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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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ON mTH-COMMUTATORS AND ANTI-COMMUTATORS INVOLVING GENERALIZED DERIVATIONS IN PRIME RINGS

Mohd Arif Raza

Abstract. In this paper, we study the m^{th}-commutator and anti-commutator involving generalized derivations on some suitable subsets of rings. We attain the information about the structure of rings and the behaviour of the generalized derivation in the form of multiplication by some specific element of the Utumi quotient ring which satisfies certain differential identities.
Keywords: prime ring; Generalized derivation, Generalized polynomial identity.

1. Motivation

It was shown by Herstein [10] that if d is a nonzero derivation of $\mathcal{R}$, a prime ring with a characteristic different from 2 such that $[d(x), d(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. Later, Bell and Daif [5] proved that if $\mathcal{R}$ is a semiprime ring, $\mathcal{I}$ is a nonzero right ideal of $\mathcal{R}$ and d is a derivation of $\mathcal{R}$ such that $[d(x), d(y)] = [x, y]$ for all $x, y \in \mathcal{I}$, then $\mathcal{I} \subseteq \mathcal{Z}(\mathcal{R})$. Motivated by the above result, Huang [11] obtained the commutativity of prime ring $\mathcal{R}$ with characteristic different from 2 satisfies $[d(x), d(y)]_m = [x, y]^n$, for all $x, y \in \mathcal{I}$, a nonzero ideal of $\mathcal{R}$, where $1 \leq m, n \in \mathbb{Z}^+$. In [2], Ashraf and Rehman studied anti-commutator involving derivation, i.e., $d(x) \circ d(y) = x \circ y$ and obtained the same conclusion.

On the other hand, Daif and Bell [7] proved that if $\mathcal{R}$ is a semiprime ring and d is a nonzero derivation of $\mathcal{R}$ such that $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In this direction, Ashraf and Rehman [2] discussed the commutativity of prime ring $\mathcal{R}$ whenever $\mathcal{R}$ satisfies $d(x \circ y) = x \circ y$ for all $x, y \in \mathcal{I}$, a nonzero ideal of $\mathcal{R}$. In recent years, several algebraist studied various generalizations of above mentioned identities and obtained the structure of rings and behaviour of derivations and generalized derivations on rings (see [1, 8, 12, 19, 20, 21] and references therein).

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In this note we shall examine the action of derivations and generalized derivations having m-th commutator and anti-commutator on prime rings. More precisely, we study the differential identities which involves both commutator and anti-commutator on some appropriate subset of rings and obtain the information about the structure of rings and the behaviour of generalized derivation in the form of multiplication by some specific element of Utumi quotient ring.

Throughout this note, unless specifically stated, $\mathcal{R}$ denotes a prime ring, i.e., for $a, b \in \mathcal{R}$, $a\mathcal{R}b = (0)$ implies that either $a = 0$ or $b = 0$. A ring $\mathcal{R}$ is said to be a left (right) faithful ring if for $a \in \mathcal{R}$, $a\mathcal{R} = (0)$ ($\mathcal{R}a = (0)$ resp.) implies $a = 0$. For a left faithful ring $\mathcal{R}$, the right Utumi quotient ring of $\mathcal{R}$ can be characterized as the ring $\mathcal{U}_r(\mathcal{R})$ (up to isomorphisms fixing $\mathcal{R}$) satisfying the following properties: (1) $\mathcal{R}$ is a subring of $\mathcal{U}_r(\mathcal{R})$; (2) For each $a \in \mathcal{U}_r(\mathcal{R})$, there exists a dense right ideal ρ of $\mathcal{R}$ such that $a\rho \subseteq \mathcal{R}$; (3) If $a \in \mathcal{U}_r(\mathcal{R})$ and $a\rho = 0$ for some dense right ideal ρ of $\mathcal{R}$, then $a = 0$; (4) For any dense right ideal ρ of $\mathcal{R}$ and for any right $\mathcal{R}$-module map $\varphi : \rho_{\mathcal{R}} \to \mathcal{R}_{\mathcal{R}}$, there exists $a \in \mathcal{U}_r(\mathcal{R})$ such that $\varphi(x) = ax$ for all $x \in \rho$. Analogously, for a right faithful ring $\mathcal{R}$ we may define $\mathcal{U}_l(\mathcal{R})$ the left Utumi quotient ring of $\mathcal{R}$ in terms of dense left ideals of $\mathcal{R}$. Let $\mathcal{R}$ be a left and right faithful ring. The two-sided Utumi quotient ring $\mathcal{U}$ of $\mathcal{R}$ is the subring of $\mathcal{U}_r(\mathcal{R})$ defined as follows: $\mathcal{U} = \{x \in U_r(\mathcal{R}) | \lambda x \subseteq \mathcal{R}$ for some dense left ideal λ of $\mathcal{R}\}$. In [6, Theorem 2], Chuang proved that if $\mathcal{R}$ is a prime ring, then each dense right ideal and $\mathcal{U}$ satisfy the same generalized polynomial identities (GPIs) with coefficients in $\mathcal{U}$. In any case, when $\mathcal{R}$ is a prime ring, all we need about $\mathcal{U}$ is that (1) $\mathcal{R} \subseteq \mathcal{U}$; (2) $\mathcal{U}$ is a prime ring; (3) The center of $\mathcal{U}$, denoted by $\mathcal{C}$, is a field which is called the extended centroid of $\mathcal{R}$. The axiomatic formulations and the properties of this quotient ring $\mathcal{U}$ can be found in [3]. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$, and inductively $x \circ_m y = (x \circ_{m-1} y) \circ y$ for $m > 1$. Again we set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and inductively $[x, y]_m = [[x, y]_{m-1}, y]$ for $m > 1$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation on $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, d is an inner derivation induced by an element $q \in \mathcal{R}$ if $d(x) = [q, x]$ holds for all $x \in \mathcal{R}$. An additive mapping $F : \mathcal{R} \to \mathcal{R}$ is called generalized derivation of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

2. Main results

We begin our discussion with the following remark as it is very crucial in developing the proof for our main results.

Remark 2.1. ([4, Lemma 7.1]) *Let ${}_{\mathcal{D}}\mathcal{M}$ be a left vector space over a division ring $\mathcal{D}$ with $dim_{\mathcal{D}}\mathcal{M} \geq 2$ and $\mathcal{T} \in End(\mathcal{M})$. If x and $\mathcal{T}x$ are $\mathcal{D}$-dependent for every $x \in \mathcal{M}$, then there exists $\lambda \in \mathcal{D}$ such that $\mathcal{T}x = \lambda x$ for all $x \in \mathcal{M}$.*

Now we prove our main results.

Theorem 2.1. *Let $1 \leq m, n \in \mathbb{Z}^+$. Next, let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$ and F be a nonzero generalized derivation associated with a derivation d of R. If $F([x,y]_m) = d(x) \circ_n d(y)$ for all $x, y \in \mathcal{I}$, then either R is commutative or $d = 0$ and there exist $a \in \mathcal{U}$ such that $F(x) = ax$ for all $x \in R$.*

Proof. By [16, Theorem 3], there exists an element $a \in \mathcal{U}$ and a derivation d on $\mathcal{U}$ such that $F(x) = ax + d(x)$ for all $x \in \mathcal{R}$. In view of our hypothesis, we have $a([x,y]_m) + d([x,y]_m) = d(x) \circ_n d(y)$ which is rewritten as

$$\begin{aligned} a\left([x,y]_m\right) \quad &+ \quad \sum_{k=1}^{m}(-1)^k\binom{m}{k}\left(\sum_{i+j=k-1} y^i d(y) y^j\right) x y^{m-k} \\ &+ \quad \sum_{k=0}^{m}(-1)^k\binom{m}{k} y^k d(x) y^{m-k} \\ &+ \quad \sum_{k=0}^{m-1}(-1)^k\binom{m}{k} y^k x\left(\sum_{r+s=m-k-1} y^r d(y) y^s\right) = d(x) \circ_n d(y) \end{aligned}$$

for all $x, y \in \mathcal{I}$. In the light of Kharchenko's theory [14], we split our proof into two cases. Firstly, we assume that d is an $\mathcal{U}$-inner derivation induced by an element $q \in \mathcal{U}$, i.e., $d(x) = [q,x]$ for all $x \in \mathcal{R}$, then we have $a(x \circ_m y) + [q, x \circ_m y] = [[q,x],[q,y]]_n$ for all $x, y \in \mathcal{I}$. By Chuang [6, Theorem 1], the last identity is also satisfied by $\mathcal{U}$. If $q \in C$, then $a(x \circ_m y) + [q, x \circ_m y] = [[q,x],[q,y]]_n$ reduces to $a(x \circ_m y) = 0$ for all $x, y \in \mathcal{U}$. This a polynomial identity and by Lanski[15, Lemma 2], there exists a field $\mathbb{F}$ such that $\mathcal{U} \subseteq \mathcal{M}_k(\mathbb{F})$, the ring of $k \times k$ matrices over a field $\mathbb{F}$, where $k \geq 1$. Moreover, $\mathcal{U}$ and $\mathcal{M}_k(\mathbb{F})$ satisfy the same polynomial identity[15, Lemma 1], i.e., $a(x \circ_m y) = 0$ for all $x, y \in \mathcal{M}_k(\mathbb{F})$. Now, we assuming $x = e_{12}$ and $y = e_{22}$, we have $0 = ae_{12}$ which implies that $a_{11} = a_{21} = 0$. Similarly, assuming $x = e_{21}$ and $y = e_{11}$ we can prove that $a_{22} = a_{12} = 0$, i.e., $a = 0$. Thus in all, $a(x \circ_m y) + [q, x \circ_m y] = [[q,x],[q,y]]_n$ is a non-trivial generalized polynomial identity (GPI) as $q \notin \mathcal{C}$. If the center $\mathcal{C}$ of $\mathcal{U}$ is infinite, then we have $a(x \circ_m y) + [q, x \circ_m y] = [[q,x],[q,y]]_n$ for all $x, y \in \mathcal{U} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is algebraic closure of $\mathcal{C}$. Since both $\mathcal{U}$ and $\mathcal{U} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ are prime and centrally closed [9, Theorem 2.5 and Theorem 3.5], we may replace $\mathcal{R}$ by $\mathcal{U}$ or $\mathcal{U} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite. Thus, we may assume that $\mathcal{R}$ is centrally closed over $\mathcal{C}$ (i.e., $\mathcal{RC} = \mathcal{R}$) which is either finite or algebraically closed and $a(x \circ_m y) + [q, x \circ_m y] = [[q,x],[q,y]]_n$ for all $x, y \in \mathcal{R}$. By Martindale [17, Theorem 3], $\mathcal{RC}$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle $\mathcal{H}$ with $\mathcal{C}$ as the associated division ring. Hence, by Jacobson's theorem [13, p.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{C}$ and $\mathcal{H}$ consists of finite rank linear transformations in $\mathcal{R}$. If $\mathcal{V}$ is finite dimensional over $\mathcal{C}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_m(\mathcal{C})$, where $m = dim_{\mathcal{C}}\mathcal{V}$.

Suppose that $dim_{\mathcal{C}}\mathcal{V} \geq 3$ such that v and qv are linearly $\mathcal{C}$-independent for all $v \in \mathcal{V}$. By density of $\mathcal{R}$, there exists $u \in \mathcal{V}$ such that v, qv and u are linearly

$\mathcal{C}$-independent and $x, y \in \mathcal{R}$ such that

$$\begin{array}{llll} xv = 0, & xqv = -u, & xu = v, & xqu = 0 \\ yv = 0, & yqv = -v, & yu = 0, & yqu = -u. \end{array}$$

Applying density theorem, we see that

$$0 = (a([x, y]_m) + [q, x] \circ_m [q, y] - [q, [x, y]_n])v = 2^m u,$$

a contradiction, as $char(\mathcal{R}) \neq 2$. Hence, we conclude that $\{v, qv\}$ is linearly $\mathcal{C}$-dependent for all $v \in \mathcal{V}$. Thus, by Remark 2.1, there exists $\lambda \in \mathcal{C}$ such that $qv = v\lambda$ for any $v \in \mathcal{V}$.

For $r \in \mathcal{R}, v \in \mathcal{V}$, we can write, $qv = v\lambda$, $r(qv) = r(v\lambda)$, and also $q(rv) = (rv)\lambda$. Thus $0 = [q, r]v$ for any $v \in \mathcal{V}$, i.e., $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible $\mathcal{R}$-module, we have $[q, r] = 0$ for all $r \in \mathcal{R}$, i.e., $q \in Z(\mathcal{R})$ which gives $d = 0$ and hence $F(x) = ax$ for all $x \in \mathcal{R}$.

Now suppose that $dim_{\mathcal{C}}\mathcal{V} \leq 2$. In this case $\mathcal{R}$ is a simple GPI-ring with 1 and so it is a central simple algebra finite dimensional over its center. By Lanski[15, Lemma 2], it follows that there exists a suitable field $\mathbb{F}$ such that $\mathcal{R} \subseteq \mathcal{M}_m(\mathbb{F})$ the ring of $m \times m$ matrices over $\mathbb{F}$ and moreover, $\mathcal{M}_m(\mathbb{F})$ satisfy the same GPI as $\mathcal{R}$. Assume $m \geq 3$, then by the same argument as above we get the conclusion. Obviously if $m = 1$, then $\mathcal{R}$ is commutative. Thus we may assume that $m = 2$, i.e., $\mathcal{R} \subseteq \mathcal{M}_2(\mathbb{F})$, where $\mathcal{M}_2(\mathbb{F})$ satisfies $a([x, y]_m) + [q, x] \circ_m [q, y] - [q, [x, y]_n] = 0$. Denote by e_{ij} the usual unit matrix with 1 at (i, j)-entry and zero elsewhere. By putting $x = y = e_{12}$ and $q = \sum\limits_{i,j} q_{ij}e_{ij}$ in the above identity and then right multiplying by e_{12}, one can easily get $(e_{12}q)^{m+1}e_{12} = 0$. It follows easily that $\begin{pmatrix} 0 & q_{21}^{m+1} \\ 0 & 0 \end{pmatrix} = 0$ implies that $q_{21} = 0$. Similarly we can get $q_{12} = 0$. Thus in all, we see that q is a diagonal matrix in $\mathcal{M}_2(\mathbb{F})$. Let $\psi \in Aut(\mathcal{M}_2(\mathbb{F}))$. Since $\psi(a)([\psi(x), \psi(y)]_m) + [\psi(q), \psi(x)] \circ_m [\psi(q), \psi(y)] - [\psi(q), [\psi(x), \psi(y)]_n] = 0$, $\psi(q)$ must be a diagonal matrix in $\mathcal{M}_2(\mathbb{F})$. In particular, let $\psi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\psi(q) = q + (q_{ii} - q_{jj})e_{ij}$, i.e., $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $\mathcal{M}_2(\mathbb{F})$, which leads to $d = 0$. Now lastly, we assume that d is $\mathcal{U}$-outer derivation, then $\mathcal{I}$ satisfies the polynomial identity

$$\begin{aligned} a\left([x, y]_m\right) \quad &+ \quad \sum_{k=1}^{m}(-1)^k \binom{m}{k} \left(\sum_{i+j=k-1} y^i z y^j \right) x y^{m-k} + \sum_{k=0}^{m}(-1)^k \binom{m}{k} y^k w y^{m-k} \\ &+ \quad \sum_{k=0}^{m-1}(-1)^k \binom{m}{k} y^k x \left(\sum_{r+s=m-k-1} y^r z y^s \right) = w \circ_n z \end{aligned}$$

for all $x, y, z, w \in \mathcal{I}$. In particular, if we take $x = z = 0$, then $\mathcal{I}$ satisfies the polynomial identity

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} y^k w y^{m-k} = 0$$

for all $y, w \in \mathcal{I}$. That is, $[w, y]_m = 0$ for all $w, y \in \mathcal{I}$, which can be written as $[\mathcal{I}_w(y), y]_{m-1} = 0$ for all $w, y \in \mathcal{I}$, where $\mathcal{I}_w(y)$ is an inner derivation determined by w. By Lanski [15, Theorem 1], either R is commutative or $\mathcal{I}_w = 0$ i.e., $\mathcal{I} \subseteq Z(\mathcal{R})$ in which case $\mathcal{R}$ is also commutative by Mayne [18, Lemma 3]. □

Theorem 2.2. *Let $1 \leq m, n \in \mathbb{Z}^+$. Next, let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$ and F be a nonzero generalized derivation associated with a derivation d of R. If $F(x \circ_m y) = [d(x), d(y)]_n$ for all $x, y \in \mathcal{I}$, then either R is commutative or $d = 0$ and there exists $a \in \mathcal{U}$ such that $F(x) = ax$ for all $x \in \mathcal{R}$.*

Proof. By the given hypothesis and [16, Theorem 3], we have $a(x \circ_m y) + d(x \circ_m y) = [d(x), d(y)]_n$ which is rewritten as

$$\begin{aligned} a\,(x \circ_m y) \quad &+ \quad \sum_{k=1}^{m} \binom{m}{k} \left(\sum_{i+j=k-1} y^i d(y) y^j \right) x y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} y^k d(x) y^{m-k} \\ &+ \quad \sum_{k=0}^{m-1} \binom{m}{k} y^k x \left(\sum_{r+s=m-k-1} y^r d(y) y^s \right) = [d(x), d(y)]_n \end{aligned}$$

for all $x, y \in \mathcal{I}$. In view of Kharchenko's theory [14], we divide the proof into two cases:

Case 1. If d is $\mathcal{U}$-outer, then $\mathcal{I}$ satisfies the polynomial identity

$$\begin{aligned} a\,(x \circ_m y) \quad &+ \quad \sum_{k=1}^{m} \binom{m}{k} \left(\sum_{i+j=k-1} y^i z y^j \right) x y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} y^k w y^{m-k} \\ &+ \quad \sum_{k=0}^{m-1} \binom{m}{k} y^k x \left(\sum_{r+s=m-k-1} y^r z y^s \right) = [w, z]_n \end{aligned}$$

for all $x, y, z, w \in \mathcal{I}$. In particular if we take $x = z = 0$, then $\mathcal{I}$ satisfies the polynomial identity

$$\sum_{k=0}^{m} \binom{m}{k} y^k w y^{m-k} = 0$$

for all $y, w \in \mathcal{I}$. That is $w \circ_m y = 0$ for all $w, y \in \mathcal{I}$. Using the same argument as used in Theorem 2.1 and by choosing $w = e_{12},\ y = e_{11}$, we see that $w \circ_m y = e_{12} \neq 0$, a contradiction.

Case 2. If d is $\mathcal{U}$-inner derivation induced by an element $q \in \mathcal{U}$, i.e., $d(x) = [q, x]$ for all $x \in \mathcal{R}$, then we have $a(x \circ_m y) + [q, x \circ_m y] = [[q, x], [q, y]]_n$ for all $x, y \in \mathcal{I}$. By Chuang [6, Theorem 1], $\mathcal{I}$ and $\mathcal{U}$ satisfy same generalized polynomial identities (GPIs), i.e., $a(x \circ_m y) + [q, x \circ_m y] = [[q, x], [q, y]]_n$ for all $x, y \in \mathcal{U}$. Using the similar techniques with necessary variations as used in the proof of Theorem 2.1, we see that if $\mathcal{V}$ is finite dimensional over $\mathcal{C}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_m(\mathcal{C})$, where $m = dim_{\mathcal{C}}\mathcal{V}$.

Suppose that $dim_{\mathcal{C}}\mathcal{V} \geq 2$. Now, we want to show that v and qv are linearly $\mathcal{C}$-dependent for all $v \in \mathcal{V}$. If $qv = 0$, then $\{v, qv\}$ is linearly $\mathcal{C}$-dependent. Suppose on the contrary that v and qv are linearly $\mathcal{C}$-independent for some $v \in \mathcal{V}$.

If $q^2v \notin Span_{\mathcal{C}}\{v, qv\}$, then the set $\{v, qv, q^2v\}$ is linearly $\mathcal{C}$-independent. Since v and qv are linearly $\mathcal{C}$-independent, by the density of $\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{array}{lll} xv = v, & xqv = 0, & xq^2v = 0; \\ yv = 0, & yqv = -v, & yq^2v = 0. \end{array}$$

When $m = n = 1$, then we see that

$$0 = (a(x \circ_m y) + [q, x \circ_m y] - [[q, x], [q, y]]_n)v = 2qv - v.$$

Moreover, when $m, n > 1$, we have

$$0 = (a(x \circ_m y) + [q, x \circ_m y] = [[q, x], [q, y]]_n)v = 2^m qv.$$

In both the cases we get a contradiction as characteristic of $\mathcal{R}$ is different from 2.

If $q^2v \in Span_{\mathcal{C}}\{v, qv\}$, then $q^2v = v\beta + qv\gamma$ for some $\beta, \gamma \in \mathcal{C}$. By the density of $\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{array}{ll} xv = v, & xqv = 0; \\ yv = 0, & yqv = -v. \end{array}$$

For this, first we take $m = n = 1$, we see that

$$0 = (a(x \circ_m y) + [q, x \circ_m y] - [[q, x], [q, y]]_n)v = 2qv - v\gamma - v.$$

Now, when $m, n > 1$, we have

$$0 = (a(x \circ_m y) + [q, x \circ_m y] - [[q, x], [q, y]]_n)v = 2^m qv - 2^{m-1}v\gamma.$$

Using an argument similar to that mentioned above, we get a contradiction in both cases. So, we conclude that $\{v, qv\}$ is linearly $\mathcal{C}$-dependent for all $v \in \mathcal{V}$. Thus, by Remark 2.1, there exists $\lambda \in \mathcal{C}$ such that $qv = v\lambda$ for any $v \in \mathcal{V}$.

For $r \in \mathcal{R}, v \in \mathcal{V}$, we can write, $qv = v\lambda$, $r(qv) = r(v\lambda)$, and also $q(rv) = (rv)\lambda$. Thus $0 = [q, r]v$ for any $v \in \mathcal{V}$, i.e., $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible $\mathcal{R}$-module, we have $[q, r] = 0$ for all $r \in \mathcal{R}$, i.e., $q \in Z(\mathcal{R})$ and hence $d = 0$. This completes the proof. □

In view of Theorem 2.1 and Theorem 2.2, we can write the following corollaries (proofs are omitted for sake of brevity)

Corollary 2.1. *Let $1 \leq m \in \mathbb{Z}^{+}$. Next, let $\mathcal{R}$ be a prime ring of a characteristic different from 2, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$ and d be a derivation of R. If $d(x) \circ_m d(y) = 0$ for all $x, y \in \mathcal{I}$, then either R is commutative or $d = 0$.*

Corollary 2.2. *Let $1 \leq m \in \mathbb{Z}$. Next, let $\mathcal{R}$ be a prime ring of a characteristic different from 2, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$ and d be a derivation of R. If $[d(x) \circ d(y)]_m = 0$ for all $x, y \in \mathcal{I}$, then either R is commutative or $d = 0$.*

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REFERENCES

1. N. Argaç and H. G. Inceboz: *Derivation of prime and semiprime rings.* J. Korean Math. Soc., **46** (5) (2009), 997-1005.
2. M. Ashraf and N. Rehman: *On commutativity of rings with derivations.* Results Math., **42** (1-2) (2002), 3-8.
3. K. I. Beidar, W. S. Martindale III and A. V. Mikhalev: *Rings with Generalized Identities.* Pure and Applied Mathematics, Marcel Dekker 196, New York, 1996.
4. K. I. Beidar and M. Brešar: *Extended Jacobson density theorem for rings with automorphisms and derivations*, Israel J. Math., **122** (2001), 317-346.
5. H. E. Bell and M. N. Daif: *On commutativity and strong commutativity-preserving maps.* Cand. Math. Bull., **37** (1994), 443-447.
6. C. L. Chuang: *GPIs having coefficients in Utumi quotient rings.* Proc. Amer. Math. Soc. **103** (1988), 723-728.
7. M. N. Daif and H. E. Bell: *Remarks on derivations on semiprime rings.* Internt. J. Math. Math. Sci. **15** (1992), 205-206.
8. Q. Deng and M. Ashraf:*On strong commutativity preserving mappings.* Results Math. **30** (1996), 259-263.
9. T. S. Erickson, W. S. Martindale III and J. M. Osborn: *Prime nonassociative algebras.* Pacific. J. Math. **60** (1975), 49-63.
10. I. N. Herstein: *A note on derivations.* Canad. Math. Bull. **21** (1978), 241-242.
11. S. Huang: *Derivation with Engel conditions in prime and semiprime rings.* Czechoslovak Math. J. **61** (136)(2011), 1135-1140.
12. S. Huang: *Generalized derivations of prime and semiprime rings.* Taiwanese J. Math., **16** (2) (2012), 771-776.
13. N. Jacobson: *Structure of Rings.* Colloquium Publications 37, Amer. Math. Soc. VII, Provindence, RI 1956.
14. V. K. Kharchenko: *Differential identities of prime rings.* Algebra Logic **17** (1979), 155-168.
15. C. Lanski: *An Engel condition with derivation.* Proc. Amer. Math. Soc. **118** (1993), 731-734.

16. T. K. Lee: *Generalized derivations of left faithful rings*. Comm. Algebra **27** (8)(1998), 4057-4073.

17. W. S. Martindale III: *Prime rings satisfying a generalized polynomial identity*. J. Algebra **12** (1969), 576-584.

18. J. H. Mayne: *Centralizing mappings of prime rings*. Canad. Math. Bull. **27** (1) (1984), 122-126.

19. M. A. Raza and N. Rehman: *On prime and semiprime rings with generalized Derivations and non-commutative Banach algebras*. Proc. Indian Acad. Sci. (Math. Sci.) **126** (3) (2016), 389-398.

20. M. A. Raza and N. Rehman: *On generalized derivation in rings and Banach algebras*. Kragujevac J. Math. **41** (1) (2017), 105-120.

21. N. Rehman, M. A. Raza and T. Bano: *On commutativity of rings with generalized derivations*. J. Egyptian Math. Soc. **24** (2) (2016), 151-155.

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BOUNDEDNESS FOR TOEPLITZ TYPE OPERATOR ASSOCIATED WITH SINGULAR INTEGRAL OPERATOR WITH VARIABLE CALDERÓN-ZYGMUND KERNEL

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Abstract. In this paper, we establish sharp maximal function inequalities for the Toeplitz-type operator associated with the singular integral operator with a variable Calderón-Zygmund kernel. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.
Keywords: function inequalities; Toeplitz-type operator; singular integral operator.

1. Introduction and Preliminaries

As the development of singular integral operators(see [6, 21]), their commutators have been well studied. In [3, 19, 20], the authors prove that the commutators generated by singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by fractional integral operators. In [7, 16], the boundedness for the commutators generated by singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1], Calderón and Zygmund introduce some singular integral operators with a variable kernel and discuss their boundedness. In [11, 12, 13, 22], the authors obtain the boundedness for the commutators generated by singular integral operators with a variable kernel and BMO functions. In [14], the authors prove the boundedness for the multilinear oscillatory singular integral operators generated by operators and BMO functions. In [8, 9], some Toeplitz-type operators associated with singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz-type operator generated by the singular integral operator with a variable Calderón-Zygmund kernel and Lipschitz and BMO functions.

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First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [6, 21])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||M^{\#}(f)||_{L^\infty}$. It has been known that (see [21])

$$||f - f_{2^k Q}||_{BMO} \leqslant Ck||f||_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

The A_p weight is defined by (see [6])

$$A_p = \left\{ w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\},$$

$1 < p < \infty$. and

$$A_1 = \{ w \in L^p_{loc}(R^n) : M(w)(x) \leqslant Cw(x), a.e. \}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be a homogeneous Triebel-Lizorkin space(see [16]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leqslant D\varphi(t) \quad \text{for} \quad t \geqslant 0.$$

Let f be a locally integrable function on R^n. Set, for $1 \leqslant p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x \in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\varphi}(R^n) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [17, 18]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue spaces.

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [4, 5, 10, 15]).

In this paper, we will study some singular integral operators as follows(see [1]).

Definition 2. Let $K(x) = \Omega(x)/|x|^n \ : \ R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel if
(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;
(b) Ω is homogeneous of degree zero;
(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n \ : \ |x| = 1\}$ is the unit sphere of R^n.

Definition 3. Let $K(x, y) = \Omega(x, y)/|y|^n \ : \ R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if
(d) $K(x, \cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;
(e) $max_{|\gamma| \le 2n} \left|\left| \frac{\partial^\gamma}{\partial^\gamma y} \Omega(x, y) \right|\right|_{L^\infty(R^n \times \Sigma)} = M < \infty$.

Moreover, let b be a locally integrable function on R^n and T be a singular integral operator with a variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x - y)f(y)dy,$$

where $K(x, x-y) = \frac{\Omega(x,x-y)}{|x-y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernel. The Toeplitz-type operator associated with T is defined by

$$T_b = \sum_{k=1}^{m} (T^{k,1} M_b I_\alpha T^{k,2} + T^{k,3} I_\alpha M_b T^{k,4}),$$

where $T^{k,1}$ is the singular integral operator with a variable Calderón-Zygmund kernel T or $\pm I$(the identity operator), $T^{k,2}$ and $T^{k,4}$ are linear operators, $T^{k,3} = \pm I$, $k = 1, ..., m$, $M_b(f) = bf$ and I_α is the fractional integral operator($0 < \alpha < n$)(see [2]).

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz-type operator T_b. The Toeplitz-type operator T_b are non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [19, 20]). The main purpose of this paper is to prove sharp maximal inequalities for the Toeplitz-type operator T_b. As the application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz-type operator T_b.

2. Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < s < \infty$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^{\#}(T_b(f))(\tilde{x}) \leq C||b||_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Theorem 2. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < s < \infty$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$\begin{aligned} \sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - c|\, dx \quad &\leqslant \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) \\ &+ \quad M_{\alpha,s}(T^{k,4}(f))(\tilde{x})). \end{aligned}$$

Theorem 3. Let T be a singular integral operator as **Definition 3**, $1 < s < \infty$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^{\#}(T_b(f))(\tilde{x}) \leq C||b||_{BMO} \sum_{k=1}^{m} (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Theorem 4. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Theorem 5. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 < D < 2^n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^{p,\varphi}(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^{p,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.

Theorem 6. Let T be a singular integral operator as **Definition 3**, $0 < \beta < 1$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any

$g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $\dot{F}_q^{\beta,\infty}(R^n)$.

Theorem 7. Let T be a singular integral operator as **Definition 3**, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$.

Theorem 8. Let T be a singular integral operator as **Definition 3**, $0 < D < 2^n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are bounded operators on $L^{p,\varphi}(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^{p,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.

To prove the theorems, we need the following lemmas.

Lemma 1.(see [1]) Let T be a singular integral operator as **Definition 3**. Then T is bounded on $L^p(R^n)$ for $1 < p < \infty$.

Lemma 2.(see [16]). For $0 < \beta < 1$ and $1 < p < \infty$, we have

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_{Q \ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(y) - f_Q| dy\right|\right|_{L^p} \\ &\approx \left|\left|\sup_{Q \ni x} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(y) - c| dy\right|\right|_{L^p}, \end{aligned}$$

where the sup is taken all cubes Q containing $x \in R^n$.

Lemma 3.(see [6]). Let $0 < p < \infty$ and $w \in \cup_{1 \le r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M(f)(x)^p w(x) dx \leqslant C \int_{R^n} M^{\#}(f)(x)^p w(x) dx.$$

Lemma 4.(see [2, 6]). Suppose that $0 < \alpha < n$, $1 \le s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then

$$||I_\alpha(f)||_{L^q} \leqslant C||f||_{L^p}$$

and

$$||M_{\alpha,s}(f)||_{L^q} \leqslant C||f||_{L^p}.$$

Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,

$$||M(f)||_{L^{p,\varphi}} \leqslant C||M^{\#}(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube Q by [6]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned}
&\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx \\
\leqslant\ & \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \leqslant C \int_{R^n} M^{\#}(f)(x)^p M(\chi_Q)(x) dx \\
\leqslant\ & C\left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx\right) \\
\leqslant\ & C||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} 2^{-kn}\varphi(2^{k+1}d) \\
\leqslant\ & C||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\
\leqslant\ & C||M^{\#}(f)||^p_{L^{p,\varphi}} \varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(d)}\int_Q M^{\#}(f)(x)^p dx\right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leqslant C||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof.

Lemma 6. Let $0 < \alpha < n$, $0 < D < 2^n$, $1 \leqslant s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then

$$||I_\alpha(f)||_{L^{q,\varphi}} \leqslant C||f||_{L^{p,\varphi}}$$

and

$$||M_{\alpha,s}(f)||_{L^{r,\varphi}} \leqslant C||f||_{L^{p,\varphi}}.$$

The proof of the Lemma is similar to that of Lemma 5 by Lemma 4, we omit the details.

3. Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|\, dx \leqslant C||b||_{Lip_\beta} \sum_{k=1}^{m} (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T (k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T_1(g) = 0$,

$$\begin{aligned}
T_b(f)(x) &= \sum_{k=1}^{m} T^{k,1} M_b I_\alpha T^{k,2}(f)(x) + \sum_{k=1}^{m} T^{k,3} I_\alpha M_b T^{k,4}(f)(x) \\
&= A_b(x) + B_b(x) = A_{b-b_Q}(x) + B_{b-b_Q}(x),
\end{aligned}$$

where

$$\begin{aligned} A_{b-b_Q}(x) &= \sum_{k=1}^{m} T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x) + \sum_{k=1}^{m} T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha T^{k,2}(f)(x) \\ &= A_1(x) + A_2(x) \end{aligned}$$

and

$$\begin{aligned} B_{b-b_Q}(x) &= \sum_{k=1}^{m} T^{k,3} I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,4}(f)(x) + \sum_{k=1}^{m} T^{k,3} I_\alpha M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,4}(f)(x) \\ &= B_1(x) + B_2(x). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{|Q|}\int_Q \left|A_{b-b_Q}(f)(x) - A_2(x_0)\right| dx &\leqslant \frac{1}{|Q|}\int_Q |A_1(x)|dx + \frac{1}{|Q|}\int_Q |A_2(x) - A_2(x_0)|dx \\ &= I_1 + I_2 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{|Q|}\int_Q \left|B_{b-b_Q}(f)(x) - B_2(x_0)\right| dx &\leqslant \frac{1}{|Q|}\int_Q |B_1(x)|dx + \frac{1}{|Q|}\int_Q |B_2(x) - B_2(x_0)|dx \\ &= I_3 + I_4. \end{aligned}$$

For I_1, by Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned} &\frac{1}{|Q|}\int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|dx \\ \leqslant\ & \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s} \\ \leqslant\ & C|Q|^{-1/s}\left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s} \\ \leqslant\ & C|Q|^{-1/s}\left(\int_{2Q} (|b(x) - b_Q||I_\alpha T^{k,2}(f)(x)|)^s dx\right)^{1/s} \\ \leqslant\ & C|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s} \\ \leq\ & C||b||_{Lip_\beta} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$\begin{aligned} I_1 &\leq \sum_{k=1}^{m} \frac{1}{|Q|}\int_{R^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|dx \\ &\leq C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}). \end{aligned}$$

For I_2, by [1][14], we know that

$$T(f)(x)=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}a_{uv}(x)\int_{R^n}\frac{Y_{uv}(x-y)}{|x-y|^n}f(y)dy,$$

where $g_u\leqslant Cu^{n-2}$, $||a_{uv}||_{L^\infty}\leqslant Cu^{-2n}$, $|Y_{uv}(x-y)|\leqslant Cu^{n/2-1}$ and

$$\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|\leqslant Cu^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

for $|x-y|>2|x_0-x|>0$, we get, for $x\in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x)-T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x_0)|\\
\leqslant\ &\int_{(2Q)^c}|b(y)-b_{2Q}||K(x,x-y)-K(x_0,x_0-y)||I_\alpha T^{k,2}(f)(y)|dy\\
=\ &\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|H_1|I_\alpha T^{k,2}(f)(y)|dy\\
\leqslant\ &C\sum_{j=1}^{\infty}||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\le|y-x_0|<2^{j+1}d}|I_\alpha T^{k,2}(f)(y)|H_2dy\\
\le\ &C||b||_{Lip_\beta}\sum_{u=1}^{\infty}u^{-2n}\cdot u^{n/2}\sum_{j=1}^{\infty}|2^{j+1}Q|^{\beta/n}\int_{2^jd\le|y-x_0|<2^{j+1}d}H_3dy\\
\le\ &C||b||_{Lip_\beta}\sum_{j=1}^{\infty}2^{-j}\left(\frac{1}{|2^{j+1}Q|^{1-\beta/n}}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|dy\right)\\
\le\ &C||b||_{Lip_\beta}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{-j}\\
\le\ &C||b||_{Lip_\beta}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

where

$$H_1=\left|\frac{\Omega(x,x-y)}{|x-y|^n}-\frac{\Omega(x_0,x_0-y)}{|x_0-y|^n}\right|,H_3=\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,2}(f)(y)|,$$

$$H_2=\sum_{u=1}^{\infty}\sum_{v=1}^{g_u}|a_{uv}(x)|\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right|,$$

thus

$$\begin{aligned}
I_2\ &\leqslant\ \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}|H_4|dx\\
&\leqslant\ C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

where

$$H_4 = T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x_0).$$

Similarly, by Lemma 4, for $1/r = 1/s - \alpha/n$,

$$\begin{aligned}
I_3 &\leq \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{k,4}(f)(x)|^r dx\right)^{1/r} \\
&\leqslant C\sum_{k=1}^{m}|Q|^{-1/r}\left(\int_{2Q}(|b(x)-b_Q||T^{k,4}(f)(x)|)^s dx\right)^{1/s} \\
&\leqslant C||b||_{Lip_\beta}\sum_{k=1}^{m}|Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n}H_5 \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}$$

where

$$H_5 = \left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s}.$$

$$\begin{aligned}
I_4 &\leq \sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right||T^{k,4}(f)(y)|dydx \\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\sum_{j=1}^{\infty}(2^jd)^\beta d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\beta-\alpha} \\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s(\beta+\alpha)/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^s dy\right)^{1/s} \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{-j} \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q|T_b(f)(x)-C_0|\,dx \leqslant C||b||_{Lip_\beta}\sum_{k=1}^{m}(M_s(I_\alpha T^{k,2}(f))(\tilde{x})+M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k=1,...,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |T_b(f)(x)-A_2(x_0)-B_2(x_0)|\,dx\\
\leq\ &\frac{1}{|Q|^{1+\beta/n}}\int_Q |A_1(x)|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q |A_2(x)-A_2(x_0)|dx\\
+\ &\frac{1}{|Q|^{1+\beta/n}}\int_Q |B_1(x)|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q |B_2(x)-B_2(x_0)|dx\\
=\ &I_5+I_6+I_7+I_8.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get, for $1/r=1/s-\alpha/n$,

$$\begin{aligned}
I_5 &\leq |Q|^{-\beta/n}\sum_{k=1}^m\left(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_Q)\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\leqslant C|Q|^{-\beta/n}\sum_{k=1}^m|Q|^{-1/s}\left(\int_{2Q}(|b(x)-b_Q||I_\alpha T^{k,2}(f)(x)|)^s dx\right)^{1/s}\\
&\leqslant C|Q|^{-\beta/n}\sum_{k=1}^m|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{1/s}\left(\frac{1}{|Q|}\int_{2Q}|I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m M_s(I_\alpha T^{k,2}(f))(\tilde{x}),\\
I_6 &\leq |Q|^{-\beta/n}\sum_{k=1}^m\frac{1}{|Q|}\int_Q\sum_{j=1}^\infty\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\\
&\quad\times|K(x,x-y)-K(x_0,x_0-y)||I_\alpha T^{k,2}(f)(y)|dydx\\
&\leq |Q|^{-\beta/n}\sum_{k=1}^m\frac{C}{|Q|}\int_Q\sum_{j=1}^\infty||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\sum_{u=1}^\infty\sum_{v=1}^{g_u}|a_{uv}(x)|\\
&\quad\times\left|\frac{Y_{uv}(x-y)}{|x-y|^n}-\frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right||I_\alpha T^{k,2}(f)(y)|dydx\\
&\leq C||b||_{Lip_\beta}|Q|^{-\beta/n}\sum_{k=1}^m\frac{1}{|Q|}\\
&\quad\times\int_Q\sum_{j=1}^\infty|2^{j+1}Q|^{\beta/n}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,2}(f)(y)|dydx
\end{aligned}$$

$$\begin{aligned}
&\le C\|b\|_{Lip_\beta} d^{-\beta}\sum_{k=1}^{m}\sum_{j=1}^{\infty}(2^j d)^\beta \frac{d}{(2^j d)^{n+1}}(2^j d)^n\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\le C\|b\|_{Lip_\beta}\sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{j(\beta-1)}\\
&\le C\|b\|_{Lip_\beta}\sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_7 &\le |Q|^{-\beta/n}\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{k,4}(f)(x)|^r dx\right)^{1/r}\\
&\leqslant C|Q|^{-\beta/n-1/r}\sum_{k=1}^{m}\left(\int_{2Q}(|b(x)-b_Q||T^{k,4}(f)(x)|)^s dx\right)^{1/s}\\
&\leqslant C\|b\|_{Lip_\beta}\sum_{k=1}^{m}|Q|^{-\beta/n-1/r}|2Q|^{\beta/n}|Q|^{1/s-\alpha/n}\\
&\quad\times\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s}\\
&\le C\|b\|_{Lip_\beta}\sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_8 &\le |Q|^{-\beta/n-1}\sum_{k=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\\
&\quad\times\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right||T^{k,4}(f)(y)|dydx\\
&\le C|Q|^{-\beta/n}\sum_{k=1}^{m}\sum_{j=1}^{\infty}\|b\|_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\\
&\quad\times\int_{2^j d\le|y-x_0|<2^{j+1}d}\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy\\
&\le C\|b\|_{Lip_\beta}\sum_{k=1}^{m}\sum_{j=1}^{\infty}d^{-\beta}(2^j d)^\beta d(2^j d)^{-n+\alpha-1}(2^j d)^{n(1-1/s)}(2^j d)^{n/s-\alpha}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^s dy\right)^{1/s}\\
&\le C\|b\|_{Lip_\beta}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{j(\beta-1)}\\
&\le C\|b\|_{Lip_\beta}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 2.

Proof of Theorem 3. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|\,dx \leqslant C||b||_{BMO}\sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_b(f)(x) - A_2(x_0) - B_2(x_0)|\,dx \le \frac{1}{|Q|}\int_Q |A_1(x)|dx \\
&+\frac{1}{|Q|}\int_Q |A_2(x) - A_2(x_0)|dx + \frac{1}{|Q|}\int_Q |B_1(x)|dx + \frac{1}{|Q|}\int_Q |B_2(x) - B_2(x_0)|dx \\
&= I_9 + I_{10} + I_{11} + I_{12}.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get, for $1 < r_1 < s$, $1 < p < \min(s, n/\alpha)$ with $1/r_2 = 1/p - \alpha/n$,

$$\begin{aligned}
I_9 &\le \sum_{k=1}^m \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^{r_1}dx\right)^{1/r_1} \\
&\leqslant C\sum_{k=1}^m |Q|^{-1/r_1}\left(\int_{2Q} (|b(x) - b_Q||I_\alpha T^{k,2}(f)(x)|)^{r_1}dx\right)^{1/r_1} \\
&\leqslant C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\left(\frac{1}{|Q|}\int_{2Q} |b(x) - b_Q|^{sr_1/(s-r_1)}dx\right)^{(s-r_1)/sr_1} \\
&\le C||b||_{BMO}\sum_{k=1}^m M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_{10} &\le \sum_{k=1}^m \frac{1}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y) - b_{2Q}||K(x, x-y) - K(x_0, x_0-y)||I_\alpha T^{k,2}(f)(y)|dydx \\
&\le \sum_{k=1}^m \frac{C}{|Q|}\int_Q \sum_{j=1}^\infty \int_{2^jd\le|y-x_0|<2^{j+1}d} |b(y) - b_{2Q}|\sum_{u=1}^\infty\sum_{v=1}^{g_u} |a_{uv}(x)| \\
&\quad\times\left|\frac{Y_{uv}(x-y)}{|x-y|^n} - \frac{Y_{uv}(x_0-y)}{|x_0-y|^n}\right||I_\alpha T^{k,2}(f)(y)|dydx
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1}^{m}\frac{C}{|Q|}\int_Q\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{|x-x_0|}{|x_0-y|^{n+1}}|I_\alpha T^{k,2}(f)(y)|dydx\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\frac{d}{(2^jd)^{n+1}}\left(\int_{2^{j+1}Q}|b(y)-b_Q|^{s'}dy\right)^{1/s'}\\
&\quad\times\left(\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s}dx\\
&\le C||b||_{BMO}\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\le C||b||_{BMO}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j}\\
&\le C||b||_{BMO}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_{11} &\le \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{k,4}(f)(x)|^{r_2}dx\right)^{1/r_2}\\
&\leqslant C|Q|^{-1/r_2}\sum_{k=1}^{m}\left(\int_{2Q}(|b(x)-b_Q||T^{k,4}(f)(x)|)^p dx\right)^{1/p}\\
&\leqslant C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{2Q}|b(x)-b_Q|^{ps/(s-p)}dx\right)^{(s-p)/ps}\left(\frac{1}{|Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s}\\
&\le C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &\le |Q|^{-1}\sum_{k=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right||T^{k,4}(f)(y)|dydx\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy\\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\alpha}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_Q|^{s'}dy\right)^{1/s'}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^s dy\right)^{1/s}\\
&\le C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j}\le C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemmas 3 and 4,

$$\begin{aligned}
&||T_b(f)||_{L^q} \leqslant \|M(T_b(f))\|_{L^q} \leqslant C\|M^{\#}(T_b(f))\|_{L^q} \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_{\beta,s}(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\beta+\alpha,s}(T^{k,4}(f))\|_{L^q}) \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^r} + \|T^{k,4}(f)\|_{L^p}) \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \leq C||b||_{Lip_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the proof of Theorem 4.

Proof of Theorem 5. Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemmas 5 and 6,

$$\begin{aligned}
&||T_b(f)||_{L^{q,\varphi}} \leqslant \|M(T_b(f))\|_{L^{q,\varphi}} \leqslant C\|M^{\#}(T_b(f))\|_{L^{q,\varphi}} \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_{\beta,s}(I_\alpha T^{k,2}(f))\|_{L^{q,\varphi}} + \|M_{\beta+\alpha,s}(T^{k,4}(f))\|_{L^{q,\varphi}}) \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^{r,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\varphi}}) \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}}) \leq C||b||_{Lip_\beta} \|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof of Theorem 5.

Proof of Theorem 6. Choose $1 < s < p$ in Theorem 2. We have, by Lemmas 2, 3 and 4,

$$\begin{aligned}
&||T_b(f)||_{\dot{F}_q^{\beta,\infty}} \leqslant C \left\| \sup_{Q \ni x} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(y) - C_0| \, dy \right\|_{L^q} \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_s(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^q}) \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^q} + \|T^{k,4}(f)\|_{L^p}) \\
\leqslant\;& C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \leqslant C||b||_{Lip_\beta} ||f||_{L^p}.
\end{aligned}$$

This completes the proof of the theorem.

Proof of Theorem 7. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 3 and 4,

$$\begin{aligned} & ||T_b(f)||_{L^q} \leqslant \|M(T_b(f))\|_{L^q} \leqslant C\|M^{\#}(T_b(f))\|_{L^q} \\ \leqslant & \; C||b||_{BMO}\sum_{k=1}^{m}(\|M_s(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^q}) \\ \leqslant & \; C||b||_{BMO}\sum_{k=1}^{m}(\|I_\alpha T^{k,2}(f)\|_{L^q} + \|T^{k,4}(f)\|_{L^p}) \\ \leqslant & \; C||b||_{BMO}\sum_{k=1}^{m}(\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \leqslant C||b||_{BMO}\|f\|_{L^p}. \end{aligned}$$

This completes the proof of Theorem 7.

Proof of Theorem 8. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 5 and 6,

$$\begin{aligned} & ||T_b(f)||_{L^{q,\varphi}} \leqslant \|M(T_b(f))\|_{L^{q,\varphi}} \leqslant C\|M^{\#}(T_b(f))\|_{L^{q,\varphi}} \\ \leqslant & \; C||b||_{BMO}\sum_{k=1}^{m}(\|M_s(I_\alpha T^{k,2}(f))\|_{L^{q,\varphi}} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^{q,\varphi}}) \\ \leqslant & \; C||b||_{BMO}\sum_{k=1}^{m}(\|I_\alpha T^{k,2}(f)\|_{L^{q,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\varphi}}) \\ \leqslant & \; C||b||_{BMO}\sum_{k=1}^{m}(\|T^{k,2}(f)\|_{L^{p,\varphi}} + \|f\|_{L^{p,\varphi}}) \leqslant C||b||_{BMO}\|f\|_{L^{p,\varphi}}. \end{aligned}$$

This completes the proof of Theorem 8.

REFERENCES

1. A. P. Calderón and A. Zygmund: *On singular integrals with variable kernels.* Appl. Anal. **7**(1978), 221–238.
2. S. Chanillo: *A note on commutators.* Indiana Univ. Math. J. **31**(1982), 7–16.
3. R. R. Coifman, R. Rochberg and G. Weiss: *Factorization theorems for Hardy spaces in several variables.* Ann. of Math. **103**(1976), 611–635.
4. G. D. FaZio and M. A. Ragusa: *Commutators and Morrey spaces.* Boll. Un. Mat. Ital. **5-A**(1991), 323–332.
5. G. D. Fazio and M. A. Ragusa: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients.* J. Func. Anal. **112**(1993), 241–256.
6. J. Garcia-Cuerva and J. L. Rubio de Francia: *Weighted norm inequalities and related topics.* North-Holland Math. Stud. 116(1985)
7. S. Janson: *Mean oscillation and commutators of singular integral operators.* Ark. Math. **16**(1978), 263–270.

8. S. Krantz and S. Li: *Boundedness and compactness of integral operators on spaces of homogeneous type and applications*. J. of Math. Anal. Appl. **258**(2001), 629–641.
9. Y. Lin and S. Z. Lu: *Toeplitz type operators associated to strongly singular integral operator*. Sci. China(Ser.A). **36**(2006), 615–630.
10. L. Z. Liu: *Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators*. Acta Math. Scientia. **25(B)**(2005), 89–94.
11. L. Z. Liu: *The continuity for multilinear singular integral operators with variable Calderón-Zygmund kernel on Hardy and Herz spaces*. Siberia Electronic Math. Reports. **2**(2005), 156–166.
12. L. Z. Liu: *Good λ estimate for multilinear singular integral operators with variable Calderón-Zygmund kernel*. Kragujevac J. of Math. **27**(2005), 19–30.
13. L. Z. Liu: *Weighted estimates of multilinear singular integral operators with variable Calderón-Zygmund kernel for the extreme cases*. Vietnam J. of Math. **34**(2006), 51–61.
14. S. Z. Lu, D. C. Yang and Z. S. Zhou: *Oscillatory singular integral operators with Calderón-Zygmund kernels*. Southeast Asian Bull. of Math. **23**(1999), 457–470.
15. T. Mizuhara: *Boundedness of some classical operators on generalized Morrey spaces, in "Harmonic Analysis"*. In: Proceedings of a conference held in Sendai, Japan, 1990, pp. 183–189.
16. M. Paluszynski: *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*. Indiana Univ. Math. J. **44**(1995), 1–17.
17. J. Peetre: *On convolution operators leaving $L^{p,\lambda}$-spaces invariant*. Ann. Mat. Pura. Appl. **72**(1966), 295–304.
18. J. Peetre: *On the theory of $L^{p,\lambda}$-spaces*. J. Func. Anal. **4**(1969), 71–87.
19. C. Pérez: *Endpoint estimate for commutators of singular integral operators*. J. Func. Anal. **128**(1995), 163–185.
20. C. Pérez **and** R. Trujillo-Gonzalez: *Trujillo-Gonzalez, Sharp weighted estimates for multilinear commutators*. J. London Math. Soc. **65**(2002), 672–692.
21. E. M. Stein: *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*. Princeton Univ. Press, Princeton NJ, 1993.
22. H. Xu **and** L. Z. Liu: *Weighted boundedness for multilinear singular integral operator with variable Calderón-Zygmund kernel*. African Diaspora J. of Math. **6**(2008), 1–12.

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(CLR)-PROPERTY ON QUASI-PARTIAL METRIC SPACES AND RELATED FIXED POINT THEOREMS

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Abstract. In this paper, we introduce the concept of common limit range ($(CLR)-$property) in the framework of quasi-partial metric spaces. By using this concept, some fixed point theorems involving two pairs of contraction mappings are proved without using the completeness condition of the whole space. Our results extend some results in literature, such as Nazir and Abbas [8] and Vetro et al. [11].
Keywords: quasi-partial metric spaces; $(CLR)-$property; contraction mappings.

1. Introduction

The connotation of partial metric spaces (PMS for short) was defined by Matthews in [9]. He amended metric spaces via setting self-distances to be not always identical to zero. Additionally, he relocated the Banach contraction principle in the setting of (PMS). Since then, there has been extensive research into fixed point results related to partial metric spaces (see [2, 3, 4, 7]). By dropping the symmetry condition, in 2013 Karapinar et al. [6] defined the notation of quasi-partial metric spaces (QPMS for short) and established some fixed point results in these spaces.

Let us first present some definitions and consequences which we need in the sequel.

Definition 1.1. [6] *The function $\sigma : X \times X \to [0, \infty)$ is a quasi-partial metric if the following conditions are satisfied for all $\gamma, \omega, \delta \in X$:*
(1) If $0 \leq \sigma(\gamma, \gamma) = \sigma(\gamma, \omega) = \sigma(\omega, \omega) \Rightarrow \gamma = \omega$;
(2) $\sigma(\gamma, \omega) \geq \sigma(\gamma, \gamma)$;
(3) $\sigma(\omega, \gamma) \geq \sigma(\gamma, \gamma)$;
(4) $\sigma(\gamma, \delta) \leq \sigma(\gamma, \omega) + \sigma(\omega, \delta) - \sigma(\omega, \omega)$.
The couple (X, σ) is known as a (QPMS).

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For each partial metric p on X, the function $d_p : X \times X \to [0, \infty)$ defined by

$$d_p(\gamma, \omega) = 2p(\gamma, \omega) - p(\gamma, \gamma) - p(\omega, \omega), \tag{1.1}$$

is a metric on X. Similarly, if (X, σ) is a (QPMS), then the function $d_\sigma : X \times X \to [0, \infty)$ defined by

$$d_\sigma(\gamma, \omega) = \sigma(\gamma, \omega) + \sigma(\omega, \gamma) - \sigma(\gamma, \gamma) - \sigma(\omega, \omega), \tag{1.2}$$

is also a metric on X.

Definition 1.2. [6] *Let (X, σ) be a quasi-partial metric space.*
1. A sequence $\{x_n\}$ is called convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} \sigma(x_n, x) = \lim_{n\to\infty} \sigma(x, x_n) = \lim_{n\to\infty} \sigma(x_n, x_n) = \sigma(x, x)$;
2. A sequence $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ and $\lim_{n,m\to\infty} \sigma(x_m, x_n)$ exist and are finite;
3. (X, σ) is called complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some $x \in X$. Further, $\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n,m\to\infty} \sigma(x_m, x_n) = \sigma(x, x)$.

In 1996, Jungck [5] introduced the concept of weakly compatible mappings (w-compatible for short).

Definition 1.3. [5] *Let X be a nonempty set. Given $S, H : X \to X$. The mappings H and S are w-compatible if and only if $SH\mu = HS\mu$ for $\mu \in C(S, H)$, where $C(S, H) = \{u,\ fu = gu\}$.*

Definition 1.4. [1] *Let S and H be two self-mappings on a metric space (X, d). The mappings S and H fulfill the $(E.A)$-property if there exists a sequence $\{a_n\}$ in X such that*

$$\lim_{n\to\infty} Ha_n = \lim_{n\to\infty} Sa_n = \mu$$

for $\mu \in X$.

Note that the $(E.A)$-property exchanges the completeness condition of the space with closedness of the range. The connotation of (CLR)-property was defined by Sintunavarat and Kumam in [10]. Its significance is that one does no longer refer to the closeness condition of the range of subspaces.

Definition 1.5. [10] *Let (X, d) be a metric space and S, H be two self-mappings on X. These maps satisfy the (CLR_S)-property, if there exists a sequence $\{a_n\}$ in X so that*

$$\lim_{n\to\infty} Ha_n = \lim_{n\to\infty} Sa_n = \mu,$$

where $\mu \in S(X)$.

Currently, Nazir and Abbas [8] established some fixed point results via the (E.A)-property in the class of (PMS). However, we see that the circumstance $p(t,t)=0$ in [4, Definition 1.7] is superfluous. In our current work, we shall give the definition of (CLR)-property (for two pairs of self-mappings) on (QPMS). Additionally, by using this concept, we employ a different method compared with that in the proof of [4, Theorem 2.1] in order to prove our main results in the class of (QPMS). Some illustrated examples are also given.

2. Main results

First, let $\psi:[0,\infty)\to[0,\infty)$ be a function such that
(a) ψ is nondecreasing and continuous;
(b) $\psi(\mu)=0 \Leftrightarrow \mu=0$.
Denote $\mathcal{F}$ (resp. $\mathcal{G}$) the set of functions verifying the conditions (a) and (b) (resp. (b) and (c): ψ is lower-semicontinuous).

Now, we introduce the concept of (CLR)-property first for a single pair and after for a double pair of self-mappings on a (QPMS).

Definition 2.1. *Let (X,σ) be a (QPMS). The pair of self-mappings (f,S) on X satisfies the (CLR_S)-property, if there exists $\{x_n\}\subset X$ such that*

$$\lim_{n\to\infty}\sigma(fx_n,w)=\lim_{n\to\infty}\sigma(w,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,w)=\lim_{n\to\infty}\sigma(w,Sx_n)=\sigma(w,w),\ w\in SX.$$

Example 2.1. *Let $X=(0,\infty)$ and $\sigma(x,y)=|x-y|+x$ for all $x,y\in X$. Clearly, (X,σ) is a (QPMS). Let (f,S) be a pair of self-mappings on X such that $fx=\frac{3x+2}{2}$ and $Sx=2x$. Choose $\{x_n\}=\{\frac{2n+1}{n}\}$. We have*

$$\lim_{n\to\infty}\sigma(fx_n,4)=\lim_{n\to\infty}\sigma(4,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,4)=\lim_{n\to\infty}\sigma(4,Sx_n)=\sigma(4,4)=S2=4.$$

Hence the pair (f,S) satisfies the (CLR_S)-property.

Definition 2.2. *Let (X,σ) be a (QPMS). The pairs of self-mappings (f,S) and (g,H) on X satisfy the (CLR_{SH})-property, if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\begin{aligned}\lim_{n\to\infty}\sigma(fx_n,w) &= \lim_{n\to\infty}\sigma(w,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,w)=\lim_{n\to\infty}\sigma(w,Sx_n)\\ &= \lim_{n\to\infty}\sigma(w,gy_n)=\lim_{n\to\infty}\sigma(gy_n,w)\\ &= \lim_{n\to\infty}\sigma(Hy_n,w)=\lim_{n\to\infty}\sigma(w,Hy_n)=\sigma(w,w),\ \ w\in SX\cap HX.\end{aligned}$$

We illustrate Definition 2.2 by the following example.

Example 2.2. *Let $X=(0,2)$ be equipped with the quasi-partial metric $\sigma(x,y)=|x-y|+x$ for all $x,y\in X$. Let (f,S) and (g,H) be two pairs of self-mappings on X defined as*

$$fx=\begin{cases} 1 & ; x\in(0,1] \\ \frac{4}{3} & ; x\in(1,2) \end{cases} \qquad gx=\begin{cases} 1 & ; x\in(0,1] \\ \frac{3}{2} & ; x\in(1,2) \end{cases}$$

$$Sx=\begin{cases} x^2 & ; x\in(0,1] \\ x-1 & ; x\in(1,2) \end{cases} \qquad Hx=\begin{cases} x & ; x\in(0,1] \\ 2-x & ; x\in(1,2). \end{cases}$$

Consider $\{x_n\}=\{1-\frac{1}{n}\}$ and $\{y_n\}=\{\frac{5n^2-4}{5n^2+2}\}$. We have

$$\lim_{n\to\infty}\sigma(fx_n,1)=\lim_{n\to\infty}\sigma(1,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,1)=\lim_{n\to\infty}\sigma(1,Sx_n)=\sigma(1,1)=S1=1.$$

Moreover,

$$\lim_{n\to\infty}\sigma(gy_n,1)=\lim_{n\to\infty}\sigma(1,gy_n)=\lim_{n\to\infty}\sigma(Hy_n,1)=\lim_{n\to\infty}\sigma(1,Hy_n)=\sigma(1,1)=H1=1.$$

Hence the two pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property.

The following lemma is crucial in order to prove our main result (Theorem 2.1).

Lemma 2.1. *Let (X,σ) be a (QPMS). Suppose that the self-mappings $f,g,S,H : X\to X$ are such that*
(i) $fX\subseteq HX$ (or $gX\subseteq SX$);
(ii) the pair (f,S) satisfies the (CLR_S)-property (or (g,H) satisfies the (CLR_H)-property);
(iii) HX (or SX) is closed;
(iv) $\{gy_n\}$ (or $\{fy_n\}$) is bounded for every sequence $\{y_n\}$ in X;
(v) there exist $\beta\in\mathcal{F}$ and $\alpha\in\mathcal{G}$ such that

$$\beta(\sigma(fa,gb))\le\beta(\Lambda(a,b))-\alpha(\Lambda(a,b)), \tag{2.1}$$

where $\Lambda(a,b)=\max\{\sigma(Sa,Hb),\sigma(fa,Sa),\sigma(Hb,gb),\sigma(fa,Hb),\sigma(Sa,gb)\}$. Then the pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property.

