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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

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NEW INEQUALITIES OF WIRTINGER TYPE FOR CONVEX AND MN-CONVEX FUNCTIONS

Tatjana Z. Mirković

Abstract. In this paper, we obtain some inequalities of Wirtinger type by using some classical inequalities and means for convex functions and establish some applications to special means for positive real numbers.

Keywords. Inequalities; inequalities of Wirtinger type; convex functions.

1. Introduction

Let f be a periodic function with period 2π and let $f' \in L^2$. Then, if $\int\limits_0^{2\pi} f(x)\,dx = 0$, the following inequality holds

$$\int_0^{2\pi} f^2(x)\,dx \leqslant \int_0^{2\pi} f'^2(x)\,dx, \tag{1.1}$$

with equality if and only if $f(x) = A\cos x + B\sin x$, where A and B are constants.

Inequality (1.1) is known in the literature as Witinger's inequality. The proof of W. Wirtinger was first published in 1916 in the book (see [1]) by W. Blaschke. There are many studies which generalize and extend Wirtinger's inequality in the literature, (see [2], [3]). However, Inequality (1.1) was known before this, though with other conditions on the function f. For example, in 1905, E.Almansi proved that

$$\int_a^b f^2(x)\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b \left[f'(x)\right]^2 dx, \tag{1.2}$$

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under the condition that f and f' are continuous on the interval (a, b), that $f(a) = f(b)$ and that $\int\limits_a^b f(x)\,dx = 0$.

Theorem 1.1. *(Hölder inequality) Let $f(x)$ and $g(x)$ be positive continuous functions on $[a, b]$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int\limits_a^b f(x)\,g(x)dx \leqslant \left(\int\limits_a^b f(x)^p\,dx\right)^{\frac{1}{p}} \left(\int\limits_a^b g(x)^q\,dx\right)^{\frac{1}{q}}. \tag{1.3}$$

Theorem 1.2. *(Reverse Hölder inequality) For two positive functions f and g satisfying $0 < m \leqslant \frac{f^p}{g^q} \leqslant M < \infty$, on the set X, and for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get*

$$\left(\int\limits_X f^p d\mu\right)^{\frac{1}{p}} \left(\int\limits_X g^q d\mu\right)^{\frac{1}{q}} \leqslant \left(\frac{M}{m}\right)^{\frac{1}{pq}} \int\limits_X fg\,d\mu.$$

Definition 1.1. *A function $I \subseteq R \to R$ is said to be convex (concave) if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:*

$$f(tx + (1-t)y) \leqslant (\geq)\, tf(x) + (1-t)f(y). \tag{1.4}$$

Anderson mentioned mean function in [4] as follows:

Definition 1.2. *A function $M : (0, \infty) \to (0, \infty)$ is called a mean function if*

(a) Symmetry: $M(x, y) = M(y, x)$;

(b) Reflexivity: $M(x, x) = x$;

(c) Monotonicity: $\min\{x, y\} \leqslant M(x, y) \leqslant \max\{x, y\}$;

(d) Homogeneity: $M(\lambda x, \lambda y) = \lambda M(x, y)$, for any positive scalar λ.

Definition 1.3. *Let $I \to (0, \infty)$ be continuous, where I is a subinterval of $(0, \infty)$. Let M and N be any two mean functions. We say f is MN-convex (concave) if $f(M(x, y)) \leqslant (\geqslant)\, N(f(x), f(y))$ for all $x, y \in I$.*

Taking into account Definition 1.3, MN-convex function will be defined by the formulas:

1. f is AA-convex iff (1.4) holds;
2. f is AG-convex iff

$$f(t\alpha + (1-t)) \leqslant [f(\alpha)]^t\,[f(\beta)]^t, \quad 0 \leqslant t \leqslant 1;$$

3. f is AH-convex iff

$$f\left((1-t)\alpha+t\beta\right)\leqslant\frac{f(\alpha)f(\beta)}{tf(\alpha)+(1-t)f(\beta)},\qquad 0\leqslant t\leqslant 1;$$

4. f is GA-convex iff

$$f\left(\alpha^{t}\beta^{1-t}\right)\leqslant tf(\alpha)+(1-t)f(\beta),\qquad 0\leqslant t\leqslant 1;$$

5. f is GG-convex iff

$$f\left(\alpha^{t}\beta^{1-t}\right)\leqslant[f(\alpha)]^{t}[f(\beta)]^{1-t},\qquad 0\leqslant t\leqslant 1;$$

6. f is GH-convex iff

$$f\left(\alpha^{1-t}\beta^{t}\right)\leqslant\frac{f(\alpha)f(\beta)}{tf(\alpha)+(1-t)f(\beta)},\qquad 0\leqslant t\leqslant 1;$$

7. f is HA-convex iff

$$f\left(\frac{\alpha\beta}{(1-t)\alpha+t\beta}\right)\leqslant tf(\alpha)+(1-t)f(\beta),\qquad 0\leqslant t\leqslant 1;$$

8. f is HG-convex iff

$$f\left(\frac{\alpha\beta}{(1-t)\alpha+t\beta}\right)\leqslant[f(\alpha)]^{t}[f(\beta)]^{1-t},\qquad 0\leqslant t\leqslant 1;$$

9. f is HH-convex iff

$$f\left(\frac{\alpha\beta}{(1-t)\alpha+t\beta}\right)\leqslant\frac{f(\alpha)f(\beta)}{(1-t)f(\alpha)+tf(\beta)},\qquad 0\leqslant t\leqslant 1.$$

The main aim of this paper is to prove some new Wirtinger-type integral inequalities for convex and MN-convex functions.

2. Main Results

Theorem 2.1. *Let f and f' be continuous functions on the interval (a,b), with $a<b$, $f(a)=f(b)$ and $\int_a^b f(x)\,dx=0$. If $(f')^2$ is convex on $[a,b]$, then*

$$\int_a^b[f(x)]^2\,dx\leqslant\frac{(b-a)^3}{(2\pi)^2}\frac{[f'(a)]^2+[f'(b)]^2}{2}\tag{2.1}$$

Proof: Since $(f')^2$ is a convex function on $[a,b]$, therefore for $t \in [0,1]$ we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b [f'(x)]^2\,dx &= \int_0^1 [f'(ta+(1-t)\,b)]^2\,dt \\
&\leqslant \int_0^1 \left[t[f'(a)]^2 + (1-t)\,[f'(b)]^2\right] dt = \frac{[f'(a)]^2+[f'(b)]^2}{2}.
\end{aligned}$$

Now multiplying both sides of the above inequality by $\frac{(b-a)^3}{(2\pi)^2}$ and with (1.2), we get the desired inequality in (2.1). $\square$

Theorem 2.2. *Let f and f' be continuous functions on the interval (a,b), with $a<b$, $f(a)=f(b)$ and $\int_a^b f(x)\,dx = 0$. If f' is convex on $[a,b]$, then*

$$\int_a^b [f(x)]^2\,dx \leqslant \frac{(b-a)^3}{(2\pi)^2}\left\{\frac{(f'(a))^2+(f'(a))\,(f'(a))+(f'(b))^2}{3}\right\}. \tag{2.2}$$

Proof: We have

$$\begin{aligned}
\left(\frac{b-a}{2\pi}\right)^2\int_a^b [f'(x)]^2\,dx &= \frac{(b-a)^3}{(2\pi)^2}\int_0^1 [f'(ta+(1-t)\,b)]^2\,dt \\
&\leqslant \frac{(b-a)^3}{(2\pi)^2}\int_0^1 [tf'(a)+(1-t)\,f'(b)]^2\,dt \\
&= \frac{(b-a)^3}{(2\pi)^2}\left\{\frac{(f'(a))^2+(f'(a))\,(f'(a))+(f'(b))^2}{3}\right\}.
\end{aligned}$$

By applying (1.2), we get (2.2). $\square$

Theorem 2.3. *Let f and f' be continuous functions on the interval (a,b) with $a<b$, $f(a)=f(b)$ and $\int_a^b f(x)\,dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are convex on $[a,b]$, then the following inequality holds*

$$\int_a^b [f(x)]^2\,dx \leqslant \alpha\,(b-a)^3\,\frac{[f'(a)]^{\frac{1}{\alpha}}+[f'(b)]^{\frac{1}{\alpha}}}{8\pi^2} + \beta\,(b-a)^3\,\frac{[f'(a)]^{\frac{1}{\beta}}+[f'(b)]^{\frac{1}{\beta}}}{8\pi^2}, \tag{2.3}$$

where $\alpha, \beta > 0$ and $\alpha+\beta = 1$.

Proof: By using the well-known inequality $cd \leqslant \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$ ($\alpha, \beta, c, d > 0$ and $\alpha + \beta = 1$), the convexity of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$, we get

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 f'(ta+(1-t)b)\,f'(ta+(1-t)b)\,dt$$
$$\leqslant \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha \int_0^1 [f'(ta+(1-t)b)]^{\frac{1}{\alpha}}\,dt + \beta \int_0^1 [f'(ta+(1-t)b)]^{\frac{1}{\beta}}\,dt\right\}$$
$$\leqslant \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha \int_0^1 [tf'(a)+(1-t)f'(b)]^{\frac{1}{\alpha}}\,dt + \beta \int_0^1 [tf'(a)+(1-t)f'(b)]^{\frac{1}{\beta}}\,dt\right\}$$
$$\leqslant \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha \int_0^1 \left[t(f'(a))^{\frac{1}{\alpha}} + (1-t)(f'(b))^{\frac{1}{\alpha}}\right]dt\right.$$
$$\left. + \beta \int_0^1 \left[t(f'(a))^{\frac{1}{\beta}} + (1-t)(f'(b))^{\frac{1}{\alpha}}\right]dt\right\}$$
$$= \frac{(b-a)^3}{(2\pi)^2}\left\{\alpha \frac{(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}}}{2} + \beta \frac{(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}}}{2}\right\}.$$

Combining with (1.2), we get the required inequality. □

Theorem 2.4. *Let f and f' be continuous on the interval (a,b), with $a < b$, $f(a) = f(b)$, $\int_a^b f(x)\,dx = 0$ and $f > 0$. Let $0 < m \leqslant \frac{|f|^p}{|f|^q} \leqslant M < \infty$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f|^p$, $|f|^q$ are concave on $[a,b]$ then*

$$\left(\frac{m}{M}\right)^{\frac{1}{pq}} [f(a)+f(b)]^2 \leqslant \frac{b-a}{\pi^2} \int_a^b [f'(x)]^2\,dx. \tag{2.4}$$

Proof: Making changes to the variable, using the reverse Hölder inequality and inequality $|u+v|^r \leqslant 2^{r-1}(|u|^r + |v|^r)$, $u, v \in \mathbb{R}$, we have

$$\frac{1}{b-a}\int_a^b [f(x)]^2\,dx = \int_0^1 [f(ta+(1-t)b)]^2\,dt$$
$$\geqslant \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_0^1 |f(ta+(1-t)b)|^p\,dt\right)^{\frac{1}{p}} \left(\int_0^1 |f(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}$$

$$\geqslant \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_0^1 \left[t|f(a)|^p + (1-t)|f(b)|^p\right] dt\right)^{\frac{1}{p}}$$

$$\cdot \left(\int_0^1 \left[t|f(a)|^q + (1-t)|f(b)|^q\right] dt\right)^{\frac{1}{q}}$$

$$= \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\frac{|f(a)|^p + |f(b)|^p}{2}\right)^{\frac{1}{p}} \left(\frac{|f(a)|^q + |f(b)|^q}{2}\right)^{\frac{1}{q}}$$

$$\geqslant \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\frac{|f(a) + f(b)|^p}{2^p}\right)^{\frac{1}{p}} \left(\frac{|f(a) + f(b)|^q}{2^q}\right)^{\frac{1}{q}}$$

$$= \left(\frac{m}{M}\right)^{\frac{1}{pq}} \frac{(f(a) + f(b))^2}{4}. \quad \square$$

By (1.2), we get the inequality (2.4). $\square$

Theorem 2.5. *Let f and f' be continuous on the interval (a,b), with $a < b$, $f(a) = f(b)$ and $\int_a^b f(x)\,dx = 0$. Then:*

1. *If $|f'|$ is AG convex, then*

$$\int_a^b [f(x)]^2\,dx \leqslant \frac{(b-a)^3}{8\pi^2} \frac{[f'(a)\,f'(b)]^2 - 1}{\ln[f'(a)\,f'(b)]};$$

2. *If $|f'|$ is AH convex, then*

$$\int_a^b [f(x)]^2\,dx \leqslant \frac{(b-a)^3}{(2\pi)^2} f'(a)\,f'(b);$$

3. *If $|f'|$ is GA convex, then*

$$\int_a^b [f(x)]^2\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \left\{ \left[-a + \frac{2a}{\ln\frac{a}{b}} + 2\frac{b-a}{\ln^2\frac{a}{b}}\right] [f'(a)]^2 \right.$$

$$+ \left[-2(a+b)\frac{1}{\ln\frac{a}{b}} - 4\frac{b-a}{\ln^2\frac{a}{b}}\right] f'(a)\,f'(b)$$

$$\left. + \left[b + \frac{2b}{\ln\frac{a}{b}} + \frac{2(b-a)}{\ln^2\frac{a}{b}}\right] [f'(b)]^2 \right\};$$

4. If $|f'|$ is GG convex, then

$$\int_a^b [f(x)]^2\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\,\frac{b[f'(b)]^2 - a[f'(a)]^2}{\ln a[f'(a)]^2 - \ln b[f'(b)]^2};$$

5. If $|f'|$ is HA convex, then

$$\begin{aligned}\int_a^b [f(x)]^2\,dx \leqslant \frac{1}{(2\pi)^2}\Bigg\{&\left[a(b-a)\left(b+a(b-a)^2\right) - a^2 b\ln\frac{b}{a}\right][f'(a)]^2\\ &+\left[ab(b-a)\left(1-(b-a)^2\right) - ab(a+b)\ln\frac{b}{a}\right] f'(a)\,f'(b)\\ &+\left[b(b-a)\left(a+b(b-a)^2\right) - ab^2\ln\frac{b}{a}\right][f'(b)]^2\Bigg\}.\end{aligned}$$

Proof.

1. From (1.2) and by using the AG convexity of $|f'|$ we have

$$\begin{aligned}\int_a^b [f(x)]^2\,dx &\leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx = \frac{(b-a)^3}{(2\pi)^2}\int_0^1 [f'(ta+(1-t)b)]^2\,dt\\ &\leqslant \frac{(b-a)^3}{(2\pi)^2}\int_0^1 \left[(f'(a))^t (f'(b))^t\right]^2 dt = \frac{(b-a)^3}{(2\pi)^2}\int_0^1 [(f'(a))(f'(b))]^{2t}\,dt\\ &= \frac{(b-a)^3}{8\pi^2}\,\frac{[f'(a)\,f'(b)]^2 - 1}{\ln[f'(a)\,f'(b)]};\end{aligned}$$

2. Since $|f'|$ is an AH-convex function, we can write

$$\begin{aligned}\int_a^b [f(x)]^2\,dx &\leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx = \frac{(b-a)^3}{(2\pi)^2}\int_0^1 [f'((1-t)a+tb)]^2\,dt\\ &\leqslant \frac{(b-a)^3}{(2\pi)^2}\int_0^1 \left[\frac{f'(a)\,f'(b)}{tf'(a)+(1-t)\,f'(b)}\right]^2 dt = \frac{(b-a)^3}{(2\pi)^2} f'(a)\,f'(b);\end{aligned}$$

3. Taking into account that $|f'|$ is GA- convex, we have

$$\int_a^b [f(x)]^2\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx = \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\int_1^0 \left[f'\left(a^t b^{1-t}\right)\right]^2 a^t b^{1-t} dt$$

$$\leqslant \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\int_1^0 [tf'(a)+(1-t)f'(b)]^2 a^t b^{1-t} dt$$

$$= b\left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\left\{\left[(f'(a))^2 - 2f'(a)f'(b) + (f'(b))^2\right]\int_1^0 t^2\left(\frac{a}{b}\right)^t dt\right.$$

$$\left. + \left[2f'(a)f'(b) - 2(f'(b))^2\right]\int_1^0 t\left(\frac{a}{b}\right)^t dt + (f'(b))^2\int_1^0 \left(\frac{a}{b}\right)^t dt\right\}$$

$$= \left(\frac{b-a}{2\pi}\right)^2 \left\{\left[-a + \frac{2a}{\ln\frac{a}{b}} + 2\frac{b-a}{\ln^2\frac{a}{b}}\right][f'(a)]^2\right.$$

$$+ \left[-2(a+b)\frac{1}{\ln\frac{a}{b}} - 4\frac{b-a}{\ln^2\frac{a}{b}}\right] f'(a)f'(b)$$

$$\left. + \left[b + \frac{2b}{\ln\frac{a}{b}} + \frac{2(b-a)}{\ln^2\frac{a}{b}}\right][f'(b)]^2\right\};$$

4. Since $|f'|$ is GG-convex, we have

$$\int_a^b [f(x)]^2\,dx \leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx$$

$$= \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\int_1^0 \left[f'\left(a^t b^{1-t}\right)\right]^2 a^t b^{1-t} dt$$

$$\leqslant \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\int_1^0 \left\{[f'(a)]^t [f'(b)]^{1-t}\right\}^2 a^t b^{1-t} dt$$

$$= b\left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}[f'(b)]^2\int_1^0 \left\{\frac{a[f'(a)]^2}{b[f'(b)]^2}\right\}^t dt$$

$$= \left(\frac{b-a}{2\pi}\right)^2 \ln\frac{a}{b}\,\frac{b[f'(b)]^2 - a(f'(a))^2}{\ln a[f'(a)]^2 - \ln b[f'(b)]^2};$$

5. Since $|f'|$ is HA-convex, we have

$$\begin{aligned}
\int_a^b [f(x)]^2\,dx &\leqslant \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2\,dx \\
&= ab\frac{(b-a)^3}{(2\pi)^2}\int_0^1 \left[f'\left(\frac{ab}{(1-t)\,a+tb}\right)\right]^2 \frac{1}{[(1-t)\,a+tb]^2}dt \\
&\leqslant ab\frac{(b-a)^3}{(2\pi)^2}\int_0^1 \left[\left(\frac{tf'(a)+(1-t)\,f'(b)}{(1-t)\,a+tb}\right)\right]^2 dt \\
&= \frac{1}{(2\pi)^2}\left\{\left[a\,(b-a)\left(b+a\,(b-a)^2\right)-a^2 b\ln\frac{b}{a}\right][f'(a)]^2\right. \\
&\quad + \left[ab\,(b-a)\left(1-(b-a)^2\right)-ab\,(a+b)\ln\frac{b}{a}\right]f'(a)\,f'(b) \\
&\quad \left. + \left[b\,(b-a)\left(a+b\,(b-a)^2\right)-ab^2\ln\frac{b}{a}\right][f'(b)]^2\right\}. \qquad \square
\end{aligned}$$

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THE COMPARABLY ALMOST (S,T)- STABILITY FOR RANDOM JUNGCK-TYPE ITERATIVE SCHEMES

Dhekra M. Albaqeri and Rashwan A. Rashwan

Abstract. In this paper, we introduce the concept of generalized ϕ - weakly contractive random operators and study a new type of stability introduced by Kim [15] which is called a comparably almost stability and then prove the comparably almost (S,T)- stability for the Jungck-type random iterative schemes. Our results extend and improve the recent results in [15], [18], [32] and many others. We also give stochastic version of many important known results.

Keywords. Weakly contractive random operators; stability; Jungck-type random iterative schemes.

1. Introduction

The theory of random operator is an important branch of probabilistic analysis which plays a key role in many applied areas. The study of random fixed points forms a central topic in this area. Research of this direction was initiated by the Prague School of probabilists in connection with random operator theory [7, 8, 29]. Random fixed point theory has attracted much attention in recent times since the publication of the survey article by Bharucha-Reid [6] in 1976, in which the stochastic versions of some well-known fixed point theorems were proved. A lot of efforts have been devoted to random fixed point theory and applications (see e.g. [2, 3, 4, 5, 13, 24, 30]) and many others.

In (1953) Mann [16] introduced an iterative scheme and employed it to approximate the solution of a fixed point problem defined by non-expansive mapping where Picard iterative scheme failed to converge. After that in (1974) Ishikawa [12] introduced an iterative scheme and employed it to obtain the convergence of a Lipschitzian pseudo-contractive operator when Manns iterative scheme is not applicable. Later in (2000) Noor [17] introduced the iterative algorithm to solve variational inequality problems. Recently, Phuengrattana and Suantai [25] introduced

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SP iterative scheme and proved that it has a better convergence rate as compared to Mann, Ishikawa and Noor iterative schemes.
About Jungck iterative, in (1976), Jungck [14] introduced the Jungck iterative process as follows:
Suppose that X is a Banach space, Y an arbitrary set and $S, T : Y \to X$ are such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, consider the iterative scheme:

$$Sx_{n+1} = Tx_n, n = 0, 1, \dots .$$

He used this iterative process to approximate the common fixed points of the mappings S and T satisfying the Jungck contraction. Clearly, this iterative process reduces to the Picard iteration when $S = I_d$ (identity mapping) and Y = X. Later, Singh et al. [28] introduced the Jungck- Mann iterative process as:

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n,\ \ \alpha_n \in [0, 1].$$

For $\alpha_n, \beta_n, \gamma_n \in [0, 1]$, Olatinwo [21] defined the Jungck-Ishikawa and Jungck-Noor iterative processes as follows:

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n Tx_n. \end{aligned}$$

$$\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Ty_n, \\ Sy_n &= (1 - \beta_n)Sx_n + \beta_n Tz_n, \\ Sz_n &= (1 - \gamma_n)Sx_n + \gamma_n Tx_n. \end{aligned}$$

The concept of the ϕ- weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [27] in 2001, extended the results of [1] to metric spaces. In 2016, Xue [31] introduced a kind of generalized ϕ-weak contraction as follows:

Definition 1.1. [31]. Let (X,d) be a metric space. A mapping $T : X \to X$ is a generalized ϕ-weak contraction if there exists a continuous and nondecreasing function $\phi : [0, \infty] \to [0, \infty]$ with $\phi(0) = 0$ such that

$$d(Tx, Ty) \leqslant d(x, y) - \phi(d(Tx, Ty)), \forall x, y \in X. \tag{1.1}$$

The concept of stable fixed point iterative scheme was introduced and studied by Harder [9], Harder and Hicks [10, 11]. Many other stability results for several fixed point iterative schemes and various classes of nonlinear mappings were obtained.

Definition 1.2. [11] Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and $x_0 \in X$. Assume that the iterative scheme

$$x_{n+1} = f(T, x_n), n \geq 0. \tag{1.2}$$

converges to a fixed point p of T. Let z_n be an arbitrary sequence in X and define

$$\varepsilon_n = d(z_{n+1}, f(T, z_n)), n \geq 0. \tag{1.3}$$

The iterative scheme defined by (1.2) is said to be T-stable or stable with respect to T if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} z_n = p. \tag{1.4}$$

Osilike [23] introduced a weaker concept of stability.

Definition 1.3. [23] Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and $x_0 \in X$. Assume that the iterative scheme (1.2) converges to a fixed point p of T. Let z_n be an arbitrary sequence in X and defined by (1.3). The iterative scheme defined by (1.2) is said to be almost T-stable or almost stable with respect to T if and only if

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty \Rightarrow \lim_{n\to\infty} z_n = p. \tag{1.5}$$

Remark 1.1. It is obvious that any stable iterative scheme is also almost stable but the reverse is not true in general. For examples see [23].

The definition of (S, T)-stability can be found in Singh et al. [28].

Definition 1.4. [28] Let $S, T : Y \to X$ be non-self operators for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and p a point of coincidence of S and T. Let $\{Sx_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure

$$Sx_{n+1} = f(T, x_n), n = 0, 1, 2, ..., \tag{1.6}$$

where $x_0 \in X$ is the initial approximation and f is some functions. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let$\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\varepsilon_n = d(Sy_n, f(T, y_n)), n = 0, 1, 2, ...$$

Then, the iterative procedure (1.6) is said to be (S,T)-stable if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies $\lim_{n\infty} Sy_n = p$.

In 2017, Kim [15] introduced a new concept of stability which is called comparably almost T- stability defined as:

Definition 1.5. Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and $x_0 \in X$. Assume that the iterative scheme (1.2) converges to a fixed point p of T. Let z_n be an arbitrary sequence in X and defined by (1.3). The iterative scheme defined by (1.2) is said to be comparably almost T-stable or comparably almost stable with respect to T if and only if

$$\sum_{n=0}^{\infty} (\theta_n + \varepsilon_n) < \infty, \theta_n \geq 0 \Rightarrow \lim_{n\to\infty} z_n = p, \lim_{n\to\infty} \theta_n = 0. \tag{1.7}$$

Also, he proved some convergence results of Mann and Ishikawa iterative schemes containing a generalized ϕ- weak contractive self maps defined as in (1.1).

Remark 1.2. 1. It is obvious that any almost stable iterative scheme is also comparably almost stable. See [15].

2. If $\theta_n = 0$ in (1.7), then (1.7) reduces to (1.5). So an almost stable iterative scheme is a special case of comparably almost stable iterative scheme.

The aim of this paper is to introduce the concept of generalized ϕ- weakly contractive random operators and study a new type of stability which is called comparably almost stability and then prove the comparably almost (S,T)- stability for the Jungck- type and SP-Jungck-type random iterative schemes. Our results extend, improve and unify the recent results in [15], [18], [32] and many others. We also give the stochastic version of many important known results.

2. Preliminaries

Let (Ω, Σ) be a measurable space, E be nonempty subset of a separable Banach space X. A mapping $\xi : \Omega \to E$ is called measurable if $\xi^{-1}(B \cap E) \in \Sigma$ for every Borel subset B of X. A mapping $T : \Omega \times E \to E$ is said to be random mapping if for each fixed $x \in E$, the mapping $T(.,x) : \Omega \to E$ is measurable. A measurable mapping $\xi^* : \Omega \to E$ is called a random fixed point of the random mapping $T : \Omega \times E \to E$ if $T(\omega, \xi^*(\omega)) = \xi^*(\omega)$ for each $\omega \in \Omega$. Let $S, T : \Omega \times E \to E$ be two random self-maps. A measurable map ξ^* is called a common random fixed point of the pair (S,T) if $\xi^*(\omega) = S(\omega, \xi^*(\omega)) = T(\omega, \xi^*(\omega))$, for each $\omega \in \Omega$ and some $\xi^*(\omega) \in E$.
let $S, T : \Omega \times E \leftrightarrow E$ be two random operator defined on E and E a nonempty subset of a separable Banach space X. Let $x_0(w) \in E$ be arbitrary measurable mapping for $w \in \Omega, n = 0, 1, ...$ with $T(w, X) \subseteq S(w, X)$, S is injective.
The Jungck-Noor type random iterative scheme is a sequence $\{S(w, x_n(\omega))\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} S(w,x_{n+1}(w)) &= (1-\alpha_n)S(w,x_n(w))+\alpha_n T(w,y_n(w)), \\ S(w,y_n(w)) &= (1-\beta_n)S(w,x_n(w))+\beta_n T(w,z_n(w)), \\ S(w,z_n(w)) &= (1-\gamma_n)S(w,x_n(w))+\gamma_n T(w,x_n(w)), \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in (0,1).
The Jungck-SP type random iterative scheme is a sequence $\{S(w, x_n(\omega))\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} S(w,x_{n+1}(w)) &= (1-\alpha_n)S(w,y_n(w))+\alpha_n T(w,y_n(w)), \\ S(w,y_n(w)) &= (1-\beta_n)S(w,z_n(w))+\beta_n T(w,z_n(w)), \\ S(w,z_n(w)) &= (1-\gamma_n)S(w,x_n(w))+\gamma_n T(w,x_n(w)), \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in (0,1).

Remark 2.1. 1. If $\gamma_n = 0$ for each $n \in \mathbb{N}$ in (2.1), then the Jungck-Noor type random iterative scheme reduce to Jungck-Ishikawa type random iterative scheme.

$$\begin{aligned} S(w, x_{n+1}(w)) &= (1-\alpha_n)S(w, x_n(w)) + \alpha_n T(w, y_n(w)), \\ S(w, y_n(w)) &= (1-\beta_n)S(w, x_n(w)) + \beta_n T(w, x_n(w)), \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in (0,1).

2. If $\beta_n = \gamma_n = 0$ for each $n \in \mathbb{N}$ in (2.1), then the Jungck-Noor type random iterative scheme reduce to Jungck-Mann type random iterative scheme.

$$S(w, x_{n+1}(w)) = (1-\alpha_n)S(w, x_n(w)) + \alpha_n T(w, x_n(w)), \tag{2.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is real sequence in (0,1).

Zhang et al. [32] in (2011), studied the almost sure T-stability and convergence of Ishikawa-type and Mann-type random iterative processes for certain ϕ- weakly contractive-type random operators in a separable Banach space. The following is the contractive condition studied by Zhang et al. [32].

Definition 2.1. [32] Let (Ω, Σ, μ) be a complete probability measure space and E be a nonempty subset of a separable Banach space X. A random operator $T : \Omega \times E \leftrightarrow E$ is called a ϕ- weakly contractive-type random operator if there exists a continuous and non- decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0, \infty)$ and $\phi(0) = 0$ such that for each $x, y \in E, \omega \in \Omega$,

$$\int_\Omega \|T(w,x) - T(w,y)\| d\mu(w) \leq \int_\Omega \|x-y\| d\mu(w) - \phi\left(\int_\Omega \|x-y\| d\mu(w)\right) \tag{2.5}$$

Recently, in (2015) Okeke and Abbas [18] introduced the concept of generalized ϕ- weakly contraction random operators and then proved the convergence and almost sure T-stability of Mann-type and Ishikawa-type random iterative schemes. Their results improved the results of Zhang et al. [32] and Olatinwo [22] and others. The generalized ϕ- weakly contraction is defined as follows:

Definition 2.2. [18] Let (Ω, Σ, μ) be a complete probability measure space and E be a nonempty subset of a separable Banach space X. A random operator $T : \Omega \times E \leftrightarrow E$ is called a ϕ- weakly contractive-type random operator if there exists $L(w) \geq 0$ and a continuous and non- decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0, \infty)$ and $\phi(0) = 0$ such that for each $x, y \in E, \omega \in \Omega$,

$$\int_\Omega \|T(w,x) - T(w,y)\| d\mu(w) \leq e^{L(w)\|x-y\|}\left(\int_\Omega \|x-y\| d\mu(w) - \phi\left(\int_\Omega \|x-y\| d\mu(w)\right)\right) \tag{2.6}$$

If $L(w) = 0$ for each $w \in \Omega$ in (2.6), then it reduces to condition (2.5).
Furthermore, Okeke and Kim in [19] introduced the random Picard-Mann hybrid iterative process. They established strong convergence theorems and summable almost T-stability of the random PicardMann hybrid iterative process and the random Mann-type iterative process generated by a generalized class of random operators in separable Banach spaces. Their results improved and generalized several well-known

deterministic stability results in a stochastic version. In addition, Okeke and Kim [20] proved some convergence and (S,T)- stability results for random Jungck-Mann type and random Ishikawa type iterative processes. Rashwan et al. [26] studied the convergence and almost sure (S,T)- stability for the random Jungck-Noor type and the random Jungck-SP type under some contractive conditions.
Keeping in mind the generalized ϕ-weakly contractive conditions (1.1) and (2.6), we introduce the following generalized ϕ-weakly contractive condition:

Definition 2.3. Let (Ω, Σ) be a measurable space and E be a nonempty subset of a separable Banach space X. Let $S, T : \Omega \times E \leftrightarrow E$ be random operators such that $T(w, X) \subseteq S(w, X)$. Then the random operators S and T are satisfying the following generalized ϕ- weakly contractive-type if there exist $L(w) \geq 0$ and a continuous and non- decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for each $t \in (0, \infty)$ and $\phi(0) = 0$ such that for each $x, y \in E, \omega \in \Omega$,

$$\|T(w,x)-T(w,y)\| \leq e^{L(w)\|S(w,x)-T(w,x)\|}(\|S(w,x)-S(w,y)\|-\phi(\|T(w,x)-T(w,y)\|)) \tag{2.7}$$

If $L(w) = 0$ for each $\omega \in \Omega$ and $S = I_d$ (identity random mapping) in the condition (2.7), then it reduces to the stochastic version of the condition (1.1).
Motivated by the definition of a comparably almost stability in [15] together with the definition of (S,T)-stability in [28], we state the stochastic version of the comparably almost (S,T)- stability as follows:

Definition 2.4. Let (Ω, Σ) be a measurable space and E be a nonempty subset of a separable Banach space X. Let $S, T : \Omega \times E \leftrightarrow E$ be random operators such that $T(w, X) \subseteq S(w, X)$ and $\xi^*(\omega)$ be a common random fixed point of S and T. For any given random variable $x_0 : \Omega \to E$. Define a random iterative scheme with the functions $\{S(\omega, x_n(\omega))\}_{n=0}^{\infty}$ as follows:

$$S(\omega, x_{n+1}(\omega)) = f(T; x_n(\omega)) \ n = 0, 1, 2, ..., \tag{2.8}$$

where f is some function measurable in the second variable.
Suppose that $\{S(\omega, x_n(\omega))\}_{n=0}^{\infty}$ converges to $\xi^*(\omega)$, and Let $\{S(\omega, \xi_n(\omega))\}_{n=0}^{\infty} \subset E$ be an arbitrary sequence of a random variable. Denote by

$$\varepsilon_n(\omega) = \|S(\omega, \xi_{n+1}(\omega)) - f(T; \xi_n(\omega))\|.$$

Then the iterative scheme (2.8) is a comparably almost (S,T)- stable or comparably almost stable with respect to (S,T) if and only if for $\omega \in \Omega$,

$$\sum_{n=0}^{\infty}(\theta_n(\omega) + \varepsilon_n(\omega)) < \infty, \ \theta_n(\omega) \geq 0 \Rightarrow S(\omega, \xi_n(\omega)) \to \xi^*, \ \theta_n(\omega) \to 0 \ as \ n \to \infty.$$

The following lemma is useful for proving our results

Lemma 2.1. *[1] Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be two sequences of nonnegative real numbers and $\{\sigma_n\}$ be a sequence of positive numbers satisfying*

$$\lambda_{n+1} \leq \lambda_n - \sigma_n \phi(\lambda_n) + \gamma_n, \quad \forall n \geq 1,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$. If $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$ and $\lim_{n\to\infty} \frac{\gamma_n}{\sigma_n} = 0$, then $\{\lambda_n\}$ converges to 0 as $n \to \infty$.

3. Main Results

In this section, we present our main results. First, we prove the comparably almost (S,T)- stability of the Jungck-Noor type random iterative scheme.

Theorem 3.1. *Let (Ω, Σ) be a measurable space and E be a nonempty subset of a separable Banach space X and let $S, T : \Omega \times E \leftrightarrow E$ be two random operators defined on E satisfying a generalized ϕ- weakly contractive-type (2.7) with $T(w, X) \subseteq S(w, X)$. Let $\xi^*(\omega)$ be a common random fixed point of (S,T) and $\{S(\omega, x_n(\omega))\}_{n=0}^{\infty}$ be a Jungck-Noor type random iterative scheme defined by (2.1) converging strongly to $\xi^*(\omega)$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0,1] satisfying*

- $\sum_{n=1}^{\infty} \alpha_n \beta_n \gamma_n = \infty$,
- $\alpha_n(1 + \beta_n + \beta_n \gamma_n) \leq 1$.

Let $\{S(w, \xi_n(w))\}_{n=0}^{\infty}$ be any sequence of random variable in E and define

$$\begin{aligned} \varepsilon_n &= \|S(w, \xi_{n+1}(w)) - (1 - \alpha_n)S(w, \xi_n(w)) - \alpha_n T(w, \eta_n(w))\|, \\ S(w, \eta_n(w)) &= (1-\beta_n)S(w,\xi_n(w))+\beta_n T(w,\zeta_n(w)), \\ S(w, \zeta_n(w)) &= (1-\gamma_n)S(w,\xi_n(w))+\gamma_n T(w,\xi_n(w)). \end{aligned}$$

Then

1. *If $\sum_{n=0}^{\infty}(\theta_n + \varepsilon_n) < \infty$, where*

   $$\begin{aligned} \theta_n = {} & \phi(\|S(w, \xi_n(w)) - \xi^*(w)\|) - \alpha_n \beta_n \gamma_n \phi(\|T(w, \xi_n(w)) - \xi^*(w)\|) \\ & -\alpha_n \beta_n \phi(\|T(w, \zeta_n(w)) - \xi^*(w)\|) - \alpha_n \phi(\|T(w, \eta_n(w)) - \xi^*(w)\|). \end{aligned}$$

   Then the Jungck-Noor type random iterative scheme $\{S(w, x_n(w))\}_{n=0}^{\infty}$ is a comparably almost (S,T)- stable.

2. *If the sequence $\{S(w, \xi_n(w))\}_{n=0}^{\infty}$ converge to the fixed point $\xi^*(w)$ of (S,T), then $\lim_{n\to\infty} \varepsilon_n = 0$.*

Proof. Using the random Jungck-Noor iterative scheme (2.1) and the sequence $\{S(w,\xi_n(w))\}_{n=0}^{\infty}$ defined in (3.1), we have

$$
\begin{aligned}
\|S(w,\xi_{n+1}(w))-\xi^*(w)\| &\leq \|S(w,\xi_{n+1}(w))-(1-\alpha_n)S(w,\xi_n(w))-\alpha_n T(w,\eta_n(w))\| \\
&+ (1-\alpha_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\alpha_n\|T(w,\eta_n(w))-\xi^*(w)\| \\
&= \varepsilon_n+(1-\alpha_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\alpha_n\|T(w,\eta_n(w))-\xi^*(w)\|
\end{aligned}
$$

(3.1)

Now, we compute the last estimate of (3.1) by using (2.7) and (3.1)

$$
\begin{aligned}
\|T(w,\eta_n(w))-\xi^*(w)\| &= \|T(w,\xi^*(w))-T(w,\eta_n(w))\| \\
&\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}\big(\|S(w,\xi^*(w))-S(w,\eta_n(w))\| \\
&- \phi(\|T(w,\xi^*(w))-T(w,\eta_n(w))\|)\big) \\
&= \|\xi^*(w)-S(w,\eta_n(w))\|-\phi(\|T(w,\xi^*(w))-T(w,\eta_n(w))\|) \\
&\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n\|T(w,\zeta_n(w))-\xi^*(w)\| \\
&- \phi(\|\xi^*(w)-T(w,\eta_n(w))\|) \\
&\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n[e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|} \\
&\Big(\|S(w,\xi^*(w))-S(w,\zeta_n(w))\|-\phi(\|T(\omega,\xi^*(\omega))-T(w,\zeta_n(w))\|))] \\
&- \phi(\|\xi^*(w)-T(w,\eta_n(w))\|) \\
&= (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n\|\xi^*(w)-S(w,\zeta_n(w))\| \\
&- \beta_n\phi(\|\xi^*(w)-T(w,\zeta_n(w))\|)-\phi(\|\xi^*(w)-T(w,\eta_n(w))\|) \\
&\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n[(1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\| \\
&+ \gamma_n\|T(w,\xi_n(w))-\xi^*(w)\|]-\beta_n\phi(\|\xi^*(w)-T(w,\zeta_n(w))\|) \\
&- \phi(\|\xi^*(w)-T(w,\eta_n(w))\|) \\
&\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n(1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\| \\
&+ \beta_n\gamma_n[e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi_n(w))-\xi^*(w)\| \\
&- \phi(\|T(w,\xi_n(w))-\xi^*(w)\|)]-\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|) \\
&- \phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= (1-\beta_n+\beta_n-\beta_n\gamma_n+\beta_n\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\| \\
&- \beta_n\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|)-\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|) \\
&- \phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= \|S(w,\xi_n(w))-\xi^*(w)\|-\beta_n\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|) \\
&- \beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)
\end{aligned}
$$

(3.2)

Applying (3.2) in (3.1), we obtain

$$\begin{aligned}
\|S(w,\xi_{n+1}(w))-\xi^*(w)\| &\leq \varepsilon_n+(1-\alpha_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\alpha_n\|S(w,\xi_n(w))-\xi^*(w)\| \\
&- \alpha_n\beta_n\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|) \\
&- \alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= \|S(w,\xi_n(w))-\xi^*(w)\|-\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)+(\varepsilon_n+\theta_n),
\end{aligned} \tag{3.3}$$

where, $\theta_n=\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\beta_n\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)$ $-\ \alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)$.
Now, we want to prove that $\theta_n \geq 0$, note that

$$\begin{aligned}
\|T(w,\xi_n(w))-\xi^*(w)\| &\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\xi_n(w))\| \\
&- \phi(\|T(w,\xi^*(w))-T(w,\xi_n(w))\|)) \\
&\leq \|S(w,\xi_n(w))-\xi^*(w)\|.
\end{aligned} \tag{3.4}$$

Also, we have by (3.4)

$$\begin{aligned}
\|T(w,\zeta_n(w))-\xi^*(w)\| &= \|T(w,\xi^*(w))-T(w,\zeta_n(w))\| \\
&\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\zeta_n(w))\| \\
&- \phi(\|T(w,\xi^*(w))-T(w,\zeta_n(w))\|)) \\
&\leq \|S(w,\zeta_n(w))-\xi^*(w)\| \\
&\leq (1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\gamma_n\|T(w,\xi_n(w))-\xi^*(w)\| \\
&\leq (1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\gamma_n\|S(w,\xi_n(w))-\xi^*(w)\| \\
&= \|S(w,\xi_n(w))-\xi^*(w)\|.
\end{aligned} \tag{3.5}$$

Similarly, from (3.5), we get

$$\begin{aligned}
\|T(w,\eta_n(w))-\xi^*(w)\| &= \|T(w,\xi^*(w))-T(w,\eta_n(w))\| \\
&\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\eta_n(w))\| \\
&- \phi(\|T(w,\xi^*(w))-T(w,\eta_n(w))\|)) \\
&\leq \|S(w,\eta_n(w))-\xi^*(w)\| \\
&\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n\|T(w,\zeta_n(w))-\xi^*(w)\| \\
&\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n\|S(w,\xi_n(w))-\xi^*(w)\| \\
&= \|S(w,\xi_n(w))-\xi^*(w)\|.
\end{aligned} \tag{3.6}$$

Now, we can study the sign of θ_n by using (3.4), (3.5), (3.6) and the condition

$\alpha_n(1+\beta_n+\beta_n\gamma_n) \leq 1$ as:

$$\begin{aligned} \theta_n &= \phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\beta_n\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|) \\ &- \alpha_n\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\ &\geq \phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\beta_n\gamma_n\phi(\|S(w,\xi_n(w))-\xi^*(w)\|) \\ &- \alpha_n\beta_n\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|S(w,\xi_n(w))-\xi^*(w)\|) \\ &= [1-\alpha_n(1+\beta_n+\beta_n\gamma_n)]\phi(\|S(w,\xi_n(w))-\xi^*(w)\|) \\ &\geq 0. \end{aligned}$$

Since $\sum_{n=0}^{\infty}(\theta_n+\varepsilon_n) < \infty$, we have $\lim_{n\to\infty}(\theta_n+\varepsilon_n) = 0$. Back to the relation (3.3) and by Lemma 2.1, we get

$$\lim_{n\to\infty}\|S(w,\xi_n(w))-\xi^*(w)\|=0 \ or \ S(w,\xi_n(w))\to\xi^*(w) \ as \ n\to\infty. \tag{3.7}$$

From (3.4) and (3.7), we get

$$0\leq\|T(w,\xi_n(w))-\xi^*(w)\|\leq\|S(w,\xi_n(w))-\xi^*(w)\|\to 0 \text{ as } n\to\infty. \tag{3.8}$$

Similarly, from (3.5), (3.6) and using (3.7)

$$0\leq\|T(w,\zeta_n(w))-\xi^*(w)\|\leq\|S(w,\xi_n(w))-\xi^*(w)\|\to 0 \text{ as } n\to\infty. \tag{3.9}$$

$$0\leq\|T(w,\eta_n(w))-\xi^*(w)\|\leq\|S(w,,\xi_n(w))-\xi^*(w)\|\to 0 \text{ as } n\to\infty. \tag{3.10}$$

Since ϕ is continuous, from (3.7)-(3.10), we obtain

$$\begin{aligned} &\lim_{n\to\infty}\theta_n=\lim_{n\to\infty}[\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\beta_n\gamma_n\phi(\|T(w,\xi_n(\omega))-\xi^*(w)\|) \\ -\ &\alpha_n\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)] \\ =\ &0. \end{aligned}$$

Hence the Jungck-Noor type random iterative scheme $\{S(w,x_n(w))\}_{n=0}^{\infty}$ is a comparably almost (S,T)- stable.
Next, suppose that $S(w,\xi_n(w)) \to \xi^*(w)\ as\ n\to\infty$, and using (3.6) and (3.7), then we can write

$$\begin{aligned} \varepsilon_n &= \|S(w,\xi_{n+1}(w))-(1-\alpha_n)S(w,\xi_n(w))-\alpha_nT(w,\eta_n(w))\| \\ &\leq \|S(w,\xi_{n+1}(w))-\xi^*(w)\|+(1-\alpha_n)\|S(w,\xi_n(\omega))-\xi^*(w)\| \\ &+ \alpha_n\|T(w,\eta_n(w))-\xi^*(w)\| \\ &\leq \|S(w,\xi_{n+1}(w))-\xi^*(w)\|+(1-\alpha_n)\|S(w,\xi_n(\omega))-\xi^*(w)\| \\ &+ \alpha_n\|S(w,\xi_n(w))-\xi^*(w)\| \\ &= \|S(w,\xi_{n+1}(w))-\xi^*(w)\|+\|S(w,\xi_n(\omega))-\xi^*(w)\|. \end{aligned}$$

Hence, we get $\varepsilon_n \to 0$ as $n\to\infty$. $\square$

From Theorem 3.1, we can present the following corollaries.

Corollary 3.1. *Let (Ω, Σ) be a measurable space and E be a nonempty subset of a separable Banach space X and let $S, T : \Omega \times E \leftrightarrow E$ be two random operators defined on E satisfying a generalized ϕ- weakly contractive-type (2.7) with $T(w, X) \subseteq S(w, X)$. Let $\xi^*(w)$ be a common random fixed point of (S,T) and $\{S(w, x_n(w))\}_{n=0}^{\infty}$ be a Jungck-Ishikawa type random iterative scheme defined by (2.3) converging strongly to $\xi^*(w)$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in [0,1] satisfying*

- $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$,
- $\alpha_n(1 + \beta_n) \leq 1$.

Let $\{S(w, \xi_n(w))\}_{n=0}^{\infty}$ be any sequence of random variable in E and define

$$\begin{aligned} \varepsilon_n &= \|S(w, \xi_{n+1}(w)) - (1 - \alpha_n)S(w, \xi_n(w)) - \alpha_n T(w, \eta_n(w))\|, \\ S(w, \eta_n(w)) &= (1 - \beta_n)S(w, \xi_n(w)) + \beta_n T(w, \xi_n(w)). \end{aligned}$$

Then

1. *If $\sum_{n=0}^{\infty} (\theta_n + \varepsilon_n) < \infty$, where*

   $$\theta_n = \phi(\|S(w,\xi_n(w)) - \xi^*(w)\|) - \alpha_n\beta_n\phi(\|T(w,\xi_n(w)) - \xi^*(w)\|) - \alpha_n\phi(\|T(w,\eta_n(w)) - \xi^*(w)\|).$$

   Then the Jungck-Ishikawa type random iterative scheme $\{S(w, x_n(w))\}_{n=0}^{\infty}$ is a comparably almost (S,T)- stable.

2. *If the sequence $\{S(w, \xi_n(w))\}_{n=0}^{\infty}$ converge to the fixed point $\xi^*(w)$ of (S,T), then $\lim_{n\to\infty} \varepsilon_n = 0$.*

Proof. Putting $\gamma_n = 0$ in the Jungck-Noor type random iterative scheme in Theorem 3.1. Then we obtain the Jungck-Ishikawa type random iterative scheme and then can be prove the Corollary 3.1 by following the same steps of proofing of Theorem 3.1. □

Corollary 3.2. *Let (Ω, Σ) be a measurable space and E be a nonempty subset of a separable Banach space X and let $S, T : \Omega \times E \leftrightarrow E$ be two random operators defined on E satisfying a generalized ϕ- weakly contractive-type (2.7) with $T(w, X) \subseteq S(w, X)$. Let $\xi^*(w)$ be a common random fixed point of (S,T) and $\{S(w, x_n(w))\}_{n=0}^{\infty}$ be a Jungck-Mann type random iterative scheme defined by (2.4) converging strongly to $\xi^*(w)$, where $\{\alpha_n\}$ is a sequence of positive numbers in [0,1] such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{S(w, \xi_n(w))\}_{n=0}^{\infty}$ be any sequence of random variable in E and define*

$$\varepsilon_n = \|S(w, \xi_{n+1}(w)) - (1 - \alpha_n)S(w, \xi_n(w)) - \alpha_n T(w, \xi_n(w))\|,$$

Then

1. *If* $\sum_{n=0}^{\infty}(\theta_n+\varepsilon_n)<\infty$, *where*

$$\theta_n=\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|).$$

Then the Jungck-Mann iterative scheme $\{S(w,x_n(w))\}_{n=0}^{\infty}$ *is a comparably almost (S,T)- stable.*

2. *If the sequence* $\{S(w,\xi_n(w))\}_{n=0}^{\infty}$ *converge to the fixed point* $\xi^*(w)$ *of (S,T), then* $\lim_{n\to\infty}\varepsilon_n=0$.

Proof. If $\gamma_n=\beta_n=0$ in the Jungck-Noor type random iterative scheme in Theorem 3.1. Then we obtain the Jungck-Mann type random iterative and then the proof of the Corollary 3.2 is similar to that of Theorem 3.1. □

Remark 3.1. If the random mapping $S=I_d$ (Identity random mapping) and $L(\omega)=0$ in Corollary 3.1 and Corollary 3.2. Then Corollary 3.1 and Corollary 3.2 are random versions of Theorem 3.2 and Corollary 3.3 respectively of Kim in [15].

Next, we prove that the Jungck- SP type random iterative scheme $\{S(w,x_n(w))\}_{n=0}^{\infty}$ is a comparably almost (S,T)- stable.

Theorem 3.2. *Let* (Ω,Σ) *be a measurable space and* E *be a nonempty subset of a separable Banach space* X *and let* $S,T:\Omega\times E\leftrightarrow E$ *be two random operators defined on E satisfying a generalized* ϕ*- weakly contractive-type (2.7) with* $T(w,X)\subseteq S(w,X)$. *Let* $\xi^*(w)$ *be a common random fixed point of (S,T) and* $\{S(w,x_n(w))\}_{n=0}^{\infty}$ *be a Jungck-SP type random iterative scheme defined by (2.2) converging strongly to* $\xi^*(w)$, *where* $\{\alpha_n\},\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences of positive numbers in [0,1] satisfying*

- $\sum_{n=1}^{\infty}\alpha_n=\infty$ *or* $\sum_{n=1}^{\infty}\beta_n=\infty$ *or* $\sum_{n=1}^{\infty}\gamma_n=\infty$.
- $\alpha_n(1+\beta_n+\gamma_n)\le 1$.

Let $\{S(w,\xi_n(w))\}_{n=0}^{\infty}$ *be any sequence of random variable in E and define*

$$\begin{aligned}\varepsilon_n &= \|S(w,\xi_{n+1}(w))-(1-\alpha_n)S(w,\eta_n(\omega))-\alpha_nT(w,\eta_n(w))\|,\\ S(w,\eta_n(w)) &= (1-\beta_n)S(w,\zeta_n(w))+\beta_nT(w,\zeta_n(w)),\\ S(w,\zeta_n(w)) &= (1-\gamma_n)S(w,\xi_n(w))+\gamma_nT(w,\xi_n(w)).\end{aligned} \tag{3.11}$$

Then

1. *If* $\sum_{n=0}^{\infty}(\theta_n+\varepsilon_n)<\infty$, *where*

$$\theta_n=\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|)$$
$$-\alpha_n\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|).$$

Then the Jungck-SP iterative scheme $\{S(w,x_n(w))\}_{n=0}^{\infty}$ *is a comparably almost (S,T)- stable.*

2. *If the sequence* $\{S(w,\xi_n(w))\}_{n=0}^{\infty}$ *converge to the fixed point* $\xi^*(w)$ *of (S,T), then* $\lim_{n\to\infty}\varepsilon_n = 0$.

Proof. By the same steps of proofing of Theorem 3.1, using the random Jungck-SP iterative scheme (2.2) and the sequence $\{S(w,\xi_n(w))\}_{n=0}^{\infty}$ defined in (3.11), we have

$$\begin{aligned}\|S(w,\xi_{n+1}(w))-\xi^*(w)\| &\leq \|S(w,\xi_{n+1}(w))-(1-\alpha_n)S(w,\eta_n(w))-\alpha_n T(w,\eta_n(w))\|\\ &+ (1-\alpha_n)\|S(w,\eta_n(w))-\xi^*(w)\|+\alpha_n\|T(w,\eta_n(w))-\xi^*(w)\|\\ &= \varepsilon_n+(1-\alpha_n)\|S(w,\eta_n(w))-\xi^*(w)\|+\alpha_n\|T(w,\eta_n(w))-\xi^*(w)\|\end{aligned}$$

(3.12)

Using (2.7) to compute the following

$$\begin{aligned}\|T(w,\eta_n(w))-\xi^*(w)\| &= \|T(w,\xi^*(w))-T(w,\eta_n(w))\|\\ &\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}\big(\|S(w,\xi^*(w))-S(w,\eta_n(w))\|\\ &- \phi(\|T(\omega,\xi^*(w))-T(w,\eta_n(w))\|)\big)\\ &= \|S(w,\eta_n(w))-\xi^*(w)\|-\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)\end{aligned}$$

(3.13)

Applying (3.13) in (3.12), we obtain

$$\begin{aligned}
\|S(w,\xi_{n+1}(w))-\xi^*(w)\| &\leq \varepsilon_n+(1-\alpha_n)\|S(w,\eta_n(w))-\xi^*(w)\| \\
&+ \alpha_n[\|S(w,\eta_n(w))-\xi^*(w)\|-\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)] \\
&= \varepsilon_n+\|S(w,\eta_n(w))-\xi^*(w)\|-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&\leq \varepsilon_n+(1-\beta_n)\|S(w,\zeta_n(w))-\xi^*(w)\|+\beta_n\|T(w,\zeta_n(w))-\xi^*(w)\| \\
&- \alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&\leq \varepsilon_n+(1-\beta_n)\|S(w,\zeta_n(w))-\xi^*(w)\|+\beta_n[e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|} \\
&(\|S(w,\xi^*(w))-S(w,\zeta_n(w))\|-\phi(\|T(\omega,\xi^*(w))-T(w,\zeta_n(w))\|))] \\
&- \alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= \varepsilon_n+(1-\beta_n)\|S(w,\zeta_n(w))-\xi^*(w)\|+\beta_n\|S(w,\zeta_n(w))-\xi^*(w)\| \\
&- \beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= \varepsilon_n+\|S(w,\zeta_n(w))-\xi^*(w)\|-\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|) \\
&- \alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&\leq \varepsilon_n+(1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\gamma_n\|T(w,\xi_n(w))-\xi^*(w)\| \\
&- \beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&\leq \varepsilon_n+(1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\| \\
&+ \gamma_n[e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\xi_n(w))\| \\
&- \phi(\|T(w,\xi^*(w))-T(w,\xi_n(w))\|))]-\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|) \\
&- \alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= \varepsilon_n+\|S(w,\xi_n(w))-\xi^*(w)\|-\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|) \\
&- \beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|) \\
&= \|S(w,\xi_n(w))-\xi^*(w)\|-\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)+(\theta_n+\varepsilon_n) \qquad (3.14)
\end{aligned}$$

where

$$\begin{aligned}
\theta_n=&\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|) \\
&-\beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|).
\end{aligned}$$

Note that,

$$\begin{aligned}
\|T(w,\xi_n(w))-\xi^*(w)\| &\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\xi_n(w))\| \\
&- \phi(\|T(w,\xi^*(w))-T(w,\xi_n(w))\|)) \\
&\leq \|S(w,\xi_n(w))-\xi^*(w)\|. \qquad (3.15)
\end{aligned}$$

Also, from (3.15) , we get

$$\begin{aligned}\|T(w,\zeta_n(w))-\xi^*(w)\| &= \|T(w,\xi^*(w))-T(w,\zeta_n(w))\|\\ &\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\zeta_n(w))\|\\ &- \phi(\|T(w,\xi^*(w))-T(w,\zeta_n(w))\|))\\ &\leq \|S(w,\zeta_n(w))-\xi^*(w)\|\\ &\leq (1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\gamma_n\|T(w,\xi_n(w))-\xi^*(w)\|\\ &\leq (1-\gamma_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\gamma_n\|S(w,\xi_n(w))-\xi^*(w)\|\\ &= \|S(w,\xi_n(w))-\xi^*(w)\|\end{aligned}$$

(3.16)

Similarly, from (3.16), we get,

$$\begin{aligned}\|T(w,\eta_n(w))-\xi^*(w)\| &= \|T(w,\xi^*(w))-T(w,\eta_n(w))\|\\ &\leq e^{L(w)\|S(w,\xi^*(w))-T(w,\xi^*(w))\|}(\|S(w,\xi^*(w))-S(w,\eta_n(w))\|\\ &- \phi(\|T(w,\xi^*(w))-T(w,\eta_n(w))\|))\\ &\leq \|S(w,\eta_n(w))-\xi^*(w)\|\\ &\leq (1-\beta_n)\|S(w,\zeta_n(w))-\xi^*(w)\|+\beta_n\|T(w,\zeta_n(w))-\xi^*(w)\|\\ &\leq (1-\beta_n)\|S(w,\xi_n(w))-\xi^*(w)\|+\beta_n\|S(w,\xi_n(w))-\xi^*(w)\|\\ &= \|S(w,\xi_n(w))-\xi^*(w)\|\end{aligned}$$

(3.17)

Using (3.15), (3.16) and (3.17) with the condition $\alpha_n+\beta_n+\gamma_n\leq 1$ we obtain,

$$\begin{aligned}\theta_n &= \phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|)\\ &- \beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)\\ &\geq \phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\gamma_n\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)\\ &- \beta_n\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)\\ &= [1-(\alpha_n+\beta_n+\gamma_n)]\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)\\ &\geq 0\end{aligned}$$

Since $\sum_{n=0}^{\infty}(\theta_n+\varepsilon_n)<\infty$, then $\lim_{n\to\infty}(\theta_n+\varepsilon_n)=0$ and by Lemma 2.1, we get

(3.18) $\lim_{n\to\infty}\|S(w,\xi_n(w))-\xi^*(w)\|=0$ *or* $S(w,\xi_n(w))\to\xi^*(w)$ *as* $n\to\infty$.

Also, we have by using (3.15), (3.16), (3.17) and (3.18)

(3.19) $0\leq\|T(w,\xi_n(w))-\xi^*(w)\|\leq\|S(w,\xi_n(w))-\xi^*(w)\|\to 0$ as $n\to\infty$.

(3.20) $0\leq\|T(w,\zeta_n(w))-\xi^*(w)\|\leq\|S(w,\xi_n(w))-\xi^*(w)\|\to 0$ as $n\to\infty$.

(3.21) $0\leq\|T(w,\eta_n(w))-\xi^*(w)\|\leq\|S(w,\xi_n(w))-\xi^*(w)\|\to 0$ as $n\to\infty$.

Since ϕ is continuous, from (3.18)- (3.21), we obtain

$$\begin{aligned}
\lim_{n\to\infty}\theta_n &= \lim_{n\to\infty}[\phi(\|S(w,\xi_n(w))-\xi^*(w)\|)-\gamma_n\phi(\|T(w,\xi_n(w))-\xi^*(w)\|)\\
&- \beta_n\phi(\|T(w,\zeta_n(w))-\xi^*(w)\|)-\alpha_n\phi(\|T(w,\eta_n(w))-\xi^*(w)\|)]\\
&= 0.
\end{aligned}$$

Hence the Jungck-SP type random iterative scheme $\{S(w,x_n(w))\}_{n=0}^{\infty}$ is a comparably almost (S,T)- stable.
Next, suppose that $S(w,\xi_n(w)) \to \xi^*(w)\ as\ n \to \infty$, and using (3.21), then we obtain

$$\begin{aligned}
\varepsilon_n &= \|S(w,\xi_{n+1}(w))-(1-\alpha_n)S(w,\eta_n(w))-\alpha_nT(w,\eta_n(w))\|\\
&\leq \|S(w,\xi_{n+1}(w))-\xi^*(w)\|+(1-\alpha_n)\|S(w,\eta_n(\omega))-\xi^*(w)\|\\
&+ \alpha_n\|T(w,\eta_n(w))-\xi^*(w)\|\\
&\leq \|S(w,\xi_{n+1}(w))-\xi^*(w)\|+(1-\alpha_n)\|S(w,\xi_n(\omega))-\xi^*(w)\|\\
&+ \alpha_n\|S(w,\xi_n(w))-\xi^*(w)\|\\
&= \|S(w,\xi_{n+1}(w))-\xi^*(w)\|+\|S(w,\xi_n(\omega))-\xi^*(w)\|\\
&\to 0\ as\ n\to\infty.
\end{aligned}$$

□

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FIXED POINT OF MULTIVALUED CONTRACTIONS IN ORTHOGONAL MODULAR METRIC SPACES

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Abstract. In this paper we generalize the notion of $O-$set and establish some fixed point theorems for $\perp - \alpha - \psi-$contraction multifunction in the setting of orthogonal modular metric spaces. As consequences of these results we deduce some theorems in orthogonal modular metric spaces endowed with a graph and partial order. Finally, we establish some theorems for integral type contraction multifunctions and give some examples to demonstrate the validity of the results.
Keywords. Fixed point theorem; metric space; contraction; partial order.

1. Introduction and Preliminaries

In order to generalize the well-known Banach contraction principle, Nadler [15] introduced the Banach contraction principle for multivalued mappings in complete metric spaces. It is known that the theorem by Nadler has been extended and generalized in various directions by several authors, see [1, 2, 3, 9, 10] and the references therein. On the other hand, modular metric spaces are a natural and interesting generalization of classical modulars over linear spaces such as Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and others. The concept of modular metric spaces was introduced in [6, 7]. Here, we look at the modular metric space as the nonlinear version of the classical one introduced by Nakano [16] on the vector space and the modular function space introduced by Musielak [14] and Orlicz [17].

Recently, many authors studied the behavior of the electrorheological fluids, sometimes referred to as "smart fluids" (e.g., lithium polymetachrylate). A perfect model for these fluids is obtained by using Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, in the case p is a function [8]. In this paper, we generalize the notion of $O-$sets and then establish some fixed point theorems for $\perp - \alpha - \psi-$contraction

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multifunction in the setting of orthogonal modular metric spaces. As consequences of these results, we deduce some theorems in orthogonal modular metric spaces endowed with a graph and partial order. In the end, we establish some theorems for integral type contraction multifunctions and give some examples to demonstrate the validity of the results.

Let X be a nonempty set and $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ be a function. For reasons of simplicity we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. [6, 7] A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If in the above definition we utilize the condition

(i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$;

instead of (i) then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \quad \text{for some} \quad \lambda > 0.$$

Again ω is called convex if for $\lambda, \mu > 0$ and $x, y, z \in X$ holds the inequality

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu(z, y).$$

Remark 1.1. Note that if ω is a pseudomodular metric on a set X then the function $\lambda \to \omega_\lambda(x, y)$ is decreasing on $(0, +\infty)$ for all $x, y \in X$. That is, if $0 < \mu < \lambda$ then

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Definition 1.2. [6, 7] Suppose that ω be a pseudomodular on X and $x_0 \in X$ and fixed. So the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \quad \text{such that} \quad \omega_\lambda(x, x_0) < +\infty\}.$$

X_ω and X^*_ω are called modular spaces (around x_0).

