y \\ &+ g(Y,Z)Q^*X - g(X,Z)Q^*Y\}. \end{aligned} \tag{3.45}$$

By using (3.4), (3.6) and (3.7) in (3.45), we get

$$\begin{aligned} & g(R(X,Y)Z,W) + g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ = & \frac{1}{2n-1}[\{S(Y,Z) + 4ng(Y,Z)\}g(X,W) \\ - & \{S(X,Z) + 4ng(X,Z)\}g(Y,W) \\ + & S(X,W)g(Y,Z) - S(Y,W)g(X,Z)]. \end{aligned} \tag{3.46}$$

Taking $W = \xi$ in (3.46), we obtain

$$S(Y,Z)\eta(X) - S(X,Z)\eta(Y) - 2n\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} = 0. \tag{3.47}$$

Taking $X = \xi$ in (3.47), we get

$$S(Y,Z) = -2ng(Y,Z). \tag{3.48}$$

Contracting the equation (3.48), we get

$$r = -2n(2n+1). \tag{3.49}$$

Using (3.48) in (3.45), we have $R^* = 0$.

Thus we state the following :

Theorem 3.10. *Let M be a Kenmotsu manifold admitting the Schouten-van Kampen connection. In M, the vanishing of the conharmonic curvature tensor with respect to the Schouten-van Kampen connection leads to the vanishing of the curvature tensor with respect to the Schouten-van Kampen connection.*

4. Ricci solitons in Kenmotsu manifold admitting Schouten-van Kampen connection

Suppose the Kenmotsu manifold M admits a Ricci soliton with respect to the Schouten-van Kampen connection ∇^*. Then

$$(L^*_V g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0. \tag{4.1}$$

If the potential vector field V is the structure vector field ξ, then since ξ is a parallel vector field with respect to the Schouten-van Kampen connection (from (3.3)), the first term in the equation (4.1) becomes zero, hence M reduces to an Einstein manifold. In this case, the results in Theorem (3.6) and (3.9) hold.
If V is pointwise collinear with the structure vector field ξ, i.e. $V = b\xi$, where b is a function on M, then the equation (1.1) implies that

$$\begin{aligned} & bg(\nabla^*_X\xi, Y) + (Xb)\eta(Y) + bg(X, \nabla^*_Y\xi) + (Yb)\eta(X) + \\ & 2S^*(X,Y) + 2\lambda g(X,Y) = 0. \end{aligned} \tag{4.2}$$

Using (3.3) and (3.6) in (4.2), it follows that

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) + 2\{2n+\lambda\}g(X,Y) = 0. \tag{4.3}$$

By setting $Y = \xi$ in (4.3) and using (2.7), we obtain

$$(Xb) = -\{2\lambda + \xi b\}\eta(X). \tag{4.4}$$

Again replacing X by ξ in (4.4), we get

$$(\xi b) = -\lambda. \tag{4.5}$$

Substituting this in (4.4), we have

$$(Xb) = -\lambda\eta(X). \tag{4.6}$$

By applying d on (4.6), we get

$$\lambda d\eta = 0. \tag{4.7}$$

Since $d\eta \neq 0$ from (4.7), we have

$$\lambda = 0. \tag{4.8}$$

Substituting (4.8) in (4.6), we conclude that b is a constant. Hence it is verified from (4.3) that

$$S(X,Y) = -(2n+\lambda)g(X,Y) + \lambda\eta(X)\eta(Y). \tag{4.9}$$

This leads to the following:

Theorem 4.1. *If a Kenmotsu manifold with respect to the Schouten-van Kampen connection admits a Ricci soliton (g, V, λ) with V, pointwise collinear with ξ, then the manifold is an η-Einstein manifold and the Ricci soliton is steady.*

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SOME SYMMETRIC PROPERTIES OF KENMOTSU MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The objective of the present paper is to study some symmetric properties of the Kenmotsu manifold endowed with a semi-symmetric metric connection. Here we consider pseudo-symmetric, Ricci pseudo-symmetric, projective pseudo-symmetric and ϕ-projective semi-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection. Finally, we provide an example of the 3-dimensional Kenmotsu manifold admitting a semi-symmetric metric connection which verifies our result.
Keywords: Kenmotsu manifold; projective curvature tensor; semi-symmetric metric connection; η-Einstein manifold.

1. Introduction

In 1932, Hayden [12] introduced the idea of metric connection with a torsion on a Riemannian manifold. By considering the torsion tensor of a linear connection, Friedmann and Schouten [11] gave a new connection called semi-symmetric connection. The torsion tensor with respect to the semi-symmetric connection $\bar{\nabla}$ is given by

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]. \tag{1.1}$$

The connection $\bar{\nabla}$ is called a semi-symmetric metric connection [12] if $\bar{\nabla} g = 0$, otherwise, non-metric connection. A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on (M,g) established by Yano [18] is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{1.2}$$

Semi-symmetric manifolds form a subclass of the class of pseudo-symmetric manifolds. The concept of pseudo-symmetric manifold was introduced by Chaki and

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Chaki [8] and Deszcz [10] in two different ways. Here we study the properties of pseudo-symmetric manifolds with a semi-symmetric metric connection in the Deszcz sense. An n-dimensional Riemannian manifold M is called pseudo-symmetric in the sense of Deszcz [10] if the Riemannian curvature tensor R satisfies the following relation

$$(R(X,Y)\cdot R)(U,V)W = L_R((X\wedge_g Y)\cdot R)(U,V)W, \tag{1.3}$$

for all the vector fields $X,Y,Z,U,V,W \in TM$. Where L_R is a smooth function on M and $X\wedge_g Y$ is an endomorphism defined by

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{1.4}$$

The notion of semi-symmetric metric connection has been weakened by many geometers such as [2, 3, 5, 9, 15, 17] etc., with different structures of manifolds and submanifolds. In particular, De [1] and Bagewadi et. al. [4] studied semi-symmetric metric connection on Kenmotsu manifolds with a projective curvature tensor. Also in [16], Singh et. al. studied the semi-symmetric metric connection in an ϵ-Kenmotsu manifold.

The projective curvature tensor $\bar{P}$ with respect to the semi-symmetric metric connection on a Kenmotsu manifold is defined by [1]

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}[\bar{S}(Y,Z)X - \bar{S}(X,Y)Z], \tag{1.5}$$

for $X,Y,Z \in \chi(M)$. Here $\bar{S}$ is the Ricci tensor with respect to the semi-symmetric metric connection.

Further, a relation between the curvature tensor $\bar{R}$ of the semi-symmetric metric connection $\bar{\nabla}$ and the curvature tensor R of the Levi-Civita connection ∇ is given by [18]

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y \\ &- g(Y,Z)LX + g(X,Z)LY,\end{aligned} \tag{1.6}$$

where α is a tensor field of type (0,2) and L is a tensor field of type (1,1) which is given by

$$\alpha(Y,Z) = g(LY,Z) = (\nabla_Y\eta)(Z) - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y,Z), \tag{1.7}$$

for any vector fields $X,Y,Z \in \chi(M)$. From (1.6), it follows that

$$\bar{S}(Y,Z) = S(Y,Z) - (n-2)\alpha(Y,Z) - ag(Y,Z), \tag{1.8}$$

where $\bar{S}$ denotes the Ricci tensor with respect to $\bar{\nabla}$ and a=trace of α.

Motivated by these studies, we investigate the semi-symmetric metric connection due to Yano [18] on Kenmotsu manifolds. The paper is organized as follows. After giving preliminaries and basic results of the Kenmotsu manifold in Section

2, in Section 3 we study pseudo-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection, proving that either $L_{\bar{R}} = -2$ or the manifold is η-Einstein. In the next section we prove that in a Ricci pseudo-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection, either $L_{\bar{S}} = -2$ or the manifold is η-Einstein. Sections 5 and 6 are devoted to the study of projective pseudo-symmetric and ϕ-projective semi-symmetric Kenmotsu manifolds with respect to the semi-symmetric metric connection. Finally, we construct an example of a 3-dimensional Kenmotsu manifold admitting the semi-symmetric metric connection and verify the results.

2. Preliminaries

Let M be an n-dimensional almost contact Riemannian manifold equipped with the almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a characteristic vector field, η is a 1-form and g is the Riemannian metric satisfying the following conditions [7];

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y on M. If an almost contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

then M is called a Kenmotsu manifold [14]. Here ∇ denotes the operator of covariant differentiation with respect to g. From (2.3), it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.5}$$

In a Kenmotsu manifold M, the following relations hold:

$$\eta(R(X, Y)Z) = [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.6}$$

$$(a)\ R(\xi, X)Y = [\eta(Y)X - g(X, Y)\xi], \ \ (b)\ R(X, Y)\xi = [\eta(X)Y - \eta(Y)X], \tag{2.7}$$

$$(a)\ S(X, Y) = -(n-1)g(X, Y), \ (b)\ QX = -(n-1)X, \tag{2.8}$$

$$(a)\ S(X, \xi) = -(n-1)\eta(X), \ (b)\ S(\xi, \xi) = -(n-1), \ (c)\ Q\xi = -(n-1)\xi, \tag{2.9}$$

$$(\nabla_W R)(X, Y)\xi = g(W, X)Y - g(W, Y)X - R(X, Y)W, \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.11}$$

Now by using (1.7), (2.1) and (2.5) in (1.6), we have the following relation

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - 3[g(Y, Z)X - g(X, Z)Y] + 2[\eta(Y)X \\ &- \eta(X)Y]\eta(Z) + 2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi. \end{aligned} \tag{2.12}$$

Contracting X in (2.12), we get

$$\bar{S}(Y, Z) = S(Y, Z) - (3n-5)g(Y, Z) + 2(n-2)\eta(Y)\eta(Z). \tag{2.13}$$

Again contracting Y and Z in (2.13), we get

$$\bar{r} = r - (n-1)(3n-4), \tag{2.14}$$

where $\bar{r}$ and r are the scalar curvatures with respect to the semi-symmetric metric connection and the Levi-Civita connection respectively.

3. Pseudo-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection

`Definition:` An n-dimensional Kenmotsu manifold M is said to be pseudosymmetric with respect to semi-symmetric metric connection if the curvature tensor $\bar{R}$ of $\bar{\nabla}$ satisfies the condition

$$(\bar{R}(X,Y)\cdot\bar{R})(U,V)W = L_{\bar{R}}((X\wedge_g Y)\cdot\bar{R})(U,V)W, \tag{3.1}$$

where $L_{\bar{R}}$ is a function on M. From (3.1), we have

$$\begin{aligned}&\bar{R}(X,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(X,Y)U,V)W - \bar{R}(U,\bar{R}(X,Y)V)W\\ &-\bar{R}(U,V)(\bar{R}(X,Y)W) = L_{\bar{R}}[(X\wedge_g Y)(\bar{R}(U,V)W) - \bar{R}((X\wedge_g Y)U,V)W\\ &-\bar{R}(U,(X\wedge_g Y)V)W - \bar{R}(U,V)(X\wedge_g Y)W].\end{aligned} \tag{3.2}$$

Replacing X by ξ in (3.2), we get

$$\begin{aligned}&\bar{R}(\xi,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(\xi,Y)U,V)W - \bar{R}(U,\bar{R}(\xi,Y)V)W\\ &-\bar{R}(U,V)(\bar{R}(\xi,Y)W) = L_{\bar{R}}[(\xi\wedge_g Y)(\bar{R}(U,V)W) - \bar{R}((\xi\wedge_g Y)U,V)W\\ &-\bar{R}(U,(\xi\wedge_g Y)V)W - \bar{R}(U,V)(\xi\wedge_g Y)W].\end{aligned} \tag{3.3}$$

Using (1.4) and (2.12) in (3.3) and then taking the inner product with ξ, we obtain

$$\begin{aligned}&(L_{\bar{R}}+2)[-\bar{R}(U,V,W,Y) + \eta(\bar{R}(U,V)W)\eta(Y) + 2g(Y,U)\eta(V)\eta(W)\\ &-2g(Y,U)g(V,W) - \eta(\bar{R}(Y,V)W)\eta(U) - 2g(Y,V)\eta(U)\eta(W)\\ &+2g(Y,V)g(U,W) - \eta(\bar{R}(U,Y)W)\eta(V) - \eta(\bar{R}(U,V)Y)\eta(W)] = 0.\end{aligned} \tag{3.4}$$

On plugging $U = Y = e_i$ in (3.4) and taking summation over i, we get

$$(L_{\bar{R}}+2)[S(V,W) - (n-5)g(V,W) + 2(n-1)\eta(V)\eta(W)] = 0. \tag{3.5}$$

This implies that either $L_{\bar{R}} = -2$ or

$$S(V,W) = (n-5)g(V,W) + 2(1-n)\eta(V)\eta(W). \tag{3.6}$$

On contracting (3.6), we get

$$r = n(n-7) + 2. \tag{3.7}$$

Hence we can state the following:

`Theorem 3.1.` Let M be an n-dimensional pseudo-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection. Then either $L_{\bar{R}} = -2$ or the manifold is η-Einstein with constant scalar curvature $r = n(n-7)+2$ with respect to Levi-Civita connection.

4. Ricci pseudo-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection

Definition: An n-dimensional Kenmotsu manifold M is said to be Ricci pseudo-symmetric with respect to semi-symmetric metric connection, if

$$(\bar{R}(X,Y)\cdot\bar{S})(Z,U) = L_{\bar{S}}Q(g,\bar{S})(Z,U;X,Y), \tag{4.1}$$

holds true on M, where $L_{\bar{S}}$ is some function and $Q(g,S)$ is the Tachibana tensor on M. From (4.1), it follows that

$$\begin{aligned}&\bar{S}(\bar{R}(X,Y)Z,U)+\bar{S}(Z,\bar{R}(X,Y)U)\\ &= L_{\bar{S}}[\bar{S}((X\wedge_g Y)Z,U)+\bar{S}(Z,(X\wedge_g Y)U)].\end{aligned} \tag{4.2}$$

Putting $Y=U=\xi$ in (4.2), we have

$$\bar{S}(\bar{R}(X,\xi)Z,\xi)+\bar{S}(Z,\bar{R}(X,\xi)\xi) = L_{\bar{S}}[\bar{S}((X\wedge\xi)Z,\xi)+\bar{S}(Z,(X\wedge\xi)\xi)]. \tag{4.3}$$

Using (1.4), (2.12), (2.13) and (2.7) in (4.3), we can get

$$(L_{\bar{S}}+2)[S(X,Z)-(n-3)g(X,Z)+2(n-2)\eta(X)\eta(Z)] = 0. \tag{4.4}$$

This implies that either $L_{\bar{S}}=-2$ or

$$S(X,Z) = (n-3)g(X,Z)+2(2-n)\eta(X)\eta(Z). \tag{4.5}$$

On contracting (4.5) over X and Z, we get

$$r = (n-1)(n-4). \tag{4.6}$$

Thus we can state the following theorem:
Theorem 4.1. If a Kenmotsu manifold M is Ricci pseudo-symmetric with respect to semi-symmetric metric connection, then either $L_{\bar{S}}=-2$ or the manifold is η-Einstein with constant scalar curvature $r=(n-1)(n-4)$ with respect to Levi-Civita connection.

5. Projective pseudo-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection

Definition: An n-dimensional Kenmotsu manifold M is said to be projective pseudo-symmetric with respect to semi-symmetric metric connection if

$$(\bar{R}(X,Y)\cdot\bar{P})(U,V)W = L_{\bar{P}}((X\wedge_g Y)\cdot\bar{P})(U,V)W, \tag{5.1}$$

holds on M. Putting $Y=W=\xi$ in (5.1), we get

$$(\bar{R}(X,\xi)\cdot\bar{P})(U,V)\xi = L_{\bar{P}}((X\wedge_g\xi)\cdot\bar{P})(U,V)\xi. \tag{5.2}$$

Now right hand side of (5.2) can be written as

$$
\begin{aligned}
L_{\bar{P}}((X \wedge_g \xi) \cdot \bar{P})(U,V)\xi &= L_{\bar{P}}[((X \wedge_g \xi)\bar{P})(U,V)\xi - \bar{P}((X \wedge_g \xi)U,V)\xi \\
&- \bar{P}(U,(X \wedge_g \xi)V)\xi - \bar{P}(U,V)(X \wedge_g \xi)\xi].
\end{aligned} \tag{5.3}
$$

By virtue of (1.4), (1.5), (2.12), (2.13) and (2.7) in (5.3), we obtain

$$L_{\bar{P}}((X \wedge_g \xi) \cdot \bar{P})(U,V)\xi = -L_{\bar{P}} \cdot \bar{P}(U,V)X. \tag{5.4}$$

Next by considering left hand side of (5.2), we have

$$
\begin{aligned}
(\bar{R}(X,\xi) \cdot \bar{P})(U,V)\xi &= \bar{R}(X,\xi)\bar{P}(U,V)\xi - \bar{P}(\bar{R}(X,\xi)U,V)\xi \\
&- \bar{P}(U,\bar{R}(X,\xi)V)\xi - \bar{P}(U,V)\bar{R}(X,\xi)\xi.
\end{aligned} \tag{5.5}
$$

Again using (1.5), (2.12), (2.13) and (2.7) in (5.5), we get

$$(\bar{R}(X,\xi) \cdot \bar{P})(U,V)\xi = 2\bar{P}(U,V)X. \tag{5.6}$$

Substituting (5.4) and (5.6) in (5.2), we obtain

$$(L_{\bar{P}} + 2)\bar{P}(U,V)X = 0. \tag{5.7}$$

This leads us to the following:

`Theorem 5.1.` If an n-dimensional Kenmotsu manifold is projective pseudo-symmetric with respect to the semi-symmetric metric connection, then either $L_{\bar{P}} = -2$ or the manifold is projectively flat.

Also, in a Kenmotsu manifold, Bagewadi, Prakasha and Venkatesha [4] proved the following:

`Lemma 5.1.` [4] If the projective curvature tensor of a Kenmotsu manifold M admitting the semi-symmetric metric connection vanishes, then M reduces to an Einstein manifold with the constant scalar curvature $-n(n-1)$.

Hence from `Theorem 5.1.` and `Lemma 5.1.`, we conclude that:

`Corollary 5.1.` A projective pseudo-symmetric Kenmotsu manifold admitting the semi-symmetric metric connection is an Einstein manifold with the constant scalar curvature with respect to the Levi-Civita connection provided $L_{\bar{P}} \neq -2$.

6. ϕ-projective semi-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection

`Definition:` An n-dimensional Kenmotsu manifold M is said to be ϕ-projectively semi-symmetric with respect to the semi-symmetric metric connection if $\bar{P}(X,Y) \cdot \phi = 0$.

Let us consider an n-dimensional Kenmotsu manifold M which is ϕ-projective semi-symmetric. Then we have

$$\bar{P}(X,Y)\phi Z - \phi\bar{P}(X,Y)Z = 0, \tag{6.1}$$

for any vector fields X, Y and Z on M.
By virtue of (1.5) in (6.1) gives

$$\bar{R}(X,Y)\phi Z - \phi\bar{R}(X,Y)Z + \frac{1}{n-1}[\bar{S}(Y,\phi Z)X - \bar{S}(X,\phi Z)Y + \bar{S}(Y,Z)\phi X - \bar{S}(X,Z)\phi Y] = 0. \tag{6.2}$$

On plugging $Y = \xi$ in (6.2) and then using (2.12), (2.13) and (2.7), we obtain

$$2g(X,\phi Z)\xi - \frac{1}{n-1}\bar{S}(X,\phi Z)\xi = 0. \tag{6.3}$$

Now taking the inner product of the above equation with ξ, we get

$$2g(X,\phi Z) - \frac{1}{n-1}\bar{S}(X,\phi Z) = 0. \tag{6.4}$$

Replacing Z by ϕZ in (6.4) and then by virtue of (2.1) and (2.13), we obtain

$$S(X,Z) = Ag(X,Z) + B\eta(X)\eta(Z), \tag{6.5}$$

where $A = 5n - 7$ and $B = -2(3n - 5)$.
Hence we can state the following:
Theorem 6.1. An n-dimensional ϕ-projective semi-symmetric Kenmotsu manifold with respect to the semi-symmetric metric connection is η-Einstein with respect to the Levi-Civita connection.

7. Example

Consider a $3-$dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$E_1 = -e^{-z}\frac{\partial}{\partial x}, \qquad E_2 = e^{-z}\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta, \tag{7.1}$$

where η is the $1-$form defined by $\eta(X) = g(X, E_3)$, for any vector field X on M. Then $\{E_1, E_2, E_3\}$ is an orthonormal basis of M. We define a $(1,1)$ tensor field ϕ as

$$\phi\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right) + Z\frac{\partial}{\partial z} = \left(Y\frac{\partial}{\partial x} - X\frac{\partial}{\partial y}\right). \tag{7.2}$$

Thus, we have

$$\phi(E_1) = E_2, \;\; \phi(E_2) = -E_1 \quad \text{and} \quad \phi(E_3) = 0. \tag{7.3}$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= 1, \qquad \phi^2 X = -X + \eta(X) E_3, \\ g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), \end{aligned}$$

for any vector fields X, Y on M.

Moreover, we get

$$[E_i, \xi] = E_i, \qquad [E_i, E_j] = 0, \qquad i,j = 1,2$$

Using Koszul's formula, we obtain

$$\nabla_{E_i} E_i = -\xi, \qquad \nabla_{E_i}\xi = E_i, \qquad i = 1,2.$$

and others are zero. Thus for $E_3 = \xi$, $M(\phi, \xi, \eta, g)$ is a Kenmotsu manifold. Now, the non-zero terms of the semi-symmetric metric connection on M become

$$\bar{\nabla}_{E_i} E_i = -2\xi, \quad \bar{\nabla}_{E_i}\xi = 2E_i \quad i = 1,2. \tag{7.4}$$

With the help of the above results it can be easily verified that

$$\begin{array}{lll} R(E_1,E_2)E_3 = 0, & R(E_2,E_3)E_3 = -E_2, & R(E_1,E_3)E_3 = -E_1, \\ R(E_1,E_2)E_2 = -E_1, & R(E_2,E_3)E_2 = E_3, & R(E_1,E_3)E_2 = 0, \\ R(E_1,E_2)E_1 = E_2, & R(E_2,E_3)E_1 = 0, & R(E_1,E_3)E_1 = E_3. \end{array}$$

and

$$\begin{array}{lll} \bar{R}(E_1,E_2)E_3 = 0, & \bar{R}(E_2,E_3)E_3 = -2E_2, & \bar{R}(E_1,E_3)E_3 = -2E_1, \\ \bar{R}(E_1,E_2)E_2 = -4E_1, & \bar{R}(E_2,E_3)E_2 = 2E_3, & \bar{R}(E_1,E_3)E_2 = 0, \\ \bar{R}(E_1,E_2)E_1 = 4E_2, & \bar{R}(E_2,E_3)E_1 = 0, & \bar{R}(E_1,E_3)E_1 = 2E_3. \end{array} \tag{7.5}$$

In view of (1.1), one can obtain the torsion tensor $\bar{T}$ with respect to the semi-symmetric metric connection as

$$\begin{gathered} \bar{T}(E_i, E_i) = 0 \quad for \quad i = 1,2,3; \\ \bar{T}(E_1,E_2) = 0, \quad \bar{T}(E_1,E_3) = E_1, \quad \bar{T}(E_2,E_3) = E_2. \end{gathered}$$

Since E_1, E_2, E_3 forms a basis, the vector fields $X, Y, Z \in \chi(M)$ can be written as

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \tag{7.6}$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1,2,3$. Using the expressions of the curvature tensors, we find values of the Riemannian curvature and Ricci curvature with respect to the semi-symmetric metric connection as;

$$\begin{aligned} \bar{R}(X,Y)Z &= [-4\{a_1b_2 - b_1a_2\}b_3 + 2\{c_1a_2 - a_1c_2\}c_3]E_1 \\ &+ [-4\{b_1a_2 - a_1b_2\}a_3 + 2\{c_1b_2 - b_1c_2\}c_3]E_2 \\ &+ [-2\{c_1a_2 - a_1c_2\}a_3 - 2\{c_1b_2 - b_1c_2\}b_3]E_3, \end{aligned} \tag{7.7}$$

$$\bar{S}(E_1,E_1) = \bar{S}(E_2,E_2) = -6, \ \bar{S}(E_3,E_3) = -4. \tag{7.8}$$

In view of the expression of the endomorphism $(E_i \wedge_g E_j)E_w = g(E_j, E_w)E_i - g(E_i, E_w)E_j$ for $1 \leq i, j, w \leq 3$ and equations (7.5) and (7.8), one can easily verify that

$$\begin{aligned} \bar{S}(\bar{R}(E_i, E_3)E_j, E_3) + \bar{S}(E_j, \bar{R}(E_i, E_3)E_3) &= -2[\bar{S}((E_i \wedge_g E_3)E_j, E_3) \\ &+ \bar{S}(E_j, (E_i \wedge_g E_3)E_3)], \end{aligned} \tag{7.9}$$

in view of the above equation `Theorem 4.1.` is verified.

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η-RICCI SOLITONS IN (ε, δ)-TRANS-SASAKIAN MANIFOLDS

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Abstract. The objective of the present paper is to study (ε, δ)-trans-Sasakian manifolds admitting η-Ricci solitons. It is shown that a symmetric second order covariant tensor in an (ε, δ)-trans-Sasakian manifold is a constant multiple of the metric tensor. Also, an example of an η-Ricci soliton in a 3-diemsional (ε, δ)-trans-Sasakian manifold is provided in the region where (ε, δ)-Trans Sasakian manifold is expanding.
Keywords: Sasakian manifolds; Ricci soliton; Tensor.

1. Introduction

In 1985, J. A. Oubina [22] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times \mathbb{R}$ belongs to the class W_4, where the classification of almost Hermition manifolds appears as a class W_4 of Hermitian manifolds which are closely related to locally conformal Kähler manifolds studied by Gray and Hervella [14]. The class $C_5 \oplus C_6$ [22] coincides with the class of trans-Sasakian structure of type (α, β). This class consists of both Sasakian and Kenmotsu structures. If $\alpha = 1$, $\beta = 0$ then the class turn into Sasakian and when $\alpha = 0$, $\beta = 1$ then it turn into Kenmotsu. The above manifolds are studied by many authors like D. E. Blair and J. C. Marrero [1], K. Kenmotsu [17], C. S. Bagewadi and Venkatesha [8], U. C. De and M. M. Tripathi [12].

The differential geometry of manifolds with indefinite metric plays an interesting role in physics. Manifolds with indefinite metric have been studied by several authors. The concept of (ϵ)-Sasakian manifolds was initiated by A. Bejancu and K. L. Duggal [2] and further investigation was taken up by X. Xufeng and C. Xiaoli [30]. U. C. De and A. Sarkar [11] studied (ε)-Kenmotsu manifolds with indefinite metric. S. S. Shukla and D. D. Singh [25] extended with indefinite metric which is a natural generalization of both (ε)-Sasakian and (ε)-Kenmotsu manifolds. The

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authors H. G. Nagaraja et al. [20] studied (ε, δ)-trans-Sasakian manifolds which are extensions of (ε)-trans-Sasakian manifolds. M. D. Siddiqi et al. also studied some properties of (ε, δ)-trans-Sasakian manifolds in [26].

In 1982, R. S. Hamilton [15] stated that Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow which is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. The Ricci soliton is said to be shrinking, steady or expanding depending on whether $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, Levy [18] obtained necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [24] initiated a study of Ricci solitons in contact Riemannian geometry . After that, Tripathi [27], Nagaraja et al. [21] and others like C. S. Bagewadi et al. ([7], [16]) extensively studied Ricci solitons in almost (ϵ)-contact metric manifolds. In 2009, J. T. Cho and M. Kimura [10] introduced the notion of η-Ricci soliton and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later η-Ricci solitons in (ε)-almost paracontact metric manifolds were studied by A. M. Blaga et. al. in [5]. Moreover, η-Ricci solitons have been studied by various authors for different structures (see [3], [4], [23], [9], [28]). Recently, K. Venu et al. [29] studied the η-Ricci solitons in trans-Sasakian manifolds. Motivated by these studies in the present paper we investigate η-Ricci solitons in 3-dimensional (ε, δ)-trans-Sasakian manifolds and derive the expression for the scalar curvature.

1.1. Preliminaries

Let M be an almost contact metric manifold equipped with the almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \qquad \eta(X) = \varepsilon g(X, \xi), \qquad g(\xi, \xi) = \varepsilon, \tag{1.4}$$

for all X, Y vector fields on M, where ε is 1 or -1 according as ξ is space-like or time-like. In particular, if the metric g is positive definite, then the (ε)-almost contact metric manifold is the usual almost contact metric manifold [25].

An (ε)-almost contact metric metric manifold is called an (ε)-trans Sasakian manifold [25] if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \varepsilon\eta(Y)\phi X) \tag{1.5}$$

holds for some smooth functions α and β on M. According to the characteristic vector field ξ we have two classes of (ε)-trans-Sasakian manifolds. When $\varepsilon = -1$ and index of g is odd, then M is a time-like trans-Sasakian manifold and when $\varepsilon = 1$ and index of g is even, then M is a space-like trans-Sasakian manifold. Further, M is a usual trans-Sasakian manifold for $\varepsilon = 1$ and the index of g is 0 and M is a Lorentzian trans-Sasakian manifold for $\varepsilon = -1$ and the index of g is 1. An ε-almost contact metric manifold is said to be a (ε, δ)-trans-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \varepsilon\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X) \tag{1.6}$$

holds for some smooth functions α and β on M, where ε is 1 or -1 according as ξ is space-like or time-like and δ is alike ε.
From (1.6), we have

$$\nabla_X \xi = -\varepsilon\alpha\phi X - \delta\beta\phi^2 X, \tag{1.7}$$

and

$$(\nabla_X \eta)Y = \delta\beta[\varepsilon g(X, Y) - \eta(X)\eta(Y)] - \alpha g(\phi X, Y). \tag{1.8}$$

In (ε, δ)-trans-Sasakian manifold M, we have the following relations [7]:

$$\begin{aligned} R(X, Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] \\ & +2\varepsilon\delta\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ & +\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] \\ & +\delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \\ & +2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi, \end{aligned} \tag{1.9}$$

$$\begin{aligned} S(X, \xi) = {} & [((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(X) \\ & -\varepsilon((\phi X)\alpha) - (n-2)\varepsilon(X\beta)), \end{aligned} \tag{1.10}$$

$$Q\xi = ((n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta))\xi + \varepsilon\phi(grad\alpha) - \varepsilon(n-2)(grad\beta), \tag{1.11}$$

where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by $S(X, Y) = g(QX, Y)$.

Further in a (ε, δ)-trans-Sasakian manifold, we have

$$\varepsilon\phi(grad\alpha) = \varepsilon(n-2)(grad\beta), \tag{1.12}$$

and

$$\varepsilon(\xi\alpha) + 2\varepsilon\delta\alpha\beta = 0. \tag{1.13}$$

Using (1.9) and (1.12), for constants α and β , we have

$$R(\xi,X)Y=(\alpha^2-\beta^2)[\varepsilon g(X,Y)\xi-\eta(Y)X], \tag{1.14}$$

$$R(X,Y)\xi=(\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y], \tag{1.15}$$

$$\eta(R(X,Y)Z)=(\alpha^2-\beta^2)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)], \tag{1.16}$$

$$S(X,\xi)=[((n-1)(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\eta(X), \tag{1.17}$$

$$Q\xi=[(n-1)(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\xi. \tag{1.18}$$

An important consequence of (1.7) is that ξ is a geodesic vector field

$$\nabla_\xi\xi=0. \tag{1.19}$$

For an arbitrary vector field X, we have that

$$d\eta(\xi,X)=0. \tag{1.20}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (1.15), we have

$$K_\xi=g(R(\xi,X),\xi,X)=(\alpha^2-\beta^2)-\delta(\xi\beta). \tag{1.21}$$

It follows from (1.21) that ξ-sectional curvature does not depend on X.

1.2. η-Ricci solitons on (M,ϕ,ξ,η,g)

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇, that is, $\nabla h=0$. Applying the Ricci commutation identity [20]

$$\nabla^2h(X,Y;Z,W)-\nabla^2h(X,Y;W,Z)=0, \tag{1.22}$$

we obtain the relation

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0. \tag{1.23}$$

Replacing $Z=W=\xi$ in (1.23) and using (1.9) and the symmetry of h, we have

$$\begin{aligned} &2(\alpha^2-\beta^2)[\eta(Y)h(X,\xi)-\eta(X)h(Y,\xi)] \\ &+2\varepsilon[(Y\alpha)h(\phi X,\xi)-(X\alpha)h(\phi Y,\xi)]+2\delta[(Y\beta)h(\phi^2X,\xi)-(X\beta)h(\phi^2Y,\xi)] \\ &+4\varepsilon\delta\alpha\beta[\eta(Y)h(\phi X,\xi)-\eta(X)h(\phi Y,\xi)]+4\alpha\beta(\delta-\varepsilon)g(\phi X,Y)h(\xi,\xi)=0. \end{aligned} \tag{1.24}$$

Putting $X=\xi$ in (1.24) and by virtue of (1.3), we obtain

$$-2[\varepsilon(\xi\alpha)+2\varepsilon\delta\alpha\beta]h(\phi Y,\xi) \tag{1.25}$$

$$+2[(\alpha^2 - \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0.$$

By using (1.13) in (1.25), we have

$$[(\alpha^2 - \beta^2) - \delta(\xi\beta)][\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{1.26}$$

Suppose $(\alpha^2 - \beta^2) - \delta(\xi\beta) \neq 0$; it results in

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{1.27}$$

Now, we can call a regular (ε, δ)-trans-Sasakian manifold if $(\alpha^2 - \beta^2) - \delta(\xi\beta) \neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of the (ε, δ)-trans-Sasakian manifold.

Differentiating (1.27) covariantly with respect to X, we have

$$(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X\xi) \tag{1.28}$$

$$= [\varepsilon g(\nabla_X Y,\xi) + \varepsilon g(Y,\nabla_X\xi)]h(\xi,\xi)$$

$$+\eta(Y)[(\nabla_X h)(Y,\xi) + 2h(\nabla_X\xi,\xi)].$$

By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X\xi) = 0$ and by virtue of (1.27) in (1.28), we get

$$h(Y,\nabla_X\xi) = \varepsilon g(Y,\nabla_X\xi)h(\xi,\xi).$$

Now using (1.7) in the above equation, we get

$$-\varepsilon\alpha h(Y,\phi X) + \delta\beta h(Y,X) = -\alpha g(Y,\phi X)h(\xi,\xi) + \varepsilon\delta\beta g(Y,X)h(\xi,\xi). \tag{1.29}$$

Replacing $X = \phi X$ in (1.29) and after simplification, we get

$$h(X,Y) = \varepsilon g(X,Y)h(\xi,\xi), \tag{1.30}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (1.27). Now by considering the above equations, we can give the conclusion:

Theorem 1.1. *Let (M,ϕ,ξ,η,g) be a (ε,δ)-trans-Sasakian manifold with a non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in \Gamma T_2^0(M)$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇, then it is a constant multiple of the metric tensor g.*

Let (M,ϕ,ξ,η,g) be an (ε)-almost contact metric manifold. Consider the equation [10]

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = 0, \tag{1.31}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_X \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{1.32}$$

for any $X, Y \in \chi(M)$.

Definition 1.2. The data (g, ξ, λ, μ) which satisfy the equation (3.10) is said to be η- *Ricci soliton* on M [10]; in particular, if $\mu = 0$ then (g, ξ, λ) is the Ricci soliton [10] and it is called shrinking, steady or expanding following $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively [10].

