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[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

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PROCEEDINGS OF THE THIRD CONFERENCE ON COMPUTATIONAL ALGEBRA, COMPUTATIONAL NUMBER THEORY AND APPLICATIONS
University of Kashan, Kashan, December 12–14, 2018

Ali Reza Ashrafi and Hassan Daghigh (Guest Editors)

Abstract. This issue of *Facta Universitatis, Series: Mathematica and Informatics* includes a selection of 20 papers that were presented at Third Conference on Computational Algebra, Computational Number Theory and Applications in the University of Kashan, Kashan, Iran. This is the 3th in the series of computational algebra conferences held in Iran organized mainly by the University of Kashan, first in 2014.

Keywords: Computational algebra; computational number theory.

Foreword

In the period from 12th to 14th December, 2018, the Third Conference on Computational Algebra, Computational Number Theory and Applications (CACNA 2018) had taken place in Kashan, a city with 7000 years history in the center of Iran. The conference was organized by the Faculty of Mathematical Sciences, University of Kashan. The scope of the conference covered various topics related to computational algebra and computational number theory, including computational group theory, computational number theory, algebraic programming languages, computer algebra, algebraic cryptography, coding theory, algebraic combinatorics and information theory.

The conference was attended by near 100 researchers from Iran, Iraq and Serbia who held 20, 40 and 60-minutes lectures. The keynote speakers of the conference were:

- Bijan Davvaz (Yazd University, Yazd),

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- Ali Reza Moghaddamfar (KNT University of Technology, Tehran),
- Predrag S. Stanimirovic (University of Nis, Serbia).

The conference had also two invited speakers: Majid Arezoomand (University of Larestan) and Ashraf Daneshkhah (Bu-Ali Sina University).

The members of the Scientific and Organizing Committee of CACNA 2018 were:

- Alireza Abdollahi (University of Isfahan),
- Seyed Hassan Alavi (Bu-Ali Sina University),
- Jalal Askari (University of Kashan),
- Ali Reza Ashrafi (University of Kashan),
- Modjtaba Bahramian (University of Kashan),
- Behnam Bazigaran (University of Kashan),
- Hassan Daghigh (University of Kashan),
- Mohammad Reza Darafsheh (University of Tehran),
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- Modjtaba Ghorbani (Shahid Rajaee Teacher Training University),
- Ali Iranmanesh (Tarbiat Modares University),
- Mohammad Ali Iranmanesh (Yazd University),
- Reza Jahani-Nezhad (University of Kashan),
- Reza Kahkeshani (University of Kashan),
- Fatemeh Koorepazan-Moftakhar (University of Kashan),
- Hamid Mousavi (University of Tabriz),
- Majid Mazroei (University of Kashan),
- Akbar Mohebi (University of Kashan),
- Marzieh Pourbabaee (University of Kashan),
- Reza Orfi (Kwarazmi University),
- Farhad Rahmati (Amir Kabir University of Technology).

This meeting was a continuation of a series of conferences on computational algebra organized mainly by Faculty of Mathematical Sciences at the University of Kashan and we gratefully acknowledge the financial support provided by this university. This issue of Facta Universitatis, Series: Mathematica and Informatics includes a selection of 20 papers from CACNA 2018. We are very thankful from Professor Predrag Stanimirovic, the Editor-in-Chief of this journal for providing us with the opportunity to be the Guest Editors of this issue. We would also like to thank all the referees for the time they allocated and their help.

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STOCHASTIC EVOLUUTION EQUATIONS WITH MONOTONE NONLINEARITY IN $Ł^p$ SPACES

Majid Amintorabi and Ruhollah Jahanipur

Abstract. In this paper, we study semilinear stochastic evolution equations with semimonotone nonlinearity and multiplicative noise in L^p spaces for $2 \leq p < \infty$. We do not impose any coercivity or Lipschitz condition on the nonlinear part of equations. We prove the existence, uniqueness and measurability of the mild solutions. The proofs of the existence and uniqueness are based on a version of the Itô type inequality which is stronger than analogous inequalities.
Keywords. Semilinear stochastic evolution equations; semimonotone nonlinearity; multiplicative noise; Lipschitz condition.

1. Introduction

Stochastic evolution equations (SEE's for short) describe the evolution in time of the stochastic phenomena and use to model dynamical systems with random effects such as problems arising in biology, chemistry, quantum mechanics, statistical physics, economics, etc. There are two approaches in the study of nonlinear SEE's. The first which is called the variational method, considers Hilbert space valued solutions in the framework of Gelfand triple under certain monotonicity and coercivity conditions on coefficients; see e.g., [20], [26] and [27]. The second approach, the one adopted in this paper, is the semigroup method in which we use the tools of semigroup theory to study mild solutions of semilinear SEE's. This approach gives a unified treatment of a wide class of parabolic, hyperbolic and functional stochastic partial differential equations. Furthermore, its advantage over the variational method is in that one does not require the coercivity condition. In semigroup method, one usually investigate existence, uniqueness and stability of mild solutions of semilinear SEE's under standard Lipshitz-type assumptions on coefficients. The Hilbert space theory of this case has been studied by many authors; see e.g., [8]

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and references therein. Brzeźniak extended a number of results of the same type to martingale type 2-spaces [3], [4]. van Neerven, Veraar and Weis studied stochastic equations in the setting of UMD Banach spaces [23].

On the other hand, some authors use semigroup method to study more general semilinear SEE's with (semi)monotone nonlinear drift instead of Lipschitz one. This approach has first followed by Browder [2] and Kato [18] for deterministic monotone-type semilinear evolution equations. Zangeneh [33, 35] applied this approach to prove the existence and uniqueness of mild solutions of monotone-type semilinear SEE's with multiplicative noise and studied [34] the measurability of mild solutions of these equations. Following this program, Jahanipur and Zangeneh [13] studied (sample-path and p-th mean) exponential asymptotic stability of solutions and Jahanipur [14] proved similar stability theorems for stochastic delay evolution equations. Hamedani and Zangeneh [10] considered a stopped version of monotone-type equations and obtained the existence, uniqueness and measurability of the solutions. Using the tools of random fixed point theory, Jahanipur [15, 16, 17] generalized this approach to study stochastic functional evolution equations. Moreover, Salavati and Zangeneh [28, 30] extended this method to investigate semilinear SEE's with Lévy (jump) noise.

In this paper, we consider monotone-type semilinear SEE's with multiplicative noise in $L^p(\mathbb{R}), 2 \leq p < \infty$, and we prove existence and uniqueness of mild solutions. Our results are remarkable from two points of view. First, we relax Lipschitz condition on nonlinearity drift to semimonotone one without imposing the coercivity hypothesis. Furthermore, while all the results for the semilinear SEE's obtained under our assumptions, have been restricted to the Hilbert space setting, we study the problem in the more general case $L^p(\mathbb{R}), 2 \leq p < \infty$, and therefore we extend some of the results mentioned above.

We make an iterative method to prove the existence and uniqueness of mild solutions in r-th moment for $r \geq 2$. This method is based on a version of Itô type inequality. This is a pathwise inequality for powers $r \geq 2$ of stochastic convolution integrals in $L^p(\mathbb{R}), 2 \leq p < \infty$, and generalizes corresponding inequalities (for example, Theorem 2 of [13]). We adopt the same approach as in [15] and we use a method based on random fixed point theory.

The organization of the paper is as follows. We begin by recalling some preliminary materials in the Section 2. Section 3 is devoted to prove an Itô type inequality inequality. In Section 4, we study the measurability of the solutions of the random integral equation. In Section 5, we introduce the semilinear SEE of monotone-type and prove the existence and uniqueness of it's mild solution.

2. Preliminaries

Throughout the paper, $(\Omega, \mathcal{F}, \mathcal{P})$ denotes a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, $p \geq 2$, $T > 0$ and $r \geq 2$ are given constants. The conjugate exponent of p will be denoted by q and we will simply write L^p for $L^p(\mathbb{R})$. Moreover, H is a real separable Hilbert space with inner product $\langle .\,,.\rangle_H$, E is a real Banach space with dual E^*, and $\mathcal{L}(H,E)$ stands for the space of all bounded linear operators from H to E. We recall that the duality mapping $J : E \longrightarrow E^*$ is defined for every $x \in E$ by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2 \},$$

in which $\langle x\,,\,x^*\rangle$ is the duality pairing between E and E^*. It is well-known that if E is uniformly convex, then J is single valued and continuous. Also, $W_H := (W_H(t))_{t\in[0,T]}$ denotes an H-cylindrical Brownian motion, i.e., $W_H(t)$ is a bounded operator from H to $L^2(\Omega)$, for each $h \in H$ the process $W_H h := (W_H(t)h)_{t\in[0,T]}$ is a real Brownian motion, and for all $h_1, h_2 \in H$ and $t_1, t_2 \in [0,T]$ we have

$$\mathbb{E}(W_H(t_1)h_1 \cdot W_H(t_2)h_2) = (t_1 \wedge t_2)\langle h_1, h_2\rangle_H \cdot$$

Furthermore, we assume that $A : D(A) \subseteq L^p \longrightarrow L^p$ is the generator of a C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ of bounded linear operators satisfying an exponential growth condition with parameter $\lambda > 0$; that is,

$$\|S(t)\| \leq e^{\lambda t} \qquad \forall t \geq 0\cdot$$

If $\|S(t)\| \leq 1$ for all $t \geq 0$, then $S(t)$ is called a contraction semigroup.

2.1. Derivative of L^p-norm

Here we calculate the first and second Fréchet derivatives of L^p-norm function. These results are used in the next sections. Let

$$h(x) = \|x\|_{L^p}^r \qquad \forall x \in L^p(\mathbb{R})\cdot$$

The first and second Fréchet derivatives of h at the point $x \in L^p(\mathbb{R})$ are defined as mappings $Dh(x) : L^p \longrightarrow \mathbb{R}$ and $\big(D^2h(x)\big)(y) : L^p \longrightarrow \mathbb{R}$ such that for any $y, z \in L^p$,

$$\begin{aligned}
\langle y, Dh(x)\rangle &= \lim_{t\downarrow 0} \frac{1}{t}\big(\|x+ty\|_{L^p}^r - \|x\|_{L^p}^r\big)\\
&= \lim_{t\downarrow 0} \frac{r}{p}\Big(\int |x+ty|^p\Big)^{\frac{r}{p}-1} \cdot p\int |x+ty|^{p-2}(x+ty)y\\
&= r\|x\|_{L^p}^{r-p}\int |x|^{p-2}xy = r\|x\|_{L^p}^{r-2}\int \|x\|_{L^p}^{2-p}|x|^{p-2}xy\\
&= r\|x\|_{L^p}^{r-2}\langle y, J(x)\rangle,
\end{aligned}$$

where $J(x)$ is the value at x of the duality mapping of J, and similarly

$$\big\langle z, \big(D^2h(x)\big)(y)\big\rangle = \big\langle z, D(r\|x\|_{L^p}^{r-2}\langle y, J(x)\rangle)\big\rangle$$

$$\begin{aligned}
&= \lim_{t\downarrow 0}\frac{1}{t}\Bigg[\Big(r\|x+tz\|_{L^p}^{r-p}\int|x+tz|^{p-2}(x+tz)y\Big)\\
&\qquad -\Big(r\|x\|_{L^p}^{r-p}\int|x|^{p-2}xy\Big)\Bigg]\\
&= \lim_{t\downarrow 0} r\Bigg[\Big(\frac{d}{dt}\|x+tz\|_{L^p}^{r-p}\Big)\int|x+tz|^{p-2}(x+tz)y\\
&\qquad +\|x+tz\|_{L^p}^{r-p}\Big(\frac{d}{dt}\int|x+tz|^{p-2}(x+tz)y\Big)\Bigg]\\
&= r(r-p)\|x\|_{L^p}^{r-2p}\int|x|^{p-2}xz\int|x|^{p-2}xy\\
&\qquad +r(p-1)\|x\|_{L^p}^{r-p}\int|x|^{p-2}zy\\
&= r(r-p)\|x\|_{L^p}^{r-4}\langle z,J(x)\rangle\langle y,J(x)\rangle\\
&\qquad +r(p-1)\|x\|_{L^p}^{r-p}\int|x|^{p-2}zy.
\end{aligned}$$

So, by Hölder's inequality

$$\Big|\Big\langle z,\big(D^2h(x)\big)(y)\Big\rangle\Big| \le r(r-p)\|x\|_{L^p}^{r-2}\|z\|_{L^p}\|y\|_{L^P}+r(p-1)\|x\|_{L^p}^{r-2}\|z\|_{L^p}\|y\|_{L^p}$$

and therefore,

$$\|D^2(h(x))\| \le r(r-1)\|x\|_{L^p}^{r-2}. \tag{2.1}$$

2.2. γ-radonifying operators

Suppose $(\gamma_n)_{n\ge 1}$ is a Gaussian sequence; i.e., a sequence of independent real-valued standard Gaussian random variables. A linear operator $R : H \longrightarrow E$ is called γ-radonifying if for some (and consequently for every) orthonormal basis $(h_n)_{n\ge1}$ of H, the series $\sum_{n=1}^{\infty}\gamma_n Rh_n$ converges in $L^2(\Omega, E)$. We denote by $\gamma(H,E)$ the set of all γ-radonifying operators from H to E. For any $R \in \gamma(H,E)$ the norm of R is defined by

$$\|R\|_{\gamma(H,E)} := \Big(\mathbb{E}\Big\|\sum_{n=1}^{\infty}\gamma_n Rh_n\Big\|^2\Big)^{\frac{1}{2}}.$$

Note that $\|\cdot\|_{\gamma(H,E)}$ is independent of the orthonormal basis $(h_n)_{n\ge1}$ for H. Endowed with this norm, $\gamma(H,E)$ is a Banach space. If $R \in \gamma(H,E)$, then R is bounded and $\|R\| \le \|R\|_{\gamma(H,E)}$. If E is also a Hilbert space, then $\gamma(H,E)$ is isometrically isomorphic to $\mathcal{L}_2(H,E)$, where $\mathcal{L}_2(H,E)$ denotes the space of all Hilbert-Schmidt operators from H to E. Specially if E is finite dimensional, then γ-radonifying norm is the same as operator norm. For more information about γ-radonifying operators and their properties, see [24].

2.3. Itô formula in UMD Banach spaces

A Banach space E is said to have the unconditional martingale difference property, or briefly, E is a UMD space, if for some (equivalently, for all) $p \in (1, \infty)$ there exists a real positive constant C such that

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^p \le C \Big\| \sum_{n=1}^{N} d_n \Big\|^p \qquad \forall N \ge 1$$

for all $(\varepsilon_n)_{n=1}^N \in \{-1, 1\}^N$ and every L^p-integrable E-valued martingale difference sequence $(d_n)_{n\ge 1}$. For example every Hilbert space is a UMD space. Also, the spaces $L^p(S)$ for $1 < p < \infty$ and σ-finite measure space $(S, \mathcal{A}, \mu)$ are UMD spaces. If E is a UMD Banach space, then it is well-known that for a suitable class of functions $\Phi : [0,T] \times \Omega \longrightarrow \gamma(H, E)$ the stochastic integral with respect to W_H is well-defined (see, e.g., [22]).

Let E and F be two normed linear spaces and $h : [0,T] \times E \longrightarrow F$ be a function. We say that h is of class $C^{1,2}$ if h is Fréchet differentiable with respect to the first variable and twice Fréchet differentiable with respect to the second variable and h, $D_1 h$, $D_2 h$ and $D_2^2 h$ are continuous functions on $[0,T] \times \Omega$. Now, we recall the main result of [6].

Theorem 2.1. *(Itô formula) Let E and F be UMD spaces. Assume that $h : [0,T] \times E \longrightarrow F$ is of class $C^{1,2}$. Let $\Phi : [0,T] \times \Omega \longrightarrow \mathcal{L}(H, E)$ be an H-strongly measurable and adapted process which is stochastically integrable with respect to W_H and assume that the paths of Φ belong to $L^2(0,T; \gamma(H,E))$ almost surely. Let $\psi : [0,T] \times \Omega \longrightarrow E$ be strongly measurable and adapted with paths in $L^1(0,T; E)$ almost surely. Let $\xi : \Omega \longrightarrow E$ be strongly $\mathcal{F}_0$-measurable. Define $\zeta : [0,T] \times \Omega \longrightarrow E$ by*

$$\zeta = \xi + \int_0^{\cdot} \psi(s) ds + \int_0^{\cdot} \Phi(s) dW_H(s)\cdot$$

Then $s \longmapsto D_2 h(s, \zeta(s))\Phi(s)$ is stochastically integrable and almost surely we have, for all $t \in [0,T]$,

$$\begin{aligned} h(t, \zeta(t)) - h(0, \zeta(0)) &= \int_0^t D_1 h(s, \zeta(s)) ds + \int_0^t D_2 h(s, \zeta(s)) \psi(s) ds \\ &+ \int_0^t D_2 h(s, \zeta(s)) \Phi(s) dW_H(s) + \frac{1}{2} \int_0^t Tr_{\Phi(s)} \big(D_2^2 h(s, \zeta(s)) \big) ds\cdot \end{aligned}$$

Moreover,

$$\int_0^t \Big\| Tr_{\Phi(s)} \Big(D_2^2 h(s, \xi(s)) \Big) \Big\| ds \le \int_0^t \Big\| D_2^2 h(s, \xi(s)) \Big\| \Big\| \Phi(s) \Big\|^2_{\gamma(H,E)} ds\cdot \tag{2.2}$$

The following theorem is a maximal inequality for stochastic convolution integrals to which we refer several times in the next sections. we recall it from [25].

Theorem 2.2. *Let E be a 2-smooth Banach space and let Φ be a progressively measurable process in $\gamma(H,E)$.If*

$$\int_0^T \left\|\Phi(t)\right\|^2_{\gamma(H,E)} dt < \infty \qquad a.s.,$$

then the stochastic convolution process $X(t) = \int_0^t S(t-s)\Phi(s)dW_H(s)$ is well-defined and has a continuous version. Moreover, for all real positive b there exists a constant D, depending only on b and E, such that

$$\mathbb{E} \sup_{0\le t\le T} \left\|X(t)\right\|^b \le D^b \mathbb{E}\Big(\int_0^T \left\|\Phi(t)\right\|^2_{\gamma(H,E)} dt\Big)^{\frac{b}{2}}. \tag{2.3}$$

We conclude this section by recalling the well-known Burkholder-Davis-Gundy inequality for stochastic integrals in UMD Banach spaces from [32].

Theorem 2.3. *(B.D.G inequality) Let E be a UMD Banach space and $\Phi : [0,T]\times \Omega \longrightarrow \gamma(H,E)$ be an H-strongly measurable and $\mathcal{F}_t$-adapted process which is scalarly in $L^0\big(\Omega, L^2(0,T;H)\big)$. If Φ is stochastically integrable with respect to W_H, then for $0 < b < \infty$ we have*

$$\mathbb{E} \sup_{0\le t\le T} \left\| \int_0^t \Phi(s)dW_H(s)\right\|^b \le C_{p,E}\mathbb{E}\left\|\Phi\right\|^b_{\gamma(L^2(0,T;H),E)}, \tag{2.4}$$

where $C_{p,E}$ is a constant depending only on E and p.

3. Itô type inequality

In this section, we prove an Itô type inequality. This is a pathwise inequality for the norm of the stochastic convolution integral. We use this result to prove the existence and uniqueness of the mild solutions of stochastic evolution equations. One of the first attempts to obtain inequalities for the stochastic convolution integrals was the one made by Kotelenez [19], where he considered Hilbert space valued processes and power $r = 2$ for stochastic convolution integral.Tubaro [31] extended this result to exponents $r \ge 2$ and Ichikawa [11] proved it for the case $0 < r < 2$. van Neerven [25] and Brzeźniak [5] considered such inequalities for processes with values in some Banach spaces of special kind (Theorem 2.2).

While all of these inequalities are for moments and involve expectations, we need a pathwise inequality for studying monotone-type semilinear SEE's. There exist several results of this type for Hilbert space valued processes. In particular, Zangeneh [33] proved a pathwise inequality for the square of the norm of stochastic convolution integral in a Hilbert space. Jahanipur and Zangeneh [13] extended this inequality to the powers $r \geq 2$ in a special case that the stochastic convolution integral is an Itô integral with respect to the Wiener process. Salavati and Zangeneh [29] proved more general case where integrator is a general martingale.

We adopt the same approach as in [13] to prove a pathwise inequality for powers $r \geq 2$ of the norm of stochastic convolution integral in L^p, $p \geq 2$. First, we recall our main assumptions.

Hypothesis 3.1. *(a) X_0 is an $\mathcal{F}_0$-measurable random variable.*

(b) $f : [0,T] \times \Omega \longrightarrow L^p$ is strongly measurable and adapted process with paths in $L^1(0,T;L^p)$ almost surly and $\int_0^T \mathbb{E}\|f(t)\|^r dt < \infty$.

(c) $g : [0,T] \times \Omega \longrightarrow \mathcal{L}(H, L^p)$ is an H-strongly measurable and adapted process which is stochastically integrable with respect to W_H, almost every path of g belong to $L^2(0,T;\gamma(H,L^p))$ and $\int_0^T \mathbb{E}\|g(t)\|^r_{\gamma(H,L^p)} < \infty$.

Theorem 3.2. *(Itô type inequality) Let hypotheses 3.1 hold and*

$$X(t) := S(t)X_0 + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dW_H(s), \quad 0 \leq t \leq T.$$

Then for all $t \in [0,T]$ we have

$$\begin{aligned} \|X(t)\|^r_{L^p} \quad &\leq e^{r\lambda t}\|X_0\|^r_{L^p} + r\int_0^t e^{r\lambda(t-s)}\|X(s)\|^{r-2}_{L^p}\langle f(s), J(X(s))\rangle ds \\ &+ r\int_0^t e^{r\lambda(t-s)}\|X(s)\|^{r-2}_{L^p}\langle g(s), J(X(s))\rangle dW_H(s) \\ &+ \tfrac{1}{2}r(r-1)\int_0^t e^{r\lambda(t-s)}\|X(s)\|^{r-2}_{L^p}\|g(s)\|^2_{\gamma(H,L^p)}ds, \end{aligned} \tag{3.1}$$

where $J(X(s))$ denotes the value of the duality mapping J at $X(s)$.

It is easy to see that by an appropriate transformation, we may assume that $\lambda = 0$ (see, e.g., Lemma 1 of [13]). Then according to the Lumer-Phillips theorem, we have $\langle Ax, J(x)\rangle \leq 0$ for each $x \in D(A)$.

The main idea of the proof is to approximate $X(t)$ using the Yosida method. For each $n \in \mathbb{N}$ we define the mapping $R_n : L^p \longrightarrow D(A)$ by $R_n = nR(n,A)$ where $R(n,A) = (nI - A)^{-1}$; hence $\|R_n\| \leq 1$. Let

$$X_0^n = R_n X_0, \qquad f_n = R_n f, \qquad g_n = R_n g$$

and define

$$X_n(t) = S(t)X_0^n + \int_0^t S(t-s)f_n(s)ds + \int_0^t S(t-s)g_n(s)dW_H(s)\cdot$$

Now, we state and prove some lemmas.

Lemma 3.1. *Under the above conditions,*

$$\big\|X_n(t) - X(t)\big\|_\infty := \sup_{0\le t\le T} \big\|X_n(t) - X(t)\big\|_{L^p} \longrightarrow 0$$

in L^r as $n \to \infty$. Moreover, there exists a subsequence, again denoted by $\{X_n\}$, such that

$$\mathbb{E}\int_0^T \Big|\big\|X_n(t)\big\|_{L^p}^r - \big\|X(t)\big\|_{L^p}^r\Big| dt \longrightarrow 0\cdot$$

Proof. By Theorem 2.2, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t S(t-s)(g_n(s)-g(s))dW_H(s)\Big\|_{L^p}^r\Big] \\
&\le D\mathbb{E}\Big[\int_0^T \big\|g_n(s)-g(s)\big\|_{\gamma(H,L^p)}^2\Big]^{\frac{r}{2}}.
\end{aligned}$$

Since

$$\big\|g_n(s)-g(s)\big\|_{\gamma(H,L^p)} \le \big\|R_n - I\big\|\big\|g(s)\big\|_{\gamma(H,L^p)} \le 2\big\|g(s)\big\|_{\gamma(H,L^p)} \quad a.s.$$

by Hypothesis 3.1(c) and the fact that $R_n \longrightarrow I$ strongly, the dominated convergence theorem implies

$$\mathbb{E}\Big[\int_0^T \big\|g_n(s)-g(s)\big\|_{\gamma(H,L^p)}^2\Big]^{r/2} \longrightarrow 0.$$

So,

$$\Big\|\int_0^\cdot S(t-s)(g_n(s)-g(s))dW_H(s)\Big\|_\infty \longrightarrow 0 \quad in \quad L^r\cdot$$

On the other hand, by Hölder's inequality,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big\|\int_0^t S(t-s)(f_n(s)-f(s))ds\Big\|_{L^p}^r\Big] \le T^{r-1}\mathbb{E}\int_0^T \|f_n(s)-f(s)\|_{L^p}^r ds$$

and the right hand side of the above inequality tends to zero by the dominated convergence theorem. Hence,

$$\Big\|\int_0^\cdot S(t-s)(f_n(s)-f(s))ds\Big\|_\infty \longrightarrow 0, \qquad in \quad L^r\cdot$$

Moreover,

$$\big\|S(t)(X_0^n - X_0)\big\|_{L^p}^r \le \|X_0^n - X_0\|_{L^p}^r \longrightarrow 0 \qquad boundedly.$$

From (1) we imply that there exists a subsequence, again denoted by $\{X_n\}$, such that for each $t \in [0,T]$,

$$\|X_n(t)\|_{L^p}^r \longrightarrow \|X(t)\|_{L^p}^r \quad a.s.$$

and

$$\|X_n(\omega) - X(\omega)\|_\infty^r \longrightarrow 0 \quad a.s.$$

Furthermore, we have

$$\begin{aligned}\Big| \big\|X_n(t,\omega)\big\|_{L^p}^r - \big\|X(t,\omega)\big\|_{L^p}^r \Big| \;\leq\; & 2^{r-1}\big\|X_n(t,\omega) - X(t,\omega)\big\|_{L^p}^r \\ & +(2^{r-1}+1)\big\|X(t,\omega)\big\|_{L^p}^r.\end{aligned}$$

Now, Lemma 3 of [13] yields the result. □

Lemma 3.2. *Let $J(X_n)$, $J(X)$ denote the values of duality mapping at X_n and X, respectively. Then, after choosing a subsequence if it is necessary, we have*

$$\mathbb{E}\int_0^T \big\|J(X_n(s)) - J(X(s))\big\|_{L^q}^r ds \longrightarrow 0.$$

Proof. By Lemma 3.1, one can find a subsequence denoted by the same notation $\{X_n\}$, such that

$$\big\|X_n(s) - X(s)\big\|_{L^p} \longrightarrow 0 \;\; a.s., \qquad for \;\; all \; s \in [0,T].$$

Hence, from the continuity of the duality mapping we imply that

$$\big\|J(X_n(s)) - J(X(s))\big\|_{L^q}^r \longrightarrow 0 \;\; a.s., \qquad for \;\; all \; s \in [0,T].$$

On the other hand

$$\big\|J(X_n(s)) - J(X(s))\big\|_{L^q}^r \leq 2^r\Big(\big\|X_n(s)\big\|_{L^p}^r + \big\|X(s)\big\|_{L^p}^r\Big).$$

Now, applying Lemma 3 of [13] and Lemma 3.1 we obtain the desired result. □

Lemma 3.3. *$\{X_n\}$ is a $D(A)$-valued process, the process $\{AX_n\}$ has integrable paths almost surely and we have for all $t \in [0,T]$ that*

$$X_n(t) = X_0^n + \int_0^t AX_n(s)ds + \int_0^t f_n(s)ds + \int_0^t g_n(s)dW_H(s), \tag{3.2}$$

and

$$\begin{aligned}\|X_n(t)\|_{L^p}^r \leq \;\; & \|X_0\|_{L^p}^r + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle f_n(s), J(X_n(s))\rangle ds \\ & +r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle g_n(s), J(X_n(s))\rangle dW_H(s) \\ & +\tfrac{1}{2}r(r-1)\int_0^t \|X_n(s)\|_{L^p}^{r-2}\|g(s)\|_{\gamma(H,L^p)}^2 ds.\end{aligned} \tag{3.3}$$

Proof. Note that

$$\int_0^t AX_n(\theta)d\theta = \underbrace{\int_0^t AS(\theta)X_0^n d\theta}_{T_1} + \underbrace{\int_0^t A\Big(\int_0^\theta S(\theta-s)f_n(s)ds\Big)d\theta}_{T_2} + \underbrace{\int_0^t A\Big(\int_0^\theta S(\theta-s)g_n(s)dW_H(s)\Big)d\theta}_{T_3}.$$

Furthermore, we have

$$T_1 = S(t)X_0^n - X_0^n,$$

and by the Fubini theorem,

$$T_2 = \int_0^t S(t-s)f_n(s)ds - \int_0^t f_n(s)ds.$$

Also, by the Fubini theorem for stochastic integrals in UMD Banach spaces [21], we have

$$T_3 = \int_0^t S(t-s)g_n(s)dW_H(s) - \int_0^t g_n(s)dW_H(s).$$

Hence (3.2) is obtained. Now we apply Itô formula (Theorem 2.1) to $h(X_n(\cdot))$ where $h(x) = \|x\|_{L^p}^r$. We find

$$\begin{aligned}\|X_n(t)\|_{L^p}^r = \quad & \|X_0^n\|_{L^p}^r + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle f_n(s), J(X_n(s))\rangle ds \\ & + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle AX_n(s), J(X_n(s))\rangle ds \\ & + r\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle g_n(s), J(X_n(s))\rangle dW_H(s) \\ & + \frac{1}{2}\int_0^t Tr_{g_n(s)}\big(D^2(\|X_n(s)\|_{L^p}^r)\big)ds,\end{aligned}$$

where $J(X_n(s))$ denotes the value of duality mapping at $X_n(s)$. Here we have used the first and second Fréchet derivatives of $\|\cdot\|_{L^p}^r$. Since $\|X_0^n\|_{L^p}^r \le \|X_0\|_{L^p}^r$, $\langle Ax, J(x)\rangle \le 0$ for all $x \in D(A)$, and

$$\|g_n(s)\|_{\gamma(H,L^p)} \le \|g(s)\|_{\gamma(H,L^p)},$$

we can apply the inequalities (2.1) and (2.2) to conclude the result. □

Proof of Theorem 3.2. It is enough to prove that the right hand side of (3.3) (after choosing a subsequence) converges term by term to that of (3.1), in

probability. We prove this in three steps:
Step 1: Note that

$$\Big|\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle f_n(s), J(X_n(s))\rangle ds - \int_0^t \|X(s)\|_{L^p}^{r-2}\langle f(s), J(X(s))\rangle ds\Big|$$

$$\le \underbrace{\Big|\int_0^t \big(\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\big)\langle f_n(s), J(X_n(s))\rangle ds\Big|}_{A_n(t)}$$

$$+ \underbrace{\Big|\int_0^t \|X(s)\|_{L^p}^{r-2}\langle f_n(s) - f(s), J(X_n(s))\rangle ds\Big|}_{B_n(t)}$$

$$+ \underbrace{\Big|\int_0^t \|X(s)\|_{L^p}^{r-2}\langle f(s), J(X_n(s)) - J(X(s))\rangle ds\Big|}_{C_n(t)}.$$

By Hölder's inequality and elementary inequality $|a-b|^k \le |a^k - b^k|$ which is true for all non-negative numbers a, b and all $k \ge 1$, we obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T} A_n(t)\Big] \le \mathbb{E}\int_0^T \Big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big|\Big\|f(s)\Big\|_{L^p}\Big\|X_n(s)\Big\|_{L^p} ds$$

$$\le \Big[\mathbb{E}\int_0^T \Big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big|^{\frac{r}{r-2}} ds\Big]^{\frac{r-2}{r}} \times$$

$$\Big[\mathbb{E}\int_0^T \Big\|f(s)\Big\|_{L^p}^r ds\Big]^{\frac{1}{r}}\Big[\mathbb{E}\int_0^T \Big\|X_n(s)\Big\|_{L^p}^r ds\Big]^{\frac{1}{r}}$$

$$\le \Big[\mathbb{E}\int_0^T \Big|\|X_n(s)\|_{L^p}^{r} - \|X(s)\|_{L^p}^{r}\Big| ds\Big]^{\frac{r-2}{r}}\Big[\mathbb{E}\int_0^T \|f(s)\|_{L^p}^r ds\Big]^{\frac{1}{r}}\Big[T\mathbb{E}\|X_n\|_\infty^r\Big]^{\frac{1}{r}}.$$