Proof. From Condition (ii), if (f,S) satisfies the (CLR_S)-property, then there exists $\{x_n\}\subset X$, so that

$$\lim_{n\to\infty}\sigma(fx_n,w)=\lim_{n\to\infty}\sigma(w,fx_n)=\lim_{n\to\infty}\sigma(Sx_n,w)=\lim_{n\to\infty}\sigma(w,Sx_n)=\sigma(w,w);\ \ w\in SX. \tag{2.2}$$

Since $fX\subseteq HX$, there exists $\{y_n\}$ such that

$$fx_n=Hy_n. \tag{2.3}$$

Due to (2.2) and (2.3), we write $\lim_{n\to\infty}\sigma(Hy_n,w)=\sigma(w,w)$, so from the closedness condition of HX, we have

$$w\in SX\cap HX.$$

Now, we want to prove that $gy_n \to w$ as $n \to \infty$. We have

$$\begin{aligned}\sigma(fx_n, gy_n) &\leq \sigma(fx_n, Sx_n) + \sigma(Sx_n, gy_n) - \sigma(Sx_n, Sx_n)\\ &\leq \sigma(fx_n, w) + \sigma(w, Sx_n) - \sigma(w, w) + \sigma(Sx_n, gy_n) - \sigma(Sx_n, Sx_n).\end{aligned}$$

By (2.2), $\lim_{n\to\infty} \sigma(Sx_n, Sx_n) = \sigma(w, w)$. We also get

$$\limsup_{n\to\infty} \sigma(fx_n, gy_n) - \limsup_{n\to\infty} \sigma(Sx_n, gy_n) \leq 0. \tag{2.4}$$

Again, by (2.2), $\lim_{n\to\infty} \sigma(fx_n, fx_n) = \sigma(w, w)$, so similarly,

$$\limsup_{n\to\infty} \sigma(Sx_n, gy_n) - \limsup_{n\to\infty} \sigma(fx_n, gy_n) \leq 0. \tag{2.5}$$

As $\{gy_n\}$ is bounded, $\limsup_{n\to\infty} \sigma(fx_n, gy_n)$ and $\limsup_{n\to\infty} \sigma(Sx_n, gy_n)$ are finite numbers. Using (2.4) and (2.5), there exists $\delta \geq 0$ such that one writes

$$\limsup_{n\to\infty} \sigma(Sx_n, gy_n) = \limsup_{n\to\infty} \sigma(fx_n, gy_n) = \delta. \tag{2.6}$$

So there are subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that

$$\lim_{k\to\infty} \sigma(Sx_{n_k}, gy_{n_k}) = \lim_{k\to\infty} \sigma(fx_{n_k}, gy_{n_k}) = \delta. \tag{2.7}$$

Clearly, by (2.2),

$$\sigma(w, w) = \lim_{k\to\infty} \sigma(fx_{n_k}, Sx_{n_k}) = \lim_{k\to\infty} \sigma(Sx_{n_k}, fx_{n_k}). \tag{2.8}$$

Since $\sigma(fx_{n_k}, fx_{n_k}) \leq \sigma(fx_{n_k}, Sx_{n_k})$, passing to the limit as $k \to \infty$, we obtain

$$\sigma(w, w) \leq \delta. \tag{2.9}$$

We have

$$\begin{aligned}&\Lambda(fx_{n_k}, y_{n_k})\\ &= \max\{\sigma(Sx_{n_k}, Hy_{n_k}), \sigma(fx_{n_k}, Sx_{n_k}), \sigma(Hy_{n_k}, gy_{n_k}), \sigma(fx_{n_k}, Hy_{n_k}), \sigma(Sx_{n_k}, gy_{n_k})\}\\ &= \max\{\sigma(Sx_{n_k}, fx_{n_k}), \sigma(fx_{n_k}, Sx_{n_k}), \sigma(fx_{n_k}, gy_{n_k}), \sigma(fx_{n_k}, fx_{n_k}), \sigma(Sx_{n_k}, gy_{n_k})\}.\end{aligned}$$

Passing to the limit as $k \to \infty$, we get due to (2.9)

$$\lim_{k\to\infty} \Lambda(fx_{n_k}, y_{n_k}) = \max\{\sigma(w, w), \sigma(w, w), \delta, \sigma(w, w), \delta\} = \delta. \tag{2.10}$$

By using (2.1),

$$\beta(\sigma(fx_{n_k}, gy_{n_k})) \leq \beta(\Lambda(x_{n_k}, y_{n_k})) - \alpha(\Lambda(x_{n_k}, y_{n_k})).$$

Taking the upper limit as $k \to \infty$ and using (2.8) and (2.10),

$$\beta(\delta) \leq \beta(\delta) - \alpha(\delta),$$

i.e., $\alpha(\delta) = 0$, which yields that $\delta = 0$. Thus $\sigma(w, w) = \delta = 0$. So, by (2.6), we have

$$\lim_{n\to\infty} \sigma(fx_n, gy_n) = 0.$$

Consequently,

$$\lim_{k\to\infty} \sigma(gy_{n_k}, gy_{n_k}) = 0 = \sigma(w, w).$$

We obtained

$$\lim_{n\to\infty} \sigma(w, gy_n) = \lim_{n\to\infty} \sigma(gy_n, w) = \lim_{n\to\infty} \sigma(Hy_n, w) = \lim_{n\to\infty} \sigma(w, Hy_n) = \sigma(w, w).$$

So the pairs (f, S) and (g, H) satisfy the $(CLR_{SH})-$property. $\square$

Now, we introduce and prove our main result by using the concept of (CLR)-property on the class of quasi-partial metric spaces.

Theorem 2.1. *Let f, g, H and S be self-mappings on a (QPMS) (X, σ) satisfying the condition (v) of Lemma 2.1. If the pairs (f, S) and (g, H) satisfy the (CLR_{SH})-property, then there exists $x \in X$ such that $fx = gx = Sx = Hx$. Furthermore, if (f, S) and (g, H) are w-compatible, then such x is the unique common fixed point of f, g, H and S.*

Proof. As (f, S) and (g, H) verify the (CLR_{SH})-property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}\lim_{n\to\infty} \sigma(fx_n, w) &= \lim_{n\to\infty} \sigma(w, fx_n) = \lim_{n\to\infty} \sigma(Sx_n, w) = \lim_{n\to\infty} \sigma(w, Sx_n)\\ &= \lim_{n\to\infty} \sigma(w, gy_n) = \lim_{n\to\infty} \sigma(gy_n, w)\\ &= \lim_{n\to\infty} \sigma(Hy_n, w) = \lim_{n\to\infty} \sigma(w, Hy_n) = \sigma(w, w);\quad w \in SX \cap HX.\end{aligned}$$

Since $w \in SX$, there exists $k \in X$ such that $Sk = w$. Now, we want to prove that $fk = Sk$. Suppose that $fk \neq Sk$. Obviously,

$$\lim_{n\to\infty} \sigma(Hy_n, gy_n) = \sigma(w, w), \tag{2.11}$$

and

$$\lim_{n\to\infty} \sigma(fk, Hy_n) = \lim_{n\to\infty} \sigma(fk, gy_n) = \sigma(fk, w). \tag{2.12}$$

From (2.1),

$$\beta(\sigma(fk, gy_n)) \leq \beta(\Lambda(k, y_n)) - \alpha(\Lambda((k, y_n)), \tag{2.13}$$

where

$$\Lambda(k, y_n) = \max\{\sigma(Sk, Hy_n), \sigma(fk, Sk), \sigma(Hy_n, gy_n), \sigma(fk, Hy_n), \sigma(Sk, gy_n)\}.$$

Taking the limit as $n \to \infty$ and using the equations (2.11) and (2.12), we get

$$\begin{aligned}(2.14)\ \lim_{n\to\infty} \Lambda(k, y_n) = \quad & \max\{\sigma(w,w), \sigma(fk,w), \sigma(w,w), \sigma(fk,w), \sigma(w,w)\}\\ &= \sigma(fk,w).\end{aligned}$$

Letting $n \to \infty$ in (2.13), by (2.12) and (2.14), we get

$$\beta(\sigma(fk,w)) \le \beta(\sigma(fk,w)) - \alpha(\sigma(fk,w)).$$

So $\alpha(\sigma(fk,w)) = 0$, that is, $\sigma(fk,w) = 0$, i.e.,

$$fk = Sk = w. \tag{2.15}$$

Since $w \in HX$, there exists $\nu \in X$ such that $H\nu = w$. As (2.11) and (2.12), we may write

$$\lim_{n\to\infty} \sigma(fx_n, Sx_n) = \sigma(w,w), \tag{2.16}$$

and

$$\lim_{n\to\infty} \sigma(Sy_n, g\nu) = \lim_{n\to\infty} \sigma(fx_n, g\nu) = \sigma(w, g\nu). \tag{2.17}$$

By (2.1),

$$\beta(\sigma(fx_n, g\nu)) \quad \le \quad \beta(\Lambda(x_n,\nu)) - \alpha(\Lambda(x_n,\nu)),$$

where

$$\Lambda(x_n,\nu) \quad = \quad \max\{\sigma(Sx_n, H\nu), \sigma(fx_n, Sx_n), \sigma(H\nu, g\nu), \sigma(fx_n, H\nu), \sigma(Sx_n, g\nu)\}.$$

Due to (2.16) and (2.17),

$$\begin{aligned}(2.18)\ \lim_{n\to\infty} \Lambda(x_n,\nu) \quad & = \max\{\sigma(w,w), \sigma(w,w), \sigma(w,g\nu), \sigma(w,w), \sigma(w,g\nu)\}\\ &= \sigma(w, g\nu).\end{aligned}$$

By (2.17) and (2.18),

$$\beta(\sigma(w,g\nu)) \le \beta(\sigma(w,g\nu)) - \alpha(\sigma(w,g\nu)).$$

This gives that $\alpha(\sigma(w,g\nu)) = 0$, hence $\sigma(w,g\nu)) = 0$. So $H\nu = g\nu = w$. The w-compatibility of (f,S) together with $fk = Sk$ implies that

$$fw = fSk = Sfk = Sw.$$

We shall prove that $fw = Sw = w$. We have

$$\beta(\sigma(fw,w)) = \beta(\sigma(fw,g\nu)) \quad \le \quad \beta(\Lambda(w,\nu)) - \alpha(\Lambda(w,\nu)),$$

where

$$\begin{aligned}\Lambda(w,\nu) &= \max\{\sigma(Sw,H\nu),\sigma(fw,Sw),\sigma(H\nu,g\nu),\sigma(fw,H\nu),\sigma(Sw,g\nu)\}\\ &= \max\{\sigma(fw,w),\sigma(fw,fw),\sigma(w,w),\sigma(fw,w),\sigma(fw,w)\}\\ &= \sigma(fw,w).\end{aligned}$$

Then

$$\beta(\sigma(fw,g\nu)) \le \beta(\sigma(fw,g\nu)) - \alpha(\sigma(fw,g\nu)).$$

This implies that $\alpha(\sigma(fw,w)) = 0$, that is, $\sigma(fw,w) = 0$, so $fw = w = Sw$. Again the w-compatibility condition of (g,H) and the fact that $g\nu = H\nu$ imply that $gw = gH\nu = Hg\nu = Hw$. Again, using (2.1),

$$\beta(\sigma(w,gw)) = \beta(\sigma(fk,gw)) \quad\le\quad \beta(\Lambda(k,w)) - \alpha(\Lambda(k,w)),$$

where

$$\begin{aligned}\Lambda(k,w) &= \max\{\sigma(Sk,Hw),\sigma(fk,Sk),\sigma(Hw,gw),\sigma(fk,Hk),\sigma(Sk,gk)\}\\ &= \max\{\sigma(w,gw),\sigma(w,w),\sigma(gw,gw),\sigma(w,gw),\sigma(w,gw)\}\\ &= \sigma(w,gw).\end{aligned}$$

Then

$$\beta(\sigma(w,gw)) = \beta(\sigma(fk,gw)) \le \beta(\sigma(w,gw)) - \alpha(\sigma(w,gw)),$$

hence, $\alpha(\sigma(w,gw)) = 0$. Thus $\sigma(w,gw) = 0$, so $w = gw = Hw$.

Finally, we shall show that w is unique. Consider that $\lambda = f\lambda = g\lambda = S\lambda = H\lambda$. From (2.1),

$$\beta(\sigma(w,\lambda)) = \beta(\sigma(fw,g\lambda)) \le \beta(\Lambda(w,\lambda)) - \alpha(\Lambda(w,w)).$$

Since

$$\begin{aligned}\Lambda(w,\lambda) &= \max\{\sigma(Sw,H\lambda),\sigma(fw,Sw),\sigma(H\lambda,g\lambda),\sigma(fw,H\lambda),\sigma(Sw,g\lambda)\}\\ &= \max\{\sigma(w,\lambda),\sigma(w,w),\sigma(\lambda,\lambda),\sigma(w,\lambda),\sigma(w,\lambda)\}\\ &= \sigma(w,\lambda),\end{aligned}$$

we get

$$\beta(\sigma(w,\lambda)) = \beta(\sigma(fw,g\lambda)) \le \beta(\sigma(w,\lambda)) - \alpha(\sigma(w,\lambda)).$$

Therefore, $\alpha(\sigma(w,\lambda)) = 0$, that is, $\sigma(w,\lambda) = 0$, hence $w = \lambda$. The proof is completed. □

Example 2.3. *Take $A = [0,1]$. Consider the quasi-partial metric on A defined by*

$$\sigma(c,d) = |c-d| + c.$$

Given $f,g,H,S : A \to A$ as

$$f(d) = 0, \quad g(d) = \frac{1}{8}d, \quad S(d) = \frac{1}{2}d, \quad H(d) = \frac{1}{3}d.$$

It is clear that $fA \subset HA$, $gA \subset SA$ and the pairs (f,S) and (g,H) satisfy the (CLR_{SH})-property. Take $\beta(t) = 8t$ and $\alpha(t) = t$. We will prove that (2.1) holds. First,

$$\beta(\sigma(fc,gd)) = \beta(|fc-gd|+fc) = \beta(\frac{1}{8}d) = d. \tag{2.19}$$

Moreover,

$$\begin{aligned}\Lambda(c,d) &= \max\{\sigma(Sc,Hd),\sigma(fc,Sc),\sigma(Hd,gd),\sigma(fc,Hd),\sigma(Sc,gd)\}\\ &= \max\,\{\sigma(\frac{1}{2}c,\frac{1}{3}d),\sigma(0,\frac{1}{2}c),\sigma(\frac{1}{3}d,\frac{1}{8}d),\sigma(0,\frac{1}{3}d)\sigma(\frac{1}{2}c,\frac{1}{8}d)\}\\ &= \max\{|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c,\frac{1}{2}c,\frac{13}{24}d,\frac{1}{3}d,|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c\}.\end{aligned}$$

Case 1. Let $\Lambda(c,d) = \frac{13}{24}d$. We obtain

$$\beta(\Lambda(c,d)) - \alpha(\Lambda(c,d)) = \frac{13}{3}d - \frac{13}{24}d = \frac{91}{24}d > d = \beta(\sigma(fc,gd)). \tag{2.20}$$

Case 2. Let $\Lambda(c,d) = |\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c$. We have

$$\begin{aligned}\beta(\Lambda(c,d)) - \alpha(\Lambda(c,d)) &= 8(|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c) - (|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c)\\ &= 7(|\frac{1}{2}c-\frac{1}{3}d|+\frac{1}{2}c) > 7(\frac{13}{24}d) > d = \beta(\sigma(fc,gd)). \qquad (2.21)\end{aligned}$$

Case 3. Let $\Lambda(c,d) = |\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c$. We have

$$\begin{aligned}\beta(\Lambda(c,d)) - \alpha(\Lambda(c,d)) &= 8(|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c) - (|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c)\\ &= 7(|\frac{1}{2}c-\frac{1}{8}d|+\frac{1}{2}c) > 7(\frac{13}{24}d) > d = \beta(\sigma(fc,gd)). \qquad (2.22)\end{aligned}$$

From (2.20) to (2.21), the condition (2.1) holds. Here, 0 is the unique common fixed point, that is, $f0 = g0 = S0 = H0 = 0$.

Example 2.4. *Let $X = [0,7)$ and $\sigma(x,y) = |x-y|+x$ for all $x,y \in X$. (X,σ) is a (QPMS). Define (f,S) and (g,H) as two pairs of self-mappings on X, where*

$$f(x) = \begin{cases} 0 \ ; x \in \{0\}\cup[5,7) \\ 2 \ ; x \in (0,5), \end{cases} \qquad g(x) = \begin{cases} 0 \ ; x \in \{0\}\cup[5,7) \\ 4 \ ; x \in (0,5) \end{cases}$$

$$S(x) = \begin{cases} 0 \ ; x \in \{0\} \\ 5 \ ; x \in (0,5) \\ \frac{x+5}{2} \ ; x \in [5,7), \end{cases} \qquad H(x) = \begin{cases} 0 \ ; x \in \{0\} \\ 6 \ ; x \in (0,5) \\ x-5 \ ; x \in [5,7). \end{cases}$$

Also, define $\beta(t) = 8t$ and $\alpha(t) = \frac{t}{10}$. Choose $\{x_n\} = \{0\}$ and $\{y_n\} = \{5+\frac{1}{n}\}$. Then

$$\lim_{n\to\infty}\sigma(f(x_n),0) = \lim_{n\to\infty}\sigma(0,f(x_n)) = \lim_{n\to\infty}\sigma(S(x_n),0) = \lim_{n\to\infty}\sigma(0,S(x_n)) = \sigma(0,0) = S(0) = 0.$$

Also

$$\begin{aligned}\lim_{n\to\infty}\sigma(g(y_n),0) &= \lim_{n\to\infty}\sigma(0,g(y_n)) = \lim_{n\to\infty}\sigma(H(y_n),0)\\ &= \lim_{n\to\infty}\sigma(0,H(y_n)) = \sigma(0,0) = H(0) = 0.\end{aligned}$$

Hence the two pairs (f,S) *and* (g,H) *satisfy the* (CLR_{SH})*-property. Now, we will show that the contraction condition (2.1) holds. For this, we distinguish the following cases.*

Case 1. $x,y\in\{0\}\cup[5,7)$. *Here, we have*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(0,0)) = 0 \le \beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)).$$

Case 2. $x\in\{0\}$ *and* $y\in(0,5)$. *We have*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(0,4)) = 32.$$

Also,

$$\begin{aligned}\Lambda(x,y) &= \max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}\\ &= \max\{\sigma(0,6),\sigma(0,0),\sigma(6,4),\sigma(0,6),\sigma(0,4)\}\\ &= \max\{6,0,8,6,4\} = 8.\end{aligned}$$

Hence,

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 64 - \frac{4}{5} > 32 = \beta(\sigma(fx,gy)).$$

Case 3. $x\in(0,5)$ *and* $y\in[5,7)$. *We have*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(2,0)) = 32.$$

Moreover,

$$\begin{aligned}\Lambda(x,y) &= \max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}\\ &= \max\{\sigma(5,y-5),\sigma(2,5),\sigma(y-5,0),\sigma(2,y-5),\sigma(5,0)\}\\ &= \max\{|10-y|+5,5,2y-10,|7-y|+2,10\} = 10.\end{aligned}$$

Then

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 79 > 32 = \beta(\sigma(fx,gy)).$$

Case 4. $x\in(0,5)$ *and* $y=0$. *In this case,*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(2,0)) = 32.$$

Then

$$\begin{aligned}\Lambda(x,y) &= \max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}\\ &= \max\{\sigma(5,0),\sigma(2,5),\sigma(0,0),\sigma(2,0),\sigma(5,0)\}\\ &= \max\{10,5,0,4,10\} = 10,\end{aligned}$$

that is,

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 79 > 32 = \beta(\sigma(fx,gy)).$$

Case 5. $x, y \in (0,5)$. *Here,*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(2,4)) = 32.$$

Also,

$$\begin{aligned}\Lambda(x,y) \quad &= \max\{\sigma(Sx,Hy), \sigma(fx,Sx), \sigma(Hy,gy), \sigma(fx,Hy), \sigma(Sx,gy)\}\\ &= \max\{\sigma(5,6), \sigma(2,5), \sigma(6,4), \sigma(2,6), \sigma(5,4)\}\\ &= \max\{6,5,8,6,6\} = 8.\end{aligned}$$

Then

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 64 - \frac{4}{5} > 32 = \beta(\sigma(fx,gy)).$$

Case 6. $x \in [5,7)$ *and* $y \in (0,5)$. *We have*

$$\beta(\sigma(fx,gy)) = \beta(\sigma(0,4)) = 32.$$

Also,

$$\begin{aligned}\Lambda(x,y) \quad &= \max\{\sigma(Sx,Hy), \sigma(fx,Sx), \sigma(Hy,gy), \sigma(fx,Hy), \sigma(Sx,gy)\}\\ &= \max\{\sigma(\frac{x+5}{2},6), \sigma(0,\frac{x+5}{2}), \sigma(6,4), \sigma(0,6), \sigma(\frac{x+5}{2},4)\}\\ &= \max\{6, \frac{x+5}{2}, 8, 6, |\frac{x+5}{2} - 4| + \frac{x+5}{2}\} = 8.\end{aligned}$$

Hence,

$$\beta(\Lambda(x,y)) - \alpha(\Lambda(x,y)) = 64 - \frac{4}{5} = \beta(\sigma(fx,gy)).$$

Therefore, all conditions of Theorem 2.1 are satisfied. So, the mappings f, g, H *and* S *have a common fixed point, which is 0.*

On the other hand, $fX = \{0,2\} \nsubseteq SX = \{0\} \cup [5,6)$ *and* $gX = \{0,4\} \nsubseteq HX = \{6\} \cup [0,2)$. *Note that the result of Nazir and Abbas [8] is not applicable because the hypothesis of containment among ranges of the mappings* f, g, S, H *in [[8], Theorem 2.1] does not hold here.*

Corollary 2.1. *Let* (X,σ) *be a (QPMS). Assume that* $f, S, g, H : X \to X$ *verify all conditions in Lemma 2.1. Suppose, in addition, that the pairs* (f,S) *and* (g,T) *are w-compatible. Then there exists a unique common fixed point of* f, g, H *and* S.

Proof. From Lemma 2.1, (f,S) and (g,H) share the (CLR_{SH})-property. All conditions of Theorem 2.1 are fulfilled. Then exists a unique $x \in X$ such that $fx = Sx = gx = Hx = x$. □

By taking $\beta(t) = \int_0^t \eta(s)ds$ in Lemma 2.1 and Theorem 2.1, where $\eta : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable summable mapping such that $\int_0^\epsilon \eta(t)dt > 0$ for $\epsilon > 0$, we state the following.

Corollary 2.2. *Let f, S, g and H be self-mappings on a (QPMS) (X,σ) such that*

$$\int_0^{\sigma(fx,gy))} \eta(s)ds \le \Lambda(x,y) - \alpha(\Lambda(x,y)), \tag{2.23}$$

where $\Lambda(x,y) = \int_0^{\max\{\sigma(Sx,Hy),\sigma(fx,Sx),\sigma(Hy,gy),\sigma(fx,Hy),\sigma(Sx,gy)\}} \eta(s)ds$. Assume that (f,S) and (g,H) fulfill the (CLR_{SH})-property. Then $fx = Sx = gx = Hx$. Furthermore, if (f,S) and (g,T) are w-compatible, there exists only one point $x \in X$ so that $fx = Sx = gx = Hx = x$.

Remark 2.1. *Corollary 2.2 extends the paper by Vetro et al.* [11] *from metric spaces to (QPMS).*

REFERENCES

1. A. Aamri and S. E. Moutawakil: *Some new common fixed point theorems under strict contractive conditions,* J. Math. Anal. Appl. 270 (2002), 181–188.
2. H. Aydi, M. Abbas and C. Vetro: *Partial Hausdorff Metric and Nadler's Fixed Point Theorem on Partial Metric Spaces,* Topology and its Applications, 159 (2012), 3234-3242.
3. H. Aydi, E. Karapinar and Sh. Rezapour: *A generalized Meir-Keeler type contraction on partial metric spaces, Abstract and Applied Analysis,* Volume 2012, Article ID 287127, 11 pages.
4. L. j. Ciric, B. Samet, H. Aydi and C. Vetro: *Common fixed points of generalized contractions on partial metric spaces and an application,* Applied Mathematics and Computation, 218 (2011) 2398-2406.
5. G. Jungck: *Common fixed points for noncontinuous nonself maps on a nonnumeric space,* Far East J. Math. Sci. 4 (1996), 195-215.
6. E. Karapinar, I. M. Erhan and A. Ozturk: *Fixed point theorems on quasi partial metric spaces,* Mathematical and Computer Modelling, 57 (2013), 2442–2448.
7. E. Karapinar, R. P. Agarwal and H. Aydi: *Interpolative Reich-Rus-Ciric type contractions on partial metric spaces,* Mathematics, 2018, 6, 256.
8. T. Nazir and M. Abbas: *Common fixed points of two pairs of mappings satisfying (E.A)-property in partial metric spaces,* J. Inequal. Appl. 2014:237.
9. S. G. Matthews: *partial metric topology, 8th Summer conference on general topology and application,* Ann. New York Acad. Sci. 728 (1994), 183-197.
10. W. Sintunavarat and P. Kumam: *Common fixed point theorems for a pair of Weakly compatible mappings in fuzzy metric spaces,* J. Appl. Math. 3 (2012), 225-237.
11. C. Vetro, S. Chauhan, E. Karapinar and W. Shatanawi: *Fixed points of weakly compatible mappings satisfying generalized $\phi-$weak contractions,* Bull. Malays. Math. Sci. Soc. 38 (2015), 1085-1105.

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A NOTE ON OPERATORS CONSISTENT IN INVERTIBILITY

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Abstract. We generalize the notion of consistency in invertibility to Banach algebras and prove that the set of all elements consistent in invertibility is an upper semiregularity. In the case of bounded liner operators on a Hilbert space, we give a complete answer when the set of all CI operators will be a regularity. Analogous results are obtained for Fredholm consistent operators.
Keywords: Banach algebra; invertibility; semiregularity; Hilbert space.

1. Notations, motivations and preliminaries

For a closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ we use the symbol $P_{\mathcal{M}}$ to denote the orthogonal projection onto $\mathcal{M}$. For a given operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively, while $n(A) = \dim \mathcal{N}(A)$ and $d(A) = \dim \mathcal{R}(A)^{\perp}$
The notion of operators consistent in invertibility, CI for short, was introduced by Gong and Han in [7]. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is consistent in invertibility (CI) if for each $A \in \mathcal{B}(\mathcal{H})$, AT is invertible if and only if TA is invertible. A characterization of CI operators is given by the next Theorem:

Theorem 1.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is CI operator if and only if one of the three mutually exclusive cases hold:*

(i) T is invertible;

(ii) $\mathcal{R}(T)$ is not closed;

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(iii) $\mathcal{N}(T) \neq \{0\}$ *and* $\mathcal{R}(T) = \overline{\mathcal{R}(T)} \neq \mathcal{H}$.

It is easy to see that an operator $T \in \mathcal{B}(\mathcal{H})$ is not CI if and only if T is left invertible but not right invertible, or right invertible but not left invertible. The CI spectrum of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\sigma_{CI}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not } CI\}$$

Results concerning CI operators were obtained in [8, 9] and [1, 2, 10]. It is fairly easy to see that if A and B are CI operators, then AB is a CI operator, it would be of interest to determine whether the set of all CI operators is a regularity. We will prove that in general this is not the case.
The notion of consistency has been generalised, and explored in other cases, such as Fredholm consistency (FC) ([1, 2]). Using a characterization of FC operators used in [2] given in the following Theorem we will answer the same questions we did in the case of CI operators in $\mathcal{B}(\mathcal{H})$:

Theorem 1.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T if Fredholm consistent (FC) if and only if one of the following conditions is satisfied:*

(i) T *is Fredholm,*

(ii) $\mathcal{R}(T)$ *is closed,* $n(T) = d(T) = \infty$,

(iii) $\mathcal{R}(T)$ *is not closed.*

It is easy to see that an operator $T \in \mathcal{B}(\mathcal{H})$ is not Fredholms consistent if and only if T is left Fredholm, but not right Fredhlom, or it is right Fredholm, but not left Fredholm. Some other recent results on Fredholm operators can be found in

Let us now recall the definition of a regularity (upper semiregularity) in a Banach algebra:

Definition 1.1. [4] Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a regularity if

(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in R \Leftrightarrow a^n \in R$,

(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$.

Definition 1.2. [5] Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called an upper semiregularity if

(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in R \Rightarrow a^n \in R$,

(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac+bd = 1_{\mathcal{A}}$, and $a, b \in R$, then $ab \in R$.

(3) R contains a neighborhood of the unit element $1_{\mathcal{A}}$.

Some important examples of regularities include sets of all invertible (left invertible, right invertible) operators, Fredholm (left Fredholm, right Fredholm) operators etc.

2. Consistency in invertibility

We introduce CI elements in Banach algebras in the same manner. Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{A}^{-1}$ the group of all invertible elements. We say that $a \in \mathcal{A}$ is consistent in invertibility (CI) if for all $c \in \mathcal{A}$

$$ac \in \mathcal{A}^{-1} \Leftrightarrow ca \in \mathcal{A}^{-1}.$$

First we prove a lemma which gives a characterisation of CI elements similar to the characterisation of CI operators:

Lemma 2.1. *A Banach algebra element a is not CI if and only if $a \in \mathcal{A}_l^{-1} \setminus \mathcal{A}_r^{-1}$ or $a \in \mathcal{A}_r^{-1} \setminus \mathcal{A}_l^{-1}$.*

Proof. Assume $a \in \mathcal{A}$ is not CI. Then there exists an element $c \in \mathcal{A}$ such that $ac \in \mathcal{A}^{-1}$ and $ca \notin \mathcal{A}^{-1}$, or $ca \in \mathcal{A}^{-1}$ and $ac \notin \mathcal{A}^{-1}$. If the first statement is correct, since $ac \in \mathcal{A}^{-1}$ we have that a must be right invertible. If a were left invertible as well, then c would be invertible, and ca would be invertible as well. From this contradiction we see that $a \in \mathcal{A}_r^{-1} \setminus \mathcal{A}_l^{-1}$. We analogously conclude that in the other case $a \in \mathcal{A}_l^{-1} \setminus \mathcal{A}_r^{-1}$. If $a \in \mathcal{A}_l^{-1} \setminus \mathcal{A}_r^{-1}$ we have that $a_l^{-1}a = 1_{\mathcal{A}}$ and $aa_l^{1} \notin \mathcal{A}^{-1}$ for an arbitrary left inverse of a, so a is not CI. We analogously conclude that a is not CI when $a \in \mathcal{A}_r^{-1} \setminus \mathcal{A}_l^{-1}$ as well. □

Theorem 2.1. *The set of all CI elements in $\mathcal{A}$ is an upper semiregularity.*

Proof. If a, b are commuting CI elements and $c \in \mathcal{A}$ arbitrary we have that

$$abc \text{ is invertible} \Leftrightarrow bca \text{ is invertible} \Leftrightarrow$$

$$\Leftrightarrow cab \text{ is invertible}$$

This stronger statement implies that conditions (1), and (2) of Definition 1.2 are satisfied.
Since invertible elements are CI, and we know that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are invertible. We conclude that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are CI. This completes the proof. □

As a corollary of the previous Theorem we have:

Corollary 2.1. *The set of all CI operators in $\mathcal{B}(\mathcal{H})$ is an upper semiregularity*

Since all invertible elements in a Banach algebra are CI have that $\sigma_{CI}(a) \subseteq \sigma(a)$, where

$$\sigma_{CI}(a) = \{\lambda \in \mathbb{C} : \ \lambda - a \text{ is not } \ CI\}.$$

Recall the following Theorem from [5]:

Theorem 2.2. *[5] Let $\mathcal{R} \subset \mathcal{A}$ be an upper semiregularity. Suppose that $\mathcal{R}$ satisfies the condition*

$$b \in \mathcal{R} \cap \mathcal{A}^{-1} \Rightarrow b^{-1} \in \mathcal{R}.$$

Then $\sigma_{\mathcal{R}}(f(a)) \subset f(\sigma_{\mathcal{R}}(a))$ for all $a \in \mathcal{A}$ and all locally non-constant functions f analytic on a neighborhood of $\sigma(a) \cup \sigma_{\mathcal{R}}(a)$.
Further, $\sigma_{\mathcal{R}}(f(a)) \subset f(\sigma_{\mathcal{R}}(a) \cup \sigma(a))$ for all functions f analytic on a neighborhood of $\sigma_{\mathcal{R}}(a) \cup \sigma(a)$.

Since $\sigma_{CI}(a) \subseteq \sigma(a)$ (and thus $\sigma_{CI}(a) \cup \sigma(a) = \sigma(a)$) we get that the following Theorem holds:

Theorem 2.3. *For every $a \in \mathcal{A}$ $\sigma_{CI}(f(a)) \subseteq f(\sigma_{CI}(a))$ for all locally non-constant functions f analytic on a neighborhood of $\sigma(a) \cup \sigma_{CI}(a) = \sigma(a)$, and $f(\sigma_{CI}(a)) \subseteq f(\sigma(a))$ for all functions f analytic on a neighborhood of $\sigma(a)$.*

It is now only natural to ask what further properties does the set of all bounded linear operators (Banach algebra elements) consistent in invertibility satisfy, and under which conditions it will be a regularity.
Remark: We from lemma 2.1 we see that

$$\sigma_{CI}(a) = (\sigma_l(a) \setminus \sigma_r(a)) \cup (\sigma_r(a) \setminus \sigma_l(a)).$$

In the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ this implies that the consistency spectrum of a bounded linear operator can be empty. For example, self-adjoint (normal) operators on Hilbert spaces will have an empty CI spectrum.
It would be natural to check whether the CI spectrum is closed, and from the following example we will see that this is generally not the case.

Example 1 Define the operator T on $\mathcal{B}(l^2 \oplus l^2)$ by

$$T = 2S \oplus (I - S^*) : l^2 \oplus l^2 \to l^2 \oplus l^2$$

where S is the right shift operator on l^2. Let $(\lambda_n)_n$ be a sequence of complex numbers such that

$$\lim_{n\to\infty} \lambda_n = 2,\ \lambda_n \in B(0,2) \setminus B(1,1),$$

where $B(\lambda, r)$ is the open ball with radius r and center λ. Recall that $S - \lambda I$ is right, but not left invertible for $|\lambda| < 1$, and $S - \lambda I$ is left, but not right invertible for $|\lambda| = 1$, and $S - \lambda I$ is invertible for $|\lambda| > 1$. We have that each $\lambda_n \in \sigma_{CI}(T)$ because $2S - \lambda_n I$ is right, but not left invertible, and $(1 - \lambda_n)I - S^*$ is invertible, T is left, but not right invertible. However, since $2S - 2I$ is not right invertible and $I - S^* - 2I = -(S^* + I)$ is not left invertible (as the Hilbert adjoint of an operator which is not right invertible), we see that $T - 2I$ is neither left nor right invertible, so $T - 2I$ is CI. We get that $\sigma_{CI}(T)$ is not closed.

It is easy to see that $T \in \mathcal{B}(\mathcal{H})$ is CI if and only T^n is CI for $n \geq 1$ so it is natural to investigate whether the set of all CI operators forms a regularity. The following examples will serve as motivation for the answer:

Example: 2. Let T and P_M be operators $\mathcal{B}(l^2)$ defined in the following way, $T = S^2$, where the S is the right shift operator on l^2 and P_M the orthogonal projection on the subspace

$$M = \{x = (x_1, x_2, \dots, x_n, \dots) \in l^2 : \; x_{2n-1} = x_{2n}, \; n \in \mathbb{N}\}.$$

Let $x = (x_1, x_2, \dots, x_n, \dots) \in l^2$ be arbitrary, then $(x_1, x_1, x_2, x_2, \dots, x_n, x_n, \dots)$ is an elements of M, so M is a non-trivial subspace of l^2. It is easy to verify that M is closed. It is easy to see that T commutes with P_M and $P_{M^\perp}$. We have that

$$2P_{M^\perp} + 2P_M - T = 2I - T,$$

which is invertible. For an $x \in l^2$ we have

$$(2P_M - T)x = (x_1 + x_2, x_1 + x_2, x_3 + x_4 - x_1, x_3 + x_4 - x_2, \dots).$$

Since $(1, 0, \dots, 0, \dots) \notin \mathcal{R}(2P_M - T)$ we have that $2P_M - T$ is not right invertible. Assume now that $(2P_M - T)x = 0$ for some $x = (x_1, x_2, \dots) \in l^2$. This means that

$$\begin{aligned} &x_1 + x_2 = 0, \\ &x_3 + x_4 - x_1 = 0, \\ &x_3 + x_4 - x_2 = 0, \\ &\vdots \end{aligned}$$

From the first three equations we get that $x_1 = x_2 = 0$, similarly we conclude that $x_3 = x_4 = 0$, and then $x_{2k-1} = x_{2k} = 0$, for $k \in \mathbb{N}$. It is easy to establish that $2P_M - T$ has closed range. This means that $2P_M - T$ is left, but not right invertible. It is easy to check that $(2I - T)^{-1}$ commutes with $P_{M^\perp}$ and $2P_M - T$. Finally we have the following:

$$(2I - T)^{-1}P_{M^\perp} + (2I - T)^{-1}(2P_M - T) = I,$$

and all the operators in question commute, $P_{M^\perp}$ is a CI operator since $\mathcal{N}(P_M) = \mathcal{R}(P_M)^\perp \neq \{0\}$, $2P_M - T$ is not a CI operator because he is left but not right invertible and

$$2P_{M^\perp}(2P_M - T) = (2P_M - T)(2P_{M^\perp}) = -2TP_{M^\perp}$$

is neither left nor right invertible, so it is a CI operator . This means that condition (2) in Definition (1.1) is not satisfied, so the set of all CI operators on l^2 is **not a regularity**.

Example 3. Any complex matrix $T \in \mathbb{C}^{n\times n}$ is a CI operator since it is either invertible or $\{0\} \neq \mathcal{N}(T), \mathcal{R}(T) \neq \mathbb{C}^n$. This means that the set of all CI matrices coincides with $\mathbb{C}^{n\times n}$ (which is equivalent to saying $\sigma_{CI}(T) = \emptyset$ for all $T \in \mathbb{C}^{n\times n}$)

We can now characterize when the set of all CI operators on a Hilbert space will be a regularity

Theorem 2.4. *The set of all CI operators in $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a regularity of and only if $\mathcal{H}$ is finite dimensional.*

Proof. If $\mathcal{H}$ is finite dimensional, it is isomorphic to $\mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$. From the previous example we see that in this case the set of all CI operators will forms a regularity.
Conversely, assume that $\mathcal{H}$ is not finite dimensional. If $\mathcal{H}$ is separable, then it is isomorphic to l^2 so we can conclude from Example 2 that the set of all CI operators in $\mathcal{B}(\mathcal{H})$ is not a regularity. If $\mathcal{H}$ is not separable, then it contains a separable closed subspace $\mathcal{K}$. We have that $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. We also know that $\mathcal{K}$ is isomorphic to l^2. From Example 2 we have a pair of commuting operators which do not satisfy condition 2. from Definition 1.1. Without loss of generality let us denote them by $2P_M^\perp$ and $2P_M - T$ as well. Then the operators

$$A = 2P_M^\perp \oplus 0,\ \ B = 2P_M - T \oplus I_{\mathcal{K}^\perp}$$

commute, and there exist operators C, D such that $AC + BD = I_{\mathcal{H}}$ which commute with A and B as well. Furthermore, A is a CI operator, B is not a CI operator, but their product is a CI operator. This is in contradiction with condition 2. of Definition 1.1, so the set of all CI operators is not a regularity. $\square$

3. Fredholm consistency

As in the case of CI operators, the notion of Fredholm consistency gan be generalized to Banach algebras as well. In [6] T-Fredholm elements of a Banach algebra were introduced. If $T : \mathcal{A} \to \mathcal{B}$ is a bounded algebra homomorphism between complex Banach algebras $\mathcal{A}$ and $\mathcal{B}$ where $1_{\mathcal{A}} \neq 0_{\mathcal{A}}(1_{\mathcal{B}} \neq 0_{\mathcal{B}})$ we say that $a \in \mathcal{A}$ is T-Fredholm (left T-Fredholm, right T-Fredholm) if and only if $T(a) \in \mathcal{B}^{-1}(\mathcal{B}_l^{-1}, \mathcal{B}_r^{-1})$. We can now say that $a \in \mathcal{A}$ is T-Fredhom consistent (T-FC) if for each $c \in \mathcal{A}$

$$ac \text{ is } T-\text{Fredholm } \Leftrightarrow ca \text{ is } T-\text{Fredholm}.$$

In a matter analogous to Lemma 2.1 and Theorems 2.1 and 3.3 we get the following results:

Lemma 3.1. *A Banach algebra element a is not T-FC if and only if a is left T-Fredholm but not right T-Fredholm, or a is right T-Fredholm but not left T-Fredholm.*

Proof. Assume $a \in \mathcal{A}$ is not $T - FC$. Then there exists an element $c \in \mathcal{A}$ such that $T(ac) \in \mathcal{B}^{-1}$ and $T(ca) \notin \mathcal{B}^{-1}$, or $T(ca) \in \mathcal{B}^{-1}$ and $T(ac) \notin \mathcal{B}^{-1}$. If the first statement is correct, since $T(ac) = T(a)T(c) \in \mathcal{B}^{-1}$ we have that $T(a)$ must be right invertible. If $T(a)$ were left invertible as well, then $T(c)$ would be invertible, and $T(ca)$ would be invertible as well. From this contradiction we see that $T(a) \in \mathcal{B}_r^{-1} \setminus \mathcal{B}_l^{-1}$, which means that a is right T-Fredholm but not left T-Fredholm. We analogously conclude that in the other case $T(a) \in \mathcal{B}_l^{-1} \setminus \mathcal{B}_r^{-1}$. If

a is left T-Fredholm but not right T-Fredholm we have that $T(a) \in \mathcal{B}_l^{-1} \setminus \mathcal{B}_r^{-1}$ we have that $T(a)_l^{-1}T(a) = 1_{\mathcal{B}}$ and $T(a)T(a)_l^{-1} \notin \mathcal{B}^{-1}$ for an arbitrary left inverse of $T(a)$, so a is not $T - FC$. We analogously conclude that a is not $T - FC$ when a is left T-Fredholm but not right T-Fredholm. □

Corollary 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras such that $1_{\mathcal{A}} \neq 0_{\mathcal{A}}$($1_{\mathcal{B}} \neq 0_{\mathcal{B}}$), and $T : \mathcal{A} \to \mathcal{B}$ a bounded algebra homomorphism. Then, $a \in \mathcal{A}$ is $T - FC$ if and only if $T(a)$ is CI.*

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras such that $1_{\mathcal{A}} \neq 0_{\mathcal{A}}$($1_{\mathcal{B}} \neq 0_{\mathcal{B}}$), and $T : \mathcal{A} \to \mathcal{B}$ a bounded algebra homomorphism. The set of all T-Fredholm consistent elements is an upper semiregularity.*

Proof. Let $a, b \in \mathcal{A}$ be commuting T-Fredholm consistent elements and $c \in \mathcal{A}$ arbitrary. We have that

$$abc \text{ is T-Fredholm} \Leftrightarrow bca \text{ is T-Fredholm} \Leftrightarrow$$

$$\Leftrightarrow cab \text{ T-Fredholm.}$$

Since invertible elements are $T - FC$, and we know that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are invertible. We conclude that there exists an open neighborhood of $1_{\mathcal{A}}$ where all elements are $T - FC$. This completes the proof. □

Corollary 3.2. *The set of all Fredholm consistent operators in $\mathcal{B}(\mathcal{H})$ is an upper semiregularity.*

Since invertible elements of a Banach algebra are T-FC we see that a Theorem analogous to Theorem 2.3 will hold for the T-FC spectrum as well where

$$\sigma_{TFC}(a) = \{\lambda \in \mathbb{C} : \; a - \lambda \text{ is not } T - FC\}$$

Theorem 3.2. *For every $a \in \mathcal{A}$ $\sigma_{TFC}(f(a)) \subseteq f(\sigma_{TFC}(a))$ for all locally non-constant functions f analytic on a neighborhood of $\sigma(a) \cup \sigma_{TFC}(a) = \sigma(a)$, and $f(\sigma_{CI}(a)) \subseteq f(\sigma(a))$ for all functions f analytic on a neighborhood of $\sigma(a)$.*

Again, in the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and when we observe Fredholm operators, self-adjoint operators have an empty FC spectrum. The following examples will show that the set of all Fredholm consistent operators in $\mathcal{B}(\mathcal{H})$ is not generally a regularity, and that the FC spectrum is generally not closed:

Example 4. Let $A \in \mathcal{B}(l^2)$ be defined in the following way:

$$Ax = (x_1, 0, x_2, 0, x_3, 0, \dots), \; x = (x_1, x_2, x_3, \dots) \in l^2.$$

In other words, $Ae_n = e_{2n-1}$ where e_n is the n-th vector in the standard orthonormal basis. It is easy to see that A is left invertible, but not right invertible and $d(A) = \infty$. This means that A is left Fredholm but not right Fredholm so A is not Fredholm consistent. On the other hand for

$$(I - A)x = (0, x_2, x_3 - x_2, x_4, x_5 - x_3, x_6, \dots),\ x = (x_1, x_2, x_3, \dots) \in l^2$$

we have

$$\mathcal{N}(I - A) = \mathcal{R}(I - A)^{\perp} = \{x \in l^2 : x_n = 0, n \geq 2\}.$$

The last part in the equation follows from the fact that for

$$(I - A^*)x = (0, x_2 - x_3, x_3 - x_5, x_4 - x_7, x_5 - x_9, \dots)$$

we have that $x \in \mathcal{N}(I - A^*)$ if

$$x_n = x_{2n-1} = x_{4n-3} = \dots$$

so

$$\mathcal{N}(I - A^*) = \{x \in l^2 : x_n = 0, n \geq 2\}.$$

We see $n(I - A) = d(I - A) = 1$ which means that $I - A$ is Fredholm, and thus FC. Now we define an operator $T \in \mathcal{B}(l^2 \oplus l^2)$ as

$$T = A \oplus I_{l^2}.$$

We have that T is also not Fredholm consistent and that

$$I_{l^2 \oplus l^2} - T = (I_{l^2} - A) \oplus 0$$

so $n(I_{l^2 \oplus l^2} - T) = d(I_{l^2 \oplus l^2} - T) = \infty$ which means that $I - T$ is Fredholm consistent in $\mathcal{B}(l^2 \oplus l^2)$. For $(I_{l^2 \oplus l^2} - T)T$ we also have that $n((I_{l^2 \oplus l^2} - T)T) = d((I_{l^2 \oplus l^2} - T)T) = \infty$ so this operator is Fredholm consistent in $\mathcal{B}(l^2 \oplus l^2)$ as well. Finally, since $(I_{l^2 \oplus l^2} - T) + T = I_{l^2 \oplus l^2}$, and $I_{l^2 \oplus l^2} - T$ and T trivially commute we see that the condition 2. from Definition 1.1 isn't satisfied from which we conclude that the set of all Fredholm consistent operators in $\mathcal{B}(l^2 \oplus l^2)$ is not a regularity.

Example 5. Let $\mathcal{H}$ be separable Hilbert space. Then $\mathcal{H}$ can be represented as an orthogonal direct sum of closed infinite dimensional subspaces M_n, $n \in \mathbb{N}$ ($\mathcal{H} = \bigoplus_{n=1}^{\infty} M_n$). To see that such subspaces exists we can do the following. Since $\mathcal{H}$ is separable,let M_1 be a closed infinite dimensional subspace of $\mathcal{H}$ with infinite codimension. We have that $M_1^{\perp}$ is also a separable infinite dimensional Hilbert space. Let M_2 be the closed subspace of $M_1^{\perp}$ isomorphic to the subspace M_1. Continuing this process we construct the subspaces M_n, $n \in \mathbb{N}$. Let $(\lambda_n)_n$ be a sequence of complex numbers that converges to 0. For each $n \in \mathbb{N}$ there exists a bounded linear operator $T_n \in \mathcal{B}(\mathcal{M}_n)$ such that $T_n, T_n - \lambda_m$, $m \in \mathbb{N} \setminus \{n\}$ are invertible and $n(T_n - \lambda_n) = \infty$ and $\mathcal{R}(T_n - \lambda_n) = M_n$. This means that $\lambda_n \in \sigma_{FC}(T_n)$ and $0, \lambda_m \notin \sigma_{FC}(T_n)$, $m \in \mathbb{N} \setminus \{n\}$. Furthermore we can select these operators in such a way that the family of operators T_n is uniformly bounded. We have that $T = \bigoplus_{n=1}^{\infty} T_n$ is a invertible bounded linear operator on $\mathcal{H}$ such that

$$n\,(T - \lambda_n) = \infty,\ \mathcal{R}\,(T - \lambda_n) = \mathcal{H},\ n \in \mathbb{N}.$$

This means that $\lambda_n \in \sigma_{FC}\,(T)\,,\ n \in \mathbb{N}$, but $0 \notin \sigma_{FC}\,(T)$. We conclude that $\sigma_{FC}\,(T)$ is not closed. To see that the operators T_n indeed exists we can construct them now. For each $n \in \mathbb{N}$ there exists $r_n > 0$ such that $\lambda_m \notin B(\lambda_n, r_n)$ for $m \neq n$. It follows that $|r_n| < |\lambda_n|$ and that $0 \notin B(\lambda_n, r_n)$. Furthermore, for each $n \in \mathbb{N}$ there exists a subspace $\mathcal{K}_n$ such that $\mathcal{M}_n = \mathcal{K}_n \oplus \mathcal{K}_n^{\perp}$ and $\dim \mathcal{K}_n = \dim \mathcal{K}_n^{\perp} = \infty$. We have that $\mathcal{K}_n$ is isomorphic to $\mathcal{M}_n$, let us denote the isomorphism by J_n'. Without loss of generality we can assume that J_n' is unitary. This isomorphism is naturally extended to $J_n \in \mathcal{B}(\mathcal{M}_n)$ by

$$J_n x = \begin{cases} J_n' x, & x \in K_n \\ 0, & x \in \mathcal{K}_n^{\perp} \end{cases}.$$

We have that $\mathcal{N}(J_n) = \mathcal{K}_n^{\perp}$, and $\mathcal{R}(J_n) = \mathcal{M}_n$. Define T_n by

$$T_n = r_n J_n + \lambda_n.$$

We have that $T_n - \lambda_n = r_n J_n$, so $n\,(T_n - \lambda_n) = n\,(J_n) = \infty$ and $\mathcal{R}\,(T_n - \lambda_n) = \mathcal{R}(J_n) = \mathcal{M}_n$, so $\lambda_n \in \sigma_{FC}\,(T_n)$. Since $|\lambda_n|, |\lambda_n - \lambda_m| > |r_n| = \|r_n J_n\|$ for $m \neq n$ we have that T_n and $T_n - \lambda_m$, $m \neq n$ are invertible, and $\|T_n\| \leq r_n + \lambda_n \leq 1 + M$ for $n \in \mathbb{N}$ where M is any upper bound for the convergent sequence $(\lambda_n)_n$ which proves that the family $(T_n)_n$ is uniformly bounded.

Since $\sigma_{FC}(T) = \emptyset$ for all $T \in \mathcal{B}(\mathcal{H})$ when $\mathcal{H}$ is finite dimensional the set of Fredholm consistent operators will coincide with $\mathcal{B}(\mathcal{H})$ and will thus be a regularity. We have that the following Theorem analogous to Theorem 2.4 holds:

Theorem 3.3. *The set of all Fredholm consistent operators in $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a regularity of and only if $\mathcal{H}$ is finite dimensional.*

REFERENCES

1. D. Cvetković-Ilić, M. Kostadinov, *Consistency in invertibility and Fredholmness of operator matrices*, Journal of Spectral Theory, DOI: 10.4171/JST/258.
2. D.S. Cvetkovi-Ilić, *Fredholm consistence of upper-triangular operator matrices*, Journal of Spectral Theory, 7(4), (2017) 1023-1038.
3. D.S. Cvetkovi-Ilić, G. Hai and A. Chen, *Some results on Fredholmness and boundedness below of an upper triangular operator matrix* , Journal of Math. Anal. Appl., 425(2) (2015) 1071-1082.
4. V. Kordula, V. Müller, *On the axiomatic theory of spectrum*, Studia Math. 119(1996), 109-128.
5. V. Müller, *Axiomatic theory of spectrum III: semiregularities*, Studia Math. 142(2), (2000), 159-169.

6. R. Harte, *Fredholm Theory relative to a Banach algebra homomorphism*, Math. Z. 179, (1982), 431-436.
7. W. Gong, D. Han, *Spectrum of the Products of Operators and Compact Perturbations*, Proc. Amer. Math. Soc., 120 (3) (1994), 755-760.
8. X. Cao, H. Zhang, Y. Zhang, *Consistent invertibility and Weyl's theorem*, J. Math. Anal. Appl. 369 (2010), 258-264.
9. X. Cao, Q. Xin, *Consistent invertibility and perturbations of the generalized property* (ω), Acta Mathematica Sinica, 55(1) (2012), 92-100.
10. G. Hai, A. Chen, *Consistent invertibility of upper-triangular operator matrices*, Linear Alg. Appl. 455 (2014) 22-31.

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NONLINEAR SINGULAR STURM-LIOUVILLE PROBLEMS WITH IMPULSIVE CONDITIONS

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Abstract. In this paper, we consider a non-linear impulsive Sturm-Liouville problem on semiinfinite intervals in which the limit-circle case holds at infinity for THE Sturm-Liouville expression. We prove the existence and uniqueness theorems for this problem.
Keywords: Impulsive Sturm-Liouville problem; Singular point; Weyl limit-circle case; Completely continuous operator; Fixed point theorems.

1. Introduction

The theory of differential equations with impulses describes processes that are subjected to abrupt changes in their states at certain moments. Such processes arise in many fields of science and technology: chemical technology, biotechnology, theoretical physics, industrial robotics, etc. For an introduction to the basic theory of differential equations with impulses see Bainov and Simeonov ([3], [4], [5]), Benchohra, Henderson and Ntouyas ([6]), Lakshmikantham, Bainov and Simeonov ([18]) Samoilenko and Perestyuk ([31]) and the references therein.

Recently, much work has been done on the existence of solutions to impulsive Sturm-Liouville equations; for regular impulsive Sturm-Liouville problems see [2, 7, 9, 12-15, 25-27, 30, 33], for singular impulsive Sturm-Liouville equations see [1, 10, 18-19, 21-24, 29]. However, there is no paper concerned with the existence of solutions to singular impulsive non-linear Sturm-Liouville problems that the limit-circle case holds at infinity. In this paper, we fill the gap by using a special way to pose boundary conditions at infinity.

Let us consider the following nonlinear Sturm-Liouville equation

$$l(y) := -(p(x)y')' + q(x)y = f(x, y), \ x \in I, \tag{1.1}$$

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where $I := I_1 \cup I_2$, $I_1 := [a, c)$, $I_2 := (c, +\infty)$, $-\infty < a < c < +\infty$, and $y = y(x)$ is a desired solution.

Let $L^2(I)$ be a Hilbert space which is composed of all complex-valued functions y satisfying

$$\int_a^\infty |y(x)|^2 \, dx < \infty$$

in relation to the inner product

$$(y, z) := \int_a^\infty y(x) \overline{z(x)} dx.$$

Denote by $\mathcal{D}$ the linear set of all functions $y \in L^2(I)$ such that y, py' are locally absolutely continuous functions on I, one-sided limits $y(c\pm)$, $(py')(c\pm)$ exist and are finite and $l(y) \in L^2(I)$. The operator L defined by $Ly = l(y)$ is called the maximal operator on $L^2(I)$.

For two arbitrary functions $y, z \in \mathcal{D}$, we have Green's formula

$$\int_a^\infty l(y) \overline{z} dx - \int_a^\infty y \overline{l(z)} dx = [y, z]_{c-} - [y, z]_a + [y, z]_\infty - [y, z]_{c+}, \tag{1.2}$$

where $[y, z]_x = y(x)\overline{(pz')(x)} - (py')(x)\overline{z(x)}$ $(x \in I)$.

We assume that the following conditions are satisfied.

(A1) The points a and c are regular for the differential expression l. p and q are real-valued, Lebesgue measurable functions on I and $\frac{1}{p}, q \in L^1_{loc}(I)$. The point c is regular if $\frac{1}{p}, q \in L^1[c-\epsilon, c+\epsilon]$ for some $\epsilon > 0$. Moreover, the functions p and q are such that all solutions of the the equation

$$l(y) = 0 \tag{1.3}$$

belong to $L^2(I)$, i.e., Weyl limit-circle case holds for the differential expression l (see [1-3]).

(A2) The function $f(x, y)$ is real-valued and continuous in $(x, \zeta) \in I \times \mathbb{R}$, and

$$|f(x, \zeta)| \le g(x) + \vartheta |\zeta| \tag{1.4}$$

for all (x, ζ) in $I \times \mathbb{R}$, where $g(x) \ge 0$, $g \in L^2(I)$, and ϑ is a positive constant.

If we define the operator F taking each function $y(.)$ to the function $f(., y(.))$, then the condition (4) is necessary and sufficient for F to map $L^2(I)$ into itself (see ([17], Chapter 1)).

Denote by

$$u := u(x) = \begin{cases} u^{(1)}(x), & x \in I_1 \\ u^{(2)}(x), & x \in I_2 \end{cases}, \quad v := v(x) = \begin{cases} v^{(1)}(x), & x \in I_1 \\ v^{(2)}(x), & x \in I_2 \end{cases}$$

the solutions to the equation (1.3) satisfying the initial conditions

$$u^{(1)}(a) = 0, \ (pu^{(1)\prime})(a) = 1, \ v^{(1)}(a) = -1, \ (pv^{(1)\prime})(a) = 0, \tag{1.5}$$

and impulsive conditions

$$(1.6)\qquad \begin{array}{c} U(c+) = CU(c-),\ U(x) := \begin{pmatrix} u(x) \\ (pu')(x) \end{pmatrix}, \\ V(c+) = CV(c-),\ V(x) := \begin{pmatrix} v(x) \\ (pv')(x) \end{pmatrix}, \end{array}$$

$$C \in M_2(\mathbb{R}),\ \det C = \rho > 0,$$

where $M_2(\mathbb{R})$ denotes the 2×2 matrices with entries from $\mathbb{R}$.

Now, we introduce the Hilbert space $H = L^2(I_1) \dot{+} L^2(I_2)$ with the inner product

$$\langle y, z \rangle := \int_a^c y^{(1)} \overline{z^{(1)}} dx + \gamma \int_c^\infty y^{(2)} \overline{z^{(2)}} dx,\ \gamma = \frac{1}{\rho},$$

where

$$y(x) = \begin{cases} y^{(1)}(x), & x \in I_1 \\ y^{(2)}(x), & x \in I_2 \end{cases},\quad z(x) = \begin{cases} z^{(1)}(x), & x \in I_1 \\ z^{(2)}(x), & x \in I_2. \end{cases}$$

We set $W_x^{(i)} := W_x\left(u^{(i)}, v^{(i)}\right) = u^{(i)}(x)(pv^{(i)\prime})(x) - (pu^{(i)\prime})(x)v^{(i)}(x)\ (x \in I_i, i = 1, 2)$. Then the equality $W_x^{(1)} = \rho W_x^{(2)}$ holds. For convenience, we denote $W_x := W_x^{(1)} = \rho W_x^{(2)}$. Since the wronskian of any two solutions of Equation (1.3) is constant, we have $W_x(u, v) = 1$. Then, u and v are linearly independent and they form a fundamental system of solutions of equation (1.3). By the condition A1, we get u, $v \in L^2(I)$ and moreover, u, $v \in \mathcal{D}$. So, the values $[y, u]_\infty$ and $[y, v]_\infty$ exist and are finite for every $y \in \mathcal{D}$. By using Green's formula (1.2) and the conditions (1.5)-(1.6), we can get

$$(1.7)\qquad \begin{array}{c} [y, u]_\infty = y(a) + \int_a^\infty u(x)\, \overline{l(y(x))} dx, \\ [y, v]_\infty = (py')(a) + \int_a^\infty v(x)\, \overline{l(y(x))} dx. \end{array}$$

Now, we will add to problem (1.1) the boundary conditions

$$(1.8)\qquad \begin{array}{c} y(a)\cos\alpha + (py')(a)\sin\alpha = d_1, \\ [y, u]_\infty \cos\beta + [y, v]_\infty \sin\beta = d_2, \end{array}$$

and impulsive conditions

$$(1.9)\qquad Y(c+) = CY(c-),\ Y = \begin{pmatrix} y \\ py' \end{pmatrix},\ \det C = \rho > 0,$$

where $\alpha, \beta \in \mathbb{R}$, and d_1, d_2 are arbitrary given real numbers, and

$$(\mathbf{A3})\qquad \omega := \cos\alpha \sin\beta - \cos\beta \sin\alpha \neq 0.$$

Since the function y in (1.8) satisfies Equation (1.1), we have

$$[y, u]_\infty = y(a) + \int_a^\infty u(x)\, f(x, y(x))\, dx,$$

$$[y, v]_\infty = (py')(a) + \int_a^\infty v(x)\, f(x, y(x))\, dx.$$

2. Green's function

In this section, we construct an appropriate Green's function. So, we will reduce the boundary-value problem (1.1), (1.8), (1.9) to a fixed point problem.

Let us consider the linear boundary value problem

$$-(p(x)y')' + q(x)y = h\left(x\right),\ x \in I,\ h \in H, \tag{2.1}$$

$$\left.\begin{array}{c} y\left(a\right)\cos\alpha + \left(py'\right)\left(a\right)\sin\alpha = 0, \\ \left[y,u\right]_{\infty}\cos\beta + \left[y,v\right]_{\infty}\sin\beta = 0,\ \alpha,\beta \in \mathbb{R}, \\ Y\left(c+\right) = CY\left(c-\right),\ Y := \begin{pmatrix} y \\ py' \end{pmatrix},\ \det C = \rho > 0, \end{array}\right\} \tag{2.2}$$

where y is a desired solution, u and v are solutions to the equation (1.3) under the conditions (1.5)-(1.6).

Define

$$\varphi\left(x\right) = \cos\alpha u\left(x\right) + \sin\alpha v\left(x\right),\ \psi\left(x\right) = \cos\beta u\left(x\right) + \sin\beta v\left(x\right), \tag{2.3}$$

where $W_x\left(\varphi,\psi\right) = \omega$. It is clear that these functions are solutions to the equation (1.3) and are in H. Further, we have

$$\begin{aligned} &[\varphi,u]_x = \varphi\left(a\right) = -\sin\alpha,\ [\varphi,v]_x = \left(p\varphi\right)'\left(a\right) = \cos\alpha,\ (x \in I_1), \\ &[\psi,u]_x = \psi\left(a\right) = -\sin\beta,\ [\psi,v]_x = \left(p\psi\right)'\left(a\right) = \cos\beta,\ (x \in I_1), \\ &[\psi,u]_{\infty} = -\rho\sin\beta,\ [\psi,v]_{\infty} = \rho\cos\beta, \\ &\Phi\left(c+\right) = C\Phi\left(c-\right),\ \Phi(x) := \begin{pmatrix} \varphi\left(x\right) \\ \left(p\varphi'\right)\left(x\right) \end{pmatrix}, \\ &\Psi\left(c+\right) = C\Psi\left(c-\right),\ \Psi(x) := \begin{pmatrix} \psi\left(x\right) \\ \left(p\psi'\right)\left(x\right) \end{pmatrix}. \end{aligned}$$

Let us introduce the function

$$G\left(x,t\right) = \begin{cases} \frac{\varphi(x)\psi(t)}{\omega}, & \text{if } a \leq x \leq t < \infty,\ x \neq c,\ t \neq c, \\ \frac{\varphi(t)\psi(x)}{\omega}, & \text{if } a \leq t \leq x < \infty,\ x \neq c,\ t \neq c. \end{cases} \tag{2.4}$$

$G\left(x,t\right)$ is called the Green's function of the boundary-value problem (2.1)-(2.2). Since $\varphi,\psi \in H$, we have

$$\int_a^{\infty}\int_a^{\infty}\left|G\left(x,t\right)\right|^2 dxdt < \infty, \tag{2.5}$$

i.e., $G\left(x,t\right)$ is a Hilbert-Schmidt kernel.