Now, from (1.7) , the equation (1.31) becomes:

$$S(X,Y) = -(\lambda + \delta\beta)g(X,Y) + (\varepsilon\delta\beta - \mu)\eta(X)\eta(Y). \tag{1.33}$$

The above equations yields

$$S(X,\xi) = -[(\lambda + \mu) + (1-\varepsilon)\delta\beta]\eta(X) \tag{1.34}$$

$$QX = -(\lambda + \beta\delta)X + (\varepsilon\delta\beta - \mu)\xi \tag{1.35}$$

$$Q\xi = -[(\lambda + \mu) + (1-\varepsilon)\delta\beta]\xi \tag{1.36}$$

$$r = -\lambda n - (n-1)\varepsilon\delta\beta - \mu, \tag{1.37}$$

where r is the scalar curvature. Off the two natural situations regrading the vector field V: $V \in Span\xi$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y) = 2\delta\beta[g(X,Y) - \varepsilon\eta(X)\eta(Y)]. \tag{1.38}$$

In a 3-dimensional (ε, δ)-trans-Sasakian manifold the Riemannian curvature tensor is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \tag{1.39}$$

$$-\frac{r}{2}[g(Y,Z)X - g(X,Z)Y].$$

Putting $Z = \xi$ in (1.39) and using (1.9) and (1.10) for 3-dimensional (ε, δ)-trans-Sasakian manifold, we get

$$(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\varepsilon\delta\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{1.40}$$

$$+\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$

$$+2(\delta - \varepsilon)\alpha\beta g(\phi X, Y)$$

$$= \varepsilon[(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(Y)X - \eta(X)Y]$$

$$+\varepsilon\eta(Y)QX - \varepsilon\eta(X)QY - \varepsilon[((\phi Y)\alpha)X + (Y\beta)X] + \varepsilon[((\phi X)\alpha)Y + (X\beta)Y].$$

Again, putting $Y = \xi$ in (1.40) and using (1.3) and (1.13), we obtain

$$QX = \left[\frac{r}{2} - 2(\varepsilon\alpha^2 - \delta\beta^2) + \varepsilon(\alpha^2 - \beta^2)\right] X. \tag{1.41}$$

$$+ \left[4(\varepsilon\alpha^2 - \delta\beta^2) - \frac{r}{2} - (\alpha^2 - \beta^2)\right] \eta(X)\xi$$

From (1.41), we have

$$S(X, Y) = \left[\frac{r}{2} - 2(\varepsilon\alpha^2 - \delta\beta^2) + \varepsilon(\alpha^2 - \beta^2)\right] g(X, Y) \tag{1.42}$$

$$+ \left[4(\varepsilon\alpha^2 - \delta\beta^2) - \frac{r}{2} - (\alpha^2 - \beta^2)\right] \varepsilon\eta(X)\eta(Y).$$

Equation (1.42) shows that a 3-dimensional (ε, δ)-trans-Sasakian manifold is *η-Einstein.*
Next, we consider the equation

$$h(X, Y) = (\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y). \tag{1.43}$$

By Using (1.48) and (1.42) in (1.43), we have

$$h(X, Y) = \left[r - 4(\varepsilon\alpha^2 - \delta\beta^2) + 2\varepsilon(\alpha^2 - \beta^2) + 2\delta\beta\right] g(X, Y) \tag{1.44}$$

$$+ \left[8(\varepsilon\alpha^2 - \delta\beta^2) - 2\varepsilon(\alpha^2 - \beta^2) - 2\delta\beta - r\right] \varepsilon\eta(X)\eta(Y) + 2\mu\eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in (1.5), we get

$$h(\xi, \xi) = 2[2\varepsilon(\varepsilon\alpha^2 - \delta\beta^2) - 2\mu]. \tag{1.45}$$

Now, (1.30) becomes

$$h(X, Y) = 2[2\varepsilon(\varepsilon\alpha^2 - \delta\beta^2) - 2\mu]\varepsilon g(X, Y). \tag{1.46}$$

From (1.43) and (1.46), it follows that (g, ξ, μ) is an η-Ricci soliton.
Therefore, we can state as:

Theorem 1.2. *Let (M, ϕ, ξ, η, g) be a 3-dimensional (ε, δ)-trans-Sasakian manifold. Then (g, ξ, μ) yields an η-Ricci soliton on M.*

Let V be pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function on the 3-dimensional (ε, δ)-trans-Sasakian manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0$$

or

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X)$$

$$+2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (1.7), we obtain

$$bg(-\varepsilon\alpha\phi X - \delta\beta(-X + \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(-\varepsilon\alpha\phi Y - \delta\beta(-Y + \eta(Y)\xi, X)$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

which yields

$$2b\delta\beta g(X,Y) - 2b\delta\beta\eta(X)\eta(Y) + (Xb)\eta(Y) \tag{1.47}$$

$$+(Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (1.47), we obtain

$$(Xb) + (\xi b)\eta(X) + 2[2(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta) + \lambda + \mu]\eta(X) = 0. \tag{1.48}$$

Again putting $X = \xi$ in (1.48), we obtain

$$\xi b = -2(\varepsilon\alpha^2 - \delta\beta^2) + (\xi\beta) - \lambda - \mu.$$

Plugging this in (1.48), we get

$$(Xb) + 2[2(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta) + \lambda + \mu]\eta(X) = 0,$$

or

$$db = -\left\{\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2)\right\}\eta = 0. \tag{1.49}$$

Applying d on (1.49), we get $\left\{\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2)\right\} d\eta$. Since $d\eta \neq 0$ we have

$$\lambda + \mu + (\xi\beta) + 2(\varepsilon\alpha^2 - \delta\beta^2) = 0. \tag{1.50}$$

Equation (1.50) in (1.49) yields b as a constant. Therefore from (1.47), it follows that

$$S(X,Y) = -(\lambda + \delta\beta)g(X,Y) + (\varepsilon\delta b\beta - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for the constant $\delta\beta$. This leads to the following:

Theorem 1.3. *If in a 3-dimensional (ε, δ)-trans-Sasakian manifold the metric g is an η-Ricci soliton and V is pointwise collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $\delta\beta$ is a constant.*

Tanking $X = Y = \xi$ in (1.30) and (1.42) and comparing, we get

$$\lambda = -2(\epsilon\alpha^2 - \delta\beta^2) + (\xi\beta) + \mu = -2K_\xi - \mu. \tag{1.51}$$

From (1.37) and (1.51), we obtain

$$r = 6(\epsilon\alpha^2 - \delta\beta^2) + 3(\xi\beta) - 2\varepsilon\delta\beta + 2\mu. \tag{1.52}$$

Since λ is a constant, it follows from (1.51) that K_ξ is a constant.

Theorem 1.4. *Let (g, ξ, μ) be an η-Ricci soliton in the 3-dimensional (ε, δ)-trans Sasaakian manifold (M, ϕ, ξ, η, g). Then the scalar $\lambda + \mu = -2K_\xi$, $r = 6K_\xi + 2\mu + 3(\xi\beta) - 2\varepsilon\delta\beta$.*

Remark 1.1. For $\mu = 0$, (1.51) reduces to $\lambda = -2K_\xi$, so the Ricci soliton in a 3-dimensional (ε, δ)-trans-Sasakian manifold is shrinking.

2. Example of η-Ricci solitons on (ε, δ)-Trans-Sasakian manifolds

Example 2.1. Consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 z \neq 0\}$, where (x, y, z) are the cartesian coordinates in $\mathbb{R}^3$ and let the vector fields

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \qquad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \qquad e_3 = \frac{-(\epsilon + \delta)}{2}\frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannian metric defined by
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \varepsilon$, $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$,
where $\epsilon = \pm 1$.

Let η be the 1-form defined by $\eta(X) = \varepsilon g(X, \xi)$, for any vector field X on M, let ϕ be the (1,1)-tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$, for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (ε)-almost contact structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.

$\nabla_{e_1} e_3 = -\frac{(\varepsilon+\delta)}{z} e_1, \qquad \nabla_{e_2} e_3 = -\frac{(\varepsilon+\delta)}{z} e_2, \qquad \nabla_{e_1} e_2 = 0,$
using the above relation, for any vector X on M, we have $\nabla_X \xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ, ξ, η, g) structure defines the (ε, δ)-tran-Sasakian structure in $\mathbb{R}^3$.

Here ∇ is the Levi-Civita connection with respect to the metric g , so we have $[e_1, e_2] = 0, \qquad [e_1, e_3] = -\frac{(\varepsilon+\delta)}{z} e_1, \quad [e_2, e_3] = -\frac{(\varepsilon+\delta)}{z} e_2.$

Thus we have

$$\nabla_{e_1} e_3 = -\frac{(\varepsilon+\delta)}{z} e_1 + e_2, \nabla_{e_1} e_2 = 0$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\frac{(\varepsilon+\delta)}{z} e_2, \quad \nabla_{e_2} e_3 = -\frac{(\varepsilon+\delta)}{z} e_2 e_1$$

$$\nabla_{e_3} e_1 = 0, \qquad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -\frac{(\varepsilon+\delta)}{z} e_1 + e_2.$$

The manifold M satisfies (1.7) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is a (ε, δ)-trans-Sasakian manifolds. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1, e_3)e_3 = \frac{(\varepsilon+\delta)}{z^2} e_1, \quad R(e_3, e_1)e_3 = -\frac{(\varepsilon+\delta)}{z^2} e_1, \qquad R(e_1, e_2)e_2 = \frac{(\varepsilon+\delta)}{z^2} e_1$$

$$R(e_2, e_3)e_3 = \frac{(\varepsilon+\delta)}{z^2} e_1, \quad R(e_3, e_2)e_3 = -\frac{(\varepsilon+\delta)}{z^2} e_1, \; R(e_2, e_1)e_1 = -\frac{(\varepsilon+\delta)}{z^2} e_1.$$

From the above expression of the curvature tensor we can also obtain

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \frac{(\varepsilon^2 + \delta\varepsilon)}{z^2}$$

since $g(e_1, e_3) = g(e_1, e_2) = 0$.
Therefore, we have

$$S(e_i, e_i) = -\frac{(\varepsilon+\delta)}{z^2} g(e_i, e_i),$$

for $i = 1, 2, 3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence M is also an *Einstein* manifold. In this case, from (1.32), we have

$$2\delta\beta[g(e_i, e_i - \varepsilon\eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + 2\lambda g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i) = 0. \tag{2.1}$$

Now, from (2.1), we get $\lambda = \frac{\varepsilon[\delta(1+z)-\varepsilon]}{z^2}$ (i.e, $\lambda > 0$) and $\mu = -\frac{\varepsilon[\varepsilon^2-\varepsilon-\delta(1+\varepsilon+\varepsilon z)]}{z^2}$, the data (g, ξ, λ, μ) is an η-Ricci soliton on (M, ϕ, ξ, η, g) i. e., expanding.

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DETERMINING SOLUTIONS OF FUZZY CELLULAR NEURAL NETWORKS WITH FLUCTUATING DELAYS

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Abstract. This paper deals with the problem of nonperiodic arrangements for fuzzy cell neural systems with fluctuating delays. By utilizing compression mapping and Krasnoselski's settled point hypothesis and developing some appropriate Lyapunov functionals, adequate conditions are set up for the presence and worldwide exponential solidness of solutions of FCNNs with fluctuating delays. In addition, illustrative examples are set up to exhibit a model.
Keywords. Cellular neural networks; fuzzy; fluctuating delays; nonperiodic solutions.

1. Introduction

Celluar neural nets (CNNs), initially presented in [1], have pulled in much consideration lately. This is generally on the grounds that they have the extensive variety of promising applications in the fields of related memory, parallel figuring, design acknowledgment, flag handling and streamlining. CNNs are portrayed by essential circuit units called cells. Every unit forms a few information flags and delivers a yield flag which is gotten by different units associated with it including itself.

In the execution of a flag or impact going through neural systems, time delays do exist and influence dynamical behavior of a working neural network. As of late there have been a few outcomes about dynamical practices of deferred neural systems including worldwide exponential steadiness of balance focuses, intermittent and relatively occasional arrangements [2, 3].

Other than defer impacts, it has been seen that numerous transformative procedures, including those identified with neural systems, may display incautious impacts. In these developmental procedures, the arrangements of framework are not consistent but rather present hops which can cause shakiness of dynamical frameworks. Thus, numerous neural systems with motivations have been contemplated broadly, and a lot of writing are engaged on the issue of the presence and steadiness

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of a balance point [4]. The presence and dependability of periodic solution of neural network with impulses are researched extensively by many authors [5, 6].

In [7], another compose cell neural systems display called fuzzy cell neural systems (FCNNs) is introduced. FCNNs joined fuzzy task with cell neural systems.

In any case, it is important that Takagi-Sugeno (T-S) fuzzy neural systems are not quite the same as FCNNs. T-S fuzzy neural systems depend on an arrangement of fuzzy guidelines to depict nonlinear framework. As of late analysts have discovered that FCNNs are helpful in picture preparing, and many fascinating outcomes have been introduced on steadiness of FCNNs. For instance, in [8], applying straight network imbalance (LMI) approach, contemplated presence, uniqueness and worldwide asymptotic steadiness of fuzzy cell neural systems with asymptotic relentlessness of cushioned cell neural frameworks with spillage delay under imprudent annoyances. The authors in [9] acquired the outcomes of asymptotic steadiness for fuzzy cell neural systems with time-shifting postponements. In [10], the steadiness of fuzzy cell neural systems is examined with time-changing delay in spillage term without accepting the boundedness of initiation function. Other related works readers can refer to [11].

However, in applied sciences, the existence of nonperiodic arrangements assumes a key job in portraying the conduct of nonlinear differential conditions. For instance, hostile to intermittent trigonometric polynomials are vital for the investigation of addition issues, against occasional wavelets and simple voltage transmission are frequently against intermittent process, in this way it is profitable to consider nonperiodic solutions. Meanwhile, anti-periodic solution, as a special case of periodic solution, has an important research value in dynamic behavior of the neural networks. In recent years, the problem of nonperiodic solution of CNNs, Hopfield neural nets and recurrent neural nets has been studied by many scholars (see [12, 13, 14] and references therein). For example, in [12], the author studied the presence and exponential security of the counter occasional arrangements of intermittent neural systems with time-differing and persistent dispersed deferrals. In [13], applying imbalance procedure and dependent on Lyapunov practical hypothesis, the authors examined the presence and worldwide exponential security of against intermittent answer for defer CNNs with hasty impacts. In any case, to the best of our insight, there are not very many outcomes on the issues of against occasional answers for fuzzy cell neural systems (FCNNs) with fluctuating delays and hasty impacts.

It is reasonable to proceed the examination of the presence and stability of nonperiodic arrangements for FCNNs with period-varying delays and impulsive effects. Here, we are concerned with the next model:

$$
(1.1)\qquad \begin{cases} x_i'(t) & = & -a_i(t)x_i(t) + \sum_{j=1}^{n} d_{ij}(t)f_j(x_j(t)) \\ & & +\bigwedge_{j=1}^{n} a_{ij}(t)g_j(x_j(t-t_{ij}(t))) \\ & & +\bigvee_{j=1}^{n} b_{ij}(t)g_j(x_j(t-t_{ij}(t))) \\ & & +E_i(t)]\,, t \geq 0, t \neq t_k, k \in N^+, \\ \Delta(x_i(t_k)) & = & x_i(t_k^+) - x_i(t_k^-) = I_{ik}(t_k, x_i(t_k)), \\ x_i(t) & = & \varrho_i(t), t \in [-t, 0], i = 1, 2, \cdots, n. \end{cases}
$$

where n is the amount of elements in the net. $x_i(t)$ is the activations of the i-th neuron at the time t. $a_i(t), d_{ij}(t), a_{ij}(t), b_{ij}(t), E_i(t), f_j(t), g_j(t), t_{ij}(t)$ are continuous functions on R. $a_i(t) > 0$ represents the amplification function. $d_{ij}(t)$ denotes the synaptic connection weight of the unit j on the unit i at time t. Thus, $a_{ij}(t)$ and $b_{ij}(t)$ are elements of fuzzy feedback MIN and MAX template, correspondingly. $\bigwedge$ and $\bigvee$ represent the fuzzy AND and OR operation, correspondingly. $E_i(t)$ denotes the i-th component of an external input source introduced from outside the network to the ith cell. $t_{ij}(t)$ is time-varying delay satisfying $0 \leq t_{ij}(t) \leq t$, t is a positive constant. $f_j(\cdot)$ and $g_j(\cdot)$ are the activation functions. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-), x_i(t_k^+) = \lim_{h\to 0+} x_i(t_k + h), x_i(t_k^-) = \lim_{h\to 0-} x_i(t_k+h), (i = 1, 2, \cdots, n, k = 1, 2, \cdots)$. $\{t_k\}$ is a sequence of real numbers such that $t_1 < t_2 < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$.

The primary motivation behind this paper is to think about the presence and worldwide exponential solidness of hostile to occasional arrangements of (1).

The framework of this paper is as per the following. In Sect. 2, we present a few definitions and lemmas. In Sect. 3, we set up new adequate conditions for the presence of the counter occasional arrangements of framework (1). In Faction 4, by building reasonable Lyapunov practical, we infer adequate conditions for the worldwide exponential strength of hostile to intermittent arrangements of framework (1). A numerical model is given to demonstrate the adequacy of our outcomes in Sect. 5. At last a general end is attracted Sect. 6.

2. Preliminaries

Let us present the following:

$$
a_i^- = \min_{t\in[0,\omega]} |a_i(t)|, a^+ = \max_{1\leqslant i\leqslant n} \max_{t\in[0,\omega]} |a_i(t)|,
$$

$$
\overline{d}_{ij} = \max_{t\in[0,\omega]} |d_{ij}(t)|, \overline{d} = \max_{1\leqslant i\leqslant n} \max_{t\in[0,\omega]} |d_{ij}(t)|,
$$

$$\overline{a}_{ij} = \max_{t\in[0,\omega]} |a_{ij}(t)|, \overline{a} = \max_{1\leqslant i\leqslant n} \max_{t\in[0,\omega]} |a_{ij}(t)|,$$

$$\overline{b}_{ij} = \max_{t\in[0,\omega]} |b_{ij}(t)|, \overline{b} = \max_{1\leqslant i\leqslant n} \max_{t\in[0,\omega]} |b_{ij}(t)|,$$

$$\overline{E} = \max_{1\leqslant i\leqslant n} \max_{t\in[0,\omega]} |E_i(t)|, \chi_i = e^{\int_0^\omega a_i(\phi)d\phi}.$$

Here, the next assumptions are made

(**A1**) For $i, j = 1, 2, \cdots, n, k = 1, 2, \cdots$, there exist $\omega > 0$ such that for $\Omega \in R$

$$a_i(t+\omega) = a_i(t), t_{ij}(t+\omega) = t_{ij}(t),$$

$$a_{ij}(t+\omega)g_j(-\Omega) = -a_{ij}(t)g_j(\Omega),$$
$$b_{ij}(t+\omega)g_j(-\Omega) = -b_{ij}(t)g_j(\Omega),$$
$$d_{ij}(t+\omega)f_j(-\Omega) = -d_{ij}(t)f_j(\Omega),$$
$$E_i(t+\omega) = -E_i(t), I_{ik}(t+\omega, \Omega) = -I_{ik}(t, -\Omega).$$

(**A2**) $f_j(\cdot), g_j(\cdot) \in C(R\times R, R)$, and the nonnegative values $M_f, M_g, m_j, n_j (j = 1, 2, \cdots, n)$ exist such that, for $u, \Omega \in R$,

$$f_j(0) = 0, \quad |f_j(t,u)| \leqslant M_f, \quad |f_j(u) - f_j(\Omega)| \leqslant m_j|u-\Omega|,$$

$$g_j(0) = 0, \quad |g_j(t,u)| \leqslant M_g, \quad |g_j(u) - g_j(\Omega)| \leqslant n_j|u-\Omega|.$$

(**A3**) For $i, j = 1, 2, \cdots, n, k = 1, 2, \cdots$, there exists a positive integer q such that

$$I_{i(k+q)} = I_{ik},\ t_{k+q} = t_k + \omega.$$

(**A4**) For $i, j = 1, 2, \cdots, n, k = 1, 2, \cdots$, there exist $c_{ik} > 0$ such that

$$|I_{ik}(t,u) - I_{ik}(t,\Omega)| \leqslant c_{ik}|u-\Omega|,\ \forall t \in [0,\omega], u, \Omega \in R.$$

Remark 2.1 In assumption (A2), the activating functions $f_j, g_j, j = 1, 2, \cdots, n$, are typically assumed to be bounded and Lipchtiz continuous and need not to be differential.

Consider $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$, whereat T is the transpositioning. The starting assumptions based on (1) are determined by:

$$x(t) = \varphi(t), \quad t \in [-t, 0],$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T \in R^n, \varphi_i(i = 1, 2, \cdots, n)$ are continuous with norm

$$\|\varphi\| = \sup_{t\in[-t,0]} (\sum_{i=1}^n |\varphi_i(t)|^2)^{\frac{1}{2}}.$$

Definition 2.1 A resolution $x(t)$ of (1) is an ω nonperiodic solution, if

$$x(t+\omega) = -x(t), \quad t \neq t_k.$$

$$x(t_k+\omega)^+ = -x(t_k^+), \quad k = 1, 2, \cdots,$$

and the smallest positive number ω is called ω anti-periodic of function $x(t)$.

Define $PC(R^n) = \{x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T : R \to R^n, x|_{(t_k,t_{k+1}]} \in C((t_k, t_{k+1}], R^n), x(t_k^+), x(t_k)$ exist, and $x(t_k^-) = x(t_k), k = 1, 2, \cdots\}$. Set $X = \{x : x \in PC(R^n), x(t+\omega) = -x(t), t \in R\}$. It is easy to see X is a Banach space with norm $\|x\| = \sup_{t\in[-t,0]}(\sum_{i=1}^n |x_i(t)|^2)^{\frac{1}{2}}$.

Next, It is similar to [13], we have the following lemma.

Lemma 2.1. *Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ be an ω anti-periodic solution of system (1). For $i = 1, 2, \cdots, n$, we have*

$$\begin{aligned} x_i(t) &= \int_t^{t+\omega} H_i(t,s)\Bigg[\sum_{j=1}^n d_{ij}(s) f_j(x_j(s)) \\ &\quad + \bigwedge_{j=1}^n a_{ij}(s) g_j(x_j(s - t_{ij}(s))) + E_i(s) \\ &\quad + \bigvee_{j=1}^n b_{ij}(s) g_j(x_j(s - t_{ij}(s)))\Bigg] ds \\ &\quad + \sum_{t_k\in[t,t+\omega]} H_i(t,t_k) I_{ik}(t_k, x_i(t_k)), \end{aligned} \tag{2.1}$$

where, for $i = 1, 2, \cdots, n$,

$$H_i(t,s) = -\frac{e^{\int_t^s a_i(\phi)d\phi}}{e^{\int_0^\omega a_i(\phi)d\phi} + 1}, \quad s \in [t, t+\omega]. \tag{2.2}$$

Lemma 2.2. *[15] Let Ω be a closed convex and nonempty subset of a Banach space X. Let Π, Σ be the operators such that*
(i) $\Pi x + \Sigma y \in \Omega$ whenever $x, y \in \Omega$;
(ii) Π is compact and continuous;
(iii) Σ is a contraction mapping.
Then there exists $z \in \Omega$ such that $z = \Pi z + \Sigma z$.

Lemma 2.3. *[13] Let $p, q, t, c_k, k = 1, 2, \cdots,$ be constants and $q \geq 0, t > 0, c_k > 0$, and assume that $x(t)$ is piece continuous nonnegative function. Suppose Ω is a closed* *and nonempty subset of a Banach space X. Give Π, Σ a chance to be the administrators such that*

(I) $\Pi x + \Sigma y \in \Omega$ *at whatever point* $x, y \in \Omega$ *;*

(ii) Π *is minimal and continuous;*

(iii) Σ *is a compression mapping.*

At that point there exists $z \in \Omega$ *with the end goal that* $z = \Pi z + \Sigma z$.

Lemma 2.4. *[13] Let* $p, q, t, c_k, k = 1, 2, \cdots$ *, be constants and* $q \geq 0, t > 0, c_k > 0$, *and accept that* $x(t)$ *is piece consistent nonnegative capacity fulfilling*

$$\begin{cases} D^+x(t) & \leqslant & px(t) + q\bar{x}(t), t \geqslant t_0, t \neq t_k, \\ x(t_k^+) & \leqslant & c_k(x(t_k)), k = 1, 2, \cdots, \\ x(t) & = & \varphi(t), t \in [t_0 - t, t_0]. \end{cases} \tag{2.3}$$

If there exist c *such that for* $k = 1, 2, \cdots$,

$$\ln c_k \leqslant c(t_k - t_{k-1}). \tag{2.4}$$

and

$$p + cq + c < 0. \tag{2.5}$$

Then

$$x(t) \leqslant c \sup_{t \in [t_0 - t, t_0]} |\varphi(t)| e^{-\lambda(t - t_0)}, \tag{2.6}$$

where $\bar{x}(t) = \sup_{s \in [t-t, t]} x(s)$,

$$c = \sup_{1 \leqslant k < +\infty} \left\{ e^{c(t_k - t_{k-1})}, \frac{1}{e^{c(t_k - t_{k-1})}} \right\},$$

λ *is a sole nonnegative resolution of* $\lambda + p + cqe^{\lambda t} + c = 0$.

Lemma 2.5. *[7] Let* u *and* Ω *be two states of system (1), then we have*

$$\left| \bigwedge_{j=1}^{n} a_{ij}(t) g_j(u) \quad - \quad \bigwedge_{j=1}^{n} a_{ij}(t) g_j(\Omega) \right| \leqslant q \sum_{j=1}^{n} |a_{ij}(t)| |g_j(u) - g_j(\Omega)|,$$

and

$$\left| \bigvee_{j=1}^{n} b_{ij}(t) g_j(u) \quad - \quad \bigvee_{j=1}^{n} b_{ij}(t) g_j(\Omega) \right| \leqslant \sum_{j=1}^{n} |b_{ij}(t)| |g_j(u) - g_j(\Omega)|.$$

3. Existence and stability of a non-periodic solution

Here, we derive some sufficient conditions of existence of anti periodic resolution of (1).

Define the operator

$$\begin{cases} (\Pi x)(t) = ((\Pi_1 x)(t), (\Pi_2 x)(t), \cdots, (\Pi_n x)(t))^T, \\ (\Sigma x)(t) = ((\Sigma_1 x)(t), (\Sigma_2 x)(t), \cdots, (\Sigma_n x)(t))^T. \end{cases} \tag{3.1}$$

where

$$\begin{aligned} (\Pi_i x)(t) \quad = \quad & \int_t^{t+\omega} H_i(t,s) \left[\sum_{j=1}^{n} d_{ij}(s) f_j(x_j(s)) \right. \\ & + \bigwedge_{j=1}^{n} a_{ij}(s) g_j(x_j(s - t_{ij}(s))) \\ & + \bigvee_{j=1}^{n} b_{ij}(s) g_j(x_j(s - t_{ij}(s))) \\ & \left. + E_i(s) \right] ds \end{aligned} \tag{3.2}$$

$$(\Sigma_i x)(t) = \sum_{t_k \in [t, t+\omega]} H_i(t, t_k) I_{ik}(t_k, x_i(t_k)), \ i = 1, 2, \cdots, n. \tag{3.3}$$

where $H_i(t,s), i = 1, 2, \ldots, n$, are defined by (4), it is easy to get, for $i = 1, 2, \cdots, n$,

$$\frac{1}{1+\chi_i} \leqslant |H_i(t,s)| \leqslant \frac{\chi_i}{\chi_i + 1}, s \in [t, t+\omega].$$

where $\chi_i = e^{\int_0^{\omega} a_i(\phi) d\phi}$.

Theorem 3.1. *Suppose that* $(A1)-(A4)$ *is valid, if the next assumption is satisfied (A5):*

$$\omega \left[\sum_{i=1}^{n} (\Upsilon_i)^2 \right]^{\frac{1}{2}} + \omega \left[\sum_{i=1}^{n} (\Upsilon_i')^2 \right]^{\frac{1}{2}} + \sum_{k=1}^{q} \left[\sum_{i=1}^{n} \left(\frac{\chi_i c_{ik}}{\chi_i + 1} \right)^2 \right]^{\frac{1}{2}} < 1, \tag{3.4}$$

where

$$\Upsilon_i = \frac{\chi_i}{\chi_i + 1} \left[\sum_{j=1}^{n} (\overline{d}_{ij} m_j)^2 \right]^{\frac{1}{2}},$$

$$\Upsilon_i' = \frac{\chi_i}{\chi_i+1}\left[\sum_{j=1}^{n}((\overline{a}_{ij}+\overline{b}_{ij})n_j)^2\right]^{\frac{1}{2}},$$

then (1) has a unique ω nonperiodic solution.

Theorem 3.2. *Assume that (A1)-(A4) hold, if the following assumption is satisfied*
(A6)

$$\sum_{k=1}^{q}\left[\sum_{i=1}^{n}\left(\frac{\chi_i c_{ik}}{\chi_i+1}\right)^2\right]^{\frac{1}{2}} < 1, \tag{3.5}$$

it is valid that (1.1) possesses more than ω nonperiodic resolutions.

Proof. We define the operator Π, Σ as (8). Choosing

$$\rho \;\geqslant\; \frac{(n\omega\overline{d}M_f + n\omega(\overline{a}+\overline{b})M_g + \omega\overline{E} + q\overline{I})\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\right)^2\right]^{\frac{1}{2}}}{1-\sum_{k=1}^{q}\left[\sum_{i=1}^{n}\left(\frac{\chi_i c_{ik}}{\chi_i+1}\right)^2\right]^{\frac{1}{2}}} > 0 \tag{3.6}$$

For $x, y \in B_\rho = \{x \in X : \|x\| \leqslant \rho\}$, we get

$$\begin{aligned}
&\|(\Pi x)(t) + (\Sigma y)(t)\| \\
= &\sup_{t\in[0,\omega]}\left\{\sum_{i=1}^{n}\left|\int_t^{t+\omega} H_i(t,s)\left[\sum_{j=1}^{n} d_{ij}(s)f_j(x_j(s)) \;+\bigwedge_{j=1}^{n} a_{ij}(s)g_j(x_j(s-t_{ij}(s)))\right.\right.\right.\\
&\left.+\bigvee_{j=1}^{n} b_{ij}(s)g_j(x_j(s-t_{ij}(s))) + E_i(s)\right] ds\\
&\left.\left.+\sum_{t_k\in[t,t+\omega]} H_i(t,t_k)I_{ik}(t_k,y_i(t_k))\right|^2\right\}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
= & \sup_{t\in[0,\omega]}\left\{\sum_{i=1}^{n}\left|\int_{t}^{t+\omega}H_i(t,s)\right.\right.\\
& \times\left[\sum_{j=1}^{n}d_{ij}(s)f_j(x_j(s))-\sum_{j=1}^{n}d_{ij}(s)f_j(0)\right.\\
& +\bigwedge_{j=1}^{n}a_{ij}(s)g_j(x_j(s-t_{ij}(s)))-\bigwedge_{j=1}^{n}a_{ij}(s)g_j(0)\\
& \left.+\bigvee_{j=1}^{n}b_{ij}(s)g_j(x_j(s-t_{ij}(s)))\;-\bigvee_{j=1}^{n}b_{ij}(s)g_j(0)+E_i(s)\right]ds\\
& \left.\left.+\sum_{t_k\in[t,t+\omega]}H_i(t,t_k)I_{ik}(t_k,y_i(t_k))\right|^2\right\}^{\frac{1}{2}}\\
\leqslant & \left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\int_0^{\omega}\sum_{j=1}^{n}\overline{d}_{ij}|f_j(x_j(s))|ds|\right)^2\right]^{\frac{1}{2}}+\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\int_0^{\omega}\sum_{j=1}^{n}\overline{(a}_{ij}+\overline{b}_{ij})\right.\right.\\
& \left.\left.\times\ |g_j(x_j(s-t_{ij}(s)))|ds|\right)^2\right]^{\frac{1}{2}}+\omega\overline{E}\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\right)^2\right]^{\frac{1}{2}}\\
& +\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\sum_{k=1}^{q}|I_{ik}(t_k,y_i(t_k))-I_{ik}(t_k,0)|\right)^2\right]^{\frac{1}{2}}\\
& +\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\sum_{k=1}^{q}|I_{ik}(t_k,0)|\right)^2\right]^{\frac{1}{2}}\\
\leqslant & \left\{\sum_{k=1}^{q}\left[\sum_{i=1}^{n}\left(\frac{\chi_i c_{ik}}{\chi_i+1}\right)^2\right]^{\frac{1}{2}}\right\}\rho\\
& +(n\omega\overline{d}M_f+n\omega(\overline{a}+\overline{b})M_g+\omega\overline{E}+q\overline{I})\\
& \times\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\right)^2\right]^{\frac{1}{2}}\\
\leqslant & \ \rho
\end{aligned}
$$

Therefore, $\Pi x+\Sigma y\in B_\rho$. Since $f_j(\cdot),g_j(\cdot),j=1,2,\cdots,n$, are continuous. Thus the operator Π is continuous. For $x\in B_\rho$, we have

$$
\|\Pi x\|\leqslant(n\omega\overline{d}M_f+n\omega(\overline{a}+\overline{b})M_g+\omega\overline{E})\left[\sum_{i=1}^{n}\left(\frac{\chi_i}{\chi_i+1}\right)^2\right]^{\frac{1}{2}} \tag{3.7}
$$

i.e. Π is uniformly surrounded on B_ρ. Then, let us show the accuracy of Π. For $t_1, t_2 \in [0,\omega]$, it is valid that

$$\begin{aligned}
&\|(\Pi x)(t_1) - (\Pi x)(t_2)\| \\
\leqslant\ & \left[\sum_{i=1}^{n} \left| \int_0^\omega |H_i(t_1,s) - H_i(t_2,s)| \left[\sum_{j=1}^{n} d_{ij}(s) f_j(x_j(s)) \right.\right.\right. \\
&+ \bigwedge_{j=1}^{n} a_{ij}(s) g_j(x_j(s - t_{ij}(s))) \\
&\left.\left.\left. + \bigvee_{j=1}^{n} a_{ij}(s) g_j(x_j(s - t_{ij}(s))) + E_i(s)\right] ds \right|^2 \right]^{\frac{1}{2}} \\
\leqslant\ & \sum_{i=1}^{n} \frac{1}{\chi_i + 1} \int_0^\omega \left| e^{\int_{t_1}^{s} a_i(\phi)d\phi} - e^{\int_{t_2}^{s} a_i(\phi)d\phi} \right| \\
&\times \left[\sum_{j=1}^{n} \overline{d}_{ij} M_f + \bigwedge_{j=1}^{n} \overline{a}_{ij} M_g + \bigvee_{j=1}^{n} \overline{b}_{ij} M_g + \overline{E}\right] ds \\
\leqslant\ & |t_1 - t_2| \left[\sum_{j=1}^{n} \overline{d}_{ij} M_f + \bigwedge_{j=1}^{n} \overline{a}_{ij} M_g \right. \\
&\left. + \bigvee_{j=1}^{n} \overline{b}_{ij} M_g + \overline{E}\right] \omega a^+ \sum_{i=1}^{n} \frac{\chi_i}{\chi_i + 1} \\
\leqslant\ & |t_1 - t_2| [n\overline{d} M_f + n(\overline{a} + \overline{b}) M_g + \overline{E}] \omega a^+ \sum_{i=1}^{n} \frac{\chi_i}{\chi_i + 1}
\end{aligned}$$

Consequently, by methods for Arzela-Ascoli hypothesis, Π is reduced on B_ρ. By presumption (A6), plainly Σ is constriction mapping. Utilizing Lemma 2.2, framework (1) has in any event ω against occasional arrangement.

Assume that $x^*(t) = (x_1^*(t),, \cdots, x_n^*(t))^T$ is an ω-occasional arrangement of framework (1). In this area, we will develop some appropriate Lyapunov practical to demonstrate the worldwide exponential security of this enemy of occasional arrangement.