The second and third terms on the right, are bounded and according to Lemma 3.1, after choosing a subsequence, the first term tends to zero. So, for this subsequence we have $\sup\limits_{0\le t\le T} A_n(t) \longrightarrow 0$ in L^1 and hence in probability. Also, the Hölder inequality implies that

$$\sup_{0\le t\le T} B_n(t) \le \int_0^T \|X(s)\|_{L^p}^{r-2}\big\|f_n(s) - f(s)\big\|_{L^p}\big\|X_n(s)\big\|_{L^p} ds$$

$$\le \big(T\|X\|_\infty^r\big)^{1-\frac{2}{r}}\Big(\int_0^T \|f_n(s) - f(s)\|_{L^p}^r ds\Big)^{\frac{1}{r}}\big(T\|X_n\|_\infty^r\big)^{\frac{1}{r}}.$$

The first and third terms on the right are bounded and the second term tends to zero almost surely by the dominated convergence theorem. Hence, $\sup_{0\le t\le T} B_n(t) \longrightarrow 0$ almost surely and so in probability. Moreover, by Hölder's inequality we have

$$\mathbb{E}\Big[\sup_{0\le t\le T} C_n(t)\Big] \le \mathbb{E}\int_0^T \Big\|X(s)\|_{L^p}^{r-2}\Big\|f(s)\Big\|_{L^p}\Big\|J(X_n(s)) - J(X(s))\Big\|_{L^q} ds$$

$$\leq \Big(T\mathbb{E}\|X\|_\infty^r\Big)^{1-\frac{2}{r}}\left(\mathbb{E}\int_0^T \|f(s)\|_{L^p}^r ds\right)^{\frac{1}{r}}\left(\mathbb{E}\int_0^T \|J(X_n(s))-J(X(s))\|_{L^q}^r ds\right)^{\frac{1}{r}}.$$

By Lemma 3.2, after choosing a subsequence, the right hand side tends to zero. So, for this subsequence we get $\sup_{0\leq t\leq T} C_n(t) \longrightarrow 0$ in L^1 and hence in probability.
Step 2: We have

$$\begin{aligned}
&\left|\int_0^t \|X_n(s)\|_{L^p}^{r-2}\langle g_n(s),(X_n(s))\rangle dW_H(s) - \int_0^t \|X(s)\|_{L^p}^{r-2}\langle g(s),J(X(s))\rangle dW_H(s)\right| \\
&\leq \left|\int_0^t \underbrace{\Big(\|X_n(s)\|_{L^p}^{r-2}-\|X(s)\|_{L^p}^{r-2}\Big)\langle g_n(s),J(X_n(s))\rangle}_{\varphi_n(s)} dW_H(s)\right| \\
&\quad + \left|\int_0^t \underbrace{\|X(s)\|_{L^p}^{r-2}\langle g_n(s)-g(s),J(X_n(s))\rangle}_{\psi_n(s)} dW_H(s)\right| \\
&\quad + \left|\int_0^t \underbrace{\|X(s)\|_{L^p}^{r-2}\langle g(s),J(X_n(s))-J(X(s))\rangle}_{\rho_n(s)} dW_H(s)\right| \\
&= D_n(t)+E_n(t)+F_n(t).
\end{aligned}$$

Since the γ-radonifying norm and the operator norm are equal in finite dimensional spaces, By B.D.G inequality (Theorem 2.3) for $b=1$ and the Hölder inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T} D_n(t)\Big] \leq C\ \mathbb{E}\big\|\varphi_n\big\|_{\gamma(L^2(0,T;H),\mathbb{R})} = C\ \mathbb{E}\big\|\varphi_n\big\| \\
&= C\ \mathbb{E}\sup_{\|f\|\leq 1}\left(\int_0^T \big|\varphi_n(s)f(s)\big| ds\right) \\
&\leq C\mathbb{E}\sup_{\|f\|\leq 1}\left(\int_0^T \|\varphi_n(s)\|^2 ds\right)^{1/2}\|f\|_{L^2(0,T;H)} \\
&\leq C\ \mathbb{E}\left[\int_0^T \big|\|X_n(s)\|_{L^p}^{r-2}-\|X(s)\|_{L^p}^{r-2}\big|^2\big\|X_n(s)\big\|_{L^p}^2\big\|g_n(s)\big\|_{\gamma(H,L^p)}^2 ds\right]^{1/2} \\
&\leq C\Big[\mathbb{E}\|X_n\|_\infty^2\big(\|X_n\|_\infty^{r-2}+\|X\|_\infty^{r-2}\big)\Big]^{1/2}\times \\
&\qquad \left[\mathbb{E}\int_0^T \big|\|X_n(s)\|_{L^p}^{r-2}-\|X(s)\|_{L^p}^{r-2}\big|\|g(s)\|_{\gamma(H,L^p)}^2 ds\right]^{1/2} \\
&\leq C\Big[\mathbb{E}\|X_n\|_\infty^r + \big(\mathbb{E}\|X_n\|_\infty^r\big)^{\frac{2}{r}}\big(\mathbb{E}\|X\|_\infty^r\big)^{\frac{r-2}{r}}\Big]^{1/2}\times \\
&\quad \left[\mathbb{E}\int_0^T \big|\|X_n(s)\|_{L^p}^r-\|X(s)\|_{L^p}^r\big| ds\right]^{\frac{r-2}{2r}}\left[\mathbb{E}\int_0^T \|g(s)\|_{\gamma(H,L^p)}^r\right]^{\frac{1}{r}},
\end{aligned}$$

where C is the same constant as in (2.4). Since $\mathbb{E}\|X_n\|_\infty^r$, $\mathbb{E}\|X\|_\infty^r$ are bounded by a constant independent of n, Hypothesis 3.1(c) and Lemma 3.1 imply that the right hand side tends to zero along some subsequence. So, after choosing a subsequence if necessary, we get $\sup_{0\le t\le T} D_n(t) \longrightarrow 0$ in L^1. Similarly, one can see that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T} E_n(t)\Big] \quad &\le C\ \mathbb{E}\|\psi_n\|_{\gamma(L^2(0,T;H),\mathbb{R})} \le C\ \mathbb{E}\Big(\int_0^T \|\psi_n(s)\|^2 ds\Big)^{1/2} \\
&\le C\ \mathbb{E}\Big[\int_0^T \|X(s)\|_{L^p}^{2r-2}\|g_n(s)-g(s)\|^2_{\gamma(H,L^p)}ds\Big]^{1/2} \\
&\le C\ \mathbb{E}\Big[\|X\|_\infty^{r-1}\Big(\int_0^T \|g_n(s)-g(s)\|^2_{\gamma(H,L^p)}ds\Big)^{1/2}\ \Big] \\
&\le CT^{\frac{r-2}{2r}}\Big(\mathbb{E}\|X\|_\infty^r\Big)^{\frac{r-1}{r}}\Big(\mathbb{E}\int_0^T \|g_n(s)-g(s)\|^r_{\gamma(H,L^p)}\Big)^{\frac{1}{r}}.
\end{aligned}$$

But by the dominated convergence theorem,

$$\mathbb{E}\int_0^T \|g_n(s)-g(s)\|^r_{\gamma(H,L^p)} \longrightarrow 0.$$

Therefore, $\sup_{0\le t\le T} E_n(t) \longrightarrow 0$ in L^1. Also by Hölder's inequality, we find

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} F_n(t)\Big] \le C\mathbb{E}\Big(\int_0^T \|\rho_n(s)\|^2 ds\Big)^{1/2} \\
&\le C\mathbb{E}\Big[\int_0^T \|X(s)\|_{L^p}^{2r-4}\|J(X_n(s))-J(X(s))\|^2_{L^q}\|g(s)\|^2_{\gamma(H,L^p)}ds\Big]^{1/2} \\
&\le C\mathbb{E}\Big[\|X\|_\infty^{r-2}\Big\|\|J(X_n(\cdot))-J(X(\cdot))\|_{L^q}\Big\|_{L^2(0,T)}\Big\|\|g(\cdot)\|_{\gamma(H,L^p)}\Big\|_{L^2(0,T)}\Big] \\
&\le CK\mathbb{E}\Big[\|X\|_\infty^{r-2}\Big\|\|J(X_n(\cdot))-J(X(\cdot))\|_{L^q}\Big\|_{L^r(0,T)}\Big\|\|g(\cdot)\|_{\gamma(H,L^p)}\Big\|_{L^r(0,T)}\Big] \\
&\le CK\big(\mathbb{E}\|X\|_\infty^r\big)^{\frac{r-2}{r}}\times \\
&\quad \Big(\mathbb{E}\int_0^T \|J(X_n(s))-J(X(s))\|^r_{L^q}ds\Big)^{\frac{1}{r}}\Big(\mathbb{E}\int_0^T \|g(s)\|^r_{\gamma(H,L^p)}ds\Big)^{\frac{1}{r}},
\end{aligned}$$

where K is a constant. By Lemma 3.2, the right hand side approaches zero after choosing a subsequence. Hence, $\sup_{0\le t\le T} F_n(t) \longrightarrow 0$ in L^1 along some subsequence.

Step 3: By Hölder's inequality,

$$\begin{aligned}
&\sup_{0\le t\le T}\Big|\int_0^t \|X_n(s)\|_{L^p}^{r-2}\|g(s)\|^2_{\gamma(H,L^p)}ds - \int_0^t \|X(s)\|_{L^p}^{r-2}\|g(s)\|^2_{\gamma(H,L^p)}ds\Big| \\
&\le \int_0^T \Big|\|X_n(s)\|_{L^p}^{r-2} - \|X(s)\|_{L^p}^{r-2}\Big|\|g(s)\|^2_{\gamma(H,L^p)}ds
\end{aligned}$$

$$\leq \left[\int_0^T \Big| \|X_n(s)\|_{L^p}^r - \|X(s)\|_{L^p}^r \Big| ds \right]^{\frac{r-2}{r}} \left[\int_0^T \|g(s)\|_{\gamma(H,L^p)}^r ds \right]^{\frac{2}{r}}.$$

Lemma 3.1 implies that by passing to a subsequence, the right hand side tends to zero in L^1. □

4. Measurability of the solutions

In this section, we stablish the existence, uniqueness and measurability of the solution to the integral equation

$$X(t,\omega) = S(t-s)X_0(\omega) + \int_0^t S(t-s) f\big(s,\omega,X(s,\omega)\big) ds + V(t,\omega), \tag{4.1}$$

on $[0,T]$ with $X_0 : \Omega \longrightarrow L^p$. Suppose that $V : [0,T] \times \Omega \longrightarrow L^p$ satisfies the Carathéodory condition; i.e., $V(\cdot,\omega)$ is continuous on $[0,T]$ for each $\omega \in \Omega$ and $V(t,\cdot)$ is measurable on Ω into $(L^p, \mathcal{B})$ for all $t \in [0,T]$, where $\mathcal{B}$ denotes the Borel σ-field of subsets of L^p. Moreover, we assume that $V(0,\omega) = 0$ for all $\omega \in \Omega$. This equation appears in the next section when we use an iterative method to prove the existence of the mild solutions to semilinear SEE's. In fact, existence and measurability of the solution of (4.1) is necessary in each step of iteration. We proceed as in [15] and we use the method based on random fixed point theory.

We say that the mapping $h : [0,T] \times L^p \longrightarrow L^p$ is weakly closed as a Nemytskii operator, if whenever $x_n \rightharpoonup x$ weakly in $L^2(0,T;L^p)$ and $h(\cdot, x_n(\cdot)) \rightharpoonup \xi(\cdot)$ weakly in $L^2(0,T;L^p)$, then $\xi(\cdot) = h(\cdot, x(\cdot))$.

The following are the relevant hypotheses on nonlinear part f of (4.1).

Hypothesis 4.1. *(a) The function $f : [0,T] \times \Omega \times L^p \longrightarrow L^p$ is jointly measurable.*

(b) For each $\omega \in \Omega$, the mapping $(t,x) \longmapsto f(t,\omega,x)$ is weakly closed as a Nemytskii operator.

(c) There exists a nonnegative measurable function $M : \Omega \longrightarrow \mathbb{R}$ such that for each $t \in [0,T]$ and $\omega \in \Omega$, the function $x \longmapsto f(t,\omega,x)$ is semimonotone with parameter $M(\omega)$; i.e.,

$$\big\langle f(t,\omega,x) - f(t,\omega,y), J(x-y) \big\rangle \leq M(\omega) \|x-y\|_{L^p}^2.$$

(d) There exists a constant C such that $\|f(t,\omega,x)\|_{L^p} \leq C(1 + \|x\|_{L^p})$ for all $t \in [0,T]$, $\omega \in \Omega$ and $x \in L^p$.

We first consider (4.1) in finite dimensions. It is well-known that the space L^p has a Schauder basis (see, e.g., [9]); i.e., there exists a sequence $(a_n, x_n)_{n\geq 0}$ in $(L^p)^* \times L^p$ such that

$$x = \sum_{n=1}^{\infty} \langle x, a_n \rangle x_n \ ,$$

for all $x \in L^p$. Let $N \in \mathbb{N}$ and $E_N = \text{span}\{x_1, x_2, ..., x_N\}$. Then $\{E_N\}_{N=1}^{\infty}$ is an increasing sequence of finite dimensional subspaces of L^p such that $\bigcup_{N=1}^{\infty} E_N$ is dense in L^p. We recall that the natural projection $P_N : L^p \longrightarrow E_N$ is defined by

$$P_N(x) = \sum_{n=1}^{N} \langle x, a_n \rangle x_n \ .$$

Theorem 4.2. *If we substitute L^p by E_N, then under Hypothesis 4.1, the integral equation*

$$X_0 = 0, \ X(t,\omega) = \int_0^t f(s,\omega,X(s,\omega))ds \tag{4.2}$$

has a unique measurable solution.

Before proceeding in the proof, we recall two results. First we give the following simple but useful lamma, the proof of which is similar to that of Lemma 2 of [36].

Lemma 4.1. *If $a(\cdot)$ is an L^p-valued integrable function on $[0,T]$, $x \in L^p$ and $X(t) = x + \int_0^t a(s)ds$, then*

$$\|X(t)\|_{L^p}^2 = \|x\|_{L^p}^2 + 2\int_0^t \langle a(s), J(X(s))\rangle ds.$$

Theorem 4.3. [12] *Let K be a closed, convex and separable subset of a Banach space. Then any continuous compact random operator $h : \Omega \times K \longrightarrow K$ has a random fixed point.*

Proof of Theorem 4.2: Let

$$\mathcal{K} = \Big\{x \in C\big([0,T], E_N\big) \Big| \ \|x(t)\| \leq e^{Ct} - 1 \quad for \ \ all \ t \in [0,T]\Big\},$$

where C is the constant appeared in Hypothesis 4.1. Define h on $\Omega \times \mathcal{K} \longrightarrow E_N$ by

$$h(\omega, x)(t) = \int_0^t f(s,\omega,x(s))ds\cdot$$

Then $\mathcal{K}$ is the closed and convex subset of the separable Banach space $C\big([0,T], E_N\big)$. Hypothesis 4.1(d) shows that h is a map into $\mathcal{K}$ and by Hypothesis 4.1(a), for each $x \in \mathcal{K}$, $h(\cdot, x)$ is measurable. Now fix $\omega \in \Omega$. We show that $h(\omega, \cdot)$ is a continuous and compact operator on $\mathcal{K}$. Let $(x_n) \subseteq \mathcal{K}$ be a sequence strongly convergent to x; i.e.,

$$\sup_{0\le t\le T} \|x_n(t) - x(t)\|_{E_N} \longrightarrow 0.$$

Then, there exists $M > 0$ such that

$$\sup_{0\le t\le T} \|x_n(t)\| \le M \qquad for \;\; all \;\; n \in \mathbb{N}.$$

Consider an arbitrary subsequence of $\{x_n\}$ which we denote it by the same symbol $\{x_n\}$. From Hypothesis 4.1 it follows that $f(\cdot,\omega,x_n(\cdot))$ is a bounded sequence in $L^2(0,T;E_N)$ and so it has a subsequence $f(\cdot,\omega,x_{n_k}(\cdot))$ which is weakly convergent in $L^2(0,T;E_N)$. Therefore, by Hypothesis 4.1(b) $f(\cdot,\omega,x_{n_k}(\cdot)) \rightharpoonup f(\cdot,\omega,x(\cdot))$ weakly in $L^2(0,T;E_N)$. Hence, the whole sequence $f(\cdot,\omega,x_n(\cdot))$ is in fact weakly convergent to $f(\cdot,\omega,x(\cdot))$ in $L^2(0,T;E_N)$. For each $t \in [0,T]$, since $f(t,\omega,x_n(t)) \rightharpoonup f(t,\omega,x(t))$ weakly in E_N and E_N is finite dimensional, $f(t,\omega,x_n(t)) \longrightarrow f(t,\omega,x(t))$ strongly in E_N. Now, by Hypothesis 4.1(d) and the dominated convergence theorem, we have

$$\sup_{0\le t\le T} \big\|h(\omega,x_n)(t) - h(\omega,x)(t)\big\| \le \int_0^T \big\|f(s,\omega,x_n(s)) - f(s,\omega,x(s))\big\|_{E_N} ds,$$

the right-hand side of which goes to zero as $n \to \infty$. Thus, $h(\omega,\cdot)$ is continuous. To prove the compactness of h, we note first that for each $x \in \mathcal{K}$ and all $t \in [0,T]$,

$$\big\|h(\omega,x)(t)\big\| \le \int_0^t \big\|f(s,\omega,x(s))\big\| ds \le C\int_0^t \big(1 + \|x(s)\|\big) ds \le (e^{Ct} - 1).$$

Hence, $h(\omega,\cdot)$ is uniformly bounded. Moreover, for $0 \le t_1 < t_2 \le T$ and $x \in \mathcal{K}$ we have

$$\big\|h(\omega,x)(t_2) - h(\omega,x)(t_1)\big\| \le \int_{t_1}^{t_2} \big\|f(s,\omega,x(s))\big\| ds \le (t_2 - t_1)(e^{Ct_2} - e^{Ct_1}).$$

So, $h(\omega,\cdot)$ is an equicontinuous family on $[0,T]$. Therefore, $h(\omega,\cdot)$ is a compact operator. Now by Theorem 4.3, there exists a measurable function $\xi : \Omega \longrightarrow K$ such that

$$\xi(\omega)(t) = \int_0^t f(s,\omega,\xi(\omega)(s)) ds, \qquad \forall t \in [0,T]\cdot$$

According to Proposition 5.1 of [15], if we define $X : [0,T]\times\Omega \longrightarrow E_N$ by $X(t,\omega) = \xi(\omega)(t)$, then X is jointly measurable and X is a solution of problem (4.2). It remains to show the uniqueness of the solution. Let X and Y be two solutions of (4.2). We have

$$X(t,\omega) - Y(t,\omega) = \int_0^t \Big(f\big(s,\omega,X(s,\omega)\big) - f\big(s,\omega,Y(s,\omega)\big)\Big) ds.$$

By Lemma 4.1 and Hypothesis 4.1(c), for each $\omega \in \Omega$ we obtain

$$
\begin{aligned}
&\big\|X(t,\omega) - Y(t,\omega)\big\|^2 \\
&= 2\int_0^t \big\langle f(s,\omega,X(s,\omega)) - f(s,\omega,Y(s,\omega)), J(X(s,\omega) - Y(s,\omega))\big\rangle ds \\
&\leq 2M(\omega)\int_0^t \big\|X(s,\omega) - Y(s,\omega)\big\|^2 ds.
\end{aligned}
$$

Hence,

$$
\mathbb{E}\Big(\sup_{0\leq s\leq t}\Big\|X(s,\omega) - Y(s,\omega)\Big\|^2\Big) \leq 2M(\omega)\int_0^t \mathbb{E}\Big(\sup_{0\leq\theta\leq s}\Big\|X(\theta,\omega) - Y(\theta,\omega)\Big\|^2\Big) ds.
$$

Thus, by the Gronwall inequality

$$
\mathbb{E}\Big(\sup_{0\leq s\leq t}\big\|X(s,\omega) - Y(s,\omega)\big\|^2\Big) = 0 \qquad \forall t \in [0,T];
$$

that is, $X = Y$.

Theorem 4.4. *Assume that f satisfies Hypothesis 4.1. Then the equation*

$$
X(0) = 0, \quad X(t,\omega) = \int_0^t f(s,\omega,X(s,\omega))ds
$$

has a unique measurable solution.

Proof. The uniqueness follows as in the proof of previous theorem. Let $P_n : L^p \longrightarrow E_n$ be the natural projection of L^p onto E_n. By Theorem 4.2, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, the equation

$$
X_0 = 0, \quad X(t,\omega) = \int_0^t P_n f(s,\omega,X(s,\omega))ds
$$

has a unique measurable solution $X_n(t,\omega)$. Due to Lemma 4.1 and Hypothesis 4.1(c), we obtain

$$
\begin{aligned}
&\big\|X_n(t,\omega)\big\|_{L^p}^2 = 2\int_0^t \big\langle P_n f(s,\omega,X_n(s,\omega)), J(X_n(s,\omega))\big\rangle ds \\
&= 2\int_0^t \big\langle P_n f(s,\omega,X_n(s,\omega)) - P_n f(s,\omega,0), J(X_n(s,\omega))\big\rangle ds \\
&\quad + 2\int_0^t \big\langle P_n f(s,\omega,0), J(X_n(s,\omega))\big\rangle ds \\
&\leq 2M(\omega)\int_0^t \big\|X_n(s,\omega)\big\|_{L^p}^2 ds + 2\int_0^t \big\|f(s,\omega,0)\big\|_{L^p}\big\|X_n(s,\omega)\big\|_{L^p} ds \\
&\leq (2M(\omega)+1)\int_0^t \big\|X_n(s,\omega)\big\|_{L^p}^2 ds + \int_0^t \big\|f(s,\omega,0)\big\|_{L^p}^2 ds.
\end{aligned}
$$

So, by the Gronwall inequality,

$$\sup_{0\le t\le T} \big\|X_n(t,\omega)\big\|_{L^p} \le e^{(2M(\omega)+1)T} \int_0^T \big\|f(s,\omega,0)\big\|_{L^p}^2 ds \le Te^{(2M(\omega)+1)T}.$$

Now fix $\omega \in \Omega$. The above inequality shows that $\{X_n(\cdot,\omega)\}$ is a bounded sequence in $L^2(0,T;L^p)$. Also by Hypothesis 4.1(c), the sequence $f(\cdot,\omega,X_n(\cdot,\omega))$ is bounded in $L^2(0,T;L^p)$. Therefore, there exists a subsequence, again denoted by $\big(X_n(\cdot,\omega)\big)$, such that $\big(X_n(\cdot,\omega)\big)$ and $f(\cdot,\omega,X_n(\cdot,\omega))$ are both weakly convergent in $L^2(0,T;L^p)$. Let $X(\cdot,\omega)$ be the weak limit of $\big(X_n(\cdot,\omega)\big)$. Then, by Hypothesis 4.1(b), $f(\cdot,\omega,X(\cdot,\omega))$ is the weak limit of $f(\cdot,\omega,X_n(\cdot,\omega))$. Since L^p is a reflexive Banach space, by Theorem 3.2.13 of [1], (a_n,x_n) is a shrinking basis; that is, for each $v\in L^q$, $\|P_n^*v - v\| \longrightarrow 0$ as $n \longrightarrow 0$. So, we have

$$\begin{aligned}
&\big\langle P_n f(t,\omega,X_n(t,\omega)),v\big\rangle\\
&= \big\langle P_n f(t,\omega,X_n(t,\omega)) - f(t,\omega,X_n(t,\omega)),v\big\rangle + \big\langle f(t,\omega,X_n(t,\omega)),v\big\rangle\\
&= \big\langle f(t,\omega,X_n(t,\omega)),P_n^*v - v\big\rangle + \big\langle f(t,\omega,X_n(t,\omega)),v\big\rangle\\
&\ \longrightarrow \big\langle f(t,\omega,X(t,\omega)),v\big\rangle, \qquad as \quad n\to\infty.
\end{aligned}$$

Thus,

$$\big\langle X_n(t,\omega),v\big\rangle = \int_0^t \big\langle P_n f(t,\omega,X_n(s,\omega)),v\big\rangle ds \longrightarrow \Big\langle \int_0^t f(s,\omega,X(s,\omega))ds, v\Big\rangle,$$

and hence,

$$\big\langle X(t,\omega),v\big\rangle = \Big\langle \int_0^t f(s,\omega,X(s,\omega))ds, v\Big\rangle\cdot$$

Therefore,

$$X(t,\omega) = \int_0^t f(s,\omega,X(s,\omega))ds\cdot$$

It remains to show that $X(\cdot,\cdot)$ is measurable on $[0,T]\times\Omega$. For arbitrary $v\in L^q$, we see that

$$\int_0^t \big\langle X_n(s,\omega),v\big\rangle ds \longrightarrow \int_0^t \big\langle X(s,\omega),v\big\rangle ds,$$

and the function $(t,\omega) \longmapsto \displaystyle\int_0^t \big\langle X_n(s,\omega),v\big\rangle ds$ is measurable. Hence, the function $(t,\omega) \longmapsto \displaystyle\int_0^t \big\langle X(s,\omega),v\big\rangle ds$ is also measurable and $\big\langle X(\cdot,\omega),v\big\rangle$ is continuous. So, we can differentiate and obtain

$$\big\langle X(t,\omega),v\big\rangle = \frac{d}{dt}\int_0^t \big\langle X(s,\omega),v\big\rangle ds,$$

which shows that $\big\langle X(t,\omega),v\big\rangle$ is measurable in $(t,\omega)\in[0,T]\times\Omega$. By separability of L^p, this implies the measurability of $X(\cdot,\cdot)$ on $[0,T]\times\Omega$. $\square$

Now, we are ready to state and prove our main result in this section.

Theorem 4.5. *Assume that X_0 is an L^p-valued random variable and $\big(S(t)\big)$ is a C_0-semigroup on L^p satisfying an exponential growth condition with generator A. Let V satisfies the Carathéodory condition and $V(0,\omega) = 0$ for all $\omega \in \Omega$. Furthermore, let Hypothesis 4.1 holds. Then (4.1) has a unique measurable solution.*

Proof. One can easily see that it suffices to prove Theorem 4.5 in the case that $\lambda = 0$, $X_0 = 0$ and $V = 0$.
Uniqueness. Assume that X and Y are two solutions of (4.1) and fix any $\omega \in \Omega$. Then using the Itô type inequality (Theorem 3.2) with $g = 0$ and $r = 2$, and Hypothesis 4.1(c), we obtain

$$
\begin{aligned}
&\big\|X(t,\omega) - Y(t,\omega)\big\|^2_{L^p} \\
&\leq 2\int_0^t \big\langle f(s,\omega,X(s,\omega)) - f(s,\omega,Y(s,\omega)), J(X(s,\omega) - Y(s,\omega))\big\rangle ds \\
&\leq 2M(\omega)\int_0^t \big\|X(s,\omega) - Y(s,\omega)\big\|^2_{L^p} ds.
\end{aligned}
$$

So,

$$
\mathbb{E}\bigg(\sup_{0\leq s\leq t}\Big\|X(s,\omega) - Y(s,\omega)\Big\|^2_{L^p}\bigg) \leq 2M(\omega)\int_0^t \mathbb{E}\bigg(\sup_{0\leq\theta\leq s}\Big\|X(\theta,\omega) - Y(\theta,\omega)\Big\|^2_{L^p}\bigg)ds.
$$

Hence, by the Gronwall inequality, we conclude that $X = Y$.
Existence. Consider the Yosida approximations

$$
R_n := n(nI - A)^{-1} : L^p \longrightarrow D(A), \quad A_n := AR_n
$$

and $f_n(t,\omega,x) = A_n x + f(t,\omega,x)$. First, let us show that f_n satisfies Hypothesis 4.1. It is clear that f_n is jointly measurable. Moreover, A_n is continuous and so weakly closed. Hence f_n is weakly closed as a Nemytskii operator. Since L^p is a reflexive and strictly convex Banach space and A is maximal monotone, A_n is a monotone operator. Thus, for all $x, y \in L^p$ we have

$$
\begin{aligned}
\big\langle f_n(t,\omega,x) - f_n(t,\omega,y), J(x-y)\big\rangle &= \underbrace{\big\langle A_n x - A_n y, J(x-y)\big\rangle}_{\leq 0} \\
&\quad + \big\langle f(t,\omega,x) - f(t,\omega,y), J(x-y)\big\rangle \\
&\leq M(\omega)\|x-y\|^2_{L^p}
\end{aligned}
$$

So, $f_n(\omega)$ is semimonotone. Note also that $\|A_n\| \leq n$ and therefore,

$$
\big\|f_n(t,\omega,x)\big\|_{L^p} \leq n\|x\|_{L^p} + C(1+\|x\|_{L^p}) \leq (n+C)(1+\|x\|_{L^p}).
$$

Now, by Theorem 4.4, for each $n \in \mathbb{N}$ the integral equation

$$X_0 = 0, \quad X(t,\omega) = \int_0^t f_n(s,\omega,X(s,\omega))ds,$$

has a unique measurable solution $X_n(t,\omega)$. According to Lemma 4.1, Hypothesis 4.1(c) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\big\|X_n(t,\omega)\big\|_{L^p}^2 &= 2\int_0^t \underbrace{\big\langle A_n X_n(s,\omega), J(X_n(s,\omega))\big\rangle}_{\leq 0} ds \\
&\quad + 2\int_0^t \big\langle f(s,\omega,X_n(s,\omega)), J(X_n(s,\omega))\big\rangle ds \\
&\leq 2M(\omega)\int_0^t \Big\|X_n(s,\omega)\Big\|_{L^p}^2 ds + 2\int_0^t \Big\|f(s,\omega,0)\Big\|_{L^p} \Big\|X_n(s,\omega)\Big\|_{L^p} ds \\
&\leq (2M(\omega)+1)\int_0^t \big\|X_n(s,\omega)\big\|_{L^p}^2 ds + \int_0^t \big\|f(s,\omega,0)\big\|_{L^p}^2 ds.
\end{aligned}$$

Thus, by Gronwall's inequality

$$\sup_{0\leq t\leq T} \big\|X_n(t,\omega)\big\|_{L^p}^2 \leq e^{(2M(\omega)+1)T} \int_0^T \big\|f(s,\omega,0)\big\|_{L^p}^2 ds\cdot \tag{4.3}$$

On the other hand, A_n is bounded and generates the uniformly continuous contraction semigroup $\big(S_n(t)\big)$. We claim that for each $x \in L^p$, $S_n(t)x \longrightarrow S(t)x$. Since $D(A)$ is dense in L^p, it suffices to prove this for $x \in D(A)$. For $x \in D(A)$, we have

$$\begin{aligned}
S(t)x - S_n(t)x \qquad &= \int_0^t \frac{d}{d\theta}\big(S_n(t-\theta)S(\theta)x\big)d\theta \\
&= \int_0^t S_n(t-\theta)[A_n - A]S(\theta)xd\theta,
\end{aligned}$$

and

$$\big\|S_n(t-\theta)[A_n - A]S(\theta)x\big\| \leq \big\|(A_n - A)S(\theta)x\big\|.$$

Also,

$$(A_n - A)S(\theta)x = \big[(I - n^{-1}A)^{-1} - I\big]AS(\theta)x = (R_n - I)AS(\theta)x \longrightarrow 0,$$

$\theta \longmapsto AS(\theta)x$ is continuous and so bounded on $[0,T]$, and $\|R_n - I\| \leq 2$. Hence, by the dominated convergence theorem, $S_n(t)x \longrightarrow S(t)x$. We claim that

$$X_n(t,\omega) = \int_0^t S_n(t-s)f(s,\omega,X_n(s,\omega))ds\cdot \tag{4.4}$$

In fact, for any fixed $\omega \in \Omega$, by Theorem 2.38 of [7], the problem

$$Y(0,\omega) = 0, \quad \frac{dY}{dt} = A_n Y(t,\omega) + f(t,\omega,X_n(t,\omega)),$$

has a unique solution

$$Y(t,\omega) = \int_0^t S_n(t-s) f(s,\omega,X_n(s,\omega))ds.$$

That is,

$$Y(t,\omega) = \int_0^t A_n Y(s,\omega)ds + \int_0^t f(s,\omega,X_n(s,\omega))ds.$$

Thus,

$$X_n(t,\omega) - Y(t,\omega) = \int_0^t \big(A_n X_n(s,\omega) - A_n Y(s,\omega)\big)ds.$$

and so, by Lemma 4.1 and monotonicity of A_n,

$$\Big\|X_n(t,\omega) - Y(t,\omega)\Big\|_{L^p}^2 = 2\int_0^t \Big\langle A_n X_n(s,\omega) - A_n Y(s,\omega), J(X_n(s,\omega) - Y(s,\omega))\Big\rangle ds \le 0\cdot$$

Hence, $\sup_{0\le t\le T} \big\|X_n(t,\omega) - Y(t,\omega)\big\|_{L^p}^2 = 0$; i.e, $X_n = Y$ and we obtain (4.4). Now, we are going to use the method of the proof of Theorem 4.4. Fix $\omega \in \Omega$. From (4.3) and Hypothesis 4.1 (b) and (d), it follows that there exists a subsequence $\big(X_{n_k}(\cdot,\omega)\big)$ and an element $X(\cdot,\omega) \in L^2(0,T;L^p)$, such that $\big(X_{n_k}(\cdot,\omega)\big)$ and $f(\cdot,\omega,X_{n_k}(\cdot,\omega))$ are respectively weakly convergent to $X(\cdot,\omega)$ and $f(\cdot,\omega,X(\cdot,\omega))$ in $L^2(0,T;L^p)$. So, for each $v \in L^q$ we have as $n \to \infty$ that

$$\begin{aligned}
&\big\langle S_n(t-s) f(s,\omega,X_n(s,\omega)), v\big\rangle \\
&= \big\langle S_n(t-s) f(s,\omega,X_n(s,\omega)) - S(t-s) f(s,\omega,X_n(s,\omega)), v\big\rangle \\
&\quad + \big\langle S(t-s) f(s,\omega,X_n(s,\omega)), v\big\rangle \longrightarrow \big\langle S(t-s) f(s,\omega,X(s,\omega)), v\big\rangle
\end{aligned}$$

and thus,

$$\big\langle X_n(t,\omega), v\big\rangle = \int_0^t \big\langle S_n(t-s) f(s,\omega,X_n(s,\omega)), v\big\rangle ds$$

tends to $\int_0^t \big\langle S(t-s) f(s,\omega,X(s,\omega)), v\big\rangle ds$ as $n \to \infty$. Hence,

$$\big\langle X(t,\omega), v\big\rangle = \Big\langle \int_0^t S(t-s) f(s,\omega,X(s,\omega))ds, v\Big\rangle,$$

and therefore,

$$X(t,\omega) = \int_0^t S(t-s) f(s,\omega,X(s,\omega))ds\cdot$$

This finishes the proof of the existence.