It is evident that $X_\omega \subset X_\omega^*$ but this inclusion may be proper in general. Assume that ω be a modular on X, from [6, 7] we derive that the modular space X_ω can be equipped with a (nontrivial) metric induced by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\} \quad \text{for all} \quad x,y \in X_\omega.$$

Note that if ω is a convex modular on X then according to [6, 7] the two modular spaces coincide, i.e., $X_\omega^* = X_\omega$, and this common set can be endowed with the metric d_ω^* given by

$$d_\omega^*(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\} \quad \text{for all} \quad x,y \in X_\omega.$$

Such distances are called Luxemburg distances.

Example 2.1 presented by Abdou and Khamsi [1] is an important motivation for developing the modular metric spaces theory. Other examples may be found in [6, 7].

Definition 1.3. [13] Assume X_ω is a modular metric space, M a subset of X_ω and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X_ω. Therefore,

(1) $(x_n)_{n\in\mathbb{N}}$ is called ω-convergent to $x \in X_\omega$ if and only if $\omega_\lambda(x_n,x) \to 0$, as $n \to +\infty$ for all $\lambda > 0$. x will be called the ω-limit of (x_n).

(2) $(x_n)_{n\in N}$ is called ω-Cauchy if $\omega_\lambda(x_m,x_n) \to 0$, as $m,n \to +\infty$ for all $\lambda > 0$.

(3) M is called ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.

(4) M is called ω-complete if any ω-Cauchy sequence in M is ω-convergent to a point of M.

(5) M is called ω-bounded if for all $\lambda > 0$ we have $\delta_\omega(M) = \sup\{\omega_\lambda(x,y); x,y \in M\} < +\infty$.

Definition 1.4. [6, 7] ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\} \subseteq X_\omega$ with $\lim_{n\to\infty} \omega_1(x_n,x) = 0$, we have

$$\omega_1(x,y) \leq \liminf_{n\to\infty} \omega_1(x_n,y)$$

for all $y \in X_\omega$.

But here we utilize the following version of the Fatou property.

Definition 1.5. ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\} \subseteq X_\omega$, ω-convergent to x, we get

$$\omega_\lambda(x,y) \leq \liminf_{n\to\infty} \omega_\lambda(x_n,y)$$

for all $y \in X_\omega$ and $\lambda > 0$.

Also we say ω satisfies the Δ_2-condition (see [2]), if $\lim_{n\to\infty}\omega(x_n,x)=0$ for some $\lambda>0$ implies $\lim_{n\to\infty}\omega(x_n,x)=0$ for all $\lambda>0$.

Definition 1.6. [5] Let M be a subset of the modular metric space X_ω.

- $CB(M)=\{C:\ C$ is nonempty ω-closed and ω-bounded subset of $M\}$
- $K(M)=\{C:\ C$ is nonempty ω-compact subset of $M\}$
- A Hausdorff modular metric $\Omega_\lambda(A,B)$ is defined on $CB(M)$ by

$$\Omega_\lambda(A,B)=\max\{\sup_{x\in A}\omega_\lambda(x,B),\sup_{y\in B}\omega_\lambda(A,y)\}$$

where $\omega_\lambda(x,B)=\inf_{y\in B}\omega_\lambda(x,y)$.

Furthermore, let $T:M\to CB(M)$ be a multifunction. We say $x\in M$ is fixed point of T whence $x\in Tx$. We denote all fixed points of T by $Fix(T)$.

Lemma 1.1. *[5] Suppose that $A,B\in CB(X_\omega)$ and $a\in A$. Thus for $\epsilon>0$, there exists $b_\epsilon\in B$ such that*

$$\omega_\lambda(a,b_\epsilon)\le\Omega_\lambda(A,B)+\epsilon$$

for all $\lambda>0$.

Asl et al. [3] defined the notion of α_*-admissible multifunction as follows.

Definition 1.7. Let $T:X\to 2^X$ and $\alpha:X\times X\to\mathbb{R}_+$. We say that T is α_*-admissible mapping if

$$\alpha(x,y)\ge 1\quad\text{implies}\quad\alpha_*(Tx,Ty)\ge 1,\quad x,y\in X$$

where

$$\alpha_*(A,B)=\inf_{x\in A,\ y\in B}\alpha(x,y).$$

Denote Ψ the family of strictly increasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\Sigma_{n=1}^{\infty}\psi^n(t)<\infty$ for all $t>0$.

Eshaghi et al. [9] introduced the notion of orthogonal set and gave a real generalization of Banach's fixed point theorem in orthogonal metric spaces (For more details on orthogonal set, also see [4]).

Definition 1.8. [9] Let $X\neq\emptyset$ and $\perp\in X\times X$ be a binary relation. Assume that there exists $x_0\in X$ such that $x_0\perp x$ or $x\perp x_0$ for all $x\in X$. Then we say that X is an orthogonal set (briefly O-set). We denote the orthogonal set by $(X,\perp)$. Also suppose that $(X,\perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if $(\forall n;x_n\perp x_{n+1})$ or $(\forall n;x_{n+1}\perp x_n)$.

Definition 1.9. [9] We say a metric space X is an orthogonal metric space if $(X, \perp)$ is an O-set. Also $T : X \to X$ is $\perp-$continuous in $x \in X$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X if $\lim_{n\to\infty} d(x_n, x) = 0$, then $\lim_{n\to\infty} d(Tx_n, Tx) = 0$. Furthermore T is $\perp$-continuous if T is $\perp$-continuous in each $x \in X$. Also we say T is $\perp$-preserving if $Tx\perp Ty$ whence $x\perp y$. Finally X is orthogonally complete (in brief O-complete) if every Cauchy O-sequence is convergent.

Now we generalize the concept of $O-$set and introduce the notion of $O^\star-$modular metric space in the following ways.

Definition 1.10. Let $X \neq \emptyset$ and $\perp \in X \times X$ be a binary relation.

- Assume that there exists $x_0 \in X$ such that $x_0\perp x$ for all $x \in X \setminus \{x_0\}$. Then we say that X is an orthogonal star set(briefly $O^\star$-set). We denote $O^\star$-set by $(X, \perp)$.
- We say x_0 is center of X and we denote the set of all centers of X by $\mathcal{C}(X)$.
- Also suppose that $(X, \perp)$ be an $O^\star$-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $O^\star$-sequence if $x_n\perp x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 1.11. Let X_ω be a modular metric space and $M \subseteq X_\omega$.

- M is an $O^\star-$modular metric space if $(M, \perp)$ is an $O^\star$-set.
- $T : M \to M$ is $\perp^\star-$continuous in $x \in M$ if for each $O^\star$-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$, implies $\lim_{n\to\infty} \omega_\lambda(Tx_n, Tx) = 0$ for all $\lambda > 0$. Furthermore T is $\perp^\star$-continuous when T is $\perp^\star$-continuous in each $x \in M$.
- $T : M \to CB(M)$ is $\perp^{\star\star}-$continuous in $x \in M$ if for each $O^\star$-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$, implies $\lim_{n\to\infty} \Omega_\lambda(Tx_n, Tx) = 0$ for all $\lambda > 0$. Also T is $\perp^{\star\star}$-continuous when T is $\perp^{\star\star}$-continuous in each $x \in M$.
- $T : M \to M$ is $\perp^\star$-preserving if $Tx\perp Ty$ whence $x\perp y$.
- $T : M \to CB(M)$ is $\perp^{\star\star}$-preserving, when $x\perp y$ implies $u\perp v$ for all $u \in Tx$ and $v \in Ty$.
- Finally X_ω is $\omega - O^\star$-complete if every ω-Cauchy $O^\star$-sequence is convergent.

If $x_0\perp y$ for all $y \in X$ then evidently $x_0\perp y$ for all $y \in X \setminus \{x_0\}$. That is every O-set $(X, \perp)$ is an $O^\star-$set, but the converse is not true. The following simple example shows this fact.

Example 1.1. Let $X = [0, \infty)$. For $x, y \in X$, assume $x\perp y$ if $x < y$. Then by putting $x_0 = 0$, X is an $O^\star-$set. In fact $x_0 = 0 < x$ for all $x \in [0, \infty) \setminus \{x_0 = 0\}$. But $0 \not< 0$. That is $(X, \perp)$ is not O-set.

2. Main Results

To demonstrate our main theorems we need the following lemmas.

Lemma 2.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let B be an $\omega-$closed subset of X_ω. Then $x \notin B$ if and only if $\omega_\lambda(x, B) > 0$ for all $\lambda > 0$.*

Proof. Let $\omega_\lambda(x, B) > 0$ for all $\lambda > 0$. Now if $x \in B$ then $\omega_\lambda(x, B) = \inf_{y \in B} \omega_\lambda(x, y) = 0$ for all $\lambda > 0$, which is a contradiction. Hence $x \notin B$.

Let $x \notin B$. Now assume there exists $\lambda_0 > 0$ such that $\omega_{\lambda_0}(x, B) = \inf_{y \in B} \omega_{\lambda_0}(x, y) = 0$. Then there exists a sequence $\{y_n\}_{n \geq 0} \subseteq B$ such that $\lim_{n\to\infty} \omega_{\lambda_0}(x, y_n) = 0$. Δ_2-condition implies $\lim_{n\to\infty} \omega_\lambda(x, y_n) = 0$ for all $\lambda > 0$. That is $y_n \to z$ as $n \to \infty$. Now since B is $\omega-$closed, then $x \in B$, which is a contradiction. □

Lemma 2.2. *Let X_ω be a modular metric space such that ω satisfies the Fatou property. Let A, B be two subsets of X_ω where B is $\omega-$compact. Then for each $x \in A$ there exists $y \in B$ such that $\omega_\lambda(x, y) \leq \Omega_\lambda(A, B)$ for all $\lambda > 0$.*

Proof. Let $x \in A$. Then by using lemma 1.1 we can say for each $n \geq 1$ there exists $y_n \in B$ such that

$$\omega_\lambda(x, y_n) \leq \Omega_\lambda(A, B) + \frac{1}{n}.$$

On the other hand B is $\omega-$compact. Thus we may assume that $\{y_n\}$ $\omega-$converges to $y \in B$. Since ω satisfies the Fatou property, we get

$$\omega_\lambda(x, y) \leq \liminf_{n\to\infty} \omega_\lambda(x, y_n) \leq \Omega_\lambda(A, B),$$

for all $\lambda > 0$. □

Lemma 2.3. *Let X_ω be a modular metric space and $\emptyset \neq M \subseteq X_\omega$. Let $A, B \in CB(M)$ and $q > 1$. Then for each $x \in A$ there exists $y \in B$ such that $\omega_\lambda(x, y) < q\Omega_\lambda(A, B)$ for all $\lambda > 0$.*

Proof. If in lemma 1.1 we take $\epsilon = \frac{1}{2}(q-1)\Omega_\lambda(A, B)$ then for each $x \in A$ there exists $y \in B$ such that

$$\begin{aligned}\omega_\lambda(x, y) \leq \Omega_\lambda(A, B) + \epsilon \quad &= \Omega_\lambda(A, B) + \frac{1}{2}(q-1)\Omega_\lambda(A, B) < \Omega_\lambda(A, B) + (q-1)\Omega_\lambda(A, B) \\ &= q\Omega_\lambda(A, B).\end{aligned}$$

□

Now we are ready to prove our first theorem.

Theorem 2.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega-O^*$-complete subset of X_ω. Let $T: M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\left\{ \begin{array}{l} x \perp y \\ \alpha(x,y) \geq 1 \end{array} \right. \Longrightarrow \Omega_\lambda(Tx,Ty) \leq \psi\big(\omega_\lambda(x,y)\big). \tag{2.1}$$

Also suppose that the following assertion holds:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$,

(ii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Proof. From (i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. We know that $x_0 \in \mathcal{C}(M)$, that is $x_0 \perp y$ for all $y \in M \setminus \{x_0\}$. If $x_0 = x_1$ then x_0 is a fixed point of T. Hence we assume that $x_0 \neq x_1$. So $x_0 \perp x_1$. Therefore from (2.1) we have

$$\Omega_\lambda(Tx_0,Tx_1) \leq \psi(\omega_\lambda(x_0,x_1)). \tag{2.2}$$

Also if $x_1 \in Tx_1$ then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$. Then by using lemma 2.1 we have

$$0 < \omega_\lambda(x_1,Tx_1) \text{ for all } \lambda > 0. \tag{2.3}$$

Now if $q > 1$ then from lemma 2.3 there exists $x_2 \in Tx_1$ such that

$$\omega_\lambda(x_1,x_2) < q\Omega_\lambda(Tx_0,Tx_1) \text{ for all } \lambda > 0. \tag{2.4}$$

Since $\omega_\lambda(x_1,Tx_1) \leq \omega_\lambda(x_1,x_2)$, for all $\lambda > 0$ then from (2.3) and (2.4) we obtain

$$0 < \omega_\lambda(x_1,Tx_1) \leq \omega_\lambda(x_1,x_2) < q\Omega_\lambda(Tx_0,Tx_1) \text{ for all } \lambda > 0.$$

And so by (2.2) we get

$$0 < \omega_\lambda(x_1,Tx_1) \leq \omega_\lambda(x_1,x_2) < q\Omega_\lambda(Tx_0,Tx_1) \leq q\psi(\omega_\lambda(x_0,x_1)).$$

That is

$$0 < \omega_\lambda(x_1,x_2) < q\psi(\omega_\lambda(x_0,x_1)). \tag{2.5}$$

Note that $x_1 \neq x_2$ (since $x_1 \notin Tx_1$). Also since T is an α_*-admissible then $\alpha_*(Tx_0,Tx_1) \geq 1$. This implies

$$\alpha(x_1,x_2) \geq \alpha_*(Tx_0,Tx_1) \geq 1.$$

Further since T is an $\perp^{\star\star}-$preserving then $x_0 \perp x_1$ implies $u \perp v$ for all $u \in Tx_0$ and $v \in Tx_1$. This implies $x_1 \perp x_2$.

Therefore from (2.1) we have

$$\Omega_\lambda(Tx_1, Tx_2) \leq \psi(\omega_\lambda(x_1, x_2)). \tag{2.6}$$

Put $t_0 = \omega_\lambda(x_0, x_1)$. We know that $x_0 \neq x_1$. Let $B = \{x_1\}$. Then lemma 2.1 implies that $\omega_\lambda(x_0, x_1) > 0$ for all $\lambda > 0$. That is $t_0 > 0$. So from (2.5) we have $\omega_\lambda(x_1, x_2) < q\psi(t_0)$ where $t_0 > 0$. Now since ψ is strictly increasing then $\psi(\omega_\lambda(x_1, x_2)) < \psi(q\psi(t_0))$. Put

$$q_1 = \frac{\psi(q\psi(t_0))}{\psi(\omega_\lambda(x_1, x_2))}$$

and so $q_1 > 1$. If $x_2 \in Tx_2$ then x_2 is a fixed point of T. Hence we suppose that $x_2 \notin Tx_2$. Then

$$0 < \omega_\lambda(x_2, Tx_2) \text{ for all } \lambda > 0.$$

So there exists $x_3 \in Tx_2$ such that

$$0 < \omega_\lambda(x_2, x_3) < q_1 \Omega_\lambda(Tx_1, Tx_2)$$

and then from (2.6) we get

$$0 < \omega_\lambda(x_2, x_3) < q_1 \Omega_\lambda(Tx_1, Tx_2) \leq q_1 \psi(\omega_\lambda(x_1, x_2)) = \psi(q\psi(t_0)).$$

Again since ψ is strictly increasing, then $\psi(\omega_\lambda(x_2, x_3)) < \psi(\psi(q\psi(t_0)))$. Put

$$q_2 = \frac{\psi(\psi(q\psi(t_0)))}{\psi(\omega_\lambda(x_2, x_3))}.$$

So $q_2 > 1$. If $x_3 \in Tx_3$ then x_3 is a fixed point of T. Hence we assume $x_3 \notin Tx_3$. Then

$$0 < \omega_\lambda(x_3, Tx_3) \text{ for all } \lambda > 0,$$

and so there exists $x_4 \in Tx_3$ such that

$$0 < \omega_\lambda(x_3, x_4) < q_2 \Omega_\lambda(Tx_2, Tx_3). \tag{2.7}$$

Clearly $x_2 \neq x_3$. Also again since T is α_*-admissible and $\perp-$preserving then

$$\alpha(x_2, x_3) \geq 1 \text{ and } x_2 \perp x_3.$$

Then from (2.1) we have

$$\Omega_\lambda(Tx_2, Tx_3) \leq \psi(\omega_\lambda(x_2, x_3)),$$

and so from (2.7) we deduce that

$$\omega_\lambda(x_3, x_4) < q_2 \Omega_\lambda(Tx_2, Tx_3) \leq q_2 \psi(\omega_\lambda(x_2, x_3)) = \psi(\psi(q\psi(t_0))).$$

By continuing this process we obtain a sequence $\{x_n\}$ in X_ω such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$, $x_{n-1} \perp x_n$, $\alpha(x_{n-1}, x_n) \geq 1$ and $\omega_1(x_n, x_{n+1}) \leq \psi^{n-1}(q\psi(t_0))$ for all $n \in \mathbb{N}$. Let p be a given positive integer. Now we can write

$$\omega_\lambda(x_n, x_{n+p}) = \omega_{p\frac{\lambda}{p}}(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} \omega_{\frac{\lambda}{p}}(x_k, x_{k+1}) \leq \sum_{k=n}^{n+p-1} \psi^{k-1}(q\psi(t_0)).$$

Therefore $\{x_n\}$ is an $\omega-$Cauchy sequence. Since X_ω is an $\omega-$complete modular metric space then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is $\perp^{\star\star}-$continuous then

$$\lim_{n\to\infty} \Omega_\lambda(Tx_{n-1}, Tz) = 0$$

for all $\lambda > 0$. Let $q > 1$. From lemma 2.3 for each $x_n \in Tx_{n-1}$ there exist $y_n \in Tz$ such that

$$\omega_\lambda(x_n, y_n) < q\Omega_\lambda(Tx_{n-1}, Tz)$$

for all $\lambda > 0$. Then $\lim_{n\to\infty} \omega_\lambda(x_n, y_n) = 0$ for all $\lambda > 0$. Therefore

$$\omega_\lambda(z, y_n) \leq \omega_{\frac{\lambda}{2}}(z, x_n) + \omega_{\frac{\lambda}{2}}(x_n, y_n).$$

By taking limit as $n \to \infty$ in the above inequality we get $\omega_\lambda(z, y_n) = 0$, for all $\lambda > 0$. That is the sequence $\{y_n\}$ $\omega-$converges to z. Since Tz is $\omega-$closed then $z \in Tz$. $\square$

For multifunction T that is not $\perp^{\star\star}-$continuous we prove the following theorem.

Theorem 2.2. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ \alpha(x, y) \geq 1 \end{cases} \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big). \tag{2.8}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$\alpha(x_n, x) \geq 1 \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. As in the proof of theorem 2.1 we deduce an $O^\star$-sequence $\{x_n\}$ starting at x_0 is $\omega-$Cauchy and so $\omega-$converges to a point $z \in X_\omega$. Then from (ii) we have

$$\alpha(x_n, z) \geq 1 \text{ and } x_n \perp z.$$

So from (2.9) we have

$$\Omega_\lambda(Tx_{n-1}, Tz) \leq \psi(\omega_\lambda(x_{n-1}, z))$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequalities we get

$$\lim_{n\to\infty} \Omega_\lambda(Tx_{n-1}, Tz) = 0.$$

Now as in the proof of theorem 2.1 we get $z \in Tz$. $\square$

Example 2.1. let $X = \{1, 2, 3\}$ and define modular metric ω on X be defined by

$$\omega_\lambda(x,y) = \omega_\lambda(y,x) = \begin{cases} 0 & x = y, \\ \dfrac{1}{4\lambda} & x, y \in X \setminus \{2\}, \\ \dfrac{1}{2\lambda} & x, y \in X \setminus \{3\}, \\ \dfrac{5}{8\lambda} & x, y \in X \setminus \{1\}. \end{cases}$$

Suppose $T2 = \{1\}$ and $Tx = \{3\}$ for $x \neq 2$, $\alpha(x,y) = 1$ and $x \perp y$ if and only if $x < y$. Let $\psi(t) = \frac{t}{2}$. For $x \perp y$, we consider to the following cases:

- Let $x = 1$ and $y = 2$, then,
$$\Omega_\lambda(T1, T2) = \omega_\lambda(1,3) = \frac{1}{4\lambda} = \psi(\omega_\lambda(1,2)).$$
- Let $x = 1$ and $y = 3$, then,
$$\Omega_\lambda(T1, T3) = \omega_\lambda(3,3) = 0 \leq \psi(\omega_\lambda(1,3)).$$
- Let $x = 2$ and $y = 3$, then,
$$\Omega_\lambda(T2, T3) = \omega_\lambda(1,3) = \frac{1}{4} \leq \frac{5}{16} = \psi(\omega_\lambda(2,3)).$$

Therefore all conditions of theorem 2.2 holds and T has a fixed point.

Example 2.2. Let $X = \mathbb{R}$, $M = [0, \infty)$ and $\omega_\lambda(x,y) = \frac{1}{\lambda}|x-y|$. Define $T : M \longrightarrow CB(M)$ by

$$Tx = \begin{cases} [\dfrac{x}{4}, \dfrac{x}{2}] & 0 \leq x \leq 1 \\ [\frac{e^{-x}}{2}, e^{-x}] & x > 0 \end{cases}$$

and $\alpha : M \times M \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 3 & x,y \in [0,1], \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that T is an α_*-admissible. Let $\psi(t) = \frac{7t}{8}$ for all $t \geq 0$ and $x \perp y$ if $x \leq y$. Let $x \perp y$ and $\alpha(x,y) \geq 1$. Then $x, y \in [0,1]$ and $0 \leq x \leq y \leq 1$. Then we write

$$\Omega_\lambda([\frac{x}{4}, \frac{x}{2}], [\frac{y}{4}, \frac{y}{2}]) = \frac{1}{2\lambda}\omega_\lambda(x,y) \leq \frac{7}{8\lambda}\omega_\lambda(x,y) = \psi(\omega_\lambda(x,y)).$$

If $\{x_n\} \subset X$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to +\infty$, then $x \in [0,1]$ and $x_n \leq x$ for all $n \geq 0$. That is $\alpha(x_n, x) \geq 1$ and $x_n \perp x$. Hence all conditions of theorem 2.2 holds and T has a fixed point. Let $x = 0$ and $y = 1$. So for usual metric $d(x,y) = |x - y|$ we have

$$\alpha(0,1)H(T0,T1) = \frac{3}{2} > 1 = d(0,1) > \psi(d(0,1)).$$

Therefore theorem 2.1 of [3] can not be applied for this example.

If in theorem we take $\alpha(x,y) = 1$, then we obtain the following corollary.

Corollary 2.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$x \perp y \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x,y)\big).$$

Then T has a fixed point.

Corollary 2.2. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible multifunction. Assume that for $\psi \in \Psi$,*

$$\alpha(x,y) \geq 1 \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x,y)\big). \tag{2.9}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) if $\{x_n\}$ be a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in M$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define a binary relation $\perp \in M \times M$ by $x \perp y$ if $(x,y) \in M \times M$. Then $x \perp y$ for all $x, y \in M$. That is $(M, \perp)$ is an $O^\star-$set and $\mathcal{C}(M) = M$. Clearly $(x,y) \in M \times M$ and $(u,v) \in M \times M$ for all $x, y \in M$ and all $u \in Tx$ and $v \in Ty$.

That is $x \perp y$ and $u \perp v$ for all $x, y \in M$ and all $u \in Tx$ and $v \in Ty$. Then T is a $\perp^{\star\star}-$preserving multifunction. From (i) there exist $x_0 \in M = \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that $\{x_n\}$ be an $O^\star-$sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in M$ as $n \to \infty$. Thus from (ii) we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Also clearly $(x_n, x) \in M \times M$ for all $n \in \mathbb{N}$. Now if $x \perp y$ and $\alpha(x, y) \geq 1$ then $(x, y) \in M \times M$ and $\alpha(x, y) \geq 1$ and so from (2.9) we get $\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big)$. Hence all conditions of theorem 2.2 hold and T has a fixed point. $\square$

If in corollary we take $\alpha(x, y) = 1$ then we obtain the following result.

Corollary 2.3. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be a multifunction. Assume that for $\psi \in \Psi$,*

$$\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big).$$

holds for all $x, y \in M$. Then T has a fixed point.

If in the above corollary we take $\psi(t) = rt$ where $r \in [0, 1)$ then we deduce the following result.

Corollary 2.4. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be a multifunction. Assume that for $r \in [0, 1)$,*

$$\Omega_\lambda(Tx, Ty) \leq r\omega_\lambda(x, y).$$

holds for all $x, y \in M$. Then T has a fixed point.

The following corollary is Theorem 2.1 of Asl et al. [3] in the setting of modular metric spaces.

Corollary 2.5. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let M be a nonempty ω-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible multifunction. Assume that for $\psi \in \Psi$*

$$\alpha(x, y)\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big) \tag{2.10}$$

holds for all $x, y \in M$. Also suppose that the following assertions hold:

(i) there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) if $\{x_n\}$ be a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ with $x_n \to x \in M$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Let $\alpha(x, y) \geq 1$. Then from (2.10) we get $\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big)$. Hence all conditions of corollary 2. hold and T has a fixed point. $\square$

3. Some Results in Modular Metric spaces endowed with a graph

As in [11], let (X_ω, ω) be a modular metric space and Δ denotes the diagonal of the cartesian product of $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover we may treat G as a weighted graph (see [12], p. 309) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x,\ x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$.

Definition 3.1. [11] Let (X, d) be a metric space endowed with a graph G. We say that a self-mapping $T : X \to X$ is a Banach G-contraction or simply a G-contraction if T preserves the edges of G, that is

$$\text{for all } x,\, y \in X, \quad (x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$$

and T decreases the weights of the edges of G in the following way:

$$\exists\, \alpha \in (0,1) \text{ such that for all } x, y \in X, \quad (x, y) \in E(G) \Longrightarrow d(Tx, Ty) \leqslant \alpha d(x, y).$$

Definition 3.2. [11] A mapping $T : X \to X$ is called G-continuous if given $x \in X$ and sequence $\{x_n\}$

$$x_n \to x \,\text{as}\, n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \,\text{for all}\, n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$

In this section we assert some $\perp-\psi-$contraction multifunction type fixed point results in O^*-modular metric spaces endowed with a graph G which can be deduced easily from our presented theorems.

Theorem 3.1. *Let X_ω be a modular metric space endowed with a graph G such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be a $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\left\{\begin{array}{l} x \perp y \\ (x, y) \in E(G) \end{array}\right. \Longrightarrow \Omega_\lambda(Tx, Ty) \le \psi\big(\omega_\lambda(x, y)\big). \tag{3.1}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$,

(ii) if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$,

(iii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Proof. Define $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ by $\alpha(x,y) = \begin{cases} 2, & \text{if } (x,y) \in E(G) \\ 0, & \text{otherwise} \end{cases}$.
First we show that T is an α_*-admissible multifunction. Let $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. From (ii) we have $(u,v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$. Then $\alpha_*(Tx,Ty) = \inf\{\alpha(u,v) : u \in Tx, v \in Ty\} = 2 \geq 1$. Thus T is an α_*-admissible multifunction. From (i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. That is $\alpha(x_0, x_1) \geq 1$. Assume that $x \perp y$ and $\alpha(x,y) \geq 1$. Thus $x \perp y$ and $(x,y) \in E(G)$. Hence from (4.1) we have $\Omega_\lambda(Tx,Ty) \leq \psi\big(\omega_\lambda(x,y)\big)$. Therefore all conditions of theorem 2.1 hold and T has a fixed point. □

Theorem 3.2. *Let X_ω be a modular metric space endowed with a graph G such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ (x,y) \in E(G) \end{cases} \Longrightarrow \Omega_\lambda(Tx,Ty) \leq \psi\big(\omega_\lambda(x,y)\big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$,

(ii) if $(x,y) \in E(G)$, then $(u,v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$,

(iii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$(x_n, x) \in E(G) \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ as in the proof of theorem 3.1. Let $\{x_n\}$ be a $O^\star-$sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$. From (iii) we get $(x_n, x) \in E(G)$ and $x_n \perp x$. That is $\alpha(x_n, x) \geq 1$ and $x_n \perp x$ for all $n \in \mathbb{N} \cup \{0\}$. Similar to the proof of theorem 3.1 we can prove that other conditions of theorem 2.2 are satisfied. Therefore all conditions of theorem 2.2 hold and T has a fixed point. □

4. Some Results in Modular Metric spaces endowed with a partial order

The existence of fixed points in partially ordered sets has been considered in [18]. Let X_ω be a nonempty set. If X_ω be a modular metric space and $(X_\omega, \preceq)$ be a partially ordered set then X_ω is called a partially ordered modular metric space. Two elements $x, y \in X_\omega$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

In this section we will show that some $\perp - \psi-$contraction multifunction type fixed point results in O^*-modular metric spaces endowed with a partial order $\preceq$ can be deduced easily from our presented theorems.

Theorem 4.1. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\left\{\begin{array}{l} x \perp y \\ x \preceq y \end{array}\right. \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big). \tag{4.1}$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Proof. Define $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ by $\alpha(x, y) = \left\{\begin{array}{ll} 2, & \text{if } x \preceq y \\ 0, & \text{otherwise} \end{array}\right.$. Let $\alpha(x, y) \geq 1$ then $x \preceq y$. From (ii) we have $u \preceq v$ for all $u \in Tx$ and $v \in Ty$. Then $\alpha_*(Tx, Ty) = \inf\{\alpha(u, v) : \ u \in Tx, \ v \in Ty\} = 2 \geq 1$. Thus T is an α_*-admissible multifunction. From (i) there exists $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$. That is $\alpha(x_0, x_1) \geq 1$. Assume that $x \perp y$ and $\alpha(x, y) \geq 1$. Thus $x \perp y$ and $x \preceq y$. Hence from (4.1) we have $\Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big)$. Therefore all conditions of Theorem 2.1 hold and T has a fixed point. $\square$

Theorem 4.2. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\left\{\begin{array}{l} x \perp y \\ x \preceq y \end{array}\right. \Longrightarrow \Omega_\lambda(Tx, Ty) \leq \psi\big(\omega_\lambda(x, y)\big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$x_n \preceq x \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ as in the proof of theorem 3.1. Let $\{x_n\}$ be a $O^\star-$sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (iii) we get $x_n \preceq x$ and $x_n \perp x$. That is $\alpha(x_n, x) \geq 1$ and $x_n \perp x$ for all $n \in \mathbb{N} \cup \{0\}$. Similar to the proof of theorem 3.1 we can prove that other conditions of theorem 2.2 are satisfied. Therefore all conditions of Theorem 2.2 hold and T has a fixed point. □

5. Some Integral type contractions

Let Φ denote the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

- every $\phi \in \Phi$ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$,
- for any $\phi \in \Phi$ and any $\epsilon > 0$, $\int_0^\epsilon \phi(\tau)d\tau > 0$.

Following arguments similar to those in Theorem 2.1 and 2.2, we can prove the following theorems.

Theorem 5.1. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ \alpha(x, y) \geq 1 \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertion holds:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,

(ii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Theorem 5.2. *Let X_ω be a modular metric space such that ω satisfies Δ_2-condition. Let $(M, \perp)$ be a nonempty $\omega - O^*$-complete subset of X_ω. Let $T : M \to CB(M)$ be an α_*-admissible and $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x \perp y \\ \alpha(x, y) \geq 1 \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) *there exist* $x_0 \in \mathcal{C}(M)$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geq 1$,

(ii) *if* $\{x_n\}$ *be an* $O-$*sequence in* X_ω *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *with* $x_n \to x$ *as* $n \to \infty$, *then*

$$\alpha(x_n, x) \geq 1 \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T *has a fixed point.*

As consequences of the above theorems we can deduce the following results in the setting of $O^\star-$modular metric space endowed with a graph G or a partial order $\preceq$.

Theorem 5.3. *Let* X_ω *be a modular metric space endowed with graph* G *such that* ω *satisfies* Δ_2-*condition. Let* $(M, \perp)$ *be a nonempty* $\omega - O^*$*-complete subset of* X_ω. *Let* $T : M \to CB(M)$ *be an* $\perp^{\star\star}-$*preserving multifunction. Assume that for* $\psi \in \Psi$,

$$\begin{cases} x \perp y \\ (x, y) \in E(G) \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) *there exist* $x_0 \in \mathcal{C}(M)$ *and* $x_1 \in Tx_0$ *such that* $(x_0, x_1) \in E(G)$,

(ii) *if* $(x, y) \in E(G)$, *then* $(u, v) \in E(G)$ *for all* $u \in Tx$ *and* $v \in Ty$,

(iii) T *is* $\perp^{\star\star}-$*continuous.*

Then T *has a fixed point.*

Theorem 5.4. *Let* X_ω *be a modular metric space endowed with graph* G *such that* ω *satisfies* Δ_2-*condition. Let* $(M, \perp)$ *be a nonempty* $\omega - O^*$*-complete subset of* X_ω. *Let* $T : M \to CB(M)$ *be an* $\perp^{\star\star}-$*preserving multifunction. Assume that for* $\psi \in \Psi$,

$$\begin{cases} x \perp y \\ (x, y) \in E(G) \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) *there exist* $x_0 \in \mathcal{C}(M)$ *and* $x_1 \in Tx_0$ *such that* $(x_0, x_1) \in E(G)$,

(ii) *if* $(x, y) \in E(G)$, *then* $(u, v) \in E(G)$ *for all* $u \in Tx$ *and* $v \in Ty$,

(iii) *if* $\{x_n\}$ *be an* $O-$*sequence in* X_ω *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}$ *with* $x_n \to x$ *as* $n \to \infty$, *then*

$$(x_n, x) \in E(G) \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

Theorem 5.5. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega-O^*$-complete subset of X_ω. Let $T: M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x\perp y \\ x \preceq y \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) T is $\perp^{\star\star}-$continuous.

Then T has a fixed point.

Theorem 5.6. *Let X_ω be a modular metric space endowed with a partial order $\preceq$ such that ω satisfies Δ_2-condition. Let $(M,\perp)$ be a nonempty $\omega-O^*$-complete subset of X_ω. Let $T: M \to CB(M)$ be an $\perp^{\star\star}-$preserving multifunction. Assume that for $\psi \in \Psi$,*

$$\begin{cases} x\perp y \\ x \preceq y \end{cases} \Longrightarrow \int_0^{\Omega_\lambda(Tx,Ty)} \phi(\tau)d\tau \leq \psi\Big(\int_0^{\omega_\lambda(x,y)} \phi(\tau)d\tau\Big).$$

Also suppose that the following assertions hold:

(i) there exist $x_0 \in \mathcal{C}(M)$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$,

(ii) if $x \preceq y$, then $u \preceq v$ for all $u \in Tx$ and $v \in Ty$,

(iii) if $\{x_n\}$ be an $O-$sequence in X_ω such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ with $x_n \to x$ as $n \to \infty$, then

$$x_n \preceq x \text{ and } x_n \perp x$$

hold for all $n \in \mathbb{N}$.

Then T has a fixed point.

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ON PARAMETRIZED HERMITE-HADAMARD TYPE INEQUALITIES

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Abstract. In recent years, many results have been devoted to the well-known Hermite-Hadamard inequality. This inequality has many applications in the area of pure and applied mathematics. In this paper, our main aim is to give a parametrized inequality of the Hermite-Hadamard type and its applications to f-divergence measures and means. First, we prove the identity associated with the right side of the Hermite-Hadamard inequality. By using this identity, the convexity of the function and some well-known inequalities, we obtain several results for the inequality. The inequalities derived here also point out some known results as their special cases.

Keywords. Hermite-Hadamard inequality; parametrized inequality; convex function.

1. Introduction

Almost no mathematician in applied mathematics, especially in nonlinear programming and optimization theory, can ignore the significant role of convex sets and convex functions. For the class of convex functions, many inequalities such as Jensen's, Hermite-Hadamard and Slater's inequalities have been introduced since this idea was introduced for the first time more than a century ago. Among the introduced inequalities, the most prominent is the so called Hermite-Hadamard's inequality. The statement of this inequality is (see [15]):

Let I be an interval in $\mathbb{R}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on I such that $a, b \in I$ with $a < b$. Then the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

hold. If the function f is concave on I, then both the inequalities in (1.1) hold in the reverse direction. It gives an estimate from both sides of the mean value of a

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convex function and also ensure the integrability of the convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f. These inequalities for convex functions play a crucial role in mathematical analysis and other areas of pure and applied mathematics.

For more recent results, generalizations, improvements and refinements related to Hermite-Hadamard inequality see [2, 3, 9, 10, 11, 12, 13, 14, 24, 30, 23, 22] and the references cited therein.
In 2010, Havva Kavurmaci et al. proved the following important lemma:

Lemma 1.1. [18] *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following identity holds:*

$$\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt. \tag{1.2}$$

Here I° denotes the interior of I.

The following results are the ultimate consequences of Lemma 1.1, which have been presented in [18] .

Theorem 1.1. *Under the assumptions of Lemma 1.1 and if $|f'|$ is convex on $[a, b]$, then we have the following inequality:*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left[\frac{|f'(x)|+2|f'(a)|}{6}\right] + \frac{(b-x)^2}{b-a}\left[\frac{|f'(x)|+2|f'(b)|}{6}\right].$$

Theorem 1.2. *Suppose the conditions of Lemma 1.1 are satisfied and if the new mapping $|f'|^q$ $(q > 1)$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2\left[|f'(x)|^q+2|f'(a)|^q\right]^{\frac{1}{q}} + (b-x)^2\left[|f'(x)|^q+2|f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right].$$

Theorem 1.3. *Suppose the conditions of Lemma 1.1 hold and if the mapping $|f'|^q$ $(q \geq 1)$ is concave on $[a, b]$, then the following inequality is valid:*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}\left[\frac{(x-a)^2\left|f'\left(\frac{x+2a}{3}\right)\right| + (b-x)^2\left|f'\left(\frac{x+2b}{3}\right)\right|}{b-a}\right].$$

The main purpose of this paper is to present a parametrized inequality of the Hermite-Hadamard type for functions whose first derivative absolute values are convex. We prove the identity for the right side of the inequality and discuss their particular case (Corollaries 2.2, 2.4, 2.6). By applying Jensen's inequality, power mean inequality and the convexity of functions in the identity, we obtain inequalities for the right side of the Hermite-Hadamard inequality. As applications, some new inequalities for f-divergence measures and means are established.

2. Main Results

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $\epsilon \in \mathbb{R}$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-\epsilon)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (\epsilon-t)f'(tx+(1-t)b)dt. \tag{2.1}$$

Proof. It suffices to note that

$$\begin{aligned} I_1 &= \frac{(x-a)^2}{b-a}\int_0^1 (t-\epsilon)f'(tx+(1-t)a)dt \\ &= \frac{(x-a)^2}{b-a}\left[\frac{(t-\epsilon)f(tx+(1-t)a)}{x-a}\Big|_0^1 - \int_0^1 \frac{f(tx+(1-t)a)dt}{x-a}\right] \\ &= \frac{(x-a)^2}{b-a}\left[\frac{(1-\epsilon)f(x)+\epsilon f(a)}{x-a} - \int_0^1 \frac{f(tx+(1-t)a)dt}{x-a}\right]. \end{aligned}$$

By substituting $u = tx + (1-t)a$ in (2.2) we have

$$\begin{aligned} I_1 &= \frac{(x-a)^2}{b-a}\left[\frac{(1-\epsilon)f(x)+\epsilon f(a)}{x-a} - \int_a^x \frac{f(u)du}{(x-a)^2}\right] \\ &= \frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]}{b-a} - \frac{1}{b-a}\int_a^x f(u)du, \end{aligned} \tag{2.2}$$

similarly

$$\begin{aligned} I_2 &= \frac{(b-x)^2}{b-a}\int_0^1 (\epsilon - t)f'(tx+(1-t)b)dt \\ &= \frac{(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_x^b f(u)du, \end{aligned} \tag{2.3}$$

now by adding (2.2) and (2.3) we get (2.1). □

Remark 2.1. If we choose $\epsilon = 1$, then from Lemma 2.1 we obtain Lemma 1.1.

Lemma 2.2. *Let ϵ be a real number. Then*

$$\int_0^1 |\epsilon - t|dt = \left\{ \begin{array}{lr} \frac{2\epsilon-1}{2}, & \epsilon \geq 1 \\ \frac{2\epsilon^2-2\epsilon+1}{2}, & 0<\epsilon<1 \\ \frac{1-2\epsilon}{2}, & \epsilon \leq 0. \end{array} \right\}$$

Proof. **Case 1.** If $\epsilon \geq 1$, then $\int_0^1 |\epsilon - t|dt = \int_0^1 (\epsilon - t)dt = \frac{2\epsilon-1}{2}$.

Case 2. If $0 < \epsilon < 1$, then
$\int_0^1 |\epsilon - t|dt = \int_0^\epsilon (\epsilon - t)dt + \int_\epsilon^1 (t-\epsilon)dt = \frac{2\epsilon^2-2\epsilon+1}{2}$.

Case 3. If $\epsilon \leq 0$, then
$\int_0^1 |\epsilon - t|dt = \int_0^1 (t-\epsilon)dt = \frac{1-2\epsilon}{2}$.
□

Lemma 2.3. *Let ϵ be a real number. Then*

$$\int_0^1 |t-\epsilon|tdt = \left\{ \begin{array}{lr} \frac{3\epsilon-2}{6}, & \epsilon \geq 1 \\ \frac{2\epsilon^3-3\epsilon+2}{6}, & 0<\epsilon<1 \\ \frac{2-3\epsilon}{6}, & \epsilon \leq 0. \end{array} \right\}$$

Lemma 2.4. *Let ϵ be a real number. Then*

$$\int_0^1 |t-\epsilon|(1-t)dt = \left\{ \begin{array}{lr} \frac{3\epsilon-1}{6}, & \epsilon \geq 1 \\ \frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}, & 0<\epsilon<1 \\ \frac{1-3\epsilon}{6}, & \epsilon \leq 0. \end{array} \right\}$$

Theorem 2.1. *Let $\epsilon \in \mathbb{R}$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function on $I^\circ, a, b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds :*

$$\left| \frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(x-a)^2}{b-a} \left\{ \begin{array}{lr} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0 \end{array} \right\}$$

$$+\frac{(b-x)^2}{b-a} \left\{ \begin{array}{lr} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0. \end{array} \right\}$$

Proof. It follows from the convexity of $|f'|$ and Lemma 2.1 that

$$\left| \frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\epsilon||f'(tx+(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 |\epsilon-t||f'(tx+(1-t)b)|dt$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\epsilon|[t|f'(x)|+(1-t)|f'(a)|]dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1 |\epsilon-t|[t|f'(x)|+(1-t)|f'(b)|]dt$$

$$= \frac{(x-a)^2}{b-a} \left\{ \begin{array}{lr} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0 \end{array} \right\}$$

$$+\frac{(b-x)^2}{b-a} \left\{ \begin{array}{lr} |f'(x)|\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|\left(\frac{3\epsilon-1}{6}\right) & \text{if } \epsilon \geq 1 \\ |f'(x)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) & \text{if } 0<\epsilon<1 \\ |f'(x)|\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|\left(\frac{1-3\epsilon}{6}\right) & \text{if } \epsilon \leq 0. \end{array} \right\}$$

□

Corollary 2.1. *Under the assumption of Theorem 2.1 if we choose* $x = \frac{a+b}{2}$*, we have*

$$\left|\frac{\epsilon f(a) + \epsilon f(b) + 2(1-\epsilon) f\left(\frac{a+b}{2}\right)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\left\{\begin{array}{l} |f'\left(\frac{a+b}{2}\right)|\left(\frac{3\epsilon-2}{6}\right) + (|f'(a)| + |f'(b)|)\left(\frac{3\epsilon-1}{6}\right) \ \ \textit{if } \epsilon \geq 1, \\ |f'\left(\frac{a+b}{2}\right)|\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right) + (|f'(a)| + |f'(b)|)\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) \\ \textit{if } 0 < \epsilon < 1, \\ |f'\left(\frac{a+b}{2}\right)|\left(\frac{2-3\epsilon}{6}\right) + (|f'(a)| + |f'(b)|)\left(\frac{1-3\epsilon}{6}\right) \ \ \textit{if } \epsilon \leq 0. \end{array}\right\}$$

Corollary 2.2. *Under the assumption of Theorem 2.1 if we choose* $x = \frac{a+b}{2}$ *and* $\epsilon = 1$*, we have*

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{12}\left(|f'(a)| + |f'\left(\frac{a+b}{2}\right)| + |f'(b)|\right) \\ &\leq \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right). \end{aligned}$$

Proof. The second inequality is obtained by using the convexity of $|f'|$. □

Theorem 2.2. *Let* $\epsilon \in \mathbb{R}$ *and let* $f : I^\circ \to \mathbb{R}$ *be a differentiable function on* $I^\circ, a, b \in I^\circ$ *with* $a < b$ *such that* $f' \in L[a,b]$*. If* $|f'|^q, q \geq 1$ *is convex on* $[a,b]$*, then the following inequality holds:*

$$\left|\frac{(x-a)[(1-\epsilon)f(x) + \epsilon f(a)] + (b-x)[(1-\epsilon)f(x) + \epsilon f(b)]}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\left\{\begin{array}{l} \left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right) + |f'(a)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \ \ \textit{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right) + |f'(a)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}} \\ \textit{if } 0 < \epsilon < 1 \\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right) + |f'(a)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \ \ \textit{if } \epsilon \leq 0 \end{array}\right\}$$

$$+\frac{(b-x)^2}{b-a}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \quad if\ \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ if\ 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \quad if\ \epsilon\leq 0.\end{array}\right\}$$

Proof. Using Lemma 2.1 and the Power mean inequality, we have

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1|t-\epsilon||f'(tx+(1-t)a)|dt+\frac{(b-x)^2}{b-a}\int_0^1|\epsilon-t||f'(tx+(1-t)b)|dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1|t-\epsilon|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(|t-\epsilon|)|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1|\epsilon-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\epsilon-t||f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1|t-\epsilon|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t-\epsilon|[t|f'(x)|^q+(1-t)|f'(a)|^q]dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1|\epsilon-t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|\epsilon-t|[t|f'(x)|^q+(1-t)|f'(b)|^q]dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-a)^2}{b-a}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \text{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \quad \text{if } \epsilon\leq 0\end{array}\right\}$$

$$+\frac{(b-x)^2}{b-a}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \text{ if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \text{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'(x)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \text{ if } \epsilon\leq 0.\end{array}\right\}$$

□

Corollary 2.3. *Under the assumption of Theorem 2.1 if we choose* $x=\frac{a+b}{2}$*, we have*

$$\left|\frac{\epsilon f(a)+\epsilon f(b)+2(1-\epsilon)f\left(\frac{a+b}{2}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{2}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(a)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(a)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \textit{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(a)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\leq 0\end{array}\right\}$$

$$+\frac{b-a}{2}\left\{\begin{array}{l}\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{3\epsilon-2}{6}\right)+|f'(b)|^q\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\geq 1\\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)+|f'(b)|^q\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}}\\ \textit{if } 0<\epsilon<1\\ \left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(|f'\left(\frac{a+b}{2}\right)|^q\left(\frac{2-3\epsilon}{6}\right)+|f'(b)|^q\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \textit{ if } \epsilon\leq 0.\end{array}\right\}$$

Corollary 2.4. *Under the assumption of Theorem 2.1 if we choose* $x=\frac{a+b}{2}$ *and* $\epsilon=1$*, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(2|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(2|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right)(b-a)(|f'(a)|+|f'(b)|).$$

Proof. The second inequality is obtained using the convexity of $|f'|^q$ and the fact that $\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s$ for $0 \leq s < 1$,$a_1, a_2, ...a_n \geq 0$, $b_1, b_2, ...b_n \geq 0$. □

Theorem 2.3. *Let $\epsilon \in \mathbb{R}$ and let $f : I^\circ \to \mathbb{R}$ be a differentiable function on $I^\circ, a, b \in I^\circ$ with $a < b$ such that $f' \in L[a,b]$. If $|f'|^q, q \geq 1$ is concave on $[a,b]$, then the following inequality holds:*

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)a}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)a}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0 < \epsilon < 1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)a}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{array}\right\}$$
$$+\frac{(b-x)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)b}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)b}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0 < \epsilon < 1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)b}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0. \end{array}\right\}$$

Proof. By concavity of $|f'|^q$ and the power mean inequality we may write

$|f'(\lambda x+(1-\lambda)y)|^q \geq \lambda|f'(x)|^q+(1-\lambda)|f'(y)|^q \geq (\lambda|f'(x)|+(1-\lambda)|f'(y)|)^q.$

Hence

$$|f'(\lambda x+(1-\lambda)y)| \geq \lambda|f'(x)|+(1-\lambda)|f'(y)|,$$

so $|f'|$ is also concave. Now by applying triangular inequality and Lemma 2.1 we have:

$$\left|\frac{(x-a)[(1-\epsilon)f(x)+\epsilon f(a)]+(b-x)[(1-\epsilon)f(x)+\epsilon f(b)]}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 |t-\epsilon||f'(tx+(1-t)a)|dt + \frac{(b-x)^2}{b-a}\int_0^1 |\epsilon-t||f'(tx+(1-t)b)|dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 |t-\epsilon|dt\right)\left|f'\left(\frac{\int_0^1 |t-\epsilon|(tx+(1-t)a)dt}{\int_0^1 |t-\epsilon|dt}\right)\right|$$

$$+ \frac{(b-x)^2}{b-a}\left(\int_0^1 |\epsilon-t|dt\right)\left|f'\left(\frac{\int_0^1 |\epsilon-t|(tx+(1-t)b)dt}{\int_0^1 |\epsilon-t|dt}\right)\right|$$

$$= \frac{(x-a)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)a}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon\geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)a}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)a}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon\leq 0 \end{array}\right\}$$

$$+ \frac{(b-x)^2}{b-a}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(3\epsilon-2)x+(3\epsilon-1)b}{3(2\epsilon-1)}\right)\right| & \text{if } \epsilon\geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(2\epsilon^3-3\epsilon+2)x+(-2\epsilon^3+6\epsilon^2-3\epsilon+1)b}{3(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(2-3\epsilon)x+(1-3\epsilon)b}{3(1-2\epsilon)}\right)\right| & \text{if } \epsilon\leq 0. \end{array}\right\}$$

□

Corollary 2.5. *Under the assumption of Theorem 2.3 if we choose* $x=\frac{a+b}{2}$, *we have*

$$\left|\frac{\epsilon f(a)+\epsilon f(b)+2(1-\epsilon)f\left(\frac{a+b}{2}\right)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(9\epsilon-4)a+(3\epsilon-2)b}{6(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)a+(2\epsilon^3-3\epsilon+2)b}{6(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(4-9\epsilon)a+(2-3\epsilon)b}{6(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{array}\right\}$$

$$+ \frac{b-a}{2}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|f'\left(\frac{(9\epsilon-4)b+(3\epsilon-2)a}{6(2\epsilon-1)}\right)\right| & \text{if } \epsilon \geq 1 \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|f'\left(\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)b+(2\epsilon^3-3\epsilon+2)a}{6(2\epsilon^2-2\epsilon+1)}\right)\right| & \text{if } 0<\epsilon<1 \\ \left(\frac{1-2\epsilon}{2}\right)\left|f'\left(\frac{(4-9\epsilon)b+(2-3\epsilon)a}{6(1-2\epsilon)}\right)\right| & \text{if } \epsilon \leq 0 \end{array}\right\}$$

Corollary 2.6. *Under the assumption of Theorem 2.3 if we choose* $x = \frac{a+b}{2}$ *and* $\epsilon = 1$*, we have*

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{8}\left[\left|f'\left(\frac{5a+b}{6}\right)\right| + \left|f'\left(\frac{a+5b}{6}\right)\right|\right] \\ &\leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \end{aligned}$$

Proof. The second inequality is obtained by using the concavity of $|f'|^q$. □

3. Applications to f-Divergence Measures

One of the basic problem in various applications of Probability Theory is finding an appropriate measure of distance between any two probability distributions. A lot of divergence measures for this purpose have been proposed and extensively studied by Kullback and Leibler [25], Renyi [29], Havrda and Charvat [16], Burbea and Rao [5], Lin [26], Csisźar[8], Ali and Silvey [1], Shioya and Da-te [31] and others (see for example [17] and the references therein). But here we will take only two of them and for this purpose define the following terms.

Let the set χ and the σ-finite measure μ be given and consider the set of all probability densities on μ to be defined on $\Omega := \{p|\, p : \chi \to \mathbb{R},\, p(x) > 0,\, \int_\chi p(x)d\mu(x) = 1\}$.

Let $f:(0,\infty)\to\mathbb{R}$ be given function and consider $D_f(p,q)$ be defined by

$$D_f(p,q) := \int_\chi p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p,q\in\Omega. \tag{3.1}$$

If f is convex function, then (3.1) is known as the Csisźar f-divergence [8].

In [31], Shioya and Da-te introduced the Hermite-Hadamard (HH) divergence

$$D^f_{HH}(p,q) := \int_\chi p(x) \frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1} d\mu(x), \quad p,\ q\in\Omega, \tag{3.2}$$

where f is convex function on $(0,\infty)$ with $f(1)=0$. In [31] the authors gave the property of HH divergence that $D^f_{HH}(p,q)\geq 0$ with the equality holds if and only if $p=q$.

Proposition 3.1. *Let all the assumptions of Theorem 2.1 hold with $I=(0,\infty)$ and $f(1)=0$. If $p,q\in\Omega$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{2}D_f(p,q) - D^f_{HH}(p,q)\right| \\ \text{(3.3)} \leq\ & \frac{1}{8}\left[|f'(1)|\int_\chi |q(x)-p(x)|d\mu(x) + \int_\chi |q(x)-p(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right| d\mu(x)\right]. \end{aligned}$$

Proof. Let $X_1=\{x\in\chi : q(x)>p(x)\}$, $X_2=\{x\in\chi : q(x)<p(x)\}$ and $X_3=\{x\in\chi : q(x)=p(x)\}$.

If $x\in X_3$, then obviously the equality holds in (3.3).
Now if $x\in X_1$, then by using Corollary 2.2 for $a=1$, $b=\frac{q(x)}{p(x)}$, multiplying both hand sides of the obtained results by $p(x)$ and then integrating over X_1, we get

$$\begin{aligned} &\left|\frac{1}{2}\int_{X_1} p(x)f\left[\frac{q(x)}{p(x)}\right]d\mu(x) - \int_{X_1} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1}d\mu(x)\right| \\ \leq\ & \frac{1}{8}\left[|f'(1)|\int_{X_1}|q(x)-p(x)|d\mu(x) + \int_{X_1}|q(x)-p(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right|d\mu(x)\right]. \end{aligned} \tag{3.4}$$

Similarly, if $x\in X_2$, then by using for $a=\frac{q(x)}{p(x)}$, $b=1$, multiplying both sides by $p(x)$ and then integrating over X_2, we get

$$\begin{aligned} &\left|\frac{1}{2}\int_{X_2} p(x)f\left[\frac{q(x)}{p(x)}\right]d\mu(x) - \int_{X_2} p(x)\frac{\int_1^{\frac{q(x)}{p(x)}} f(t)dt}{\frac{q(x)}{p(x)}-1}d\mu(x)\right| \\ \leq\ & \frac{1}{8}\left[|f'(1)|\int_{X_2}|p(x)-q(x)|d\mu(x) + \int_{X_2}|p(x)-q(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right|d\mu(x)\right]. \end{aligned} \tag{3.5}$$

By adding the inequalities (3.4) and (3.5) and then using the triangular inequality we get (3.3). □

Proposition 3.2. *Let all the assumptions of Theorem 2.2 hold with* $I = (0, \infty)$ *and* $f(1) = 0$*. If* $p, q \in \Omega$*, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{2}D_f(p,q) - D_{HH}^f(p,q)\right| \\ \leq\ & \left(\frac{3^{1-\frac{1}{q}}}{8}\right)\left[|f'(1)|\int_{\chi}|q(x)-p(x)|d\mu(x)\right. \\ +\ & \left.\int_{\chi}|q(x)-p(x)|\left|f'\left(\frac{q(x)}{p(x)}\right)\right|d\mu(x)\right]. \end{aligned} \tag{3.6}$$

Proof. The proof is similar to the proof of Proposition 3.1 but use Corollary 2.4 instead of Corollary 2.2. □

Proposition 3.3. *Let all the assumptions of Theorem 2.3 hold with* $I = (0, \infty)$ *and* $f(1) = 0$*. If* $p, q \in \Omega$*, then we have the inequality*

$$\begin{aligned} &\left|\frac{1}{2}D_f(p,q) - D_{HH}^f(p,q)\right| \\ \leq\ & \frac{1}{4}\left[\int_{\chi}|q(x)-p(x)|\left|f'\left(\frac{p(x)+q(x)}{2p(x)}\right)\right|d\mu(x)\right]. \end{aligned} \tag{3.7}$$

Proof. The proof is similar to the proof of Proposition 3.1 but use Corollary 2.6 instead of Corollary 2.2. □

As in [10], we will consider the following particular means for any $a, b, c \in \mathbb{R}, a \neq b \neq c$ which are well known in the literature:

$$\begin{aligned} A(a,b,c;w_a,w_b,w_c) &= \frac{w_a a + w_b b + w_c c}{w_a + w_b + w_c} \qquad a,b,c>0, \\ \bar{L}(a,b) &= \frac{b-a}{\ln b - \ln a} \qquad a \neq b,\ b,b>0, \\ L_n(a,b) &= \left[\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}} \qquad a,b\in\mathbb{R}, a<b, n\neq -1,0, n\in\mathbb{R}. \end{aligned}$$

Proposition 3.4. *Let* $0 < a < b < c$*,* $n \in \mathbb{R}$*,and* $n > 2$*. Then the inequality*

$$\begin{aligned} &|A(a^n, b^n, \left(\frac{a+b}{2}\right)^n; \epsilon, \epsilon, 2(1-\epsilon)) - L_n(a,b)^n| \\ \leq\ & \frac{n(b-a)}{2}\left\{\begin{array}{l} \left|\frac{a+b}{2}\right|^{n-1}\left(\frac{3\epsilon-2}{6}\right) + \left(|a|^{n-1}+|b|^{n-1}\right)\left(\frac{3\epsilon-1}{6}\right) \ \textit{if } \epsilon \geq 1, \\ \left|\frac{a+b}{2}\right|^{n-1}\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right) + \left(|a|^{n-1}+|b|^{n-1}\right)\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right) \\ \textit{if } 0<\epsilon<1, \\ \left|\frac{a+b}{2}\right|^{n-1}\left(\frac{2-3\epsilon}{6}\right) + \left(|a|^{n-1}+|b|^{n-1}\right)\left(\frac{1-3\epsilon}{6}\right) \ \textit{if } \epsilon \leq 0, \end{array}\right\} \end{aligned}$$

holds.

Proof. By using the function $f(s) = s^n, s > 0, n > 2$, the proof can be obtained from Corollary 2.1. □

Proposition 3.5. *Let* $0 < a < b < c$, $n \in \mathbb{R}$,*and* $n > 2$. *Then the inequality*

$$
\begin{aligned}
&\left|A\left(a^n, b^n, \left(\frac{a+b}{2}\right)^n; \epsilon, \epsilon, 2(1-\epsilon)\right) - L_n(a,b)^n\right| \\
\leq\ & \frac{n(b-a)}{2}\left\{\begin{array}{l}
\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{3\epsilon-2}{6}\right) + |a|^{(n-1)q}\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \quad \textit{if } \epsilon \geq 1, \\
\left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)\right. \\
\left. +|a|^{(n-1)q}\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}} \quad \textit{if } 0 < \epsilon < 1, \\
\left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2-3\epsilon}{6}\right) + |a|^{(n-1)q}\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \quad \textit{if } \epsilon \leq 0,
\end{array}\right\} \\
&+\frac{n(b-a)}{2}\left\{\begin{array}{l}
\left(\frac{2\epsilon-1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{3\epsilon-2}{6}\right) + |b|^{(n-1)q}\left(\frac{3\epsilon-1}{6}\right)\right)^{\frac{1}{q}} \quad \textit{if } \epsilon \geq 1, \\
\left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2\epsilon^3-3\epsilon+2}{6}\right)\right. \\
\left. +|b|^{(n-1)q}\left(\frac{-2\epsilon^3+6\epsilon^2-3\epsilon+1}{6}\right)\right)^{\frac{1}{q}} \quad \textit{if } 0 < \epsilon < 1, \\
\left(\frac{1-2\epsilon}{2}\right)^{1-\frac{1}{q}}\left(\left|\frac{a+b}{2}\right|^{(n-1)q}\left(\frac{2-3\epsilon}{6}\right) + |b|^{(n-1)q}\left(\frac{1-3\epsilon}{6}\right)\right)^{\frac{1}{q}} \quad \textit{if } \epsilon \leq 0,
\end{array}\right\}
\end{aligned}
$$

holds.

Proof. By using the function $f(s) = s^n, s > 0, n > 2$, the proof can be obtained from Corollary 2.3. □

Proposition 3.6. *Let* $0 < a < b < c$, $n \in \mathbb{R}$,*and* $1 < n < 2$. *Then the inequality*

$$
\begin{aligned}
&\left|A\left(a^n, b^n, \left(\frac{a+b}{2}\right)^n; \epsilon, \epsilon, 2(1-\epsilon)\right) - L_n(a,b)^n\right| \\
\leq\ & \frac{|n|(b-a)}{2}\left\{\begin{array}{ll}
\left(\frac{2\epsilon-1}{2}\right)\left|\frac{(9\epsilon-4)a+(3\epsilon-2)b}{6(2\epsilon-1)}\right|^{n-1} & \textit{if } \epsilon \geq 1, \\
\left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)a+(2\epsilon^3-3\epsilon+2)b}{6(2\epsilon^2-2\epsilon+1)}\right|^{n-1} & \textit{if } 0 < \epsilon < 1, \\
\left(\frac{1-2\epsilon}{2}\right)\left|\frac{(4-9\epsilon)a+(2-3\epsilon)b}{6(1-2\epsilon)}\right|^{n-1} & \textit{if } \epsilon \leq 0,
\end{array}\right\}
\end{aligned}
$$

$$+\quad \frac{|n|(b-a)}{2}\left\{\begin{array}{ll} \left(\frac{2\epsilon-1}{2}\right)\left|\frac{(9\epsilon-4)b+(3\epsilon-2)a}{6(2\epsilon-1)}\right|^{n-1} & \textit{if } \epsilon\geq 1, \\ \left(\frac{2\epsilon^2-2\epsilon+1}{2}\right)\left|\frac{(-2\epsilon^3+12\epsilon^2-9\epsilon+4)b+(2\epsilon^3-3\epsilon+2)a}{6(2\epsilon^2-2\epsilon+1)}\right|^{n-1} & \textit{if } 0<\epsilon<1, \\ \left(\frac{1-2\epsilon}{2}\right)\left|\frac{(4-9\epsilon)b+(2-3\epsilon)a}{6(1-2\epsilon)}\right|^{n-1} & \textit{if } \epsilon\leq 0, \end{array}\right\}$$

holds.