Theorem 2.1. *The function*

$$y\left(x\right) = \int_a^c G\left(x,t\right)h\left(t\right)dt + \gamma\int_c^{\infty} G\left(x,t\right)h\left(t\right)dt,\ x \in I, \tag{2.6}$$

is the solution of the boundary-value problem (2.1)-(2.2).

Proof. By a variation of constants formula, the general solution of the equation (2.1) has the form

$$y(x)=\begin{cases} k_1\varphi^{(1)}(x)+k_2\psi^{(1)}(x) \\ +\frac{\psi^{(1)}(x)}{\omega}\int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\varphi^{(1)}(x)}{\omega}\int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x\in I_1, \\ k_3\varphi^{(2)}(x)+k_4\psi^{(2)}(x) \\ +\frac{\gamma}{\omega}\psi^{(2)}(x)\int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}\varphi^{(2)}(x)\int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x\in I_2, \end{cases} \tag{2.7}$$

where k_1,k_2,k_3 and k_4 are arbitrary constants.

By (2.7), we get

$$(py)'(x)=\begin{cases} k_1\left(p\varphi^{(1)}\right)'(x)+k_2(p\psi^{(1)})'(x) \\ +\frac{\left(p\psi^{(1)}\right)'(x)}{\omega}\int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\left(p\varphi^{(1)}\right)'(x)}{\omega}\int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x\in I_1, \\ k_3\left(p\varphi^{(2)}\right)'(x)+k_4(p\psi^{(2)})'(x) \\ +\frac{\gamma}{\omega}\left(p\psi^{(2)}\right)'(x)\int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}\left(p\varphi^{(2)}\right)'(x)\int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x\in I_2. \end{cases}$$

Hence, we have

$$\begin{aligned} y(a)&=k_1\varphi^{(1)}(a)+k_2\psi^{(1)}(a)+\tfrac{\varphi^{(1)}(a)}{\omega}\int_a^c \psi^{(1)}(t)\,h(t)\,dt \\ &=-k_1\sin\alpha-k_2\sin\beta-\tfrac{1}{\omega}\sin\alpha\int_a^c \varphi^{(1)}(t)\,h(t)\,dt, \\ (py)'(a)&=k_1\left(p\varphi^{(1)}\right)'(a)+k_2(p\psi^{(1)})'(a) \\ &\quad+\tfrac{1}{\omega}\left(p\varphi^{(1)}\right)'(a)\int_a^c \psi^{(1)}(t)\,h(t)\,dt \\ &=k_1\cos\alpha+k_2\cos\beta+\tfrac{1}{\omega}\cos\alpha\int_a^c \varphi^{(1)}(t)\,h(t)\,dt. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.2), we get

$$k_2(\cos\alpha\sin\beta-\sin\alpha\cos\beta)=0,\ k_2\omega=0,$$

i.e., $k_2 = 0$. Further, we have

$$[y,u]_x = y(x)(pu')(x) - (py')(x)u(x)$$

$$= \begin{cases} k_1[\varphi^{(1)},u]_x + \frac{1}{\omega}[[\psi^{(1)}(x),u]_x \int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{1}{\omega}[\varphi^{(1)}(x),u]_x \int_x^c \psi^{(1)}(t)\,h(t)\,dt, \ x \in I_1, \\ k_3[\varphi^{(2)},u]_x + k_4[\psi^{(2)},u]_x \\ +\frac{\gamma}{\omega}[\psi^{(2)},u]_x \int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}[\varphi^{(2)},u]_x \int_x^\infty \psi^{(2)}(t)\,h(t)\,dt, \ x \in I_2. \end{cases}$$

Thus

$$[y,u]_\infty = -k_3\rho\sin\alpha - k_4\rho\sin\beta - \frac{\gamma}{\omega}\rho\sin\beta \int_c^\infty \varphi^{(2)}(t)\,h(t)\,dt.$$

Similarly, we get

$$[y,v]_x = y(x)(pv')(x) - (py')(x)v(x)$$

$$= \begin{cases} k_1[\varphi^{(1)},v]_x \\ +\frac{1}{\omega}[[\psi^{(1)}(x),v]_x \int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{1}{\omega}[\varphi^{(1)}(x),v]_x \int_x^c \psi^{(1)}(t)\,h(t)\,dt, \ x \in I_1, \\ k_3[\varphi^{(2)},v]_x + k_4[\psi^{(2)},v]_x \\ +\frac{\gamma}{\omega}[\psi^{(2)},v]_x \int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}[\varphi^{(2)},v]_x \int_x^\infty \psi^{(2)}(t)\,h(t)\,dt, \ x \in I_2, \end{cases}$$

and

$$[y,v]_\infty = k_3\rho\cos\alpha + k_4\rho\cos\beta + \frac{\gamma}{\omega}\rho\cos\beta \int_c^\infty \varphi^{(2)}(t)\,h(t)\,dt.$$

From the conditions (2.2), we obtain

$$k_3(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = 0.$$

Hence, $k_3 = 0$. Similarly, we have

$$Y(c+) = \begin{pmatrix} y(c+) \\ (py')(c+) \end{pmatrix} = \begin{pmatrix} k_4\psi^{(2)}(c+) \\ k_4(p\psi^{(2)})'(c+) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\gamma}{\omega}\varphi^{(2)}(c+)\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \\ \\ \frac{\gamma}{\omega}\left(p\varphi^{(2)}\right)'(c+)\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \end{pmatrix}$$

$$= k_4 \begin{pmatrix} \psi^{(2)}(c+) \\ (p\psi^{(2)})'(c+) \end{pmatrix}$$

$$+ \frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \begin{pmatrix} \varphi^{(2)}(c+) \\ \left(p\varphi^{(2)}\right)'(c+) \end{pmatrix}$$

$$= k_4\Psi(c+) + \left\{\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt\right\}\Phi(c+)$$

and

$$Y(c-) = \begin{pmatrix} y(c-) \\ (py')(c-) \end{pmatrix}$$

$$= \begin{pmatrix} k_1\varphi^{(1)}(c-) \\ k_1\left(p\varphi^{(1)}\right)'(c-) \end{pmatrix} + \begin{pmatrix} \frac{\psi^{(1)}(c-)}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \\ \\ \frac{\left(p\psi^{(1)}\right)'(c-)}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \end{pmatrix}$$

$$= k_1 \begin{pmatrix} \varphi^{(1)}(c-) \\ \left(p\varphi^{(1)}\right)'(c-) \end{pmatrix} + \frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \begin{pmatrix} \psi^{(1)}(c-) \\ \left(p\psi^{(1)}\right)'(c-) \end{pmatrix}$$

$$= k_1\Phi(c-) + \left\{\frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt\right\}\Psi(c-).$$

By the conditions (2.2), we obtain

$$k_4\Psi(c+) + \left\{\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt\right\}\Phi(c+)$$

$$= C\left\{k_1\Phi(c-) + \left\{\frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt\right\}\Psi(c-)\right\}.$$

Using the conditions (2.) and (2.), we get

$$\Phi(c-)\left\{\frac{\gamma}{\omega}\int_c^{\infty}\psi^{(2)}(t)\,h(t)\,dt-k_1\right\}$$

$$=\Psi(c-)\left\{\frac{1}{\omega}\int_a^{c}\varphi^{(1)}(t)\,h(t)\,dt-k_4\right\},$$

$$\begin{pmatrix}\varphi^{(1)}(c-)\\ (p\varphi^{(1)\prime})(c-)\end{pmatrix}\left\{\frac{\gamma}{\omega}\int_c^{\infty}\psi^{(2)}(t)\,h(t)\,dt-k_1\right\}$$

$$=\begin{pmatrix}\psi^{(1)}(c-)\\ (p\psi^{(1)\prime})(c-)\end{pmatrix}\left\{\frac{1}{\omega}\int_a^{c}\varphi^{(1)}(t)\,h(t)\,dt-k_4\right\}.$$

So, we have the following linear equation system

$$k_4\psi^{(1)}(c-)-k_1\varphi^{(1)}(c-)$$

$$=\left\{\frac{1}{\omega}\int_a^{c}\varphi^{(1)}(t)\,h(t)\,dt\right\}\psi^{(1)}(c-)$$

$$-\left\{\frac{\gamma}{\omega}\int_c^{\infty}\psi^{(2)}(t)\,h(t)\,dt\right\}\varphi^{(1)}(c-),$$

$$k_4(p\psi^{(1)\prime})(c-)-k_1(p\varphi^{(1)\prime})(c-)$$

$$=\left\{\frac{1}{\omega}\int_a^{c}\varphi^{(1)}(t)\,h(t)\,dt\right\}(p\psi^{(1)\prime})(c-)$$

$$-\left\{\frac{\gamma}{\omega}\int_c^{\infty}\psi^{(2)}(t)\,h(t)\,dt\right\}(p\varphi^{(1)\prime})(c-),$$

i.e.,

$$\begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{pmatrix}\begin{pmatrix} k_4 \\ -k_1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \\ \\ -\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \end{pmatrix}.$$

Hence, we have the following determinant of this linear equation system

$$\begin{vmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{vmatrix} = -\omega.$$

Since this determinant is different from zero, the solution of this system is unique. If we solve this system, we have the following equalities

$$k_1 = \frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt, k_4 = \frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt.$$

From what has already been done, we have

$$y(x) = \begin{cases} \varphi^{(1)}(x)\,\frac{\gamma}{\omega}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \\ +\frac{\psi^{(1)}(x)}{\omega}\int_a^x \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\varphi^{(1)}(x)}{\omega}\int_x^c \psi^{(1)}(t)\,h(t)\,dt,\ x\in I_1, \\ \psi^{(2)}(x)\,\frac{1}{\omega}\int_a^c \varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}\psi^{(2)}(x)\int_c^x \varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\gamma}{\omega}\varphi^{(2)}(x)\int_x^\infty \psi^{(2)}(t)\,h(t)\,dt,\ x\in I_2, \end{cases}$$

i.e., (2.4) and (2.6) hold. □

Thus we have a

Theorem 2.2. *The unique solution to the equation (2.1) under the conditions (1.8)-(1.9) is given by the formula*

$$y(x) = w(x) + \langle G(x,.)\,,\overline{h(.)}\rangle,$$

where

$$w(x) = \frac{d_1}{\omega}\varphi(x) - \frac{d_2}{\omega}\psi(x).$$

Proof. By the conditions (2.)-(2.), the function $w(x)$ is a unique solution of the equation (1.3) satisfying the conditions (1.8)-(1.9). By Theorem 1 the function $\langle G(x,.), \overline{h(.)}\rangle$ a unique solution to the equation (2.1) satisfying the conditions (2.2). This finishes the proof. □

From Theorem 2, the boundary-value problem (1.1), (1.8), (1.9) in H is equivalent to the non-linear integral equation

$$y(x) = w(x) + \langle G(x,.), f(., y(.))\rangle,\ x \in I, \tag{2.9}$$

where the functions $w(x)$ and $G(x,t)$ are defined above. Hence, we shall study the equation (2.9).

By (1.4) and (2.5), we can define the operator $T : H \to H$ by the formula

$$(Ty)(x) = w(x) + \langle G(x,.), f(., y(.))\rangle,\ x \in I, \tag{2.10}$$

where $y, w \in H$. Then the equation (2.9) can be written as $y = Ty$.

Now, we search the fixed points of the operator T because it is equivalent to solving the equation (2.9).

3. The fixed points of the operator T

In this section, we investigate the fixed points of the operator T by using the following Banach fixed point theorem:

Definition 3.1. [[16]]Let A be a mapping of a metric space R into itself. Then x is called a fixed point of A if $Ax = x$. Suppose there exists a number $\alpha < 1$ such that

$$\rho(Ax, Ay) \le \alpha\rho(x,y)$$

for every pair of points $x, y \in R$. Then A is said to be a contraction mapping.

Theorem 3.1. [16] *Every contraction mapping A defined on a complete metric space R has a unique fixed point.*

Theorem 3.2. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. Further, let the function $f(x,y)$ satisfy the following Lipschitz condition: there* *exists a constant $K > 0$ such that*

$$\int_a^c \left| f^{(1)}\left(x, y^{(1)}(x)\right) - f^{(1)}\left(x, z^{(1)}(x)\right)\right|^2 dx$$

$$+\gamma \int_c^\infty \left| f^{(2)}\left(x, y^{(2)}(x)\right) - f^{(2)}\left(x, z^{(2)}(x)\right)\right|^2 dx$$

$$\leq K^2 \left(\int_a^c \left| y^{(1)}(x) - z^{(1)}(x)\right|^2 dx + \gamma \int_c^\infty \left| y^{(2)}(x) - z^{(2)}(x)\right|^2 dx \right)$$

$$= K^2 \left\| y - z \right\|^2$$

for all $y, z \in H$. If

$$K \left(\int_a^c \int_a^c |G(x,t)|^2 \, dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 \, dxdt \right) < 1, \tag{3.1}$$

then the boundary-value problem (1.1), (1.8), (1.9) has a unique solution in H.

Proof. It suffices to prove that the operator T is a contraction operator. For $y, z \in H$, we have

$$\begin{aligned} |Ty(x) - Tz(x)|^2 &= |\langle G(x,.), [f(.,y(.)) - f(.,z(.))]\rangle|^2 \\ &\leq \|G(x,.)\|^2 \, \|f(.,y(.)) - f(.,z(.))\|^2 \\ &\leq K^2 \|G(x,.)\|^2 \, \|y - z\|^2, \ x \in I. \end{aligned}$$

Thus, we get

$$\|Ty - Tz\| \leq \alpha \|y - z\|,$$

where

$$\alpha = K \left(\int_a^c \int_a^c |G(x,t)|^2 \, dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 \, dxdt \right)^{\frac{1}{2}} < 1,$$

i.e., T is a contraction mapping. □

Now, our next claim is that the function $f(x, y)$ satisfies a Lipschitz condition on a subset of H but not of the whole space.

Theorem 3.3. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let the function $f(x, y)$ satisfy the following Lipschitz condition: there* *exist constants* M, $K>0$ *such that*

$$\int_a^c \left|f^{(1)}\left(x,y^{(1)}(x)\right)-f^{(1)}\left(x,z^{(1)}(x)\right)\right|^2 dx$$

$$+\gamma\int_c^\infty \left|f^{(2)}\left(x,y^{(2)}(x)\right)-f^{(2)}\left(x,z^{(2)}(x)\right)\right|^2 dx$$

$$\leq K^2\left(\int_a^c \left|y^{(1)}(x)-z^{(1)}(x)\right|^2 dx+\gamma\int_c^\infty \left|y^{(2)}(x)-z^{(2)}(x)\right|^2 dx\right)$$

$$=K^2\left\|y-z\right\|^2$$

for all y *and* z *in* $S_M=\{t\in H:\|t\|\leq M\}$, *where* K *may depend on* M. *If*

$$\left\{\int_a^c \left|w^{(1)}(x)\right|^2 dx+\gamma\int_c^\infty \left|w^{(2)}(x)\right|^2 dx\right\}^{1/2}$$

$$+\left(\int_a^c\int_a^c \left|G(x,t)\right|^2 dxdt+\gamma\int_c^\infty\int_c^\infty \left|G(x,t)\right|^2 dxdt\right)^{\frac{1}{2}}$$

$$\times\sup_{y\in S_M}\left\{\begin{array}{c}\int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right)-f^{(1)}\left(t,z^{(1)}(t)\right)\right|^2 dt\\ +\gamma\int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right)-f^{(2)}\left(t,z^{(2)}(t)\right)\right|^2 dt\end{array}\right\}^{1/2}$$

$$\leq M$$

and

$$K\left(\int_a^c\int_a^c \left|G(x,t)\right|^2 dxdt+\gamma\int_c^\infty\int_c^\infty \left|G(x,t)\right|^2 dxdt\right)^{\frac{1}{2}}<1, \tag{3.2}$$

then the boundary-value problem (1.1), (1.8), (1.9) has a unique solution with

$$\int_a^c \left|y^{(1)}(x)\right|^2 dx+\gamma\int_c^\infty \left|y^{(2)}(x)\right|^2 dx\leq M^2.$$

Proof. It is clear that S_M is a closed set of H. We first prove that the operator T

maps S_M into itself. For $y \in S_M$ we have

$$\|Ty\| = \|w(x) + \langle G(x,.), f(., y(.))\rangle\| \leq \|w\| + \|\langle G(x,.), f(., y(.))\rangle\|$$

$$\leq \|w\| + \left(\int_a^c \int_a^c |G(x,t)|^2 \, dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 \, dxdt\right)^{\frac{1}{2}}$$

$$\times \sup_{y \in S_M} \left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^{1/2} \leq M.$$

Consequently, $T : S_M \to S_M$.

We can now proceed analogously to the proof of Theorem 5. So, we can get

$$\|Ty - Tz\| \leq \alpha \|y - z\|, \ y, z \in S_M.$$

If we apply the Banach fixed point theorem, then we obtain a unique solution of the boundary-value problem (1.1), (1.8), (1.9) in S_M. □

4. An existence theorem without uniqueness

In this section, we get an existence theorem without uniqueness of solution. Therefore, we will use the following Schauder fixed point theorem:

Definition 4.1. [[11]]An operator acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Theorem 4.1. [11] *Let* $\mathbf{B}$ *be a Banach space and* $\mathbf{S}$ *a nonemty bounded, convex, and closed subset of* $\mathbf{B}$. *Assume* $A : \mathbf{B} \to \mathbf{B}$ *is a completely continuous operator. If the operator* A *leaves the set* $\mathbf{S}$ *invariant, i.e., if* $A(\mathbf{S}) \subset \mathbf{S}$, *then* A *has at least one fixed point in* $\mathbf{S}$.

A set $S \subset H$ is relatively compact iff S is bounded and for every $\varepsilon > 0$ (i) there exists $\delta > 0$ such that $\|y(x+h) - y(x)\| < \varepsilon$ for all $y \in S$ and all $h \geq 0$ with $h < \delta$, (ii) there exists a number $N > 0$ such that $\int_N^\infty |y(x)|^2 dx < \varepsilon$ for all $y \in S$ ([11]).

Now, we give

Theorem 4.2. *The operator* T *defined by (2.10) is completely continuous operator under the conditions (A1), (A2) and (A3).*

Proof. Let $y_0 \in H$. Then, we have

$$\begin{aligned}
&\left|(Ty)(x) - (Ty_0)(x)\right|^2 \\
&= \left|\langle G(x,.), [f(.,y(.)) - f(.,y_0(.))]\rangle\right|^2 \\
&\leq \|G(x,.)\|^2 \left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right) - f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right) - f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
&\|Ty - Ty_0\|^2 \\
&\leq K \left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right) - f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right) - f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^2 ,
\end{aligned}$$

where

$$K = \left(\int_a^c \int_a^c |G(x,t)|^2 \, dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 \, dxdt \right).$$

We know that an operator F defined by $Fy(x) = f(x,y(x))$ is continuous in H under the condition (A2) (see [17]). Hence, for a given $\epsilon > 0$, we can find a $\delta > 0$ such that $\|y - y_0\| < \delta$ implies

$$\left\{ \begin{array}{c} \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right) - f^{(1)}(t,y_0^{(1)}(t))\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right) - f^{(2)}(t,y_0^{(2)}(t))\right|^2 dt \end{array} \right\} < \frac{\epsilon^2}{K^2}.$$

From (4.), we get

$$\|Ty - Ty_0\| < \epsilon,$$

i.e., T is continuous.

Set $Y = \{y \in H : \|y\| \leq m\}$. By (3.3), we have

$$\|Ty\| \leq \|w\| + \left\{ \begin{array}{c} K \int_a^c \left|f^{(1)}\left(t,y^{(1)}(t)\right)\right|^2 dt \\ +\gamma K \int_c^\infty \left|f^{(2)}\left(t,y^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^{1/2}, \text{ for all } y \in Y.$$

Furthermore, using (1.4), we get

$$\int_a^c \left|f^{(1)}\left(t, y^{(1)}(t)\right)\right|^2 dt + \gamma \int_c^\infty \left|f^{(2)}\left(t, y^{(2)}(t)\right)\right|^2 dt$$

$$\leq \int_a^c \left[g^{(1)}(t) + \vartheta\left|y^{(1)}(t)\right|\right]^2 dt + \gamma \int_c^\infty \left[g^{(2)}(t) + \vartheta\left|y^{(2)}(t)\right|\right]^2 dt$$

$$\leq 2\int_a^c [\left(g^{(1)}\right)^2(t) + \vartheta^2\left|y^{(1)}(t)\right|^2]dt$$

$$+ 2\gamma \int_c^\infty [\left(g^{(2)}\right)^2(t) + \vartheta^2\left|y^{(2)}(t)\right|^2]dt$$

$$= 2(\|g\|^2 + \vartheta^2\|y\|^2) \leq 2(\|g\|^2 + \vartheta^2 m^2).$$

Thus, for all $y \in Y$, we obtain

$$\|Ty\| \leq \|w\| + \left[2K\left(\|g\|^2 + \vartheta^2 m\right)\right]^{1/2},$$

i.e., $T(Y)$ is a bounded set in H.

Moreover, for all $y \in Y$, we have

$$\begin{aligned}
&\int_a^c \left|(Ty^{(1)})(x+h) - (Ty^{(1)})(x)\right|^2 dx \\
&+ \gamma \int_c^\infty \left|(Ty^{(2)})(x+h) - (Ty^{(2)})(x)\right|^2 dx \\
&= \|\langle [G(x+h,.) - G(x,.)], f(., y(.))\rangle\|^2 \\
&\leq \begin{pmatrix} \int_a^c \int_a^c |G(x+h,t) - G(x,t)|^2 dxdt \\ +\gamma \int_c^\infty \int_c^\infty |G(x+h,t) - G(x,t)|^2 dxdt \end{pmatrix} \\
&\times \begin{Bmatrix} \int_a^c \left|f^{(1)}\left(t, y^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t, y^{(2)}(t)\right)\right|^2 dt \end{Bmatrix}^2 \\
&\leq 2\left(\|g\|^2 + \vartheta^2 m\right) \begin{pmatrix} \int_a^c \int_a^c |G(x+h,t) - G(x,t)|^2 dxdt \\ +\gamma \int_c^\infty \int_c^\infty |G(x+h,t) - G(x,t)|^2 dxdt \end{pmatrix}
\end{aligned}$$

From (2.5), there exists a $\delta > 0$ such that

$$\int_a^c \left|Ty^{(1)}(x+h) - Ty^{(1)}(x)\right|^2 dx$$

$$+\gamma \int_c^\infty \left|Ty^{(2)}(x+h) - Ty^{(2)}(x)\right|^2 dx < \epsilon^2,$$

for given $\epsilon > 0$, all $y \in Y$ and all $h < \delta$.

Further, for all $y \in Y$, we have $(N > c)$

$$\int_N^\infty \left|\left(Ty^{(2)}\right)(x)\right|^2 dx$$

$$\leq \int_N^\infty \left|w^{(2)}(x)\right|^2 dx + 2\left(\|g\|^2 + \vartheta^2 m\right) \int_N^\infty \|G(x,.)\|^2 dx.$$

So, from (2.5), we see that for a given $\epsilon > 0$ there exists a positive number N, depending only on ϵ such that

$$\int_N^\infty \left|\left(Ty^{(2)}\right)(x)\right|^2 dx < \epsilon^2,$$

for all $y \in Y$.

Thus $T(Y)$ is a relatively compact in H, i.e., the operator T is completely continuous. $\square$

Theorem 4.3. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let there exist constants $M > 0$ such that*

$$\left\{\int_a^c \left|w^{(1)}(x)\right|^2 dx + \gamma \int_c^\infty \left|w^{(2)}(x)\right|^2 dx\right\}^{1/2}$$

$$+\left(\int_a^c \int_a^c |G(x,t)|^2 dxdt + \gamma \int_c^\infty \int_c^\infty |G(x,t)|^2 dxdt\right)$$

$$\times \sup_{y \in S_M} \left\{\begin{array}{c} \int_a^c \left|f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right)\right|^2 dt \\ +\gamma \int_c^\infty \left|f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right)\right|^2 dt \end{array}\right\}^{1/2}$$

$$\leq M,$$

where $S_M = \{y \in H : \|y\| \leq M\}$. Then the boundary-value problem (1.1), (1.8), (1.9) has at least one solution with

$$\int_a^c \left|y^{(1)}(x)\right|^2 dx + \gamma \int_c^\infty \left|y^{(2)}(x)\right|^2 dx \leq M^2.$$

Proof. Let us define an operator $T : H \to H$ by (2.10). From theorems 6, 9 and (4.3), we conclude that T maps the set S_M into itself. It is clear that the set S_M is bounded, convex and closed. Using Theorem 8, the theorem follows. $\square$

REFERENCES

1. R. P. Agarwal and D. O'Regan: Boundary value problems of nonsingular type on the semi-infinite interval, Tohoku Math. Journal, 51, 3 (1999), 391–397.
2. R. P. Agarwal and D. O'Regan: Multiple nonnegative solutions for second order impulsive differential equations, Appl. Math. Comput., 114 (2000), 51-59.
3. D. D. Bainov and P. S. Simeonov: Impulsive Differential Equations: Periodic Solutions and Applications, Longman: Harlow, 1993.
4. D. D. Bainov and P. S. Simeonov: Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific: Singapore, 1995.
5. D. D. Bainov and P. S. Simeonov: Oscillation Theory of Impulsive Differeential Equations, International Publications: Orlando, 1998.
6. M. Benchohra, J. Henderson and S. Ntouyas: Impulsive Differential Equations and Inclusions, Hindawi Publishing Corporation: New York, 2006.
7. W. Ding and M. Han: Periodic boundary value problem for the second order impulsive functional differential equations, Appl. Math. Comput., 155, 3 (2004), 709-726.
8. N. Dunford and J. T. Schwartz: Linear Operators. Part II, Interscience, New York, 1964.
9. M. Feng and D. Xie: Multiple positive solutions of multi-point boundary value problem for second order impulsive differential equations, J. Comput. Appl. Math., 223, 1 (2009), 438–448.
10. D. Guo: Second order impulsive integro-differential equations on unbounded domains in Banach spaces, Nonlinear Analysis: Theory Meth. Appl., 35, 4 (1999), 413–423.
11. G. Sh. Guseinov and I. Yaslan: Boundary value problems for second order nonlinear differential equations on infinite intervals. J. Math. Anal. Appl., 290 (2004), 620–638.
12. S. G. Hristova and D. D. Bainov: Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations, J. Math. Anal. Appl., 197, 1 (1996), 1-13.
13. T. Jankowski: Positive solutions to second order four-point boundary value problems for impulsive differential equations, Appl. Math. Comput., 202, 2 (2008), 550–561.
14. T. Jankowski: Positive solutions of three-point boundary value problems for second order impulsive differential equations with advanced arguments, Appl. Math. Comput., 197, 1 (2008), 179–189.
15. T. Jankowski: Existence of solutions for second order impulsive differential equations with deviating arguments, Nonlinear Analysis: Theory Meth. Appl., 67, 6 (2007), 1764–1774.
16. A. N. Kolmogorov and S. V. Fomin: Introductory Real Analysis, Translated by R.A. Silverman, Dover Publications, New York, 1970.

17. M. A. Krasnosel'skii: Topological Methods in the Theory of Nonlinear Integral Equations, Gostekhteoretizdat, Moscow, 1956, English transl. Pergamon Press, New York, 1964.

18. V. Lakshmikantham D. D. Bainov and P. S. Simeonov: Theory of Impulsive Differential Equations. World Scientific: Singapore, 1989.

19. I. K. C. Leung and P. N. Shivakumar: On the eigenvalue problem $-y'' + f(x)\,y = \lambda y$ on a semi infinite interval. Math. Comput. Mod., 46 (2007), 316–330.

20. B. M. Levitan and I. S. Sargsjan: Sturm-Liouville and Dirac Operators. Mathematics and its Applications (Soviet Series, translated from the Russian), Kluwer Academic Publishers Group, Dordrecht, 1991.

21. J. Li and J. Shen: Existence of positive solution for second-order impulsive boundary value problems on infinity intervals. Bound. Value Probl., vol. 2006, Article ID 14594, (2006), 11 pages.

22. J. Li and J. J. Nieto: Existence of positive solutions for multipoint boundary value problem on the half-line with impulses, Bound. Value Probl., vol. 2009, Article ID 834158, (2009), 12 pages.

23. H. Lian and W. Ge: Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line, J. Math. Anal. Appl., 321 (2006), 781–792.

24. H. Lian, H. Pang and W. Ge: Triple positive solutions for boundary value problems on infinite intervals, Nonlinear Anal., 67 (2007) 2199–2207.

25. X. Lin and D. Jiang: Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Math. Anal. Appl., 321, 2 (2006), 501–514.

26. L. Liu and F. Y. Li: Multiple positive solution of nonlinear two-point boundary value problems, J. Math. Anal. Appl., 203 (1996), 610-625.

27. X. Liu and D. Guo: Periodic Boundary value problems for a class of second-Order impulsive integro-differential equations in Banach spaces, J. Math. Anal. Appl., 216, 1 (1997), 284-302.

28. Y. Liu: Boundary value problems for second order differential equations on unbounded domains in a Banach space, Appl. Math. Comput., 135, 2-3 (2003), 569–583.

29. R. Ma: Existence of positive solutions for second-order boundary value problems on infinity intervals. Appl. Math. Letters, 16, 1 (2003), 33–39.

30. I. Rachunkova and J. Tomecek: Impulsive BVPs with nonlinear boundary conditions for the second order differential equations without growth restrictions, J Math Anal Appl., 292, 2 (2004), 525-539.

31. A. M. Samoilenko and N. A. Perestyuk: Impulsive Differential Equations, World Scientific: Singapore, 1995.

32. E. C. Titchmarsh: Eigenfunction Expansions Associated with Second-Order Differential Equations. Part I. Second Edition, Clarendon Press, Oxford, 1962

33. Z. Wei: Periodic boundary value problems for second order impulsive integrodifferential equations of mixed type in Banach spaces, J Math Anal Appl., 195, 1 (1995), 214-229.

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FIXED-CIRCLE PROBLEM ON S-METRIC SPACES WITH A GEOMETRIC VIEWPOINT

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Abstract. Recently, a new geometric approach called the fixed-circle problem has been introduced to fixed-point theory. The problem has been studied using different techniques on metric spaces. In this paper, we consider the fixed-circle problem on S-metric spaces. We investigate existence and uniqueness conditions for fixed circles of self-mappings on an S-metric space. Some examples of self-mappings having fixed circles are also given.
Keywords: fixed-circle problem; self-mapping; S-metric space.

1. Introduction

The existence and uniqueness theorems of fixed points of self-mappings satisfying some contractive conditions have been extensively studied since the time of Stefan Banach (see [1, 2]). Many authors have investigated new fixed-point theorems on metric spaces or generalizations of metric spaces. For example, Sedghi, Shobe and Aliouche obtained Banach's contraction principle on S-metric spaces [12]. We studied some generalizations of Banach's contraction principle on an S-metric space [8] and investigated new fixed-point theorems for the following contractive condition (which is called Rhoades' condition [11]) (see [6, 14]):

$$\begin{aligned}(S25) \quad \mathcal{S}(Tx,Tx,Ty) \quad &< \quad \max\{\mathcal{S}(x,x,y),\mathcal{S}(Tx,Tx,x),\mathcal{S}(Ty,Ty,y),\\ &\qquad \mathcal{S}(Ty,Ty,x),\mathcal{S}(Tx,Tx,y)\},\end{aligned}$$

for each $x,y \in X$, $x \neq y$. We then gave the concept of diameter and obtained a new contractive condition using this notion as follows [6]:

$$(S25a) \quad \mathcal{S}(Tx,Tx,Ty) < diam\{U_x \cup U_y\},$$

for each $x,y \in X$ $(x \neq y)$, where $U_x = \{T^n x : n \in \mathbb{N}\}$, $U_y = \{T^n y : n \in \mathbb{N}\}$, $diam\{U_x\} < \infty$ and $diam\{U_y\} < \infty$.

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Although the existence of fixed points of functions has been studied on various metric spaces, there is no study on the existence of fixed circles. Therefore, the fixed-circle problem arises naturally. There are some examples of functions with a fixed circle on some special metric spaces. For example, let $\mathbb{C}$ be an S-metric space with the S-metric

$$\mathcal{S}(z,w,t)=\frac{|z-t|+|w-t|}{2},$$

for all $z,w,t\in\mathbb{C}$. Let the mapping T be defined as

$$Tz=\frac{1}{\overline{z}},$$

for all $z\in\mathbb{C}\backslash\{0\}$. The mapping T fixes the unit circle $C_{0,1}^{S}=\{x\in X:\mathcal{S}(x,x,0)=1\}$.

Recently, Özdemir, İskender and Özgür used new types of activation functions having a fixed circle for a complex valued neural network [5]. The usage of these types activation functions leads us to guarantee the existence of fixed points of the complex valued Hopfield neural network (see [5] for more details).

Hence it is important to investigate some fixed-circle theorems on various metric spaces. In [9], we obtained some fixed-circle theorems on metric spaces. We studied some existence theorems for fixed circles with a geometric interpretation and gave necessary conditions for the uniqueness of fixed circles. Also, we provided some examples of self-mappings with fixed circles. On the other hand, we proved new fixed-circle results and applied the obtained results to the discontinuity problem and discontinuous activation functions [10].

Motivated by the above studies, our aim in this paper is to obtain some fixed-circle theorems for self-mappings on S-metric spaces. In Section 2., we recall some necessary definitions, lemmas and basic facts. In Section 3., we introduce the notion of a fixed circle on an S-metric space and then obtain some existence and uniqueness theorems for self-mappings having fixed circles via different techniques. We investigate the case in which the number of fixed circles is infinitely many. Some examples of self-mappings with fixed circles are given with a geometric viewpoint. Using Mathematica (Wolfram Research, Inc., Mathematica, Trial Version, Champaign, IL (2016)), we draw some figures related to the given examples.

2. Preliminaries

Definition 2.1. [12] Let X be a nonempty set and $\mathcal{S}:X^3\to[0,\infty)$ be a function satisfying the following conditions for all $x,y,z,a\in X$.

1. $\mathcal{S}(x,y,z)=0$ if and only if $x=y=z$,
2. $\mathcal{S}(x,y,z)\leq\mathcal{S}(x,x,a)+\mathcal{S}(y,y,a)+\mathcal{S}(z,z,a)$.

Then S is called an S-metric on X and the pair $(X, \mathcal{S})$ is called an S-metric space.

The following lemma can be considered as the symmetry condition and it will be used in the proofs of some theorems.

Lemma 2.1. *[12] Let $(X, \mathcal{S})$ be an S-metric space. Then we have*

$$\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x).$$

The relationships between a metric and an S-metric was given in what follows.

Lemma 2.2. *[4] Let (X, d) be a metric space. Then the following properties are satisfied:*

1. *$\mathcal{S}_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in $(X, \mathcal{S}_d)$.*
3. *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in $(X, \mathcal{S}_d)$.*
4. *(X, d) is complete if and only if $(X, \mathcal{S}_d)$ is complete.*

The metric $\mathcal{S}_d$ was called an S-metric generated by d [7]. We know some examples of an S-metric which are not generated by any metric (see [4, 7, 14] for more details).

On the other hand, Gupta claimed that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x, y) = S(x, x, y) + S(y, y, x), \tag{2.1}$$

for all $x, y \in X$ [3]. However, the function $d_S(x, y)$ defined in (2.1) does not always define a metric because the triangle inequality is not satisfied for all elements of X everywhere (see [7] for more details).

The notions of an open ball, a closed ball and diameter were introduced on S-metric spaces as the following definitions.

Definition 2.2. [12] Let $(X, \mathcal{S})$ be an S-metric space. The open ball $B_S(x_0, r)$ and closed ball $B_S[x_0, r]$ with a center x_0 and a radius r are defined by

$$B_S(x_0, r) = \{x \in X : \mathcal{S}(x, x, x_0) < r\}$$

and

$$B_S[x_0, r] = \{x \in X : \mathcal{S}(x, x, x_0) \leq r\},$$

for $r > 0$ and $x_0 \in X$.

Definition 2.3. [6] Let $(X, \mathcal{S})$ be an S-metric space and A be a nonempty subset of X. The diameter of A is defined by

$$diam\{A\} = sup\{\mathcal{S}(x, x, y) : x, y \in A\}.$$

If A is S-bounded, then we will write $diam\{A\} < \infty$.

Now we define the notion of a circle on an S-metric space.

Definition 2.4. Let $(X, \mathcal{S})$ be an S-metric space and $x_0 \in X$, $r \in (0, \infty)$. We define the circle centered at x_0 with the radius r as

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = r\}.$$

3. Some Fixed-Circle Theorems on S-Metric Spaces

In this section, we introduce the notion of a fixed circle on an S-metric space. Then we investigate some existence and uniqueness theorems for self-mappings having fixed circles.

Definition 3.1. Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be a circle on X and $T : X \to X$ be a self-mapping. If $Tx = x$ for all $x \in C^S_{x_0,r}$ then the circle $C^S_{x_0,r}$ is said to be a fixed circle of T.

3.1. The existence of fixed circles

We obtain some existence theorems for fixed circles of self-mappings.

Theorem 3.1. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let us define the mapping*

$$\varphi : X \to [0, \infty), \varphi(x) = \mathcal{S}(x, x, x_0), \tag{3.1}$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

$$\mathcal{S}(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r \tag{3.2}$$

and

$$\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) \leq r, \tag{3.3}$$

for all $x \in C^S_{x_0,r}$, then $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$. Then using the conditions (3.2), (3.3), Lemma 2.1 and the triangle inequality, we get

$$\begin{array}{rcl} \mathcal{S}(x,x,Tx) & \leq & \varphi(x)+\varphi(Tx)-2r \\ & = & \mathcal{S}(x,x,x_0)+\mathcal{S}(Tx,Tx,x_0)-2r \\ & \leq & \mathcal{S}(x,x,Tx)+\mathcal{S}(x,x,Tx)+\mathcal{S}(Tx,Tx,x_0)+\mathcal{S}(Tx,Tx,x_0)-2r \\ & = & 2\mathcal{S}(x,x,Tx)+2\mathcal{S}(Tx,Tx,x_0)-2r \\ & \leq & 2r-2r=0 \end{array}$$

and so

$$\mathcal{S}(x,x,Tx)=0,$$

which implies $Tx = x$. Consequently, $C^S_{x_0,r}$ is a fixed circle of T. □

Remark 3.1. 1) Notice that the condition (3.2) guarantees that Tx is not in the interior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Similarly, the condition (3.3) guarantees that Tx is not the exterior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Hence $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$ and so we get $T(C^S_{x_0,r}) \subset C^S_{x_0,r}$.

2) If an S-metric is generated by any metric d, then Theorem 3.1 can be used on the corresponding metric space.

3) The converse statement of Theorem 3.1 is also true.

Now we give an example of a self-mapping with a fixed circle.

Example 3.1. Let $X = \mathbb{R}$ and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z)=|x-z|+|y-z|,$$

for all $x, y, z \in \mathbb{R}$ [13]. Then $(X,\mathcal{S})$ is called the usual S-metric space. This S-metric is generated by the usual metric on $\mathbb{R}$. Let us consider the circle $C^S_{0,2}$ and define the self-mapping $T_1 : \mathbb{R} \to \mathbb{R}$ as

$$T_1x=\left\{\begin{array}{ccc} x & if & x\in\{-1,1\} \\ 10 & & otherwise \end{array}\right. ,$$

for all $x \in \mathbb{R}$. Then the self-mapping T_1 satisfies the conditions (3.2) and (3.3). Hence $C^S_{0,2} = \{-1,1\}$ is a fixed circle of T_1.

Notice that $C^S_{\frac{9}{2},11} = \{-1,10\}$ is another fixed circle of T_1 and so the fixed circle is not unique for a giving self-mapping.

On the other hand, if we consider the usual metric d on $\mathbb{R}$ then we obtain $C_{0,2} = \{-2,2\}$. The circle $C_{0,2}$ is not a fixed circle of T_1.

Example 3.2. Let $X = \mathbb{R}^2$ and let the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z)=\sum_{i=1}^{2}\left(|x_i-z_i|+|x_i+z_i-2y_i|\right),$$

for all $x=(x_1,x_2)$, $y=(y_1,y_2)$ and $z=(z_1,z_2)$. Then it can be easily seen that $\mathcal{S}$ is an S-metric on $\mathbb{R}^2$, which is not generated by any metric, and the pair $\left(\mathbb{R}^2,\mathcal{S}\right)$ is an S-metric space.

Let us consider the unit circle $C_{0,1}^S$ and define the self-mapping $T_2:\mathbb{R}\to\mathbb{R}$ as

$$T_2x=\begin{cases} x & if \quad x\in C_{0,1}^S \\ (1,0) & \quad otherwise \end{cases},$$

for all $x\in\mathbb{R}^2$. Then the self-mapping T_2 satisfies the conditions (3.2) and (3.3). Therefore $C_{0,1}^S$ is a fixed circle of T_2 as shown in Figure 3.1.

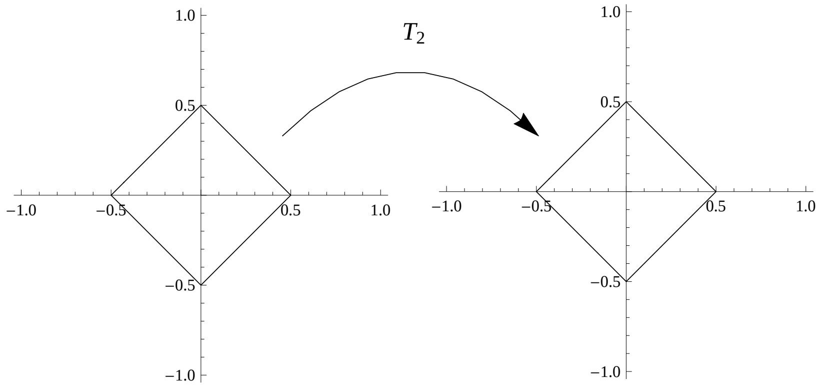

FIG. 3.1: The fixed circle of T_2.

In the following example, we give an example of a self-mapping which satisfies the condition (3.2) and does not satisfy the condition (3.3).

Example 3.3. Let $X=\mathbb{R}$ and the function $\mathcal{S}:X^3\to[0,\infty)$ be defined by

$$\mathcal{S}(x,y,z)=|x-z|+|x+z-2y|\,,$$

for all $x,y,z\in\mathbb{R}$ [7]. Then $\mathcal{S}$ is an S-metric which is not generated by any metric and $(X,\mathcal{S})$ is an S-metric space. Let us consider the circle $C_{0,3}^S$ and define the self-mapping $T_3:\mathbb{R}\to\mathbb{R}$ as

$$T_3x=\begin{cases} -\frac{7}{2} & if \quad x=-\frac{3}{2} \\ \frac{7}{2} & if \quad x=\frac{3}{2} \\ 7 & \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_3 satisfies the condition (3.2) but does not satisfy the condition (3.3). Clearly T_3 does not fix the circle $C^S_{0,3}$.

In the following example, we give an example of a self-mapping which satisfies the condition (3.3) and does not satisfy the condition (3.2).

Example 3.4. Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be a circle on X and the self-mapping $T_4 : X \to X$ be defined as

$$T_4x = x_0,$$

for all $x \in X$. Then the self-mapping T_4 satisfies the condition (3.3) but does not satisfy the condition (3.2). Clearly T_4 does not fix the circle $C^S_{x_0,r}$.

Now we give another existence theorem for fixed circles.

Theorem 3.2. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let the mapping φ be defined as (3.1). If there exists a self-mapping $T : X \to X$ satisfying*

$$\mathcal{S}(x, x, Tx) \leq \varphi(x) - \varphi(Tx) \tag{3.4}$$

and

$$h\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) \geq r, \tag{3.5}$$

for all $x \in C^S_{x_0,r}$ and some $h \in [0, 1)$, then $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$. On the contrary, assume that $x \neq Tx$. Then using the conditions (3.4) and (3.5), we obtain

$$\begin{aligned}
\mathcal{S}(x, x, Tx) &\leq \varphi(x) - \varphi(Tx) \\
&= \mathcal{S}(x, x, x_0) - \mathcal{S}(Tx, Tx, x_0) \\
&= r - \mathcal{S}(Tx, Tx, x_0) \\
&\leq h\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) - \mathcal{S}(Tx, Tx, x_0) \\
&= h\mathcal{S}(x, x, Tx),
\end{aligned}$$

which is a contradiction since $h \in [0, 1)$. Hence we get $Tx = x$ and $C^S_{x_0,r}$ is a fixed circle of T. □

Remark 3.2. 1) Notice that the condition (3.4) guarantees that Tx is not in the exterior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Similarly, the condition (3.5) shows that Tx can lie on either the exterior or the interior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Hence Tx should lie on the interior of the circle $C^S_{x_0,r}$.

2) If an S-metric is generated by any metric d, then Theorem 3.2 can be used on the corresponding metric space.

3) The converse statement of Theorem 3.2 is also true.

Now we give some examples of self-mappings which have a fixed-circle.

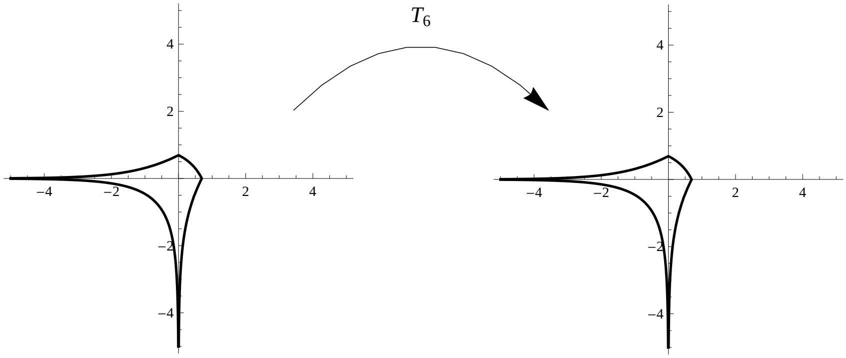

FIG. 3.2: The fixed circle of T_6.

Example 3.5. Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be the usual S-metric space. Let us consider the circle $C_{1,2}^S = \{0, 2\}$ and define the self-mapping $T_5 : \mathbb{R} \to \mathbb{R}$ as

$$T_5 x = \begin{cases} e^x - 1 & if \quad x = 0 \\ 2x - 2 & if \quad x = 2 \\ 3 & \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_5 satisfies the conditions (3.4) and (3.5). Hence $C_{1,2}^S$ is a fixed circle of T_5.

On the other hand, if we consider the usual metric d on $\mathbb{R}$ then we have $C_{1,2} = \{-1, 3\}$. The circle $C_{1,2}$ is not a fixed circle of T_5. But $C_{1,1} = \{0, 2\}$ is a fixed circle of T_5 on (X, d).

Example 3.6. Let $X = \mathbb{R}^2$ and let the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = \sum_{i=1}^{2} \left(|e^{x_i} - e^{z_i}| + |e^{x_i} + e^{z_i} - 2e^{y_i}| \right),$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then it can be easily checked that $\mathcal{S}$ is an S-metric on $\mathbb{R}^2$, which is not generated by any metric, and the pair $\left(\mathbb{R}^2, \mathcal{S}\right)$ is an S-metric space.

Let us consider the circle $C_{x_0,r}^S$ centered at $x_0 = (0, 0)$ with the radius $r = 2$ and define the self-mapping $T_6 : \mathbb{R} \to \mathbb{R}$ as

$$T_6 x = \begin{cases} x & if \quad x \in C_{0,2}^S \\ (\ln 2, 0) & \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}^2$. Then the self-mapping T_6 satisfies the conditions (3.4) and (3.5). Therefore $C^S_{0,2}$ is the fixed circle of T_6 as shown in Figure 3.2.

In the following example, we give an example of a self-mapping which satisfies the condition (3.4) and does not satisfy the condition (3.5).

Example 3.7. Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be a circle on X. If we consider the self-mapping $T_4x = x_0$, then the self-mapping T_4 satisfies the condition (3.4) but does not satisfy the condition (3.5). It can be easily seen that T_4 does not fix a circle $C^S_{x_0,r}$.

In the following example, we give an example of a self-mapping which satisfies the condition (3.5) and does not satisfy the condition (3.4).

Example 3.8. Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be an S-metric space with an S-metric defined as in Example 3.3. Let us consider the unit circle $C^S_{0,1}$ and define the self-mapping $T_7 : \mathbb{R} \to \mathbb{R}$ as

$$T_7x = 1,$$

for all $x \in \mathbb{R}$. Then the self-mapping T_7 satisfies the condition (3.5) but does not satisfy the condition (3.4). It can be easily shown that T_7 does not fix the unit circle $C^S_{0,1}$.

Let $I_X : X \to X$ be the identity map defined as $I_X(x) = x$ for all $x \in X$. Notice that the identity map satisfies the conditions (3.2) and (3.3) (resp. (3.4) and (3.5)) in Theorem 3.1 (resp. Theorem 3.2) for any circle. Now we determine a condition which excludes the I_X from Theorem 3.1 and Theorem 3.2. For this purpose, we give the following theorem.

Theorem 3.3. *Let $(X, \mathcal{S})$ be an S-metric space, $T : X \to X$ be a self mapping having a fixed circle $C^S_{x_0,r}$ and the mapping φ be defined as (3.1). The self-mapping T satisfies the condition*

$$(I_S) \quad \mathcal{S}(x, x, Tx) \leq \frac{\varphi(x) - \varphi(Tx)}{h},$$

for all $x \in X$ and some $h > 2$ if and only if $T = I_X$.

Proof. Let $x \in X$ be an arbitrary element. Then using the inequality (I_S), Lemma 2.1 and triangle inequality, we obtain

$$\begin{aligned} h\mathcal{S}(x, x, Tx) &\leq \varphi(x) - \varphi(Tx) \\ &= \mathcal{S}(x, x, x_0) - \mathcal{S}(Tx, Tx, x_0) \\ &\leq 2\mathcal{S}(x, x, Tx) + \mathcal{S}(Tx, Tx, x_0) - \mathcal{S}(Tx, Tx, x_0) \\ &= 2\mathcal{S}(x, x, Tx) \end{aligned}$$

and so

$$(h - 2)\mathcal{S}(x, x, Tx) \leq 0.$$

Since $h > 2$ it should be $\mathcal{S}(x, x, Tx) = 0$ and so $Tx = x$. Consequently, we obtain $T = I_X$.

Conversely, it is clear that the identity map I_X satisfies the condition (I_S). $\square$

Remark 3.3. 1) If a self-mapping T, which has a fixed circle, satisfies the conditions (3.2) and (3.3) (resp. (3.4) and (3.5)) in Theorem 3.1 (resp. Theorem 3.2) but does not satisfy the condition (I_S) in Theorem 3.3 then the self-mapping T cannot be an identity map.

2) If an S-metric is generated by any metric d, then Theorem 3.3 can be used on the corresponding metric space.

3.2. The uniqueness of fixed circles

We investigate the uniqueness conditions of fixed circles given in the existence theorems. For any given circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ on X, we notice that there exists at least one self-mapping T of X such that T fixes the circles $C^S_{x_0,r}$, $C^S_{x_1,\rho}$. Indeed let us define the mappings $\varphi_1, \varphi_2 : X \to [0, \infty)$ as

$$\varphi_1(x) = \mathcal{S}(x, x, x_0)$$

and

$$\varphi_2(x) = \mathcal{S}(x, x, x_1),$$

for all $x \in X$. If we define the self-mapping $T_8 : X \to X$ as

$$T_8 x = \begin{cases} x & if \quad x \in C^S_{x_0,r} \cup C^S_{x_1,\rho} \\ \alpha & \qquad otherwise \end{cases},$$

for all $x \in X$, where α is a constant satisfying $S(\alpha, \alpha, x_0) \neq r$ and $S(\alpha, \alpha, x_1) \neq \rho$, it can be easily seen that the self-mapping $T_8 : X \to X$ satisfies the conditions (3.2) and (3.3) in Theorem 3.1 (resp. (3.4) and (3.5) in Theorem 3.2) for the circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ using the mappings φ_1 and φ_2, respectively. Hence T_8 fixes both of the circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. In this way, the number of fixed circles can be extended to any positive integer n using the same arguments.

In the following example, the self-mapping T_9 has two fixed circle.

Example 3.9. Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be an S-metric space with the S-metric defined in Example 3.3. Let us consider the circles $C^S_{0,2}$, $C^S_{0,4}$ and define the self-mapping $T_9 : \mathbb{R} \to \mathbb{R}$ as

$$T_9 x = \begin{cases} x & if \quad x \in \{-2, -1, 1, 2\} \\ \alpha & \qquad otherwise \end{cases},$$

for all $x \in X$ where $\alpha \in X$. Then the conditions (3.2) and (3.3) are satisfied by T_9 for the circles $C^S_{0,2}$ and $C^S_{0,4}$, respectively. Consequently, $C^S_{0,2}$ and $C^S_{0,4}$ are the fixed circles of T_9.

Now we investigate the uniqueness conditions for the fixed circles in Theorem 3.1 using Rhoades' contractive condition on S-metric spaces.

Theorem 3.4. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping satisfying the conditions (3.2) and (3.3) given in Theorem 3.1. If the contractive condition*

$$\begin{aligned} \mathcal{S}(Tx, Tx, Ty) < \max\{&\mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Ty, Ty, y), \\ &\mathcal{S}(Ty, Ty, x), \mathcal{S}(Tx, Tx, y)\}, \end{aligned} \tag{3.6}$$

is satisfied for all $x \in C^S_{x_0,r}$, $y \in X \setminus C^S_{x_0,r}$ by T, then $C^S_{x_0,r}$ is a unique fixed circle of T.

Proof. Suppose that there exist two fixed circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions (3.2) and (3.3) for each circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_1,\rho}$ be arbitrary points with $x \neq y$. Using the contractive condition (3.6), we obtain

$$\begin{aligned} \mathcal{S}(x, x, y) &= \mathcal{S}(Tx, Tx, Ty) < \max\{\mathcal{S}(x, x, y), \mathcal{S}(Tx, Tx, x), \mathcal{S}(Ty, Ty, y), \\ &\quad \mathcal{S}(Ty, Ty, x), \mathcal{S}(Tx, Tx, y)\} \\ &= \mathcal{S}(x, x, y), \end{aligned}$$

which is a contradiction. Hence it should be $x = y$. Consequently, $C^S_{x_0,r}$ is the unique fixed circle of T. □

The following example shows that the circle $C^S_{x_0,r}$ is not necessarily unique in Theorem 3.2.

Example 3.10. Let $(X, \mathcal{S})$ be an S-metric space and $C_{x_1,r_1}, \cdots, C_{x_n,r_n}$ be any circles on X. Let us define the self-mapping $T_{10} : X \to X$ as

$$T_{10}x = \begin{cases} x & if \quad x \in \bigcup\limits_{i=1}^{n} C_{x_i,r_i} \\ x_0 & \qquad otherwise \end{cases},$$

for all $x \in X$, where x_0 is a constant in X. Then it can be easily checked that the conditions (3.4) and (3.5) are satisfied by T_{10} for the circles $C_{x_1,r_1}, \cdots, C_{x_n,r_n}$, respectively. Consequently, the circles $C_{x_1,r_1}, \cdots, C_{x_n,r_n}$ are fixed circles of T_{10}. Notice that these circles do not have to be disjoint.

Now we give the following uniqueness theorem for the fixed circles in Theorem 3.2 using the notion of diameter on S-metric spaces.

Theorem 3.5. *Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be any circle on X, $U_x = \{T^n x : n \in \mathbb{N}\}$, $U_y = \{T^n y : n \in \mathbb{N}\}$, $diam\{U_x\} < \infty$ and $diam\{U_y\} < \infty$. Let*

$T : X \to X$ be a self-mapping satisfying the conditions (3.4) and (3.5) given in Theorem 3.2. If the contractive condition

$$\mathcal{S}(Tx,Tx,Ty) < diam\{U_x \cup U_y\}, \tag{3.7}$$

is satisfied for all $x \in C^S_{x_0,r}$, $y \in X \setminus C^S_{x_0,r}$ by T, then $C^S_{x_0,r}$ is the unique fixed circle of T.

Proof. Assume that there exist two fixed circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$ of the self-mapping T, that is, T satisfies the conditions (3.4) and (3.5) for each circles $C^S_{x_0,r}$ and $C^S_{x_1,\rho}$. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_1,\rho}$ be arbitrary points with $x \neq y$. Using the contractive condition (3.7), we obtain

$$\mathcal{S}(x,x,y) = \mathcal{S}(Tx,Tx,Ty) < diam\{U_x \cup U_y\} = \mathcal{S}(x,x,y),$$

which is a contradiction. Hence it should be $x = y$. Consequently, $C^S_{x_0,r}$ is the unique fixed circle of T. $\square$

3.3. Infinity of fixed circles

We give a new approach to obtain fixed-circle results. To do this, let us denote by $R_S(x,y)$ the right side of the inequality (S25). Using the number $R_S(x,y)$, we obtain the following theorem. This theorem generates many (finite or infinite) fixed circles for a given self-mapping.

Theorem 3.6. *Let $(X,\mathcal{S})$ be an S-metric space, $T : X \to X$ be a self-mapping and $r = \inf\{\mathcal{S}(Tx,Tx,x) : Tx \neq x\}$. If there exists a point $x_0 \in X$ satisfying*

$$\mathcal{S}(x,x,Tx) < R_S(x,x_0) \tag{3.8}$$

for all $x \in X$ when $\mathcal{S}(Tx,Tx,x) > 0$ and

$$\mathcal{S}(Tx,Tx,x_0) = r \tag{3.9}$$

for all $x \in C^S_{x_0,r}$, then $C^S_{x_0,r}$ is a fixed circle of T. The self-mapping T also fixes the closed ball $B_S[x_0,r]$.

Proof. Let $x \in C^S_{x_0,r}$ and $Tx \neq x$. Then using the inequality (3.8) and Lemma 2.1, we get

$$\begin{aligned} &\mathcal{S}(x,x,Tx) < R_S(x,x_0) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(x,x,x_0), \mathcal{S}(Tx,Tx,x), \mathcal{S}(Tx_0,Tx_0,x_0), \\ \mathcal{S}(Tx_0,Tx_0,x), \mathcal{S}(Tx,Tx,x_0) \end{array} \right\}. \end{aligned} \tag{3.10}$$

At first, using the inequality (3.10) and Lemma 2.1, we show $Tx_0 = x_0$. Suppose that $Tx_0 \neq x_0$. For $x = x_0$, we obtain

$$\begin{aligned} \mathcal{S}(x_0,x_0,Tx_0) &< R_S(x_0,x_0) \\ &= \max\left\{ \begin{array}{c} \mathcal{S}(x_0,x_0,x_0), \mathcal{S}(Tx_0,Tx_0,x_0), \mathcal{S}(Tx_0,Tx_0,x_0), \\ \mathcal{S}(Tx_0,Tx_0,x_0), \mathcal{S}(Tx_0,Tx_0,x_0) \end{array} \right\} \\ &= \mathcal{S}(Tx_0,Tx_0,x_0) = \mathcal{S}(x_0,x_0,Tx_0), \end{aligned}$$

a contradiction. It should be $Tx_0 = x_0$. Then by the inequality (3.10), the condition (3.9), definition of r and Lemma 2.1, we have

$$\begin{aligned} \mathcal{S}(x,x,Tx) &< \max\left\{ \begin{array}{c} \mathcal{S}(x,x,x_0), \mathcal{S}(Tx,Tx,x), \mathcal{S}(x_0,x_0,x_0), \\ \mathcal{S}(x_0,x_0,x), \mathcal{S}(Tx,Tx,x_0) \end{array} \right\} \\ &= \max\{r, \mathcal{S}(Tx,Tx,x)\} = \mathcal{S}(Tx,Tx,x) = \mathcal{S}(x,x,Tx), \end{aligned}$$

a contradiction. Therefore we get $Tx = x$, that is, $C^S_{x_0,r}$ is a fixed circle of T.

Finally we prove that T fixes the closed ball $B_S[x_0, r]$. To do this, we show that T fixes any circle $C^S_{x_0,\rho}$ with $\rho < r$. Let $x \in C^S_{x_0,\rho}$ and $Tx \neq x$. From the similar arguments used in the above, we have $Tx = x$. □

We give the following example.

Example 3.11. Let $X = \mathbb{R}$ be the usual S-metric space. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \left\{ \begin{array}{ccc} x & if & |x| < 3 \\ x+2 & if & |x| \geq 3 \end{array} \right.,$$

for all $x \in \mathbb{R}$. The self-mapping T satisfies the conditions of Theorem 3.6 with $x_0 = 0$. Indeed, we get

$$\mathcal{S}(x,x,Tx) = 2|x - Tx| = 4 > 0,$$

for all $x \in \mathbb{R}$ such that $|x| \geq 3$. Then we have

$$\begin{aligned} R_S(x,0) &= \max\{\mathcal{S}(x,x,0), \mathcal{S}(Tx,Tx,x), \mathcal{S}(0,0,0), \mathcal{S}(0,0,x), \mathcal{S}(Tx,Tx,0)\} \\ &= \max\{2|x|, 4, 0, 2|x|, 2|x+2|\} \\ &= \max\{2|x|, 2|x+2|\} \end{aligned}$$

and so

$$\mathcal{S}(x,x,Tx) < R_S(x,0).$$

Therefore the condition (3.8) is satisfied. We also obtain

$$r = \min\{\mathcal{S}(Tx,Tx,x) : Tx \neq x\} = 4.$$

It can be easily seen that the condition (3.9) is satisfied by T. Consequently, T fixes the circle $C^S_{0,4} = \{x \in \mathbb{R} : |x| = 2\}$ and the closed ball $B_S[0,4] = \{x \in \mathbb{R} : |x| \leq 2\}$.

Remark 3.4. 1) Notice that the condition (3.9) guarantees that $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$ and so $T(C^S_{x_0,r}) \subset C^S_{x_0,r}$.

2) The self-mapping T defined in Example 3.11 has other fixed circles. Theorem 3.6 gives us some of these circles.