Theorem 3.3. *Suppose that assumptions* $(A1) - (A5)$ *hold. If the following assumptions are satisfied*
(**A7**) *there exist* $c, \overline{c}_{ik} \geq 0, i = 1, 2, \cdots, n, k = 1, 2, \cdots,$ *such that*

$$|u + I_{ik}(t,u) - \Omega - I_{ik}(t,\Omega)| \leqslant \overline{c}_{ik} |u - \Omega|, t \in [0,\omega], u, \Omega \in R, \tag{3.8}$$

and for $k = 1, 2, \cdots,$

$$2 \ln c_k \leqslant c(t_k - t_{k-1}). \tag{3.9}$$

(**A8**) *there exist* $c_i > 0$ *and* $\delta_{ij}, \eta_{ij}, \vartheta_{ij}, \xi_{ij} \in R, i = 1, 2, \cdots, n$ *such that*

$$-\Theta_1 + c\Theta_2 + c = 0 \tag{3.10}$$

where

$$\begin{aligned}\Theta_1 &= \min_{1\leqslant i\leqslant n}\Bigg\{2a_i^- - \sum_{j=1}^{n}(\overline{d}_{ij})^{2\delta_{ij}}m_j^{2\eta_{ij}} \\ &\quad - \sum_{j=1}^{n}\frac{c_j}{c_i}(\overline{d}_{ji})^{2(1-\delta_{ij})}m_j^{2(1-\eta_{ij})} \\ &\quad - \sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij})^{2\vartheta_{ij}}n_j^{2\xi_{ij}}\Bigg\}.\end{aligned}$$

$$\Theta_2 = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\frac{c_j}{c_i}(\overline{a}_{ji}+\overline{b}_{ji})^{2(1-\vartheta_{ij})}m_j^{2(1-\xi_{ij})}\right\},$$

$$c_k = \max_{1\leqslant i\leqslant n}\{\overline{c}_{ik}\},$$

$$c = \max_{1\leqslant k<+\infty}\left\{e^{c(t_k-t_{k-1})}, \frac{1}{e^{c(t_k-t_{k-1})}}\right\}$$

then ω *anti periodic solution of system (1) is globally exponentially stable with convergence rate* $\lambda/2$, *and* λ *is an unique positive solution of* $\lambda - \Theta_1 + c\Theta_2 e^{\lambda t} + c = 0$.

Proof. Suppose $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ is an ω nonperiodic arrangement of (1). $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is an arrangement of (1). Set $y(t) = x(t) - x^*(t)$. Then, for $k = 1, 2, \cdots, i = 1, 2, \cdots, n$,

$$\begin{cases} y_i'(t) &= -a_i(t)(x_i(t) - x_i^*(t)) \\ &\quad + \sum_{j=1}^n d_{ij}(t)[f_j(x_j(t)) - f_j(x_j^*(t))] \\ &\quad + \bigwedge_{j=1}^n a_{ij}(t)g_j(x_j(t-t_{ij}(t))) \\ &\quad - \bigwedge_{j=1}^n a_{ij}(t)g_j(x_j^*(t-t_{ij}(t))) \\ &\quad + \bigvee_{j=1}^n b_{ij}(t)g_j(x_j(t-t_{ij}(t))) \\ &\quad - \bigvee_{j=1}^n b_{ij}(t)g_j(x_j^*(t-t_{ij}(t))), t \geq 0, t \neq t_k \\ y_i(t_k^+) &= x_i(t_k) - x_i^*(t_k) + I_{ik}(t_k, x_i(t_k)) \\ &\quad - I_{ik}(t_k, x_i^*(t_k)). \end{cases} \tag{3.11}$$

Considering the following function

$$\Omega(t) = \sum_{i=1}^{n} c_i |y_i(t)|^2. \tag{3.12}$$

Computing the above right derivative of $\Omega(t)$, for $t \neq t_k$,

$$\begin{aligned}
& D^+\Omega(t) \\
= \; & \sum_{i=1}^{n} 2c_i D^+ |y_i(t)| \\
\leqslant \; & \sum_{i=1}^{n} -2c_i a_i(t)|y_i(t)||y_i(t)| \\
& + \sum_{i=1}^{n} 2c_i \sum_{j=1}^{n} |d_{ij}(t)||y_i(t)||f_j(x_j(t)) - f_j(x_j^*(t))| \\
& + \sum_{i=1}^{n} 2c_i \sum_{j=1}^{n} (|a_{ij}(t)| + |b_{ij}(t)|)|y_i(t)| \\
& \times |g_j(x_j(t - t_{ij}(t))) - g_j(x_j^*(t - t_{ij}(t)))| \\
\leqslant \; & \sum_{i=1}^{n} -2c_i a_i^- |y_i(t)|^2 + \sum_{i=1}^{n} 2c_i \sum_{j=1}^{n} \overline{d}_{ij} |y_i(t)| m_j |y_j(t)| \\
& + \sum_{i=1}^{n} 2c_i \sum_{j=1}^{n} (\overline{a}_{ij} + \overline{b}_{ij})|y_i(t)| n_j |y_j(t - t_{ij}(t))|
\end{aligned} \tag{3.13}$$

Using inequality $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$, we have

$$\begin{aligned}
& \sum_{j=1}^{n} \overline{d}_{ij} |y_i(t)| m_j |y_j(t)| \\
= \; & \sum_{j=1}^{n} [(\overline{d}_{ij})^{\delta_{ij}} m_j^{\eta_{ij}} |y_i(t)|][(\overline{d}_{ij})^{1-\delta_{ij}} m_j^{1-\eta_{ij}} |y_j(t)|] \\
\leqslant \; & \sum_{j=1}^{n} \left[\frac{1}{2} (\overline{d}_{ij})^{2\delta_{ij}} m_j^{2\eta_{ij}} |y_i(t)|^2 \right. \\
& \left. + \frac{1}{2} (\overline{d}_{ij})^{2(1-\delta_{ij})} m_j^{2(1-\eta_{ij})} |y_j(t)|^2 \right]
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij})|y_i(t)|n_j|y_j(t-t_{ij}(t))| \\ \leqslant\ &\sum_{j=1}^{n}\left[\frac{1}{2}(\overline{a}_{ij}+\overline{b}_{ij})^{2\vartheta_{ij}}n_j^{2\xi_{ij}}|y_i(t)|^2\right. \\ &+\frac{1}{2}(\overline{a}_{ij}+\overline{b}_{ij})^{2(1-\vartheta_{ij})}m_j^{2(1-\xi_{ij})} \\ &\left.\times|y_j(t-t_{ij}(t))|^2\right] \end{aligned} \tag{3.15}$$

Substituting (21) and (22) into (20), we have, for $t \neq t_k$,

$$\begin{aligned} &D^+\Omega(t) \\ \leqslant\ &\sum_{i=1}^{n}c_i\left\{-2a_i^-|y_i(t)|+\sum_{j=1}^{n}\left[(\overline{d}_{ij})^{2\delta_{ij}}m_j^{2\eta_{ij}}|y_i(t)|^2\right.\right. \\ &\left.+(\overline{d}_{ij})^{2(1-\delta_{ij})}m_j^{2(1-\eta_{ij})}|y_j(t)|^2\right] \\ &+\sum_{j=1}^{n}\left[(\overline{a}_{ij}+\overline{b}_{ij})^{2\vartheta_{ij}}n_j^{2\xi_{ij}}|y_i(t)|^2\right. \\ &\left.\left.+(\overline{a}_{ij}+\overline{b}_{ij})^{2(1-\vartheta_{ij})}m_j^{2(1-\xi_{ij})}|y_j(t-t_{ij}(t))|^2\right]\right\} \\ =\ &\sum_{i=1}^{n}c_i\left\{\left[-2a_i^-+\sum_{j=1}^{n}(\overline{d}_{ij})^{2\delta_{ij}}m_j^{2\eta_{ij}}\right.\right. \\ &+\sum_{j=1}^{n}\frac{c_j}{c_i}(\overline{d}_{ji})^{2(1-\delta_{ij})}m_j^{2(1-\eta_{ij})} \\ &\left.+\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij})^{2\vartheta_{ij}}n_j^{2\xi_{ij}}\right]|y_i(t)|^2 \\ &+\sum_{j=1}^{n}\frac{c_j}{c_i}(\overline{a}_{ji}+\overline{b}_{ji})^{2(1-\vartheta_{ij})}m_j^{2(1-\xi_{ij})} \\ &\left.\times|y_j(t-t_{ij}(t))|^2\right\} \\ \leqslant\ &-\Theta_1\Omega(t)+\Theta_2\overline{\Omega}(t) \end{aligned} \tag{3.16}$$

where $\Omega(t)=\sup_{t-t\leqslant\eta\leqslant t}\Omega(\eta)$. From (A6), we have

$$\Omega(t_k^+)=\sum_{i=1}^{n}c_i|y_i(t_k^+)|^2\leqslant\sum_{i=1}^{n}c_i\overline{c}_{ik}^2|y_i(t_k)|^2<c_k^2\Omega(t_k). \tag{3.17}$$

From Lemma 2.3, there is $c > 1$ satisfying

$$\Omega(t) \leqslant c(\sup_{-t\leqslant t\leqslant 0} \Omega(t))e^{-\lambda t} \tag{3.18}$$

Thus

$$\|x(t) - x^*(t)\| \leqslant \left(\frac{c\max_{1\leqslant i\leqslant n}(c_i)}{\min_{1\leqslant i\leqslant n}(c_i)}\right)^{\frac{1}{2}} \|\varphi - \varphi^*\|e^{-\lambda t/2}. \tag{3.19}$$

The validity of the theorem is completed.

The global exponential stability of FCNNs is important dynamical behavior. Time delays and impulsive effects often cause system instability or oscillatory behaviour. It is clear that the results obtained are related with the time delay and impulses for justifying global exponentially stability of ω anti periodic solution of system (1).

4. A numerical example

In this segment, a precedent is given to demonstrate adequacy of results acquired.

Example 5.1 Consider the accompanying FCNNs with time-changing deferral and hasty impacts.

$$\left\{\begin{array}{lcl} x_i'(t) & = & -a_i(t)x_i(t) + \sum_{j=1}^{2} d_{ij}(t)f_j(x_j(t)) \\ & & +\bigwedge_{j=1}^{2} a_{ij}(t)g_j(x_j(t-t_{ij}(t))) \\ & & +\bigvee_{j=1}^{2} b_{ij}(t)g_j(x_j(t-t_{ij}(t))) \\ & & +E_i(t), t \neq \frac{k\pi}{2}, k = 1, 2, \cdots, \\ \Delta x_i(t_k)) & = & -\frac{2}{3}x_i(t_k), t = t_k = \frac{k\pi}{2}, i = 1, 2, \end{array}\right. \tag{4.1}$$

where $a_1(t) = a_2(t) = \frac{1}{8}$, $f_j(x) = g_j(x) = \arctan x (j = 1, 2)$.

$$(d_{ij}(t))_{2\times 2} = \begin{pmatrix} 1/4 & 1/8 \\ 1/6 & 1/3 \end{pmatrix},$$

$$(a_{ij}(t))_{2\times 2} = \begin{pmatrix} 1/8 & 1/6 \\ 1/6 & 1/8 \end{pmatrix},$$

$$(b_{ij}(t))_{2\times 2} = \begin{pmatrix} 1/16 & 1/4 \\ 1/4 & 1/16 \end{pmatrix},$$

$$(E_i(t))_{2\times 1} = \begin{pmatrix} 1/4\sin t \\ 1/3\cos t \end{pmatrix}.$$

impulsive functions $I_{1k}(t,x) = I_{2k}(t,x) = -\frac{2}{3}x$, impulsive points $t_k = \frac{k\pi}{2}, t_{11}(t) = t_{21}(t) = |\sin(2\pi t)|, t_{12}(t) = t_{22}(t) = |\cos(2\pi t)|$, then, we can easily check that $u = \Omega = \frac{\pi}{2}, c_{1k} = c_{2k} = \frac{2}{3}, \overline{c}_{1k} = \overline{c}_{2k} = \frac{1}{3}, c_k = \frac{1}{3}, c = 1, c_1 = c_2 = e^{\frac{\pi}{8}}$, Taking $\delta_{ij} = \eta_{ij} = \vartheta_{ij} = \xi_{ij} = \frac{1}{2}(i = 1, 2), \frac{2\ln c_k}{t_k - t_{k-1}} \leqslant -1.39 = c.$

It is easy to conclude that assumptions (A6) and (A8) hold true. Numerical arrangement $x(t) = (x_1(t), x_2(t))^T$ of frameworks (27) for introductory esteem $\varphi(s) = (0.5, -0.4)^T$,$s \in [-2, 0]$.

5. Conclusion

In this paper, the presence and internationally exponential solidness of the counter intermittent answer for fuzzy cell neural systems with time-differing delays are considered. Some adequate conditions set up here are effortlessly confirmed what's more, these conditions are related with parameters of the framework (1). The acquired criteria can be connected to plan all around exponential stable of hostile to occasional ceaseless fuzzy cell neural systems.

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GENERALIZED FUGLEDE-PUTNAM THEOREM AND m-QUASI-CLASS $A(k)$ OPERATORS

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Abstract. For a bounded linear operator T acting on an complex infinite dimensional Hilbert space $\mathcal{H}$, we say that T is an m-quasi-class $A(k)$ operator for $k > 0$ and m is a positive integer (abbreviation $T \in \mathbb{Q}(A(k), m)$) if $T^{*m}\left((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T|^2\right)T^m \geq 0$. The famous *Fuglede-Putnam theorem* asserts that: the operator equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal operators. In this paper, we prove that if $T \in \mathbb{Q}(A(k), m)$ and S^* is an operator of class $A(k)$ for $k > 0$. Then $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$ is an injective with a dense range which implies $XT^* = S^*X$.
Keywords. Bounded linear operator; Hilbert space; Fuglede-Putnam theorem; Normal operator.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on $\mathcal{H}$. Throughout this paper, the range and the null space of an operator T will be denoted by $ran(T)$ and $\ker(T)$, respectively. Let $\overline{\mathcal{M}}$ and $\mathcal{M}^\perp$ be the norm closure and the orthogonal complement of the subspace $\mathcal{M}$ of $\mathcal{H}$. The classical Fuglede-Putnam theorem [12, Problem 152] asserts that if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ are normal operators such that $TX = XS$ for some operator $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS^*$. The references [16, 17, 18, 19, 20, 21] are among the various extensions of this celebrated theorem for non-normal operators.

Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([7], [12], [14] and [31]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *positive*, $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

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An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators ([1, 4, 5, 8, 9] and [13]). An operator T is said to be *p-hyponormal* if$(T^*T)^p \geq (TT^*)^p$ for $p \in (0,1]$ and an operator T is said to be log-*hyponormal* if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal* if it satisfies the following norm inequality

$$\|T^2\|\|x\| \geq \|Tx\|^2$$

for all $x \in \mathcal{H}$. Ando [3] proved that every log-hyponormal operator is paranormal. It was originally introduced as an intermediate class between hyponormal and normaloid operators.

In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators, Furuta el al. [9] introduced a class A defined by $|T^2| \geq |T|^2$ and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example [9, 10]. Fujii et al. [10] introduced a new class $A(t,s)$ of operators: For $t > 0$ and $s > 0$, the operator T belongs to class $A(s,t)$ if it satisfies the operator inequality

$$\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{t+s}} \geq |T^*|^{2t}.$$

Furuta el al. [9] introduced class $A(k)$ for $k > 0$ as a class of operators including p-hyponormal and log-hyponormal operators, where $A(1)$ coincides with class A operator. We say that an operator T is class $A(k)$, $k > 0$ (Abbreviation, $T \in A(k)$) if $(T^*|T^{2k}|T)^{\frac{1}{k+1}} \geq |T|^2$. The inclusion relations among these classes are known as follows:

$$\begin{aligned}\{hyponormal\ operators\} &\subset \{p - hyponormal\ operators\ for\ 0 < p \leq 1\}\\ &\subset \{class\ A(s,t)\ operators\ for\ s,t \in [0,1]\}\\ &\subset \{class\ A\ operators\}\\ &\subset \{paranormal\ operators\}.\end{aligned}$$

and

$$\begin{aligned}\{hyponormal\ operators\} &\subset \{p - hyponormal\ operators\ for\ 0 < p \leq 1\}\\ &\subset \{class\ A\ operators\}\\ &\subset \{class\ A(k)\ for\ k \geq 1\}.\end{aligned}$$

2. Spectral properties of k-quasi class $A(m)$ operators

Throughout this article we would like to present some known results as propositions which will be used in the sequel. Firstly, we begin with the following definition.

Definition 2.1. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is of m-quasi class $A(k)$ (abbreviate $\mathbb{Q}(A_k, m)$), if

$$T^{*m}(T^*|T|^{2k}T)^{1/(k+1)}T^m \geq T^{m*}|T|^2T^m,$$

where m is a positive integers and $k > 0$. If $m = 1$, then T is called a quasi-class $A(k)$ and if $k = m = 1$, then $\mathbb{Q}(A_k, m)$ coincides with quasi-class A operator.

Lemma 2.1. *[6, Hansen's Inequality] If $A, B \in \mathcal{B}(\mathcal{H})$ satisfying $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\alpha \geq B^*A^\alpha B \qquad \forall \alpha \in (0, 1].$$

Proposition 2.1. *[23, Lemma 2.2]Let $T \in \mathbb{Q}(A_k, m)$ and T^m not have a dense range. Then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ran(T^m)} \oplus \ker(T^{*m}),$$

where $T_1 = T|_{\overline{ran(T^m)}}$ is the restriction of T to $\overline{ran(T^m)}$, and $T_1 \in A(k)$ and T_3 is nilpotent of nilpotency m. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proposition 2.2. *[23, Theorem 2.3] Let $T \in \mathcal{B}(\mathcal{H})$ be a $\mathbb{Q}(A_k, m)$ operator and $\mathcal{M}$ be its invariant subspace. Then the restriction $T|_\mathcal{M}$ of T to $\mathcal{M}$ is also $\mathbb{Q}(A_k, m)$ operator.*

Proposition 2.3. *[23, Theorem 2.4] Let $T \in \mathbb{Q}(A_k, m)$. Then the following assertions holds:*

(a) If $\mathcal{M}$ is an invariant subspace of T and $T|_\mathcal{M}$ is an injective normal operator, then $\mathcal{M}$ reduces T.

(b) If $(T - \lambda)x = 0$, then $(T - \lambda)^*x = 0$ for all $\lambda \neq 0$.

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in \mathcal{H}$ such that $(T - \lambda)x = 0$. If, in addition, $(T^* - \overline{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly, $\sigma_p(T) \subseteq \sigma_{jp}(T)$. In general, $\sigma_p(T) \neq \sigma_{jp}(T)$.

In [33], Xia showed that if T is a semi-hyponormal operator, then $\sigma_p(T) = \sigma_{jp}(T)$; Tanahashi extended this result to log-hyponormal operators in [27]. Aluthge [2] showed that if T is w-hyponormal, then nonzero points of $\sigma_p(T)$ and $\sigma_{jp}(T)$ are identical; Uchiyama extended this result to class A operators in [28]. In the following, we will point out that if T is a quasi-$*$-class (A, k) operator for a positive integer k, then nonzero points of $\sigma_{jp}(T)$ and $\sigma_p(T)$ are also identical and the eigenspaces corresponding to distinct eigenvalues of T are mutually orthogonal.

Corollary 2.1. *If $T \in \mathbb{Q}(A_k, m)$, then $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.*

Corollary 2.2. *If $T \in \mathbb{Q}(A_k, m)$ and $\alpha, \beta \in \sigma_p(T) \setminus \{0\}$ with $\alpha \neq \beta$. Then* $\ker(T-\alpha) \perp \ker(T-\beta)$.

Proof. Let $x \in \ker(T-\alpha)$ and $y \in \ker(T-\beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha\langle x, y\rangle = \langle \alpha x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\beta} y\rangle = \beta\langle x, y\rangle.$$

Hence $\alpha\langle x, y\rangle = \beta\langle x, y\rangle$ and so $(\alpha-\beta)\langle x, y\rangle = 0$. But $\alpha \neq \beta$, hence $\langle x, y\rangle = 0$. Consequently, $\ker(T-\alpha) \perp \ker(T-\beta)$. $\square$

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{Q}(A_k, m)$ with a dense range, then T is a class $A(k)$ operator for $k > 0$.*

Proof. Since T has a dense range, $\overline{ran(T^m)} = \mathcal{H}$. Then there exists a sequence $\{x_n\} \subset \mathcal{H}$ such that $\lim\limits_{n\to\infty} T^m x_n = y$. Since $T \in \mathbb{Q}(A_k, m)$, we have

$$\begin{aligned}
\langle T^{*m}(T^*|T|^{2k}T)^{\frac{1}{k+1}}T^m x_n, x_n\rangle &\geq \langle T^{*m}|T|^2T^m x_n, x_n\rangle \\
\langle T^{*m}(T^*|T|^{2k}T)^{\frac{1}{k+1}}T^m x_n, x_n\rangle &\geq \langle T^{*m}|T|^2T^m x_n, x_n\rangle \\
\langle (T^*|T|^{2k}T)^{\frac{1}{k+1}}T^m x_n, T^m x_n\rangle &\geq \langle |T|^2T^m x_n, T^m x_n\rangle \ \forall n \in \mathbb{N}
\end{aligned}$$

By the continuity of the inner product, we have

$$\langle ((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T|^2)y, y\rangle \geq 0,$$

for all $y \in \mathcal{H}$. Therefore T is a class $A(k)$ operator for $k > 0$. $\square$

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{Q}(A_k, m)$ and not class $A(k)$, then T is not invertible.*

3. Generalized Fuglede-Putnam Theorem

For $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, we say that the FP-theorem holds for the pair (T, S) if $TX = XS$ implies $T^*X = XS^*$, $\overline{ran(X)}$ reduces T, and $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{ran(X)}}$ and $S|_{\ker(X)^\perp}$ are unitary equivalent normal operators for all $X \in \mathcal{B}(\mathcal{H})$. The following result is very useful in the sequel.

Proposition 3.1. *[26] Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent.*

1. *If $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS^*$,*
2. *If $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$, then $\overline{ran(X)}$ reduces T, $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{ran(X)}}$ and $S|_{\ker(S)^\perp}$ are normal.*

The numerical range of an operator T, denoted by $W(T)$, is the set defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

In general, the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(T)}$ do not imply that T is normal. If $T = SB$, where S is positive and invertible, B is self-adjoint, and S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal. Therefore the following question arises naturally.
Question: Which operator T satisfying the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ is normal?

In 1966, Sheth [24] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for any operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint. We extend the result of Sheth to the class $A(k), k > 0$ operators as follows.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{K})$. If T or T^* belongs to class $A(k)$ for every $k > 0$ and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

To prove Theorem 3.1 we need the following Lemmas.

Lemma 3.1. *[30] If $T \in \mathcal{B}(\mathcal{H})$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let T belongs to the class $A(s,t)$ for some $s > 0$ and $t > 0$, we have*

(a) If $\widetilde{T}_{s,t}$ is normal, then T is normal [29].

(b) If $m_2(\sigma(T)) = 0$, where m_2 means the planer Lebsegue measure, then T is normal [22].

Proof. [Proof of Theorem 3.1] Suppose that T or T^* is a class $A(k), k > 0$ operator. Since $\sigma(T) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Apply Lemma 3.1 to T^* to get $\sigma(T^*) \subseteq \mathbb{R}$. Then $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subseteq \mathbb{R}$. Thus $m_2(\sigma(T)) = m_2(\sigma(T^*))) = 0$ for the planer Lebesgue measure m_2. It follows from Lemma 3.2 that T or T^* is normal. Since $\sigma(T) = \sigma(T^*) \subseteq \mathbb{R}$. Therefore, T is self-adjoint. □

We can extend the result of Theorem 3.1 to the class of $\mathbb{Q}(A_k, m)$ as follows:

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathbb{Q}(A_k, m)$ and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is a direct sum of self-adjoint and nilpotent operator.*

Proof. Since T is m-quasi-class $A(k)$. then by Proposition 2.1, T has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ran(T^m)} \oplus \ker(T^{*m}),$$

where $T_1 = T|_{\overline{ran(T^m)}}$ is the restriction of T to $\overline{ran(T^m)}$, and T_1 is a class $A(k)$ and T_3 is nilpotent of nilpotency m. Since $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, we have $\sigma(T) \subseteq \mathbb{R}$ by Lemma 3.1. Therefore $\sigma(T_1) \subseteq \mathbb{R}$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$ and hence T_1 is self-adjoint by Theorem 3.1 because T_1 belongs to class $A(k)$. Now let Q be the orthogonal projection of $\mathcal{H}$ onto $\overline{ran(T^m)}$. Since $T \in \mathbb{Q}(A_k, m)$ we have

$$\begin{aligned}
\begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix} &= Q|T|^2Q \le Q(T^*|T|^{2k}T)^{1/(k+1)}Q \\
&\le \left(QT^*|T|^{2k}T)Q\right)^{1/(k+1)} \\
&\le \left(QT^*(QT^*TQ)^kT)Q\right)^{1/(k+1)} = \begin{pmatrix} (T_1^*|T_1|^{2k}T_1)^{\frac{1}{k+1}} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

by Lemma 2.1. Therefore,

$$Q(T^*|T|^{2k}T)^{1/(k+1)}Q = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = Q|T|^2Q.$$

Since S is normal, we can write $(T^*|T|^{2k}T)^{1/(k+1)} = \begin{pmatrix} |T_1|^2 & C \\ C^* & D \end{pmatrix}$. Since

$$\begin{pmatrix} |T_1|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix} = Q(T^*|T|^{2k}T)Q = Q((T^*|T|^{2k}T)^{k+1})^{1/(k+1)}Q,$$

we can easily show that $C = 0$. Therefore,

$$(T^*|T|^{2k}T)^{1/(k+1)} = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & D \end{pmatrix}$$

and hence

$$T^*|T|^{2k}T = \begin{pmatrix} |T_1|^{2(k+1)} & 0 \\ 0 & D^{k+1} \end{pmatrix} = T^*(T^*T)^kT.$$

This implies that $D = (T_3^*|T_3|^{2k}T_3)^{1/(k+1)}$, and by the matrix representation of T we also have

$$T^*T = \begin{pmatrix} T_1T_1^* & T_1^*T_2 \\ T_2^*T_1 + T_3^*T_3 & T_2^*T_2 \end{pmatrix}.$$

Therefore $T_2^*T_2 = 0$ and hence $T_2 = 0$, which completes the proof. $\square$

The following corollary is an extension of the result of Theorem 3.1 to the class of quasi-class $A(k)$ operators.

Corollary 3.1. *If T is a quasi-class $A(k)$ operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

Proof. If T is a quasi-class $A(k)$ operator, T has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \overline{ran(T)} \oplus \ker(T^*),$$

where T_1 is a class $A(k)$ on $\overline{ran(T)}$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since T_1 is self-adjoint and $T_2 = 0$ by Theorem 3.2, T is also self-adjoint. $\square$

In 1976, Stampfli and Wadhwa [25] showed that if $T^* \in \mathcal{B}(\mathcal{H})$ is hyponormal, $S \in \mathcal{B}(\mathcal{H})$ is dominant, $X \in \mathcal{B}(\mathcal{H})$ is injective and has a dense range, and if $XT = SX$, then T and S are normal. on the other hand, in 1981, Gupta and Ramanujan [11] showed that if $T \in \mathcal{B}(\mathcal{H})$ is k-quasihyponormal operator and $S \in \mathcal{B}(\mathcal{H})$ is normal operator for which $TY = YS$ where $Y \in \mathcal{B}(\mathcal{H})$ is injective with dense range, then T is normal operator unitarily equivalent to S. In the following theorem , we extend the result of Gupta and Ramanujan to the *class* $\mathbb{Q}(A_k, m)$ operators. We need the following Lemmas.

Lemma 3.3. [15] Let T, S be normal operators. If there exist injective operators X and Y such that $XT = SX$ and $YS = TY$, then T and S are unitarily equivalent.

Lemma 3.4. *Let $T = U|T|$ be the polar decomposition of T which belong to class $A(p,p)$ for $p > 0$. Then $\widetilde{T}_{p,p} = |T|^p U |T|^p$ is semi-hyponormal and $\widetilde{\widetilde{T}}_{p,p}$ is hyponormal.*

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be class $A(k)$ and $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. If $X \in \mathcal{B}(\mathcal{H})$ has dense range and satisfies $TX = XN$, then T is also a normal operator.*

Proof. Since $TX = XN$ and X has dense range, we have $\overline{X\overline{ran(N)}} = \overline{ran(T)}$. If we denote the restriction of X to $\overline{ran(N)}$ by X_1, then $X_1 : \overline{ran(N)} \to \overline{ran(T)}$ has dense range and for every $x \in \overline{ran(N)}$

$$X_1 N x = XNx = TXx = TX_1 x$$

so that $X_1 N = TX_1$. Since T is of class $A(k)$ then T belongs to class $A(p,p)$, where $p = \max\{1, k\}$. Hence it follows from Lemma 3.4 that $\widetilde{T}_{p,p}$ is semi-hyponormal and hence there is a quasiaffinity Y such that $\widetilde{T}_{p,p} Y = YT$. Thus we have

$$\widetilde{T}_{p,p} Y X_1 = YTX_1 = YX_1 N$$

since YX_1 has dense range, $\widetilde{T}_{p,p}$ is normal, and so T is normal by Lemma 3.2. $\square$

Theorem 3.4. *Let $T^* \in \mathcal{B}(\mathcal{H})$ be of class $A(k)$ for $k > 0$ and let $S \in \mathcal{B}(\mathcal{H})$ be of class $A(k)$ for $k > 0$. If $XT = SX$, where $X : \mathcal{H} \to \mathcal{H}$ is an injective bounded linear operator with dense range, then T is a normal operator unitarily equivalent to S.*

Proof. Since T^* and S are class $A(k)$, then T^* and S are class $A(p,p)$, where $p = \max\{1, k\}$. Now, decompose S and T^* into their normal and pure parts by $S = W \oplus J$ and $T^* = L^* \oplus Q^*$. Let $X_1 = \widetilde{\widetilde{X}} = |\tilde{J}_{p,p}|^{\frac{1}{2}} |\tilde{J}_{p,p}|^{\frac{1}{2}} X |\tilde{Q^*}_{p,p}|^{\frac{1}{2}} |\widetilde{Q^*}_{p,p}|^{\frac{1}{2}}$.

Since $XQ = JX$, $X_1\tilde{\tilde{Q}}_{p,p} = \tilde{\tilde{J}}_{p,p}X_1$, where $\tilde{\tilde{Q}}_{p,p}$, $\tilde{\tilde{J}}_{p,p}$ are hyponormal operators by Lemma 3.4 and X_1 is quasi-affinity. Now by Fuglede-Putnam Theorem for hyponormal operators, $X_1\tilde{\tilde{Q}}_{p,p} = \tilde{\tilde{J}}^*{}_{p,p}X_1$ and $\overline{ran(X_1)}$ reduces $\tilde{\tilde{J}}_{p,p}$ and $(\ker X_1)^{\perp}$ reduces $\tilde{\tilde{Q}}_{p,p}$ and $\tilde{\tilde{J}}_{p,p}|_{\overline{ran(X_1)}}$ and $\tilde{\tilde{Q}}_{p,p}|_{(\ker X_1)^{\perp}}$ are unitarily equivalent normal operators. Since X_1 is quasiaffinity, then $\overline{ran(X_1)} = \mathcal{H}$ and $(\ker X_1)^{\perp} = \{0\}$ and $\tilde{\tilde{Q}}_{p,p}$ and $\tilde{\tilde{J}}_{p,p}$ are unitarily equivalent normal operators. In particular, $\tilde{\tilde{Q}}_{p,p}$ and $\tilde{\tilde{J}}_{p,p}$ are normal operators and by Lemmas 3.3, 3.3, the result follows. □

Theorem 3.5. *If $T^* \in \mathcal{B}(\mathcal{H})$ is of class $A(k)$ for $k > 0$, $S \in \mathcal{B}(\mathcal{H})$ is of class $A(k)$ for $k > 0$ and $XT = SX$ for $X \in \mathcal{B}(\mathcal{H})$ is quasiaffinity, then $XT^* = S^*X$*

Proof. Since by assumption $XT = SX$, we can see that $(\ker(X))^{\perp}$and $\overline{ran(X)}$ are invariant subspaces of T^* and S , respectively. Then $T^*|_{(\ker X)^{\perp}}$ is of class $A(k)$ and $S|_{\overline{ran(X)}}$ is also of class $A(k)$. Now consider the decomposition $\mathcal{H} = (\ker X)^{\perp} \oplus \ker X$ and $\mathcal{H} = \overline{ran(X)} \oplus (\overline{ran(X)})^{\perp}$. Then we have the following matrix representation:

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where T_1^* is of class $A(k)$, S_1 is of class $A(k)$ and X_1 is injective with dense range. Therefore, we have $X_1T_1x = XTx = SXx = S_1X_1x$ for $x \in (\ker X)^{\perp}$. That is, $X_1T_1 = S_1X_1$ and T_1 and S_1 are normal by Theorem 3.4. By Fuglede-Putnam theorem we have $X_1T_1^* = S_1^*X_1$. Therefore, $(\ker X)^{\perp}$ and $(\overline{ran(X)})$ reduces T^* and S, respectively. Hence, we obtain the $XT^* = S^*X$. □

Theorem 3.6. *Let $T \in \mathbb{Q}(A_k, m)$ and let S^* be an operator of class $A(k)$ for $k > 0$. If $TX = XS$, where $X \in \mathcal{B}(\mathcal{H})$ is an injective with dense range. Then $XT^* = S^*X$.*

Proof. Let $T_1 = T|_{\overline{ran(T^m)}}$ and $S_1 = S|_{\overline{ran(S^m)}}$. Then we have the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

where T_1 is class $A(k)$, $T_3^m = 0$ and $S_1^* = 0$. Notice that $T^mX = XS^m$ for all positive integer m. Thus $X(\overline{ran(S^m)}) = \overline{ran(T^m)}$. If we denote the restriction of X to $\overline{ran(S^m)}$ by N then $N : \overline{ran(S^m)} \to \overline{ran(S^m)}$ is an injective and has a dense range. Since $NS_1x = XSx = TXx = T_1Nx$ for all $x \in \overline{ran(S^m)}$, it follows that $NS_1 = T_1N$. On the other hand, since T_1 and S_1^* are belong to class $A(k)$, it follows from Theorem 3.5 that T_1 is a normal operator unitarily equivalent to S_1. Now let E be the orthogonal projection of $\mathcal{H}$ onto $\overline{ran(T^m)}$. Since $T \in \mathbb{Q}(A_k, m)$ and T_1 is a normal operator, from the argument of the proof of Theorem 3.2 we have

$T_2 = 0$ and hence $\overline{ran(T^m)}$ reduces T. Since $X^*(\ker(T^{m^*})) \subseteq \ker(S^{m^*}) = \ker(S^*)$, we have that for each $x \in \ker(T^{m^*})$,

$$X^*T_3^*x = X^*T^*x = S^*X^*x = 0. \tag{3.2}$$

But since X has a dense range, X^* is an injective and hence $T_3^*x = 0$ for every $x \in \ker(T^{k^*})$. Thus $T_3 = 0$, so that $T = T_1 \oplus 0$. Therefore, the proof is achieved. □

Theorem 3.7. *If $T^* \in \mathcal{B}(\mathcal{H})$ is of class $A(k)$ for $k > 0$, $S \in \mathcal{B}(\mathcal{H})$ is injective m-quasi-class $A(k)$, and if $XT = SX$ for $X \in \mathcal{B}(\mathcal{H})$, then $XT^* = S^*X$.*

Proof. Since by assumption $XT = SX$, we can see that $(\ker X)^{\perp}$ and $\overline{ranX}$ are invariant subspace of T^* and S, respectively. Therefore, by Lemma 2.2 we have that $T^*|_{(\ker X)^{\perp}}$ is class $A(k)$ and $S|_{\overline{ran(X)}} \in \mathbb{Q}(A_k, m)$. Now consider the decomposition $\mathcal{H} = (\ker X)^{\perp} \oplus \ker X$. Then we have the matrix representations:

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & T_3 \end{bmatrix}, S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix}, X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

where T_1^* is of class $A(k)$ and S_1 is injective m-quasi-class $A(k)$ and X_1 is an injective with dense range. Therefore, we have

$$X_1T_1x = XTx = SXx = S_1X_1x \quad for \quad x \in (\ker X)^{\perp}. \tag{3.4}$$

that is, $X_1T_1 = S_1X_1$ and hence, T_1 and S_1 are normal by Theorem 3.6 and $X_1T_1^* = S_1^*X_1$ by the Fuglede-Putnam Theorem. Therefore, it follows from Lemma 2.3 that $(\ker X)^{\perp}$ and $\overline{ran(X)}$ reduces T^* and S , respectively. Hence, we obtain the $XT^* = S^*X$. □

Let $T \in \mathcal{B}(\mathcal{H})$ be compact, and let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if

$$\|T\|_p = \left(\sum_{j=1}^{\infty}(s_j(T))^p\right)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty, 1 \leq p < \infty,$$

where $tr(.)$ denote the trace functional. Hence $C_1(\mathcal{H})$ is the trace class, $C_2(\mathcal{H})$ is the Hilbert-Schmidt class, and C_∞ is the class of compact operator with $\|T\|_\infty = s_1(T)$ denoting the usual norm.