Proof. By using the function $f(s)=s^n, s>0, 1<n<2$, the proof can be obtained from Corollary 2.5. □

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SOME COMMON FIXED POINT RESULTS FOR RATIONAL CONTRACTION TYPE VIA THE $\mathcal{C}$-CLASS FUNCTIONS ON METRIC SPACES

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Abstract. The purpose of this paper is to prove some common fixed point results for rational contraction type via the $\mathcal{C}$-class functions on metric spaces. As an application, we study the existence of solutions to the system of nonlinear integral equations.

Keywords. Common fixed point; rational contraction mappings; triangular α-orbital admissible mapping; contraction; integral equations.

1. Introduction

The Banach contraction principle [2] is a basic tool in studying the existence of solutions to many problems in mathematics and many different fields. In recent times, the contraction principle has been extended in many various directions. Geraghty's theorem [7] is one of the generalized result. In 2013, Cho *et al.* [5] introduced the notion of α-Geraghty contraction type maps and proved some fixed point theorems for such maps in complete metric spaces. In 2014, Popescu [14] extended the results in [5] by proving certain fixed point theorems for generalized α-Geraghty contraction type maps. Later, Karapina [12] introduced the notion of α-ψ-Geraghty contraction type maps and proved the existence and uniqueness of fixed points for such maps in metric spaces. In 2016, Chuadchawna *et al.* [6] improved and generalized the results in [12, 14] by proving some fixed point theorems for α-η-ψ-Greraghty contraction type maps in α-η complete metric spaces. Recently, Ansari and Kaewcharoen [1] extended the results in [12] and proved the fixed point theorems for α-η-ψ-φ-F contraction type maps in α-η complete metric spaces by using the $\mathcal{C}$-class function.

In 1977, Jaggi [11] also extended the Banach contraction principle by proving some fixed point theorems for a contractive condition of rational type in metric

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spaces. After that, some authors extended the main results in [11] by many different ways. Furthermore, certain fixed point results for rational contractions were established in metric spaces and generalized metric spaces (see, for example, [4, 8, 9, 13] and the references therein).

In this paper, we state some common fixed point theorems for rational contraction type via the $\mathcal{C}$-class functions on metric spaces. The obtained results are generalizations of the main results in [1, 12, 14]. In addition, we study the existence of solutions to the system of nonlinear integral equations.

2. Preliminaries

First, we recall some symbols that

1. $\mathcal{C}$ is the family of all functions $F : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ such that for all $s, t \in [0, \infty)$,
   (a) F is continuous.
   (b) $F(s, t) \leq s$.
   (c) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.
2. Ψ is the family of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that
   (a) ψ is nondecreasing and continuous.
   (b) $\psi(t) = 0$ if and only if $t = 0$.
3. Φ the family of all functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that
   (a) φ is continuous.
   (b) $\varphi(t) > 0$ for all $t > 0$.

In [1], the authors gave some functions which are elements in $\mathcal{C}$.

Example 2.1. ([1], Example 1.12) The following functions $F : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ are elements in $\mathcal{C}$.

1. $F(s, t) = s - t$ for all $s, t \in [0, \infty)$.
2. $F(s, t) = ms$ for all $s, t \in [0, \infty)$ where $0 < m < 1$.
3. $F(s, t) = \dfrac{s}{(1 + t)^r}$ for all $s, t \in [0, \infty)$ where $r \in (0, \infty)$.
4. $F(s, t) = s\beta(s)$ for all $s, t \in [0, \infty)$ where $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function.
5. $F(s, t) = s - \varphi(s)$ for all $s, t \in [0, \infty)$ where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ iff $t = 0$.

In 2014, Popescu [14] introduced the notion of α-orbital admissible mappings and triangular α-orbital admissible mappings as follows.

Definition 2.1. ([14], Definition 5) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be two mappings. Then f is called an *α-orbital admissible mapping* if for all $x \in X$,

$$\alpha(x, fx) \geq 1 \text{ implies } \alpha(fx, f^2x) \geq 1.$$

Definition 2.2. ([14], Definition 6) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be two mappings. Then f is called a *triangular α-orbital admissible mapping* if

1. f is an α-orbital admissible.
2. For all $x, y \in X$, $\alpha(x, y) \geq 1, \alpha(y, fy) \geq 1$ imply $\alpha(x, fy) \geq 1$.

In 2016, Chuadchawna *et al* [6] introduced the notion of α-orbital admissible mappings respect to η and triangular α-orbital admissible mappings respect to η as follows.

Definition 2.3. ([6], Definition 2.1) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be two mappings. Then f is called an *α-orbital admissible mapping respect to η* if for all $x \in X$,

$$\alpha(x, fx) \geq \eta(x, fx) \text{ implies } \alpha(fx, f^2x) \geq \eta(fx, f^2x).$$

Definition 2.4. ([6], Definition 2.2) Let X be a non-empty set and $f : X \longrightarrow X$, $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings. Then f is called a *triangular α-orbital admissible mapping respect to η* if

1. f is an α-orbital admissible respect to η.
2. For all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y), \alpha(y, fy) \geq \eta(y, fy)$ imply $\alpha(x, fy) \geq \eta(x, fy)$.

In 2014, Hussain *et al.* [10] introduced the notion of α-η-complete metric spaces and α-η-continuous functions.

Definition 2.5. ([10], Definition 4) Let (X, d) be a metric space, $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings. Then

1. (X, d) is called *α-η-complete* if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ is a convergent sequence in (X, d).
2. (X, d) is called *α-complete* if X is α-η-complete with $\eta(x, y) = 1$ for all $x, y \in X$.

Remark 2.1. Every complete metric space is an α-η-complete metric space. However, [6, Example 1.12] proves that there exists an α-η-complete metric space which is not a complete metric space.

Definition 2.6. ([10], Definition 7) Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be mappings. Then f is called an *α-η-continuous mapping* on (X,d) if for all $x \in X$, $\lim\limits_{n\to\infty} x_n = x, \alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ imply $\lim\limits_{n\to\infty} fx_n = fx$.

Remark 2.2.

1. Every continuous mapping is an α-η-continuous mapping. However, there exists an α-η-continuous mapping is not a continuous mapping, (see [6, Example 1.14]).
2. T is called α-continuous if T is α-η-continuous with $\eta(x,y) = 1$ for all $x, y \in X$.

In 2016, Ansari and Kaewcharoen [1] introduced the notion of a generalized α-η-ψ-φ-F-contraction type and stated some fixed point results for such contraction type in metric spaces as follows.

Definition 2.7. ([1], Definition 2.1) Let (X,d) be a metric space, $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ and $f : X \longrightarrow X$ be mappings. Then f is called a *generalized α-η-ψ-φ-F-contraction type* if there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x, y \in X$ with $\alpha(x,y) \geqslant \eta(x,y)$, we have

$$\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(M(x,y)), \varphi(M(x,y))\big)$$

where

$$M(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy)\Big\}.$$

Theorem 2.1. ([1], Theorem 2.3, Theorem 2.4, Theorem 2.5) *Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-η-complete metric space.*
2. *f is triangular α-orbital admissible respect to η.*
3. *f is an α-η-ψ-φ-F-contraction type.*
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.*
5. *(a) Either f is α-η-continuous or*

   (b) If $\{x_n\}$ is a sequence in X and $\lim\limits_{n\to\infty} x_n = x$ such that $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant \eta(x_{n(k)}, x)$ for all $k \in \mathbb{N}$.

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$ there exists $z \in X$ such that $\alpha(z, fz) \geq \eta(z, fz), \alpha(x,z) \geq \eta(x,z)$ and $\alpha(y,z) \geq \eta(y,z)$, then f has a unique fixed point.

3. Main results

First, we generalize the notion of triangular α-orbital admissible mappings to a pair of mappings as follows.

Definition 3.1. Let X be a non-empty set, $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings. Then the pair (f, g) is called *triangular α-orbital admissible* if for all $x, y, z \in X$,

1. (L1) $\alpha(x, fx) \geq 1$ implies $\alpha(fx, gfx) \geq 1$.
2. (L2) $\alpha(x, y) \geq 1$ and $\alpha(y, fy) \geq 1$ imply $\alpha(x, fy) \geq 1$.
3. (L3) $\alpha(x, gx) \geq 1$ implies $\alpha(gx, fgx) \geq 1$.
4. (L4) $\alpha(x, y) \geq 1$ and $\alpha(y, gy) \geq 1$ imply $\alpha(x, gy) \geq 1$.

Lemma 3.1. *Let X be a non-empty set, $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *The pair (f, g) is triangular α-orbital admissible.*
2. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

Then the sequence $\{x_n\}$ defined by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ satisfies $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m \neq n$.

Proof. Since $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$ and the property (L1) of the pair (f, g), we obtain $\alpha(x_1, x_2) = \alpha(fx_0, gfx_0) \geq 1$. Since $\alpha(x_1, x_2) \geqslant 1$ and $x_2 = gx_1$, we get $\alpha(x_1, gx_1) \geq 1$. By using the property (L3) of the pair (f, g), we obtain $\alpha(gx_1, fgx_1) \geq 1$. This implies that $\alpha(x_2, x_3) \geqslant 1$. Since $x_3 = fx_2$, we obtain $\alpha(x_2, fx_2) \geq 1$. By using the property (L1) of the pair (f, g), we obtain $\alpha(fx_2, gfx_2) \geq 1$. This implies $\alpha(x_3, x_4) \geqslant 1$. By continuing the process as above, we obtain $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Now, suppose that $\alpha(x_n, x_m) \geq 1$ for $m > n$. We will prove that $\alpha(x_n, x_{m+1}) \geq 1$ for $m > n$. If m is odd, $\alpha(x_m, gx_m) = \alpha(x_m, x_{m+1}) \geq 1$. Note that $\alpha(x_n, x_m) \geq 1$. From the property (L4) of the pair (f, g), we have $\alpha(x_n, x_{m+1}) = \alpha(x_n, gx_m) \geq 1$. If m is even, $\alpha(x_m, fx_m) = \alpha(x_m, x_{m+1}) \geq 1$. Note that $\alpha(x_n, x_m) \geq 1$. From the property (L2) of the pair (f, g), we get $\alpha(x_n, x_{m+1}) = \alpha(x_n, fx_m) \geq 1$. Therefore, $\alpha(x_n, x_m) \geq 1$ for all $m > n$. □

Next, we introduce the notion of a pair of ψ-φ-F-rational contraction type mappings in metric space.

Definition 3.2. Let (X,d) be a metric space, $\alpha : X \times X \longrightarrow [0,\infty)$ and $f,g : X \longrightarrow X$ be mappings. Then the pair (f,g) is called a *ψ-φ-F-rational contraction type* if there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x,y \in X, x \neq y$ with $\alpha(x,y) \geqslant 1$, we have

$$\psi\big(d(fx,gy)\big) \leqslant F\big(\psi(H(x,y)), \varphi(H(x,y))\big) \tag{3.1}$$

where

$$H(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,gy)}{d(x,y)}\Big\}.$$

Definition 3.3. Let (X,d) be a metric space, $\alpha : X \times X \longrightarrow [0,\infty)$ and $f,g : X \longrightarrow X$ be mappings. Then the pair (f,g) is called a *ψ-φ-F_k-rational contraction type* if there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that for all $x,y \in X$ with $\alpha(x,y) \geqslant 1$, we have

$$\psi\big(d(fx,gy)\big) \leqslant F\big(\psi(H_k(x,y)), \varphi(H_k(x,y))\big) \tag{3.2}$$

where

$$H_k(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,gy)}{k+d(x,y)}\Big\}.$$

The first main result is a sufficient condition for the existence of a common fixed point of a pair of mappings satisfying ψ-φ-F-rational contraction type in metric spaces.

Theorem 3.1. *Let (X,d) be a metric space, $f,g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-complete metric space.*
2. *The pair (f,g) is triangular α-orbital admissible.*
3. *The pair (f,g) is a ψ-φ-F-rational contraction type.*
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
5. *f and g are α-continuous.*

Then f or g has a fixed point, or f and g have a common fixed point.

Proof. We define a sequence $\{x_n\}$ in X by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}$, where $\alpha(x_0, fx_0) \geq 1$. If there exists $n \in \mathbb{N}$ such that $x_{2n} = x_{2n+1}$, then $x_{2n} = fx_{2n}$, that is, x_{2n} is a fixed point of f. Similarly, if there exists $n \in \mathbb{N}$ such that $x_{2n+1} = x_{2n+2}$, then $x_{2n+1} = gx_{2n+1}$, that is, x_{2n+1} is a fixed point of g. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since the pair (f,g) is triangular α-orbital admissible, by using Lemma 3.1, we obtain the following for all $m, n \in \mathbb{N}, m > n$,

$$\alpha(x_n, x_m) \geq 1. \tag{3.3}$$

Since (f, g) is a ψ-φ-F-rational contraction type and using (3.3), we have

$$\begin{aligned} \psi\big(d(x_{2n+1}, x_{2n+2})\big) &= \psi\big(d(fx_{2n}, gx_{2n+1})\big) \\ &\leq F\big(\psi(H(x_{2n}, x_{2n+1})), \varphi(H(x_{2n}, x_{2n+1}))\big) \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} & H(x_{2n}, x_{2n+1}) \\ = \; & \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \\ & \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2}, \frac{d(x_{2n}, fx_{2n})d(x_{2n+1}, gx_{2n+1})}{d(x_{2n}, x_{2n+1})}\Big\} \\ = \; & \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\Big\} \\ \leq \; & \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\Big\} \\ = \; & \max\big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\big\}. \end{aligned}$$

If there exists $n \in \mathbb{N}$ such that

$$\max\big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\big\} = d(x_{2n+1}, x_{2n+2}) > \; 0,$$

then (3.4) becomes

$$\begin{aligned} \psi\big(d(x_{2n+1}, x_{2n+2})\big) &\leq F\big(\psi(d(x_{2n+1}, x_{2n+2})), \varphi(d(x_{2n+1}, x_{2n+2}))\big) \\ &< \psi\big(d(x_{2n+1}, x_{2n+2})\big). \end{aligned}$$

It is a contradiction. Therefore,

$$\max\big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\big\} = d(x_{2n}, x_{2n+1}) > 0$$

for all $n \in \mathbb{N}$. Then (3.4) becomes

$$\begin{aligned} \psi\big(d(x_{2n+1}, x_{2n+2})\big) &\leq F\big(\psi(d(x_{2n}, x_{2n+1})), \varphi(d(x_{2n}, x_{2n+1}))\big) \\ &< \psi\big(d(x_{2n}, x_{2n+1})\big) \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Moreover, since ψ is nondecreasing, we have

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) \tag{3.6}$$

for all $n \in \mathbb{N}$. Also, from (3.3) and(f, g) is a ψ-φ-F-rational contraction type, we have

$$\begin{aligned} \psi\big(d(x_{2n+1}, x_{2n})\big) &= \psi\big(d(fx_{2n}, gx_{2n-1})\big) \\ &\leq F\big(\psi(H(x_{2n}, x_{2n-1})), \varphi(H(x_{2n}, x_{2n-1}))\big). \end{aligned}$$

Similar to the above arguments, we also have

$$d(x_{2n+1}, x_{2n}) < d(x_{2n}, x_{2n-1}) \tag{3.7}$$

for all $n \in \mathbb{N}, n \geq 1$. Therefore, from (3.6) and (3.7), we obtain $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers. Hence, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Then, taking the limit as $n \to \infty$ in (3.5), we obtain $\psi(r) \leq F\big(\psi(r), \varphi(r)\big)$. This implies that $F\big(\psi(r), \varphi(r)\big) = \psi(r)$. Then, $\psi(r) = 0$ or $\varphi(r) = 0$. So, we have $r = 0$. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ where $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(x_{2m(k)}, x_{2n(k)}) \geq \varepsilon. \tag{3.9}$$

It implies that

$$d(x_{2m(k)}, x_{2n(k)-2}) < \varepsilon. \tag{3.10}$$

Then, from (3.9) and (3.10), we have

$$\begin{aligned} \varepsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \varepsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}). \end{aligned} \tag{3.11}$$

Taking the limit as $k \to \infty$ in (3.11) and using (3.8), we get

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \varepsilon. \tag{3.12}$$

Moreover, we have

$$|d(x_{2m(k)}, x_{2n(k)-1}) - d(x_{2m(k)}, x_{2n(k)})| \leq d(x_{2n(k)-1}, x_{2n(k)}). \tag{3.13}$$

$$|d(x_{2m(k)}, x_{2n(k)-1}) - d(x_{2n(k)-1}, x_{2m(k)+1})| \leq d(x_{2m(k)}, x_{2m(k)+1}). \tag{3.14}$$

$$|d(x_{2m(k)+1}, x_{2n(k)}) - d(x_{2m(k)+1}, x_{2n(k)-1})| \leq d(x_{2n(k)}, x_{2n(k)-1}). \tag{3.15}$$

Taking the limit as $k \to \infty$ in (3.13), (3.14), (3.15) and using (3.8), (3.12) we obtain

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)-1}) = \varepsilon. \tag{3.16}$$

$$\lim_{k\to\infty} d(x_{2n(k)-1}, x_{2m(k)+1}) = \varepsilon. \tag{3.17}$$

$$\lim_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)}) = \varepsilon. \tag{3.18}$$

Since $\lim\limits_{k\to\infty} d(x_{2m(k)}, x_{2n(k)-1}) = \varepsilon > 0$, we have $d(x_{2m(k)}, x_{2n(k)-1}) > 0$ for all $k \geqslant k_0$ with some $k_0 \in \mathbb{N}$. For all $k \geqslant k_0$, since $2m(k) < 2n(k) - 1$ and using (3.3), we obtain $\alpha(x_{2m(k)}, x_{2n(k)-1}) \geq 1$. By using (3.1), we have

$$\begin{aligned} \psi\big(d(x_{2m(k)+1}, x_{2n(k)})\big) &= \psi\big(d(fx_{2m(k)}, gx_{2n(k)-1})\big) \\ &\leq F\big(\psi(H(x_{2m(k)}, x_{2n(k)-1})), \varphi(H(x_{2m(k)}, x_{2n(k)-1}))\big) \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} &H(x_{2m(k)}, x_{2n(k)-1}) \\ = \; &\max\Big\{d(x_{2m(k)}, x_{2n(k)-1}), d(x_{2m(k)}, x_{2m(k)+1}), d(x_{2n(k)-1}, x_{2n(k)}), \\ &\qquad \frac{d(x_{2m(k)}, x_{2n(k)}) + d(x_{2n(k)-1}, x_{2m(k)+1})}{2}, \\ &\qquad \frac{d(x_{2m(k)}, x_{2m(k)+1}) d(x_{2n(k)-1}, x_{2n(k)})}{d(x_{2m(k)}, x_{2n(k)-1})}\Big\}. \end{aligned} \tag{3.20}$$

Taking the limit as $k \to \infty$ in (3.20) and using (3.8), (3.12), (3.16), (3.17), we obtain

$$\lim_{k\to\infty} H(x_{2m(k)}, x_{2n(k)-1}) = \max\Big\{\varepsilon, 0, 0, \frac{\varepsilon + \varepsilon}{2}, 0\Big\} = \varepsilon. \tag{3.21}$$

Taking the limit as $k \to \infty$ in (3.19), using the continuity of F, ψ, φ and (3.18), (3.21), we have

$$\psi(\varepsilon) \leq F\big(\psi(\varepsilon), \varphi(\varepsilon)\big).$$

It follows from the property of F that $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. This implies that $\varepsilon = 0$ which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is an α-complete metric space and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$. Since f and g are α-continuous mappings, we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = f(\lim_{n\to\infty} x_{2n}) = fx$$

and

$$x = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} gx_{2n+1} = g(\lim_{n\to\infty} x_{2n+1}) = gx.$$

This implies that x is a common fixed point of f and g. □

The second main result is a sufficient condition for the existence of a common fixed point of a pair of mappings satisfying ψ-φ-F_k-rational contraction type in metric spaces.

Theorem 3.2. *Let (X, d) be a metric space, $f, g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *(X, d) is an α-complete metric space.*
2. *The pair (f, g) is triangular α-orbital admissible.*
3. *The pair (f, g) is a ψ-φ-F_k-rational contraction type.*
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
5. *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_{2n}, x) \geq 1$ and $\alpha(x, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$.*

Then f or g has a fixed point, or f and g have a common fixed point.

Proof. As in the proof of Theorem 3.1, we conclude that either f or g has a fixed point or the sequence $\{x_n\}$ defined by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \in \mathbb{N}$ satisfies

$$\alpha(x_n, x_m) \geqslant 1, \tag{3.22}$$

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{3.23}$$

for all $n, m \in \mathbb{N}$ with $n > m$ and there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x. \tag{3.24}$$

Then, from the assumption (5), we obtain $\alpha(x_{2n}, x) \geq 1$ and $\alpha(x, x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$. Since $\alpha(x_{2n}, x) \geq 1$, (f, g) is triangular α-orbital admissible, we have

$$\psi\big(d(x_{2n+1}, gx)\big) = \psi\big(d(fx_{2n}, gx)\big) \leq F\big(\psi(H(x_{2n}, x), \varphi H(x_{2n}, x))\big) \tag{3.25}$$

where

$$\begin{aligned} H(x_{2n}, x) \quad = \quad & \max\Big\{d(x_{2n}, x), d(x_{2n}, x_{2n+1}), d(x, gx), \\ & \frac{d(x_{2n}, gx) + d(x, x_{2n+1})}{2}, \frac{d(x_{2n}, x_{2n+1})d(x, gx)}{k + d(x, x_{2n})}\Big\}. \end{aligned} \tag{3.26}$$

Taking the limit as $n \to \infty$ in (3.26) and using (3.23), (3.24), we get

$$\lim_{n\to\infty} H(x_{2n}, x) = d(x, gx). \tag{3.27}$$

Taking the limit as $n \to \infty$ in (3.25), using the continuity of F, ψ, φ and (3.27), we obtain

$$\psi\big(d(x, gx)\big) \leq F\big(\psi(d(x, gx), \varphi(d(x, gx))\big).$$

By using the property of F, we have $\psi\big(d(x, gx)\big) = 0$ or $\varphi\big(d(x, gx)\big) = 0$. This implies that $d(x, gx) = 0$. Hence, $gx = x$. Similarly, we also have $fx = x$. Therefore, x is a common fixed point of f and g. $\square$

The following theorems are the sufficient conditions for the existence of a unique common fixed point of the pair of mappings satisfying ψ-φ-F-rational contraction type and ψ-φ-F_k-rational contraction type in metric spaces.

Theorem 3.3. *Suppose all assumptions of Theorem 3.1 hold. Assume that for all $x, y \in X, x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1$, $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Then f or g has a fixed point or f and g have a unique common fixed point.*

Proof. By Theorem 3.1, f or g has a fixed point or f and g have a common fixed point. Suppose that x, y are two common fixed point of f, g such that $x \neq y$. By the assumption, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$. Since (f, g) is triangular α-orbital admissible, we have $\alpha(x, fz) \geq 1$. Since $\alpha(z, fz) \geq 1$ and using Theorem 3.1, we deduce that

$$\lim_{n\to\infty} z_n = z^* \tag{3.28}$$

where $z^* \in X$ and $\{z_n\}$ is defined by $z_0 = z, z_{2n+1} = fz_{2n}$ and $z_{2n+2} = gz_{2n+1}$ for all $n \in \mathbb{N}$.

Moreover, $\alpha(x, z_1) = \alpha(x, fz) \geq 1$, and (f, g) is triangular α-orbital admissible, we have $\alpha(x, z_2) = \alpha(fx, gz_1) \geq 1$. This implies that $\alpha(x, z_3) = \alpha(gx, fz_2) \geq 1$. Continue this process, we have $\alpha(x, z_n) \geq 1$ for all $n \in \mathbb{N}$. We consider two following cases.

Case 1. If there exists $z_{n_0} \in X$ such that $z_{n_0} = x$, then

$$\lim_{n\to\infty} z_n = x. \tag{3.29}$$

By using (3.28) and (3.29), we obtain $x = z^*$.

Case 2. If $z_n \neq x$ for all $n \in \mathbb{N}$, then using (3.1), we obtain

$$\begin{aligned} \psi\big(d(x, z_{2n+2})\big) &= \psi\big(d(fx, gz_{2n+1})\big) \\ &\leq F\big(\psi(H(x, z_{2n+1})), \varphi(H(x, z_{2n+1}))\big) \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} H(x, z_{2n+1}) &= \max\Big\{d(x, z_{2n+1}), d(x, fx), d(z_{2n+1}, gz_{2n+1}), \\ &\qquad \frac{d(x, gz_{2n+1}) + d(z_{2n+1}, fx)}{2}, \frac{d(x, fx)d(z_{2n+1}, gz_{2n+1})}{d(x, z_{2n+1})}\Big\} \\ &= \max\Big\{d(x, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(x, z_{2n+2}) + d(z_{2n+1}, x)}{2}\Big\}. \end{aligned} \tag{3.31}$$

Taking the limit as $n \to \infty$ in (3.31), we have

$$\lim_{n\to\infty} H(x, z_{n+1}) = d(x, z^*). \tag{3.32}$$

Taking the limit as $n \to \infty$ in (3.30) and (3.32), we have

$$\psi\big(d(x,z^*)\big) \le F\big(\psi(d(x,z^*)), \varphi(d(x,z^*))\big). \tag{3.33}$$

By using the property of F, we have $\psi(d(x,z^*)) = 0$ or $\varphi(d(x,z^*)) = 0$. This implies that $d(x,z^*) = 0$. This means $x = z^*$.

From the above cases, we conclude that $x = z^*$. Similar, we also obtain $y = z^*$. Therefore, $x = y$ and hence the common fixed point of f and g is unique. □

Theorem 3.4. *Suppose all assumptions of Theorem 3.2 hold. Assume that for all $x, y \in X, x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1$, $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Then f or g has a fixed point or f and g have a unique common fixed point.*

Proof. The proof is similar to the proof of Theorem 3.3. □

By choosing $f = g$ in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, we get the following results.

Corollary 3.1. *Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-complete metric space.*
2. *f is a triangular α-orbital admissible mapping.*
3. *For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geq 1$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*

$$\alpha(x,y)\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H^f(x,y)), \varphi(H^f(x,y))\big)$$

where

$$H^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{d(x,y)}\Big\}.$$

4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
5. *f is α-continuous.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z,fz) \geq 1, \alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$, then f has a unique fixed point.

Corollary 3.2. *Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-complete metric space.*

2. *f is a triangular α-orbital admissible mapping.*

3. *For all $x, y \in X$ with $\alpha(x,y) \geq 1$, there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*

$$\alpha(x,y)\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H_k^f(x,y)),\varphi(H_k^f(x,y))\big)$$

where

$$H_k^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{k+d(x,y)}\Big\}.$$

4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

5. *If $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$, then f has a unique fixed point.

By using the arguments as in the proof of [3, Theorem 2.2], from Corollary 3.1 and Corollary 3.2, we obtain the following results. These can be viewed as extending analogues of Theorem 2.1.

Corollary 3.3. *Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-η-complete metric space.*

2. *f is a triangular α-orbital admissible mapping respect to η.*

3. *For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geq \eta(x,y)$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*

$$\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H^f(x,y)),\varphi(H^f(x,y))\big)$$

where

$$H^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{d(x,y)}\Big\}.$$

4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.*

5. *f is α-η-continuous.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq \eta(z, fz), \alpha(x,z) \geq \eta(x,z)$ and $\alpha(y,z) \geq \eta(y,z)$, then f has a unique fixed point.

Corollary 3.4. *Let (X,d) be a metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *(X,d) is an α-η-complete metric space.*
2. *f is a triangular α-orbital admissible mapping respect to η.*
3. *For all $x, y \in X$ with $\alpha(x,y) \geq \eta(x,y)$, there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $F \in \mathcal{C}$ such that*
$$\psi\big(d(fx,fy)\big) \leqslant F\big(\psi(H_k^f(x,y)), \varphi(H_k^f(x,y))\big)$$
where
$$H_k^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{k+d(x,y)}\Big\}.$$
4. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.*
5. *If $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z,fz) \geq \eta(z,fz), \alpha(x,z) \geq \eta(x,z)$ and $\alpha(y,z) \geq \eta(y,z)$, then f has a unique fixed point.

In Corollary 3.1 and Corollary 3.2, by choosing $F(s,t) = s\beta(s)$ for all $s, t \in [0,\infty)$ where $\beta : [0,\infty) \longrightarrow [0,1)$ is a continuous function, we obtain the following corollaries. These results can be viewed as the extending analogues of [12, 14] with the condition " $\lim\limits_{n\to\infty} \beta(t_n) = 0$ implying that $\lim\limits_{n\to\infty} t_n = 1$" replaced by "$\beta$ is continuous".

Corollary 3.5. *Let (X,d) be a complete metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty)$ be mappings such that*

1. *f is a triangular α-orbital admissible mapping.*
2. *For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geq 1$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $\beta : [0,\infty) \longrightarrow [0,1)$ is a continuous function such that*
$$\alpha(x,y)\psi\big(d(fx,fy)\big) \leqslant \psi\big(H^f(x,y)\big).\beta\big(\psi(H^f(x,y)\big)$$
where
$$H^f(x,y) = \max\Big\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}, \frac{d(x,fx)d(y,fy)}{d(x,y)}\Big\}.$$
3. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*

4. *f is continuous.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then f has a unique fixed point.

Corollary 3.6. *Let (X, d) be a complete metric space, $f : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *f is a triangular α-orbital admissible mapping.*
2. *For all $x, y \in X$ with $\alpha(x, y) \geq 1$, there exist $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ and $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function such that*

$$\alpha(x, y)\psi\big(d(fx, fy)\big) \leqslant \psi\big(H_k^f(x, y)\big).\beta\big(\psi(H_k^f(x, y)\big)$$

where

$$H_k^f(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{k + d(x, y)}\right\}.$$

3. *There exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$.*
4. *If $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} x_n = x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then f has a fixed point. Moreover, if for all $x, y \in X$, $x \neq y$, there exists $z \in X$ such that $\alpha(z, fz) \geq 1, \alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then f has a unique fixed point.

In Corollary 3.3 and Corollary 3.4, by choosing $F(s, t) = s\beta(s)$ for all $s, t \in [0, \infty)$ where $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function, we obtain the following corollaries. These results can be viewed as extending analogues of [6, Theorem 2.7, Theorem 2.8, Theorem 2.9] with the condition "$\lim_{n \to \infty} \beta(t_n) = 0$ implies $\lim_{n \to \infty} t_n = 1$" replaced by "$\beta$ is continuous" .

Corollary 3.7. *Let (X, d) be a complete metric space, $f : X \longrightarrow X$ and $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ be mappings such that*

1. *f is a triangular α-orbital admissible mapping respect to η.*
2. *For all $x, y \in X, x \neq y$ with $\alpha(x, y) \geq \eta(x, y)$, there exist $\psi \in \Psi, \varphi \in \Phi$ and $\beta : [0, \infty) \longrightarrow [0, 1)$ is a continuous function such that*

$$\psi\big(d(fx, fy)\big) \leqslant \psi\big(H^f(x, y)\big).\beta\big(\psi(H^f(x, y)\big)$$

where

$$H^f(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{d(x, y)}\right\}.$$

3. *There exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.

4. f *is continuous.*

Then f *has a fixed point. Moreover, if for all* $x, y \in X$, $x \neq y$, *there exists* $z \in X$ *such that* $\alpha(z, fz) \geq \eta(z, fz), \alpha(x, z) \geq \eta(x, z)$ *and* $\alpha(y, z) \geq \eta(y, z)$ *then* f *has a unique fixed point.*

Corollary 3.8. *Let* (X, d) *be a complete metric space,* $f : X \longrightarrow X$ *and* $\alpha, \eta : X \times X \longrightarrow [0, \infty)$ *be mappings such that*

1. f *is a triangular* α*-orbital admissible mapping respect to* η.

2. *For all* $x, y \in X$ *with* $\alpha(x, y) \geq \eta(x, y)$, *there exist* $k > 0$, $\psi \in \Psi, \varphi \in \Phi$ *and* $\beta : [0, \infty) \longrightarrow [0, 1)$ *is a continuous function such that*

$$\psi\big(d(fx, fy)\big) \leqslant \psi\big(H_k^f(x, y)\big).\beta\big(\psi(H_k^f(x, y)\big)$$

where

$$H_k^f(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}, \frac{d(x, fx)d(y, fy)}{k + d(x, y)}\Big\}.$$

3. *There exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$.

4. *If* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} x_n = x$ *and* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point. Moreover, if for all* $x, y \in X$, $x \neq y$, *there exists* $z \in X$ *such that* $\alpha(z, fz) \geq \eta(z, fz), \alpha(x, z) \geq \eta(x, z)$ *and* $\alpha(y, z) \geq \eta(y, z)$, *then* f *has a unique fixed point.*

The following example shows that there exist $f, F, \alpha, \eta, \psi, \varphi$ such that Corollary 3.3 can be applied.

Example 3.1. Let $X = \{1, 2, 3, 4, 5\}$ and metric d on X as follows.

$$d(x, y) = \begin{cases} \dfrac{1}{4} & \text{if } (x, y) \in \{(2, 4); (3, 4); (3, 5); (4, 2); (4, 3); (4, 5); (5; 3); (5, 4)\} \\ 0 & \text{if } x = y \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

Define $f : X \longrightarrow X, \alpha, \eta : X \times X \longrightarrow [0, \infty)$, $F : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ and $\varphi, \psi : [0, \infty) \longrightarrow [0, \infty)$ by

$$f1 = f4 = 1; f2 = 3; f3 = f5 = 2,$$

$$\alpha(x, y) = \begin{cases} \dfrac{1}{2} & \text{if } (x, y) \in \{(1, 1); (3, 5); (4, 1); (4, 2); (4, 3); (4, 5); (5, 3)\} \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta(x,y) = \frac{1}{2} \text{ for all } x, y \in X,$$
$$F(s,t) = s - t \text{ for all } s, t \in [0,\infty),$$
$$\psi(t) = t, \varphi(t) = \frac{t^2}{4} \text{ for all } t \in [0,\infty).$$

Then Corollary 3.3 can be applied to $f, F, \alpha, \eta, \psi, \varphi$.

Proof. For all $x, y \in X, x \neq y$ with $\alpha(x,y) \geqslant \eta(x,y)$, we obtain

$$(x,y) \in \{(3,5); (4,1); (4,2); (4,3); (4,5); (5,3)\}.$$

We consider the following cases.

Case 1. $(x,y) \in \{(3,5); (4,1); (5,3)\}$. Then $\psi\big(d(f(x), f(y))\big) = 0$ and

$$\begin{aligned} H^f(3,5) &= \max\Big\{d(3,5), d(3,f3), d(5,f5), \frac{d(3,f5)+d(5,f3)}{2}, \frac{d(3,f3)d(5,f5)}{d(3,5)}\Big\} \\ &= \max\Big\{\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\Big\} \\ &= 1, \end{aligned}$$

$$\begin{aligned} H^f(4,1) &= \max\Big\{d(4,1), d(4,f4), d(1,f1), \frac{d(4,f1)+d(1,f4)}{2}, \frac{d(4,f4)d(1,f1)}{d(4,1)}\Big\} \\ &= \max\Big\{\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{4}, 0\Big\} \\ &= \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} H^f(5,3) &= \max\Big\{d(5,3), d(5,f5), d(3,f3), \frac{d(5,f3)+d(3,f5)}{2}, \frac{d(5,f5)d(3,f3)}{d(5,3)}\Big\} \\ &= \max\Big\{\frac{1}{4}, \frac{1}{2},, \frac{1}{2},, \frac{1}{2}, 1\Big\} \\ &= 1. \end{aligned}$$

Therefore, $F\big(\psi(H^f(x,y)), \varphi(H^f(x,y))\big) = H^f(x,y) - \frac{(H^f(x,y))^2}{4} > 0 = \psi\big(d(f(x), f(y))\big)$.

Case 2. $(x,y) \in \{(4,2); (4,3); (4,5)\}$. Then

$$\psi\big(d(f4, f2)\big) = \psi\big(d(1,3)\big) = \frac{1}{2},$$

$$\psi\big(d(f4, f3)\big) = \psi\big(d(1,2)\big) = \frac{1}{2},$$

$$\psi\big(d(f4,f5)\big)=\psi\big(d(1,2)\big)=\frac{1}{2}.$$

$$\begin{aligned}H^f(4,2) &= \max\Big\{d(4,2),d(4,f4),d(2,f2),\frac{d(4,f2)+d(2,f4)}{2},\frac{d(4,f4)d(2,f2)}{d(4,2)}\Big\}\\ &= \max\Big\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{8},1\Big\}\\ &= 1.\end{aligned}$$

$$\begin{aligned}H^f(4,3) &= \max\Big\{d(4,3),d(4,f4),d(3,f3),\frac{d(4,f3)+d(3,f4)}{2},\frac{d(4,f4)d(3,f3)}{d(4,3)}\Big\}\\ &= \max\Big\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{8},1\Big\}\\ &= 1.\end{aligned}$$

$$\begin{aligned}H^f(4,5) &= \max\Big\{d(4,5),d(4,f4),d(5,f5),\frac{d(4,f5)+d(5,f4)}{2},\frac{d(4,f4)d(5,f5)}{d(4,5)}\Big\}\\ &= \max\Big\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{8},1\Big\}\\ &= 1.\end{aligned}$$

Therefore

$$F\big(\psi(H^f(x,y)),\varphi(H^f(x,y))\big)=H^f(x,y)-\frac{(H^f(x,y))^2}{4}=\frac{3}{4}>\frac{1}{2}=\psi\big(d(f(x),f(y))\big).$$

Hence, the inequality (3.1) is satisfied for all $x,y\in X, x\neq y$ with $\alpha(x,y)\geqslant\eta(x,y)$.

Next, we claim that f is a triangular α-orbital admissible respect to η. Indeed, since $\alpha(x,fx)\geqslant\eta(x,fx)$, we have $x=1$ or $x=4$. Then

$$\alpha(f1,f^21)=\alpha(1,1)\geqslant\eta(1,1)=\eta(f1,f^21),\alpha(f4,f^24)=\alpha(1,1)\geqslant\eta(1,1)=\eta(f4,f^24).$$

Hence, f is an α-orbital admissible respect to η. Since $\alpha(x,y)\geqslant\eta(x,y)$, $\alpha(y,fy)\geqslant\eta(y,fy)$ implies $(x,y)=(1,1)$ or $(x,y)=(4,1)$. Then,

$$\alpha(4,f1)=\alpha(4,1)\geqslant\eta(4,1)=\eta(4,f1),\alpha(1,f4)=\alpha(1,1)\geqslant\eta(1,1)=\eta(1,f1).$$

Hence, f is a triangular α-orbital admissible respect to η. Furthermore, all assumptions in Corollary 3.3 are satisfied. Then Corollary 3.3 can be applied to $f,F,\alpha,\eta,\psi,\varphi$ given. $\square$

Finally, we apply Theorem 3.2 to study the existence of solutions to the system of nonlinear integral equations.

Theorem 3.5. *Let $C[a,b]$ be a set of all continuous functions on $[a,b]$ and d be a metric defined by*

$$d(u,v) = \sup_{t\in[a,b]} |u(t) - v(t)|$$

for all $u, v \in C[a,b]$. Consider the system of nonlinear integral equations

$$\left\{ \begin{array}{l} u(t) = \varphi(t) + \int_a^b K_1(t,s,u(s))ds \\ u(t) = \varphi(t) + \int_a^b K_2(t,s,u(s))ds \end{array} \right. \tag{3.34}$$

where $t \in [a,b], \varphi : [a,b] \longrightarrow \mathbb{R}, K_1, K_2 : [a,b] \times [a,b] \times [a,b] \longrightarrow \mathbb{R}$. Suppose that the following statements hold.

1. *$K_1(t,s,u(s))$ and $K_2(t,s,u(s))$ are integrable with respect to s on $[a,b]$.*
2. *$fu, gu \in C[a,b]$ for all $u \in C[a,b]$, where*

$$fu(t) = \varphi(t) + \int_a^b K_1(t,s,u(s))ds,$$

$$gu(t) = \varphi(t) + \int_a^b K_2(t,s,u(s))ds$$

   for all $t \in [a,b]$.
3. *For all $u \in C[a,b]$ such that $u(t) \geq 0$ for all $t \in [a,b]$, we have $fu(t) \geq 0$ and $gu(t) \geq 0$ for all $t \in [a,b]$.*
4. *For all $s,t \in [a,b]$ and $u,v \in C[a,b]$ such that $u(t) \neq v(t)$ and $u(t), v(t) \in [0,\infty)$, we have*

$$\begin{aligned} & |K_1(t,s,u(s)) - K_2(t,s,v(s))| \\ \leq\ & \phi(t,s) \max\Big\{ |u(s)-v(s)|, |u(s)-fu(s)|, |v(s)-gv(s)|, \\ & \qquad \frac{|u(s)-gv(s)| + |v(s)-fu(s)|}{2}, \frac{|u(s)-fu(s)||v(s)-gv(s)|}{1+|u(s)-v(s)|} \Big\} \end{aligned}$$

   where $\phi : [a,b] \times [a,b] \longrightarrow [0,\infty)$ is a continuous function satisfying

$$0 < \sup_{t\in[a,b]} \Big(\int_a^b \phi(t,s)ds \Big) < 1.$$

5. *There exists $u_0 \in C[a,b]$ such that $u_0(t) \geq 0$ for all $t \in [a,b]$.*

Then the system of nonlinear integral equations (3.34) has a solution $u \in C[a,b]$.

Proof. Consider $f, g : C[a,b] \longrightarrow C[a,b]$ defined by

$$fu(t) = \varphi(t) + \int_a^b K_1(t,s,u(s))ds \quad \text{and} \quad gu(t) = \varphi(t) + \int_a^b K_2(t,s,u(s))ds$$

for all $u \in C[a,b]$ and $t \in [a,b]$. It follows from assumptions (1) and (2) that f and g are well-defined. Notice that the existence of a solution to (3.34) is equivalent to the existence of the common fixed point of f and g. Now, we shall prove that all assumptions of Theorem 3.2 are satisfied.

Define a mapping $\alpha : C[a,b] \times C[a,b] \longrightarrow \mathbb{R}$ by

$$\alpha(u,v) = \begin{cases} 1 & \text{if } u(t), v(t) \in [0,\infty) \text{ for all } t \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

(1) Since $(C[a,b], d)$ is a complete metric space, $(C[a,b], d)$ is a α-complete metric space.

(2) We claim that the pair (f,g) is triangular α-orbital admissible. Indeed,

(L1) For all $u \in C[a,b]$ such that $\alpha(u, fu) \geq 1$, we have $u(t), fu(t) \in [0,\infty)$ for all $t \in [a,b]$. It follows from assumption (3), we conclude that $gfu(t) \geq 0$ for all $t \in [a,b]$. Therefore, $\alpha(fu, gfu) \geq 1$.

(L2) For all $u, v \in C[a,b]$ such that $\alpha(u,v) \geq 1$ and $\alpha(v, fv) \geq 1$, we obtain $u(t), fv(t) \in [0,\infty)$. Thus, $\alpha(u, fv) \geq 1$.

(L3) For all $u \in C[a,b]$ such that $\alpha(u, gu) \geq 1$, we have $u(t), gu(t) \in [0,\infty)$ for all $t \in [a,b]$. It follows from assumption (3), we conclude that $fgu(t) \geq 0$ for all $t \in [a,b]$. Therefore, $\alpha(gu, fgu) \geq 1$.

(L4) For all $u, v \in C[a,b]$ such that $\alpha(u,v) \geq 1$ and $\alpha(v, gv) \geq 1$, we obtain $u(t), gv(t) \in [0,\infty)$. Thus, $\alpha(u, gv) \geq 1$.

From the above, we conclude that the pair (f,g) is triangular α-orbital admissible.

(3) We claim that the pair (f,g) is a ψ-φ-F-rational contraction mapping with $F(s,t) = \lambda s$ for all $s,t \in [a,b]$ and $0 < \lambda < 1$. Indeed, let $u, v \in C[a,b]$ with $u \neq v$ and $\alpha(u,v) \geq 1$. Then $u(t), v(t) \in [0,\infty)$ for all $t \in [a,b]$. Therefore, from assumption (4), we have

$$\begin{aligned} |fu(t) - gv(t)| &\leq \int_a^b |K_1(t,s,u(s)) - K_2(t,s,v(s))| ds \\ &\leq \int_a^b \Big(\phi(t,s) \max \Big\{ |u(s) - v(s)|, |u(s) - fu(s)|, |v(s) - gv(s)|, \\ &\qquad \frac{|u(s) - gv(s)| + |v(s) - fu(s)|}{2}, \frac{|u(s) - fu(s)||v(s) - gv(s)|}{1 + |u(s) - v(s)|} \Big\} \Big) ds \\ &\leq H(u,v) \int_a^b \phi(t,s) ds \\ &\leq \lambda H(u,v). \end{aligned}$$

where $\lambda = \sup_{t\in[a,b]} \Big(\int_a^b \alpha(t,s)ds \Big)$ and $H(u,v)$ defined by (3.1). It implies that

$$d(fu, fv) \le \lambda H(u, v).$$

Therefore, the pair (f, g) is a ψ-φ-F-rational contraction mapping with $\psi(t) = t$, $F(s,t) = \lambda s$ for all $s,t \in [0,\infty)$, $0 < \lambda < 1$.

(4) We claim that there exists $u_0 \in C[a,b]$ such that $\alpha(u_0, fu_0) \ge 1$. Indeed, from assumption (5), there exists $u_0 \in C[a,b]$ such that $u_0(t) \ge 0$ for all $t \in [a,b]$. By using assumption (3), we see that $fu_0(t) \ge 0$ for all $t \in [a,b]$. Therefore, $\alpha(u_0, fu_0) \ge 1$.

(5) We claim that assumption (5) in Theorem 3.2 holds. Indeed, let $\{u_n\}$ be a sequence in $C[a,b]$ such that $\lim_{n\to\infty} u_n = u$ and $\alpha(u_n, u_{n+1}) \ge 1$. Then $u(t) \ge 0$ and $u_n(t) \ge 0$ for all $t \in [a,b]$ and $n \in \mathbb{N}$. Therefore, $\alpha(u_{2n}, u) \ge 1$ and $\alpha(u, u_{2n+1}) \ge 1$.

By the above, all assumptions of Theorem 3.2 are satisfied. Then, f and g have a common fixed point $u \in C[a,b]$ and the system of integral equations (3.34) has a solution $u \in C[a,b]$. $\square$

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TUBULAR SURFACES WITH DARBOUX FRAME IN GALILEAN 3-SPACE

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Abstract. In this paper, we define tubular surface by using a Darboux frame instead of a Frenet frame. Subsequently, we compute the Gaussian curvature and the mean curvature of the tubular surface with a Darboux frame. Moreover, we obtain some characterizations for special curves on this tubular surface in a Galilean 3-space.
Keywords. Tubular surface; Darboux frame; Frenet frame; Gaussian curvature.

1. Introduction

A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to classical mechanics. On the other hand, the Galilean space-time plays an important role in non-relativistic physics. The fact is that the fundamental concepts such as velocity, momentum, kinetic energy, etc., and the principles, laws of motion and conservation laws of classical physics are expressed in terms of the Galilean space[7].

As it is well known, the geometry of space is associated with a mathematical group. The idea of invariance of geometry under transformation groups may imply that on some spacetimes of maximum symmetry there should be a principle of relativity, which requires the invariance of physical laws without gravity under transformations among inertial systems. Surface theory has been a popular topic for many researchers in many aspects[3, 9, 8]. Furthermore, canal surfaces are more popular in computer aided geometric design (CAGD), including designing models of internal and external organs, preparing terrain infrastructures, constructing blending surfaces, reconstructing shape or robotic path planning. Several geometers have studied canal surfaces and tube surfaces and have obtained many interesting results[4, 1, 10, 2, 5]. Maekawa [6] et al carried out a research on the necessary and

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sufficient conditions for the regularity of tube surfaces. Besides, Ro and Yoon [10] studied the tubes of the Weingarten type in a Euclidean 3-space. M. Dede [2] studied tube surfaces in a Galilean 3-space. Recently, Dogan and Yayli [4] investigated tubes with a Darboux frame in a Euclidean 3-space. In this study, we investigate tubular surfaces by taking a Darboux frame instead of a Frenet frame in a Galilean 3-space.

2. Preliminaries

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$. The absolute figure of the Galilean space consists of an ordered triple $\{w, f, I\}$, where w is the ideal(absolute) plane, f is the line(absolute line) in w and I is the fixed elliptic involution of points of f.

In the non-homogeneous coordinates the similarity group H_8 has the form

$$\begin{aligned} \bar{x} &= a_{11} + a_{12}x \\ \bar{y} &= a_{21} + a_{22}x + a_{23}y\cos\theta + a_{23}z\sin\theta \\ \bar{z} &= a_{31} + a_{32}x - a_{23}y\sin\theta + a_{23}z\cos\theta \end{aligned} \tag{2.1}$$

where a_{ij} and θ are real numbers[7]. In what follows, the real numbers a_{12} and a_{23} will play the special role. In particular, for $a_{12} = a_{23} = 1$, (1) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3.

Planes $x = \textit{constant}$ are Euclidean and so is the plane ω. Other planes are isotropic. A vector $\mathbf{u} = (u_1, u_2, u_3)$ is said to be non-isotropic if $u_1 \neq 0$. All unit non-isotropic vectors are of the form $\mathbf{u} = (1, u_2, u_3)$. For isotropic vectors, $u_1 = 0$ holds [7].

Since $x = 0$ plane is a Euclidean in Galilean space, it is easy to see that isotropic vectors are on the Euclidean plane.

Definition 1.1. Let $\mathbf{a} = (x, y, z)$ and $\mathbf{b} = (x_1, y_1, z_1)$ be vectors in a Galilean space. The scalar product is defined as

$$< \mathbf{a}, \mathbf{b} >= x_1 x \tag{2.2}$$

and the scalar product of two isotropic vectors, $\mathbf{p} = (0, y, z)$ and $\mathbf{q} = (0, y_1, z_1)$, is defined as

$$< \mathbf{p}, \mathbf{q} >_1= yy_1 + zz_1 \tag{2.3}$$

Definition 1.2. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in a Galilean space [8]. The cross-product of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as follows:

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \tag{2.4}$$

The curve $\alpha : I \subset \mathbb{R} \to G_3$ of the class C^∞ in the Galilean space G_3 is defined by the parametrization

$$\alpha(s) = (s, y(s), z(s)),$$

where s is a Galilean invariant arc-length of α. Then the curvature $\kappa(s)$ and the torsion $\tau(s)$ are given by, respectively

$$\kappa(s) = \sqrt{\ddot{y}(s)^2 + \ddot{z}(s)^2}, \tau(s) = \frac{\det((\dot{\alpha}(s), \ddot{\alpha}(s), \dddot{\alpha}(s))}{\kappa^2(s)}$$

On the other hand, the Frenet vectors of $\alpha(s)$ in G_3 are defined by

$$\begin{aligned} \mathbf{t} &= \dot{\alpha}(s) = ((1, \dot{y}(s), \dot{z}(s)), \\ \mathbf{n} &= \tfrac{1}{\kappa(s)}\ddot{\alpha}(s) = \tfrac{1}{\kappa(s)}((0, \ddot{y}(s), \ddot{z}(s)), \\ \mathbf{b} &= \tfrac{1}{\kappa(s)}((0, -\ddot{z}(s), \ddot{y}(s)). \end{aligned}$$

The vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are called the vector of tangent, principal normal and binormal of α, respectively. For their derivatives the following Frenet formula is satisfied[8]

$$\begin{aligned} \mathbf{t}^{'}(s) &= \kappa(s)\mathbf{n}, \\ \mathbf{n}^{'}(s) &= \tau(s)\mathbf{b}, \\ \mathbf{b}^{'}(s) &= -\tau(s)\mathbf{n}. \end{aligned} \tag{2.5}$$

Since the curve $\alpha(s)$ lies on the surface M, there exists another frame along the curve . This new frame is called Darboux frame and is denoted by $\{\mathbf{T}, \mathbf{Y}, \mathbf{N}\}$ where $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{N}$ is the unit normal of the surface M along the curve $\alpha(s)$ and $\mathbf{Y}$ is a unit vector given by $\mathbf{Y} = \mathbf{N} \times \mathbf{T}$. This frame gives us an opportunity to investigate the properties of the curve according to the surface. Since the unit tangent $\mathbf{T}$ is common in both Frenet frame and Darboux frame, the vectors $\mathbf{n}$, $\mathbf{b}$, $\mathbf{Y}$ and $\mathbf{N}$ lie on the same plane. The derivative formulae of the Darboux frame of $\alpha(s)$ is given as[9]

$$\begin{aligned} \mathbf{T}^{'}(s) &= k_g(s)\mathbf{Y} + k_n\mathbf{N}, \\ \mathbf{Y}^{'}(s) &= t_r(s)\mathbf{N}, \\ \mathbf{N}^{'}(s) &= t_r(s)\mathbf{Y}. \end{aligned} \tag{2.6}$$

where k_g, k_n and t_r are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively.

Now, we shall mention the surface theory in the Galilean space G_3.

Let us consider the surface M given by the parametrization

$$\varphi(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2)) \tag{2.7}$$

where $x(u^1,u^2), y(u^1,u^2), z(u^1,u^2) \in C^3$ and $u^1, u^2 \in \mathbb{R}$.

The isotropic unit normal vector field $\mathbf{Z}$ is given by

$$\mathbf{Z} = \frac{\varphi_{,1} \wedge \varphi_{,2}}{\|\varphi_{,1} \wedge \varphi_{,2}\|_1} \tag{2.8}$$

where $w = \|\varphi_{,1} \wedge \varphi_{,2}\|_1$ and the partial differentiation with respect to $u^{2.1}$ and $u^{2.2}$ denoted by suffixes 1 and 2, respectively.

The first fundamental form of the surface is defined as

$$I = (g_{ij} + \varepsilon h_{ij}) du^i du^j$$

where

$$g_{ij} = \langle \varphi_{,i}, \varphi_{,j} \rangle , h_{ij} = \langle \varphi_{,i}, \varphi_{,j} \rangle_1 \tag{2.9}$$

and ε is

$$\epsilon = \begin{cases} 0, & dv^1 : dv^2 \quad \text{non-isotropic} \\ 1, & dv^1 : dv^2 \quad \text{isotropic} \end{cases}$$

The coefficients L_{ij} of the second fundamental form are given by

$$L_{ij} =< \frac{\varphi_{,ij} x_{,1} - x_{,ij} \varphi_{,1}}{x_{,1}}, \mathbf{Z} >_1 \tag{2.10}$$

Finally, the Gauss curvature K and the mean curvature H of the surface M are defined as

$$K = \frac{\det L_{ij}}{w^2} = \frac{L_{11}L_{22} - L_{12}}{w^2} \tag{2.11}$$

$$2H = g^{ij} L_{ij} = g^{11} L_{11} + g^{12} L_{12} + g^{22} L_{22} \tag{2.12}$$

respectively, where

$$g^{11} = \frac{g_{22}}{w^2},\ g^{12} = -\frac{g_{12}}{w^2}, g^{22} = -\frac{g_{11}}{w^2} \tag{2.13}$$

3. Tubular surface with Darboux frame in G_3

M.Dede defined the tubular surface by using a Frenet frame in a Galilean 3-space. In this section, we define tubular surface by using a Darboux frame instead of a Frenet frame.

Let the center curve $\alpha(s)$ be on the surface M. The characteristic circles of the canal surface lie in the plane which is perpendicular to the tangent of the center curve $\alpha(s)$. In light of this definition, the tubular surface can be defined by using a Darboux frame as

$$M(s, \beta) = \alpha(s) + r(\cos\beta \mathbf{Y} + \sin\beta \mathbf{N}) \tag{3.1}$$

Using (2.6) we get partial derivatives of M with respect to s and β as follows:

$$M_s = T + r(\cos\beta(t_r\mathbf{N}) - \sin\beta(t_r Y)) \tag{3.2}$$

$$M_\beta = -r\sin\beta\mathbf{Y} + r\cos\beta\mathbf{N} \tag{3.3}$$

The cross-product of these two vectors is given as

$$M_s \wedge M_\beta = -r\cos\beta\mathbf{Y} - r\sin\beta\mathbf{N} \tag{3.4}$$

Using (2.8) and (3.5), we obtain an isotropic normal vector of the tubular surface as

$$\mathbf{Z} = -\cos\beta\mathbf{Y} - \sin\beta\mathbf{N} \tag{3.5}$$

From (2.13) and (3.5), we obtain the first fundamental form of the tubular surface with a Darboux frame in a Galilean space as

$$I = ds^2 + \epsilon r^2 d\beta^2 \tag{3.6}$$

where ϵ is

$$\epsilon = \begin{cases} 0, & du \neq 0 \\ 1, & du = 0 \end{cases} \tag{3.7}$$

The second order partial differentials of the surface M are obtained as

$$M_{ss} = (k_g - r\cos\beta t_r^2 - r\sin\beta t_r^{'})\mathbf{Y} + (k_n + r\cos\beta t_r^{'} - r\sin\beta t_r^2)\mathbf{N}, \tag{3.8}$$

$$M_{\beta s} = (-r\cos\beta t_r)\mathbf{Y} + (-r\sin\beta t_r)\mathbf{N}, \tag{3.9}$$

$$M_{\beta\beta} = -r\cos\beta\mathbf{Y} - r\sin\beta\mathbf{N} \tag{3.10}$$

Equations (3.12), (3.13) and (3.1) lead to the coefficients of the second fundamental form obtained by,

$$L_{11} = -\cos\beta k_g - k_n\sin\beta + rt_r^2, \tag{3.11}$$

$$L_{12} = rt_r, \tag{3.12}$$

$$L_{22} = r. \tag{3.13}$$

Thus, the Gaussian curvature K is given by:

$$K = \frac{-\cos\beta k_g - k_n\sin\beta}{r} \tag{3.14}$$

From the equations (2.11), (3.4) and (3.5), we get

$$g^{11} = g^{12} = 0,\ g^{22} = \frac{1}{r^2} \tag{3.15}$$

Then, substituting (3.14), (3.15) and (3.16) into (3.1), we obtain the mean curvature of the tubular surface as

$$H = \frac{1}{2r} \tag{3.16}$$

4. Some Characterizations for Special Curves on Tubular Surfaces in G_3

In this section, we investigate the relation between parameter curves and special curves such as geodesic curves, asymptotic curves, and lines of curvature on this tube surface $M(s,\beta)$ in a Galilean space

Theorem 3.1 Let $M(s,\beta)$ be a tubular surface in G_3. Then

i) $\beta-$ parameter curves are also geodesic.

ii) $s-$ parameter curves are also geodesic if and only if k_g, k_n and t_r of $\alpha(s)$ satisfy the equation:

$$-\cos\beta k_n + \sin\beta k_g - rt_r^{'} = 0. \tag{4.1}$$

Proof:

i) For $s-$ and $\beta-$ parameter curves, we get

$$\begin{aligned} \mathbf{Z}\wedge M_{\beta\beta} &= (-\cos\beta\mathbf{Y} - \sin\beta\mathbf{N}) \wedge (-r\cos\beta\mathbf{Y} - r\sin\beta\mathbf{N}) \\ &= r\cos\beta\sin\beta T - r\sin\beta\cos\beta T = 0. \end{aligned} \tag{4.2}$$

Since $Z\wedge M_{\beta\beta} = 0$, it follows that $\beta-$ parameter curves are geodesic.

ii)

$$\mathbf{Z}\wedge M_{ss} = (-\cos\beta k_n + \sin\beta k_g - rt_r^{'})\mathbf{T}(s) \tag{4.3}$$

It is easy to see that $\mathbf{Z}\wedge M_{ss} = 0$ if and only if $-\cos\beta k_n + \sin\beta k_g - rt_r^{'} = 0$. This completes the proof.

Corollary 3.1 Let $\alpha(s)$ be a geodesic curve on the tubular surface $M(s,\beta)$ in G_3. If $s-$ parameter curves are also geodesic on $M(s,\beta)$, then the curvatures κ and τ of $\alpha(s)$ satisfy the equation:

$$\cos\beta\kappa - r\tau^{'} = 0.$$

Proof: Since the center curve $\alpha(s)$ is geodesic curve, we have $k_g = 0$, $k_n = \kappa$ and $t_r = \tau$. Substituting $k_g = 0$, $k_n = \kappa$ and $t_r = \tau$ in (4.1) the equation, we get

$$\cos\beta\kappa - r\tau^{'} = 0.$$

Hence, the proof is completed.

Corollary 3.2 Let $\alpha(s)$ be an asymptotic curve on the tubular surface $M(s,\beta)$ in G_3. If $s-$ parameter curves are also geodesic on $M(s,\beta)$, then the curvatures κ and τ of $\alpha(s)$ satisfy the equation:

$$\sin\beta\kappa - r\tau^{'} = 0.$$

Proof: Since the center curve $\alpha(s)$ is an asymptotic curve, we have $k_n = 0$, $k_g = \kappa$ and $t_r = \tau$. Substituting $k_n = 0$, $k_g = \kappa$ and $t_r = \tau$ in (4.1), we get

$$\cos\beta\kappa - r\tau^{'} = 0.$$

Hence, the proof is completed.

Theorem 3.2 For the tubular surface $M(s,\beta)$ in G_3.

i) $\beta-$ parameter curves cannot be asymptotic curves.

ii) $s-$ parameter curves are also geodesic if and only if $M(s,\beta)$ is generated by a moving sphere with the radius function

$$r = \frac{\cos\beta k_g + \sin\beta k_n}{t_r'}. \tag{4.4}$$

Proof: i) Since $\langle \mathbf{Z}, M_{\beta\beta}\rangle = r \neq 0$, $\beta-$ parameter curves cannot be asymptotic curves on $M(s,\beta)$

ii) $s-$parameter curves are also asymptotic curve on $M(s,\beta)$ if and only if

$$\langle \mathbf{Z}, M_{ss}\rangle = -\cos\beta k_g - \sin\beta k_n + rt_r^{'} = 0. \tag{4.5}$$

Thus, we get the radius function:

$$r = \frac{\cos\beta k_g + \sin\beta k_n}{t_r'}.$$

This completes the proof.

Corollary 3.3 Let $s-$ parameter curves are also asymptotic curves on $M(s,\beta)$ in G_3.

i) If the center curve $\alpha(s)$ is a geodesic curve on $M(s,\beta)$,then

$$r = \frac{\sin\beta\kappa}{\tau^{'}}$$

ii) If the center curve $\alpha(s)$ is an asymptotic curve on $M(s,\beta)$,then

$$r = \frac{\cos\beta\kappa}{\tau^{'}}$$

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BERTRAND-B CURVES IN THE THREE DIMENSIONAL SPHERE

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Abstract. We define a Bertrand-B curve α in the three dimensional sphere $\mathbb{S}^3(r)$ such that there exists an isometry ϕ of $\mathbb{S}^3(r)$, satisfying $(\phi \circ \beta)(s) = X(s, t(s))$ for another curve β and both curves have common binormal geodesics at corresponding points. We analyze the condition of being Bertrand-B curves in $\mathbb{S}^3(r)$ and prove that the immersed curve with curvatures $\varepsilon_1, \varepsilon_2$ in $\mathbb{S}^3(r)$ is a Bertrand-B curve if and only if it satisfies $\varepsilon_1^2 + \varepsilon_2^2 = 1$. Also, we analyze some conclusions about a pair of Bertrand-B curves in $\mathbb{S}^3(r)$. As an application, we give an example that the conclusions are verified.
Keywords. Bertrand-B curve; isometry; curvature.