3) A self-mapping T can fix infinitely many circles (see Example 3.11).

The converse statement is not always true as seen in the following example.

Example 3.12. Let $x_0 \in X$ be any point. If we define the self-mapping $T : X \to X$ as

$$Tx = \left\{ \begin{array}{ccc} x & if & x \in B_S[x_0,\mu] \\ x_0 & if & x \notin B_S[x_0,\mu] \end{array} \right.,$$

for all $x \in X$ with $\mu > 0$, then T does not satisfies the condition (3.8), but T fixes every circle $C^S_{x_0,\rho}$ with $\rho \leq \mu$.

REFERENCES

1. S. Banach: *Sur les operations dans les ensembles abstraits et leur application aux equations integrals*. Fund. Math. **2** (1922), 133–181.
2. K. Ciesielski: *On Stefan Banach and some of his results*. Banach J. Math. Anal. **1 (1)** (2007), 1–10.
3. A. Gupta: *Cyclic contraction on S-metric space*. Int. J. Anal. Appl. **3 (2)** (2013), 119–130.
4. N. T. Hieu, N. T. Ly and N. V. Dung: *A Generalization of Ciric Quasi-Contractions for Maps on S-Metric Spaces*. Thai J. Math. **13 (2)** (2015), 369–380.
5. N. Özdemir, B. B. İskender and N. Y. Özgür: *Complex valued neural network with Möbius activation function*. Commun. Nonlinear Sci. Numer. Simul. **16 (12)** (2011), 4698–4703.
6. N. Y. Özgür and N. Taş: *Some fixed point theorems on S-metric spaces*. Mat. Vesnik **69 (1)** (2017), 39–52.
7. N. Y. Özgür and N. Taş: *Some new contractive mappings on S-metric spaces and their relationships with the mapping* ($S25$). Math. Sci. **11 (1)** (2017), 7–16.
8. N. Y. Özgür and N. Taş: *Some generalizations of fixed point theorems on S-metric spaces*. Essays in Mathematics and Its Applications in Honor of Vladimir Arnold, New York, Springer, 2016.
9. N. Y. Özgür and N. Taş: *Some fixed-circle theorems on metric spaces*. Bull. Malays. Math. Sci. Soc. (2017). https://doi.org/10.1007/s40840-017-0555-z
10. N. Y. Özgür and N. Taş: *Some fixed-circle theorems and discontinuity at fixed circle*. AIP Conference Proceedings **1926**, 020048 (2018).
11. B. E. Rhoades: *A comparison of various definitions of contractive mappings*. Trans. Amer. Math. Soc. **226** (1977), 257–290.
12. S. Sedghi, N. Shobe and A. Aliouche: *A generalization of fixed point theorems in S-metric spaces*: Mat. Vesnik **64 (3)** (2012), 258–266.
13. S. Sedghi and N. V. Dung: *Fixed point theorems on S-metric spaces*. Mat. Vesnik **66 (1)** (2014), 113–124.
14. N. Taş: *Fixed point theorems and their various applications*. Ph. D. Thesis, University of Balıkesir, Balıkesir, 2017.

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THE CLASSICAL BERNOULLI-EULER ELASTIC CURVE IN A MANIFOLD

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Abstract. In this study, we describe the classical Bernoulli-Euler elastic curve in a manifold by the property that the velocity vector field of the curve is harmonic. Then, a condition is obtained for the elastic curve in a manifold. Finally, we give an example which provides the condition mentioned in this paper and illustrate it with a figure.
Keywords: Energy; energy of a unit vector field; elastic curve.

1. Introduction

The history of the elastica or the elastic curve is very old and many researchers have worked on this issue, for example [6, 11]. One can study a bent thin rod and consider the energy it stores. The classical Euler-Bernoulli model assigns a numerical value to this energy, which is proportional to $\int_0^s k^2(u)du$. The elastica is the critical point for this total squared curvature functional on regular curves with given boundary conditions [8].

In [1] the author calculated the energy of the Frenet vector fields in R^n, showing that the energy of the velocity vector field was $\mathcal{E}(V_1(s)) = \frac{1}{2}\int_a^s k_1^2(u)du$. By means of this result, we have seen that the speed vector field of the Bernoulli-Euler elastic curve is harmonic.

In this paper, using the above result, we give a condition for elastica on a manifold.

Definition 1.1. Let (M, g) be a Riemann manifold and $\alpha : I \to M$, be a unit speed curve.

If $\{E_i\}_{i=1}^r$ is an orthonormal frame along α and

$$E_1 = \frac{d\alpha}{ds},$$

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$$\nabla^{\alpha}_{\frac{\partial}{\partial s}} E_1 = k_1 E_2,$$
$$\nabla^{\alpha}_{\frac{\partial}{\partial s}} E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad \forall i = 2, ..., r-1$$
$$\nabla^{\alpha}_{\frac{\partial}{\partial s}} E_r = -k_{r-1} E_{r-1},$$

where $k_1, ..., k_{r-1}$ are positive functions with a real value on I, then α is said to be an r-th order Frenet curve. These functions are called the curvature functions of the curve α.

Proposition 1.1. The connection map $K : T(T^1M) \to T^1M$ verifies the following conditions.

1) $\pi \circ K = \pi \circ d\pi$ and $\pi \circ K = \pi \circ \widetilde{\pi}$, where $\widetilde{\pi} : T(T^1M) \to T^1M$ is the tangent bundle projection.

2) For $\omega \in T_xM$ and a section $\xi : M \to T^1M$, we have

$$K(d\xi(\omega)) = \nabla_\omega \xi$$

where T^1M is the unit tangent bundle and ∇ is the Levi-Civita covariant derivative [3].

Definition 1.2. For $\eta_1, \eta_2 \in T_\xi(T^1M)$, we define

$$g_{\mathcal{S}}(\eta_1, \eta_2) = < d\pi(\eta_1), d\pi(\eta_2) > + < K(\eta_1), K(\eta_2) > . \tag{1.1}$$

This gives a Riemannian metric on tangent bundle TM. As mentioned, $g_{\mathcal{S}}$ is called *the Sasaki metric*. The metric g_s makes the projection $\pi : T^1M \to M$ a Riemannian submersion [3, 10].

Definition 1.3. Let $f : (M, <,>) \to (N, h)$ be a differentiable map between Riemannian manifolds. The energy of f is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_M (\sum_{a=1}^{n} h(df(e_a), df(e_a))\upsilon \tag{1.2}$$

where υ is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (see [12, 4], for example).

By a (smooth) variation of f we mean a smooth map $f : M \times (-\epsilon, \epsilon) \to N$, $(x, t) \to f_t(x)$ $(\epsilon > 0)$ such that $f_0 = f$. We can think of $\{f_t\}$ as a family of smooth mappings which depend 'smoothly' on a parameter $t \in (-\epsilon, \epsilon)$.

Definition 1.4. A smooth map $f : (M, g) \to (N, h)$ is said to be harmonic if

$$\frac{d}{dt} \mathcal{E}(f_t; D)|_{t=0} = o$$

where $\mathcal{E}(f; D) = \frac{1}{2} \int_D (\sum_{a=1}^{n} h(df(e_a), df(e_a))\upsilon_g$, for all compact domains D and all smooth variations f_t of f supported in D, [2].

Definition 1.5. Let $\alpha : [a,b] \to R^n$ be a regular curve. Elastica is defined for the curve α over the each point on a fixed interval $[a,b]$ as a minimizer of the bending energy:

$$\mathcal{E}_B = \frac{1}{2}\int_a^b k_1^2(s)ds, \tag{1.3}$$

with some boundary conditions [5, 7].

The right side of Equation (1.3) is the energy of the velocity vector field according to [1]. By combining this resultant with the definition 1.4 we can give the following definition

2. Elastica in a Manifold

Definition 2.1. A curve on a manifold is called a classical Bernoulli-Euler elastic curve if the velocity vector field of the curve is harmonic.

Theorem 2.1. Let M be a Riemann manifold, α be r-th order Frenet curve in M and $\alpha(a) = p,\ \ \alpha(b) = q$. If α is classical elastic curve, then the following equation is satisfied,

$$\int_a^b \lambda(s)k_1(s)k_1^{'}(s)ds = 0 \tag{2.1}$$

where k_1 is the 1^{th} curvature function and λ is the real-valued function on $[a,b]$.

Proof . Let $\alpha : I \to M$ be the r-th order Frenet curve C on $\varphi(U) \subset M$ and $\alpha = \varphi \circ \gamma,\ \gamma = (\gamma_1, ..., \gamma_m), \gamma : I \to U \subset R^m; \varphi : U \to M$. Let $(\{E_i\}_{i=1}^r)$ be the Frenet frame field on α.

We define the λ and v_i functions to create a curve family between two fixed points on the manifold. The functions are: $\lambda : [a,b] \subset I \to R,\ \lambda(s) = (s-a)(b-s)$, $\lambda(a) = 0,\ \lambda(b) = 0$ and $\lambda(s) \neq 0$ for all $s \in (a,b)$, of class C^2 and

$$\lambda(s)\ E_1(s) = (v_1(s), v_2(s), ..., v_n(s)).\ \ v_i : [a,b] \to R.$$

Since $\{\varphi_1(\gamma(s)), ..., \varphi_m(\gamma(s))\}$ is a local basis of the tangent space, where $\varphi_1, ..., \varphi_m$ are first-order partial derivatives, we have

$$\lambda(s)E_1(s) = \Sigma_{i=1}^m v_i(s)\varphi_i(\gamma(s));\quad where\ \ v_i : [a,b] \to R. \tag{2.2}$$

Let the collection of the curve be

$$\alpha^t(s) = \varphi(\gamma_1(s) + tv_1(s), ..., \gamma_m(s) + tv_m(s)), \tag{2.3}$$

for $t=0$, $\alpha^0(s)=\alpha(s)$ and

$$(\varphi^{-1}\circ\alpha^t)(s)=\gamma^t(s)=(\gamma_1(s)+tv_1(s),...,\gamma_m(s)+tv_m(s)).$$

From (2.2) we get $\lambda(a)E_1(a)=\Sigma_{i=1}^m v_i(a)\varphi_i(\gamma(a))$. Since $\lambda(a)=0$ we have $v_i(a)=0$ and

$$\gamma^t(a)=(\gamma_1(a)+tv_1(a),...,\gamma_m(a)+tv_m(a)=(\gamma_1(a),...,\gamma_m(a))=\gamma(a).$$

Similarly, we get $\gamma^t(b)=\gamma(b)$. Using these results in (2.3) we obtain

$\alpha^t(a)=(\varphi\circ\gamma^t)(a)=\alpha(a)=p$ and $\alpha^t(b)=(\varphi\circ\gamma^t)(b)=\alpha(b)=q$.

These results show that α^t is a curve segment from p to q on M. Take this collection $\alpha^t(s)=\alpha(s,t)$ for all curves. The expression for the energy of the velocity vector field E_{1t} of α^t from p to q on M becomes $\mathcal{E}(E_{1t})$.

Let TC_t be the tangent bundle. So we have $E_{1_t}:C_t\to TC_t$, where $TC_t=\cup_{j\in I}T_{\alpha^t(j)}C_t$, $C_t=\alpha^t(I)$ and $T_{\alpha^t(j)}C_t$ is the straight line through the point $\alpha^t(j)$ in the E_{1_t} direction. Let $\pi:TC_t\to C_t$ be the bundle projection. By using Equation (1.2) we calculate the energy of E_{1_t} as

$$\mathcal{E}(E_{1_t})=\frac{1}{2}\int_a^b g_{\mathcal{S}}(dE_{1_t}(E_{1_t}(\alpha(s,t)),dE_{1_t}(E_{1_t}(\alpha(s,t)))ds \tag{2.4}$$

where ds is the element arc length. From (1.1) we have

$$\begin{aligned}g_{\mathcal{S}}(dE_{1_t}(E_{1_t}),dE_{1_t}(E_{1_t}))=&<d\pi(dE_{1_t}(E_{1_t})),d\pi(dE_{1_t}(E_{1_t}))>\\&+<K(dE_{1_t}(E_{1_t})),K(dE_{1_t}(E_{1_t}))>.\end{aligned}$$

Since E_{1_t} is a section, we have $d(\pi)\circ d(E_{1_t})=d(\pi\circ E_{1_t})=d(id_{C_t})=id_{TC_t}$. By Proposition 1.1, we also have that

$$K(dE_{1_t}(E_{1_t}))=\nabla^{\alpha}_{E_{1_t}}E_{1_t}=E^{'}_{1_t}=\frac{\partial E_{1_t}}{\partial s},$$

giving

$$g_{\mathcal{S}}(dE_{1_t}(E_{1_t}),dE_{1_t}(E_{1_t}))=<E_{1_t},E_{1_t}>+<E^{'}_{1_t},E^{'}_{1_t}>.$$

Using these results in (2.4) we get

$$\mathcal{E}(E_{1_t})=\frac{1}{2}\int_a^b(<E_{1_t},E_{1_t}>+<E^{'}_{1_t},E^{'}_{1_t}>)ds \tag{2.5}$$

By Definition 1.4, if E_{1_t} is a harmonic, then $t=0$ should be the critical point of $\mathcal{E}(E_{1_t})$. Supposing that $\frac{\partial\mathcal{E}(E_{1_t})}{\partial t}_{|t=0}=0$, from (2.5) we obtain:

$$\begin{aligned}\frac{\partial\mathcal{E}(E_{1_t})}{\partial t}&=\frac{\partial}{\partial t}[\frac{1}{2}\int_a^b(<E_{1_t},E_{1_t}>+<E^{'}_{1_t},E^{'}_{1_t}>)ds]\\&=\frac{1}{2}[\int_a^b\frac{\partial}{\partial t}[(<E_{1_t},E_{1_t}>+<\frac{\partial E_{1_t}}{\partial s},\frac{\partial E_{1_t}}{\partial s}>]ds.\end{aligned}$$

Since $< E_{1_t}, E_{1_t} >= 1$ we have $\frac{\partial}{\partial t} < E_{1_t}, E_{1_t} >= 0$ and we get

$$(2.6)\ \frac{\partial \mathcal{E}(E_{1_t})}{\partial t} = \frac{1}{2}\int_a^b \frac{\partial}{\partial t} < \frac{\partial E_{1_t}}{\partial s}, \frac{\partial E_{1_t}}{\partial s} > ds = \int_a^b < \frac{\partial^2 E_{1_t}}{\partial s \partial t}, \frac{\partial E_{1_t}}{\partial s} > ds.$$

We can write

$$\frac{\partial}{\partial s} < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial E_{1_t}}{\partial s} >=< \frac{\partial^2 E_{1_t}}{\partial s \partial t}, \frac{\partial E_{1_t}}{\partial s} > + < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial^2 E_{1_t}}{\partial s^2} > .$$

Thus, we can deduce,

$$(2.7) \qquad < \frac{\partial^2 E_{1_t}}{\partial s \partial t}, \frac{\partial E_{1_t}}{\partial s} >= \frac{\partial}{\partial s} < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial E_{1_t}}{\partial s} > - < \frac{\partial E_{1_t}}{\partial t}, \frac{\partial^2 E_{1_t}}{\partial s^2} >$$

Substituting (2.7) in (2.6), for, $t = 0$, we have

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = \int_a^b [\frac{\partial}{\partial s} < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_k}}{\partial s}(s,0) > - < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) >]ds$$

and

$$(2.8) \qquad \frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_t}}{\partial s}(s,0) >|_a^b - \int_a^b < \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) > ds.$$

From (2.2) and (2.3), we obtain,

$$(2.9) \qquad \frac{\partial \alpha}{\partial t}(s,t) = \lambda(s) E_{1_t}(s).$$

and

$$(2.10) \qquad \frac{\partial \alpha}{\partial s}(s,t)_{|t=0} = \alpha^{'}(s) = E_1(s).$$

Now we calculate the partial derivatives of (2.10) with respect to s and t; using Frenet formulas, we get

$$(2.11) \qquad \frac{\partial E_{1_t}}{\partial s}(s) = \frac{\partial^2 \alpha}{\partial s^2}(s,t)_{|t=0} = \alpha^{''}(s) = E_1^{'}(s) = k_1(s) E_2(s)$$

and

$$\frac{\partial E_{1_t}}{\partial t}(s,t) = \frac{\partial^2 \alpha}{\partial s \partial t}(s,t) = \frac{\partial^2 \alpha}{\partial t \partial s}(s,t).$$

From (2.9), we have

$$(2.12) \qquad \frac{\partial E_{1_t}}{\partial t}(s,t)_{|t=0} = \frac{\partial E_{1_t}}{\partial t}(s,0) = \lambda^{'}(s) E_1(s) + \lambda(s) k_1(s) E_2(s).$$

It follows from (2.11) and (2.12) that

$$< \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_t}}{\partial s}(s,0) >= \lambda(s)k_1^2(s).$$

Considering the candidate function $\lambda(a) = \lambda(b) = 0$, we get:

$$< \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial E_{1_t}}{\partial s}(s,0) >|_a^b = \lambda(b)k_1^2(b) - \lambda(a)k_1^2(a) = 0. \tag{2.13}$$

From (2.11), we get

$$\frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) = -k_1^2(s)E_1(s) + k_1^{'}(s)E_2(s) + k_1(s)k_2(s)E_3(s) \tag{2.14}$$

Therefore, (2.12) and (2.14) gives

$$< \frac{\partial E_{1_t}}{\partial t}(s,0), \frac{\partial^2 E_{1_t}}{\partial s^2}(s,0) >= [-\lambda(s)k_1^2(s)]^{'} + 3\lambda(s)k_1(s)k_1^{'}(s) \tag{2.15}$$

Substituting (2.13) and (2.15) in (2.8) yields

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = -\int_a^b ([-\lambda(s)k_1^2(s)]^{'} + 3\lambda(s)k_1(s)k_1^{'}(s))ds = 0$$

and

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = [\lambda(s)k_1^2(s)] \,|_a^b -3\int_a^b \lambda(s)k_1(s)k_1^{'}(s)ds = 0$$

We are looking the candidate function $\lambda(a) = \lambda(b) = 0$,
which given $[\lambda(s)k_1^2(s)] \,|_a^b = 0$ and

$$\frac{\partial \mathcal{E}(E_{1_t})}{\partial t}_{|t=0} = -3\int_a^b \lambda(s)k_1(s)k_1^{'}(s)ds = 0$$

This completes the proof of the theorem. ∎

Example 1. Let $\varphi : R^2 \to R^3, \varphi = (x, y, \frac{1}{3}xy), \varphi(R^2) = M$ and $\alpha(s) = (3s, s^2, s^3)$. If we can choose $\lambda : [-10, 10] \to R, \lambda(s) = 10^2 - s^2$ then $\lambda(-10) = 0\lambda(10) = 0$ and $\lambda(s) \neq 0$ for all $s \in (-10, 10)$. We calculate

$$k_1(s) = \frac{6\sqrt{s^4 + 9s^2 + 1}}{(\sqrt{9s^4 + 4s^2 + 9})^3},$$

$$k_1^{'}(s) = 6\frac{\frac{2s^3+9s}{\sqrt{s^4+9s^2+1}}(\sqrt{9s^4 + 4s^2 + 9})^3 - 3\sqrt{s^4 + 9s^2 + 1}(\sqrt{9s^4 + 4s^2 + 9})^2(35s^3 + 8s)}{(9s^4 + 4s^2 + 9)^3},$$

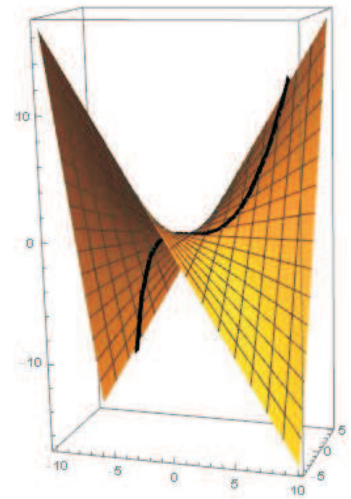

FIG. 2.1:

and

$$\frac{\partial \mathcal{E}(T_k)}{\partial k}_{|k=0} = -\int_{-10}^{10} (10^2 - s^2) k_1(s) k_1^{'}(s) ds = 0.$$

Thus α is an elastica on M, Figure 2.1.

Conclusion. In this paper, we have determined the classical Bernoulli-Euler elastic curve that is the harmonic of the velocity vector field of the curve on a manifold. We have obtained the collection of curves passing through p and q points using λ and v_i functions on the manifold. We have also proposed a novel condition to be the classical Bernoulli-Euler elastic curve in the collection of curves. In the end, we have given an example of the elastic curve satisfying the novel condition on a two-dimensional manifold and shown the graphs of both the manifold and the elastic curve.

REFERENCES

1. A. ALTIN, *On the Energy of Frenet Vectors Fields in R^n*. Cogent Mathematics, 2017.
2. P. BAIRD and J. C. WOOD: *Harmonic Morphisms Between Riemannian Manifold.* Clarendos press, Oxford, 2003.
3. P. M. CHACÓN, A. M. NAVEIRA, and J. M. WESTON: On the Energy of Distributions, with Application to the Quarternionic Hopf Fibration. Monatshefte fûr Mathematik, 133, 281-294. 2001.

4. P. A. Chacón and A. M. Naveira: *Corrected Energy of Distributions on Riemannian Manifold*. Osaka Journal Mathematics, Vol 41, 97-105, 2004.
5. L. Euler: *Additamentum de curvis elasticis. In Methodus Inveniendi Lineas Curvas Maximi Minimive Probprietate Gaudentes. Lausanne*, 1744.
6. H. A Goldstine: *A History of the Calculus of Variations From the 17th Through the 19th Century*. Springer, New York, 1980.
7. J. Guven, D. M.Valencia and J. Vazquez-Montejo: *Environmental bias and elastic curves on surfaces*. Phys. A: Math Theory. 47 355201,2014.
8. R. Huang: *A Note on the p-elastica in a Constant Sectional Curvature Manifold*. Journal of Geometry and Physics, Vol. 49, pp. 343-349, 2004.
9. B. O'Neill: *Elementary Diffrential Geometry*. Academic Press Inc., 1966.
10. T. Sakai, *Riemannian Geometry*. American Mathematical Society, 1996.
11. D. H. Steinberg: *Elastic Curves in Hyperbolic Space*. Doctoral Thesis, Case Western Reserve University, UMI Microform 9607925, 72p. 1995,
12. C. M. Wood: *On the Energy of a Unit Vector Field*. Geometrae Dedicata, 64, 319-330, 1997.

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ON THE WALKS ON CAYLEY GRAPHS

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Abstract. Let G be a group and S be an inverse-closed subset of G which does not contain the identity element of G. The Cayley graph of G with respect to S, $\mathrm{Cay}(G,S)$, is a graph with the vertex set G and the edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. In this paper, we compute the number of walks of any length between two arbitrary vertices of $\mathrm{Cay}(G,S)$ in terms of complex irreducible representations of G. Using our main result, we give exact formulas for the number of walks of any length between two vertices in complete graphs, cycles, complete bipartite graphs, Hamming graphs and complete transposition graphs.
Keywords: Cayley graph; Hamming graphs; complete transposition graphs.

1. Introduction

Let G be a finite group and S be an inverse-closed subset of G not containing the identity element of G. The Cayley graph on G with respect to S, $\mathrm{Cay}(G,S)$, is a graph with the vertex set G and the edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. $\mathrm{Cay}(G,S)$ is an undirected loop-free regular graph of valency $|S|$. Many famous regular graphs can be represented as Cayley graphs. For example, cycles, complete graphs, Hamming graphs and complete transposition graphs are Cayley graphs. Some chemical graphs are Cayley graphs as well. For instance, the Buckyball, a soccer ball like molecule which consists of 60 carbon atoms, is a Cayley graph on the alternating group A_5 on 5 symbols with the connection set $\{(12345), (54321), (12)(23)\}$ [5, p. 209]. Also, the honeycomb toroidal graph is a Cayley graph on a generalized dihedral group [1, Theorem 3.4]. Since Cayley graphs possess many properties such as low degree, low diameter, symmetry, low congestion, high connectivity, high fault tolerance, and efficient routing algorithms, in the past several years there has been a spurt of research on using Cayley graphs in constructions of interconnection networks. For more details see [7].

A *walk* of length r from vertex x to vertex y in a graph Γ is a sequence of vertices $(v_0, v_1, \ldots, v_r)$ such that $v_0 = x$, $v_r = y$ and v_{i-1} is adjacent to v_i for all

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$1 \leq i \leq r$. If $x = y$ then the walk is called a closed walk of length r at vertex x. The number of walks in a graph is often necessary in, for instance, network analysis, epidemiology (requiring slow diffusion of viruses) and network design (aiming for fast data propagation) [3]. Also walks in molecular graphs and their counts for a long time have found applications in theoretical chemistry [6]. Furthermore, using counting closed walks, many non-Cayley vertex-transitive graphs are constructed [10, 11, 12, 13]. So it seems that computing the number of walks in Cayley graphs is important in graph theory. In this paper, we give an exact formula for the number of walks of any length between two vertices of a Cayley graph on a group G in terms of irreducible representations of G. For the representation group's theoretic and graph theoretic terminology not defined here, we refer the reader to [9] and [5], respectively.

2. Main Results

Let G be a finite group and $\mathbb{C}[G]$ be the complex vector space of dimension $|G|$ with basis $\{e_g \mid g \in G\}$. We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Thus a function $\varphi : G \to \mathbb{C}$ corresponds to the vector $\varphi = \sum_{g \in G} \varphi(g) e_g$ and vice versa. In particular, the vector e_g, where $g \in G$, of the standard basis corresponds to the function e_g, where

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

Let $A = [a_{x,y}]_{x,y \in G}$ be the adjacency matrix of $\Gamma = \mathrm{Cay}(G, S)$, $S = S^{-1} \subseteq G \setminus \{1\}$, where

$$a_{x,y} = \begin{cases} 1 & xy^{-1} \in S \\ 0 & xy^{-1} \notin S \end{cases}.$$

Then viewing A as a linear map on $\mathbb{C}[G]$, we have

$$Ae_x = \sum_{y \in G} a_{y,x} e_y = \sum_{y \in G, yx^{-1} \in S} e_y = \sum_{s \in S} e_{sx}. \tag{2.1}$$

Let $\omega_r(\Gamma; x, y)$ be the number of walks of length k from the vertex x to the vertex y in a graph Γ. We denote this by $\omega_r(x, y)$ when there is no ambiguity. Recall that for a graph Γ with adjacency matrix A, $\omega_r(\Gamma; x, y)$ is the xy-entry of A^r [5, Lemma 8.1.2]. In particular, $\omega_r(\Gamma) := \sum_{x \in V(\Gamma)} \omega_r(\Gamma; x, x)$, the total number of closed walks of length r, is the trace of A which is equal to the sum of rth powers of the adjacency eigenvalues of Γ [5, p. 165]. Let us start with an important lemma:

Lemma 2.1. *Let A be the adjacency matrix of $\Gamma = \mathrm{Cay}(G, S)$. Then*

$$A^r e_x = \sum_{y \in G} \omega_r(x, y) e_y.$$

Proof. We use induction on r. Since by (2.1), $Ae_x = \sum_{s\in S} e_{sx}$, and

$$\omega_1(x,y) = \begin{cases} 1 & yx^{-1} \in S \\ 0 & yx^{-1} \notin S, \end{cases}$$

the induction holds for $r = 1$. Now let $r \geq 2$ and the result hold for $r-1$. Since there exists a walk of length r from x to y if and only if there exists a walk of length $r-1$ of x to z where $yz^{-1} \in S$, we have

$$\omega_r(x,y) = \sum_{s\in S} \omega_{r-1}(x, s^{-1}y). \tag{2.2}$$

Now we have

$$\begin{aligned} A^r e_x &= A(A^{r-1}e_x) \\ &= A\Big(\sum_{y\in G} \omega_{r-1}(x,y)e_y\Big) \quad \text{(by induction hypothesis)} \\ &= \sum_{y\in G} \omega_{r-1}(x,y)Ae_y \\ &= \sum_{y\in G} \omega_{r-1}(x,y)\Big(\sum_{s\in S} e_{sy}\Big) \quad \text{(by (2.1))} \\ &= \sum_{z\in G}\sum_{s\in S} \omega_{r-1}(x,s^{-1}z)e_z \\ &= \sum_{z\in G} \omega_r(x,z)e_z, \quad \text{(by (2.2))} \end{aligned}$$

which completes the proof. □

Lemma 2.2. *Let A be the adjacency matrix of $\Gamma = \mathrm{Cay}(G,S)$. Then*

$$A^r e_x = \sum_{s_1,\ldots,s_r\in S} e_{s_r s_{r-1}\ldots s_1 x}.$$

Proof. We prove the result by induction. By 2.1, we have $Ae_x = \sum_{s\in S} e_{sx}$ which proves the result for $r = 1$. Let $r \geq 2$ and the result holds for $r-1$. Then

$$\begin{aligned} A^r e_x &= A(A^{r-1}e_x) \\ &= A\Big(\sum_{s_1,\ldots,s_{r-1}\in S} e_{s_{r-1}s_{r-2}\ldots s_1 x}\Big) \quad \text{(by induction hypothesis)} \\ &= \sum_{s_1,\ldots,s_{r-1}\in S} Ae_{s_{r-1}s_{r-2}\ldots s_1 x} \\ &= \sum_{s_1,\ldots,s_{r-1}\in S}\sum_{s_r\in S} e_{s_r(s_{r-1}\ldots s_1 x)} \quad \text{(by (2.1))} \\ &= \sum_{s_1,\ldots,s_r\in S} e_{s_r s_{r-1}\ldots s_1 x}, \end{aligned}$$

which completes the proof. □

Let $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$ be the set of all irreducible inequivalent $\mathbb{C}$-representations of G. Let d_k and $\varrho^{(k)}$ be the degree and a unitary matrix representation of ρ_k, $k = 1, \ldots, m$, respectively. We keep these notations throughout the paper. In the following lemma, which seems to be well-known, the authors constructed an orthogonal basis for $\mathbb{C}[G]$ using the matrix representations $\varrho^{(k)}$, $1 \leq k \leq m$.

Lemma 2.3. ([2, Lemma 1]) *Let $\varrho^{(k)}_{ij}(g)$ be the ijth entry of $\varrho^{(k)}(g)$, $1 \leq i, j \leq d_k$, and $\bar{\varrho}^{(k)}_{ij} = \sum_{g\in G} \overline{\varrho^{(k)}_{ij}(g)} e_g$. Then*

(i) $\{\bar{\varrho}^{(k)}_{ij} \mid 1 \leq k \leq m, 1 \leq i, j \leq d_k\}$ form an orthogonal basis for $\mathbb{C}[G]$,

(ii) $\rho_{\mathrm{reg}}(g)\bar{\varrho}^{(k)}_{ij} = \sum_{l=1}^{d_k} \varrho^{(k)}_{li}(g)\bar{\varrho}^{(k)}_{lj}$, for all $g \in G$ and $1 \leq i, j \leq d_k$, $1 \leq k \leq m$, where ρ_{reg} is the left regular representation of G,

(iii) $\mathbb{C}[G] = \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{d_k} W^{(k)}_j$, where $W^{(k)}_j = \langle \bar{\varrho}^{(k)}_{ij} \mid 1 \leq i \leq d_k\rangle$ which is a ρ_{reg}-invariant subspace of $\mathbb{C}[G]$ of dimension d_k.

Now we are ready to prove our main result. Let us denote the ij entry of a matrix X by $[X]_{ij}$. Then we have the following theorem.

Theorem 2.1. *Let $\Gamma = \mathrm{Cay}(G, S)$, $1 \notin S = S^{-1}$ and $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$. Then*

$$\omega_r(x, y) = \frac{1}{|G|} \sum_{k=1}^{m} \sum_{i,j=1}^{d_k} d_k \Big[\big(\sum_{s\in S} \varrho^{(k)}(s)\big)^r\Big]_{ij} \Big[\varrho^{(k)}(xy^{-1})\Big]_{ji}.$$

Proof. First, recall that the adjacency matrix A of Γ can be viewed as a linear map on $\mathbb{C}[G]$ and by Lemma $\mathbb{C}[G] = \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{d_k} W^{(k)}_j$, where $W^{(k)}_j = \langle \bar{\varrho}^{(k)}_{ij} \mid 1 \leq i \leq d_k\rangle$ which is a ρ_{reg}-invariant subspace of $\mathbb{C}[G]$ of dimension d_k. Since $A^r e_x \in \mathbb{C}[G]$, there exist complex numbers $\alpha^{(k)}_{ij}$, $1 \leq i, j \leq d_k$ such that

$$A^r e_x = \sum_{k=1}^{m} \sum_{i,j=1}^{d_k} \alpha^{(k)}_{ij} \bar{\varrho}^{(k)}_{ij}. \tag{2.3}$$

On the other hand, $\alpha^{(k)}_{ij} = \frac{\langle A^r e_x, \bar{\varrho}^{(k)}_{ij}\rangle}{\langle \bar{\varrho}^{(k)}_{ij}, \bar{\varrho}^{(k)}_{ij}\rangle}$, where $\langle u, v\rangle$ denotes the usual inner product of u and v in complex field vector spaces. Furthermore,

$$\begin{aligned}
\langle A^r e_x, \bar{\varrho}_{ij}^{(k)}\rangle &= \langle \sum_{s_1,\ldots,s_r\in S} e_{s_r s_{r-1}\ldots s_1 x}, \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)} e_g\rangle \quad \text{(by Lemma 2.2)} \\
&= \sum_{s_1,\ldots,s_r\in S} \langle e_{s_r s_{r-1}\ldots s_1 x}, \sum_{g\in G} \overline{\varrho_{ij}^{(k)}(g)} e_g\rangle \\
&= \sum_{s_1,\ldots,s_r\in S}\sum_{g\in G} \varrho_{ij}^{(k)}(g)\langle e_{s_r s_{r-1}\ldots s_1 x}, e_g\rangle \\
&= \sum_{s_1,\ldots,s_r\in S} \varrho_{ij}^{(k)}(s_r s_{r-1}\ldots s_1 x) \\
&= \sum_{s_1,\ldots,s_r\in S} \Big[\varrho^{(k)}(s_r)\ldots\varrho^{(k)}(s_1)\varrho^{(k)}(x)\Big]_{ij} \quad \text{(since } \varrho^{(k)} \text{ is a homomorphism)} \\
&= \Big[\sum_{s_1,\ldots,s_r\in S} \varrho^{(k)}(s_r)\ldots\varrho^{(k)}(s_1)\varrho^{(k)}(x)\Big]_{ij} \\
&= \Big[(\sum_{s_r\in S}\varrho^{(k)}(s_r))\ldots(\sum_{s_1\in S}\varrho^{(k)}(s_1))\varrho^{(k)}(x)\Big]_{ij} \\
&= \Big[(\sum_{s\in S}\varrho^{(k)}(s))^r\varrho^{(k)}(x)\Big]_{ij}.
\end{aligned}$$

Also

$$\begin{aligned}
\langle\bar{\varrho}_{ij}^{(k)}, \bar{\varrho}_{ij}^{(k)}\rangle &= \langle\sum_{g\in G}\overline{\varrho_{ij}^{(k)}(g)}e_g, \sum_{h\in G}\overline{\varrho_{ij}^{(k)}(h)}e_h\rangle \\
&= \sum_{g\in G}\overline{\varrho_{ij}^{(k)}(g)}\sum_{h\in G}\varrho_{ij}^{(k)}(h)\langle e_g, e_h\rangle \\
&= \sum_{g\in G}\overline{\varrho_{ij}^{(k)}(g)}\varrho_{ij}^{(k)}(g) \\
&= \sum_{g\in G}\varrho_{ji}^{(k)}(g^{-1})\varrho_{ij}^{(k)}(g) \quad \text{(since } \varrho^{(k)} \text{ is unitary)} \\
&= \frac{|G|}{d_k} \quad \text{(by Schur's relations).}
\end{aligned}$$

Hence $\alpha_{ij}^{(k)} = \frac{d_k}{|G|}\Big[(\sum_{s\in S}\varrho^{(k)}(s))^r\varrho^{(k)}(x)\Big]_{ij}$. Now from the equality (2.3), Lemma 2.1 and this fact that $\bar{\varrho}_{ij}^{(k)} = \sum_{g\in G}\varrho_{ji}^{(k)}(g^{-1})e_g$, we have

$$\omega_r(x,y) = \frac{1}{|G|}\sum_{k=1}^{m}\sum_{i,j=1}^{d_k} d_k\Big[(\sum_{s\in S}\varrho^{(k)}(s))^r\Big]_{ij}\Big[\varrho^{(k)}(xy^{-1})\Big]_{ji},$$

which completes the proof. □

Keeping the notations of Theorem 2.1, since $\varrho^{(k)}(1) = I_{d_k}$, we have the following direct consequence.

Corollary 2.1.

$$\omega_r(\Gamma : x, x) = \frac{1}{|G|}\sum_{k=1}^{m} d_k \mathrm{Tr}[(\sum_{s\in S}\varrho^{(k)}(s))^r],$$

where $\mathrm{Tr}[X]$ *denotes the trace of matrix* X. *In particular,*

$$\omega_r(\Gamma) = \sum_{k=1}^{m} d_k \mathrm{Tr}[(\sum_{s\in S}\varrho^{(k)}(s))^r].$$

Corollary 2.2. *([15, Theorem 2]) Let* $\Gamma = \mathrm{Cay}(G, S)$ *and* $1 \notin S = S^{-1}$ *be a union of conjugacy classes of* G. *Then*

$$\omega_r(x, y) = \frac{1}{|G|}\sum_{k=1}^{m} \frac{(\sum_{s\in S}\chi_k(s))^r \chi_k(xy^{-1})}{d_k^{r-1}}.$$

In particular, if G *is abelian then*

$$\omega_r(x, y) = \frac{1}{|G|}\sum_{k=1}^{|G|}(\sum_{s\in S}\chi_k(s))^r \chi_k(xy^{-1}).$$

Proof. First, note that S is a union of conjugacy classes if and only if for all $g \in G$ we have $g^{-1}Sg = S$. Thus for all $g \in G$, we have

$$\begin{aligned} \varrho^{(k)}(g^{-1})(\sum_{s\in S}\varrho^{(k)}(s))\varrho^{(k)}(g) &= \sum_{s\in S}\varrho^{(k)}(g^{-1}sg) \\ &= \sum_{s\in S}\varrho^{(k)}(s) \quad (\text{since } g^{-1}Sg = S). \end{aligned}$$

Hence by Schur's Lemma, $\sum_{s\in S}\varrho^{(k)}(s) = \frac{1}{d_k}\mathrm{Tr}(\sum_{s\in S}\varrho^{(k)}(s))I_{d_k} = \frac{\sum_{s\in S}\chi_k(s)}{d_k}I_{d_k}$. Now the result follows from Theorem 2.1. □

Let $G = \langle a\rangle \cong \mathbb{Z}_n$ be a cyclic group of order n. Then $\mathrm{Irr}(G) = \{\chi_i \mid i = 0, \ldots, n-1\}$, where $\chi_k(a^r) = \exp(2\pi ikr/n)$.

Corollary 2.3. *(See also [14]) Let* K_n *be a complete graph with* n *vertices. Then*

$$\omega_r(K_n; x, y) = \begin{cases} \frac{1}{n}((n-1)^r - (-1)^r) & x \neq y \\ \frac{n-1}{n}((n-1)^{r-1} - (-1)^{r-1}) & x = y. \end{cases}$$

Proof. Let $G = \langle a\rangle$ be a cyclic group of order n and $S = G \setminus \{1\}$. Then for all $g \in G$, $g^{-1}Sg = S$ and $K_n = \mathrm{Cay}(G, S)$. Hence, by Corollary 2.2,

$$\omega_r(K_n; x, y) = \frac{1}{n}\sum_{k=0}^{n-1}(\sum_{s\in S}\chi_k(s))^r \chi_k(xy^{-1}).$$

On the other hand

$$\sum_{s\in S}\chi_k(s)=\begin{cases}-1 & k\neq 0\\ n-1 & k=0\end{cases}$$

Let $x=a^l$ and $y=a^{l'}$. Then $\chi_k(xy^{-1})=\exp(2k(l-l')\pi i/n)$, $k=0,\ldots,n-1$. It is clear that if $x=y$ then $\sum_{k=0}^{n-1}(\sum_{s\in S}\chi_k(s))^r\chi_k(xy^{-1})=(n-1)^r+(n-1)(-1)^r$. Since $z+z^2+\ldots+z^{n-1}=-1$ whenever z is a nth root of unity, we conclude that if $x\neq y$ then $\sum_{k=0}^{n-1}(\sum_{s\in S}\chi_k(s))^r\chi_k(xy^{-1})=(n-1)^r-(-1)^r$, which completes the proof. □

Corollary 2.4. *Let C_n be an n-cycle. Then $C_n=\mathrm{Cay}(G,S)$ where $G=\langle a\rangle$ and $S=\{a,a^{-1}\}$. Furthermore,*

$$\omega_r(C_n;a^l,a^{l'})=\frac{2^r}{n}\sum_{k=0}^{n-1}\cos^r(\frac{2\pi k}{n})\cos(\frac{2\pi k(l-l')}{n}).$$

Proof. Let $\chi_k\in\mathrm{Irr}(G)$. Then $\chi_k(a)+\chi_k(a^{-1})=2\cos(\frac{2\pi k}{n})$. Also $\chi_k(xy^{-1})=\cos(\frac{2\pi k(l-l')}{n})+i\sin(\frac{2\pi k(l-l')}{n})$. Furthermore, $\sum_{k=0}^{n-1}\cos(\frac{2\pi k}{n})^r\sin(\frac{2\pi k(l-l')}{n})=0$. Now the result follows immediately from Corollary 2.2. □

Corollary 2.5. *Let $K_{n,n}$ be the complete bipartite graph with $2n$ vertices, where $n\geq 3$. Then $K_{n,n}=\mathrm{Cay}(G,S)$, where $G=\langle a\rangle\cong\mathbb{Z}_{2n}$ and $S=\{a,a^3,\ldots,a^{2n-1}\}$.*

$$\omega_r(K_{n,n};a^l,a^{l'})=\frac{n^r+(-n)^r(-1)^{l-l'}}{2n}.$$

Proof. Let $w_k=\exp(\pi ik/n)$. Then irreducible characters of G are χ_k, $k=0,\ldots,2n-1$, where $\chi_k(a^l)=w_k^l$. For $k\neq 0,n$ we have $w_k+w_k^3+\ldots+w_k^{2n-1}=0$. Thus

$$\sum_{s\in S}\chi_k(s)=\begin{cases}0 & k\neq 0,n\\ n & k=0\\ -n & k=n\end{cases}.$$

Let $x=a^l$ and $y=a^{l'}$. Then $\chi_k(xy^{-1})=w_k^{l-l'}$ which completes the proof. □

Recall that the Hamming graph $H(n,m)$ is the graph whose vertex set is the Cartesian product of n copies of a set with m elements, where two vertices are adjacent if they differ in precisely one coordinate. $H(n,2)=Q_n$ is the familiar n-dimensional hypercube. It is well-known that $\Gamma=\mathrm{Cay}(G_1\times\ldots\times G_n,S)$ where $G_i=\langle a\rangle$, $i=1,\ldots,n$, is of order m and S is the set of all elements of $G_1\times\ldots\times G_n$ with exactly one non-identity coordinate. In the following example, we compute the number of walks between any two vertices in the Hamming graphs.

Corollary 2.6. *Let $\Gamma = H(n,m)$. Then*

$$\omega_r(\Gamma;x,y) = \frac{1}{m^n}\sum_{0\le j_1,\ldots,j_n\le m-1}\Big(n(m-1)-mc(j_1,\ldots,j_n)\Big)^r\tau^{(r_1-s_1)j_1+\ldots+(r_n-s_n)j_n},$$

where $x=(a^{r_1},\ldots,a^{r_n})$, $y=(a^{s_1},\ldots,a^{s_n})$ and $c(j_1,\ldots,j_n)$ is the number of non-zero coordinates of $(j_1,\ldots,j_n)$.. In particular,

$$\omega_r(Q_n;x,y) = \frac{1}{2^n}\sum_{0\le j_1,\ldots,j_n\le 1}\Big(n-2c(j_1,\ldots,j_n)\Big)^r\tau^{(r_1-s_1)j_1+\ldots+(r_n-s_n)j_n},$$

where $x=(a^{r_1},\ldots,a^{r_n})$ and $y=(a^{s_1},\ldots,a^{s_n})$.

Proof. Let $\chi\in\mathrm{Irr}(G_1\times\ldots\times G_n)$ and $g=(a^{i_1},\ldots,a^{i_n})\in G_1\times\ldots\times G_n$. Then there exist $(j_1,\ldots,j_n)$, where $0\le j_i\le m-1$, such that $\chi(g)=\tau^{i_1j_1+\ldots+i_nj_n}$, where $\tau=\exp(2\pi i/m)$. Hence every irreducible character of $G_1\times\ldots\times G_n$ completely determined by an n-tuple $(j_1,\ldots,j_n)$, where $0\le j_i\le m-1$. Let us denote the corresponding character of this tuple by $\chi_{(j_1,\ldots,j_n)}$.

Let $x=a^i\neq 1$ and $x^{(j)}$ be a $1\times n$ vector that its only non-identity element is x at the jth position. Let $s\in S$. Then $s=(a^i)^{(k)}$ for some $1\le i\le m-1$ and $1\le k\le n$. Hence $\chi_{(j_1,\ldots,j_n)}(s)=\tau^{ij_k}$ which implies that $\sum_{s\in S}\chi_{(j_1,\ldots,j_n)}(s)=\sum_{k=1}^{n}\sum_{i=1}^{m-1}\tau^{ij_k}$. On the other hand,

$$\sum_{i=1}^{m-1}(\tau^{j_k})^i=\begin{cases} m-1 & j_k=0\\ -1 & j_k\neq 0\end{cases}.$$

Let $c(j_1,\ldots,j_n)$ be the number of non-zero coordinates of $(j_1,\ldots,j_n)$. Then $\sum_{s\in S}\chi_{(j_1,\ldots,j_n)}(s)=n(m-1)-mc(j_1,\ldots,j_n)$. Now, by Corollary 2.2,

$$\omega_r(x,y)=\frac{1}{m^n}\sum_{0\le j_1,\ldots,j_n\le m-1}\Big(n(m-1)-mc(j_1,\ldots,j_n)\Big)^r\tau^{(r_1-s_1)j_1+\ldots+(r_n-s_n)j_n},$$

where $x=(a^{r_1},\ldots,a^{r_n})$ and $y=(a^{s_1},\ldots,a^{s_n})$. This completes the proof. $\square$

Recall that a *partition* of a positive integer n is a sequence $\lambda=(\lambda_1,\ldots,\lambda_m)$ of positive integers such that $\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_m$ and $\sum_{i=1}^m\lambda_i=n$. We write $\lambda\vdash n$ to indicate that λ is a partition of n. Since the inequivalent irreducible representations of the symmetric group S_n on n letters are conveniently by partitions of n, we write ρ_λ, χ_λ and d_λ for the irreducible representation, the character and the degree of the representation associated with $\lambda\vdash n$.

For $\lambda=(\lambda_1,\ldots,\lambda_m)\vdash n$, put $l_i=\lambda_i+m-i$, $1\le i\le m$. If $m=1$ then $d_\lambda=1$ and whenever $m>1$, by [4, equality (4.11)] we have

$$d_\lambda=\frac{n!}{l_1!l_2!\ldots l_m!}\prod_{i<j}(l_i-l_j). \tag{2.4}$$

Furthermore,

(1) if $\tau \in S_n$ is a transposition, then by [8, equality (5.1)],

$$\chi_\lambda(\tau) = \frac{M_2(\lambda)}{n(n-1)} d_\lambda, \tag{2.5}$$

(2) if $\tau \in S_n$ is a 3-cycle, then by [8, equality (5.2)]

$$\chi_\lambda(\tau) = \frac{M_3(\lambda) - 3n(n-1)}{2n(n-1)(n-2)} d_\lambda, \tag{2.6}$$

(3) if τ is a product of two disjoint transpositions, then by [8, equality (5.5)]

$$\chi_\lambda(\tau) = \frac{M_2(\lambda)^2 - 2M_3(\lambda) + 4n(n-1)}{n(n-1)(n-2)(n-3)} d_\lambda, \tag{2.7}$$

where

$$M_2(\lambda) = \sum_{j=1}^{m} \Big((\lambda_j - j)(\lambda_j - j + 1) - j(j-1) \Big)$$

and

$$M_3(\lambda) = \sum_{j=1}^{m} \Big((\lambda_j - j)(\lambda_j - j + 1)(2\lambda_j - 2j + 1) + j(j-1)(2j-1) \Big).$$

Corollary 2.7. *Let $\Gamma = \mathrm{Cay}(S_n, S)$, be the complete transposition graph, where S is the set of all transpositions of $\{1, \dots, n\}$. Then for all $x \in S_n$, we have*

$$\omega_r(x, x) = \frac{1}{n!2^r} \sum_{\lambda \vdash n} d_\lambda^2 M_2(\lambda)^r.$$

Furthermore, if $x \neq y$ be two non-disjoint transpositions then

$$\omega_r(x, y) = \frac{1}{n!2^{r+1} n(n-1)(n-2)} \sum_{\lambda \vdash n} d_\lambda^2 M_2(\lambda)^r (M_3(\lambda) - 3n(n-1)),$$

and if they are disjoint, then

$$\omega_r(x, y) = \frac{1}{n!2^r n(n-1)(n-2)(n-3)} \sum_{\lambda \vdash n} d_\lambda^2 M_2(\lambda)^r (M_2(\lambda)^2 - 2M_3(\lambda) + 4n(n-1)).$$

Proof. Since S is the set of all transpositions of S_n, it is a conjugacy class of S_n with $\frac{n(n-1)}{2}$ elements. On the other hand, by Equality (2.5), for any $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$ we have

$$\sum_{s \in S} \chi_\lambda(s) = |S| \chi_\lambda((1, 2)) = \frac{M_2(\lambda)}{2} d_\lambda.$$

Let $x, y \in S_n$. If $x = y$ then $xy^{-1} = 1$ and $\chi_\lambda(xy^{-1}) = \chi_\lambda(1) = d_\lambda$. If $x \neq y$ and they are not disjoint transpositions then xy^{-1} is a 3-cycle. Now the result follows immediately from Corollary 2.2 and equalities (2.6) and (2.7). $\square$

REFERENCES

1. B. Alspach and M. Dean: *Honeycomb toroidal graphs are Cayley graphs*, Inform. Process. Letters, **109**(13) (2009) 705–708.
2. M. Arezoomand and B. Taeri: *On the characteristic polynomial of n-Cayley digraphs*, The Electron. J. Combin., **20**(3) (2013), # P57.
3. A. Farrugia and I. Sciriha: *The main eigenvalues and number of walks in self-complementary graphs*, Linear Multilinear Algebra, **62**(10) (2014) 1346–1360.
4. W. Fulton and J. Harris, *Representation theory, A first course*, Graduate Texts in Mathematics 129, Springer-Verlag, New York, 1991.
5. G. Godsil and G. Royle: *Algebraic graph theory*, Springer-Verlag, New-York, 2001.
6. I. Gutman, C. R ücker and G. Rücker: *On walks in molecular graphs*, J. Chem. Inf. Comput. Sci. **41** (2001) 739–745.
7. M. C. Heydemann: *Cayley graphs and interconnection networks*, In: Hahn G., Sabidussi G. (eds) Graph Symmetry. NATO ASI Series (Series C: Mathematical and Physical Sciences), vol 497. Springer, Dordrecht, 1997.
8. R. E. Ingram: *Some characters of the symmetric group*, Proc. Am. Math. Soc. **1**(3) (1950) 358–369.
9. G. James and M. Liebeck: *Representations and characters of groups*, Cambridge University Press, Second Edition, 2001.
10. R. Jajcay and J. Širăn:, *A construction of vertex-transitive non-Cayley graphs*, Australas. J. Combin. **10** (1994) 105–114.
11. R. Jajcay and J. Širăn: *More constructions of vertex-transitive non-Cayley graphs based on counting closed walks*, Australas. J. Combin. **14** (1996) 121-132.
12. R. Jajcay, A. Malnič and D. Marušič: *On the number of closed walks in vertex-transitive graphs*, Discrete Math. **307** (2007) 484–493.
13. R. Kurcz: *Closed walks in coset graphs and vertex-transitive non-Cayley graphs*, Acta Math. Univ. Comenian. **68** (1) (1999) 127-135.
14. R. P. Stanley: *Topic in Algebraic Combinatorics*, Version of 1 February 2013.
15. P.-H Zieschang: *Cayley graphs of finite groups*, J. Algebra, **118** (1998) 447-454.

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GENERALIZED BERNSTEIN-KANTOROVICH OPERATORS OF BLENDING TYPE

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Abstract. In this note, we derive some approximation properties of the generalized Bernstein-Kantorovich-type operators based on two nonnegative parameters considered by A. Kajla [Appl. Math. Comput. 2018]. We establish a Voronovskaja-type asymptotic theorem for these operators. The rate of convergence for differential functions whose derivatives are of bounded variation is also derived. Finally, we show the convergence of the operators to certain functions by illustrative graphics using Mathematica software.

Keywords: Approximation; Bernstein-Kantorovich type operators; convergence.

1. Introduction

For $f \in C(I)$, with $I = [0,1]$, the classical Bernstein polynomials are defined as follows:

$$\mathcal{B}_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis.

Also for $f : I \to \mathbb{R}$ an integrable function, the classical Bernstein-Kantorovich operators are defined by

$$M_n(f;x) = n \sum_{k=0}^{n} p_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)dt, \ x \in [0,1], \ n \in \mathbb{N}.$$

The above operators M_n can also be written as follows:

$$M_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f\left(\frac{k+t}{n}\right) dt. \tag{1.1}$$

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Stancu [31] introduced the Bernstein-type operators involving two parameters $r, s \in \mathbb{N} \cup \{0\}$, as follows:

$$(S_{n,r,s}) f(x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^{s} p_{s,k}(x) f\left(\frac{\mu + kr}{n}\right). \tag{1.2}$$

For $r = s = 0$, these operators reduces to Bernstein operators $\mathcal{B}_n(f; x)$.
Abel and Heilmann [1] investigated the complete asymptotic expansion of Bernstein-Durrmeyer operators. Gonska and Paltanea [16] presented genuine Bernstein-Durrmeyer operators based on one parameter family of linear positive operators and study the simultaneous approximation for these operators. Cárdenas-Morales and Gupta [12] derived a two-parameter family of Bernstein-Durrmeyer-type operators based on the Polya distribution and gave a Voronovskaja-type asymptotic theorem. In [9], Agrawal et al. introduced the Kantorovich-type generalization of Luaps operators and obtained the local and global approximation properties of these operators. Abel et al. [2] considered the Durrmeyer-type modification of the operators (1.2) defined by

$$\mathcal{S}_{n,r,s}(f; x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^{s} p_{s,k}(x)(n+1) \int_0^1 p_{n,\mu+kr}(t) f(t) dt. \tag{1.3}$$

The authors studied a complete asymptotic expansion and derived some basic approximation theorems for these operators. Gupta et al. [18] considered the Durmeyer variant of Baskakov operators based on the inverse Pòlya-Eggenberger distribution and studied the local and global approximation properties. Many researchers have contributed to this area of approximation theory [cf. [3–8, 10, 11, 13–15, 17–20, 22, 24–30] etc.] and the references therein.

For $f \in C(I)$, Kajla [23] defined the following Stancu-Kantorovich-type operators based on two nonnegative parameters:

$$\mathcal{K}_{n,r,s}(f; x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^{s} p_{s,k}(x) \int_0^1 f\left(\frac{\mu + kr + t}{n}\right) dt. \tag{1.4}$$

The approximation behaviour of $\mathcal{K}_{n,r,s}$ was examined in the paper [23].
In this article, we prove the Voronovskaja-type asymptotic theorem for these operators. The rate of convergence for differential functions whose derivatives are of bounded variation is also obtained. Finally, we show the convergence of the operators by illustrative graphics in Mathematica software to certain functions.

Let $e_i(x) = x^i, i = 0, 1, 2 \cdots$

Lemma 1.1. *[23] For the operators $\mathcal{K}_{n,r,s}(f; x)$, we have*

(i) $\mathcal{K}_{n,r,s}(e_0; x) = 1$;

(ii) $\mathcal{K}_{n,r,s}(e_1;x) = x + \dfrac{1}{2n}$;

(iii) $\mathcal{K}_{n,r,s}(e_2;x) = x^2 + \dfrac{x(1-x)}{n}\left(1 + \dfrac{sr(r-1)}{n}\right) + \dfrac{x}{n} + \dfrac{1}{3n^2}$;

(iv) $$\mathcal{K}_{n,r,s}(e_3;x) = x^3 + \frac{3x(3-2x)}{2n} + \frac{x(7-9x)+6rsx^2(r-1)(1-x)+4x^3}{2n^2} + \frac{1-2rsx\Big((5+9x-4x^2)-3r(1-x^2)-2r^2(1-3x+8x^2)\Big)}{4n^3};$$

(v) $$\begin{aligned}\mathcal{K}_{n,r,s}(e_4;x) &= x^4 + \frac{x^4}{5n^4}\Big[55n^2 - 30n^3 + 30n^2rs - 30(-1+r)rs - 30n^2r^2s + \\ &15(r-1)r^2(s-2)s - 15(r-1)r^2s(s+2) + 10n(-3+(r-1)r(7+4r)s)\Big] + \\ &\frac{x^3}{5n^4}\Big[40n^3 - 80n - 120n^2 - 30n^2rs + 80(r-1)rs + 30n^22r^2s + 50(r-1)r^2s - \\ &30n(r-1)rs(2r+5) - 30r^3s(r-1)(s-2) + 15r^2s^2(r-1) + 15r^2s^2(r-1)(s+2)\Big] \\ &+ \frac{x^2}{5n^4}\Big[75n^2 - 75n - 75rs(r-1) - 65r^2s(r-1) - 5r^3s(r-1) + 20nrs(r-1) + \\ &15r^3s(s-2) - 15r^2s^2(r-1)\Big] \\ &+ \frac{x}{5n^4}\Big[30n + 25rs(r-1) + 15r^2s(r-1) + 5r^3s(r-1)\Big] + \frac{1}{5n^4}.\end{aligned}$$

Let $e_i^x(t) = (t-x)^i, i = 1,2,4.$

Lemma 1.2. *[23] For the operators $\mathcal{K}_{n,r,s}(f;x)$, we get*

(i) $\mathcal{K}_{n,r,s}(e_1^x(t);x) = \dfrac{1}{2n}$;

(ii) $\mathcal{K}_{n,r,s}(e_2^x(t);x) = \dfrac{x(1-x)}{n}\left(1 + \dfrac{sr(r-1)}{n}\right) + \dfrac{1}{3n^2}$.

Lemma 1.3. *[23] For $f \in C(I)$, we have*

$$\|\mathcal{K}_{n,r,s}(f;x)\| \leq \|f\|.$$

Remark 1.1. For every $x \in I$, we have

$$\begin{aligned}\lim_{n\to\infty} n\ \mathcal{K}_{n,r,s}(e_1^x(t);x) &= \frac{1}{2}, \\ \lim_{n\to\infty} n\ \mathcal{K}_{n,r,s}(e_2^x(t);x) &= x(1-x), \\ \lim_{n\to\infty} n^2\ \mathcal{K}_{n,r,s}(e_4^x(t);x) &= 3x^2(1-x)^2.\end{aligned}$$

Lemma 1.4. *For $n \in \mathbb{N}$, we obtain*

$$\mathcal{K}_{n,r,s}(e_2^x(t);x) \leq \frac{\mathcal{X}_{r,s} \quad x(1-x)}{n},$$

where $\mathcal{X}_{r,s}$ is a positive constant depending only on r, s.

Theorem 1.1. *[23] Let $f \in C(I)$. Then $\lim_{n\to\infty} \mathcal{K}_{n,r,s}(f;x) = f(x)$, uniformly in I.*

2. Voronovskaja type theorem

The aim of this section, we prove the Voronvoskaja-type theorem for the operators $\mathcal{K}_{n,r,s}$.

Theorem 2.1. *Let $f \in C(I)$. If f'' exists at a point $x \in I$, then we have*

$$\lim_{n\to\infty} n\left[\mathcal{K}_{n,r,s}(f;x) - f(x)\right] = \frac{1}{2}f'(x) + \frac{x(1-x)}{2}f''(x).$$

Proof. By Taylor's formula of f, we get

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varpi(t,x)(t-x)^2, \tag{2.1}$$

where $\lim_{t\to x} \varpi(t,x) = 0$. By applying the linearity of the operator $\mathcal{K}_{n,r,s}$, we obtain

$$\begin{aligned}\mathcal{K}_{n,r,s}(f;x) - f(x) \quad &= \mathcal{K}_{n,r,s}((t-x);x)f'(x) + \tfrac{1}{2}\mathcal{K}_{n,r,s}((t-x)^2;x)f''(x)\\ &\qquad + \mathcal{K}_{n,r,s}(\varpi(t,x)(t-x)^2;x).\end{aligned}$$

Now, applying the Cauchy-Schwarz property, we can get

$$n\mathcal{K}_{n,r,s}(\varpi(t,x)(t-x)^2;x) \leq \sqrt{\mathcal{K}_{n,r,s}(\varpi^2(t,x);x)}\sqrt{n^2\mathcal{K}_{n,r,s}((t-x)^4;x)}.$$

From Theorem 1.1, we have $\lim_{n\to\infty} \mathcal{K}_{n,r,s}(\varpi^2(t,x);x) = \varpi^2(x,x) = 0$, since $\varpi(t,x) \to 0$ as $t \to x$, and Remark 1.1 for every $x \in I$, we may write

$$\lim_{n\to\infty} n^2\mathcal{K}_{n,r,s}\left((t-x)^4;x\right) = 3x^2(1-x)^2. \tag{2.2}$$

Hence,

$$n\mathcal{K}_{n,r,s}(\varpi(t,x)(t-x)^2;x) = 0.$$

Applying Remark 1.1, we get

$$\begin{aligned}&\lim_{n\to\infty} n\mathcal{K}_{n,r,s}\left(t-x;x\right) = \frac{1}{2},\\ &\lim_{n\to\infty} n\mathcal{K}_{n,r,s}\left((t-x)^2;x\right) = x(1-x).\end{aligned} \tag{2.3}$$

Collecting the results from the above theorem is completed. □

3. Rate of convergence

$DBV(I)$ denotes the class of all absolutely continuous functions f defined on I, having on I a derivative $f^{'}$ equivalent to a function of bounded variation on I. We notice that the functions $f \in DBV(I)$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0)$$

where $g \in BV(I)$, i.e., g is a function of bounded variation on I.