For each pairs of operators A and B in $\mathcal{B}(\mathcal{H})$, an operator τ in $(B_2(\mathcal{H}))$ is defined by

$$\tau X = AXB.$$

Evidently $\|\tau\| \leq \|A\|\|B\|$. And the adjoint of τ is given by the formula $\tau^*X = A^*XB^*$. In particular, if A and B are both positive, then τ is positive and $\tau^{\frac{1}{2}} = A^{\frac{1}{2}}XB^{\frac{1}{2}}$, as one sees from the calculation

$$\begin{aligned} \langle \tau X, X\rangle &= tr(AXBX^*) = tr(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}}) \\ &= tr\left((A^{\frac{1}{2}}XB^{\frac{1}{2}})(A^{\frac{1}{2}}XB^{\frac{1}{2}})^*\right) \geq 0. \end{aligned}$$

Since $|\tau|^2X = |A|^2X|B^*|^2$ and $|\tau^*|^2X = |A^*|^2X|B|^2$, we have

$$|\tau|^{\frac{1}{2^n}} = |A|^{\frac{1}{2^n}}X|B^*|^{\frac{1}{2^n}}$$

and

$$|\tau^*|^{\frac{1}{2^n}} = |A^*|^{\frac{1}{2^n}}X|B|^{\frac{1}{2^n}}$$

for each integer $n \geq 1$.

Now, we need the following lemma.

Lemma 3.5. *Let A and B be operators in $\mathcal{B}(\mathcal{H})$. If A and B^* are m-quasi-class $A(k)$ for $k > 0$. Then the operator $\tau : C_2(\mathcal{H}) \to C_2(\mathcal{H})$ defined by $\tau X = AXB$ is m-quasi-class $A(k)$ for $k > 0$.*

Proof. For $X \in C_2(\mathcal{H})$, we have

$$\begin{aligned} \tau^{*m}\left(\left(\tau^*|\tau|^{2k}\tau\right)^{\frac{1}{k+1}} \right. &- \left.|\tau|^2\right)\tau^m X \\ &= A^{*m}\left[(A^*|A|^{2k}A)^{\frac{1}{k+1}} - |A|^2\right]A^mXB^m\left(B|B^*|^{2k}B^*\right)^{\frac{1}{k+1}}B^{*m} \\ &+ A^{*m}|A|^2A^mXB^m\left((B|B^*|^{2k}B^*)^{\frac{1}{k+1}} - |B^*|^2\right)B^{*m} \end{aligned}$$

Since A and B^* are m-quasi-class $A(k)$ operators, we have

$$\tau^{*m}\left(\left(\tau^*|\tau|^{2k}\tau\right)^{\frac{1}{k+1}} - |\tau|^2\right)\tau^m \geq 0.$$

□

Theorem 3.8. *Let A be m-quasi-class $A(k)$ operator for $k > 0$ and B^* be an invertible class $A(k)$ operator for $k > 0$. If $AX = XB$ for $X \in C_2(\mathcal{H})$, then $A^*X = XB^*$.*

Proof. Let τ be defined on $C_2(\mathcal{H})$ by $\tau X = AXB^{-1}$. Since B^* is an invertible class $A(k)$ operator, then it follows that B^* is also a class $A(k)$ operator for $k > 0$. Since A is an m-quasi-class $A(k)$ operator and $(B^{-1})^* = (B^*)^{-1}$ is an m-quasi-class $A(k)$ operator, we have that τ is an m-quasi-class $A(k)$ operator on $B_2(\mathcal{H})$ by Lemma 3.5. Moreover, we have $\tau X = AXB^{-1} = X$ because of $AX = XB$. Hence X is an eigenvector of τ. By Proposition 2.3 part (b), we have $\tau^*X = A^*X(B^{-1})^* = X$, that is, $A^*X = XB^*$. So, the proof is achieved. □

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ON DERIVATIONS SATISFYING CERTAIN IDENTITIES ON RINGS AND ALGEBRAS

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Abstract. The present paper deals with the commutativity of an associative ring R and a unital Banach Algebra A via derivations. Precisely, the study of multiplicative (generalized)-derivations F and G of semiprime (prime) ring R satisfying the identities $G(xy) \pm [F(x), y] \pm [x, y] \in Z(R)$ and $G(xy) \pm [x, F(y)] \pm [x, y] \in Z(R)$ has been carried out. Moreover, we prove that a unital prime Banach algebra A admitting continuous linear generalized derivations F and G is commutative if for any integer $n > 1$ either $G((xy)^n) + [F(x^n), y^n] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [F(x^n), y^n] - [x^n, y^n] \in Z(A)$.
Keywords. Banach algebra; Associative ring; Generalized derivations.

1. Multiplicative (generalized)-derivations on rings

All throughout this paper $Z(R)$ stands for the center of an associative ring R. Recall that if $aRb = (0)$ (resp. $aRa = (0)$) implies either $a = 0$ or $b = 0$ (resp. $a = 0$) then R is called a prime (resp. semi-prime) ring for all $a, b \in R$. For a positive integer n, a ring R is called n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. The symbol $[x, y]_n = [[x, y]_{n-1}, y]$ represents the nth commutator where $[x, y]_1 = [x, y] = xy - yx$. A mapping $\delta : R \to R$ satisfying $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$ is called a derivation of R. The notion of derivations has been generalized in many ways for instance local derivations, skew derivations, (θ, ϕ)-derivations, Lie derivations, Jordan derivations, multiplicative derivations etc. A set $A_R(S)$ $=\{a \in R : as = sa = 0$ for all $s \in S\}$ is called the annihilator of a non-empty subset S of R. By a left centralizer, we mean an additive mapping $H : R \to R$ such that $H(xy) = H(x)y$ for all $x, y \in R$. A mapping $f : R \to R$ is called centralizing (resp. commuting) on R if $[f(a), a] \in Z(R)$ (resp. $[f(a), a] = 0$) for all $a \in R$. There has been a significant interest in the study of centralizing and commuting mappings in associative rings (for example, see [5], [6] , [19] and references therein).

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Let us turn to the earlier investigation of multiplicative derivation and its generalizations. A map $\delta : R \to R$ is called a multiplicative derivation of R if it satisfies the *Leibniz rule* on R i.e.; $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. Of course these mappings are not necessarily additive. The idea of such mappings was introduced by Daif [8] inspired by the work of Martindale [18]. Further Goldmann and Šemrl [12] provided a complete study of these maps. The following example shows the existence of multiplicative derivation; let $R = C[0, 1]$ be the ring of all continuous real (or complex) valued functions and a map $\delta : R \to R$ defined as:

$$\delta(h)(u) = \left\{ \begin{array}{ll} h(u)log|h(u)| & \text{if } \ h(u) \neq 0 \\ 0 & \text{if } \ h(u) = 0 \end{array} \right\}$$

It is easy to verify that the map δ is not additive but it satisfies the *Leibnitz's rule*. Further, Daif and Tammam-El-Sayiad [10] amplified this notion of multiplicative derivation to multiplicative generalized derivation as; A mapping $D : R \to R$ is said to be a multiplicative generalized derivation if it is uniquely determined by a derivation $\delta : R \to R$ such that $D(ab) = D(a)b + a\delta(b)$ for all $a, b \in R$. Recently, Dhara and Ali [11] made a slight generalization in the definition of multiplicative generalized derivation and hence introduced the notion of multiplicative (generalized)-derivation. Accordingly, a mapping $F : R \to R$ (not necessarily additive) is called multiplicative (generalized)-derivation associated with a map $f : R \to R$ (not necessarily additive nor a derivation) if $F(ab) = F(a)b + af(b)$ for all $a, b \in R$. Very recently, Camci and Aydin [7] proved that if F is a multiplicative (generalized)-derivation of a semiprime ring associated with a map f, then f is a multiplicative derivation. For our convenience, we denote a multiplicative (generalized)-derivation as (F, f) throughout this paper. The multiplicative (generalized)-derivation looks more appropriate than multiplicative generalized derivation as it covers both the concept of multiplicative derivation and multiplicative left multiplier.

During the last two decades, the commutativity of associative rings with derivations have become one of the focus point of several authors and a significant work has been done in this direction (for the references one can see [3], [5], [9], [14], [17], [19], [20], [4] and references therein). In [14], Hongan proved that if d is a derivation of a prime ring R such that $d([x, y]) \pm [x, y] \in Z(R)$ for all $x, y \in I$, where I is a nonzero ideal of R, then R is commutative. Further, Qadri et al. [20] extended this result by proving it for generalized derivations of prime rings. In [4], Ashraf et al. explored the commutativity of prime rings that admit generalized derivations satisfying several differential identities on appropriate subsets. Precisely, they proved the following: *Let R be a prime ring and I be a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d satisfying any one of the identities: (i) $F(xy)xy \in Z(R)$; (ii) $F(xy) + xy \in Z(R)$; (iii) $F(xy)yx \in Z(R)$; (iv) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, then R is commutative.* Very recently, Tiwari et al. [23] discussed the commutativity of prime rings by studying the following conditions: (i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$; (ii) $G(xy) \pm F(y)F(x) \pm xy \in Z(R)$; (iii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$; (iv) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$; (v) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$ for all $x, y \in I$, where I is a nonzero ideal of R and F, G are the generalized derivation of R.

Clearly, a generalized derivation is a multiplicative (generalized)- derivation but the converse is not true. Thus, it would be a fact of interest to think about the results of generalized derivations for multiplicative (generalized)-derivations. In this direction, the initial results are due to Dhara and Ali [11], where they extended the theorems of Ashraf et al. [4] to the class of multiplicative (generalized)-derivations of semiprime rings. Moreover, Khan [15] studied the following differential identities: (i) $d(x) \circ F(y) \pm (x \circ y) = 0$; (ii) $d(x) \circ F(y) \pm [x, y] = 0$; (iii) $d(x) \circ F(y) = 0$; (iv) $[d(x), F(y)] \pm [x, y] = 0$; (v) $[d(x), F(y)] \pm (x \circ y) = 0$; (vi) $[d(x), F(y)] = 0$ for all x, y in an appropriate subset of a semiprime ring R and (F, d) the multiplicative (generalized)-derivation of R. For a good cross section of this subject, we refer the reader to [1], [16], [7], [21] and references therein. In this paper, our aim is to explore the nature of multiplicative derivations acting on a semiprime rings. More specifically, we investigate the following differential identities:

(i) $G(xy) \pm [F(x), y] \pm [x, y] \in Z(R)$;

(ii) $G(xy) \pm [x, F(y)] \pm [x, y] \in Z(R)$,

where (F, d) and (G, g) are the multiplicative (generalized)-derivations of a semiprime ring R.

1.1. Preliminaries

To achieve our objectives, we make utilization of the following commutator identities: $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$. We also use the following well known results:

Lemma 1.1. [[17] THEOREM 2. (II)] *Let R be a prime ring and I be a nonzero ideal of R. If there exist a derivation d of R such that $x[[d(x), x], x] = 0$ for all $x \in I$, then either $d = 0$ or R is commutative.*

Lemma 1.2. [[6] THEOREM 4.] *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in I$, then R is commutative.*

1.2. Main Results

Theorem 1.1. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R such that $G(xy)+[F(x), y]\pm[x, y] \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = 0$ and $z[f(z), z]_2 = 0$ for all $z \in I$.*

Proof. By our hypothesis

$$G(xy) + [F(x), y] \pm [x, y] \in Z(R) \quad \text{for all } x, y \in I. \tag{1.1}$$

On replacing y by yz in (1.1), we get $(G(xy) + [F(x), y] \pm [x, y])z + xyg(z) + y[F(x), z] \pm y[x, z] \in Z(R)$ for any $x, y, z \in I$. On commuting with z and using given hypothesis we obtain

$$[xyg(z), z] + [y[F(x), z], z] \pm [y[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.2}$$

Put zy in the place of y in (1.2) and we find

$$[xzyg(z), z] + z[y[F(x), z], z] \pm z[y[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.3}$$

Left multiply (1.2) by z and subtract from (1.3) to obtain

$$[[x, z]yg(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.4}$$

Replacing x by xt in (1.4) and we get

$$[x[t, z]yg(z), z] + [[x, z]tyg(z), z] = 0 \quad \text{for all } x, y, z, t \in I. \tag{1.5}$$

Put $y = ty$ in (1.4) and subtract from (1.5), we get $0 = [x[t, z]yg(z), z] = x[[t, z]y g(z), z] + [x, z][t, z]yg(z)$ for any $x, y, z, t \in I$. Using (1.4), we obtain

$$[x, z][t, z]yg(z) = 0 \quad \text{for all } x, y, z, t \in I. \tag{1.6}$$

Substituting tk for t in (1.6) in order to get

$$[x, z]t[k, z]yg(z) + [x, z][t, z]kyg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.7}$$

Replace y by ky in (1.6) and subtract from (1.7), we obtain

$$[x, z]t[k, z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.8}$$

Put $x = xg(z)$ in (1.8) and we have

$$x[g(z), z]t[k, z]yg(z) + [x, z]g(z)t[k, z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.9}$$

Replace t by $g(z)t$ in (1.8) and subtract from (1.9) to get

$$x[g(z), z]t[k, z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.10}$$

Putting $kg(z)$ for k in (1.10) and we find

$$x[g(z), z]tk[g(z), z]yg(z) + x[g(z), z]t[k, z]g(z)yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.11}$$

Replace y by $g(z)y$ in (1.10) and subtract from (1.11), we have

$$x[g(z), z]tk[g(z), z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.12}$$

Substitute $k = g(z)zk$ in (1.12) and we obtain

$$x[g(z), z]tg(z)zk[g(z), z]yg(z) = 0 \quad \text{for all } x, y, z, t, k \in I. \tag{1.13}$$

Replacing t by $tzg(z)$ in (1.12) to get

$$x[g(z),z]tzg(z)k[g(z),z]yg(z) = 0 \quad \text{for all } x,y,z,t,k \in I. \tag{1.14}$$

Subtract (1.13) and (1.14), we get $x[g(z),z]t[g(z),z]k[g(z),z]yg(z) = 0$ for all $x,y,z,t,k \in I$. It implies that $x[g(z),z]t[g(z),z]k[g(z),z]y[g(z),z] = 0$ for all $x,y,z,t,k \in I$. In particular, $(I[g(z),z])^4 = (0)$ for all $z \in I$. Since R is semiprime ring, so we must have $I[g(z),z] = (0)$ for all $w \in I$. Therefore, semiprimeness of I yields that $[g(z),z] = 0$ for all $z \in I$.

Now, substitute $y = yz$ in (1.2), we get

$$[xyzg(z),z] + [yz[F(x),z],z] \pm [yz[x,z],z] = 0 \quad \text{for all } x,y,z \in I. \tag{1.15}$$

Right multiply (1.2) by z and subtract from (1.15) and using the fact that $[g(z),z] = 0$, we get

$$[y[[F(x),z],z],z] \pm [y[[x,z],z],z] = 0 \quad \text{for all } x,y,z \in I. \tag{1.16}$$

Replace x by xz in (1.16) in order to obtain

$$[y[[F(x),z],z],z]z + [y[[xf(z),z],z],z] \pm [y[[x,z],z],z]z = 0, \tag{1.17}$$

for all $x,y,z \in I$. Right multiply (1.16) by z and subtract from (1.17), we get

$$[y[[xf(z),z],z],z] = 0 \quad \text{for all } x,y,z \in I. \tag{1.18}$$

Replace y by $[xf(z),z]y$ in (1.18) and we find $[xf(z),z][y[[xf(z),z],z],z] + [[xf(z),z],z]y[[xf(z),z],z] = 0$ for any $x,y,z \in I$. Using (1.18), we get $[[xf(z),z],z]y[[xf(z),z],z] = 0$ for all $x,y,z \in I$. That is, $(I[[xf(z),z],z])^2 = 0$ but R is a semi-prime ring so we must have $I[[xf(z),z],z] =$ for each $x,z \in I$. Semi-primeness of I implies that $[[xf(z),z],z] = 0$ for all $x,z \in I$. In particular, we obtain $z[f(z),z]_2 = 0$ for all $z \in I$, as desired. □

In Theorem 1.1, substitute $G = -G$ and $g = -g$ we get the following theorem:

Theorem 1.2. *Let I be a nonzero ideal of a semiprime ring R. If (F,f) and (G,g) are multiplicative (generalized)-derivations of R such that $G(xy) - [F(x),y] \pm [x,y] \in Z(R)$ holds for all $x,y \in I$, then $[g(z),z] = 0$ and $z[f(z),z]_2 = 0$ for all $z \in I$.*

Corollary 1.1. *Let I be a nonzero ideal of a prime ring R. If (F,f) and (G,g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x),y] \pm [x,y] \in Z(R)$ holds for all $x,y \in I$, then either $f = 0 = g$ or R is commutative.*

Proof. Observe that in Theorem 1.1 and 1.2, if R is prime and f,g are derivations of R, by Lemma 1.1 and Lemma 1.2 the equations $z[[f(z),z],z] = 0$ and $[g(z),z] = 0$ for all $z \in I$ respectively implies that either $f = 0 = g$ or R is commutative. □

In Corollary 1.1, substitute $G \mp I_d$ for G we get the following result:

Corollary 1.2. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm yx \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

In Corollary 1.2, substitute $F \pm I_d$ for F we get the following result:

Corollary 1.3. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm xy \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

Theorem 1.3. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R such that $G(xy)+[x, F(y)]\pm[x, y] \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = -[f(z), z]$ for all $z \in I$.*

Proof. Let us assume that

$$G(xy) + [x, F(y)] \pm [x, y] \in Z(R) \quad \text{for all } x, y \in I. \tag{1.19}$$

Put $y = yz$ in (1.19) and we find $(G(xy) + [x, F(y)] \pm [x, y])z + xyg(z) + F(y)[x, z] + [x, yf(z)] \pm y[x, z] \in Z(R)$ for all $x, y, z \in I$. On commuting with z and using our hypothesis, we obtain

$$[xyg(z), z] + [F(y)[x, z], z] + [[x, yf(z)], z] \pm [y[x, z], z] = 0, \tag{1.20}$$

for all $x, y, z \in I$. Replacing x by xz in (1.20), we get

$$[xzyg(z), z] + [F(y)[x, z], z]z + [[x, yf(z)], z]z + [x[z, yf(z)], z] \pm [y[x, z], z]z = 0, \tag{1.21}$$

for all $x, y, z \in I$. Right multiply (1.20) by z and subtract from (1.21), we find $[x[z, yg(z)], z] + [x[z, yf(z)], z] = 0$ where $x, y, z \in I$. That is

$$[x[z, y(g(z) + f(z))], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.22}$$

On substituting ry in the place of y, where $r \in R$ in (1.22), we get

$$[xr[z, y(g(z) + f(z))], z] + [x[z, r]y(g(z) + f(z)), z] = 0, \tag{1.23}$$

for all $x, y, z \in I$, $r \in R$. Replacing x by xr in (1.22) and subtract from (1.23), we get

$$[x[z, r]y(g(z) + f(z)), z] = 0 \quad \text{for all } x, y, z \in I, r \in R. \tag{1.24}$$

Put sx in the place of x, where $s \in R$ in (1.24) in order to find $s[x[z, r]y(g(z) + f(z)), z] + [s, z]x[z, r]y(g(z) + f(z)) = 0$ for all $x, y, z \in I$ and $r, s \in R$. Eq. (1.24) reduces it to

$$[s, z]x[r, z]y(g(z) + f(z)) = 0 \quad \text{for all } x, y, z \in I, r, s \in R. \tag{1.25}$$

Replace y by yz in (1.25), we get

$$[s, z]x[r, z]yz(g(z) + f(z)) = 0 \quad \text{for all } x, y, z \in I, r, s \in R. \tag{1.26}$$

Right multiply (1.25) by z and subtract from (1.26), we get $[s,z]x[r,z]y[(g(z)+f(z)),z] = 0$ for each $x,y,z \in I$ and $r,s \in R$. In particular, we have $(I[(g(z)+f(z)),z])^3 = (0)$ for all $z \in I$. Since R is semiprime ring, so we must have $I[(g(z)+f(z)),z] = (0)$ for all $z \in I$. Therefore, $[(g(z)+f(z)),z] \in I \cap A_R(I) = (0)$ for any $z \in I$. Hence $[g(z),z] = -[f(z),z]$ for all $z \in I$, as desired. □

In Theorem 1.3, substitute $G = -G$ and $g = -g$ we get the following theorem:

Theorem 1.4. *Let I be a nonzero ideal of a semiprime ring R. If (F,f) and (G,g) are multiplicative (generalized)-derivations of R such that $G(xy)-[x,F(y)]\pm[x,y] \in Z(R)$ holds for all $x,y \in I$, then $[g(z),z] = [f(z),z]$ for all $z \in I$.*

Corollary 1.4. *Let R be a prime ring. If (F,f) and (G,g) are multiplicative generalized derivations of R such that $G(xy) - [x,F(y)] \pm [x,y] \in Z(R)$ holds for all $x,y \in R$ then either $g = f$ or R is commutative.*

Proof. From Theorem 1.4 we have, $[(-g+f)(z),z] = 0$ for all $z \in R$. We know that sum of two derivations is a derivation so Posner's second theorem [19] yields that either $g = f$ or R is commutative. □

Corollary 1.5. *Let R be a prime ring with a nonzero ideal I. Suppose that (F,f) and (G,g) are multiplicative generalized derivations of R. If $G(xy)-[x,F(y)]\pm yx \in Z(R)$ holds for all $x,y \in I$ then either $f = g$ or R is commutative.*

Proof. It is easy to check that if G is a multiplicative (generalized)-derivation on R associated with a map g, then $(G \mp I_d)$ is also a multiplicative (generalized)-derivation on R associated with map g. On replacing G by $(G \mp I_d)$ in Theorem 1.4, we obtain that $[(-g+f)(z),z] = 0$ for the situation $G(xy) - [F(x),y] \mp yx \in Z(R)$ for all $x,y \in I$. If we assume that F and G are multiplicative generalized derivations associated with non-zero derivations f and g respectively same conclusion i.e.; $(-g+f)$ is commuting on I holds. Hence, Lemma 1.2 implies that either $f = g$ or R is commutative. □

In Corollary 1.5, substitute $F \pm I_d$ for F we get the following results:

Corollary 1.6. *Let I be a nonzero ideal of a prime ring R. If (F,f) and (G,g) are multiplicative generalized derivations of R such that $G(xy) - [x,F(y)] \pm xy \in Z(R)$ holds for all $x,y \in I$, then either $f = g$ or R is commutative.*

Theorem 1.5. *Let I be a nonzero ideal of a semiprime ring R. If (F,f) and (G,g) are multiplicative (generalized)-derivations of R and H is a left centralizer of R such that $G(xy) + [F(x),y] \pm H(xy) \in Z(R)$ holds for all $x,y \in I$, then $[g(z),z] = 0$ and $z[f(z),z]_2 = 0$ for all $z \in I$.*

Proof. By the hypothesis, we have

$$G(xy) + [F(x), y] \pm H(xy) \in Z(R) \quad \text{for all } x, y \in I. \tag{1.27}$$

Taking yz instead of y with $z \in I$ in (1.27), we get $(G(xy) + [F(x), y] \pm H(xy))z + xyg(z) + y[F(x), z] \in Z(R)$ for all $x, y, z \in I$. On commuting with z and using the hypothesis, we get

$$[xyg(z), z] + [y[F(x), z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.28}$$

Replacing y by zy in (1.28), so we have

$$[xzyg(z), z] + z[y[F(x), z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.29}$$

Left multiply (1.28) by z and subtract from (1.29), we obtain

$$[[x, z]yg(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.30}$$

So, same equation with the (1.4) was obtained. Similar proof shows that $[g(z), z] = 0$, for all $z \in I$. If we replace y by yz in (1.28), we get

$$[xyzg(z), z] + [yz[F(x), z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.31}$$

Right multiply (1.28) by z and subtract from (1.31) and using the $[g(z), z] = 0$, we get

$$[y[[F(x), z], z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.32}$$

Replace x by xz and using (1.32), we have

$$[[y[[xf(z), z], z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{1.33}$$

So, same equation with the (1.18) has obtained. Similar operations applied after this shows that $z[[f(z), z], z] = 0$ for all $z \in I$. □

In Theorem 1.5, substitute $G = -G$ and $g = -g$ we get the following theorem.

Theorem 1.6. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations of R and H is a left centralizer of R such that $G(xy) - [F(x), y] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = 0$ and $z[f(z), z]_2 = 0$ for all $z \in I$.*

Corollary 1.7. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy) \pm [F(x), y] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then either $f = 0 = g$ or R is commutative.*

By using the similar technique, we obtain the following results. For the sake of brevity, we omit the proofs here.

Theorem 1.7. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations and H is a left centralizer of R such that $G(xy) + [x, F(y)] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = -[f(z), z]$ for all $z \in I$.*

In Theorem 1.7, substitute $G = -G$ and $g = -g$ we get the following theorem.

Theorem 1.8. *Let I be a nonzero ideal of a semiprime ring R. If (F, f) and (G, g) are multiplicative (generalized)-derivations and H is a left centralizer of R such that $G(xy) - [x, F(y)] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then $[g(z), z] = [f(z), z]$ for all $z \in I$.*

Corollary 1.8. *Let I be a nonzero ideal of a prime ring R. If (F, f) and (G, g) are multiplicative generalized derivations of R such that $G(xy)-[x, F(y)] \pm H(xy) \in Z(R)$ holds for all $x, y \in I$, then either $f = g$ or R is commutative.*

2. Generalized derivations on Banach algebras

In order to extend the scope of this work, we discuss the commutativity of unital prime Banach algebras with derivations which is directly motivated by the work of Yood [24] and Ali [2]. Since we have already proved that (as in Corollary 1.1 and 1.4) if constraints $G(xy) + [F(x), y] + [x, y] \in Z(R)$ and $G(xy) - [x, F(y)] - [x, y] \in Z(R)$ hold on a prime ring R where F and G are generalized derivations associated with non-zero non-equal derivations f and g respectively, then R is commutative. For an integer $n > 1$, it is natural to consider the constraints: 1. either $G((xy)^n) + [F(x^n), y^n] + [x^n, y^n] \in Z(R)$ or $G((xy)^n) + [y^n, F(x^n)] + [y^n, x^n] \in Z(R)$ and 2. $G((xy)^n) + [x^n, F(y^n)] + [x^n, y^n] \in Z(R)$ or $G((xy)^n) + [x^n, F(y^n)] + [x^n, y^n] \in Z(R)$ on Banach Algebra.

2.1. Preliminaries

Lemma 2.1. [[24]]*Let A is a Banach algebra and M be a closed linear subspace of A. If $p(t) = a_1t + a_2t^2 + ... + a_nt^n$ be a polynomial in real variable t over A such that $p(t) \in M$, then each $a_i \in M$.*

Lemma 2.2. [Open problem 1, [22]] *Let A be a unital prime Banach algebra with non-trivial center $Z(A)$. If $d : A \to A$ be a derivation of A, then $d(e) \in Z(A)$.*

Proof. Let $0 \neq c \in Z(A)$. It is easy to check that $d(c) \in Z(A)$. That means for all $a \in A$, $0 = [d(c), a] = [d(ce), a] = [d(c)e, a] + [cd(e), a] = c[d(e), a]$. Therefore, $cA[d(e), b] = (0)$ for all $b \in A$. Since $c \neq 0$, we get $d(e) \in Z(A)$. □

Lemma 2.3. [Theorem 2, [19]] *A prime ring R admitting a non-zero centralizing derivation is commutative.*

Lemma 2.4. *Let A be a unital prime algebra and $F : A \to A$ be a generalized derivation associated with a derivation f such that $[F(x), x] \in Z(A)$ for all $x \in A$, $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Then A is commutative.*

Proof. By hypothesis, for each $x \in A$, $[F(x), x] \in Z(A)$. Linearizing this relation in order to obtain $[F(x), y] + [F(y), x] \in Z(A)$. Replace x by $xF(e)$ we obtain $([F(x), y] + [F(y), x])F(e) + [x, y]f(F(e)) \in Z(A)$. As $Z(A)$ is a linear subspace of A, we left with $[x, y]f(F(e)) \in Z(A)$. Since $f(F(e)) \neq 0$, we have $[x, y] \in Z(A)$. That means, $0 = [[y, x], z] = [I_y(x), z]$ for all $x, y, z \in A$, where I_y is an inner derivation of A. Hence, Lemma 2.3 completes the proof. □

2.2. Main Results

Theorem 2.1. *Let $F, G : A \to A$ are continuous linear generalized derivations of a unital prime Banach Algebra A associated with non-zero continuous linear derivations $f, g : A \to A$ respectively such that $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Suppose that $G((xy)^n) + [F(x^n), y^n] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [F(x^n), y^n] - [x^n, y^n] \in Z(A)$ for all $x \in P_1$ and $y \in P_2$, where P_1, P_2 are open sets in A and $n = n(x, y) > 1$ is an integer. Then A is commutative.*

Proof. Firstly, we set $\phi_1(x, y, n)=G((xy)^n) + [F(x^n), y^n] + [x^n, y^n]$ and $\phi_2(x, y, n)=G((xy)^n) + [y^n, F(x^n)] + [y^n, x^n]$. By our hypothesis, $\phi_1(x, y, n) \in Z(A)$ and $\phi_2(x, y, n) \in Z(A)$ for all $x \in P_1$ and $y \in P_2$. For an arbitrary fixed element $x \in P_1$, we construct a set $E_n = \{y \in A : \phi_1(x, y, n) \notin Z(A), \phi_2(x, y, n) \notin Z(A)\}$. We claim that E_n is open. For this, we choose a sequence $< s_k >$ in E_n^c that converges to s and prove that $s \in E_n^c$. By our assumption, $s_k \in E_n^c$ i.e. $\phi_1(x, s_k, n) \in Z(A)$ or $\phi_2(x, s_k, n) \in Z(A)$. On making k arbitrarily large, the continuity of G implies that $\phi_1(x, s, n) \in Z(A)$ or $\phi_2(x, s, n) \in Z(A)$. That means, $s \in E_n^c$. Hence, E_n is open. By the Baire Category theorem; if every E_n is dense, then so is their intersection, which contradicts the existence of P_2. Therefore, there must exist a positive integer $m = m(x) > 1$ such that E_m is not dense. Let P_3 be a nonzero open set in E_m^c such that $\phi_1(x, y, m) \in Z(A)$ or $\phi_2(x, y, m) \in Z(A)$ for all $y \in P_3$. Take $q_0 \in P_3$ and $w \in A$ for sufficiently small real t, $q_0 + tw \in P_3$. Therefore, we have

$$\phi_1(x, q_0 + tw, m) \in Z(A) \tag{2.1}$$

or

$$\phi_2(x, q_0 + tw, m) \in Z(A) \tag{2.2}$$

One of these relations must hold for infinitely many real t. If (2.1) holds, the corresponding binomial expansion is a polynomial in t. In the light Lemma 2.1, each coefficient of the polynomial must be in $Z(A)$. On taking the coefficients of t^m, we get $\phi_1(x, w, m) \in Z(A)$. Similarly, if (2.2) holds, $\phi_2(x, w, m) \in Z(A)$. That means, for given $x \in P_1$ there exist an integer $m = m(x) > 1$ such that for each $w \in A$ either $\phi_1(x, w, m) \in Z(A)$ or $\phi_2(x, w, m) \in Z(A)$.

Next, let $y \in A$ be an arbitrary element. Now we want to show that there exists an integer $r > 1$ depending on y such that for each $u \in A$, either $\phi_1(u, y, r) \in Z(A)$ or $\phi_2(u, y, r) \in Z(A)$. Fix $y \in A$ and for each integer $p(y) > 1$, we consider a set $V_p = \{v \in A : \phi_1(v, y, p) \notin Z(A), \phi_2(v, y, p) \notin Z(A)\}$. It is easy to see that V_p is open. The application of the Baire category theorem forces that there exists an integer $r = r(y) > 1$ such that V_r is not dense in A. Let P_4 be a non-empty open subset of V_r^c such that either $\phi_1(x, y, r) \in Z(A)$ or $\phi_2(x, y, r) \in Z(A)$ for all $x \in P_4$. Take $x_0 \in P_4$ and $u \in A$ then $x_0 + tu \in P_4$ for all sufficiently small real t and either $\phi_1(x_0 + tu, y, r) \in Z(A)$ or $\phi_2(x_0 + tu, y, r) \in Z(A)$ for all $u \in A$ and $x_0 \in P_4$. Applying the same argument, we obtain that either $\phi_1(u, y, r) \in Z(A)$ or $\phi_2(u, y, r) \in Z(A)$ for all $u \in A$.

Now, we construct a set $T_j = \{y \in A : \phi_1(w, y, j) \in Z(A)$ or $\phi_2(w, y, j) \in Z(A)$ for all $w \in A\}$. By our above arguments it is clear that $\cup T_j = A$ and each T_j is closed i.e.; each T_j^c is open. Again by the Baire category theorem, if each T_j^c is dense, then their intersection is also dense, which is again a contradiction to the existence of P_2. Thus there must exist an integer $l > 1$ such that T_l contains a non-empty open set P_5 and either $\phi_1(w, y_0, l) \in Z(A)$ or $\phi_1(w, y_0, l) \in Z(A)$ for all $y_0 \in P_5$. If $y_0 \in P_5$ and $z \in A$ then $y_0 + tz \in P_5$ for all sufficiently small real t. Therefore, either $\phi_1(w, y_0 + tz, l) \in Z(A)$ or $\phi_2(w, y_0 + tz, l) \in Z(A)$ for all $w, z \in A$ and $y_0 \in P_5$. By repeating the same argument as earlier, we get either $\phi_1(w, z, l) \in Z(A)$ or $\phi_2(w, z, l) \in Z(A)$ for all $w, z \in A$ and an integer $l > 1$.

As we assumed A a prime Banach algebra with unity and from what that just has been shown, we obtain either $\phi_1(e + tx, y, n) \in Z(A)$ or $\phi_2(e + tx, y, n) \in Z(A)$ for all $x, y \in A$. Explicitly, we have either $G(((e+tx)y)^n) + [F((e+tx)^n), y^n] + [(e+tx)^n, y^n] \in Z(A)$ or $G(((e+tx)y)^n) + [y^n, F((e+tx)^n)] + [y^n, (e+tx)^n] \in Z(A)$ for all $x, y \in A$. The expansions of these expressions are the polynomials in t. Using Lemma 2.1 and taking the coefficients of t, we get either $G(nxy^n) + [F(nx), y^n] + [nx, y^n] \in Z(A)$ or $G(nxy^n) + [y^n, F(nx)] + [y^n, nx] \in Z(A)$ for all $x, y \in A$. Note that $nxy^n = xy^n + \sum_{i=1}^{n-1} y^i xy^{n-i} = xy^n + Q$ where $Q = \sum_{i=1}^{n-1} y^i xy^{n-i}$. Therefore, we have either

$$G(xy^n + Q) + n[F(x), y^n] + n[x, y^n] \in Z(A) \tag{2.3}$$

or

$$G(xy^n + Q) + n[y^n, F(x)] + n[y^n, x] \in Z(A) \tag{2.4}$$

for all $x, y \in A$. Taking $y(e + tx)$ in the place of $(e + tx)y$ and note that $ny^n x = y^n x + Q$, we find either

$$G(y^n x + Q) + n[F(y^n), x] + n[y^n, x] \in Z(A) \tag{2.5}$$

or

$$G(y^n x + Q) + n[x, F(y^n)] + n[x, y^n] \in Z(A) \tag{2.6}$$

for all $x, y \in A$. Thus one of the pair of equations (2.3)-(2.5),(2.3)-(2.6),(**??**)-(2.5) and (2.4)-(2.6) must hold on A. On subtracting these pairs we get either

$$G[x, y^n] + n[(F - i_d)(x), (F + i_d)(y^n)] + 2n[x, y^n] \in Z(A) \tag{2.7}$$

or

$$G[x,y^n] - n[(F-i_d)(x),(F+i_d)(y^n)] - 2n[x,y^n] \in Z(A) \tag{2.8}$$

or

$$G[x,y^n] \pm n[(F-i_d)(x),(F-i_d)(y^n)] \in Z(A) \tag{2.9}$$

holds for all $x, y \in A$ where i_d is the identity map. Firstly, we consider $G[x,y^n] + n[(F-i_d)(x),(F+i_d)(y^n)] + 2n[x,y^n] \in Z(A)$ for all $x, y \in A$. Replacing y by $e+ty$ in this relation. Using Lemma 2.1 and collecting the coefficients of t, we find that $G[x,y] + n[(F-i_d)(x),(F+i_d)(y)] + 2n[x,y] \in Z(A)$ where x, y varies over A. It is easy to check that $F-i_d$ and $F+i_d$ are continuous linear generalized derivations associated with nonzero continuous linear derivations f. Set $F-i_d = H$ and $F+i_d = K$. For each $x, y \in A$, we have $G[x,y]+n[H(x),K(y)]+2n[x,y] \in Z(A)$. Substitute $yF(e)$ for y in the last expression, we get $(G[x,y]+n[H(x),K(y)]+2n[x,y])F(e) + [x,y]g(F(e)) + n[H(x),y]f(F(e)) \in Z(A)$ where $x, y \in A$. Since $Z(A)$ is a linear subspace of A, last relation reduces to $[x,y]g(F(e)) + n[H(x),y]f(F(e)) \in Z(A)$ for all $x, y \in A$. In particular, put $x = y$, we have with $n[H(x),x]f(F(e)) \in Z(A)$ where $x, y \in A$. Since $0 \neq f(F(e)) \in Z(A)$, we have $n[H(x),x] \in Z(A)$. That is, for each $x \in A$, $[H(x),x] \in Z(A)$. By Lemma 2.4, A is commutative.