Measurability. Similar to the proof of Theorem 4.4, one can see that $X(\cdot,\cdot)$ is measurable on $[0,T]\times\Omega$. $\square$

5. Existence and uniqueness of mild solutions

In this section, we use semigroup theory to make an iterative method in order to prove the existence and uniqueness of the mild solutions of monotone-type semilinear SEE's. The Itô type inequality (Theorem 3.2) is a key tool to study both existence and uniqueness. Consider the following semilinear stochastic evolution equation on L^p $(p \geq 2)$:

$$\begin{cases} dX(t) = AX(t)dt + f(t, X(t))dt + g(t, X(t))dW_H(t), \ t \in [0, T] \\ X(0) = X_0, \end{cases} \tag{5.1}$$

where the initial data X_0 is an L^p-valued $\mathcal{F}_0$-measurable random variable and $\mathbb{E}\|X_0\|^r_{L^p} < \infty$. Our hypotheses on A, g and nonlinear part f are as follows.

Hypothesis 5.1. *(a) $A : D(A) \subseteq L^p \longrightarrow L^p$ is the generator of a C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ of linear operators satisfying an exponential growth condition; i.e., there exists $\lambda \geq 0$ such that*

$$\|S(t)\| \leq e^{\lambda t} \qquad \forall t \geq 0\cdot$$

(b) f satisfies Hypothesis 4.1 with the constant M which is independent of $\omega \in \Omega$.

(c) $g : [0, T] \times \Omega \times L^p \longrightarrow \gamma(H, L^p)$ is a progressively measurable process such that for all $t \in [0, T]$, $\omega \in \Omega$ and $x, y \in L^p$

$$\big\|g(t, \omega, x) - g(t, \omega, y)\big\|_{\gamma(H,L^p)} \leq C\|x - y\|_{L^p},$$

where C is the constant appeared in Hypothesis 4.1(d). Moreover,

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \|g(s, 0)\|^r_{\gamma(H,L^p)}\Big) < \infty, \quad \forall t \in [0, T]\cdot$$

Definition 5.2. *An adapted process $X : [0, T] \times \Omega \longrightarrow L^p$ is called a mild solution of (5.1) if it satisfies the integral equation*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)f(s, X(s))ds + \int_0^t S(t-s)g(s, X(s))dW_H(s). \tag{5.2}$$

Theorem 5.3. *If Hypothesis 5.1 holds, then (5.1) has a unique continuous mild solution X such that*

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \|X(s)\|^r_{L^p}\Big) < \infty, \qquad r \geq 2, t \in [0, T]\cdot$$

Proof. One can easily see that it suffices to prove the theorem in the case that X_0 and λ are zero.

Uniqueness: Let $X(t)$ and $Y(t)$ be two continuous mild solutions of (5.1) with initial data $X(0) = Y(0) = 0$. Then we have

$$\begin{aligned} X(t) - Y(t) = &\int_0^t S(t-s)\big(f(s,X(s)) - f(s,Y(s))\big)ds \\ &+ \int_0^t S(t-s)\big(g(s,X(s)) - g(s,Y(s))\big)dW_H(s). \end{aligned}$$

We can apply Itô-type inequality (Theorem 3.2) with $r = 2$ and find that

$$\begin{aligned} \big\|X(t) - Y(t)\big\|_{L^p}^2 \leq\ & 2\int_0^t \big\langle f(s,X(s)) - f(s,Y(s)), J(X(s) - Y(s))\big\rangle ds \\ +\ & 2\int_0^t \big\langle g(s,X(s)) - g(s,Y(s)), J(X(s) - Y(s))\big\rangle dW_H(s) \\ +\ & \int_0^t \big\|g(s,X(s)) - g(s,Y(s))\big\|_{\gamma(H,L^p)}^2 ds. \end{aligned}$$

By Hypothesis 5.1 (b) and (c), we have

$$\int_0^t \Big\langle f(s,X(s)) - f(s,Y(s)), J(X(s) - Y(s))\Big\rangle ds \leq M\int_0^t \Big\|X(s) - Y(s)\Big\|_{L^p}^2 ds, \tag{5.3}$$

and

$$\int_0^t \Big\|g(s,X(s)) - g(s,Y(s))\Big\|_{\gamma(H,L^p)}^2 ds \leq C^2 \int_0^t \Big\|X(s) - Y(s)\Big\|_{L^p}^2 ds. \tag{5.4}$$

Also, using B.D.G inequality (Theorem2.3) with $b = 1$, Hypothesis 5.1(c) and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &\mathbb{E}\sup_{0\leq\rho\leq t}\left|\int_0^\rho \underbrace{\Big\langle g(s,X(s)) - g(s,Y(s)), J(X(s) - Y(s))\Big\rangle}_{\phi(s)} dW_H(s)\right| \\ &\leq C_1\mathbb{E}\Big\|\phi\Big\|_{\gamma(L^2(0,t;H),\mathbb{R})} = C_1\mathbb{E}\Big\|\phi\Big\| = C_1\sup_{\|f\|\leq 1}\Big[\phi, f\Big]_{L^2(0,t;H)} \\ &\leq C_2\mathbb{E}\left(\int_0^t \Big\|\phi(s)\Big\|^2 ds\right)^{1/2} \\ &\leq C_2\mathbb{E}\left[\sup_{0\leq s\leq t}\|X(s) - Y(s)\|_{L^p}\left(\int_0^t \|X(s) - Y(s)\|_{L^p}^2 ds\right)^{1/2}\right] \\ &\leq C_2\Big[\mathbb{E}\Big(\sup_{0\leq s\leq t}\|X(s) - Y(s)\|_{L^p}^2\Big)\Big]^{1/2}\Big[\mathbb{E}\int_0^t \|X(s) - Y(s)\|_{L^p}^2 ds\Big]^{1/2} \\ &\leq \frac{1}{4}\mathbb{E}\Big(\sup_{0\leq s\leq t}\|X(s) - Y(s)\|_{L^p}^2\Big) + 2{C_2}^2\,\mathbb{E}\int_0^t \|X(s) - Y(s)\|_{L^p}^2 ds. \end{aligned} \tag{5.5}$$

Here, C_1 is the constant appeared in inequality 2.4 and we have used the inequality $ab \leq \frac{1}{2}(\frac{1}{k}a^2 + kb^2)$ for any $a, b \in \mathbb{R}$ and any $k > 0$, with $k = 2C_2$. From (5.3), (5.4) and (5.5) we obtain

$$\frac{1}{2}\mathbb{E}\Big(\sup_{0\le s\le t}\|X(s)-Y(s)\|_{L^p}^2\Big)\le A\int_0^t \mathbb{E}\Big(\sup_{0\le\theta\le s}\|X(\theta)-Y(\theta)\|_{L^p}^2\Big)ds,$$

where $A=2M+C^2+4C_2^2$. Hence, by the Gronwall inequality

$$\mathbb{E}\Big(\sup_{0\le s\le t}\|X(s)-Y(s)\|_{L^p}^2\Big)=0 \quad \textit{for all } t\in[0,T].$$

So, $X=Y$ on $[0,T]$ almost surly.
Existence: Let $X_1(t)=0$ and define $X_n(t)$ by induction. Assume $X_n(t)$ is defined. Theorem 4.5 implies that there exists a continuous adapted solution X_{n+1} of

$$X_{n+1}(t)=\int_0^t S(t-s)f(s,X_{n+1}(s))ds+V_n(t),$$

where

$$V_n(t)=\int_0^t S(t-s)g(s,X_n(s))dW_H(s)\cdot$$

We claim that

$$\mathbb{E}\Big(\sup_{0\le s\le t}\|X_n(s)\|_{L^p}^r\Big)<\infty,\quad \forall n\in\mathbb{N},\ \forall t\in[0,T], \tag{5.6}$$

the proof of which is by induction on n. By Hypothesis 5.1(b),

$$\|X_{n+1}(t)\|_{L^p}^2\le 4C^2\int_0^t\big(1+\|X_{n+1}(s)\|_{L^p}^2\big)ds+2\|V_n(t)\|_{L^p}^2.$$

Hence,

$$\sup_{0\le s\le t}\|X_{n+1}(s)\|_{L^p}^2\le 4C^2t+4C^2\int_0^t\sup_{0\le\theta\le s}\|X_{n+1}(\theta)\|_{L^p}^2ds+2\sup_{0\le s\le t}\|V_n(s)\|_{L^p}^2.$$

So, by Gronwall's inequality we obtain

$$\sup_{0\le s\le t}\|X_{n+1}(s)\|_{L^p}^2\le\Big[4C^2t+2\sup_{0\le s\le t}\|V_n(s)\|_{L^p}^2\Big]e^{4C^2t},$$

and thus,

$$\sup_{0\le s\le t}\|X_{n+1}(s)\|_{L^p}^r\le 2^{r/2}\Big[(4C^2t)^{r/2}+2^{r/2}\sup_{0\le s\le t}\|V_n(s)\|_{L^p}^r\Big]e^{2rC^2t}.$$

Therefore, to get (5.6) it suffices to prove that

$$\mathbb{E}\sup_{0\le s\le t}\|V_n(s)\|_{L^p}^r<\infty\cdot$$

By Theorem 2.2 there exists a constant K such that

$$\mathbb{E}\sup_{0\le s\le t}\|V_n(s)\|_{L^p}^r \le K^r\mathbb{E}\bigg[\int_0^T \big\|g(t,X_n(t))\big\|_{\gamma(H,L^p)}^2 dt\bigg]^{r/2}.$$

By Hypothesis 5.1(c) and Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\|V_n(s)\|_{L^p}^r &\le K^r\mathbb{E}\bigg[2C\int_0^T \|X_n(t)\|_{L^p}^2 dt + 2\int_0^T \|g(t,0)\|_{\gamma(H,L^p)}^2 dt\bigg]^{r/2}\\ &\le 2^r K^r\bigg[C^{r/2}T\,\mathbb{E}\sup_{0\le t\le T}\|X_n(t)\|_{L^p}^r + \int_0^T \mathbb{E}\|g(t,0)\|_{\gamma(H,L^p)}^r dt\bigg],\end{aligned}$$

which is finite by induction. Next, we are going to prove the convergence of sequence $\{X_n\}$ to a mild solution of (5.1). Note that

$$\begin{aligned}X_{n+1}(t)-X_n(t) &= \int_0^t S(t-s)\big(f(s,X_{n+1}(s))-f(s,X_n(s))\big)ds\\ &+ \int_0^t S(t-s)\big(g(s,X_n(s))-g(s,X_{n-1}(s))\big)dW_H(s).\end{aligned}$$

Therefore, Itô type inequality (Theorem 3.2) implies that

$$\begin{aligned}&\|X_{n+1}(t)-X_n(t)\|_{L^p}^r \le\\ &r\int_0^t \|X_{n+1}(s)-X_n(s)\|_{L^p}^{r-2}\Big\langle f(s,X_{n+1}(s))-f(s,X_n(s)),J(X_{n+1}(s)-X_n(s))\Big\rangle ds\\ &+r\int_0^t \underbrace{\|X_{n+1}(s)-X_n(s)\|_{L^p}^{r-2}\Big\langle g(s,X_n(s))-g(s,X_{n-1}(s)),J(X_{n+1}(s)-X_n(s))\Big\rangle}_{\phi(s)} dW_H(s)\\ &+\frac{r(r-1)}{2}\int_0^t \|X_{n+1}(s)-X_n(s)\|_{L^p}^{r-2}\Big\|g(s,X_n(s))-g(s,X_{n-1}(s))\Big\|_{\gamma(H,L^p)}^2 ds\end{aligned}$$

$$= A_n(t)+B_n(t)+C_n(t). \tag{5.7}$$

Using Hypothesis 5.1(b) for the first term, $A_n(t)$, we find

$$A_n(t)\le rM\int_0^t \|X_{n+1}(s)-X_n(s)\|_{L^p}^r ds\cdot \tag{5.8}$$

Moreover, using Theorem 2.2 for the second term, $B_n(t)$, yields us

$$\mathbb{E}\sup_{0\le\theta\le t}|B_n(\theta)|\le rD\mathbb{E}\|\phi\|_{\gamma(L^2(0,t;H),\mathbb{R})},$$

in which D is a constant. By an argument similar to that of the proof of Theorem 3.2 (Step 2), one can see that

$$\mathbb{E}\sup_{0\le\theta\le t}|B_n(\theta)|\le rD\mathbb{E}\bigg(\int_0^t \|\phi(s)\|^2 ds\bigg)^{1/2}.$$

From Hypothesis 5.1(c), we obtain that the right hand side is

$$
\begin{aligned}
&\le rD\mathbb{E}\Big[\int_0^t \big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{2r-2}\big\|X_n(s)-X_{n-1}(s)\big\|_{L^p}^2 ds\Big]^{1/2}\\
&\le rD\mathbb{E}\Big[\sup_{0\le s\le t}\|X_{n+1}(s)-X_n(s)\|_{L^p}^{r/2}\times\\
&\qquad\Big(\int_0^t \|X_{n+1}(s)-X_n(s)\|_{L^p}^{r-2}\|X_n(s)-X_{n-1}(s)\|_{L^p}^2 ds\Big)^{1/2}\Big]
\end{aligned}
$$

Using the elementary inequality $ab \le \frac{1}{2}(k^{-1}a^2+kb^2)$ which is true for any $a, b \in \mathbb{R}$ and $k > 0$ with $k = rD$, we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le\theta\le t}|B_n(\theta)| \le \frac{1}{2}\,\mathbb{E}\sup_{0\le s\le t}\big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^r\\
+&\quad\frac{r^2D^2}{2}\,\mathbb{E}\int_0^t \big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^{r-2}\big\|X_n(s)-X_{n-1}(s)\big\|_{L^p}^2 ds.
\end{aligned}
$$

Applying the inequality $u^{1-\alpha}v^{\alpha} \le (1-\alpha)u + \alpha v$ which holds for all $u, v \ge 0$ and $0 \le \alpha \le 1$, we deduce that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le\theta\le t}|B_n(\theta)| &\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le s\le t}\big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^r\Big)\\
&+\frac{r(r-2)D^2}{2}\int_0^t\mathbb{E}\Big(\sup_{0\le\theta\le s}\big\|X_{n+1}(\theta)-X_n(\theta)\big\|_{L^p}^r\Big)ds\\
&+ rD^2\int_0^t\mathbb{E}\Big(\sup_{0\le\theta\le s}\big\|X_n(\theta)-X_{n-1}(\theta)\big\|_{L^p}^r\Big)ds. \qquad (5.9)
\end{aligned}
$$

Similarly, by Hypothesis 5.1(c) one can show that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le\theta\le t}|C_n(\theta)| &\le \frac{(r-1)(r-2)}{2}\,C^2\int_0^t\mathbb{E}\Big(\sup_{0\le\theta\le s}\Big\|X_{n+1}(\theta)-X_n(\theta)\Big\|_{L^p}^r\Big)ds\\
&+(r-1)C^2\int_0^t\mathbb{E}\Big(\sup_{0\le\theta\le s}\Big\|X_n(\theta)-X_{n-1}(\theta)\Big\|_{L^p}^r\Big)ds. \qquad (5.10)
\end{aligned}
$$

Now, we define

$$h_n(t) = \mathbb{E}\Big(\sup_{0\le s\le t}\big\|X_{n+1}(s)-X_n(s)\big\|_{L^p}^r\Big), \quad t\in[0,T].$$

Note that $h_n(t) < \infty$ for all $t \in [0, T]$ and hence by substituting (5.8), (5.9) and (5.10) in the right hand side of (5.7), we obtain

$$h_n(t) \le \alpha\int_0^t h_n(s)ds + \beta\int_0^t h_{n-1}(s)ds,$$

where

$$\alpha = 2rM + r(r-2)D^2 + (r-1)(r-2)C^2,$$

and

$$\beta = 2rD^2 + 2(r-1)C^2.$$

Therefore, by Gronwall's inequality

$$h_n(t) \leq \beta e^{\alpha t} \int_0^t h_{n-1}(s) ds\cdot$$

We know $h_0(t) \leq h_0(T) = \mathbb{E} \sup_{0\leq s\leq T} \|X_1(s)\|_{L^p}^r < \infty$. Thus, if $\gamma = h_0(T)$ it follows inductively that

$$h_n(t) \leq \gamma \ \frac{(\beta e^{\alpha T} t)^n}{n!}, \quad n \geq 1.$$

Hence, $\{X_n\}$ is a Cauchy sequence in $L^r(\Omega, C(0,T; L^p))$ and so there exists a continuous adapted process $X(t)$ with

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} \big\|X(s)\big\|_{L^p}^r \Big) < \infty,$$

such that $\mathbb{E}\Big(\sup_{0\leq s\leq t} \big\|X_n(s) - X(s)\big\|_{L^p}^r \Big) \longrightarrow 0$. To complete the proof, we show that $X(t)$ is the mild solution of (5.1). Consider

$$R(t) = X(t) - \int_0^t S(t-s) f(s, X(s)) ds - \int_0^t S(t-s) g(s, X(s)) dW_H(s),$$

and

$$R_n(t) = X_{n+1}(t) - \int_0^t S(t-s) f(s, X_{n+1}(s)) ds - \int_0^t S(t-s) g(s, X_n(s)) dW_H(s).$$

We know that $R_n(t) = 0$ for all $t \in [0,T]$. Let $x \in L^q$ and $t \in [0,T]$. We show that $\langle R(t), x\rangle = 0$ *a.s.*, which implies $R(t) = 0$ *a.s.* Then letting t ranges over all rational numbers and using continuity of R, it follows that $R(t) = 0$ for all $t \in [0,T]$ w.p.1. First, by passing to a subsequence if necessary, we may assume that

$$\sup_{0\leq s\leq T} \big\|X_n(s) - X(s)\big\|_{L^p} \longrightarrow 0, \quad a.s.$$

Consequently

(5.11) $$\langle X_{n+1}(t), x\rangle \longrightarrow \langle X(t), x\rangle, \quad a.s.$$

Now, since

$$\int_0^T \|X_{n+1}(s) - X(s)\|_{L^p}^2 ds \leq T \sup_{0\leq s\leq T} \|X_{n+1}(s) - X_n(s)\|_{L^p}^2 \longrightarrow 0, \quad a.s.$$

we have

$$X_{n+1} \longrightarrow X \qquad in \ \ L^2(0,T; L^p) \quad a.s.$$

Moreover, by Hypothesis 5.1(b),

$$\begin{aligned}\int_0^T \big\|f(s,X_{n+1}(s))\big\|_{L^p}^2 ds &\leq TC\big(1+\sup_{0\leq s\leq T}\|X_{n+1}(s)\|_{L^p}\big)^2\\ &\leq TC\big(2+\sup_{0\leq s\leq T}\|X(s)\|_{L^p}\big)^2<\infty,\end{aligned}$$

for large enough n. This shows that $f(\cdot,X_{n+1}(\cdot))$ is a bounded sequence in $L^2(0,T;L^p)$. So, by passing to a subsequence, we may assume that $f(\cdot,X_{n+1}(\cdot))$ is weakly convergence in $L^2(0,T;L^p)$. Hence, it follows from weakly closedness of f (Hypothesis 5.1(b)) that

$$f\big(\cdot,X_{n+1}(\cdot)\big)\rightharpoonup f(\cdot,X(\cdot))$$

weakly in $L^2(0,T;L^p)$. Therefore,

$$\int_0^T \big\langle f(s,X_{n+1}(s))-f(s,X(s)),v(s)\big\rangle \longrightarrow 0,$$

for all $v\in L^2(0,T;L^p)$. Thus,

$$\begin{aligned}&\int_0^t \Big\langle S(t-s)\Big(f(s,X_{n+1}(s))-f(s,X(s))\Big),x\Big\rangle ds\\ &=\int_0^T \Big\langle f(s,X_{n+1}(s))-f(s,X(s)),S^*(t-s)x1_{[0,t]}(s)\Big\rangle ds\longrightarrow 0.\end{aligned}\tag{5.12}$$

At last, by Theorem 2.2 and Hypothesis 5.1(c), there exist constants K and C such that

$$\begin{aligned}&\mathbb{E}\Big\|\int_0^t S(t-s)\big(g(s,X_n(s))-g(s,X(s))\big)dW_H(s)\Big\|_{L^p}^r\\ &\leq K\mathbb{E}\Big(\int_0^T \big\|g(s,X_n(s))-g(s,X(s))\big\|_{\gamma(H,L^p)}^2\Big)^{r/2}\\ &\leq KC\ \mathbb{E}\Big(\sup_{0\leq s\leq T}\|X_n(s)-X(s)\|_{L^p}^r\Big)\longrightarrow 0.\end{aligned}$$

Consequently, after choosing a subsequence, we have

$$\Big\langle\int_0^t S(t-s)\big(g(s,X_n(s))-g(s,X(s))\big)dW_H(s),x\Big\rangle\longrightarrow 0.\tag{5.13}$$

From (5.11) , (5.12) and (5.13), we get that

$$\langle R(t),x\rangle=\lim_{n\to\infty}\langle R_n(t),x\rangle=0\ \cdot$$

The proof is now complete. □

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ON CAPABLE GROUPS OF ORDER p^4 *

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Abstract. A group H is said to be capable, if there exists another group G such that $\frac{G}{Z(G)} \cong H$, where $Z(G)$ denotes the center of G. In a recent paper [5], the authors considered the problem of capability of five non-abelian $p-$groups of order p^4 into account. In this paper, we try to solve the problem of capability by considering three other groups of order p^4. It is proved that the group

$$H_6 = \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = xyz, yz = zy\rangle$$

is not capable. Moreover, if $p > 3$ is a prime number and $d \not\equiv 0, 1 \ (mod\ p)$ then the following groups are not capable:

$$\begin{aligned}
H_7^1 &= \langle x, y, z \mid x^9 = y^3 = 1, z^3 = x^3, yx = x^4y, zx = xyz, zy = yz\rangle,\\
H_7^2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = x^{p+1}yz, zy = x^p yz\rangle,\\
H_8^1 &= \langle x, y, z \mid x^9 = y^3 = 1, z^3 = x^{-3}, yx = x^4y, zx = xyz, zy = yz\rangle,\\
H_8^2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yx = x^{p+1}y, zx = x^{dp+1}yz, zy = x^{dp}yz\rangle.
\end{aligned}$$

Keywords: Capable group; $p-$group; non-abelian $p-$groups; center.

1. Introduction

A group H is said to be capable if there exists another group G such that $\frac{G}{Z(G)} \cong H$, or equivalently H can be represented as the inner automorphism group of a given group G. The capability of groups was first studied by Baer [1] who was asked the question "which conditions a group H must fulfill in order to be the group of inner automorphisms of a group G?". In the mentioned paper, he determined all capable groups which are direct products of cyclic groups. Since the time that

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Hall and Senior published their inovating work [3], such groups are called capable. It is well-known that the classification of capable groups is the first step towards the classification of prime power order groups [4]. The following theorem of Baer is well-known in the context of capable groups.

Theorem 1.1. *Let A be a finite abelian group written as $A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ such that each integer n_{i+1} is divisible by n_i. Then A is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.*

Burnside [2] was classified all $p-$groups of order p^4 which p is an odd prime number. This classification is expressed in the following theorem:

Theorem 1.2. *Suppose p is an odd prime number and $d \not\equiv 0, 1$ (mod p). Then there are fifteen different groups of order p^4 up to isomorphisms. Five of those are abelian and the non-abelian groups are in the list below.*

$$
\begin{aligned}
H_1 &= \langle x, y \mid x^{p^3} = y^p = 1, yxy^{-1} = x^{p^2+1} \rangle, \\
H_2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, xy = yx, xz = zx, zyz^{-1} = x^p y \rangle, \\
H_3 &= \langle x, y \mid x^{p^2} = y^{p^2} = 1, yxy^{-1} = x^{p+1} \rangle, \\
H_4 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, xy = yx, yz = zy, zxz^{-1} = x^{p+1} \rangle, \\
H_5 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, xy = yx, yz = zy, zxz^{-1} = xy \rangle, \\
H_6 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yxy^{-1} = x^{p+1}, zxz^{-1} = xy, yz = zy \rangle, \\
H_7^1 &= \langle x, y, z \mid x^9 = y^3 = 1, [y, z] = 1, z^3 = x^3, y^{-1}xy = x^4, z^{-1}xz = xy^{-1} \rangle, \\
H_7^2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yxy^{-1} = x^{p+1}, zxz^{-1} = x^{p+1}y, zyz^{-1} = x^p y \rangle \qquad p > 3, \\
H_8^1 &= \langle x, y, z \mid x^9 = y^3 = 1, [y, z] = 1, z^3 = x^{-3}, y^{-1}xy = x^4, z^{-1}xz = xy^{-1} \rangle, \\
H_8^2 &= \langle x, y, z \mid x^{p^2} = y^p = z^p = 1, yxy^{-1} = x^{p+1}, zxz^{-1} = x^{dp+1}y, zyz^{-1} = x^{dp} y \rangle \qquad p > 3, \\
H_9 &= \langle x, y, z, t \mid x^p = y^p = z^p = t^p = [x, y] = [x, z] = [x, t] = [y, z] = [y, t] = 1, tzt^{-1} = xz \rangle, \\
H_{10}^1 &= \langle x, y, z \mid x^9 = y^3 = z^3 = 1, xy = yx, z^{-1}xz = xy, z^{-1}yz = x^{-3}y \rangle, \\
H_{10}^2 &= \langle x, y, z, t \mid x^p = y^p = z^p = t^p = [x, y] = [x, z] = [x, t] = [y, z] = [t, y]x^{-1} = [t, z]y^{-1} = 1 \rangle \qquad p > 3.
\end{aligned}
$$

Zainal et al. [5] examined the capability of five groups out of ten non-abelian groups of order p^4 and proved that among first five groups the previous theorem, only the group number 3 is capable. We record this result in the following theorem:

Theorem 1.3. (*See [5]*) *The groups H_i, $1 \leq i \leq 5$, is capable if and only if $i = 3$.*

2. Main Results

Our aim in this section is to prove the groups numbers $6, 7$ and 8 in Theorem 1.2 are not capable.

Theorem 2.1. *The group H_6 is not capable.*

Proof. By definition of H_6 and some calculations we have the following equations,

$$y^j x^i = x^{ijp+i} y^j \tag{2.1}$$

$$z^k x^i = x^{\frac{i(i-1)}{2}kp+i} y^{ik} z^k \tag{2.2}$$

We put $i = p$ and $j = k = 1$ in Equations 2.1 and 2.2. Since p is odd and $x^{p^2} = y^p = 1$, $yx^p = x^p y$ and $zx^p = x^p z$. Thus $\langle x^p \rangle \leq Z(H_6)$ and $|Z(H_6)| = p$ or p^2. Suppose $|Z(H_6)| = p^2$. Then for every $h \in H_6 \setminus Z(H_6)$, $Z(H_6)\langle C_{H_6}(h) \rangle \leq H_6$ and so $|C_{H_6}(h)| = p^3$. This proves that the conjugacy class h^{H_6} has size p. Choose j, k with this condition that $0 \leq j, k \leq p-1$. Since x is not central and by Equations 2.1 and 2.2, $y^j x y^{-j} = x^{jp+1}$ and $z^k x z^{-k} = xy^k$, we find that $|x^{H_6}| > p$ which is not possible. Therefore $|Z(H_6)| = p$ and $Z(H_6) = \langle x^p \rangle$.

If H_6 is capable then there exists a non-abelian group G with center Z such that $H_6 \cong \frac{G}{Z}$. Since G is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^{p^2} = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \\ (cZ)(aZ) = (aZ)(bZ)(cZ), (bZ)(cZ) = (cZ)(bZ) \end{array} \right\rangle.$$

By definition, $a^{p^2}, b^p, c^p \in Z$ and by Equation 2.1 one can see the following equation:

$$ba^p = a^p b. \tag{2.3}$$

By Equation 2.2 and some calculations, we have:

$$(aZcZ)^n = (aZ)^{t_n p}(aZ)^n (bZ)^{\frac{n(n-1)}{2}} (cZ)^n \tag{2.4}$$

in which $t_n = \frac{n(n-1)(n-2)}{6}$. By substituting $n = p$ in Equation 2.4, we obtain the following equality:

$$(aZcZ)^p = (aZ)^{t_p p}(aZ)^p. \tag{2.5}$$

We now consider two cases that $p = 3$ or $p > 3$.

1. $p > 3$. Then $p \mid t_p$ and so by Equation 2.5 and this fact that $a^{p^2} \in Z$,

$$\begin{aligned} (ac)^p Z &= (acZ)^p \\ &= (aZcZ)^p \\ &= (aZ)^{t_p p}(aZ)^p \\ &= (aZ)^p \\ &= a^p Z. \end{aligned}$$

Hence there exists $z \in Z$ such that $(ac)^p = a^p z$ and so $ca^p = a^p c$. Finally, we apply Equation 2.3 to conclude that $a^p \in Z$ which is a contradiction.

2. $p = 3$. Then $t_p = 1$ and by Equation 2.5, $(ac)^3Z = (aZcZ)^3 = (aZ)^3(aZ)^3 = (aZ)^6 = a^6Z$. Hence there exists $z \in Z$ such that $(ac)^3 = a^6z$ and so $ca^6 = a^6c$. By these equations and and Equation 2.3, we conclude that $a^6 \in Z$ which is our final contradiction.