1. Introduction

The theory of curves examines the geometric property of the plane and space curves by means of algebraic and calculus methods. The most common application areas of these methods are special curves such as helices, Bertrand curves, Mannheim curves, etc. The special curves in ambient spaces (semi-Euclidean space $\mathbb{R}_v^{n+1}$, Galilean space G^3, etc.) are generally characterized by the algebraic equations relating their curvature and torsion functions [1],[2],[3],[4]. For instance, Bertrand curves and Mannheim curves in the three dimensional Euclidean space $\mathbb{R}^3$ are characterized by, respectively;

$$\lambda\kappa + \mu\tau = 1 \text{ and } \kappa = \lambda\left(\kappa^2 + \tau^2\right)$$

where $\lambda \neq 0$ and μ are some constants, κ and τ are the curvature and torsion functions of these special curves, respectively [5],[6].

Naturally, it gives rise to the following question: Is it possible to extend the studies concerning the mentioned curves to 3-dimensional Riemannian or Lorentzian space forms? As an answer to this question, Choi et al. have given a definition

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of Bertrand curves in 3-dimensional Riemannian space forms M and they have proved that a Frenet curve α with the curvature κ and the torsion τ in M is a Bertrand curve. It satisfies $\tau = 0$ or $\kappa + a\tau = b$ for constants a and $b \neq 0$ [7] as a necessary and sufficient condition. Recently, a definition of Mannheim curves in both Riemannian and Lorentzian space forms has also been given [8],[9]. All of the surveys mentioned above are done with reference to the Frenet-Serret frame which was adopted to formulate a space curve in ambient spaces. On the other hand, it is known that there are other frames in which all invariant properties of a space curve are investigated. These are called *Bishop frames*. The basic idea of creating such frames is to provide minimum bending. The minimum bending of space curves has a wide range of applications such as the creation of a continuous robot model from the analysis of the DNA structure [10], [11]. Thus, it is appropriate to expect all new types of curves that can be introduced on the Bishop frames to contribute to such areas of application.

The notion of Bertrand B-curves in 3-dimensional Euclidean space has been defined by Yerlikaya et al. and has been given the characterizations of Bertrand B-curves related to mate [12]. In this paper, we expand the definition of Bertrand B-curves to the three-dimensional sphere $\mathbb{S}^3(r)$ and give the algebraic qualification of Bertrand B-curves in $\mathbb{S}^3(r)$.

2. Basic definitions and notations

Let $\mathbb{S}^3(r)$ denote a three-dimensional sphere with the constant curvature $c = 1$, defined by

$$\mathbb{S}^3(r) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^{4} x_i^2 = r^2 \right\}, \qquad r > 0.$$

Note that we regard $S^3(r)$ as a subcase of $\mathbb{R}^4$ equipped with the inner product for $x, y \in T_p S^3(r)$:

$$\langle x, y \rangle \; = \; x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

where $T_p S^3(r)$ denotes the tangent space of $S^3(r)$ at $p \in \mathbb{R}^4$. We also need to note the definition of wedge product (or cross product) in $\mathbb{R}^4$. If $x, y, z \in \mathbb{R}^4$, the vector $\langle x \times y \times z, w \rangle$ is defined as a unique one that satisfies $\langle x \times y \times z, w \rangle = \det(x, y, z, w)$ for every $w \in \mathbb{R}^4$

Let $\alpha = \alpha(s) : I \subset \mathbb{R} \to S^3(r)$ be an immersed curve and suppose, without loss of generality, that α is parametrized by the arc-length parameter and there exists an orthonormal frame $\{T, N, B\}$ with functions $\{\kappa, \tau\}$ (called the curvature and torsion of α) along α (called the Frenet-Serret frame), satisfying the derivative formula

$$\begin{aligned} \nabla_T T &= \kappa N - \frac{1}{r^2}\alpha\,, \\ \nabla_T N &= -\kappa T + \tau B\,, \\ \nabla_T B &= -\tau N\,. \end{aligned} \tag{2.1}$$

where ∇ symbolises the Levi-Civita connection of $\mathbb{R}^4$.

On the other hand, a new version of the Bishop frame in a three dimensional Euclidean space $\mathbb{R}^3$ is introduced by Yılmaz and Turgut such that type-2 Bishop and Frenet-Serret frames have a mutual vector field, i.e. *binormal vector fields* [13]. They present both a relationship between Frenet and Bishop vectors and type-2 Bishop derivative equation. From now on, it is possible to ask the following question in the light of the mentioned work.

Question: Is it possible to get the covariant derivative equations of type-2 Bishop for an immersed curve in $S^3(r)$?

As an answer to the above question, we need to be reminded of the following expressions:

Definition 2.1. The rotation matrix for two arbitrary vectors in the Euclidean plane is defined by the following expressions, respectively:

$$\begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix}$$

or

$$\begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix}$$

where $\theta(s)$ is the angle between two vectors [14].

Definition 2.2. Let α be an immersed curve in the three dimensional sphere $S^3(r)$. Then, the Gauss formula of $S^3(r)$ along α is given by the following equation for any vector field X:

$$X' = \triangledown_s X - \langle X, \alpha' \rangle \, \alpha$$

where $'$ and $\triangledown_s$ are symbolised by the natural differentiation of $\mathbb{R}^4$ and the covariant derivative of $S^3(r)$ along α, respectively [14].

We can now express the covariant derivative equation of type-2 Bishop using the above definitions in $S^3(r)$ as follows:

$$\begin{aligned} \nabla_s \xi_1 &= -\varepsilon_1 B - \cos\theta(s)\, \alpha, \\ \nabla_s \xi_2 &= -\varepsilon_2 B + \sin\theta(s)\, \alpha, \\ \nabla_s B &= \varepsilon_1 \xi_1 + \varepsilon_2 \xi_2 \end{aligned}$$

Remark 2.1.

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \cos\theta(s) & \sin\theta(s) & 0 \\ -\sin\theta(s) & \cos\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}, \tag{2.2}$$

$$\kappa = -\theta'(s) \tag{2.3}$$

and

$$\cos\theta(s)\, \varepsilon_1(s) = -\sin\theta(s)\, \varepsilon_2(s). \tag{2.4}$$

3. Bertrand-B curves in 3-dimensional sphere

We begin by getting a crucial definition that is used to expand the concept of Bertrand-B curves to the sphere $S^3(r)$.

Definition 3.1. For $v \in T_pS^3(r)$, let γ_v denote a unique maximal geodesic in $S^3(r)$ with the initial velocity $\gamma_v'(0) = v$. Let

$$U = \left\{ v \in TS^3(r) : 1 \in \text{domain } \gamma_v \right\}$$

and let $\exp : U \to S^3(r)$ be defined by

$$\exp_p(v) = \gamma_v(1).$$

exp is called the exponential map of $S^3(r)$,where U is an open set in $TS^3(r)$ and p is the start point of γ_v [14].

We can now give the definition of Bertrand-B curves:

Definition 3.2. Let $\alpha(s)$ be an immersed curve in a 3-dimensional simply connected space form $S^3(r)$ and $\{\xi_{1_\alpha}, \xi_{2_\alpha}, B_\alpha\}$ be type-2 Bishop of α. With the aim of exponential map, a ruled surface X_{B_α} is defined such that

$$X_{B_\alpha}(s,t) = \exp_{\alpha(s)}\left(tB_\alpha(s)\right).$$

An immersed curve $\beta = \beta(s)$ in $S^3(r)$ is said to be a Bertrand-B mate of α if the binormal vector field of β determined by $\beta(s) = X_{B_\alpha}\left(s, t(s)\right)$ is congruent to $B_\alpha(s_0)$ or $-B_\alpha(s_0)$ for each s_0. By the time an immersed curve α in $S^3(r)$ accepts its Bertrand-B mate, we call α a Bertrand-B curve in $S^3(r)$.

Another concept related to this exponential map: parallelism can be used to transport tangent vectors from one point of a surface to another. Accordingly, for $p \in S^3(r)$ and $v \in T_pS^3(r)$ with $\|v\| = 1$ are considered as vectors in $\mathbb{R}^4$, a relationship between the exponential map and the parallel transport $P^t(v)$ as follows:

$$\exp_p(tv) \;=\; \cos t\, p + \sin t\, v$$

and

$$P^t(v) \;=\; -\,\sin t\, p \,+\, \cos t\, v,$$

In the light of the concepts described above, our goal is to find the condition of being a Bertrand-B curve for an arbitrary immersed curve in the three dimensional sphere:

For $\alpha = \alpha(s)$ let there be an immersed curve parametrized by arc-lenght in $S^3(r)$, let $\beta = \beta(\bar{s})$ with $\|\beta'(\bar{s})\| = 1$ be a Bertrand-B mate of α. Note that we can assume, without loss of generality, that $\frac{d\bar{s}}{ds} > 0$ and the curve $\beta(\bar{s})$ and $B_\beta(\bar{s})$, called its binormal vector field, stated by

$$\begin{aligned}\beta(\bar{s}) &= \exp_{\alpha(s)}(t(s)B_\alpha(s)) \\ &= \cos(t(s))\,\alpha(s) + \sin(t(s))B_\alpha(s)\end{aligned} \tag{3.1}$$

and $B_\beta(\bar{s}) = P^{t(s)}(B_\alpha(s))$, where $\beta(\bar{s})$ is the point in β corresponding to $\alpha(s)$.

By taking the derivative of the equation 3.1 in $\mathbb{R}^4$ and applying the Gauss formula and the Bishop type-2 equation of α, we get

$$\begin{aligned}\beta'(\bar{s}) &= \{\cos(t(s))\}'\,\alpha(s) + \tfrac{ds}{d\bar{s}}\{\cos(t(s))\ \cos\theta(s) + \varepsilon_{1_\alpha}(s)\ \sin(t(s))\}\,\xi_{1_\alpha}(s) \\ &+ \frac{ds}{d\bar{s}}\{\cos(t(s))\ \sin\theta(s) + \varepsilon_{2_\alpha}(s)\ \sin(t(s))\}\,\xi_{2_\alpha}(s) + \{\sin(t(s))\}'\ B_\alpha(s).\end{aligned} \tag{3.2}$$

Considering the fact that

$$\langle \beta', B_\beta\rangle = 0,\ \ \langle \beta', \beta\rangle = 0$$

and

$$B_\beta = -\sin(t(s))\,\alpha + \cos(t(s))\,B_\alpha \tag{3.3}$$

β' is orthogonal to α and B_α in $\mathbb{R}^4$. Thus from 3.2, we easily get

$$\{\cos(t(s))\}' = \{\sin(t(s))\}' = 0. \tag{3.4}$$

Now that t is a non-zero smooth function, $t(\bar{s})$ designate for $\mu \neq 0$. Besides, 3.1 and 3.2 are respectively determined by

$$\beta(\bar{s}) = \cos\mu\,\alpha(s) + \sin\mu\,B_\alpha(s)$$

and

$$\beta'(\bar{s}) = \frac{ds}{d\bar{s}}\left\{\begin{array}{c}\cos\mu\cos\theta(s) \\ +\varepsilon_{1_\alpha}(s)\sin\mu\end{array}\right\}\xi_{1_\alpha}(s) + \frac{ds}{d\bar{s}}\left\{\begin{array}{c}\sin\mu\,\varepsilon_{2_\alpha}(s) \\ +\cos\mu\sin\theta(s)\end{array}\right\}\xi_{2_\alpha}(\bar{s}) \tag{3.5}$$

from which,

$$d\bar{s}/ds = \sqrt{(\cos\mu\cos\theta(s) + \varepsilon_{1_\alpha}(s)\sin\mu)^2 + (\sin\mu\,\varepsilon_{2_\alpha}(s) + \cos\mu\sin\theta(s))^2}$$

or equivalently,

$$d\bar{s}/ds = \sqrt{\cos^2\mu + \sin^2\mu\left(\varepsilon_{1_\alpha}^2(s) + \varepsilon_{2_\alpha}^2(s)\right)}. \tag{3.6}$$

Now we can calculate the tangent vector of β with regard to the Frenet vectors of α that is

$$T_\beta(\bar{s}) = c_1(s)\, T_\alpha(s) + c_2(s)\, N_\alpha(s) \tag{3.7}$$

or bearing in mind that the equation 2.2

$$T_\beta(\bar{s}) = \{c_1(s)\cos\theta(s) - c_2(s)\sin\theta(s)\}\, \xi_{1_\alpha}(s) + \{c_1(s)\sin\theta(s) + c_2(s)\cos\theta(s)\}\, \xi_{2_\alpha}(s). \tag{3.8}$$

Equating the coefficients of the equations 3.5 and 3.8, we get a linear equation system as follows:

$$\begin{aligned}
\cos\theta(s)\, c_1(s) - \sin\theta(s)\, c_2(s) &= \frac{ds}{d\bar{s}}\left(\cos\mu \cos\theta(s) + \varepsilon_{1_\alpha}(s)\sin\mu\right)\\
\sin\theta(s)\, c_1(s) + \cos\theta(s)\, c_2(s) &= \frac{ds}{d\bar{s}}\left(\sin\mu\, \varepsilon_{2_\alpha}(s) + \cos\mu \sin\theta(s)\right).
\end{aligned}$$

Solving this system according to the cramer method, the functions c_1 and c_2 are determined such that

$$c_1(s) = \frac{ds}{d\bar{s}}\cos\mu \tag{3.9}$$

$$c_2(s) = \frac{ds}{d\bar{s}}\sin\mu\left(\cos\theta(s)\, \varepsilon_{2_\alpha}(s) - \sin\theta(s)\, \varepsilon_{1_\alpha}(s)\right) \tag{3.10}$$

By taking the covariant derivative of 3.7 with regard to $\bar{s}$ in $\mathbb{R}^4$ and using the chain rule, the Gauss formula and the Frenet-Serret equation of α, we get

$$\begin{aligned}
\nabla_{\bar{s}} T_\beta(\bar{s}) = &\left\{\cos\mu - c_1(s)\frac{ds}{d\bar{s}}\right\}\alpha(s) + \left\{c_1{}'(s) - c_2(s)\kappa_\alpha(s)\frac{ds}{d\bar{s}}\right\} T_\alpha(s)\\
&+ \left\{c_2{}'(s) + c_1(s)\kappa_\alpha(s)\frac{ds}{d\bar{s}}\right\} N_\alpha(s) + \left\{\sin\mu + c_2(s)\sqrt{\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2}\frac{ds}{d\bar{s}}\right\} B_\alpha(s).
\end{aligned}$$

In what follows, since $\nabla_{\bar{s}} T_\beta(\bar{s})$ is proportional to

$$N_\beta(\bar{s}) = c_3(s)T_\alpha(s) + c_4(s)N_\alpha(s),$$

it reduces to

$$\nabla_{\bar{s}} T_\beta(\bar{s}) = \left\{c_1{}'(s) - c_2(s)\kappa_\alpha(s)\frac{ds}{d\bar{s}}\right\} T_\alpha(s) + \left\{c_2{}'(s) + c_1(s)\kappa_\alpha(s)\frac{ds}{d\bar{s}}\right\} N_\alpha(s). \tag{3.11}$$

Lemma 3.1. *Let $\alpha(s)$ be an immersed curve parametrized by arc-length and let $\beta(\bar{s})$ be a Bertrand-B mate with $\|\beta'(\bar{s})\| = 1$ in the three dimensional sphere $S^3(r)$. Then, the following equalities hold:*

1. $\cos\mu - c_1(s)\frac{ds}{d\bar{s}} = 0$

2. $\sin\mu + c_2(s)\sqrt{\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2}\frac{ds}{d\bar{s}} = 0.$

We can now evaluate two cases from the lemma 3.1 as follows:

Case 1.From lemma 3.1, we have

$$\cos\mu - c_1(s)\frac{ds}{d\bar{s}} = 0.$$

Then, from the equations 3.6 and 3.9 and the necessary arrangement, we can write

$$\frac{\sin^2\mu\,\cos\mu\left((\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2) - 1\right)}{\cos^2\mu + \sin^2\mu\,\left(\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2\right)} = 0.$$

Subcase 1.1. Let $\cos\mu = 0$. 3.6 and 3.9- 3.10 is reduced to $c_1 = 0$, $c_2 = \pm 1$ and $d\bar{s}/ds = \sqrt{\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2}$. According to the previous expressions, from 3.7, we get $T_\beta = \pm N_\alpha$. Then, apply these to 3.11:

$$\nabla_{\bar{s}}T_\beta(\bar{s}) = \pm\frac{\kappa_\alpha}{\sqrt{\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2}}T_\alpha(s). \tag{3.12}$$

We distinguish four subcases according to the sign of the vector field.

Subsubcase 1.1.S1. $(T_\beta = N_\alpha, \nabla_{\bar{s}}T_\beta(\bar{s}) > 0)$. Eq. 3.12 becomes

$$\nabla_{\bar{s}}T_\beta(\bar{s}) = \frac{\kappa_\alpha}{\sqrt{\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2}}T_\alpha(s)$$

from which, $\|\nabla_{\bar{s}}T_\beta(\bar{s})\| = \frac{\kappa_\alpha}{\sqrt{\varepsilon_{1_\alpha}^2+\varepsilon_{2_\alpha}^2}}$ and $N_\beta = T_\alpha$. From the wedge product in E^4, B_β is given by

$$B_\beta = -\sin\mu\,\alpha + \cos\mu\,B_\alpha = P^\mu(B_\alpha).$$

Thus β is a Bertrand-B mate of α.

Reasoning as in the subsubcase 1.1, one says whether β is a Bertrand-B mate of α or not.

Subcase 1.2. Let $\sin\mu = 0$, say $\mu = \pi k$, $k \in \mathbb{Z}$. Thus, β is isometric to α.

Subcase 1.3. Let $\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2 = 1$. Eqs. (3.6) and (3.9)-(3.10) is reduced to $ds = d\bar{s}$, $c_1 = \cos\mu$ and $c_2 = \pm\sin\mu$. According to the previous expressions, from 3.7, we get $T_\beta(\bar{s}) = \cos\mu\,T_\alpha(s) \pm \sin\mu\,N_\alpha(s)$. Then, apply these to 3.11:

$$\nabla_{\bar{s}}T_\beta(\bar{s}) = \kappa_\alpha(s)\left\{\pm\sin\mu\,T_\alpha(s) + \cos\mu N_\alpha(s)\right\} \tag{3.13}$$

from which,

$$\|\nabla_{\bar{s}} T_\beta(\bar{s})\| = |\kappa_\alpha(s)|$$

Note that if $\kappa_\alpha(s) = 0$, then $\nabla_{\bar{s}} T_\beta(\bar{s}) = 0$, that is, β is a geodesic in S^3. Its principal normal vector field N_β is given by

$$N_\beta(\bar{s}) = \pm \sin\mu \; T_\alpha(s) + \cos\mu \; N_\alpha(s).$$

Considering the wedge product in E^4, the binormal vector field B_β is obtained by

$$\begin{aligned} B_\beta(\bar{s}) &= \sin\mu\, \alpha(s) - \cos\mu\, B_\alpha(s) \\ &= P^\mu\left(B_\alpha((s))\right). \end{aligned}$$

Thus β is a Bertrand-B mate of α.

Case 2. From the lemma 3.1, we have

$$\sin\mu + c_2(s)\sqrt{\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2}\,\frac{ds}{d\bar{s}} = 0.$$

Then, from the equations 3.6 and 3.10 and the necessary arrangement, we can write

$$\frac{\sin\mu \cos^2\mu \left(1 - (\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2)\right)}{\cos^2\mu + \sin^2\mu \,\left(\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2\right)} = 0.$$

In this case, it is clear that the curve β is again a Bertrand-B mate of α, examined as in the case 1.

Proposition 3.1. *Let $\alpha = \alpha(s)$ be an immersed curve parametrized by arc-lenght in $S^3(r)$ with curvatures ε_{1_α} and ε_{2_α} and $\beta(\bar{s}) = \cos\mu\, \alpha(s) + \sin\mu\, B_\alpha(s)$. Then, we have*

- *When $0 < \mu < \frac{\pi}{2}$, β is not a Bertrand-B mate of α.*
- *When $\frac{\pi}{2} < \mu < \pi$, β is a Bertrand-B mate of α*
- *If $\kappa_\alpha = 0$ then β is a geodesic in $S^3(r)$.*

Theorem 3.1. *Let $\alpha = \alpha(s)$ be an immersed curve in the 3-dimensional sphere $S^3(r)$ with curvatures ε_{1_α} and ε_{2_α}. Then, α is a Bertrand-B curve if and only if $\varepsilon_{1_\alpha}^2 + \varepsilon_{2_\alpha}^2 = 1$.*

After finding the condition of being a Bertrand-B curve in the three dimensional sphere $S^3(r)$, we can now give results concerning a pair of Bertrand-B curves:

Let $\alpha(s)$ and $\beta(\bar{s})$ be a pair of Bertrand-B curves having the Bishop type-2 frames $\{\xi_{1_\alpha}, \xi_{2_\alpha}, B_\alpha\}$ and $\left\{\xi_{1_\beta}, \xi_{2_\beta}, B_\beta\right\}$, respectively, then there exists a differentiable function $t(\bar{s})$ such that

$$\beta\left(\bar{s}\right) = \cos\left(t(s)\right)\alpha\left(s\right) + \sin\left(t(s)\right)B_\alpha\left(s\right) \tag{3.14}$$

where $\beta(\bar{s})$ is the point in β corresponding to $\alpha(s)$.

Proposition 3.2. *Let α and β be a pair of Bertrand-B curves in $\mathbb{S}^3$. Then the following properties hold:*

1. *The function $t(\bar{s})$ is constant.*
2. *The angle between the tangent vectors $T_\alpha(s)$ and $T_\beta(\bar{s})$ at corresponding points equals to μ.*
3. *The angle between ξ_{1_α} and ξ_{1_β} vectors at corresponding points is constant.*
4. *The angle between ξ_{2_α} and ξ_{2_β} vectors at corresponding points is constant.*

Proof. (1) It can be seen that the function $t(\bar{s})$ is the constant from Eq. (3.4), which completes the proof.

(2) Taking the derivative of Eq. (3.1) with respect to s in $\mathbb{R}^4$, we have

$$T_\beta(\bar{s})\ \frac{d\bar{s}}{ds} = \cos\mu\ T_\alpha(s) + \sin\mu\ \left\{\xi_{1_\alpha}(s)\ \varepsilon_{1_\alpha}(s) + \xi_{2_\alpha}(s)\ \varepsilon_{2_\alpha}(s)\right\}.$$

By multiplying the previous equation with $T\alpha(s)$, we get

$$\langle T_\beta(\bar{s}),\ T_\alpha(s)\rangle = \cos\mu + \sin\mu\ \left\{\ \varepsilon_{1_\alpha}(s)\ \ \cos\theta\left(s\right) + \varepsilon_{2_\alpha}(s)\ \ \sin\theta\left(s\right)\right\}$$

where we use $\langle\xi_{1_\alpha}(s),\ T_\alpha(s)\rangle = \cos\theta(s)$ and $\langle\xi_{2_\alpha}(s),\ T_\alpha(s)\rangle = \sin\theta(s)$. Finally, taking into account Eq.(2.4), we deduce (2).

(3) By a straightforward computation, we get

$$\begin{aligned}\tfrac{d}{ds}\left\langle\xi_{1_\alpha}(s),\ \xi_{1_\beta}(\bar{s})\right\rangle = {} & -\varepsilon_{1_\alpha}(s)\ \left\langle B_\alpha(s),\ \xi_{1_\beta}(\bar{s})\right\rangle - \cos\theta(s)\ \left\langle\alpha(s),\ \xi_{1_\beta}(\bar{s})\right\rangle \\ & - \tfrac{d\bar{s}}{ds}\ \varepsilon_{1_\beta}(\bar{s})\ \langle\xi_{1_\alpha}(s),\ B_\beta(\bar{s})\rangle - \tfrac{d\bar{s}}{ds}\ \cos\theta(s)\ \langle\xi_{1_\alpha}(s),\ \beta(\bar{s})\rangle,\end{aligned}$$

that jointly with (3.1), (3.3) and $\xi_{1_\beta} \in Sp\left\{T_\alpha, N_\alpha\right\}$ yields

$$\frac{d}{ds}\left\langle\xi_{1_\alpha}(s),\ \xi_{1_\beta}(\bar{s})\right\rangle = 0,$$

which completes the claim.

(4) Similarly as in the item (b), one can see that the proof of the claim can be ended.

□

Theorem 3.2. *Let α and β be a pair of Bertrand-B curves in $\mathbb{S}^3$ and let κ_α and κ_β be the curvatures of the pair, respectively. Then there exists a constant μ and η such that the following relations hold:*

1. $\kappa_\beta^2(\bar{s}) = \cos^2\mu\ \kappa_\alpha^2(s) + \sin^2\mu\ \left\{\varepsilon_{1_\alpha}^{'2}(s) + \varepsilon_{2_\alpha}^{'2}(s)\right\}$
2. $\kappa_\alpha^2(s) = \cos^2\mu\ \kappa_\beta^2(\bar{s}) + \sin^2\mu\ \left\{\varepsilon_{1_\beta}^{'2}(\bar{s}) + \varepsilon_{2_\beta}^{'2}(\bar{s})\right\}$
3. $\cos 2\eta = \cos 2\mu\ \left\{\varepsilon_{1_\alpha}(s)\ \varepsilon_{1_\beta}(\bar{s}) + \varepsilon_{2_\alpha}(s)\ \varepsilon_{2_\beta}(\bar{s})\right\}$

where ε_{1_α}, ε_{2_α}, ε_{1_β} and ε_{2_β} stand for the curvatures of α and β, respectively.

Proof. (1) By the covariant derivative of Eq. (3.1), we get Eq. (3.5). Using Eq. (2.2) for the curve β, we have the following equation

$$\begin{aligned}\left\{\cos\theta(\bar{s})\ \xi_{1_\beta}(\bar{s}) + \sin\theta(\bar{s})\ \xi_{2_\beta}(\bar{s})\right\}\tfrac{d\bar{s}}{ds} &= \{\cos\mu\ \cos\theta(s) + \sin\mu\ \varepsilon_{1_\alpha}(s)\}\,\xi_{1_\alpha}(s)\\ &\quad + \{\cos\mu\ \sin\theta(s) + \sin\mu\ \varepsilon_{2_\alpha}(s)\}\,\xi_{2_\alpha}(s).\end{aligned}$$

On the other hand, we have a constant angle η because of the items (3) and (4) of Proposition (3.2), thus we can write

$$\cos[\theta(\bar{s}) - \eta] = \frac{ds}{d\bar{s}}\{\cos\mu\ \cos\theta(s) + \sin\mu\ \varepsilon_{1_\alpha}(s)\} \tag{3.15}$$

$$\sin[\theta(\bar{s}) - \eta] = \frac{ds}{d\bar{s}}\{\cos\mu\ \sin\theta(s) + \sin\mu\ \varepsilon_{2_\alpha}(s)\}. \tag{3.16}$$

By taking the derivative of Eqs. (3.15) and (3.16) wrt s in $\mathbb{R}^4$ and applying Eqs. (2.3) and (2.4), we deduce (1). This ends the proof.

(2) Now we have to hold Eq. (3.1) according to the curve α:

$$\alpha(s) = \cos\mu\,\beta(\bar{s}) + \sin\mu\,B_\beta(\bar{s}).$$

Thus, a straightforward computation leads to the following two equations:

$$\cos[\theta(s) + \eta] = \frac{d\bar{s}}{ds}\left\{\cos\mu\ \cos\theta(\bar{s}) - \sin\mu\ \varepsilon_{1_\beta}(\bar{s})\right\} \tag{3.17}$$

$$\sin[\theta(s) + \eta] = \frac{d\bar{s}}{ds}\left\{\cos\mu\ \sin\theta(\bar{s}) + \sin\mu\ \varepsilon_{2_\beta}(\bar{s})\right\}. \tag{3.18}$$

By following a similar path in the item (a), one can easily see that the claim concludes.

(3) It is a consequence of the way followed by the multiplication of the Eqs. (3.15-3.17) and (3.16-3.18). □

Example 3.1. Let $\alpha = \alpha(s)$ be a model helix in the 3-dimensional sphere $\mathbb{S}^3(r)$, given by

$$\alpha(s) = (\cos\phi\ \cos(as),\ \cos\phi\ \sin(as),\ \sin\phi\ \cos(bs),\ \sin\phi\ \sin(bs)\ ),$$

where s is arc-length when $a^2\cos^2\phi + b^2\sin^2\phi = 1$. Also, a straightforward computation gives us the following Frenet apparatus of α:

$$\begin{cases} T_\alpha(s) = (-a\ \cos\phi\sin(as),\ a\ \cos\phi\cos(as),\ -b\ \sin\phi\sin(bs),\ b\ \sin\phi\cos(bs)\ ) \\ N_\alpha(s) = (-\sin\phi\cos(as),\ -\sin\phi\sin(as),\ \cos\phi\cos(bs),\ \cos\phi\sin(bs)\ ) \\ B_\alpha(s) = (-b\ \sin\phi\sin(as),\ b\ \sin\phi\cos(as),\ a\ \cos\phi\sin(bs),\ -a\ \cos\phi\cos(bs)\ ) \\ \kappa_\alpha = \sqrt{(a^2-1)(1-b^2)} \\ \tau_\alpha = ab \end{cases}$$

By applying the first two equations of the apparatus to (2.2), we get the Bishop-type 2 vector field as follows:

$$\begin{aligned} \xi_{1_\alpha}(s) = \Big(&-f(s)\sin(as) + \sin\phi\sin\theta(s)\cos(as), f(s)\cos(as) + \sin\phi\sin\theta(s)\sin(as), \\ &- g(s)\sin(bs) - \cos\phi\sin\theta(s)\cos(bs), \\ &g(s)\cos(bs) - \cos\phi\sin\theta(s)\sin(bs)\Big), \end{aligned}$$

where $f(s) = a\cos\phi\cos\theta(s)$ and $g(s) = b\sin\phi\cos\theta(s)$. Similarly, we get the remaining vector field $\xi_{2_\alpha}(s)$. Note a case that if $b = \frac{1}{a}$, then the curve α is a helix in $\mathbb{S}^3(r)$ parametrized by arc-length with $\tau = 1$. In such a case, by taking the covariant derivative $\xi_{1_\alpha}(s)$ and $\xi_{2_\alpha}(s)$ with respect to s and using the derivative formula of the Bishop type 2, we obtain the following Bishop type 2 curvatures:

$$\varepsilon_{1_\alpha}(s) = -\sin\theta(s), \quad \varepsilon_{2_\alpha}(s) = -\cos\theta(s) \tag{3.19}$$

from which

$$\varepsilon_{1_\alpha}^2(s) + \varepsilon_{2_\alpha}^2(s) = 1.$$

This means that the curve α with $\tau = 1$ is a Bertrand-B curve in $\mathbb{S}^3(r)$. Morever, the Bertrand-B partner curve β of α is given by $\beta(\bar{s}) = \cos\mu\,\alpha(\bar{s}) + \sin\mu\, B_\alpha(\bar{s})$ with

$\bar{s} = \sqrt{\cos^2\mu + \sin^2\mu\ \left\{\varepsilon_{1_\alpha}^2(s) + \varepsilon_{2_\alpha}^2(s)\right\}}\ s$. Observe that the last equation supports Eq. (3.6). Furthermore, by calculating the curvature of β, it is easy see that the item (1) of the theorem (3.2) is satisfied when the curve α is a Bertrand-B curve, i.e. for $ab = 1$:

$$\kappa_\beta(s) = \sqrt{(a^2-1)(1-b^2) + \sin^2\mu\ \left\{\left((ab)^2 - 1\right)(a^2+b^2-2)\right\}}$$

Consequently, we have the following proposition:

Proposition 3.3. *A model helix with $\tau = 1$ in the 3-dimensional sphere $\mathbb{S}^3(r)$ is a Bertrand-B curve. Moreover, the Bertrand-B partner curve of a model helix in $\mathbb{S}^3(r)$ is also a Bertrand-B curve.*

4. Conclusion

In this paper, we obtained a lemma which states the condition of what it takes to be a Bertrand-B curve. In creating this lemma, we used another curve (mentioned as $\beta(\bar{s})$)as its mate and saw that it is possible for these curves to be Bertrand-B curves only if their mates exist. In addition, some conclusions about a pair of Bertrand-B curves in the three dimensional sphere (called a special Riemannian manifold) are stated. On the other hand, recent studies show that Bishop frames have attracted the attention of many scientists and geometers due to various applications in areas from engineering to computer graphics. Hence, we hope that the results of this study will serve the areas of application associated with Bishop frames.

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RADICAL TRANSVERSAL SCR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of radical transversal screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds giving a characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2, D and $D^{\perp}$ on radical transversal SCR-lightlike submanifolds of an indefinite Kaehler manifold have been obtained. Further, we obtain necessary and sufficient conditions for foliations determined by the above distributions to be totally geodesic.
Keywords. Semi-Riemannian manifold, degenerate metric, radical distribution, screen distribution, screen transversal vector bundle, lightlike transversal vector bundle, Gauss and Weingarten formulae.

1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu ([7]). Various classes of lightlike submanifolds of indefinite Kaehler manifolds are defined according to the behaviour of distributions on these submanifolds with respect to the action of (1, 1) tensor field $\overline{J}$ in Kaehler structure of the ambient manifolds. Such submanifolds have been studied by Duggal and Sahin in ([8], [10]). In [9], Duggal and Sahin introduced the notion of generalized CR-lightlike submanifolds of an indefinite Kaehler manifold which contains CR-lightlike and SCR-lightlike submanifolds as its sub-cases. In [3], Sahin and Gunes studied geodesic CR-lightlike submanifolds and found some geometric properties of CR-lightlike submanifolds of an indefinite Kaehler manifold.

However, all these submanifolds of an indefinite Kaehler manifold mentioned above have invariant radical distribution on their tangent bundles i.e $\overline{J}(RadTM) \subset TM$, where $RadTM$ is the radical distribution and TM is the tangent bundle.

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In [2], Sahin introduced radical transversal and transversal lightlike submanifolds of an indefinite Kaehler manifold for which the action of (1, 1) tensor field $\overline{J}$ on radical distribution of such submanifolds does not belong to the tangent bundle, more precisely, $\overline{J}(Rad(TM)) = ltr(TM)$, where $ltr(TM)$ is the lightlike transversal bundle of lightlike submanifolds.

Thus motivated sufficiently, we introduce the notion of radical transversal screen Cauchy-Riemann (SCR)-lightlike submanifolds of an indefinite Kaehler manifold. This new class of lightlike submanifolds of an indefinite Kaehler manifold includes invariant, screen real, screen Cauchy-Riemann, radical transversal, totally real and generalized transversal lightlike submanifolds as its sub-cases. The paper is arranged as follows. There are some basic results in Section 2. In Section 3, we study radical transversal screen Cauchy-Riemann (SCR)-lightlike submanifolds of an indefinite Kaehler manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions D_1, D_2, D and $D^{\perp}$ involved in the definition of the above submanifolds of an indefinite Kaehler manifold.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold [7] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Now consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be a complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}), \tag{2.4}$$

where $\oplus$ denotes the direct sum and $\oplus_{orth}$ denotes the orthogonal direct sum. Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^{\perp})\big)$:

Case.1 r-lightlike if $r < min\,(m, n)$, Case.2 co-isotropic if $r = n < m$, $S\left(TM^{\perp}\right) = \{0\}$,

Case.3 isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$,

Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall V \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and the vector bundle $tr(TM)$, respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l\left(X, Y\right) + h^s\left(X, Y\right), \quad \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X\left(N\right) + D^s\left(X, N\right), \quad \forall N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X\left(W\right) + D^l\left(X, W\right), \quad \forall W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $h^l(X, Y) = L\left(h(X, Y)\right)$, $h^s(X, Y) = S\left(h(X, Y)\right)$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. $\nabla^l$and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M, respectively. For any vector field X tangent to M, we put

$$\overline{J}X = PX + FX, \tag{2.10}$$

where PX and FX are tangential and transversal parts of $\overline{J}X$, respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.11}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \tag{2.12}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.13}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \quad \xi \in \Gamma(RadTM). \tag{2.14}$$

By using the above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X,\overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X,\overline{P}Y), \tag{2.16}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \qquad A^*_\xi \xi = 0. \tag{2.17}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.7), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{2.18}$$

An indefinite almost Hermitian manifold $(\overline{M},\overline{g},\overline{J})$ is a 2m-dimensional semi-Riemannian manifold $\overline{M}$ with a semi-Riemannian metric $\overline{g}$ of the constant index q, $0 < q < 2m$ and a (1, 1) tensor field $\overline{J}$ on $\overline{M}$ such that the following conditions are satisfied:

$$\overline{J}^2 X = -X, \qquad \forall X \in \Gamma(T\overline{M}), \tag{2.19}$$

$$\overline{g}(\overline{J}X,\overline{J}Y) = \overline{g}(X,Y), \tag{2.20}$$

for all $X, Y \in \Gamma(T\overline{M})$.

An indefinite almost Hermitian manifold $(\overline{M},\overline{g},\overline{J})$ is called an indefinite Kaehler manifold if $\overline{J}$ is parallel with respect to $\overline{\nabla}$, i.e.,

$$(\overline{\nabla}_X \overline{J})Y = 0, \tag{2.21}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection with respect to $\overline{g}$.

A plane section S in tangent space $T_x\overline{M}$ at a point x of a Kaehler manifold $\overline{M}$ is called a holomorphic section if it is spanned by a unit vector X and $\overline{J}X$, where X is a non-zero vector field on $\overline{M}$. The sectional curvature $K(X,\overline{J}X)$ of a holomorphic section is called a holomorphic sectional curvature. A simply connected complete Kaehler manifold $\overline{M}$ of the constant sectional curvature c is called a complex space-form and denoted by $\overline{M}(c)$. The curvature tensor of the complex space-form $\overline{M}(c)$ is given by ([12])

$$\begin{aligned}\overline{R}(X,Y)Z =& \frac{c}{4}[\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(\overline{J}Y,Z)\overline{J}X \\ &- \overline{g}(\overline{J}X,Z)\overline{J}Y + 2\overline{g}(X,\overline{J}Y)\overline{J}Z],\end{aligned} \tag{2.22}$$

for any smooth vector fields X, Y and Z on $\overline{M}$. This result is also true for an indefinite Kaehler manifold $\overline{M}$.

3. Radical Transversal SCR-Lightlike Submanifolds

In this section, we introduce the notion of radical transversal SCR-lightlike submanifolds of an indefinite Kaehler manifold.

Definition 3.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then we say that M is the radical transversal SCR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(i) there exist orthogonal distributions D_1, D_2, D and $D^{\perp}$ on M such that $RadTM = D_1 \oplus_{orth} D_2$ and $S(TM) = D \oplus_{orth} D^{\perp}$,

(ii) the distributions D_1 and D are invariant distributions with respect to $\overline{J}$, i.e. $\overline{J}D_1 = D_1$ and $\overline{J}D = D$,

(iii) the distributions D_2 and $D^{\perp}$ are transversal distributions with respect to $\overline{J}$, i.e. $\overline{J}D_2 \subset \Gamma(ltr(TM))$ and $\overline{J}D^{\perp} \subset \Gamma S(TM^{\perp})$.

From the above definition, we have the following decomposition

$$TM = D_1 \oplus_{orth} D_2 \oplus_{orth} D \oplus_{orth} D^{\perp}. \tag{3.1}$$

In particular, we have

(i) if $D_1 = 0$, then M is a generalized transversal lightlike submanifold,

(ii) if $D_1 = 0$ and $D = 0$, then M is a transversal lightlike submanifold,

(iii) if $D_1 = 0$ and $D^{\perp} = 0$, then M is a radical transversal lightlike submanifold,

(iv) if $D_2 = 0$, then M is a screen CR-lightlike submanifold,

(v) if $D_2 = 0$ and $D = 0$, then M is a screen real lightlike submanifold,

(vi) if $D_2 = 0$ and $D^{\perp} = 0$, then M is an invariant lightlike submanifold.

Thus this new class of lightlike submanifolds of an indefinite Kaehler manifold includes radical transversal, transversal, generalized transversal, invariant, screen real, screen Cauchy-Riemann lightlike submanifolds which have been studied in ([2], [8], [10], [15]) as its sub-cases.

Let $(\mathbb{R}^{2m}_{2q}, \overline{g}, \overline{J})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual Kaehler structure given by

$\overline{g} = \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i)$,

$\overline{J}(\sum_{i=1}^{m}(X_i \partial x_i + Y_i \partial y_i)) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i)$,

where (x^i, y^i) are the cartesian coordinates on $\mathbb{R}^{2m}_{2q}$.

Now, we construct some examples of radical transversal SCR-lightlike submanifolds of an indefinite Kaehler manifold.

Example 1. Let $(\mathbb{R}^{16}_{4}, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+,-,-,+,+,+,+,+,+)$ with respect to $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial y_8\}$.

Suppose M is a submanifold of $\mathbb{R}^{16}_{4}$ given by $x^1 = -y^3 = u_1$, $x^3 = y^1 = u_2$, $x^1 = -y^4 = u_3$, $x^4 = -y^1 = u_4$, $x^5 = -y^6 = u_5$, $x^6 = y^5 = u_6$, $x^7 = y^8 = u_7$, $x^8 = y^7 = u_8$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where

$Z_1 = 2(\partial x_1 - \partial y_3)$, $\quad Z_2 = 2(\partial x_3 + \partial y_1)$,

$Z_3 = 2(\partial x_1 - \partial y_4)$, $\quad Z_4 = 2(\partial x_4 - \partial y_1)$,

$Z_5 = 2(\partial x_5 - \partial y_6)$, $\quad Z_6 = 2(\partial x_6 + \partial y_5)$,

$Z_7 = 2(\partial x_7 + \partial y_8)$, $\quad Z_8 = 2(\partial x_8 + \partial y_7)$.

Hence $RadTM = span\{Z_1, Z_2, Z_3, Z_4\}$ and $S(TM) = span\{Z_5, Z_6, Z_7, Z_8\}$.

Now $ltr(TM)$ is spanned by $N_1 = \partial x_1 + \partial y_3$, $N_2 = \partial x_3 - \partial y_1$, $N_3 = 2(\partial x_1 + \partial y_4), N_4 = 2(\partial x_4 + \partial y_1)$ and $S(TM^{\perp})$ is spanned by

$W_1 = 2(\partial x_5 + \partial y_6)$, $\quad W_2 = 2(\partial x_6 - \partial y_5)$,

$W_3 = 2(\partial x_7 - \partial y_8)$, $\quad W_4 = 2(\partial x_8 - \partial y_7)$.

It follows that $D_1 = span\{Z_1, Z_2\}$ such that $\overline{J}Z_1 = -Z_2$, $\overline{J}Z_2 = Z_1$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = span\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = -N_4$, $\overline{J}Z_4 = -N_3$, which implies that $\overline{J}D_2 \subset ltr(TM)$. On the other hand, we can see that $D = span\{Z_5, Z_6\}$ such that $\overline{J}Z_5 = -Z_6$, $\overline{J}Z_6 = Z_5$, which implies that D is invariant with respect to $\overline{J}$ and $D^{\perp} = span\{Z_7, Z_8\}$ such that $\overline{J}Z_7 = W_4$, $\overline{J}Z_8 = W_3$, which implies that $D^{\perp}$ is anti-invariant with respect to $\overline{J}$. Hence M is a radical transversal SCR-lightlike submanifold of $\mathbb{R}^{16}_4$.

Example 2. Let $(\mathbb{R}^{16}_4, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,-,+,+,+,+,+,+,-,-,+,+,+,+,+,+)$ with respect to $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial y_8\}$.

Suppose M is a submanifold of $\mathbb{R}^{16}_4$ given by $x^3 = u_1$, $y^3 = u_2$, $x^2 = u_1 \cos\alpha - u_2 \sin\alpha$, $y^2 = u_1 \sin\alpha + u_2 \cos\alpha$, $x^2 = u_3$, $y^2 = -u_4$, $x^4 = u_3 \cos\beta - u_4 \sin\beta$, $y^4 = u_3 \sin\beta + u_4 \cos\beta$, $x^5 = u_5 \cos\gamma$, $y^6 = u_5 \sin\gamma$, $x^6 = u_6 \sin\gamma$, $y^5 = -u_6 \cos\gamma$, $x^7 = u_8 \cos\delta$, $y^8 = u_8 \sin\delta$, $x^8 = u_7 \cos\delta$, $y^7 = u_7 \sin\delta$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where

$Z_1 = 2(\partial x_3 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2)$, $\quad Z_2 = 2(\partial y_3 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2)$,

$Z_3 = 2(\partial x_2 + \cos\beta \partial x_4 + \sin\beta \partial y_4)$, $\quad Z_4 = 2(-\partial y_2 - \sin\beta \partial x_4 + \cos\beta \partial y_4)$,

$Z_5 = 2(\cos\gamma \partial x_5 + \sin\gamma \partial y_6)$, $\quad Z_6 = 2(\sin\gamma \partial x_6 - \cos\gamma \partial y_5)$,

$Z_7 = 2(\cos\delta \partial x_8 + \sin\delta \partial y_7)$, $\quad Z_8 = 2(\cos\delta \partial x_7 + \sin\delta \partial y_8)$.

Hence $RadTM = span\{Z_1, Z_2, Z_3, Z_4\}$ and $S(TM) = span\{Z_5, Z_6, Z_7, Z_8\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_3 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2$, $N_2 = -\partial y_3 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2$, $N_3 = 2(-\partial x_2 + \cos\beta \partial x_4 + \sin\beta \partial y_4)$, $N_4 = 2(-\partial y_2 + \sin\beta \partial x_4 - \cos\beta \partial y_4)$ and $S(TM^{\perp})$ is spanned by

$W_1 = 2(\sin\gamma \partial x_5 - \cos\gamma \partial y_6)$, $\quad W_2 = 2(\cos\gamma \partial x_6 + \sin\gamma \partial y_5)$,

$W_3 = 2(\sin\delta \partial x_8 - \cos\delta \partial y_7)$, $\quad W_4 = 2(\sin\delta \partial x_7 - \cos\delta \partial y_8)$.

It follows that $D_1 = span\{Z_1, Z_2\}$ such that $\overline{J}Z_1 = -Z_2$, $\overline{J}Z_2 = Z_1$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = span\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = N_4$, $\overline{J}Z_4 = N_3$, which implies that $\overline{J}D_2 \subset ltr(TM)$. On the other hand, we can see that $D = span\{Z_5, Z_6\}$ such that $\overline{J}Z_5 = Z_6$, $\overline{J}Z_6 = -Z_5$, which implies that D is invariant with respect to $\overline{J}$ and $D^{\perp} = span\{Z_7, Z_8\}$ such that $\overline{J}Z_7 = W_4$,

$\overline{J}Z_8 = W_3$, which implies that $D^{\perp}$ is anti-invariant with respect to $\overline{J}$. Hence M is a radical transversal SCR-lightlike submanifold of $\mathbb{R}^{16}_4$.

Now, we denote the projection morphisms on D_1, D_2, D and $D^{\perp}$ in TM by P_1, P_2, P_3 and P_4 respectively. Similarly, we denote the projection morphisms of $tr(TM)$ on ν, $\overline{J}D_2$, μ and $\overline{J}D^{\perp}$ by Q_1, Q_2, Q_3 and Q_4 respectively, where ν and μ are orthogonal complementry distributions of $\overline{J}D_2$ and $\overline{J}D^{\perp}$ in $ltr(TM)$ and $S(TM^{\perp})$ respectively. Then, we get

$$X = P_1X + P_2X + P_3X + P_4X, \qquad \forall X \in \Gamma(TM). \tag{3.2}$$

Now applying $\overline{J}$ to (3.2), we have

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X + \overline{J}P_4X. \tag{3.3}$$

Thus we get $\overline{J}P_1X \in D_1 \subset RadTM$, $\overline{J}P_2X \in \overline{J}D_2 \subset ltr(TM)$, $\overline{J}P_3X \in D \subset S(TM)$, $\overline{J}P_4X \in \overline{J}D^{\perp} \subset S(TM^{\perp})$. Also, we have

$$W = Q_1W + Q_2W + Q_3W + Q_4W, \qquad \forall W \in \Gamma(tr(TM)). \tag{3.4}$$

Applying $\overline{J}$ to (3.4), we obtain

$$\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W + \overline{J}Q_3W + \overline{J}Q_4W. \tag{3.5}$$

Thus we get $\overline{J}Q_1W \in \nu \subset ltr(TM), \overline{J}Q_2W \in D_2 \subset RadTM$, $\overline{J}Q_3W \in \mu \subset S(TM^{\perp})$ and $\overline{J}Q_4W \in D^{\perp} \subset S(TM)$.

Now, by using (2.21), (3.3), (3.5) and (2.7)-(2.9) and identifying the components on D_1, D_2, D, $D^{\perp}$, ν, $\overline{J}D_2$, μ and $\overline{J}D^{\perp}$, we obtain

$$\begin{aligned} P_1(\nabla_X\overline{J}P_1Y) + P_1(\nabla_X\overline{J}P_3Y) - P_1(A_{\overline{J}P_2Y}X) - P_1(A_{\overline{J}P_4Y}X)& \\ = \overline{J}P_1\nabla_XY,& \end{aligned} \tag{3.6}$$

$$\begin{aligned} P_2(\nabla_X\overline{J}P_1Y) + P_2(\nabla_X\overline{J}P_3Y) - P_2(A_{\overline{J}P_2Y}X) - P_2(A_{\overline{J}P_4Y}X)& \\ = \overline{J}Q_2h^l(X,Y),& \end{aligned} \tag{3.7}$$

$$\begin{aligned} P_3(\nabla_X\overline{J}P_1Y) + P_3(\nabla_X\overline{J}P_3Y) - P_3(A_{\overline{J}P_2Y}X) - P_3(A_{\overline{J}P_4Y}X)& \\ = \overline{J}P_3\nabla_XY,& \end{aligned} \tag{3.8}$$

$$\begin{aligned} P_4(\nabla_X\overline{J}P_1Y) + P_4(\nabla_X\overline{J}P_3Y) - P_4(A_{\overline{J}P_2Y}X) - P_4(A_{\overline{J}P_4Y}X)& \\ = \overline{J}Q_4h^s(X,Y),& \end{aligned} \tag{3.9}$$

$$\begin{aligned} Q_1h^l(X,\overline{J}P_1Y) + Q_1h^l(X,\overline{J}P_3Y) + Q_1\nabla^l_X\overline{J}P_2Y + Q_1D^l(X,\overline{J}P_4Y)& \\ = \overline{J}Q_1h^l(X,Y),& \end{aligned} \tag{3.10}$$

$$\begin{aligned} Q_2h^l(X,\overline{J}P_1Y) + Q_2h^l(X,\overline{J}P_3Y) + Q_2\nabla^l_X\overline{J}P_2Y + Q_2D^l(X,\overline{J}P_4Y)& \\ = \overline{J}P_2\nabla_XY,& \end{aligned} \tag{3.11}$$

$$Q_3 h^s(X, \overline{J}P_1Y) + Q_3 h^s(X, \overline{J}P_3Y) + Q_3 \nabla^s_X \overline{J}P_4Y + Q_3 D^s(X, \overline{J}P_2Y) = \overline{J}Q_3 h^s(X, Y), \tag{3.12}$$

$$Q_4 h^s(X, \overline{J}P_1Y) + Q_4 h^s(X, \overline{J}P_3Y) + Q_4 \nabla^s_X \overline{J}P_4Y + Q_4 D^s(X, \overline{J}P_2Y) = \overline{J}P_4 \nabla_X Y. \tag{3.13}$$

Theorem 3.1. *Let M be a radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then μ is an invariant distribution with respect to $\overline{J}$.*

Proof. Let M be a radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. For any $X \in \Gamma(\mu)$, $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$, we have $\overline{g}(\overline{J}X, \xi) = -\overline{g}(X, \overline{J}\xi) = 0$ and $\overline{g}(\overline{J}X, N) = -\overline{g}(X, \overline{J}N) = 0$. Thus $\overline{J}X$ has no components in $RadTM$ and $ltr(TM)$.

Now, for $X \in \Gamma(\mu)$ and $Y \in \Gamma(D^\perp)$, we have $\overline{g}(\overline{J}X, Y) = -\overline{g}(X, \overline{J}Y) = 0$, as $\overline{J}Y \in \Gamma(\overline{J}D^\perp)$, which implies that $\overline{J}X$ has no components in $D^\perp$. Hence $\overline{J}(\mu) \subset \Gamma(\mu)$, which complete the proof.

Now we give a characterization theorem for radical transversal SCR-lightlike submanifold.

Theorem 3.2. *Let M be a lightlike submanifold of an indefinite complex space-form $(\overline{M}(c), \overline{g})$, $c \neq 0$. Then M is a radical transversal SCR-lightlike submanifold if and only if*

(i) the maximal invariant subspace of T_pM, $p \in M$ defines a distribution $\overline{D} = D_1 \oplus D$, where $RadTM = D_1 \oplus D_2$ and D is a non-degenerate invariant distribution on M,

(ii) $\overline{g}(\overline{R}(\xi, N)\xi_1, \xi_2) \neq 0$, for all $\xi \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$ and $\xi_1, \xi_2 \in \Gamma(D_2)$,

(iii) $\overline{g}(\overline{R}(X, Y)Z, W) = 0$, for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$, where $D^\perp$ is the complementry distribution of D in $S(TM)$.

Proof. Let M be a radical transversal SCR-lightlike submanifold of an indefinite complex space-form $(\overline{M}(c), \overline{g})$, $c \neq 0$. Then proof of (i) follows from the definition of radical transversal SCR-lightlike submanifold. For $\xi \in \Gamma(D_1), N \in \Gamma ltr(TM)$ and $\xi_1, \xi_2 \in \Gamma(D_2)$, from (2.22), we have

$$\overline{g}(\overline{R}(\xi, N)\xi_1, \xi_2) = \frac{c}{2}\overline{g}(\overline{J}\xi, N)\overline{g}(\xi_1, \overline{J}\xi_2). \tag{3.14}$$

Since D_1 is invariant distribution, we obtain $\overline{g}(\overline{J}\xi, N) \neq 0$, $\forall \xi \in \Gamma(D_1), N \in \Gamma ltr(TM)$. Also $\overline{J}D_2 \subset ltr(TM)$, so we get $\overline{g}(\xi_1, \overline{J}\xi_2) \neq 0$, $\forall \xi_1, \xi_2 \in \Gamma(D_2)$. Hence $\overline{g}(\overline{R}(\xi, N)\xi_1, \xi_2) \neq 0$ for all $\xi \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$ and $\xi_1, \xi_2 \in \Gamma(D_2)$, which proves (ii). For $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$, from (2.22), we have

$$\overline{g}(\overline{R}(X, Y)Z, W) = \frac{c}{2}g(\overline{J}X, Y)\overline{g}(Z, \overline{J}W). \tag{3.15}$$

In view of $\overline{J}W \in S(TM^{\perp})$, we have $\overline{g}(Z, \overline{J}W) = 0$, $\forall Z, W \in \Gamma(D^{\perp})$. Hence $\overline{g}(\overline{R}(X,Y)Z, W) = 0$, which proves (iii).

Now, conversely suppose that the conditions (i), (ii), (iii) are satisfied. Since D_1 is invariant distribution, we have $\overline{g}(\overline{J}\xi, N) \neq 0$ $\forall \xi \in \Gamma(D_1)$, $N \in \Gamma(ltr(TM))$. Thus from (ii) and (3.14), we have $\overline{g}(\xi_1, \overline{J}\xi_2) \neq 0$ $\forall \xi_1, \xi_2 \in \Gamma(D_2)$, which implies $\overline{J}D_2 \subset ltr(TM)$.

Further, since D is non-degenerate invariant distribution, we may choose $X, Y \in \Gamma(D)$ such that $g(\overline{J}X, Y) \neq 0$. Thus from (iii) and (3.15), we have $\overline{g}(Z, \overline{J}W) = 0$, $\forall Z, W \in \Gamma(D^{\perp})$, which implies that $\overline{J}W$ have no components in $(D^{\perp})$. For any $X \in \Gamma(D)$, we have $\overline{g}(\overline{J}W, X) = -\overline{g}(W, \overline{J}X) = 0$, which implies that $\overline{J}W$ have no components in D.

Now, form (i) and (ii), we also have $\overline{g}(\overline{J}W, \xi) = -\overline{g}(W, \overline{J}\xi) = 0$ and $\overline{g}(\overline{J}W, N) = -\overline{g}(W, \overline{J}N) = 0$, $\forall \xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$, which implies that $\overline{J}W$ have no components in $RadTM$ and $ltr(TM)$. Thus, we get $\overline{J}D^{\perp} \subseteq S(TM^{\perp})$, which completes the proof.

Theorem 3.3. *Let M be a radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if $P_3\nabla_X\overline{J}P_1\xi = P_3A_{\overline{J}P_2\xi}X$, $Q_4h^s(X, \overline{J}P_1\xi) = 0$ and $Q_4D^s(X, \overline{J}P_2\xi) = 0$, $\forall X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.*

Proof. Let M be a radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ ([10]), i.e. $\nabla_X\xi \in \Gamma(RadTM)$, $\forall X \in \Gamma(TM), \forall \xi \in \Gamma(RadTM)$. From (2.21), we have

$$\overline{\nabla}_X\overline{J}\xi = \overline{J}\overline{\nabla}_X\xi \quad \forall X \in \Gamma(TM), \forall \xi \in \Gamma(RadTM). \tag{3.16}$$

From (2.7), (2.8), (2.19) and (3.16), we obtain

$$\begin{aligned}\overline{J}\nabla_X\overline{J}P_1\xi + \overline{J}h^l(X, \overline{J}P_1\xi) + \overline{J}h^s(X, \overline{J}P_1\xi) - \overline{J}A_{\overline{J}P_2\xi}X + \\ \overline{J}\nabla^l_X\overline{J}P_2\xi + \overline{J}D^s(X, \overline{J}P_2\xi) + \nabla_X\xi + h^l(X,\xi) + h^s(X,\xi) = 0.\end{aligned} \tag{3.17}$$

Now, taking tangential components in (3.17), we get

$$\begin{aligned}\overline{J}P_1\nabla_X\overline{J}P_1\xi + \overline{J}P_3\nabla_X\overline{J}P_1\xi + \overline{J}Q_2h^l(X, \overline{J}P_1\xi) + \overline{J}Q_4h^s(X, \overline{J}P_1\xi) - \\ \overline{J}P_1A_{\overline{J}P_2\xi}X - \overline{J}P_3A_{\overline{J}P_2\xi}X + \overline{J}Q_2\nabla^l_X\overline{J}P_2\xi + \overline{J}Q_4D^s(X, \overline{J}P_2\xi) + \nabla_X\xi = 0.\end{aligned} \tag{3.18}$$

Thus $\nabla_X\xi = \overline{J}P_1A_{\overline{J}P_2\xi}X - \overline{J}P_1\nabla_X\overline{J}P_1\xi - \overline{J}Q_2h^l(X, \overline{J}P_1\xi) - \overline{J}Q_2\nabla^l_X\overline{J}P_2\xi \in \Gamma(RadTM)$ if and only if $P_3\nabla_X\overline{J}P_1\xi = P_3A_{\overline{J}P_2\xi}X$, $Q_4h^s(X, \overline{J}P_1\xi) = 0$ and $Q_4D^s(X, \overline{J}P_2\xi) = 0$, $\forall X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, which completes the proof.

Theorem 3.4. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 is integrable if and only if*

(i) $Q_2h^l(Y,\overline{J}P_1X) = Q_2h^l(X,\overline{J}P_1Y)$ *and* $Q_4h^s(Y,\overline{J}P_1X) = Q_4h^s(X,\overline{J}P_1Y)$,

(ii) $P_3(\nabla_X\overline{J}P_1Y) = P_3(\nabla_Y\overline{J}P_1X), \quad \forall X,Y \in \Gamma(D_1)$.

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X,Y \in \Gamma(D_1)$. From (3.8), we get $P_3(\nabla_X\overline{J}P_1Y) = \overline{J}P_3\nabla_XY$, which gives $P_3(\nabla_X\overline{J}P_1Y) - P_3(\nabla_Y\overline{J}P_1X) = \overline{J}P_3[X,Y]$. In view of (3.11), we have $Q_2h^l(X,\overline{J}P_1Y) = \overline{J}P_2\nabla_XY$. Thus $Q_2h^l(X,\overline{J}P_1Y) - Q_2h^l(Y,\overline{J}P_1X) = \overline{J}P_2[X,Y]$. Also from (3.13), we obtain $Q_4h^s(X,\overline{J}P_1Y) = \overline{J}P_4\nabla_XY$, which gives $Q_4h^s(X,\overline{J}P_1Y) - Q_4h^s(Y,\overline{J}P_1X) = \overline{J}P_4[X,Y]$. This concludes the theorem.

Theorem 3.5. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 is integrable if and only if*

(i) $P_1(A_{\overline{J}P_2Y}X) = P_1(A_{Y\overline{J}P_2X}Y)$ *and* $P_3(A_{\overline{J}P_2Y}X) = P_3(A_{\overline{J}P_2X}Y)$,

(ii) $Q_4D^s(Y,\overline{J}P_2X) = Q_4D^s(X,\overline{J}P_2Y), \quad \forall X,Y \in \Gamma(D_2)$.

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X,Y \in \Gamma(D_2)$. From (3.6), we get $P_1(A_{\overline{J}P_2Y}X) = -\overline{J}P_1\nabla_XY$, which gives $P_1(A_{\overline{J}P_2X}Y) - P_1(A_{\overline{J}P_2Y}X) = \overline{J}P_1[X,Y]$. In view of (3.8), we obtain $P_3(A_{\overline{J}P_2Y}X) = -\overline{J}P_3\nabla_XY$, which implies $P_3(A_{\overline{J}P_2X}Y) - P_3(A_{\overline{J}P_2Y}X) = \overline{J}P_3[X,Y]$. Also from (3.13), we have $Q_4D^s(X,\overline{J}P_2Y) = \overline{J}P_4\nabla_XY$, which gives $Q_4D^s(X,\overline{J}P_2Y) - Q_4D^s(Y,\overline{J}P_2X) = \overline{J}P_4[X,Y]$. This completes the proof.

Theorem 3.6. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D is integrable if and only if*

(i) $Q_2h^l(Y,\overline{J}P_3X) = Q_2h^l(X,\overline{J}P_3Y)$ *and* $Q_4h^s(Y,\overline{J}P_3X) = Q_4h^s(X,\overline{J}P_3Y)$,

(ii) $P_1(\nabla_X\overline{J}P_3Y) = P_1(\nabla_Y\overline{J}P_3X), \quad \forall X,Y \in \Gamma(D)$.

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X,Y \in \Gamma(D)$. From (3.6), we get $P_1(\nabla_X\overline{J}P_3Y) = \overline{J}P_1\nabla_XY$, which gives $P_1(\nabla_X\overline{J}P_3Y) - P_1(\nabla_Y\overline{J}P_3X) = \overline{J}P_1[X,Y]$. In view of (3.11), we have $Q_2h^l(X,\overline{J}P_3Y) = \overline{J}P_2\nabla_XY$. Thus $Q_2h^l(X,\overline{J}P_3Y) - Q_2h^l(Y,\overline{J}P_3X) = \overline{J}P_2[X,Y]$. Also from (3.13), we obtain $Q_4h^s(X,\overline{J}P_3Y) = \overline{J}P_4\nabla_XY$, which gives $Q_4h^s(X,\overline{J}P_3Y) - Q_4h^s(Y,\overline{J}P_3X) = \overline{J}P_4[X,Y]$. Thus, we obtain the required results.