The operators $\mathcal{K}_{n,r,s}(f;x)$ also admit the integral representation

$$\mathcal{K}_{n,r,s}(f;x) = \int_0^1 \mathcal{W}_{n,r,s}(x,t)f(t)dt, \tag{3.1}$$

where the kernel $\mathcal{W}_{n,r,s}(x,t)$ is given by

$$\mathcal{W}_{n,r,s}(x,t) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{k=0}^{s} p_{s,k}(x)\chi_{n,k}(t),$$

where $\chi_{n,k}(t)$ is the characteristic function of the interval $[k/n,(k+1)/n]$ with respect to I.

Lemma 3.1. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

(i) $\beta_{n,r,s}(x,y) = \int_0^y \mathcal{W}_{n,r,s}(x,t)dt \leq \dfrac{\mathcal{X}_{r,s} \quad x(1-x)}{n(x-y)^2}$, $0 \leq y < x$,

(ii) $1 - \beta_{n,r,s}(x,z) = \int_z^1 \mathcal{W}_{n,r,s}(x,t)dt \leq \dfrac{\mathcal{X}_{r,s} \quad x(1-x)}{n(x-y)^2} n(z-x)^2$, $x < z < 1$.

Proof. (i) Using Lemma 1.2 we get

$$\begin{aligned}\beta_{n,r,s}(x,y) &= \int_0^y \mathcal{W}_{n,r,s}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 \mathcal{W}_{n,r,s}(x,t)dt \\ &= \mathcal{K}_{n,r,s}((t-x)^2;x)(x-y)^{-2} \leq \frac{\mathcal{X}_{r,s} \quad x(1-x)}{n(x-y)^2}.\end{aligned}$$

As the proof of (ii) is similar, the details are omitted. □

Theorem 3.1. *Let $f \in DBV(I)$. Then for every $x \in (0,1)$ and sufficiently large*

n, we have

$$
\begin{aligned}
|D_n^{*(1/n)}(f;x)-f(x)| &\le \frac{|f'(x+)+f'(x-)|}{4n}+\sqrt{\frac{\mathcal{X}_{r,s}\;\;x(1-x)}{n}}\frac{|f'(x+)-f'(x-)|}{2}\\
&+\frac{\mathcal{X}_{r,s}\;\;(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x)\\
&+\frac{\mathcal{X}_{r,s}\;\;x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+((1-x)/k)}(f'_x)+\frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x)/\sqrt{n})}(f'_x),
\end{aligned}
$$

where $\bigvee_a^b(f'_x)$ denotes the total variation of f'_x on $[a,b]$ and f'_x is defined by

$$
f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\le t<x\\ 0, & t=x\\ f'(t)-f'(x+) & x<t<1. \end{cases} \tag{3.2}
$$

Proof. Since $\mathcal{K}_{n,r,s}(1;x)=1$, by using Lemma 1.1, for every $x\in(0,1)$ we get

$$
\begin{aligned}
\mathcal{K}_{n,r,s}(f;x)-f(x) &= \int_0^1 \mathcal{W}_{n,r,s}(x,t)(f(t)-f(x))dt\\
&= \int_0^1 \mathcal{W}_{n,r,s}(x,t)\int_x^t f'(u)dudt.
\end{aligned} \tag{3.3}
$$

For any $f\in DBV(I)$, by (3.2) we can write

$$
\begin{aligned}
f'(u) &= f'_x(u)+\frac{1}{2}(f'(x+)+f'(x-))+\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\\
&\quad+\delta_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))],
\end{aligned} \tag{3.4}
$$

where

$$
\delta_x(u)=\begin{cases} 1\;, & u=x\\ 0\;, & u\neq x. \end{cases}
$$

Obviously,

$$
\int_0^1\left(\int_x^t\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)du\right)\mathcal{W}_{n,r,s}(x,t)dt=0.
$$

By (3.1) and a straightforward calculation we have

$$
\begin{aligned}
\int_0^1\left(\int_x^t\frac{1}{2}(f'(x+)+f'(x-))du\right)\mathcal{W}_{n,r,s}(x,t)dt &= \frac{1}{2}(f'(x+)+f'(x-))\int_0^1(t-x)\mathcal{W}_{n,r,s}(x,t)dt\\
&= \frac{1}{2}(f'(x+)+f'(x-))\mathcal{K}_{n,r,s}((t-x);x)
\end{aligned}
$$

and

$$\Big|\int_0^1 \mathcal{W}_{n,r,s}(x,t)\Big(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)du\Big)dt\Big|$$

$$\begin{aligned}&\leq \frac{1}{2}\mid f'(x+)-f'(x-)\mid \int_0^1 |t-x|\mathcal{W}_{n,r,s}(x,t)dt\\&\leq \frac{1}{2}\mid f'(x+)-f'(x-)\mid \mathcal{K}_{n,r,s}(|t-x|;x)\\&\leq \frac{1}{2}\mid f'(x+)-f'(x-)\mid \Big(\mathcal{K}_{n,r,s}((t-x)^2;x)\Big)^{1/2}.\end{aligned}$$

Applying the lemmas 1.2 and 1.4 and using (3.3),(3.4) we obtain the following estimate

$$\begin{aligned}|\mathcal{K}_{n,r,s}(f;x)-f(x)| \leq\ & \frac{1}{4n}|f'(x+)+f'(x-)|\\&+\frac{1}{2}|f'(x+)-f'(x-)|\sqrt{\frac{\mathcal{X}_{r,s}\ \ x(1-x)}{n}}\\&+\Big|\int_0^x\Big(\int_x^t f'_x(u)du\Big)\mathcal{W}_{n,r,s}(x,t)dt\\&+\int_x^1\Big(\int_x^t f'_x(u)du\Big)\mathcal{W}_{n,r,s}(x,t)dt\Big|.\end{aligned}\tag{3.5}$$

Let

$$A_{n,r,s}(f'_x,x)=\int_0^x\Big(\int_x^t f'_x(u)du\Big)\mathcal{W}_{n,r,s}(x,t)dt,$$
$$B_{n,r,s}(f'_x,x)=\int_x^1\Big(\int_x^t f'_x(u)du\Big)\mathcal{W}_{n,r,s}(x,t)dt.$$

To complete the proof, it is sufficient to estimate the terms $A_{n,r,s}(f'_x,x)$ and $B_{n,r,s}(f'_x,x)$. Since $\int_a^b d_t\beta_{n,r,s}(x,t)\leq 1$ for all $[a,b]\subseteq[0,1]$, using integration by parts and applying Lemma 3.1 with $y=x-(x/\sqrt{n})$, we have

$$\begin{aligned}|A_{n,r,s}(f'_x,x)| &= \Big|\int_0^x\Big(\int_x^t f'_x(u)du\Big)d_t\beta_{n,r,s}(x,t)\Big|\\&= \Big|\int_0^x \beta_{n,r,s}(x,t)f'_x(t)dt\Big|\\&\leq \Big(\int_0^y+\int_y^x\Big)|f'_x(t)|\ |\beta_{n,r,s}(x,t)|dt\\&\leq \frac{\mathcal{X}_{r,s}\ \ x(1-x)}{n}\int_0^y\bigvee_t^x(f'_x)(x-t)^{-2}dt+\int_y^x\bigvee_t^x(f'_x)dt\\&\leq \frac{\mathcal{X}_{r,s}\ \ x(1-x)}{n}\int_0^y\bigvee_t^x(f'_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x).\end{aligned}$$

By the substitution of $u = x/(x-t)$, we obtain

$$\begin{aligned}\frac{\mathcal{X}_{r,s}\quad x(1-x)}{n}\int_0^{x-(x/\sqrt{n})}(x-t)^{-2}\bigvee_t^x(f'_x)dt &= \frac{\mathcal{X}_{r,s}\quad(1-x)}{n}\int_1^{\sqrt{n}}\bigvee_{x-(x/u)}^{x}(f'_x)du\\ &\le \frac{\mathcal{X}_{r,s}\quad(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-(x/k)}^{x}(f'_x)du\\ &\le \frac{\mathcal{X}_{r,s}\quad(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x).\end{aligned}$$

Thus,

$$(3.6)\quad |A_{n,r,s}(f'_x,x)| \le \frac{\mathcal{X}_{r,s}\quad(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x) + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x).$$

Using integration by parts and applying Lemma 3.1 with $z = x + ((1-x)/\sqrt{n})$, we have
$|B_{n,r,s}(f'_x,x)|$

$$\begin{aligned}&= \left|\int_x^1\left(\int_x^t f'_x(u)du\right)\mathcal{W}_{n,r,s}(x,t)dt\right|\\ &= \left|\int_x^z\left(\int_x^t f'_x(u)du\right)d_t(1-\beta_{n,r,s}(x,t)) + \int_z^1\left(\int_x^t f'_x(u)du\right)d_t(1-\beta_{n,r,s}(x,t))\right|\\ &= \left|\left[\int_x^t f'_x(u)(1-\beta_{n,r,s}(x,t))du\right]_x^z - \int_x^z f'_x(t)(1-\beta_{n,r,s}(x,t))dt\right.\\ &\quad\left.+\int_z^1\left(\int_x^t f'_x(u)du\right)d_t(1-\beta_{n,r,s}(x,t))\right|\\ &= \left|\int_x^z f'_x(u)du(1-\beta_{n,r,s}(x,z)) - \int_x^z f'_x(t)(1-\beta_{n,r,s}(x,t))dt\right.\\ &+ \left.\left[\int_x^t f'_x(u)du(1-\beta_{n,r,s}(x,t))\right]_z^1 - \int_z^1 f'_x(t)(1-\beta_{n,r,s}(x,t))dt\right|\\ &= \left|\int_x^z f'_x(t)(1-\beta_{n,r,s}(x,t))dt + \int_z^1 f'_x(t)(1-\beta_{n,r,s}(x,t))dt\right|\\ &\le \frac{\mathcal{X}_{r,s}\quad x(1-x)}{n}\int_z^1\bigvee_x^t(f'_x)(t-x)^{-2}dt + \int_x^z\bigvee_x^t(f'_x)dt\\ &= \frac{\mathcal{X}_{r,s}\quad x(1-x)}{n}\int_{x+((1-x)/\sqrt{n})}^1\bigvee_x^t(f'_x)(t-x)^{-2}dt + \frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x).\end{aligned}$$

By the substitution of $v=(1-x)/(t-x)$, we get

$$\begin{aligned}
|B_{n,r,s}(f'_x,x)| &\leq \frac{\mathcal{X}_{r,s}\ \ x(1-x)}{n}\int_1^{\sqrt{n}}\bigvee_x^{x+((1-x)/v)}(f'_x)(1-x)^{-1}dv+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x)\\
&\leq \frac{\mathcal{X}_{r,s}\ \ x}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_x^{x+((1-x)/v)}(f'_x)dv+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x)\\
&= \frac{\mathcal{X}_{r,s}\ \ x}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+((1-x)/k)}(f'_x)+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x))/\sqrt{n}}(f'_x). \qquad (3.7)
\end{aligned}$$

Collecting the estimates (3.5)-(3.7), we get the required result. This completes the proof of the theorem. □

4. Numerical Examples.

Example 4.1. In Figure 1, for $n=10, r=1, s=1$, the comparison of convergence of $\mathcal{K}_{n,r,s}(f;x)$ (yellow) and Bernstein-Kantorovich $M_n(f;x)$ (blue) operators to $f(x)=e^{x^3}$ (red) is illustrated. It is observed that the $\mathcal{K}_{n,r,s}(f;x)$ gives a better approximation to $f(x)$ than Bernstein-Kantorovich $M_n(f;x)$ operators for $n=10, r=1, s=1$.

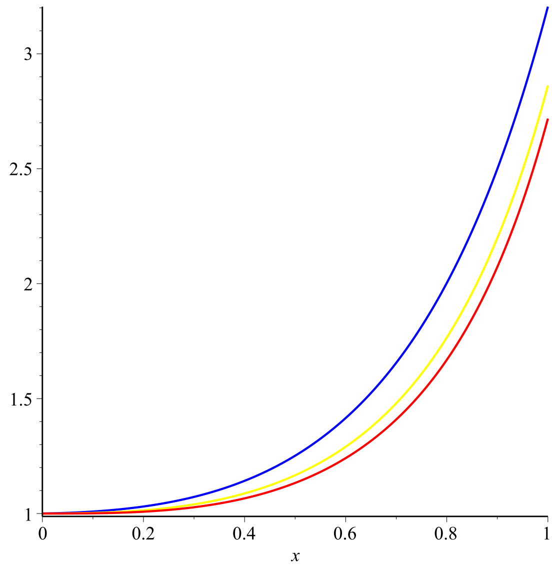

Figure 1.The convergence of $M_{10}(f;x)$ and $\mathcal{K}_{10,1,1}(f;x)$ to $f(x)$

Example 4.2. In Figure 2, for $n=50, r=1, s=1$, the comparison of convergence of $\mathcal{K}_{n,r,s}(f;x)$ (yellow) and Bernstein-Kantorovich $M_n(f;x)$ (blue) operators to $f(x)=x^2\sin\left(\frac{2x}{\pi}\right)$ (red) is illustrated. It is observed that the $\mathcal{K}_{n,r,s}(f;x)$ gives a better approximation to $f(x)$ than Bernstein-Kantorovich $M_n(f;x)$ operators for $n=50, r=1, s=1$.

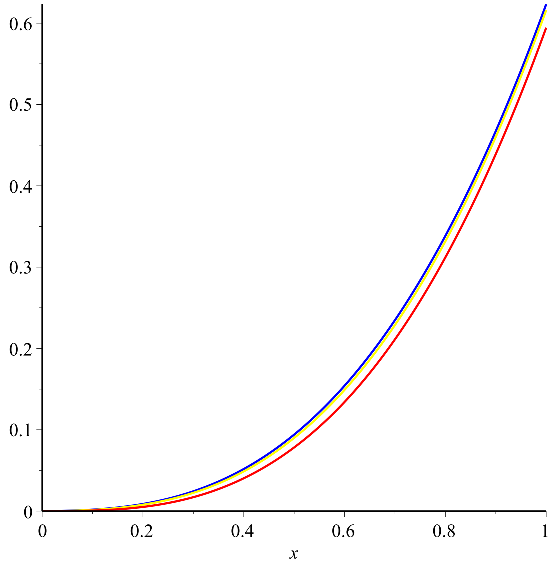

Figure 2.The convergence of $M_{50}(f;x)$ and $\mathcal{K}_{50,1,1}(f;x)$ to $f(x)$

REFERENCES

1. U. Abel and M. Heilmann, *The complete asymptotic expansion for Bernstein-Durrmeyer operators with Jacobi weights*, Mediterr. J.Math. 1 (2004) 487-499.
2. U. Abel, M. Ivan and R. Paltanea, *The Durrmeyer variant of an operator defined by D.D. Stancu*, Appl. Math. Comput. 259 (2015) 116-123.
3. T. Acar, A. Aral and S. A. Mohiuddine, *Approximation by bivariate (p,q)-Bernstein-Kantorovich operators*, Iran. J. Sci. Technol. Trans. Sci. 42 (2018) 655-662.
4. T. Acar, A. Aral and I. Raşa, *Approximation by k-th order modifications of Szász-Mirakyan operators*, Studia Sci. Math. Hungar. 53 (2016) 379–398.
5. T. Acar, A. Aral and S. A. Mohiuddine, *On Kantorovich modification of (p,q)-Baskakov operators*, J. Inequal. Appl. 98 2016, (2016) 1-14.
6. T. Acar, A. Aral and I. Raşa, *Modified Bernstein-Durrmeyer operators*, General Math. 22 (1) (2014) 27-41.
7. T. Acar, *Asymptotic formulas for generalized Szász-Mirakyan operators*, Appl. Math. Comput. 263 (2015), 223-239.
8. A. M. Acu, *Stancu-Schurer-Kantorovich operators based on $q-$integers*, Appl. Math. Comput. 259 (2015) 896-907.
9. P. N. Agrawal, N. Ispir and A. Kajla, *Approximation properties of Lupas-Kantorovich operators based on Polya distribution*, Rend. Circ. Mat. Palermo 65 (2016) 185-208.
10. P. N. Agrawal, M. Goyal and A. Kajla, *$q-$Bernstein-Schurer-Kantorovich type operators*, Boll. Unione Mat. Ital. 8 (2015) 169-180.
11. D. Barbosu, *Kantorovich-Stancu type operators*, J. Inequal. Pure Appl. Math. 5 (2004) 1-6.

12. D. Cárdenas-Morales and V. Gupta, *Two families of Bernstein-Durrmeyer type operators*, Appl. Math. Comput. 248 (2014) 342-353.

13. D. Costarelli and G. Vinti, *A Quantitative estimate for the sampling Kantorovich series in terms of the modulus of continuity in orlicz spaces*, Constr. Math. Anal., 2 (1) (2019), 8-14.

14. S. G. Gal and S. Trifa, *Quantitative Estimates for L^p-Approximation by Bernstein-Kantorovich-Choquet Polynomials with Respect to Distorted Lebesgue Measures*, Constr. Math. Anal. 2 (1) (2019) 15-21.

15. H. Gonska, M. Heilmann and I. Rasa, *Kantorovich operators of order k*, Numer. Funct. Anal. Optim. 32 (2011) 717-738.

16. H. Gonska and R. Paltanea, *Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions*, Czech. Math. J. 60 (2010) 783-799.

17. V. Gupta and R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer (2014).

18. V. Gupta, A. M. Acu and D. F. Sofonea, *Approximation of Baskakov type Pòlya-Durrmeyer operators*, Appl. Math. Comput. 294 (2017) 318-331.

19. V. Gupta, T. M. Rassias, P. N. Agrawal and A. M. Acu, *Recent Advances in Constructive Approximation Theory*, Springer, 2018.

20. V. Gupta and T. M. Rassias, *Lupas-Durrmeyer operators based on Polya distribution*, Banach J. Math. Anal. 8 (2) (2014) 146-155.

21. V. Gupta and G. Tachev, *Approximation with Positive Linear Operators and Linear Combinations*, Springer, Cham, 2017.

22. V. Gupta, G. Tachev, and A.M. Acu, *Modified Kantorovich operators with better approximation properties*, Numerical Algorithms, DOI: 10.1007/s11075-018-0538-7.

23. A. Kajla, *The Kantorovich variant of an operator defined by D. D. Stancu*, Appl. Math. Comput. 316 (2018) 400-408.

24. A. Kajla and T. Acar, *A new modification of Durrmeyer type mixed hybrid operators*, Carpathian J. Math. 34 (1) (2018) 47-56.

25. A. Kajla and T. Acar, *Blending type approximation by generalized Bernstein-Durrmeyer type operators*, Miskolc Math Notes 19 (1) (2018) 319-336.

26. A. Kajla and P. N. Agrawal, *Szász-Durrmeyer type operators based on Charlier polynomials*, Appl. Math. Comput. 268 (2015) 1001-1014.

27. R. Maurya, H. Sharma and C. Gupta, *Approximation properties of Kantorovich type modifications of (p,q)-Meyer-König-Zeller operators*, Constr. Math. Anal., 1 (1) (2018) 58-72.

28. D. Miclaus, *The generalization of certain results for Kantorovich operators*, General Math 21 (2) (2013) 47-56.

29. D. Miclăuş and O. T. Pop, *The Voronovskaja theorem for some linear positive operators defined by infinite sum*, Creat. Math. Inform, 20 (2011) 55-61.

30. S. A. Mohiuddine T. Acar, A. Alotaibi, *Construction of a new family of Bernstein-Kantorovich operators*, Math. Meth. Appl. Sci. 40 (2017) 7749-7759.

31. D. D. Stancu, *The remainder in the approximation by a generalized Bernstein operator: a representation by a convex combination of second-order divided differences*, Calcolo 35 (1998) 53-62.

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RICCI SOLITONS AND GRADIENT RICCI SOLITONS ON NEARLY KENMOTSU MANIFOLDS

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Abstract. In this paper, we study nearly Kenmotsu manifolds with a Ricci soliton and we obtain certain conditions about curvature tensors.
Keywords: Contact manifold, Nearly Kenmotsu Manifold, Ricci Solitons.

1. Introduction and Preliminaries

Ricci solitons $\frac{\partial}{\partial t}g = -2S$ reflected on the modulo diffeomorphisms and scales from the space of the metrics are fixed points of the Ricci flow and mostly explosive limits for the Ricci flow in compact manifolds. Generally, physicists have studied Ricci solitons in relation with string theory. In particular, in differential geometry we use a Ricci soliton as a special type of the Riemannian metric. Such metrics builds from the Ricci flow only by symmetries of the flow so they can be viewed as generalizations of Einstein metrics. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of the Einstein metric such that [12]

$$\pounds Vg + 2S + 2\lambda g = 0 \tag{1.1}$$

where S is a Ricci tensor and $\pounds V$ is the Lie derivative along the vector field V on M and λ is a real number.

Depending on whether λ is negative, zero or positive, a Ricci soliton is named shrinking, steady or expanding, respectively. In addition, if the vector field V is the gradient of a potential function $-f$, then the metric g is called a gradient Ricci soliton. We can regulate the (1.1) as

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

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Ricci solitons firstly become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. [19] In particular, after Sharma had studied the Ricci solitons in contact geometry, Ricci flows in contact geometry gained a significant attention. They have been studied extensively ever since. The geometry of Ricci solitons in contact metric manifolds have been studied by authors such as Bagewadi, Bejan and Crasmareanu, Blaga, Hui et al., Chen, Deshmukh et al., Nagaraja and Premalatta, Tripathi and many others. In [11], Ricci solitons in K-contact manifolds were studied by Sharma. Ghosh, Sharma and Cho [11] studied gradient Ricci solitons in non-Sasakian (k, μ)-contact manifolds. In addition, in [21], Tripathi showed gradient Ricci solitons, compact Ricci solitons in $N(k)$-contact metric manifolds and (k, μ)-manifolds. Recently in [1], B. Barua and U. C. De focused on some properties of Ricci solitons in Riemannian manifolds.

Einstein solitons are open examples of Ricci solitons, where g is an Einstein metric and X is a Killing vector field. On a compact manifold, a Ricci soliton has a constant curvature, especially in dimension 2 and in dimension 3 [12, 13]. For details about these studies, we refer the reader to Chow and Knopf [8] and Derdzinski [10]. An important result by Perelman shows that on a compact manifold, the Ricci soliton is a gradient Ricci soliton.

Based on these studies, in this paper we review Ricci solitons $(R.S)$ and gradient Ricci solitons $(G.R.S)$ in a nearly Kenmotsu manifold. The paper progresses as follows. After some preliminary information and definitions in Section 2, we consider the case that in a nearly Kenmotsu manifold, if g admits a $(R.S)$ in the form of (g, V, λ) and V is point-wise collinear with ξ, then the manifold is an η-Einstein manifold. Furthermore, we show that if a nearly Kenmotsu manifold admits a compact $(R.S)$, then the manifold is Einstein. Finally, in the last section, we prove that when an η-Einstein nearly Kenmotsu manifold admits a $(G.R.S)$, the manifold transforms into an Einstein manifold under certain conditions.

Let M be an n-dimensional nearly Kenmotsu manifold with the (ϕ, ξ, η, g) structure that ϕ is a $(1,1)$ type tensor field, ξ is a contravariant vector field, η is a 1-form and g is a Riemannian metric. Then by definition, it satisfies the following relation [15]

$$\eta(\xi) = 1, \qquad \phi^2 = -I + \eta \otimes \xi, \tag{1.3}$$

$$\phi\xi = 0, \quad \eta\phi = 0, \quad \nabla_X \xi = X - \eta(X)\xi, \tag{1.4}$$

$$\eta(X) = g(\xi, X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.5}$$

$$(\nabla_X \eta)(Y) = \Omega(Y, X), \quad \Omega(X, Y) = \Omega(Y, X) \quad (\Omega(Y, X) = g(\phi Y, X)), \tag{1.6}$$

$$(\nabla_Z \Omega)(X, Y) = \{g(X, Z) + \eta(X)\eta(Z)\}\eta(Y) + \{g(Y, Z) + \eta(Y)\eta(Z)\}\eta(X) \tag{1.7}$$

for any vector fields X, Y and Z on M, where the $\otimes$ is the tensor product and I shows the identity map on T_pM.

In an n-dimensional nearly Kenmotsu manifold with (ϕ, ξ, η, g) structure, the following relations hold.

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad S(X, \xi) = -(n-1)\eta(X), \tag{1.8}$$

$$R(\xi, Y)X = -g(Y, X)\xi + \eta(X)Y, \quad R(Y, X)\xi = \eta(Y)X - \eta(X)Y, \tag{1.9}$$

$$\phi(R(X, \phi Y)Z) = R(X, Y)Z + 2\{\eta(Y)X - \eta(X)Y\}\eta(Z) + 2\{g(X, Z)\eta(Y) \\ -g(Y, Z)\eta(X)\} + \Omega(X, Z)\phi(Y) - \Omega(Y, Z)\phi X + g(Y, Z)X - g(X, Z)Y, \tag{1.10}$$

where R is the curvature tensor and S is the Ricci tensor with respect to g. If the Ricci tensor S satisfies the condition

$$S = ag + b\eta \otimes \eta, \tag{1.11}$$

then an n-dimensional nearly Kenmotsu manifold is said to be η-Einstein. In the Ricci tensor equation, a and b are smooth functions on M.

In an η-Einstein nearly Kenmotsu manifold, the Ricci tensor S and Ricci operator Q are shown in the form below.

$$S(X, Y) = \left[\frac{r}{n-1} - 1\right] g(X, Y) + \left[\frac{r}{n-1} - n\right] \eta(X)\eta(Y) \tag{1.12}$$

$$QX = \left[\frac{r}{n-1} - 1\right] X + \left[\frac{r}{n-1} - n\right] \eta(X)\xi. \tag{1.13}$$

2. $(R.S)$ on Nearly Kenmotsu Manifolds

Suppose that a nearly Kenmotsu manifold admits a $(R.S)$. Considering the properties of nearly Kenmotsu manifolds with $(R.S)$, we know that $\nabla g = 0$. Since λ in the $(R.S)$ equation is a constant, we can specify that $\nabla \lambda g = 0$. Because of this, it is easy to say that $\pounds_V g + 2S$ is parallel.

It was proved in [16] that if a nearly Kenmotsu manifold with a symmetric parallel $(0, 2)$ type tensor, then the tensor is a constant multiple of the metric tensor. As a result of this theorem, we can say that $\pounds_V g + 2S$ is a constant multiple of metric tensors g, i.e., and $\pounds_V g + 2S = ag$, such that a is constant.

From the above equations, we can write $\pounds_V g + 2S + 2\lambda g$ as $(a + 2\lambda)g$. Then using $(R.S)$, we get $\lambda = -a/2$.

Based on these results we can write the following proposition.

Proposition 2.1. *In a nearly Kenmotsu manifold, depending on whwther a is positive or negative, $(R.S)$ with the form of (g, λ, V) is shrinking or expanding. Particularly, let V be point-wise collinear with ξ i.e. $V = b\xi$, where b is a function on a nearly Kenmotsu manifold. Then*

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{2.1}$$

which adds up to

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

or,

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0.$$

Using (1.4), we get

$$2bg(X,Y) - 2b\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{2.2}$$

Then putting $Y = \xi$ in (2.2) we obtain

$$(Xb) + \eta(X)\xi b + 2(1-n)\eta(X) + 2\lambda\eta(X)$$

or,

$$(Xb) = (1 - n - \lambda)\eta(X). \tag{2.3}$$

We know that in a nearly Kenmotsu manifold $d\eta = 0$ and from (2.3) we get

$$Xb = 0$$

if

$$\lambda = 1 - n.$$

Theorem 2.1. *If in a nearly Kenmotsu manifold, the metric g is a $(R.S)$ and V is point-wise collinear with ξ, then V is a constant multiple of ξ on condition that $\lambda = 1 - n$. Especially, if we take $V = \xi$. Then*

$$(\pounds_V g + 2S + 2\lambda g)(X,Y) = 0,$$

implies that

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{2.4}$$

Substituting $X = \xi$, we get $\lambda = -(n-1) < 0$. Because it is negative, we can say that the $(R.S)$ is shrinking.

Particularly, if the manifold is a nearly Kenmotsu manifold, then we have

$$(\nabla_X \eta)(Y) = g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.5}$$

Hence using (2.3), (2.5) Equation (2.2) becomes

$$S(X,Y) = (n-2)g(X,Y) + \eta(X)\eta(Y), \tag{2.6}$$

that is, it is an $\eta-$Einstein manifold. In addition, we have the following theorem.

Theorem 2.2. *If in a nearly Kenmotsu manifold the metric g is a $(R.S)$ and V is point-wise collinear with ξ, then the manifold is an η-Einstein manifold.*

Conversely, if we have a nearly Kenmotsu η-Einstein manifold M with the following form in which γ and δ constants

$$S(X,Y) = \delta g(X,Y) + \gamma\eta(X)\eta(Y), \tag{2.7}$$

then taking $V = \xi$ in (2.1) and using the above equation, we obtain

$$\begin{aligned}(\pounds\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) \\ = 2(1+\lambda+\delta)g(X,Y) + 2(\gamma-1)\eta(X)\eta(Y).\end{aligned} \tag{2.8}$$

From Equation (2.8) it follows that M with a $(R.S)$ with the form of (g,ξ,λ) such that $\lambda = \gamma - \delta$.

So we have the following theorem.

Theorem 2.3. *If a nearly Kenmotsu Manifold is η-Einstein, then the manifold admits a $(R.S)$ of type $(g,\xi,(\gamma-\delta))$.*

Again, as a result of some adjustments, we get from (2.6)

$$r = (n-1)^2 = constant.$$

By the last equation, the scalar curvature is constant.

In [11] Sharma proved that a compact Ricci soliton with a constant scalar curvature is Einstein. Therefore, from this theorem, we give the following result.

Corollary 2.1. *Let M be a nearly Kenmotsu manifold with a compact $(R.S)$, then the manifold is Einstein.*

3. $(G.R.S)$ on Nearly Kenmotsu Manifolds

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci Soliton and we can regulate (1.1) as

$$\nabla\nabla f = S + \lambda g. \tag{3.1}$$

This can be written as

$$\nabla_Y Df = QY + \lambda Y, \tag{3.2}$$

where D shows the gradient operator of g. From (3.2) it is clear that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{3.3}$$

This implies that

$$g(R(\xi,Y)Df,\xi) = g((\nabla\xi Q)Y,\xi) - g((\nabla_Y Q)\xi,\xi). \tag{3.4}$$

Now using (1.13) and (1.4) we have

$$(\nabla_Y Q)(X) = \left[\frac{r}{1-n} - n\right](-2\eta(X)\eta(Y)\xi + g(X,Y)\xi + \eta(X)Y). \tag{3.5}$$

Then clearly

$$g((\nabla_X Q)\xi - (\nabla\xi Q)X, \xi) = 0. \tag{3.6}$$

Then we have from (3.4)

$$g(R(\xi, X)Df, \xi) = 0. \tag{3.7}$$

From (1.9) and (3.7) we get

$$g(R(\xi, Y)Df, \xi) = -g(Y, Df) + \eta(Df)\eta(Y) = 0.$$

Hence

$$Df = \eta(Df)\xi = g(Df, \xi)\xi = (\xi f)\xi. \tag{3.8}$$

Using (3.8) in (3.2) we get

$$S(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X) + \xi f g(\phi X, \phi Y). \tag{3.9}$$

Putting $X = \xi$ in (3.9) and using (2.3) we get

$$Y(\xi f) = (1 - n + \lambda)\eta(Y). \tag{3.10}$$

With this equation, it is clear that if $\lambda = n - 1$.

So from here, $\xi f = constant$. Then using (3.8) we have

$$Df = (\xi f)\xi = c\xi.$$

Particularly, taking a frame field $\xi f = 0$, we get from (3.8), $f = constant$. Therefore, Equation (3.1) can be shown as

$$S(X,Y) = (1-n)g(X,Y),$$

that is M is an Einstein manifold.

Theorem 3.1. *If an η-Einstein nearly Kenmotsu manifold admits a $(G.R.S)$ then the manifold transforms to an Einstein manifold provided $\lambda = 1 - n$ and with the frame field $\xi f = 0$.*

REFERENCES

1. B. Barua and U. C. De: *Characterizations of a Riemannian manifold admitting Ricci solitons.* Facta Universitatis(NIS) Ser. Math. Inform. **28**(2) (2013), 127–132.
2. A. M. Blaga and M. Crasmareanu: *Torse forming $\eta-$Ricci solitons in almost paracontact $\eta-$Einstein geometry.* Filomat, (to appear).

3. D. E. Blair: *Riemannian geometry of contact and symplectic manifolds*. Progress in Mathematics, 203. Birkhuser Boston, Inc., Boston, MA, 2002.

4. M. M. Boothby and R. C. Wong: *On contact manifolds*. Ann. Math. **68** (1958), 421–450.

5. H. D. Cao: *Geometry of Ricci solitons*. Chinese Ann. Math. Ser. B **27** (2006), 121–142.

6. T. Chave and G. Valent: *Quasi-Einstein metrics and their renoirmalizability properties*. Helv. Phys. Acta. **69** (1996), 344–347.

7. B. Y. Chen: *Geometry of Submanifolds*. Marcel Dekker, Inc. New York, 1973.

8. B. Chow and D. Knopf: *The Ricci flow: An introduction*. Mathematical Surveys and Monographs 110. American Math. Soc., 2004.

9. U. C. De: *Ricci soliton and gradient Ricci soliton on P-Sasakian manifolds*. The Aligarh Bull. of Maths. **29** (2010), 29–34.

10. A. Derdzinski: *Compact Ricci solitons*. Preprint.

11. A. Ghosh, R. Sharma and J. T. Cho: *Contact metric manifolds with $\eta-$parallel torsion tensor*. Ann. Glob. Anal. Geom. **34** (2008), 287–299.

12. R. S. Hamilton: *The Ricci flow on surfaces*. Math. and General Relativity, Contemporary Math. **71** (1988), 237–262.

13. T. Ivey: *Ricci solitons on compact 3-manifolds*. Differential Geo. Appl. **3** (1993), 301–307.

14. K. Kenmotsu: *A class of almost contact Riemannian manifolds*. Tohoku Math. J. **24** (1972), 93–103.

15. B. Najafi and N. H. Kashani: *On nearly Kenmotsu Manifolds*. Turkish Journal of Mathematics **37** (6) (2013), 1040–1047.

16. R. H. Ojha: *A note on the M-projective curvature tensor*. Indian J. Pure Appl. Math. **8**(12) (1975), 1531–1534.

17. G. Perelman: *The entopy formula for the Ricci flow and its geometric applications*. Preprint, http://arxiv.org/abs/math.DG/02111159.

18. G. P. Pokhariyal and R. S. Mishra: *Curvature tensor and their relativistic significance II*. Yokohama Mathematical Journal **19** (1971), 97–103.

19. R. Sharma: *Certain results on k-contact and (k,μ)-contact manifolds*. J. of Geom. **89** (2008), 138–147.

20. S. Tanno: *Curvature tensors and non-existence of killing vectors*. Tensor N.S. **22** (1971), 387–394.

21. M. M. Tripathi: *Ricci solitons in contact metric manifolds*. arXiv:0801.4222v1 [math.DG] 28 Jan 2008.

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THE NEW WEIGHTED INVERSE RAYLEIGH DISTRIBUTION AND ITS APPLICATION

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Abstract. In this study, a new weighted version of the inverse Rayleigh distribution based on two different weight functions is introduced. Some statistical and reliability properties of the introduced distribution including the moments, moment generating function, entropy measures (i.e., Shannon and Rényi) and survival and hazard rate functions are derived. The maximum likelihood estimators of the unknown parameters cannot be obtained in explicit forms. So, a numerical method has been required to compute maximum likelihood estimates. Finally, the daily mean wind speed data set has been analysed to show the usability of the new weighted inverse Rayleigh distribution.

Keywords: New weighted inverse Rayleigh distribution; Shannon entropy; hazard rate function; Fisher information matrix; wind speed data.

1. Introduction

The accuracy of procedures in the statistical analysis depends on the suitableness of a distribution used in modeling a data set. Therefore, many statistical distributions have been proposed in the literature because it is very important to determine the distribution which provides the best fit to a data set.

One of the widely-used statistical distributions in the context of reliability studies is the inverse Rayleigh (IR) distribution introduced by Trayer [24]. Sherina and Oluyede [25] stated that the distribution of lifetimes of several types of experimental units can be modeled by the IR distribution. Various extensions of this distribution have been proposed in the literature: transmuted IR distribution [1], modified IR distribution [10], kumaraswamy IR distribution [21] and beta IR distribution [12].

On the other hand, the theory of weighted distributions introduced by Rao [17] and Fisher [3] provides a unifying approach to deal with the problems of model

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specification and data interpretation (see [9]). There are more studies on weighted distributions and their applications in various fields including ecology and reliability (see [6], [7], [16], [14], [15], [19], [13] and [4] among the others). Fatima and Ahmad [8] also introduced a weighted *IR* (*WIR*) distribution with a single weight function $w(x) = x^k$ where $k \geq 0$, and they studied several of its properties.

The objective of the paper is to introduce a new weighted version of *IR* distribution obtained by using two different weight functions and to discuss its basic characteristics.

The rest of the paper is organized as follows. The new *WIR* (*NWIR*) distribution is introduced in Section 2. Some of its statistical and reliability properties are given in Section 3. Equations of maximum likelihood estimates of parameters and a Fisher information matrix are obtained in Section 4. In Section 5, an application of the distribution to real data is presented. Finally, the paper ends with a conclusion.

2. The New Weighted Inverse Rayleigh Distribution

Suppose that X is a non-negative random variable with its probability density function (*pdf*), and $w(x)$ is weight function where $E(w(x)) < \infty$. The *pdf* of weighted distribution of X can be defined as

$$f_w(x) = \frac{w(x) f(x)}{E(w(x))}. \tag{2.1}$$

It should be noted that a general class of weight functions $w(x)$ can be defined by

$$w(x) = x^i e^{jx} F^k(x) (1 - F(x))^l,$$

see [23]. Weight functions can be determined for a different combination of i, j, k and l values. If we take $w(x) = x^i$, then the obtained distribution is called size-biased distribution, and it is length-biased distribution for $i = 1$.

Let X be a random variable with the *IR* distribution having the scale parameter λ. The *pdf* and cumulative density function (*cdf*) of the *IR* distribution are given by

$$\begin{aligned} f(x) &= 2\lambda x^{-3} e^{-\lambda x^{-2}}, \; x > 0, \; \lambda > 0, \\ F(x) &= e^{-\lambda x^{-2}}, \; x > 0, \; \lambda > 0, \end{aligned}$$

respectively. Now, substituting the multiplication of weighted functions, $w_1(x) = x^{-\alpha}$ and $w_2(x) = e^{-\alpha x^{-2}}$, and *pdf* of *IR* distribution in (2.1), the *pdf* of the *NWIR* distribution is defined by

$$\begin{aligned} f_w(x) &= \frac{w_1(x) w_2(x) f(x)}{E(w_1(x) w_2(x))} \\ &= \frac{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}+1\right)} x^{-(\alpha+3)} e^{-(\alpha+\lambda)x^{-2}}, x > 0, \lambda > 0, \alpha > 0, \end{aligned} \tag{2.2}$$

where

$$E\left(w_1\left(x\right)w_2\left(x\right)\right) = \int_0^{\infty} 2\lambda x^{-(\alpha+3)}e^{-(\alpha+\lambda)x^{-2}}dx = \frac{\lambda\Gamma\left(\frac{\alpha}{2}+1\right)}{\left(\alpha+\lambda\right)^{\frac{\alpha}{2}+1}} < \infty.$$

It should be noted that the following transformation is applied in order to calculate $E\left(w_1\left(x\right)w_2\left(x\right)\right)$

$$u = \left(\alpha+\lambda\right)x^{-2} \Longrightarrow x = \sqrt{\frac{\alpha+\lambda}{u}} \Longrightarrow du = -2\left(\alpha+\lambda\right)x^{-3}dx. \tag{2.3}$$

The corresponding *cdf* of the *NWIR* distribution is

$$F_w\left(x\right) = \frac{\Gamma\left(\frac{\alpha}{2}+1, \frac{\alpha+\lambda}{x^2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)} = 1 - \frac{\gamma\left(\frac{\alpha}{2}+1, \frac{\alpha+\lambda}{x^2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}. \tag{2.4}$$

Here $\Gamma\left(\frac{\alpha}{2}+1, \frac{\alpha+\lambda}{x^2}\right)$ is an upper incomplete Gamma function defined by

$$\Gamma\left(a,x\right) = \int_x^{\infty} t^{a-1}e^{-t}dt.$$

$$\Gamma\left(a,x\right) = \Gamma\left(a\right) - \gamma\left(a,x\right),$$

where $\gamma\left(a,x\right)$ is a lower incomplete Gamma function as

$$\gamma\left(a,x\right) = \int_0^{x} t^{a-1}e^{-t}dt.$$

In FIG. 2.1, different *pdf* and *cdf* plots of the *NWIR* distribution are presented for the selected values of parameters α and λ. Now, let $Y = \left(\alpha+\lambda\right)X^{-2}$, where X has the *NWIR* distribution with parameters α and λ. The *pdf* of the random variable Y becomes

$$f\left(y\right) = \frac{1}{\Gamma\left(\frac{\alpha}{2}+1\right)}y^{\frac{\alpha}{2}}e^{-y}$$

for $y > 0$. Thus, the random variable Y has a Gamma distribution shown as $Y \sim Gamma\left(\frac{\alpha}{2}+1, 1\right)$.

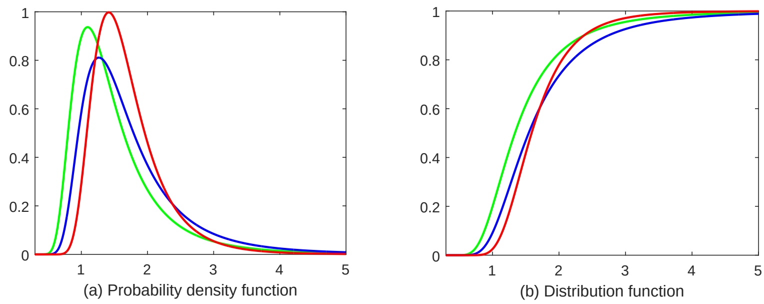

FIG. 2.1: Plots of the *pdf* and *cdf* of the *NWIR* distribution where $\alpha = 2, \lambda = 1$ (green line); $\alpha = 2, \lambda = 2$ (blue line); $\alpha = 5, \lambda = 3$ (red line)

3. Statistical and Reliability Properties

In this section we consider some statistical and reliability properties of the *NWIR* distribution.

3.1. r^{th} moments

If a random variable X has the *NWIR* distribution with a scale parameter λ and shape parameter α, then the r^{th} moment of the *NWIR* distributed random variable X is obtained as

$$E\left(X^r\right) = \int_0^\infty \frac{2\left(\alpha+\lambda\right)^{\frac{\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}+1\right)} x^{r-\alpha-3} e^{-(\alpha+\lambda)x^{-2}} dx.$$

In order to calculate $E\left(X^r\right)$, using the transformation in (2.3), we obtain

$$E\left(X^r\right) = \left(\alpha+\lambda\right)^{\frac{r}{2}} \frac{\Gamma\left(\frac{\alpha-r}{2}+1\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

Hence, from the r^{th} moment of the *NWIR* distribution, the first four moments can be easily calculated to obtain the mean, variance, coefficient of skewness and the coefficient of kurtosis of the *NWIR* distribution as follows

$$\begin{aligned}
E\left(X\right) &= \left(\alpha+\lambda\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}, \\
E\left(X^2\right) &= \frac{2\left(\alpha+\lambda\right)}{\alpha}, \\
E\left(X^3\right) &= \left(\alpha+\lambda\right)^{\frac{3}{2}} \frac{\Gamma\left(\frac{\alpha-3}{2}+1\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)},
\end{aligned}$$

and

$$E\left(X^4\right) = (\alpha+\lambda)^2 \frac{\Gamma\left(\frac{\alpha-4}{2}+1\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$$

3.2. Moment generating function

The moment generating function of the *NWIR* distribution is given as follows. formula

$$\begin{aligned} M_X(t) &= E\left(e^{tx}\right) \\ &= \int_0^\infty e^{tx} \frac{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}+1\right)} x^{-(\alpha+3)} e^{-(\alpha+\lambda)x^{-2}} dx. \end{aligned}$$

By applying the Maclaurin series $e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$ and setting the transformation in (2.3), we finally get

$$M_X(t) = \frac{1}{\Gamma\left(\frac{\alpha}{2}+1\right)} \sum_{n=0}^{\infty} \frac{t^n}{n!} (\alpha+\lambda)^{\frac{n}{2}} \Gamma\left(\frac{\alpha-n}{2}+1\right).$$

3.3. Quantile function

The quantile function of the *NWIR* distribution is obtained by

$$x_q = F_w^{-1}(q), 0 < q < 1, \tag{3.1}$$

where $F_w^{-1}(q)$ is the inverse of *cdf* in (2.4). The median of the *NWIR* distributed random variable X can be found by putting $q = 0.5$ in (3.1). $F_w^{-1}(q)$ can be computed numerically via some mathematical and statistical software packages since it does not have a closed-form expression. Moreover, the equation in (3.1) can be used in order to generate a random number from the proposed distribution.

3.4. Mode

Now, the natural logarithm of the $f_w(x)$ in (2.2) is given by

$$\ln f_w(x) \propto -(\alpha+3)\ln x - (\alpha+\lambda) x^{-2}. \tag{3.2}$$

Using the differentiating equation (3.2) with respect to x, we obtain as

$$\frac{d}{dx} \ln f_w(x) = -(\alpha+3) x^{-1} + 2(\alpha+\lambda) x^{-3}. \tag{3.3}$$

If the equation (3.3) is equal to 0 and solve for x, then the mode of the *NWIR* distribution has the following expression

$$X_M = \sqrt{\frac{2(\alpha+\lambda)}{\alpha+3}}$$

for $\alpha > 0$ and $\lambda > 0$. Note that $f_w(x)$ is increasing when $x \in (0, X_M)$ and is decreasing when $x \in (X_M, \infty)$.

3.5. Shannon entropy

The statistical entropy introduced by Shannon [22] is defined as a measure of the information content associated with the outcome of a random variable (see [2]). The Shannon entropy of the *NWIR* distribution is expressed by

$$\begin{aligned} I_S(\alpha,\lambda) &= -E(\ln f_w(x)) \\ &= \ln\left(\frac{\Gamma\left(\frac{\alpha}{2}+1\right)}{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}\right) + (\alpha+3)E(\ln x) \\ &\quad + (\alpha+\lambda)E\left(x^{-2}\right). \end{aligned} \tag{3.4}$$

To calculate $E(\ln x)$, if we use the transformation in (2.3), then we have

$$\begin{aligned} E(\ln x) &= \frac{1}{2\Gamma\left(\frac{\alpha}{2}+1\right)} \int_0^\infty u^{\frac{\alpha}{2}} (\ln(\alpha+\lambda) - \ln u) e^{-u} du \\ &= \frac{1}{2}\left(\ln(\alpha+\lambda) - \Psi\left(\frac{\alpha}{2}+1\right)\right), \end{aligned} \tag{3.5}$$

where Ψ is a digamma function with

$$\Psi(r) = \frac{d}{dr}\ln\Gamma(r) = \frac{\Gamma'(r)}{\Gamma(r)}, r > 0$$

defined as the logarithmic derivative of the Gamma function. It is also well known that the derivative of $\Gamma(r)$ is

$$\Gamma'(r) = \int_0^\infty t^{r-1}(\ln t)e^{-t}dt.$$

Substituting $E\left(x^{-2}\right) = \frac{\frac{\alpha}{2}+1}{\alpha+\lambda}$ and (3.5) into (3.4), Shannon entropy of the *NWIR* distribution $I_S(\alpha,\lambda)$ becomes

$$\begin{aligned} I_S(\alpha,\lambda) &= \ln\left(\frac{\Gamma\left(\frac{\alpha}{2}+1\right)}{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}\right) + \left(\frac{\alpha}{2}+1\right) \\ &\quad + \frac{(\alpha-3)}{2}\left(\ln(\alpha+\lambda) - \Psi\left(\frac{\alpha}{2}+1\right)\right). \end{aligned}$$

3.6. Rényi entropy

Rényi entropy considered by Rényi [18] is a generalization of the Shannon entropy. The Rényi entropy of the *NWIR* distribution is expressed by

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta}\ln\int_0^\infty f_w^\delta(x)\,dx \\ &= \frac{1}{1-\delta}\ln\int_0^\infty \left(\frac{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}+1\right)}x^{-(\alpha+3)}e^{-(\alpha+\lambda)x^{-2}}\right)^\delta dx \\ &= \frac{1}{1-\delta}\left(\delta\ln\frac{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}+1\right)}+\ln\int_0^\infty x^{-\delta(\alpha+3)}e^{-\delta(\alpha+\lambda)x^{-2}}dx\right), \end{aligned}$$

where $\delta \neq 1$ and $\delta > 0$. By using the transformation in (2.3), we obtain that

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta}\left(\ln 2^{\delta-1}+\left(\frac{1-\delta}{2}\right)\ln(\alpha+\lambda)-\delta\ln\Gamma\left(\frac{\alpha}{2}+1\right)\right) \\ &\quad+\frac{1}{1-\delta}\left(\ln\Gamma\left(\frac{\delta(\alpha+3)-1}{2}\right)-\frac{\delta(\alpha+3)-1}{2}\ln\delta\right). \end{aligned}$$

3.7. Survival and hazard rate functions

The survival and hazard rate functions of the *NWIR* distribution are defined by

$$\begin{aligned} S(x) &= 1-F_w(x) \\ &= \frac{\gamma\left(\frac{\alpha}{2}+1,\frac{\alpha+\lambda}{x^2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}, \end{aligned}$$

and

$$\begin{aligned} H(x) &= \frac{f_w(x)}{S(x)} \\ &= \frac{2(\alpha+\lambda)^{\frac{\alpha}{2}+1}}{\gamma\left(\frac{\alpha}{2}+1,\frac{\alpha+\lambda}{x^2}\right)}x^{-(\alpha+3)}e^{-(\alpha+\lambda)x^{-2}} \end{aligned}$$

for $x > 0$, respectively. In FIG. 3.1, the graphs of the survival and hazard rate functions, which are plotted against different values of the parameters α and λ, are demonstrated.

Then, to determine the behavior of the hazard rate function of the *NWIR* distribution, the lemma established by Glaser [5] is used. Now, we define

$$\begin{aligned} \eta(x) &= -\frac{f_w'(x)}{f_w(x)} \\ &= (\alpha+3)x^{-1}-2(\alpha+\lambda)x^{-3}, \end{aligned}$$

and

$$\eta^{'}(x) = -(\alpha+3)\,x^{-2} + 6\,(\alpha+\lambda)\,x^{-4},$$

where $f_w^{'}(x)$ is derivative of *pdf* of the *NWIR* distribution with respect to x. Thus, $\eta^{'}(x) = 0$ provides when $x_0 = \sqrt{\frac{6(\alpha+\lambda)}{\alpha+3}}$ for $\lambda > 0$, $\alpha > 0$. Note that, $\eta^{'}(x) > 0$ and $\eta^{'}(x_0) = 0$ when $0 < x < x_0$ and $\eta^{'}(x) < 0$ when $x > x_0$. Therefore, the hazard rate function of the *NWIR* distribution is an upside down bathtub shape (see [19] and [23]).

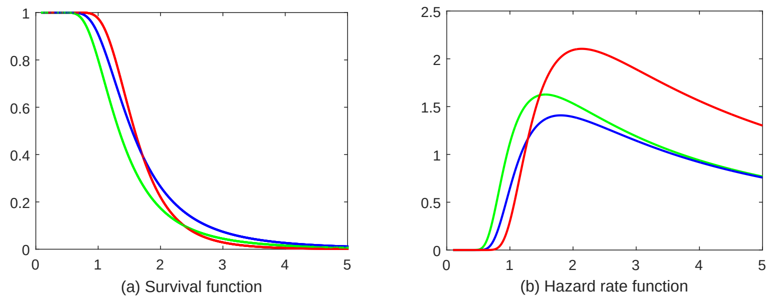

FIG. 3.1: Plots of the survival and hazard rate functions of the *NWIR* distribution where $\alpha = 2, \lambda = 1$ (green line); $\alpha = 2, \lambda = 2$ (blue line); $\alpha = 5, \lambda = 3$ (red line)

3.8. Order statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order statistics of a random sample $X_1, X_2, \ldots, X_n$ from the *NWIR* distribution. It is well known that the *pdf* of r^{th} order statistic $X_{(r)}$ $(r = 1, 2, \ldots, n)$ is given as:

$$f_{r:n}(x;\alpha,\lambda) = r\binom{n}{r} f(x)\,(F(x))^{r-1}\,(1-F(x))^{n-r}\,. \tag{3.6}$$

Applying the binomial series expansion of $(1-F(x))^{n-r}$ in (3.6), we get

$$f_{r:n}(x;\alpha,\lambda) = \sum_{k=0}^{n-r} r\binom{n}{r}\binom{n-r}{k}(-1)^k f(x)\,(F(x))^{r+k-1}\,. \tag{3.7}$$

After substituting (2.2) and (2.4) into (3.7), if we put the binomial series expansion of $(F(x))^{r+k-1}$ in (3.7), then we have

$$f_{r:n}(x;\alpha,\lambda) = \sum_{k=0}^{n-r}\sum_{t=0}^{r+k-1} 2(-1)^{r+2k-1} \times \left[r\binom{n}{r}\binom{n-r}{k}\binom{r+k-1}{t}\right] \times \left[\frac{(\alpha+\lambda)^{\frac{\alpha}{2}+1}\gamma^{r+k-1}\left(\frac{\alpha}{2}+1,\frac{\alpha+\lambda}{x^2}\right)}{\Gamma^{r+k}\left(\frac{\alpha}{2}+1\right)}\right] \times \left[x^{-(\alpha+3)}e^{-(\alpha+\lambda)x^{-2}}\right]. \tag{3.8}$$

Thus, the *pdf*s of the smallest order statistic $X_{(1)}$ and largest order statistic $X_{(n)}$ can be obtained by writing the $r = 1$ and $r = n$ in (3.8), respectively.

4. Estimation

Let $\{X_1, X_2, \ldots, X_n\}$ be a random sample from the *NWIR* distribution. The log-likelihood function of the sample is

$$\ln L(\alpha,\lambda \mid \underline{x}) = n\ln 2 + n\left(\frac{\alpha}{2}+1\right)\ln(\alpha+\lambda) - n\ln\Gamma\left(\frac{\alpha}{2}+1\right) - (\alpha+3)\sum_{i=1}^{n}\ln x_i - (\alpha+\lambda)\sum_{i=1}^{n}x_i^{-2}. \tag{4.1}$$

By differentiating (4.1) with respect to parameters α and λ, we have normal equations as

$$\frac{\partial \ln L(\alpha,\lambda \mid \underline{x})}{\partial\alpha} = \frac{n}{2}\ln(\alpha+\lambda) + n\frac{\left(\frac{\alpha}{2}+1\right)}{\alpha+\lambda} - \frac{n}{2}\Psi\left(\frac{\alpha}{2}+1\right) - \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}x_i^{-2} = 0 \tag{4.2}$$

$$\frac{\partial \ln L(\alpha,\lambda \mid \underline{x})}{\partial\lambda} = n\frac{\left(\frac{\alpha}{2}+1\right)}{\alpha+\lambda} - \sum_{i=1}^{n}x_i^{-2} = 0, \tag{4.3}$$

where $\Psi\left(\frac{\alpha}{2}+1\right) = \frac{d}{d\alpha}\ln\Gamma\left(\frac{\alpha}{2}+1\right) = \frac{\Gamma'\left(\frac{\alpha}{2}+1\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}$. Note that the solution of the equations in (4.2)-(4.3) gives maximum likelihood estimators $\widehat{\alpha}$ and $\widehat{\lambda}$ of parameters α and λ. However, they do not have a closed form solution, and we must use numerical methods to solve them. Now, to give asymptotically a lower bound for the covariance matrix of $\widehat{\alpha}$ and $\widehat{\lambda}$, the Fisher information matrix is provided as a minus expected value of the second-order partial derivatives of the log-likelihood function under the regularity conditions, see [11]. It is defined by

$$I_n(\alpha,\lambda)=\begin{bmatrix} -E\left(\frac{\partial^2 \ln L(\alpha,\lambda|\underline{x})}{\partial\alpha^2}\right) & -E\left(\frac{\partial^2 \ln L(\alpha,\lambda|\underline{x})}{\partial\alpha\partial\lambda}\right) \\ -E\left(\frac{\partial^2 \ln L(\alpha,\lambda|\underline{x})}{\partial\lambda\partial\alpha}\right) & -E\left(\frac{\partial^2 \ln L(\alpha,\lambda|\underline{x})}{\partial\lambda^2}\right) \end{bmatrix},$$

and the elements of the matrix are obtained as follows

$$\begin{aligned} E\left(\frac{\partial^2 \ln L(\alpha,\lambda\mid\underline{x})}{\partial\alpha^2}\right) &= \frac{n}{(\alpha+\lambda)}-n\frac{\left(\frac{\lambda}{2}+1\right)}{(\alpha+\lambda)^2}-\frac{n}{4}\Psi'\left(\frac{\alpha}{2}+1\right) \\ E\left(\frac{\partial^2 \ln L(\alpha,\lambda\mid\underline{x})}{\partial\lambda^2}\right) &= -n\frac{\left(\frac{\alpha}{2}+1\right)}{(\alpha+\lambda)^2} \\ E\left(\frac{\partial^2 \ln L(\alpha,\lambda\mid\underline{x})}{\partial\alpha\partial\lambda}\right) &= n\frac{\left(\frac{\lambda}{2}-1\right)}{(\alpha+\lambda)^2}, \end{aligned}$$

where $\Psi'\left(\frac{\alpha}{2}+1\right)$ is first derivative of $\Psi\left(\frac{\alpha}{2}+1\right)$ with respect to α. Therefore, maximum likelihood estimators of parameters α and λ have asymptotically normal distribution with mean vector $\underline{0}$ and the covariance matrix $I_n^{-1}(\alpha,\lambda)$ as

$$\sqrt{n}\left(\widehat{\alpha}-\alpha,\widehat{\lambda}-\lambda\right)\rightarrow N_2\left(\underline{0},I_n^{-1}(\alpha,\lambda)\right),$$

where $I_n^{-1}(\alpha,\lambda)$ is inverse of $I_n(\alpha,\lambda)$.

5. An Application

In this section, we consider a real data set, which is the daily mean wind speed data for March, taken in 2015 from the Turkish Meteorological Services for Sinop, Turkey, to demonstrate the practicability of the proposed distribution over the *IR* and *WIR* (proposed by Fatima and Ahmad [8]) distributions, see Table 5.1.

Table 5.1: The daily mean wind speed data

2.8	1.8	3.2	5.0	2.4	4.8	2.9	2.9
2.3	3.2	2.3	2.0	1.9	3.3	4.4	6.7
4.3	1.9	2.2	3.3	2.1	4.0	2.0	3.1
3.8	3.1	3.2	3.4	2.8	2.1	3.1	

The Kolmogorov-Smirnov (*K-S*) test, which is the one of the widely used goodness of fit tests, has been applied to verify that distributions fit to the real data set. The results of the *K-S* test indicate that the *NWIR*, *WIR* and *IR* distributions are suitable for modeling the data set since the computed *K-S* test values are less than theoretical *K-S* test value ($K\text{-}S_{0.05;31}=0.24$), see Table 5.2.

Then, we determined which distribution better fits the real data set using model evaluating tests, i.e., the root mean square error ($RMSE$), the coefficient of determination (R^2), ln-likelihood ($\ln L$) and the Akaike information criterion (AIC).

The tests results demonstrate that the $NWIR$ distribution gives a better fit to the data set compared to the WIR and IR distributions because it has minimum $RMSE$ and AIC and maximum R^2 and $\ln L$ values among the other distributions (see Table 5.2 and FIG. 5.1). Additionally, it was observed that there is no difference between the fitting performances of the WIR and IR distributions for the wind speed data (see FIG. 5.1).

Table 5.2: The ML estimates of parameters and results of the K-S test, $RMSE$, R^2, $\ln L$ and AIC for the wind speed data

Distribution	$\hat{\alpha}$	$\hat{\lambda}$	K-S	$RMSE$	R^2	$\ln L$	AIC
$NWIR$	3.7934	17.1586	0.0971	0.0532	0.9687	-41.2814	86.5629
WIR	0.0100	7.1969	0.2398	0.1162	0.6691	-48.7263	101.4525
IR	-	7.2331	0.2393	0.1158	0.6729	-48.6648	101.3290

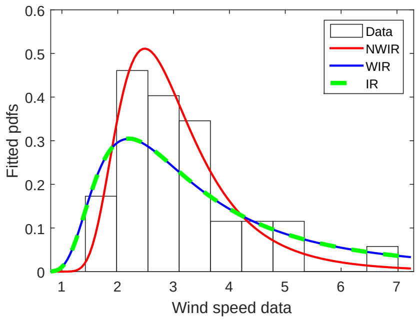

FIG. 5.1: Fitted plots and histogram for the data

6. Conclusion

In this study, a new weighted IR distribution based on two different weight functions has been introduced. Moments, the moment generating function, survival and hazard rate functions, order statistics and entropy measures of the new distribution have been derived. The estimating equations have been provided in order to obtain ML estimates of the individual parameters, and the Fisher information matrix has been derived in order to obtain approximate confidence intervals of the parameters. The relationship between the $NWIR$ distribution and the Gamma distribution has also been proved.

The applicability and superiority of the proposed distribution over the *WIR* and *IR* distributions have been illustrated with real data. Therefore, the *NWIR* distribution can be considered as an alternative model for the statistical data analysis in wind speed studies and other fields.