Therefore, the group H_6 is not capable. $\square$

Theorem 2.2. *The group H_7^1 is not capable.*

Proof. By definition of H_7^1 and some tedious calculations, one can see that

$$y^jx^i = x^{3ij+i}y^j \tag{2.6}$$

$$z^kx^i = x^{3k\frac{i(i-1)}{2}+i}y^{ik}z^k \tag{2.7}$$

We put $i = 3$ and $j = k = 1$ in Equations 2.6 and 2.7. Since $x^9 = y^3 = 1$, $yx^3 = x^3y$ and $zx^3 = x^3z$ and so $\langle x^3\rangle \leq Z(H_7^1)$. On the other hand, $|H_7^1| = 3^4$ and hence $|Z(H_7^1)| = 3$ or 9. Suppose $|Z(H_7^1)| = 9$. Then for every $h \in H_7^1 \setminus Z(H_7^1)$, $Z(H_7^1)\langle C_{H_7^1}(h)\rangle \leq H_7^1$ which implies that $|C_{H_7^1}(h)| = 3^3$ or equivalently $|h^{H_7^1}| = 3$. Note that $x \in H_7^1 \setminus Z(H_7^1)$. Choose j, k such that $0 \leq j, k \leq 2$. By Equations 2.6 and 2.7, $y^jxy^{-j} = x^{3j+1}$ and $z^kxz^{-k} = xy^k$ which shows that $|x^{H_7^1}| > 3$. This contradiction implies that $|Z(H_7^1)| = 3$ and $Z(H_7^1) = \langle x^3\rangle$. If H_7^1 is capable, there is a non-abelian group G with center Z such that $H_7^1 \cong \frac{G}{Z}$. Since G is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^9 = (bZ)^3 = 1, (cZ)^3 = (aZ)^3, (bZ)(aZ) = (aZ)^4(bZ), \\ (cZ)(aZ) = (aZ)(bZ)(cZ), (cZ)(bZ) = (bZ)(cZ) \end{array} \right\rangle.$$

Obviously $a^9, b^3, c^9 \in Z$ and by Equation 2.6,

$$(aZbZ)^n = (aZ)^{3\frac{n(n-1)}{2}}(aZ)^n(bZ)^n.$$

In above equation, we put $n = 3$. Since $a^9, b^3 \in Z$, $(ab)^3Z = (abZ)^3 = (aZbZ)^3 = (aZ)^9(aZ)^3(bZ)^3 = (aZ)^3 = a^3Z$ and so there exists $z \in Z$ such that $(ab)^3 = a^3z$. Therefore,

$$ba^3 = a^3b \tag{2.8}$$

On the other hand, $a^3Z = c^3Z$ and so there exists $z_1 \in Z$ such that

$$a^3 = c^3z_1 \tag{2.9}$$

Put $k = 1$ and $i = 3$ in Equation 2.7. Since $o(aZ) = 9$ and $o(bZ) = 3$,

$$\begin{aligned} ca^3Z &= (cZ)(aZ)^3 \\ &= (aZ)^9(aZ)^3(bZ)^3(cZ) \\ &= (aZ)^3(cZ) \\ &= a^3cZ. \end{aligned}$$

Thus there exists $z_2 \in Z$ such that

$$ca^3 = a^3 c z_2. \tag{2.10}$$

Now by inserting the Equation 2.9 in 2.10, $cc^3 z_1 = c^3 z_1 c z_2$ which shows that $z_2 = 1$. Apply again Equation 2.10 to conclude that $ca^3 = a^3 c$. Now by Equation 2.8 $a^3 \in Z$ and hence $(aZ)^3 = Z$ which is our final contradiction. $\square$

Theorem 2.3. *The group H_7^2 is not capable.*

Proof. By presentation of H_7^2 and some tedious calculations one can see that

$$\begin{aligned} y^j x^i &= x^{ijp+i} y^j, & (2.11)\\ z^k x^i &= x^{\frac{i(i+1)}{2}kp + \frac{k(k-1)}{2}ip + i} y^{ik} z^k, & (2.12)\\ z^k y^j &= x^{jkp} y^j z^k. \end{aligned}$$

By substituting $i = p$ and $j = k = 1$ in Equations 2.11 and 2.12 we have $yx^p = x^p y$ and $zx^p = x^p z$. Hence $\langle x^p \rangle \leq Z(H_7^2)$ and arguments similar to the proof of Theorem 2.1 show that $Z(H_7^2) = \langle x^p \rangle$. If H_7^2 is capable, there is a non-abelian group G with center Z such that and $H_7^2 \cong \frac{G}{Z}$. Since G is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^{p^2} = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \\ (cZ)(aZ) = (aZ)^{p+1}(bZ)(cZ), (cZ)(bZ) = (aZ)^p (bZ)(cZ) \end{array} \right\rangle.$$

Thus $a^{p^2}, b^p, c^p \in Z$. Now by Equation 2.11 and a similar argument as Theorem 2.1,

$$ba^p = a^p b. \tag{2.13}$$

Apply Equation 2.12 to conclude that

$$(aZcZ)^n = (aZ)^{k_n p}(aZ)^n (bZ)^{\frac{n(n-1)}{2}} (cZ)^n$$

in which $k_n = \frac{n(n-1)(2n-1)}{6}$. Next we assume that $n = p$. Since b^p, c^p are central,

$$\begin{aligned} (ac)^p Z &= (acZ)^p = (aZcZ)^p \\ &= (aZ)^{k_p p}(aZ)^p (bZ)^{\frac{p(p-1)}{2}} (cZ)^p \\ &= (aZ)^{(k_p+1)p} = a^{(k_p+1)p} Z. \end{aligned}$$

Hence there exists $z \in Z$ such that

$$(ac)^p = a^{(k_p+1)p} z. \tag{2.14}$$

It is clear that $p \mid 6k_p$. Since $p > 3$, $p \mid k_p$ and so $p \nmid k_p + 1$. Since $(ac)^p(ac) = (ac)(ac)^p$, Equation 2.14 implies that $ca^{(k_p+1)p} = a^{(k_p+1)p} c$ and by Equation 2.13, $a^{(k_p+1)p} \in Z$. So, $(aZ)^{(k_p+1)p} = Z$. But $o(aZ) = p^2$ and hence $p^2 \mid (k_p + 1)p$ which implies that $p \mid k_p + 1$. This contradiction completes the proof. $\square$

Theorem 2.4. *The group H_8^1 is not capable.*

Proof. By presentation of H_8^1 we have:

$$y^j x^i = x^{3ij+i} y^j, \tag{2.15}$$

$$z^k x^i = x^{3k\frac{i(i-1)}{2}+i} y^{ik} z^k. \tag{2.16}$$

Again substitute $i = 3$ and $j = k = 1$ in Equations 2.15 and 2.16. Since $x^9 = y^3 = 1$, $yx^3 = x^3y$ and $zx^3 = x^3z$. Thus $\langle x^3\rangle \leq Z(H_8^1)$. Similar to the proof of Theorem 2.2, $Z(H_8^1) = \langle x^3\rangle$. If H_8^1 is capable, there is a non-abelian group G with center Z such that $H_8^1 \cong \frac{G}{Z}$. Since G is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{c} aZ, bZ, cZ \mid (aZ)^9 = (bZ)^3 = 1, (cZ)^3 = (aZ)^{-3}, (bZ)(aZ) = (aZ)^4(bZ), \\ (cZ)(aZ) = (aZ)(bZ)(cZ), (cZ)(bZ) = (bZ)(cZ) \end{array} \right\rangle.$$

Obviously, $a^9, b^3, c^9 \in Z$ and by Equation 2.15,

$$(aZbZ)^n = (aZ)^{3\frac{n(n-1)}{2}}(aZ)^n(bZ)^n.$$

Put $n = 3$. Since $a^9, b^3 \in Z$,

$$(ab)^3Z = (abZ)^3 = (aZbZ)^3 = (aZ)^9(aZ)^3(bZ)^3 = (aZ)^3 = a^3Z.$$

Hence there exists $z \in Z$ such that $(ab)^3 = a^3z$ and so

$$ba^3 = a^3b. \tag{2.17}$$

On the other hand, $c^3Z = a^{-3}Z$ and so there exists $z_1 \in Z$ such that

$$a^3 = c^{-3}z_1. \tag{2.18}$$

Since $o(aZ) = 9$ and $o(bZ) = 3$, by Equation 2.16 and substituting $k = 1$ and $i = 3$, we can see that

$$\begin{aligned} ca^3Z &= (cZ)(aZ)^3 \\ &= (aZ)^9(aZ)^3(bZ)^3(cZ) \\ &= (aZ)^3(cZ) = a^3cZ \end{aligned}$$

and so there exists $z_2 \in Z$ such that

$$ca^3 = a^3cz_2. \tag{2.19}$$

We now insert Equation 2.18 in our last equation to deduce that $cc^{-3}z_1 = c^{-3}z_1cz_2$. Thus $z_2 = 1$ and by Equation 2.19, $ca^3 = a^3c$. Therefore, $a^3 \in Z$ and hence $9 = o(aZ) \mid 3$, which is impossible. This completes the proof. $\square$

Theorem 2.5. *The group H_8^2 is not capable.*

Proof. By presentation of H_8^2 and some tedious calculations, we have

$$y^j x^i = x^{ijp+i} y^j, \tag{2.20}$$

$$\begin{aligned} z^k x^i &= x^{\frac{i(i-1)}{2}kp+\frac{k(k+1)}{2}idp+i} y^{ik} z^k, \\ z^k y^j &= x^{jkdp} y^j z^k. \end{aligned} \tag{2.21}$$

In Equations 2.20 and 2.21, we insert $i = p$ and $j = k = 1$. It is clear that $yx^p = x^p y$ and $zx^p = x^p z$ and so $\langle x^p \rangle \leq Z(H_8^2)$. Similar to Theorem 2.1, we can see that $Z(H_8^2) = \langle x^p \rangle$. If H_8^2 is capable, there is a non-abelian group G with center Z such that $H_8^2 \cong \frac{G}{Z}$. Since G is not centerless, there are elements $a, b, c \in G \setminus Z$ such that

$$\frac{G}{Z} = \left\langle \begin{array}{r} aZ, bZ, cZ \mid (aZ)^{p^2} = (bZ)^p = (cZ)^p = 1, (bZ)(aZ) = (aZ)^{p+1}(bZ), \\ (cZ)(aZ) = (aZ)^{dp+1}(bZ)(cZ), (cZ)(bZ) = (aZ)^{dp}(bZ)(cZ) \end{array} \right\rangle,$$

where $d \not\equiv 0, 1 (mod\, p)$. It is obvious that $a^{p^2}, b^p, c^p \in Z$ and by Equations 2.20 and a similar argument used in the proof of the Theorem 2.1,

$$ba^p = a^p b. \tag{2.22}$$

Moreover, by Equation 2.21,

$$(aZcZ)^n = (aZ)^{s_n dp}(aZ)^{t_n p}(aZ)^n (bZ)^{\frac{n(n-1)}{2}}(cZ)^n \tag{2.23}$$

in which $s_n = \frac{n(n-1)(n+1)}{6}$ and $t_n = \frac{n(n-1)(n-2)}{6}$. It is easy to see that $p \mid s_p$ and $p \mid t_p$. Also by inserting $n = 1$ in Equation 2.23,

$$\begin{aligned} (ac)^p Z &= (acZ)^p = (aZcZ)^p \\ &= (aZ)^{s_p dp}(aZ)^{t_p p}(aZ)^p (bZ)^{\frac{p(p-1)}{2}}(cZ)^p \\ &= (aZ)^p = a^p Z. \end{aligned}$$

Hence there exists $z \in Z$ such that $(ac)^p = a^p z$ and so $ca^p = a^p c$. This implies that $a^p \in Z$ and therefore $p^2 = o(aZ) \mid p$, which is our final contradiction. $\square$

3. Concluding Remarks

In this paper the authors continued a recently published paper of Zainal et al. [5] in investigating finite $p-$groups of order p^4. It is proved that three non-abelian groups of this order are not capable. By results of [5] and our results to complete the classification of capable group of order p^4 it is enough to investigate the groups H_9 and H_{10} in Theorem 1.2. Our calculations with computer algebra software GAP in working with small groups of order p^4 suggests the following conjecture:

Conjecture 3.1. *The groups H_9 and H_{10} are not capable.*

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A CHARACTERIZATION OF $U_4(2)$ BY NSE*

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Abstract. Let G be a finite group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and m_k be the number of elements of order k in G. Let $nse(G) = \{m_k | k \in \omega(G)\}$. The aim of this paper is to prove that, if G is a finite group such that nse(G)=nse($U_4(2)$), then $G \cong U_4(2)$.
Keywords. element order; number of elements of the same order; projective special unitary group; simple K_n - group.

1. Introduction

This section contains the relevant definitions, some standard facts on nse, and a brief exposition of nse history. Throughout this paper, G is a finite group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, and by $\omega(G)$, we introduce the set of order of elements from G. Set $m_k = m_k(G) = |\{g \in G | o(g) = k\}|$, and nse($G$)=$\{m_k | k \in \omega(G)\}$. In fact, m_k is the number of elements of order k in G and $nse(G)$ is the set of sizes of elements with the same order in G.

To the world's mathematics and researchers, one of the important problems in group theory is characterization of a group by a given property, that is, to prove there exists only one group with a given property (up to isomorphism). Until now, different characterizations are investigated for finite simple groups. For instance, in [21, 22] motivated by one of the Thompson's problem, the authors introduced a new characterization for the finite simple group G by $nse(G)$ and $|G|$. In fact, they proved that if G is a simple K_i- group ($i = 3, 4$), then G is characterizable by nse(G) and $|G|$ (The simple group G is called simple K_n-group if $|\pi(G)| = n$). Following this result, several groups were characterized by nse and order. For example, in [5, 11], it is proved that Suzuki group, and sporadic groups are characterizable by nse and order.

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We remark here that not all groups can be characterized by their group orders and the set nse. As an illustration, let $H_1 = C_4 \times C_4$ and $H_2 = C_2 \times Q_8$, where C_2 and C_4 are cyclic groups of order 2 and 4 respectively, and Q_8 is a quaternion group of order 8. It is easy to see that $nse(H_1) = nse(H_2) = \{1, 3, 12\}$ and $|H_1| = |H_2| = 16$ but $H_1 \ncong H_2$.

However, it is claimed that some simple groups could be characterized by exactly the set nse without considering the order of group. In fact, a finite non-abelian simple group H is called characterizable by nse, if every finite group G with $nse(G) = nse(H)$ implies that $G \cong H$. In [7, 8, 9, 10, 12, 13, 24] it is proved that the alternating groups A_n, where $n \in \{7, 8\}$, the symmetric groups S_n where $n \in \{3, 4, 5, 6, 7\}$, M_{12}, $L_2(27)$, $L_2(q)$ where $q \in \{16, 17, 19, 23\}$, $L_2(q)$ where $q \in \{7, 8, 11, 13\}$, $L_2(q)$ where $q \in \{17, 27, 29\}$, are uniquely determined by nse(G). Besides, in [1, 14, 15, 16] it is proved that $U_3(4)$, $L_3(4)$, $U_3(5)$, $L_3(5)$, are uniquely determined by nse(G). Recently, in [3, 6, 18, 19], it is proved that the simple groups $U_3(3)$, $L_3(3)$, $G_2(4)$, $L_2(3^n)$, where $|\pi(L_2(3^n))| = 4$, and $L_2(2^m)$, where $|\pi(L_2(2^m))| = 4$, are uniquely determined by nse(G). Therefore, it is natural to ask what happens with other kinds of simple groups.

In an effort to fill some of the empty ground about the characterization of simple groups by nse, in this paper we will prove the following main theorem.

Main Theorem. Let G be a group such that $nse(G) = nse(U_4(2))$. Then G is isomorphic to $U_4(2)$.

2. Notation and Preliminaries

Before we get started, let us fix some notations that will be used throughout the paper. For a natural number n by $\pi(n)$, we mean the set of all prime divisors of n, so it is obvious that if G is a finite group, then $\pi(G) = \pi(|G|)$. A Sylow r-subgroup of G is denoted by P_r and by $n_r(G)$, we mean the number of Sylow r- subgroup of G. Also the largest element order of P_r is signified by $exp(P_r)$. Moreover, we denote by ϕ the Euler function. In the following, we bring some useful lemmas which be used in the proof of the main theorem.

Lemma 2.1. *[25]. Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leqslant s(s^2 - 1)$.*

Lemma 2.2. *[24]. Let G be a group. If $1 \neq n \in nse(G)$ and $2 \nmid n$, then the following statements hold:*
(1) $2||G|$;
(2) $m_2 = n$;
(3) for any $2 < t \in \omega(G)$, $m_t \neq n$.

Lemma 2.3. *[2]. Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m||L_m(G)|$.*

Lemma 2.4. *[23]. Let G be a group and P be a cyclic Sylow p-group of G of order p^α. If there is a prime r such that $p^\alpha r \in \omega(G)$, then $m_{p^\alpha r} = m_r(C_G(P))m_{p^\alpha}$. In particular $\phi(r)m_{p^\alpha}|m_{p^\alpha r}$, where $\phi(r)$ is the Eular function of r.*

Lemma 2.5. *[17]. Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic group and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

We say that a group G acts semi regularly on set X if G acts on X in such a way that $G_x = 1$ for all $x \in X$.

Lemma 2.6. *[20]. Let the finite group G acts on the finite set X. If the action is semi regular, then $|G| \mid |X|$.*

Let us mention the structure of simple K_3-groups, that will be needed in Section 3.

Lemma 2.7. *[4]. If G is a simple K_3-group, then G is isomorphic to one of the following groups:*

$$A_5,\ A_6,\ L_2(7),\ L_2(8),\ L_2(17),\ L_3(3),\ U_3(3),\ U_4(2).$$

Lemma 2.8. *[22]. Let G be a group and M a simple K_3-group. Then $G \cong M$ if and only if the following hold: (1) $|G| = |M|$, (2) $nse(G) = nse(M)$.*

3. Main Theorem and its Proof

Suppose G is a group such that $nse(G) = nse(U_4(2))$. By Lemma 2.1, we can assume that G is finite. Let m_n be the number of elements of order n. We notice that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. In addition, we notice that if $n > 2$, then $\phi(n)$ is even. If $n \in \omega(G)$, then by Lemma 2.3 and the above argument, we have

$$\begin{cases} \phi(n)|m_n \\ n|\sum_{d|n} m_d \end{cases} \qquad (3.1)$$

In the proof of the main theorem, we often apply formula (3.1) and the above comments.

Proof of the Main Theorem. Let G be a group with

$$nse(G) = nse(U_4(2)) = \{1, 315, 800, 3780, 4320, 5184, 5760\}$$

where $U_4(2))$ is the projective special unitary group of degree 4 over field of order 2. We have divided the proof into a sequence of lemmas.

Remark 3.1. Let $2 \neq p \in \pi(G)$, by formula (3.1), $p|(1+m_p)$ and $(p-1)|m_p$, which implies that $p \in \{3,5,7,17,19\}$.

In the following lemma, we prove some basic properties of group G:

Lemma 3.1. *If $p \in \pi(G)$ and $p \in \{2,3,5\}$, then*

(1) $2 \in \pi(G)$ *and* $m_2 = 315$*;*

(2) $m_3 = 800$, $m_5 = 5184$*;*

(3) $\{5^2, 3^6, 2^9\} \bigcap \omega(G) = \emptyset$*;*

(4) $|P_2| | 2^9$.

Proof. The proof is straightforward according to Lemma 2.2, Lemma 2.3, and formula (3.1). □

Lemma 3.2. $\{17, 19\} \bigcap \pi(G) = \emptyset$.

Proof. We prove that $17 \notin \pi(G)$. Conversely, suppose that $17 \in \pi(G)$. Then formula (3.1) implies $m_{17} = 5184$. On the other hand, by formula (3.1), we conclude that if $2.17 \in \omega(G)$, then $m_{2.17} \in \{800, 4320, 5184, 5760\}$ and $2.17|1 + m_2 + m_{17} + m_{2.17} (= 6300, 9820, 10684, 11260)$, which is a contradiction, and hence $2.17 \notin \omega(G)$. Since $2.17 \notin \omega(G)$, the group P_{17} acts fixed point freely on the set of elements of order 2 of G and by Lemma 2.6, $|P_{17}||m_2$, which is a contradiction. Hence, $17 \notin \pi(G)$. Similarly, we can prove that $19 \notin \pi(G)$. □

To remove the prime 7, let us first show that $5 \in \pi(G)$.

Lemma 3.3. $\{5\} \bigcap \pi(G) = \{5\}$.

Proof. Assume that $5 \notin \pi(G)$.
• If $3, 7 \notin \pi(G)$, then G is a 2-group. Since $2^9 \notin \omega(G)$, we have $\omega(G) \subseteq \{1, 2, 2^2, \cdots, 2^8\}$. Hence $|G| = 2^m = 20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$, where k_1, k_2, k_3, k_4, k_5 and m are non-negative integers, and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 2$. It is obvious that $20160 \leqslant |G| \leqslant 20160 + (k_1 + k_2 + k_3 + k_4 + k_5)5760$ and so $20160 \leqslant |G| \leqslant 20160 + 2.5760$. Now, it is easily seen that the equation has no solution.
Hence 3 or 7 belongs to $\pi(G)$, and the following cases are considered.
• If $7 \in \pi(G)$, by formula (3.1) $m_7 = 5760$, then as $exp(P_7) = 7$, $|P_7||1 + m_7$ and so $|P_7| = 7$. Since $n_7 = \frac{m_7}{\phi(7)} = 2^6.3.5||G|$, it follows that $5 \in \pi(G)$, which is a contradiction.
• If $3 \in \pi(G)$, then $exp(P_3) = 3, 3^2, 3^3, 3^4, 3^5$.

⋆ If $exp(P_3) = 3$, then by Lemma 2.3, $|P_3||(1 + m_3)$ and so $|P_3||3^2$. We will consider two cases for $|P_3|$.
Case 1 If $|P_3| = 3$, then since $n_3 = \frac{m_3}{\phi(3)} = 2^3.5^3||G|$, $5 \in \pi(G)$ which is a contradiction.

Case 2 If $|P_3| = 3^2$, then since $5, 7 \notin \pi(G)$ and $\pi(G) \subseteq \{2, 3, 5, 7\}$, we can assume that $\{2\} \subseteq \pi(G) \subseteq \{2, 3\}$, and so we have

$$\omega(G) \subseteq \{1, 2, \cdots 2^8\} \cup \{3, 3.2, 3.2^2, 3.2^3, \cdots, 3.2^7\}$$

($2^8.3 \notin \omega(G)$ by formula (3.1)) and $|\omega(G)| \leqslant 17$. Therefore $20160+800k_1+3780k_2+4320k_3+5184k_4+5760k_5 = |G| = 2^a.9$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 10$. Since $20160 \leqslant 2^a.9 \leqslant 20160 + 10.5760$, we have $a = 12$, or $a = 13$. If $a = 12$, then since $|P_2|||2^9$, we have a contradiction. Similarly, we can rule out $a = 13$.

$\star$ If $exp(P_3) = 3^2$, then, by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2})$ and so $|P_3|||3^8$. (for example, when $m_9 = 5760$). We will consider seven cases for $|P_3|$.
Case 1. If $|P_3| = 3^2$, then $n_3 = \frac{m_9}{\phi(9)}$, since $m_9 \in \{3780, 4320, 5184, 5760\}$, $n_3 = 3^2.2.5.7$, $n_3 = 2^4.3^2.5$, or $n_3 = 2^6.3.5$, and so $5 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3^3$, since $n_3 \not\equiv 1 \ (mod\, 3)$, we have a contradiction.
Case 2. If $|P_3| = 3^3$, then since $5, 7 \notin \pi(G)$, we can assume that $\{2\} \subseteq \pi(G) \subseteq \{2, 3\}$ and so we have $\omega(G) \subseteq \{1, 2, \cdots 2^8\} \cup \{3, 3.2, 3.2^2, \cdots, 3.2^7\} \cup \{3^2, 3^2.2, 3^2.2^2, \cdots, 3^2.2^7\}$ ($2^8.3 \notin \omega(G)$, $2^8.3^2 \notin \omega(G)$ by formula (3.1)) and $|\omega(G)| \leqslant 25$. Therefore $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.27$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 18$. Since $20160 \leqslant 2^a.27 \leqslant 20160 + 18.5760$, we have $a = 10$, $a = 11$, or $a = 12$.
If $a = 10$, then since $|P_2|||2^9$, we have a contradiction. Similarly, we can rule out $a = 11$ and $a = 12$.
Case 3. If $|P_3| = 3^4$, then since $exp(P_3) = 3^2$ and $2^8.3, 2^8.9 \notin \omega(G)$, $\omega(G) \subseteq \{1, \cdots, 2^8\} \cup \{3, \cdots, 3.2^7\} \cup \{3^2, \cdots, 3^2.2^7\}$. On the other hand, if $2^8 \in \omega(G)$ since $2^8.3 \notin \omega(G)$, the group P_3 acts fixed point freely on the set of elements of order 2^8. Hence $|P_3||m_{2^8} = 5760$, which is a contradiction. Hence $2^8 \notin \omega(G)$ and $|\omega(G)| \leqslant 24$. Therefore $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.81$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 17$. Since $20160 \leqslant 2^a.81 \leqslant 20160 + 17.5760$, we have $a = 8$, $a = 9$, or $a = 10$.
If $a = 8$, then $576 = 800k_1 + 3760k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 17$. By a computer calculation, it is easy to see this equation has no solution.
If $a = 9$, then $21312 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 17$. The only solution of this equation is $(0, 0, 0, 3, 1)$. We show this is impossible. Since $|\omega(G)| = 11$ and $2^8 \notin \omega(G)$, $exp(P_2) = 2^i$, where $3 \leqslant i \leqslant 7$. Hence, if $exp(P_2) = 2^i$ where $3 \leqslant i \leqslant 7$, then $|P_2||(1+m_2+m_4+\cdots+m_{2^i})$ by Lemma 2.3. In fact $|P_2||(1+315+800t_1+3780t_2+4320t_3+5184t_4+5760t_5)$ where t_1, t_2, t_3, t_4, t_5, are non-negative integers and $0 \leqslant t_1 + t_2 + t_3 + t_4 + t_5 \leqslant 6$. Because $k_1 = 0$ and $m_3 = 800$, $m_{2^i} \neq 800$ for $1 \leqslant i \leqslant 7$, $t_1 = 0$. Since $k_2 = 0$, $0 \leqslant t_2 \leqslant 1$. We claim $t_2 = 0$. Suppose, contrary to our claim, $t_2 = 1$. If $m_4 = 3780$, then since $m_9 \in \{3780, 4320, 5184, 5760\}$, we have a contradiction and so $t_2 = 0$. If $m_4 \neq 3780$, then by a computer calculation $m_8 = 3780$, since $m_9 \in \{3780, 4320, 5184, 5760\}$, we have a contradiction and so $t_2 = 0$. Also $k_3 = 0, k_4 = 3$, and $k_5 = 1$, thus $0 \leqslant t_3 \leqslant 1$, $0 \leqslant t_4 \leqslant 4$, and $0 \leqslant t_5 \leqslant 2$. By an easy computer calculation, this is impossible.

If $a = 10$, then since $|P_2||2^9$, we have a contradiction.
Similarly, we can rule out the other cases.

$\star$ If $exp(P_3) = 3^3$, then by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3})$ and so $|P_3||3^4$ (for example when $m_9 = 5184$ and $m_{27} = 5760$). We will consider two cases for $|P_3|$.

Case 1. If $|P_3| = 3^3$, then $n_3 = \frac{m_{27}}{\phi(27)}$, since $m_{27} \in \{3780, 4320, 5184, 5760\}$, $n_3 = 2.3.5.7$, $n_3 = 2^4.3.5$, or $n_3 = 2^6.5$, and so $5 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3^2$, since $n_3 \not\equiv 1 \ (mod\, 3)$, we have a contradiction.

Case 2. If $|P_3| = 3^4$, then by Lemma 2.5, $27|m_{27}$. Since $(27 \not| 5760)$, it is understood that $m_{27} \in \{3780, 4320, 5184\}$). Since $2^8.3 \notin \omega(G)$, $2^8.3^2 \notin \omega(G)$, $2^8.3^3 \notin \omega(G)$, and $2^8 \notin \omega(G)$, $|\omega(G)| \leqslant 32$. Therefore $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.81$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 25$. Since $20160 \leqslant 2^a.81 \leqslant 20160 + 25.5760$, we have $a = 8$, $a = 9$, or $a = 10$.

If $a = 8$, then $576 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 25$. By a computer calculation, it is easily seen that the equation has no solution.

If $a = 9$, then $21312 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 25$. By a computer calculation, the only solution of this equation is $(0, 0, 0, 3, 1)$. We show this is impossible. Since $|\omega(G)| = 11$ and $2^8 \notin \omega(G)$, $exp(P_2) = 2^i$, where $3 \leqslant i \leqslant 7$. Hence, if $exp(P_2) = 2^i$, where $3 \leqslant i \leqslant 7$ then $|P_2||(1 + m_2 + m_4 + \cdots + m_{2^i})$ by Lemma 2.3. In fact $|P_2||(1 + 315 + 800t_1 + 3780t_2 + 4320t_3 + 5184t_4 + 5760t_5)$ where t_1, t_2, t_3, t_4, t_5, are non-negative integers and $0 \leqslant t_1 + t_2 + t_3 + t_4 + t_5 \leqslant 6$. Because $k_1 = 0$ and $m_3 = 800$, $m_{2^i} \neq 800$ for $1 \leqslant i \leqslant 7$, $t_1 = 0$. Since $k_2 = 0$, $0 \leqslant t_2 \leqslant 1$. We claim $t_2 = 0$. Suppose, contrary to our claim, $t_2 = 1$. If $m_4 = 3780$, then since $m_{27} \in \{3780, 4320, 5184\}$, we have a contradiction and so $t_2 = 0$. If $m_4 \neq 3780$, then by computer calculation $m_8 = 3780$, since $m_{27} \in \{3780, 4320, 5184\}$, we have a contradiction and so $t_2 = 0$. Also $k_3 = 0, k_4 = 3$, and $k_5 = 1$, thus $0 \leqslant t_3 \leqslant 1$, $0 \leqslant t_4 \leqslant 4$, and $0 \leqslant t_5 \leqslant 2$. By an easy computer calculation, this is impossible.

If $a = 10$, then since $|P_2||2^9$, we have a contradiction.

$\star$ If $exp(P_3) = 3^4$, then by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4})$ and so $|P_3||3^4$ (for example when $m_9 = 5760$, $m_{27} = 3780$,and $m_{81} = 4320$).

If $|P_3| = 3^4$, then $n_3 = \frac{m_{81}}{\phi(81)}$, since $m_{81} \in \{3780, 4320, 5184\}$, $n_3 = 2^4.5$, or $n_3 = 2.5.7$, and so $5 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3$, since a cyclic group of order 81 has two elements of order 3, $m_3 \leqslant 2^5.3.2 = 192$, which is a contradiction.

$\star$ If $exp(P_3) = 3^5$, then by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4} + m_{3^5})$ and so $|P_3||3^5$ (for example when $m_9 = 5184$, $m_{27} = 5760$,and $m_{81} = m_{243} = 5184$). In a similar way we have a contradiction. Therefore, $5 \in \pi(G)$. $\square$

Lemma 3.4. $\{7\} \bigcap \pi(G) = \emptyset$.

Proof. By Lemma 2.3 $|P_5||1 + m_5$ and so $|P_5| = 5$. In the following, that the prime 7 do not belong to $\pi(G)$ is proved. Let $7 \in \pi(G)$. Then formula (3.1) implies

$m_{7.5} \in \{4320, 5184, 5760\}$ and $7.5|1+m_5+m_7+m_{5.7}(= 15256, 16129, 16705)$, which is a contradiction, and hence $5.7 \notin \omega(G)$. It follows that the Sylow 7-subgroup of G acts fixed point freely on the set of elements of order 5 and so $|P_7||m_5$, which is a contradiction. Hence $7 \notin \pi(G)$. □

From what has already been proved, we conclude $2{,}5 \in \pi(G)$, so the following cases will be considered $\{2, 5\}, \{2, 3, 5\}$.

Lemma 3.5. $\pi(G) = \{2, 3, 5\}$.

Proof. If $\pi(G) = \{2, 5\}$, since $exp(P_5) = 5$, then by Lemma 2.3, $|P_5||1 + m_5$, and so $|P_5| = 5$. Since $n_5 = \frac{m_5}{\phi(5)} = 2^4.3^4$, it follows that 3 belongs to $\pi(G)$, which is a contradiction. Hence $\pi(G) = \{2, 3, 5\}$. The proof is completed by showing that $|G| = |U_4(2)|$. □

Lemma 3.6. $G \cong U_4(2)$.