Theorem 3.7. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D^\perp$ is integrable if and only if*

(i) $P_1(A_{\overline{J}P_4Y}X) = P_1(A_{Y\overline{J}P_4X}Y)$ *and* $P_3(A_{\overline{J}P_4Y}X) = P_3(A_{\overline{J}P_4X}Y)$,

(ii) $Q_2D^l(Y,\overline{J}P_4X) = Q_2D^l(X,\overline{J}P_4Y), \quad \forall X,Y \in \Gamma(D^\perp)$.

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D^{\perp})$. From (3.6), we get $P_1(A_{\overline{J}P_4Y}X) = -\overline{J}P_1\nabla_X Y$, which gives $P_1(A_{\overline{J}P_4X}Y) - P_1(A_{\overline{J}P_4Y}X) = \overline{J}P_1[X,Y]$. In view of (3.8), we have $P_3(A_{\overline{J}P_4Y}X) = -\overline{J}P_3\nabla_X Y$, which implies $P_3(A_{\overline{J}P_4X}Y) - P_3(A_{\overline{J}P_4Y}X) = \overline{J}P_3[X,Y]$. Also from (3.11), we obtain $Q_2D^l(X, \overline{J}P_4Y) = \overline{J}P_2\nabla_X Y$, which gives $Q_2D^l(X, \overline{J}P_4Y) - Q_2D^l(Y, \overline{J}P_4X) = \overline{J}P_2[X,Y]$. This proves the theorem.

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions to be totally geodesic on the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold.

Theorem 4.1. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ defines a totally geodesic foliation in M if and only if*

(i) $h^l(X, \overline{J}Z) = 0$ and $D^l(X, \overline{J}W) = 0$,

(ii) $\nabla_X \overline{J}Z$ and $A_{\overline{J}W}X$ have no components in $Rad\,TM$,

$\forall X \in \Gamma(RadTM)$, $Z \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$.

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution RadTM defines a totally geodesic foliation if and only if $\nabla_X Y \in \mathrm{Rad}TM$, $\forall X, Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric a connection, from (2.7), (2.20) and (2.21), for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(D)$, we have $\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}Z)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y) = -\overline{g}(\nabla_X \overline{J}Z, \overline{J}P_2Y) - \overline{g}(h^l(X, \overline{J}Z), \overline{J}P_1Y)$. In view of (2.7), (2.20) and (2.21), for any $X, Y \in \Gamma(RadTM)$ and $W \in \Gamma(D^{\perp})$, we obtain $\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}W)$, which implies $\overline{g}(\nabla_X Y, W) = \overline{g}(A_{\overline{J}W}X, \overline{J}P_2Y) - \overline{g}(D^l(X, \overline{J}W), \overline{J}P_1Y)$. This completes the proof.

Theorem 4.2. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D defines a totally geodesic foliation in M if and only if $A_{\overline{J}W}X$, $A_{\overline{J}Q_1N}X$ and $A^*_{\overline{J}Q_2N}X$ have no components in D, $\forall X \in \Gamma(D)$, $\forall N \in \Gamma(ltr(TM))$ and $\forall W \in \Gamma(D^{\perp})$.*

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D defines a totally geodesic foliation if and only if $\nabla_X Y \in D$, $\forall X, Y \in \Gamma(D)$. Since $\overline{\nabla}$ is metric a connection, from (2.7), (2.20) and (2.21), for any $X, Y \in \Gamma(D)$ and $W \in \Gamma(D^{\perp})$, we have $\overline{g}(\nabla_X Y, W) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}W)$, which gives $\overline{g}(\nabla_X Y, W) = -\overline{g}(\overline{\nabla}_X \overline{J}W, \overline{J}Y) = \overline{g}(A_{\overline{J}W}X, \overline{J}Y)$. In view of (2.7), (2.20) and (2.21), for any $X, Y \in \Gamma(D)$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X(\overline{J}Q_1N + \overline{J}Q_2N))$. This concludes the theorem.

Theorem 4.3. *Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $D^{\perp}$ defines a totally geodesic foliation in M if and only if*

(i) $D^s(X, \overline{J}Q_1N) = 0$ *and* $h^s(X, \overline{J}Q_2N) = 0$, $\forall N \in \Gamma(ltr(TM))$,

(ii)$h^s(X, \overline{J}Z) = 0$, $\forall X \in \Gamma(D^{\perp})$ *and* $\forall Z \in \Gamma(D)$.

Proof. Let M be the radical transversal SCR-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution $D^{\perp}$ defines a totally geodesic foliation if and only if $\nabla_X Y \in D^{\perp}$, $\forall X, Y \in \Gamma(D^{\perp})$. Since $\overline{\nabla}$ is metric a connection, in view of (2.7), (2.20) and (2.21), for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$, we have $\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}Z)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y) = \overline{g}(h^s(X, \overline{J}Z), \overline{J}Y)$. From (2.7), (2.20) and (2.21), for any $X, Y \in \Gamma(D^{\perp})$ and $N \in \Gamma(ltr(TM))$, we obtain $\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \overline{J}Y, \overline{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{\nabla}_X(\overline{J}Q_1N + \overline{J}Q_2N), \overline{J}Y) = -\overline{g}(h^s(X, \overline{J}Q_2N) + D^s(X, \overline{J}Q_1N), \overline{J}Y)$. Thus, we obtain the required results.

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CROSS-MOMENTS COMPUTATION FOR STOCHASTIC CONTEXT-FREE GRAMMARS *

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Abstract. In this paper we consider the problem of efficient computation of cross-moments of a vector random variable represented by a stochastic context-free grammar. Two types of cross-moments are discussed. The sample space for the first one is the set of all derivations of the context-free grammar, and the sample space for the second one is the set of all derivations which generate a string belonging to the language of the grammar. In the past, this problem was widely studied, but mainly for the cross-moments of scalar variables and up to the second order. This paper presents new algorithms for computing the cross-moments of an arbitrary order, while the previously developed ones are derived as special cases.

Keywords. Stochastic context-free grammar; cross-moments; semiring; moment-generating function; partition function; inside-outside algorithm.

1. Introduction

The cross-moments of random variables modeled with stochastic context-free grammars (SCFG) are important quantities in the SCFG modeling [10] and statistics [1]. They are defined as the expected value of the product of integer powers of the entries of a random vector variable, which can represent string or derivation length, the number of rule occurrences in a derivation or uncertainty associated with the occurring rule. The expectation can be taken either with respect to the sample space of all SCFG derivations or with respect to the sample space of all derivations which generate a string belonging to the language of the grammar. Throughout this paper, the name *cross-moments* is usually used in the former case, while in the latter case the term *conditional cross-moments* is used.

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The computation of cross-moments may become demanding if the sample space is large. In the past, this problem was widely studied, but mainly for the cross-moments of scalar variables (called simply *moments*) and up to the second order. The first order moments computation, such as expected length of derivations and expected string length, is given in [26]. The computation of SCFG entropy is considered in [17]. The procedure for computing the moments of string and derivation length is given in [10], where the explicit formulas for the moments up to the second order are derived. First order conditional cross-moments are considered in [11], where the algorithm for conditional SCFG entropy is derived. A more general algorithm for computing the conditional cross-moments of a vector variable of the second order is derived in [16].

In this paper we give the recursive formulas for computing the cross-moments and the conditional cross-moments of an arbitrary order, for a vector variable which factorizes according to a certain rule which is satisfied in the case of string or derivation length, the number of rule occurrences in the derivation or uncertainty associated with the occurring rule. The formulas are derived by differentiation of the recursive equations for the moment-generating function [22], which are obtained from the algorithms for computing the partition function of a SCFG [18] for the cross-moments and with the inside algorithm [15], [8] for the conditional cross-moments.

The paper is organized as follows. Section 2 introduces multi-index notation which is used throughout the paper, and reviews some preliminary notions about generalized Leibniz's formula, basic algebraic structures, and context free grammars. In Section 3 we give the formal definition of SCFG cross-moments and derive the recursive equations for cross-moments computation. The conditional cross moments are considered in Section 4.

2. Preliminaries

This section provides some basic definitions and theorems which are used in the paper. We review the multi-index formulation of the *Generalized Leibniz's* formula [21], and basic notions from the theory of weighted context free grammars, according to [18] and [19].

2.1. Multiindexes, Multinomial theorem and Generalized Leibniz's formula

Multi-indexes. A multi-index is defined as a tuple of nonnegative integers $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. We define its dimension as $\dim(\boldsymbol{\alpha}) = d$ and its length as the sum $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. The multi-index factorial is $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_d!$. The zero multi-index is $\mathbf{0} = (0, \dots, 0)$.

If $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$, we write $\boldsymbol{\beta} < \boldsymbol{\alpha}$ if $\beta_i < \alpha_i$ for $i = 1, \dots, d$. We write $\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}$ provided $\beta_i \leqslant \alpha_i$ for $i = 1, \dots, d$. The sum and difference of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined to be $\boldsymbol{\alpha} \pm \boldsymbol{\beta} = (\alpha_1 \pm \beta_1, \dots, \alpha_d \pm \beta_d)$.

If $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N$ are multi-indexes and $\boldsymbol{\beta}_1 + \cdots + \boldsymbol{\beta}_N = \boldsymbol{\alpha}$, we define the multinomial coefficients to be

$$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N} = \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}_1! \cdots \boldsymbol{\beta}_N!}.$$

For vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N) \in \mathbb{R}^d$ and a multi-index $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, the multi-index power is defined to be

$$\boldsymbol{z}^{\boldsymbol{\beta}} = z_1^{\beta_1} \cdots z_d^{\beta_d}.$$

Multinomial theorem and Generalized Leibniz's formula. With these settings, the multinomial theorem [20] can be expressed as

$$\Big(\sum_{i=1}^{N} \boldsymbol{z}_i\Big)^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}_1 + \cdots + \boldsymbol{\beta}_N = \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N} \prod_{i=1}^{N} \boldsymbol{z}_i^{\boldsymbol{\beta}_i},$$

for a vector $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and let $C_{\boldsymbol{\alpha}}$ denote the set of all functions $u : \mathbb{R}^d \to \mathbb{R}$ having $\boldsymbol{\alpha}$-th partial derivative at zero. For a function $u : \mathbb{R}^d \to \mathbb{R}$, we define the partial derivative at zero of an order $\boldsymbol{\alpha}$ as

$$\mathcal{D}^{(\boldsymbol{\alpha})}\{u\} = \frac{\partial^{|\boldsymbol{\alpha}|} u(t_1, \ldots, t_d)}{\partial^{\alpha_1} t_1 \ldots \partial^{\alpha_d} t_d}\bigg|_{t=0}.$$

Note that $\mathcal{D}^{(\mathbf{0})}\{u\} = u(\boldsymbol{t})$. According to the *generalized Leibniz's formula* [21], the following equality holds:

$$\mathcal{D}^{(\boldsymbol{\alpha})}\{FG\} = \sum_{\mathbf{0} \leqslant \boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathcal{D}^{(\boldsymbol{\beta})}\{F\} \cdot \mathcal{D}^{(\boldsymbol{\alpha}-\boldsymbol{\beta})}\{G\}, \tag{2.1}$$

for all $F, G \in C_{\boldsymbol{\alpha}}$. The derivative of the product of more than two functions can be found according to [25]

$$\mathcal{D}^{(\boldsymbol{\alpha})}\Big\{\prod_{i=1}^{N} F_i\Big\} = \sum_{\boldsymbol{\beta}_1 + \cdot + \boldsymbol{\beta}_m = \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N} \prod_{i=1}^{N} \mathcal{D}^{(\boldsymbol{\beta}_i)}\{F_i\}, \tag{2.2}$$

for all $F_i \in C_{\boldsymbol{\alpha}};\ i = 1, \ldots, N$.

Tuples of elements indexed with multi-indexes. The set of all multi-indexes lower than or equal to $\boldsymbol{\nu}$ is denoted with $\mathcal{A}_{\boldsymbol{\nu}}$,

$$\mathcal{A}_{\boldsymbol{\nu}} = \big\{\boldsymbol{\alpha} \in \mathbb{N}_0^{\dim(\boldsymbol{\nu})} \mid \boldsymbol{\alpha} \leqslant \boldsymbol{\nu}\big\},$$

and $|\mathcal{A}_{\boldsymbol{\nu}}|$ denotes its cardinality.

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d)$, we define the lexicographic order relation $\prec$, so that $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ if

$$\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n \text{ and } \alpha_{n+1} < \beta_{n+1}.$$

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ be a multi-index and $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{|\mathcal{A}_{\boldsymbol{\nu}}|}$ be multi-indexes from $\mathcal{A}_{\boldsymbol{\nu}}$ such that $\mathbf{0} = \boldsymbol{\alpha}_1 \prec \boldsymbol{\alpha}_2 \prec \cdots \prec \boldsymbol{\alpha}_{|\mathcal{A}_{\boldsymbol{\nu}}|} = \boldsymbol{\nu}$. Let $\boldsymbol{z}$ be a function which associates a real number $z^{(\boldsymbol{\alpha}_i)}$ to each $\boldsymbol{\alpha}_i$ from $\mathcal{A}_{\boldsymbol{\nu}}$, i.e. $\boldsymbol{z}$ is a vector from $\mathbb{R}^{|\mathcal{A}_\nu|}$ indexed by multindexes. We use the following notation for the vector $\boldsymbol{z}$:

$$\boldsymbol{z} = \left(\ z^{(\boldsymbol{\alpha})}\ \right)_{\boldsymbol{\alpha} \in \mathcal{A}_{\boldsymbol{\nu}}} = (z^{(\boldsymbol{\alpha}_1)}, \ldots, z^{(\boldsymbol{\alpha}_{|\mathcal{A}_\nu|})}).$$

2.2. Semirings

A *monoid* is a triple $(\mathbb{K}, \oplus, 0)$, where $\oplus$ is an associative binary operation on the set $\mathbb{K}$ and 0 is the identity element for $\oplus$, i.e. $a \otimes 0 = 0 \oplus a = a$, for all $a \in \mathbb{K}$. A monoid is commutative if the operation $\oplus$ is commutative.

Example 2.1. Let Σ be a non-empty set. The *free monoid* $\boldsymbol{\Sigma}^* = (\Sigma, \cdot, \epsilon)$ over Σ is a monoid, where the carrier set $\Sigma^* = \{\, a_1 \ldots a_n \,|\, n \in \mathbb{N}_0,\ a_i \in \Sigma\, (1 \leqslant i \leqslant n)\}$ is the set of all *strings* over Σ and ϵ is the (unique) empty string of length zero. The operation $\cdot$ denotes the composition (concatenation) of strings defined by $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \boldsymbol{u}_1\boldsymbol{u}_2$ for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \Sigma^*$.

A *semiring* is a tuple $(\mathbb{K}, \oplus, \otimes, 0, 1)$ such that

1. $(\mathbb{K}, \oplus, 0)$ is a commutative monoid with 0 as the identity element for $\oplus$,
2. $(\mathbb{K}, \otimes, 1)$ is a monoid with 1 as the identity element for $\otimes$,
3. $\otimes$ distributes over $\oplus$, i.e. $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$, for all a, b, c in $\mathbb{K}$,
4. 0 is an annihilator for $\otimes$, i.e. $a \otimes 0 = 0 \otimes a = 0$, for every a in $\mathbb{K}$.

A semiring is commutative if the operation $\otimes$ is commutative. The operations $\oplus$ and $\otimes$ are called the addition and the multiplication in $\mathbb{K}$. For a topology τ we define the topological semiring as a pair $\big(\mathbb{K}, \tau\big)$.

Example 2.2. If $C_{\boldsymbol{\alpha}}$ denotes the set of all functions $u : \mathbb{R}^d \to \mathbb{R}$ having the $\boldsymbol{\alpha}$-th partial derivative at zero, $\mathbf{1}$ is identity function and $\mathbf{0}$ is zero function, then we can define a semiring of $\boldsymbol{\alpha}$-continuous functions as $(C_{\boldsymbol{\alpha}}, \cdot, +, \mathbf{0}, \mathbf{1})$.

2.3. Weighted and stochastic context-free grammars

By a *weighted context-free grammar (WCFG)* over a commutative semiring $\big(\mathbb{K}, +, \cdot, 1, 0\big)$ we mean a tuple $G = \big(\Sigma, \mathcal{N}, S, \mathcal{R}, w\big)$, where

- $\Sigma = \big\{w_1, \ldots, w_{|\Sigma|}\big\}$ is a finite set of *terminals*,
- $\mathcal{N} = \big\{A_1, \ldots, A_{|\mathcal{N}|}\big\}$ is a finite set of *nonterminals* disjoint with Σ,
- $S \in \mathcal{N}$ is called the *start symbol* (throughout the paper it is usually assumed that $S = A_1$),

- $\mathcal{R} \subseteq \mathcal{N} \times (\Sigma \cup \mathcal{N})^*$ is a finite set of rules. A rule $(A, \alpha) \in \mathcal{R}$ is commonly written as $A \to \alpha$, where the nonterminal A is called the *premise*. The set of all rules $A_i \to B_{i,j}$, $B_{i,j} \in (\mathcal{N} \cup \Sigma)^*$ will be denoted by $\mathcal{R}_i$.

- $w : \mathcal{R} \to \mathbb{K}$ is the function called *weight*.

The *left-most rewriting relation* $\Rightarrow$ associated with G is defined as the set of triples $\big(\alpha, \pi, \beta\big) \in (\Sigma \cup \mathcal{N})^* \times \mathcal{R} \times (\Sigma \cup \mathcal{N})^*$, for which there is a terminal string $\boldsymbol{u} \in \Sigma^*$ and a nonterminal string $\delta \in (\Sigma \cup \mathcal{N})^*$, along with a nonterminal $A \in \mathcal{N}$ and a string $\gamma \in (\Sigma \cup \mathcal{N})^*$ such that $\alpha = \boldsymbol{u}A\delta$, $\beta = \boldsymbol{u}\gamma\delta$, and $\pi = A \to \gamma$ is a rule from $\mathcal{R}$. The left-most relation triple $\big(\alpha, \pi, \beta\big)$ will be denoted by $\alpha \overset{\pi}{\Rightarrow} \beta$. The *left-most derivation* (hereinafter the *derivation*) in this grammar is a string $\pi_1, \ldots, \pi_n \in \mathcal{R}^*$ for which there are grammar symbols $\alpha, \beta \in \Sigma \cup \mathcal{N}$ such that we can derive β from α by applying the rewriting rules $\pi_1, \ldots, \pi_n$: $\alpha \overset{\pi_1}{\Rightarrow} \cdots \overset{\pi_n}{\Rightarrow} \beta$. The weight function is extended to derivations such that $w\ (\pi_1 \cdots \pi_N) = w(\pi_1) \cdots w(\pi_N)$, for all $\pi_1 \cdots \pi_N \in R^*$. A nonterminal A is *productive* if there exists a derivation $\pi_1 \cdots \pi_k$ such that $A \overset{\pi_1}{\Rightarrow} \cdots \overset{\pi_k}{\Rightarrow} \boldsymbol{u}, \boldsymbol{u} \in \Sigma^*$. A nonterminal A is *accessible* from a nonterminal B if there exist derivations $\pi_1 \cdots \pi_k$ such that $B \overset{\pi_1}{\Rightarrow} \cdots \overset{\pi_k}{\Rightarrow} \eta A \xi$ where $\eta, \xi \in (\Sigma \cup \mathcal{N})^*$ (if A is accessible from S, then it is simply accessible). A nonterminal A is *useful* if it is accessible and productive (otherwise, it is *useless*). We say that a WCFG has a cycle if there is a derivation $\pi_1, \ldots, \pi_n$, such that for a nonterminal A it holds that $A \overset{\pi_1}{\Rightarrow} \cdots \overset{\pi_n}{\Rightarrow} A$. Otherwise, the WCFG is cycle-free.

A weighted context-free grammar $G = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, p\big)$ over the probability semiring $\big(\mathbb{R}_+, +, \cdot, 0, 1\big)$ is called a *stochastic context-free grammar* (SCFG) if the weight p maps all rules to the real unit interval $[0, 1]$. A *SCFG* is *reduced* if $p(A \to \gamma) > 0$ for all $A \to \gamma \in \mathcal{R}$ and each nonterminal A, and all nonterminals are useful. In this paper we consider only the reduced *SCFGs*. In addition, we assume that the *SCFG* is *proper*, which means that the weight function p gives us a probability distribution over the rules that we can apply, i.e. $\sum_{j=1}^{|\mathcal{R}_i|} p\big(A_i \to B_{i,j}\big) = 1$ for all $1 \leqslant i \leqslant |\mathcal{N}|$.

For a stochastic context-free grammar $G = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, p\big)$ we define the subgrammar $G_i = \big(\Sigma, \mathcal{N}_i, A_i, \mathcal{R}_i, p_i\big)$ with the start symbol A_i, where $\mathcal{N}_i$ is the set which consists of A_i and nonterminals accessible from A_i and $\mathcal{R}_i \subseteq \mathcal{R}$ is the set of rules in which only nonterminals from $\mathcal{N}_i$ appear as premises and p_i is restriction of p to $\mathcal{R}_i$, such that $p_i(\pi) = p(\pi)$ for each $\pi \in \mathcal{R}_i$. Note that if G is reduced, then G_i also has this property.

Let $G = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, p\big)$ be a stochastic context-free grammar and Ω the set of all derivations in G. The grammar G is *consistent* if

$$\sum_{\boldsymbol{\pi} \in \Omega} p(\boldsymbol{\pi}) = 1.$$

Booth and Thompson [2] gave the consistency condition by the following theorem.

Theorem 2.1. *A reduced stochastic context-free grammar G is consistent if $\rho(M) < 1$, where $\rho(M)$ is the absolute value of the largest eigenvalue of the expectation matrix $M = [M_{i,n}], 1 \leqslant i, n \leqslant |\mathcal{N}|$ defined by*

$$M_{i,n} = \sum_{j=1}^{|\mathcal{R}_i|} p\big(A_i \to B_{i,j}\big) r_n(i,j), \tag{2.3}$$

where $r_n(i,j)$ denotes the number of times the nonterminal A_n appears on the right-hand side of the rule $\pi = A_i \to B_{i,j}$.

Note that if G is reduced, then G_i also has this property. In addition, the expectation matrices $\mathrm{M}^{(i)}$ of all subgrammars G_i are the principal submatrices of M, and according to [9] (Corollary 8.1.20), $\rho(\mathrm{M}^{(i)}) \leqslant \rho(\mathrm{M})$. Thus, the conditions from Theorem 2.1 are satisfied and G_i are also consistent, i.e.

$$\sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi}) = 1, \tag{2.4}$$

for $1 \leqslant i \leqslant |\mathcal{N}|$, where Ω_i is the set of all derivations starting at $A_i \in \mathcal{N}$.

3. Cross-moments computation of SCFG

In this section we first define cross-moments and the moment-generating function of SCFG. After that, we provide formulas for the computation of cross moments up to an arbitrary order. In the end, we retrieve, as special cases, the formulas for the first and the second order moments, previously derived in [2] and [10].

3.1. Cross-moments and moment-generating function of SCFG

Let $G = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, p\big)$ be reduced and consistent SCFG, and let $\boldsymbol{X}_i = \big[X_{i,1}, \ldots, X_{i,D}\big]^T : \Omega_i \to \mathbb{R}^D$ be random variables distributed according to the p_i, which are restrictions of p to $\mathcal{R}_i$ $(p_i(\boldsymbol{\pi}) = p(\boldsymbol{\pi}))$, for $1 \leqslant i \leqslant |\mathcal{N}|$. The *$i$-th cross-moment of an order* $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_D)$ of $\boldsymbol{X}_i$ is defined with

$$\mu^{(\nu)}_{p,\boldsymbol{X}_i} = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi}) \cdot X_{i,1}(\boldsymbol{\pi})^{\nu_1} \cdots X_{i,D}(\boldsymbol{\pi})^{\nu_d} = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi})\ \boldsymbol{X}_i(\boldsymbol{\pi})^{\boldsymbol{\nu}}. \tag{3.1}$$

In this paper we consider random vectors $\boldsymbol{X}_i : \Omega_i \to \mathbb{R}^D$, which can be represented as the sum of random vectors $\boldsymbol{Y} : \mathcal{R} \to \mathbb{R}^D$:

$$\boldsymbol{X}_i(\pi_1 \cdots \pi_N) = \boldsymbol{Y}\big(\pi_1\big) + \cdots + \boldsymbol{Y}\big(\pi_N\big), \tag{3.2}$$

for all $\pi_1 \cdots \pi_N \in \Omega_i$. This assumption may seem too restrictive, but it holds in some important cases: (1) If $\boldsymbol{X}(\boldsymbol{\pi})$ represents derivation length starting from a nonterminal A_i, then $\boldsymbol{Y}(\pi_i) = 1$; (2) if $\boldsymbol{X}(\boldsymbol{\pi})$ is the length of a string derived from a nonterminal A_i, then $\boldsymbol{Y}(\pi_i)$ equals the number of terminals on the right-hand side of π_i; (3) if $\boldsymbol{X}(\boldsymbol{\pi})$ represents the self-information of derivation $\boldsymbol{\pi}$ [11], then $\boldsymbol{Y}(\pi_i) = -\log p(\pi_i)$.

Following Proposition 6 from [4], it can be shown that the cross-moments are bounded if the factorization (4.5) holds and, for all $\boldsymbol{t} = (t_1, \ldots, t_D)$; $|t_i| < 1$, we have

$$\begin{aligned} &\sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi})\ \left(\boldsymbol{t}^T \boldsymbol{X}_i(\boldsymbol{\pi})\right)^{\boldsymbol{\nu}} < \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi})\ \boldsymbol{X}_i(\boldsymbol{\pi})^{\boldsymbol{\nu}} < C < \infty \\ &\Rightarrow \quad \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi})\left(\boldsymbol{t}^T \boldsymbol{X}_i(\boldsymbol{\pi})\right)^k = \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T \boldsymbol{X}_i(\boldsymbol{\pi})}. \end{aligned} \tag{3.3}$$

Accordingly, we can define the *i-th moment-generating function (MGF)*, as the function $M_{p,\boldsymbol{X}_i} : \mathbb{R}^D \to \mathbb{R}$, where

$$M_{p,\boldsymbol{X}_i}(\boldsymbol{t}) = \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T \boldsymbol{X}_i(\boldsymbol{\pi})}, \tag{3.4}$$

for all $\boldsymbol{t} \in \mathbb{R}^D$, and the cross-moments can be retrieved from the *MGF* by differentiating:

$$\mu^{(\boldsymbol{\nu})}_{p,\boldsymbol{X}_i} = \frac{\partial^{|\boldsymbol{\nu}|} M_{p,\boldsymbol{X}_i}(\boldsymbol{t})}{\partial^{\nu_1} t_1 \ldots \partial^{\nu_D} t_d}\bigg|_{\boldsymbol{t}=\boldsymbol{0}} = \mathcal{D}_{\boldsymbol{\nu}}\Big\{M_{p,\boldsymbol{X}_i}\Big\} = \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi})\ \boldsymbol{X}_i(\boldsymbol{\pi})^{\boldsymbol{\nu}}. \tag{3.5}$$

Note that

$$\mu^{(0)}_{p,\boldsymbol{X}_i} = \mathcal{D}_{\boldsymbol{0}}\big\{M_{p,\boldsymbol{X}_i}\big\} = \Big(\sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T \boldsymbol{X}(\boldsymbol{\pi})}\Big)\Big|_{\boldsymbol{t}=\boldsymbol{0}} = \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi}) = 1. \tag{3.6}$$

The direct cross-moments computation by enumerating all derivations is inefficient since it requires the $\mathcal{O}(|\Omega|)$ operations, and it even becomes infeasible when Ω is an infinite set. On the other hand, if we can derive an expressions for efficient computation of the moment-generating function (4.2), the moment can be retrieved by differentiation.

3.2. Cross-moments computation of SCFG

For the grammar $G = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, p\big)$, we define the *moment-generating grammar* $\widetilde{G} = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, w\big)$ endowed with a topology induced by the supremum norm and with the weight function taking values from the semiring of $\boldsymbol{\alpha}$ continuous functions, $w : \mathcal{R} \to C_{\boldsymbol{\alpha}}$, defined with

$$w(\pi) = p(\pi) e^{\boldsymbol{t}^T \boldsymbol{Y}(\pi)}, \tag{3.7}$$

for all $\pi \in \mathcal{R}$. A derivation $\boldsymbol{\pi} = \pi_1 \cdots \pi_N$ in G with the weight $p(\boldsymbol{\pi}) = p(\pi_1) \cdots p(\pi_N)$ is also a derivation in $\widetilde{G}$, for which the weight is given with

$$w(\boldsymbol{\pi}) = w(\pi_1) \cdots w(\pi_N) = p(\pi_1) e^{\boldsymbol{t}^T \boldsymbol{Y}(\pi_1)} \cdots p(\pi_N) e^{\boldsymbol{t}^T \boldsymbol{Y}(\pi_N)} = p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T \boldsymbol{X}(\boldsymbol{\pi})}. \tag{3.8}$$

The i-th *MGF* can now be expressed as the weights sum of derivation in Ω_i as

$$M_{p,\boldsymbol{X}_i}(\boldsymbol{t}) = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T \boldsymbol{X}_i(\boldsymbol{\pi})} = \sum_{\boldsymbol{\pi} \in \Omega_i} w(\boldsymbol{\pi}). \tag{3.9}$$

Thus, the i-th *MGF* represents the i-th *partition function* of grammar $\widetilde{G}$ and the problem of *MGF* computation is reduced to the problem of the *partition function* computation. By factoring out the first rewriting of each derivation in the sum, using the distributive law, the partition function can be expressed with the system [18]:

$$M_{p,\boldsymbol{X}_i} = \sum_{j=1}^{|\mathcal{R}_i|} w(A_i \to B_{i,j}) \cdot \prod_{k=1}^{|\mathcal{N}|} M_{p,\boldsymbol{X}_k}^{r_k(i,j)}, \tag{3.10}$$

where $1 \leqslant i \leqslant |\mathcal{N}|$ and $r_k(i,j)$ denotes the number of times the nonterminal A_k appears on the right-hand side of the rule $A_i \to B_{i,j}$.

The cross-moments, $\mu_{p,\boldsymbol{X}_i}^{(\boldsymbol{\alpha})} = \mathcal{D}_{\boldsymbol{\alpha}}\Big\{M_{p,\boldsymbol{X}_i}\Big\}$, can be obtained by applying the generalized Leibniz's formula (2.1) to (3.10), which leads us to the following system:

$$\mu_{p,\boldsymbol{X}_i}^{(\boldsymbol{\alpha})} = \sum_{j=1}^{|\mathcal{R}_i|} \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathcal{D}_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\big\{w\big(A_i \to B_{i,j}\big)\big\} \cdot \mathcal{D}_{\boldsymbol{\beta}}\Big\{\prod_{k=1}^{|\mathcal{N}|} M_{p,\boldsymbol{X}_k}^{r_k(i,j)}\Big\}, \tag{3.11}$$

where

$$\mathcal{D}_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\big\{w\big(A_i \to B_{i,j}\big)\big\} = p\big(A_i \to B_{i,j}\big) \cdot \boldsymbol{Y}\big(A_i \to B_{i,j}\big)^{\boldsymbol{\alpha}-\boldsymbol{\beta}}, \tag{3.12}$$

since $w(\pi) = p(\pi) e^{\boldsymbol{t}^T \cdot \boldsymbol{Y}}$, for $\pi \in \mathcal{R}$. According to the generalized Leibniz's rule (2.2), we have

$$\mathcal{D}_{\boldsymbol{\beta}}\Big\{\prod_{k=1}^{|\mathcal{N}|} M_{p,\boldsymbol{X}_k}^{r_k(i,j)}\Big\} = \sum_{\boldsymbol{\gamma}_1 + \cdots + \boldsymbol{\gamma}_{|\mathcal{N}|} = \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_{|\mathcal{N}|}} \prod_{k=1}^{|\mathcal{N}|} \mathcal{D}_{\boldsymbol{\gamma}_k}\big\{M_{p,\boldsymbol{X}_k}^{r_k(i,j)}\big\} \tag{3.13}$$

and

$$\mathcal{D}_{\boldsymbol{\gamma}_k}\big\{M_{p,\boldsymbol{X}_k}^{r_k(i,j)}\big\} = \mathcal{D}_{\boldsymbol{\gamma}_k}\Big\{\prod_{l=1}^{r_k(i,j)} M_{p,\boldsymbol{X}_k}\Big\} = \sum_{\boldsymbol{\delta}_1 + \cdots + \boldsymbol{\delta}_{r_k(i,j)} = \boldsymbol{\gamma}_k} \binom{\boldsymbol{\gamma}_k}{\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{r_k(i,j)}} \prod_{l=1}^{r_k(i,j)} \mu_{p,\boldsymbol{X}_k}^{(\boldsymbol{\delta}_l)}. \tag{3.14}$$

By substituting (3.14) and (3.13) in (3.11), we obtain:

$$\mu_{p,\boldsymbol{X}_i}^{(\alpha)} = \sum_{j=1}^{|\mathcal{R}_i|} \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} Q_{i,j}(\boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{3.15}$$

where

$$(3.16)\quad Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} p(A_i \to B_{i,j}) \cdot \boldsymbol{Y}(A_i \to B_{i,j})^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \cdot$$

$$\sum_{\boldsymbol{\gamma}_1+\cdots+\boldsymbol{\gamma}_{|\mathcal{N}|}=\boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}} \prod_{k=1}^{|\mathcal{N}|} \sum_{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_k(i,j)}=\boldsymbol{\gamma}_k} \binom{\boldsymbol{\gamma}_k}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_k(i,j)}} \prod_{l=1}^{r_k(i,j)} \mu_{p,\boldsymbol{X}_k}^{(\delta_l)}.$$

To solve the system (3.15), we split it into two parts: one dependent and the other not dependent on $\mu_{p,\boldsymbol{X}_i}^{(\alpha)}$:

$$(3.17)\qquad \mu_{p,\boldsymbol{X}_i}^{(\alpha)} = \sum_{j=1}^{|\mathcal{R}_i|} Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\alpha}) + \sum_{j=1}^{|\mathcal{R}_i|} \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where

$$(3.18)\qquad Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\alpha}) = p(A_i \to B_{i,j}) \cdot W_{i,j}(\boldsymbol{\alpha})$$

and
(3.19)

$$W_{i,j}(\boldsymbol{\alpha}) = \sum_{\gamma_1+\cdots+\gamma_{|\mathcal{N}|}=\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}} \prod_{k=1}^{|\mathcal{N}|} \sum_{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_k(i,j)}=\boldsymbol{\gamma}_k} \binom{\boldsymbol{\gamma}_k}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_k(i,j)}} \prod_{l=1}^{r_k(i,j)} \mu_{p,\boldsymbol{X}_k}^{(\delta_l)}.$$

Further, if we set
(3.20)

$$H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) = \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}} \prod_{k=1}^{|\mathcal{N}|} \sum_{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_k(i,j)}=\boldsymbol{\gamma}_k} \binom{\boldsymbol{\gamma}_k}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_k(i,j)}} \prod_{l=1}^{r_k(i,j)} \mu_{p,\boldsymbol{X}_k}^{(\delta_l)},$$

the expression for $W_{\boldsymbol{\alpha}}(B_{i,j})$ can be rewritten as:

$$(3.21)\qquad W_{i,j}(\boldsymbol{\alpha}) = \sum_{n=1}^{|\mathcal{N}|} H_{i,j}^{(n)}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) + \sum_{\substack{\boldsymbol{\gamma}_1+\cdots+\boldsymbol{\gamma}_{|\mathcal{N}|}=\boldsymbol{\alpha} \\ \boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}<\boldsymbol{\alpha}}} H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}),$$

where $H_{i,j}^{(n)}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|})$ stands for $H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|})$ with $\boldsymbol{\gamma}_n = \boldsymbol{\alpha}$ and all other $\boldsymbol{\gamma}$-s equal zero, which, according to (3.20), is
(3.22)

$$H_{i,j}^{(n)}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) = \sum_{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\delta_l)} \cdot \prod_{\substack{k=1 \\ k\neq n}}^{|\mathcal{N}|} \prod_{l=1}^{r_k(i,j)} \mu_{p,\boldsymbol{X}_k}^{(0)}.$$

Finally, after using of $\mu_{p,\boldsymbol{X}_k}^{(0)} = 1$, we obtain

$$(3.23)\qquad H_{i,j}^{(n)}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) = \sum_{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\delta}_l)},$$

which can be rewritten using the same procedure as

(3.24)
$$H_{i,j}^{(n)}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) = \sum_{s=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\alpha})} + \sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_k(i,j)}=\boldsymbol{\alpha} \\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_m}^{(\boldsymbol{\delta}_l)}$$
$$= r_n(i,j)\cdot \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\alpha})} + \sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha} \\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\delta}_l)}.$$

By substitution of (3.24) in (3.21), it follows that:

$$\text{(3.25)} \quad W_{i,j}(\boldsymbol{\alpha}) = \sum_{n=1}^{|\mathcal{N}|} r_n(i,j)\cdot \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\alpha})} +$$
$$\sum_{n=1}^{|\mathcal{N}|} \sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha} \\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\delta}_l)} + \sum_{\substack{\boldsymbol{\gamma}_1+\cdots+\boldsymbol{\gamma}_{|\mathcal{N}|}=\boldsymbol{\alpha} \\ \boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}<\boldsymbol{\alpha}}} H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}).$$

Further, by substitution of (3.25) and (3.18) in (3.17), the moment can be expressed with:

$$\text{(3.26)} \quad \mu_{p,\boldsymbol{X}_i}^{(\boldsymbol{\alpha})} = \sum_{j=1}^{|\mathcal{R}_i|} p(A_i \to B_{i,j}) \sum_{n=1}^{|\mathcal{N}|} r_n(i,j)\cdot \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\alpha})} +$$
$$\sum_{j=1}^{|\mathcal{R}_i|} p(A_i \to B_{i,j}) \sum_{n=1}^{|\mathcal{N}|} \sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha} \\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\delta}_l)} +$$
$$\sum_{j=1}^{|\mathcal{R}_i|} p(A_i \to B_{i,j}) \sum_{\substack{\boldsymbol{\gamma}_1+\cdots+\boldsymbol{\gamma}_{|\mathcal{N}|}=\boldsymbol{\alpha} \\ \boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}<\boldsymbol{\alpha}}} H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) + \sum_{j=1}^{|\mathcal{R}_i|} \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where $H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|})$ and $Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\beta})$ are given with (3.20) and (3.16). Finally, if we introduce

(3.27)
$$c_i^{(\boldsymbol{\alpha})} = \sum_{j=1}^{|\mathcal{R}_i|} p(A_i \to B_{i,j}) \sum_{n=1}^{|\mathcal{N}|} \sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha} \\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<\boldsymbol{\alpha}}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}} \prod_{l=1}^{r_n(i,j)} \mu_{p,\boldsymbol{X}_n}^{(\boldsymbol{\delta}_l)} +$$
$$\sum_{j=1}^{|\mathcal{R}_i|} p(A_i \to B_{i,j}) \sum_{\substack{\boldsymbol{\gamma}_1+\cdots+\boldsymbol{\gamma}_{|\mathcal{N}|}=\boldsymbol{\alpha} \\ \boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}<\boldsymbol{\alpha}}} H_{i,j}(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}) + \sum_{j=1}^{|\mathcal{R}_i|} \sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}} Q_{i,j}(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

the equation (3.26) can be more compactly written as:

$$\mu^{(\boldsymbol{\alpha})}_{p,\boldsymbol{X}_i} = \sum_{n=1}^{|\mathcal{N}|} \mathrm{M}_{i,n} \cdot \mu^{(\boldsymbol{\alpha})}_{p,\boldsymbol{X}_n} + c_i^{(\boldsymbol{\alpha})}, \tag{3.28}$$

or in a matrix form:

$$\mu_p^{(\boldsymbol{\alpha})} = \mathrm{M} \cdot \mu_p^{(\boldsymbol{\alpha})} + \boldsymbol{c}^{(\boldsymbol{\alpha})}, \tag{3.29}$$

where $\mu_p^{(\boldsymbol{\alpha})} = \big[\mu^{(\alpha)}_{p,\boldsymbol{X}_1}, \ldots, \mu^{(\alpha)}_{p,\boldsymbol{X}_{|\mathcal{N}|}}\big]^T$ is the *cross-moment vector* $\boldsymbol{c}^{(\boldsymbol{\alpha})} = \big[c_1^{(\boldsymbol{\alpha})}, \ldots, c_{|\mathcal{N}|}^{(\boldsymbol{\alpha})}\big]^T$ and M is the momentum matrix defined in Theorem 2.1. Since we assume that the condition $\rho(\mathrm{M}) < 1$ given in Theorem 2.1 is satisfied, $\mathrm{I} - \mathrm{M}$ is invertible, and the matrix equation has a unique solution given with

$$\mu_p^{(\boldsymbol{\alpha})} = \big(\mathrm{I} - \mathrm{M}\big)^{-1} \boldsymbol{c}^{(\boldsymbol{\alpha})}. \tag{3.30}$$

Provided that the we have computed the inverse $\big(\mathrm{I} - \mathrm{M}\big)^{-1}$, which does not depend on $\boldsymbol{\alpha}$, the cross-moment is completely determined by the term $\boldsymbol{c}^{(\boldsymbol{\alpha})}$, which depends on all cross-moments of the order lower than $\boldsymbol{\alpha}$ and can be computed using (3.27). In the following sections, we derive $\boldsymbol{c}^{(\boldsymbol{\alpha})}$ for scalar random variables up to the second order, and retrieve the previous results for the first and second order moments [2], [10] as a special case of the equation (3.30).

3.3. First order moments computation of SCFG

In the case of the first order moments $\boldsymbol{\alpha} = (1)$ and the expression (3.9) reduces to the expectation of $\boldsymbol{X}_i$,

$$\mu^{(1)}_{p,\boldsymbol{X}_i} = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi}) \boldsymbol{X}_i(\boldsymbol{\pi}). \tag{3.31}$$

The moment vector, $\mu_p^{(\boldsymbol{\alpha})} = \big[\mu^{(1)}_{p,\boldsymbol{X}_1}, \ldots, \mu^{(1)}_{p,\boldsymbol{X}_{|\mathcal{N}|}}\big]$, is computed as in the equation (3.30),

$$\mu_p^{(1)} = \big(\mathrm{I} - \mathrm{M}\big)^{-1} \boldsymbol{c}^{(1)}, \tag{3.32}$$

where $c^{(1)} = \big[c_1^{(1)}, \ldots, c_{|\mathcal{N}|}^{(1)}\big]^T$. The first and second sum in the expression (3.27) for $c_i^{(\boldsymbol{\alpha})}$ reduce to zero and $c_i^{(1)} = \sum_{j=1}^{|\mathcal{R}_i|} Q_{i,j}(1,0)$, or, after the use of the expression (3.16) for $Q_{i,j}(\boldsymbol{\alpha}, \boldsymbol{\beta})$,

$$c_i^{(1)} = \sum_{j=1}^{|\mathcal{R}_i|} p\big(A_i \to B_{i,j}\big) \cdot \boldsymbol{Y}\big(A_i \to B_{i,j}\big). \tag{3.33}$$

Let $\pi_1 \cdots \pi_N$ be a derivation starting at the start symbol A_1 and ending with a string $\boldsymbol{u} \in \Sigma^*$. If we set $\boldsymbol{Y}\big(A_i \to B_{i,j}\big) = 1$, according to (4.5), we have

$\boldsymbol{X}_1(\pi_1\cdots\pi_N)=\sum_{n=1}^{N}\boldsymbol{Y}\big(\pi_n\big)=N$, i.e. $\boldsymbol{X}_1$ is the length of the derivation. According to the expression (3.31), the moment $\mu^{(1)}_{p,\boldsymbol{X}_1}$ is the expected derivation length which agrees with [2] and [10].

Similarly, if we set $\boldsymbol{Y}\big(A_i\to B_{i,j}\big)=\sum_{n=1}^{|\Sigma|}t_n(i,j)$, where $t_n(i,j)$ denotes the number of terminals in the string $B_{i,j}$, the variable $\boldsymbol{X}_1(\pi_1\cdots\pi_N)$ reduces to the length of the word derived from $\pi_1\cdots\pi_N$. In this case, the moment $\mu^{(1)}_{p,\boldsymbol{X}_1}$ reduces to the expected string length and the formula (3.33) reduces to the result from [2].

3.4. Second order moments computation of SCFG

The formula for the second order moments is somewhat more complicated. In the case when $\boldsymbol{\alpha}=(2)$, $c_i^{(\boldsymbol{\alpha})}$ is reduced to:

(3.34)
$$c_i^{(2)}=\sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{n=1}^{|\mathcal{N}|}\sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=\boldsymbol{\alpha}\\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<2}}\binom{2}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}}\prod_{l=1}^{r_n(i,j)}\mu^{(\boldsymbol{\delta}_l)}_{p,\boldsymbol{X}_n}+$$
$$\sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{\substack{\gamma_1+\cdots+\gamma_{|R|}=2\\ \gamma_1,\ldots,\gamma_{|R|}<2}}H_{i,j}\big(\gamma_1,\ldots,\gamma_{|R|}\big)+\sum_{j=1}^{|\mathcal{R}_i|}Q_{i,j}\big(2,0\big)+\sum_{j=1}^{|\mathcal{R}_i|}Q_{i,j}\big(2,1\big).$$

The first sum in the previous expression can be transformed to:

$$\sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{n=1}^{|\mathcal{N}|}\sum_{\substack{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_n(i,j)}=2\\ \boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}<2}}\binom{2}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_n(i,j)}}\prod_{l=1}^{r_n(i,j)}\mu^{(\boldsymbol{\delta}_l)}_{p,\boldsymbol{X}_n}=$$
$$\sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{n=1}^{|\mathcal{N}|}r_n(i,j)\big(r_n(i,j)-1\big)\cdot\big(\mu^{(1)}_{p,\boldsymbol{X}_n}\big)^2. \tag{3.35}$$

To compute the second sum we introduce $H^{(a,b)}_{i,j}\big(\gamma_1,\ldots,\gamma_{|\mathcal{N}|}\big)$, which is $H_{i,j}\big(\gamma_1,\ldots,\gamma_{|\mathcal{N}|}\big)$ with $\boldsymbol{\gamma}_a=\boldsymbol{\gamma}_b=1$ and with all other $\boldsymbol{\gamma}$-s equals to zero. We have:

$$H^{(a,b)}_{i,j}\big(\gamma_1,\ldots,\gamma_{|\mathcal{N}|}\big)=2\cdot\sum_{\delta_1+\cdots+\delta_{r_a(i,j)}=\boldsymbol{\gamma}_a}\binom{\gamma_a}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_a(i,j)}}\prod_{l=1}^{r_a(i,j)}\mu^{(\boldsymbol{\delta}_l)}_{p,\boldsymbol{X}_k}$$
$$\sum_{\delta_1+\cdots+\delta_{r_b(i,j)}=\boldsymbol{\gamma}_b}\binom{\gamma_b}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_b(i,j)}}\prod_{l=1}^{r_b(i,j)}\mu^{(\boldsymbol{\delta}_l)}_{p,\boldsymbol{X}_a}\prod_{\substack{k=1\\ k\neq a,b}}^{|\mathcal{N}|}\sum_{\boldsymbol{\delta}_1+\cdots+\boldsymbol{\delta}_{r_k(i,j)}=\boldsymbol{\gamma}_k}\binom{\gamma_k}{\boldsymbol{\delta}_1,\ldots,\boldsymbol{\delta}_{r_k(i,j)}}\prod_{l=1}^{r_k(i,j)}\mu^{(\boldsymbol{\delta}_l)}_{p,\boldsymbol{X}_b} \tag{3.36}$$

and
(3.37)

$$H_{i,j}^{(a,b)}\big(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}\big) = 2\cdot\sum_{c=1}^{r_a(i,j)}\mu_{p,\boldsymbol{X}_k}^{(1)}\cdot\sum_{d=1}^{r_b(i,j)}\mu_{p,\boldsymbol{X}_k}^{(1)} = 2\cdot r_a(i,j)\cdot r_b(i,j)\cdot\mu_{p,\boldsymbol{X}_a}^{(1)}\mu_{p,\boldsymbol{X}_b}^{(1)}.$$

By substitution of the second sum in (3.34),

(3.38)

$$\begin{aligned}\sum_{j=1}^{|R_i|}p(A_i\to B_{i,j})\sum_{\substack{\gamma_1+\cdots+\gamma_{|\mathcal{N}|}=2\\ \gamma_1,\ldots,\gamma_{|\mathcal{N}|}<2}}H_{i,j}\big(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}\big) &= \sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{a=1}^{|\mathcal{N}|}\sum_{b=a+1}^{|\mathcal{N}|}H_{i,j}^{(a,b)}\big(\boldsymbol{\gamma}_1,\ldots,\boldsymbol{\gamma}_{|\mathcal{N}|}\big)\\ &= 2\cdot\sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{a=1}^{|\mathcal{N}|}\sum_{b=a+1}^{|\mathcal{N}|}r_a(i,j)\cdot r_b(i,j)\cdot\mu_{p,\boldsymbol{X}_a}^{(1)}\mu_{p,\boldsymbol{X}_b}^{(1)} =\\ &= \sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{a=1}^{|\mathcal{N}|}\sum_{b=1}^{|\mathcal{N}|}r_a(i,j)\cdot r_b(i,j)\cdot\mu_{p,\boldsymbol{X}_a}^{(1)}\mu_{p,\boldsymbol{X}_b}^{(1)} - \sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{n=1}^{|\mathcal{N}|}r_n(i,j)^2\big(\mu_{p,\boldsymbol{X}_n}^{(1)}\big)^2.\end{aligned}$$

Now, (3.34) reduces to

$$c_i^{(2)} = CR_i + \sum_{j=1}^{|\mathcal{R}_i|}Q_{i,j}\big(2,0\big) + \sum_{j=1}^{|\mathcal{R}_i|}Q_{i,j}\big(2,1\big), \tag{3.39}$$

where

(3.40)

$$CR_i = \sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{a=1}^{|\mathcal{N}|}\sum_{b=1}^{|\mathcal{N}|}r_a(i,j)\cdot r_b(i,j)\cdot\mu_{p,\boldsymbol{X}_a}^{(1)}\mu_{p,\boldsymbol{X}_b}^{(1)} - \sum_{j=1}^{|\mathcal{R}_i|}p(A_i\to B_{i,j})\sum_{n=1}^{|\mathcal{N}|}r_n(i,j)\big(\mu_{p,\boldsymbol{X}_n}^{(1)}\big)^2$$

and

$$Q_{i,j}\big(2,0\big) = p\big(A_i\to B_{i,j}\big)\cdot Y\big(A_i\to B_{i,j}\big)^2, \tag{3.41}$$

$$\begin{aligned}Q_{i,j}\big(2,1\big) &= 2\cdot p\big(A_i\to B_{i,j}\big)\cdot Y\big(A_i\to B_{i,j}\big)\sum_{n=1}^{|\mathcal{N}|}\sum_{a=1}^{r_n(i,j)}\mu_{p,\boldsymbol{X}_n}^{(1)}\\ &= 2\cdot p\big(A_i\to B_{i,j}\big)\cdot Y\big(A_i\to B_{i,j}\big)\sum_{n=1}^{|\mathcal{N}|}r_n(i,j)\mu_{p,\boldsymbol{X}_n}^{(1)}.\end{aligned} \tag{3.42}$$

If we set $Y\big(A_i\to B_{i,j}\big) = 1$ for all $A_i\to B_{i,j}\in\mathcal{R}$, $\boldsymbol{X}_1$ becomes derivation length. The formula for computing the second order moments of derivation length is given in [10] and it can be derived from the equation (3.39), since

$$\sum_{j=1}^{|\mathcal{R}_i|} Q_{i,j}(2,0) = \sum_{j=1}^{|\mathcal{R}_i|} p\big(A_i \to B_{i,j}\big) \cdot Y\big(A_i \to B_{i,j}\big)^2 = \tag{3.43}$$
$$= \sum_{j=1}^{|\mathcal{R}_i|} p\big(A_i \to B_{i,j}\big) = 1,$$

$$\sum_{j=1}^{|\mathcal{R}_i|} Q_{i,j}(2,1) = 2 \cdot \sum_{n=1}^{|\mathcal{N}|} \Big(\sum_{j=1}^{|\mathcal{R}_i|} p\big(A_i \to B_{i,j}\big) \cdot r_n(i,j) \Big) \mu_{p,\boldsymbol{X}_n}^{(1)} \tag{3.44}$$
$$= 2 \cdot \sum_{n=1}^{|\mathcal{N}|} e_{i,n} \mu_{p,\boldsymbol{X}_n}^{(1)} = 2 \cdot \mu_{p,\boldsymbol{X}_n}^{(1)} - 2,$$

where the last equation follows from (3.30), and

$$c_i^{(2)} = CR_i + 2 \cdot \mu_{p,\boldsymbol{X}_n}^{(1)} - 1. \tag{3.45}$$

Finally, by substituting (3.45) in (3.30), we obtain

$$\mathrm{m}^{(\boldsymbol{\alpha})}\big\{\boldsymbol{X}\big\} = \big(\mathrm{I} - \mathrm{M}\big)^{-1} \cdot \Big(\mathrm{CR}_i + 2 \cdot \mathrm{m}_1 - \mathbf{1}\Big), \tag{3.46}$$

where $\mathrm{CR}_i = \big[CR_1, \ldots, CR_{|\mathcal{N}|}\big]$ and $\mathbf{1} = \big[1, \ldots, 1\big]$, in agreement with [10].

4. Conditional cross-moments computation of SCFG

In this section we consider the computation of conditional SCFG cross moments. We derive two equivalent versions of the algorithm for the computation, based on inside algorithm. The first one is obtained in the same manner as in Section 3 via the conditional moment-generating function. In the second version we use dynamic programming over the binomial semiring, which relates the algorithm to the previously developed special cases by Hwa [11], [5], for the first order moments, and by Li and Eisner [16] for higher order cross-moments.

4.1. Conditional cross-moments and conditional moment-generating function of SCFG

Let $G = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, p\big)$ be reduced SCFG and let $\Omega_i(\boldsymbol{u})$ be a set of derivations starting at $A_i \in \mathcal{N}$ and finishing at $\boldsymbol{u}$. Let $\boldsymbol{X}_i = \big[X_{i,1}, \ldots, X_{i,D}\big]^T : \Omega_i(\boldsymbol{u}) \to \mathbb{R}^D$, where $\boldsymbol{u} \in \Sigma^*$, be random variables distributed according to the p_i, which are restrictions of p to $\mathcal{R}_i$ ($p_i(\boldsymbol{\pi}) = p(\boldsymbol{\pi})$), for $1 \leqslant i \leqslant |\mathcal{N}|$. The *i-th conditional cross-moment of an order* $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_D)$ of $\boldsymbol{X}_i$ conditioned on $\boldsymbol{u}$ is defined with

$$\mu_{p,\boldsymbol{X}_i|\boldsymbol{u}}^{(\nu)} = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi}) \cdot X_{i,1}(\boldsymbol{\pi})^{\nu_1} \cdots X_{i,D}(\boldsymbol{\pi})^{\nu_d} = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi})\ \boldsymbol{X}_i(\boldsymbol{\pi})^{\boldsymbol{\nu}}. \tag{4.1}$$

The *i-th conditional moment-generating function (MGF)*, conditioned on $\boldsymbol{u}$, $M_{p,\boldsymbol{X}_i|\boldsymbol{u}} : \mathbb{R}^D \to \mathbb{R}$, is defined as:

$$M_{p,\boldsymbol{X}_i|\boldsymbol{u}}(\boldsymbol{t}) = \sum_{\boldsymbol{\pi} \in \Omega_i} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T \boldsymbol{X}_i(\boldsymbol{\pi})}, \tag{4.2}$$

for all $\boldsymbol{t} \in \mathbb{R}^D$, and the cross-moments can be retrieved from the *MGF* by differentiating:

$$\mu^{(\boldsymbol{\nu})}_{p,\boldsymbol{X}_i|\boldsymbol{u}} = \frac{\partial^{|\boldsymbol{\nu}|} M_{p,\boldsymbol{X}_i|\boldsymbol{u}}(\boldsymbol{t})}{\partial^{\nu_1} t_1 \ldots \partial^{\nu_D} t_d}\bigg|_{\boldsymbol{t}=\boldsymbol{0}} = \mathcal{D}_{\boldsymbol{\nu}}\Big\{M_{p,\boldsymbol{X}_i|\boldsymbol{u}}\Big\} = \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi})\ \boldsymbol{X}_i(\boldsymbol{\pi})^{\boldsymbol{\nu}}. \tag{4.3}$$

and

$$\mu^{(0)}_{p,\boldsymbol{X}_i|=}\mathcal{D}_{\boldsymbol{0}}\big\{M_{p,\boldsymbol{X}_i}\big\} = \Big(\sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T\boldsymbol{X}(\boldsymbol{\pi})}\Big)\Big|_{\boldsymbol{t}=\boldsymbol{0}} = \sum_{\boldsymbol{\pi}\in\Omega_i} p(\boldsymbol{\pi}) = 1. \tag{4.4}$$

Let $\boldsymbol{X}_i : \Omega_i(\boldsymbol{u}) \to \mathbb{R}^D$ be represented as the sum of random vectors $\boldsymbol{Y} : \mathcal{R} \to \mathbb{R}^D$:

$$\boldsymbol{X}_i(\pi_1 \cdots \pi_N) = \boldsymbol{Y}\big(\pi_1\big) + \cdots + \boldsymbol{Y}\big(\pi_N\big), \tag{4.5}$$

for all $\pi_1 \cdots \pi_N \in \Omega_i$.

Then, for G we can construct the conditional MGF grammar $\widetilde{G} = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, w\big)$ with a weight function which takes values from the semiring of $\boldsymbol{\nu}$ continuous functions $w : \mathcal{R} \to C_{\boldsymbol{\nu}}$, defined as:

$$w(\pi) = p(\pi) e^{\boldsymbol{t}^T\boldsymbol{Y}(\pi)}, \tag{4.6}$$

for all $\pi \in \mathcal{R}$.

A derivation $\boldsymbol{\pi} = \pi_1 \cdots \pi_N$ in G with the weight function $p(\boldsymbol{\pi}) = p(\pi_1) \cdots p(\pi_N)$ is a derivation in $\widetilde{G}$, with the weight function

$$w(\boldsymbol{\pi}) = w(\pi_1) \cdots w(\pi_N) = p(\pi_1) e^{\boldsymbol{t}^T\boldsymbol{Y}(\pi_1)} \cdots p(\pi_N) e^{\boldsymbol{t}^T\boldsymbol{Y}(\pi_N)} = p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T\boldsymbol{X}_i(\boldsymbol{\pi})}. \tag{4.7}$$

The i-th conditional MGF of the vector random variable $\boldsymbol{X}_i$ can now be computed as a sum of derivations in $\widetilde{G}$:

$$M_{p,\boldsymbol{X}_i|\boldsymbol{u}}(\boldsymbol{t}) = \sum_{\boldsymbol{\pi}\in\Omega(\boldsymbol{u})} p(\boldsymbol{\pi}) e^{\boldsymbol{t}^T\boldsymbol{X}_i(\boldsymbol{\pi})} = \sum_{\boldsymbol{\pi}\in\Omega(\boldsymbol{u})} w(\boldsymbol{\pi}). \tag{4.8}$$

In this manner, the computation of conditional *MGF* can be performed using the inside algorithm [8] over the semiring of $\boldsymbol{\nu}$-continuous functions at zero, which is done in the following section.

4.2. Conditional cross-moments computation of SCFG

Let $\widetilde{G} = \big(\Sigma, \mathcal{N}, A_1, \mathcal{R}, w\big)$ be a weighted context-free grammar over a commutative semiring $\big(\mathbb{K}, +, \cdot, 1, 0\big)$, and $\Omega_i(\boldsymbol{u})$ be a set of all derivations which derive $\boldsymbol{u} \in \Sigma^*$ starting from a nonterminal A_i. The i-th *inside weight* of the weighted grammar $\widetilde{G}$ is the function $\sigma_i : \Sigma^* \to \mathbb{K}$, defined as the sum of weights of all derivations starting from $\Omega_i(\boldsymbol{u})$:

$$\sigma_i(\boldsymbol{u}) = \sum_{\boldsymbol{\pi} \in \Omega_i(\boldsymbol{u})} w(\boldsymbol{\pi}), \tag{4.9}$$

for $1 \leqslant i \leqslant |\mathcal{R}|$ and $\boldsymbol{u} \in \Sigma^*$. Let $A_i \to B_{i,j} \in \mathcal{R}$ and

$$B_{i,j} = \boldsymbol{v}_1 A_{i_1} \boldsymbol{v}_2 A_{i_2} \cdots \boldsymbol{v}_k A_{i_k} v_{k+1}, \tag{4.10}$$

where $\boldsymbol{v}_i \in \Sigma^*$ and $A_{i_n} \in \mathcal{N}$. For the cycle-free reduced grammars the inside weight can be computed using the *inside algorithm* [8] and [24] which, after recursive application of

$$\sigma_i(\boldsymbol{u}) = \sum_{j=1}^{|\mathcal{R}_i|} \sum_{\substack{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \Sigma^* \\ \boldsymbol{u} = \boldsymbol{v}_1 \boldsymbol{u}_1 \boldsymbol{v}_2 \cdots \boldsymbol{v}_k \boldsymbol{u}_k \boldsymbol{v}_{k+1}}} w(A_i \to B_{i,j}) \cdot \prod_{j=1}^{k} \sigma_{i_j}(\boldsymbol{u}_j), \tag{4.11}$$

ends with the equation in which only rules $A_i \to \boldsymbol{u}, \boldsymbol{u} \in \Sigma^*$ appear on the right-hand side:

$$\sigma_i(\boldsymbol{u}) = w(A_i \to \boldsymbol{u}). \tag{4.12}$$

If $\widetilde{G}$ is the moment-generating grammar for G as defined in Section 4.1, the value of the i-th inside weight corresponds to the i-th conditional moment-generating function,

$$\sigma_i(\boldsymbol{u}) = M_{p, \boldsymbol{X}_i|\boldsymbol{u}}(\boldsymbol{t}) \quad \text{and} \quad \mu^{(\boldsymbol{\alpha})}_{p, \boldsymbol{X}_i|\boldsymbol{u}} = \mathcal{D}_{\boldsymbol{\alpha}}\{\sigma_i(\boldsymbol{u})\}, \tag{4.13}$$

for all $1 \leqslant i \leqslant |\mathcal{N}|$. Thus, an efficient computation procedure of the higher order cross-moments can be obtained by applying the generalized Leibniz's formula (2.1) to (4.11), which leads us to the recursive equations:

$$\begin{aligned} \mu^{(\boldsymbol{\alpha})}_{p, \boldsymbol{X}_i|\boldsymbol{u}} = & \sum_{j=1}^{|\mathcal{N}|} \sum_{\substack{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \in \Sigma \\ \boldsymbol{u} = \boldsymbol{v}_1 \boldsymbol{u}_1 \boldsymbol{v}_2 \cdots \boldsymbol{u}_k \boldsymbol{v}_k \boldsymbol{u}_{k+1}}} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} p(A_i \to B_{i,j}) \cdot \boldsymbol{Y}(A_i \to B_{i,j})^{\boldsymbol{\alpha} - \boldsymbol{\beta}} \\ & \times \sum_{\boldsymbol{\gamma}_1 + \cdots \gamma_k = \beta} \prod_{j=1}^{k} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_k} \mu^{(\gamma_j)}_{p, \boldsymbol{X}_{i_j}|\boldsymbol{u_j}} \end{aligned} \tag{4.14}$$

with the base case

$$\mu^{(\gamma)}_{p, \boldsymbol{X}_i|\boldsymbol{u}} = p(A_i \to \boldsymbol{u}) \cdot \boldsymbol{Y}(A_i \to \boldsymbol{u})^{\gamma}. \tag{4.15}$$

Note that the recursive algorithm (4.14)-(4.15) can always be implemented in the iterative manner using some of the procedures considered in [8].