REFERENCES

1. A. Ahmad , S. P. Ahmad and A. Ahmed: *Transmuted inverse Rayleigh distribution: a generalization of the inverse Rayleigh distribution*. Mathematical Theory and Modeling **4**(7) (2014), 90–98.
2. S. F. Bush: *Nanoscale Communication Networks*. Artech House, 2010.
3. R. A. Fisher: *The effects of methods of ascertainment upon the estimation of frequencies*. Annals of Eugenics **6** (1934), 13–25.
4. M. E. Ghitany , F. Alqallaf , D. K. Al-Mutairi and H. A. Husain: *A two-parameter weighted Lindley distribution and its applications to survival data*. Mathematics and computers in Simulation **81**(6) (2011), 1190–1201.
5. R. E. Glaser: *Bathtub and related failure rate characterizations*. Journal of American Statistical Association **75** (1980), 667–672.
6. R. C. Gupta and J. P. Keating: *Relations for reliability measures under length biased sampling*. Scandinavian Journal of Statistics **13** (1986), 49–56.
7. R. C. Gupta and S. N. Kirmani: *The role of weighted distributions in stochastic modeling*. Communications in Statistics-Theory and methods **19**(9) (1990), 3147–3162.
8. K. Fatima and S. P. Ahmad: *Weighted inverse Rayleigh distribution*. International Journal of Statistics and Systems **12**(1) (2017), 119–137.
9. J. X. Kersey: *Weighted inverse Weibull and beta-inverse Weibull distribution*. Georgia Southern University, 2010.
10. M. S. Khan: *Modified inverse Rayleigh distribution*. International Journal of Computer Applications **87**(13) (2014), 28–33.
11. A. Klein and G. Mélard: *Computation of the Fisher information matrix for time series models*. Journal of Computational and Applied Mathematics **64**(1–2) (1995), 57–68.
12. J. Leao , H. Saulo , M. Bourguignon , R. Cintra , L. Rgo and G. Cordeiro: *On some properties of the beta inverse Rayleigh distribution*. Chilean Journal of Statistics **4**(2) (2013), 111–131.
13. V. Leiva , A. Sanhueza and J. M. Angulo: *A length-biased version of the Birnbaum-Saunders distribution with application in water quality*. Stochastic Environmental Research and Risk Assessment **23**(3) (2009), 299–307.
14. B. O. Oluyede: *On inequalities and selection of experiments for length biased distributions*. Probability in the Engineering and Informational Sciences **13**(2) (1999), 169–185.
15. A. K. Nanda and K. Jain: *Some weighted distribution results on univariate and bivariate cases*. Journal of Statistical Planning and Inference **77**(2) (1999), 169–180.
16. G. P. Patil: *Encountered data, statistical ecology, environmental statistics, and weighted distribution methods*. Environmetrics **2**(4) (1991), 377–423.

17. C. R. Rao: *On discrete distributions arising out of methods of ascertainment*. The Indian Journal of Statistics, Series A, **27** (1965), 311–324.

18. A. Rényi: *On measures of information and entropy*. Statistics and Probability **1** (1961), 547–561.

19. R. Roman: *Theoretical properties and estimation in weighted Weibull and related distributions*. M. S. Thesis, Georgia Southern University, Georgia, 2010.

20. A. Saghir , S. Tazeem and I. Ahmad: *The length-biased weighted exponentiated inverted Weibull distribution*. Cogent Mathematics **3**(1) (2016), 1267299.

21. M. Q. Shahbaz , S. Shahbaz and N. S. Butt: *The Kumaraswamy inverse Weibull distribution*. Pakistan Journal of Statistics and Operation Research **8**(3) (2012), 479–489.

22. E. Shannon: *A mathematical theory of communication*. The Bell System Technical Journal **27**(10) (1948), 379–423.

23. V. Sherina and B. O. Oluyede: *Weighted inverse Weibull distribution: statistical properties and applications*. Theoretical Mathematics and Applications **4**(2) (2014), 1–30.

24. V. N. Trayer: *Doklady Acad.* Nauk, Belorus, 1964.

25. V. G. Voda: *On the inverse Rayleigh random variable*. Reports of Statistical Application Research **19** (1972), 13–21.

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CONDITIONAL LEAST SQUARES ESTIMATION OF THE PARAMETERS OF HIGHER ORDER RANDOM ENVIRONMENT INAR MODElS

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Abstract. Two different random environment INAR models of higher order, precisely RrNGINARmax(p) and RrNGINAR$_1$(p), are presented as a new approach to modeling non-stationary nonnegative integer-valued autoregressive processes. The interpretation of these models is given in order to better understand the circumstances of their application to random environment counting processes. The estimation statistics, defined using the Conditional Least Squares (CLS) method, is introduced and the properties are tested on the replicated simulated data obtained by RrNGINAR models with different parameter values. The obtained CLS estimates are presented and discussed.
Keywords: Random environment; INAR(p); RrNGINAR; negative binomial thinning; geometric marginals; conditional least squares.

1. Introduction

One of the latest and most significant approaches to the modeling of count processes was designed by introducing integer-valued autoregressive (INAR) models almost simultaneously by [7] and [2]. This breakthrough in the analysis of integer-valued time series was a consequence of using a new thinning operator. Namely, the deterministic part of a process random variable was calculated using the realization of a Bernoulli counting sequence limited by the process realization in the preceding moment. This way of modeling was simply more natural and intuitively justified, so it led to much better results in fitting the counting processes than other models known at that time. This was followed by many modifications and generalizations. Some authors considered the thinning operator ([3], [6], [17, 18] and [13]), while others focused on marginal distributions ([8], [1], [4] and [5]). Also, as an alternative to the NGINAR(1) process from [13], a zero-inflated NGINAR(1) process was considered, which is given in [14]. In order to obtain more suitable models for processes of higher correlation between distant elements, INAR models of higher

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order were introduced. The most operative approach was developed in [16], where X_n as a process value at time n was defined using p possible preceding random values X_{n-i}, for $i \in \{1, 2, \ldots, p\}$, each with a certain probability. This inspired the construction of models presented in [10] and [9]. So, the evolution of INAR models continued.

All the models listed above corresponded only to stationary counting processes. In many applications, this was found as a frequent limitation. Recently, random environment INAR models, whose marginal distribution depends on random circumstances, have been introduced (more details about these models are given below). However, the conditional least squares (CLS) estimators of random environment INAR models parameters have not been considered so far. Therefore, in this paper, we obtain CLS estimators and test them on the simulated values from the corresponding random INAR model.

Using as a starting point some ideas from [15], [11] defined the r-states random environment integer-valued autoregressive process of order 1, denoted as $(RrINAR(1))$. It is given by

$$X_n(Z_n) = \sum_{i=1}^{X_{n-1}(Z_{n-1})} U_i + \varepsilon_n(Z_{n-1}, Z_n), \ n \in \mathbb{N},$$

where

$$X_n(Z_n) = \sum_{z=1}^{r} X_n(z) I_{\{Z_n = z\}},$$

$$\varepsilon_n(Z_{n-1}, Z_n) = \sum_{z_1=1}^{r} \sum_{z_2=1}^{r} \varepsilon_n(z_1, z_2) I_{\{Z_{n-1}=z_1, Z_n=z_2\}},$$

$\{U_i\}$, $i \in \mathbb{N}$, is a counting sequence of independent and identically distributed (i.i.d.) random variables generating a thinning operator, $\{Z_n\}$, $n \in \mathbb{N}_0$ is an r-states random environment process defined as a Markov chain taking values in $E_r = \{1, 2, \ldots, r\}$. Further, $\{\varepsilon_n(i,j)\}$, $n \in \mathbb{N}_0$, $i, j \in E_r$, are sequences of i.i.d. random variables, for which $\{Z_n\}$, $\{\varepsilon_n(1,1)\}$, $\{\varepsilon_n(1,2)\}$, $\ldots$, $\{\varepsilon_n(r,r)\}$, are mutually independent, for all $n \in \mathbb{N}_0$, and Z_m and $\varepsilon_m(i,j)$ are independent of $X_n(l)$, for $n < m$ and any $i, j, l \in E_r$. In order to obtain more efficient INAR modeling, a new random environment INAR(1) process with one-step-ahead determined marginal distribution was introduced in [11]. As can be seen, this process is non-stationary, which makes it more applicable in practice. Adapting the process to more dynamical counting data, the authors specify geometric marginals and the negative binomial thinning operator $\alpha *$, which was utilized for construction of the NGINAR(1) model introduced in [13]. This resulted in the r-states random environment $INAR(1)$ process with determined (z_n–guided) geometric marginal distribution based on the negative binomial thinning operator $(RrNGINAR(1))$ given by

$$X_n(z_n) = \alpha * X_{n-1}(z_{n-1}) + \varepsilon_n(z_{n-1}, z_n), \ n \in \mathbb{N}, \tag{1.1}$$

where $\alpha \in (0,1)$, the counting sequence $\{U_i\}$, $i \in \mathbb{N}$, incorporated in $\alpha *$, makes a sequence of i.i.d. random variables with the probability mass function (pmf) given by

$$P(U_i = u) = \frac{\alpha^u}{(1+\alpha)^{u+1}}\,,\ u \in \mathbb{N}_0,$$

and finally the process pmf is defined as

$$P(X_n(z_n) = x) = \frac{\mu_{z_n}^x}{(1+\mu_{z_n})^{x+1}}\,,\ x \in \mathbb{N}_0, \tag{1.2}$$

where $\mu_{z_n} \in \{\mu_1, \mu_2, \dots, \mu_r\}$ and $r \in \mathbb{N}$.

1.1. Interpretation of the random environment INAR processes of higher order

Continuing the efforts towards the optimal fitting of the counting processes, models of higher order were introduced in [12]. Two approaches were used, which we discuss in what follows.

Definition 1. *Let z_n be the realization of a random environment process $\{Z_n\}$ at the moment $n \geqslant 0$. We say that $\{X_n(z_n)\}_{n\in\mathbb{N}_0}$ is an INAR process with r-states random environment guided geometric marginals based on the negative binomial thinning operator of maximal order p (RrNGINARmax(p)), $p \in \mathbb{N}$, if the random variable $X_n(z_n)$ is defined as*

$$X_n(z_n) = \begin{cases} \alpha * X_{n-1}(z_{n-1}) + \varepsilon_n(z_{n-1}, z_n), & w.p.\ \phi_1^{(p_n)}, \\ \alpha * X_{n-2}(z_{n-2}) + \varepsilon_n(z_{n-2}, z_n), & w.p.\ \phi_2^{(p_n)}, \\ \vdots & \vdots \\ \alpha * X_{n-p_n}(z_{n-p_n}) + \varepsilon_n(z_{n-p_n}, z_n), & w.p.\ \phi_{p_n}^{(p_n)}, \end{cases} \tag{1.3}$$

for $n \geqslant 1$, where

$$p_n = \begin{cases} p, & p_n^* \geq p, \\ p_n^*, & p_n^* < p, \end{cases}$$

$p_n^* = \max\{i \in \{1, 2, \dots, n\} : z_{n-1} = z_{n-2} = \dots = z_{n-i}\}$ *and the following conditions are satisfied:*

1. $\phi_i^{(p_n)} \geqslant 0$, $i \in \{1, 2, \dots, p_n\}$, $\sum_{i=1}^{p_n} \phi_i^{(p_n)} = 1$,

2. $\alpha \in (0,1)$ *and the counting sequence* $\{U_i\}_{i\in\mathbb{N}}$ *of the negative binomial thinning operator* $\alpha *$ *has pmf* $P(U_i = u) = \frac{\alpha^u}{(1+\alpha)^{u+1}}$, $u \in \{0, 1, 2, \dots\}$,

3. $P(X_n(z_n) = x) = \frac{\mu_{z_n}^x}{(1+\mu_{z_n})^{x+1}}$, $x \in \{0, 1, 2, \dots\}$, *where* $\mu_{z_n} \in \{\mu_1, \mu_2, \dots, \mu_r\}$, $\mu_i > 0$, $i \in \{1, 2, \dots, r\}$ *and* $r \in \mathbb{N}$ *is the number of states of the random environment process* $\{Z_n\}$,

4. *for fixed* $i, j \in E_r = \{1, 2, .., r\}$, $\{\varepsilon_n(i, j)\}_{n\in\mathbb{N}}$ *is a sequence of i.i.d. random variables,*

5. *$\{Z_n\}, \{\varepsilon_n(1,1)\}, \{\varepsilon_n(1,2)\}, \ldots, \{\varepsilon_n(r,r)\}$ are mutually independent sequences of random variables,*

6. *$X_n(l)$ is independent of Z_m and $\varepsilon_m(i,j)$, for $0 \leq n < m$ and any $i,j,l \in E_r$.*

Definition 2. *Let z_n be the realization of a random environment process $\{Z_n\}$ at the moment $n \geqslant 0$. We say that $\{X_n(z_n)\}_{n\in\mathbb{N}_0}$ is an INAR process with r-states random environment guided geometric marginals based on the negative binomial thinning operator of order p ($RrNGINAR_1(p)$) if the random variable $X_n(z_n)$ is defined as*

$$X_n(z_n) = \begin{cases} \alpha * X_{n-1}(z_{n-1}) + \varepsilon_n(z_{n-1}, z_n), & w.p.\ \phi_1^{(p_n)}, \\ \alpha * X_{n-2}(z_{n-2}) + \varepsilon_n(z_{n-2}, z_n), & w.p.\ \phi_2^{(p_n)}, \\ \vdots & \vdots \\ \alpha * X_{n-p_n}(z_{n-p_n}) + \varepsilon_n(z_{n-p_n}, z_n), & w.p.\ \phi_{p_n}^{(p_n)}, \end{cases}$$

for $n \geqslant 1$, where

$$p_n = \begin{cases} p, & p_n^* \geq p, \\ 1, & p_n^* < p, \end{cases}$$

$p_n^* = \max\{i \in \{1,2,\ldots,n\} : z_{n-1} = z_{n-2} = \cdots = z_{n-i}\}$ and conditions $1-6$ from Definition 1 are satisfied.

Since the distribution parameter values of the processes may vary over time, it could happen that each of the equations (1.3) and (2), at a certain moment, contains differently distributed X_n random variables, which would make the models pretty complicated to work with. In order to avoid this, each of these models is defined with the ability of changing the number of possibilities (possible expressions) on the right side of the equation. So, the process introduced by Definition 1 has a fully variable order, possibly taking all the values from 1 to p. When the process random state changes, then the order of the process becomes equal to 1 and then starts rising successively, until it reaches p (when the process takes shape of the model of fixed order), or the state changes again. However, for the process given by Definition 2, the order takes one of two possible values. Namely, every time the state changes, the order becomes equal to 1 and it remains the same until there is a series of enough (p) previous process elements corresponding to the same state, when the order becomes equal to p. By virtue of these qualities, these processes are the most suitable for counting, for example, some elements of the observed unstable system or some random events recorded in a variable environment. In each case, certain area conditions or random circumstances may affect the dynamics of the interactions in the observed populations, which further affects the values of counts. So, the finite number of possible combinations of circumstances in which the population is observed is represented by the finite number (r) of random states and is modeled by the Markov process $\{Z_n\}$. Its realization $\{z_n\}$ directly determines the value of the selected marginal distribution. Hence, while being in the same state z_n, the process behaves as a stationary one with the marginal parameter value μ_{z_n}.

Nevertheless, its non-stationarity comes from changing its mean parameter value μ_{z_n}, which is directly guided by $\{z_n\}$. So, the counting process is basically piece-by-piece stationary, where each piece is as long as the random process $\{Z_n\}$ remains in the same state, i.e. the population circumstances do not change.

2. Conditional least squares estimators

Let $\{X_n(z_n)\}$ be the RrNGINARmax(p) or RrNGINAR$_1(p)$ time series model. In order to apply Theorem 2 from [12] we have to suppose conditions from that theorem. Let $\mu_1 > 0$, $\mu_2 > 0$, $\dots$, $\mu_r > 0$ and let us suppose that $0 \leq \alpha \leq \min\left\{\frac{\mu_l}{1+\mu_k}, k, l \in E_r\right\}$, $z_n = j$ and $z_{n-1} = i$, for $i, j \in E_r$. Now, recalling the mentioned theorem, the conditional expectation of the random variable X_n for given $X_{n-1}, X_{n-2}, ..., X_{n-p_n}$ is

$$E(X_n|H_{n-1}) = \mu_j - \alpha\mu_i + \alpha\sum_{l=1}^{p_n}\phi_l^{(p_n)}X_{n-l},$$

where H_{n-1} represents σ-algebra generated by $X_{n-1}, X_{n-2}, \dots$. Now, if we define new parameters as $\theta_l^{(z_n)} = \alpha\phi_l^{(p_n)}$, for $l \in \{1, 2, ..., p_n\}$, then $\alpha = \sum_{l=1}^{p_n}\theta_l^{(p_n)}$ and consequently

$$\begin{aligned} E(X_n|H_{n-1}) &= \mu_j - \alpha\mu_i + \theta_1^{(p_n)}X_{n-1} + \theta_2^{(p_n)}X_{n-2} + \dots + \theta_{p_n}^{(p_n)}X_{n-p_n} \\ &= \mu_j - \sum_{l=1}^{p_n}\theta_l^{(p_n)}\mu_i + \theta_1^{(p_n)}X_{n-1} + \theta_2^{(p_n)}X_{n-2} + \dots + \theta_{p_n}^{(p_n)}X_{n-p_n}. \end{aligned}$$

Let $k \in E_r$, $p_n = p$ and $J_k = \{n \in \mathbb{N}|X_n, X_{n-1}, ..., X_{n-p_k} \in U^{(k)}\}$, where $U^{(k)}$ represents the process subsample which consists of all the elements corresponding to the same state k. In conducting the conditional least squares (CLS) estimation, the aim is to minimize the following sum of squares

(2.1)
$$Q_N^{(k)}(\mathbf{a}) = \sum_{n\in J_k}\left(X_n - \mu_j - \sum_{l=1}^{p}\theta_l^{(p)}\mu_i - \theta_1^{(p)}X_{n-1} - \theta_2^{(p)}X_{n-2} - \dots - \theta_p^{(p)}X_{n-p}\right)^2,$$

with respect to the vector $\mathbf{a} = (\theta_1^{(p)}, \theta_2^{(p)}, ..., \theta_p^{(p)}, \mu_k)'$. This is achieved by solving the system $\frac{\partial Q_N}{\partial\theta_1^{(p)}} = 0$, $\frac{\partial Q_N}{\partial\theta_2^{(p)}} = 0, ..., \frac{\partial Q_N}{\partial\theta_p^{(p)}} = 0$, $\frac{\partial Q_N}{\partial\mu_k} = 0$. Since the summation in the previous expression is over the set J_k, it holds that $X_n, X_{n-1}, ..., X_{n-p} \in U^{(k)}$ and $z_n = z_{n-1} = ... = z_{n-p} = k$. So, considering the process on the subsample $U^{(k)}$, we deal with the CGINAR(p) model introduced in [10]. Therefore, the corresponding results and equations obtained for the CGINAR(p) model can be used here. Thus, we have

$$\mu_{k,p} = \frac{1}{1 - \sum_{i=1}^{p}\theta_{i,p}^{(k)}}\left(\overline{X}^{(0)} - \sum_{i=1}^{p}\theta_{i,p}^{(k)}\overline{X}^{(i)}\right), \tag{2.2}$$

where

$$\overline{X}^{(i)} = \frac{1}{|J_k|}\sum_{n\in J_k} X_{n-j}, \quad j\in\{0,1,...,p\}.$$

Replacing (2.2) in (2.1) the system becomes

$$\sum_{j=1}^{p}\theta_j^{(p)}\widehat{\gamma}^*(|l-j|) = \widehat{\gamma}^*(l), \quad l=1,2,...,p, \tag{2.3}$$

where

$$\widehat{\gamma}^*(|l-j|) = \frac{1}{|J_k|}\sum_{n\in J_k} X_{n-l}X_{n-j} - \overline{X}^{(l)}\overline{X}^{(j)}.$$

Solving it gives us $\widehat{\theta}_j^{(p)} = \frac{D_j^*}{D^*}$, $j=1,2,...,p$, where D_j^* and D^* are the appropriate determinants from Kramer's method. Substituting the last equations in (2.2) we get

$$\widehat{\mu}_k^{CLS} = \frac{1}{1-\sum_{i=1}^{p}\frac{D_i^*}{D^*}}\left(\frac{1}{|J_k|}\sum_{n\in J_k}X_n - \sum_{j=1}^{p}\frac{D_i^*}{D^*}\cdot\frac{1}{|J_k|}\sum_{n\in J_k}X_{n-j}\right).$$

Therefore,

$$\begin{aligned}\widehat{\alpha}^{(k),CLS} &= \frac{\sum_{j=1}^{p}D_j^*}{D^*},\\ \widehat{\phi}_{i,p}^{(k),CLS} &= \frac{D_i^*}{\sum_{j=1}^{p}D_j}, \quad i\in\{1,2,...,p\}.\end{aligned}\tag{2.4}$$

Finally, using the preceding results for each $k\in\{1,2,\ldots,r\}$, it is only left to calculate the weighted thinning parameter and the weighted probabilities, respectively, as

$$\widehat{\alpha}^{CLS} = \frac{\sum_{k=1}^{r}|J_k|\widehat{\alpha}^{(k),CLS}}{\sum_{k=1}^{r}|J_k|}, \tag{2.5}$$

$$\widehat{\phi}_{i,p}^{CLS} = \frac{\sum_{k=1}^{r}|J_k|\widehat{\phi}_{i,p}^{(k),CLS}}{\sum_{k=1}^{r}|J_k|}, \tag{2.6}$$

which represent the required estimators.

Based on Lemma 6, from [10], the estimators $\widehat{\alpha}^{CLS}$, $\widehat{\mu}_k^{CLS}$ and $\widehat{\phi}_{i,p}^{CLS}$ are asymptotically almost surely equivalent to the corresponding Yule-Walker estimators. So, the strong consistence of the Yule-Walker estimators, proved in [12], implies the strong consistence of the here observed CLS estimators.

3. Simulation results

In this section we try to confirm the correctness of the introduced CLS estimators. With that in mind, we have simulated 100 replicates of realizations of the processes RrNGINARmax(p) and RrNGINAR$_1$(p), each of size 10000. Parameter values for $\alpha, p, r, \boldsymbol{\mu}, \mathbf{p}_{mat}$ and $\boldsymbol{\phi}$ are chosen and then the corresponding models are simulated. The transition probability matrix of the random environment process is denoted by $\mathbf{p}_{mat}$, and $\boldsymbol{\mu}$ is a vector of means. In the case of RrNGINARmax(p) model, the p_nth row, $p_n \in \{2, \dots, p\}$, of the matrix $\boldsymbol{\phi}$ contains probabilities $\phi_i^{(p_n)}$, $i \in \{1, 2, \dots, p_n\}$ and in the case of RrNGINAR$_1$(p) model, the last row represents probabilities $\phi_i^{(p)}$, $i \in \{1, 2, \dots, p\}$. The simulated realization of random environment process, $\{z_n\}$, is obtained using $\mathbf{p}_{mat}$ and then the sequence $\{p_n\}$ is specified based on the corresponding definition. We have considered six different cases of chosen parameter values and presented all the results in the appropriate tables. Also, we have decided for the same parameter values as in the case of Yule-Walker parameter estimation discussed in [12]. There are three tables. In the first one we have $p = 2, r = 2$, in the second $p = 3, r = 2$ and in the last $p = 3, r = 3$. In the first table, for $r = p = 2$ we considered different choices of other parameters. The larger α gives better estimates for probabilities $\phi_i^{(p_n)}$. The higher diagonal values of p_{mat} ensures longer subsamples and, consequently, better results. Also, the higher values of p and r implies more subsamples and, therefore, a larger number of them and smaller sizes, which gives us worse results for the same samples size. Finally, for the small sample sizes it is possible to have very small subsamples and to get bad results.

Table 3.1: $r = 2$, $p = 2$

True values $\boldsymbol{\mu} = (1, 2)$, $\alpha = 0.3$, $\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 \\ 0.6 & 0.4 \end{bmatrix}$, $\mathbf{p}_{mat} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$

n	$\hat{\mu}_1^{CLS}$	$\hat{\mu}_2^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$
500	1.0100	1.9964	0.3359	0.6900	0.3100	0.2963	0.6207	0.3793
SE	0.1195	0.2214	0.1751	0.2451	0.2451	0.1557	0.3836	0.3836
1000	1.0119	1.9976	0.3307	0.6248	0.3752	0.2896	0.6127	0.3873
SE	0.0797	0.1373	0.1176	0.1364	0.1364	0.1187	0.1229	0.1229
5000	1.0024	2.0047	0.3026	0.6048	0.3952	0.2978	0.5984	0.4016
SE	0.0354	0.0600	0.0478	0.0595	0.0595	0.0565	0.0579	0.0579
10000	1.0016	2.0072	0.3020	0.5990	0.4010	0.2956	0.6029	0.3971
SE	0.0249	0.0429	0.036	0.0386	0.0386	0.0406	0.0393	0.0393

Table 3.2: $r = 2$, $p = 2$

True values $\boldsymbol{\mu} = (1, 2)$, $\alpha = 0.15$, $\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$, $\mathbf{p}_{mat} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$

n	$\hat{\mu}_1^{CLS}$	$\hat{\mu}_2^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$
500	0.99609	2.0089	0.1735	-0.3945	1.3945	0.1679	0.2434	0.7566
SE	0.1014	0.1588	0.1451	15.064	15.064	0.1389	2.2148	2.2148
1000	0.9977	2.0143	0.1475	0.5418	0.4582	0.1547	0.3885	0.6115
SE	0.0666	0.1259	0.0966	0.4849	0.4849	0.0878	0.9751	0.9751
5000	1.0045	1.9993	0.1505	0.4970	0.5030	0.1508	0.4893	0.5107
SE	0.0360	0.0618	0.037	0.1008	0.1008	0.0384	0.1113	0.1113
10000	1.0024	1.9981	0.1494	0.5039	0.4961	0.1514	0.4964	0.5036
SE	0.0252	0.0425	0.027	0.0702	0.0702	0.0297	0.0682	0.0682

Table 3.3: $r = 2$, $p = 2$

True values $\boldsymbol{\mu} = (1, 2)$, $\alpha = 0.3$, $\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 \\ 0.6 & 0.4 \end{bmatrix}$, $\mathbf{p}_{mat} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

n	$\hat{\mu}_1^{CLS}$	$\hat{\mu}_2^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$
500	0.9957	2.0003	0.3322	0.7480	0.252	0.3075	0.6919	0.3081
SE	0.0996	0.1919	0.1192	2.8876	2.8876	0.0988	0.5416	0.5416
1000	0.9928	2.0004	0.3108	0.6208	0.3792	0.3036	0.6556	0.3444
SE	0.0732	0.1345	0.0892	0.5292	0.5292	0.0837	0.3017	0.3017
5000	1.0019	2.0008	0.3037	0.5973	0.4027	0.2976	0.5931	0.4069
SE	0.0414	0.06231	0.0380	0.0894	0.0894	0.0387	0.0818	0.0818
10000	0.9993	2.0030	0.3020	0.5904	0.4096	0.2985	0.5929	0.4071
SE	0.0245	0.0418	0.0264	0.0702	0.0702	0.0284	0.0633	0.0633

Table 3.4: $r = 2$, $p = 2$

True values $\boldsymbol{\mu} = (4, 5)$, $\alpha = 0.5$, $\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 \\ 0.6 & 0.4 \end{bmatrix}$, $\mathbf{p}_{mat} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$

n	$\hat{\mu}_1^{CLS}$	$\hat{\mu}_2^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$	$\hat{\alpha}^{CLS}$	$\hat{\phi}_{2,1}^{CLS}$	$\hat{\phi}_{2,2}^{CLS}$
500	3.9973	5.0266	0.5277	0.6094	0.3906	0.5151	0.6108	0.3892
SE	0.4207	0.5014	0.1589	0.1595	0.1595	0.1364	0.1420	0.1420
1000	3.9776	5.0367	0.5166	0.5973	0.4027	0.5109	0.5927	0.4073
SE	] 0.3344	0.3312	0.1009	0.0975	0.0975	0.1100	0.0935	0.0935
5000	3.9923	5.0179	0.4960	0.5944	0.4056	0.5031	0.5867	0.4133
SE	0.1340	0.1635	0.0559	0.0417	0.0417	0.0638	0.0513	0.0513
10000	3.9947	5.0157	0.4997	0.5931	0.4069	0.5050	0.5935	0.4065
SE	0.0985	0.1157	0.0398	0.0285	0.0285	0.0428	0.0391	0.0391

Table 3.5: $r = 2$, $p = 3$

True values $\boldsymbol{\mu} = (1, 2)$, $\alpha = 0.3$, $\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 0.4 & 0 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$, $\mathbf{p}_{mat} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$

n	$\widehat{\mu}_1^{CLS}$	$\widehat{\mu}_2^{CLS}$	$\widehat{\alpha}^{CLS}$	$\widehat{\phi}_{2,1}^{CLS}$	$\widehat{\phi}_{2,2}^{CLS}$	$\widehat{\phi}_{3,1}^{CLS}$	$\widehat{\phi}_{3,2}^{CLS}$	$\widehat{\phi}_{3,3}^{CLS}$	$\widehat{\alpha}^{CLS}$	$\widehat{\phi}_{3,1}^{CLS}$	$\widehat{\phi}_{3,2}^{CLS}$	$\widehat{\phi}_{3,3}^{CLS}$
500	1.0025	1.9650	0.3139	0.5793	0.4207	0.6083	0.2945	0.0972	0.3022	0.0761	0.3122	0.6118
SE	0.1186	0.1981	0.171	1.5608	1.5608	0.7284	0.2673	0.7066	0.1032	3.5502	1.1115	3.6095
1000	1.0023	2.0057	0.3094	0.6659	0.3341	0.5140	0.3125	0.1735	0.2958	0.5485	0.2607	0.1908
SE	0.0808	0.1338	0.0988	0.5715	0.5715	0.1731	0.1701	0.1455	0.0735	0.2608	0.2818	0.1948
5000	0.9951	2.0011	0.3058	0.6155	0.3845	0.4902	0.3026	0.2072	0.2985	0.4941	0.3095	0.1964
SE	0.0335	0.0652	0.0477	0.1239	0.1239	0.0669	0.0610	0.0677	0.0347	0.0729	0.0751	0.0591
10000	0.9995	2.0019	0.3009	0.5924	0.4076	0.4970	0.2972	0.2058	0.2983	0.4943	0.3113	0.1944
SE	0.0257	0.0461	0.0329	0.0787	0.0787	0.0506	0.0460	0.0514	0.0248	0.0500	0.0503	0.0434

Table 3.6: $r = 3$, $p = 3$

True values $\boldsymbol{\mu} = (1, 1.5, 2)$, $\alpha = 0.3$, $\boldsymbol{\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6 & 0.4 & 0 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$, $\mathbf{p}_{mat} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$

n	$\widehat{\mu}_1^{CLS}$	$\widehat{\mu}_2^{CLS}$	$\widehat{\mu}_3^{CLS}$	$\widehat{\alpha}^{CLS}$	$\widehat{\phi}_{2,1}^{CLS}$	$\widehat{\phi}_{2,2}^{CLS}$	$\widehat{\phi}_{3,1}^{CLS}$	$\widehat{\phi}_{3,2}^{CLS}$	$\widehat{\phi}_{3,3}^{CLS}$	$\widehat{\alpha}^{CLS}$	$\widehat{\phi}_{3,1}^{CLS}$	$\widehat{\phi}_{3,2}^{CLS}$	$\widehat{\phi}_{3,3}^{CLS}$
500	0.9749	1.5187	2.0151	0.3331	0.9811	0.0189	0.9537	0.3139	-0.2675	0.3027	0.5286	0.2473	0.2241
SE	0.1527	0.1659	0.2519	0.1341	2.1042	2.1042	2.8729	1.6246	4.3460	0.0990	0.5116	0.8627	0.7784
1000	0.9886	1.5182	1.9855	0.3143	0.7626	0.2374	0.4260	0.6499	-0.0759	0.3010	0.5560	0.2774	0.1666
SE	0.1051	0.1161	0.1819	0.1000	0.8830	0.8830	0.7415	2.3627	1.9789	0.0638	0.5080	0.4014	0.5072
5000	1.0025	1.5043	1.9918	0.3047	0.6003	0.3997	0.5133	0.2923	0.1944	0.3018	0.5003	0.3050	0.1947
SE	0.0516	0.0572	0.0785	0.0458	0.1328	0.1328	0.1031	0.0999	0.0982	0.0271	0.0970	0.1020	0.1070
10000	1.0038	1.4999	1.9961	0.2988	0.5984	0.4016	0.4998	0.2970	0.2032	0.3032	0.4955	0.3087	0.1958
SE	0.0335	0.0390	0.0562	0.0290	0.0874	0.0874	0.0572	0.0635	0.0561	0.0191	0.0714	0.0678	0.0629

4. Conclusion

Varying the sizes of the simulated samples, we have noticed quite a similar behavior of the here obtained estimates compared to those obtained by the Yule-Walker statistics, thus confirming the asymptotical equivalence mentioned at the end of Section 2. Also, the convergence of the obtained estimations to the real parameter values, which is easy to observe in all the following tables, confirms the strong consistency of the conditional least squares estimators.

Some negative values for $\widehat{\phi}_{3,3}^{CLS}$ are obtained when the sample size is small, which is induced by the model properties. Namely, $\phi_{3,3}$ represents the probability that $X_n(z_n)$ will depend on $X_{n-3}(z_{n-3})$. In this case $\phi_{3,3} = 0.2$, so the portion of the data from which we can obtain $\widehat{\phi}_{3,3}^{CLS}$ is approximately 0.2. However, another "reduction" of the data occurs since all estimators are defined on the subsamples with the same state. So, in this case, the subsample is too small to get a good result. By enlarging the data size, $\widehat{\phi}_{3,3}^{CLS}$ converge to the true value. This effect of the small subsample also results in the large values of standard deviations for the small sample size.

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REFERENCES

1. Al-Osh, M.A., Aly, E.E.A.A.: *First order autoregressive time series with negative binomial and geometric marginals*, Commun. Statist. Theory Meth.(1992) 21, 2483–2492.
2. Al-Osh, M.A., Alzaid, A.A.: *First-order integer-valued autoregressive (INAR(1)) process*, J. Time Ser. Anal. (1987) 8, 261–275.
3. Aly, E.E.A.A., Bouzar, N.: *On Some Integer-Valued Autoregressive Moving Average Models*, Journal of Multivariate Analysis (1994) 50, 132–151.
4. Alzaid, A. A., Al-Osh, M.A.: *Some autoregressive moving average processes with generalized Poisson marginal distributions*, Ann. Inst. Statist. Math. (1993) 45, 223–232.
5. Bakouch, H.S., Ristić, M.M. *Zero Truncated Poisson Integer Valued AR(1) Model*, Metrika (2010) 72(2), 265-280.
6. Latour, A.: *Existence and stochastic structure of a non-negative integer-valued autoregressive process*, J. Time Ser. Anal. (1998) 19, 439–455

7. McKenzie, E. : *Some simple models for discrete variate time series*, Water Resour. Bull.(1985) 21, 645–650.
8. McKenzie, E.: *Autoregressive moving-average processes with negative binomial and geometric distributions*, Adv. Appl. Prob. (1986) 18, 679–705.
9. Nastić, A.S., Ristić, M.M.: *Some geometric mixed integer-valued autoregressive (INAR) models*, Statistics and Probability Letters (2012) 82, 805-811.
10. Nastić, A.S., Ristić, M.M., Bakouch, H.S.: *A combined geometric INAR(p) model based on negative binomial thinning*, Mathematical and Computer Modelling (2012) 55, 1665–1672.
11. Nastić, A.S., Laketa, P.N., Ristić, M.M.: *Random Environment Integer-Valued Autoregressive process*, J. Time Ser. Anal.(2016) 37, 267–287.
12. Nastić, A.S., Laketa, P.N., Ristić, M.M.: *Random environment INAR models of higher order*, RevStat: Statistical Journal (2017)Volume 17, Number 1, January 2019
13. Ristić, M.M., Bakouch, H.S., Nastić, A.S.: *A new geometric first-order integer-valued autoregressive (NGINAR(1)) process*, J. Stat. Plan. Inference (2009) 139, 2218–2226.
14. Ristić, M.M., Bourguignon, M., Nastić, A.S.: *Zero-Inflated NGINAR(1) process*, J. Communications in Statistics - Theory and Methods (2018)
15. Tang, M., Wang, Y.: *Asymptotic Behaviour of Random Coefficient INAR Model under Random Environment defined by Difference Equation, Advances in Difference Equations*, 2014(1), 1–9.
16. Weiss, C.H.: *The combined INAR*(p) *models for time series of counts*, Statist. Probab. Lett. (2008) 72, 1817–1822.
17. Zheng, H., Basawa, I.V., Datta, S.: *Inference for pth-order random coefficient integer-valued autoregressive processes*, J. Time Ser. Anal. (2006) 27, 411–440.
18. Zheng, H., Basawa, I.V., Datta, S.: *First-order random coefficient integer-valued autoregressive processes*, J. Stat. Plann. Inf. (2007) 137, 212–229.

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QUASI MAPPING SINGULARITIES

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Abstract. We obtain a list of all simple classes of singularities of curves (irreducible and reducible) in real spaces of any dimension with respect to the quasi equivalence relation.

Keywords: Singularities; curve; quasi equivalence relation.

1. Introduction

Motivated by the importance of the locus of points on a hypersurface where a given vector field is not transversal to it, Vladimir Zakalyukin introduced a new equivalence relation on projections of hypersurfaces which he named quasi equivalence [9]. The relation is more rough than the standard group of diffeomorphisms preserving a given projection [8]. The difference between the $\mathcal{A}$-equivalence relation and the quasi relation is illustrated as follows: Let Λ be the graph of a map f from $\mathbb{R}^m$ to $\mathbb{R}^n$ and let π be a trivial fibration structure. If p_1 and p_2 are two points on Λ lying on the same fibre of the projection then they are mapped by π to the same image. This property persists for the $\mathcal{A}$-equivalent maps f_i,$i = 1, 2$. However, this is not the case for the quasi equivalence as p_1 and p_2 might be mapped by a diffeomorphism to different fibres and hence they are mapped by π to different images.

The locus of the points on the hypersurface where a given vector field is not transversal to it is of importance. One of the possible and interesting applications for the quasi-projection equivalence relation is used in partial differential equations (PDE) with boundary value problems. Consider the characteristic method solving the simplest Cauchy problem for first order linear PDE: $\sum a_i(x)\frac{\partial u}{\partial x_i} = 0$, where $u(x)$ is an unknown function with $x \in \mathbb{R}^m$ and $a_i(x)$ are given functions. The problem includes the boundary hypersurface $S \subset \mathbb{R}^m$ and the boundary values

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$U|_S = U_0$. Generically, the characteristic vector field $v = a_i \frac{\partial}{\partial x_i}$ is tangent to S at some points which are called characteristic. Outside the set K of characteristic points, the problem has a unique local solution. So the geometry of the set K is an essential feature of the problem. If we rectify the vector field getting, say $\frac{\partial}{\partial x_1}$, then the problem of classifying K is exactly to find critical points of the projection of S along parallel rays. Similarly, in many other complicated PDE boundary value problems, mainly in continuum mechanics, the generalisation of the Neumann boundary condition is used.

In [3], the first steps in the study of the quasi-equivalence of projections of graphs of maps were taken within the approach similar to the one introduced by Zakalyukin [9]. Two cases were investigated there: maps from $\mathbb{R}$ to $\mathbb{R}^2$ and maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ (see [6] and [8] for the corresponding results for the $\mathcal{A}$-equivalence). In the current paper, we consider irreducible and reducible curve singularities in a linear real space of any dimension and give the list of stably simple classes with respect to the quasi equivalence (see [2] and [5] for the corresponding results for the $\mathcal{A}$-equivalence).

The paper is organized as follows. In Section 2 we review the definition of the quasi-equivalence relation of the projections of hypersurfaces and recall the main results from [9] which are needed in the next sections. In Section 3 we introduce the main definition of the quasi-equivalence of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ and derive an algebraic expression for the respective tangent space to a quasi class of mapping. Then, we recall the classification of quasi-simple singularities of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ from [3], giving detailed proofs. After that, we classify quasi-stably simple classes of irreducible curves in $\mathbb{R}^n$. Finally, in Section 4 we classify stably simple reducible curve singularities with respect to the quasi-equivalence relation.

2. Quasi projections of hypersurfaces

Consider germs of subvarieties V in the space $\mathbb{R}^p = \{(x, y) : x \in \mathbb{R}^m, y \in \mathbb{R}^n\}$, equipped with the trivial fibration structure, given by the projection $\pi : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(x, y) \mapsto y$. When the distinction between x and y is not crucial, we will be using the notation $w = (x, y)$ for the whole set of coordinates on $\mathbb{R}^p$.

Consider germs of $\mathbf{C}^\infty$ functions $f : (\mathbb{R}^p, 0) \to \mathbb{R}$ and denote by $\mathbb{C}_w$ the ring of all such germs at the origin and by $\mathbb{M}_w$ the maximal ideal in $\mathbb{C}_w$.

Definition 2.1. [9] A point $b \in V$ is called *critical* if the fiber containing b is not transversal to V at b. In particular, b can be a singular point of V.

Definition 2.2. [9] Two subvarieties V_0 and V_1 in $\mathbb{R}^p$ are called *pseudo equivalent* if there exists a diffeomorphism $\Phi : \mathbb{R}^p \to \mathbb{R}^p$, such that:

1. $\Phi(V_1) = V_0$.

2. The set of critical points of V_1 is mapped by Φ onto the set of critical points of V_0.

3. The derivative of Φ at any critical point of V_1 maps the direction of the projection to that at the image of the point.

In the current section we consider only the case of analytic hypersurfaces V given by a single equation $f = 0$. Also, we assume the fibers are one dimensional $x \in \mathbb{R},\ m = 1$.

Now, suppose that all germs of hypersurfaces in a smooth family $V_t = \big\{f_t = 0\big\}$ are pseudo-equivalent to $V_0 = \big\{f_0 = 0\big\}$, $h_t(f_t \circ \theta_t) = f_0,\ t \in [0,1]$, with respect to a smooth family $\Phi_t : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ of germs of diffeomorphisms such that $\Phi_0 = id_{\mathbb{R}^p}$, $h_0 = 1$ and $t \in [0,1]$. Therefore, the respective homological equation is

$$-\frac{\partial f_t}{\partial t} = f_t A_t + \frac{\partial f_t}{\partial x}\dot{X}(t) + \sum_{i=1}^{n} \frac{\partial f_t}{\partial y_i}\dot{Y}_i(t),$$

where the vector field

$$v_t = \dot{X}(t)\frac{\partial}{\partial x} + \sum_{i=1}^{n} \dot{Y}_i(t)\frac{\partial}{\partial y_i}$$

generates the phase flow Φ_t and $A_t \in \mathbb{C}_w$.

Let J_{f_t} be the ideal in $\mathbb{C}_w$ generated by $\frac{\partial f_t}{\partial x}$ and f_t. Denote by $Rad(J_{f_t})$ the *radical* of J_{f_t}. Recall that the radical of an ideal is the set of all elements in $\mathbb{C}_w$, vanishing on the set of common zeros of germs from that ideal. Denote by IJ_{f_t} and $IRad(J_{f_t})$ the *integral* of J_{f_t} and $Rad(J_{f_t})$, consisting of all function germs φ such that the partial derivative of φ with respect to x belongs to J_{f_t} and $Rad(J_{f_t})$, respectively.

Proposition 2.3. [9] *The components of v_t satisfy the following*

$$\dot{X}(t) \in \mathbb{C}_w \qquad \textit{and} \qquad \dot{Y}_i(t) \in IRad(J_{f_t}).$$

In general, the radical of an ideal behaves badly when the ideal depends on a parameter (see [4]). Therefore, we modify the pseudo-equivalence relation since it does not satisfy the properties of a geometrical subgroup of equivalences in J. Damon sense and hence the versatility theorem can fail [7]. Namely, we replace $Rad(J_{f_t})$ by the ideal J_{f_t} itself in the equivalence definition.

Definition 2.4. [9] Two subvarieties $V_0 = \{f_0 = 0\}$ and $V_1 = \{f_1 = 0\}$ in $\mathbb{R}^p$ are called *quasi equivalent* if there is a family of smooth functions h_t which depends continuously on parameter $t \in [0,1]$ and a continuous piece-wise smooth family of diffeomorphisms $\Phi_t : \mathbb{R}^p \to \mathbb{R}^p$ also depending on $t \in [0,1]$ such that:

1. $h_t(f_t \circ \Phi_t) = f_0$, $\Phi_0 = id_{\mathbb{R}^p}$, $h_0 = 1$.
2. The set of critical points of V_t is mapped by Φ_t onto the set of critical points of V_0.
3. The components of the vector field v_t generating Φ_t on each segment of smoothness satisfy the following: $\dot{X}(t) \in \mathbb{C}_w$ and $\dot{Y}_i(t) \in IJ_{f_t}$.

Remarks 2.5.

1. The module IJ_{f_t} is defined precisely as the set of elements of the form
$$e_i + \int_0^x (f_t a_i + \frac{\partial f_t}{\partial x} b_i)dx,$$
where $a_i, b_i \in \mathbb{C}_{x,y}$ and $e_i \in \mathbb{C}_y$.
2. If two subvarieties are equivalent with respect to the standard projection equivalence then they are quasi-equivalent, since functions independent of x are in IJ_f for any f.

The classification of simple classes of quasi-projections of hypersurfaces in low dimensions is given by the following theorems, the proof of which is based on the classification of V.V. Goryunov [8].

Theorem 2.6. [9] *For $n = 1$ the list of simple classes is the same as for the standard group of foliation-preserving diffeomorphisms of the plane acting on the germs of curves:*
$$\begin{array}{lll} A_k: & f = x^{k+1} + y, & k \geqslant 0, \\ B_k: & f = x^2 \pm y^k, & k \geqslant 2, \\ C_k: & f = xy + x^k, & k \geqslant 3, \\ F_4: & f = x^3 + y^2. & \end{array}$$

Theorem 2.7. [9] *For $n = 2$ the list of simple quasi-projections of regular hypersurface singularities consists of*
$$\begin{array}{lll} \widetilde{A}_k: & f = x^{k+1} + y_1 x + y_2, & k \geq 0, \\ \widetilde{B}_k: & f = x^3 + y_1^k x + y_2, & k \geq 2, \\ \widetilde{C}_k: & f = x^{k+1} + y_1^2 x + y_2, & k \geq 3, \\ \widetilde{F}_4: & f = x^4 + y_1^2 x + y_2. & \end{array}$$
The list of simple quasi projections of singular hypersurfaces is
$$A_k^*,\ k \geqslant 0,\quad D_\ell^*,\ \ell \geqslant 4,\quad E_s^*,\ s = 6, 7, 8: \qquad f = x^2 + g(y_1, y_2)$$
where g is one of the standard simple ADE function germs in y,
$$\begin{array}{ll} A_2^{**}: & f = x^3 + y_1 x + y_2^2, \\ A_2^{(k)}: & f = x^3 + y_1^k x + y_1^2 + y_2^2,\ k \geq 2. \end{array}$$

3. Quasi equivalence relation of maps from $\mathbb{R}^m$ to $\mathbb{R}^n$

Consider a $\mathbf{C}^\infty$ map germ $F : (\mathbb{R}^m, 0) \to \mathbb{R}^n$, $x = (x_1, \ldots, x_m) \mapsto y = (y_1, \ldots, y_n)$, $y_i = f_i(x)$, where $f_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ is a smooth function-germ. Denote by $\mathbb{C}^m_n$ the space of all such maps. Since $\mathbb{C}^m_n$ is a vector space, sometimes its elements will be written as column vectors:

$$f = (f_1, f_2, \ldots, f_n)^t = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Let Λ_F be the graph of F, that is $\Lambda_F = \big\{(x, y) : y_i = f_i(x), i = 1, 2, \ldots, n\big\} \subset \mathbb{R}^p$.

Definition 3.1. Two map germs F_0 and F_1 are called *quasi equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$, such that $\Phi(\Lambda_{F_1}) = \Lambda_{F_0}$ and the derivative of Φ preserves the direction of the projection at the points which lie on Λ_{F_1}.

Remarks 3.2.

1. The quasi-equivalence is an equivalence relation.
2. Clearly, if two map germs F_0 and F_1 are $\mathcal{A}$-equivalent then they are quasi-equivalent.

Denote by Q_F the quasi-equivalence class of a map germ F and call it a *quasi orbit.* Then, the tangent space TQ_F to Q_F has the following description.

Lemma 3.3. *TQ_F is the set of all expressions of the form*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{pmatrix} + \begin{pmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \vdots \\ \dot{Y}_n \end{pmatrix},$$

where

$$\frac{\partial \dot{Y}_i}{\partial x_j} = \sum_{r=1}^{n} A_{ir} \frac{\partial f_r}{\partial x_j}, \quad \text{and} \quad \dot{X}_1, \dot{X}_2, \ldots, \dot{X}_m \in \mathbb{C}_x,$$

with $A_{ir} \in \mathbb{C}_x$ for all i and j.

Proof. Introduce a family Φ_t of diffeomorphism germs depending on a parameter $t \in [0,1]$ of the form

$$\Phi_t : (\mathbb{R}^m \times \mathbb{R}^n, 0) \to (\mathbb{R}^m \times \mathbb{R}^n, 0), w \mapsto \Big(X_1(t), \ldots, X_m(t), Y_1(t), \ldots, Y_n(t)\Big),$$

such that $\Phi_0 = id_{\mathbb{R}^m \times \mathbb{R}^n}$. Let $V_t = \sum\limits_{i=1}^{m} \dot{X}_i \frac{\partial}{\partial x_i} + \sum\limits_{i=1}^{n} \dot{Y}_i \frac{\partial}{\partial y_i}$ be the vector field generating Φ_t, where $\dot{X}_i = \frac{\partial X_i}{\partial t}$ and $\dot{Y}_i = \frac{\partial Y_i}{\partial t}$. Let $a_1 = \frac{\partial}{\partial x_1}, a_2 = \frac{\partial}{\partial x_2}, \ldots, a_m = \frac{\partial}{\partial x_m}$ be the basis of the vector space $\mathbb{R}^m$. Then, the family of the vector fields Φ_t^* preserves the direction of the projection if the following relation is satisfied

$$\Phi_t^*(a_i) = \sum_{j=1}^{m} \lambda_j a_j, \tag{3.1}$$

where $\lambda_j \in \mathbb{C}_w$ and also depending on $t \in [0,1]$. Let $V_0 = \sum\limits_{i=1}^{m} \dot{X}_i(0) \frac{\partial}{\partial x_i} + \sum\limits_{i=1}^{n} \dot{Y}_i(0) \frac{\partial}{\partial y_i}$, where $\dot{X}_i(0) = \frac{\partial X_i}{\partial t}\big|_{t=0}$ and $\dot{Y}_i(0) = \frac{\partial Y_i}{\partial t}\big|_{t=0}$.

If we differentiate (3.1) with respect to t and substitute $t = 0$ then we obtain

$$[V_0, a_i] = \sum_{j=1}^{m} \lambda(0)_j a_j, \tag{3.2}$$

where $[.,.]$ is the Lie bracket and $\lambda_i(0) = \frac{\partial \lambda_i}{\partial t}\big|_{t=0}$. In fact, (3.2) is equivalent to

$$-\left(\sum_{r=1}^{m} \frac{\partial \dot{X}_r(0)}{\partial x_i} \frac{\partial}{\partial x_r} + \sum_{s=1}^{n} \frac{\partial \dot{Y}_s(0)}{\partial x_i} \frac{\partial}{\partial y_s}\right) = \sum_{j=1}^{m} \lambda(0)_j a_j. \tag{3.3}$$

Therefore, (3.3) implies that $\dot{X}_r(0) \in \mathbb{C}_w$ and $\frac{\partial \dot{Y}_s(0)}{\partial x_i} = 0$, for all r and s.

Now assume that all map germs in a smooth family F_t depending on $t \in [0,1]$ are quasi equivalent to F_0, with respect to Φ_t. Then, from Definition 3.1 we see that derivatives $\frac{\partial \dot{Y}_s(0)}{\partial x_i}$ belong to the radical of the ideal defining the graph Λ_0 of F_0. Therefore,

$$\frac{\partial \dot{Y}_s(0)}{\partial x_i} \in Rad(I),$$

where I is the ideal generated by $y_j - f_j$, $j = 1, 2, \ldots, n$. Note that $Rad(I) = I$. Hence, we have

$$\frac{\partial \dot{Y}_s(0)}{\partial x_i} = \sum_{j=1}^{n} (y_j - f_j) B_{sj}, \tag{3.4}$$

where $B_{sj} \in \mathbb{C}_w$.

Denote by I^2 the square of the ideal I. Using the Hadamard Lemma, we can always write

$$\dot{Y}_s(0) = \widetilde{Y}_s + \sum_{j=1}^{n}(y_j - f_j)A_{sj} + \psi, \tag{3.5}$$

where $\widetilde{Y}_s \in \mathbb{C}_x$, $A_{sj} \in \mathbb{C}_w$ and $\psi \in I^2$. Differentiation of (3.5) with respect to x_i and using (3.5) followed by the restriction of $\frac{\partial \dot{Y}_s(0)}{\partial x_i}$ to the surface by setting $y_j = f_j$ yield that

$$\frac{\partial \widetilde{Y}_s}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} \widetilde{A}_{sj}$$

where $\widetilde{A}_{sj} \in \mathbb{C}_x$, as required. $\square$

Following [1], we call a map germ $F : (\mathbb{R}^m, 0) \to \mathbb{R}^n$ *simple* if its sufficiently small neighbourhood in the space of all map germs from $(\mathbb{R}^m, 0)$ to $\mathbb{R}^n$ contains only a finite number of quasi-equivalence classes.

3.1. Classification of simple mappings

We start this subsection with recalling the classification of simple singularities of quasi-mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ from [3], giving details of proofs of main results. After that, we classify simple irreducible curve singularities in $\mathbb{R}^m$ with respect to the quasi-stably equivalence relation.

3.1.1. Simple quasi classes of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

Classification of simple quasi-singularities of mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ is as follows.

Theorem 3.4. [3] *Let a map germ $F : (\mathbb{R}^2, 0) \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (y_1, y_2)$, be simple with respect to the quasi-equivalence relation. Then, F is quasi-equivalent to one of the following:*

Notation	*Normal form*	*Restrictions*
$\widetilde{A}_k$	$(x_2, x_1^{k+1} + x_1x_2)$	$k \geq 0,$
$\widetilde{B}_k$	$(x_2, x_1^3 + x_2^k x_1)$	$k \geq 2$
$\widetilde{C}_k$	$(x_2, x_1^{k+1} + x_1^2 x_2)$	$k \geq 2$
$\widetilde{F}_4$	$(x_2, x_1^4 + x_2^2 x_1)$	
$\mathcal{A}_2^{\pm}$	$(x_1^2 \pm x_2^2, x_1x_2)$	
$\mathcal{A}_3$	$(x_1x_2, x_1^2 + x_2^3)$	

To prove Theorem 3.4, we need the following auxiliary results.

We first treat the case when the co-rank of F is one. In this case and up to the $\mathcal{A}$-equivalence relation, we will assume that F has the form $\big(x_2, f\big)$, where $f \in \mathbb{M}_x^2$.

Let $F_t : (\mathbb{R}^2, 0) \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (x_2, f_t)$, be a family of quasi-equivalent map germs at the origin, preserving the first component, where $t \in [0,1]$ and $f_t \in \mathbb{M}_x^2$. Consider the family of regular germs $V_t = \big\{(x_1, y_1, y_2) : y_1 = x_2, y_2 = f_t\big\}$, equipped with trivial fibration structure $\pi : \mathbb{R}_{x_1} \times \mathbb{R}_y^2 \to \mathbb{R}_y^2$.

Lemma 3.5. *The quasi classifications of (x_2, f_t) reduces to the classifications of (V_t, π) with respect to the quasi-equivalence relation, introduced in Definition 2.4.*

Proof. Note that the $\dot{Y}_i$ summands in TQ_{F_t} satisfy the following

$$\frac{\partial \dot{Y}_i}{\partial x_1} = \frac{\partial f_t}{\partial x_1} B_i \tag{3.6}$$

and

$$\frac{\partial \dot{Y}_i}{\partial x_2} = A_i + \frac{\partial f_t}{\partial x_2} B_i, \tag{3.7}$$

for some $A_i, B_i \in \mathbb{C}_x$ and $i \in \{1, 2\}$. Since A_i is an arbitrary smooth function, (3.6) and (3.7) imply

$$\dot{Y}_i = D_i + \int_0^{x_1} \frac{\partial f_t}{\partial x_1} B_i \; dx_1,$$

where $D_i \in \mathbb{C}_{x_2}$. On the other hand, from the first row of the homological equation $-\frac{\partial F_t}{\partial t} = M$, $M \in TQ_{F_t}$, we have $\dot{Y}_1 = -\dot{X}_2$, where $\dot{X}_2 \in \mathbb{C}_x$. Hence, the second row takes the form

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x_1} \dot{X}_1 - \frac{\partial f_t}{\partial y_1} \dot{Y}_1 + \dot{Y}_2, \tag{3.8}$$

where $\dot{X}_1 \in \mathbb{C}_x$. Note that the elements on the right side of (3.8) are exactly those belonging to the tangent space $T\widetilde{Q}_{f_t}$ at the regular germs (V_t, π) with respect to the quasi-equivalence relation, and the result follows. $\square$

Now assume that F has co-rank 2. Then, using the $\mathcal{A}$-equivalence relation, one can show the following.

Lemma 3.6. *The adjacency of the 2-jets of map germs F is*

$$I^{\pm} : (x_1^2 \pm x_2^2, x_1 x_2) \leftarrow II : (x_1 x_2, x_1^2) \leftarrow (III)^{\pm} : (x_1^2 \pm x_2^2, 0) \leftarrow V : (x_1^2, 0) \leftarrow IV : (0, 0).$$

Remark 3.1. Classes in Lemma 3.6 remain non-quasi-equivalent.

Lemma 3.7. *1. If the 2-jet of F is equivalent to $(x_1^2 \pm x_2^2, 0)$ then F is non-simple with respect to the quasi equivalence relation.*

2. If the 4-jet of F is equivalent to $(x_1x_2, x_1^2 + \alpha x_1 x_2^2 + \beta x_2^4)$, $\alpha \neq 0, \beta \neq 0$, then F is non-simple with respect to the quasi-equivalence relation.

Proof. For the first part of the Lemma, consider the homogenous mapping $F_3 = (x_1^2 \pm x_2^2, f_3)$ where $f_3 = x_1^3 + \alpha x_1^2 x_2 + \beta x_1 x_2^2 + \gamma x_2^3$. Then, TQ_{F_3} is the set of all expressions of the form

$$\begin{pmatrix} 2x_1\dot{X}_1 \pm 2x_2\dot{X}_2 + \dot{Y}_1 \\ \frac{\partial f_3}{\partial x_1}\dot{X}_1 + \frac{\partial f_3}{\partial x_2}\dot{X}_2 + \dot{Y}_2 \end{pmatrix}, \tag{$*$}$$

where $\dot{X}_1, \dot{X}_2 \in \mathbb{C}_x$ and the $\dot{Y}_i$ summands satisfy the following constraints

$$\frac{\partial \dot{Y}_i}{\partial x_1} = 2x_1 A_i + \frac{\partial f_3}{\partial x_1} B_i \quad \text{and} \quad \frac{\partial \dot{Y}_i}{\partial x_2} = \pm 2x_2 A_i + \frac{\partial f_3}{\partial x_2} B_i,$$

for some $A_i, B_i \in \mathbb{C}_x$. Notice that the 3-jet of $\dot{Y}_i$ is $a_i(x_1^2 \pm x_2^2) + b_i f_3$, where $a_i, b_i \in \mathbb{R}$. Therefore, the 3-jet of TQ_{F_3} is generated by the vectors:

$$\begin{aligned} v_1 &= (2x_1^2, x_1\frac{\partial f_3}{\partial x_1}), v_2 = (2x_1x_2, x_2\frac{\partial f_3}{\partial x_1}), v_3 = (\pm 2x_2^2, x_2\frac{\partial f_3}{\partial x_2}), \\ v_4 &= (\pm 2x_1x_2, x_1\frac{\partial f_3}{\partial x_2}), v_5 = (0, f_3), v_6 = (x_1^3, 0), v_7 = (x_1^2 x_2, 0), \\ v_8 &= (x_1x_2^2, 0), v_9 = (x_2^3, 0), v_{10} = (x_1^2 \pm x_2^2, 0), v_{11} = (0, x_1^2 \pm x_2^2). \end{aligned}$$

These vectors form a subspace of dimension at most 11. The dimension of the space of the 3-jets of co-rank 2 mappings is 14 which is greater than the subspace dimension. This means that the germ F_3 is non-simple with respect to the quasi equivalence relation.

Similarly, we can prove the second part of the Lemma. □

Proof of Theorem 3.4. Firstly, suppose that the co-rank of F is one. Then, Lemma 3.5 and Theorem 2.7 imply that if F is simple then it is quasi equivalent to one of the following: $(x_2, x_1^{k+1} + x_1x_2)$, $k \geq 0$, $(x_2, x_1^3 + x_2^k x_1)$, $k \geq 2$, $(x_2, x_1^{k+1} + x_1^2 x_2), k \geq 2$ and $(x_2, x_1^4 + x_2^2 x_1)$.

Next, let the co-rank of F be two. Then, Lemma 3.6 and Lemma 3.7 yield that all simple quasi singularities are among map germs whose 2-jets are quasi equivalent to either $(x_1^2 \pm x_2^2, x_1x_2)$ or (x_1x_2, x_1^2). Using Arnold's spectral sequence method [1], one can easily prove the results below.

- If F is a map germ with the 2-jet $(x_1^2 \pm x_2^2, x_1x_2)$, then F is quasi equivalent to
$\mathcal{A}_2^{\pm} : (x_1^2 \pm x_2^2, x_1x_2)$.

- Let $F = \big(x_1x_2 + f, x_1^2 + g\big)$, where $f, g \in \mathbb{M}_x^3$. If g contains a term ax_2, then F is quasi equivalent to $\mathcal{A}_3 : \big(x_1x_2, x_1^2 + x_2^3\big)$. Otherwise, in the most general case, F is equivalent to a non-simple germ, by Lemma 3.7. This finishes the proof of Theorem. □

3.1.2. Quasi-stably simple classes of irreducible curves in $\mathbb{R}^n$

Recall that an irreducible curve at the origin in $\mathbb{R}^n$ can be described by a germ of an analytic map $F : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, $x \mapsto y = \big(y_1 = f_1(x), y_2 = f_2(x), \ldots, y_n = f_n(x)\big)$. Following Arnold in [2], we introduce the following.

Definition 3.8. An irreducible curve is called *quasi-stably simple* if it is simple with respect to the quasi-equivalence relation and remains simple when the ambient space is embedded into a larger space. Two curves which are obtained one from the other by such embedding are called *quasi-stably equivalent.*

Remark 3.2. By the codimension here and below, we mean the codimension in the space of the Taylor series with zero constant terms.

The classification of quasi-stably simple classes is as follows.

Theorem 3.9. *Assume that the curve F is quasi-stably simple. Then, F is quasi-stably equivalent to one of the lines* $\mathbb{A}_k : \big(x^k, 0\big)$, $k \geq 1$.

Remarks 3.10.