Proof. First, we show that $|G| = |U_4(2)|$. From the above arguments, we have $|P_5| = 5$. Now, we prove $10 \notin \omega(G)$. Conversely, suppose that $10 \in \omega(G)$. Then formula (3.1) implies $m_{10} \in \{800, 3780, 4320, 5760\}$. On the other hand, if $2.5 \in \omega(G)$, then by Lemma 2.4, $m_{2.5} = m_5.\phi(2).t$ for some integer t, which is a contradiction and hence $2.5 \notin \omega(G)$. Since $2.5 \notin \omega(G)$, the group P_2 acts fixed point freely on the set of elements of order 5, and so $|P_2||m_5$, hence $|P_2||3^4.2^6$. In fact $|P_2||2^6$. In the same way, since $15 \notin \omega(G)$, $|P_3||m_5$ and hence $|P_3||3^4.2^6$. In fact $|P_3||3^4$. Therefore we have $|G| = 2^m.3^n.5$. Since $20160 = 2^6.3^2.5.7 \leqslant |G| = 2^m.3^n.5$, $|G| = 2^6.3^4.5$. Hence $|G| = 2^6.3^4.5 = |U_4(2)|$ and by assumption $nse(G) = nse(U_4(2))$, so by Lemma 2.8, $G \cong U_4(2)$ and the proof is completed. □

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ON THE PARTIAL DIFFERENCE SETS IN CAYLEY DERANGEMENT GRAPHS

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Abstract. Let G be a finite group. The set $D \subseteq G$ with $|D| = k$ is called a (n, k, λ, μ)-partial difference set (PDS) in G if the differences $d_1 d_2^{-1}, d_2, d_2 \in D, d_1 \neq d_2$, represent each non-identity element in D exactly λ times and each non-identity element in $G - \{D\}$ exactly μ times. In the present paper, we determine for which group $G \in \{D_{2n}, T_{4n}, U_{6n}, V_{8n}\}$ the derangement set is a PDS. We also prove that the derangement set of a Frobenius group is a PDS.
Keywords. Finite group; Frobenius group; derangement set.

1. Introduction

Let G be a finite group. A symmetric subset of group G is a subset $S \subseteq G$, where $1 \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G, S)$ with respect to S is a graph whose vertex set is $V(\Gamma) = G$ and two vertices $x, y \in V(\Gamma)$ are adjacent if and only if $yx^{-1} \in S$. It is a well-known fact that a Cayley graph is connected if and only if $G = \langle S \rangle$. Also a Cayley graph is a regular graph (every vertex has the same degree).

A derangement is a permutation with no fixed points. The set $\mathcal{D}$ of permutation group is derangement if all elements of $\mathcal{D}$ are derangements. Suppose G is a permutation group and $\mathcal{D} \subseteq G$ is a derangement set. The derangement graph $\Gamma_G = Cay(G, \mathcal{D})$ has the elements of G as its vertices and two vertices are adjacent if and only if they do not intersect.

Suppose G is a permutation group of degree n. A subset S of G is said to be intersecting if for any pair of permutations $\sigma, \tau \in S$ there exists $i \in \{1, 2, \ldots, n\}$ such that $\sigma\tau^{-1}(i) = i$. A group G has the Erdös-Ko-Rado (***ekr***) property, if for any intersecting subset $S \subseteq G$, $|S|$ is bounded above by the size of the largest point stabilizer in G. The maximal intersecting set is one with maximum size. A group can have the property under one action while it fails to have this property under

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another action. We refer to [1, 2, 8, 9, 13, 17] for background information about the history of this intresting problem.

Section 2 includes the ***ekr*** properties of well-known groups. In section 3, the derangement set of well-known groups are studied.

2. Erdös-Ko-Rado property

For the subgroup H of group G and the element $g \in G$, the conjugate of subgroup H in G is denoted by $H^g = g^{-1}Hg$. Suppose $G \leq Sym(n)$ is a transitive permutation group, then G is called a Frobenius group if it has a non-trivial subgroup H, where $H \cap H^g = \{1\}$, for all $g \in G \setminus H$. The kernel of Frobenius group G is defined as

$$K = (G \setminus \cup_{g \in G} H^g) \cup \{1\}.$$

It is not difficult to see that all non-identity elements of K are all derangement elements of G. In other words, let G be a non-trivial permutation group and $G^* = G - \{1\}$. If G is a Frobenius group then for all $g \in G^*$, $|fix(g)| \leq 1$ and at least there exist an element $g_0 \in G^*$ such that $|fix(g_0)| = 1$.

Theorem 2.1. *[16] (Frobenius Theorem) Suppose H is a proper non-identity subgroup of G such that for all $g \in G \setminus H$, we have $H \cap g^{-1}Hg = \{1\}$. Let $K = G \setminus \cup_{g \in G} g^{-1}(H \setminus \{1\})g$, then $K \lhd G$, $G = KH$ and $H \cap K = \{1\}$.*

Proposition 2.1. *[2] Every Frobenius group has the **ekr** property.*

Theorem 2.2. *Let $G \leq Sym(n)$ and the derangement graph $Cay(G, \mathcal{D})$ be the disjoint union of n-cliques. Then G has the **ekr** property.*

Proof. Let $\{k_1, k_2, \ldots, k_{n-1}\}$ be the set of derangements of G and $\{g_i, g_ik_1, \ldots, g_i k_{n-1}\}$ be the vertices of the i-th clique in derangemen graph $Cay(G, \mathcal{D})$, where $g_i \in G$. Since each clique has size n and G acts on n elements, every elemen of each clique has exactly one fixed point and every pair of elements in a clique has no same fixed point. Let H be the set of all vertices in $Cay(G, \mathcal{D})$ that fixes point x. Suppose $1 \neq g_rk_t \in H$ and $(g_rk_t)^g \in H$, where $g \in G - H$. So $g^{-1}g_rk_tg(x) = x$ and thus $g_rk_tg(x) = g(x)$. This means that g_rk_t fixes $g(x)$ while $g(x) \neq x$, a contradiction. The proof is completed. □

A group G acting on a set X is transitive if for every pair of points $(a, b) \in X$ there exist $x \in G$ such that $x.a = b$. The permutation group G is regular if G acts transitively on X and for all $x \in X$, $G_x = 1$. A group G is 2–transitive if for any two ordered pairs $(a, r), (b, s) \in X$, with $a \neq r$ and $b \neq s$ there exists $x \in G$ such that $x.a = b$ and $x.r = s$. We say that G is sharply 2-transitive if G is 2-transitive and for any two points $x, y \in X$, $G_{x,y} = 1$. In this paper by, $(G|X)$ we mean that the group G acts on the set X.

Theorem 2.3. *[5] Let $(G|X)$ be transitive and $x \in X$. Then $(G|X)$ is 2-transitive if and only if G_x acts transitvely on the set $X - \{x\}$.*

Theorem 2.4. *[5] (The orbit-stabilizer property) Let $(G|X)$ and $x \in X$. If G is finite, then $|x^G||G_x| = |G|$.*

Theorem 2.5. *[5] (Galois Theorem). Let $(G|X)$ be a transitive permutation group of degree a prime number. Then the group G is solvable if and only if for all $x, y \in X, x \neq y$, we have $G_{x,y} = 1$.*

Theorem 2.6. *Let $(G|X)$ be a 2-transitive permutation group of degree n and $(x_1, x_2) \in X^2$. Then $|G| = n(n-1)|G_{x_1,x_2}|$.*

Proof. Suppose the group G acts on X, transitively. So the action of G on X has one orbit. Then by Theorem 2.4, $|G| = n|G_{x_1}|$. On the other hand, by Theorem 2.3 group G_{x_1} acts transitively on the set $X - \{x_1\}$, and by the orbit-stabilizer property $|G_{x_1}| = (n-1)|G_{x_1,x_2}|$. This completes the proof. □

Theorem 2.7. *Let $(G|X)$ be a transitive non-regular group of degree a prime number. If G is solvable then G has the **ekr** property.*

Proof. Since G is non-regular, there exist $x \in X$ such that $G_x \neq 1$. By Theorem 2.5, for $x, y \in X$ we have $G_{x,y} = 1$ and this means that every non-identity element of G fixes at most one element. If every non-identity element of G fixes no element of X, then $|G| = |X|$ and it is contradict with the non-regularity of G. So there exist at least one $1 \neq x \in X$ such that $|G_x| = 1$. Hence, G is Frobenius group and by Proposition 2.1, it has the ***ekr*** property. □

Theorem 2.8. *Let $(G|X)$ be a transitive permutation group such that the action G is non-regular and for all $x, y \in X, x \neq y$, we have $G_{x,y} = 1$. Then G has the **ekr** property.*

Proof. Similar to the proof of theorm 2.7, we can conclude that G is Frobenius group and the result follows. □

Theorem 2.9. *[5] Let $(G|X)$ and the act of G be 2-transitive. Then the action of G on X is sharply 2-transitive if and only if $|G| = n(n-1)$.*

Theorem 2.10. *Let $(G|X)$ be 2-transitive non-regular permutation group of degree n such that $|G| = n(n-1)$. Then G has the **ekr** property.*

Proof. By Theorem 2.9, G is a sharply 2-transitive group and so for $x, y \in X (x \neq y)$, we have $G_{x,y} = 1$. Now, similar to the proof of Theorem 2.7, G is a Frobenius group and thus it has the ***ekr*** property. □

Let $\rho : G \to GL(n, \mathbb{F})$ be a representation with $\rho(g) = [g]_\beta$. The character $\chi_\rho : G \to \mathbb{C}$ of ρ is defined as $\chi_\rho(g) = tr([g]_\beta)$ for some basis β. The character χ of an irreducible representation is called the irreducible character and χ is linear, if $\chi(1) = 1$. The set of all irreducible characters of group G is denoted by $Irr(G)$.

Let $(G|X)$ and $fix(g) = \{x \in X | g(x) = x\}$. The character π such that $\pi(g) = |fix(g)|$ is called permutation character and the character $\chi = |fix(g)| - 1$ is called standard character.

Theorem 2.11. *[12] Let G be 2-transitive group, then the standard character of G is irreducible character.*

Theorem 2.12. *[6] Let G be a finite group with a normal symmetric subset S. Let A be the adjacency matrix of graph $Cay(G, S)$. Then the eigenvalues of A are given by*

$$[\lambda_\chi]^{\chi(1)^2}, \ \chi \in Irr(G)$$

where $\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$.

Theorem 2.13. *The derangement graph of any 2-transitive group is not a bipartite graph.*

Proof. Let G acts 2-transitive on n elementsa and complete bipartite graph $K_{r,s}$ be the derangement graph of G. Since the derangement graph is a regular graph, we have $r = s$. The eigenvalues of $K_{r,r}$ are $\{[-r]^1, [0]^{2r-2}, [r]^1\}$. On the other hand by Theorem 2.11, the standard character π of a 2-transitive group is irreducible. So by Theorem 2.12, we have $\lambda_\chi = \frac{-|\mathcal{D}|}{\chi(1)} = \frac{-r}{n-1}$. Since the rational eigenvalues of a graph are integers, we have $n = 2$ and then $G \cong \mathbb{Z}_2$ or $G \cong \{1\}$. $\square$

3. Partial difference set

Let G be a finite group and $D \subseteq G$. Then D is a (n, k, λ, μ)-partial difference set (PDS) in G if and only if $DD^{-1} = \gamma 1_G + \lambda D + \mu(G - D)$, where $\gamma = k - \mu$ if $1_G \notin D$ and $\gamma = k - \lambda$ if $1_G \in D$. We will usually assume that $1_G \notin D$ and $D^{(-1)} = D$, in which case, we have

$$D^2 = (k - \mu)1_G + (\lambda - \mu)D + \mu G.$$

Partial difference sets were named by I. M. Chakravarti, 1969 [4], but introduced by Bose and Cameron, 1965 [3] in their studies of calibration designs and the bridge tournament problem. D is called abelian if G is abelian. It is well known that a PDS D with $1 \notin D$ and $\{d^{-1} \ : d \in D\} = D$ is equivalent to a strongly regular Cayley graph, such a PDS is called regular. The study of partial difference sets is closely related to partial geometries, Schur rings, strongly regular Cayley graphs and two-weight codes. Asurvey of Ma [15] contains very detailed descriptions of these connections.

Theorem 3.1. *Let $G = HK \leq Sym(n)$ be a Frobenius group with kernel K. The derangement set of G is a $(n|H|, n-1, n-2, 0)$-PDS.*

Proof. We know that $|K| = n$. Every non-identity element of kernel G is a derangement of G and $\mathcal{D} \cup \{1\}$ is a subgroup. This implies that the derangement set of G is a $(n|H|, n-1, n-2, 0)$-PDS. □

Theorem 3.2. *Consider the dihedral group D_{2n} with derangement set $\mathcal{D}$. If n is odd, then $\mathcal{D}$ is a PDS and if n is even, then $\mathcal{D}$ is not a PDS.*

Proof. Consider the dihedral group $D_{2n} = \langle a, b | a^n = b^2 = 1, aba^{-1} = a^{-1}\rangle$. If n is odd, then D_{2n} is a Frobenius group and by Theorem 3.1 the derangement set is a PDS. Now, let n be even. Suppose that $a = (1, 2, 3, \ldots, n)$ and $b = (1, 2)(3, n) \ldots (\frac{n}{2} + 1, \frac{n}{2} + 2)$ is permutation presentation of generators of D_{2n}. The derangement set of D_{2n} is

$$\mathcal{D} = \{a, a^2, \ldots, a^{n-1}, b, a^2b, a^4b, \ldots, a^{n-2}b\}.$$

If $a^ia^{-j} = a^2$, then $i - j \equiv 2(mod\ n)$ and $\{(3, 1), (4, 2), \ldots, (n-1, n-3)\}$ are $n-3$ solutions for (i, j). On the other hand, if $(a^ib)(a^jb)^{-1} = a^2(i, j$ are even), then $a^ia^{-j} = a^2$ and so $i - j \equiv 2(mod\ n)$. Thus $\{(4, 2), (6, 4), \ldots, (n-2, n-4)\}$ are $n/2 - 2$ solutions for (i, j). One can see that $a(a^{n-1})^{-1} = a^2$, $b(a^{n-2}b)^{-1} = a^2$ and $(a^2b)b^{-1} = a^2$. Let $(a^ib)a^{-j} = a^2$, by using the relation of group, we have $a^{i-j}b = a^2$ and this is impossible. The equation $a^i(a^jb)^{-1} = a^2$ is impossible, too. So if $d_i, d_j \in \mathcal{D}$, then $d_id_j^{-1} = a^2$ has $(3n/2) - 2$ solutions. If $a^ia^{-j} = a$, then $i - j \equiv 1(mod\ n)$ and $\{(2, 1), (3, 2), \ldots, (n-1, n-2)\}$ are the solutions for (i, j). By the relation of D_{2n}, there is no other solutions for $d_id_j^{-1} = a$. So in this case there are $n - 2$ solutions. Then we conclude that the derangement set of dihedral group in this case is not a PDS. □

Consider the dicyclic group T_{4n}, U_{6n} and V_{8n} by the following presentations:

$$T_{4n} = \langle a, b | a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1}\rangle,$$

$$U_{6n} = \langle a, b | a^{2n} = b^3 = e, a^n = b^2, a^{-1}ba = b^{-1}\rangle,$$

$$V_{8n} = \langle a, b | a^{2n} = b^4 = e, aba = b^{-1}, ab^{-1}a = b^{-1}\rangle.$$

Theorem 3.3. *The derangement set of dicyclic group T_{4n} is a $(4n, 4n-1, 4n-2, 0)$-PDS.*

Proof. In [7] Darafsheh proved that two elements $a = (1, 2, 3, \ldots, 2n)(2n+1, 2n+2, 2n+3, \ldots, 4n)$ and $b = (1, 2n+1, n+1, 3n+1)(2, 4n, n+2, 3n)(3, 4n-1n+3, 3n-1), \ldots, (n-1, 3n+3, 2n-1, 2n+3)(n, 3n+2, 2n, 2n+2)$ are the generators of T_{4n}. All elements of T_{4n} have no fixed point. Then $\mathcal{D} = T_{4n} - \{e\}$ which is a $(4n, 4n-1, 4n-2, 0)$-PDS. □

Theorem 3.4. *The derangement set of* $U_{6n}(n \geq 4)$ *is not a PDS set.*

Proof. Let $a = (1, 2, 3, \ldots, 2n)(2n+1, 2n+2)$ and $b = (2n+1, 2n+2, 2n+3)$ be the permutation peresentations of generators of U_{6n} [7]. One can see that the derangement set of U_{6n} is $\mathcal{D} = \{a^i b, a^i b^2 | 2 \leq i \leq 2n-2 \text{ and } i \text{ is even}\}$. Let $a^i b^j, a^r b^s \in \mathcal{D}$ and $(a^i b^j)(a^r b^s)^{-1} = b$. Then we have $a^i b^{j-s} a^{-r} = b$ and so $a^{-i} b a^r = b^{j-s}$. Thus $a^{r-i} a^{-r} b a^r = b^{j-s}$ and by using the relation of U_{6n}, we have $a^{r-i} b^{(-1)^r} = b^{j-s}$. This yields that

$$\begin{cases} r \equiv i \ (mod\ 2n) \\ j - s = 1 \end{cases}.$$

Hence the relation $(a^i b^j)(a^r b^s)^{-1} = b$ has $n-1$ solutions. On the other hand $(a^i b^j)(a^r b^s)^{-1} = a$ has no solution and thus $\mathcal{D}$ is not a PDS set. □

Theorem 3.5. *The derangement set of* V_{8n} $(n \geq 3)$ *is not a PDS set.*

Proof. For group V_{8n} we can consider two following cases:

• Case 1. Suppose n is an odd number. Let $a = (1, 2, 3, \ldots, 2n)(2n+1, 2n+2, \ldots, 4n)$ and $b = (1, 2, 2n+1, 2n+2)(3, 2n, 2n+3, 4n)(4, 4n-1, 2n+4, 2n-1) \ldots (n+1, 3n+2, 3n+1, n+2)$ be the permutation peresentations of generators of V_{8n} [7]. One can see that the derangement set of V_{8n} is

$$\mathcal{D} = \{a, a^2, \ldots, a^{2n-1}, b, b^2, b^3, a^i b, a^i b^2, a^i b^3, a^r b^2\},$$

where $2 \leq i \leq 2n-2$ (i is even) and $1 \leq r \leq 2n-1$ (r is odd).

We are going to show that the number of elements of $A = \{d_i, d_j \in \mathcal{D} | d_i d_j^{-1} = a\}$ and $B = \{d_i, d_j \in \mathcal{D} | d_i, d_j^{-1} = a^2\}$ are not equal. By considering $i - j \equiv 1 \ (mod\ 2n)$, the equation $a^i (a^j)^{-1} = a$ has $2n-2$ solutions. Similarly, the equation $(a^i b^2)(a^j b^2)^{-1} = a$ has $2n-2$ solutions. On the other hand, we have $b^2 (a^{2n-1} b^2)^{-1} = a$ and $(ab^2)(b^2)^{-1} = a$. So the set A has $4n-2$ elements. Now, we compute the elements of the set B. By considering $i - j \equiv 2 \ (mod\ 2n)$, the equation $a^i (a^j)^{-1} = a^2$ has $2n-3$ solutions. Also, $(a^i b^2)(a^j b^2)^{-1} = a^2$ has $2n-3$ solutions. Suppose that $4 \leq i \leq 2n-2$ (i is even) and $j \equiv i - 2 \ (mod\ 2n)$, then we have $(a^i b)(a^j b)^{-1} = a^2$ and $(a^i b^3)(a^j b^3)^{-1} = a^2$. This means that each of this equations has $n-2$ solutions. On can see that $b^i (a^{2n-2} b^i)^{-1} = a^2$ for $i = 1, 2, 3$. On the other hand, we have $(a^2 b^i)(b^{-i}) = a^2 (i = 1, 2, 3)$, $(ab^2)(a^{2n-1} b^2) = a^2$ and $a(a^{2n-1})^{-1} = a^2$. Then the set B has $6n-2$ elements and the derangement set of V_{8n}(n is odd) is not a PDS set.

• Case 2. Suppose n is even number. Let $a = (1, 2, 3, \ldots, 2n)(2n+1, 2n+2, \ldots, 4n)$ and $b = (1, 2, 2n+1, 2n+2)(3, 2n, 2n+3, 4n)(4, 4n-1, 2n+4, 2n-1) \ldots (n, 3n+3, 3n, n+3)(n+1, n+2, 3n+1, 3n+2)$ be the permutation peresentations of generators of V_{8n} [7]. One can see that the derangement set of V_{8n} is

$$\mathcal{D} = \{a, a^2, \ldots, a^{2n-1}, b, b^2, b^3, a^i b, a^i b^2, a^i b^3, a^r b, a^r b^2, a^s b^2, a^s b^3\},$$

where $2 \leq i \leq 2n-2$ (i is even), $r \in \{1, 5, 9, \ldots, 2n-3\}$ and $s \in \{3, 7, 11, \ldots, 2n-1\}$.

Now, we show that the number of elements of $E = \{d_i, d_j \in \mathcal{D} | d_i d_j^{-1} = a\}$ and $F = \{d_i, d_j \in \mathcal{D} | d_i d_j^{-1} = a^4\}$ are not equal. By regarding $i - j \equiv 1$ $(mod\ 2n)$, the equation $a^i(a^j)^{-1} = a$ has $2n-2$ solutions. If $j \equiv i-1$ $(mod\ n)$ and $i \in \{2, 5, 6, 9, 10, \ldots, 2n-2\}$, then the equation $(a^i b^s)(a^j b^s)^{-1} = a$, where $s \in \{1, 2\}$ has $n-1$ solutions. If $j \equiv i-1$ $(mod\ n)$ and $i \in \{3, 4, 7, 8, 11, \ldots, 2n-1\}$, then the equation $(a^i b^s)(a^j b^s)^{-1} = a$, where $s \in \{2, 3\}$ has $n-1$ solutions. One can see that $(ab^t)(b^t)^{-1} = a$, where $t \in \{1, 2\}$ and $b^t(a^{2n-1}b^t)^{-1} = a$, where $t \in \{2, 3\}$. Then the set E has $6n-2$ elements. Now, we compute the elements of the set F. By considering $i - j \equiv 4$ $(mod\ 2n)$ the equation $a^i(a^j)^{-1} = a^4$ has $2n-5$ solutions. It is clear that $a^1(a^{2n-3})^{-1} = a^2(a^{2n-2})^{-1} = a^3(a^{2n-1})^{-1} = a^4$. One can see that if $t \in \{1, 2, 3\}$ then $(a^4 b^t)(b^t)^{-1} = a^4$, and $b^t(a^{2n-4}b^t)^{-1} = a^4$. Let i, j be even, $i - j \equiv 4$ $(mod\ 2n)$ and $r \in \{1, 2, 3\}$. Then $(a^i b^r)(a^j b^r)^{-1} = a^4$ yields $3(n-1)$ solutions. Let i be odd, $i - j \equiv 4$ $(mod\ 2n)$ and $r \in \{5, 9, 13, \ldots, 2n-3\}$. Then by using $(a^i b^r)(a^j b^r)^{-1} = a^4$ we get $n-2$ solutions for this equation. Let i be odd, $i - j \equiv 4$ $(mod\ 2n)$ and $r \in \{7, 11, 15, \ldots, 2n-1\}$. Again by $(a^i b^r)(a^j b^r)^{-1} = a^4$ we acheive $n-2$ solutions. If $i \in \{1, 2\}$ then $(ab^i)(a^{2n-3}b^i)^{-1} = a^4$. If $i \in \{2, 3\}$ then $(a^3 b^i)(a^{2n-1}b^i)^{-1} = a^4$ and if $i \in \{1, 2, 3\}$ then $(a^2 b^i)(a^{2n-2}b^i)^{-1} = a^4$. So the set F has $7n-2$ elements. Then the derangement set of V_{8n}(n is odd) is not a PDS set. □

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ON THE ROOTS OF TOTAL DOMINATION POLYNOMIAL OF GRAPHS, II

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Abstract. Let $G = (V, E)$ be a simple graph of order n. The total dominating set of G is a subset D of V that every vertex of V is adjacent to some vertices of D. The total domination number of G is equal to minimum cardinality of total dominating set in G and is denoted by $\gamma_t(G)$. The total domination polynomial of G is the polynomial $D_t(G, x) = \sum_{i=\gamma_t(G)}^{n} d_t(G, i)x^i$, where $d_t(G, i)$ is the number of total dominating sets of G of size i. A root of $D_t(G, x)$ is called a total domination root of G. The set of total domination roots of graph G is denoted by $Z(D_t(G, x))$. In this paper, we show that $D_t(G, x)$ has $\delta - 2$ non-real roots and if all roots of $D_t(G, x)$ are real, then $\delta \leq 2$, where δ is the minimum degree of vertices of G. Also we show that if $\delta \geq 3$ and $D_t(G, x)$ has exactly three distinct roots, then $Z(D_t(G, x)) \subseteq \{0, -2 \pm \sqrt{2}i, \frac{-3\pm\sqrt{3}i}{2}\}$. Finally we study the location roots of total domination polynomial of some families of graphs.
Keywords. graph; total domination number; total domination polynomial; root.

1. Introduction

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is the set $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. A leaf (end-vertex) of a graph is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The set $D \subset V$ is a total dominating set if every vertex of V is adjacent to some vertices of D, or equivalently, $N(D) = V$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. A total dominating set with cardinality $\gamma_t(G)$ is called a γ_t-set. An i-subset of V is a subset of V of cardinality i. Let

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$\mathcal{D}_t(G,i)$ be the family of total dominating sets of G which are i-subsets and let $d_t(G,i) = |\mathcal{D}_t(G,i)|$. The polynomial $D_t(G;x) = \sum_{i=1}^{n} d_t(G,i)x^i$ is defined as total domination polynomial of G. As an example, $D_t(K_n,x) = (x+1)^n - nx - 1$ and $D_t(K_{1,n},x) = x((x+1)^n - 1)$. A root of $D_t(G,x)$ is called a total domination root of G. The set of total domination roots of graph G is denoted by $Z(D_t(G,x))$. For many graph polynomials, their roots have attracted considerable attention. For example in [5] Brown, Hickman, and Nowakowski proved that the real roots of the independence polynomials are dense in the interval $(-\infty, 0]$, while the complex roots are dense in the complex plane. For matching polynomial, in [14] was proved that all roots of the matching polynomials are real. Also it was shown that if a graph has a Hamiltonian path, then all roots of its matching polynomial are simple (see Theorem 4.5 of [15]). For domination polynomial, Brown and Tufts in [4] studied the location of domination roots and they proved that the set of all domination roots is dense in the complex plane. For graphs with few domination roots see [1]. Related to the roots of total domination polynomials there are a few papers. See [2, 16] for more details. Recently authors in [16] shown that all roots of $D_t(G,x)$ lie in the circle with center $(-1,0)$ and radius $\sqrt[\delta]{2^n - 1}$, where δ is the minimum degree of G and n is the order of G. As a consequence, they proved that if $\delta \geq \frac{2n}{3}$, then every integer root of $D_t(G,x)$ lies in the set $\{-3,-2,-1,0\}$.

In this paper we show that $D_t(G,x)$ has $\delta - 2$ non-real roots and if all roots of $D_t(G,x)$ are real, then $\delta \leq 2$. Also we show that if $\delta \geq 3$ and $D_t(G,x)$ has exactly three distinct roots, then $Z(D_t(G,x)) \subseteq \{0, -2 \pm \sqrt{2}i, \frac{-3 \pm \sqrt{3}i}{2}\}$. Finally we study the location roots of total domination polynomial of some families of graphs.

2. Main results

In this section we obtain some results on total domination roots. Oboudi in [20] has studied graphs whose domination polynomials have only real roots. More precisely he obtained the number of non-real roots of domination polynomial of graphs. Similarly, we do it for total domination roots, in the next theorem.

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$.*

i) If all roots of G are real, then $\delta = 1$ or 2.

ii) The polynomial $D_t(G,x)$ has at least $\delta - 2$ non-real roots.

Proof. Let $g(x) = D_t(G,x)$ and $g^{(m)}(x)$ be the m-th derivative of $g(x)$ with respect to x. It is easy to see that if $i \geq n - \delta + 1$, then $d_t(G,i) = \binom{n}{i}$ and if $i \leq n - \delta$, then $d_t(G,i) < \binom{n}{i}$, where $d_t(G,i)$ is the number of total dominating sets of G with cardinality i, for every natural number i. Thus there exists a polynomial $f(x)$ with positive coefficients and with degree $n - \delta$ such that $D_t(G,x) = (x+1)^n - f(x)$. Since all roots of $g(x)$ are real, by Rolle's theorem we conclude that all roots of $g^{(n-\delta)}(x)$ are real as well. On the other hand $g^{(n-\delta)}(x) = \frac{n!}{\delta!}(x+1)^\delta - a(n-\delta)!$, where a is the coefficient of $x^{n-\delta}$ in $f(x)$. Since all roots of $g^{(n-\delta)}(x)$ are real, this shows that $\delta \leq 2$. Since G is connected, so $\delta = 1$ or 2.

Now suppose that $g(x)$ has exactly r real roots. Using Rolle's theorem one can see that $g^{(n-\delta)}(x)$ has at least $r-(n-\delta)$ real roots. On the other hand $g^{(n-\delta)}(x) = \frac{n!}{\delta!}(x+1)^{\delta} - a(n-\delta)!$. Thus $r-(n-\delta) \leq 2$. Therefore $g(x)$ has at least $\delta - 2$ non-real roots. □

`Theorem 2.2.` [2] *If $G = (V, E)$ is a graph of order n with r support vertices, then $d_t(G, n-1) = n-r$.*

`Theorem 2.3.` [15] *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{n}{2}$.*

The study of graphs which their polynomials have few roots can give sometimes a surprising information about the structure of the graph. If A is the adjacency matrix of G, then the eigenvalues of A, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are said to be the eigenvalues of the graph G. These are the roots of the characteristic polynomial $\phi(G, \lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$. For more details on the characteristic polynomials. The characterization of graphs with few distinct roots of characteristic polynomials (i.e. graphs with few distinct eigenvalues) have been the subject of many researches. Graphs with three adjacency eigenvalues have been studied by Bridges and Mena [3] and Klin and Muzychuk [17]. Also van Dam studied graphs with three and four distinct eigenvalues [6, 7, 8, 9]. Graphs with three distinct eigenvalues and index less than 8 were studied by Chuang and Omidi in [18]. Graphs with few domination roots were studied by Akbari, Alikhani and Peng in [1]. In [2], authors studied graphs with exactly two total domination roots $\{-3, 0\}$, $\{-2, 0\}$ and $\{-1, 0\}$. Here we study graphs with three distinct total domination roots.

`Theorem 2.4.` *Let G be a graph with $\delta \geq 3$. If $D_t(G, x)$ has exactly three distinct roots, then*

$$Z(D_t(G, x)) \subseteq \{0, -2 \pm \sqrt{2}i, \frac{-3 \pm \sqrt{3}i}{2}\}.$$

Proof. Let G be a connected graph of order n and $Z(D_t(G, x)) = \{0, a, b\}$ that $a \neq b$. Therefore $D_t(G, x) = x^i(x-a)^j(x-b)^k$, for some i, j, k. So by Theorem 2.2, we have

$$-(ja + kb) = n. \tag{2.1}$$

Also because $d_t(G, i) = \binom{n}{i}$ for $i \geq n - \delta + 1$, we have

$$\binom{j}{2}a^2 + \binom{k}{2}b^2 + jkab = d_t(G, n-2) = \binom{n}{2}. \tag{2.2}$$

Let $P(x)$ be the minimal polynomial of a over $\mathbb{Q}$. Clearly, all roots of $P(x)$ are simple. This implies that $deg(P(x)) = 1$ *or* 2. We consider two cases.

Case 1. $deg(P(x)) = 1$. So $D_t(G, x) = x^i(x-a)^j(x-b)^k$, where $-a, -b \in \mathbb{N}$. By Theorem 2.1, we have $\delta = 1$ or 2, a contradiction.

Case 2. $deg(P(x)) = 2$. In this case since $D_t(G,x)$ has three distinct roots, the minimal polynomial of b over $\mathbb{Q}$ is also $P(x)$, Thus we have $D_t(G,x) = x^i(x^2+rx+s)^j$, where $P(x) = x^2+rx+s$. We have $i+2j = n$, and also by (2.1), $-(a+b)j = n$. By Theorem 2.3, $i \le \frac{n}{2}$. Therefore $j \ge \frac{n}{4}$. Since $-(a+b)j = n$ and $a+b$ is an integer, we have $-(a+b) \in \{1,2,3,4\}$. We consider four cases:

Subcase 2.1. If $a+b = -1$, then $j = n$, a contradiction.

Subcase 2.2. If $a+b = -2$, then $j = \frac{n}{2}$, a contradiction.