The equations (4.14)-(4.15) can also be expressed in semiring dynamic programming form, as shown in the section below.

4.3. Conditional SCFG cross-moments computation using binomial semiring dynamic programming

Let us introduce the mapping $\mathcal{B}^{(\boldsymbol{\nu})} : C_{\boldsymbol{\nu}} \to \mathbb{R}^{|A_{\boldsymbol{\nu}}|}$, which associates an ordered tuple to any $f \in C_{\boldsymbol{\nu}}$:

$$\mathcal{B}^{(\boldsymbol{\nu})}(f) = \left(\mathcal{D}_{\boldsymbol{\alpha}}(f)\right)_{\boldsymbol{\alpha}\in A_{\boldsymbol{\nu}}}. \tag{4.16}$$

In accordance to Leibniz's formulae we obtain

$$\mathcal{B}^{(\boldsymbol{\nu})}(f+g) = \mathcal{B}^{(\boldsymbol{\nu})}(f) \oplus \mathcal{B}^{(\boldsymbol{\nu})}(g), \tag{4.17}$$

$$\mathcal{B}^{(\boldsymbol{\nu})}(f\cdot g) = \mathcal{B}^{(\boldsymbol{\nu})}(f) \otimes \mathcal{B}^{(\boldsymbol{\nu})}(g), \tag{4.18}$$

where the $\oplus$ and $\otimes$ are defined with

$$u \oplus v = \left(u^{(\boldsymbol{\alpha})} + v^{(\boldsymbol{\alpha})}\right)_{\boldsymbol{\alpha}\in A_{\boldsymbol{\nu}}}, \tag{4.19}$$

$$u \otimes v = \left(\textstyle\sum_{\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} u^{(\boldsymbol{\beta})}\cdot v^{(\boldsymbol{\alpha}-\boldsymbol{\beta})} \right)_{\boldsymbol{\alpha}\in A_{\boldsymbol{\nu}}}, \tag{4.20}$$

for all $u, v \in \mathbb{R}^{|\mathcal{A}_{\boldsymbol{\nu}}|}$. Therefore, the mapping $\mathcal{B}^{(\boldsymbol{\nu})}$ maps the semiring of $\boldsymbol{\nu}$ continuous functions in the binomial semiring of an order $\boldsymbol{\nu}$ [23], which is defined as the tuple $(\mathbb{R}^{|\mathcal{A}_{\boldsymbol{\nu}}|}, \oplus, \otimes, \mathbf{0}, \mathbf{1})$, where the identities for $\oplus$ and $\otimes$ are respectively given with

$$\begin{aligned} \mathbf{0} &= (\underbrace{0, 0, \ldots, 0}_{|\mathcal{A}_{\boldsymbol{\nu}}|\text{ times}}) \\ \mathbf{1} &= (1, \underbrace{0, \ldots, 0}_{|\mathcal{A}_{\boldsymbol{\nu}}|-1\text{ times}}). \end{aligned}$$

By using of $\mathcal{B}^{(\boldsymbol{\nu})}$, all the cross moments can be represented as an order tuple

$$\mathcal{B}^{(\boldsymbol{\nu})}\{\sigma_i(\boldsymbol{u})\} = \left(\mu^{(\boldsymbol{\alpha})}_{p,\boldsymbol{X}_i|\boldsymbol{u}} \right)_{\boldsymbol{\alpha}\in A_{\boldsymbol{\nu}}}, \tag{4.21}$$

where $\sigma_i(\boldsymbol{u})$ stands for the inside weight. In this way, the equations for the cross moments computation can be represented in the binomial semiring dynamic form

$$\mathcal{B}^{(\boldsymbol{\nu})}\{\sigma_i(\boldsymbol{u})\} = \bigoplus_{j=1}^{|\mathcal{R}_i|} \bigoplus_{\substack{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\in\Sigma^* \\ \boldsymbol{u}=\boldsymbol{v}_1\boldsymbol{u}_1\boldsymbol{v}_2\cdots\boldsymbol{v}_k\boldsymbol{u}_k\boldsymbol{v}_{k+1}}} w(A_i \to B_{i,j}) \otimes \bigotimes_{j=1}^{k} \mathcal{B}^{(\boldsymbol{\nu})}\{\sigma_{i_j}(\boldsymbol{u}_j)\}, \tag{4.22}$$

with the base case

$$\mathcal{B}^{(\boldsymbol{\nu})}\{\sigma_i(\boldsymbol{u})\} = \mathcal{B}^{(\boldsymbol{\nu})}\{w(A_i \to \boldsymbol{u})\}. \tag{4.23}$$

The algorithm given by equations (4.14)-(4.15) or, equivalently, (4.22)-(4.23) can be considered as a generalization of the algorithms by Li and Eisner [16] for the cross-moments of order $\boldsymbol{\alpha} = (1,1)$ and by Hwa [11] for the cross-moments of order $\boldsymbol{\alpha} = (1)$. Li and Eisner introduced the second order entropy semiring [13], which is the binomial semiring of the order $(1,1)$, and ran the inside algorithm on it. The algorithm for the moments of order $\boldsymbol{\alpha} = (1)$ is provided by Hwa [11], where conditional entropy is considered. As noted in [5], Hwa's algorithm can be obtained by running the inside algorithm over the first order entropy semiring [7], [13], [12], [14] which is the binomial semiring of the order (1). Hwa's algorithm is considered in the following subsection.

4.4. First order conditional moments computation of SCFG

In the case of first order conditional moments $\boldsymbol{\alpha} = (1)$, the i-th conditional cross-moment (4.1) is the expectation of $\boldsymbol{X}_i$,

$$\mu^{(1)}_{p,\boldsymbol{X}_1|\boldsymbol{u}} = \sum_{\boldsymbol{\pi}\in\Omega_i(\boldsymbol{u})} p(\boldsymbol{\pi})\boldsymbol{X}_i(\boldsymbol{\pi}). \tag{4.24}$$

In this case, the recursive equations (4.14)-(4.15) reduce to

$$\mu^{(0)}_{p,\boldsymbol{X}_i|\boldsymbol{u}} = \sum_{j=1}^{|\mathcal{N}|} \sum_{\substack{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\in\Sigma \\ \boldsymbol{u}=\boldsymbol{v}_1\boldsymbol{u}_1\boldsymbol{v}_2\cdots\boldsymbol{u}_k\boldsymbol{v}_k\boldsymbol{u}_{k+1}}} p\big(A_i \to B_{i,j}\big)\cdot\prod_{j=1}^{k}\mu^{(0)}_{p,\boldsymbol{X}_{i_j}|\boldsymbol{u_j}}, \tag{4.25}$$

(4.26)

$$\mu^{(1)}_{p,\boldsymbol{X}_i|\boldsymbol{u}} = \sum_{j=1}^{|\mathcal{N}|} \sum_{\substack{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\in\Sigma \\ \boldsymbol{u}=\boldsymbol{v}_1\boldsymbol{u}_1\boldsymbol{v}_2\cdots\boldsymbol{u}_k\boldsymbol{v}_k\boldsymbol{u}_{k+1}}} p\big(A_i \to B_{i,j}\big)\cdot\boldsymbol{Y}\big(A_i \to B_{i,j}\big)\cdot\prod_{j=1}^{k}\mu^{(0)}_{p,\boldsymbol{X}_{i_j}|\boldsymbol{u_j}} + p\big(A_i \to B_{i,j}\big)\cdot\sum_{n=1}^{k}\mu^{(1)}_{p,\boldsymbol{X}_{i_n}|\boldsymbol{u_n}}\prod_{\substack{j=1\\ j\neq n}}^{k}\mu^{(0)}_{p,\boldsymbol{X}_{i_j}|\boldsymbol{u_j}},$$

with the base case:

$$\mu^{(0)}_{p,\boldsymbol{X}_i|\boldsymbol{u}} = p\big(A_i \to \boldsymbol{u}\big), \qquad \mu^{(1)}_{p,\boldsymbol{X}_i|\boldsymbol{u}} = p\big(A_i \to \boldsymbol{u}\big)\cdot Y\big(A_i \to \boldsymbol{u}\big). \tag{4.27}$$

In [11], Hwa considered the conditional entropy of the grammar given in Chomsky form for which $B_{i,j} = v_1 A_{i_1} v_2 A_{i_2} v_3$ and v_1, v_2, v_3 are equal to the empty string. The conditional entropy is obtained as the moment $\mu^{(1)}_{p,\boldsymbol{X}_1|\boldsymbol{u}}$, where $\boldsymbol{X}_1(\boldsymbol{\pi}) = -\log p(\boldsymbol{\pi})$, for all $\boldsymbol{\pi} \in \Omega_1$, and Hwa's algorithm can be retrieved by imposing Chomsky form condition in (4.14)-(4.15), with $\boldsymbol{Y}(\pi_i) = -\log p(\pi_i)$.

5. Conclusion

In this paper we considered the problem of computing the cross-moments and the conditional cross-moments of a vector variable represented by a stochastic context-free grammar. We proposed new algorithms, derived by differentiation of the recursive equations for the moment-generating function [22], which are obtained from the algorithms for computing the partition function of a SCFG [18] for the cross-moments and with the inside algorithm [15], [8] for the conditional cross-moments. In this way, we obtained the algorithms which can be considered as a generalization of the previously developed formulas for moments [10], [26] and conditional cross-moments [11], [16].

The computation of cross-moments may be demanding and often infeasible. The proposed method for its solution via the computation of moment-generating function turned out to be very elegant and powerful. In the future, we hope that this idea can successfully be reused in the theory of formal languages for the computation of cross moments of string and tree automata [3], [6].

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NULL CONTROLLABILITY OF DEGENERATE NONAUTONOMOUS PARABOLIC EQUATIONS

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Abstract. In this paper we are interested in the study of the null controllability for the one dimensional degenerate nonautonomous parabolic equation

$$u_t - M(t)(a(x)u_x)_x = h\chi_\omega, \qquad (x,t) \in Q = (0,1) \times (0,T),$$

where $\omega = (x_1, x_2)$ is a small nonempty open subset in $(0,1)$, $h \in L^2(\omega \times (0,T))$, the diffusion coefficients $a(\cdot)$ is degenerate at $x = 0$ and $M(\cdot)$ is nondegenerate on $[0,T]$. Also, the boundary conditions are considered to be Dirichlet- or Neumann-type related to the degeneracy rate of $a(\cdot)$. Under some conditions on the functions $a(\cdot)$ and $M(\cdot)$, we prove some global Carleman estimates which will yield the observability inequality of the associated adjoint system and, equivalently, the null controllability of our parabolic equation.

Keywords. Null controllability; nonautonomous parabolic equation; Carleman estimates.

1. Introduction

The purpose of this paper is to establish the null controllability for the linear nonautonomous degenerate parabolic equation

$$(1.1) \qquad \begin{cases} u_t - M(t)(a(x)u_x)_x = h\chi_\omega, & (x,t) \in Q \\ u(1,t) = u(0,t) = 0, \quad t \in (0,T) \\ \text{or} \\ u(1,t) = (au_x)(0,t) = 0, \quad t \in (0,T) \\ u(x,0) = u_0(x), \qquad x \in (0,1), \end{cases}$$

where $\omega = (x_1, x_2)$ is a nonempty open subinterval of $(0,1)$, $Q = (0,1) \times (0,T)$, $a(\cdot)$ and $M(\cdot)$ are time and space diffusion coefficients, the initial condition u_0 is given

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in $L^2(0,1)$, and $h \in L^2(\omega \times (0,T))$ is the control function acting on ω.
The null controllability of nondegenerate parabolic equations have been widely studied in the last years (see in particular [6], [13], [14], [18], [20]). On the other hand, very few results are known in the case of autonomous ($M(t) = 1$) degenerate equations; see [3], [4], [5], [8], [19]. The main tool to study the null controllability of the above parabolic equations is the Carleman estimates. These last estimates are used to show the observability inequality of the adjoint parabolic equations, which is equivalent to the null controllability of the above parabolic equations. The Carleman estimates are the main results of the above references. Recently in [21], the authors established a new Carleman estimate for the autonomous degenerate equations under some general conditions on the degenerate diffusion coefficient a.

The main objective of this paper is the null controllability of a one-dimensional parabolic equation when the diffusion coefficient is allowed to be degenerate at the boundary point $x = 0$ of the interval $I = (0,1)$, and it might be non-autonomous. This can help to study a local null controllability result for a nonlinear degenerate parabolic PDE with nonlocal nonlinearities which has important physical motivations. In particular there exists several examples of real world physical models where nonlocal terms appear naturally:

• In the case of migration of populations, for instance bacteria in a container, we may have instead of M:

$$M(t) = \tilde{M}\left(\int_0^1 u(x,t)\,dx\right)$$

Other more general M can also be found in practice, for instance

$$M(t) = \tilde{M}\left(\int_0^1 u(x,t)\,dx, \int_0^1 u_x(x,t)\,dx\right)$$

• In the context of reaction-diffusion systems, terms of this kind

$$M(t) = \tilde{M}\left(\int_0^1 |u_x(x,t)|^2\,dx\right)$$

appear in the parabolic Kirchhoff equation (see **[10]**).

2. Assumptions and Preliminary Results

In order to study the null controllability of equations 1.1, we make the following assumptions on the coefficients $M(\cdot)$ and $a(\cdot)$.

Hypothesis 1.

1. *M is continuous on $(0,T)$ and there exist two positive constants α_0, β_0 independent of T such that*

$$0 < \alpha_0 \le M(t) \le \beta_0, \quad t \in (0,T),$$

2. *M is derivable on $(0,T)$ and there exists a positive constant γ_0 independent of T such that*
$$|M'(t)| \leq \gamma_0, \quad t \in (0,T).$$

Hypothesis 2.

1. *$a \in C([0,1]) \cap C^1((0,1])$, $a(x) > 0$ in $(0,1]$ and $a(0) = 0$,*
2. *there exists $\alpha \in (0,2)$ such that $xa'(x) \leq \alpha a(x)$ for every $x \in [0,1]$,*
3. *if $\alpha \in [1,2)$, there exist $m > 0$ and $\delta_0 > 0$ such that for every $x \in [0,\delta_0]$, we have*
$$a(x) \geq m \sup_{0 \leq y \leq x} a(y).$$

Remark 2.1. It should be noted that Hypothesis 2 appeared for the first time in [21]. It is weaker than the condition given in [5]. In [21] the author also proved that under Hypothesis 2 the classical Hardy-inequality does not hold in general, (see [21, Example 3]) and they proposed an improved Hardy inequality (see Proposition 2.2).

As in [5, 21, 24], for the well-posedness of the problem, the natural setting involves the space
$$H_a^1(0,1) := \{u \in L^2(0,1) \cap H_{loc}^1(0,1) : \int_0^1 a(x) u_x^2 dx < \infty\},$$
which is a Hilbert space for the scalar product
$$\langle u, v\rangle := \int_0^1 uv + a(x) u_x v_x dx, \quad u, v \in H_a^1(0,1). \tag{2.1}$$

For any $u \in H_a^1(0,1)$, the trace of u at $x = 1$ obviously makes sense, which allows us to consider the homogeneous Dirichlet condition at $x = 1$. On the other hand, the trace of u at $x = 0$ only makes sense when $0 \leq \alpha < 1$. However, for $\alpha \geq 1$, the trace at $x = 0$ does not make sense anymore, so one chooses a suitable Neumann boundary condition in this case (see, for example, Lemma 10 of [21]). This leads to the introduction of the following space $H_{a,0}^1(0,1)$ depending on the value of α:

1. For $0 \leq \alpha < 1$,
$$H_{a,0}^1(0,1) := \{u \in H_a^1(0,1) : u(1) = u(0) = 0\}.$$
2. For $1 \leq \alpha < 2$,
$$H_{a,0}^1(0,1) := \{u \in H_a^1(0,1) : u(1) = 0\}.$$

In order to study the well-posedeness of 1.1, we define the operator $(A(t), D(A(t)))$ by

$$A(t)u := M(t)Au := M(t)(a(x)u_x)_x, \tag{2.2}$$

endowed with the domain

$$D(A(t)) = D(A) = \{u \in H^1_{a,0}(0,1) \cap H^2_{\text{loc}}((0,1]) : (a(x)u_x)_x \in L^2(0,1)\}, t \in [0,T].$$

Remark 2.2. The domain $D(A)$ may also be characterized in the case of $\alpha \in [0,1)$ by

$$D(A) := \{u \in L^2(0,1) \cap H^2_{\text{loc}}((0,1]) : a(x)u_x \in H^1(0,1) \text{ and } \quad u(0) = u(1) = 0\},$$

and in the case of $\alpha \in [1,2)$ by

$$D(A) := \{u \in L^2(0,1) \cap H^2_{\text{loc}}((0,1]) : a(x)u_x \in H^1(0,1) \text{ and } \quad (a(x)u_x)(0) = 0 = u(1)\}.$$

Some properties of the operator A are given in the following proposition, see [7].

Proposition 2.1. *The operator $(A, D(A))$ is closed, self-adjoint and negative with the dense domain in $L^2(0,1)$. Hence A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0,1)$.*

From the assumptions on $M(\cdot)$, we can check that the family of operators $(A(t), D(A(t))), 0 \leq t \leq T$, satisfies the Acquistapace-Terreni conditions (see [1, 2]), thereby generating an evolution family $U(t,s), t \geq s \geq 0$. More precisely, for $t \geq s$ the map $(t,s) \mapsto U(t,s) \in \mathcal{L}(L^2(0,1))$ is continuous and continuously differentiable in t, $U(t,s)L^2(0,1) \subset D(A(t))$, and $\partial U(t,s) = A(t)U(t,s)$. We further have $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = I$ for $t \geq s \geq r \geq 0$. Moreover, for $s \in \mathbb{R}$ and $x \in D(A(s))$, the function $t \mapsto u(t) = U(t,s)x$ is continuous at $t = s$ and u is the unique solution in $C([s,\infty), L^2(0,1)) \cap C^1((s,\infty), L^2(0,1))$ of the Cauchy problem $u'(t) = A(t)u(t), t > s, u(s) = x$. These facts have been established in [1, 2].

The problem 1.1 is well-posed in the sense of the following theorem.

Theorem 2.1. *For all $h \in L^2(\omega \times (0,T))$ and $u_0 \in L^2(0,1)$, the problem 1.1 has a unique weak solution*

$$u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_a(0,1)).$$

Moreover, if $u_0 \in D(A)$, then

$$u \in H^1(0,T; L^2(0,1)) \cap L^2(0,T; D(A)) \cap C([0,T]; H^1_a(0,1)).$$

Throughout this paper we use the following improved Hardy inequality taken from [21, Theorem 2.1], which will be the key ingredient in the proof of our Carleman estimate.

Proposition 2.2. *For all $\eta > 0$ and $0 < \gamma < 2 - \alpha$, there exists some positive constant $C_0(a, \alpha, \gamma, \eta) > 0$ such that for all $u \in H^1_{a,0}(0,1)$, the following inequality holds*

$$\int_0^1 a(x)u_x^2 dx + C_0 \int_0^1 u^2 dx \geq \frac{a(1)(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx + \eta \int_0^1 \frac{u^2}{x^\gamma} dx. \tag{2.3}$$

3. Carleman Estimates

In this section, we prove a crucial Carleman estimate, which will be useful for proving the observability inequality for the adjoint problem of 1.1. For this purpose, let us consider the parabolic problem

$$(3.1)\qquad \begin{cases} v_t + A(t)v = f, & (x,t) \in Q \\ v(1,t) = v(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in (0,1) \\ v(1,t) = (av_x)(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in [1,2), \\ v(x,T) = v_T(x), & x \in (0,1). \end{cases}$$

Now, we consider $0 < \gamma < 2 - \alpha$ and $\varphi(x,t) = \theta(t)p(x)$. Here

$$(3.2)\qquad \theta(t) = [t(T-t)]^{-k}, k = 1 + 2/\gamma, \quad p(x) = \frac{c_1}{2-\alpha}\Big(\int_0^x \frac{y}{a(y)}dy - c_2\Big)$$

where $c_1 > 0$ and $c_2 > \frac{1}{a(1)(2-\alpha)}$ such that $p(x) < 0$ for all $x \in [0,1]$. Observe that there exists some constant $c = c(T) > 0$ such that

$$(3.3)\qquad |\theta_t| \le c\theta^{1+1/k}, \quad |\theta_{tt}| \le c\theta^{1+2/k} \quad \text{in} \quad (0,T).$$

We have the following main result.

Theorem 3.1. *Assume that the functions $a(\cdot)$ and $M(\cdot)$ satisfy Hypotheses 1 and 2 and let $T > 0$. For every $0 < \gamma < 2-\alpha$ there exists $s_0 = s_0(T, a, \alpha, \gamma, \beta_0, \alpha_0, \gamma_0) > 0$ such that for all $s \ge s_0$ and all solutions v of (3.1), we have*

$$\frac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3 \frac{x^2}{a(x)} v^2 e^{2s\varphi}dxdt + s\int\int_Q \theta a(x) v_x^2 e^{2s\varphi}dxdt + sa(1)(1-\alpha)^2\int\int_Q \theta \frac{v^2}{x^{2-\alpha}} e^{2s\varphi}dxdt$$
$$+s\int\int_Q \theta\frac{v^2}{x^\gamma}e^{2s\varphi}dxdt \le \frac{18}{\alpha_0^2}\Big(\int\int_Q f^2 e^{2s\varphi}dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta v_x^2(1,t)e^{2s\varphi(1,t)}dt\Big).$$

Proof For the proof, let us define the function $w = e^{s\varphi}v$, where $s > 0$ and v is the solution to (3.1). Then w satisfies

$$\begin{cases} (e^{-s\varphi}w)_t + M(t)\Big(a(x)(e^{-s\varphi}w)_x\Big)_x = f, & (x,t) \in Q, \\ w(1,t) = w(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in (0,1), \\ w(1,t) = (aw_x)(0,t) = s(\varphi_x a w)(0,t) = 0, \quad t \in (0,T), & \text{in the case } \alpha \in [1,2), \\ w(x,T) = w(x,0) = 0, & x \in (0,1). \end{cases}$$
(3.4)

Set

$$Lv := v_t + M(t)(a(x)v_x)_x, \quad L_s w := e^{s\varphi}L(e^{-s\varphi}w).$$

$$L_s w := L_1 w + L_2 w$$

where

$$\begin{aligned} L_1w &:= M(t)(a(x)w_x)_x - s\varphi_t w + s^2 M(t)a(x)\varphi_x^2 w, \\ L_2w &:= w_t - 2sM(t)a(x)\varphi_x w_x - sM(t)(a(x)\varphi_x)_x w. \end{aligned} \tag{3.5}$$

Therefore, we have

$$2\langle L_1w, L_2w\rangle \le \|L_1w + L_2w\|^2 = \|fe^{s\varphi}\|^2, \tag{3.6}$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the usual norm and scalar product in $L^2(Q)$, respectively. The proof of Theorem 3.1 is based on the computation of the scalar product (L_1w, L_2w) which comes in the following lemma.

Lemma 3.1. *The scalar product $\langle L_1w, L_2w\rangle$ may be written as a sum of the distributed term $(d.t)$ and boundary term $(b.t)$, where the distributed term $(d.t)$ is given by*

$$\begin{aligned} (d.t) \quad = \quad & -2s^2 \int\!\!\int_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\int\!\!\int_Q \theta_{tt} p w^2 dxdt \\ & + s\int\!\!\int_Q \theta(2ap_{xx} + a'p_x)a(x)M^2(t)w_x^2 dxdt \\ & + s^3 \int\!\!\int_Q \theta^3(2ap_{xx} + a'p_x)a(x)p_x^2 M^2(t)w^2 dxdt \\ & + \frac{1}{2}\int\!\!\int_Q M'(t)a(x)w_x^2 dxdt - \frac{s^2}{2}\int\!\!\int_Q M'(t)\theta^2 a(x)p_x^2 w^2 dxdt \end{aligned} \tag{3.7}$$

whereas the boundary term $(b.t)$ is given by

$$(b.t) = -s\int_0^T \Big[M^2(t)\theta p_x (a(x)w_x)^2\Big]_0^1 dt. \tag{3.8}$$

Proof To simplify the notation, we will denote by $(L_iw)_j, (1 \le i \le 2, 1 \le j \le 3)$ the j^{th} term in the expression of L_iw given in (3.5). We will develop nine terms appearing in the product scalar $\langle L_1w, L_2w\rangle$. For this, we will integrate by parts several times respect to the space and time variables. First we have

$$\begin{aligned} \langle (L_1w)_1, (L_2w)_1\rangle &= \int\!\!\int_Q M(t)(a(x)w_x)_x w_t dxdt \\ \text{(3.9)} \quad &= \int_0^T \Big[M(t)a(x)w_x w_t\Big]_0^1 dt - \int\!\!\int_Q M(t)a(x)w_x w_{tx} dxdt \\ &= \int_0^T \Big[M(t)a(x)w_x w_t\Big]_0^1 dt - \frac{1}{2}\int_0^1 \Big[M(t)a(x)w_x^2\Big]_0^T dx + \frac{1}{2}\int\!\!\int_Q M'(t)a(x)w_x^2 dxdt. \end{aligned}$$

Then

$$\begin{aligned}\langle (L_1w)_2,(L_2w)_1\rangle &= -s\int\int_Q \varphi_t w w_t dxdt\\ &= -\frac{s}{2}\int_0^1 \Big[\varphi_t w^2\Big]_0^T dx + \frac{s}{2}\int\int_Q \varphi_{tt} w^2 dxdt \qquad (3.10)\\ &= -\frac{s}{2}\int_0^1 \Big[\varphi_t w^2\Big]_0^T dx + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt.\end{aligned}$$

We also have

$$\begin{aligned}\langle (L_1w)_3,(L_2w)_1\rangle &= s^2\int\int_Q a(x)M(t)\varphi_x^2 w w_t dxdt = \tfrac{s^2}{2}\int_0^1 \Big[a(x)M(t)\varphi_x^2 w^2\Big]_0^T dx\\ &-s^2\int\int_Q a(x)M(t)\varphi_x\varphi_{xt} w^2 dxdt - \tfrac{s^2}{2}\int\int_Q a(x)M'(t)\varphi_x^2 w^2 dxdt\\ &= \tfrac{s^2}{2}\int_0^1 \Big[a(x)M(t)\varphi_x^2 w^2\Big]_0^T dx - s^2\int\int_Q a(x)M(t)p_x^2\theta\theta_t w^2 dxdt \qquad (3.11)\\ &-\tfrac{s^2}{2}\int\int_Q a(x)M'(t)\theta^2 p_x^2 w^2 dxdt.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle (L_1w)_1,(L_2w)_2\rangle &= -2s\int\int_Q M^2(t)\varphi_x(a(x)w_x)(a(x)w_x)_x dxdt\\ &= -s\int_0^T \Big[M^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt + s\int\int_Q M^2(t)\varphi_{xx}a^2(x)w_x^2 dxdt \qquad (3.12)\\ &= -s\int_0^T \Big[M^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt + s\int\int_Q M^2(t)\theta p_{xx}a^2(x)w_x^2 dxdt.\end{aligned}$$

We also have

$$\begin{aligned}\langle (L_1w)_2,(L_2w)_2\rangle &= 2s^2\int\int_Q M(t)a(x)\varphi_x\varphi_t w w_x dxdt\\ &= s^2\int_0^T \Big[M(t)a(x)\varphi_t\varphi_x w^2\Big]_0^1 dt - s^2\int\int_Q M(t)a(x)\varphi_{tx}\varphi_x w^2 dxdt\\ &-s^2\int\int_Q M(t)\varphi_t(a(x)\varphi_x)_x w^2 dxdt\end{aligned}$$

$$= s^2 \int_0^T \Big[M(t)a(x)\varphi_t\varphi_x w^2\Big]_0^1 dt - s^2 \int\int_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt \tag{3.13}$$
$$-s^2 \int\int_Q M(t)\theta_t p(a(x)\varphi_x)_x w^2 dxdt.$$

Additionally, we find that

$$\langle (L_1w)_3, (L_2w)_2\rangle = -2s^3 \int\int_Q M^2(t)a^2(x)\varphi_x^3\varphi_t w w_x dxdt \tag{3.14}$$

$$= -s^3 \int_0^T \Big[M^2(t)a^2(x)\varphi_x^3 w^2\Big]_0^1 dt + s^3 \int\int_Q M^2(t)\Big[2aa'\varphi_x + 3a^2\varphi_{xx}\Big]\varphi_x^2 w^2 dxdt.$$

Let us now consider the scalar product

$$\langle (L_1w)_1, (L_2w)_3\rangle = -s \int\int_Q M^2(t)(a(x)w_x)_x(a(x)\varphi_x)_x w dxdt \tag{3.15}$$

$$= -s \int_0^T \Big[M^2(t)(a(x)\varphi_x)_x a(x)w_x w\Big]_0^1 dt + s \int\int_Q M^2(t)(a(x)\varphi_x)_{xx} a(x) w w_x dxdt$$
$$+s \int\int_Q M^2(t)(a(x)\varphi_x)_x a(x) w_x^2 dxdt$$

$$= -s \int_0^T \Big[M^2(t)(a(x)\varphi_x)_x a(x) w w_x\Big]_0^1 dt + s \int\int_Q M^2(t)(a(x)\varphi_x)_x a(x) w_x^2 dxdt,$$

since $(a(x)\varphi_x)_{xx} = 0$.
Furthemore

$$\langle (L_1w)_2, (L_2w)_3\rangle = s^2 \int\int_Q M(t)\varphi_t(a(x)\varphi_x)_x w^2 dxdt. \tag{3.16}$$

Finally, we have

$$\langle (L_1w)_3, (L_2w)_3\rangle = -s^3 \int\int_Q M^2(t)a(x)\varphi_x^2(a(x)\varphi_x)_x w^2 dxdt. \tag{3.17}$$

Additionally (3.9)-(3.17), we find that

$$(d.t) \quad = \quad -2s^2 \int\int_Q M(t)a(x)\theta\theta_t p_x^2 w^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt$$

$$+s\int\int_Q \theta(2ap_{xx}+a'p_x)a(x)M^2(t)w_x^2dxdt$$

$$+s^3\int\int_Q \theta^3(2ap_{xx}+a'p_x)a(x)p_x^2M^2(t)w^2dxdt \tag{3.18}$$

$$+\frac{1}{2}\int\int_Q M'(t)a(x)w_x^2dxdt-\frac{s^2}{2}\int\int_Q M'(t)\theta^2a(x)p_x^2w^2dxdt,$$

and

$$\begin{aligned}(b.t) \;&= \int_0^T \Big[M(t)a(x)w_xw_t - sM^2(t)\varphi_x(a(x)w_x)^2+s^2M(t)a(x)\varphi_t\varphi_xw^2 \\ &\quad -s^3M^2(t)a^2(x)\varphi_x^3w^2 - sM^2(t)(a(x)\varphi_x)_xa(x)ww_x\Big]_0^1 dt \\ &\quad +\int_0^1\Big[-\tfrac{1}{2}M(t)a(x)w_x^2-\tfrac{s}{2}\varphi_tw^2+\tfrac{s^2}{2}a(x)M(t)\varphi_x^2w^2\Big]_0^T dx \\ &= -\int_0^T\Big[sM^2(t)\varphi_x(a(x)w_x)^2\Big]_0^1 dt.\end{aligned} \tag{3.19}$$

The proof of (3.19) is similar to that in [5] and the fact was used that $M(\cdot)$ is a bounded function. Now we put $(d.t)=A+B$, where

$$\begin{aligned}A = \;& -2s^2\int\int_Q M(t)a(x)\theta\theta_tp_x^2w^2dxdt+\tfrac{s}{2}\int\int_Q\theta_{tt}pw^2dxdt \\ &+s\int\int_Q\theta(2ap_{xx}+a'p_x)a(x)M^2(t)w_x^2dxdt \\ &+s^3\int\int_Q\theta^3(2ap_{xx}+a'p_x)a(x)p_x^2M^2(t)w^2dxdt,\end{aligned} \tag{3.20}$$

and

$$B=\frac{1}{2}\int\int_Q M'(t)a(x)w_x^2dxdt-\frac{s^2}{2}\int\int_Q M'(t)\theta^2a(x)p_x^2w^2dxdt. \tag{3.21}$$

Observe that

$$A+B\le\frac{1}{2}\|fe^{s\varphi}\|^2-(b.t). \tag{3.22}$$

The crucial step is to prove the following estimate.

Lemma 3.2. *There exists a positive constant $s_1=s_1(T,a,\alpha,\alpha_0,\beta_0,\gamma,\gamma_0)>0$ such that for all $s\ge s_1$ we have,*

$$\begin{aligned}A+B\ge\;& \tfrac{s^3\alpha_0^2}{4(2-\alpha)^2}\int\int_Q\theta^3\tfrac{x^2}{a(x)}w^2dxdt+s\tfrac{\alpha_0^2}{4}\int\int_Q\theta a(x)w_x^2dxdt \\ &+\tfrac{sa(1)(1-\alpha)^2\alpha_0^2}{4}\int\int_Q\theta\tfrac{w^2}{x^{2-\alpha}}dxdt+\tfrac{s}{4}\alpha_0^2\int\int_Q\theta\tfrac{w^2}{x^\gamma}dxdt.\end{aligned} \tag{3.23}$$

Proof By the assumption $xa'(x) \leq \alpha a(x)$ and the fact that $p_x = \frac{c_1 x}{(2-\alpha)a(x)}$, and the observation that

$$\begin{aligned} 2ap_{xx} + a'p_x &= \tfrac{c_1}{2-\alpha}\left(\tfrac{2a(x)-xa'(x)}{a(x)}\right) \\ &\geq \tfrac{c_1}{2-\alpha}\left(\tfrac{2a(x)-\alpha a(x)}{a(x)}\right) = c_1 \end{aligned} \tag{3.24}$$

one can estimate A in the following way

$$\begin{aligned} A \geq &-\frac{2s^2c_1^2}{(2-\alpha)^2}\beta_0 \int\int_Q \theta\theta_t \frac{x^2}{a(x)} w^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt \\ &+ sc_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2}\int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt. \end{aligned} \tag{3.25}$$

According to the relation (3.3), we know that $|\theta\theta_t| \leq c\theta^{2+1/k} \leq c'\theta^3$ and we obtain

$$\begin{aligned} A \geq &\left(\frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2} - \frac{2s^2c_1^2c'}{(2-\alpha)^2}\beta_0\right)\int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ &+ sc_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \frac{s}{2}\int\int_Q \theta_{tt} p w^2 dxdt. \end{aligned} \tag{3.26}$$

Let

$$A_1 = c_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \int\int_Q \theta_{tt} p w^2 dxdt. \tag{3.27}$$

Therefore

$$\begin{aligned} A \geq &\left(\frac{s^3c_1^3\alpha_0^2}{(2-\alpha)^2} - \frac{2s^2c_1^2c'}{(2-\alpha)^2}\beta_0\right)\int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ &+ \frac{s}{2}c_1\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt + \frac{s}{2}A_1. \end{aligned} \tag{3.28}$$

We apply the improved Hardy inequality (2.3), with $\eta = 1$, which gives

$$\int_0^1 a(x) w_x^2 dx + c_0 \int_0^1 w^2 dx \geq \frac{a(1)(1-\alpha)^2}{4}\int_0^1 \frac{w^2}{x^{2-\alpha}} dx + \int_0^1 \frac{w^2}{x^\gamma} dx, \tag{3.29}$$

for suitable $c_0 = c_0(a, \alpha, \gamma)$. Therefore, we can write

$$\begin{aligned} A_1 \geq &\frac{a(1)(1-\alpha)^2 c_1\alpha_0^2}{4}\int\int_Q \theta \frac{w^2}{x^{2-\alpha}} dxdt + c_1\alpha_0^2 \int\int_Q \theta \frac{w^2}{x^\gamma} dxdt \\ &- c_0c_1\alpha_0^2 \int\int_Q \theta w^2 dxdt + \int\int_Q \theta_{tt} p w^2 dxdt. \end{aligned} \tag{3.30}$$

Finally, we need to estimate the term

$$A_2 = \int\int_Q \theta_{tt} p w^2 dxdt - c_0 c_1 \alpha_0^2 \int\int_Q \theta w^2 dxdt. \tag{3.31}$$

By (3.3), there exists a positive constant c_3 such that

$$|A_2| \le c_3 \int\int_Q \theta^{1+2/k} w^2 dxdt. \tag{3.32}$$

Now, we consider $q = \frac{k}{k-1}$ and $q' = k$, so that $\frac{1}{q} + \frac{1}{q'} = 1$. Using the Young inequality, we have for all $\varepsilon > 0$

$$\begin{aligned} |A_2| &\le c_3 \int\int_Q \left(\theta^{1+2/k-\frac{3}{q'}} a^{\frac{1}{q'}} x^{\frac{-2}{q'}} w^{\frac{2}{q}}\right)\left(\theta^{\frac{3}{q'}} a^{\frac{-1}{q'}} x^{\frac{2}{q'}} w^{\frac{2}{q'}}\right) dxdt \\ &\le c_3 \varepsilon \int\int_Q \theta^{(1+2/k-\frac{3}{q'})q} a^{\frac{q}{q'}} x^{\frac{-2q}{q'}} w^2 dxdt + c_3 c(\varepsilon) \int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt, \end{aligned} \tag{3.33}$$

where $c(\varepsilon) = \frac{1}{q'}(\varepsilon q)^{\frac{-q'}{q}}$. Observe that

$$(1 + 2/k - \frac{3}{q'})q = 1, \quad \frac{2q}{q'} = \gamma. \tag{3.34}$$

Using the fact that $a(\cdot)$ is continuous on $[0,1]$, there exists a positive constant c_4 such that $(a(x))^{\frac{q}{q'}} \le c_4$ for every $x \in [0,1]$, and then

$$A_2 \ge -c_3 c_4 \varepsilon \int\int_Q \theta \frac{w^2}{x^\gamma} dxdt - c_3 c(\varepsilon) \int\int_Q \theta^3 \frac{x^2}{a(x)} w^2 dxdt. \tag{3.35}$$

Putting the estimate (3.35) in (3.30) and using (3.28), we obtain

$$\begin{aligned} A &\ge \left(\tfrac{s^3 c_1^3 \alpha_0^2}{(2-\alpha)^2} - \tfrac{2s^2 c_1^2 c'}{(2-\alpha)^2}\beta_0 - \tfrac{s c_3 c(\varepsilon)}{2}\right) \int\int_Q \theta^3 \tfrac{x^2}{a(x)} w^2 dxdt + \tfrac{s}{2} c_1 \alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt \\ &\quad + \tfrac{s a(1)(1-\alpha)^2 c_1 \alpha_0^2}{8} \int\int_Q \theta \tfrac{w^2}{x^{2-\alpha}} dxdt + \tfrac{s}{2}\left(c_1 \alpha_0^2 - c_3 c_4 \varepsilon\right) \int\int_Q \theta \tfrac{w^2}{x^\gamma} dxdt. \end{aligned} \tag{3.36}$$

Now, take $c_1 = 2$ and $\varepsilon = \varepsilon(a, \alpha, \alpha_0, \gamma) = \frac{3\alpha_0^2}{2c_3 c_4}$. Thus there exists $s_2 = s_2(T, a, \alpha, \alpha_0, \beta_0, \gamma) > 0$ such that for all $s \ge s_2$

$$\begin{aligned} A &\ge \tfrac{s^3 \alpha_0^2}{(2-\alpha)^2} \int\int_Q \theta^3 \tfrac{x^2}{a(x)} w^2 dxdt + s\alpha_0^2 \int\int_Q \theta a(x) w_x^2 dxdt \\ &\quad + \tfrac{s a(1)(1-\alpha)^2 \alpha_0^2}{4} \int\int_Q \theta \tfrac{w^2}{x^{2-\alpha}} dxdt + \tfrac{s}{4}\alpha_0^2 \int\int_Q \theta \tfrac{w^2}{x^\gamma} dxdt. \end{aligned} \tag{3.37}$$

On the other hand, we have

$$\begin{aligned}|B| &\leq \tfrac{1}{2}\int\int_Q |M'(t)|a(x)w_x^2dxdt + \tfrac{s^2}{2}\int\int_Q |M'(t)|\theta^2 a(x)p_x^2w^2dxdt \\ &\leq \tfrac{\gamma_0}{2}\int\int_Q a(x)w_x^2dxdt + \tfrac{2s^2\gamma_0}{(2-\alpha)^2}\int\int_Q \theta^2\tfrac{x^2}{a(x)}w^2dxdt \\ &\leq 2\gamma_0\Big(\int\int_Q a(x)w_x^2dxdt + \tfrac{s^2}{(2-\alpha)^2}\int\int_Q \theta^2\tfrac{x^2}{a(x)}w^2dxdt\Big) \\ &\leq 2c_5\gamma_0\Big(\int\int_Q \theta a(x)w_x^2dxdt + \tfrac{s^2}{(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt\Big) \\ &\leq \tfrac{3\alpha_0^2}{4}\Big(s\int\int_Q \theta a(x)w_x^2dxdt + \tfrac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt\Big) \qquad (3.38)\end{aligned}$$

for all $s \geq \frac{8c_5\gamma_0}{3\alpha_0^2}$. Therefore,

$$B \geq -s\frac{3\alpha_0^2}{4}\int\int_Q \theta a(x)w_x^2dxdt - \frac{3s^3\alpha_0^2}{4(2-\alpha)^2}\int\int_Q \theta^3\frac{x^2}{a(x)}w^2dxdt. \qquad (3.39)$$

By adding (3.37) and (3.39), for $s \geq s_1(a,\alpha,\gamma,\beta_0,\alpha_0,\gamma_0) > 0$, with $s_1 = \max\{s_2, \frac{8c_5\gamma_0}{3\alpha_0^2}\}$, we obtain the complet proof of Lemma 3.2.

Now, using the fact that $\int_0^T \Big[sM^2(t)\varphi_x(a(x)w_x)^2\Big]_0 dt$ is non-negative, the right hand of (3.22) becomes

$$\frac{1}{2}\|fe^{s\varphi}\|^2 - (b.t) \leq \frac{1}{2}\int\int_Q f^2e^{2s\varphi}dxdt + \frac{2sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta w_x^2(1,t)dt. \qquad (3.40)$$

From (3.22), (3.40) and Lemma 3.2, we obtain

$$\begin{aligned}&\tfrac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3\tfrac{x^2}{a(x)}w^2dxdt + s\int\int_Q \theta a(x)w_x^2dxdt + sa(1)(1-\alpha)^2\int\int_Q \theta\tfrac{w^2}{x^{2-\alpha}}dxdt \\ &+s\int\int_Q \theta\tfrac{w^2}{x^\gamma}dx \leq \tfrac{2}{\alpha_0^2}\Big(\int\int_Q f^2e^{2s\varphi}dxdt + \tfrac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta w_x^2(1,t)dt\Big)\end{aligned} \qquad (3.41)$$

for all $s \geq s_1$. Finally, we turn back to our original function $v = e^{-s\varphi}w$. Using that

$$v_x = \Big(-s\theta\frac{2}{2-\alpha}\frac{x}{a(x)}w + w_x\Big)e^{-s\varphi},$$

by the Young inequality, we find

$$\begin{aligned}s\int\int_Q \theta a(x)v_x^2e^{2s\varphi}dxdt &\leq 8\frac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3\frac{x^2}{a(x)}w^2dxdt \\ &+2s\int\int_Q \theta a(x)w_x^2dxdt. \qquad (3.42)\end{aligned}$$

Also, we have

$$w_x(1,t) = \Big(s\varphi_x v(1,t) + v_x(1,t)\Big)e^{s\varphi(1,t)} \tag{3.43}$$
$$= v_x(1,t)e^{s\varphi(1,t)}.$$

Consequently, from 3.41-3.43, we have

$$\frac{s^3}{(2-\alpha)^2}\int\int_Q \theta^3 \frac{x^2}{a(x)} v^2 e^{2s\varphi}dxdt + s\int\int_Q \theta a(x)v_x^2 e^{2s\varphi}dxdt + sa(1)(1-\alpha)^2\int\int_Q \theta\frac{v^2}{x^{2-\alpha}}e^{2s\varphi}dxdt$$
$$+s\int\int_Q \theta\frac{v^2}{x^\gamma}e^{2s\varphi}dxdt \le \frac{18}{\alpha_0^2}\Big(\int\int_Q f^2e^{2s\varphi}dxdt + \frac{4sa(1)\beta_0^2}{2-\alpha}\int_0^T \theta v_x^2(1,t)e^{2s\varphi(1,t)}dt\Big)$$

for all $s \ge s_0$, with $s_0 = s_1$

4. Observability Inequality and null controllability

In order to prove the controllability of (1.1), we first need to derive the observability inequality for the following adjoint problem

$$\begin{cases} v_t + A(t)v = 0, & (x,t)\in Q \\ v(1,t) = v(0,t) = 0, & \text{in the case} \quad \alpha\in(0,1) \quad t\in(0,T) \\ v(1,t) = (av_x)(0,t) = 0, & \text{in the case} \quad \alpha\in[1,2) \quad t\in(0,T) \\ v(x,T) = v_T(x), & x\in(0,1). \end{cases} \tag{4.1}$$

More precisely, we need to prove the following inequality

Proposition 4.1. *Assume that the coefficients $a(\cdot)$ and $M(\cdot)$ satisfiy the hypothesis (2) and (1), respectivly, and let $T>0$ be given and ω be a nonempty subinterval of $(0,1)$. Then there existe a positive constant $C = C(T,a,\alpha,M)$ such that the following observability inequality is valid for every solution v of (4.1)*

$$\int_0^1 v^2(x,0)dx \le C\int_0^T\int_\omega v^2(x,t)dxdt. \tag{4.2}$$

Now, by standard arguments, a null controllability result follows.

Theorem 4.1. *Let $T>0$ be given, and ω be a nonempty subinterval of $(0,1)$. Then for all $u_0\in L^2(0,1)$, there exists $h\in L^2(\omega\times(0,T))$ such that the solution u of (1.1) satisfies $u(x,T)=0$, for every $x\in(0,1)$. Furthermore, we have the estimate*

$$\|h\|_{L^2(\omega\times(0,T))} \le C\|u_0\|_{L^2(0,1)} \tag{4.3}$$

for some constant C.

To prove the observability inequality, we need the following lemma.

Lemma 4.1. *(Caccioppoli's inequality) Let $\omega_0 \Subset \omega$ be a nonempty open set. Then, there exists a positive constant $\tilde{c}$ such that for every solution of (4.1)*

$$\int_0^T \int_{\omega_0} v_x^2 e^{2s\varphi} dxdt \le \tilde{c} \int_0^T \int_{\omega} v^2 dxdt.$$

Proof Let us consider a smooth function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & \forall x \in \mathbb{R}, \\ \xi(x) = 1, & x \in \omega_0 \\ \xi(x) = 0, & x \notin \bar{\omega} \end{cases} \tag{4.4}$$

and $\xi > 0$ for $x \in \omega$. Then

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \int_0^1 \xi^2 e^{2s\varphi} v^2 dxdt \\ &= 2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + 2 \int\int_Q \xi^2 e^{2s\varphi} v v_t dxdt \\ &= 2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt - 2 \int\int_Q \xi^2 M(t) e^{2s\varphi} v (a(x) v_x)_x dxdt \\ &= 2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + 2 \int\int_Q M(t) (\xi^2 e^{2s\varphi})_x a(x) v v_x dxdt + 2 \int\int_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt. \end{aligned}$$

Hence,

$$\begin{aligned} 2 \int\int_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt &= -2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt - 2 \int\int_Q M(t) (\xi^2 e^{2s\varphi})_x a(x) v v_x dxdt \\ &\le -2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + \frac{\beta_0^2}{\alpha_0} \int\int_Q \left(\sqrt{a} \frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} v \right)^2 dxdt \\ &\quad + \alpha_0 \int\int_Q \left(\sqrt{a} \xi e^{s\varphi} v_x \right)^2 dxdt. \end{aligned} \tag{4.5}$$

In other hand we have

$$2\alpha_0 \int\int_Q \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt \le 2 \int\int_Q M(t) \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt. \tag{4.6}$$

Using (4.5) and (4.6), we obtain

$$\begin{aligned} &\alpha_0 \int\int_Q \xi^2 a(x) v_x^2 e^{2s\varphi} dxdt \\ &\le -2s \int\int_Q \xi^2 \varphi_t e^{2s\varphi} v^2 dxdt + \frac{\beta_0^2}{\alpha_0} \int\int_Q \left(\sqrt{a} \frac{(\xi^2 e^{2s\varphi})_x}{\xi e^{s\varphi}} \right) v^2 dxdt. \end{aligned} \tag{4.7}$$

Due to the definition of ξ and the fact that $\varphi_t e^{s\varphi}$ and $\varphi_t e^{s\varphi}$ are bounded functions on $\omega \times (0, T)$, the inequality (4.7) implies that there exists a positive constant $\tilde{c_1}$

such that

$$\min_{x\in\omega_0}(a(x))\int_0^T\int_{\omega_0} v_x^2 e^{2s\varphi}dxdt \le \int_0^T\int_{\omega_0} a(x)v_x^2e^{2s\varphi}dxdt \le \int\int_Q \xi^2 a(x)v_x^2e^{2s\varphi}dxdt$$
$$\le \tilde{c_1}\int_0^T\int_\omega v^2dxdt.$$

We deduce that

$$\int_0^T\int_{\omega_0} v_x^2e^{2s\varphi}dxdt \le \tilde{c}\int_0^T\int_\omega v^2dxdt, \tag{4.8}$$

with

$$\tilde{c}=\frac{\tilde{c_1}}{\min_{x\in\omega_0}(a(x))}.$$

The proof of the observability inequality (4.2). The proof can be derived in three steps.
Step 1: We consider $\omega_0=(x_1',x_2')\Subset\omega=(x_1,x_2)$ and a smooth cut-off function $0\le\xi\le 1$ such that

$$\begin{cases}\xi(x)=1, & x\in(0,x_1')\\ \xi(x)=0, & x\in(x_2',1)).\end{cases} \tag{4.9}$$

The function $w:=\xi v$, where v is the solution to (4.1), satisfies the following problem

$$\begin{cases} w_t+M(t)(a(x)w_x)_x=M(t)(2a(x)\xi'v_x+(a(x)\xi')'v):=f, & (x,t)\in Q\\ w(1,t)=w(0,t)=0, \quad t\in(0,T), \quad \text{in the case } \alpha\in(0,1), & \\ w(1,t)=(aw_x)(0,t)=0, \quad t\in(0,T), \quad \text{in the case } \alpha\in[1,2), & \\ w(x,T)=w_T(x), \quad x\in(0,1). & \end{cases} \tag{4.10}$$

Applying Theorem 4.1 with $\gamma=\frac{2-\alpha}{2}$ and observe that $w_x(1,t)=0$, we get

$$\begin{aligned} s_0\int\int_Q \theta w^2e^{2s_0\varphi}dxdt &\le s_0\int\int_Q\theta\frac{w^2}{x^\gamma}e^{2s_0\varphi}dxdt\\ &\le \frac{18}{\alpha_0^2}\int\int_Q M^2(t)(2a(x)\xi'v_x+(a(x)\xi')'v)^2e^{2s_0\varphi}dxdt\\ &\le c\int_0^T\int_{\omega_0}(v_x^2+v^2)e^{2s_0\varphi}dxdt.\end{aligned}$$

According to Lemma 4.1, we obtain

$$s_0\int\int_Q\theta w^2e^{2s_0\varphi}dxdt\le\check{c}\int_0^T\int_\omega v^2dxdt.$$

Next, using the definition of ξ, we obtain

$$\int_0^T \int_0^{x_1} \theta v^2 e^{2s_0\varphi} dxdt \leq \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt.$$

Using the fact that $p(x)$ and θ satisfies the following inequality

$$\theta(t) \leq \Big(\frac{3T^2}{16}\Big)^{-k}, t \in [T/4, 3T/4],$$

and

$$|p(x)| \leq \frac{2c_2}{2-\alpha}, \quad \text{for all} \quad x \in [0,1].$$

Then there exists a positive constant $c = c(T, a, \alpha)$ such that

$$e^{-cs_0} \int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt \leq \Big(\frac{T^2}{4}\Big)^k \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt,$$

which implies

$$\int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt \leq e^{cs_0} \Big(\frac{T^2}{4}\Big)^k \frac{\check{c}}{s_0} \int_0^T \int_\omega v^2 dxdt.$$

Step 2: We define $z = (1-\xi)v$. Then, z satisfies the folowing problem

$$\begin{cases} z_t + M(t)(a(x)z_x)_x = M(t)(2a(x)(1-\xi)'v_x + (a(x)(1-\xi)')'v) := f, & (x,t) \in (x_1', 1) \times (0,T) \\ z(1,t) = z(x_1', t) = 0, \quad t \in (0,T), \\ z(x,T) = z_T(x), \qquad x \in (x_1', 1). \end{cases} \tag{4.11}$$

In this case, we use classical Carleman estimates, since the operator $(a(x)z_x)_x$ is nondegenerate on $(x_1', 1)$. Then v can be estimated on $(x_2, 1) \subset (x_1', 1)$ in the same way, see [14]. Therefore

$$\begin{aligned} \int_{T/4}^{3T/4} \int_0^1 v^2 dxdt &= \textstyle\int_{T/4}^{3T/4} \int_0^{x_1} v^2 dxdt + \int_{T/4}^{3T/4} \int_\omega v^2 dxdt + \int_{T/4}^{3T/4} \int_{x_2}^1 v^2 dxdt \\ &\leq C \textstyle\int_0^T \int_\omega v^2 dxdt. \end{aligned} \tag{4.12}$$

Step 3: Multiplying both sides of (4.1) by v and integrate on $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2 dx = M(t) \int_0^1 a(x) v_x^2 dx \geq 0, \quad t \in (0,T).$$

Hence, we deduce that

$$\|v(\cdot,0)\|^2_{L^2(0,1)} \leq \|v(\cdot,t)\|^2_{L^2(0,1)} \quad \text{for all} \quad t \in (0,T). \tag{4.13}$$

Then integrate (4.13) on $(T/4, 3T/4)$ and use (4.13) to obtain

$$\int_0^1 v^2(x,0)dx \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_0^1 v^2 dxdt \leq \tilde{C} \int_0^T \int_\omega v^2 dxdt. \tag{4.14}$$

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OCCURRENCE OF STABLE ROUGH FRACTIONAL INTEGRAL INCLUSION

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Abstract. In the present paper, we generalize the Fredholm type integral operator, by using the fractional rough kernel. We also deal with the Ulam-Hyers stability for rough fractional integral inclusion and utilize the weakly Picard operator method as well as the generalized Covitz-Nadler fixed point theorem.
Keywords. Integral operator; Picard operator method; Fixed point theorem.

1. Introduction

The Ulam stability and its generalizations of different functional equations have been studied by various researchers (for recent studies see [2]-[13]). The generalized Ulam-Hyers product-sum stability of the Cauchy type additive functional equation has been investigated by Rassias [14].

Differential equations of arbitrary order were presumed to be models for nonlinear differential equations which played important roles in science, engineering and economics. The studies described computational processes and systems. Consequently, considerable attention has been viewed in the results of fractional differential equations, integral equations, fractional diffeo-integral equations, and fractional partial differential equations of physical phenomena. Most of the studies are concerned with the stability of the solutions [11]-[17].

In this paper, we generalize the Fredholm type integral operator by using the fractional rough kernel and also deal with the Ulam-Hyers stability for rough fractional integral inclusion. We utilize the weakly Picard operator method as well as the generalized Covitz-Nadler fixed point theorem.

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2. Preliminaries

Let (E,d) be a metric space and let us define the following classes of E

$$\mathcal{P}(E) := \{S \mid S \neq \emptyset\}, \quad \mathcal{P}_b(E) := \{S \in \mathcal{P}(E) \mid S \text{ is bounded}\},$$

$$\mathcal{P}_{cl}(E) := \{S \in \mathcal{P}(E) \mid S \text{ is closed}\}, \quad \mathcal{P}_{cp}(E) := \{S \in \mathcal{P}(E) \mid S \text{ is compact}\},$$

$$\mathcal{P}_{cv}(E) := \{S \in \mathcal{P}(E) | S \text{ is convex}\}.$$

Let $B(e_0, r) := \{e \in E | d(e_0, e) < r,\, r > 0\}$ be the open ball centered at $e_0 \in E$. Moreover, denoted by $\overline{B}(e_0, r)$, the closure of $B(e_0, r)$ and $\widetilde{B}(e_0, r) := \{e \in E | d(e_0, e) \leq r\}$ the closed ball. Define the *gap functional* in $\mathcal{P}(E)$ by

$$\mathcal{D}_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}_+,\ \mathcal{D}_d(X,Y) = \inf\{d(x,y) | x \in X,\, y \in Y\}.$$

Also let

$$\mathcal{H}_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}_+,\ \mathcal{H}_d(X,Y) = \max\{\sup_{x\in X} \mathcal{D}_d(x,Y), \sup_{y\in Y} \mathcal{D}_d(y,X) | x \in X,\, y \in Y\},$$

where $\mathcal{D}_d(y,X) := \mathcal{D}_d(\{y\},X)$ and $\mathcal{D}_d(x,Y) := \mathcal{D}_d(\{x\},Y)$.

If $G : E \to \mathcal{P}(E)$ is a multivalued operator, then $g \in E$ is called a *fixed point* for G iff $g \in G$. The set $Fix(G) := \{g \in E | g \in G\}$ is called the *fixed point set* of G. In addition, the set $SFix(G) := \{g \in E | \{g\} \equiv G\}$ is called the *strict fixed point* set of G. For the multi-valued operator $G : E \to \mathcal{P}(E)$, the *graph* of G is defined by

$$Gra(G) := \{(\phi, \psi) \in E \times S : \psi \in G\}.$$

Notice that $\gamma : E \to S$ is a *selection* for $G : E \to \mathcal{P}(S)$ if $\gamma(x) \in G(x)$, $x \in E$. We need the following concepts and out comes in the sequel.

Definition 2.1 Let $\varphi : E \to E$ be an operator and (E,d) be a metric space. Then φ is called a *weakly Picard operator* if the sequence $< \varphi^n >_{n\in\mathbb{N}}$ of approximations of φ converges and its limit is a fixed point of φ.

Definition 2.2 Let $(E,\mathfrak{d})$ be a metric space and $\varphi : E \to E$ be an operator and $\kappa > 0$ be a positive constant. Then φ is called a $\kappa-$ *weakly Picard operator* if and only if

$$\mathfrak{d}\Big(\chi, \varphi^\infty(\chi)\Big) \leq \kappa \mathfrak{d}\Big(\chi, \varphi(\chi)\Big), \quad \forall \chi \in E,$$

where

$$\varphi^\infty : E \to E, \qquad \varphi^\infty(\chi) := \lim_{n\to\infty} \varphi^n(\chi).$$

Definition 2.3 Let $G : E \to \mathcal{P}_{cl}(E)$ be a multivalued operator on the metric space $(E,\mathfrak{d})$. Then G is called a *multivalued weakly Picard operator* if for all $\chi \in E$ and $g \in G(\chi)$ there exists a sequence $< \chi_n >_{n\in\mathbb{N}}$ such that

1. $\chi_0 = \chi,\ \chi_1 = g;$
2. $\chi_{n+1} \in G(\chi_n), \quad n \in \mathbb{N};$
3. $<\chi_n>_{n\in\mathbb{N}} \to g,\ g \in Fix(G).$

Definition 2.4 Let $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous at 0 and $\sigma(0) = 0$ and $(E, \mathfrak{d})$ be a metric space. Then $G : E \to \mathcal{P}(E)$ is said to be $\sigma-$*weakly Picard operator* if it is a multi-valued weakly Picard operator and there exists a selection $\varphi^\infty : Gra(G) \to Fix(G)$ such that

$$\mathfrak{d}\big(\chi, \varphi^\infty(\chi, \iota)\big) \leq \sigma\big(\mathfrak{d}(\chi, \iota)\big), \quad (\chi, \iota) \in Gra(G).$$

If there exists a constant $\kappa > 0$ such that $\sigma(t) := \kappa t$ for each $t \in \mathbb{R}_+$, then G is called a *multi-valued* $\kappa-$*weakly Picard operator.*

Definition 2.5 $G : E \to \mathcal{P}_{cl}(E)$ is called a *multi-valued* $\lambda-$*contraction* if $\lambda \in [0, 1)$ and

$$\mathcal{H}_{\mathfrak{d}}(G(\chi), G(\iota)) \leq \lambda \mathfrak{d}(\chi, \iota), \quad \forall\, \chi, \iota \in E,$$

where $(E, \mathfrak{d})$ is a metric space.

Definition 2.6 Let $G : E \to \mathcal{P}(E)$ be a multivalued operator, where $(E, \mathfrak{d})$ is a metric space. The fixed point inclusion

$$u \in G(u), \quad u \in E \tag{2.1}$$

is called generalized *Ulam-Hyers stable* if and only if there exists $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and continuous at 0 and $\sigma(0) = 0$ such that for each $\epsilon > 0$ and for each solution $\iota^* \in E$ of the inequality

$$\mathcal{D}_{\mathfrak{d}}(\iota, G(\iota)) \leq \epsilon, \quad \iota \in E \tag{2.2}$$

there exists a solution u^* of (2.1) such that

$$\mathfrak{d}(u^*, \iota^*) \leq \sigma(\epsilon).$$

If for $\kappa > 0$, $\sigma(t) = \kappa t$, $t \in \mathbb{R}_+$ then the fixed point inclusion (2.1) is said to be $\kappa-$*Ulam-Hyers stable.*

The following theorem (see Rus [15]) deals with the Ulam-Hyers stability of the fixed point inclusion (2.1).

Theorem 2.1 Let $G : E \to \mathcal{P}_{cp}(E)$ be a multivalued $\sigma-$ weakly Picard operator and $(E, \mathfrak{d})$ be a metric space. Then the fixed point inclusion (2.1) is generalized Ulam-Hyers stable.

The following result is a generalization of the Covitz-Nadler fixed point theorem, which can be found in [12] :

Theorem 2.2 Let (E, d) be a complete metric space and $G : E \to \mathcal{P}_{cl}(E)$ be a multivalued $\lambda-$ contraction operator $\mathcal{H}_d(G(u_1), G(u_2)) \leq \lambda d(u_1, u_2), \quad \forall u_1, u_2 \in E$. Then $Fix(G)$ is nonempty and for $u_0 \in E$ there exists a sequence of approximations of G starting from u_0 which converges to a fixed point of G.

Next Ulam-Hyers stability result, which is very useful for applications, was introduced in [10].

Theorem 2.3 Let $(E, \mathfrak{d})$ be a complete metric space and $G : E \to \mathcal{P}_{cl}(E)$ be a multi-valued $\lambda-$ contraction operator. Then

1. G is a multi-valued weakly Picard operator;
2. If $\psi(at) \leq a\psi(t)$ for every $t \in \mathbb{R}_+$, $a > 1$ and the series $\tau(t) := \sum_{n=1}^{\infty} \varphi^n(t)$ converges to the point $t = 0$, then G is a $\sigma-$multivalued weakly Picard operator with $\sigma(t) := t + \tau(t)$ and $t \in \mathbb{R}_+$;
3. Let $Q : E \to \mathcal{P}_{cl}(E)$ be a multi-valued $\lambda-$contraction and $b > 0$ such that $\mathcal{H}(Q(\chi), G(\chi)) \leq b$, $\chi \in E$. Suppose that $\lambda(at) \leq a\lambda(t)$, $t \in \mathbb{R}_+$, $a > 1$ and the series $\tau(t)$ converges uniformly to the point $t = 0$. Then $\mathcal{H}\Big(Fix(Q), Fix(G)\Big) \leq \sigma(b)$.