In the same way, we can prove the same conclusion for the equation (2.8) and (2.9). □

Theorem 2.2. *Let $F, G : A \to A$ are continuous linear generalized derivations of a unital prime Banach Algebra A associated with nonzero continuous linear derivations $f, g : A \to A$ respectively such that $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Suppose that $G((xy)^n) + [x^n, F(y^n)] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [x^n, F(y^n)] - [x^n, y^n] \in Z(A)$ for all $x \in P_1$ and $y \in P_2$, where P_1, P_2 are open sets in A and $n = n(x,y) > 1$ is an integer. Then A is commutative.*

Proof. By following the same argument with some necessary variations as in Theorem 2.1, we find either

$$G[x,y^n] + n[(F+i_d)(x),(F+i_d)(y^n)] + 2n[x,y^n] \in Z(A) \tag{2.10}$$

or

$$G[x,y^n] - n[(F+i_d)(x),(F+i_d)(y^n)] - 2n[x,y^n] \in Z(A) \tag{2.11}$$

or

$$G[x,y^n] + n[(F-i_d)(x),(F-i_d)(y^n)] \in Z(A) \tag{2.12}$$

for all $x, y \in A$ and an integer $n > 1$. Again from Theorem 2.1 we can get the desired outcomes. □

Theorem 2.3. *Let $F, G : A \to A$ are continuous linear generalized derivations of a unital prime Banach Algebra A associated with nonzero continuous linear derivations $f, g : A \to A$ respectively such that $F(e) \in Z(A)$ and $f(F(e)) \neq 0$. Suppose that $G((xy)^n) + [F(x^n), F(y^n)] + [x^n, y^n] \in Z(A)$ or $G((xy)^n) - [F(x^n), F(y^n)] - [x^n, y^n] \in Z(A)$ for all $x \in P_1$ and $y \in P_2$, where P_1, P_2 are open sets in A and $n = n(x,y) > 1$ is an integer. Then A is commutative.*

Proof. By following the same argument with some necessary variations as in Theorem 2.1, we find either

$$G[x, y^n] + 2n[F(x), F(y^n)] + 2n[x, y^n] \in Z(A) \tag{2.13}$$

or

$$G[x, y^n] - 2n[F(x), F(y^n)] - 2n[x, y^n] \in Z(A) \tag{2.14}$$

or

$$G[x, y^n] \in Z(A) \tag{2.15}$$

for all $x, y \in A$ and an integer $n > 1$. Let us consider for each $x, y \in A$, $G[x, y^n] \in Z(A)$. This situation is the same as in [Eq. (15), [22]], hence the conclusion follows. For the remaining identities, by applying the same procedure as in Theorem 2.1, we can get the required results. □

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TENSOR PRODUCT OF THE POWER GRAPHS OF SOME FINITE RINGS *

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Abstract. Suppose R is a ring. The multiplicative power graph $\mathcal{P}(R)$ of R is the graph whose vertices are elements of R, where two distinct vertices x and y are adjacent if and only if there exists a positive integer n such that $x^n = y$ or $y^n = x$. In this paper, the tensor product of the power graphs of some finite rings are studied.
Keywords: Power graph; bipartite graph; finite rings; tensor product.

1. Introduction

All graphs considered here are assumed to be undirected and simple and the vertex and edge set of such a graph G will be denoted by $V(G)$ and $E(G)$, respectively. An edge connecting two vertices x and y in G is denoted by xy. We first state some definitions and notations that will be kept throughout the paper.

Given a semigroup S, the *undirected power graph* $\mathcal{P}(S)$ has a vertex set S and two distinct vertices x and y are adjacent if and only if $x^n = y$ or $y^n = x$, for a positive integer n [4]. The directed version of this graph was introduced by Kelarev and Quinn in an innovating work [11]. These authors continued their work on this graph in papers [8, 9, 10]. We also recommend that the authors should be consulted for the survey article [1] and references therein for more information on this topic. In [14], the authors proved a number of results that relate the structure of the group to the structure of its power graph. Among other things, they presented a counterexample to a conjecture of Charkabarty, Ghosh and Sen. In [3], it was proved that the only finite group whose automorphism group is the same as that of its power graph is the Klein group of order 4.

Mary Flagg [6], in her interesting paper studied the power graph of rings. Since a ring R has two binary operations "+" and "×", there will be two different power

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graphs $\mathcal{P}^+(R)$ and $\mathcal{P}^\times(R)$ that can be associated to R. The power graphs $\mathcal{P}^+(R)$ and $\mathcal{P}^\times(R)$ are called the additive and multiplicative power graphs of R, respectively.

Recall that a graph is said to be *connected* if for each pair of distinct vertices x and y, there is a finite sequence of distinct vertices $x = x_1, \cdots, x_n = y$ such that each pair (x_i, x_{i+1}) is an edge. A graph without edges is called *totally disconnected.* For distinct vertices x and y, let $\mathrm{d}(x, y)$ be the shortest length of a path connecting x and y and let $d(x, y) = \infty$ if no such path exists. The *diameter* of G is defined as $diam(G) = max\{d(x, y) \mid x, y \in V(G)\}$.

For a graph G, the *degree* of a vertex x in G is the number of edges of G incident with x, denoted by $\deg(x)$. A *regular graph* is a graph that every vertex has the same degree. The graph G is called *bipartite* with vertex bipartition $\{V_1, V_2\}$ if the set of all vertices of G is $V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and each edge of G joins a vertex from V_1 to a vertex of V_2. A *complete bipartite* graph is a bipartite graph containing all edges joining the vertices of V_1 and V_2. A complete bipartite graph on vertex sets of sizes m and n is denoted by $K_{m,n}$. If $m = 1$ then the resulting graph $K_{1,n}$ is called a *star graph.*

Suppose G and H are two graphs. We say that G is a *subgraph* of H, when $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. A *cycle* in G is a subgraph that by deleting one of its edge the resulting subgraph is a path. The *girth* of G, written $gr(G)$, is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycle. A *connected component* of an undirected graph is a subgraph in which any two vertices are connected to each other by at least one path and the number of connected components of G is denoted by $\mathcal{C}(G)$.

The *tensor product* of graphs G and H is denoted by $G \otimes H$, whose vertex set is $V(G) \times V(H)$ and for which vertices (g, h) and (g', h') are adjacent precisely when $gg' \in E(G)$ and $hh' \in E(H)$, see [7] for details.

Suppose p is a prime. Fine [5], classified all rings of order p^2 as follows:

$$\begin{array}{rcl}
A_p &=& \langle a : p^2a = 0, a^2 = a\rangle \cong \mathbb{Z}_{p^2},\\
B_p &=& \langle a : p^2a = 0, a^2 = pa\rangle,\\
C_p &=& \langle a : p^2 = 0, a^2 = 0\rangle,\\
D_p &=& \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0\rangle = \mathbb{Z}_p \oplus \mathbb{Z}_p,\\
E_p &=& \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle,\\
F_p &=& \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle,\\
G_p &=& \langle a, b : pa = pb = 0, a^2 = 0, b^2 = 0, ab = ba = a\rangle,\\
H_p &=& \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0\rangle,\\
I_p &=& \langle a, b : pa = pb = 0, a^2 = b, ab = 0\rangle,\\
J_p &=& \langle a, b : pa = pb = 0, a^2 = b^2 = 0\rangle,\\
K_p &=& \left\{\begin{array}{ll} \langle a, b : 2a = 2b = 0, a^2 = a, b^2 = a + b, ab = b, ba = b\rangle & p = 2\\ \langle a, b : pa = pb = 0, a^2 = a, b^2 = ja, ab = ba = b\rangle & p \neq 2\end{array}\right.
\end{array}$$

where j is not a square in $\mathbb{Z}_p$.

Throughout this paper the *cardinality* of a set A will be denoted by $|A|$ and K_n and $U(\mathbb{Z}_{p^2})$ stand for the complete graph on n vertices and the group of multiplicative units of $\mathbb{Z}_{p^2}$, respectively. Our other notations are standard and can be obtained from the books [2, 12, 13].

2. The Number of Components

By [3, Theorem 1], the additive power graph of a ring determines the additive structure of the ring and so we will focus on the multiplicative power graph $\mathcal{P}(R) = \mathcal{P}^{\times}(R)$. In this section we investigate the number of components of the tensor products of two rings R and S. Note that the tensor product of graphs are commutative so in this paper we will avoid the repeated cases. If $x \in R$, $y \in S$, $A \subseteq R$ and $B \subseteq S$ then we define:

$$\begin{aligned}(x, B) &= \{(x, b) \mid b \in S\},\\ (A, y) &= \{(a, y) \mid a \in A\}.\end{aligned}$$

Let p be a prime and R be a ring of order p. Then as an additive group, $R \cong \mathbb{Z}_p$. This implies that there are two rings of order p, the ring $\mathbb{Z}_p$ and the zero ring on the additive group, denoted by N_p.

Theorem 2.1. Suppose p, q are primes, R_p and R_q denote arbitrary rings of order p and q, respectively, and $\Gamma = \mathcal{P}(R_p) \otimes \mathcal{P}(R_q)$. Then one of the following statements is hold:

(1) The graph Γ has two components, one of them is isomorphic to a complete bipartite graph $K_{(p-1),(q-1)}$ and another one is the star graph $K_{1,(p-1)(q-1)}$.

(2) Γ has one or two components and $p + q - 1$ isolated vertices.

(3) Γ has a bipartite component and q isolated vertices.

(4) Γ has two components of the form $K_{1,(q-1)}$ and q isolated vertices.

(5) The graph Γ is totally disconnected.

Proof . Since there are two non-isomorphic rings of a prime order, it is enough to consider the graphs $\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(\mathbb{Z}_q)$, $\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(N_q)$ and $\mathcal{P}(N_p) \otimes \mathcal{P}(N_q)$. Our main proof will consider three separate cases as follows:

1. If $\Gamma = \mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(\mathbb{Z}_q)$, then for $p = 2$ and any prime q the graph $\mathcal{P}(\mathbb{Z}_2) \otimes \mathcal{P}(\mathbb{Z}_q)$ is totally disconnected, since $\mathcal{P}(\mathbb{Z}_2)$ is totally disconnected. If $p, q \neq 2$ except the case that $p = q = 3$, then $V(\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(\mathbb{Z}_q))$ has a subset $A = \{(0, 0), (0, v), (u, 0) \mid u \in V(\mathcal{P}(\mathbb{Z}_p)), v \in V(\mathcal{P}(\mathbb{Z}_q))\}$ of size $\mid A \mid= p + q - 1$ as its isolated vertices. We claim that all other vertices form a component. For every vertex $(x, y) \in V(\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(\mathbb{Z}_q)) - A$, we have the following two cases:

(a) $x, y \neq 1$. Then it is clear that (x, y) and $(1, 1)$ are adjacent.

(b) $x = 1$ *and* $y \neq 1$ or vice versa. In this case, a vertex $(x', y') \in V(\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(\mathbb{Z}_q)) - A$ exists such that (x, y) is adjacent with (x', y') and the last one is adjacent to $(1, 1)$. Note that if $p = q = 3$, then $\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(\mathbb{Z}_q)$ has exactly five isolated vertices and two components isomorphic to K_2.

2. If $\Gamma = \mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(N_q)$, then it is clear that $\mathcal{P}(\mathbb{Z}_2) \otimes \mathcal{P}(N_q)$ is a totally disconnected graph. Let $p \neq 2$. Then $\mathcal{P}(\mathbb{Z}_p) \otimes \mathcal{P}(N_q)$ have q isolated vertices and a bipartite connected component such that one part contains all vertices of the form $(V(\mathcal{P}(\mathbb{Z}_p)) - \{0\}, 0)$ and another part contains all vertices of the form $(V(\mathcal{P}(\mathbb{Z}_p)) - \{0\}, V(\mathcal{P}(N_q)) - \{0\})$. Note that if $p = q = 3$ then we obtain three isolated vertices and two star $K_{1,q-1}$ as components.

3. If $\Gamma = \mathcal{P}(N_p) \otimes \mathcal{P}(N_q)$, then in this case the component corresponding to the vertex $(0, 0)$ is a star graph $K_{1,(p-1)(q-1)}$, since the vertex 0 is adjacent to all other vertices in $\mathcal{P}(N_p)$. It is now straightforward to verify that the second component is $K_{(p-1),(q-1)}$.

This completes the proof. ∎

There are 11 non-isomorphic rings of order p^2 and the power graph of these rings have already described by Flagg in [6]. By [6, Corollary 3.1] and [6, Corollary 3.2], $\mathcal{P}(A_p) \cong \mathcal{P}(G_p)$, $\mathcal{P}(B_p) \cong \mathcal{P}(I_p)$, $\mathcal{P}(C_p) \cong \mathcal{P}(J_p)$ and $\mathcal{P}(E_p) \cong \mathcal{P}(F_p)$. Accordingly, it is sufficient to consider the rings A_p, B_p, C_p, D_p, E_p, H_p and K_p in order to investigate the tensor product of the power graphs of two rings of order p^2.

Theorem 2.2. Let R_q be a ring of order q^2. Then

$$\mathcal{C}(\mathcal{P}(A_p) \otimes \mathcal{P}(R_q)) \in \{2, 3, 5, 6, 8, 9, 11, 12, 16, p^2 + 2, p^2 + 4\}.$$

Proof . Our main proof will consider seven cases as follows:

1. $R_q \cong A_q$. We claim that the tensor product graph has five components with the following vertex sets:

$$\begin{aligned}
M_1 &= \{(0,0), (npa, mqa) \mid 1 \leq n \leq p-1,\ 1 \leq m \leq q-1\},\\
M_2 &= \{(npa, 0), (0, mqa) \mid\ 1 \leq n \leq p-1,\ 1 \leq m \leq q-1\},\\
M_3 &= \{(npa, ma) \mid\ m \in U(\mathbb{Z}_{q^2}),\ \ n \in \mathbb{N}\},\\
M_4 &= \{(ma, nqa) \mid\ m \in U(\mathbb{Z}_{p^2}), n \in \mathbb{N}\},\\
M_5 &= \{(m_1 a, m_2 a) \mid\ m_1 \in U(\mathbb{Z}_{p^2}),\ m_2 \in U(\mathbb{Z}_{q^2})\}.
\end{aligned}$$

To prove our claim, we first notice that $M_i \cap M_j = \varnothing$, $1 \leq i \neq j \leq 5$. Since in $\mathcal{P}(A_p)$ we have $(npa)^2 = 0$ and also all of the vertices of $U(\mathbb{Z}_{p^2})$ are just connected to some vertices in $U(\mathbb{Z}_{p^2})$, it is clear that M_i for $1 \leq i \leq 5$ composes a component. Note that if $p = q = 2$, one can easily check that each of sets M_1 and M_2 composes a component and other sets are split into two components. Also if $p = 2$ and $q \geq 3$, then M_4 split into two components and each of the other sets makes a component.

2. $R_q \cong B_q$. In this case, we have two components with the following vertex sets:

$$\begin{aligned} M_1 &= \{(u,v) \mid u \in U(\mathbb{Z}_{p^2}), \quad v \in V(\mathcal{P}(B_q))\}, \\ M_2 &= \{(u,v) \mid u \in V(\mathcal{P}(A_p)) - U(\mathbb{Z}_{p^2}), v \in V(\mathcal{P}(B_q))\}. \end{aligned}$$

It is clear that $M_1 \cap M_2 = \varnothing$. Since vertices in $U(\mathbb{Z}_{p^2})$ are in a component of $\mathcal{P}(A_p)$, the graph $\mathcal{P}(B_q)$ is a connected graph, every vertex $u \in V(\mathcal{P}(A_p)) - U(\mathbb{Z}_{p^2})$ has the form npa, where a is a generator of A_p and $n \in \mathbb{N}$. Moreover, there is an edge $0(npa)$ in $\mathcal{P}(A_p)$, where $1 \le n \le p-1$. Thus each of M_1 and M_2 makes a component.

3. $R_q \cong C_q$. We claim that $\mathcal{P}(A_p) \otimes \mathcal{P}(C_q)$ has three connected components as follows:

   (a) $K_{1,|A|(q^2-1)}$, where $A = \{npa \mid \ 1 \le n \le p-1\}$ in which a is a generator of A_p.

   (b) A complete bipartite graph in which one part is containing all vertices of the form $(0, ka')$, where $1 \le k \le q^2-1$ and a' is a generator of C_q. Another part will be the set of all $(npa, 0)$, $1 \le n \le p-1$.

   (c) A bipartite graph in which one part is

   $$\{(u,0) \mid u \in \mathcal{P}(A_p), u \neq npa, \quad 0 \le n \le p-1\}$$

   and another part is

   $$\{(u,ka') \mid u \in \mathcal{P}(A_p), u \neq npa, \quad 0 \le n \le p-1 \quad, 1 \le k \le q^2-1\}.$$

To prove our claim we note that $(npa)^2 = 0$ in $\mathcal{P}(A_p)$ and in $\mathcal{P}(C_q)$ the vertex 0 is connected to all other vertices of the form ka', for all $1 \le k \le q^2-1$. Thus, the graph has the edges $(npa, ka')(0,0)$ and $(0,ka')(npa,0)$, where $1 \le n \le p-1$. In $\mathcal{P}(A_p)$ all vertices that are not multiple of p are in one connected component that completes the assertion. The only exception in this case occurs when $p = q = 2$. This special case has the same components and the presented component of (c) is split into two components.

4. $R_q \cong D_q$. Suppose $\{a', b'\}$ is a generating set for D_q. It is clear that $\mathcal{P}(A_p) \otimes \mathcal{P}(D_q)$ has p^2 isolated vertices, since the vertex 0 of $\mathcal{P}(D_q)$ is an isolated vertex. Consider the set of vertices $\{(ka, ia') \mid 0 \le k \le p^2-1, 1 \le i \le q-1\}$. Then $(0, ia')$ is adjacent with (npa, ja') for all $1 \le n \le p-1$ and $1 \le j \le q-1$. So, these vertices compose a component except for some values of p and q presenting at the end of this case, the rest vertices of this set make another component corresponding to (a,a). The set $\{(ka, ia'+jb') \mid 0 \le k \le p^2-1, 1 \le i,j \le q-1\}$ of vertices is partitioned into two sets $\{(ma, ia'+jb') \mid m \in U(\mathbb{Z}_{p^2}), \ \ 1 \le i,j \le q-1\}$ and $\{(npa, ia'+jb') \mid 0 \le n \le p-1, 1 \le i,j \le q-1\}$ and each of them composes a component. One can easily check that the set

$\{(ka, ib') \mid 0 \leq k \leq p^2 - 1, 1 \leq i \leq q - 1\}$ also makes a component. Note that there are several exception in this case such that all of them have the same isolated vertices but some differences exist. Now we mention these exceptions. If $p = q = 2$ then the tensor product graph is a totally disconnected graph, and also if $p = 2$ and $q = 3$ then the second set mentioned above is broken up to four star graph $K_{1,3}$, also each of two other sets is partitioned into four components isomorphic to K_2.

5. $R_q \cong E_q$. Suppose $\{a', b'\}$ is a generating set for D_q. In this case, the tensor product graph has eleven components as follows:

    (a) Consider the set $\{(npa, ia' + jb') \mid 1 \leq n \leq p-1,\ \ 1 \leq i, j \leq q-1, i+j = q\}$. Since 0 is adjacent with npa in $\mathcal{P}(A_p)$, $1 \leq n \leq p-1$, and there are the edges $(ia' + jb')0$ in $\mathcal{P}(E_q)$, $1 \leq i, j \leq q-1$; $i + j = q$, this set of vertices only connected to the vertex $(0, 0)$. Hence we get a star $K_{1,|A||B|}$, where $A = \{npa \mid 1 \leq n \leq p-1\}$ is a subset of $V(\mathcal{P}(A_p))$ and $B = \{ia' + jb' \mid 1 \leq i, j \leq q-1, i+j = q\}$ is a subset of $V(\mathcal{P}(E_q))$.

    (b) It is easy to check that the vertex set $\{(npa, ia') \mid 1 \leq i \leq q-1, n \in \mathbb{N}\}$ makes a component.

    (c) In E_q the element b' has the same property as a', so we obtain a component from $\{(npa, ib') \mid 1 \leq i \leq q-1, n \in \mathbb{N}\}$.

    (d) The set $\{(0, ia' + jb'), (npa, 0) \mid 1 \leq n \leq p-1, 1 \leq i, j \leq q-1, i+j = q\}$ forms a component.

    (e) The set $\{(npa, ia' + ib') \mid 1 \leq i \leq q-1, n \in \mathbb{N}\}$ composes a component. Since in $\mathcal{P}(A_p)$ the vertex 0 is adjacent with npa, where $1 \leq n \leq p-1$ and on the other hand for every vertex $ia' + ib'$ in $\mathcal{P}(E_q)$, $1 \leq i \leq q-1$, we have $(ia' + ib')^2 = 2i^2a' + 2i^2b'$. According to the presentation of the ring E_q, if $1 \leq i \leq q-1$, then $1 \leq 2i^2 \leq q-1$ and so we can set $i' = 2i^2$. Thus in $\mathcal{P}(E_q)$ we have the edges $(ia' + ib')(i'a' + i'b')$; $1 \leq i, i' \leq q-1$.

    (f) The set $\{(ma, 0), (ma, ia' + jb') \mid m \in U(\mathbb{Z}_{p^2}), 1 \leq i, j \leq q-1, i+j = q\}$ makes a bipartite component with parts V_1 and V_2 such that $\mid V_1 \mid= t$ and $\mid V_2 \mid= t \mid A \mid$, where $A = \{ia' + jb' \mid 1 \leq i, j \leq q-1, i+j = q\}$ and $t =\mid U(\mathbb{Z}_{p^2}) \mid$.

    (g) The set $\{(ma, ka') \mid\ m \in U(\mathbb{Z}_{p^2}), 1 \leq k \leq q-1\}$ is a component.

    (h) Similar to the case (g) the set $\{(ma, kb') \mid\ m \in U(\mathbb{Z}_{p^2}), 1 \leq k \leq q-1\}$ makes an isomorphic component.

    (i) The set $\{(ma, ia' + ib') \mid m \in U(\mathbb{Z}_{p^2}), 1 \leq i \leq q-1\}$ is a component, since in $\mathcal{P}(A_p)$ if ma is adjacent to x, then $x = m'a$, where $m' \in U(\mathbb{Z}_{p^2})$. On the other hand, by using the same argument as in (e), one can show that $(ia' + ib')$ is adjacent to $(i'a' + i'b')$, where $1 \leq i, i' \leq q-1$.

    (j) The component corresponding to the set

$$\{(npa, ia' + jb') \mid\ \ 1 \leq i, j \leq q-1, i+j \neq q, i \neq j, n \in \mathbb{N}\}$$

is a bipartite graph with parts V_1 and V_2 such that $|V_1| = |A|$ and $|V_2| = |A|(|B|-1)$, where $A = \{ia' + jb' \mid 1 \leq i, j \leq q-1, i+j \neq q\}$ and $B = \{npa \mid n \in \mathbb{N}\}$.

(k) It is obvious that the vertices of the form $(ma, ia' + jb')$ where $m \in U(\mathbb{Z}_{p^2}), 1 \leq i \neq j \leq q-1, i+j \neq p$ make a component.

Note that in this case if $p = q = 2$, then the tensor product graph has 8 isolated vertices and four connected components isomorphic to K_2. If $p = q = 3$, then the tensor product graph has exactly nine components.

6. $R_q \cong H_q$ *and* $p = q = 2$. The graph only contains eight components isomorphic to K_2. Also if $p = 2$ and $q = 3$, then the connected component will be introduced in (6.d) splits into two connected components. For other values of p and q it is straightforward to check that one of the following cases will be occurred for the components of $\mathcal{P}(A_p) \otimes \mathcal{P}(H_q)$.

   (a) A star graph $K_{1,(q^2-1)}$ containing the vertex $(0, 0)$ and vertices (npa, ma') for all $1 \leq n \leq p-1$ and $1 \leq m \leq q-1$.

   (b) A complete bipartite graph corresponding to the set

   $$\{(npa, 0), (0, ma') \mid 1 \leq n \leq p-1,\ 1 \leq m \leq q-1\}$$

   of vertices.

   (c) The set $\{(np, v) \mid 0 \leq n \leq p-1, 0 \leq m \leq q-1, v \in V(\mathcal{P}(H_q)) - \{ma'\}\}$ makes a bipartite component.

   (d) For every $m \in U(\mathbb{Z}_{p^2})$, the vertices of the form $(ma, 0)$ is connected to the vertices of the form $(m'a, ka')$, where $m \neq m' \in U(\mathbb{Z}_{p^2}), 1 \leq k \leq q-1$. So, these vertices form a component.

   (e) The component corresponding to the set

   $$\{(u, v) \mid u \in U(\mathbb{Z}_{p^2}), 0 \leq m \leq q-1, v \in V(\mathcal{P}(H_q)) - \{ma'\}\}.$$

7. $R_q \cong K_q$. Then the tensor product graph contain p^2 isolated vertices and two other components.

This completes our argument. ■

Theorem 2.3. Let R_q be a ring of order q^2. Then

$$\mathcal{C}(\mathcal{P}(B_p) \otimes \mathcal{P}(R_q)) \in \{1, 2, 4, 5, 9, 16, p^2+1, p^2+3\}.$$

Proof . Suppose a is a generator of B_p. Our main proof will consider six cases as follows:

1. *$R_q \cong B_q$ and a' is a generator of B_q.* In this case, the graph vertices can be partitioned into the parts $M_i, 1 \le i \le 5$.

$$\begin{aligned}
M_1 &= \{(u,v) \mid u \in V(\mathcal{P}(B_p)), v \in V(\mathcal{P}(B_q)), v \neq tqa', t \in \mathbb{N}\},\\
M_2 &= \{(npa, mqa') \mid 0 \le n \le p-1, 0 \le m \le q-1\},\\
M_3 &= \{(u, mqa') \mid u \in V(\mathcal{P}(B_p)) - \{npa\}, 0 \le n \le p-1, 1 \le m \le q-1\}\}\\
M_4 &= \{(a \text{ } or \text{ } kpa, lqa') \mid 1 \le k \le p-1, 1 \le l \le q-1\},\\
M_5 &= \{(u, lqa') \mid u \in V(\mathcal{P}(B_p)) - \{a, kpa\}, 1 \le k \le p-1, 1 \le l \le q-1\},
\end{aligned}$$

where n and m are squares modulo p and q, respectively. Moreover, k and l are not squares modulo p and q, respectively. It is easy to check that $M_i \cap M_j = \varnothing$ and there are vertices $u_i \in M_i$ and $u_j \in M_j$ which are adjacent in $B_p \times B_q$, $1 \le i, j \le 5$. Hence this graph is connected.

2. *$R_q \cong C_q$.* In this case, $\mathcal{P}(B_p) \otimes \mathcal{P}(C_q)$ is a bipartite graph with parts V_1 and V_2 such that $|V_1| = p^2(q^2-1)$ and $|V_2| = p^2$. Hence this graph is connected.

3. *$R_q \cong D_q$ and $\{a', b'\}$ is a generating set for D_q.* In this case, we claim that the tensor product graph has p^2+3 components. Since 0 is an isolated vertex of $\mathcal{P}(D_q)$, the tensor product graph has p^2 isolated vertices. On the other hand, this graph has a bipartite component for $q \neq 2$, such that one part is containing all vertices of the form $(u, a'+b')$ in which $u \in V(\mathcal{P}(B_p))$ and another part is the set of all vertices of the form (u, v) such that $u \in V(\mathcal{P}(B_p))$ and $v \in V(\mathcal{P}(D_q)) - A$, where $A = \{0, ma', mb', a'+b' \mid \ 1 \le m \le q-1\}$ is a subset of $V(\mathcal{P}(D_q))$. Note that $\mathcal{P}(B_p)$ is a connected graph but it is not regular. Also the adjacent vertices to $a'+b' \in V(\mathcal{P}(D_q)$ are the group of unit elements of the ring D_q. These are all elements of the form $ia'+jb'$, $1 \le i, j \le q-1$, and so each vertex of $V(\mathcal{P}(B_p)) - A$ is adjacent to $a'+b'$. Therefore, this component is a non-complete bipartite subgraph. It is clear that the sets $\{(u, ma') \mid \ u \in V(\mathcal{P}(B_p)), 1 \le m \le q-1\}$ and $\{(u, mb') \mid \ u \in V(\mathcal{P}(B_p)), 1 \le m \le q-1\}$ are different components for the graph. Thus, we get exactly p^2+3 components. One can see that if $p = q = 2$ then the tensor product graph is a totally disconnected graph on sixteen vertices.

4. *$R_q \cong E_q$ and $\{a', b'\}$ is a generating set for E_q.* One can see that the graph has five connected components that two of them make from the vertices of the form (u, ma') and (u, mb'), respectively, where $m \in V(\mathcal{P}(B_p))$, $1 \le m \le q-1$. Three other components are corresponding to three set of vertices as follows:

$$\begin{aligned}
&\{(u, ia'+ib') \mid \ u \in V(\mathcal{P}(B_p)),\ 1 \le i \le q-1\},\\
&\{(u, ia'+jb') \mid \ u \in V(\mathcal{P}(B_p)),\ 1 \le i \neq j \le q-1, i+j \neq q\},\\
&\{(u,0), (u, ia'+jb') \mid \ u \in V(\mathcal{P}(B_p)),\ 1 \le i \neq j \le q-1, i+j = q\}.
\end{aligned}$$

If $p = q = 2$, then the graph has eight isolated vertices and just a component. If $p = 2$ and $q = 3$ this graph has four components since it does note have the component corresponding to

$$\{(u, ia'+jb') \mid \ u \in V(\mathcal{P}(B_p)),\ 1 \le i \neq j \le q-1, i+j \neq q\}.$$

5. $R_q \cong H_q$. Suppose that a' is a generator of H_q. In this case, the tensor product graph has two components which one of them is containing all vertices in the form $(u, 0)$ and (u, ma'); $u \in V(\mathcal{P}(B_p))$ and $1 \leq m \leq q-1$. The remaining vertices will make another component.

6. $R_q \cong K_q$. In this case, all vertices of the form $(u, 0)$ where $u \in V(\mathcal{P}(B_p))$ are isolated vertices of the tensor product graph, since 0 is an isolated vertex of $\mathcal{P}(K_q)$. These are p^2 isolated vertices. On the other hand, all of non-zero vertices are connected to each other in $\mathcal{P}(K_q)$, hence all the remaining vertices put together another component.

Hence the result. ■

Theorem 2.4. Let R_q be a ring of order q^2. Then

$$\mathcal{C}(\mathcal{P}(C_p) \otimes \mathcal{P}(R_q)) \in \{2, 3, 4, 6, 10, p^2+1, p^2+3, p^2+6, 4p^2\}.$$

Proof . Suppose a is a generator of the ring C_p. Our main proof will consider some cases as follows:

1. $R_q \cong C_q$. The tensor product graph has a star component isomorphic to $K_{1,3(n^2-1)}$ such that $n = max\{p, q\}$. Hence the vertex 0 in $\mathcal{P}(C_p)$ is adjacent to every other non-zero vertex. So, $(0, 0)$ is adjacent to all vertices in which the first and the second entries are non-zero. Therefore, we obtain a star and a complete bipartite component isomorphic to K_{p^2-1,q^2-1}. Note that all the non-zero vertices of $\mathcal{P}(C_p)$ are connected only with the vertex 0 and so the tensor product has the edges $(u, 0)(0, v)$, where $u, v \in V(\mathcal{P}(C_p)) - \{0\}$.

2. $R_q \cong D_q$. Choose a generating set $\{a', b'\}$ for D_q. The tensor product graph has p^2 isolated vertices and one can check that the set

   $$\{(u, ma') \mid \ u \in V(\mathcal{P}(C_p)),\ 1 \leq m \leq q-1\}$$

   makes a component. Since the elements a' and b' in D_q have the same properties, the set $\{(u, mb') \mid \ u \in V(\mathcal{P}(C_p)),\ 1 \leq m \leq q-1\}$ also makes a component isomorphic to last one. So far we do not have considered the vertices of the form $(u, ia'+jb')$, where $u \in V(\mathcal{P}(C_p))$ and $1 \leq i, j \leq q-1$. These vertices put together another component. In this case, if $q = 2$ then the graph is totally disconnected. If $q = 3$ then it is clear that the tensor product has p^2 isolated vertices and six components.

3. $R_q \cong E_q$. Suppose $\{a', b'\}$ is a generating set for D_q. This graph has six components as follows:

   (a) A star graph corresponding to the vertex $(0, 0)$.

   (b) A complete bipartite graph.

   (c) The subgraph induced by $\{(u, ma') \mid u \in V(\mathcal{P}(C_p)), 1 \leq m \leq q-1\}$.

(d) The subgraph induced by $\{(u, mb') \mid u \in V(\mathcal{P}(C_p)), 1 \leq m \leq q-1\}$.
(e) $\{(u, 0), (0, ia' + jb') \ \mid u \in V(\mathcal{P}(C_p)) - \{0\}, 1 \leq i, j \leq q-1, i+j = q\}$.
(f) $\{(u, ia' + jb') \mid u \in V(\mathcal{P}(C_p)), 1 \leq i, j \leq q-1, i \neq j, i+j \neq q\}$.

Note that if $p = q = 2$ then the graph has ten components.

4. $R_q \cong H_q$. There is a component corresponding to the vertex $(0, 0)$ that is adjacent to all other vertices of the form (na, ma'), where $1 \leq n \leq p^2 - 1$ and $1 \leq m \leq q-1$. Also the graph has two other components such that each of them can be induced by one of the following subsets:

(a) $\{(0, ma'), (na, 0) \mid 1 \leq m \leq q-1, 1 \leq n \leq p^2 - 1\}$.
(b) $\{(na, u) \mid u \in V(\mathcal{P}(H_q)) - \{0, ma\}, 0 \leq n \leq p^2 - 1, 1 \leq m \leq q-1\}$.

If $q = 2$ then the component corresponding to the part (b) will be divided into two new components.

5. $R_q \cong K_q$. All the vertices of the from $(u, 0)$ where $u \in V(\mathcal{P}(C_p))$ are isolated vertices and all of the remaining vertices make only a component.

This completes our argument. ■

Define:

$$T_1 = \{16, 18, 27, 33, p^2 + q^2 + 8, 4q^2, q^2 + 12, q^2 + 7, p^2 + q^2 + 2, 4p^2, q^2 + 9, 2p^2 + 7\}.$$

Theorem 2.5. Let R_q be a ring of order q^2. Then, $\mathcal{C}(\mathcal{P}(D_p) \otimes \mathcal{P}(R_q)) \in T_1$.