Subcase 2.3. If $a+b = -3$, then $i = j = \frac{n}{3}$, so we have $D_t(G,x) = x^{\frac{n}{3}}(x^2+rx+s)^{\frac{n}{3}}$. Now, by (2.2) we have

$$\binom{\frac{n}{3}}{2}(a^2+b^2) + \frac{n^2ab}{9} = \binom{n}{2}.$$

In the other hand, since $a+b = -3$, we conclude that $a^2+b^2 = 9-2ab$. Thus by simple calculation we obtain $nab = 3n$. Therefore $ab = 3$. By using $a+b = -3$, we have

$$a \in \{\frac{-3 \pm \sqrt{3}i}{2}\}$$

Subcase 2.4. Now, suppose that $a+b = -4$. Then $i = \frac{n}{2}$ and $j = \frac{n}{4}$. With the same calculations, we have $ab = 6$. Using the fact that $a+b = -4$, we have $a \in \{-2 \pm \sqrt{2}i\}$. □

□

As noted before, in [2], authors studied graphs with exactly two total domination roots $\{-3,0\}$, $\{-2,0\}$ and $\{-1,0\}$. Here we present a family of graphs whose total domination roots are -1 and 0.

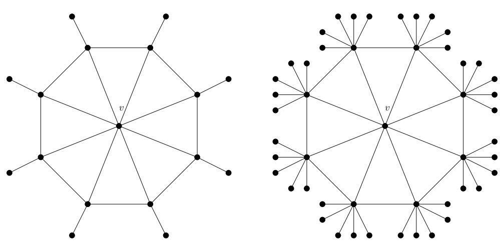

FIG. 2.1: Helm graph H_8 and generalized helm graph $H_{8,5}$, respectively.

The helm graph H_n is obtained from the wheel graph W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel. We define generalized helm graph $H_{n,m}$, the graph is obtained from the wheel graph W_n by attaching m pendent edges at each vertex of the n-cycle of the wheel (Figure 2.1). We recall that corona product of two graphs G and H is denoted by $G \circ H$ and was introduced by Harary [12, 13]. This graph formed from one copy of G and $|V(G)|$ copies of H, where the i-th vertex of G is adjacent to every vertex in the i-th copy of H. We need the following theorems:

Theorem 2.5. [10] *Let $G = (V, E)$ be a graph and $u, v \in V$ two non-adjacent vertices of the graph with $N(u) \subseteq N(v)$. Then*

$$D_t(G, x) = D_t(G \setminus v, x) + xD_t(G/v, x) + x^2 \sum_{w \in N(v) \cap N(u)} D_t(G \setminus N[\{v, w\}], x).$$

Theorem 2.6. [16] *For any graph G of order $n \geq 2$, $D_t(G \circ \overline{K_m}, x) = x^n(1 + x)^{mn}$.*

Theorem 2.7. *For every natural number n, m, we have*

i) $D_t(H_n, x) = x^n(x + 1)^{n+1}$,

ii) $D_t(H_{n,m}, x) = x^n(1 + x)^{mn+1}$.

Proof. Let v be the center vertex of wheel in helm graph H_n and $H_{n,m}$. By Theorems 2.5 and 2.6 we have

i) $D_t(H_n, x) = D_t(C_n \circ K_1, x) + xD_t(K_n \circ K_1, x) = (1 + x)(x(1 + x))^n$,

ii) $D_t(H_{n,m}, x) = D_t(C_n \circ \overline{K_m}, x) + xD_t(K_n \circ \overline{K_m}, x) = (1 + x)(x(1 + x)^m)^n$.

So we have the result. □

The lexicographic product is also known as graph substitution, a name that bears witness to the fact that $G[H]$ can be obtained from G by substituting a copy H_u of H for every vertex u of G and then joining all vertices of H_u with all vertices of H_v if $\{u, v\} \in E(G)$.

Theorem 2.8. *Let K_m, K_n be complete graphs of order m and n. The total domination polynomial of lexicographic product of K_m and K_n is*

$$D_t(K_m[K_n], x) = D_t(K_m, D(K_n, x)) + mD_t(K_n, x).$$

Proof. Note that $K_m[K_n] \cong K_{mn}$, So the result is obtained. □

The generalized friendship graph $F_{n,q}$ is a collection of n cycles (all of order q), meeting at a common vertex (see Figure 2.4). The generalized friendship graph may also be referred to as a flower [19]. For $q = 3$ the graph $F_{n,q}$ is denoted simply by F_n and is friendship graph. The total domination polynomial of F_n and its roots studied in [16]. Here, we compute the total domination number of $F_{n,4}$. To study the total domination roots of $F_{n,4}$ we first obtain a formula for the total domination polynomial of graph G_n depicted in Figure 2.2. We need the following theorem:

Theorem 2.9. [10]

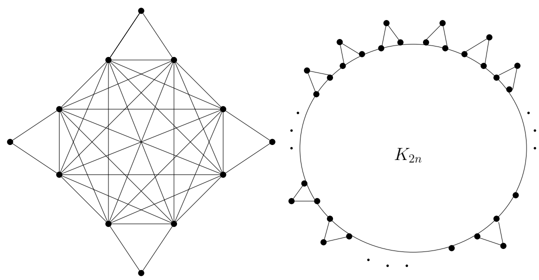

FIG. 2.2: Graphs G_4 and G_n in proof of Theorem 2.9, respectively.

(i) For any vertex u in the graph G we have

$$\begin{aligned} D_t(G,x) &= D_t(G \setminus u, x) + xD_t(G/u, x) + x^2 \sum_{v \in N(u)} D_t(G \setminus N[\{u,v\}], x) \\ &\quad -(1+x)p_u(G), \end{aligned}$$

where $p_u(G,x)$ is the polynomial counting the total dominating sets of $G \setminus u$ which do not contain any vertex of $N(u)$ in G.

(ii) Let $u, v \in V(G)$ be two non-adjacent vertices of G with $N(v) \subseteq N(u)$. Then $D_t(G,x)$

$$= D_t(G \setminus u, x) + xD_t(G/u, x) + x^2 \textstyle\sum_{w \in N(u) \cap N(v)} D_t(G \setminus N[\{u,w\}], x).$$

`Theorem 2.10.` *For any $n \in \mathbb{N}$, $D_t(G_n, x) = (x(x+1)(x+2))^n$.*

Proof. Consider the graph G_n shown in Figure 2.2 and v be a vertex of degree two of this graph. By Theorem 2.9(i) and the fact that $p_v(G_n, x) = D_t(G_{n-1}, x)$ and $G_n - v \cong G_n/v$, we have

$$D_t(G_n, x) = (x+1)D_t(G_n - v, x) - (x+1)D_t(G_{n-1}, x).$$

Now by Theorem 2.9(ii) for graph $G_n - v$ and the vertex u of this graph (see figure 2.3):

$$D_t(G_n, x) = (x+1)^2 D_t(G_n - v/u, x) - (x+1)D_t(G_{n-1}, x).$$

Again by Theorem 2.9(ii) for the vertex w of the graph $G_n - v/u$ shown in figure

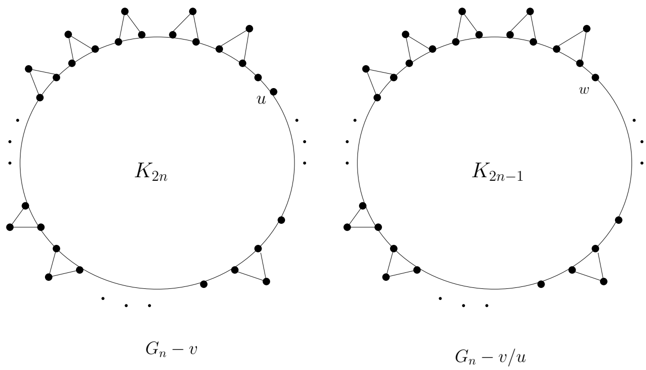

FIG. 2.3: Graphs in proof of Theorem 2..

2.3, we have the following equations.

$$\begin{aligned}
D_t(G_n, x) &= (x+1)^2 D_t(G_n - v/u, x) - (x+1) D_t(G_{n-1}, x) \\
&= (x+1)^3 D_t(G_{n-1}, x) - (x+1) D_t(G_{n-1}, x) \\
&= x(x+1)(x+2) D_t(G_{n-1}, x) \\
&= (x(x+1)(x+2))^n
\end{aligned}$$

So we have result. □

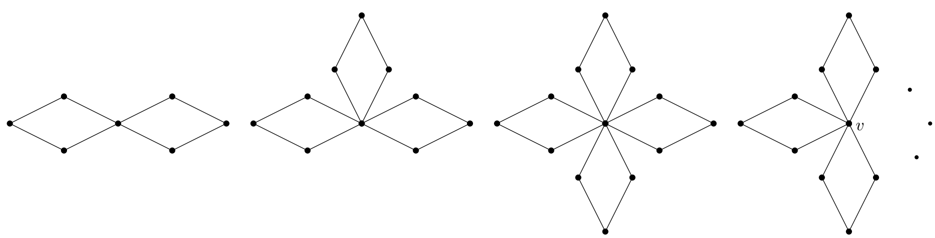

FIG. 2.4: Friendship graphs $F_{2,4}, F_{3,4}, F_{4,4}$ and $F_{n,4}$, respectively.

Theorem 2.11. *For every natural number n, total domination polynomial of generalize friendship graph $F_{n,4}$ is*

$$D_t(F_{n,4}, x) = x^{n+1}(x+2)^n((x+1)^n + x^{n-1}).$$

Proof. Let v be center vertex of $F_{n,4}$. By theorem 2.5 we have

$$D_t(F_{n,4}, x) = (D_t(P_3, x))^n + x D_t(G_n, x)$$

where G_n is graph in Figure 2.2 and so by Theorem 2.10 we have the result. □

We need the following lemma to obtain more results:

Lemma 2.12. [4] $lim_{n\to\infty} ln(n)\Big(\frac{ln(n)-1}{ln(n)}\Big)^n = 0.$

The basic idea of the following result follows from the proof of Theorem 8 in [4].

Theorem 2.13. *For natural number $n \geq 2$,*

i) The total domination polynomial of the generalized friendship graph, $D_t(F_{n,4}, x)$, has a real root in the interval $(-1, 0)$

ii) The total domination polynomial of the generalized friendship graph, $D_t(F_{n,4}, x)$, has a real root in the interval $(-n, -ln(n))$, for n sufficiently large.

Proof. i) Let $f(x) = (x+1)^n + x^{n-1}$. So $f(0) = 1$ and $f(-1) = (-1)^{n-1} = -1$. By the intermediate value theorem, we have result.

ii) Suppose that

$$f_{2n}(x) = x^{n+1}((x+1)^n + x^{n-1}).$$

Observe that

$$f_{2n}(x) = x^{2n+1} + (n+1)x^{2n} + \binom{n}{n-2}x^{2n-1} + \binom{n}{n-3}x^{2n-2} + ... + nx^{n+2} + x^{n+1}.$$

Consider

$$f_{2n}(-n) = (-1)^{2n+1}n^{2n+1}\Big(1 - \frac{n+1}{n} + \frac{\binom{n}{2}}{(n)^2} - ... + \frac{(-1)^n}{(n)^n}\Big).$$

So $f_{2n}(-n) < 0$ for n sufficiently large, because the following inequality is true for n sufficiently large,

$$\frac{n+1}{n} - \frac{\binom{n}{2}}{(n)^2} + ... - \frac{(-1)^n}{(n)^n} < 1.$$

Now consider

$$\begin{aligned} f_{2n}(-ln(n)) &= (-ln(n))^{n+1}(1 - ln(n))^n + (-ln(n))^{2n} \\ &= (ln(n))^{2n}\Big(1 - ln(n)\Big(\frac{ln(n)-1}{ln(n)}\Big)^n\Big). \end{aligned}$$

From Lemma 2.12, we have $ln(n)\Big(\frac{ln(n)-1}{ln(n)}\Big)^n \to 0$, as $n \to \infty$ which implies that $f_{2n}(-ln(n)) > 0$. By the Intermediate Value Theorem, for sufficiently large n, $f_{2n}(x) = D_t(F_n, x)$ has a real root in the interval $(-n, -ln(n))$.

□

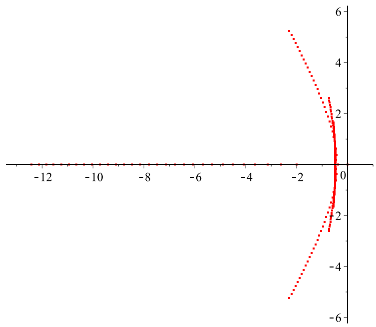

FIG. 2.5: Total domination roots of $F_{n,4}$, for $2 \leq n \leq 30$.

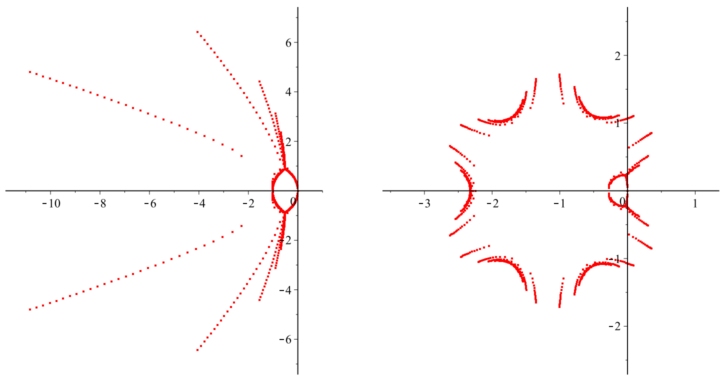

FIG. 2.6: Total domination roots of $K_{1,n}[K_2]$ and $K_{1,n}[K_7]$, for $2 \leq n \leq 30$, respectively.

Figure 2.5 shows the total domination roots of $F_{n,4}$ for $2 \leq n \leq 30$.

Theorem 2.14. *Let G and H be two graphs of order m and n, respectively. The total domination polynomial of join of these two graphs is*

$$D_t(G \vee H) = ((1+x)^m - 1)((1+x)^n - 1) + D_t(G, x) + D_t(H, x).$$

Theorem 2.15. *For every natural numbers m, n,*

$$D_t(K_{1,n}[K_m], x) = (1+x)^{mn}((1+x)^m - 1) + ((1+x)^m - mx - 1)^n - mx.$$

Proof. For two natural numbers m, n, $K_{1,n}[K_m] \cong k_m \vee nK_m$. So by Theorem 2.14, it is easy to see the equation is true. $\square$

Using Maple we think that for two natural numbers m,n, if m and n are even or n is odd, then the total domination polynomial of $K_{1,n}[K_m]$ has no real roots.

However, until now all attempts to prove this failed. See the total domination roots of $K_{1,n}[K_2]$ and $K_{1,n}[K_7]$ for $2 \leq n \leq 30$ in Figure 2.6.

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PARTITION-EQUIVALENT n-POINTS CONFIGURATIONS WITH TWO DISTANCES *

Ali Asghar Rezaei

Abstract. In this paper we define an equivalence relation on the set of all possible geometrical models of $M(n, k)$ containing n points in 3D Euclidean space having k distinct distances. We investigate the number of geometrical models for $M(4, 2)$, $M(5, 2)$ and $M(6, 2)$ up to the mentioned equivalence relation.
Keywords. Constructible models; distinct distances; partition-equivalent; geometrical model.

1. Introduction

Distance geometry has considered two main problems since its inception. One of these problems is the study of the embedding of a semimetric space in an Euclidean space. From the empirical point of view, $\mathbb{R}^3$ is the most important Euclidean space, especially in several applications such as molecular conformation, wireless sensor networks, statics, dimensionality reduction, and robotics. In these applications input data is a set of points and their pair-wise distances(a semimetric space) and the output is a set of points in $\mathbb{R}^3$ realizing those given distances.

Our task is to focus on the number of distinct distances in a semimetric space. A semimetric space with n points and k distinct distances may be embedable in $\mathbb{R}^3$ or not. Such space is denoted by $M(n, k)$ and if it can be embedded in $\mathbb{R}^3$, we say that $M(n, k)$ is constructible. Such problems have been extensively researched, and yet in many cases are still wide open (see for example [4, 3, 7]). Some computational theorems have been proved in [5] for $M(n, k)$. In [6] an equivalence relation was introduced for all models of $M(n, k)$ and the author classified all possible models for $M(4, 2)$ and $M(5, 2)$. In this paper we define a new equivalence relation in term

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of the partitions of a natural number and find the number of the equivalence classes for $M(4,2)$, $M(5,2)$, and $M(6,2)$.

The paper has been organized as follow: First we provide some preliminaries and definitions. Using the partition of a number, we then define an equivalence relation to classify the models of $M(n,k)$. Finally we will investigate $M(6,2)$ in term of the mentioned equivalence relation.

2. Definitions and Notations

In this section we introduce the basic concepts which are used through the paper. Some of these preliminaries have been defined in [1, 6, 5].

Definition 2.1. A *semimetric* on a set S is a function $d : S \times S \to [0,\infty)$ which satisfies the following properties:

- $d(x,y) = d(y,x)$ for all $x, y \in S$.
- $d(x,y) = 0$ if and only if $x = y$.

A *semimetric space* is a pair (S,d) where S is a set and d is a semimetric on it.

When d is understood, we usually omit mention of it and just say "S is a semimetric space." In some literature d is called the distance function. The distance between two points p and q is denoted in both notations $d(p,q)$ or pq.

The problem of embedding an arbitrary semimetric space isometrically into $\mathbb{R}^3$ is an interesting task in *Distance Geometry*. A necessary condition for embedding can be stated in term of Cayley-Menger determinant.

Definition 2.2. Let $\{p_0, p_1, \ldots, p_k\}$ be a semimetric space. The *Cayley-Menger determinant* for this $k+1$-tuple is defined as

$$D(p_0, ..., p_k) = \begin{vmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & p_0p_1^2 & ... & p_0p_k^2 \\ ... & ... & ... & ... & ... \\ 1 & p_kp_0^2 & p_kp_1^2 & ... & 0 \end{vmatrix}$$

Theorem 2.1. *[1] A necessary condition that a semimetric $r+1$-tuple $\{p_0, p_1, \ldots, p_r\}$ be isometrically embedded in an Euclidean space $\mathbb{R}^n$ is that for every $k = 1, 2, \ldots, r$ the determinant $D(p_0, ..., p_k)$ either vanish or have the sign of $(-1)^{k+1}$. If $n < r$, then $D(p_0, ..., p_k) = 0$ $(n < k \leqslant r)$.*

We will consider this theorem for $n = 3$ and $r = 5$. Note that if a five-points semimetric space $\{p_0, \ldots, p_4\}$ can an embedded in $\mathbb{R}^3$, then a necessary condition for embedding six points $\{p_0, p_1, \ldots, p_5\}$ in $\mathbb{R}^3$ is that $D(p_0, \ldots, p_5) = 0$.

Definition 2.3. Let $S = \{p_1, p_2, \ldots, p_n\}$ be a semimetric space such that

$$card\{d(p_i, p_j) \mid i \neq j,\ i, j = 1, 2, \ldots, n\} = k$$

(d is the distance function). Then S is called a model with n points and k distances and is denoted by $M(n, k)$. If S can be isometrically embedded in $\mathbb{R}^3$, then we say $M(n, k)$ is *constructible* .

For example $M(5, 1)$ is not constractible, while $M(4, 1)$ is constructible. In [6] $M(n, 2)$ was investigated for $n \leqslant 5$. Here we consider the case $n = 6$. Note that $M(n, 2)$ is not constructible for $n > 6$ [2].

Definition 2.4. [5] Let $m, m_1, m_2, \ldots, m_k$ are natural numbers such that

$$m = m_1 + m_2 + \cdots + m_k, \quad 1 \leqslant m_1 \leqslant m_2 \leqslant \ldots \leqslant m_k.$$

Then the summand $m_1 + m_2 + \cdots + m_k$ is called a *k-partition* for m.

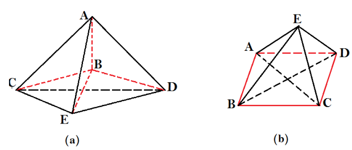
For example 1+9, 2+8, 3+7, 4+6, and 5+5 are 2-partitions for 10. Similarly, 2-partitions for 15 are 1+14, 2+13, and so on.
Notation. We correspond to each model $M(n, k)$, a k-partition of $m = n(n-1)/2$ (the number of edges) as follow. Let $d_1, d_2, \ldots, d_k$ be the distances in this model and m_j be the number of edge with length d_j. Without loss of generality we can assume $m_1 \leqslant m_2 \leqslant \ldots \leqslant m_k$. Then the number of all edges is

$$m = m_1 + m_2 + \cdots + m_k.$$

We also correspond to each model $M(n, k)$, a colored graph with n vertices in which the edges with same length have same color.

Definition 2.5. Two models for $M(n, k)$ are said to be *partition-equivalent* if their k-partitions are same.

For example the following models for $M(5, 2)$ are partition-equivalent with 2-partitions 4+6:



In (a), the points A, C, E, and D are vertices of a regular tetrahedron and B is its center, while in (b), the points A, B, C, D, and E are the vertices of a right pyramid whose base is a unit squar with edge length equal to $\sqrt{2}$.

One can easily show that partition-equivalence is an equivalence relation on the set of all constructible models for $M(n, k)$.

It was shown that all 2-partitions of 10 concerning to $M(5, 2)$ are constructible [6]. So up to partition-equivalence there are exactly 5 geometrical models for $M(5, 2)$. Similarly all 2-partitions of 6 concerning to $M(4, 2)$ are constructible, so there are exactly 3 geometrical model for $M(4, 2)$ up to partition-equivalence. In the next section we complete our task to classify all models for $M(n, 2)$ up to partition-equivalence by considering $M(6, 2)$.

3. Partition-Equivalent Models for $M(6,2)$

For 6 points in $\mathbb{R}^3$ the number of edges is

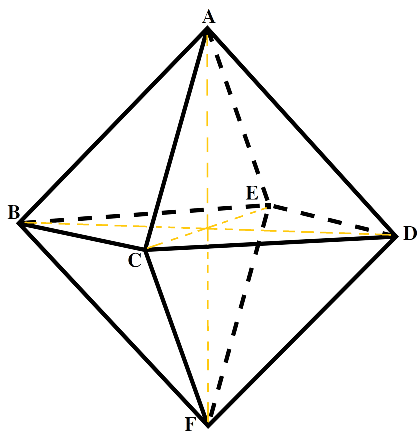
$$\binom{6}{2} = 15.$$

As we see before, there are seven 2-partitions for 15: 1+14, 2+13, 3+12, 4+11, 5+10, 6+9, and 7+8. So one can say that there are at most seven constructible model for $M(6,2)$ up to partition-equivalence. But as we will see some of these partitions are not constructible. Our goal is to determine which of these partitions are constructible. The construction is as follow: starting with all possible graphs for each partition $m+n=15$, we next investigate which graph is constructible. In the rest of paper we use the notation $P(m,n)$ for the partition $m+n=15$.

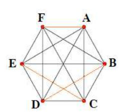
First we show that $P(3,12)$, $P(5,10)$, and $P(6,9)$ are constructible.

Proposition 3.1. *$P(3,12)$ is constructible for $M(6,2)$.*

Proof. Let the points A, B, C, D, E, and F are the vertices of a regular octahedron as follows:



If we take $d(A,B)=1$ as other edges, then $d(B,D)=d(C,E)=d(A,F)=\sqrt{2}$. In fact this shape is the geometrical realization for the following graph whose 2-partition is $P(3,12)$.

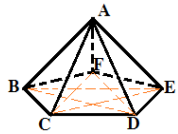


This completes our argument. □

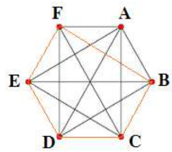
Note that for $P(3,12)$ we have some other graphs. But to be constructible, it is sufficient to find one graph having a geometric realization as above.

Proposition 3.2. *$P(5,10)$ is constructible for $M(6,2)$.*

Proof. Take the points A, B, C, D, E, and F as the vertices of a regular pyramid whose base is a regular pentagon as follow.



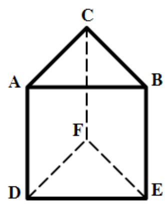
If we assume for example $d(A, B) = 1$ (and the same for other edges), then $d(B, D) = d(C, E) = d(A, D) = d(B, E) = d(A, C) = \sqrt{2 - 2\cos 3\pi/5}$. So this is a geometrical realization for the foolowing graph of $P(5, 10)$.



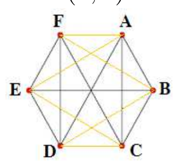
This completes our argument. □

Proposition 3.3. *$P(6, 9)$ is constructible for $M(6, 2)$*

Proof. Let A, B, C, D, E, and F are the vertices of a right prism as follow:



Take $d(A, B) = 1$ (and the same for other edges), then $d(B, D) = d(A, E) = d(A, D) = d(B, F) = d(E, C) = d(A, F) = d(C, D) = \sqrt{2}$. The above shape is a realization for the following graph of $P(6, 9)$.

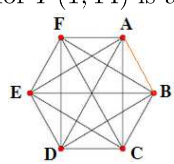


This completes our argument. □

Now we continue with non-constructible partitions.

Proposition 3.4. *$P(1, 14)$ is not constructible for $M(6, 2)$.*

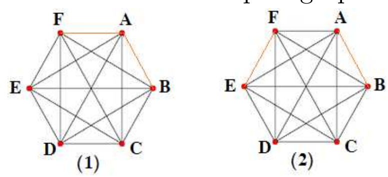
Proof. The corresponding graph for $P(1, 14)$ is as follow.



If this graph have a geometric realization, then the five points A, C, D, E, F have equal pair-wise distances, which is impossible, because in $\mathbb{R}^3$ there are at most four points with this property. □

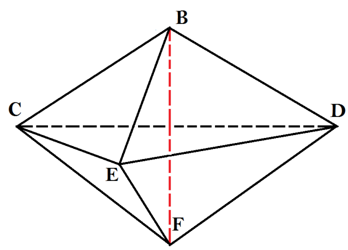
Proposition 3.5. *$P(2,13)$ is not constructible for $M(6,2)$*

Proof. For $P(2,13)$ there are two non-isomorphic graphs as follows:



Graph (1) has no geometrical realization, because it is impossible that five points B, C, D, E, and F have same pair-wise distances.

We show the same statement for graph (2). If graph (2) has a geometrical realization then the points B, C, D, and E are vertices of a regular tetrahedron and so are F, D, C, and B.These two pyramids have the common triangle BCD as a common face and hence the only possible geometric structure for these five points is as follows:

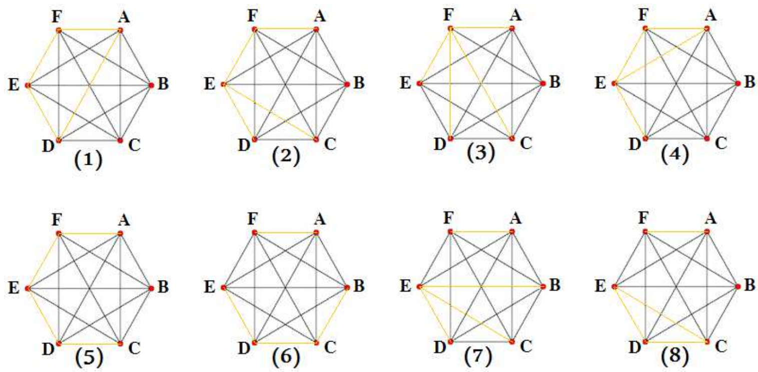


If we assume $d(B,D) = 1$, then $d(B,F) = 2\sqrt{2/3}$. It means that $d(B,D) \neq d(B,F)$, while these two edges in graph (2) have same length. So the graph (2) has no geometrical realization. □

The argument used in the above proposition will be used in next proposition. We will recall this argument as *two regular pyramids with a common face.*

Proposition 3.6. *$P(4,11)$ is not constructible for $M(6,2)$.*

Proof. All possible graphs for $P(4,11)$ have been presented in the following figure:



We show that these graphs have no realizations in $\mathbb{R}^3$. In graphs (2), (4), and (7) we have two regular pyramids with a common face. Due to the length of the other edges, it follows that these graphs have no realization in $\mathbb{R}^3$.

Graphs (3) is not constructible since the points A, B, C, D, and E have same pair-wise distances which is impossible in $\mathbb{R}^3$.

Now consider the graph (1). We have two regular pyramids $ABCE$ and $BCDF$ with common edge BC. Since $ED = AF$, the position of two pyramids is symmetric. Without lose of generality to construct these pyramids one can take the vertices as follows:

$$A = (\tfrac{1}{2}, 0, \tfrac{\sqrt{2}}{2}), \qquad B = (0, -\tfrac{1}{2}, 0), \qquad C = (0, \tfrac{1}{2}, 0),$$
$$D = (-\tfrac{1}{2}, 0, -\tfrac{\sqrt{2}}{2}), \quad E = (-\tfrac{1}{2}, 0, \tfrac{\sqrt{2}}{2}), \quad F = (\tfrac{1}{2}, 0, -\tfrac{\sqrt{2}}{2}).$$

By simple calculation one can see that $AF = ED = \sqrt{2}$ and $AD = \sqrt{3/2}$, so $AD \neq AF$, while in (1) we have $AD = AF$.

For the remaining graphs, we use Theorem 2.1. First consider the graph (5). Omit the point E for a moment, the remaining points A, B, C, D, and F are vertices of a right pyramid whose base is a square of side 1 (B is the apex of pyramid), so $AF = DC = \sqrt{2}$. If this garph has a realization in $\mathbb{R}^3$, then its Cayley-Menger determinant must be zero. But we have

$$D(A,B,C,D,E,F) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 & 1 & 2 & 0 \end{vmatrix} = 5 \neq 0.$$

Same argument can be applied for the graphs (6) and (8) by disregarding the point D. It is easy to see that $D(A,B,C,D,E,F) = -4$ for graph (6), and $D(A,B,C,D,E,F) = -16$ for graph (8). So the necessary condition in Theorem 2.1 does not hold for these cases. $\square$

The only partition which has not been specified is $P(7,8)$. Because of its variety, the investigation of $P(7,8)$ requires a separate research (there are at least 19 non-isomorphic graphs for $P(7,8)$). The author's research for $P(7,8)$ has been led to the following conjecture:

Conjecture 3.1. *$P(7,8)$ is not constructible for $M(6,2)$.*

Regardless of whether the above conjecture is correct or not we have already proved the following important theorem:

Theorem 3.1. *Up to partition-equivalence, there are at least 3 constructible models for $M(6,2)$.*

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A CHARACTERIZATION OF PSL(4,p) BY SOME CHARACTER DEGREE

Younes Rezayi and Ali Iranmanesh

Abstract. Let G be a finite group and cd(G) be the set of irreducible character degree of G. In this paper we prove that if p is a prime number, then the simple group PSL(4,p) are uniquely determined by its order and some its character degrees.
Keywords. Character degrees; order; projective special linear group.

1. Introduction

All groups considered are finite and all characters are complex characters. Let G be a group. Denote by Irr(G) the set of all irreducible characters of G. Let cd(G) be the set of all irreducible character degree of G

Many authors were recently concerned with the following question:
What can be said about the structurs of a finite group G , if some information is known about the arithmetical structure of the degree of the irreducible characters of G (see[16,17])?A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors.Yan et all. in[16] and [17] proved that all simple k_3-group and the Mathieu groups are uniquely determined by their orders and some its character degrees. Also Khosravi et all. in [9] and [10] proved that the simple groups $\mathrm{PSL}(2, p)$ and $\mathrm{PSL}(2, p^2)$ are uniquely determined by its order and its largest and second largest irreducible character degrees, where p is an odd prime. Also Hung and Thamson in[13] proved that the simple group $\mathrm{PSL}(4, q)$ whit $q \geq 13$ are determined by the set of their character degrees.

The goal of this paper is to introduce a new characterization for the finite group $\mathrm{PSL}(4, p)$, where p is prime, by its order and some its character degrees. In fact we prove the following theorem.

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Theorem 1.1. *(Main Theorem) Let $p > 7$ be a prime. If G is a finite group such that the following statments hold, then G is isomorphic to $PSL(4, p)$.*

(i) $|G| = |PSL(4, p)|$

(ii) $kp^6 \in cd(G)$ if only if $k = 1$, where k is an integer number.

(iii) $p(p^2 + p + 1)$ is the smallest nonlinear character degree of G

(iv) $\{p(p + 1)^2(p^2 + 1), (p + 1)(p^2 + 1)\} \subset cd(PSL(4, p))$.

2. Notation and Preliminary

We know that if p is an odd prime, then

$$|PSL(4, p)| = \frac{p^6(p^2 - 1)(p^3 - 1)(p^4 - 1)}{(4, p - 1)}.$$

and

$$\{p^6, p(p^2 + p + 1), p(p + 1)^2(p^2 + 1), (p + 1)(p^2 + 1)\} \subset cd(PSL(4, p)).$$

and the smallest nonlinear character degrees of $PSL(4, p)$ is $p(p^2 + P + 1)$.

If n is an integer and r is a prime number, then we write $r^\alpha || n$, when $r^\alpha | n$ but $r^{\alpha+1} \mid n$. All other notations are standard and we refer to [1].