1. Any irreducible curve is either quasi-stably simple (and hence is quasi-stably equivalent to one of lines, stated in the theorem) or belongs to the subset of infinite codimension in the space of all curves.

2. The codimension of the class $\mathbb{A}_k$ is $kn - 1$.

Proof of Theorem 3.9. Up to the $\mathcal{A}$-equivalence relation, we may assume that any irreducible curve has the form $F = \big(x^k, f_2, \ldots, f_n\big)$, where $k \geq 1$ and $f_i \in \mathbb{M}_x^{k+1}$. Notice that the derivatives of the $\dot{Y}_i$ summands in TQ_F with respect to x belong to the ideal generated by x^{k-1} and hence $\dot{Y}_i = x^k A_i$, for some $A_i \in \mathbb{C}_x$. By Arnold's spectral sequence method, one can easily show that F is quasi-stably equivalent to the germ $\mathbb{A}_k : \big(x^k, 0\big)$, $k \geq 1$.

4. The quasi classification of some multi-germs of curves in $\mathbb{R}^n$

We start with recalling the standard notions and basic definitions concerning multi-germs of curves from [5].

A reducible curve at the origin in $\mathbb{R}^n$ is determined by a collection of maps

$$(\mathbb{R},0)\to(\mathbb{R}^n,0), x\mapsto y=(y_1,\dots,y_n).$$

Definitions 4.1. A multi-germ of curves in $\mathbb{R}^n$ is a set $G=(F_1,\dots,F_r)$ of germs of analytic maps $F_i:(\mathbb{R},0)\to(\mathbb{R}^n,0)$, where Im$(F_i)\cap$ Im $(F_j)=\{0\}$ for $i\neq j$ ($F_1,F_2,\dots$ and F_r are called components of the multi-germ G).

The group of $\mathcal{A}$-equivalences $\mathcal{A}=\mathcal{L}\times\mathcal{R}_1\times\mathcal{R}_2\times\cdots\times\mathcal{R}_r$, where $\mathcal{R}_i$ is the i-th copy of the group of the standard right equivalences, acts on the space of multi-germs $G=(F_1,\dots,F_r)$ by the formula

$$(\phi,\varphi_1,\dots,\varphi_r).(F_1,\dots,F_r)=(\phi\circ F_1\circ\varphi_1^{-1},\dots,\phi\circ F_r\circ\varphi_r^{-1}),$$

where $\phi\in\mathcal{L}$ and $\varphi_i\in\mathcal{R}_i$.

Definitions 4.2. A multi-germ G is called *simple* if there exists a neighbourhood of G in the space of multi-germs which intersects only the finite number of $\mathcal{A}$-orbits. It is *stably simple*, if it remains simple when the ambient space is immersed in a larger space.

Definitions 4.3. Two multi-germs G and $\widetilde{G}$ in $\mathbb{R}^n$ are equivalent if they lie in one orbit of the $\mathcal{A}$-action.

The tangent space $T\mathcal{A}.G$ to the orbit $\mathcal{A}.G$ is equal to $T\mathcal{R}.G+T\mathcal{L}.G$. The first set is the direct sum $\bigoplus\limits_{i=1}^{r}\mathbb{M}_x(\frac{\partial F_i}{\partial x})$ and its elements denoted by matrices where the i-th column of which corresponds to an element of $T\mathcal{R}.F_i$. On the other hand, $T\mathcal{L}.G$ is the set of matrices of the form

$$\begin{bmatrix} \dot{Y}_{11} & \dot{Y}_{12} & \dots & \dot{Y}_{1r}\\ \dot{Y}_{21} & \dot{Y}_{22} & \dots & \dot{Y}_{2r}\\ \vdots & \vdots & \dots & \vdots\\ \dot{Y}_{n1} & \dot{Y}_{n2} & \dots & \dot{Y}_{nr} \end{bmatrix},$$

where $\dot{Y}_{ij}=U_i\circ F_j$ and $U_i\in\mathbb{M}_y$.

The quasi-equivalence of multi-germs of curves is defined as follows.

Let $F_j : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$, $x \mapsto y = (y_1, \dots, y_n)$, $y_i = f_{ij}(x)$, $i = 1, \dots, n$ and denote by Λ_j its graph.

Definition 4.4. Two multi-germs $G = (F_1, \dots, F_r)$ and $\widetilde{G} = (\widetilde{F}_1, \dots, \widetilde{F}_r)$ in $\mathbb{R}^n$ are called *quasi equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R} \times \mathbb{R}^n, 0)$, such that $\Phi(\Lambda_j) = \widetilde{\Lambda}_j$, for all j, and the derivative of Φ preserves the direction of the projection at the points which lie on Λ_j.

Obviously, the quasi-equivalence of multi-germs of curves is an equivalence relation. By similar consideration and technique which were used in the proof of Lemma 3.3, we obtain the following description of the tangent space $TQ.G$ to the quasi class $Q.G$ of a multi-germ G.

Lemma 4.5. *$TQ.G = T\mathcal{R}.G + T\mathcal{Q}.G$, where $T\mathcal{R}.G = \bigoplus_{i=1}^{r} \mathbb{M}_x(\frac{\partial F_i}{\partial x})$ and $T\mathcal{Q}.G$ is the set of matrices of the form*

$$\begin{bmatrix} \dot{Y}_{11} & \dot{Y}_{12} & \dots & \dot{Y}_{1r} \\ \dot{Y}_{21} & Y_{22} & \dots & \dot{Y}_{2r} \\ \vdots & \vdots & \dots & \vdots \\ \dot{Y}_{n1} & \dot{Y}_{n2} & \dots & \dot{Y}_{nr} \end{bmatrix}.$$

which satisfy the following

$$\begin{bmatrix} \dot{Y}'_{11} & \dot{Y}'_{12} & \dots & \dot{Y}'_{1r} \\ \dot{Y}'_{21} & \dot{Y}'_{22} & \dots & \dot{Y}'_{2r} \\ \vdots & \vdots & \dots & \vdots \\ \dot{Y}'_{n1} & \dot{Y}'_{n2} & \dots & \dot{Y}'_{nr} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} f'_{11} & f'_{12} & \dots & f'_{1r} \\ f'_{21} & f'_{22} & \dots & f'_{2r} \\ \vdots & \vdots & \dots & \vdots \\ f'_{n1} & f'_{n2} & \dots & f'_{nr} \end{bmatrix}.$$

where $A_{ij} \in \mathbb{C}_x$, $f'_{ij} = \frac{df_{ij}}{dx}$ and $\dot{Y}'_{ij} = \frac{d\dot{Y}_{ij}}{dx}$.

Proposition 4.6. *$T\mathcal{A}.G \subset TQ.G$.*

Proof. Let $V \in T\mathcal{A}.G$. Then, we can write $V = V_1 + V_2$, where $V_1 \in T\mathcal{R}.G$ and $V_2 \in T\mathcal{L}.G$. Hence, $V_2 = (\dot{Y}_{ij})$, where $\dot{Y}_{ij} = U_i \circ F_j$ and $U_i \in \mathbb{M}_y$. Notice that $\frac{d\dot{Y}_{ij}}{dx} = \sum_{k=1}^{n} \frac{df_{kj}}{dx}\frac{\partial U_i}{\partial y_k}$. Moreover, if we let $F'_j = (f'_{1j}, f'_{2j}, \dots, f'_{2n})$, where $f'_{ij} = \frac{df_{ij}}{dx}$ and denote by $(F'_j)^T$ the transpose of F'_j, then we have

$$\sum_{k=1}^{n} \frac{df_{kj}}{dx}\frac{\partial U_i}{\partial y_k} = A_i(F'_j)^T,$$

where $A_i = (A_{i1}, A_{i2}, \ldots, A_{in})$ with $A_{ik} = \frac{\partial U_i}{\partial y_k}$, $f'_{kj} = \frac{df_{kj}}{dx}$ and the result follows. $\square$

Remark 4.1. For the standard $\mathcal{A}$-equivalences of multi-germs, we are free to change the coordinates about each point independently of the associated branch in the source, whereas in the target the same coordinate change must be applied to each branch. On the other hand, for the quasi-equivalence, we are still free to change the coordinates in the source about each point independently of the associated branch, but in the target if a quasi-change of the coordinates Y_{ij} occurs on a certain branch F_j and the derivative of $\dot{Y}_{ij}$ is equal to $A_i(F'_j)^T$, then the same factor A_i must be applied to all quasi-changes of the coordinates on other branches.

Definition 4.7. A multi-germ G is called *simple* with respect to the quasi-equivalence relation if there exists a neighbourhood of G in the space of multi-germs which intersects only finite number of quasi-classes. Moreover, it is called *quasi-stably simple* if it remains simple when the ambient space is immersed in a larger space.

We will only consider bi-germs (multi-germs with two components) of curves and give the beginning of the classifications with respect to the quasi-equivalence relation.

Theorem 4.8. *Let G be a quasi-stably simple bi-germ. Then, up to permutation of curves, G is quasi-equivalent to one of the bi-germs $\big(F_1, F_2\big)$, described in the following table.*

Notation	F_1	F_2	*Restrictions*
$\mathcal{A}_k$	$(x, 0)$	$(0, x^k)$	$k \geq 1$
$\mathcal{B}_{k,l}$	$(x, 0)$	(x^k, x^l)	$l > k \geq 1$
$\mathcal{C}_2$	$(x^2, 0)$	$(0, x^2)$	
$\mathcal{C}_3$	$(x^2, 0)$	$(0, 0, x^3)$	
$\mathcal{D}_{2,3}$	$(x^2, 0)$	$(x^2, 0, x^3)$	

To prove Theorem 4.8, we use the spectral sequence method [1] together with the following auxiliary results.

Consider a pair G of curves with a regular first component which will be written in the normal form $(x, 0, \ldots, 0)$ or equivalently as $(x, 0)$. Introduce a family of quasi-equivalent pairs $G_t = \big((x, 0), F_2(t)\big)$, preserving the first component, where $F_2(t) = \big(f_1(t), f_2(t), \ldots, f_m(t)\big)$, $f_i \in \mathbb{C}_x$ and $t \in [0, t]$ such that $G_0 = G$. Let $f'_i = \frac{df_i}{dx}$ and denote by Ω the ideal generated by $f'_1, f'_2, f'_3, \ldots, f'_n$, and by $\widetilde{\Omega}$ the ideal generated by $f'_2, f'_3, \ldots, f'_n$.

Lemma 4.9. *The homological equation of G_t is*

$$\begin{bmatrix} 0 & \dot{f}_1 \\ 0 & \dot{f}_2 \\ \vdots & \vdots \\ 0 & \dot{f}_n \end{bmatrix} = \begin{bmatrix} H_1 & f_1' H_2 \\ 0 & f_2' H_2 \\ \vdots & \vdots \\ 0 & f_n' H_2 \end{bmatrix} + \begin{bmatrix} \dot{Y}_{11} & \dot{Y}_{12} \\ \dot{Y}_{21} & \dot{Y}_{22} \\ \vdots & \vdots \\ \dot{Y}_{n1} & \dot{Y}_{n2} \end{bmatrix},$$

such that $\dot{Y}_{11} \in \mathbb{M}_x$, $\dot{Y}_{i1} = 0$, $\dot{Y}_{12}' \in \Omega$, and $\dot{Y}_{i2}' \in \widetilde{\Omega}$ for all $i \in \{2, 3, \ldots, n\}$. Here, $\dot{f}_i = \frac{df_i}{dt}$ and $H_1, H_2 \in \mathbb{M}_x$.

Proof. By differentiating G_t with respect to t, we obtain the homological equation described in Lemma. Moreover, Lemma 4.5 implies that

$$\begin{bmatrix} \dot{Y}_{11}' & \dot{Y}_{12}' \\ \dot{Y}_{21}' & \dot{Y}_{22}' \\ \vdots & \vdots \\ \dot{Y}_{n1}' & \dot{Y}_{n2}' \end{bmatrix} = \begin{bmatrix} A_{11} & \sum_{k=1}^{n} A_{1k} f_k' \\ A_{21} & \sum_{k=1}^{n} A_{2k} f_k' \\ \vdots & \vdots \\ A_{n1} & \sum_{k=1}^{n} A_{nk} f_k' \end{bmatrix}. \tag{4.1}$$

Comparing the columns of the homological equation and (4.1) yields that $\dot{Y}_{11} = -H_1$, $\dot{Y}_{i1} = 0$, and therefore $A_{11} = -\frac{dH_1}{dx}$, $A_{i1} = 0$ for all $i \in \{2, 3, \ldots, m\}$. As H_1 is an arbitrary germ, we have that $\dot{Y}_{12}' \in \Omega$ and $\dot{Y}_{i2}' \in \widetilde{\Omega}$ for all $i \in \{2, 3, \ldots, n\}$, as required. □

Now suppose that both components of G are singular. Then,

Lemma 4.10. [5] *The 2-jet of G is $\mathcal{A}$-equivalent to either $\big((x^2, 0), (0, x^2)\big)$ or $\big((x^2, 0), (x^2, 0)\big)$.*

Moreover,

Lemma 4.11. *A pair of curves with the 3-jet $\big((x^2, x^3), (x^2, \alpha x^3)\big)$, where $\alpha \neq 1$, is not simple with respect to quasi-equivalence.*

Proof. Let $G_\alpha = \big((x^2, x^3), (x^2, \alpha x^3)\big)$. Then, the 3-jet in $TQ.G_\alpha$ is generated by the following 10 vectors: $v_1 = \big((2x^2, 3x^3), (0, 0)\big)$, $v_2 = \big((0, 0), (2x^2, 3\alpha x^3)\big)$, $v_3 = \big((2x^3, 0), (0, 0)\big)$, $v_4 = \big((0, 0), (2x^3, 0)\big)$, $v_5 = \big((x^2, 0), (x^2, 0)\big)$, $v_6 = \big((0, x^2), (0, x^2)\big)$, $v_7 = \big((2x^3, 0), (2x^3, 0)\big)$, $v_8 = \big((0, 2x^3), (0, 2x^3)\big)$, $v_9 = \big((x^3, 0), (\alpha x^3, 0)\big)$, $v_{10} = \big((0, x^3), (0, \alpha x^3)\big)$. Notice that $v_3 + v_4 = v_7$, $2av_1 + 2v_2 - 4\alpha v_5 = 3\alpha v_8$ and $v_1 + v_2 - 2v_5 = 3v_9$. Therefore, the vectors v_7, v_8 and v_9 can be removed from the list above. The remaining vectors form a subspace of dimension at most 7. The dimension of the space of all 3-jets of bi-germs with two singular components is 8 which is greater than the subspace dimension. This means that the germ G_α is non-simple. □

4.1. Proof of the main Theorem 4.8

We distinguish the following cases.

1. Pairs of curves with a regular first component. In this case the pair takes the form $G = ((x, 0), F)$. Therefore, we classify the second component using Lemma 4.9 as follows.
   - Assume the 1-jet of F is nontrivial and equal to $(\alpha x, \beta x)$, with $\alpha, \beta \in \mathbb{R}$, and hence is equivalent to either $(0, x)$ or $(x, 0)$. Consider the first case. Then, G is quasi equivalent to $\mathcal{A}_1 : \big((x, 0), (0, x)\big)$. Next, if k be the minimal number such that the k-jet of F is not $(x, 0)$ then G is quasi-equivalent to $\mathcal{B}_{1,k} : \big((x, 0), (x, x^k)\big)$ where $k \geq 2$.
   - Consider the case when F is singular and its multiplicity is k. Then, the k-jet of F is equivalent to either $(0, x^k)$ or $(x^k, 0)$. Suppose that l is the minimal number such that the l-jet of F is not $(x^k, 0)$ then G is quasi equivalent to $\mathcal{B}_{k,l} : \big((x, 0), (x^k, x^l)\big)$ where $l > k \geq 2$. Next, if the k-jet of F is $(0, x^k)$ then G is quasi-equivalent to $\mathcal{A}_k : \big((x, 0), (0, x^k)\big)$, with $k \geq 2$.
2. Pairs of curves with singular components. In this case the nontrivial 2-jet of G is equivalent to either $\big((x^2, 0), (0, x^2)\big)$ or $\big((x^2, 0), (x^2, 0)\big)$.
   - Consider the case when the 2-jet is $\big((x^2, 0), (0, x^2)\big)$. Then, G is quasi-equivalent $\mathcal{C}_2 : \big((x^2, 0), (0, x^2)\big)$.
   - If the 2-jet is $\big((x^2, 0), (x^2, 0)\big)$ then Lemma 4.11 yields that all quasi-stably simple singularities are among pairs with the 3-jet is either $\big((x^2, x^3, 0), (x^2, 0, x^3)\big)$ or $\big((x^2, x^3, 0), (0, 0, x^3)\big)$. In such cases, we obtain $\mathcal{C}_3 : \big((x^2, 0), (0, 0, x^3)\big)$ and $\mathcal{D}_{2,3} : \big((x^2, 0), (x^2, 0, x^3)\big)$, respectively. Pairs from other cases are adjacent to the family $\big((x^2, x^3), (x^2, \alpha x^3)\big)$, where $\alpha \neq 1$.

REFERENCES

1. V. I. Arnold and S. M. Gusein-Zade and A. N. Varchenko: *Singularities of differentiable maps.* Vol. I. Monographs in Mathematics 82, BirkhäuserBoston, Boston, 1985.
2. V. I. Arnold: *Simple singularities of curves.* Trudy Mat. Inst. Steklov. 226 (1999), 27-35; English transl., Proc. Steklov Inst. Math
3. F. D. Alharbi: *Classification of singularities of functions and mappings via non standard equivalence relations.* PhD Thesis, University of Liverpool, 2011, 256pp.
4. F. D. Alharbi: *Quasi cusp singularities.* The Journal of Singularities, volume 12(2015), 1–18.
5. P. A. Kolgushkin and R. R. Sadykov: *Simple singularities of multigerms of curves.* Revista Matematica Complutense (2001) vol. XIV, num. 2, 311-344.

6. J. W. Bruce and T. J. Gaffney: *Simple singularities of mappings* $\mathbf{C}, 0 \to \mathbf{C}^2, 0$. Proceedings of the London Mathematical Society (2) **26** (1982), no. 3, 465–474.
7. J. Damon: *The unfolding and determinacy theorems for subgroups of A and K*. Proceedings of the American Mathematical Society **50** (1984), no. 306, 233-254.
8. V. V. Goryunov: *Singularities of projections of complete intersections.* (Russian) Current problems in mathematics, Vol. 22, Itogi Nauki i Tekhniki, VINITI, Moscow, 1983, 167–206; English translation in Journal of Soviet Mathematics **27** (1984), no. 3, 2785–2811.
9. V. M. Zakalyukin: *Quasi-projections.* Proceedings of the Steklov Institute of Mathematics 259 (2007), no. 1, 273–280.

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ON BIVARIATE RETARDED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, we obtain some retarded integral inequalities in two independent variables which can be used as tools in the theory of partial differential and integral equations with time delays. The presented inequalities are of new forms compared with the existing ones so far in the literature. In order to illustrate the validity of the theorems we give one application for them for the solution to certain fractional order differential equations.
Keywords: integral inequalities; differential equations; time delay.

1. Introduction

As it is well known integral inequalities play a significant role in the qualitative analysis of differential and integral equations theory. Over the years, various investigators have discovered many useful integral inequalities in order to achieve a diversity of desired goals, see [1]-[12] and the references given therein. In a recent paper [8] Pachpatte presented a retarded inequality which has very good characters. A large number of papers have been presented dealing with various extensions and generalizations of this inequality. Some of the results may be found in [8], but let us first recall the main results of [8] as follows:

In what follows, $\mathbb{R}$ denotes a set of real numbers, $\mathbb{R}_+ = [0,\infty)$, $J_1 = [x_0, X)$, $J_2 = [y_0, Y)$ are given subsets of $\mathbb{R}$, $\Delta = J_1 \times J_2$ and $'$ denotes the derivative.

Theorem 1.1. *Let* $u(x,y)$, $a(x,y) \in C(\Delta, \mathbb{R}_+)$, $b(x,y,s,t) \in C(\Delta^2, \mathbb{R}_+)$, *for* $x_0 \le s \le x \le X$, $y_0 \le t \le y \le Y$, $\alpha(x) \in C^1(J_1, J_1)$, $\beta(y) \in C^1(J_2, J_2)$ *be non-decreasing with* $\alpha(x) \le x$ *on* J_1, $\beta(y) \le y$ *on* J_2 *and* $k \ge 0$ *be a constant.*

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(A_1) If

$$u(x,y) \leq k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t)u(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta)u(\sigma,\eta)d\sigma d\eta \right] dtds, \tag{1.1}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq k \exp(A(x,y)), \tag{1.2}$$

for $(x,y) \in \Delta$, where

$$A(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta)d\sigma d\eta \right] dtds, \tag{1.3}$$

for $(x,y) \in \Delta$.

(A_2) Let $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing function with $g(u) > 0$ for $u > 0$. If

$$u(x,y) \leq k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t)g(u(s,t)) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta)g(u(\sigma,\eta))d\sigma d\eta \right] dtds, \tag{1.4}$$

for $(x,y) \in \Delta$, then for $x_0 \leq x \leq x_1,\ y_0 \leq y \leq y_1$,

$$u(x,y) \leq G^{-1}\left[G(k) + A(x,y)\right], \tag{1.5}$$

where $A(x,y)$ is defined by (1.3), G^{-1} is the inverse function of

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)},\ r > 0,\ r_0 > 0$$

and $x_1 \in J_1, y_1 \in J_2$ are chosen so that

$$G(k) + A(x,y) \in Dom(G^{-1}),$$

for all x and y lying in $[x_0, x]$ and $[y_0, y]$ respectively.

The purpose of this paper is to explore two independent retarded versions of the above integral inequalities which can be used as tools in the theory of partial differential and integral equations with time delays. Applications are also given to convey the significance of our results.

2. Main Results

The first section of this paper will present some new non-linear retarded integral inequalities in two independent variables which can be used as effective tools in the study on non-linear partial differential equations with time delay.

Theorem 2.1. *If $u(x,y),\ p(x,y),\ a(x,y)$ are real valued non-negative continuous functions and $u(x,y) \geq 2p(x,y)$ is defined for $x \geq 0,\ y \geq 0,\ b(x,y,s,t)$ are continuous non-decreasing in x and y for t,s. $0 \leq \alpha(x) \leq x,\ 0 \leq \beta(y) \leq y,\ \alpha'(x), \beta'(y) \geq 0$ are real valued continuous functions defined for $x \geq 0,\ y \geq 0$, that satisfy*
(2.1)

$$u(x,y) \leq p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t)u(s,t) + \int_0^s \int_0^t b(s,t,\sigma,\eta)u(\sigma,\eta)d\sigma d\eta \right] dtds,$$

then

$$u(x,y) \leq p(x,y) \times \left(1 + e^{\int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t) + \int_0^s \int_0^t b(s,t,\sigma,\eta)d\sigma d\eta \right] dtds} \right). \tag{2.2}$$

Proof. First of all let $z(x,y)$ denote the function on the right hand side of 2.1, that is,

$$z(x,y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t)u(s,t) + \int_0^s \int_0^t b(s,t,\sigma,\eta)u(\sigma,\eta)d\sigma d\eta \right] dtds,$$

then $z(0,y) = z(x,0) = 0$ and our assumption on a, b, u, α and β imply that z is a non-decreasing positive function for $x \geq 0,\ y \geq 0$ and $x \in [0,T_1],\ y \in [0,T_2]$ we have

$$z_{xy}(x,y) = \alpha'(x)\beta'(y) \left[a(\alpha(x),\beta(y))u(\alpha(x),\beta(y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta)u(\sigma,\eta)d\sigma d\eta \right]$$

$$\leq \alpha'(x)\beta'(y) \left[a(\alpha(x),\beta(y)) \left(p(\alpha(x),\beta(y)) + z(\alpha(x),\beta(y)) \right) \right.$$

$$\left. + \int_0^{\alpha(x)} \int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta) \left(p(\sigma,\eta) + z(\sigma,\eta) \right) d\sigma d\eta \right].$$

Then by rearranging the above inequality we obtain

$$z_{xy}(x,y) \leq z(T_1,T_2) \left(\alpha'(x)\beta'(y) \left[a(\alpha(x),\beta(y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta)d\sigma d\eta \right] \right)$$

$$+ \left(\alpha'(x)\beta'(y) \left[a(\alpha(x),\beta(y))p(\alpha(x),\beta(y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta)p(\sigma,\eta)d\sigma d\eta \right] \right).$$

As $0 \leq \alpha(x) \leq x$ and $0 \leq \beta(y) \leq y$ and $z(x,y)$ is non-decreasing with respect to x, y we get

$$\frac{z_{xy}(x,y)}{z(T_1,T_2)} \leq 2\left(\frac{\partial^2}{\partial x \partial y}\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right). \tag{2.3}$$

On the other hand,

$$\frac{\partial}{\partial y}\left(\frac{z_x(x,y)}{z(T_1,T_2)}\right) \leq \frac{z_{xy}(x,y)}{z(T_1,T_2)}. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\frac{\partial}{\partial y}\left(\frac{z_x(x,y)}{z(T_1,T_2)}\right) \leq 2\left(\frac{\partial^2}{\partial x \partial y}\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right).$$

Integrating both sides of the above inequality with respect to y from 0 to y, we get

$$\frac{z_x(x,y)}{z(T_1,T_2)} \leq 2\left(\frac{\partial}{\partial x}\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right),$$

then again integrating the above inequality with respect to x from 0 to x we obtain

$$\ln|z(T_1,T_2)| \leq \ln|p(T_1,T_2)| + \int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds$$

for $x \in [0,T_1]$, $y \in [0,T_2]$. Thus we have

$$z(T_1,T_2) \leq p(T_1,T_2) \times e^{\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds}. \tag{2.5}$$

Let $x = T_1$, $y = T_2$ in (2.5), we obtain

$$z(T_1,T_2) \leq p(T_1,T_2) \times e^{\int_0^{\alpha(T_1)}\int_0^{\beta(T_2)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds}.$$

From the definition of $z(x,y)$, we have $u(x,y) \leq p(x,y) + z(x,y)$. As a result, we get the required inequality in (2.2). □

Corollary 2.1. *Assume that a, b, α, β are as in Theorem 2.1 and $p(x,y) \equiv p > 0$, if $u \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ satisfying (2.1), then*

$$u(x,y) \quad \leq \quad p + pe^{\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds}, \quad x \geq 0, \ y \geq 0.$$

Corollary 2.2. *Assume that* a, b, α, β *are as in Theorem 2.1 and* $p(x, y) \equiv p > 0$. *Suppose* $u \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *is a solution to the integral equation*

$$u(x,y) = p + \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t)u(s,t) + \int_0^s \int_0^t b(s,t,\sigma,\eta)u(\sigma,\eta)d\sigma d\eta \right] dtds, \quad x \geq 0, \ y \geq 0.$$

If

$$\lim_{x\to\infty} \left(\lim_{y\to\infty} \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t) + \int_0^s \int_0^t b(s,t,\sigma,\eta)d\sigma d\eta \right] dtds \right) < \infty,$$

then u *is bounded.*

Theorem 2.2. *Assume that* p, a, b, α, β *are as in Theorem 2.1 and* $g(r)$ *is a positive continuous non-decreasing function for* $r > 0$ *with* $g(0) = 0$ *and* $\int_1^\infty \frac{dt}{g(t)} = \infty$, *if* $u \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *satisfies for* $x \geq 0, \ y \geq 0$

(2.6)

$$u(x,y) \leq p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t)g\left(u(s,t)\right) + \int_0^s \int_0^t b(s,t,\sigma,\eta)g\left(u(\sigma,\eta)\right) d\sigma d\eta \right] dtds,$$

then

(2.7)

$$u(x,y) \leq G^{-1} \left(G\left(p(x,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t) + \int_0^s \int_0^t b(s,t,\sigma,\eta)d\sigma d\eta \right] dtds \right),$$

where

$$G(r) = \int_1^r \frac{dt}{g(t)}, \quad r \geq 0.$$

Proof. Assume $T_1, T_2 > 0$ is fixed and let

$$z(x,y) = \int_0^{\alpha(x)} \int_0^{\beta(y)} \left[a(s,t)g\left(u(s,t)\right) + \int_0^s \int_0^t b(s,t,\sigma,\eta)g\left(u(\sigma,\eta)\right) d\sigma d\eta \right] dtds,$$

with the assumption on a, b, α, β imply that $z(x, y)$ is non-decreasing about x and y. Hence for $x \in [0, T_1], \ y \in [0, T_2]$ we have

$$\begin{aligned}
z_{xy}(x,y) &= \alpha'(x)\beta'(y)\left[a(\alpha(x),\beta(y))g\left(u(\alpha(x),\beta(y))\right)\right. \\
&+ \left.\int_0^{\alpha(x)}\int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta)g\left(u(\sigma,\eta)\right)d\sigma d\eta\right] \\
&\leq \alpha'(x)\beta'(y)\left[a(\alpha(x),\beta(y))\left(g\left(p(\alpha(x),\beta(y))\right)+g\left(z(\alpha(x),\beta(y))\right)\right)\right. \\
&+ \left.\int_0^{\alpha(x)}\int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta)\left(g\left(p(\sigma,\eta)\right)+g\left(z(\sigma,\eta)\right)\right)d\sigma d\eta\right] \\
&\leq g(p(T_1,T_2)+z(T_1,T_2)) \\
&\times \left(\alpha'(x)\beta'(y)\left[a(\alpha(x),\beta(y))+\int_0^{\alpha(x)}\int_0^{\beta(y)} b(\alpha(x),\beta(y),\sigma,\eta)d\sigma d\eta\right]\right).
\end{aligned}$$

Therefore, we write

$$\frac{z_{xy}(x,y)}{g(p(T_1,T_2)+z(T_1,T_2))} \leq \frac{\partial^2}{\partial x \partial y}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right).$$

Noting that

$$\frac{\partial}{\partial y}\left(\frac{z_x(x,y)}{g(p(T_1,T_2)+z(T_1,T_2))}\right) \leq \frac{z_{xy}(x,y)}{g(p(T_1,T_2)+z(T_1,T_2))}.$$

We obtain

$$\frac{\partial}{\partial y}\left(\frac{z_x(x,y)}{g(p(T_1,T_2)+z(T_1,T_2))}\right) \leq \frac{\partial^2}{\partial x \partial y}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right).$$

Integrating both sides of the above inequality with respect to y from 0 to y we get

$$\frac{z_x(x,y)}{g(p(T_1,T_2)+z(T_1,T_2))} \leq \frac{\partial}{\partial x}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right),$$

then integrating the above inequality with respect to x from 0 to x we have

(2.8)

$$G(p(T_1,T_2)+z(T_1,T_2)) \leq G\left(p(T_1,T_2)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[a(s,t)+\int_0^s\int_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds,$$

for $x \in [0, T_1],\ y \in [0, T_2]$.

In view of $\int\limits_1^\infty \frac{dt}{g(t)} = \infty$, from (2.8), we have

(2.9)

$$p(T_1, T_2) + z(T_1, T_2)) \leq G^{-1}\left(G\left(p(T_1, T_2)\right) + \int\limits_0^{\alpha(x)}\int\limits_0^{\beta(y)}\left[a(s,t) + \int\limits_0^s\int\limits_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right).$$

Let $x = T_1$, $y = T_2$ in (2.9), we obtain

$$p(T_1, T_2) + z(T_1, T_2)) \leq G^{-1}\left(G\left(p(T_1, T_2)\right) + \int\limits_0^{\alpha(T_1)}\int\limits_0^{\beta(T_2)}\left[a(s,t) + \int\limits_0^s\int\limits_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right).$$

Due to T_1, T_2 are arbitrary and $u(x, y) \leq p(x, y) + z(x, y)$, we obtain (2.7). $\square$

Corollary 2.3. *Assume that p, a, b, α, β are as in Theorem 2.2. Suppose $u \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation*

$$u(x, y) = p(x, y) + \int\limits_0^{\alpha(x)}\int\limits_0^{\beta(y)}\left[a(s,t)g\left(u(s,t)\right) + \int\limits_0^s\int\limits_0^t b(s,t,\sigma,\eta)g\left(u(\sigma,\eta)\right)d\sigma d\eta\right]dtds,$$

for $x \geq 0,\ y \geq 0$. If p is bounded and

$$\lim_{x\to\infty}\left(\lim_{y\to\infty}\int\limits_0^{\alpha(x)}\int\limits_0^{\beta(y)}\left[a(s,t) + \int\limits_0^s\int\limits_0^t b(s,t,\sigma,\eta)d\sigma d\eta\right]dtds\right) < \infty,$$

then u is bounded.

3. Basic Application

In this section, we will present some basic applications of our results to obtain the bounds on the solution to the integral equation with time delay. We would like to develop a set of benchmark applications which may be used in the theory of partial differential and integral equations with time delay so we invite other researchers to contact us with their results for these cases, and perhaps forward us their own examples.

3.1. Application:

In order to exemplify the application of Theorem 2.1 we set up the bound on the solutions of partial integral equations of the form :

(3.1)

$$u(x, y) = k(x, y) + \int\limits_0^{\alpha(x)}\int\limits_0^{\beta(y)}\left[G(x,y,s,t,u(s,t)) + \int\limits_0^s\int\limits_0^t F(x,y,s,t,\sigma,\eta,u(\sigma,\eta))d\sigma d\eta\right]dtds$$

where all the function are continuous on their respective domains of their definitions and

$$|k(x,y)| \leq p(x,y) \tag{3.2}$$

$$|G(s,t,u)| \leq a(s,t)u(s,t) \tag{3.3}$$

$$|F(s,t,\sigma,\eta,u(\sigma,\eta))| \leq b(s,t,\sigma,\eta)u(\sigma,\eta) \tag{3.4}$$

for $x \geq 0,\ y \geq 0$ where a, b, p, α, β are as in Theorem 2.1 using the equations (3.2)-(3.4) in the equation (3.1) then applying Theorem 2.1, we obtain the bound on the solution $u(x,y)$ to the equation (3.1).

In addition to this, in order to provide explicit bounds on the solution to partial differential equations of the form $u_{xy} = G(x, y, \alpha(s), \beta(y), u)$, one can use the integral inequalities which are obtain in Theorems 2.1 and 2.2.

4. Concluding Remarks

In concluding this paper, we have established some new generalized Pachpatte-type inequalities. As it can be seen from the present application, the results established are useful in researching both qualitative and quantitative properties for solutions to certain fractional order differential equations.

REFERENCES

1. D. Bainov and P. Simeonov: *Integral Inequalities and Applications*. Kluwer Academic Publishers, Dordrecht, 1992.
2. R. Bellman: *The stability of solutions of linear differential equations*. Duke Math. J., 10 (1943), 643–647.
3. I. Bihari: *generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*. Acta. Math. Acad. Sci. Hungar., 7 (1965), 81–94.
4. T. H. Gronwall: *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*. Ann. Math., 20 (1919), 292–296.
5. O. Lipovan: *A retarded Gronwall-like inequality and its applications*. J. Math. Anal. Appl., 252 (2000), 389–401.
6. B. G. Pachpatte: *Inequalities for Differential and Integral Equations*. Academic Press, New York, 1998.
7. B. G. Pachpatte: *On some new inequalities related to certain inequalities in the theory of differential equations*. J. Math. Anal. Appl. 189 (1995) 128–144.
8. B. G. Pachpatte: *On some retarded integral inequalities and applications*. J. Inequal. Pure Appl. Math., 3(2), Article 18, 2002.

9. F. Usta and M. Z. Sarikaya: *Explicit bounds on certain integral inequalities via conformable fractional calculus*. Cogent Mathematics, 4: 1277505, 2017.
10. F. Usta and M. Z. Sarikaya: *Some improvements of conformable fractional integral inequalities*. International Journal of Analysis and Applications, 14(2), (2017), 162–166.
11. F. Usta and M. Z. Sarikaya: *On generalization of pachpatte type inequalities for conformable fractional integral*. TWMS J. App. Eng. Math. 8(1), 2018, pp. 106-113.
12. F. Usta and M. Z. Sarikaya: *on generalization conformable fractional integral inequalities*. FILOMAT. (In press).

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ON THE EIGENVALUES OF N-CAYLEY GRAPHS: A SURVEY

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Abstract. A graph Γ is called an *n-Cayley graph over a group G* if $\mathrm{Aut}(\Gamma)$ contains a semi-regular subgroup isomorphic to G with n orbits. In this paper, we review some recent results and future directions around the problem of computing the eigenvalues on n-Cayley graphs.
Keywords: n-Cayley graph; eigenvalues; semi-regular subgroup.

1. Introduction

The spectrum of a graph is one of the most important algebraic invariants as it is known that numerous proofs in graph theory depend on the spectrum of graphs. In particular, eigenvalues of Cayley graphs have attracted increasing attention due to their prominent roles in algebraic graph theory and applications in many areas such as expanders, chemical graph theory, quantum computing, etc [21]. This paper is a survey of the literature on the eigenvalues of graphs having a semi-regular of subgroup of their automorphism groups.

A digraph Γ is a pair (V, E) of vertices V and edges E where $E \subseteq V \times V$. A graph is a digraph with no edges of the form (α, α) and with the property that $(\alpha, \beta) \in E$ implies $(\beta, \alpha) \in E$. The set of all permutations of V which preserve the adjacency structure of Γ is called the automorphism group of Γ; it is denoted by $\mathrm{Aut}(\Gamma)$. In this paper all digraphs have no loops. For the most part our notation and terminology are standard and mainly taken from [9] (for graph theory) and [16] (for representation theory of finite groups). For the graph-theoretic and group-theoretic terminology not defined here we refer the reader to [9, 16].

Let Γ be a (di)graph with n vertices. The adjacency matrix of A of Γ is an $n \times n$ matrix with ij-entry equal to 1 if ith and jth vertices are adjacent and 0 otherwise. The *spectrum* of a graph is the multi-set of eigenvalues of its adjacency matrix. It

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is known that numerous proofs in graph theory depend on the spectrum of graphs and the spectrum of a graph is one of the most important algebraic invariants [9].

Let G be a group and S be a subset of G not containing the identity element of G. The *Cayley (di)graph of G with respect to S* is a graph with a vertex set G where (g, h) is an arc whenever $hg^{-1} \in S$. A large number of results on spectra of Cayley graphs have been produced over the last more than four decades. For a survey of the literature on eigenvalues of Cayley graphs and their applications see [21].

By a theorem of Sabidussi [26], a (di)graph Γ is a Cayley graph over a group G if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. As a generalization, a (di)graph Γ is called an *n-Cayley (di)graph over a group G* if there exists a semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G with n orbits (of equal size). Since every regular subgroup is a transitive semi-regular subgroup, every Cayley (di)graphs is a 1-Cayley (di)graph. Also a Cayley graph over a finite group G having a subgroup H of index n is an n-Cayley graph over H [1, Lemma 8]. n-Cayley graphs over cyclic groups are called n-circulant. In particular 2-Cayley and 3-Cayley graphs over cyclic groups are called bicirculant and tricirculant graphs [24], respectively. Unlike Cayley graphs, in general n-Cayley graphs are not vertex-transitive for $n \geq 2$. Furthermore, there are vertex-transitive n-Cayley graphs which are not Cayley graphs such as generalized Petersen graphs. Undirected and loop-free 2-Cayley graphs are called, by some authors, semi-Cayley graphs [25, 3] and also bi-Cayley graphs [17]. In this paper, we follow [25] to use the term semi-Cayley.

n-Cayley graphs, in particular when $n = 2$ or $n = 3$, have played an important role in many classical fields of graph theory, such as strongly regular graphs [19, 22, 23, 24, 25], automorphisms [2, 15, 28], isomorphisms [3, 5], symmetry properties of graphs [10, 11, 20] and the spectrum of graphs [1, 4, 8, 12, 13]. In this paper, we review recent results and future directions of some problems related to the spectrum of n-Cayley graphs.

2. Presentation of n-Cayley graphs

Recall that a (di)graph Γ is called an n-Cayley graph over a group G if $\mathrm{Aut}(\Gamma)$ contains a semi-regular subgroup isomorphic to G with n orbits (of equal size). It is proved that every n-Cayley graph over a group G can be presented by suitable n^2 subsets of G:

Lemma 2.1. *([1, Lemma 2]) A digraph Γ is n-Cayley digraph over G if and only if there exist subsets T_{ij} of G, where $1 \leq i, j \leq n$, such that Γ is isomorphic to a digraph X with vertex set $G \times \{1, 2, \ldots, n\}$ and edge set*

$$E(X) = \bigcup_{1 \leq i,j \leq n} \big\{((g, i), (tg, j)) \mid g \in G \ \text{ and } \ t \in T_{ij}\big\}.$$

By Lemma 2.1, an n-Cayley (di)graph is characterized by a group G and n^2 subsets T_{ij} of G (some subsets may be empty). So we denote an n-Cayley (di)graph with respect to n^2 subsets T_{ij} by $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq n)$. It is easy to see that $\mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq n)$ is undirected if and only if $T_{ij}^{-1} = T_{ji}$ for all $1 \leq i,j \leq n$. Also it is loop-free if $1 \notin T_{ii}$ for all $1 \leq i \leq n$. Let Γ be a 2-Cayley graph which is undirected and loop-free. Then there exist three subsets $R = T_{11}$, $L = T_{22}$, $S = T_{12}$ and $T_{21} = S^{-1}$ of G such that $R = R^{-1}$, $L = L^{-1}$ and $1 \notin R \cup L$ and $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq 2)$. We denote this graph with $\mathrm{SC}(G; R, L, S)$ and call it semi-Cayley graph. In the case $R = L = \emptyset$, we denote it by $\mathrm{BCay}(G; S)$ and call it bi-Cayley graph.

There are a lot of examples of n-Cayley graphs, $n \geq 2$. Here we provide some.

Example 2.1. Let P be the Petersen graph. Then $P = \mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_5$, $R = \{a, a^4\}$, $L = \{a^2, a^3\}$ and $S = \{1\}$.

Example 2.2. ([1, Lemma 8]) Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley (di)graph. Suppose that there exists a subgroup H of G with index n. If $\{t_1, \ldots, t_n\}$ is a left transversal to H in G, then $\Gamma \cong \mathrm{Cay}(H, T_{ij} \mid 1 \leq i,j \leq n)$, where $T_{ij} = \{h \in H \mid t_j^{-1} h t_i \in S\} = H \cap t_j S t_i^{-1}$.

Example 2.3. The I-graph $I(n,j,k)$ is a cubic graph of order $2n$ with vertex set $\{u_i, v_i \mid 0 \leq i \leq n-1\}$ and edge set $\{u_i u_{i+j}, u_i v_i, v_i v_{i+k}\}$. Graphs $I(n,1,k)$ are called generalized Petersen graphs. It is easy to see that $I(n,j,k) = \mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a^j, a^{-j}\}$, $L = \{a^k, a^{-k}\}$ and $S = \{1\}$.

Example 2.4. Let $RW(n,j,k)$ be a Rose Window graph, for the definition of graph see [18]. $RW(n,j,k)$ is a 4-valent bicirculant graph isomorphic to $\mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $L = \{a^j, a^{-j}\}$ and $S = \{1, a^k\}$.

Example 2.5. For given natural numbers $n \geq 3$ and $1 \leq r,j,k \leq n-1$, with $j \neq n/2$ and $r \neq k$, the *Tabačjn graph* $T(n,r,k,j)$ is a pentavalent graph with vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set

$$\{x_i x_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{y_i y_{i+j} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_{i+r} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_{i+k} \mid i \in \mathbb{Z}_n\}.$$

It is easy to see that $T(n,r,k,j) = \Gamma \cong \mathrm{SC}(G; R, L, S)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $L = \{a^j, a^{-j}\}$ and $S = \{1, a^r, a^k\}$.

Example 2.6. Let $K_{r,r,\ldots,r}$ be the n-partite complete graph. Then $K_{r,r,\ldots,r} = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq n)$, where G is a finite group of order r, and for all $1 \leq i,j \leq n$ where $j \neq i$, $T_{ii} = \emptyset$ and $T_{ij} = G$.

3. Eigenvalues of n-Cayley (di)graphs

In 2007, the spectrum of bi-Cayley graphs over finite abelian groups computed in [29]:

Theorem 3.1. *Let $\Gamma = \mathrm{BCay}(G, S)$ be a bi-Cayley graph over finite abelian group $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$ with respect to S. Then eigenvalues of Γ are*

$$\pm \Big| \sum_{(i_1,\ldots,i_t) \in S} \omega_{n_1}^{r_1 i_1} \ldots \omega_{n_t}^{r_t i_t} \Big|, \quad r_j = 0, \ldots, n_j - 1, \quad j = 1, \ldots, t.$$

In 2010, Gao and Luo improved Theorem 3.1. They studied the spectrum of semi-Cayley graphs over finite abelian groups. Using matrix theory, they derived a formula of the spectrum of semi-Cayley graphs over finite abelian groups:

Theorem 3.2. *([12, Theorem 3.2]) Let* $\Gamma = \mathrm{SC}(G; R, L, S)$ *be a semi-Cayley graph over a finite abelian group* $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$. *Then* Γ *has eigenvalues*

$$\frac{\lambda^R_{r_1\ldots r_t} + \lambda^R_{r_1\ldots r_t} \pm \sqrt{(\lambda^R_{r_1\ldots r_t} - \lambda^L_{r_1\ldots r_t})^2 + 4|\lambda^S_{r_1\ldots r_t}|^2}}{2},$$

$r_j = 0, \ldots, n_j - 1$, $j = 1, \ldots, t$, *where* $\lambda^X_{r_1\ldots r_t} = \sum_{(i_1,\ldots,i_t)\in X} \omega_{n_1}^{r_1 i_1} \ldots \omega_{n_t}^{r_t i_t}$ *and* ω_n *is the primitive* n*th root of unity.*

Also the spectrum of a bi-Cayley graph of an arbitrary group with respects to a normal subset computed in [6, Theorem 2.1], a generalization of Theorem 3.1. In 2013, Theorem 3.2 extended to n-Cayley graphs, $n \geq 2$, over arbitrary groups by Arezoomand and Taeri in [1] using representation theory of finite groups. Let us recall some definitions of the latter paper. Let G be a finite group and $\mathbb{C}[G]$ be the complex vector space of dimension $|G|$ with basis $\{e_g \mid g \in G\}$. We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Thus a function $\varphi : G \to \mathbb{C}$ corresponds to the vector $\varphi = \sum_{g\in G} \varphi(g)e_g$ and vice versa. In particular, the vector e_g, where $g \in G$, of the standard basis corresponds to the function e_g, where

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

The (left) regular representation ρ_{reg} of G on $\mathbb{C}[G]$ is defined by its action on the basis $\{e_h \mid h \in G\}$; that is for all $g, h \in G$, $\rho_{\mathrm{reg}}(g)e_h = e_{gh}$. Let $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$ be the set of all irreducible inequivalent $\mathbb{C}$-representations of G and d_k be the degree of ρ_k, $k = 1, \ldots, m$. Let e^i_g be the $1 \times n|G|$ vector with n blocks, where ith block is e_g, as defined, and other blocks are $0_{1\times|G|}$ vectors. Let V be the vector space with basis $\{e^i_g \mid g \in G, 1 \leq i \leq n\}$. Clearly $V \cong \underbrace{\mathbb{C}[G] \oplus \mathbb{C}[G] \oplus \cdots \oplus \mathbb{C}[G]}_{n-\text{times}}$, as $\mathbb{C}[G] = \langle e_g \mid g \in G\rangle$. So $\dim_{\mathbb{C}} V = n \dim_{\mathbb{C}} \mathbb{C}[G] = n|G|$. Let $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ and $A = [a_{(g,i)(h,j)}]_{g,h\in G, 1\leq i,j\leq n}$ be the adjacency matrix of Γ. Viewing A as the linear map

$$\begin{aligned} A : V &\to V \\ e^i_g &\mapsto \sum_{j=1}^{n} \sum_{h\in G} a_{(h,j)(g,i)} e^j_h, \qquad 1 \leq i \leq n,\ g \in G, \end{aligned}$$

it is proved that:

Theorem 3.3. *([1, Theorem 6]) Let* $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ *be an* n*-Cayley digraph over a finite group* G *and* $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$. *For each* $l \in \{1, \ldots, m\}$, *we define $nd_l \times nd_l$ block matrix $A_l := \left[A_{ij}^{(l)}\right]$, where $A_{ij}^{(l)} = \sum_{t\in T_{ji}} \rho^{(l)}(t)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, respectively. Then $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.*

Example 3.1. ([7, Corollary 2.3]) The eigenvalues of $I(n,j,k)$ are

$$\cos(2lj\pi/n) + \cos(2lk\pi/n) \pm \sqrt{(\cos(2lj\pi/n) - \cos(2lk\pi/n))^2 + 1}, \quad l = 0, \dots, n-1.$$

Example 3.2. ([7, Corollary 2.4]) The eigenvalues of $RW(n,j,k)$ are

$$\cos(2l\pi/n)+\cos(2lj\pi/n)\pm\sqrt{(\cos(2l\pi/n) - \cos(2lj\pi/n))^2 + 2 + 2\cos(2lk\pi/n)}, \quad l = 0, \dots, n-1.$$

Example 3.3. ([7, Corollary 2.5]) The eigenvalues of $T(n,r,k,j)$ are

$$\cos(2l\pi/n) + \cos(2lj\pi/n) \pm \sqrt{(\cos(2l\pi/n) - \cos(2lj\pi/n))^2 + \alpha_l}, \quad l = 0, \dots, n-1,$$

where $\alpha_l = 3 + 2\Big(\cos(2\pi lr/n) + \cos(2\pi lk/n) + \cos(2\pi l(r-k)/n)\Big)$.

Since any Cayley graph over a group G is a 1-Cayley graph over G, as a direct consequence of Theorem 3.3, we can reprove the following result which is proved in [27]:

Corollary 3.1. *Let $\Gamma = \mathrm{Cay}(G,S)$ be a Cayley digraph over a finite group G with irreducible matrix representations $\varrho^{(1)}, \dots, \varrho^{(m)}$. Let d_l be the degree of $\varrho^{(l)}$. For each $l \in \{1, \dots, m\}$, define a $d_l \times d_l$ block matrix $A_l := \left[A_S^{(l)}\right]$, where $A_S^{(l)} = \sum_{s\in S} \varrho^{(l)}(s)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, the adjacency matrix of Γ, respectively. Then $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.*

Let G be a finite abelian group. Then by [16, Theorem 9.8], putting $n = 2$ in Theorem 3.3, Theorem 3.2 directly follows. Also Theorems 4.6 and 4.3 of [12] improved in [1]:

Corollary 3.2. *([1, Corollary 9]) Let $\Gamma = \mathrm{Cay}(G,S)$ be a Cayley digraph, $H = \langle a\rangle$ a cyclic subgroup of G of order n and of index 2 with left transversal $\{t_1, t_2\}$. Then the characteristic polynomial of the adjacency matrix of Γ is $\chi_A(\lambda) = \Pi_{k=0}^{n-1}(\lambda - \lambda_k^+)(\lambda - \lambda_k^-)$, where*

$$\lambda_k^+ = \frac{\lambda_k^{11} + \lambda_k^{22} + \sqrt{(\lambda_k^{11} - \lambda_k^{22})^2 + 4\lambda_k^{12}\lambda_k^{21}}}{2},$$
$$\lambda_k^- = \frac{\lambda_k^{11} + \lambda_k^{22} - \sqrt{(\lambda_k^{11} - \lambda_k^{22})^2 + 4\lambda_k^{12}\lambda_k^{21}}}{2},$$

$\lambda_k^{ij} = \sum_{t\in T_{ji}} \omega_n^{kt}$ and $T_{ij} = \{t \mid 0 \le t \le n-1, a^t \in t_j S t_i^{-1}\}$.

Let Γ be a k-regular graph with n vertices and adjacency matrix A and A^c be the adjacency matrix of the complement of Γ. Then $(\lambda+k+1)\chi_{A^c}(\lambda) = (-1)^n(\lambda-n+k+1)\chi_A(-\lambda-1)$, see [9, p. 20]. Despite of Cayley graphs, n-Cayley graphs $n\geq 2$, are not necessarily regular, but we have a similar relation between the characteristic polynomials of any n-Cayley graph and its complement which is given in the next theorem:

Theorem 3.4. *([1, Theorem 10]) Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1\leq i,j\leq n)$ be an n-Cayley graph over a finite group G, $n\geq 1$. Let Γ^c be the complement of Γ with adjacency matrix A^c. Then the characteristic polynomials of Γ and Γ^c are related with the following equation:*

$$\chi_{B_1}(\lambda)\chi_A(-\lambda-1) = (-1)^{|G|-1}\chi_{A_1}(-\lambda-1)\chi_{A^c}(\lambda),$$

where $B_1 = |G|J - I_n - A_1$, J is the all ones matrix of degree n, and $A_1 = [|T_{ji}|]_{1\leq i,j\leq n}$.

An eigenvector of the adjacency matrix of a graph Γ is said to be *main eigenvector* if it is not orthogonal to the all ones vector $\mathbf{j}$. An eigenvalue of a graph Γ is said to be a *main eigenvalue* if it has a main eigenvector. By Perron-Frobenius Theorem, the largest eigenvalue of a graph is a main eigenvalue. It is also well known that a graph is regular if and only if it has exactly one main eigenvalue. So for every Cayley graph $\Gamma = \mathrm{Cay}(G,S)$, $|S|$ is the only main eigenvalue of Γ. Since n-Cayley graphs, for $n\geq 2$ are not necessarily regular, determining the main eigenvalues of these graphs seems to be important. This problems reduced to determining main eigenvalues of the matrix A_1:

Theorem 3.5. *([1, Corollary 12]) Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1\leq i,j\leq n)$ be an n-Cayley graph over a finite group G and $n\geq 2$. The main eigenvalues of Γ is equal to main eigenvalues of the matrix $A_1 = [|T_{ji}|]_{1\leq i,j\leq n}$.*

4. Integrality of n-Cayley graphs

A graph Γ is called *integral* if all eigenvalues of the adjacency matrix of Γ are integers. The concept of integral graphs was first defined by Harary and Schwenk [14]. During the last forty years many mathematicians have tried to construct and classify some special classes of integral graphs including Cayley graphs(for a survey see [21]). It seems that integral graphs are very rare and determining all the integral n-Cayley graphs, even for $n=2$, is difficult. It is easy to construct integral semi-Cayley graphs over finite abelian groups, as the following corollary shows:

Corollary 4.1. *([12, Corollary 3.5]) Let $\Gamma = \mathrm{SC}(G; R, R, S)$ be a semi-Cayley graph over a finite abelian group G. If $\mathrm{Cay}(G,R)$ and $\mathrm{Cay}(G,S)$ are integral then Γ is integral.*

The study of integrality of bi-Cayley graphs started by Arezoomand and Taeri in 2015:

Theorem 4.1. *([4, Corollary 3.10]) Every bi-Cayley graph of a finite group G is integral if and only if G is isomorphic to one of the groups $\mathbb{Z}_2^k$, $k \geq 1$, $\mathbb{Z}_3$ or S_3.*

Also finite groups admitting a connected cubic integral bi-Cayley graph determined in the following theorem:

Theorem 4.2. *([8, Theorem A]) A finite group G admits a connected cubic integral bi-Cayley graph if and only if G is isomorphic to one of the groups*

$$\mathbb{Z}_2^2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathbb{Z}_2 \times \mathbb{Z}_6,\ S_3,\ A_4,\ Dic_{12}.$$

The following questions naturally arise:

Problem 4.1. Determine finite groups admitting a connected k-regular, $k \geq 4$, bi-Cayley graphs.

Problem 4.2. Let $\Gamma = \mathrm{BCay}(G, S)$. In what conditions on S, is Γ an integral?

Problem 4.3. Determine finite groups in which all bi-Cayley graphs over them of the valency at most $k \geq 2$ are integral.

Problem 4.4. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a semi-Cayley graph over a group G. In what conditions on R, L and S is Γ an integral?

5. Laplacian and signless Laplacian eigenvalues of n-Cayley graphs

Let Γ be a graph with vertex set $\{v_1, \ldots, v_n\}$. Recall that the adjacency matrix of Γ is an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = 1$ whenever v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The degree matrix of Γ is a diagonal $n \times n$ matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, where d_i is the number of vertices adjacent to v_i. The matrices $L = D - A$ and $Q = D + A$ are called *Laplacian* and *signless Laplacian* matrices of Γ, respectively. The characteristic polynomial of an $n \times n$ matrix X is $\det(\lambda I_n - X)$, where I_n is the $n \times n$ identity matrix and the roots of this polynomial are called eigenvalues of X. In this paper, the Laplacian eigenvalues and signless Laplacian eigenvalues of a graph Γ are eigenvalues of Laplacian and signless Laplacian matrices of Γ, respectively.

In 2015, the Laplacian and signless Laplacian spectrum of semi-Cayley graphs over abelian groups computed:

Theorem 5.1. *([13, Theorem 1]) Let $\Gamma = \mathrm{SC}(G; R, L, S)$ be a semi-Cayley graph over a finite abelian group $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_t}$. Then Γ has Laplacian eigenvalues (resp. signless Laplacian eigenvalues)*

$$\frac{\mu^R_{r_1\ldots r_t} + \mu^L_{r_1\ldots r_t} + 2|S| \pm \sqrt{(\mu^R_{r_1\ldots r_t} - \mu^L_{r_1\ldots r_t})^2 + 4|\lambda^S_{r_1\ldots r_t}|^2}}{2}$$

$r_j = 0, \ldots, n_j - 1$, $j = 1, \ldots, t$, where $\lambda^S_{r_1 \ldots r_t}$ are eigenvalues of $\mathrm{Cay}(G,S)$*, and $\mu^R_{r_1\ldots r_t}, \mu^L_{r_1\ldots r_t}$ are the Laplacian (resp. signless Laplacian) eigenvalues of* $\mathrm{Cay}(G,R)$ *and* $\mathrm{Cay}(G,L)$*, respectively.*

The n-*sunlet* graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n. It is easy to see that $\Gamma = \mathrm{SC}(G,R,S,T)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $S = \emptyset$ and $T = \{1\}$.

Example 5.1. Let Γ be an n-sunlet graph. Then

(1) Lpalcian eigenvalues of Γ are

$$2 - \cos\frac{2\pi l}{n} \pm \sqrt{(1 - \cos\frac{2\pi l}{n})^2 + 1},$$

where $l = 0, \ldots, n-1$.

(2) signless Laplacian eigenvalues of Γ are

$$2 + \cos\frac{2\pi l}{n} \pm \sqrt{(1 + \cos\frac{2\pi l}{n})^2 + 1},$$

where $l = 0, \ldots, n-1$.

As a corollary, one can construct semi-Cayley graphs with an integral Laplacian and signless Laplacian spectrum:

Corollary 5.1. *([13, Corollary 4.6]) Let $\Gamma = \mathrm{SC}(G; R, R, S)$ be a semi-Cayley graph over a finite abelian group G. If* $\mathrm{Cay}(G,R)$ *and* $\mathrm{Cay}(G,S)$ *are integral graphs then Γ is a Laplacian and signless Laplacian integral graph.*

We end the paper with some open problems:

Problem 5.1. Determine the Laplacian and signless Laplacian eigenvalues of semi-Cayley graphs over non-abelian groups. Also do this for n-Cayley graphs when $n \geq 3$.

Problem 5.2. In what conditions on R, L and S, $\mathrm{SC}(G; R, L, S)$ is Laplacian (and signless Laplacian) an integral?

REFERENCES

1. M. Arezoomand and B. Taeri: *On the characteristic polynomial of n-Cayley digraphs*, Electron. J. Combin., **20(3)** (2013), # P57.
2. M. Arezoomand and B. Taeri: *Normality of* 2*-Cayley digraphs*, Discrete Math., **338** (2015), 41-47.
3. M. Arezoomand and B. Taeri: *Isomorphisms of finite semi-Cayley graphs*, Acta Math. Sin. Engl. Ser., **31(4)** (2015) 715-730.

4. M. Arezoomand and B. Taeri: *A classification of finite groups with integral bi-Cayley graphs*, Trans. Combin., **4(4)** (2015) 55-61.
5. M. Arezoomand and B. Taeri: *Finite BCI-groups are solvable*, Int. J. Group Theory, **5(2)** (2016) 1-6.
6. M. Arezoomand and B. Taeri: *Which elements of a finite group are non-vanishing?*, Bull. Iranian Math. Soc., **42(5)** (2016) 1097-1106.
7. M. Arezoomand and A. R. Ashrafi: *On the spectrum of* 2*-type bicirculants*, Proceeding of 10th Graph Theory and Algebraic Combinatorics Conference, Yazd, Iran, 2017, pp. 10-13.
8. M. Arezoomand and B. Taeri: *Finite groups admitting a connected cubic integral bi-Cayley graph*, Alg. Struc. Appl., **5(2)** (2018) 35-43.
9. N. Biggs: *Algebraic Graph Theory*, Cambridge University Press, 1974.
10. S.F. Du and D. Marušič:*An infinite family of biprimitive semisymmetric graphs*, J. Graph Theory **32** (1999) 217-228.
11. S.F. Du and D. Marušič: *Biprimitive graphs of smallest order*, J. Algebraic Combin. **9** (1999) 151-156.
12. X. Gao and Y. Luo: *The spectrum of semi-Cayley graphs over abelian groups*, Linear Algebra Appl., **432** (2010) 2974-2983.
13. X. Gao, H. Lü and Y. Hao: *The Laplacian and signless Laplacian spectrum of semi-Cayley graphs over abelian groups*, J. Appl. Math. Comput., **51 (1)** (2016) 383-395.
14. F. Harary and A. J. Schwenk: *Which graphs have integral spectra?* in "Graphs and Combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973)", Lecture Notes in Mathematics 406. Springer, Berlin, 1974, pp. 45-51.
15. A. Hujdurović, K. Kutnar and D. Marušič: *On normality of n-Cayley graphs*, Appl. Math. Comput., **332** (2018) 469-476.
16. G. James and M. Liebeck: *Representations and characters of groups*, Cambridge University Press, Second Edition, 2001.
17. I. Kovács, I. Malnič, D. Marušič, and Š. Miklavič: *One-Matching bi-Cayley graphs over abelian groups*, European J. Combin. **30** (2009) 602-616.
18. I. Kovacs, B. Kuzman, A. Malnič and S. Wilson: *Characterization of edge-transitive 4-valent bicirculants*, J. Graph Theory, DOI: 10.1002/jgt.20594
19. K. Kutnar, D. Marušič, Š. Miklavič and P. Šparl: *Strongly regular tri-Cayley graphs*, European J. Combin. **30(4)** (2009) 822-832.
20. Z.P. Lu, C.Q. Wang and M.Y. Xu: *Semisymmetric cubic graphs constructed from bi-Cayley graphs of* A_n, Ars Combin. **80** (2006) 177-187.
21. X. Liu and S. Zhou: *Eigenvalues of Cayley graphs*: arXiv: 1809.09829 [math.CO].
22. K.H. Leung and S.L. Ma: *Partial difference triples*, J. Algebraic Combin. **2** (1993) 397-409.
23. L. Martinez: *Strongly regular m-Cayley circulant graphs and digraphs*: Ars Math. Contemp. **8** (2015) 195-213.
24. D. Marušič and D. Marui: *Strongly regular bicirculants and tricirculants*, Ars Combin. **25** (1988) 11-15.
25. M. J. D. Resmi and D. Jungnickel: *Strongly regular Semi-Cayley graphs*, J.Algebraic Combin. **1** (1992), 171-195.