Proof . Suppose $\{a, b\}$ is a generating set for D_p. Our main proof will consider four cases as follows:

1. $R_q \cong D_q$. The vertices of the form $(u, 0)$ and $(0, v)$, $u \in V(\mathcal{P}(D_p))$ and $v \in V(\mathcal{P}(D_q))$, are isolated. It is straightforward to show that each of the set

$$\{(na, ma') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\},$$
$$\{(nb, mb') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\},$$
$$\{(na, mb') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\},$$
$$\{(nb, ma') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\},$$

induced a component. There are two other components corresponding to the sets

$$\{(a + b, ma') \mid 1 \leq m \leq q-1\},$$
$$\{(a + b, mb') \mid 1 \leq m \leq q-1\},$$

that each of them composes a star isomorphic to $K_{1,((p-1)^2-1)n}$, where $n = deg(ma') = deg(mb')$. On the other hand, there are two new components corresponding to the sets $\{(na, a' + b') \mid 1 \leq n \leq p-1\}$ and $\{(nb, a' + b') \mid 1 \leq n \leq p-1\}$. Obviously, after composing all these connected components all of the remaining vertices are adjacent to the vertex $(a+b, a'+b')$ which gives our final component. In this case, if one of p or q is equal to 2, then the tensor product is totally disconnected and it has $4q^2$ and $4p^2$ isolated vertices, respectively. Also if $p = q = 3$ then the graph has the same components as general case other than the connected component corresponding to the vertex $(a+b, a'+b')$ is broken into two components.

2. $R_q \cong E_q$. We first notice that the vertices of the form $(0, u)$, $u \in V(\mathcal{P}(E_q))$, are isolated vertices in $\mathcal{P}(D_q) \otimes \mathcal{P}(E_q)$. The non-isolated vertices of $\mathcal{P}(D_q) \otimes \mathcal{P}(E_q)$ can be divided into the following sets:

   (a) $\{(na, mb') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\}$,
   (b) $\{(na, ma') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\}$,
   (c) $\{(nb, ma') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\}$,
   (d) $\{(nb, mb') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\}$,
   (e) $\{(ia + jb, ma') \mid 1 \leq i, j \leq p-1, 1 \leq m \leq q-1\}$,
   (f) $\{(ia + jb, mb') \mid 1 \leq i, j \leq p-1, 1 \leq m \leq q-1\}$,
   (g) $\{(na, ia' + ib') \mid 1 \leq n \leq p-1, 1 \leq i \leq q-1\}$,
   (h) $\{(nb, ia' + ib') \mid 1 \leq n \leq p-1, 1 \leq i \leq q-1\}$,
   (i) $\{(na, 0) \mid 1 \leq n \leq p-1\}$,
   (j) $\{(nb, 0) \mid 1 \leq n \leq p-1\}$,
   (k) $\{(ia + jb, i'a' + i'b') \mid 1 \leq i, j \leq p-1, 1 \leq i' \leq q-1\}$,
   (l) $\{(ia + jb, 0), (ia + jb, i'a + j'b) \mid 1 \leq i, j \leq p-1, 1 \leq i', j' \leq q-1, i' + j' = q\}$.

   One can easily check that each of these subsets induce a component in the graph. The end component is bipartite with vertex bipartization

   $$\{(ia + jb, 0) \mid 1 \leq i, j \leq p-1\},$$
   $$\{(ia + jb, i'a + j'b) \mid 1 \leq i \neq j \leq p-1, 1 \leq i', j' \leq q-1, i' + j' = q\}.$$

   We now mention some exceptions in this case. If $p = 2$ then the graph is totally disconnected with $4q^2$ vertices. If $p \geq 3$ and $q = 2$ then the tensor product graph contains $4 + 2(p^2 - 1)$ isolated vertices that they are made from the sets $(a - f)$ in above list. Also, each of the next two subsets is broken into two connected components and the remaining vertices composes another connected component. If $p = q = 3$ then we have nine isolated vertices and all of the $(a - l)$ are partitioned into two components.

3. $R_q \cong H_q$. In this case, the tensor product graph has q^2 isolated vertices $(0, u)$, $u \in V(\mathcal{P}(H_q))$. Also, it has seven components corresponding to each of the following subsets:

   (a) $\{(na, u) \mid u \in V(\mathcal{P}(H_q)) - \{ma'\}, 1 \leq n \leq p-1\}, 0 \leq m \leq q-1$.

(b) $\{(nb,u) \mid 0 \neq u \in V(\mathcal{P}(H_q)) - \{ma'\}, 1 \leq n \leq p-1\}, 0 \leq m \leq q-1$.

(c) $\{(a+b,ma'),(ia+jb,0) \mid 1 \leq m \leq q-1, 1 \leq i,j \leq p-1\}$.

(d) $\{(a+b,0),(ia+jb,ma') \mid 1 \leq m \leq q-1, 1 \leq i,j \leq p-1\}$.

(e) $\{(ia+jb,u) \mid 1 \leq i,j \leq p-1, u \in V(\mathcal{P}(H_q)) - \{ma'\}, 0 \leq m \leq q-1$.

(f) $\{(na,ma') \mid 1 \leq n \leq p-1, 0 \leq m \leq q-1\}$.

(g) $\{(nb,ma') \mid 1 \leq n \leq p-1, 0 \leq m \leq q-1\}$.

Note that in this case if $p = 2$ then the tensor product graph is totally disconnected on $4q^2$ vertices and if $p = q = 3$ then all above arguments are valid just the sets (f) and (g) above are divided into two components.

4. $R_q \cong K_q$. The tenor product graph has $p^2 + q^2 + 1$ isolated vertices and also each of the set

(a) $\{(na,u) \mid 1 \leq n \leq p-1, u \in V(\mathcal{P}(H_q)) - \{0\}\}$,

(b) $\{(nb,u) \mid 1 \leq n \leq p-1, u \in V(\mathcal{P}(H_q)) - \{0\}\}$,

will induce a component. The remaining vertices of this graph compose only one another component, so we have p^2+q^2+2 components. If $p = q = 2$, then the tensor product graph is a totally disconnected graph on sixteen vertices.

This proves the theorem. ■

Theorem 2.6. Let R_q be a ring of order q^2. Then

$$\mathcal{C}(\mathcal{P}(E_p) \otimes \mathcal{P}(R_q)) \in \{8, 10, 12, 14, 20, 21, p^2+6, 2q^2+7\}.$$

Proof . Choose a generating set $\{a,b\}$ for E_p. The proof will consider four cases as follows:

1. $R_q \cong E_q$ *and* $p = q = 2$. In this case, we have twelve isolated vertices and two components isomorphic to K_2. If $p = 2$ and $q > 2$ then the tensor product graph has $2q^2$ isolated vertices and only seven components. For other values of p and q, we don't have isolated vertices and by using the graph structure of $\mathcal{P}(E_p)$, one can check that each of the following subsets induce a unique connected component of the graph:

(a) $\{(0,0),(ia+jb,i'a'+j'b') \mid 1 \leq i,j \leq p-1, 1 \leq i',j' \leq q-1, i+j = p, i'+j' = q\}$.

(b) $\{(ia+ib,ma') \mid 1 \leq i \leq p-1, 1 \leq m \leq q-1\}$.

(c) $\{(0,ma'),(ia+jb,ma') \mid 1 \leq i,j \leq p-1, i+j = p, 1 \leq m \leq q-1\}$.

(d) $(ia+ib,mb') \mid 1 \leq i \leq p-1, 1 \leq m \leq q-1\}$.

(e) $\{(0,mb'),(ia+jb,mb') \mid 1 \leq i,j \leq p-1, i+j = p, 1 \leq m \leq q-1\}$.

(f) $\{(ia+jb,0),(0,i'a'+j'b') \mid 1 \le i,j \le p-1, 1 \le i',j' \le q-1, i+j = p, i'+j' = q\}$, that is a complete bipartite component $K_{p-1,q-1}$.

(g) $\{(ia+ib, i'a'+i'b')) \mid, 1 \le i \le p-1, 1 \le i' \le q-1\}$, that forms a bipartite component, with parts V_1 and V_2 such that $|V_1| = |V_2| = t$, where $t = \frac{(p-1)(q-1)}{2}$.

(h) $\{(i'a+j'b, ia'+jb'),(0, ia'+jb') \mid 1 \le i \ne j \le q-1, 1 \le i',j' \le p-1, i+j \ne q, i'+j' = p; deg(ia'+jb') > deg(ja'+ib') \; or \; deg(ia'+jb') = deg(ja'+ib'), i < j\}$.

(i) $\{(na, ia'+ib') \mid 1 \le n \le p-1, 1 \le i \le q-1\}$.

(j) $\{(na, ia'+jb') \mid 1 \le n \le p-1, 1 \le i \ne j \le q-1, i+j \ne q\}$.

(k) $\{(nb, ia'+ib') \mid 1 \le n \le p-1, 1 \le i \le q-1\}$.

(l) $\{(nb, ia'+jb') \mid 1 \le n \le p-1, 1 \le i \ne j \le q-1, i+j \ne q\}$.

(m) $\{(na, mb') \mid 1 \le n \le p-1, 1 \le m \le q-1\}$.

(n) $\{(nb, mb') \mid 1 \le n \le p-1, 1 \le m \le q-1\}$.

(o) $\{(na, ma') \mid 1 \le n \le p-1, 1 \le m \le q-1\}$.

(p) $\{(nb, ma') \mid 1 \le n \le p-1, 1 \le m \le q-1\}$.

(q) $\{(na, ia'+jb') \mid 1 \le n \le p-1, 1 \le i \ne j \le q-1\}$.

(r) $\{(ia+ib, i'a'+j'b') \mid 1 \le i \le p-1, 1 \le i' \ne j' \le q-1, i'+j' \ne q\}$.

(s) $\{(ia+jb,0),(ia+jb, i'a'+j'b') \mid 1 \le i,j \le p-1, 1 \le i',j' \le q-1, i+j \ne p, i'+j' = q\}$.

(t) $\{(na,0),(na, ia'+jb') \mid 1 \le n \le p-1, 1 \le i \ne j \le q-1, i+j = q\}$.

(u) $\{(nb,0),(nb, ia'+jb') \mid 1 \le n \le p-1, 1 \le i \ne j \le q-1, i+j = q\}$.

Therefore, we have twenty one connected components.

2. $R_q \cong H_q$ *and* $p = q = 2$. In this case, the graph has exactly eight isolated vertices, since the vertices a and b are not adjacent in $\mathcal{P}(E_2)$. Also, this graph has four components isomorphic to K_2. For $p = 2$ and $q = 3$ it is easy to show that the graph has eighteen isolated vertices and three other components. For other values of p and q we have a star component corresponding to the vertex $(0,0)$. This vertex is adjacent to all vertices of the form $(ia+jb, ma')$, where $1 \le i,j \le p-1$, $i+j = p$ and $1 \le m \le q-1$. Also, it has a bipartite component containing all vertices of the form $(ia+jb, u) \cup (0,u)$, where $1 \le i,j \le p-1$, $1 \le m \le q-1$ and u is a non-zero elements of H_q such that $u \ne ta', 1 \le t \le q-1$. Each of the following sets composes a component:

(a) $\{(ia+jb, ma') \mid 1 \le i,j \le p-1, j+j \ne p, 1 \le m \le q-1\}$

(b) $\{(0, ma'),(ia+jb,0) \mid 1 \le i,j \le p-1, 1 \le m \le q-1, i+j = p\}$

(c) $\{(na, u) \mid 1 \le n \le p-1, 0 \le m \le q-1, u \in V(\mathcal{P}(H_q)) - \{ma'\}\}$

(d) $\{(na, ma') \mid 1 \le n \le p-1, 0 \le m \le q-1\}$

(e) $\{(nb, ma') \mid 1 \leq n \leq p-1, 0 \leq m \leq q-1\}$

(f) $\{(nb, u) \mid 1 \leq n \leq p-1, 0 \leq m \leq q-1, u \in V(\mathcal{P}(H_q)) - \{ma'\}\}$

Therefore, the graph has exactly eight connected components.

3. $R_q \cong K_q$. In this case, we have p^2 isolated vertices and six components corresponding to the following subsets:

(a) $\{(0, u), (ia + jb, u) \mid 1 \leq i, j \leq, i + j = p\}$.

(b) $\{(na, u) \mid 1 \leq n \leq p-1\}$.

(c) $\{(nb, u) \mid 1 \leq n \leq p-1\}$.

(d) $\{(ia + ib, u) \mid 1 \leq i \leq p-1\}$.

(e) $\{(ia + jb, u) \mid 1 \leq i \neq j \leq p-1, i + j \neq p\}$.

(f) $\{(ja + ib, u) \mid (ia + jb, u) \in (e)\}$,

where $u \in V(\mathcal{P}(K_q)) - \{0\}$. If $p = 2$ and $q = 2$ or 3 then the graph has exactly 10 or 20 connected components, respectively.

Hence the result. ■

Theorem 2.7. Let R_q be a ring of order q^2 and p be a prime. Then

$$\mathcal{C}(\mathcal{P}(H_p) \otimes \mathcal{P}(R_q)) \in \{5, 6, 8, p^2 + 2\}.$$

Proof . Choose the generating set $\{a, b\}$ for H_p. It is enough to consider two cases that $R_q \cong H_q$ or $R_q \cong K_q$.

1. $R_q \cong H_q$. Consider the following subsets of $H_p \otimes R_q$:

(a) $\{(0, 0), (na, ma') \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\}$,

(b) $\{(u, ma') \mid 0 \leq m \leq q-1\}$,

(c) $\{(u, v)\}$,

(d) $\{(na, v) \mid 0 \leq n \leq p-1\}$,

(e) $\{(0, ma'), (na, 0) \mid 1 \leq n \leq p-1, 1 \leq m \leq q-1\}$.

It is easy to see that each of these subset are connected components of $\mathcal{C}(\mathcal{P}(H_p) \otimes \mathcal{P}(R_q))$. In each of the cases that $p = q = 2$ and $p = 2, q = 3$ we have the same components that are composed of the set of vertices given in parts (a) and (e) but in the first case each of other set of vertices $(b - d)$ is partitioned into two components isomorphic to K_2 and in the second case the set of vertices in (b) contains two components.

2. $R_q \cong K_q$. We can see that $\mathcal{P}(H_p) \otimes \mathcal{P}(K_q)$ has p^2 isolated vertices and two components corresponding to the subsets:

(a) $\{(na, u) \mid 0 \leq n \leq p-1, u \in V(\mathcal{P}(K_q)) - \{0\}\}$.

(b) $\{(u, v) \mid u \in V(\mathcal{P}(H_p)) - \{na\}, 0 \leq n \leq p-1, v \in V(\mathcal{P}(K_q)) - \{0\}\}$.

Therefore, the graph has exactly five, six, eight or $p^2 + 2$ connected components.

This proves the result. ■

Theorem 2.8. Let R_q be a ring of order q^2. Then $\mathcal{C}(\mathcal{P}(K_p) \otimes \mathcal{P}(K_q)) = p^2 + q^2$.

Proof . The proof follows from analyzing the graph $\mathcal{P}(K_p)$. ■

3. Diameter and Girth

In Section 2, some information on the connectivity of the tensor product of the power graphs of some ring of order p^2 were given. In this section our purpose is to obtain diameter of these graphs when they are connected. Furthermore, we will obtain the girth of $\mathcal{P}(R_p) \otimes \mathcal{P}(R_q)$.

Theorem 3.1. Let R_p and R_q be two rings of order p^2 and q^2, respectively. Then the graph $\mathcal{P}(R_p) \otimes \mathcal{P}(R_q)$ is connected if and only if $R_p \cong B_p$ and $R_q \cong B_q$, C_q. Moreover, $diam(\mathcal{P}(B_p) \otimes \mathcal{P}(B_q)) = 3$ and $diam(\mathcal{P}(B_p) \otimes \mathcal{P}(C_q)) = 4$.

Proof . The first part is a direct consequence of Theorems 2.2-2.8. Suppose that $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are two distinct vertices of the graph $\mathcal{P}(B_p) \otimes \mathcal{P}(B_q)$. With notations as in Theorem 2.3, we have some different cases and in each case we compute $d(u, v)$. In all of the following cases we will introduce the shortest path between u and v in $\mathcal{P}(B_p) \otimes \mathcal{P}(B_q)$.

1. If $u, v \in M_1$, then we can assume that there are $1 \leq m_1, m_2, k_1, k_2 \leq q-1$ such that $y_1 = (m_1 + k_1 q)a'$ and $y_2 = (m_2 + k_2 q)a'$. Since there are not different vertices in the form $(m + kq)a$, $1 \leq m, k \leq q-1$, that they are connected in $\mathcal{P}(B_q)$, (x_1, y_1) is not adjacent to (x_2, y_2). Thus $1 < d(u, v)$ and we can proceed based on this fact that whether or not x_1 is adjacent with x_2 in $\mathcal{P}(B_p)$. We first assume that x_1 is adjacent with x_2 in $\mathcal{P}(B_p)$. Thus $x_1 = 0$ or $x_2 = 0$. Suppose $x_1 = 0$ and choose $1 \leq n, m, k \leq p-1, 1 \leq k' \leq q-1$. We consider some different cases as follows:

   (a) $x_2 = kpa$, *where k is square modulo p.* We consider the path $u = (x_1, y_1) = (0, (m_1 + k_1 q)a'), ((m + np)a, k'qa'), (kpa, (m_2 + k_2 q)a') = (x_2, y_2) = v$ of length two, where k' is a square modulo q.

   (b) $x_2 = kpa$, *where k is not square modulo p.* It is enough to consider the path $u = (x_1, y_1) = (0, (m_1 + k_1 q)a'), (t, 0), (0, k'qa'), (kpa, (m_2 + k_2 q)a') = (x_2, y_2) = v$ of length three, where $t \in V(\mathcal{P}(B_p)) - \{0\}$ and k' is a square modulo q.

(c) $x_2 = (m + np)a$. In this case the path $u = (x_1, y_1) = (0, (m_1 + k_1q)a')$, $(kpa, k'qa')$, $((m + np)a, (m_2 + k_2q)a') = (x_2, y_2) = v$ has length two, where k and k' are squares modulo p and q, respectively.

Suppose $x_2 = 0$. By a similar method and a case by case investigation, one can see that $d(u, v) \in \{2, 3\}$. We now assume that x_1 is not adjacent to x_2. It is clear that $x_1 \neq 0$ and $x_2 \neq 0$. Choose $1 \leq n, m, k, n', m', k' \leq p - 1$ and $1 \leq k'' \leq q - 1$, k'' is square modulo q, based on the following cases:

(a) *$x_1 = (m + np)a$ and $x_2 = kpa$, k is not square modulo p.* The path $u = (x_1, y_1) = ((m+np)a, (m_1+k_1q)a'), (0, k''qa'), (kpa, (m_2+k_2q)a') = (x_2, y_2) = v$ has length two.

(b) *$x_1 = kpa$, k is not square modulo p and $x_2 = (m + np)a$.* The path $u = (x_1, y_1) = (kpa, (m_1+k_1q)a'), (0, k''qa'), ((m+np)a, (m_2+k_2q)a') = (x_2, y_2) = v$ has length two.

(c) *$x_1 = kpa$, k is square modulo p and $x_2 = k'pa$, k' is not square modulo p.* The path $u = (x_1, y_1) = (kpa, (m_1 + k_1q)a'), (0, k''qa'), (k'pa, (m_2 + k_2q)a') = (x_2, y_2) = v$ has length two.

(d) *$x_1 = kpa$, k is not square modulo p, and $x_2 = k'pa$, k' is a square modulo p.* It can be easily seen that the path $u = (x_1, y_1) = (kpa, (m_1 + k_1q)a')$, $(0, k''qa')$, $(k'pa, (m_2 + k_2q)a') = (x_2, y_2) = v$ has length two.

(e) *$x_1 = (m + np)a$ and $x_2 = (m' + n'p)a$.* It is easy to see that the path $u = (x_1, y_1) = ((m+np)a, (m_1+k_1q)a')$, $(k'pa, k''qa')$, $((m'+n'p)a, (m_2+k_2q)a') = (x_2, y_2) = v$ has length three, where k' is a square modulo p.

(f) *$x_1 = kpa$, and $x_2 = k'pa$, k and k' are squares modulo p and q, respectively.* The path $u = (x_1, y_1) = (kpa, (m_1 + k_1q)a'), (0$ *or* $(m + np)a, k''qa'), (k'pa, (m_2 + k_2q)a') = (x_2, y_2) = v$ has length two.

(g) *$x_1 = kpa$, and $x_2 = k'pa$, k and k' are not squares modulo p and q, respectively.* The path $u = (x_1, y_1) = (kpa, (m_1 + k_1q)a')$, $(0, k''qa')$, $(k'pa, (m_2 + k_2q)a') = (x_2, y_2) = v$ has length two.

2. $u, v \in M_2$. A similar argument as in the Case 1 shows that $d(u, v) \in \{1, 2\}$.

3. $u, v \in M_4$. Since $d(u, v) > 1$, the only case that can be occurred is the case that in $\mathcal{P}(B_p)$, x_1 is not adjacent to x_2 and in $\mathcal{P}(B_q)$, x_2 is not adjacent to y_2. Choose the path $(x_1, y_1), (0, 0), (x_2, y_2)$ of length two to prove that $d(u, v) = 2$.

Note that a similar argument for the remaining cases shows that $d(u, v) \in \{1, 2, 3\}$ and so $diam(\mathcal{P}(B_p) \otimes \mathcal{P}(B_q)) = 3$. We now return to determine $diam(\mathcal{P}(B_p) \otimes \mathcal{P}(C_q))$. By Theorem 2.3, the graph $\mathcal{P}(B_p) \otimes \mathcal{P}(C_q)$ is bipartite and the parts V_1 and V_2 are defined as the set of all vertices of the form $(V(\mathcal{P}(B_p)), 0)$ and $(V(\mathcal{P}(B_p)), V(\mathcal{P}(C_q)) - \{0\})$, respectively. It is obvious that if u and v are in different parts, then $d(u, v) \in \{1, 3\}$. If $u, v \in V_1$, then one can see that $d(u, v) \in \{2, 4\}$. Assume that $u, v \in V_2$, where $u = (u_1, v_1)$ and $v = (u_2, v_2)$ such that u_1 and u_2 are non-zero. Then the path $(u_1, v_1), (0, 0), (u_2, v_2)$, connecting u and v

is a shortest path in this case. If u_1 and u_2 are zero, then we consider the path $(u_1, v_1), (kpa, 0), (u_2, v_2)$, where $1 \leq k \leq p-1$ and k is a square modulo p and so $d(x, y) = 2$. Therefore, $diam(\mathcal{P}(B_p) \otimes \mathcal{P}(C_q)) = 4$. ■

Theorem 3.2. Let R_q be a ring of order q^2. Then $gr(\mathcal{P}(A_p)\otimes\mathcal{P}(R_q)) \in \{3, 4, 6, \infty\}$.

Proof . Apply Theorems 2.3-2.8. We have the following separate cases:

1. $R_q \cong A_q$ *and* $p, q \neq 2$. It is clear that we have the cycle (u, v), (x, y), (x^{-1}, y^{-1}), (u, v), where $u \in U(\mathbb{Z}_{p^2})$, $v \in U(\mathbb{Z}_{q^2})$, x is a generator of $U(\mathbb{Z}_{p^2})$ and y is a generator of $U(\mathbb{Z}_{q^2})$. Thus, the girth of the graph is 3.

2. $R_q \cong B_q$ *and* $p, q \neq 2$. In this case let $u \in U(\mathbb{Z}_{p^2})$, v be a generator of $U(\mathbb{Z}_{p^2})$ and $1 \leq m, k \leq q-1$. Then we will have the following cycles:

   (a) $(u, kqa'), (v, 0), (v^{-1}, (m + kq)a'), (u, kqa')$,

   (b) $(u, 0), (v, kqa'), (v^{-1}, (m + kq)a'), (u, 0)$,

   (c) $(u, (m + kq)a'), (v, kqa'), (v^{-1}, 0), (u, (m + kq)a')$.

   Hence the girth of the graph will be 3.

3. $R_q \cong D_q$, $p \neq 2$ *and* $p, q \neq 3$. In this case let $u \in U(\mathbb{Z}_{p^2})$, v be a generator of $U(\mathbb{Z}_{p^2})$ and $1 \leq n, m, k \leq q-1$. Then the shortest cycles have one of the following forms:

   (a) $(u, na'), (v, ma'), (v^{-1}, ka'), (u, na')$,

   (b) $(u, nb'), (v, mb'), (v^{-1}, kb'), (u, nb')$.

   So, the girth of the graph is 3.

4. $R_q \cong E_q$, $p \neq 2$ *and* $q \neq 2, 3$. Let $u \in U(\mathbb{Z}_{p^2})$, v be a generator of $U(\mathbb{Z}_{p^2})$ and $1 \leq n, m, k \leq q-1$. Then the shortest cycles have one of the following forms:

   (a) $(u, na'), (v, ma'), (v^{-1}, ka'), (u, na')$,

   (b) $(u, nb'), (v, mb'), (v^{-1}, kb'), (u, nb')$,

   and so the girth is equal to 3.

5. $R_q \cong H_q$ *and* $p, q \neq 2$. Let $u \in U(\mathbb{Z}_{p^2})$, v be a generator of $U(\mathbb{Z}_{p^2})$. Then the cycle $(u, 2b'), (v, b'), (v^{-1}, 2a' + 2b'), (u, 2b')$ has the shortest length and so the girth is 3.

6. $R_q \cong K_q$ *and* $p \neq 2$. One can see that the cycle $(u, a'+b'), (v, a'), (v^{-1}, b'), (u, a'+b')$ has the minimum length. Thus the girth is 4.

We now present the cases that $gr(\mathcal{P}(A_p) \otimes \mathcal{P}(R_q)) = 4$.

1′ $R_q \cong A_q$, $p = 2$ *and* $q \neq 2$. The cycle $(0, u), (2a, v), (0, a'), (2a, v^{-1}), (0, u)$ has the shortest length, where $u \in U(\mathbb{Z}_{q^2})$ and v is a generator of $(U(\mathbb{Z}_{q^2}), \times)$.

2′ $R_q \cong A_q$, $q = 2$ *and* $p \neq 2$. Note that $(u, 0), (v, 2a'), (a, 0), (v^{-1}, 2a'), (u, 0)$ is a shortest cycle for the graph.

3′ $R_q \cong C_q$, $p \neq 2$. It is enough to consider the following cycles:

(a) $(u, 0)$, (v, w), $(v^{-1}, 0)$, $(v, z), (u, 0)$,
(b) $(kpa, 0)$, $(0, w)$, $(k'pa, 0)$, $(0, z)$, $(kpa, 0)$,
(c) (u, w), $(v, 0)$, (a, z), $(v^{-1}, 0), (u, w)$,

where $u \in U(\mathbb{Z}_{p^2})$, v is a generator of $(U(\mathbb{Z}_{p^2}), \times)$, $w, z \in V(\mathcal{P}(C_q)) - \{0\}$, $w \neq z$ and $1 \leq k \neq k' \leq p - 1$.

4′ $R_q \cong D_q$, $p \neq 2$ *and* $q = 3$. It is enough to choose the shortest cycle (u, a'), $(v, 2a')$, (a, a'), $(v^{-1}, 2a'), (u, a')$, where $u \in U(\mathbb{Z}_{p^2}) - \{a\}$ and v is a generator of $(U(\mathbb{Z}_{p^2}), \times)$.

5′ $R_q \cong E_q$, $p \neq 2$ *and* $q = 3$. The cycles:

(a) (u, a'), $(v, 2a')$, (a, a'), $(v^{-1}, 2a')$, (u, a'),
(b) (u, b'), $(v, 2b')$, (a, b'), $(v^{-1}, 2b')$, (u, b'),

where $u \in U(\mathbb{Z}_{p^2}) - \{a\}$, v is a generator of $U(\mathbb{Z}_{p^2})$, have length 4 and they are the shortest cycles.

6′ $R_q \cong E_q$, $p \neq 2$ *and* $q = 2$. A shortest cycle for the graph is $(u, 0)$, $(v, a' + b')$, $(a, 0)$, $(v^{-1}, a' + b'), (u, 0)$, as desired.

7′ $R_q \cong H_q$, $p \neq 2$ *and* $q = 2$. The cycle $(u, 0), (v, a'), (a, 0), (v^{-1}, a'), (u, 0)$ has the shortest length.

8′ $R_q \cong B_q$ *and* $p = 2$. The result follows from the fact that (a, qa'), $(3a, a')$, $(a, 0)$, $(3a, (q + 1)a'), (a, qa')$ is a shortest cycle of length 4.

9′ $R_q \cong C_q$ *and* $p, q \neq 2$. By Theorem 2.2, the graph $\mathcal{P}(A_p) \otimes \mathcal{P}(C_q)$ has at least one complete bipartite component and so the girth of this graph is 4.

10′ $R_q \cong H_q$ *and* $p = 2$. In this case, a shortest cycle for the graph is (a, b'), $(3a, a' + 2b')$, $(a, 2b'), (3a, 2a' + 2b'), (a, b')$.

Finally if $R_q \cong K_q$ and $p = q = 2$, then the cycles:

1. $(0, a')$, $(2a, a' + b')$, $(0, b')$, $(2a, a')$, $(0, a' + b')$, $(2a, b')$, $(0, a')$
2. (a, a'), $(3a, a' + b')$, (a, b'), $(3a, a')$, $(a, a' + b')$, $(3a, b')$, (a, a'),

are the shortest cycles of length 6. In the remaining cases, the graph is acyclic which completes the proof. ■

Theorem 3.3. Let R_q be a ring of order q^2. Then $gr(\mathcal{P}(B_q)\otimes\mathcal{P}(R_q)) \in \{3,4,\infty\}$.

Proof . The proof runs as Theorem 3.2. We first note that $(0,0)$, (a,a), (pa,qa), $(0,0)$ is a triangle in $\mathcal{P}(B_p)\otimes\mathcal{P}(B_q)$ and so $gr(\mathcal{P}(B_p)\otimes\mathcal{P}(B_q)) = 3$. Also, by [2, Theorem 1] we have $gr(\mathcal{P}(B_p)\otimes\mathcal{P}(C_q)) \in \{4,6,8\}$, but we have a square $(0,0)$, (a,a), $(pa,0)$, (a,qa), $(0,0)$ in the graph. Thus $gr(\mathcal{P}(B_p)\otimes\mathcal{P}(C_q)) = 4$. By Theorem 2.3, it is straightforward to see that $gr(\mathcal{P}(B_2)\otimes\mathcal{P}(D_2)) = \infty$ and for other values of p and q, $gr(\mathcal{P}(B_p)\otimes\mathcal{P}(D_q)) = 4$, since $(0,a'), (a,(q-1)a'), (pa,a'), ((p+1)a,(q-1)a'), (0,a')$ is a shortest cycle for the graph. Furthermore, $gr(\mathcal{P}(B_2)\otimes\mathcal{P}(E_2)) = 3$ and by Theorem 2.3, for other values of p and q we have the cycle $(kpa, ia'+ib')$, $((m+np)a, a'+b')$, $(0, i'a'+i'b'), (k'pa, ia'+jb')$, where $1 \le m,n,k,k' \le p-1$, $2 \le i,\, i' \le q-1$ and k is a square modulo p. In $\mathcal{P}(B_2)\otimes\mathcal{P}(H_2)$, we have the cycle $(a,a'), (2a,0), (3a,a'), (0,0), (a,a')$ and so $gr(\mathcal{P}(B_2)\otimes\mathcal{P}(H_2)) = 4$. For other values of p and q, we have $gr(\mathcal{P}(B_p)\otimes\mathcal{P}(H_q)) = 3$, since $(a,b'), (pa,2b'), (0,2a'+2b'), (a,b')$ is a cycle in the graph. Finally, let $1 \le m,k,k' \le p-1$ such that k' be a square modulo p. Then the cycle $(0,b'), (k'pa,a'), ((m+kp)a, a'+b'), (0,b')$ is a triangle in $\mathcal{P}(B_p)\otimes\mathcal{P}(K_q)$, which proves that $gr(\mathcal{P}(B_p)\otimes\mathcal{P}(K_q)) = 3$. ■

Theorem 3.4. Let R_q be a ring of order q^2. Then $gr(\mathcal{P}(C_p)\otimes\mathcal{P}(R_q)) \in \{4,\infty\}$.

Proof . By Theorem 2.4, it is easy to prove that if $q = 2$, then $gr(\mathcal{P}(C_p)\otimes\mathcal{P}(D_2)) = gr(\mathcal{P}(C_p)\otimes\mathcal{P}(H_2)) = \infty$ and if $p = q = 2$, then $gr(\mathcal{P}(C_p)\otimes\mathcal{P}(E_q)) = \infty$. Again by Theorem 2.4 and using the method of Theorem 3.3, we can show that in the remaining cases $gr(\mathcal{P}(C_p)\otimes\mathcal{P}(R_q)) = 4$. ■

Theorem 3.5. Let R_q be a ring of order q^2. Then $gr(\mathcal{P}(S_p)\otimes\mathcal{P}(R_q)) \in \{4,6,\infty\}$, where $S_p \cong D_p, E_p$ or H_p and $R_q \cong D_q, E_q, H_q$ or K_q. Moreover $gr(\mathcal{P}(K_p)\otimes\mathcal{P}(K_q)) = 3$.

Proof . In view of Theorems 2.5, 2.6 and 2.7, it is clear that if ($S_p \cong D_p$, $R_q \cong D_q, K_q$ and $p = 2$ *or* $q = 2$), ($S_p \cong E_p$, $R_q \cong E_q, H_q$ and $p = q = 2$) and finally ($S_p \cong H_p$, $R_q \cong H_q$ and $p = q = 2$), then $gr(\mathcal{P}(S_p)\otimes\mathcal{P}(R_q)) = \infty$ and also $gr(\mathcal{P}(D_2)\otimes\mathcal{P}(H_q)) = \infty$. Also, there is a cycle of length 6 in $\mathcal{P}(E_2)\otimes\mathcal{P}(K_2)$. Moreover, $\mathcal{P}(H_p)\otimes\mathcal{P}(H_q)$, when $p \neq 2$ and $q \neq 2$ has the girth 4. By the same way in other cases, we have a cycle of length 4 or 6. To prove the second part, it is enough to consider the triangle $(a+b, a'+b'), (b,a'), (a,b'), (a+b, a'+b')$. ■

4. Concluding Remarks

In this paper the number of connected components in the tensor product of the power graphs of some finite rings were computed. We apply our results to calculate the diameter of all such graphs when they are connected. Moreover, the girth of these graphs are also computed.

In the end of this paper, we suppose that p, q are primes and R_p, R_q denote arbitrary rings of order p^2 and q^2, respectively. Then we claim that $\mathcal{P}(R_p \times R_q) \subseteq \mathcal{P}(R_p) \otimes \mathcal{P}(R_q)$. To do this, we first note that for every edge $(a,b)(c,d) \in E(\mathcal{P}(R_p \times R_q))$, there exists $n \in \mathbb{N}$ such that $(a,b)^n = (c,d)$ or there exists $m \in \mathbb{N}$ such that $(c,d)^m = (a,b)$. Therefore, $a^n = c$, $b^n = d$ or $c^m = a$, $d^m = b$. Then $ac \in E(\mathcal{P}(R_p)), bd \in E(\mathcal{P}(R_q))$, which shows that $(a,b)(c,d) \in \mathcal{P}(R_p) \otimes \mathcal{P}(R_q)$. The Figures 1 and 2, present a counterexample which proves that another conclusion does not hold in general.

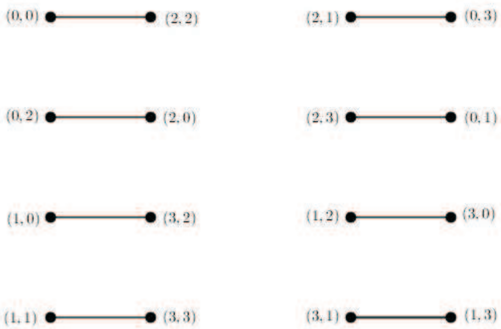

Figure 1. $\mathcal{P}(A_2) \otimes \mathcal{P}(A_2)$

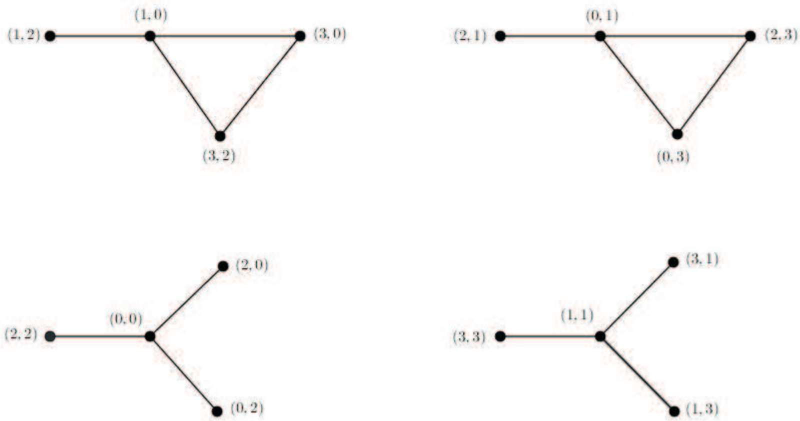

Figure 2. $\mathcal{P}(A_2 \times A_2)$

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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO A FIRST-ORDER DIFFERENTIAL EQUATION VIA FIXED POINT THEOREM IN ORTHOGONAL METRIC SPACE

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Abstract. In this paper we provide new and simple proofs for the classical existence and uniqueness theorems of solutions to the first-order differential equation using the fixed point theorem in an orthogonal metric space.
Keywords: Fixed point; Differential equation; Existence; Uniqueness; Solution; Orthogonal set.