If $N \trianglelefteq G$ and $\theta \in Irr(N)$, then the inertia group of θ in G is $I_G(\theta)$=$\{g \in G \mid \theta^g$=θ $\}$.

Lemma 2.1. *(Thompson)[13, Lemma 2.3] . Suppos that p is a prime and $p \mid \chi(1)$ for every nonlinear $\chi \in Irr(G)$. Then G has a normal p-complement.*

Lemma 2.2. *(Ghallgher's Theorem)[7, Corollary 6.17]. Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$ be such that $\chi_N = \theta \in Irr(N)$. Then the characters $\beta\chi$ for $\beta \in Irr(\frac{G}{N})$ are irreducible and distinct for distinct β and are all of the irreducible constituents of θ^G.*

Lemma 2.3. *(Ito's Theorem)[3, Corollary 6.15]. Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in Irr(G)$.*

Lemma 2.4. *([3, Theorems 6.2, 6.8, 11.29]). Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$. Let θ be an irreducible constituent of χ_N, and suppose $\theta_1 = \theta$,..,θ_t are the distinct conjugates of θ in G . Then χ_N=$e\sum_{i=1}^{t} e_i\chi_i$, where e=$[\chi_N, \theta]$ and t=$[G$:$I_G(\theta)]$. Also $\theta(1)|\chi(1)$ and $\chi(1)/\theta(1)||G$:$N|$.*

Lemma 2.5. *[16, Lemma] Let G be nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple group and $|G/K|||Out(K/H)|$.*

Lemma 2.6. *([3], Lemma 12.3 and Theorem 12.4). Let $N \trianglelefteq G$ be maximal such that G/N is solvable and nonabelian. Then one of the following holds.*

(i) G/N is a r-group for some prime r. If $\chi \in Irr(G)$ and $r \mid \chi(1)$, then $\chi\tau \in Irr(G)$ for all $\tau \in Irr(G/N)$.

(ii) G/N is a Frobenius group with an elementary abelian Frobenius kennel F/N.

Thus $|G : F| \in cd\ (G)$, $|F : N| = r^\alpha$, where a is the smallest integer such that $|G : F| \mid r^\alpha - 1$. For every $\psi \in Irr(F)$, either $|G : F|\psi(1) \in cd(G)$ or $|F : N||\psi(1)^2$. If no proper multiple of $|G : F|$ is in $cd(G)$, then $\chi(1)||G : F|$ for all $\chi \in Irr(G)$ such that $r \mid \chi(1)$.

Lemma 2.7. *(15, Lemma 2.3) In the context of (ii) of Lemma 2.5, we have*

(i) If $\chi \in Irr(G)$ such that lcm($\chi(1)$,$|G : F|$) does not divide any character degree of G, then $r^\alpha \mid \chi(1)^2$

(ii) If $\chi \in Irr(G)$ such that no proper multiple of $\chi(1)$ is a degree of G, then either $|G : F| \mid \chi(1)$ or $r^\alpha|\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G then $|G : F| = \chi(1)$ or $r^a|\chi(1)^2$.

3. Proof of The Main Theorem

In this section we present the proof of Main theorem. In fact, we prove this theorem by two steps:

Step1. First we prove that G is a nonsolvable group.We show that $G' = G''$. Assume by contradiction that $G' \neq G''$ and let $N \trianglelefteq G$ be maximal such that G/N is solvable and nonabelian.By Lemma 2.6, G/N is an r-group for some prime r or G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N.

Case 1. G/N is an r-group for some prime r. Since G/N is nonabelian, there is $\psi \in Irr(G/N)$ such that $\psi(1) = r^a > 1$. From the classification of prime power degree representations of quasi-simple group in [12], we deduce that $\psi(1) = r^a$ must be equal to the degree of the Steinberg character of H of degree p^6 and thus $r^a = p^6$, which implies that $r = p$. By Lemma 2.1, G possesses a nontrivial irreducible character χ with $p \mid \chi(1)$. Lemma 2.4 implies that $\chi_N \in Irr(N)$. Using Ghallagher's lemma, we deduce that $\chi(1)\psi(1) = p^6\chi(1)$ is a character degree of G, which is impossible with the condition (ii) of main theorem.

Case 2. G/N is a Frobenius group whit an elementary abelian Frobenius kernel F/N. Thus according to Lemma 2.6, $|G : F| \in cd(G)$, $|F : N| = r^a$, where a is the smallest integer such that $|G : F||r^a - 1$. Let χ be a character of G of degree p^6. As no proper multiple of p^6 is in $cd(G)$, Lemma 2.6 implies that either $|G : F||p^6$ or $r = p$. We consider two following subcases.

(a) $|G : F||p^6$. Then $|G : F| \in cd(G)$, by the assumption of the theorem, this implies that no multiple of $|G : F|$ is in $cd(G)$. Therefore, by Lemma 2.6, for every $\psi \in Irr(G)$ either $\psi(1)|p^6$ or $r|\psi(1)$. Taking ψ to be characters of degree $p(p^2+p+1)$ and $p(p+1)^2(p^2+1)$, we obtain that $r|\psi(1)$.This implies that r divides both $p(p^2+p+1)$ and $p(p+1)^2(p^2+1)$. This leads us to a contradiction since $((p^2+p+1)\ ,\ (p+1)^2(p^2+1))$=1

(b) $r = p$. Thus $|F : N| = p^a$ and $|G : F||p^a - 1$. Let χ be a character of G of degree $p(p+1)^2(p^2+1)$ and ψ be a character of degree $(p+1)(p^2+1)$. It follows that $\psi(1)|\chi(1)$ so that by Lemma 2.7, $|G : F| = p(p+1)^2(p^2+1)$ or $p^a|(p(p+1)^2(p^2+1))^2$, which implies that $a \leq 2$, $|G : F| \leq p^2 - 1$. This leads us to a contradiction since $min\{\chi(1)|\chi(1) > 1, \chi \in Irr(G)\} = p(p^2+p+1)$.
Therefore, G is not a solvable group.

Step 2. *Now we prove that G is isomorphic to $PSL(4,p)$.*

By the above discussion and using Lemma 2.5, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of m copies of a nonabelian simple group S and $|G/K|||Out$(K/H)|. Also p is a prime divisor of $|G|$ such that $p^6\|\,|G|$

First we prove that $p \nmid |G/K|$. On the contrary, let $p||G/K|$. We know that $Out(K/H) \cong Out(S) \wr S_m$, which implies that $p||S_m|$ or $p||Out(S)|$. If $P||S_m|$, then $m \geq p$ and so $p^6(p^2-1)(p^3-1)(p^4-1) \geq |K/H| \geq 60^p$, which is impossible. Hence $p||Out(S)|$. According to the orders of automorphism group of alternating group and sporadic simple group, we implies that S is a simple group of Lie type over $GF(q)$, where $q = p_0^f$. By assumption, $p||Out(S)| = dfg$, where d, f, and $g \leq 3$ are the orders of diagonal, field, and graph automorphisms of S respectively. Using [2], we know that if S is a simple group of Lie type over $GF(q)$, then $q(q^2-1) \leq S$ and so if $p|f$, then $2^p(2^{2p}-1) \leq q(q^2-1) \leq |S| \leq p^6(p^2-1)(p^3-1)(p^4-1)$, which is a contradiction. Hence $p|d$. Since $p > 7$, we get that $S = A_n(q)$ and $d = (n+1, q-1)$ or $S =^2 A_n(q)$ and $d = (n+1, q+1)$. In each case we get that $p|q-1$ and $n \geq 6$ or $p|q+1$ and $n \geq 6$. Then $p^7||S|$, which is a contradiction. Therefore, $p \nmid |G/K|$.

Now we prove that $p \nmid |H|$. On the contrary, let $p||H|$. So there exist six possibilities, $p\|\,|H|$ or $p^2\|\,|H|$ or $p^3\|\,|H|$ or $p^4\|\,|H|$ or $p^5\|\,|H|$ or $p^6\|\,|H|$.

Case 1. First, suppose that $p\|\,|H|$. Using the classification of finite simple group we determine all simple groups S such that $p^5||S|^5$. Now we consider two subcases:

(i) Let m=1. Then $p^5||S|$ and $|S||p^5(p^2-1)(p^3-1)(p^4-1)$.

If $S \cong A_n$, then $p \leq n$ and $n!|p^5(p^2-1)(p^3-1)(p^4-1)$. Which is impossible since $p > 7$. Also there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple groups, we get that, there is no Lie group statisfying these conditions.
Since the proofs for the other simple groups are similar, we state the proof only for a few of them for convenience.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions.

If $S \cong B_n(q)$, where $n \geq 2$, then $p|q^{2j} - 1$, for some $1 \leq j \leq n$. Therefore, $p \leq q^n + 1$. Then since $q^{2i-1} \leq q^{2i} - 1$, we get that

$$q^{n^2}.q^{2(1+2+...+n)-n} \leq |S| < p^{14} \leq (q^n + 1)^{14} \leq q^{14n+14}$$

which implies that $2n^2 < 14(n + 1)$. Therefore $n \in \{2, 3, 4, 5, 6, 7\}$. First let $n = 2$. Then $p^5|q^4(q^2-1)(q^4-1)$.It implies that $p^5|(q-1)^2$ or $p^5|(q+1)^2$ or $p^5|q^2+1$, and so $p^5 < 2q^2$. On the other hand $q^4|(p-1)^3$ or $q^4|(p+1)^2$ or $q^4|p^2+1$ or $q^4|(p^2+p+1)$, and so $q^4 < p^3$. Therefore, easily we get a contradiction. If $n \in \{3, 4, 5, 6, 7\}$, similarly we get a contadiction. If $S \cong C_n(q)$, where $n \geq 4$, then withe the same manner we get a contradiction.

If $S \cong A_n(q)$, then similary to the above, we get $n \in \{1, 2, ..., 9\}$. For example, let $n = 5$. Then

$$p^5|(q-1)^5(q+1)^3(q^2+q+1)^2(q^2-q+1)(q^4+q^3+q^2+q+1)$$

so $p^5 < 5q^4$. On the other hand

$$q^{15}|6(p-1)^3(p+1)^2(\frac{p^2+1}{2})(\frac{p^2+p+1}{3})$$

so $q^{15} < p^7$. Therefore we get a contradiction. For other case, similarly we get a contradiction. If $S \cong^2 A_n(q)$, with the same manner we get a contradiction.

If $S \cong D_n(q)$, where $n \geq 4$, then $p^5||S|$, Therefore $p|q^{2i}-1$, for some $1 \leq i \leq n-1$ or $p|(q^n - 1)$. Therefore, $p < q^n$, and since $q^{2i-1} < q^{2i} - 1$, we get that

$$q^{n(n-1)}q^{n-1}(q^{2(1+2+...+(n-1)-(n-1))} < |S| < p^{14}$$

and so $q^{(2n(n-1)} < |S| < p^{14}$. On the other hand, $p < q^n$ and hence $2(n - 1) < 14$. Therefore $n \in \{4, 5, 6, 7\}$.Let $n = 6$. Then

$$p^5|(q-1)^6(q+1)^6(q^2+q+1)^2(q^2-q+1)^2(q^2+1)^2(q^4+1)(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$$

and so $p^5 < q^7$. On the other hand

$$q^{30}|(p-1)^3(p+1)^2(p^2+1)(p^2+p+1)$$

and so, $q^{30} < p^{12}$. Therefore we get a contradiction. Fore some other cases, similarly we get a contadiction. If $S \cong^2 D_n(q)$, with the same manner we get a contradiction.

If $S \cong G_2(q)$, then $p^5||S|$, and hence $p^5 < q^3$. On the other hand,

$$q^6|6(p-1)^3(p+1)^2(\frac{p^2+1}{2})(\frac{p^2+p+1}{3})$$

so $q^6 < p^7$. Therefore we get a contradiction. If $S \cong F_4(q),^2 F_4(q), E_6(q), E_7(q)$ or $E_8(q)$, we get a contradiction similarly.

If $S \cong^2 B_2(q)$, where $q = 2^{2n+1}$, then $p^5|q-1$ or $p^5|q^2+1$. If $p^5|q-1$, then $|S| < p^{14} < (q-1)^3$, wiche is impossible. If $p^5|(q^2+1)$, then $p^5|(q^2+1)/5$, so $p^5 < q^2$. On the other hand

$$q^2|6(p-1)^3(p+1)^2(\frac{p^2+1}{2})(\frac{p^2+p+1}{3}$$

therefore, $q^2|8(p-1)^3$ or $q^2|16(p+1)^2$, so $q < p^2$, which is impossible.

If $S \cong^2 G_2(q)$, where $q = 3^{2n+1}$, then $p^5||S|$, therefore $p^5|q-1$ or $p^5|q+1$ or $p^5|q^2-q+1$, it follows that $p^5 < q^2$. On the other hand, $q^3|6(p-1)^3(p+1)^2$ or $q^3|(p^2+1)/2$ or $q^3|(p^2+p+1)/2$, it follows that $q^3 < p^7$, whis is impossible.

Therefore $m \neq 1$.

(ii)m=5. Then $p||S|$ and $|S|^5|p^5(p^2-1)(p^3-1)(p^4-1)$.
Similarly to the previous case we get a contradiction.

Case 2. Suppose that $p^2|||H|$.Therefore $p^4||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 2, 4\}$. Now we consider three subcases:

(i)Let $m = 1$. Then $p^4||S|$ and $|S||p^4(p^2-1)(p^3-1)(p^4-1)$. We claim that there is no simple group satisfying these conditions.

If $S \cong A_n$, then $p < n$ and $n!|p^4(p^2-1)(p^3-1)(p^4-1)$, which is impossible since $p > 7$. Also there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

Similarl to case 1, we deduce that, there is no nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$, satisfying the above conditions.

Hence $m \neq 1$.

(ii)Let $m = 2$

Similar to last case, we deduce $S \ncong A_n$. Also there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of caracteristic p, using the order of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence $m \neq 2$

(iii) Let $m = 4$. Then $p||S|$ and $|S|^4|p^4(p^2-1)(p^3-1)(p^4-1)$. Using the classification of finite simple group, we show that, there is no simple group satisfying these conditions.

If $S \cong A_n$, then $p \leq n$ and $(n!)^4|p^4(p^2-1)(p^3-1)(p^4-1)$, which is impossible since $p > 7$. Also there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple groups, we get that, the only possibility cases are $A_1(p)$ and $A_2(p)$.

(A) If $S \cong A_1(p)$, then $p^4(p^2-1)^4|16p^4(p-1)^3(p+1)^2(p^2+1)(p^2+p+1)$, therefore $(p-1)(p+1)^2|16(p^2+1)(p^2+p+1)$, which is impossible.

(B) If $S \cong A_2(p)$, then $|S|^4 \leq p^4(p^2-1)(p^3-1)(p^4-1)$, therefore $p^{15} < p^{13}$, which is impossible.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 4$.

Case 3. If $p^3\|\,|H|$. Therefore $p^3||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1,3\}$. Now we consider two subcases:

(i) Let $m = 1$. Then $p^3\|\,|S|$ and $|S||p^3(p^2-1)(p^3-1)(p^4-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 1$.

(ii) Let $m = 3$. Then $p||S|$ and $|S|^3|p^3(p^2-1)(p^3-1)(p^4-1)$

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above condition.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 3$.

Case 4. If $p^4\|\,|H|$. Therefore $p^2||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1,2\}$. Now we consider two subcases:

(i)Let $m = 1$. Then $p^2\|\,|S|$ and $|S||p^2(p^2-1)(p^3-1)(p^4-1)$.

If $S \cong A_n$, then similar to Case 1, we get a contradiction. Also there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 1$.

(ii) Let $m = 2$. Then $p \| |S|$ and $|S|^2 | p^2(p^2-1)(p^3-1)(p^4-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 2$.

Case 5. If $p^5 \| |H|$. Therefore $p | |K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m = 1$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Case 6. If $p^6 | |H|$, choos $\chi \in Irr(G)$, such that $\chi(1) = p^6$. Let θ be an irreducible constituent of χ_H, then $\chi(1)/\theta(1) | |G : H|$, which implies that $\theta(1) = p^6$. Therefore $\chi_H = \theta$ and by Gallagher's theorem $\beta\chi \in Irr(G)$, for each $\beta \in Irr(G/H)$. Hence $p^6\beta(1) \in \mathrm{cd(G)}$, which is contradiction.

By the above discussion, we get that $p^6 | |K/H|$. Since $p^6 \| |G|$, it follows that K/H is a nonabelian simple group say S, such that $p^6 \| |S|$ and $|S| | p^6(p^2-1)(p^3-1)(p^4-1)$ or $K/H \cong S \times S$ and $p^3 \| |S|$ and $|S| | p^6(p^2-1)(p^3-1)(p^4-1)$ or $K/H \cong S \times S \times S$ and $p^2 \| |S|$ and $|S|^3 | p^6(p^2-1)(p^3-1)(p^4-1)$ or $K/H \cong S \times S \times S \times S \times S \times S$ and $p \| |S|$ and $|S|^6 | p^6(p^2-1)(p^3-1)(p^4-1)$.

Now using the classification of finite simple groups and similar to the above argument, we get $K/H \cong PSL(4, p)$. Therefore $|H||G/K| = 1$, and hence, $H = 1$ and $G/K = 1$. Hence $G \cong PSL(4, p)$, and the main theorem is proved.

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HYPERHROUPS AND H_v-GROUPS ASSOCIATED TO ELEMENTS WITH FOUR OXIDATION STATES

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Abstract. The theory of hyperstructures is of great importance due to its connections to various fields of Science. H_v-structures are hyperstructures where the equality is replaced by the nonempty intersection. This class of the hyperstructures is very large so one can use it in order to define several objects that they are not possible to be defined in the classical hyperstructure theory. This paper attempts an exposition of the connection between hyperstructure (H_v-structure) theory and certain type of chemical reactions. In this regard, we consider elements with four oxidation states and investigate their mathematical structures.
Keywords. Hyperstructures; chemical reaction; mathematical structure; chemical reaction.

1. Introduction

The origin of hypergroups can be followed back to Marty [18] who introduced it in 1934 at the eighth Congress of Scandinavian Mathematicians. He generalized the concept of a group, where the theory of groups is the oldest branch of ordinary algebra, by considering the result of the "interaction" between two elements of a non-empty set to be a non empty set of elements. He published some notes on hypergroups, using them in different contexts as algebraic functions, rational fractions, non-commutative groups. The theory knew an important progress starting with the 70's, when its research area has enlarged and new concepts were introduced and studies such as canonical hypergroups, hyperrings, hypermodules, etc. A generalization of algebraic hyperstructures was introduced in 1990 by T. Vougiouklis [21] where he defined weak hyperstructures. Many researchers such as Corsini [6], Corsini and Leoreanu [7], Davvaz [8, 9], Davvaz and Leoreanu-Fotea [16], and Vougiouklis [20] wrote books related to hyperstructure theory and their applications.

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One of motivations for the study of hyperstructures comes from chemical reactions. In [11], Davvaz and Dehghan-Nezhad provided examples of hyperstructures associated with chain reactions. In [13], Davvaz et al. introduced examples of weak hyperstructures associated with dismutation reactions. In [15], Davvaz et al. investigated the examples of hyperstructures and weak hyperstructures associated with redox reactions. In [1], Al-Tahan et al. presented three different examples of weak hyperstructures associated to elechtrochemical cells. In [5]. Chung et al. investigated mathematical structures of chemical reactions for three consecutive oxidation states of elements. Some authors considered particular elements with four oxidation states and investigated their chemical hyperstructures. For example, Chun in [4] presented chemical hyperstructures of chemical reactions for a set of Titanium and Al-Tahan et al. in [2] studied chemical hyperstructures of chemical reactions for a set of Astatine, a set of Tellurium and a set of Bismuth. Then in [3], they studied mathematical structures of chemical reactions for arbitrary elements with four oxidation states.

In this paper, we consider an arbitrary element with four oxidation states and investigate its algebraic hyperstructures and it is organized as follows: After an Introduction, Section 2 presents some definitions and concepts related to (weak) hyperstructures that are used throughout the paper. Section 3 presents chemical hyperstructures using redox reactions of an arbitrary element with four oxidation states as algebraic hyperstructures under certain conditions. Finally, Section 4 presents some examples of elements with four oxidation states that satisfy the conditions presented in Section 3.

2. Preliminaries

In this section, we present some definitions and concepts related to (weak) hyperstructures that are used throughout the paper.

Definition 2.1. [8] Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*.

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b,\ x \circ A = \{x\} \circ A \ and\ A \circ x = A \circ \{x\}.$$

H_v-structures were introduced by T. Vougiouklis as a generalization of the well-known algebraic hyperstructures [19, 21], also see [12, 20]. Some axioms of classical algebraic hyperstructures are replaced by their corresponding weak axioms in H_v-structures. Most of H_v-structures are used in representation theory.

Definition 2.2. [20] A hypergroupoid $(H, \circ)$ is called an H_v*-semigroup* if $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$.

A subset K of an H_v*-semigroup* is an H_v-subsemigroup if K is an H_v-semigroup. An element $x \in H$ is called *idempotent* if $x^2 = x \circ x = x$ and an element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$, for all $x \in H$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$, for all $x, y \in H$.

We present an example of a commutative H_v-semigroup.

Example 2.1. [2] Let $H = \{a, b, c, d\}$ and define $(H, \star)$ by the following table:

$\star$	a	b	c	d
a	a	$\{a, c\}$	$\{a, c\}$	c
b	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	c
c	$\{a, c\}$	$\{a, c\}$	c	$\{c, d\}$
d	c	c	$\{c, d\}$	d

Then $(H, \star)$ is a commutative H_v-semigroup.

Definition 2.3. [6] A hypergroupoid $(H, \circ)$ is called a:

1. *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;
2. *quasi-hypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (The latter condition is called the reproduction axiom);
3. *hypergroup* if it is a semihypergroup and a quasi-hypergroup.

We present an example of a commutative hypergroup of four elements.

Example 2.2. [2] Let $H = \{a, b, c, d\}$ and define "$\circ$" on H by the following table:

$\circ$	a	b	c	d
a	a	$\{a, b, c\}$	$\{a, c\}$	H
b	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	H
c	$\{a, c\}$	$\{a, b, c\}$	c	$\{c, d\}$
d	H	H	$\{c, d\}$	d

Then $(H, \circ)$ is a commutative hypergroup.

Definition 2.4. [6] Two hypergroupoids $(H, \circ)$ and $(K, \star)$ are said to be *isomorphic hypergroupoids*, written as $H \cong K$, if there exists a bijective function $f : H \to K$ such that $f(x \circ y) = f(x) \star f(y)$ for all $x, y \in H$.

3. Chemical hyperstructures for elements with four oxidation states

Through the Latimer diagrams of all elements, we selected a lot of chemical species that were recorded four consecutive standard reduction potentials in acidic and/or basic solution [17].

Example 3.1. [17] The following Latimer diagram are for Astatine, Copper, Iridium, and Carbon respectively that are elements with four consecutive standard reduction potentials in acidic and/or basic solution.

$$At0_3^- \longrightarrow_{0.5} At0^- \longrightarrow_{0} At_2 \longrightarrow_{0.2} At^-;$$
$$CuO^+ \longrightarrow_{1.8} Cu^{2+} \longrightarrow_{0.159} Cu^+ \longrightarrow_{0.521} Cu;$$
$$Ir^{6+} \longrightarrow_{0.4} Ir^{4+} \longrightarrow_{0.1} Ir^{3+} \longrightarrow_{0.1} Ir;$$
$$C^{6+} \longrightarrow_{-1.01} C^{+} \longrightarrow_{-0.52} C \longrightarrow_{-0.7} C^{4-}.$$

Let A, B, C, and D be chemical species of an arbitrary element S and let n_1 be the difference of the oxidation number between D and C, n_2 be the difference of the oxidation number between C and B and n_3 be the difference of the oxidation number between B and A. Let α, β, and γ be the potential difference between D, C, B, and A like in the following Latimer diagram of S.

$$D \longrightarrow_{\alpha} C \longrightarrow_{\beta} B \longrightarrow_{\gamma} A.$$

We present the following reductions.
(1) $D \longrightarrow B$, $E_1 = \frac{\alpha n_1+\beta n_2}{n_1+n_2}$,

(2) $D \longrightarrow A$, $E_2 = \frac{\alpha n_1+\beta n_2+\gamma n_3}{n_1+n_2+n_3}$,

(3) $C \longrightarrow A$, $E_3 = \frac{\beta n_2+\gamma n_3}{n_2+n_3}$.
Let $\{x, x', y, y'\} \subseteq \{A, B, C, D\}$ such that $x \longrightarrow_a x'$ and $y \longrightarrow_b y'$ where a, b are potential differences. We get the following redox reaction:

$$x + x' \longrightarrow_{a+b} y + y'$$

If $E = a + b > 0$ then our redox reaction is spontaneous. Otherwise it is not spontaneous.
In this paper, we are concerned about the chemical hyperstructures of S under the condition $\alpha \geq \gamma \geq \beta$. We consider the four different cases, case $\alpha = \gamma = \beta$, case $\alpha > \gamma > \beta$, case $\alpha > \gamma = \beta$, and case $\alpha = \gamma > \beta$. In this regard, we consider each case separately and find the (weak) hyperstructures associated to it.

3.1. Case $\alpha = \gamma = \beta$

Let $H = \{A, B, C, D\}$, it is clear that:

$$E_1 = E_2 = E_3 = \alpha = \beta = \gamma.$$

For all $x, y \in H$, we define "$\oplus$" on H as follows: $x \oplus y = z$ where z in the product of the spontaneous redox reaction with greatest potential difference that occurs between x and y.
Then we obtain the following table for $(H, \oplus)$.

$\oplus$	A	B	C	D
A	A	{A, B}	{A, C}	{A, D}
B	{A, B}	B	{B, C}	{B, D}
C	{A, C}	{B, C}	C	{C, D}
D	{A, D}	{B, D}	{C, D}	D

Theorem 3.1. *Let $H = \{A, B, C, D\}$. Then $(H, \oplus)$ is a commutative hypergroup.*

Proof. The proof is straightforward since $(H, \oplus)$ is the *Biset hypergroup* with four elements. □

3.2. Case $\alpha > \gamma > \beta$

Let $H = \{A, B, C, D\}$, it is clear that:

$$\beta < E_1 < \alpha,\ \beta < E_2 < \alpha,\ \beta < E_3 < \gamma.$$

The following are all possible spontaneous redox combinations for H.
$A + A \longrightarrow A + A\ \ [0]$
For the spontaneous reactions of $A + B$, we consider the three cases: Case $E_2 < \gamma$, Case $E_2 > \gamma$ and Case $E_2 = \gamma$.
Case $E_2 < \gamma$. We get that

$$A + B \longrightarrow \begin{cases} C + A\ \ [\gamma - E_3] \\ D + A\ \ [\gamma - E_2] \\ B + A\ \ [0] \end{cases}$$

Case $E_2 > \gamma$. We get that

$$A + B \longrightarrow \begin{cases} C + A\ \ [\gamma - E_3] \\ B + A\ \ [0] \end{cases}$$

Case $E_2 = \gamma$. We get that

$$A + B \longrightarrow \begin{cases} C + A\ \ [\gamma - E_3] \\ B + A\ \ [0] \end{cases}$$

$A + C \longrightarrow C + A\ \ [0]$
For the spontaneous reaction $A + D$, we consider the 6 cases: Case $E_2, E_1 < \gamma$,

Case $E_2, E_1 > \gamma$, Case $E_2, E_1 = \gamma$, Case $E_1 > \gamma$, $E_2 < \gamma$, Case $E_1 > \gamma$, $E_2 = \gamma$, Case $E_1 < \gamma$, $E_2 > \gamma$, Case $E_1 < \gamma$, $E_2 = \gamma$.

Case $E_2, E_1 < \gamma$**.** We get that

$$A + D \longrightarrow \begin{cases} B + C \ \ [\alpha - \gamma] \\ C + C \ \ [\alpha - E_3] \\ D + C \ \ [\alpha - E_2] \\ D + A \ \ [0] \end{cases}$$

Case $E_2, E_1 > \gamma$**.** We get that

$$A + D \longrightarrow \begin{cases} B + C \ \ [\alpha - \gamma] \\ B + B \ \ [E_1 - \gamma] \\ B + A \ \ [E_2 - \gamma] \\ C + C \ \ [\alpha - E_3] \\ D + C \ \ [\alpha - E_2] \\ D + A \ \ [0] \end{cases}$$

Case $E_1 = \gamma, E_2 = \gamma$**.** We get that

$$A + D \longrightarrow \begin{cases} B + C \ \ [\alpha - \gamma] \\ C + C \ \ [\alpha - E_3] \\ C + B \ \ [E_1 - E_3] \\ C + A \ \ [E_2 - E_3] \\ D + C \ \ [\alpha - E_2] \\ D + A \ \ [0] \end{cases}$$

Case $E_1 > \gamma, E_2 < \gamma$**.** We get that

$$A + D \longrightarrow \begin{cases} B + C \ \ [\alpha - \gamma] \\ B + B \ \ [E_1 - \gamma] \\ C + C \ \ [\alpha - E_3] \\ D + C \ \ [\alpha - E_2] \\ D + A \ \ [0] \end{cases}$$

Case $E_1 > \gamma, E_2 = \gamma$**.** We get that

$$A + D \longrightarrow \begin{cases} B + C \ \ [\alpha - \gamma] \\ B + B \ \ [E_1 - \gamma] \\ C + C \ \ [\alpha - E_3] \\ C + A \ \ [\gamma - E_3] \\ D + C \ \ [\alpha - \gamma] \\ D + A \ \ [0] \end{cases}$$

Case $E_1 < \gamma, E_2 > \gamma$**.** We get that

$$A + D \longrightarrow \begin{cases} B + C \ \ [\alpha - \gamma] \\ B + A \ \ [E_2 - \gamma] \\ C + C \ \ [\alpha - E_3] \\ D + C \ \ [\alpha - E_2] \\ D + A \ \ [0] \end{cases}$$

Case $E_1 < \gamma, E_2 = \gamma$. We get that

$$A + D \longrightarrow \begin{cases} B + C \ [\alpha - \gamma] \\ C + C \ [\alpha - E_3] \\ C + A \ [\gamma - E_3] \\ D + C \ [\alpha - E_2] \\ D + B \ [E_1 - \gamma] \\ D + A \ [0] \end{cases}$$

For the spontaneous reaction $B + B$, we consider the three cases: Case $E_1 < \gamma$, Case $E_1 > \gamma$ and Case $E_1 = \gamma$.

Case $E_1 < \gamma$. We get that

$$B + B \longrightarrow \begin{cases} B + B \ [0] \\ A + C \ [-\beta + \gamma] \\ A + D \ [\gamma - E_1] \end{cases}$$

Case $E_1 > \gamma$. We get that

$$B + B \longrightarrow \begin{cases} B + B \ [0] \\ A + C \ [-\beta + \gamma] \end{cases}$$

Case $E_1 = \gamma$. We get that

$$B + B \longrightarrow \begin{cases} B + B \ [0] \\ A + C \ [-\beta + \gamma] \end{cases}$$

$$B + C \longrightarrow \begin{cases} C + B \ [0] \\ C + A \ [-\beta + E_3] \end{cases}$$

$$B + D \longrightarrow \begin{cases} C + C \ [-\beta + \alpha] \\ C + B \ [-\beta + E_1] \\ C + A \ [-\beta + E_2] \\ D + C \ [\alpha - E_1] \\ D + B \ [0] \end{cases}$$

$C + C \longrightarrow C + C \ [0]$
$C + D \longrightarrow D + C \ [0]$
$D + D \longrightarrow D + D \ [0]$

Remark 3.1. For all $x, y \in H$, the major product of x and y is that with the greatest potential difference.

For all $x, y \in H$, we define "$\oplus_i$" with $i = 1, 2, \ldots, 7$ on H for $E_2, E_1 < \gamma$, Case $E_2, E_1 > \gamma$, Case $E_2, E_1 = \gamma$, Case $E_1 > \gamma$, $E_2 < \gamma$, Case $E_1 > \gamma$, $E_2 = \gamma$, Case $E_1 < \gamma$, $E_2 > \gamma$, Case $E_1 < \gamma$, $E_2 = \gamma$. Where $x \oplus_i y = z_i$ and z_i is in the product of the spontaneous redox reaction with greatest potential difference that occurs between x and y.
Since the major part for all cases of E_1, E_2 under the case $\alpha > \gamma > \beta$ is the same, we get $(H, \oplus_i) = (H, \oplus_1)$ for $i = 1, 2, \ldots, 7$.