26. G. Sabidussi: *Vertex-transitive graphs*, Monatsh. Math., 68 (1964) 426–438.
27. P. Diaconis and M. Shahshahani: *Generating a random permutation with random transpositions*, Z. Wahrsch. Verw. Gebiete, **57** (1981) 159-179.
28. J. X Zhou: *Every finite group has a normal bi-Cayley graph*, Ars Math. Contemp., **14** (2018) 177-186.
29. H. Zou and J.X. Meng: *Some algebraic properties of Bi-Cayley graphs*, Acta Math. Sin. (Chin. Ser.) **50 (5)** (2007) 1075-1080.

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ON THE CHARACTERIZABILITY OF SOME FAMILIES OF FINITE GROUP OF LIE TYPE BY ORDERS AND VANISHING ELEMENT ORDERS

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Abstract. In this paper, we show that the following simple groups are uniquely determined by their orders and vanishing element orders: $A_{p-1}(2)$, where $p \neq 3$, ${}^2D_{p+1}(2)$, where $p \geq 5$, $p \neq 2^m - 1$, $A_p(2)$, $C_p(2)$, $D_p(2)$, $D_{p+1}(2)$ which for all of them p is an odd prime and $2^p - 1$ is a Mersenne prime. Also, ${}^2D_n(2)$ where $2^{n-1} + 1$ is a Fermat prime and $n > 3$, ${}^2D_n(2)$ and $C_n(2)$ where $2^n + 1$ is a Fermat prime. Then we give an almost general result to recognize the non-solvability of finite group H by an analogy between orders and vanishing element orders of H and a finite simple group of Lie type.
Keywords: simple groups; Mersenne prime; Fermat prime; Lie group.

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Keywords: Finite simple groups, vanishing element orders, prime graph.

1. Introduction

Throughout this paper G and H are two finite groups. Let X be a finite set of positive integers. The prime graph $\Pi(X)$ is a graph whose vertices are the prime divisors of elements of X, and two distinct vertices p and q are adjacent if there exists an element of X divisible by pq. For a finite group G, we denote by $\omega(G)$, the set of element orders of G. The prime graph $\Pi(\omega(G))$ is denoted by $GK(G)$ and is called the Gruenberg-Kegel graph of G. Here, $s(G)$ denotes the number of connected components of $GK(G)$. For the group G, we denote by $\rho(G)$ some independence sets in $GK(G)$ with maximal number of vertices and put $t(G) = |\rho(G)|$, independence number of $GK(G)$. $g \in G$ is called a vanishing element of G if $\chi(g) = 0$ for some $\chi \in \mathrm{Irr}(G)$. Let us denote by $\mathrm{Van}(G)$ and $\mathrm{vo}(G)$ the set of all vanishing elements

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and the set of vanishing element orders of G, respectively. Also the prime graph $\Pi(\mathrm{vo}(G))$ is denoted by $\Gamma(G)$ and is called the vanishing prime graph of G.

If n is a natural number and π is a set of primes, then we denote the set of all prime divisors of n by $\pi(n)$, and the maximal divisor t of n such that $\pi(t) \subseteq \pi$ by n_π. If $\pi(G)$ is the set of prime divisors of $|G|$, then $\pi_i(G) = \pi(m_i)$ for some positive integers m_i, $1 \leq i \leq t$, such that $|G| = m_1 m_2 \cdots m_t$ and $t = s(G)$. Also for any group with even order, $2 \in \pi_1(G)$. We set $OC(G) = \{m_1, \cdots, m_t\}$ and call the set of order components of G. A finite simple group G is said characterizable by its order components, if $G \cong H$ for each finite group H such that $OC(G) = OC(H)$. Some authors have proved that some non-abelian simple groups are recognizable by their order components. We refer the reader to [23] to find a list of papers with the OC-characterizability criterion for some finite simple groups.

It was shown in [38] that if G is a finite group such that $\mathrm{vo}(G) = \mathrm{vo}(A_5)$ then $G \cong A_5$. According to this result, M. Foroudi, A. Iranmanesh and F. Mavadatpour in [12] stated the conjecture as follows:

Conjecture 1.1. *Let G and H be two groups with the same order. If G is a non-abelian group and* $\mathrm{vo}(G) = \mathrm{vo}(H)$, *then $G \cong H$.*

First, this conjecture was proved for $L_2(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $L_3(4)$, A_7, $Sz(8)$ and $Sz(32)$ in [12]. Then they proved this conjecture in [13] for finite simple K_n-groups with $n \in \{3, 4\}$, sporadics, alternatatings and $L_2(p)$ where p is an odd prime. In [24] it has been verified that the groups $Sz(q)$ satisfy this conjecture, where $q = 2^{2n+1}$ and either $q-1$, $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ is a prime, and $F_4(q)$, where $q = 2^n$ and either $q^4 + 1$ or $q^4 - q^2 + 1$ is a prime. In this paper, we show that the above conjecture is valid for some families of simple groups of Lie type. Then we prove another result about non-solvability of some finite group using vanishing element orders. In fact, we prove the following theorems:

Theorem 1.1. *Let G and H be two groups with the same order and G be a simple group of Lie type $A_{p-1}(2)$ where $p \neq 3$, ${}^2D_{p+1}(2)$, where $p \geq 5$, $p \neq 2^m - 1$, $A_p(2)$, $C_p(2)$, $D_p(2)$, $D_{p+1}(2)$, which for all of them p is an odd prime and $2^p - 1$ is a Mersenne prime, ${}^2D_n(2)$ where $2^{n-1} + 1$ is a Fermat prime, ${}^2D_n(2)$ and $C_n(2)$ where for the last two groups $2^n + 1$ is a Fermat prime. If* $\mathrm{vo}(G) = \mathrm{vo}(H)$, *then $G \cong H$.*

Theorem 1.2. *Let G and H be two groups with the same order. Suppose G is a simple group of Lie type with $s(G) \geq 2$ except $A_2(q)$, where $(q-1)_3 \neq 3$, q is a Mersenne prime, ${}^2A_2(q)$, where $(q+1)_3 \neq 3$, q is a Fermat prime, $C_2(q)$ where $q > 2$. If* $\mathrm{vo}(G) = \mathrm{vo}(H)$, *then H is non-solvable.*

2. Preliminaries

In this section, we state some results which will be of use to the proof of the main theorems.

Definition 2.1. A group G is said to be a 2-Frobenius group if there exist two normal subgroups F and L of G with the following properties: L is a Frobenius group with kernel F, and G/F is a Frobenius group with kernel L/F.

Recall that a Frobenius group with kernel N and complement H is a semidirect product $G = H \ltimes N$ such that N is a normal subgroup in G, and $C_N(h) = 1$ for every non-identity element h of H. As is well-known, N is the Fitting subgroup of G.

Definition 2.2. G is a nearly 2-Frobenius group if there exists two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G, furthermore G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group.

Lemma 2.1. *[11]*

(a) Let G be a solvable Frobenius group with kernel F and complement H. The graph $GK(G)$ has two connected components, whose vertex sets are $\pi_1 = \pi(F)$ and $\pi_2 = \pi(H)$, and which are both complete graphs.

(b) Let G be a finite solvable group. Then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius group or a nearly 2-Frobenius group.

(c) Let G be a nearly 2-Frobenius group. If $\Gamma(G)$ is disconnected, then each connected component is a complete graph.

(d) Let G be a solvable Frobenius group with kernel F and complement H. If $F \cap Van(G) \neq \emptyset$, then $\Gamma(G) = GK(G)$, and the otherwise $\Gamma(G)$ coincides with the connected component of $GK(G)$ with vertex set $\pi(H)$.

Lemma 2.2. *[10] If G is a finite non-abelian simple group, then $GK(G) = \Gamma(G)$, unless $G \cong A_7$.*

Theorem 2.1. *[13] Let G be a finite group and let M be a simple K_3-group or a K_4-group. If $|G| = |M|$ and $\mathrm{vo}(G) = \mathrm{vo}(M)$, then $G \cong M$.*

Recall that a finite simple group G is called a K_n-group if its order has exactly n distinct prime divisors, where n is a natural number.

Theorem 2.2. *[36] Let G be a finite simple group. Then all the connected components of $GK(G)$ are cliques if and only if G is one of the following: A_5, A_6, A_7, A_9, A_{12}, A_{13}, M_{11}, M_{22}, J_1, J_2, J_3, HS, $A_1(q)$, with $q > 2$, $\mathrm{Sz}(q)$ with $q = 2^{2m+1}$, $C_2(q)$, $G_2(3^k)$, $A_2(q)$ where q is a Mersenne prime, ${}^2A_2(q)$ where q is a Fermat prime, $A_2(4)$, ${}^2A_2(9)$, ${}^2A_3(3)$, ${}^2A_5(2)$, $C_3(2)$,$D_4(2)$, ${}^3D_4(2)$.*

3. Main results

To prove Theorem 1.2, we adopt Table I by [14] of components of prime graphs of simple groups of Lie type over a field of even characteristics which in this table p is an odd prime. In Table 1, m_2 coincides with the factor for primes in the second connected component. Table 2 shows OC-characterizable groups of Lie type with their prime graph having two connected components. We also use Tables 3 and 4 for the proof of Theorem 1.3. These tables were adopted from [37] and they show the independence number of prime graphs of finite simple groups of Lie type and. In Tables 3 and 4, n and k are natural numbers. $[x]$ denotes the integral part of x. We assume that G is a finite non-abelian simple group of Lie type over a field of characteristic p and order q. We define the primitive prime divisor of $q^m - 1$ by r_m. If p is odd then we say that 2 is a primitive prime divisor of $q-1$ if $q \equiv 1 \pmod 4$ and that 2 is a primitive prime divisor of q^2-1 if $q \equiv -1 \pmod 4$.

The following lemma is a conclusion from some noteworthy properties of a simple group G with $s(G) = 2$ and the conditions of Conjecture 1.1.

Lemma 3.1. *Let G and H be two groups with the same order. Suppose that G is a non-abelian simple group with $s(G) = 2$ and $GK(H)$ is disconnected. If $\mathrm{vo}(G) = \mathrm{vo}(H)$, then $OC(G) = OC(H)$.*

Proof. The assumption $\mathrm{vo}(G) = \mathrm{vo}(H)$ and Lemma 2.2 imply $GK(G) = \Gamma(G) = \Gamma(H)$. So the set of vertices of the vanishing prime graph of H is equal to $\pi(H)$. Since $\Gamma(H) \leq GK(H)$, the prime graph of H has two connected components. Let $OC(G) = \{m_1, m_2\}$ and $OC(H) = \{n_1, n_2\}$. It is sufficient to prove $m_1 = n_1$. Assume $m_1 \neq n_1$. Therefore, $\pi_1(G) \neq \pi_1(H)$. Without loss of generality, we suppose there is a prime p in $\pi_1(G)$ such that $p \notin \pi_1(H)$. So $p \in \pi_2(H)$. The connectedness of components implies $\pi_1(G) \subseteq \pi_2(H)$, that is, $2 \in \pi_2(H)$, a contradiction. If p is an isolated vertex, then $p = 2$ because the order of G is even. Therefore $2 \in \pi_2(H)$ which is impossible. $\square$

Before bringing forward the proof of Theorem 1.2, we recall that an irreducible character χ of group G is called p-defect zero if $p \nmid |G|/\chi(1)$ where p is a prime.

3.1. Proof of Theorem 1.2

First we show that $GK(H)$ is disconnected. According to Table 1, $s(G) = 2$ and the second order component of G are prime. From $\mathrm{vo}(G) = \mathrm{vo}(H)$ and Lemma 2.2, we deduce $GK(G) = \Gamma(G) = \Gamma(H)$. The last equalities imply that $\Gamma(H)$ has a connected component with a single vertex p. On the other hand, H has a vanishing p-element. Since characters of degree not divisible by some prime number p never vanish on p-elements, it is then clear that H has a p-defect zero character, namely χ. We claim that $GK(H)$ is disconnected. We assume the assertion is false. Then there exists a non-vanishing element x of order pq in H where $q \in \pi_1(G)$. Since any p-defect zero characters vanish on elements of order divisible by p, we observe

$\chi(x) = 0$. It means that $\Gamma(H)$ is connected. This is a contradiction and hence $GK(G)$ is disconnected. Then by Lemma 3.1, $OC(G) = OC(H)$. According to Table 2, G is an OC-characterizable group with $s(G) = 2$ and therefore $G \cong H$. $\Box$

Lemma 3.1 will be of use to show the validity of Conjecture 1.1 for more OC-characterizable simple groups of Lie type that we state as a general result.

Theorem 3.1. *Let G and H be two groups with the same order. Suppose G is an OC-characterizable simple group of Lie type with $s(G) = 2$ and $GK(H)$ is disconnected. If $\mathrm{vo}(G) = \mathrm{vo}(H)$, then $G \cong H$.*

In particular, the Conjecture 1.1 is valid for any group of Table 2 with a prime m_2.

Table 1: The prime graph components of the simple groups of Lie type over the field of even characteristic.

Type	Factors for primes in π_1	m_2
$A_{p-1}(q), (p,q) \neq (3,2), (3,4)$	$q, q^i - 1, 1 \leq i \leq p-1$	$\frac{q^p-1}{(q-1)(q-1,p)}$
$A_p(q), q-1 \mid p+1$	$q, q^{p+1} - 1, q^i - 1, 1 \leq i \leq p-1$	$\frac{q^p-1}{q-1}$
$C_k(q), k = 2^n$	$q, q^k - 1, q^{2i} - 1, 1 \leq i \leq k-1$	$q^k + 1$
$C_p(q), (q-1,p) = 1$	$q, q^p + 1, q^{2i} - 1, 1 \leq i \leq p-1$	$\frac{q^p-1}{q-1}$
$D_p(q), (q-1,p) = 1$	$q, q^{2i} - 1, 1 \leq i \leq p-1$	$\frac{q^p-1}{q-1}$
$D_{p+1}(2)$	$2, 2^{2i} - 1, 1 \leq i \leq p-1,$ $2^p + 1, 2^{p+1} - 1$	$2^p - 1$
${}^2A_3(2^2)$	2,3	5
${}^2A_{p-1}(q^2)$	$q, q^i - (-1)^i, 1 \leq i \leq p-1$	$\frac{q^p+1}{(q+1)(q+1,p)}$
${}^2A_p(q^2), q+1 \mid p+1$	$q, q^{p+1} - 1, q^i - (-1)^i,$ $1 \leq i \leq p-1$	$\frac{q^p+1}{q+1}$
${}^2D_k(q), k = 2^n, n \geq 2$	$q, q^{2i} - 1, 1 \leq i \leq k-1$	$q^k + 1$
${}^2D_{k+1}(2), k = 2^n, n \geq 2$	$2, 2^{2i} - 1, 1 \leq i \leq k-1,$ $2^k - 1, 2^{k+1} + 1$	$2^k + 1$
$G_2(q), q \equiv 1 \pmod 3$	$q, q^2 - 1, q^3 - 1$	$q^2 - q + 1$
$G_2(q), q \equiv -1 \pmod 3$	$q, q^2 - 1, q^3 + 1$	$q^2 + q + 1$
${}^3D_4(q^3)$	$q, q^6 - 1$	$q^4 - q^2 + 1$
${}^2F_4(2)'$	2,3,5	13
$E_6(q), q \equiv 1 \pmod 3$	$q, q^5 - 1, q^8 - 1, q^{12} - 1$	$\frac{q^6+q^3+1}{3}$
$E_6(q), q \equiv 1 \pmod 3$	$q, q^5 - 1, q^8 - 1, q^{12} - 1$	$q^6 + q^3 + 1$
${}^2E_6(q^2), q \equiv -1 \pmod 3$	$q, q^5 + 1, q^8 - 1, q^{12} - 1$	$\frac{q^6-q^3+1}{3}$
${}^2E_6(q^2), q \equiv 1 \pmod 3$	$q, q^5 + 1, q^8 - 1, q^{12} - 1$	$q^6 - q^3 + 1$

Table 2: OC-characterizable simple groups of Lie type with their prime graphs having two connected components.

G	Restriction on G	Reference
$A_{p-1}(q)$	$p \neq 3, q \neq 2, 4$	[16, 15, 26]
$A_p(q)$	$(q-1)\|(p+1)$	[8, 34]
${}^2A_p(q)$	$(q+1)\|(p+1)$, $p \neq 3, 5$, $q \neq 2, 3$	[29]
${}^2A_{p-1}(q)$		[18, 19, 20, 30]
$B_n(q)$	$n = 2^m \geq 2,$	[22, 39, 25, 28]
$B_p(3)$		[7]
$C_n(q)$	$n = 2^m \geq 2$	[22, 39, 25, 28]
$C_p(q)$	$q = 2, 3$	[7] and Table 4 of [23]
$D_p(5)$	$p \geq 5, q = 2, 3, 5$	Table 4 of [23]
$D_{p+1}(q)$	$q = 2, 3$	[6]
${}^2D_n(q)$	$n = 2^m$	[27, 31]
${}^2D_n(2)$	$n = 2^m + 1$, $m \geq 2$	[9]
${}^2D_p(3)$	$5 \leq p \neq 2^m + 1$	[35, 5]
${}^2D_n(3)$	$n = 2^m + 1 \neq p$, $m \geq 2$	[4]
${}^3D_4(q)$		[3]
$E_6(q)$		[33]
${}^2E_6(q)$	$q > 2$	[32]
$F_4(q)$	q odd	[21, 17]
$G_2(q)$	$2 < q \equiv \varepsilon \pmod 3, \varepsilon = \pm 1$	[1, 2]

3.2. Proof of Theorem 1.3

From $\mathrm{vo}(G) = \mathrm{vo}(H)$ and Lemma 2.2, we deduce that $GK(G) = \Gamma(G) = \Gamma(H)$. Since for a simple group G with $s(G) > 2$, non-solvability of H is concluded from Lemma 2.1 (b), it is sufficient that we investigate the case $s(G) = 2$. Let H be a solvable group and G be a simple group of Lie type with $s(G) = 2$. Since $\Gamma(H)$ has two connected components, Lemma 2.1 (b) implies that H is either a Frobenius group or a nearly 2-Frobenius group. For both cases, using Lemma 2.1 (a), (b) and (c), $GK(G)$ has two clique connected components. So G is the above mentioned simple group of Theorem 2.2. According to Tables 3 and 4 for simple groups of Lie type with $s(G) = 2$ except $A_2(q)$, where $(q-1)_3 \neq 3$ and q is a Mersenne prime, ${}^2A_2(q)$, where $(q+1)_3 \neq 3$ and q is a Fermat prime, $C_2(q)$ where $q > 2$, ${}^2A_2(9)$, $C_3(2)$, $D_4(2)$ and ${}^3D_4(2)$, we have $t(G) \geq 3$. Thus, if $p, q, r \in \rho(G)$, then at least two of them lie in a component such that they are non-adjacent, which is impossible. Now, if G is one of the following groups: ${}^2A_2(9)$, $C_3(2)$, $D_4(2)$ or ${}^3D_4(2)$, then G is a K_4-group and Theorem 2.1 implies $H \cong G$. Hence the desired conclusion holds. $\square$

Table 3: Independence number and set of finite simple classical groups of Lie type.

G	Condition	$t(G)$	$\rho(G)$
$A_{n-1}(q)$	$n=2, q>3$	3	$\{p, r_1, r_2\}$
	$n=3, (q-1)_3=3$ and $q+1\neq 2^k$	4	$\{p, 3, r_2, r_3\}$
	$n=3, (q-1)_3\neq 3$ and $q+1\neq 2^k$	3	$\{p, r_2, r_3\}$
	$n=3, (q-1)_3=3$ and $q+1=2^k$	3	$\{p, 3, r_3\}$
	$n=3, (q-1)_3\neq 3$ and $q+1=2^k$	2	$\{p, r_3\}$
	$n=4$	3	$\{p, r_{n-1}, r_n\}$
	$n=5, 6, q=2$	3	$\{5, 7, 31\}$
	$7\leq n\leq 11, q=2$	$\left[\frac{n-1}{2}\right]$	$\{r_i \mid i\neq 6, \left[\frac{n}{2}\right]<i\leq n\}$
	$n\geq 5$ and $q>2$ or $n\geq 12$ and $q=2$	$\left[\frac{n+1}{2}\right]$	$\{r_i \mid \left[\frac{n}{2}\right]<i\leq n\}$
${}^2A_{n-1}(q)$	$n=3, q\neq 2, (q+1)_3=3$, and $q-1\neq 2^k$	4	$\{p, 3, r_1, r_6\}$
	$n=3, (q+1)_3\neq 3$ and $q-1\neq 2^k$	3	$\{p, r_1, r_6\}$
	$n=3, (q+1)_3=3$ and $q-1=2^k$	3	$\{p, 3, r_6\}$
	$n=3, (q+1)_3\neq 3$ and $q-1=2^k$	2	$\{p, r_6\}$
	$n=4, q=2$	2	$\{2, 5\}$
	$n=4, q>2$	3	$\{p, r_4, r_6\}$
	$n=5, q=2$	3	$\{2, 5, 11\}$
	$n\geq 5$ and $(n, q)\neq(5, 2)$	$\left[\frac{n+1}{2}\right]$	$\{r_{i/2} \mid \left[\frac{n}{2}\right]<i\leq n,$ $i\equiv 2(\mathrm{mod}\ 4)\}\cup$ $\{r_{2i} \mid \left[\frac{n}{2}\right]<i\leq n,$ $i\equiv 1(\mathrm{mod}\ 2)\}\cup$ $\{r_i \mid \left[\frac{n}{2}\right]<i\leq n,$ $i\equiv 0(\mathrm{mod}\ 4)\}$
$B_n(q)$ or $C_n(q)$	$n=2, q>2$	2	$\{p, r_4\}$
	$n=3, q=2$	2	$\{5, 7\}$
	$n=4, q=2$	3	$\{5, 7, 17\}$
	$n=5, q=2$	4	$\{7, 11, 17, 31\}$
	$n=6, q=2$	5	$\{7, 11, 13, 17, 31\}$
	$n>2, (n, q)\neq(3, 2), (4, 2), (5, 2), (6, 2)$	$\left[\frac{3n+5}{4}\right]$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right]\leq i\leq n\}\cup$ $\{r_i \mid \left[\frac{n}{2}\right]<i\leq n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$
$D_n(q)$	$n=4$ and $q=2$	2	$\{5, 7\}$
	$n=5$ and $q=2$	4	$\{5, 7, 17, 31\}$
	$n=6$ and $q=2$	4	$\{7, 11, 17, 31\}$
	$n\geq 4,$ $(n, q)\neq(4, 2), (5, 2), (6, 2)$	$\left[\frac{3n+1}{4}\right]$	$\{r_{2i} \mid \left[\frac{n+1}{2}\right]\leq i<n\}\cup$ $\{r_i \mid \left[\frac{n}{2}\right]<i\leq n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$ $\{r_{2i} \mid \left[\frac{n+1}{2}\right]\leq i<n\}\cup$ $\{r_i \mid \left[\frac{n}{2}\right]\leq i\leq n\}$
${}^2D_n(q)$	$n=4$ and $q=2$	3	$\{5, 7, 17\}$
	$n=5$ and $q=2$	3	$\{7, 11, 17\}$
	$n=6$ and $q=2$	5	$\{7, 11, 13, 17, 31\}$
	$n=7$ and $q=2$	5	$\{11, 13, 17, 31, 43\}$
	$n\geq 4, n\not\equiv 1(\mathrm{mod}\ 4),$ $(n, q)\neq(4, 2), (6, 2), (7, 2),$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right]\leq i\leq n\}\cup$ $\{r_i \mid \left[\frac{n}{2}\right]<i<n\}$ $i\equiv 1(\mathrm{mod}\ 2)\}$
	$n>4, n\equiv 1(\mathrm{mod}\ 4), (n, q)\neq(5, 2)$	$\left[\frac{3n+4}{4}\right]$	$\{r_{2i} \mid \left[\frac{n}{2}\right]<i\leq n\}\cup$ $\{r_i \mid \left[\frac{n}{2}\right]<i\leq n,$ $i\equiv 1(\mathrm{mod}\ 2)\}$

Table 4: Independence number and set of finite simple exceptional Lie-type groups.

G	Conditions	$t(G)$	$\rho(G)$
$G_2(q)$	$q > 2$	3	$\{p, r_3, r_6\}$
$F_4(q)$	$q = 2$	4	$\{5, 7, 13, 17\}$
	$q > 2$	5	$\{r_3, r_4, r_6, r_8, r_{12}\}$
$E_6(q)$	$q = 2$	5	$\{5, 13, 17, 19, 31\}$
	$q > 2$	6	$\{r_4, r_5, r_6, r_8, r_9, r_{12}\}$
${}^2E_6(q)$		5	$\{r_4, r_8, r_{10}, r_{12}, r_{18}\}$
$E_7(q)$		7	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}\}$
$E_8(q)$		11	$\{r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$
${}^3D_4(q)$	$q = 2$	2	$\{2, 13\}$
	$q > 2$	3	$\{r_3, r_6, r_{12}\}$
${}^2B_2(2^{2n+1})$	$n \geq 1$	4	$\{2, s_1, s_2, s_3\}$ where $s_1 \mid 2^{2n+1} - 1$ $s_2 \mid 2^{2n+1} - 2^{n+1} + 1$ $s_3 \mid 2^{2n+1} + 2^{n+1} + 1$
${}^2G_2(3^{2n+1})$	$n \geq 1$	5	$\{3, s_1, s_2, s_3, s_4\}$, where $s_1 \neq 2, s_1 \mid 3^{2n+1} - 1$ $s_2 \neq 2, s_2 \mid 3^{2n+1} + 1$ $s_3 \mid 3^{2n+1} - 3^{n+1} + 1$ $s_4 \mid 3^{2n+1} + 3^{n+1} + 1$
${}^2F_4(2^{2n+1})$	$n \geq 2$	5	$\{s_1, s_2, s_3, s_4, s_5\}$, where $s_1 \neq 3, s_1 \mid 2^{2n+1} + 1$ $s_2 \mid 2^{4n+2} + 1$ $s_3 \neq 3, s_3 \mid 2^{4n+2} - 2^{2n+1} + 1$ $s_4 \mid 2^{4n+2} - 2^{3n+2} + 2^{2n+1} - 2^{n+1} + 1$ $s_5 \mid 2^{4n+2} + 2^{3n+2} + 2^{2n+1} + 2^{n+1} + 1$
${}^2F_4(2)'$	none	3	$\{3, 5, 13\}$
${}^2F_4(8)$	none	4	$\{7, 19, 37, 109\}$

REFERENCES

1. G. Y. Chen: *A new characterization of $G_2(q)$*, J. Southwest China Normal Univ., **21** (1996), 47-51.
2. G. Y. Chen: *Characterization of Lie type group $G_2(q)$ by its order components,* J. Southwest China Normal Univ., (Natural Science), **26** (5) (2001), 503-509.
3. G. Y. Chen: *Characterization of ${}^3D_4(q)$*, Southeast Asian Bulletin of Math., **25**, (2001), 389-401.
4. G. Y. Chen and H. Shi: *${}^2D_n(3)$ ($9 \leq n = 2^m + 1$ not a prime) can be characterized by its order components,* J. Appl. Math. Comput., **19**, (2005), 355-362.
5. M. R. Darafsheh: *Characterizability of group ${}^2D_p(3)$ by its order components, where $p \geq 5$ is a prime number not of the form $2^m + 1$,* Acta Mathematica Sinica, English Series, **24** (7), (2008), 1117-1126.

6. M. R. Darafsheh: *Characterization of groups $D_{p+1}(2)$ and $D_{p+1}(3)$ using order components,* J. Korean Math. Soc., **47** (2), (2010), 311-329.
7. M. R. Darafsheh: *On non-isomorphic groups with the same set of order components,* J. Korean Math. Soc., **45** (1), (2008), 137-150.
8. M. R. Darafsheh and A. Mahmiani: *A quantitative characterization of the linear group $L_{p+1}(2)$ where p is a prime number,* Kumamoto J. Math., **20**, (2007), 33-50.
9. M. R. Darafsheh and A. Mahmiani: *A characterization of the group ${}^2D_n(2)$ where $n = 2^m + 1 \geq 5$,* J. Appl. Math. Comput., **31**, (2009), 447-457.
10. S. Dolfi, E. Pacifici, L. Sanus and P. Spiga: *On the vanishing prime graph of finite groups,* J. London Math. Soc.,**82** (1) (2010), 167-183.
11. S. Dolfi, E.Pacifici, L. Sanus and P. Spiga: *On the vanishing prime graph of solvable groups,* J. Group Theory, **13** (2010), 189-206.
12. M. Foroudi Ghasemabadi, A. Iranmanesh and F. Mavadatpour: *A new characterization of some finite simple groups,* Siberian Math. J., **56**, (2015), 78-82.
13. M. Foroudi Ghasemabadi, A. Iranmanesh and M. Ahanjideh: *A new characterization of some families of finite simple groups,* Rend. Sem. Mat. Univ. Padova, European Mathematical Society, **137** (2017), 57-74.
14. N. Iiyori and H. Yamaki: *Prime graph components of the simple groups of Lie type over the field of even characteristic,* Proc. Japan Acad., **67**, Ser. A, 1991.
15. A. Iranmanesh, S. H. Alavi and B. Khosravi: *A characterization of $PSL(3, q)$ where q is an odd prime power,* J. Pure Appl. Algebra, **170**, (2002), 243-254.
16. A. Iranmanesh, S. H. Alavi and B. Khosravi: *A characterization of $PSL(3, q)$ where $q = 2^n$*, Acta Math. App. Sinica (English Ser.), **18**, (2002), 463-472.
17. A. Iranmanesh and B. Khosravi: *A characterization of $F_4(q)$ where q is an odd prime power,* Lecture Notes London Math. Soc., **304**, (2003), 277-283.
18. A. Iranmanesh and B. Khosravi: *A characterization of $PSU(5, q)$,* Int. Math. J., **3**, (2003), 129-141.
19. A. Iranmanesh and B. Khosravi: *A characterization of $PSU(7, q)$,* Int. J. Appl. Math., **15**, (2004), 329-340.
20. A. Iranmanesh and B. Khosravi: *A characterization of $PSU(11, q)$,* Canad. Math. Bull., **47**, (2004), 530-539.
21. A. Iranmanesh and B. Khosravi: *A characterization of $F_4(q)$ where $q = 2^n$, $(n > 1)$,* Far East J. Math. Sci., (2000), 853-859.
22. A. Iranmanesh and B. Khosravi: *A characterization of $C_2(q)$ where $q > 5$,* Comment. Math. Univ. Carolin., **43**, (2002), 9-21.
23. M. Khademi, M. R. Darafsheh: *Characterization of the Group $D_p(5)$ by order components, where $p \geq 5$ is a prime number,* Southeast Asian Bulletin of Mathematics, **37** (2013), 867-885.
24. M. Khatami and A. Babai: *Recognition of some families of finite simple groups by order and set of orders of vanishing elements,* Czechoslovak Math. J., **68**, (2018), 121-130.

25. B. Khosravi, B. Khosravi and B. Khosravi: *The number of isomorphism classes of finite groups with the set of order components of $C_4(q)$,* Applicable Algebra in Engineering Comunication and Computing, **15**, (2005), 349-359.
26. A. Khosravi and B. Khosravi: *A new characterization of $PSL(p, q)$,* Comm. Algebra, **32** (6), (2004), 2325-2339.
27. A. Khosravi and B. Khosravi: *A characterization of ${}^2D_n(q)$, where $n = 2^m$,* Int. J. Math. Game Theory Algebra, **13**, (2003), 253-266.
28. A. Khosravi and B. Khosravi: *r-Recognizability of $B_n(q)$ and $C_n(q)$ where $n = 2^m$,* J. Pure App. Algebra, (2005), 149-165.
29. A. Khosravi and B. Khosravi: *A characterizability of $PSU(p + 1, q)$ by its order components,* Rock Mountain J. Math., **36** (5), (2006), 1555-1575.
30. B. Khosravi, B. Khosravi and B. Khosravi: *A new characterization of $PSU(p, q)$,* Acta Math. Hungar., **107** (3), (2005), 235-252.
31. B. Khosravi: *A characterization of ${}^2D_4(q)$,* Pure Math. Appl., 12, (2001), 415-424.
32. B. Khosravi and B. Khosravi: *A characterization of ${}^2E_6(q)$,* Kumamoto J. Math., **16**, (2003), 1-11.
33. B. Khosravi and B. Khosravi: *A characterization of $E_6(q)$,* Algebras Groups Geom., **19**, (2002), 225-243.
34. B. Khosravi and B. Khosravi: *Characterizability of $PSL(p + 1, q)$ by its order components,* Houston Journal of Mathematics, **32** (3), (2006), 683-700.
35. B. Khosravi and A. Iranmanesh: *A characterization of ${}^2D_p(3)$, where $p = 2^n + 1(n \geq 2)$,* Hadronic J. Suppl., **18** (4), (2003), 465-477.
36. M. Silvia Lucido and Ali Reza Moghaddamfar: *Groups with complete prime graph connected components,* J. Group Theory , (2004), 373-384.
37. A. V. Vasiliev and E. P. Vdovin: *An adjacency criterion for the prime graph of a finite simple group,* Algebra and Logic, **44** (6), (2005).
38. J. Zhang, Z. Li and C. Shao: *Finite groups whose irreducible characters vanish only on elements of prime power order,* International Electronic J. Algebra, **9** (2011), 114-123.
39. L. Zhang and W. Shi: *New characterization of $S_4(q)$ by its noncommuting graph,* Siberian Mathematical Journal, **50** (3), (2009), 533-540.

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MAHALANOBIS DISTANCE AND ITS APPLICATION FOR DETECTING MULTIVARIATE OUTLIERS

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Abstract. While methods of detecting outliers is frequently implemented by statisticians when analyzing univariate data, identifying outliers in multivariate data pose challenges that univariate data do not. In this paper, after short reviewing some tools for univariate outliers detection, the Mahalanobis distance, as a famous multivariate statistical distances, and its ability to detect multivariate outliers are discussed. As an application the univariate and multivariate outliers of a real data set has been detected using R software environment for statistical computing.
Keywords: Mahalanobis distance, multivariate normal distribution, multivariate outliers, outlier detection.

1. Introduction

The role of statistical distances when dealing with problems such as hypothesis testing, goodness of fit tests, classification techniques, clustering analysis, outlier detection and density estimation methods is of great importance. Using distance measures (or similarities) enable us to quantify the closeness between two statistical objects. These objects can be two random variables, two probability distributions, moment generating functions, an individual sample point and a probability distributions or two individual samples. There exists many statistical distance measures [38], among them the Mahalanobis distance has the advantage of its ability to detect multivariate outliers.

Outliers are those data that deviate from global behavior of majority of data. Outliers or outlying observation have different definition in texts, for example "an outlier deviates so much from other observations as to arouse suspicions that it was generated by a different mechanism", see [12]. Outliers have major influence on the statistical inference. They increase error variance and reduce the power of statistical

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tests and cause bias estimates that may be of substantive interest [22]. Therefore, the process of outlier detection is an interesting and important aspect in the data analysis, see [3] and [5]. Depending on application synonyms are often used for the outlier detection process, among them, one can mention anomaly detection, deviation detection, exception mining, fault detection in safety critical systems, fraud detection for credit cards, intrusion detection in cyber security (unauthorized access in computer networks), misuse detection, noise detection and novelty detection see [1], [9], [23] and [32].

All proximity-based techniques for identification of outliers such as k-Nearest Neighbor (k-NN) algorithm calculate the nearest neighbors of a record using a suitable distance calculation metric such as Euclidean distance, Mahalanobis distance or some other measure of dissimilarity. For large data set using the Mahalanobis distance is computationally more expensive than Euclidean distance as it require to pass through all variables in data set to calculate the underlying inter-correlation structure. An iterative Mahalanobis distance type of method for the detection of outliers in multivariate data has been proposed by [10]. Due to the masking effect, in which one outlier masks a second outlier, if the second outlier can be considered as an outlier only by itself, but not in the presence of the first outlier, detecting multiple outliers is more completed than the case where data consist of a single outlier, since masking effects might decrease the Mahalanobis distance of an outlier. This might happen because a small cluster of outliers attracts mean and inflate variance towards its direction [4]. In such cases using robust estimates of sample mean and variance, can often improve the performance of the detection procedure, see [24] and [30].

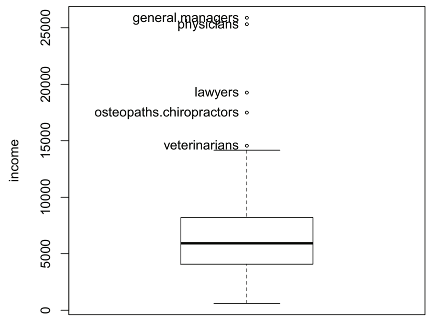
In this paper, the problems of the univariate and multivariate outlier detection has been addressed. For univariate outlier detection, the result of applying the classical visual method based on box-plot and Ven der Loo method [36] on a real data set has been compared. For multivariate outlier detection, usual and robust Mahalanobis distances has been used to find the outliers of a real data set using R software environment for statistical computing.

2. Univariate Outlier Detection

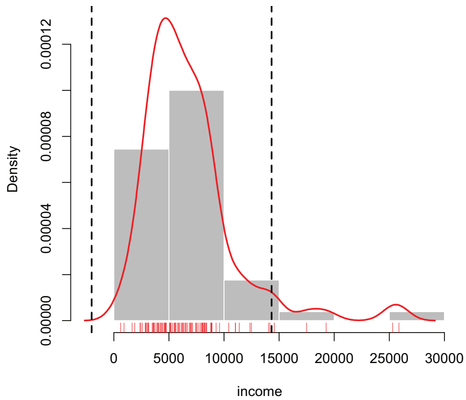
A simple visualization tools, such as scatter plot, box-and-whisker (boxplot), stem-and-leaf plot, QQ-plot, etc., can be used to discover the outliers. The box plots, first introduced by [35], are a standardized way of displaying the distribution of data based on a five number summary ("minimum", first quartile (Q_1), median, third quartile (Q_3), and "maximum"). In general, the box of a box plot shows the median and quartiles. The box plot rule declares observations as outliers if they lie outside the interval

$$Q_1 - k(Q_1 - Q_3), Q_3 + k(Q_3 - Q_1),$$

the common choices for k is 1.5 for flagging (dubbed) outliers and 3.0 for flagging outliers, see Figure 2.1, in which the whiskers are shown for $k = 1.5$. This rule differs from standard outlier identification rules, since it is not sample-size dependent, the probability of declaring outliers when none exist changes with the number of observations [29]. Moreover, for data coming from a random normal sample of size 75, the probability of labeling at least one outlier is 0.5 [13]. Many other statistical tests have been used to detect outliers, as discussed in [3].



(a) The Boxplot of jobs income and 5 jobs above the upper whisker that flagged out as outliers

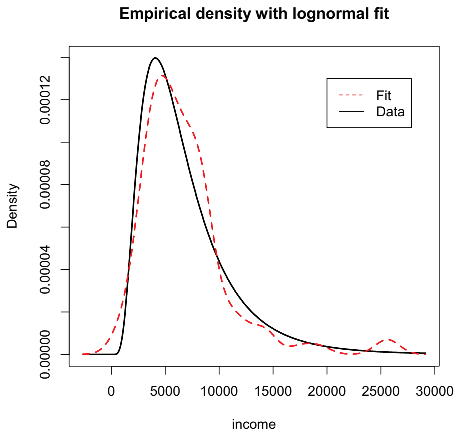


(b) The empirical density and the corresponding box plot whiskers. On the x axis, five outliers are shown that exceed the upper whisker threshold

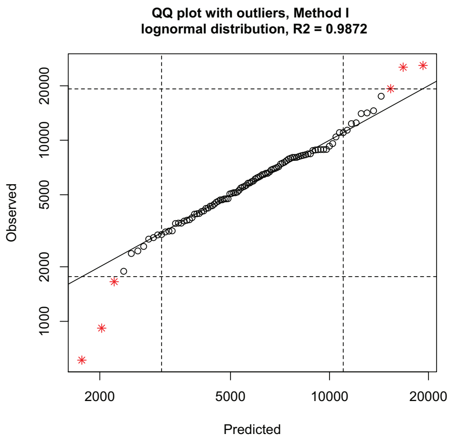
FIG. 2.1: Univariate outlier detection using the boxplot for job incomes in Prestige data set

Van der Loo [36] developed two methods to detect outliers in economic data, when an approximate data distribution is known. In the following, his first method is applied in order to detect the outliers of "income" variable (average income of incumbents, dollars, in 1971) from Prestige of Canadian Occupations data set in "car" package in R software environment [8]. The Prestige data set has 102 rows and 6 columns. This data consists of some measurment related to different occupations.

According to the Kolmogrov-Smirnov goodness-of fit test, the log-normal distribution fits well to income data (p-value=0.47), see the left panel of Figure 2.2. Therefore, the Var der Loo method was applied to detect possible outliers in this data using the plotting facilities developed in the “extremevalues” package in *R* software environment [37].



(a) The empirical distribution of job incomes and the fitted log-normal distribution



(b) Outlier detected using the first Van der Loo method, which are indicated by $*$ sign

FIG. 2.2: Model based univariate outlier detection for job incomes in Prestige data set

As it is shown in the right panel of Figure 2.2, this method detects six outliers which are located on two sides of data. The Outliers on the left down part of the Figure are case numbers 53, 63, 68, and the rest are 2, 17, 24, whereas the upper outliers on the boxplot are case numbers 2, 17, 24, 25, 26.

The study of outliers in structured situations like regression models are based on the residuals and has been studied by several authors, see [29] and references therein. Five widely used test statistics for detecting outliers have been compared using Monte Carlo method by Balasooriya and Tse [2].

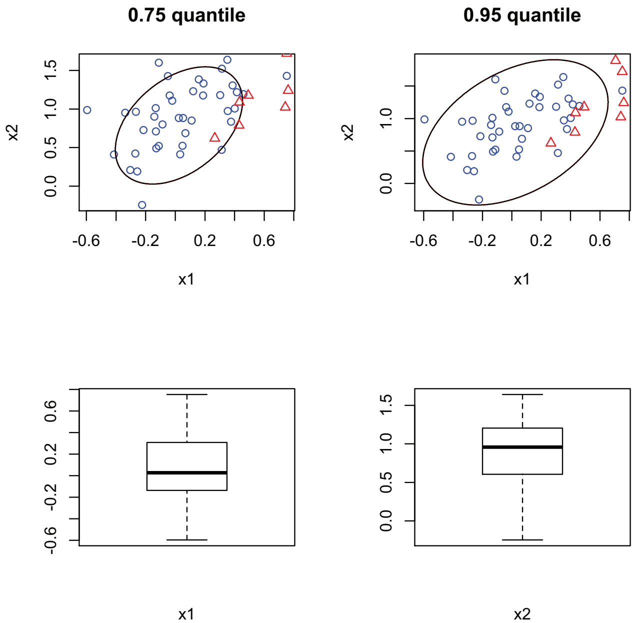


FIG. 2.3: (above) Scatter plot of two simulated samples from bivariate normal distributions, which show clear outliers out of 0.75 and 0.95 cutoffs corresponding to quantiles of the $\chi^2(2)$ distribution, (below) the box plot of margins of the same data with no points lying outside the whiskers

3. Multivariate Outliers Detection

Nowadays more and more observed data are multi-dimensional, which increase the chance of occurring unusual observations. The problem is that a few outliers is always enough to distort the results of data (by altering the mean performance, by increasing variability, etc.). Therefore, detecting outliers is a growing concern in many scientific areas, including but not limited to Psychology [18], Financial market [6] and Chemometrics [26].

In the field of multivariate statistics, the Mahalanobis distance has a major application for the detection of outliers [20]. The Mahalanobis distance is defined in the next section. Mahalanobis distance measures the number of standard deviations that an observation is from the mean of a distribution. Since outliers do not behave as normal as usuall observations at least in one dimension, this measure can be used to detect outliers. See [14] for a comparison of Mahalanobis distances with other proximity-based outlier detection techniques.

3.1. The Mahalanobis distance

From geometric point of view, the Euclidean distance between two points is the shortest possible distance between them. One problem with the Euclidean distance measure is that it does not take the correlation between highly correlated variables into account. In this situation, Euclidean distance assigns equal weight to such variables, and since these variables measure essentially the same characteristic, therefore this single characteristic gets additional weight. In effect, correlated variables gets excess weight by Euclidean distance, see [16] and [21].

An alternative approach is to scale the contribution of individual variables to the distance value according to the variability of each variable. This approach is considered by the Mahalanobis distance, which has been developed as a statistical measure by PC Mahalanobis, an Indian statistician [19]. The Mahalanobis distance finds wide applications in the field of multivariate statistics. It differs from Euclidean distance in this way that it takes into account the correlations between variables. It is a scale invariant metric and provides a measure of distance between a point $\mathbf{x} \in R^p$ generated from a given $p-$variate (probability) distribution $f_{\mathbf{X}}(.)$ and the mean $\mu = E(\mathbf{X})$ of the distribution. Assume $f_{\mathbf{X}}(.)$ has finite second order moments and denote $\Sigma = E(\mathbf{X} - \mu)$ be the covariance matrix. Then the Mahalanobis distance is defined by

$$D(\mathbf{X}, \mu) = \sqrt{(\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu)}. \tag{3.1}$$

If the covariance matrix is the identity matrix, the Mahalanobis distance reduces to the Euclidean distance. For the comparison of these two distances see Figure 3.1, in which the Euclidean and Mahalanobis distances of points located on the circles and ellipse are 1 and 2 unit far away from the center of data. The computation has been done on a data set, that are find under `geog.uoregon.edu/GeogR/data/csv/midwtf2.csv`. The observed difference stems from this fact that the Mahalanobis distance also accounts for the covariance (or correlation) structure of data.

Apart from usual application of the Mahalanobis distance in multivariate analysis techniques such as classification and clustering, discriminant analysis and pattern analysis, principal component analysis, there exists modern applications, among them financial applications [33], image processing [39], Neurocomputing [11] and Physics [31] might be mentioned.

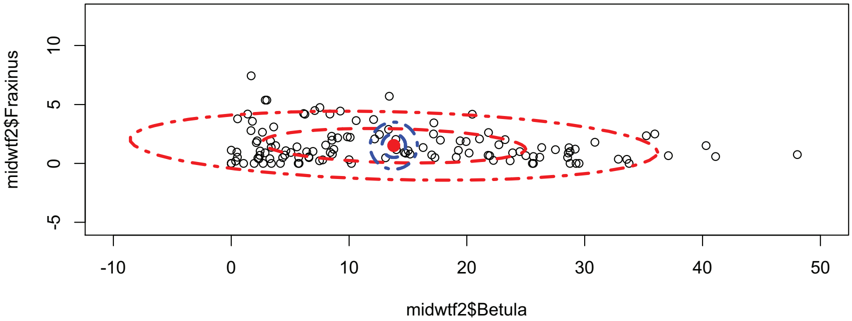


FIG. 3.1: Schematic comparison of the Mahalanobis (ellipse) and Euclidean (circle) distances calculated for a data set. The two lines, circles and ellipses, correspond to the Euclidean and the Mahalanobis distances, of one and two units apart from the center of data

3.2. Multivariate normal distribution

Recall the multivariate normal density function below, in which the parameters μ and Σ, are the mean and the covariance matrix of the distribution, respectively.

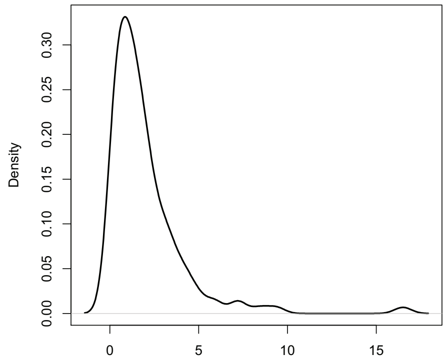
$$\phi(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^{p/2} |\Sigma|^{-1/2} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\},$$

note that this density function, $\phi(x)$, only depends on x through the following squared Mahalanobis distance in the exponent:

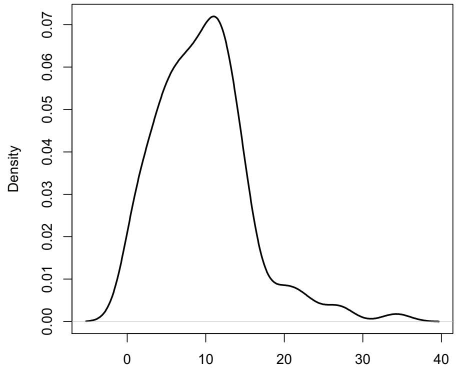
$$(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu).$$

There are some important facts about this exponent:

- All values of $\mathbf{x}$ such that $(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) = c$ for any specified constant value c have the same value of the density f(x) and thus have equal likelihood. The paths of these $\mathbf{x}$ values yielding a constant height for the density are ellipsoids. That is, the multivariate normal density is constant on surfaces where the square of the distance $(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)$ is constant. These paths are called contours, which can be constructed from the eigenvalues and eigenvectors of the covariance matrix, meaning that the direction of the ellipse axes are in the direction of the eigenvalues and the length of the ellipse axes are proportional to the constant times the eigenvectors [15].
- As the value of $(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)$ increases, the value of the density function decreases.
- The value of $(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)$ increases as the distance between $\mathbf{x}$ and μ increases.



(a) The Mahalanobis distance.



(b) The Eucleadn distance

FIG. 3.2: Emperical densities

- The Mahalanobis distance $d^2 = (\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)$ has a chi-square distribution with p degrees of freedom, see Figure 3.1.

Suppose that X, is a p-dimensional vector having multivariate normal distribution, $X \sim N_p(\mu, \Sigma)$, the Mahananobis squared distance $D^2(\mathbf{X}, \mu)$ is then distributed as a χ^2 random variable with p degrees of freedom. The classical approach of outlier detection uses the estimates of the Mahalanobis distance, by plugging in multivariate sample mean $\bar{X}$ and covariance matrix S estimates for unknown mean μ and covariance matrix Σ, and tags as outlier any observation which has a Mahalanobis squared distance $d^2(\mathbf{X}, \bar{X})$ lying above a predefined quantile of the χ^2 distribution with p degrees of freedom [7].

This method is problematic, because all relies on normality assumption and the parameters estimates are particularly sensitive to outliers. Therefore, it is important to consider robust alternatives to these estimators for calculating robust Mahalanobis distances. The most widely used estimator of this type is the minimum covariance determinant (MCD) estimator defined in [25] for which also a fast computing algorithm was constructed [27].

In the next section, a sample data has been subjected to find its multivariate outliers by calculating the robust version of the Mahalanobis distances using the R as a modern statistical software for heavy computations involved.

4. Analyzing a Sample Data

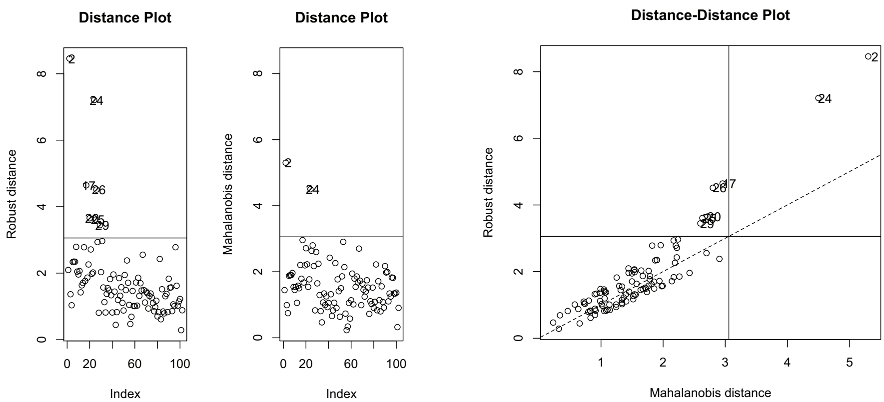
In the following, the vector of three variables of Prestige data set are considered as a multivariate observation. These variables are "education" (average education of occupational incumbents), "income" (average income of incumbents) and "prestige" (Pineo-Porter prestige score for occupation). The aim is to detect multivariate outliers in this data set using robust version of the Mahalanobis distance, the (MCD) estimator, which has been implemented in "rrcov" package in R [34]. First the mean vector and usual (classic) covariance matrix of the observation and the robust version of them are calculated. The results are:

```
-> Method:  Classical Estimator.
Estimate of Location:
education     income   prestige
    10.74    6797.90      46.83

Estimate of Covariance:
           education  income     prestige
education  7.444e+00  6.691e+03  3.991e+01
income     6.691e+03  1.803e+07  5.222e+04
prestige   3.991e+01  5.222e+04  2.960e+02

-> Method:  Robust Estimator.
Robust Estimate of Location:
education     income   prestige
     9.97    5833.96      41.64

Robust Estimate of Covariance:
           education  income     prestige
education  7.156e+00  4.355e+03  3.192e+01
income     4.355e+03  9.695e+06  3.923e+04
prestige   3.192e+01  3.923e+04  2.559e+02
```

Comparing classical and robust estimators of mean vector μ and the covariance matrix Σ, shows clear differences. These robust estimators are relatively insensitive to small changes in the bulk of the observations (inliers) or large changes in small number of observations (outliers).

In two left panels of Figure 4.1, the robust and classical Mahalanobis distances are shown in parallel. In most right panel of this figure, the distance-distance plot defined by [28] is shown, which plots the classical Mahalanobis versus robust distances and enable us to classify the observations and identify the potential outliers. The dashed line represents the points for which the robust and classical distances are equal. The horizontal and vertical lines are drawn at values $x = y = \sqrt{\chi^2_{(3,0.975)}}$. Points beyond these lines can be considered as outliers and are identified by their labels. In all panels, the outliers have large robust distances and are identified by their labels, for more details see [34].

Looking at the non-robust Mahalanobis distances at right panel of Figure 4.1 flagged out the observation number 2 and 24 as outliers, whereas robust Mahalanobis at the same panel flagged out the observation number 2, 7, 24, 25, 26 and 29 as outliers. In other words, applying the robust method enabled us to detect hidden outliers which has been masked by each other.



(a) Distance plot. (b) Chi-square Q-Q plot.

FIG. 4.1: Multivariate outlier detection using the robust Mahalanobis distances

5. Conclusion

In this paper, the Mahalanobis distance as a multivariate distance and its advantages relative to the Euclidean distance was reviewed. It made clear when dealing with correlated multivariate data the Mahalanobis distance is more suitable than the Euclidean distance because it takes the correlation into account. Moreover, It was shown how the Mahalanobis distances can be used as a tool for identifying multivariate outliers. When calculating the Mahalanobis distances one needs to estimate the theoretical mean vector and covariance matrix. Estimating these parameters using their usual empirical counterparts especially when data contain outliers yields misleading results, since these estimators are affected seriously by outliers. One reasonable solution is to use robust statistical techniques. There are different robust estimates, but distance-based methods, such as MCD are based on robust estimates of the mean and covariance matrix so that a robust Mahalanobis distance can be computed for each point. In this paper, the above mentioned methods have been applied to detect multivariate outliers in a real data set, using R software environment for statistical computing.

REFERENCES

1. C. C. Aggarwal: *Outlier Analysis, 2th Edition*. Springer, New York, 2017.
2. U. Balasooriya and Y. K. TSE: *Outlier detection in linear models: A comparative study in simple linear regression*. Communications in Statistics: Theory and Methods **15(12)** 1986, 3589–3598.
3. V. Barnett and T. Lewis: *Outliers in Statistical Data*. John Wiley and Sons, Chichester, England, 1994.
4. C. Becker and U. Gather: *The masking breakdown point of multivariate outlier identification rules*. Journal of the American Statistical Association **94(447)** (1999), 947–955.
5. V. Chandola, A. Banerjee and V. Kumar: *Anomaly detection: a survey*. ACM Comput. Surv. **41(3)** (2009), 1–58.
6. W. Dai and M. G. Genton: *Multivariate functional data visualization and outlier detection*. Journal of Computational and Graphical Statistics **27(4)** (2018), 923–934.
7. C. Fauconnier and G. Haesbroeck: *Outliers detection with the minimum covariance determinant estimator in practice*. Statistical Methodology **6(4)** (2009), 363–379.
8. J. Fox and S. Weisberg: *An R Companion to Applied Regression, 3th Edition*. SAGE Publications, Los Angeles, 2019.
9. M. Goldstein and S. Uchida: *A comparative evaluation of unsupervised anomaly detection algorithms for multivariate data*. PLoS ONE **11(4)** (2016), 1–31.
10. A. S. Hadi: *Identifying multiple outliers in multivariate data*. Journal of the Royal Statistical Society, Series B, **54** (1992), 761-771.
11. N. Haldar K. Farrukh A. Aftab and H. Abbas: *Arrhythmia classification using Mahalanobis distance based improved fuzzy C-Means clustering for mobile health monitoring systems*. Neurocomputing, **220** (2016), 221–235.
12. D. M. Hawkins: *Identification of Outliers*. Chapman and Hall, London, 1980.
13. D. C. Hoaglin, B. Iglewicz and J. W. Tukey: *Performance of some resistant rules for outlier labeling*. Journal of the American Statistical Association **81** (1986), 991–999.
14. V. J. Hodge and J. Austin: *A survey of outlier detection methodologies*. Artif. Intell. Rev. **22** (2004), 85126.
15. R. A. Johnson and D. Wichern: *Applied Multivariate Statistical Analysis*. Prentice Hall, 2007.
16. I. T. Jolliffe: *Principal Component Analysis*. Springer-Verlag (1986).

17. W. J. Krzanowski: *Principles of Multivariate Analysis: A Users Perspective*, Oxford Science Publications, 1988.
18. C. Leys, O. Klein, Y. Dominicy and C. Ley: *Detecting multivariate outliers: Use a robust variant of the Mahalanobis distance*. Journal of Experimental Social Psychology **7**4 (2018), 150–156.
19. P. C. Mahalanobis: *On the generalized distance in statistics*. Proceedings of the National Institute of Sciences (Calcutta), 1936, **2**, pp. 49–55.
20. J. Majewska: *Identification of multivariate outliers problems and challenges of visualization methods*. Informatyka i Ekonometria **4** (2015), 69–83.
21. G. M. Mimmack, S. Mason and J. Galpin: *Choice of distance matrices in cluster analysis: defining regions*. Journal of Climate **14** (2001), 2790–2797.
22. J. W. Osborne and A. Overbay: *The power of outliers (and why researchers should always check for them)*. Pract. Assess. Res. Eval. **9(6)** (2004), 1–9.
23. M. A. F. Pimentel, D. A. Clifton, L. Clifton and L. Tarassenko: *A review of novelty detection*. Signal Processing **99** (2014), 215-249.
24. D. M. Rocke and D. L. Woodruff: *Identification of outliers in multivariate data*. Journal of the American Statistical Association **91(435)** (1996), 1047–1061.
25. P. J. Rousseeuw: *Multivariate estimation with high breakdown point*. In: Mathematical Statistics and Applications (W. Grossmann, G. Pflug, I. Vincze, W. Wertz, eds.), Reidel, Dordrecht, **B**, 1985, pp 283–297.
26. P. Rousseeuw, M. Debruyne, S. Engelen and M. Hubert: *Robustness and outlier detection in Chemometrics*. Critical Reviews in Analytical Chemistry **36(3)**, (2006), 221–242.
27. P. J. Rousseeuw and K. Van Driessen: *A fast algorithm for the minimum covariance determinant estimator*. Technometrics, **41** (1999), 212–223.
28. P. J. Rousseeuw and B. C. van Zomeren: *Robust distances: simulation and cutoff Values*. In: Directions in Robust Statistics and Diagnostics, Part II. (W. Stahel, S. Weisberg, eds.), Springer-Verlag, New York, 1991.
29. T. A. Sajesh and M. R. Srinivasan: *An overview of multiple outliers in multidimensional data*. Sri Lankan Journal of Applied Statistics **14** (2013), 86–120.
30. C. J. Santos-Pereira and A. M. Pires: *Detection of outliers in multivariate data: a method based on clustering and robust estimators*. In: Compstat (W. Hrdle, B. Rnz, eds.), Physica, Heidelberg, 2002, pp 291–296.
31. N. G. Sharma, M. Silarski, T. Bednarski, P. Biaas, E. Czerwiski, A. Gajos, M. Gorgol, B. Jasiska, D. Kamiska, . Kapon, G. Korcyl, P. Kowalski, T. Kozik, W. Krzemie, E. Kubicz, S. Niedwiecki, M. Paka, L. Raczyski, Z. Rudy, O. Rundel, A. Somski, A. Strzelecki, A. Wieczorek, W. Wilicki, M. Zieliski, B. Zgardziska and P. Moskal: *Reconstruction of hit time and hit position of annihilation quanta in the J-PET detector using the Mahalanobis distance*. Nukleonika **4** (2015), 765–769.
32. K. Singh and D. S. Upadhyaya: *Outlier detection: Applications and techniques*. International Journal of Computer Applications **89(6)** (2014) 307–323.
33. S. Stckl and M. Hanke: *Financial applications of the Mahalanobis distance*, SSRN Electronic Journal **1(2)** (2014), 78–84.

34. V. Todorov and P. Filzmoser: *An object-oriented framework for robust multivariate analysis*. Journal of Statistical Software **32(3)** (2009), 1–47.

35. J. W. Tukey: *Exploratory Data Analysis*. Addison-Wesley, New York, USA, 1977.

36. M. P. J. van der Loo: *Distribution based outlier detection for univariate data*. Discussion paper **1**0003 Statistics Netherlands (2010), 3–14.

37. M. P. J. van der Loo: *Extremevalues, an R package for outlier detection in univariate data*. R package version 2.3 (2010), url = http://www.github.com/markvanderloo/extremevalues.

38. G. M. Venturini: *Statistical Distances and Probability Metrics for Multivariate Data*. Ph. D. Thesis, Charles III University of Madrid, 2015.

39. Y. Zhang, B. Du, L. Zhang and S. Wang: *A low-rank and sparse matrix decomposition-based Mahalanobis distance method for hyperspectral anomaly detection*. IEEE Transactions on Geoscience and Remote Sensing **220** (2016), 1376–1389.

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