1. Introduction

Let us consider the differential equation

$$\dot{x}(t) = v(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $v(t, x)$ is defined and differentiable (of class $C^r, r \geq 1$) in a domain U of $\mathbb{R} \times \mathbb{R}^n$.
The solution to this equation will be a function $\phi : \mathbb{R} \to \mathbb{R}^n$ where

$$\dot{\phi}(t) = v(t, \phi(t)), \quad \phi(t_0) = x_0. \tag{1.2}$$

The existence and uniqueness of solutions to first-order differential equations with given initial conditions are some of the most fundamental results of ordinary differential equations. This is stated in the two following theorems.

Theorem 1.1. [8] *(The Existence Theorem) Suppose the right-hand side v of the differential equation $\dot{x}(t) = v(t, x)$ is continuously differentiable in a neighborhood of the point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then there exists a neighborhood of the point t_0 such that a solution of the differential equation is defined in this neighborhood with the initial condition $\phi(t_0) = x_0$, where x is any point sufficiently close to x_0. Moreover, this solution depends continuously on the initial point x.*

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Theorem 1.2. *[8](The Uniqueness Theorem) Given the above conditions, there is only one possible solution for any given initial point, in the sense that all possible solutions are equal in the neighborhood under consideration.*

Previous studies have provided proofs of Theorems 1.1 and 1.2 using the concepts of Banach contraction principle [1, 7, 8], [12] and [16, 15].
Recently, M. Eshaghi et.al. [13] introduced the concept of orthogonal sets. A real extension of Banach contraction principle in orthogonal metric space has been considered in [13] (see also [9, 10, 19]). In this paper, we are interested in obtaining new and simple proofs for Theorems 1.1 and 1.2 which guarantee existence and uniqueness of the solution for any equation of the form (1.1).
This paper is organized as follows: In section 2, we state some definitions and theorems which are needed to prove the main results. Also, we recall under what conditions will any mapping on an orthogonal metric space have a unique fixed point. In section 3, we consider new concepts of tangent space to an orthogonal metric space and derivative of mapping at a point in an orthogonal metric space. This section provides a priori bound for the solution. In this section, we make use of the standard tools of the fixed point theory in orthogonal metric spaces to obtain new and simple proofs for existence and uniqueness theorems of solutions for the differential equation (1.1).

2. Preliminary definitions

First, we begin with the following definition which can be considered as the main definition of [13].

Definition 2.1. [13] Let $M \neq \phi$ and $\perp \subseteq M \times M$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0;\ ((\forall y; y\perp x_0) \quad or \quad (\forall y; x_0\perp y)),$$

it is called an orthogonal set (briefly O-set). We denote this O-set by $(M, \perp)$ (see also [9, 10, 19]).

We now give some examples of orthogonal sets.

Example 2.1. Let $M = [2, \infty)$, we define $x\perp y$ if $x \leq y$, then by putting $x_0 = 2$, $(M, \perp)$ is an O-set.

In the following example, we can see that x_0 is not necessarily unique.

Example 2.2. Suppose $\mathcal{M}(n)$ is the set of all $n \times n$ matrices and Q is a positive definite matrix. Define the relation $\perp$ on $\mathcal{M}(n)$ by

$$A\perp B \iff \exists X \in \mathcal{M}(n) \ ; \quad AX = B.$$

It is easy to see that $I\perp B$, $B\perp 0$ and $Q^{\frac{1}{2}}\perp B$ for all $B \in \mathcal{M}(n)$.

Now, we turn our consideration to the definition of orthogonal sequence.

Definition 2.2. [13] Let $(M,\perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if

$$(\forall n; x_n \perp x_{n+1}) \ \ or \ \ (\forall n; x_{n+1} \perp x_n).$$

(see also [9, 10, 19]).

Let $(M,\rho,\perp)$ be an orthogonal metric space ($(M,\perp)$ is an O-set and (M,ρ) is a metric space). We consider the notion of O-complete orthogonal metric space.

Definition 2.3. [13] M is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent (see also [9, 10, 19]).

Definition 2.4. Let $(M,\rho,\perp)$ be an orthogonal metric space and $0<\lambda<1$ (see [13]).

1. A mapping $f: M \to M$ is said to be orthogonal contraction ($\perp-$contraction) with Lipchitz constant λ if

$$\rho(fx, fy) \leq \lambda\rho(x,y) \qquad if x \perp y. \tag{2.1}$$

2. A mapping $f: M\to M$ is called orthogonality-preserving ($\perp-$preserving) if $f(x)\perp f(y)$ if $x\perp y$.

3. A mapping $f: M\to M$ is continuous orthogonal ($\perp-$continuous) in $a\in M$ if for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in M if $a_n\to a$, then $f(a_n)\to f(a)$. Also f is $\perp-$continuous on M if f is $\perp-$continuous in each $a\in M$.

(see also [9, 10, 19]).

Example 2.3. Let $M=[0,1)$ and let the metric on M be the Euclidian metric. Define $x\perp y$ if $xy\in\{x,y\}$. M is not complete but it is O-complete. Let $x\perp y$ and $xy=x$. If $\{x_k\}$ is an arbitrary Cauchy O-sequence in M, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n}=0$ for all n. It follows that $\{x_{k_n}\}$ converges to a $x\in M$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent.
Let $f: M\to M$ be a mapping defined by $f(x)=\frac{x}{2}$ if $x\in\mathbb{Q}\cap M$ and $f(x)=0$ if $x\in\mathbb{Q}^c\cap M$.

We have the following cases:

case 1) $x=0$ and $y\in\mathbb{Q}\cap M$. Then $f(x)=0$ and $f(y)=\frac{y}{2}$.

case 2) $x=0$ and $y\in\mathbb{Q}^c\cap M$. Then $f(x)=f(y)=0$.

This implies that $f(x)f(y)=f(x)$. Hence f is $\perp$-preserving.
Also, this implies that $|f(x)-f(y)|\leq\frac{1}{2}|x-y|$. Hence f is $\perp-$contraction. But f is not a contraction. To see this, for each $\lambda<1$, $|f(\frac{1}{2})-f(\frac{\sqrt{3}}{4})|>\lambda|\frac{1}{2}-\frac{\sqrt{3}}{4}|$.

If $\{x_n\}$ is an arbitrary O-sequence in M such that $\{x_n\}$ converges to $x \in M$. Since f is $\perp-$contraction, for each $n \in \mathbb{N}$ we have

$$|f(x_n) - f(x)| \leq \frac{1}{2}|x_n - x|.$$

As n goes to infinity, f is $\perp$-continuous. But it can be easily seen that f is not continuous.

We can now state the main theoretical result of [13]. Sufficient conditions under which any mapping on an orthogonal metric space will have a unique fixed point are given in the theorem.

Theorem 2.1. *Let $(M, \rho, \perp)$ be an O-complete metric space (not necessarily complete metric space) and $0 < \lambda < 1$. Let $f : M \to M$ be $\perp-$continuous, $\perp-$contraction (with Lipschitz constant λ) and $\perp-$preserving, then f has a unique fixed point x^* in M. Also, f is a Picard operator, that is,* $\lim f^n(x) = x^*$ *for all $x \in M$.*

(see also [9, 10, 19]).

Theorem 2.2. [8](chap.4,31.1) *Given a point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ consider a differential equation (1.1). Let P be a Picard mapping that takes a function $\phi : t \to x$ to the function $P\phi : t \to x$ defined by*

$$(P\phi)(t) = x_0 + \int_{t_0}^{t} v(\tau, \phi(\tau))d\tau \qquad \tau \in \mathbb{R} \tag{2.2}$$

Note that $(P\phi)(t_0) = x_0$ for any ϕ. The mapping $\phi : I \to \mathbb{R}^n$ is a solution to $\dot{x} = v(t, x)$ with the initial condition $\phi(t_0) = x_0$ if and only if $\phi = P\phi$.

Simply, the theorem states that the solution to a first-order differential equation is the "fixed point" of a Picard mapping. Theorem 2.1 gives us some conditions under which a mapping has one and only one fixed point. Thus, if we could construct a mapping that includes both types of functions in just the right way, we could take advantage of the existence and uniqueness of the fixed point of this mapping to prove the existence and uniqueness of the solution to our differential equation.

3. Main results

In this section, we are ready to state new and simple proofs of Theorems 1.1 and 1.2. To this end, we need some definitions.
Let $(M, \rho, \perp)$ be an orthogonal metric space ($(M, \perp)$ is an O-set and (M, ρ) is a metric space).

Definition 3.1. Let ϕ be a mapping of an open interval I in $\mathbb{R}$ to $(M, \rho, \perp)$. The derivative of ϕ is defined by

$$\dot{\phi}(t) := \lim_{s \to 0} \frac{\rho(\phi(t + s), \phi(t))}{s},$$

where $t \in \mathbb{R}$ is a limit point of I and $\phi(t) \perp \phi(t + s)$ if the limit exists.

We now consider the tangent space to $(M, \rho, \perp)$ at a point.

Definition 3.2. Let ϕ be a differentiable mapping of an open interval I in $\mathbb{R}$ to $(M, \rho, \perp)$. ϕ is said to leave the point x for some $x \in M$ if $\phi(0) = x$. The derivative of ϕ at the point $t = 0$ is a vector v as:

$$v = \dot{\phi}(0) = \frac{d\phi}{dt}|_{t=0}. \tag{3.1}$$

The tangent space to $(M, \perp)$ at a point x is the set of all vectors v of all such curves leaving x and denoted $T_x M$.

We turn our attention to the concept of the derivative of a mapping f at a point.

Definition 3.3. Let $f : U \to V$ be a differentiable mapping from the subset U of the orthogonal metric space $(M_1, \rho_1, \perp_1)$ into the subset V of the orthogonal metric space $(M_2, \rho_2, \perp_2)$ and let $\phi : I \to U$ be a differentiable mapping which leaves the point $x \in U$ at $t = 0$. The derivative of the mapping f at the point x is the mapping

$$f_{*x} : T_x U \to T_{f(x)} V,$$

which carries the vector v leaving the point x of the curve ϕ into the vector $f_{*x}(v)$ leaving the point $f(x)$ of the curve $f(\phi)$ i.e.

$$f_{*x}(v) = f_{*x}(\frac{d\phi}{dt}|_{t=0}) = \frac{df(\phi)}{dt}|_{t=0}. \tag{3.2}$$

Then we have the following result.

Proposition 3.1. *Let $f : U \to \mathbb{R}^n$ be a smooth mapping ($f \in C^r, r \geq 1$) from $U \subseteq (\mathbb{R}^m, \perp_1)$ to $(\mathbb{R}^n, \perp_2)$ and $x \in U$. Then f satisfies the Lipchitz condition on each convex compact subset V of U with the Lipchitz constant L equal to the supremum of the derivative of f on V:*

$$L = \sup_{x \in V} |f_{*x}|. \tag{3.3}$$

Proof. Take any two points $x, y \in V$, $x \perp_1 y$ and join them together with a line segment

$$z(t) = x + t(y - x) \ ; \quad 0 \leq t \leq 1.$$

Since V is convex, $z(t) \in V$; $\forall t \in [0, 1]$. Now, we have

$$\int_0^1 \frac{d}{dt}(f(z(t))dt = f(z(1)) - f(z(0)) = f(y) - f(x),$$

and

$$\int_0^1 \frac{d}{dt}(f(z(t))dt = \int_0^1 \frac{df}{dz}|_{z(t)} \frac{dz}{dt}(t)dt = \int_0^1 f_{*z(t)}(y - x)dt.$$

Examining the absolute magnitude of this integral, we find

$$\begin{aligned} \left|\int_0^1 f_{*z(t)}(y-x)dt\right| &\leq \int_0^1 |f_{*z(t)}(y-x)|dt \\ &\leq \int_0^1 |f_{*z(t)}||y-x|dt \\ &\leq \left(\int_0^1 |f_{*z(t)}|dt\right)\ |y-x| \\ &\leq \left(\int_0^1 Ldt\right)\ |y-x| \\ &= |L.1-L.0|\ |y-x| = L|y-x|. \end{aligned}$$

We have thus determined that for any two points $x, y \in V$,

$$|f(y)-f(x)| = \left|\int_0^1 f_{*z(\tau)}(y-x)d\tau\right| \leq L|y-x|,$$

and hence f satisfies the Lipchitz condition on V with the constant L. $\square$

Remark 3.1. In the previous proposition, since $f \in C^1$ the mapping $f_* = \frac{df}{dx}$ which takes a given x and returns the mapping f_{*x} is continuous. Since V is compact $|f_{*x}|$ actually attains its maximum value L.

Now, we are interested in obtaining a mapping that satisfies the properties of Theorem 2.1 and the fixed point of this mapping is the solution to (1.1). In this way, we prove the existence and uniqueness (Theorems 1.1 and 1.2) of the solution to (1.1).
Because v is differentiable at the point $(t_0, x_0) \in U$, there exists some neighborhood C around (t_0, x_0) such that $C \subset U$. Then there exist small enough numbers a and b such that

$$C = \{(t,x); |t-t_0| \leq a, |x-x_0| \leq b\} \subset U. \tag{3.4}$$

Clearly, C is compact and $|v|$ attains its supremum over C. Similarly, $|v_*| = |\frac{dv}{dx}|$ attains its supremum over C. Let

$$c = \sup_C |v|, \qquad L = \sup_C |v_*|. \tag{3.5}$$

We are interested in obtaining a function based on v, satisfying Lipchitz condition on each convex compact subset of U, including C with the Lipchitz constant L. Let us separate C into some subregions. There exists

$$\acute{a} = \min\{a, \frac{b}{2c}, \frac{1}{2L}\}, \tag{3.6}$$

such that

$$K_0 = \{(t,x); |t-t_0| \leq \acute{a}, |x-x_0| \leq c|t-t_0|\},$$

lies in C.
For $\acute{b} = \frac{b}{2}$ and $\acute{x}$ with $|\acute{x} - x_0| \leq \acute{b}$ another point $(t_0, \acute{x})$ can be considered such that

$$K_{\acute{x}} = \{(t, x); |t - t_0| \leq \acute{a}, |x - \acute{x}| \leq c|t - t_0|\}. \tag{3.7}$$

The following argument shows that $\acute{a}$ exists and is equal to $\min\{a, \frac{b}{2c}, \frac{1}{2L}\}$. Since $|x - x_0| \leq c|t - t_0| \leq c\acute{a}$ then $\acute{a} = \min\{a, \frac{b}{c}\}$ exists. On the other hand, by using triangle inequality, we find

$$|x - x_0| \leq |x - \acute{x}| + |\acute{x} - x_0| \leq c\acute{a} + \acute{b} = b.$$

So, let $\acute{a} = \min\{a, \frac{b}{2c}\}$. $\acute{a}$ will need one more bound later on, namely, the condition $\acute{a} < \frac{1}{L}$ (we are ignoring the trivial case $L = 0$). So, let us go ahead and put $\acute{a} = \min\{a, \frac{b}{2c}, \frac{1}{2L}\}$.
We are trying to obtain the solution $\phi_{\acute{x}} : \mathbb{R} \to \mathbb{R}^n$ of (1.1) with the initial condition $\phi_{\acute{x}}(t_0) = \acute{x}$ expressed in the form $\phi_{\acute{x}}(t) = \acute{x} + h(t, \acute{x})$, though we can now remove the prime on x:

$$\phi_x(t) = x + h(t, x). \tag{3.8}$$

Then the mapping

$$\phi : \{(t, x); |t - t_0| \leq \acute{a}, |x - x_0| \leq \acute{b}\} \to \mathbb{R}^n, \tag{3.9}$$

defined by

$$\phi(t, x) = \phi_x(t), \tag{3.10}$$

is the "general" solution of (1.1).
One may easily verify the following lemma:

Lemma 3.1. *For any solution ϕ_x, the point $(t, \phi_x(t))$ lies within K_x for all t such that $|t - t_0| \leq \acute{a}$.*

Recall that we are interested in obtaining a mapping that satisfies the properties of Theorem 2.1 and the fixed point of this mapping is the solution to (1.1). Let us first define the orthogonal metric space we will use. This space should include all the mappings which could possibly be solutions. Given some central initial condition (t_0, x_0), the mapping ϕ should take the point (t, x) from the region $|t - t_0| \leq \acute{a}, |x - x_0| \leq \acute{b}$ to $\mathbb{R}^n$.

Since ϕ_x must be a differentiable function in order to be a solution, it must be continuous on the set over which it is a solution. The space of all continuous functions $h(t, x)$ which added to x could give us a solution ϕ_x with the initial condition $\phi_x(t_0) = x$ will be considered. Denote this space by M. Since ϕ takes the point (t, x) from the region $|t - t_0| \leq \acute{a}, |x - x_0| \leq \acute{b}$ to $\mathbb{R}^n$, the map h must be over this region.

$$h : \{(t, x); |t - t_0| \leq \acute{a}, |x - x_0| \leq \acute{b}\} \to \mathbb{R}^n. \tag{3.11}$$

Note that $h(t_0, x) = 0$ for any $h \in M$, $x \in C$, where 0 is the zero vector in $\mathbb{R}^n$. In the space M, we can define a relation $\perp$ by

$$h_1 \perp h_2 \iff \|h_1\|\|h_2\| \leq c|t - t_0|(\|h_1\| \vee \|h_2\|), \tag{3.12}$$

which is an orthogonality relation on M. It shows that the space M is an orthogonal space.
Let $\rho : M \times M \to \mathbb{R}_+$ be given by

$$\rho(h_1, h_2) = \|h_1 - h_2\| = \sup |h_1(t, x) - h_2(t, x)|, \tag{3.13}$$

for all $h_1, h_2 \in M$. Then ρ is a metric on M and the orthogonal metric space M will be denoted by $(M, \rho, \perp)$. Since every h is a continuous function over a closed and bounded subset of the Euclidean space, this supremum is actually attained. Hence, the orthogonal metric space $(M, \rho, \perp)$ is complete.
In the orthogonal metric space $(M, \rho, \perp)$, a mapping $A : (M, \rho, \perp) \to (M, \rho, \perp)$ can be defined by

$$(Ah)(t, x) = \int_{t_0}^{t} v(\tau, x + h(\tau, x))d\tau, \tag{3.14}$$

for $|t - t_0| \leq \acute{a}$, $|x - x_0| \leq \acute{b}$. Clearly, $(\tau, x + h(\tau, x))$ is in the domain of v for any (τ, x) in the appropriate region but we should be careful to check that Ah is in fact an element of $(M, \rho, \perp)$.

Lemma 3.2. *For all $h \in M$, $Ah \in M$.*

Proof. Take any $h \in M$. By construction Ah is a function that satisfies (3.11). The function h is continuous for any (τ, x) in its domain, so the point $(\tau, x + h(\tau, x))$ varies continuously with (τ, x) and since v is also continuous on its domain v varies continuously with (τ, x) as well. Taking the integral will then result in a continuous function of the boundary terms taken at (t, x) and (t_0, x). Thus, Ah is a continuous function of (t, x) meaning $Ah \in M$. □

We now discuss some properties of mapping A.

1. A is $\perp$-preserving mapping.

2. A is $\perp$-contraction mapping.

3. A is $\perp$-continuous mapping.

Proof. 1. We recall that A is $\perp$-preserving, if for $h_1, h_2 \in M$, $h_1 \perp h_2$, we have $Ah_1 \perp Ah_2$.

$$|(Ah_1)(t, x)| = \left|\int_{t_0}^{t} v(\tau, x + h_1(\tau, x))d\tau\right|$$

$$\begin{aligned}
&\leq \int_{t_0}^{t} |v(\tau, x+h_1(\tau,x))| d\tau \\
&\leq \int_{t_0}^{t} c d\tau \\
&= |c.t - c.t_0| = c|t-t_0|.
\end{aligned}$$

So,

$$\|Ah_1\|\|Ah_2\| \leq c|t-t_0|\|Ah_2\|.$$

Meaning that $Ah_1 \perp Ah_2$.

2. We need to prove that for any $h_1, h_2 \in M$, $h_1 \perp h_2$, $\|Ah_1 - Ah_2\| \leq \lambda \|h_1 - h_2\|$ for some constant $0 < \lambda < 1$. Let us then construct the mapping $Ah_1 - Ah_2$.

$$|(Ah_1)(t,x)| = \Big|\int_{t_0}^{t} v(\tau, x+h_1(\tau,x))d\tau\Big| \qquad (abbreviated \int_{t_0}^{t} v_1 d\tau),$$

$$(Ah_1 - Ah_2)(t,x) = \int_{t_0}^{t} v_1 d\tau - \int_{t_0}^{t} v_2 d\tau = \int_{t_0}^{t} (v_1 - v_2) d\tau.$$

For a fixed (τ, x), v will act as a mapping that takes $h_i(\tau, x)$ to $v(\tau, x+h_i(\tau,x))$. As v was assumed to be continuously differentiable over its domain, we invoke Proposition 3.1 to find that v satisfies the Lipchitz condition on each convex compact subset of its domain and therefore on each subset C of U. Proposition 3.1 also gives us the Lipchitz constant $L(\tau) = \sup_{|x-x_0|\leq b} |v_*|$ where we have emphasized the fact that this L depends on the choice of the constant τ. Thus, for all points (τ, x),

$$|v_1(\tau,x) - v_2(\tau,x)| \leq L(\tau)\|h_1 - h_2\|.$$

As seen earlier, the magnitude of any mapping in M attains its supremum at some point in its domain, so we have

$$\|Ah_1 - Ah_2\| = \sup |Ah_1(t,x) - Ah_2(t,x)| = |Ah_1(t_m, x_m) - Ah_2(t_m, x_m)|,$$

for some $(t_m, x_m) \in C$. Therefore,

$$\begin{aligned}
\|Ah_1 - Ah_2\| &= \Big|\int_{t_0}^{t_m} (v_1(\tau, x_m) - v_2(\tau, x_m)) d\tau\Big| \\
&\leq \int_{t_0}^{t_m} |(v_1(\tau, x_m) - v_2(\tau, x_m))| d\tau \\
&\leq \int_{t_0}^{t_m} L(\tau)\|h_1 - h_2\| d\tau \\
&= \int_{t_0}^{t_m} L(\tau) d\tau \|h_1 - h_2\|.
\end{aligned}$$

In (3.5), L (without the parenthetical τ) was designated the supremum of $|v_*|$ over all of C i.e. over both the t and x domains meaning that

$$\begin{aligned}\|Ah_1 - Ah_2\| &\leq \int_{t_0}^{t_m} L(\tau)d\tau\|h_1 - h_2\| \\ &\leq \int_{t_0}^{t_m} Ld\tau\|h_1 - h_2\| \\ &= L|t_m - t_0|\|h_1 - h_2\| \\ &\leq L\acute{a}\|h_1 - h_2\|.\end{aligned}$$

Lastly, we take advantage of the extra bound we placed on $\acute{a}$ to find that $L\acute{a} \leq L\frac{1}{2L} = \frac{1}{2} < 1$. Thus, for all $h_1, h_2 \in M$, $h_1 \perp h_2$,

$$\|Ah_1 - Ah_2\| \leq L\acute{a}\|h_1 - h_2\| \quad , \quad 0 < L\acute{a} < 1,$$

making A a $\perp$-contraction mapping.

3. Suppose $\{h_n\}$ is an O-sequence in M such that $\{h_n\}$ converging to $h \in M$. Because A is $\perp$-preserving, $\{Ah_n\}$ is an O-sequence. For each $n \in \mathbb{N}$, since A is $\perp$-contraction, we have

$$\|Ah_n(t,x) - Ah(t,x)\| \leq L\acute{a}\|h_n - h\|.$$

As n goes to infinity, it follows that A is $\perp$-continuous.

□

The mapping A defined above is $\perp$-preserving, $\perp$-contraction and $\perp$-continuous mapping over an orthogonal metric space $(M, \rho, \perp)$. The mapping A satisfies the hypotheses of Theorem 2.1. Thus, the existence and uniqueness of its fixed point $h_0 \in M$ is guaranteed by Theorem 2.1. The purpose of the present paper is to incorporate this in a Picard mapping of potential solutions to (1.1). Using the existence and uniqueness of h_0 to confirm the existence and uniqueness of the fixed point of the Picard mapping, which will in turn prove our main theorems.
First, recall that we are looking for solutions expressed in the form $\phi_x(t) = x + h(t,x)$. If h is a fixed point of A, then $\phi_x(t) = x + Ah(t,x)$ and when the solution ϕ_x is the fixed point, our Picard mapping $\phi_x(t)$ will equal $(P\phi_x)(t)$. Hence,

$$\begin{aligned}(P\phi_x)(t) &= x + (Ah)(t,x) \\ &= x + \int_{t_0}^{t} v(\tau, x + h(\tau,x))d\tau \\ &= x + \int_{t_0}^{t} v(\tau, \phi_x(\tau))d\tau\end{aligned}$$

By Theorem (2.2), ϕ_x is a solution to $\dot{x} = v(t,x)$ with $\phi_x(t_0) = x$ if and only if $\phi_x = P\phi_x$. We can now conclude this section with a new proof of the forthcoming results concerning the existence and uniqueness of the solution to (1.1) satisfying any initial condition in the domain of v.

Theorem 3.1. *(The Existence Theorem) Suppose the right-hand side v of the differential equation $\dot{x}(t) = v(t, x)$ is continuously differentiable in a neighborhood of the point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then there exists a neighborhood of the point t_0 such that the solution to the differential equation is defined in this neighborhood with the initial condition $\phi(t_0) = x_0$ where x is any point sufficiently close to x_0. Moreover, this solution depends continuously on the initial point x.*

Proof. Given $v(t, x)$ as well as (t_0, x_0), demarcate a neighborhood C around the central point and use it to define the constants $\acute{a}, \acute{b}$; also, construct the orthogonal metric space $(M, \perp, \rho)$, $\perp$-preserving, $\perp$- continuous, $\perp$-contraction mapping A and a Picard mapping P as determined by v,C and the central point (t_0, x_0). Since M is an orthogonal complete metric space, the fixed point h_0 of A must exist by Theorem 2.1. The function $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g(t, x) = x + h_0(t, x),$$

is therefore always well-defined in a neighborhood of (t_0, x_0). Applying the Picard mapping

$$(Pg)(t, x) = x + (Ah_0)(t, x) = x + h_0(t, x) = g(t, x),$$

which proves that, by Theorem 2.2, g is the solution to the differential equation which satisfies the initial condition $g(t_0, x) = x$. The function which returns the value x is continuous on $\mathbb{R} \times \mathbb{R}^n$, h_0 is continuous by construction and the sum of any two continuous function is continuous over the same domain. So g, the function of t and x, is continuous over its domain. Thus, the solution depends continuously on the initial point x. □

Uniqueness immediately follows:

Theorem 3.2. *(The Uniqueness Theorem) Given the above conditions, there is only one possible solution for any given initial point, in the sense that all possible solutions are equal in the neighborhood under consideration.*

Proof. Construct a neighborhood and mapping as above but now set $\acute{b} = 0$, which restricts the initial x under our consideration to the specific point x_0. Find the solution $g(t, x_0) = x_0 + h_0(t, x_0)$. The uniqueness of the fixed point h_0 guarantees that this is the only solution with the initial condition x_0 that can be expressed in the form $x + h(t, x)$.
Now, consider any solution ϕ_{x_0} with $\phi_{x_0}(t_0) = x_0$. By Lemma 3.1, $\phi_{x_0}(t) \in K_0$ for all t in our neighborhood. Label $\phi_{x_0}(t) - x_0$ by $h_\phi(t, x_0)$. This new function also clearly satisfies (3.11) and, furthermore, since any solution ϕ must be continuous, h_ϕ is also continuous. So, $h_\phi \in M$ and $\phi_{x_0}(t) = x_0 + h_\phi(t, x_0)$. The uniqueness of h_0 shows that all possible solutions to the differential equation with a given initial condition are expressed in the form $\phi_{x_0} = x_0 + h(t, x_0)$ for $h \in M$. As there is only one such function possible, the solution g is thus unique. □

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COMPUTING TRIANGULATIONS OF THE CONVEX POLYGON IN PHP/MYSQL ENVIRONMENT

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Abstract. In this paper we implement the Block method for convex polygon triangulation in the web environment (PHP/MySQL). Our main aim is to show the advantages of the usage of web technologies in performing complex algorithm from computer graphics. The basic assumption is that once obtained, the results can be stored in a database and used for other calculations. Databases are convenient and structured methods of sharing and retrieving data. We have performed a comparative analysis of the developed program with respect to two criteria: CPU time in generating triangulation and CPU time in reading results from the database.
Keywords: Computer graphics, Polygon triangulation, Block method, PHP/MySQL.

1. Introduction and preliminaries

Polygon triangulation is an important problem applicable in computer graphics. Restricted to the convex case, the decomposition of a polygon is done into triangles by a maximal set of non-intersecting diagonals.

Let P_n denote a polygon with n vertices. The total number T_n of n-gon triangulations is

$$T_n = C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2} = \frac{(2n-4)!}{(n-1)!(n-2)!}, \quad n \geqslant 3. \tag{1.1}$$

Here, C_n represents the nth Catalan number (see e.g. [9]).

The set of all triangulations of the convex polygon P_n is denoted by $\mathcal{T}_n$. Diagonal connecting vertices i and j are denoted by $\delta_{i,j}$. An outer face edge can be considered as a diagonal, while nonadjacent vertices are connected by an *internal diagonal*.

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Many authors deal with the problem of how to generate the triangulation of a convex polygon based on some criterion. In this paper we implement the Block method for convex polygon triangulation [6] in the web environment using PHP/MySQL technologies.

The combination of PHP and MySQL is the most convenient approach to dynamic, database-driven web design application. Due to its open source roots, it is free to implement and is therefore an extremely popular option for web development.

PHP is extremely powerful and exceptionally fast it can run on even the most basic hardware, and it hardly puts a dent in the system resources. The main characteristics of PHP are described in [2].

According to the TIOBE Programming Community index[1], the PHP programming language is one of the top 10 most popular programming languages. Eighty percent of the top 10 million websites use PHP in one way or the other, including Facebook and Wikipedia.

PHP, as a scripting language, is popular among web developers because of its ability to interact with database systems.

MySQL is probably the most popular database management system for web servers.

MySQL is a fast and powerful, yet easy-to-use, database system that offers just about anything a website would need in order to find and serve up data to browsers.

The combination of PHP and MySQL can be used to build simple or complex and high traffic websites (see for e.g. [1, 7]). Similarly, the authors [4] used PHP/MySQL environment for computing the weighted Moore-Penrose inverse employing the partitioning method, as well as for storing the generated results.

This paper is organized as follows. In the Section 2 we present the main parts of the Block method for convex polygon triangulation. In Section 3 we describe the implementation of the algorithm in the PHP/MySQL environment. Section 4 includes a comparative analysis of the obtained numerical results.

2. Block method for convex polygon triangulation

Here we restate the Block method for convex polygon triangulation [6] which is the subject of our implementation.

The general strategy of the method is to decompose the problem into smaller dependant subproblems. Each subproblem is solved only once and used many times avoiding unnecessary repetitions of calculation.

The method is based on the usage of the previously generated triangulations for polygon with a smaller number of vertices. More precisely, the algorithm generates the set $\mathcal{T}_n$ using all the previously generated triangulations $\mathcal{T}_b$, where $b < n$. The set $\mathcal{T}_b$ is used as many times as necessary as a block, i.e. it is repeated several times in $\mathcal{T}_n$.

[1] https://www.tiobe.com/tiobe-index/

The formal statement of the subject method is given by the following equation

$$T_n = 2T_{n-1} + \mathrm{rest}(R_n). \tag{2.1}$$

The general idea of the Block method uses T_{n-1} to generate T_n, which is illustrated in Figure 2.1, where one case of the transformation process from a P_5 triangulation into two corresponding P_6 triangulations is presented. In part (a) we see that the diagonals $\delta_{2,4}$ and $\delta_{2,5}$ make all vertices closed except the vertices 1, 2, 5, and 6 which form a quadrilateral. The parts (b) and (c) show two ways to triangulate a quadrilateral, which gives two P_6 triangulations having a P_5 triangulation as a starting block.

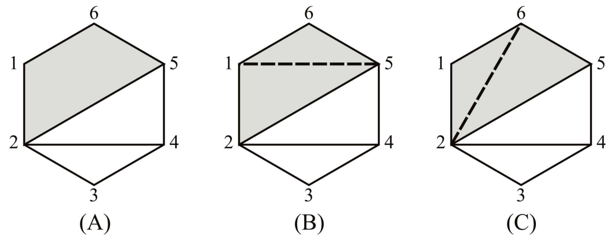

FIG. 2.1: Transformation from a P_5 into the corresponding P_6 triangulations.
$P_5 = \{(2,4),(2,5)\} \rightarrow P_6 = \{(2,4),(2,5),(1,5)\&(2,4),(2,5),(2,6)\}$

Starting from the assumption that triangulation has at least two ears and that, in the worst case, one ear can be a vertex n, then we always have at least one ear among the rest of the vertices.

For the correctness of the algorithm in the procedure used for finding and eliminating closed vertices, the authors introduced a list of ordered pairs of the form

$$L = \{(1,1),(2,2),\ldots,(n,n)\}. \tag{2.2}$$

After the elimination of $n-l$ pairs the list L becomes

$$L = \{(s,i_s),\ s = 1,\ldots,l\},\ 4 \le l \le n,\ i_l = n. \tag{2.3}$$

The values $i_s,\ s = 1,\ldots,l$ are the vertex marks, while the values $1,\ldots,l$ represent the relative vertex positions in the list L.

Here we restate two additional algorithms 2.1 - Pair elimination & 2.2 - Form a quadrilateral, which are part of the Block Method Algorithm 2.3.

Algorithm 2.1 Pair elimination

Require: List L of the form (2.3) and vertices i_p and i_q, where $d(p,q) = 2$.

1: Remove from the list L the pair placed between the pairs (p, i_p) and (q, i_q) in a circular manner.

2: Decrease by one the first pair members in the pairs following the eliminated one.

Algorithm 2.2 Form a quadrilateral

Require: List L of the form (2.2), integer n and array of $n-4$ diagonals (i.e a row in the table for $\mathcal{T}_n$).

1: Find a diagonal δ_{i_p,i_q} where $d(p,q) = 2$ in the list L.

2: Call **Algorithm 2.1** for the parameters i_p and i_q.

3: Repeat Steps 1–2 $n-4$ times.

The main algorithm for the Block method is presented below.

Algorithm 2.3 Algorithm for the Block method

Require: An integer n and $\mathcal{T}_b$ with $row_b = C_{n-3}$ rows and $col_b = n-4$ columns

1: Create an empty table for $\mathcal{T}_n$ with $row_n = C_{n-2}$ rows and $col_n = n-3$ columns.

2: Fill the table for $\mathcal{T}_n$ by the triangulations from $\mathcal{T}_b$ duplicating each row from $\mathcal{T}_b$.

3: Fill the rest of the entered blocks (the last column in the first $2row_b$ rows) in the following way.

`for` $(i = 1; i <= 2row_b; i+ = 2)$

{

Make a list L of the form (2.2).

Call **Algorithm 2.2** with row i from the table for $\mathcal{T}_n$ as a parameter.

From the remaining four vertices in the list L make a diagonal δ_{i_1,i_3} and place it in the last column of the row i and diagonal δ_{i_2,i_4} and place it in the last column of the row $i+1$.

}

4: Fill the rest of the table for $\mathcal{T}_n$ containing $T_n - 2T_b$ rows.