3. Ulam Stability

In this section, we further investigate the Ulam stability by utilizing the above mentioned concepts and results. Recently, Ibrahim and Jalab [8] established the existence of solutions for integral inclusion of fractional order in the sense of Riemann-Liouville integral operator. We consider the following fractional integral inclusion:

$$u(t) \in \int_a^b G(t, \varsigma, u(\varsigma)) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + g(t), \tag{3.1}$$

where $u \in \mathbb{R}^n$, Ω belongs to the unit sphere of $\mathbb{R}^n$, $0 < \alpha < n$ and $t, \varsigma \in J := [a, b]$, $a, b > 0$. When $\Omega \equiv 1$ and $\alpha \to n$, inclusion(3.1) reduces to the Fredholm type integral inclusion. We have the following result:

Theorem 3.1 Let $G : J \times J \times \mathbb{R}^n \to \mathcal{P}_{cl,cv}$, $\Omega : J \to \mathbb{R}^n$ and $g : J \to \mathbb{R}^n$ such that

1. There exists an integrable function $\imath : J \to \mathbb{R}^n_+$ such that $G(t, \varsigma, u) \subset \imath(\varsigma) \times B(0, 1)$, $t, \varsigma \in J$, $u \in \mathbb{R}^n = E$;

2. $G(.,.,u) : J \times J \times \mathbb{R}^n \to \mathcal{P}_{cl,cv}$ is jointly measurable for all $u \in \mathbb{R}^n$;

3. $G(.,\varsigma,u) : J \times J \times \mathbb{R}^n \to \mathcal{P}_{cl,cv}$ is lower semi-continuous for all $(\varsigma, u) \in (J, \mathbb{R}^n)$;

4. There exist a continuous function $\rho : J \times J \to \mathbb{R}_+$ with $\sup_{t\in J} \int_a^b \rho(t,\varsigma)d\varsigma \leq 1$ and a positive function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\mathcal{H}\Big(G(t,\varsigma,u), G(t,\varsigma,v)\Big) \leq \rho(t,\varsigma).\theta(|u-v|); \tag{3.2}$$

5. Ω and g are continuous;

6. $\|\Omega\| := \sup_{s\in J} |\Omega(s)|$; with $\frac{\|\Omega\|}{a^{n-\alpha}} < 1$.

Then the following conclusions hold

1. Inclusion (3.1) has at least one solution $u^* \in C(J, \mathbb{R}^n)$;

2. If the series $\sum_{n=1}^{\infty} \theta^n$ converges uniformly to $t = 0$, where $\theta(qt) \leq q\theta(t)$ for every $t \in \mathbb{R}_+$, $q > 1$, then the fractional integral inclusion (3.1) is generalized Ulam-Hyers with function σ, where $\sigma(t) = t + \varsigma(t)$ for each $t \in \mathbb{R}_+$ and $\varsigma(t) := \sum_{n=1}^{\infty} \theta^n$. Equivalently, for each $\epsilon > 0$ and $v \in C(J, \mathbb{R}^n)$ there exists $u \in C(J, \mathbb{R}^n)$ such that

$$u(t) \in \int_a^b G(t,\varsigma,v(\varsigma)) \frac{\Omega(\varsigma)}{|\varsigma|^{n-\alpha}} d\varsigma + g(t),$$

$$|u(t) - v(t)| \leq \epsilon, \quad t \in J$$

and

$$|v(t) - u^*(t)| \leq \sigma(\epsilon), \quad t \in J.$$

Proof. Define the multivalued operator $M : C(J, \mathbb{R}^n) \to \mathcal{P}(C(J, \mathbb{R}^n))$ by

$$M(u) := \left\{ v \in C(J, \mathbb{R}^n) | v(t) \in \int_a^b G(t,\varsigma,u(\varsigma)) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + g(t) \right\}.$$

Then (3.1) is equivalent to the fixed point inclusion

$$u \in M(u), \quad u \in C(J, \mathbb{R}^n). \tag{3.3}$$

The rest of the proof will be given in three steps.

Step 1. $M(u) \in \mathcal{P}_{cp}(C(J, \mathbb{R}^n))$.

By the continuity of Ω and g, we obtain $\gamma(t,\varsigma) \in G(t,\varsigma,u)$, $t,\varsigma \in J$ such that

$$v(t) := \int_a^b \gamma(t,\varsigma)\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma \in M(u).$$

In view of [1], Theorem 8.6.3, together with the hypotheses 1 and 2 we conclude that $M(u)$ is a compact set for all $u \in C(J,\mathbb{R}^n)$.

Step 2. $\mathcal{H}\big(M(u_1),M(u_2)\big) \le \theta(\|u_1-u_2\|), \quad u_1,u_2 \in C(J,\mathbb{R}^n)$.
For $u_1,u_2 \in C(J,\mathbb{R}^n)$, we let $v_1 \in M(u_1)$. Thus

$$v_1(t) \in \int_a^b G(t,\varsigma,u_1(\varsigma))\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma + g(t).$$

Therefore there is an integrable function γ_1 such that

$$v_1(t) = \int_a^b \gamma_1(t,\varsigma)\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma + g(t).$$

In virtue of the assumption 4, we conclude that

$$H\big(G(t,\varsigma,u_1),G(t,\varsigma,u_2)\big) < \rho(t,\varsigma)\theta(|u_1(\varsigma)-u_2(\varsigma)|) \le \rho(t,\varsigma)\theta(\|u_1-u_2\|).$$

So, there exists $w \in G(t,\varsigma,u_2(\varsigma))$ such that

$$|\gamma_1(t,\varsigma)-w| \le \rho(t,\varsigma)\theta(\|u_1-u_2\|), \quad t,\varsigma \in J.$$

Define a set $\Gamma(t,\varsigma)$ by

$$\Gamma(t,\varsigma) := \{w \,|\, |\gamma_1(t,\varsigma)-w| \le \rho(t,\varsigma)\theta(\|u_1-u_2\|)\}$$

and a multivalued operator by

$$\Theta(t,\varsigma) := \Gamma(t,\varsigma)\bigcap G\big(t,\varsigma,u_2(\varsigma)\big).$$

Thus according to the assumptions 2 and 3, Θ is jointly measurable and lower semi-continuous in t. Consequently, there exists $\gamma_2(t,\varsigma)$ a selection for Θ, jointly measurable, integrable in ς and lower semi-continuous in t. Hence $\gamma_2(t,\varsigma) \in G(t,\varsigma,u_2)$ and

$$|\gamma_1(t,\varsigma)-\gamma_2(t,\varsigma)| \le \rho(t,\varsigma)\theta(\|u_1-u_2\|), \quad t,\varsigma \in J.$$

Consider

$$v_2(t) \in \int_a^b G(t,\varsigma,u_2(\varsigma))\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma + g(t)$$

with

$$v_2(t) = \int_a^b \gamma_2(t,\varsigma)\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma + g(t).$$

Then, by utilizing assumption 6, we get

$$|v_1(t) - v_2(t)| \leq \frac{\|\Omega\|}{a^{n-\alpha}} \int_a^b \rho(t,\varsigma)\theta(\|u_1 - u_2\|)d\varsigma \leq \theta(\|u_1 - u_2\|).$$

Similar arguments can be used for u_1 and u_2. Hence, in view of Theorem 2.2, inclusion (3.1) has a solution.

Step 3. *Generalized Ulam-Hyers stable.*

Now, our aim is to show that the fixed point inclusion (3.3) is generalized Ulam-Hyers stable. Let $\epsilon > 0$ and $\mu \in C(J, \mathbb{R}^n)$ for which there exists $u \in C(J, \mathbb{R}^n)$ such that

$$u(t) \in \int_a^b G(t, \varsigma, \mu(\varsigma)) \frac{\Omega(\varsigma)}{|\varsigma|^{n-\alpha}} d\varsigma + g(t), \quad t \in J$$

and

$$\|u - \mu\| \leq \epsilon.$$

This implies that

$$\mathcal{D}_{\|.\|}\left(\mu, M(\mu)\right) \leq \epsilon.$$

Since M is a multivalued $\theta-$contraction and applying Theorem 2.3, we have that M is a multivalued $\sigma-$weakly Picard operator. Then according to Theorem 2.1, we conclude that the fixed point problem (3.3) is generalized Ulam-Hyers stable. Therefore, the fractional integral inclusion (3.1) is generalized Ulam-Hyers stable. For the last assertion, we utilize Theorem 2.3, which completes the proof.

Next, we consider the following fractional integral inclusion

$$u(t) \in \int_a^t K(t, \varsigma, u(\varsigma)) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + h(t), \tag{3.4}$$

where $u \in \mathbb{R}^n$, Ω belongs to the unit sphere of $\mathbb{R}^n$, $0 < \alpha < n$ and $t, \varsigma \in J = [a, b]$, $a, b > 0$. When $\Omega \equiv 1$ and $\alpha \to n$, inclusion (3.4) reduces to the Volterra type integral inclusion. In the same manner as of Theorem 3.1, we have the following

Theorem 3.2 Let $K : J \times J \times \mathbb{R}^n \to \mathcal{P}_{cl,cv}$, $\Omega : J \to \mathbb{R}^n$ and $h : J \to \mathbb{R}^n$ such that

1. There exists an integrable function $\imath : J \to \mathbb{R}^n_+$ such that $K(t, \varsigma, u) \subset \imath(\varsigma) \times B(0, 1)$, $t, \varsigma \in J$, $u \in \mathbb{R}^n = E$;
2. $K(., ., u) : J \times J \times \mathbb{R}^n \to \mathcal{P}_{cl,cv}$ is jointly measurable for all $u \in \mathbb{R}^n$;
3. $K(., \varsigma, u) : J \times J \times \mathbb{R}^n \to \mathcal{P}_{cl,cv}$ is lower semi-continuous for all $(\varsigma, u) \in (J, \mathbb{R}^n)$;
4. There exist a continuous function $\rho : J \times J \to \mathbb{R}_+$ with $\sup_{t \in J} \int_a^t \rho(t, \varsigma) d\varsigma \leq 1$ and a positive function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\mathcal{H}\Big(K(t, \varsigma, u), K(t, \varsigma, v)\Big) \leq \rho(t, \varsigma).\theta(|u - v|); \tag{3.5}$$

5. Ω and h are continuous;

6. $\|\Omega\| := \sup_{s\in J} |\Omega(s)|$; with $\frac{\|\Omega\|}{a^{n-\alpha}} < 1$.

Then the following conclusions hold:

1. Inclusion (3.4) has at least one solution $u^* \in C(J, \mathbb{R}^n)$;

2. If the series $\sum_{n=1}^{\infty} \theta^n$ converges uniformly to $t = 0$, where $\theta(qt) \leq q\theta(t)$ for every $t \in \mathbb{R}_+$, $q > 1$, then the fractional integral inclusion (3.4) is generalized Ulam-Hyers with function σ, where $\sigma(t) = t + \varsigma(t)$ for each $t \in \mathbb{R}_+$ and $\varsigma(t) := \sum_{n=1}^{\infty} \theta^n$. Equivalently, for each $\epsilon > 0$ and $v \in C(J, \mathbb{R}^n)$ there exists $u \in C(J, \mathbb{R}^n)$ such that

$$u(t) \in \int_a^b K(t, \varsigma, v(\varsigma)) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + h(t),$$

$$|u(t) - v(t)| \leq \epsilon, \quad t \in J$$

and

$$|v(t) - u^*(t)| \leq \sigma(\epsilon), \quad t \in J.$$

Proof. Define the multivalued operator $V : C(J, \mathbb{R}^n) \to \mathcal{P}(C(J, \mathbb{R}^n))$ by

$$V(u) := \left\{ v \in C(J, \mathbb{R}^n) | v(t) \in \int_a^t K(t, \varsigma, u(\varsigma)) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + h(t) \right\}.$$

Then (3.4) is equivalent to the fixed point inclusion

$$u \in V(u), \quad u \in C(J, \mathbb{R}^n). \tag{3.6}$$

In a similar method as of Theorem 3.1, we may have $V(u) \in \mathcal{P}_{cp}(C(J, \mathbb{R}^n))$. Now we proceed to show that V is $\theta-$contraction mapping on $C(J, \mathbb{R}^n)$.

For $u_1, u_2 \in C(J, \mathbb{R}^n)$, we let $v_1 \in V(u_1)$. Thus

$$v_1(t) \in \int_a^t K(t, \varsigma, u_1(\varsigma)) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + h(t).$$

Therefore, there is an integrable function γ_1 such that

$$v_1(t) = \int_a^t \gamma_1(t, \varsigma) \frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}} d\varsigma + h(t).$$

In view of the assumption 4, we conclude that

$$H\big(K(t, \varsigma, u_1), K(t, \varsigma, u_2)\big) < \rho(t, \varsigma)\theta(|u_1(\varsigma) - u_2(\varsigma)|) \leq \rho(t, \varsigma)\theta(\|u_1 - u_2\|).$$

Thus, there exists $w \in K(t,\varsigma,u_2(\varsigma))$ such that

$$|\gamma_1(t,\varsigma)-w| \leq \rho(t,\varsigma)\theta(\|u_1-u_2\|), \quad t,\varsigma \in J.$$

Define a set $\Lambda(t,\varsigma)$ by

$$\Lambda(t,\varsigma) := \{w : |\gamma_1(t,\varsigma)-w| \leq \rho(t,\varsigma)\theta(\|u_1-u_2\|)\}$$

and a multivalued operator by

$$\Psi(t,\varsigma) := \Lambda(t,\varsigma) \bigcap K\big(t,\varsigma,u_2(\varsigma)\big).$$

Therefore, according to the assumptions 2 and 3, Ψ is jointly measurable and lower semi-continuous in t. Consequently, there exists $\gamma_2(t,\varsigma)$ a selection for Ψ, jointly measurable, integrable in ς and lower semi-continuous in t. Hence, $\gamma_2(t,\varsigma) \in K(t,\varsigma,u_2)$ and

$$|\gamma_1(t,\varsigma)-\gamma_2(t,\varsigma)| \leq \rho(t,\varsigma)\theta(\|u_1-u_2\|), \quad t,\varsigma \in J.$$

Consider

$$v_2(t) \in \int_a^t K(t,\varsigma,u_2(\varsigma))\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma + h(t)$$

with

$$v_2(t) = \int_a^t \gamma_2(t,\varsigma)\frac{\Omega(\varsigma)}{\varsigma^{n-\alpha}}d\varsigma + h(t).$$

Define a norm by

$$\|u\|_{\mathcal{B}} := \sup_{t\in J}(|u(t)e^{-q(t)}|),$$

where $q(t) := \int_a^t \rho(\varsigma)d\varsigma$. Then by using the assumption 6, we get

$$\begin{aligned}
|v_1(t)-v_2(t)| &\leq \frac{\|\Omega\|}{a^{n-\alpha}}\int_a^t |\gamma_1(t,\varsigma)-\gamma_2(t,\varsigma)|)d\varsigma \\
&\leq \frac{\|\Omega\|}{a^{n-\alpha}}\int_a^t \rho(t,\varsigma)\theta(|u_1(\varsigma)-u_2(\varsigma)|)d\varsigma \\
&\leq \frac{\|\Omega\|}{a^{n-\alpha}}\int_a^t \rho(t,\varsigma)\theta(e^{q(\varsigma)}|u_1(\varsigma)-u_2(\varsigma)|e^{-q(\varsigma)})d\varsigma \\
&\leq \frac{\|\Omega\|}{a^{n-\alpha}}\int_a^t \rho(t,\varsigma)e^{q(\varsigma)}\theta(\|u_1-u_2\|_{\mathcal{B}}) \\
&\leq \frac{\|\Omega\|}{a^{n-\alpha}}\theta(\|u_1-u_2\|_{\mathcal{B}})\big(e^{q(t)}-e^{q(a)}\big) \\
&\leq \theta(\|u_1-u_2\|_{\mathcal{B}})e^{q(t)}.
\end{aligned}$$

Thus, we have

$$\|v_1-v_2\|_{\mathcal{B}} \leq \theta(\|u_1-u_2\|_{\mathcal{B}}).$$

Hence, in view of Theorem 2.2, inclusion (3.4) has a solution.

It suffices to prove that the fixed point inclusion (3.6) is generalized Ulam-Hyers stable. Let $\epsilon > 0$ and $\nu \in C(J, \mathbb{R}^n)$ for which there exists $u \in C(J, \mathbb{R}^n)$ such that

$$u(t) \in \int_a^t K(t, \varsigma, \nu(\varsigma)) \frac{\Omega(\varsigma)}{|\varsigma|^{n-\alpha}} d\varsigma + h(t), \quad t \in J$$

and

$$\|v_1 - v_2\|_{\mathcal{B}} \leq \|u - \nu\| \leq \epsilon.$$

This implies that

$$\mathcal{D}_{\|.\|_{\mathcal{B}}}(\nu, V(\nu)) \leq \epsilon.$$

Since V is a multi-valued $\theta-$contraction with respect to the norm $\|.\|_{\mathcal{B}}$, then V is a multi-valued weakly operator. Using Theorem 2.3, we have that V is a multi-valued $\sigma-$weakly Picard operator. Then according to Theorem 2.1, we conclude that the fixed point problem (3.6) is generalized Ulam-Hyers stable. This implies that there exists a solution u^* of inclusion (3.4) such that

$$\|\nu - u^*\|_{\mathcal{B}} \leq \sigma(\epsilon), \quad \epsilon > 0.$$

Thus, we have

$$|\nu - u^*| \leq \sigma(e^{tq(b)}\epsilon), \quad t \in J = [a, b].$$

Theorem 2.3 yields the last conclusion and this completes the proof.

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EXISTENCE AND STABILITY RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH TWO CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In this paper, we discuss the existence, uniqueness and stability of solutions for a nonlocal boundary value problem of nonlinear fractional differential equations with two Caputo fractional derivatives. By applying the contraction mapping and O'Regan fixed point theorem, the existence results are obtained. We also derive the Ulam-Hyers stability of solutions. Finally, some examples are given to illustrate our results.
Keywords: Caputo derivative, Fixed point, Existence, Uniqueness, Boundary value problem.

1. Introduction

Boundary value problems for fractional differential equations with nonlocal boundary conditions constitute a very interesting and important class of problems (see [4, 5]). Differential equations of fractional order with nonlocal boundary conditions arise in a variety of different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to nonlocal problems with integral boundary conditions. For more details, we refer the reader to [6, 25]. Recently, by applying different fixed point theorems such as the Banach fixed point theorem, Schaefer's fixed point theorem, Krasnoselskii's fixed point theorem, the Leray-Schauder nonlinear alternative and the fixed point theorem of O'Regan, many researchers have obtained some interesting results of the existence and uniqueness of solutions to boundary value problems for fractional differential equations with nonlocal boundary value problems [1, 2, 7, 8, 9, 14, 15, 18, 23, 24] and the references therein. Ulam's stability problem [17] has been attracted by several famous researchers. Since then, a large number of monographs have been published in connection with

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various generalizations of Ulam's type stability theory or the Ulam-Hyers stability theory. For some recent development on Ulam's type stability, we refer the reader to [3, 12, 16, 17, 19, 20, 21, 22]. The stability of fractional differential equations has been investigated by many authors [19, 21, 22].

Motivated by the above papers, we study the existence, uniqueness and stability of solutions to the following fractional boundary value problem with tow Caputo fractional derivatives involving nonlocal boundary conditions:

$$(1.1)\qquad \begin{cases} D^{\alpha}\left(D^{\beta}+\lambda\right)x\left(t\right)=f\left(t,x\left(t\right)\right)+\int_{0}^{t}\frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)}f\left(s,x\left(s\right)\right)ds,\ t\in\left[0,T\right], \\ \\ x\left(0\right)=x_{0}+g\left(x\right),\ x\left(T\right)=\theta\int_{0}^{\eta}\frac{(\eta-s)^{p-1}}{\Gamma(p)}x\left(s\right)ds,\ 0<\eta<T, \end{cases}$$

where D^{α}, D^{β} denote the Caputo fractional derivatives, with $0<\alpha,\beta\leq 1, 1<\alpha+\beta\leq 2$, $f:[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$ and $g:C\left([0,T],\mathbb{R}\right)\rightarrow\mathbb{R}$ are given continuous functions, and $\sigma,p>0$, λ,x_{0},θ are real constants In (1.1), $g\left(x\right)$ may be regarded as $g\left(x\right)=\sum_{j=0}^{m}k_{j}x\left(t_{j}\right)$, where $k_{j}, j=1,...,m$ are given constants and $0<t_{0}<...<t_{m}\leq 1$.
The paper is organized as follows: In Section 2, we recall some preliminaries and lemmas that we need in the sequel. In Section 3, we present our main results for the existence, uniqueness and stability of solutions to the fractional boundary value problem (1.1). Some examples to illustrate our results are presented in Section 4.

2. Preliminaries

In this section, we present some useful definitions and lemmas [10, 11, 13]:

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\vartheta\geq 0$, for a continuous function f on $[a,b]$ is defined as:

$$I^{\vartheta}f\left(t\right)=\frac{1}{\Gamma(\vartheta)}\int_{a}^{t}\left(t-\tau\right)^{\vartheta-1}f\left(\tau\right)d\tau,\ \vartheta>0,\ a\leq t\leq b$$

$$I^{0}f\left(t\right)=f\left(t\right),$$

where $\Gamma\left(\vartheta\right):=\int_{0}^{+\infty}e^{-u}u^{\vartheta-1}du.$

Definition 2.2. The fractional derivative of $f\in C^{n}\left([a,b]\right)$ in Caputo's sense is defined as:

$$D^{\vartheta}f\left(t\right)=\frac{1}{\Gamma(n-\vartheta)}\int_{a}^{t}\left(t-\tau\right)^{n-\vartheta-1}f^{(n)}\left(\tau\right)d\tau,\ n-1<\vartheta,\ n\in N^{*},\ a\leq t\leq b.$$

The following lemmas give some properties of Riemann-Liouville fractional integrals and the Caputo fractional derivative [10, 11]:

Lemma 2.1. *Let* $\vartheta, s > 0,\ f \in L^1([a,b])$. *Then* $I^{\vartheta} I^{s} f(t) = I^{\vartheta+s} f(t), D^{s} I^{s} f(t) = f(t),\ t \in [a,b]$.

Lemma 2.2. *Let* $s > \vartheta > 0,\ f \in L^1([a,b])$. *Then* $D^{\vartheta} I^{s} f(t) = I^{s-\vartheta} f(t),\ t \in [a,b]$.

We also give the following lemmas [10]:

Lemma 2.3. *For* $\vartheta > 0$, *the general solution to the fractional differential equation* $D^{\vartheta} x(t) = 0$ *is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1,\ n = [\vartheta] + 1.$

Lemma 2.4. *Let* $\vartheta > 0$. *Then*

$$I^{\vartheta} D^{\vartheta} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1,\ n = [\vartheta] + 1.$

We also need the following auxiliary result:

Lemma 2.5. *For a given* $h \in C([0,T], \mathbb{R})$, *the solution to the fractional boundary value problem*

$$\text{(2.1)} \qquad \begin{cases} D^{\alpha}\left(D^{\beta} + \lambda\right) x(t) = h(t),\ t \in [0,T],\ 0 < \alpha, \beta \leq 1, \\ \\ x(0) = x_0 + g(x),\ x(T) = \theta I^{p} x(\eta), \end{cases}$$

is given by

$$\begin{aligned} \text{(2.2)} \quad & x(t) \\ = {} & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\, ds - \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} x(s)\, ds \\ & - \frac{\Delta t^{\beta}}{\Gamma(\beta+1)} \int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} h(s)\, ds + \frac{\lambda \Delta t^{\beta}}{\Gamma(\beta+1)} \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} x(s)\, ds \\ & + \frac{\Delta \theta t^{\beta}}{\Gamma(\beta+1)} \int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} h(s)\, ds - \frac{\lambda \Delta \theta t^{\beta}}{\Gamma(\beta+1)} \int_0^{\eta} \frac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)} x(s)\, ds \\ & + \left(\Delta \frac{(\theta \eta^{p} - \Gamma(p+1))\, t^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)} + 1 \right) (x_0 + g(x)), \end{aligned}$$

where
(2.3)

$$\Delta = \frac{\Gamma(\beta+p+1)\Gamma(\beta+1)}{\Gamma(\beta+p+1) T^{\beta} - \Gamma(\beta+1)\theta \eta^{\beta+p}}, \Gamma(\beta+p+1) T^{\beta} \neq \Gamma(\beta+1)\theta \eta^{\beta+p}.$$

Proof. By Lemmas 5 and 6, we have

$$x(t) = I^{\alpha+\beta}h(t) - \lambda I^{\beta}x(t) - \frac{c_0}{\Gamma(\beta+1)}t^{\beta} - c_1, \tag{2.4}$$

for some arbitrary constants $c_0, c_1 \in \mathbb{R}$.

Using the boundary condition: $x(0) = x_0 + g(x)$, we obtain

$$c_1 = -(x_0 + g(x)).$$

Thanks to Lemma 3, we get

$$I^{p}x(t) = I^{\alpha+\beta+p}h(t) - \lambda I^{\beta+p}x(t) - \tfrac{c_0}{\Gamma(\beta+p+1)}t^{p+\beta} - \tfrac{c_1}{\Gamma(p+1)}t^{p}.$$

Applying the boundary condition: $x(T) = \theta I^{p}x(\eta)$, we obtain

$$\begin{aligned} c_0 &= \Delta\Big[I^{\alpha+\beta}h(T) - \lambda I^{\beta}x(T) - \theta I^{\alpha+\beta+p}h(\eta) + \lambda\theta I^{\beta+p}x(\eta) \\ &\quad - \tfrac{(\theta\eta^{p} - \Gamma(p+1))}{\Gamma(p+1)}(x_0 + g(x))\Big], \end{aligned}$$

where Δ defined by (2.3). Substituting the value of c_0 and c_1 in (2.4), we obtain the solution (2.2). $\square$

In view of Lemma 4, we define the operator: $\phi : X \to X$ as

$$\begin{aligned} &\phi x(t) \\ &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))\,ds - \lambda\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} x(s)\,ds \\ &\quad - \frac{\Delta t^{\beta}}{\Gamma(\beta+1)}\int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))\,ds + \frac{\lambda\Delta t^{\beta}}{\Gamma(\beta+1)}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} x(s)\,ds \\ &\quad + \frac{\Delta\theta t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} f(s,x(s))\,ds - \frac{\lambda\Delta\theta t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta} \tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)} x(s)\,ds \\ &\quad + \left(\Delta\frac{(\theta\eta^{p} - \Gamma(p+1))t^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)} + 1\right)(x_0 + g(x)), \end{aligned} \tag{2.5}$$

We also introduce the operators ϕ_1, $\phi_2 : X \to X$, such that

$$\begin{aligned} &\phi_1 x(t) \\ &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))\,ds - \lambda\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} x(s)\,ds \\ &\quad - \frac{\Delta t^{\beta}}{\Gamma(\beta+1)}\int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s,x(s))\,ds + \frac{\lambda\Delta t^{\beta}}{\Gamma(\beta+1)}\int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} x(s)\,ds \\ &\quad + \frac{\Delta\theta t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta} \frac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} f(s,x(s))\,ds - \frac{\lambda\Delta\theta t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta} \tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)} x(s)\,ds, \end{aligned} \tag{2.6}$$

and

$$\phi_2(x)(t) = \left(\Delta \frac{(\theta\eta^p - \Gamma(p+1))\, t^\beta}{\Gamma(p+1)\Gamma(\beta+1)} + 1\right)(x_0 + g(x)). \tag{2.7}$$

Clearly

$$\phi(x)(t) = \phi_1 x(t) + \phi_2 x(t),\ t \in [0,T]. \tag{2.8}$$

3. Main Results

We denote by $X = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$.

For computational convenience, we set the notations:

$$\begin{aligned}\Lambda \ &= \ \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} \\ &\quad + \frac{|\Delta|\, T^\beta}{\Gamma(\beta+1)}\left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}\right. \\ &\quad \left. + \frac{|\theta|\, \eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \frac{|\theta|\, \eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\right],\end{aligned} \tag{3.1}$$

$$\Lambda_1 = \frac{|\Delta(\theta\eta^p - \Gamma(p+1))|\, T^\beta}{\Gamma(p+1)\Gamma(\beta+1)} + 1, \tag{3.2}$$

$$\Lambda_2 = \frac{|\lambda|\, T^\beta}{\Gamma(\beta+1)}\left[1 + \frac{|\Delta|\, T^\beta}{\Gamma(\beta+1)} + \frac{|\Delta\theta|\, \eta^{\beta+p}}{\Gamma(\beta+p+1)}\right], \tag{3.3}$$

and

$$\begin{aligned}\rho \ &= \ \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}\right. \\ &\quad + \frac{|\Delta|\, T^\beta}{\Gamma(\beta+1)}\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}\right. \\ &\quad \left.\left. + \frac{|\theta|\, \eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \frac{|\theta|\, \eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\right)\right] \|\gamma\|.\end{aligned} \tag{3.4}$$

Now, we impose the following hypotheses:

$(H1)$: There exists a constant $\omega > 0$ such that for all $t \in [0,T]$ and $x, y \in C([0,T],\mathbb{R})$, we have $|f(t,x) - f(t,y)| \le \omega \|x - y\|$,

$(H2)$: There exists a positive constant $\varpi < \frac{1}{\Lambda_1}$ and a continuous function $\varphi : [0,\infty) \to [0,\infty)$such that $\varphi(u) \leq \varpi u$ and $|g(x) - g(y)| \leq \varpi(\|x-y\|)$, for all $x, y \in C([0,T])$.

$(H3)$: $g(0) = 0$.

$(H4)$: There exists a non-negative function $\gamma(t) \in C([0,T], \mathbb{R})$ and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, such that $|f(t,x)| \leq \gamma(t)\psi(|x|)$ for all $(t,x) \in [0,T] \times X$.

$(H5)$: $\sup\limits_{r\in(0,\infty)} \frac{r}{\rho\psi(r)+\Lambda_1|x_0|} > \frac{1}{1-(\Lambda_2+\Lambda_1\varpi)}$, where Λ_1, ρ and Λ_2 are given respectively in (3.2), (3.3) and (3.4).

3.1. Existence and uniqueness of solutions

The first result is concerned with the existence and uniqueness of solutions to fractional boundary value problems and is based on the Banach contraction principle.

Theorem 3.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that $(H1)$ and $(H2)$ hold. If the inequality*

$$\Lambda\omega + \Lambda_1\varpi < 1 - \Lambda_2, \tag{3.5}$$

is valid, then the fractional boundary value problem (1.1) has a unique solution on $[0,T]$.

Proof. For $x, y \in X$ and by $(H1)$ and $(H2)$ we have:

$$\begin{aligned}
&\|\phi(x) - \phi(y)\| \\
\leq\ & \sup_{t\in[0,T]} \left\{ \int_0^t \tfrac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s)) - f(s,y(s))|\, ds \right. \\
&+ \int_0^t \tfrac{(t-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)} |f(s,x(s)) - f(s,y(s))|\, ds \\
&+ |\lambda| \int_0^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} |x(s) - y(s)|\, ds \\
&+ \tfrac{|\Delta| t^\beta}{\Gamma(\beta+1)} \int_0^T \tfrac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s,x(s)) - f(s,y(s))|\, ds \\
&+ \tfrac{|\Delta| t^\beta}{\Gamma(\beta+1)} \int_0^T \tfrac{(T-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)} |f(s,x(s)) - f(s,y(s))|\, ds \\
&+ \tfrac{|\lambda||\Delta| t^\beta}{\Gamma(\beta+1)} \int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)} |x(s) - y(s)|\, ds \\
&+ \tfrac{|\Delta||\theta| t^\beta}{\Gamma(\beta+1)} \int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)} |f(s,x(s)) - f(s,y(s))|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{|\Delta||\theta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta}\tfrac{(\eta-s)^{\alpha+\beta+p+\sigma-1}}{\Gamma(\alpha+\beta+p+\sigma)}\left|f\left(s,x\left(s\right)\right)-f\left(s,y\left(s\right)\right)\right|ds\\
&+\tfrac{|\Delta||\theta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta}\tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)}\left|x\left(s\right)-y\left(s\right)\right|ds\\
&+\left(\tfrac{|\Delta(\theta\eta^{p}-\Gamma(p+1))|T^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)}+1\right)\left|g\left(x\right)-g\left(y\right)\right|\Big\}\\
\leq\ &\sup_{t\in[0,T]}\Big\{\Big[\int_0^t\tfrac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds+\int_0^t\tfrac{(t-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}ds\\
&+\tfrac{|\Delta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{T}\tfrac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds+\tfrac{|\Delta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{T}\tfrac{(T-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}ds\\
&+\tfrac{|\Delta\theta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta}\tfrac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}ds+\tfrac{|\Delta\theta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta}\tfrac{(\eta-s)^{\alpha+\beta+p+\sigma-1}}{\Gamma(\alpha+\beta+p+\sigma)}ds\Big]\,\omega\\
&+|\lambda|\Big[\int_0^t\tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}ds+\tfrac{|\Delta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{T}\tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\\
&+\tfrac{|\Delta\theta|t^{\beta}}{\Gamma(\beta+1)}\int_0^{\eta}\tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)}ds\Big]+\left(\tfrac{|\Delta(\theta\eta^{p}-\Gamma(p+1))|t^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)}+1\right)\varpi\Big\}\left\|x-y\right\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ &\Big\{\Big[\tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\tfrac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}+\tfrac{|\Delta|T^{\beta}}{\Gamma(\beta+1)}\Big(\tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&+\tfrac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}+\tfrac{|\theta|\eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\tfrac{|\theta|\eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\Big)\Big]\,\omega\\
&+\left[\tfrac{|\lambda|T^{\beta}}{\Gamma(\beta+1)}\left(1+\tfrac{|\Delta|T^{\beta}}{\Gamma(\beta+1)}+\tfrac{|\Delta\theta|\eta^{\beta+p}}{\Gamma(\beta+p+1)}\right)\right]+\left(\tfrac{|\Delta(\theta\eta^{p}-\Gamma(p+1))|T^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)}+1\right)\varpi\Big\}\left\|x-y\right\|\\
=\ &\left(\Lambda\omega+\Lambda_2+\Lambda_1\varpi\right)\left\|x-y\right\|.
\end{aligned}
$$

Thanks to (3.5), we conclude that ϕ is a contraction. As a consequence of the Banach fixed point theorem, we deduce that ϕ has a fixed point which is a solution to the fractional boundary value problem (1.1). □

In the next result, we prove the existence of solutions to the fractional boundary value problem by applying the following Lemma.

Lemma 3.1. *(O'Regan Lemma) [15]. Denote by V an open set in a closed, convex set C of a Banach space E. Assume $0\in V$. Also assume that $\phi\left(\bar{V}\right)$ is bounded and that $\phi:\bar{V}\to C$ is given by $\phi=\phi_1+\phi_2$, in which $\phi_1:\bar{V}\to E$ is continuous and completely continuous and $\phi_2:\bar{V}\to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\varphi:(0,\infty)\to(0,\infty)$ satisfying $\varphi\left(u\right)<u$ for $v>0$ such that $\left\|\phi_2\left(x\right)-\phi_2\left(y\right)\right\|\leq\varphi\left\|x-y\right\|$ for all $x,y\in\bar{V}$). Then, either*

(I) : ϕ has a fixed point $x\in\bar{V}$; or

(II) : there exists a point $x\in\partial V$ and $0<\mu<1$ with $x=\mu\phi\left(x\right)$, where $\bar{V}$ (respectively ∂V) represents the closure (respectively the boundary) of V.

Let

$$\Omega := \{x \in C([0,T],\mathbb{R}) : \|x\| < \delta\},$$

and denote the maximum number by

$$N_\delta := \max\{|f(t,x)| : (t,x) \in [0,T] \times [\delta,-\delta]\}.$$

Theorem 3.2. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $(H2),(H3),(H4)$ and $(H5)$ are satisfied.*

Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. Consider the operator $\phi : X \to X$ defined by:

$$\phi(x)(t) := \phi_1(x)(t) + \phi_2(x)(t),\ t \in [0,T],$$

where the operators ϕ_1 and ϕ_2 are defined respectively in (2.6) and (2.7).

From (H_5) there exists a number $\delta_0 > 0$ such that

$$\frac{\delta_0}{\rho\psi(\delta_0) + \Lambda_1|x_0|} > \frac{1}{1-(\Lambda_2+\Lambda_1\varpi)}. \tag{3.6}$$

We shall prove that the operators ϕ_1 and ϕ_2 satisfy all the conditions in Lemma 9.

*Step*1 : We show that the operator $\phi_1 : \overline{\Omega}_{\delta_0} \to X$ is continuous and completely continuous. Let us consider the set

$$\overline{\Omega}_{\delta_0} := \{x \in C([0,T],\mathbb{R}) : \|x\| \le \delta_0\}, \tag{3.7}$$

and show that $\phi_1\left(\overline{\Omega}_{\delta_0}\right)$ is bounded. For each $x \in \overline{\Omega}_{\delta_0}$, we have

$$\begin{aligned}
&\|\phi_1(x)\| \\
\le\ & \sup_{t\in[0,T]}\Bigg\{\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))|\,ds + \int_0^t \frac{(t-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}|f(s,x(s))|\,ds \\
&+|\lambda|\int_0^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}|x(s)|\,ds + \tfrac{|\Delta|t^\beta}{\Gamma(\beta+1)}\Bigg[\int_0^T \tfrac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|f(s,x(s))|\,ds \\
&+\int_0^T \tfrac{(T-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}|f(s,x(s))|\,ds + |\lambda|\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}|x(s)|\,ds \\
&+|\theta|\int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}|f(s,x(s))|\,ds + |\theta|\int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p+\sigma-1}}{\Gamma(\alpha+\beta+p+\sigma)}|f(s,x(s))|\,ds \\
&+|\lambda\theta|\int_0^\eta \tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)}|x(s)|\,ds\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\le\ & \Bigg[\tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \tfrac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} + \tfrac{|\Delta|T^\beta}{\Gamma(\beta+1)}\Bigg(\tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&+\tfrac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} + \tfrac{|\theta|\eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \tfrac{|\theta|\eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\Bigg)\Bigg] N_{\delta_0}\|\gamma\| \\
&+\tfrac{|\lambda|T^\beta}{\Gamma(\beta+1)}\left(1 + \tfrac{|\Delta|T^\beta}{\Gamma(\beta+1)} + \tfrac{|\Delta\theta|\eta^{\beta+p}}{\Gamma(\beta+p+1)}\right)\delta_0 \\
=\ & \Lambda N_{\delta_0}\|\gamma\| + \Lambda_2\delta_0.
\end{aligned}$$

Thus the operator $\phi_1\left(\overline{\Omega}_{\delta_0}\right)$ is uniformly bounded. For any $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&\left|\phi_1 x\left(t_2\right)-\phi_1 x\left(t_1\right)\right| \\
\leq\; & \int_0^{t_1} \tfrac{\left[\left(t_2-s\right)^{\alpha+\beta-1}-\left(t_1-s\right)^{\alpha+\beta-1}\right]}{\Gamma(\alpha+\beta)}\left|f\left(s, x\left(s\right)\right)\right| ds+\int_{t_1}^{t_2} \tfrac{\left(t_2-s\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f\left(s, x\left(s\right)\right)\right| ds \\
&+\int_0^{t_1} \tfrac{\left[\left(t_2-s\right)^{\alpha+\beta+\sigma-1}-\left(t_1-s\right)^{\alpha+\beta+\sigma-1}\right]}{\Gamma(\alpha+\beta+\sigma)}\left|f\left(s, x\left(s\right)\right)\right| ds+\int_{t_1}^{t_2} \tfrac{\left(t_2-s\right)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}\left|f\left(s, x\left(s\right)\right)\right| ds \\
&+|\lambda| \int_0^{t_1} \tfrac{\left[\left(t_2-s\right)^{\beta-1}-\left(t_1-s\right)^{\beta-1}\right]}{\Gamma(\beta)}\left|x\left(s\right)\right| ds+|\lambda| \int_{t_1}^{t_2} \tfrac{\left(t_2-s\right)^{\beta-1}}{\Gamma(\beta)}\left|x\left(s\right)\right| ds \\
&+\tfrac{\left|t_2^\beta-t_1^\beta\right|}{\Gamma(\beta+1)}\left[|\Delta| \int_0^T \tfrac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f\left(s, x\left(s\right)\right)\right| ds+|\Delta| \int_0^T \tfrac{(T-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}\left|f\left(s, x\left(s\right)\right)\right| ds\right. \\
&+|\lambda \Delta| \int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}\left|x\left(s\right)\right| ds+|\Delta \theta| \int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}\left|f\left(s, x\left(s\right)\right)\right| ds \\
&\left.+\;|\Delta \theta| \int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p+\sigma-1}}{\Gamma(\alpha+\beta+p+\sigma)}\left|f\left(s, x\left(s\right)\right)\right| ds+|\lambda \Delta \theta| \int_0^\eta \tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)}\left|x\left(s\right)\right| ds\right]
\end{aligned}$$

$$\begin{aligned}
\leq\; & \frac{N_{\delta_0}\|\gamma\|}{\Gamma(\alpha+\beta+1)}\left|t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right|+\frac{N_{\delta_0}\|\gamma\|}{\Gamma(\alpha+\beta+\sigma+1)}\left|t_2^{\alpha+\beta+\sigma}-t_1^{\alpha+\beta+\sigma}\right| \\
&+\left[\frac{|\Delta| N_{\delta_0}\|\gamma\|}{\Gamma(\beta+1)}\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}\right.\right. \\
&\left.+\frac{|\theta| \eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\frac{|\theta| \eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\right) \\
&\left.+\frac{|\lambda| \delta_0}{\Gamma(\beta+1)}\left(1+\frac{|\Delta| T^\beta}{\Gamma(\beta+1)}+\frac{|\Delta \theta| \eta^{\beta+p}}{\Gamma(\beta+p+1)}\right)\right]\left|t_2^\beta-t_1^\beta\right|,
\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, ϕ_1 is equicontinuous. Hence, by the Arzelà–Ascoli theorem, $\phi_1\left(\overline{\Omega}_{\delta_0}\right)$ is a relatively compact set. Now, let the sequence $x_n \subset \overline{\Omega}_{\delta_0}$ with $x_n \to x$. Then $x_n\left(t\right) \to x\left(t\right)$ uniformly valid on $[0, T]$, then for each $t \in [0, T]$, we have. From the uniform continuity of $f\left(t, x\right)$ on the compact set $[0, T] \times [\delta_0, -\delta_0]$, it follows that $\left\|f\left(t, x_n\left(t\right)\right)-f\left(t, x\left(t\right)\right)\right\| \to 0$ is uniformly valid on J. Hence $\left\|\phi_1\left(x_n\right)\left(t\right)-\phi_1\left(x\right)\left(t\right)\right\| \to 0$ as $n \to \infty$, which proves the continuity of $\phi_1\left(\overline{\Omega}_{\delta_0}\right)$.

*Step*2 : The operator $\phi_2 : \overline{\Omega}_{\delta_0} \to X$ is contractive, this is the consequence of $\left(H_2\right)$.

*Step*3 : The set $\phi_2\left(\overline{\Omega}_{\delta_0}\right)$ is bounded. For any $x \in \overline{\Omega}_{\delta_0}$ and by $\left(H_2\right)$ and $\left(H_3\right)$, we have

$$\left\|\phi_2\left(x\right)\right\| \leq \Lambda_1\left(\left|x_0\right|+\varpi \delta_0\right),$$

combining, with the set $\phi_1\left(\overline{\Omega}_{\delta_0}\right)$ being bounded, then the set $\phi\left(\overline{\Omega}_{\delta_0}\right)$ is bounded.
*Step*4 : Finally, will be show that the case (II) in Lemma 9 does not hold. On the contrary, we suppose that (II) holds. Then, there exist $\mu \in (0,1)$ and $x \in \partial \Omega_{\delta_0}$,

such that $x=\mu\phi(x)$. So we have $\|x\|=\delta_0$ and

$$\begin{aligned}
&x(t)\\
=\;&\mu\Big[\int_0^t \tfrac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))\,ds+\int_0^t \tfrac{(t-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}f(s,x(s))\,ds\\
&-\lambda\int_0^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}x(s)\,ds-\tfrac{\Delta t^\beta}{\Gamma(\beta+1)}\int_0^T \tfrac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,x(s))\,ds\\
&-\tfrac{\Delta t^\beta}{\Gamma(\beta+1)}\int_0^T \tfrac{(T-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}f(s,x(s))\,ds+\tfrac{\lambda\Delta t^\beta}{\Gamma(\beta+1)}\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}x(s)\,ds\\
&+\tfrac{\Delta\theta t^\beta}{\Gamma(\beta+1)}\int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}f(s,x(s))\,ds+\tfrac{\Delta\theta t^\beta}{\Gamma(\beta+1)}\int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p+\sigma-1}}{\Gamma(\alpha+\beta+p+\sigma)}f(s,x(s))\,ds\\
&-\tfrac{\lambda\Delta\theta t^\beta}{\Gamma(\beta+1)}\int_0^\eta \tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)}x(s)\,ds+\Big(\tfrac{\Delta(\theta\eta^p-\Gamma(p+1))t^\beta}{\Gamma(p+1)\Gamma(\beta+1)}+1\Big)(x_0+g(x))\Big],\ t\in[0,T].
\end{aligned}$$

Using the hypotheses $(H3)-(H5)$ we get

$$\begin{aligned}
\|x\| \;\le\;& \Big[\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\gamma(s)\,ds+\int_0^t \frac{(t-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}\gamma(s)\,ds\\
&+\frac{|\Delta|\,t^\beta}{\Gamma(\beta+1)}\int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\gamma(s)\,ds\\
&+\frac{|\Delta|\,t^\beta}{\Gamma(\beta+1)}\int_0^T \frac{(T-s)^{\alpha+\beta+\sigma-1}}{\Gamma(\alpha+\beta+\sigma)}\gamma(s)\,ds\\
&+\tfrac{|\Delta\theta|t^\beta}{\Gamma(\beta+1)}\int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p-1}}{\Gamma(\alpha+\beta+p)}\gamma(s)\,ds\\
&+\tfrac{|\Delta\theta|t^\beta}{\Gamma(\beta+1)}\int_0^\eta \tfrac{(\eta-s)^{\alpha+\beta+p+\sigma-1}}{\Gamma(\alpha+\beta+p+\sigma)}\gamma(s)\,ds\Big]\psi(\|x\|)\\
&+|\lambda|\Big[\int_0^t \tfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}ds+\tfrac{|\Delta|t^\beta}{\Gamma(\beta+1)}\int_0^T \tfrac{(T-s)^{\beta-1}}{\Gamma(\beta)}ds\\
&\tfrac{|\Delta\theta|t^\beta}{\Gamma(\beta+1)}\int_0^\eta \tfrac{(\eta-s)^{\beta+p-1}}{\Gamma(\beta+p)}ds\Big]\|x\|\\
&+\Big(\tfrac{|\Delta(\theta\eta^p-\Gamma(p+1))|t^\beta}{\Gamma(p+1)\Gamma(\beta+1)}+1\Big)(x_0+\varpi\|x\|).
\end{aligned}$$

By (3.3) and (3.7), we obtain

$$\begin{aligned}
\delta_0 \;\le\;& \Big[\tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\tfrac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}+\tfrac{|\Delta|T^\beta}{\Gamma(\beta+1)}\Big(\tfrac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\tfrac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}\\
&+\tfrac{|\theta|\eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\tfrac{|\theta|\eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\Big)\Big]\|\rho\|\,\psi(\delta_0)\\
&+\Big[\tfrac{|\lambda|T^\beta}{\Gamma(\beta+1)}\Big(1+\tfrac{|\Delta|T^\beta}{\Gamma(\beta+1)}+\tfrac{|\Delta\theta|\eta^{\beta+p}}{\Gamma(\beta+p+1)}\Big)+\Big(\tfrac{|\Delta(\theta\eta^p-\Gamma(p+1))|T^\beta}{\Gamma(p+1)\Gamma(\beta+1)}+1\Big)\varpi\Big]\delta_0\\
&+\Big(\tfrac{|\Delta(\theta\eta^p-\Gamma(p+1))|T^\beta}{\Gamma(p+1)\Gamma(\beta+1)}+1\Big)|x_0|,
\end{aligned}$$

which implies

$$\delta_0 \leq \rho\psi\left(\delta_0\right) + \left(\Lambda_2 + \Lambda_1\varpi\right)\delta_0 + \Lambda_1\left|x_0\right|.$$

However,

$$\frac{\delta_0}{\rho\psi\left(\delta_0\right) + \Lambda_1\left|x_0\right|} \leq \frac{1}{1 - \left(\Lambda_2 + \Lambda_1\varpi\right)},$$

which contradicts (3.6). Consequently, the operators ϕ_1 and ϕ_2 satisfy all the conditions in Lemma 9. Hence, the operator ϕ has at least one fixed point $x \in \overline{\Omega}_{\delta_0}$, which is the solution of the fractional boundary value problem (1.1). This completes the proof. □

3.2. Ulam-Hyers stability

In this section, we will study Ulam's type stability of the fractional boundary value problem (1.1).

Let $\varepsilon > 0$, we consider the equation

$$D^{\alpha}\left(D^{\beta} + \lambda\right)x\left(t\right) = f\left(t, x\left(t\right)\right) + \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f\left(s, x\left(s\right)\right)ds$$

and the following inequality

$$\left|D^{\alpha}\left(D^{\beta} + \lambda\right)y\left(t\right) - f\left(t, y\left(t\right)\right) - \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f\left(s, y\left(s\right)\right)ds\right| \leq \varepsilon,\ t \in [0, T], \tag{3.8}$$

with $y\left(0\right) = y_0 + g\left(y\right),\, y\left(T\right) = \theta I^p y\left(\eta\right).$

Definition 3.1. The fractional boundary value problem (1.1) is Ulam-Hyers stable if there exists a real number $k > 0$ such that for each solution $y \in X$ to the inequality (3.8) there exists a solution $x \in X$ of the fractional boundary value problem (1.1) with

$$\|x - y\| \leq k\varepsilon.$$

Definition 3.2. The fractional boundary value problem (1.1) is generalized Ulam-Hyers stable if there exists $z \in C\left(\mathbb{R}^+, \mathbb{R}^+\right), z\left(0\right) = 0$ such that for each solution $y \in X$ to the inequality (3.8), there exists a solution $x \in X$ of the fractional boundary value problem (1.1) with

$$\|x - y\| \leq z\left(\varepsilon\right).$$

Theorem 3.3. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $(H_1) - (H_4)$ holds. In addition, we assume that:*

(H_6) $\sup\limits_{t\in[0,T]}\left|D^{\alpha}\left(D^{\beta} + \lambda\right)x\left(t\right)\right| \geq \Lambda N_{\delta_0}\|\gamma\| + \left(\Lambda_2 + \Lambda_1\varpi\right)\delta_0 + \Lambda_1\left|x_0\right|.$

If

$$\omega < \frac{\Gamma\left(\sigma + 1\right)}{\left[\Gamma\left(\sigma + 1\right) + T^{\sigma}\right]}, \tag{3.9}$$

then the fractional boundary value problem (1.1) has the Ulam-Hyers stability in X.

Proof. For each $\varepsilon > 0$, $y \in X$, we have

$$\left| D^{\alpha}\left(D^{\beta}+\lambda\right) y(t) - f(t, y(t)) - \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, y(s))\, ds \right| \leq \varepsilon,$$

with $y(0) = y_0 + g(y)$, $y(T) = \theta I^p y(\eta)$.

Let us denote by $x \in X$ the unique solution of the fractional boundary value problem(1.1).

According to the assumptions of Theorem 8, we have

$$|x(t)| \leq \Lambda N_{\delta_0} \|\gamma\| + (\Lambda_2 + \Lambda_1 \varpi)\,\delta_0 + \Lambda_1 |x_0|, \; t \in [0, T].$$

By (H_6), we get

$$\sup_{t\in[0,T]} |x(t)| \leq \sup_{t\in[0,T]} \left| D^{\alpha}\left(D^{\beta}+\lambda\right) x(t)\right|.$$

Then

$$\begin{aligned}
\sup_{t\in[0,T]} |x(t) - y(t)| &\leq \sup_{t\in[0,T]} \left| D^{\alpha}\left(D^{\beta}+\lambda\right)(x(t) - y(t))\right| \\
&\leq \sup_{t\in[0,T]} \Big| D^{\alpha}\left(D^{\beta}+\lambda\right) x(t) - f(t, x(t)) - \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, x(s))\, ds \\
&\quad - D^{\alpha}\left(D^{\beta}+\lambda\right) y(t) + f(t, y(t)) + \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, y(s))\, ds \\
&\quad + f(t, x(t)) + \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, x(s))\, ds \\
&\quad - f(t, y(t)) - \int_0^t \tfrac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} f(s, y(s))\, ds \Big| \\
&\leq 2\varepsilon + \left(1 + \tfrac{T^{\sigma}}{\Gamma(\sigma+1)}\right) \omega \sup_{t\in[0,T]} |x(t) - y(t)|.
\end{aligned}$$

Hence

$$\|x - y\| \leq \frac{2}{1 - \left(1 + \frac{T^{\sigma}}{\Gamma(\sigma+1)}\right)\omega} \varepsilon = k\varepsilon.$$

Thus, the fractional boundary value problem (1.1) has the Ulam-Hyers stability in X. □

Remark 3.1. By putting $z(\varepsilon) = k\varepsilon, z(\varepsilon) = 0$ yields that the fractional boundary value problem (1.1) has the generalized Ulam-Hyers stability in X.

4. Examples

To illustrate our main results, we treat the following examples.

Example 4.1. Let us consider the following fractional boundary value problem:

$$(4.1)\qquad \begin{cases} D^{\frac{1}{2}}\left(D^{\frac{1}{2}}+\frac{3}{20}\right)x(t)=\left(\frac{e^{-\pi t}|x(t)|}{\left(25\sqrt{\pi}+e^{-\pi t}\right)(1+|x(t)|)}+\frac{1}{2}+\cosh\left(t^2+2\right)\right) \\ +\int_0^t \frac{(t-s)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{e^{-\pi s}|x(s)|}{\left(25\sqrt{\pi}+e^{-\pi s}\right)(1+|x(s)|)}+\frac{1}{2}+\cosh\left(s^2+2\right)\right)ds,\ t\in[0,1], \\ x(0)=\frac{\sqrt{2}}{3}+\frac{1}{19}x(\zeta),\ x(1)=\frac{5}{6}\int_0^{\frac{1}{4}}\frac{\left(\frac{1}{4}-s\right)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)}ds, 0<\zeta<1. \end{cases}$$

with $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, $\lambda=\frac{3}{20}$, $\sigma=\frac{5}{2}$, $\theta=\frac{5}{6}$, $p=\frac{4}{3}$, $\eta=\frac{1}{4}$ and

$$f(t,x)=\frac{e^{-\pi t}|x(t)|}{\left(25\sqrt{\pi}+e^{-\pi t}\right)(1+|x(t)|)}+\frac{1}{2}+\cosh\left(t^2+2\right),\ g(x)=\frac{3}{7}x(\zeta).$$

Let $x,y\in\mathbb{R}$ and $t\in[0,1]$. Then

$$|f(t,x)-f(t,y)|\leq\frac{e^{-\pi t}}{\left(25\sqrt{\pi}+e^{-\pi t}\right)}|x-y|\leq\frac{1}{25\sqrt{\pi}+e^{-\pi}}|x-y|.$$

Hence the condition (H_1) holds with $\omega=\frac{1}{25\sqrt{\pi}+e^{-\pi}}$. Also, for $x,y\in C[0,1]$, we have

$$|g(t,x)-g(t,y)|\leq\frac{1}{19}|x-y|.$$

Hence (H_2) is satisfied with $\varpi=\frac{1}{19}$. We can find that

$$\begin{aligned} \Delta &:= \frac{\Gamma(\beta+p+1)\Gamma(\beta+1)}{\Gamma(\beta+p+1)T^{\beta}-\Gamma(\beta+1)\theta\eta^{\beta+p}}=0.917\,16, \\ \Lambda &:= \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}+\frac{|\Delta|T^{\beta}}{\Gamma(\beta+1)}\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right. \\ &\quad \left.+\frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)}+\frac{|\theta|\eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)}+\frac{|\theta|\eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\right)=2.222\,1, \\ \Lambda_1 &:= \frac{|\Delta(\theta\eta^{p}-\Gamma(p+1))|T^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)}+1=1.920\,83, \\ \Lambda_2 &:= \frac{|\lambda|T^{\beta}}{\Gamma(\beta+1)}\left(1+\frac{|\Delta|T^{\beta}}{\Gamma(\beta+1)}+\frac{|\Delta\theta|\eta^{\beta+p}}{\Gamma(\beta+p+1)}\right)=0.350\,33. \end{aligned}$$

Therefore, we have

$$\Lambda\omega+\Lambda_1\varpi<1-\Lambda_2.$$

Hence, by Theorem 6, the fractional boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2. Consider the following fractional boundary value problem:

$$(4.2)\qquad \begin{cases} D^{\frac{3}{7}}\left(D^{\frac{1}{2e}}+\frac{1}{17}\right)x(t)=\frac{\tanh\left(t+e^{1-t}\right)\sqrt{(1+t)}\sin\left(x^2-1\right)}{e^{\frac{1+3t}{5}}\left(|x|+t^2+1\right)} \\ +\int_0^t \frac{(t-s)^{\frac{e^2}{3}-1}}{\Gamma\left(\frac{e^2}{3}\right)}\left(\frac{\tanh\left(t+e^{1-t}\right)\sqrt{(1+t)}\sin\left(x^2(s)-1\right)}{25e^{\frac{1+3t}{5}}\left(|x(s)|+t^2+1\right)}\right)ds, t\in[0,1], \\ x(0)=\frac{1}{9}+\frac{\ln 3}{11}x(\xi),\ x(1)=\frac{1}{10}\int_0^{\frac{2}{7}}\frac{\left(\frac{2}{7}-s\right)^{\frac{1}{7}}}{\Gamma\left(\frac{8}{7}\right)}ds, 0<\xi<1, \end{cases}$$

with $\alpha=\frac{3}{7},\beta=\frac{1}{2e},\lambda=\frac{1}{17},\sigma=\frac{e^2}{3},x_0=\frac{1}{9},\theta=\frac{1}{10},p=\frac{8}{7},\eta=\frac{2}{7}$ and $f(t,x)=\frac{\tanh\left(t+e^{1-t}\right)\sqrt{(1+t)}\sin\left(x^2-1\right)}{25e^{\frac{1+3t}{5}}\left(|x|+t^2+1\right)}, g(x)=\frac{\ln 3}{11}x(\zeta)$.

For any $x, y \in C[0,1]$, we have

$$|g(x) - g(y)| \leq \frac{\ln 3}{11} \|x - y\|,$$

which implies that the function $g(x) = \frac{\ln 3}{11} x(\zeta)$ is contractive. Moreover, $g(0) = 0$. Hence, the condition (H_3) is satisfied. Also for $x, y \in$ and $t \in [0,1]$, we have

$$|f(t,x)| \leq \frac{\tanh\left(t + e^{1-t}\right)\sqrt{(1+t)}}{25 e^{\frac{1+3t}{5}}} (|x| - 1).$$

So, we take $\gamma(t) = \frac{\tanh\left(t+e^{1-t}\right)\sqrt{(1+t)}}{25e^{\frac{1+3t}{5}}}$ and $\psi(|x|) = |x| + 1$, then the condition (H_4) is satisfied. With the given values, it is found that

$$\begin{aligned}
\|\gamma\| &= 4.5912 \times 10^{-2}, \\
\rho \ &:= \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} + \frac{|\Delta| T^{\beta}}{\Gamma(\beta+1)} \left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right.\right. \\
&\quad \left.\left. + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} + \frac{|\theta|\eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \frac{|\theta|\eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\right)\right] \|\gamma\| = 0.135\,10,
\end{aligned}$$

$$\begin{aligned}
\Delta \ &:= \frac{\Gamma(\beta+p+1)\Gamma(\beta+1)}{\Gamma(\beta+p+1)T^{\beta} - \Gamma(\beta+1)\theta\eta^{\beta+p}} = 0.936\,41, \\
\Lambda \ &:= \frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} + \frac{|\Delta| T^{\beta}}{\Gamma(\beta+1)} \left[\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right. \\
&\quad \left. + \frac{T^{\alpha+\beta+\sigma}}{\Gamma(\alpha+\beta+\sigma+1)} + \frac{|\theta|\eta^{\alpha+\beta+p}}{\Gamma(\alpha+\beta+p+1)} + \frac{|\theta|\eta^{\alpha+\beta+p+\sigma}}{\Gamma(\alpha+\beta+p+\sigma+1)}\right] = 2.\,563\,6, \\
\Lambda_1 \ &:= \frac{|\Delta(\theta\eta^{p} - \Gamma(p+1))| T^{\beta}}{\Gamma(p+1)\Gamma(\beta+1)} + 1 = 1.992\,30, \\
\Lambda_2 \ &:= \frac{|\lambda| T^{\beta}}{\Gamma(\beta+1)} \left(1 + \frac{|\Delta| T^{\beta}}{\Gamma(\beta+1)} + \frac{|\Delta\theta|\eta^{\beta+p}}{\Gamma(\beta+p+1)}\right) = 0.129\,43.
\end{aligned}$$

and the condition

$$\frac{\delta_0}{\Lambda_1|x_0| + \rho\psi(\delta_0)} > \frac{1}{1 - (\Lambda_1\varpi + \Lambda_2)},$$

implies that $\delta_0 > 0.622\,71$. Clearly all the conditions of Theorem 10 are satisfied. Hence by the conclusion of Theorem 10, the fractional boundary value problem (4.2) has a solution on $[0,1]$.

Example 4.3. Consider:

(4.3)

$$\begin{cases}
D^{\frac{2}{3}}\left(D^{\frac{5}{6}} + \frac{2}{19}\right) x(t) = \frac{1}{23(\ln(t+1)+1)}\left(\sinh t + \frac{|x(t)|}{1+|x(t)|} + |x(t)|\right) + 1 + \ln(t+3) \\
+ \int_0^t \frac{(t-s)^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \left(\frac{1}{23(\ln(t+1)+1)}\left(\sinh t + \frac{|x(s)|}{1+|x(s)|} + |x(s)|\right) + 1 + \ln(s+3)\right) ds, \ t \in [0,1], \\
x(0) = \sum_{i=1}^{n} c_i \frac{|x(t_i)|}{1+|x(t_i)|}, \ x(1) = \sqrt{2} \int_0^{\frac{1}{3}} \frac{\left(\frac{1}{3}-s\right)^{\frac{1}{6}}}{\Gamma\left(\frac{7}{6}\right)} ds.
\end{cases}$$

where $0 < t_1 < t_2 < \ldots < t_n < 1, c_i, i = 1, 2, \ldots, n$, are given positive constants with $\sum_{i=1}^{n} c_i < \frac{1}{2}$.

Consider the fractional boundary value problem (4.3), with, $\alpha = \frac{3}{2}$, $\beta = \frac{5}{6}$, $\lambda = \frac{2}{19}, \sigma = \frac{4}{3}$, $\theta = \sqrt{2}$, $p = \frac{7}{6}, \eta = \frac{1}{3}$ and $f(t,x) = \frac{e^{-\pi t}|x(t)|}{\left(25\sqrt{\pi} + e^{-\pi t}\right)(1+|x(t)|)} + \frac{1}{2} + \cosh\left(t^2 + 2\right)$, $g(x) = \sum_{i=1}^{n} c_i \frac{|x(t_i)|}{1+|x(t_i)|}$.