4.1 Filling the first $n-4$ columns in the last $row_n - 2row_b$ rows.

$i = 2 * row_b + 1$;

Make the list L of the form (2.2).

Eliminate the vertices adjacent to n calling Algorithm 2.1 for the parameters 1 and $n-1$.

Fill the current table row i by diagonals $\delta_{2,n}, \delta_{3,n}, \ldots, \delta_{n-2,n}$.

The first $n-4$ columns in the rest $row_n - 2row_b - 1$ rows should be filled with the diagonals with the last vertex n, while the first vertices are combinations of the $(n-4)$th class in the set $\{2, 3, \ldots, n-2\}$. The number of these combinations is $\binom{n-3}{n-4} = n-3$.

4.2 Filling the last column in the last $(row_n - 2row_b)$ rows.

`for` $(i = 2row_b + 2; i <= row_n; i++)$

{

Make the list L of the form (2.2).

Call **Algorithm 2.2** with the row i from the table for $\mathcal{T}_n$ as a parameter. From the remaining four vertices in the list L make a diagonal δ_{i_1,i_3} and place it in the last column of the row i.
}

3. PHP/MySQL implementation of Block method

The most used architecture for development of web applications is three-tier architecture (Figure 3.1). Three-tier web architecture is a unique system for developing web database applications which work around the three-tier model comprising the database tier at the bottom, the application tier in the middle and the client tier on top.

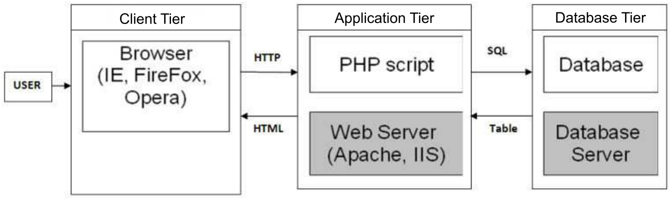

FIG. 3.1: Three-tier Web Architecture

The web interface of our application is given in Figure 3.2

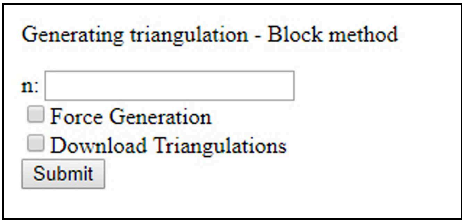

FIG. 3.2: Web interface of the application

According to the three-tier architecture, our application is organized as follows:

- On the *client tier* we have the web interface;
- Algorithm for the Block method is performed on *application tier*;
- Generated triangulations are stored on *database tier*;

In what follows, we presents a detailed view of the application scenario:

First, we have to enter the value n for which convex polygon we want to calculate triangulation.

Preconditions: $n \geq 4$

Second, when we press the submit button, Application search in database:

Case 1: Force Generation = Not marked

Have we already calculated triangulations of n in the database;

- If we have, the application displays the results of T_n in the browser;

- If we have not, the application checks if we have the results of T_{n-1} in the database:

  * If we have, then call Algorithm 2.3

  * If we do not, the preconditions of Algorithm 2.3 are not fulfilled;

Case 2: Force Generation = Marked

Have we already calculated triangulations of $n-1$ in the database;

- If we have, then call Algorithm 2.3

- If we have not, the preconditions of Algorithm 2.3 are not fulfilled;

Third, the output results can be downloaded in a `CSV` format if we mark "Download Triangulation".

Example 3.1. Let us illustrate how the application works on generating hexagon triangulations using the already known pentagon triangulations.

First, $n = 6$;

Preconditions fulfilled: $6 \geq 4$;

Second, the submit button is pressed

Case 1: Force Generation = Not marked

- The application checks if we have the results of T_5 in the database:

  * If we have, then call Algorithm 2.3

$\rightarrow$ Generating triangulations and displaying results in browsers (Figure 3.3)

```
Done: 14 triangulations were found for P_{6} in 0.345 sec.
T_1= 3-6; 3-5; 1-3
T_2= 1-3; 1-5; 3-5
T_3= 4-6; 1-3; 1-4
T_4= 1-4; 1-3; 1-5
T_5= 2-5; 2-4; 2-6
T_6= 1-5; 2-4; 2-5
T_7= 4-6; 1-4; 2-4
T_8= 1-5; 1-4; 2-4
T_9= 2-5; 3-5; 2-6
T_10= 2-5; 1-5; 3-5
T_11= 3-6; 4-6; 1-3
T_12= 2-4; 2-6; 4-6
T_13= 3-6; 3-5; 2-6
T_14= 2-6; 3-6; 4-6
```

FIG. 3.3: Generating results for T_6

4. Comparative analysis and experimental results

The main idea of our implementation is to provide an appropriate client-server web application, in the free open source PHP/MySQL development environment, utilizing the minimum of resources: an internet browser and an operating system.

For a comparative analysis in presenting the advantages of web technologies, we implement an additional algorithm from the field of computer graphics (Orbiting Triangle method [8]).

Both algorithms are based on the usage of the previously generated triangulations for a polygon with a smaller number of vertices.

The execution times with respect to two criteria are presented in Table 4.1. The table column "Speedup" shows the quotient of the values contained in the previous two columns.

The testing is performed on the following configuration*: *CPU - Inter(R) Core(TM) i5-4210U CPU @ 1.70GHz 2.40GHz, RAM memory 8GB, Graphics card: NVIDIA GeForce 820M.*

Table 4.1: The execution times of computing triangulations (in seconds)

n	**Number of triangulations**	**BM in generating**	**BM in reading from DB**	**Speedup**	**OTM in generating**	**OTM in reading from DB**	**Speedup**
5	5	0.256	0.003	**85.33**	0.067	0.001	**67.00**
6	14	0.345	0.003	**115.00**	0.088	0.001	**88.00**
7	42	0.391	0.003	**130.33**	0.123	0.001	**123.00**
8	132	0.457	0.004	**114.25**	0.185	0.002	**92.50**
9	429	0.756	0.008	**94.50**	0.927	0.002	**463.50**
10	1,430	1.606	0.019	**84.53**	1.524	0.003	**508.00**
11	4,862	3.915	0.063	**62.14**	3.182	0.008	**397.75**
12	16,796	26.657	0.461	**57.82**	10.081	0.024	**420.04**
13	58,786	185.566	2.482	**74.76**	29.713	0.075	**396.17**
14	208,012	883.726	6.802	**129.92**	121.749	0.248	**490.92**
15	742,900	4,498.768	25.697	**175.07**	536.326	0.975	**550.08**

5. Conclusion

We implemented the Block method for convex polygon triangulation in the web environment using the open source software (PHP/MySQL). With this implementation we presented the advantages of web technologies in preforming a complex algorithm from computer graphics. The research also contributes to the manner in which an MySQL database is used for storing the obtained results and utilizing them for another calculation. As presented in the comparative analysis section, we can conclude that the advantages of using a database in performing complex algorithms are justified. This way of implementation provides a good basis for further application of the web technology in computing other algorithms.

A Important source code of the implementation

Source code for creating a MySQL database:

```
CREATE DATABASE IF NOT EXISTS triangulation;';
        if ($conn->query($sql)) {
            $conn->select_db('triangulation');
        } else {
            die('Could not create database: ' . $conn->error . '<br/>');
        }
```

The source code for database connection:

```
// Connection
$conn = new mysqli('localhost', 'root', '');
if ($conn->connect_errno) {
    die('Could not connect: (' . $conn->connect_errno . ')
        ' . $conn->connect_error . '<br/>');
    }
```

In our implementation, we use only one table for storing generated triangulations.

```
CREATE TABLE IF NOT EXISTS Triangulation
        (
            n int,
            T int,
            i int,
            j int,
            INDEX Triangulation_n_idx (n),
            INDEX Triangulation_T_idx (T),
            INDEX Triangulation_i_idx (i),
            INDEX Triangulation_j_idx (j)
        );
```

The source code of the implementation of Algorithm 2.3 - step 3:

```
        // Step 3
        // diagonal \delta_{i_1 ,i_3}
        $sql .= '
        INSERT INTO Triangulation
        SELECT DISTINCT a.n,
            a.T,
            1 AS i,
            ' . ($n-1) . ' AS j
        FROM Triangulation a
        WHERE a.n=' . $n . '
            AND a.T%2=0;
        ';
        // diagonal \delta_{i_2 ,i_4}
        $sql .= '
        INSERT INTO Triangulation
        SELECT ' . $n . ' AS n,
            a.T,
```

```
            a.v AS i,
            ' . $n . ' AS j
        FROM
            (SELECT a.T,
                a.j AS v
            FROM Triangulation a
            WHERE a.n=' . $n . '
                AND a.T%2=1
                AND a.i=1
            UNION
            SELECT DISTINCT a.T,
                2 AS v
            FROM Triangulation a
            WHERE a.n=' . $n . '
                AND a.T%2=1) a
            INNER JOIN
            (SELECT a.T,
                a.i AS v
            FROM Triangulation a
            WHERE a.n=' . $n . '
                AND a.T%2=1
                AND a.j=' . ($n-1) . '
            UNION
            SELECT DISTINCT a.T,
                ' . ($n-2) . ' AS v
            FROM Triangulation a
            WHERE a.n=' . $n . '
                AND a.T%2=1) b
            ON a.T=b.T
                AND a.v=b.v;
            ';
```

The source code of the implementation of Algorithm 2.3 - step 4:

```
// Step 4
for ($k = 1; $k <= $n-4; $k++) {
    $sql .= '
    INSERT INTO Triangulation
    SELECT ' . $n . ' AS n,
        (CASE a.T
            WHEN @curTn_1 THEN @curTn
            ELSE @curTn := @curTn + SIGN(@curTn_1 := a.T) * SIGN(@curK := 1)
                            * SIGN(@lastV := ' . ($n-2) . ')
        END) + ' . $N . ' AS T,
        (CASE
            WHEN a.i=' . $n . ' THEN @lastV
            ELSE a.i
        END) AS i,
        SIGN(
            CASE WHEN a.j=' . ($n-1) . ' AND @curK = ' . ($k+1) . '
                THEN @lastV:= a.i
                ELSE 1
            END) *
            (CASE
                WHEN a.j=' . ($n-1) . ' AND @curK <= ' . $k . '
                THEN ' . $n . ' * SIGN(@curK := @curK+1)
```

```
            ELSE a.j
        END) AS j
FROM
(SELECT a.T,
    a.i,
    a.j
FROM Triangulation a
WHERE a.n=' . ($n-1) . '
    AND a.T IN
(SELECT a.T
FROM Triangulation a
WHERE a.n=' . ($n-1) . '
    AND a.j=' . ($n-1) . '
GROUP BY a.T
HAVING count(a.i)>=' . $k . ')
UNION SELECT a.T,
    ' . $n . ' AS i,
    ' . $n . ' AS j
FROM Triangulation a
WHERE a.n=' . ($n-1) . '
    AND a.j=' . ($n-1) . '
GROUP BY a.T
HAVING count(a.i)>=' . $k . ' ) a ,
(SELECT @curTn := 0, @curTn_1 := 0, @curK := 0, @lastV := ' . ($n-2) . ') b
ORDER BY a.T,
    a.i,
    a.j;
';
if ($result = $conn->query('
    SELECT a.T
    FROM Triangulation a
    WHERE a.n=' . ($n-1) . '
        AND a.j=' . ($n-1) . '
    GROUP BY a.T
    HAVING count(a.i)>=' . $k . ';
    '
    )) {
    $N += $result->num_rows;
    $result->close();
    }
    else {
        die('Could not access Triangulation table: ' . $conn->error . '<br/>');
    }
}
```

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HERMITE-HADAMARD TYPE INEQUALITIES FOR P-CONVEX FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this paper, the authors establish the Hermite-Hadamard inequality for p-convex functions via Katugampola fractional integrals, followed by proving a new identity involving Katugampola fractional integrals. By using this identity, some new Hermite-Hadamard type inequalities for classes of p-convex functions are obtained.
Keywords: p-convex function, Hermite-Hadamard type inequalities, Katugampola fractional integrals.

1. Introduction

Definition 1.1. The function $f : I \subset \mathbb{R} \to \mathbb{R}$, is said to be convex, if the following inequality holds

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right)$$

for all $x, y \in I$ and $t \in [0,1]$. We say f is concave if $(-f)$ is convex.

Now we will give a useful inequality for convex functions as below.

Let $f : I \subset\to \mathbb{R}$ be a convex function defined on an interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(x\right)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Both inequalities hold in the reserved direction, when f is concave. Hermite-Hadamard inequality for convex functions has

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received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; for example, see [1, 6, 7, 8, 10, 18, 11] and the references cited therein.

In [28], Zhang and Wan gave a definition of the p-convex function as follows.

Definition 1.2. Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be a p-convex function or belongs to class $PC(I)$, if

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \le tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Remark 1.1. [28] An interval I is said to be a $p-convex$ set, if $[tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$ for all $x, y \in I$ and $t \in [0, 1]$,where $p = 2k + 1$ or $p = n/m$, $n = 2r + 1$, $m = 2s + 1$ and $k, r, s \in \mathbb{N}$.

Remark 1.2. [9] If $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$, then for all $x, y \in I$ and $t \in [0, 1]$, $[tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$.

According to Remark 1.2, we can give a different version of the definition of the p-convex function as below.

Definition 1.3. [9] Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \le tf(x) + (1-t) f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality is reserved, then f is said to be p-concave.

According to the definition above, it can easily be seen that for $p = 1$ and $p = -1$, p-convexity reduces to ordinary convexity and harmonically convexity [12] of functions defined on $I \subset (0, \infty)$ respectively.

In [3, Theorem 5], if we take $I \subset (0, \infty)$, $p \in \mathbb{R}\backslash\{0\}$ and $h(t) = t$, then we have the following inequalities for $p - convex\ functions$.

$f : I \to \mathbb{R}$ be a p-convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then we have

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \le \frac{f(a) + f(b)}{2}. \tag{1.3}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see now [3, 9, 22, 21, 28].

In [22, Lemma 2.4], if we take $I \subset (0, \infty)$ and $p \in \mathbb{R}\backslash\{0\}$, then we have the following Lemma.

Lemma 1.1. *Let $f : I \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then we have,*

$$\frac{f(a)+f(b)}{2} - \frac{p}{b^p - a^p} \int_b^a \frac{f(x)}{x^{1-p}} dx \tag{1.4}$$

$$= \frac{p}{b^p - a^p} \int_0^1 \frac{1-2t}{\left[ta^p + (1-t)b^p\right]^{1-\frac{1}{p}}} f'\left(\left[ta^p + (1-t)b^p\right]^{\frac{1}{p}}\right) dt.$$

We recall the following special functions and inequality.(see [16, 27])

(1) The Gamma Function:

The Gamma Γ function is defined by

$$\Gamma(z) = \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

for all complex numbers z with $\mathrm{Re}(z) > 0$, respectively. The gamma function is a natural extension of the factorial from integers n to real (and complex) numbers z.

(2) The Beta Function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0,$$

(3) The Hypergeometric Function

$${}_2F_1(a,b;c,z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, c > b > 0, |z| < 1.$$

Lemma 1.2. *[24, 29] For $0 < \alpha < 1$ and $0 \le a < b$, we have*

$$|a^\alpha - b^\alpha| \le (b-a)^\alpha.$$

Definition 1.4. Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$ and $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively.(see [16])

In [26] Sarıkaya et al. proved the following theorem for Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha > 0$.

Definition 1.5. [17] Let the space $X_c^p(a,b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) of those complex-valued Lebesque measurable functions f on $[a,b]$ for which $\|f\|\, x_c^p < \infty$, where the norm is defined by,

$$\|f\|\, x_c^p - \left(\int_a^b |t^c f(t)|^p \,\frac{dt}{t}\right)^{1/p} < \infty \tag{1.6}$$

for $1 \leq p \leq \infty$, $c \in \mathbb{R}$ and for the case $p = \infty$,

$$\|f\|\, x_c^p = ess \sup_{a \leq t \leq b} \left[t^c \,|f(t)|\right] \quad (c \in \mathbb{R})\,. \tag{1.7}$$

Katugampola introduced a new fractional which generalizes the Riemann-Liouville and the Hadamard fractional integrals into a single form as follows.(see [13, 14, 15])

Definition 1.6. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left-and right-side Katugampola fractional integrals of order ($\alpha > 0$) of $f \in X_c^p(a,b)$ are defined by

$${}^{p}I_{a^+}^{\alpha} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{p-1}}{(x^p - t^p)^{1-\alpha}} f(t)dt \quad and \quad {}^{p}I_{b^-}^{\alpha} f(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{p-1}}{(t^p - x^p)^{1-\alpha}} f(t)dt$$

with $a < x < b$ and $p > 0$, if the integral exists.

For more detailed information about fractional integrals and their applications, we refer the reader to see [4, 5, 2, 20, 23, 25, 19]

The aim of this paper is to establish some new Hermite-Hadamard type inequalities for $p-convex$ function via Katugampola fractional integral.

2. Main Results

Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section,

$$K_f(\alpha,a,b) = \frac{f\,(a) + f\,(b)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2\,(b^p - a^p)^{\alpha}}\left[{}^{p}I_{a^+}^{\alpha} f(b) + {}^{p} I_{b^-}^{\alpha} f(a)\right]$$

will be taken, where $a, b \in I, \alpha > 0$ and Γ is Euler Gamma function.

Theorem 2.1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p > 0$, $\alpha > 0$ and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p^\alpha \Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}\left[{}^pI^\alpha_{a+}f(b) + {}^pI^\alpha_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Proof. Since f is p-convex function on $[a,b]$, we have for all $x, y \in [a,b]$ (with $t = \frac{1}{2}$ in 1.2)

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{f(x)+f(y)}{2}.$$

By choosing $x = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ and $y = [(1-t)a^p + tb^p]^{\frac{1}{p}}$, then we get

$$2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) + f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right). \tag{2.2}$$

Multiplying both sides of the inequality of (2.2) by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, then we obtain,

$$\begin{aligned}
\frac{2}{\alpha}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \int_0^1 t^{\alpha-1} f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dt \\
&\quad + \int_0^1 t^{\alpha-1} f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)dt \\
&= \int_b^a \left(\frac{b^p-x^p}{b^p-a^p}\right)^{\alpha-1} f(x)\frac{px^{p-1}}{a^p-b^p}dx \\
&\quad + \int_a^b \left(\frac{x^p-a^p}{b^p-a^p}\right)^{\alpha-1} f(x)\frac{px^{p-1}}{b^p-a^p}dx \\
&= \frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left[{}^pI^\alpha_{a+}f(b) + {}^pI^\alpha_{b-}f(a)\right].
\end{aligned}$$

Thus we have

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p^\alpha\Gamma(\alpha+1)}{2(b^p-a^p)^\alpha}\left[{}^pI^\alpha_{a+}f(b) + {}^pI^\alpha_{b-}f(a)\right],$$

which completes the proof of the the first inequality. For the proof of the second inequality in (2.1), by using p-convexity of a function f, we can write,

$$f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \leq tf(a) + (1-t)f(b),$$

and

$$f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\leq(1-t)f(a)+tf(b).$$

By adding these inequalities, then we have,

$$f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)+f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\leq f(a)+f(b). \tag{2.3}$$

Multiplying both sides of the equation (2.3) by $t^{\alpha-1},\alpha>0$ and then integrating the resulting inequality with t over $[0,1]$, we similarly obtain,

$$\frac{p^\alpha\Gamma(\alpha+1)}{2\left(b^p-a^p\right)^\alpha}\left[{}^pI^\alpha_{a+}f(b)+{}^pI^\alpha_{b-}f(a)\right]\leq\frac{f(a)+f(b)}{2}.$$

So the proof is completed. □

Remark 2.1. In Theorem 2.1, if we take $p=1$, then the inequality reduces to the inequality (1.5).

Remark 2.2. In Theorem 2.1, if we take $\alpha=1$, then the inequality reduces to the inequality (1.3).

Lemma 2.1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function mapping with $0\leq a<b$. If f' is differentiable on $[a,b]$, then the following inequality holds:*

$$K_f(\alpha,a,b)=\frac{b^p-a^p}{2p}\int_0^1\frac{\left[(1-t)^\alpha-t^\alpha\right]f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}dt. \tag{2.4}$$

Proof. Let $M_p=ta^p+(1-t)b^p$. It suffices to note that

$$\begin{aligned}
&K_f(\alpha,a,b)\\
=&\ \frac{b^p-a^p}{2p}\int_0^1\frac{\left[(1-t)^\alpha-t^\alpha\right]f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}dt\\
=&\ \frac{b^p-a^p}{2p}\int_0^1\frac{(1-t)^\alpha f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}dt\\
&-\frac{b^p-a^p}{2p}\int_0^1\frac{t^\alpha f'\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)}{\left[ta^p+(1-t)b^p\right]^{1-\frac{1}{p}}}dt\\
=&\ I_1+I_2.
\end{aligned} \tag{2.5}$$

By integrating the part, we have,

$$I_1=-\frac{1}{2}\left[\begin{array}{c}(1-t)^{\alpha-1}f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\Big|_0^1\\ +\alpha\int_0^1(1-t)^{\alpha-1}f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)dt\end{array}\right], \tag{2.6}$$

if we take $x = [ta^p + (1-t)b^p]^{\frac{1}{p}}$

$$\begin{aligned}
&= -\frac{1}{2}\left[-f(b) + \frac{p\alpha}{(b^p - a^p)^{\alpha}}\int_a^b \frac{(x^p - a^p)^{\alpha-1}}{x^{1-p}} f(x)dx\right] \\
&= \frac{f(b)}{2} - \frac{p\alpha}{2(b^p - a^p)^{\alpha}}\int_a^b \frac{(x^p - a^p)^{\alpha-1}}{x^{1-p}} f(x)dx \\
&= \frac{f(b)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p - a^p)^{\alpha}}\left[{}^{p}I_{b-}^{\alpha} f(a)\right]
\end{aligned}$$

and similarly we get I_2,

(2.7)

$$\begin{aligned}
I_2 &= \frac{1}{2}\left[t^{\alpha-1} f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right)\Big|_0^1 + \alpha\int_0^1 t^{\alpha-1} f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right)dt\right] \\
&= \frac{1}{2}\left[-f(a) - \frac{p\alpha}{(b^p - a^p)^{\alpha}}\int_a^b \frac{(b^p - a^p)^{\alpha-1}}{x^{1-p}} f(x)dx\right] \\
&= \frac{f(a)}{2} - \frac{p\alpha}{2(b^p - a^p)^{\alpha}}\int_a^b \frac{(b^p - x^p)^{\alpha-1}}{x^{1-p}} f(x)dx \\
&= \frac{f(a)}{2} - \frac{p^{\alpha}\Gamma(\alpha+1)}{2(b^p - a^p)^{\alpha}}\left[{}^{p}I_{a+}^{\alpha} f(b)\right].
\end{aligned}$$

By adding the results of (2.6) and (2.7) side by side in the equation (2.6), we obtain the inequality (2.4). This completes the proof. □

Remark 2.3. Also in the equation (2.4) of Lemma (2.1), if we take specially $\alpha = 1$, then the inequality reduces to the equation (1.4).

By using Lemma 2.1, we can have the following fractional inequality.

Theorem 2.2. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°,$a, b \in I$ with $a < b$, $p > 0$, and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q \geq 1$ then the following inequality for fractional integrals holds:*

(2.8)
$$|K_f(\alpha,a,b)| \leq \frac{b^p - a^p}{2p} M_1^{1-1/q}(\alpha,a,b)\left[M_2(\alpha,a,b)\left|f'(a)\right|^q + M_3(\alpha,a,b)\left|f'(b)\right|^q\right]^{1/q}$$

where

$$M_1(\alpha,a,b) = \frac{b^{1-p}}{\alpha+1}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},1;\alpha+2;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+2;1-\frac{a^p}{b^p}\right)\end{array}\right]$$

$$M_2(\alpha,a,b) = \frac{b^{1-p}}{\alpha+2}\left[\begin{array}{c} \frac{1}{\alpha+1}\,{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},\alpha+2;\alpha+3;1-\frac{a^p}{b^p}\right) \end{array}\right]$$

$$M_3(\alpha,a,b) = \frac{b^{1-p}}{\alpha+1}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},1;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +\frac{1}{\alpha+1}\,{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+3;1-\frac{a^p}{b^p}\right) \end{array}\right].$$

Proof. From Lemma 2.1 by using the property of the modulus, the power mean inequality and the p-convexity of $|f'|^q$, then we have,

$$\begin{aligned}
&|K_f(\alpha,a,b)| \\
(2.9)\quad &\leq \frac{b^p-a^p}{2p}\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
&\leq \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-1/q} \\
&\quad\times\left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1/q} \\
&\leq \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{[(1-t)^\alpha + t^\alpha]}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-1/q} \\
&\quad\times\left(\int_0^1 \frac{[(1-t)^\alpha + t^\alpha]}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left[t\left|f'(a)\right|^q+(1-t)t\left|f'(b)\right|^q\right]dt\right)^{1/q} \\
(2.10) &= \frac{b^p-a^p}{2p}M_1^{1-1/q}(\alpha,a,b)\left[M_2(\alpha,a,b)\left|f'(a)\right|^q+M_3(\alpha,a,b)\left|f'(b)\right|^q\right]^{1/q},
\end{aligned}$$

where, by simple computation, we obtain,

$$\begin{aligned}
M_1(\alpha,a,b) &= \int_0^1 \frac{[(1-t)^\alpha + t^\alpha]}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
(2.11)\qquad &= \frac{b^{1-p}}{\alpha+1}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},1;\alpha+2;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+2;1-\frac{a^p}{b^p}\right) \end{array}\right]
\end{aligned}$$

$$M_2(\alpha,a,b) = \int_0^1 \frac{[(1-t)^\alpha + t^\alpha]}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}t\,dt$$

$$(2.12)\qquad = \frac{b^{1-p}}{\alpha+2}\left[\begin{array}{c} \frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},\alpha+2;\alpha+3;1-\frac{a^p}{b^p}\right) \end{array}\right]$$

$$\begin{aligned} M_3(\alpha,a,b) &= \int_0^1 \frac{[(1-t)^\alpha+t^\alpha]}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}(1-t)dt \\ (2.13)\qquad &= \frac{b^{1-p}}{\alpha+1}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},1;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+3;1-\frac{a^p}{b^p}\right) \end{array}\right]. \end{aligned}$$

Then by using the results from the equations (2.11)-(2.13) in the equation (2.10), we have desired result (2.9). This completes the proof. □

Remark 2.4. If we specially take $\alpha = 1$, in inequality 2.9, then the inequality reduces to [22, Theorem 3.2].

When $0 < \alpha \leq 1$ by using Lemma 1.2 and Lemma 2.1, we have another fractional integral inequality for p convex functions as follows.

Theorem 2.3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°,$a,b \in I$ with $a < b$, $p > 0$, and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$(2.14)\qquad |K_f(\alpha,a,b)| \leq \frac{b^p-a^p}{2p} M_4^{1-1/q}(\alpha,a,b)\left[M_5(\alpha,a,b)\,|f'(a)|^q + M_6(\alpha,a,b)\,|f'(b)|^q\right]^{1/q}$$

where

$$\begin{aligned} M_4(\alpha,a,b) &= \frac{b^{1-p}}{\alpha+1}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+2;1-\frac{a^p}{b^p}\right) \\ -{}_2F_1\left(1-\frac{1}{p},1;\alpha+2;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},1;\alpha+2;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{array}\right] \\ M_5(\alpha,a,b) &= \frac{b^{1-p}}{\alpha+2}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},\alpha+2;\alpha+3;1-\frac{a^p}{b^p}\right) \\ -\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +\frac{1}{2(\alpha+1)}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{array}\right] \\ M_6(\alpha,a,b) &= \frac{b^{1-p}}{\alpha+2}\left[\begin{array}{c} \frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+3;1-\frac{a^p}{b^p}\right) \\ -{}_2F_1\left(1-\frac{1}{p},1;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},1;\alpha+2;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \\ -\frac{1}{2(\alpha+1)}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{array}\right] \end{aligned}$$

and $0 < \alpha \leq 1$.

Proof. From Lemma 2.1 using the property of the modulus, the power mean inequality and the p-convexity of $|f'|^q$, we have,

$$
\begin{aligned}
& |K_f(\alpha,a,b)| \\
(2.15)\quad \le\ & \frac{b^p-a^p}{2p}\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
\le\ & \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-1/q} \\
& \times\left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1/q} \\
(2.16)\quad & \\
\le\ & \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-1/q} \\
& \times\left(\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left[t\left|f'(a)\right|^q+(1-t)t\left|f'(b)\right|^q\right]dt\right)^{1/q} \\
(2.17)\quad =\ & \frac{b^p-a^p}{2p}K_4^{1-1/q}(\alpha,a,b)\left[K_5(\alpha,a,b)\left|f'(a)\right|^q+K_6(\alpha,a,b)\left|f'(b)\right|^q\right]^{1/q},
\end{aligned}
$$

where by using Lemma 1.2 and by simple calculations of integrals, we obtain,

$$
\begin{aligned}
K_4 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
&= \int_0^{1/2} \frac{(1-t)^\alpha - t^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + \int_{1/2}^1 \frac{t^\alpha-(1-t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
&= \int_0^1 \frac{t^\alpha-(1-t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt + 2\int_0^{1/2} \frac{(1-t)^\alpha - t^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
&\le \int_0^1 \frac{t^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt - \int_0^1 \frac{(1-t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
&\quad +2\int_0^{1/2} \frac{(1-2t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt
\end{aligned}
$$

$$(2.18)\qquad = M_4(\alpha,a,b) = \frac{b^{1-p}}{\alpha+1}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+2;1-\frac{a^p}{b^p}\right)- \\ {}_2F_1\left(1-\frac{1}{p},1;\alpha+2;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},1;\alpha+2;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{array}\right],$$

$$\begin{aligned} K_5 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}tdt \\ &\le \int_0^1 \frac{t^{\alpha+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt - \int_0^1 \frac{(1-t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}tdt \\ &\quad +2\int_0^{1/2} \frac{(1-2t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}tdt \end{aligned}$$

$$(2.19)\ = M_5(\alpha,a,b) = \frac{b^{1-p}}{\alpha+2}\left[\begin{array}{c} {}_2F_1\left(1-\frac{1}{p},\alpha+2;\alpha+3;1-\frac{a^p}{b^p}\right) \\ -\frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +\frac{1}{2(\alpha+1)}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{array}\right],$$

$$\begin{aligned} K_6 &= \int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}(1-t)dt \\ &\le \int_0^1 \frac{t^{\alpha}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}(1-t)dt - \int_0^1 \frac{(1-t)^{\alpha+1}}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\ &\quad +2\int_0^{1/2} \frac{(1-2t)^\alpha}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}(1-t)dt \end{aligned}$$

$$(2.20)\ = M_6(\alpha,a,b) = \frac{b^{1-p}}{\alpha+2}\left[\begin{array}{c} \frac{1}{\alpha+1}{}_2F_1\left(1-\frac{1}{p},\alpha+1;\alpha+3;1-\frac{a^p}{b^p}\right) \\ -{}_2F_1\left(1-\frac{1}{p},1;\alpha+3;1-\frac{a^p}{b^p}\right) \\ +{}_2F_1\left(1-\frac{1}{p},1;\alpha+2;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \\ -\frac{1}{2(\alpha+1)}{}_2F_1\left(1-\frac{1}{p},2;\alpha+3;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{array}\right].$$

Then by using the results from the equations (2.18)-(2.20), we have the desired inequality (2.15). This completes the proof. □

Theorem 2.4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°,$a,b \in I$ with $a < b$, $p > 0$, and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$|K_f(\alpha,a,b)| \leq \frac{b^p-a^p}{2p} M_7^{1/r}(\alpha,a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q} \tag{2.21}$$

where

$$M_7(\alpha,a,b) = \frac{b^{1-p}}{2}\,{}_2F_1\left(r-\frac{r}{p},1;2;1-\frac{a^p}{b^p}\right)$$

and $1/r + 1/q = 1$.

Proof. From Lemma 1.2 and Lemma 2.1, by using the property of the modulus, the Hölder inequality and the p-convexity of $|f'|^q$, we obtain,

$$\begin{aligned}
&|K_f(\alpha,a,b)| \\
\leq\;& \frac{b^p-a^p}{2p}\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\
\leq\;& \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt\right)^{1/r} \\
&\times\left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{1/q} \\
\leq\;& \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt\right)^{1/r} \qquad (2.22)\\
&\times\left(\int_0^1 |1-2t|^{\alpha q}\left[t\,|f'(a)|^q+(1-t)t\,|f'(b)|^q\right]dt\right)^{1/q}
\end{aligned}$$

$$= \frac{b^p-a^p}{2p} M_7^{1/r}(\alpha,a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}, \tag{2.23}$$

after calculations of integrals in the inequality (2.21) as follows,

$$M_7(\alpha,a,b) = \int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt = \frac{b^{1-p}}{2}\,{}_2F_1\left(r-\frac{r}{p},1;2;1-\frac{a^p}{b^p}\right) \tag{2.24}$$

$$\int_0^1 |1-2t|^{\alpha q}\, tdt = \int_0^{1/2} (1-2t)^{\alpha q}\, tdt + \int_{1/2}^1 (2t-1)^{\alpha q}\, tdt = \frac{1}{2(\alpha q+1)} \tag{2.25}$$

$$\int_0^1 |1-2t|^{\alpha q}\,(1-t)dt = \frac{1}{2(\alpha q+1)}. \tag{2.26}$$

Then by using the results from the equations (2.24)-(2.26) in the equation (2.23), then we have the desired result (2.21). This the completes the proof. □

Theorem 2.5. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°,$a, b \in I$ with $a < b$, $p > 0$, and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

(2.27)

$$|K_f(\alpha,a,b)| \leq \tfrac{b^p-a^p}{2p}\left(M_8^{1/r}(\alpha,a,b)+M_9^{1/r}(\alpha,a,b)\right)\left(\tfrac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}$$

where

$$\begin{aligned} M_8(\alpha,a,b) &= \frac{b^{(1-p)r}}{\alpha p+1}{}_2F_1\left(r-\frac{r}{p},1;\alpha r+2;1-\frac{a^p}{b^p}\right) \\ M_9(\alpha,a,b) &= \frac{b^{(1-p)r}}{\alpha p+1}{}_2F_1\left(r-\frac{r}{p},\alpha r+1;\alpha r+2;1-\frac{a^p}{b^p}\right) \end{aligned}$$

and $1/r+1/q=1$.

Proof. From Lemma 2.1, by using the property of the modulus, the Hölder inequality and the p-convexity of $|f'|^q$, then we obtain,

$$\begin{aligned} &|K_f(\alpha,a,b)| \\ \leq\;& \frac{b^p-a^p}{2p}\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt \\ \leq\;& \frac{b^p-a^p}{2p}\left\{\begin{array}{l} \left(\int_0^1 \frac{(1-t)^{\alpha r}}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt\right)^{1/r}\left(\int_0^1\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^1 \frac{t^{\alpha r}}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt\right)^{1/r}\left(\int_0^1\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right\} \end{aligned}$$

(2.28)

$$\leq \frac{b^p-a^p}{2p}\left(M_8^{1/r}(\alpha,a,b)+M_9^{1/r}(\alpha,a,b)\right)\left(\int_0^1 t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\,dt\right)^{\frac{1}{q}}$$

(2.29)

$$= \frac{b^p-a^p}{2p}\left(M_8^{1/r}(\alpha,a,b)+M_9^{1/r}(\alpha,a,b)\right)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q},$$

after calculations of integrals in the inequality (2.28) as follows,

(2.30)

$$M_8(\alpha,a,b)=\int_0^1 \frac{(1-t)^{\alpha r}}{\left[ta^p+(1-t)b^p\right]^{r-\frac{r}{p}}}dt=\frac{b^{(1-p)r}}{\alpha p+1}\,{}_2F_1\left(r-\frac{r}{p},1;\alpha r+2;1-\frac{a^p}{b^p}\right)$$

(2.31)

$$M_9(\alpha,a,b)=\int_0^1 \frac{t^{\alpha r}}{\left[ta^p+(1-t)b^p\right]^{r-\frac{r}{p}}}dt=\frac{b^{(1-p)r}}{\alpha p+1}\,{}_2F_1\left(r-\frac{r}{p},\alpha r+1;\alpha r+2;1-\frac{a^p}{b^p}\right).$$

Then by using the results from the equations (2.31)-(2.32) in the equation (2.29), then we have the desired inequality (2.28). This completes the proof. □

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