Let $t \in [0,1]$ and $x, y \in \mathbb{R}$. Then

$$|f(t,x) - f(t,y)| \leq \left| \frac{1}{23\,(\ln(t+1)+1)} \right| |x-y| \leq \frac{1}{23} |x-y| .$$

Hence the condition (H_1) holds with $\omega = \frac{1}{23}$. Also, for any $x, y \in C([0,1])$, we have

$$|g(x) - g(y)| \leq \sum_{i=1}^{n} c_i \, |x-y| .$$

So, (H_2) is satisfied with $\varpi = \sum_{i=1}^{n} c_i < \frac{1}{2}$.

Thus the condition

$$\omega = 4.3478 \times 10^{-2} < \frac{\Gamma(\sigma+1)}{[\Gamma(\sigma+1) + T^{\sigma}]} = 0.5435.$$

is satisfied. It follows from Theorem 8 that the fractional boundary value problem (4.3) has a unique solution on $[0,1]$, and from Theorem 13, the fractional boundary value problem (4.3) has the Ulam-Hyers stability.

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HILBERT MATRIX AND DIFFERENCE OPERATOR OF ORDER m

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Abstract. In this paper, some applications of the Hilbert matrix in image processing and cryptology are mentioned and an algorithm related to the Hilbert view of a digital image is given. New matrix domains are constructed and some of their properties are investigated. Furthermore, dual spaces of new matrix domains are computed and matrix transformations are characterized. Finally, examples of transformations of new spaces are given.
Keywords. Hilbert matrix; cryptology; image processing; matrix domains.

1. Introduction

1.1. Hilbert Matrix and Applications

We consider the matrices H and H_n as follows:

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots & \\ 1/3 & 1/4 & \cdots & & \\ 1/4 & \cdots & & & \\ \vdots & \vdots & & & \end{bmatrix}$$

$$H_n = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & & \\ 1/3 & 1/4 & \cdots & & & \\ \vdots & \vdots & & & & \\ 1/n & \cdots 1/(2n-1) & & & & \end{bmatrix}$$

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It is well known that these matrices are called the infinite Hilbert matrix and the $n \times n$ Hilbert matrix, respectively. A famous inequality of Hilbert ([8], Section 9) asserts that the matrix H determines a bounded linear operator on the Hilbert space of square summable complex sequences. Also, $n \times n$ Hilbert matrices are well-known examples of extremely ill-conditioned matrices.

Frequently, Hilbert matrices are used in both mathematics and computational science. For example, in image processing, Hilbert matrices are commonly used. Any $2D$ array of natural numbers in the range $[0, n]$ for all $n \in \mathbb{N}$ can be viewed as a greyscale digital image.

We take the Hilbert matrix $H_n(n \times n$ matrix). If we use the Mathematica, then we can write

$$hilbert = HilbertMatrix[5]//MatrixForm$$

and we can obtain

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

Now, we use **MatrixPlot** to obtain the image shown in Fig. 1..1

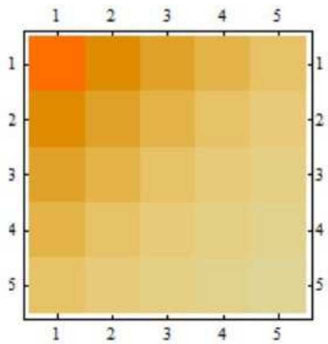

Fig. 1..1: 2D Plot

With the following algorithm that used the Mathematica script[16], we can obtain the Hilbert view of a digital image:

i. Extract an $n \times n$ subimage h from your favorite greyscale digital image.

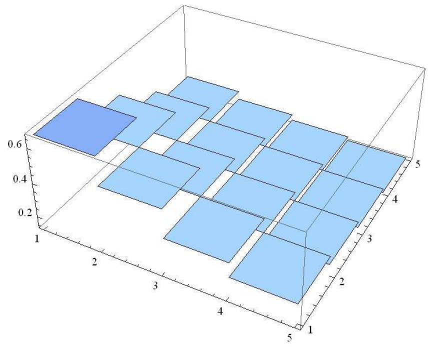

Fig. 1..2: 3D Plot

ii. Multiply the subimage h by the corresponding Hilbert $n \times n$ matrix. Let $hilbertim = h. * HilbertMatrix[n]$.

iii. Produce a $2D$ MatrixPlot for *hilbertim* like the one in Fig. 1..1

iv. Use the following scrip to produce a $3D$ plot for *hilbertim* like the one in Fig. 1..2

$$ListPlot3D[HilbertMatrix[n], \to 0, Mesh \to None].$$

Again, cryptography is an example of Hilbert matrix applications. Cryptography is a science of using mathematics to encrypt and decrypt data. A classical crypt-analysis involves an interesting combination of analytical reasoning, application of mathematical tools and pattern finding. In some studies related to cryptographic methods, the Hilbert matrix is used for authentication and confidentiality[18]. It is well known that the Hilbert matrix is very unstable [15] and so it can be used in security systems.

2. The Hilbert matrix, difference operator and new spaces

Let ω, ℓ_∞, c, c_0, ϕ denote sets of all complex, bounded, convergent, null convergent and finite sequences, respectively. Also, for the sets of convergent, bounded and absolutely convergent series, we denote cs, bs and ℓ_1.

Let $A = (a_{nk})$, $(n, k \in \mathbb{N})$ be an infinite matrix of complex numbers and X, Y be subsets of ω. We write $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)$ for all $n \in \mathbb{N}$. All the series $A_n x$ converge. The set $X_A = \{x \in \omega : Ax \in X\}$ is called the matrix domain of A in X. We write $(X : Y)$ for the space of those matrices which send the whole of sequence space X into the sequence space Y in this sense.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. For the triangle matrices A, B and a sequence x, $A(Bx) = (AB)x$ holds. We remark that a triangle matrix A uniquely has an inverse $A^{-1} = B$ and the matrix B is also a triangle.

Let X be a normed sequence space. If X contains a sequence (b_n) with the property that for every $x \in X$, then there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} \alpha_k b_k \right\| = 0.$$

Thus, (b_n) is called Schauder basis for X.

In [11], new sequence spaces are defined and some topological and structural properties are investigated. A flow chart of the stages of the newly constructed sequence spaces and the algorithms of the workings at each step are given by Kirisci [11].

The difference operator was first used in the sequence spaces by Kızmaz[12]. The idea of difference sequence spaces of Kızmaz was generalized by Çolak and Et[6, 7]. The difference matrix $\Delta = \delta_{nk}$ defined by

$$\delta_{nk} := \begin{cases} (-1)^{n-k} & , \quad (n-1 \leq k \leq n) \\ 0 & , \quad (0 < n-1 \text{ or } n > k) \end{cases}$$

The difference operator order of m is defined by $\Delta^m : \omega \to \omega$, $(\Delta^1 x)_k = (x_k - x_{k-1})$ and $\Delta^m x = (\Delta^1 x)_k \circ (\Delta^{m-1} x)_k$ for $m \geq 2$.

The triangle matrix $\Delta^{(m)} = \delta_{nk}^{(m)}$ defined by

$$\delta_{nk} := \begin{cases} (-1)^{n-k}\binom{m}{n-k} & , \quad (\max\{0, n-m\} \leq k \leq n) \\ 0 & , \quad (0 \leq k < \max\{0, n-m\} \text{ or } n > k) \end{cases}$$

for all $k, n \in \mathbb{N}$ and for any fixed $m \in \mathbb{N}$.

We can also mention Fibonacci matrices as an example of difference matrices[9].

The Hilbert matrix is defined by $H_n = [h_{ij}] = [\frac{1}{i+j-1}]_{i,j=1}^{n}$ for each $n \in \mathbb{N}$. The inverse of Hilbert matrix is defined by

$$H_n^{-1} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-1}{i-1}^2 \tag{2.1}$$

for all $k, i, j, n \in \mathbb{N}$[5]. Polat [17], has defined new spaces by using the Hilbert matrix. Let h_c, h_0, h_∞ be convergent Hilbert, null convergent Hilbert and bounded Hilbert spaces, respectively. Then, we have:

$$X = \{x \in \omega : Hx \in Y\}$$

where $X = \{h_c, h_0, h_\infty\}$ and $Y = \{c, c_0, \ell_\infty\}$.

Now, we will give new difference Hilbert sequence spaces as below:

$$\begin{aligned} h_c\left(\Delta^{(m)}\right) &= \left\{x \in \omega : \Delta^{(m)} x \in h_c\right\} \\ h_0\left(\Delta^{(m)}\right) &= \left\{x \in \omega : \Delta^{(m)} x \in h_0\right\} \\ h_\infty\left(\Delta^{(m)}\right) &= \left\{x \in \omega : \Delta^{(m)} x \in h_\infty\right\}. \end{aligned}$$

These new spaces, as the set of all sequences whose $\Delta^{(m)}-$transforms are in the Hilbert sequences spaces which are defined by Polat[17].

We also define the $H\Delta^{(m)}-$transform of a sequence, as below:

$$y_n = \left(H\Delta^{(m)} x\right)_n = \sum_{k=1}^{n} \left[\sum_{i=k}^{n} \frac{1}{n+i-1}(-1)^{i-k}\binom{m}{i-k}\right] x_k \tag{2.2}$$

for each $m, n \in \mathbb{N}$. Here and after by $H^{(m)}$, we denote the matrix $H^{(m)} = H\Delta^{(m)} = [h_{nk}]$ defined by

$$h_{nk} = \sum_{i=k}^{n} \frac{1}{n+i-1}(-1)^{i-k}\binom{m}{i-k}$$

for each $k, m, n \in \mathbb{N}$.

Theorem 2.1. *The Hilbert sequences spaces derived by the difference operator of m are isomorphic copies of the convergent, null convergent and bounded sequence spaces.*

Proof. We will only prove that the null convergent Hilbert sequence space is an isomorphic copy of the null convergent sequence space. To prove the fact $h_0(\Delta^{(m)}) \cong c_0$, we should show the existence of a linear bijection between the spaces $h_0(\Delta^{(m)})$ and c_0. Consider the transformation T, defined with the notation (2.2) from $h_0(\Delta^{(m)})$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective. Let $y \in c_0$ and define the sequence $x = (x_n)$ by

$$x_n = \sum_{k=1}^{n} \left[\sum_{i=k}^{n} \binom{m+n-i-1}{n-i} h_{ik}^{-1}\right] y_k \tag{2.3}$$

where h_{ik}^{-1} is defined by (2.1). Then,

$$\begin{aligned}\lim_{n\to\infty}\left(H\Delta^{(m)}x\right)_k &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}\Delta^{(m)}x_k \\ &= \sum_{k=1}^{n}\frac{1}{n+k-1}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i} \\ &= \sum_{k=1}^{n}\left[\sum_{i=k}^{n}\frac{1}{n+i-1}(-1)^{i-k}\binom{m}{i-k}\right]x_k = \lim_{n\to\infty}y_n = 0.\end{aligned}$$

Thus, we have that $x \in h_0(\Delta^{(m)})$. Consequently, T is surjective and is norm preserving. Hence, T is linear bijection which implies that the null convergent Hilbert sequence space is an isomorphic copy of the null convergent sequence space. □

It is well known that the spaces c, c_0 and ℓ_∞ are $BK-$spaces. Considering the fact that $\Delta^{(m)}$ is a triangle, we can say that the Hilbert sequences spaces derived by the difference operator of m are $BK-$spaces with the norm

$$\|x\|_\Delta = \|H\Delta^{(m)}x\|_\infty = \sup_n\left|\sum_{k=1}^{n}\frac{1}{n+k-1}\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}\right|. \tag{2.4}$$

In the theory of matrix domain, it is well known that the matrix domain X_A of a normed sequence space X has a basis if and only if X has a basis whenever $A = (a_{nk})$ is a triangle. Then, we have:

Corollary 2.2. *Define the sequence* $b^{(k)} = (b_n^{(k)}(\Delta^{(m)}))_{n\in\mathbb{N}}$ *by*

$$b_n^{(k)}(\Delta^{(m)}) := \begin{cases}\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h_{ik}^{-1} & , \quad (n\geq k)\\ 0 & , \quad (n<k)\end{cases}$$

for every fixed $k \in \mathbb{N}$. *The following statements hold:*

i. The sequence $b^{(k)}(\Delta^{(m)}) = (b_n^{(k)}(\Delta^{(m)}))_{n\in\mathbb{N}}$ *is a basis for the null convergent Hilbert sequence space, and for any* $x \in h_0(\Delta^{(m)})$ *has a unique representation of the form*

$$x = \sum_k\left(H\Delta^{(m)}x\right)_k b^{(k)}.$$

ii. The set $\{t, b^{(1)}, b^{(2)}, \cdots\}$ *is a basis for the convergent Hilbert sequence space, and for any* $x \in h_c(\Delta^{(m)})$ *has a unique representation of form*

$$x = st + \sum_k\left[\left(H\Delta^{(m)}x\right)_k - s\right]b^{(k)}$$

where $t = t_n(\Delta^{(m)}) = \sum_{k=1}^{n}\sum_{i=k}^{n}\binom{m+n-i-1}{n-i}h_{ik}^{-1}$ *for all* $k \in \mathbb{N}$ *and* $s = \lim_{k\to\infty}(H\Delta^{(m)}x)_k$.

If we take into consideration the fact that a space which has a Schauder basis is separable, then we can give the following corollary:

Corollary 2.3. *The convergent Hilbert and null convergent Hilbert sequence spaces are separable.*

3. Dual Spaces and Matrix Transformations

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space* of X and Y. In the special cases of $Y = \{\ell_1, cs, bs\}$, we write $x^{\alpha} = x^{-1} * \ell_1$, $x^{\beta} = x^{-1} * cs$, $x^{\gamma} = x^{-1} * bs$ and $X^{\alpha} = M(X, \ell_1)$, $X^{\beta} = M(X, cs)$, $X^{\gamma} = M(X, bs)$ for the $\alpha-$dual, $\beta-$dual, $\gamma-$dual of X. By $A_n = (a_{nk})_{k=0}^{\infty}$ we denote the sequence in the $n-$th row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ $n = (0, 1, ...)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n.

Lemma 3.1. *[4, Lemma 5.3] Let X, Y be any two sequence spaces. $A \in (X : Y_T)$ if and only if $TA \in (X : Y)$, where A an infinite matrix and T a triangle matrix.*

Lemma 3.2. *[1, Theorem 3.1] Let $U = (u_{nk})$, be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$. Let $B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and inverse of the triangle matrix $U = (u_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} a_j v_{jk}$$

for all $k, n \in \mathbb{N}$. Then,

$$X_U^{\beta} = \{a = (a_k) \in \omega : B^U \in (X : c)\}.$$

and

$$X_U^{\gamma} = \{a = (a_k) \in \omega : B^U \in (X : \ell_{\infty})\}.$$

Now, we list the following useful conditions.

Table 3.1:

To → From ↓	ℓ_∞	c	bs	cs
c_0	**1.**	**2.**	**3.**	**4.**
c	**1.**	**5.**	**3.**	**6.**
ℓ_∞	**1.**	**7.**	**3.**	**8.**

$$\sup_n \sum_k |a_{nk}| < \infty \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0 \tag{3.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} \quad \textit{exists} \tag{3.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n\to\infty} a_{nk} \right| \tag{3.4}$$

$$\lim_n a_{nk} = 0 \quad \text{for all k} \tag{3.5}$$

$$\sup_m \sum_k \left| \sum_{n=0}^{m} \right| < \infty \tag{3.6}$$

$$\sum_n a_{nk} \quad \textit{convergent for all } k \tag{3.7}$$

$$\sum_n \sum_k a_{nk} \quad \textit{convergent} \tag{3.8}$$

$$\lim_n a_{nk} \quad \text{exists for all k} \tag{3.9}$$

$$\lim_m \sum_k \left| \sum_{n=m}^{\infty} a_{nk} \right| = 0 \tag{3.10}$$

Lemma 3.3. *For the characterization of the class $(X : Y)$ with $X = \{c_0, c, \ell_\infty\}$ and $Y = \{\ell_\infty, c, cs, bs\}$, we can give the necessary and sufficient conditions from Table 3.1, where*

1. *(3.1)*	***2.*** *(3.1), (3.9)*	***3.*** *(3.6)*	***4.*** *(3.6), (3.7)*
5. *(3.1), (3.9), (3.3)*	***6.*** *(3.6), (3.7), (3.8)*	***7.*** *(3.9), (3.4)*	***8.*** *(3.10)*

Let h_{nk}^{-1} is defined by (2.1). For the proof of Theorem 3.4, we define the matrix $V = (v_{nk})$ as below:

$$v_{nk} = \left[\sum_{i=k}^{n} \binom{m+n-i-1}{n-i} h_{ik}^{-1} a_n \right] \tag{3.11}$$

Theorem 3.4. *The $\beta-$ and $\gamma-$ duals of the Hilbert sequence spaces derived by the difference operator of m defined by*

$$\left[h_{c_0}\left(\Delta^{(m)}\right)\right]^{\beta} = \{a=(a_k)\in\omega : V\in(c_0:c)\}$$
$$\left[h_{c}\left(\Delta^{(m)}\right)\right]^{\beta} = \{a=(a_k)\in\omega : V\in(c:c)\}$$
$$\left[h_{\infty}\left(\Delta^{(m)}\right)\right]^{\beta} = \{a=(a_k)\in\omega : V\in(\ell_\infty:c)\}$$
$$\left[h_{c_0}\left(\Delta^{(m)}\right)\right]^{\gamma} = \{a=(a_k)\in\omega : V\in(c_0:\ell_\infty)\}$$
$$\left[h_{c}\left(\Delta^{(m)}\right)\right]^{\gamma} = \{a=(a_k)\in\omega : V\in(c:\ell_\infty)\}$$
$$\left[h_{\infty}\left(\Delta^{(m)}\right)\right]^{\gamma} = \{a=(a_k)\in\omega : V\in(\ell_\infty:\ell_\infty)\}$$

Proof. We will only show the $\beta-$ and $\gamma-$ duals of the null convergent Hilbert sequence spaces derived by difference operator of m. Let $a=(a_k)\in\omega$. We begin the equality

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n}\sum_{i=1}^{k}\left[\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}h_{ij}^{-1}\right]a_k y_i \tag{3.12}$$

$$\begin{aligned} &= \sum_{k=1}^{n}\left\{\sum_{i=1}^{k}\left[\sum_{j=i}^{k}\binom{m+k-j-1}{k-j}h_{ij}^{-1}\right]a_i\right\}y_k \\ &= (Vy)_n \end{aligned}$$

where $V=(v_{nk})$ is defined by (3.11). Using (3.12), we can see that $ax=(a_kx_k)\in cs$ or bs whenever $x=(x_k)\in h_{c_0}(\Delta^{(m)})$ if and only if $Vy\in c$ or ℓ_∞ whenever $y=(y_k)\in c_0$. Then, from Lemma 3.1 and Lemma 3.2, we obtain the result that $a=(a_k)\in\left(h_{c_0}(\Delta^{(m)})\right)^{\beta}$ or $a=(a_k)\in\left(h_{c_0}(\Delta^{(m)})\right)^{\gamma}$ if and only if $V\in(c_0:c)$ or $V\in(c_0:\ell_\infty)$, which is what we wished to prove. □

Therefore, the $\beta-$ and $\gamma-$ duals of new spaces will help us in the characterization of the matrix transformations.

Let X and Y be arbitrary subsets of ω. We shall show that the characterizations of the classes (X,Y_T) and (X_T,Y) can be reduced to that of (X,Y), where T is a triangle.

It is well known that if $h_{c_0}(\Delta^{(m)})\cong c_0$, then the equivalence

$$x\in h_{c_0}(\Delta^{(m)})\Leftrightarrow y\in c_0$$

holds. Then, the following theorems will be proved and given some corollaries which can be obtained in a way similar to that of Theorems 3.5 and 3.6. Then, using Lemmas 3.1 and 3.2, we have:

Theorem 3.5. *Consider the infinite matrices $A = (a_{nk})$ and $D = (d_{nk})$. These matrices get associated with each other the following relations:*

$$d_{nk} = \sum_{j=k}^{\infty} \binom{m+n-j-1}{n-j} h_{jk}^{-1} a_{nj} \tag{3.13}$$

for all $k, m, n \in \mathbb{N}$. Then, the following statements are true:
i. *$A \in (h_{c_0}(\Delta^{(m)}) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [h_{c_0}(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and $D \in (c_0 : Y)$, where Y be any sequence space.*
ii. *$A \in (h_c(\Delta^{(m)}) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [h_c(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and $D \in (c : Y)$, where Y be any sequence space.*
iii. *$A \in (h_\infty(\Delta^{(m)}) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [h_\infty(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and $D \in (\ell_\infty : Y)$, where Y be any sequence space.*

Proof. We assume that the (3.13) holds between the entries of $A = (a_{nk})$ and $D = (d_{nk})$. Let us remember that from Theorem 2.1, the spaces $h_{c_0}(\Delta^{(m)})$ and c_0 are linearly isomorphic. Firstly, we choose any $y = (y_k) \in c_0$ and consider $A \in (h_{c_0}(\Delta^{(m)}) : Y)$. Then, we obtain that $DH\Delta^{(m)}$ exists and $\{a_{nk}\} \in \left(h_{c_0}\Delta^{(m)}\right)^\beta$ for all $k \in \mathbb{N}$. Therefore, the necessity of (3.13) yields and $\{d_{nk}\} \in c_0^\beta$ for all $k, n \in \mathbb{N}$. Hence, Dy exists for each $y \in c_0$. Thus, if we take $m \to \infty$ in the equality

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m} \left[\sum_{i=1}^{k} \sum_{j=i}^{k} \binom{m+k-j-1}{k-j} h_{ij}^{-1} \right] a_{nk} = \sum_{k} d_{nk}y_k$$

for all $m, n \in \mathbb{N}$, then, we understand that $Dy = Ax$. So, we obtain that $D \in (c_0 : Y)$.

Now, we consider that $\{a_{nk}\}_{k\in\mathbb{N}} \in (h_{c_0}\Delta^{(m)})^\beta$ for all $n \in \mathbb{N}$ and $D \in (c_0 : Y)$. We take any $x = (x_k) \in h_{c_0}\Delta^{(m)}$. Then, we can see that Ax exists. Therefore, for $m \to \infty$, from the equality

$$\sum_{k=1}^{m} d_{nk}y_k = \sum_{k=1}^{m} a_{nk}x_k$$

for all $n \in \mathbb{N}$, we obtain that $Ax = Dy$. Therefore, this shows that $A \in (h_{c_0}(\Delta^{(m)} : Y)$. $\square$

Theorem 3.6. *Consider that the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ with*

$$e_{nk} := \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{n+j-1} (-1)^{j-k} \binom{m}{j-k} a_{jk}. \tag{3.14}$$

Then, the following statements are true:
i. *$A = (a_{nk}) \in (X : h_{c_0}(\Delta^{(m)})$ if and only if $E \in (X : c_0)$*

ii. *$A = (a_{nk}) \in (X : h_c(\Delta^{(m)})$ if and only if $E \in (X : c)$*
iii. *$A = (a_{nk}) \in (X : h_\infty(\Delta^{(m)})$ if and only if $E \in (X : \ell_\infty)$*

Proof. We take any $z = (z_k) \in X$. Using the (3.14), we have

$$\sum_{k=1}^{m} e_{nk} z_k = \sum_{k=1}^{m} \left[\sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{n+j-1} (-1)^{j-k} \binom{m}{j-k} a_{jk} \right] z_k \tag{3.15}$$

for all $m, n \in \mathbb{N}$. Then, for $m \to \infty$, equation (3.15) gives us that $(Ez)_n = \{H\Delta^{(m)}(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in h_{c_0}(\Delta^{(m)}$ whenever $z \in X$ if and only if $Ez \in c_0$ whenever $z \in X$. Thus, the proof is completed. □

4. Examples

If we choose any sequence spaces X and Y in Theorem 3.5 and 3.6 in the previous section, then we can find several consequences in every choice. For example, if we take the space ℓ_∞ and the spaces which are isomorphic to ℓ_∞ instead of Y in Theorem 3.5, we obtain the following examples:

Example 4.1. *The Euler sequence space e^r_∞ is defined by ([3] and [2])*

$$e^r_\infty = \{x \in \omega : \sup_{n\in\mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k| < \infty\}.$$

We consider the infinite matrix $A = (a_{nk})$ and define the matrix $C = (c_{nk})$ by

$$c_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \qquad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(h_{c_0}(\Delta^{(m)}) : e^r_\infty)$ in Theorem 3.5, then, we replace the entries of the matrix A by those of the matrix C.

Example 4.2. *Let $T_n = \sum_{k=0}^{n} t_k$ and $A = (a_{nk})$ be an infinite matrix. We define the matrix $G = (g_{nk})$ by*

$$g_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk} \qquad (k, n \in \mathbb{N}).$$

Then, the necessary and sufficient conditions in order for A belongs to the class $(h_{c_0}(\Delta^{(m)}) : r^t_\infty)$ are obtained from in Theorem 3.5 by replacing the entries of the matrix A by those of the matrix G; where r^t_∞ is the space of all sequences whose R^t−transforms is in the space ℓ_∞ [14].

Example 4.3. *In the space r^t_∞, if we take $t = e$, then this space becomes a Cesaro sequence space of non-absolute type X_∞ [13]. As a special case, example 4.2 includes the characterization of class $(h_{c_0}(\Delta^{(m)}) : r^t_\infty)$.*

Example 4.4. *The Taylor sequence space t^r_∞ is defined by ([10])*

$$t^r_\infty = \{x \in \omega : \sup_{n\in\mathbb{N}} |\sum_{k=n}^{\infty} \binom{k}{n}(1-r)^{n+1} r^{k-n} x_k| < \infty\}.$$

We consider the infinite matrix $A = (a_{nk})$ and define the matrix $P = (p_{nk})$ by

$$p_{nk} = \sum_{k=n}^{\infty} \binom{k}{n}(1-r)^{n+1} r^{k-n} a_{jk} \qquad (k, n \in \mathbb{N}).$$

If we want to get necessary and sufficient conditions for the class $(h_{c_0}(\Delta^{(m)}) : t^r_\infty)$ in Theorem 3.5, then, we replace the entries of the matrix A by those of the matrix P.

If we take the spaces c, cs and bs instead of X in Theorem 3.6, or Y in Theorem 3.5 we can write the following examples. Firstly, we give some conditions and the following lemmas:

$$\lim_k a_{nk} = 0 \quad \text{for all n}, \tag{4.1}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0, \tag{4.2}$$

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0, \tag{4.3}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = 0, \tag{4.4}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty \tag{4.5}$$

$$\lim_k (a_{nk} - a_{n,k+1}) \text{ exists for all k} \tag{4.6}$$

$$\lim_{n\to\infty} \sum_k |a_{nk} - a_{n,k+1}| = \sum_k \left| \lim_{n\to\infty} (a_{nk} - a_{n,k+1}) \right| \tag{4.7}$$

$$\sup_n \left| \lim_k a_{nk} \right| < \infty \tag{4.8}$$

Lemma 4.5. *Consider that the $X \in \{\ell_\infty, c, bs, cs\}$ and $Y \in \{c_0\}$. The necessary and sufficient conditions for $A \in (X : Y)$ can be read as the following from Table 4.1:*

9. *(4.3)*	***10.*** *(3.1), (3.5), (4.2)*	***11.*** *(4.1), (4.4)*	***12.*** *(3.5), (4.5)*
13. *(4.1), (4.6), (4.7)*	***14.*** *(4.5), (3.9)*	***15.*** *(4.1), (4.5)*	***16.*** *(4.5), (4.8)*

Table 4.1:

From $\rightarrow$	ℓ_∞	c	bs	cs
To $\downarrow$				
c_0	**9.**	**10.**	**11.**	**12.**
c	**7.**	**5.**	**13.**	**14.**
ℓ_∞	**1.**	**1.**	**15.**	**16.**

5. Conclusion

In 1894, Hilbert introduced the Hilbert matrix. Hilbert matrices are notable examples of poorly conditioned (ill-conditioned) matrices, making them notoriously difficult to use in numerical computation. In other words, Hilbert matrices whose entries are specified as machine-precision numbers are difficult to invert using numerical techniques. That is why we offered some examples related to the usage of the Hilbert matrix such as image processing and cryptography. We also provided an algorithm. Further, we constructed new spaces with the Hilbert matrix and difference operator of order m. We calculated dual spaces of new spaces and characterized some matrix classes. In the last section, we gave some examples of matrix classes. Images of new spaces can be plotted using the Mathematica as a continuation of this study. Again, different applications of cryptography can be investigated.

Conflict of Interests

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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ON SOME NEW $\mathcal{P}_\delta$-TRANSFORMS OF KUMMER'S CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Abstract. The aim of our paper is to present $\mathcal{P}_\delta$-transforms of the Kummer's confluent hypergeometric functions by employing the generalized Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem obtained earlier by Lavoie, Grondin and Rathie [9]. Relevant connections of certain special cases of the main results presented here are also pointed out.
Keywords. Hypergeometric functions; Gauss's second summation theorem; gamma functions; Summation theorems.

1. Introduction

The *generalized hypergeometric function* ${}_pF_q$ with p numerator and q denominator parameters is defined as follows:

$$(1.1)\qquad {}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array}\middle| z\right] := \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!},$$

where $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, j \in \overline{1, q} := \{1, 2, \cdots, q\}$. Here and in the following text, let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, and non-positive integers, respectively. The series converges for all $z \in \mathbb{C}$ if $p \leq q$. It is divergent for all $z \neq 0$ when $p > q + 1$, unless at least one numerator parameter is a negative integer in which case (1.1) is a polynomial. Finally, if $p = q+1$, the series converges on the unit circle $|z| = 1$ when $\mathrm{Re}\left(\sum b_j - \sum a_j\right) > 0$. The importance of the hypergeometric series lies in the fact that almost all elementary functions such as exponential, binomial, trigonometric, hyperbolic, logarithmic are its special cases. It should be remarked here that whenever generalized hypergeometric functions reduce to gamma functions, the results are important from the applications point of view. Thus, the well-known classical summation theorems such as those of the Gauss

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second summation theorem, Bailey summation theorem and Kummer's summation theorem for the series ${}_2F_1$ [11] given below play an important role in the theory of hypergeometric functions.

The Gauss's second summation theorem

$$ {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(b+1)}; \tag{1.2} $$

Bailey's summation theorem

$$ {}_2F_1\left[\begin{matrix} a, 1-a \\ b \end{matrix} \,\Big|\, \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}(b+1)}{\Gamma(\frac{1}{2}(b+a))\Gamma(\frac{1}{2}(b-a+1))}; \tag{1.3} $$

Kummer's summation theorem

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix} \,\Big|\, -1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{1.4} $$

During 1992-96, in a series of three research papers, Lavoie *et al.* [7, 8, 9] have generalized various classical summation theorems such as the Gauss second, Bailey and Kummer ones for the ${}_2F_1$ series, as well as the Watson, Dixon and Whipple ones for the ${}_3F_2$ series. However, in our present investigation, we are interested in the following generalized Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem given in [9]

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix} \,\Big|\, \frac{1}{2}\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2})} \\ &\times\left\{\frac{A_i(a,b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{B_i(a,b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}i-\lfloor\frac{i}{2}\rfloor)}\right\}; \end{aligned} \tag{1.5} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a, 1-a+i \\ b \end{matrix} \,\Big|\, \frac{1}{2}\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma(1-a+\frac{1}{2}i+\frac{1}{2}|i|)} \\ &\times\left\{\frac{C_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})+\Gamma(\frac{1}{2}b+\frac{1}{2}a-\lfloor\frac{1+i}{2}\rfloor)} + \frac{D_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-\lfloor\frac{i}{2}\rfloor)}\right\}; \end{aligned} \tag{1.6} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+i \end{matrix} \,\Big|\, -1\right] &= \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma(1-b+\frac{1}{2}i+\frac{1}{2}|i|)} \\ &\times\left\{\frac{E_i(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{F_i(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}i-\lfloor\frac{i}{2}\rfloor)}\right\}, \end{aligned} \tag{1.7} $$

respectively. Here, and in what follows, $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Also, throughout the paper, as usual, $[x]$ denotes the greatest integer less than or equal to the real number x and its absolute value is denoted by $|x|$. The coefficients which appear in (1.5), (1.6) and (1.7) are listed in Tables 1-3 and that, for $i = 0$, these equations reduce, respectively to (1.2), (1.3) and (1.4).

Table 1

i	$A_i(a,b)$	$B_i(a,b)$
5	$-(a+b+6)^2+\frac{1}{2}(b-a+6)(b+a+6)+\frac{1}{4}(b-a+6)^2 11(b+a+6)-\frac{13}{2}(b-a+6)+20$	$(a+b+6)^2+\frac{1}{2}(b-a+6)(b+a+6)-\frac{1}{4}(b-a+6)^2-17(b+a+6)-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(a+b+1)(a+b-3)-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a-1)$
3	$\frac{1}{2}(b-a+4)-(b+a+4)+3$	$\frac{1}{2}(b-a+4)+(b+a+4)-7$
2	$\frac{1}{2}(b+a+3)-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a-1)$	2
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
-4	$\frac{1}{2}(a+b-3)(a+b+1)-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$
-5	$(b+a-4)^2-\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2+4(b+a-4)-\frac{7}{2}(b-a-4)$	$(b+a-4)^2+\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2+8(b+a-4)-\frac{1}{2}(b-a-4)+12$

Table 2

i	$C_i(a,b)$	$D_i(a,b)$
5	$-(4b^2-2ab-a^2-22b+13a+20)$	$4b^2+2ab-a^2-34b-a+62$
4	$2(b-2)(b-4)-(a-1)(a-4)$	$-4(b-3)$
3	$a-2b+3$	$a+2b-7$
2	$b-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	b	2
-3	$2b-a$	$a+2b+2$
-4	$2b(b+2)-a(a+3)$	$4(b+1)$
-5	$4b^2-2ab-a^2+8b-7a$	$4b^2+2ab-a^2+16b-a+12$

Table 3

i	$E_i(a,b)$	$F_i(a,b)$
5	$-4(6+a-b)^2+2b(6+a-b)+b^2-22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)-b^2-34(6+a-b)-b+62$
4	$2(a+b-3)(a-b+1)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2+16(a-b-4)-b+12$

The main objective of this paper is to derive three new interesting and general $\mathcal{P}_\delta$-transforms of the Kummer's confluent hypergeometric functions by employing the generalized Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem given in (1.5), (1.6) and (1.7), respectively. Relevant connections of certain special cases of the main results presented here with those earlier ones are also pointed out.

2. $\mathcal{P}_\delta$-transforms

The $\mathcal{P}_\delta$-transforms or pathway transforms of the function $f(t)\,(t\in\mathbb{R})$ is a function $F_{\mathcal{P}}(s)$ of a complex variable s defined by (see, *e.g.*, [6])

$$\mathcal{P}_\delta\{f(t);s\}=F_{\mathcal{P}}(s)=\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\,f(t)\,dt\ \ (\delta>1), \tag{2.1}$$

For the sufficient condition for the existence of the $\mathcal{P}_\delta$-transform (2.1) to exist, we refer the reader to [6]. The $\mathcal{P}_\delta$-transform of the power function $t^{\mu-1}$ is given by [6, p. 7, Eq. (32)]

$$\mathcal{P}_\delta\{t^{\mu-1};s\}=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right)^{\mu}\Gamma(\mu)=\frac{\Gamma(\mu)}{[\Lambda(\delta;s)]^{\mu}}\ \ (\mathrm{Re}(\mu)>0;\,\delta>1). \tag{2.2}$$

Furthermore, upon letting $\delta\mapsto 1$ in the definition (2.1), the $\mathcal{P}_\delta$-transform reduces to the classical Laplace transform (see, *e.g.*, [13]):

$$L\{f(t);s\}=\int_0^\infty e^{-st}f(t)\,dt\ \ (\,\mathrm{Re}(s)>0\,). \tag{2.3}$$

In view of the power function formula (2.2), it is easy to derive the $\mathcal{P}_\delta$-transform of the generalized hypergeometric function to obtain the following formula (see, [6, p. 8, Eq. (42)]):

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{\mu-1}\, {}_pF_q\left[\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q\end{matrix}\,\middle|\,\omega t\right] dt \tag{2.4}$$
$$= \frac{\Gamma(\mu)}{[\Lambda(\delta;s)]^\mu}\; {}_{p+1}F_q\left[\begin{matrix} a_1,\cdots,a_p,\mu \\ b_1,\cdots,b_q\end{matrix}\,\middle|\,\frac{\omega}{\Lambda(\delta;s)}\right],$$

for $p<q$, $\mathrm{Re}(\mu)>0$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)>0$ and $\delta>1$ or for $p=q$, $\mathrm{Re}(\mu)>0$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)>\mathrm{Re}(\omega)$ and $\delta>1$.

If $p=q=1$, we get the following formula :

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{\mu-1}\, {}_1F_1\left[\begin{matrix} a \\ c\end{matrix}\,\middle|\,\omega t\right] dt \tag{2.5}$$
$$= \frac{\Gamma(\mu)}{[\Lambda(\delta;s)]^\mu}\; {}_2F_1\left[\begin{matrix} a,\mu \\ c\end{matrix}\,\middle|\,\frac{\omega}{\Lambda(\delta;s)}\right],$$

for $\mathrm{Re}(c)>0$, $\mathrm{Re}(\mu)>0$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)>\mathrm{Re}(\omega)$ and $\delta>1$. In the next section, we shall demonstrate how one can obtain three rather general $\mathcal{P}_\delta$-transforms of the Kummer's confluent hypergeometric functions by employing the results (1.5), (1.6) and (1.7).

3. $\mathcal{P}_\delta$-transforms of ${}_1F_1(a;b;x)$

In this section, we establish the following integral formulas, asserted in Theorem(3.1), Theorem(3.2) and Theorem(3.3).

Theorem 3.1. *Let* $\mathrm{Re}(b)>0$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)>0$ *and* $\delta>1$. *Then*

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+i+1)\end{matrix}\,\middle|\,\frac{t\,\Lambda(\delta;s)}{2}\right] dt \tag{3.1}$$
$$= \frac{\Gamma(b)}{[\Lambda(\delta;s)]^b}\,\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2})}$$
$$\times\left\{\frac{A_i(a,b)}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)}+\frac{B_i(a,b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}i-\lfloor\frac{i}{2}\rfloor)}\right\}.$$

Theorem 3.2. *Let* $\mathrm{Re}(1-a+i)>0\,(i=0,\pm1,\pm2,\pm3,\pm4,\pm5)$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)>0$ *and* $\delta>1$. *Then*

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{-a+i}\, {}_1F_1\left[\begin{matrix} a \\ b\end{matrix}\,\middle|\,\frac{t\,\Lambda(\delta;s)}{2}\right] dt \tag{3.2}$$

$$= \frac{\Gamma(1-a+i)}{[\Lambda(\delta;s)]^{1-a+i}} \frac{\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma(1-a+\frac{1}{2}i+\frac{1}{2}|i|)}$$

$$\times\left\{\frac{C_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})+\Gamma(\frac{1}{2}b+\frac{1}{2}a-\lfloor\frac{1+i}{2}\rfloor)} + \frac{D_i(a,b)}{\Gamma(\frac{1}{2}b-\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-\lfloor\frac{i}{2}\rfloor)}\right\}.$$

Theorem 3.3. *Let* $\mathrm{Re}(b) > 0$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) > 0$ *and* $\delta > 1$. *Then*

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ 1+a-b+i \end{matrix}\;\middle|\; -t\,\Lambda(\delta;s)\right] \mathrm{d}t = \tag{3.3}$$

$$= \frac{\Gamma(b)}{[\Lambda(\delta;s)]^{b}} \frac{\Gamma(\frac{1}{2})\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma(1-b+\frac{1}{2}i+\frac{1}{2}|i|)}$$

$$\times\left\{\frac{E_i(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\lfloor\frac{1+i}{2}\rfloor)} + \frac{F_i(a,b)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}i-\lfloor\frac{i}{2}\rfloor)}\right\}.$$

Proof. In order to prove Theorem (3.1), setting $\omega = \frac{\Lambda(\delta;s)}{2}$, $\mu = b$ and $c = \frac{1}{2}(a+b+i+1)$ for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ in (2.5), we have

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+i+1) \end{matrix}\;\middle|\; \frac{t\,\Lambda(\delta;s)}{2}\right] \mathrm{d}t \tag{3.4}$$

$$= \frac{\Gamma(b)}{[\Lambda(\delta;s)]^{b}}\; {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix}\;\middle|\; \frac{1}{2}\right].$$

We observe that the ${}_2F_1$ appearing on the right-hand side of (3.4) can be evaluated with the help of generalized Gauss's second summation theorem (1.5). This yields the desired formula (3.1).

The results in Theorem (3.2) and Theorem (3.3) can also be proven in a similar way by applying summation theorems (1.6) and (1.7), respectively. □

4. Special Cases

The particular cases $i = 0$ of Theorem (3.1) to Theorem (3.3), reduce to the following interesting and presumably new results for classical ones.

Corollary 4.1. *Let* $\mathrm{Re}(b) > 0$, $\mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) > 0$ *and* $\delta > 1$. *Then*

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+1) \end{matrix}\;\middle|\; \frac{t\,\Lambda(\delta;s)}{2}\right] \mathrm{d}t \tag{4.1}$$

$$= \frac{\Gamma(b)}{[\Lambda(\delta;s)]^{b}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}.$$

Corollary 4.2. *Let* $\mathrm{Re}(1-a) > 0, \mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) > 0$ *and* $\delta > 1$. *Then*

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{-a}\, {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}\;\middle|\; \frac{t\,\Lambda(\delta;s)}{2}\right] \mathrm{d}t \tag{4.2}$$

$$= \frac{\Gamma(1-a)}{[\Lambda(\delta;s)]^{1-a}} \frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}b+\frac{1}{2}a)\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})}.$$

Corollary 4.3. *Let* $\mathrm{Re}(b) > 0, \mathrm{Re}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) > 0$ *and* $\delta > 1$*. Then*

$$\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ 1+a-b \end{matrix} \,\middle|\, -t\,\Lambda(\delta;s)\right] \mathrm{d}t = \frac{\Gamma(b)}{[\Lambda(\delta;s)]^b}\,\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}. \tag{4.3}$$

Similarly, for $i = \pm1, \pm2, \pm3, \pm4, \pm5$, other results can also be obtained.

5. Concluding remarks

By letting $\delta \mapsto 1$ in the definition (2.1), the $\mathcal{P}_\delta$-transform is reduced to the classical Laplace transform. Hence, for $\delta \mapsto 1$, the results (3.1), (3.2) and (3.3) immediately reduce to the corresponding results due to Kim *et al.* [5].

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A QUADRATIC PROGRAMMING MODEL FOR OBTAINING WEAK FUZZY SOLUTION TO FUZZY LINEAR SYSTEMS

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Abstract. Real life applications arising in various fields of engineering and science (e.g. electrical, civil, economics, dietary, etc.) can be modelled using a system of linear equations. In such models, it may happen that the values of the parameters are not known or they cannot be stated precisely and that only their estimation due to experimental data or experts knowledge is available. In such a situation it is convenient to represent such parameters by fuzzy numbers. In this paper we propose an efficient optimization model for obtaining a weak fuzzy solution to fuzzy linear systems (FLS). We solve some examples and we show that this method is always efficient.
Keywords. Experimental data; fuzzy numbers; fuzzy solution; fuzzy linear systems.

1. Introduction

Fuzzy numbers are one way to describe the vagueness and lack of precision of data. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [4] and [13] Mizumoto and Tanaka [9] and [10], Dubois and Prade [7] and Nahmias [11]. One of the major applications using fuzzy number arithmetic is treating linear systems whose parameters all are or partially represented by fuzzy numbers and called fuzzy linear systems (FLS). Many authors have investigated the solution to fuzzy linear systems([1], [2], [6] and [8]) and all of them make use of the definition given in [8] for converting non-fuzzy solutions to weak fuzzy solutions. In 1998, Friedman et al. [8] proposed a general method for obtaining a solution of a $n \times n$ FLS, whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the embedding method given in [5] and replaced the original $n \times n$ FLS by a $2n \times 2n$ crisp linear system (CLS). The new obtained system was solved and the solution vector was called either a strong fuzzy solution or a weak fuzzy solution to the original fuzzy system.

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Hitherto, many researchers have used Friedman et al.'s method. To solve $2n \times 2n$ CLS various methods have been employed along with and the mentioned definition in [8]. All researches took it for granted that the weak fuzzy solution defined by Friedman et al. is always a fuzzy number vector, i.e. all of the vector's components are fuzzy numbers.
Afterwards, T.Allahviranloo et al. [3] showed in an example that Friedman et al.'s weak solution may not be a fuzzy vector.
In this paper, we proposed a new method which guarantees that a weak fuzzy solution obtained by this method is always a fuzzy number vector.

2. Fuzzy Linear Systems

We represent an arbitrary fuzzy number as follows.

Definition 2.1. [3] The parametric form of an arbitrary fuzzy number $\tilde{a}$ is presented by an ordered pair of functions $(\underline{a}(r), \overline{a}(r))$, $0 \le r \le 1$, which satisfy the following requirements:

- $\underline{a}(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$.
- $\overline{a}(r)$ is a bounded left-continuous non-increasing function over $[0, 1]$.
- $\underline{a}(1) = \overline{a}(1)$.
- $\underline{a}(r) \le \overline{a}(r)$, $0 \le r \le 1$.

Remark 2.1. If $t \in (0, 1)$ be a fixed number and

$$\underline{a}(r) = \begin{cases} \alpha_1 + \beta_1 r, & 0 \le r \le t; \\ \alpha_1' + \beta_1' r, & t \le r \le 1; \end{cases} \quad , \ \overline{a}(r) = \begin{cases} \alpha_2 - \beta_2 r, & 0 \le r \le t; \\ \alpha_2' - \beta_2' r, & t \le r \le 1. \end{cases}$$

then based on Definition 2.1, $\tilde{a} = (\underline{a}(r), \overline{a}(r))$ is a fuzzy number iff

$$\begin{aligned} &\alpha_1 + \beta_1 t = \alpha_1' + \beta_1' t, \\ &\alpha_2 - \beta_2 t = \alpha_2' - \beta_2' t, \\ &\alpha_1' + \beta_1' = \alpha_2' - \beta_2', \\ &\alpha_1 \le \alpha_2, \alpha_1' \le \alpha_2', \\ &\beta_1, \beta_1', \beta_2, \beta_2' \ge 0. \end{aligned} \tag{2.1}$$

Example 2.1. [3] According to Definition 2.1, the number $\tilde{a} = (\underline{a}(r), \overline{a}(r))$ where

$$\underline{a}(r) = \begin{cases} 8r - 5, & 0 \le r \le \frac{1}{2}; \\ 4r - 3, & \frac{1}{2} \le r \le 1; \end{cases}$$

and

$$\overline{a}(r) = \begin{cases} 4 - 4r, & 0 \le r \le \frac{1}{2}; \\ 3 - 2r, & \frac{1}{2} \le r \le 1; \end{cases}$$

is a fuzzy number which satisfies the conditions of the remark 2.1.

Definition 2.2. [8] The following $n \times n$ linear system is called an FLS:

$$
\begin{aligned}
k_{11}x_1 + k_{12}x_2 + \ldots + k_{1n}x_n &= \tilde{b_1}, \\
k_{21}x_1 + k_{22}x_2 + \ldots + k_{2n}x_n &= \tilde{b_2}, \\
&\vdots \\
k_{n1}x_1 + k_{n2}x_2 + \ldots + k_{nn}x_n &= \tilde{b_n},
\end{aligned} \tag{2.2}
$$

where the coefficients matrix $A = (k_{ij}), 1 \leq i,j \leq n$, is a crisp $n \times n$ matrix and $\tilde{b_i}, 1 \leq i \leq n$, are fuzzy numbers.

Definition 2.3. [8] A fuzzy number vector $(x_1, x_2, \ldots, x_n)^T$ is given by $x_i = (\underline{x}_i(r), \overline{x}_i(r)),\ 1 \leq i \leq n,\ 0 \leq r \leq 1$, is called a solution of the FLS (2.2) if

$$
\underline{\sum_{j=1}^{n} k_{ij}x_j} = \sum_{j=1}^{n} \underline{k_{ij}x_j} = \underline{b_i},\ \overline{\sum_{j=1}^{n} k_{ij}x_j} = \sum_{j=1}^{n} \overline{k_{ij}x_j} = \overline{b_i}, \tag{2.3}
$$

according to the proposed model by Friedman et al. we convert the $n \times n$ FLS (2.2) to the following $2n \times 2n$ CLS:

$$
TX = B, \tag{2.4}
$$

where

$$
X = \left(\underline{x_1}, \ldots, \underline{x_n}, -\overline{x_1}, \ldots, -\overline{x_n}\right)^t,\ B = \left(\underline{b_1}, \ldots, \underline{b_n}, -\overline{b_1}, \ldots, -\overline{b_n}\right)^t,
$$

and T_{ij} determined as follows:

$$
\begin{aligned}
k_{ij} \geq 0 \quad &\Rightarrow \quad T_{ij} = k_{ij},\ T_{i+n,j+n} = k_{ij}, \\
k_{ij} < 0 \quad &\Rightarrow \quad T_{i,j+n} = -k_{ij},\ T_{i+n,j} = -k_{ij},
\end{aligned}
$$

and any T_{ij} which is not determined by the above equations is zero. Having calculated X which solves Eq.(2.4) and on the assumption T is nonsingular, Friedman et al. defined the "fuzzy solution" to the original system given by Eqs. (2.2) as below.

Definition 2.4. [8] Let the unique solution to CLS (2.4) be denoted by:

$$
X = \{(\underline{x}_i(r), -\overline{x}_i(r)),\ 1 \leq i \leq n\},
$$

then the fuzzy number vector $W = \{(\underline{w}_i(r), \overline{w}_i(r)), 1 \leq i \leq n\}$ defined by

$$
\begin{aligned}
\underline{w}_i(r) &= \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}, \\
\overline{w}_i(r) &= \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\},
\end{aligned}
$$

is called the fuzzy solution of (2.4).
If $(\underline{x}_i(r), \overline{x}_i(r)),\ 1 \leq i \leq n,\ 0 \leq r \leq 1$, are all fuzzy numbers then $\underline{w}_i(r) = \underline{x}_i(r), \overline{w}_i(r) = \overline{x}_i(r),\ 1 \leq i \leq n$, and W is called a strong fuzzy solution. Otherwise, W is called a weak fuzzy solution.

It should be noted that replacing $\overline{x}_i$ and $\underline{x}_i$ by $\overline{w}_i$ and $\underline{w}_i$ does not give the exact equality in (2.3) that is, a weak solution is not a solution to (2.2). Therefore, a weak solution does not satisfy the original problem (2.2). Based on Definition 2.4, Friedman et al. claimed that their weak solution always produces a fuzzy number vector.
T.Allahviranloo et al. in [3] gave an example that this claim is not always true.

Example 2.2. [3] consider the following FLS:

$$\begin{cases} x_1 + x_2 = \tilde{b_1}, \\ x_1 + 2x_2 = \tilde{b_2}, \end{cases}$$

where

$$\tilde{b_1} = (\underline{b_1}(r), \overline{b_1}(r)), \quad \tilde{b_2} = (\underline{b_2}(r), \overline{b_2}(r)),$$

and

$$\underline{b_1}(r) = \begin{cases} 8r - 14, & 0 \le r \le \frac{1}{2}, \\ 2r - 11, & \frac{1}{2} \le r \le 1, \end{cases} , \overline{b_1}(r) = \begin{cases} -1 - 13r, & 0 \le r \le \frac{1}{2}, \\ -6 - 3r, & \frac{1}{2} \le r \le 1, \end{cases}$$

$$\underline{b_2}(r) = \begin{cases} 12r - 24, & 0 \le r \le \frac{1}{2}, \\ 6r - 21, & \frac{1}{2} \le r \le 1, \end{cases} , \overline{b_2}(r) = \begin{cases} -2 - 18r, & 0 \le r \le \frac{1}{2}, \\ -7 - 8r, & \frac{1}{2} \le r \le 1. \end{cases}$$

The authors in [3] solved this system and obtained the following solution:

$$\underline{x_1}(r) = \begin{cases} 4r - 4, & 0 \le r \le \frac{1}{2}, \\ -2r - 1, & \frac{1}{2} \le r \le 1, \end{cases} , \overline{x_1}(r) = \begin{cases} -8r, & 0 \le r \le \frac{1}{2}, \\ 2r - 5, & \frac{1}{2} \le r \le 1, \end{cases}$$

and $\underline{x_2}(r) = 4r - 10$, $\overline{x_2}(r) = -1 - 5r$.

Obviously, x_2 is a fuzzy number and x_1 is not a fuzzy number(see figure 3.1). By use of Definition 2.4, the vector $W = (\tilde{w}_1, \tilde{w}_2)$ must be a fuzzy number, but we have

$$\underline{w_1}(r) = \begin{cases} 4r - 4, & 0 \le r \le \frac{1}{4}, \\ -3, & \frac{1}{4} \le r \le \frac{3}{8}, \\ -8r, & \frac{3}{8} \le r \le \frac{1}{2}, \\ 2r - 5, & \frac{1}{2} \le r \le 1, \end{cases} , \quad \overline{w_1}(r) = \begin{cases} -8r, & 0 \le r \le \frac{1}{3}, \\ 4r - 4, & \frac{1}{3} \le r \le \frac{1}{2}, \\ -2r - 1, & \frac{1}{2} \le r \le 1, \end{cases}$$

and $\underline{w_2}(r) = 4r - 10$, $\overline{w_2}(r) = -1 - 5r$.
It is clear that $\tilde{w}_2 = (\underline{w}_2, \overline{w}_2)$ is a fuzzy number, whereas $\tilde{w}_1 = (\underline{w}_1, \overline{w}_1)$ is not a fuzzy number. In fact $\underline{w_1}(r)$ and $\overline{w_1}(r)$ are not non-decreasing and non-increasing functions over $[\frac{3}{8}, \frac{1}{2}]$ and $[\frac{1}{3}, \frac{1}{2}]$, respectively. In the next section we propose an optimization model that can be applied for approximation solution to FLS (2.2) which is always a fuzzy number vector. Typically, we solve the example 2.2 and show that the solution which is obtained by this method is a fuzzy number vector.

3. Proposed Model

In this section, we show that the approximation solution of FLS (2.2) with this method becomes a quadratic programming(QP) problem.
A general form of QP problem is as follows:

$$
\begin{aligned}
&\min \quad q(x) = \frac{1}{2}x^T Gx + x^T c \\
&s.t. \\
&\qquad\quad Ax \leq b,
\end{aligned}
\tag{3.1}
$$

where G is a symmetric $n \times n$ matrix and A is the $m \times n$ matrix, jacobian of constraints, and b is a vector in Re^m. We solve the problem (3.1) by use of the active set method [12].

Definition 3.1. Let $v = (\underline{v}(r), \overline{v}(r))$ where

$$
\underline{v}(r) = \begin{cases} a_1 + b_1 r, & 0 \leq r \leq t, \\ a_1' + b_1' r, & t \leq r \leq 1, \end{cases} \quad , \quad \overline{v}(r) = \begin{cases} a_2 - b_2 r, & 0 \leq r \leq t, \\ a_2' - b_2' r, & t \leq r \leq 1, \end{cases}
$$

which is not satisfied with the definition 2.1, i.e. v is not a fuzzy number and we want to approximate it with a fuzzy number $u = (\underline{u}(r), \overline{u}(r))$, where

$$
\underline{u}(r) = \begin{cases} \alpha_1 + \beta_1 r, & 0 \leq r \leq t, \\ \alpha_1' + \beta_1' r, & t \leq r \leq 1, \end{cases} \quad , \quad \overline{u}(r) = \begin{cases} \alpha_2 - \beta_2 r, & 0 \leq r \leq t, \\ \alpha_2' - \beta_2' r, & t \leq r \leq 1. \end{cases}
$$

We say that u is an approximation for v iff it is a solution of the following optimization problem:

$$
\begin{aligned}
&\min \quad \|p - q\|^2 \\
&s.t. \\
&\qquad \alpha_1 + \beta_1 t = \alpha_1' + \beta_1' t, \\
&\qquad \alpha_2 - \beta_2 t = \alpha_2' - \beta_2' t, \\
&\qquad \alpha_1' + \beta_1' = \alpha_2' - \beta_2', \\
&\qquad \beta_1, \beta_1', \beta_2, \beta_2' \geq 0, \\
&\qquad \alpha_1 \leq \alpha_2, \\
&\qquad \alpha_1 \leq \alpha_2,
\end{aligned}
\tag{3.2}
$$

where $p = (\alpha_1, \beta_1, \alpha_1', \beta_1', \alpha_2, \beta_2, \alpha_2', \beta_2')$, $q = (a_1, b_1, a_1', b_1', a_2, b_2, a_2', b_2')$ are corresponding vectors with v and u respectively.

Remark 3.1. If we rewrite the problem (3.2) into a matrix form then we have:

$$
\begin{aligned}
&\min \quad f(x) = \frac{1}{2}x^T Gx + x^T c \\
&s.t. \\
&\qquad\quad Ax \leq b,
\end{aligned}
\tag{3.3}
$$

where $G = 2I_{8\times 8}$ (I is identity matrix), $c = -2\left[a_1, b_1, a_1', b_1', a_2, b_2, a_2', b_2'\right]^t$ and

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 \\ 1 & t & -1 & -t & 0 & 0 & 0 & 0 \\ -1 & -t & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & t & 1 & -t \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Now, using the model (3.3) for the x_2 in the example 2.2 and solving it with the active-set method, the weak fuzzy solution $z = (z_1, z_2)$ is obtained as follows:

$$\underline{z_1}(r) = \begin{cases} -4.8 + 3.6r, & 0 \le r \le \frac{1}{2}, \\ -3, & \frac{1}{2} \le r \le 1, \end{cases}, \quad \overline{z_1}(r) = \begin{cases} 0.8 - 7.6r, & 0 \le r \le \frac{1}{2}, \\ -3, & \frac{1}{2} \le r \le 1, \end{cases}$$

and

$$z_2 = (4r - 10, -1 - 5r).$$

By substituting z in the example 2.2 we have:

$$\begin{cases} z_1 + z_2 = \tilde{y}_1, \\ z_1 + 2z_2 = \tilde{y}_2, \end{cases}$$

where

$$\tilde{y}_1 = (\underline{y_1}(r), \overline{y_1}(r)), \quad \tilde{y}_2 = (\underline{y_2}(r), \overline{y_2}(r))$$

and

$$\overline{y_1}(r) = \begin{cases} -12.6r - 0.2, & 0 \le r \le \frac{1}{2}, \\ -5r - 4, & \frac{1}{2} \le r \le 1, \end{cases}$$

$$\underline{y_1}(r) = \begin{cases} 7.6r - 14.8, & 0 \le r \le \frac{1}{2}, \\ 4r - 13, & \frac{1}{2} \le r \le 1. \end{cases} \tag{3.4}$$

$$\overline{y_2}(r) = \begin{cases} -17.6r - 1.2, & 0 \le r \le \frac{1}{2}, \\ -10r - 5, & \frac{1}{2} \le r \le 1, \end{cases}$$

$$\underline{y_2}(r) = \begin{cases} 11.6r - 24.8, & 0 \le r \le \frac{1}{2}, \\ 8r - 23, & \frac{1}{2} \le r \le 1. \end{cases} \tag{3.5}$$

We can see x_1 and its fuzzy approximation, z_1, in Figure 3.2

Also, Figure 3.3 and Figure 3.4 show an error in the system.

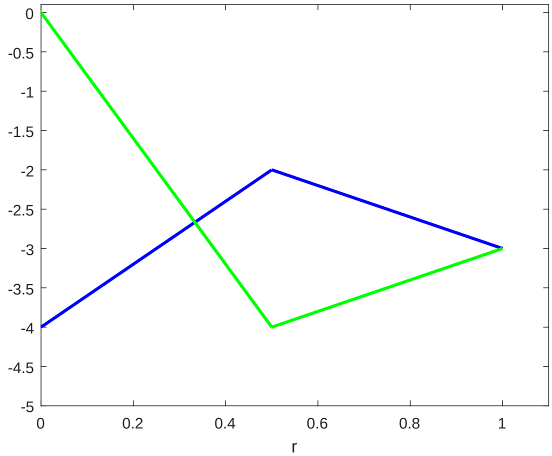

FIG. 3.1: nonfuzzy number x_1

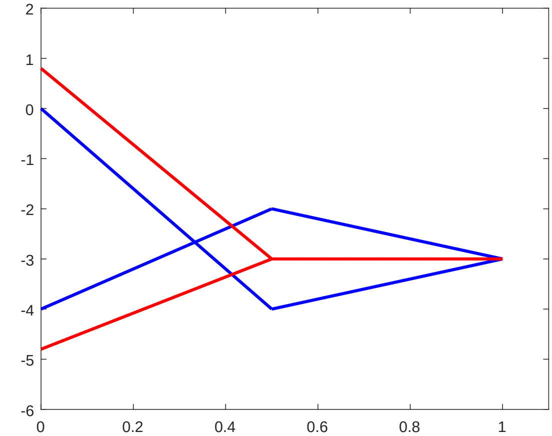

FIG. 3.2: The fuzzy number z_1 and nonfuzzy number x_1

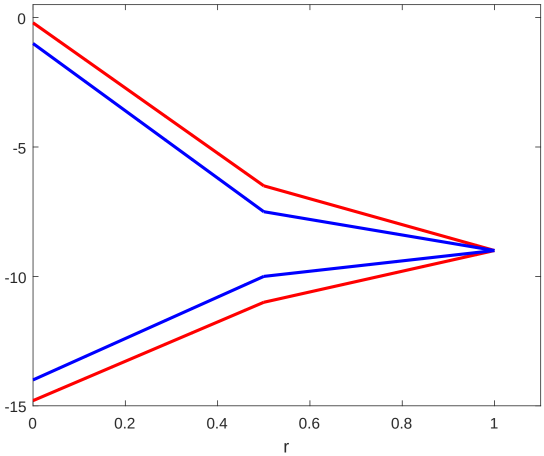

FIG. 3.3: comparing $\tilde{y}_1$ and $\tilde{b}_1$

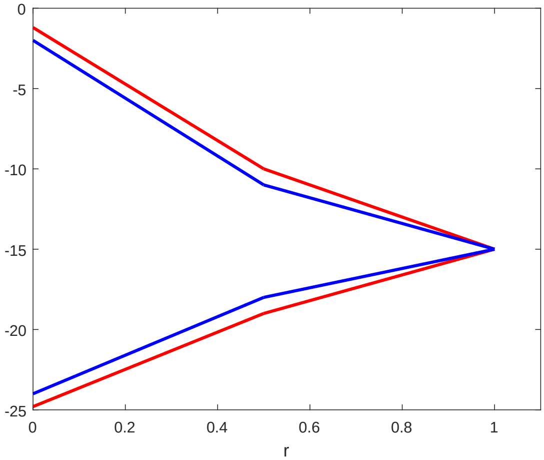

FIG. 3.4: comparing $\tilde{y}_2$ and $\tilde{b}_2$

4. Conclusion

In this paper, a new method was presented for obtaining the weak fuzzy solution to fuzzy linear systems. This method can be generalized for any number that is not fuzzy and we can approximate it with the fuzzy number by solving a quadratic programming problem.

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