$\oplus_1$	A	B	C	D
A	A	$\{A,C\}$	$\{A,C\}$	C
B	$\{A,C\}$	$\{A,C\}$	$\{A,C\}$	C
C	$\{A,C\}$	$\{A,C\}$	C	$\{C,D\}$
D	C	C	$\{C,D\}$	D

Theorem 3.2. *$(H, \oplus_1)$ is commutative H_v-semigroup.*

Proof. $(H, \oplus_1)$ is isomorphic to the H_v-semigroup $(H, \star)$ presented in Example 2.1. □

Let $H = \{A, B, C, D\}$, for all $x, y \in H$, we define "$\otimes_1$" for the case $E_1, E_2 < \gamma$, "$\otimes_2$" for the case $E_1, E_2 > \gamma$, "$\otimes_3$" for the case $E_1 < \gamma, E_2 > \gamma$, "$\otimes_4$" for the case $E_1 > \gamma, E_2 < \gamma$, "$\otimes_5$" for the case $E_1 < \gamma, E_2 = \gamma$, "$\otimes_6$"for the case $E_1 > \gamma, E_2 = \gamma$ and "$\otimes_7$" for the case $E_1 = E_2 = \gamma$. Here, $x \otimes_i y = z$, where z is in the product of any spontaneous redox reaction that occurs between x and y for $i = 1, 2, \ldots, 7$.

Remark 3.2. $(H, \otimes_2) = (H, \otimes_6) = (H, \otimes_7)$ and $(H, \otimes_3) = (H, \otimes_5)$.

$(H, \otimes_i)$ for $i = 1, 2, \ldots, 7$ are given by tables 3.1, 3.2, 3.3, 3.4.

Table 3.1: $(H, \otimes_1)$

$\otimes_1$	A	B	C	D
A	A	H	$\{A,C\}$	H
B	H	H	$\{A,B,C\}$	H
C	$\{A,C\}$	$\{A,B,C\}$	C	$\{C,D\}$
D	H	H	$\{C,D\}$	D

Table 3.2: $(H, \otimes_2)$

$\otimes_2$	A	B	C	D
A	A	$\{A,B,C\}$	$\{A,C\}$	H
B	$\{A,B,C\}$	$\{A,B,C\}$	$\{A,B,C\}$	H
C	$\{A,C\}$	$\{A,B,C\}$	C	$\{C,D\}$
D	H	H	$\{C,D\}$	D

Table 3.3: $(H, \otimes_3)$

$\otimes_3$	A	B	C	D
A	A	$\{A, B, C\}$	$\{A, C\}$	H
B	$\{A, B, C\}$	H	$\{A, B, C\}$	H
C	$\{A, C\}$	$\{A, B, C\}$	C	$\{C, D\}$
D	H	H	$\{C, D\}$	D

Table 3.4: $(H, \otimes_4)$

$\otimes_4$	A	B	C	D
A	A	H	$\{A, C\}$	H
B	H	$\{A, B, C\}$	$\{A, B, C\}$	H
C	$\{A, C\}$	$\{A, B, C\}$	C	$\{C, D\}$
D	H	H	$\{C, D\}$	D

Remark 3.3. It is clear that $(H, \otimes_i)$, $i = 1, 2, 3, 4$ are not isomorphic.

Proposition 3.1. *$(H, \otimes_1)$ is a commutative quasi-hypergroup.*

Proof. Since all elements of H are present in every row and column, it follows that $(H, \otimes_1)$ is a quasi-hypergroup. □

Proposition 3.2. *$(\{A, C\}, \otimes_1)$ and $(\{C, D\}, \otimes_1)$ are commutative semihypergroups.*

Proof. The proof is straightforward. □

Proposition 3.3. *$(H, \otimes_1)$ is a commutative semihypergroup.*

Proof. Since $(\{A, C\}, \otimes_1)$ and $(\{C, D\}, \otimes_1)$ are hypergroups, it suffices to consider the following cases for associativity:
$A \otimes_1 (D \otimes_1 z) = (A \otimes_1 D) \otimes_1 z = H$.
$A \otimes_1 (B \otimes_1 z) = (A \otimes_1 B) \otimes_1 z = H$.
$A \otimes_1 (C \otimes_1 z) = (A \otimes_1 C) \otimes_1 z = H$ for all $z \in H - \{C\}$.
$A \otimes_1 (C \otimes_1 C) = (A \otimes_1 C) \otimes_1 C = \{A, C\}$.

$$B \otimes_1 (y \otimes_1 z) = \begin{cases} \{A, B, C\} & \text{if } y = C, z = C \\ H, & \text{otherwise} \end{cases} = (B \otimes_1 y) \otimes_1 z.$$

$$C \otimes_1 (y \otimes_1 B) = \begin{cases} \{A, B, C\} & \text{if } y = C \\ H, & \text{otherwise} \end{cases} = (C \otimes_1 y) \otimes_1 B.$$

$$C \otimes_1 (B \otimes_1 z) = \begin{cases} \{A, B, C\} & \text{if } z = C \\ H & \text{otherwise} \end{cases} = (C \otimes_1 B) \otimes_1 z.$$

$D \otimes_1 (y \otimes_1 z) = (D \otimes_1 y) \otimes_1 z = G$ for all $y, z \notin \{C, D\}$. Thus, $(H, \otimes_1)$ is a semihypergroup.. □

Theorem 3.3. *$(H, \otimes_1)$ is a commutative hypergroup.*

Proof. The proof is follows from Propositions 3.1 and 3.3. □

Theorem 3.4. *$(H, \otimes_i)$ for $i = 2, 3, 5, 6, 7$ are commutative hypergroups.*

Proof. By following the same proof done in Theorem 3.3, we get that $(H, \otimes_i)$ for $i = 2, 3, 5, 6, 7$ are commutative hypergroups. □

Theorem 3.5. *$(H, \otimes_4)$ is a commutative H_v-group.*

Proof. Easy computations show that $(H, \otimes_4)$ is a commutative H_v-group. □

3.3. Case $\alpha > \gamma = \beta$

Let $H = \{A, B, C, D\}$, it is clear that:

$$\beta = \gamma < E_1 < \alpha,\ \beta = \gamma < E_2 < \alpha,\ \beta = E_3 = \gamma.$$

The following are all possible spontaneous redox combinations for H.

$A + A \longrightarrow A + A\ [0]$

$A + B \longrightarrow B + A\ [0]$

$A + C \longrightarrow C + A\ [0]$

$$A + D \longrightarrow \begin{cases} B + C\ [\alpha - \gamma] \\ B + B\ [E_1 - \gamma] \\ B + A\ [E_2 - \gamma] \\ C + C\ [\alpha - E_3] \\ D + C\ [\alpha - E_2] \\ D + A\ [0] \end{cases}$$

$B + B \longrightarrow B + B\ [0]$

$B + C \longrightarrow C + B\ [0]$

$$B + D \longrightarrow \begin{cases} C + C\ [-\beta + \alpha] \\ C + B\ [-\beta + E_1] \\ C + A\ [\beta - E_2] \\ D + C\ [\alpha - E_1] \\ D + B\ [0] \end{cases}$$

$C + C \longrightarrow C + C\ [0]$

$C + D \longrightarrow D + C\ [0]$

$D + D \longrightarrow D + D\ [0]$

For all $x, y \in H$, we define "$\oplus_8$" on H as follows: $x \oplus_8 y = z$, where z is the major product of the spontaneous redox reaction that occurs between x and y.

We obtain the following table for $(H, \oplus_8)$:

$\oplus_8$	A	B	C	D
A	A	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$
B	$\{A,B\}$	B	$\{B,C\}$	C
C	$\{A,C\}$	$\{B,C\}$	C	$\{C,D\}$
D	$\{B,C\}$	C	$\{C,D\}$	D

Proposition 3.4. *Let $H = \{A,B,C,D\}$. Then $(H,\oplus_8)$ is a commutative H_v-semigroup.*

Proof. Let $x,y,z \in H$. We consider the following cases for x,y,z.

- If $x = y = z$ then $x \in x \oplus_8 (y \oplus_8 z) \cap (x \oplus_8 y) \oplus_8 z$.
- If $x = A$ then $A \in x \oplus_8 (y \oplus_8 z) \cap (x \oplus_8 y) \oplus_8 z$ for all $(y,z) \neq (D,D)$ and $C \in x \oplus_8 (D \oplus_8 D) \cap (x \oplus_8 D) \oplus_8 D$.
- If $x = B$ then $B \in x \oplus_8 (y \oplus_8 z) \cap (x \oplus_8 y) \oplus_8 z$ for all $(y,z) \neq (D,D)$ and $C \in x \oplus_8 (D \oplus_8 D) \cap (x \oplus_8 D) \oplus_8 D$.
- If $x = C$ then $C \in x \oplus_8 (y \oplus_8 z) \cap (x \oplus_8 y) \oplus_8 z$.
- If $x = D$ and $(y,z) \neq (D,D)$ then $C \in x \oplus_8 (y \oplus_8 z) \cap (x \oplus_8 y) \oplus_8 z$ for all $(y,z) \neq (D,D)$.

Therefore, $(H,\oplus_8)$ is a commutative H_v-semigroup. □

Let $H = \{A,B,C,D\}$, for all $x,y \in H$, we define "$\otimes_8$" on H, where $x \otimes_8 y = z$ and z is the product of any spontaneous redox reaction that occurs between x and y. Then we obtain the following tables for $(H,\otimes_8)$:

$\otimes_8$	A	B	C	D
A	A	$\{A,B\}$	$\{A,C\}$	H
B	$\{A,B\}$	B	$\{B,C\}$	H
C	$\{A,C\}$	$\{B,C\}$	C	$\{C,D\}$
D	H	H	$\{C,D\}$	D

Proposition 3.5. *$(\{A,B\},\otimes_8)$, $(\{A,C\},\otimes_8)$, $(\{B,C\},\otimes_8)$, $(\{C,D\},\otimes_8)$, and $(\{A,B,C\},\otimes_8)$ are hypergroups.*

Proof. Since $(\{A,B\},\otimes_8)$, $(\{A,C\},\otimes_8)$, $(\{B,C\},\otimes_8)$, $(\{C,D\},\otimes_8)$ and $(\{A,B,C\},\otimes_8)$ are biset hypergroups, it follows that they are hypergroups. □

Remark 3.4. Every element in $(H, \otimes_8)$ is idempotent, that is $x \otimes_8 x = x$.

Proposition 3.6. *Let $H = \{A, B, C, D\}$. Then $(H, \otimes_8)$ is semihypergroup.*

Proof. Since $(\{A, B\}, \otimes_8)$, $(\{A, C\}, \otimes_8)$, $(\{B, C\}, \otimes_8)$, $(\{C, D\}, \otimes_8)$, and $(\{A, B, C\}, \otimes_8)$ are hypergroups, it suffices to consider the following cases for associativity in Table 3.5. Thus, $(H, \otimes_8)$ is semihypergroup. $\square$

Theorem 3.6. *Let $H = \{A, B, C, D\}$. Then $(H, \otimes_8)$ is a commutative hypergroup.*

Proof. The proof is followed by Propositions 3.5 and 3.6. $\square$

3.4. Case $\alpha = \gamma > \beta$

Let $H = \{A.B, C, D\}$, it's clear that:

$$\beta < E_1 < \alpha = \gamma,\ \beta < E_2 < \alpha = \gamma,\ \beta < E_3 < \gamma = \alpha.$$

The following are all possible spontaneous redox combinations of H.

$A + A \longrightarrow A + A\ \ [0]$

$$A + B \longrightarrow \begin{cases} B + A\ \ [0] \\ C + A\ \ [\gamma - E_3] \\ A + D\ \ [\gamma - E_2] \end{cases}$$

$A + C \longrightarrow C + A\ \ [0]$

We have $E_1 - E_3 = \frac{\alpha n_1 + \beta n_2}{n_1 + n_2} - \frac{\beta n_2 + \gamma n_3}{n_2 + n_3} = n_2(n_1 - n_3)(\alpha - \beta)$.

Case $n_1 > n_3$. We get that

$$A + D \longrightarrow \begin{cases} C + C\ \ [\alpha - E_3] \\ C + B\ \ [E_1 - E_3] \\ D + C\ \ [\alpha - E_2] \\ D + A\ \ [0] \end{cases}$$

Case $n_1 \leq n_3$. We get that

$$A + D \longrightarrow \begin{cases} C + C\ \ [\alpha - E_3] \\ D + C\ \ [\alpha - E_2] \\ D + A\ \ [0] \end{cases}$$

$$B + B \longrightarrow \begin{cases} B + B\ \ [0] \\ A + C\ \ [-\beta + \gamma] \\ A + D\ \ [\gamma - E_1] \end{cases}$$

$$B + C \longrightarrow \begin{cases} C + B\ \ [0] \\ C + A\ \ [\gamma - E_3] \end{cases}$$

Table 3.5: Associativity of $(H, \otimes_8)$

$D \otimes_8 (A \otimes_8 A) = D \otimes_8 A = H$	$(D \otimes_8 A) \otimes_8 A = H \otimes_8 A = H$
$A \otimes_8 (D \otimes_8 A) = A \otimes_8 H = H$	$(A \otimes_8 D) \otimes_8 A = H \otimes_8 A = H$
$A \otimes_8 (A \otimes_8 D) = A \otimes_8 H = H$	$(A \otimes_8 A) \otimes_8 D = A \otimes_8 D = H$
$D \otimes_8 (B \otimes_8 B) = D \otimes_8 B = H$	$(D \otimes_8 B) \otimes_8 B = H \otimes_8 B = H$
$B \otimes_8 (D \otimes_8 B) = B \otimes_8 H = H$	$(B \otimes_8 D) \otimes_8 B = H \otimes_8 B = H$
$B \otimes_8 (B \otimes_8 D) = B \otimes_8 H = H$	$(B \otimes_8 B) \otimes_8 D = B \otimes_8 D = H$
$D \otimes_8 (D \otimes_8 D) = D \otimes_8 D = D$	$(D \otimes_8 D) \otimes_8 D = D \otimes_8 D = D$
$A \otimes_8 (D \otimes_8 D) = A \otimes_8 D = H$	$(A \otimes_8 D) \otimes_8 D = H \otimes_8 D = H$
$B \otimes_8 (D \otimes_8 D) = B \otimes_8 D = H$	$(B \otimes_8 D) \otimes_8 D = H \otimes_8 D = H$
$D \otimes_8 (A \otimes_8 D) = D \otimes_8 H = H$	$(D \otimes_8 A) \otimes_8 D = H \otimes_8 D = H$
$D \otimes_8 (B \otimes_8 D) = D \otimes_8 H = H$	$(D \otimes_8 B) \otimes_8 D = H \otimes_8 D = H$
$D \otimes_8 (D \otimes_8 A) = D \otimes_8 H = H$	$(D \otimes_8 D) \otimes_8 A = D \otimes_8 A = H$
$D \otimes_8 (D \otimes_8 C) = D \otimes_8 \{C, D\} = \{C, D\}$	$(D \otimes_8 D) \otimes_8 C = D \otimes_8 C = \{C, D\}$
$A \otimes_8 (B \otimes_8 D) = A \otimes_8 H = H$	$(A \otimes_8 B) \otimes_8 D = \{A, B\} \otimes_8 D = H$
$A \otimes_8 (C \otimes_8 D) = A \otimes_8 \{C, D\} = H$	$(A \otimes_8 C) \otimes_8 D = \{A, C\} \otimes_8 D = H$
$A \otimes_8 (D \otimes_8 B) = A \otimes_8 H = H$	$(A \otimes_8 D) \otimes_8 B = H \otimes_8 B = H$
$A \otimes_8 (D \otimes_8 C) = A \otimes_8 \{C, D\} = H$	$(A \otimes_8 D) \otimes_8 C = H \otimes_8 C = H$
$B \otimes_8 (A \otimes_8 D) = B \otimes_8 H = H$	$(B \otimes_8 A) \otimes_8 D = H \otimes_8 D = H$
$B \otimes_8 (C \otimes_8 D) = B \otimes_8 \{C, D\} = H$	$(B \otimes_8 C) \otimes_8 D = \{B, C\} \otimes_8 D = H$
$B \otimes_8 (D \otimes_8 A) = B \otimes_8 H = H$	$(B \otimes_8 D) \otimes_8 A = H \otimes_8 A = H$
$B \otimes_8 (D \otimes_8 C) = B \otimes_8 \{C, D\} = H$	$(B \otimes_8 D) \otimes_8 C = H \otimes_8 C = H$
$C \otimes_8 (A \otimes_8 D) = C \otimes_8 H = H$	$(C \otimes_8 A) \otimes_8 D = \{A, C\} \otimes_8 D = H$
$C \otimes_8 (B \otimes_8 D) = C \otimes_8 H = H$	$(C \otimes_8 B) \otimes_8 D = \{B, C\} \otimes_8 D = H$
$C \otimes_8 (D \otimes_8 A) = C \otimes_8 H = H$	$(C \otimes_8 D) \otimes_8 A = \{C, D\} \otimes_8 A = H$
$C \otimes_8 (D \otimes_8 B) = C \otimes_8 H = H$	$(C \otimes_8 D) \otimes_8 B = \{C, D\} \otimes_8 B = H$
$D \otimes_8 (A \otimes_8 B) = D \otimes_8 \{A, B\} = H$	$(D \otimes_8 A) \otimes_8 B = H \otimes_8 B = H$
$D \otimes_8 (A \otimes_8 C) = D \otimes_8 \{A, C\} = H$	$(D \otimes_8 A) \otimes_8 C = H \otimes_8 C = H$
$D \otimes_8 (B \otimes_8 A) = D \otimes_8 H = H$	$(D \otimes_8 B) \otimes_8 A = H \otimes_8 A = H$
$D \otimes_8 (B \otimes_8 C) = D \otimes_8 \{B, C\} = H$	$(D \otimes_8 B) \otimes_8 C = H \otimes_8 C = H$
$D \otimes_8 (C \otimes_8 A) = D \otimes_8 \{A, C\} = H$	$(D \otimes_8 C) \otimes_8 A = \{C, D\} \otimes_8 A = H$
$D \otimes_8 (C \otimes_8 B) = D \otimes_8 \{B, C\} = H$	$(D \otimes_8 C) \otimes_8 B = \{C, D\} \otimes_8 B = H$

$$B + D \longrightarrow \begin{cases} C + C \ [-\beta + \alpha] \\ C + B \ [-\beta + E_1] \\ C + A \ [-\beta + E_2] \\ D + C \ [\alpha - E_1] \\ D + B \ [0] \end{cases}$$
$$C + C \longrightarrow C + C \ [0]$$
$$C + D \longrightarrow D + C \ [0]$$
$$D + D \longrightarrow D + D \ [0]$$

For all $x, y \in H$, we define "$\oplus_9$" on H for the case $n_1 > n_3$ and the case $n_1 \leq n_3$, where $x \oplus_9 y = z$, where z is the major product. We obtain the table below for $(H, \oplus_9)$:

$\oplus_9$	A	B	C	D
A	A	$\{A, C\}$	$\{A, C\}$	C
B	$\{A, C\}$	$\{A, C\}$	$\{A, C\}$	C
C	$\{A, C\}$	$\{A, C\}$	C	$\{C, D\}$
D	C	C	$\{C, D\}$	D

Theorem 3.7. *$(H, \oplus_9)$ is a commutative H_v-semigroup.*

Proof. $(H, \oplus_9)$ is isomorphic to $(H, \oplus_1)$ in Theorem 3.2. □

Let $H = \{A, B, C, D\}$, for all $x, y \in H$, we define "$\otimes_9$" on H for the case $n_1 > n_3$, "$\otimes_{10}$" for the case $n_1 \leq n_3$, where $x \otimes_9 y = z$ where z is the product of any spontaneous redox reaction that occurs between x and y. Then we obtain the tables for $(H, \otimes_9)$ and $(H, \otimes_{10})$:

$\otimes_9$	A	B	C	D
A	A	H	$\{A, C\}$	H
B	H	H	$\{A, B, C\}$	H
C	$\{A, C\}$	$\{A, B, C\}$	C	$\{C, D\}$
D	H	H	$\{C, D\}$	D

Proposition 3.7. *$(H, \otimes_9)$ is a commutative hypergroup.*

Proof. $(H, \otimes_9)$ is isomorphic to $(H, \otimes_1)$ in Theorem 3.3. □

Proposition 3.8. *$(H, \otimes_{10})$ is a commutative hypergroup.*

$\otimes_{10}$	A	B	C	D
A	A	H	$\{A, C\}$	$\{A, C, D\}$
B	H	H	$\{A, B, C\}$	H
C	$\{A, C\}$	$\{A, B, C\}$	C	$\{C, D\}$
D	$\{A, C, D\}$	H	$\{C, D\}$	D

Proof. It is clear that $(H, \otimes_{10})$ is a quasi-hypergroup since all elements of H are present in every row and column, so it remains to prove that $(H, \otimes_{10})$ is a semi-hypergroup, i.e. associative.
Since $(\{a, c\}, \otimes_{10})$ and $(\{a, c, d\}, \otimes_{10})$ are associative, then it suffices to consider the cases for associativity in a similar way that is done in Table 3.5. $\square$

Remark 3.5. In some of the above spontaneous reactions, not all spontaneous reactions are considered because their presence or absence does not affect our results.

4. Examples of Chemical hyperstructures

In this section, we present some elements with four oxidation states and identify their chemical hyperstructure. In particular, we present Copper and Astatine as an examples under the case $\alpha > \gamma > \beta$ with $E_1 > \gamma$ and $E_2 > \gamma$, Iridium under the case $\alpha > \gamma = \beta$ with $E_1, E_2 > \gamma$, and Uranium under the case $\alpha > \gamma > \beta$ with $E_1 > \gamma$ and $E_2 > \gamma$.

Example 4.1. Copper, denoted as Cu, is a soft and ductile element with very high thermal and electrical conductivity. Copper is one of the few metals that can occur in nature in a directly usable metallic form native metals. The Latimer diagram of Copper in acid solution satisfying the condition $\alpha \geq \gamma \geq \beta$ is given as follows:

$$CuO^+ \longrightarrow_{1.8} Cu^{2+} \longrightarrow_{0.159} Cu^+ \longrightarrow_{0.521} Cu.$$

Copper has four different oxidation states: $+3, +2, +1$, and 0. We denote CuO^+ by Cu^{3+}. We have $\alpha = 1.8, \beta = 0.159, \gamma = 0.521$ and $n_1 = n_2 = n_3 = 1$. Since $E_1 = \frac{(1.8)(1)+(0.159)(1)}{2} = 0.979 > \gamma$, and $E_2 = \frac{(1.8)(1)+(0.159)(1)+(0.521)(1)}{3} = 0.826 > \gamma$, it follows that:

- $(H = \{Cu, Cu^+, Cu^{2+}, Cu^{3+}\}, \oplus_2)$ is a commutative H_v-semigroup.

According to Table 4.1, Cu^{2+} is the most common oxidation state.

- $(H = \{Cu, Cu^+, Cu^{2+}, Cu^{3+}\}, \otimes_2)$ is a commutative hypergroup with the following table:

Every element e in H is identity since for all $x \in H, x \in x \otimes e \cap e \otimes x$.

Example 4.2. [2] Astatine denoted as At is a radioactive chemical element and it is the heaviest known halogen.The Latimer diagram of Astatine in base solution satisfying the condition $\alpha > \gamma > \beta$ is given as follows:

Table 4.1: $(\{Cu, Cu^+, Cu^{2+}, Cu^{3+}\}, \oplus_2)$

$\oplus_2$	Cu	Cu^+	Cu^{2+}	Cu^{3+}
Cu	Cu	$\{Cu, Cu^{2+}\}$	$\{Cu, Cu^{2+}\}$	Cu^{2+}
Cu^+	$\{Cu, Cu^{2+}\}$	$\{Cu, Cu^{2+}\}$	$\{Cu, Cu^{2+}\}$	Cu^{2+}
Cu^{2+}	$\{Cu, Cu^{2+}\}$	$\{Cu, Cu^{2+}\}$	Cu^{2+}	$\{Cu^{2+}, Cu^{3+}\}$
Cu^{3+}	Cu^{2+}	Cu^{2+}	$\{Cu^{2+}, Cu^{3+}\}$	Cu^{3+}

Table 4.2: $(\{Cu, Cu^+, Cu^{2+}, Cu^{3+}\}, \otimes_2)$

$\otimes_2$	Cu	Cu^+	Cu^{2+}	Cu^{3+}
Cu	Cu	$\{Cu, Cu^+, Cu^{2+}\}$	$\{Cu, Cu^{2+}\}$	H
Cu^+	$\{Cu, Cu^+, Cu^{2+}\}$	$\{Cu, Cu^+, Cu^{2+}\}$	$\{Cu, Cu^+, Cu^{2+}\}$	H
Cu^{2+}	$\{Cu, Cu^{2+}\}$	$\{Cu, Cu^+, Cu^{2+}\}$	Cu^{2+}	$\{Cu^{2+}, Cu^{3+}\}$
Cu^{3+}	H	H	$\{Cu^{2+}, Cu^{3+}\}$	Cu^{3+}

$$At^{5+} \longrightarrow_{0.5} At^+ \longrightarrow_0 At_2 \longrightarrow_{0.2} At^-.$$

Al-Tahan et al. in [2], studied the H_v-semigroup and hypergroup associated to Astatine. Their results on Astatine can be also concluded from our results of Section 3.

Astatine has four oxidation states: $+5, +1, 0, -1$. We have $\alpha = 0.5, \beta = 0, \gamma = 0.2$ and $n_1 = 4$, $n_2 = 1$, $n_3 = 1$. Since $E_1 = \frac{(0.5)(4)+(0)(1)}{5} = 0.4 > \gamma$ and $E_2 = \frac{(0.5)(4)+(0)(1)+(0.2)(1)}{6} = 0.367 > \gamma$, we get the following results:

• $(H = \{At^-, At_2, At^+, At^{5+}\}, \oplus_2)$ is a commutative H_v-semigroup.

Table 4.3: $(\{At^-, At_2, At^+, At^{5+}\}, \oplus_2)$

$\oplus_2$	At^-	At_2	At^+	At^{5+}
At^-	At^-	$\{At^-, At^+\}$	$\{At^-, At^+\}$	At^{5+}
At_2	$\{At^-, At^+\}$	$\{At^-, At^+\}$	$\{At^-, At^+\}$	At^+
At^+	$\{At^-, At^+\}$	$\{At^-, At^+\}$	At^+	$\{At^+, At^{5+}\}$
At^{5+}	At^+	At^+	$\{At^+, At^{5+}\}$	At^{5+}

According to Table 4.3, At^+ is the most common oxidation state.

• $(H = \{At^-, At_2, At^+, At^{5+}\}, \otimes_2)$ is a commutative $H_v - semigroup.$

According to Table 4.4, At^{5+} is the least common oxidation state.

Example 4.3. Uranium (U) is a metallic, silver-gray element that is a member of the actinide series. It is the principle fuel for nuclear reactors, but it also used in the manufacture of nuclear weapons. The Latimer diagram of Uranium in base solution satisfying the condition $\alpha > \gamma > \beta$ is given as follows:

Table 4.4: $(\{At^-, At_2, At^+, At^{5+}\}, \otimes_2)$

$\otimes_2$	At^-	At_2	At^+	At^{5+}
At^-	At^-	$\{At^-, At_2, At^+\}$	$\{At^-, At^+\}$	H
At_2	$\{At^-, At_2, At^+\}$	$\{At^-, At_2, At^+\}$	$\{At^-, At_2, At^+\}$	H
At^+	$\{At^-, At^+\}$	$\{At^-, At_2, At^+\}$	At^+	$\{At^+, At^{5+}\}$
At^{5+}	H	H	$\{At^+, At^{5+}\}$	At^{5+}

$$U^{6+} \longrightarrow_{-0.3} U^{4+} \longrightarrow_{-2.6} U^{3+} \longrightarrow_{-2.1} U.$$

Uranium has four oxidation states: $+6, +4, +3, 0$. We have $\alpha = -0.3, \beta = -2.6, \gamma = -2.1$. Since $E_1 = \frac{(-0.3)(2)+(-2.6)(1)}{3} = -1.06 > \gamma$ and $E_2 = \frac{(-0.3)(2)+(-2.6)(1)+(3)(-2.1)}{6} = -1.58 > \gamma$. We get the following results:

• $(H = \{U, U^{3+}, U^{4+}, U^{6+}\}, \oplus_2)$ is a commutative H_v-semigroup.

Table 4.5: $(\{U, U^{3+}, U^{4+}, U^{6+}\}, \oplus_2)$

$\oplus_2$	U	U^{3+}	U^{4+}	U^{6+}
U	U	$\{U, U^{4+}\}$	$\{U, U^{4+}\}$	U^{4+}
U^{3+}	$\{U, U^{4+}\}$	$\{U, U^{4+}\}$	$\{U, U^{4+}\}$	U^{4+}
U^{4+}	$\{U, U^{4+}\}$	$\{U, U^{4+}\}$	U^{4+}	$\{U^{4+}, U^{6+}\}$
U^{6+}	U^{4+}	U^{4+}	$\{U^{4+}, U^{6+}\}$	U^{6+}

According to Table 4.5, U^{4+} is the most common oxidation state.

• $(H = \{U, U^{3+}, U^{4+}, U^{6+}\}, \otimes_2)$ is a hypergroup with the following table.

Table 4.6: $(\{U, U^{3+}, U^{4+}, U^{6+}\}, \otimes_2)$

$\otimes_2$	U	U^{3+}	U^{4+}	U^{6+}
U	U	$\{U, U^{3+}, U^{4+}\}$	$\{U, U^{4+}\}$	H
U^{3+}	$\{U, U^{3+}, U^{4+}\}$	$\{U, U^{3+}, U^{4+}\}$	$\{U, U^{3+}, U^{4+}\}$	H
U^{4+}	$\{U, U^{4+}\}$	$\{U, U^{3+}, U^{4+}\}$	U^{4+}	$\{U^{4+}, U^{6+}\}$
U^{6+}	H	H	$\{U^{4+}, U^{6+}\}$	U^{6+}

According to Table 4.6, U^{6+} is the least common oxidation state.

Example 4.4. Iridium (Ir) is a chemical element, a very hard, brittle, silvery-white transition metal of the platinum group. The Latimer diagram of Iridium in base solution satisfying the condition $\alpha > \gamma = \beta$ is given as follows:

$$Ir^{6+} \longrightarrow_{0.4} Ir^{4+} \longrightarrow_{0.1} Ir^{3+} \longrightarrow_{0.1} Ir.$$

Iridium has four oxidation states $+6, +4, +3, 0$. We have $\alpha = 0.4, \beta = \gamma = 0.1$ and $n_1 = 2, n_2 = 1, n_3 = 3$. Since $E_1 = \frac{(0.4)(2)+(0.1)(1)}{3} = 0.3 > \gamma$ and $E_2 = \frac{(0.4)(2)+(0.1)(1)+(0.1)(3)}{6} = 0.2 > \gamma$, we get the following results:

• $(H = \{Ir^{6+}, Ir^{4+}, Ir^{3+}, Ir\}, \oplus_7)$ is a commutative H_v-semigroup with the following table:

Table 4.7: $(\{Ir^{6+}, Ir^{4+}, Ir^{3+}, Ir\}, \oplus_7)$

$\oplus_7$	Ir	Ir^{3+}	Ir^{4+}	Ir^{6+}
Ir	Ir	$\{Ir, Ir^{3+}\}$	$\{Ir, Ir^{4+}\}$	$\{Ir^{3+}, Ir^{4+}\}$
Ir^{3+}	$\{Ir, Ir^{3+}\}$	Ir^{3+}	$\{Ir^{3+}, Ir^{4+}\}$	Ir^{4+}
Ir^{4+}	$\{Ir, Ir^{4+}\}$	$\{Ir^{3+}, Ir^{4+}\}$	Ir^{4+}	$\{Ir^{4+}, Ir^{6+}\}$
Ir^{6+}	$\{Ir^{3+}, Ir^{4+}\}$	Ir^{4+}	$\{Ir^{4+}, Ir^{6+}\}$	Ir^{6+}

According to above table 4.7, Ir^{6+} is the least common oxidation state.

• $(H = \{Ir^{6+}, Ir^{4+}, Ir^{3+}, I\}, \otimes_7)$ is a commutative hypergroup.

Table 4.8: $(\{Ir^{6+}, Ir^{4+}, Ir^{3+}, Ir\}, \otimes_7)$

$\otimes_7$	Ir	Ir^{3+}	Ir^{4+}	Ir^{6+}
Ir	Ir	$\{Ir, Ir^{3+}\}$	$\{Ir, Ir^{4+}\}$	H
Ir^{3+}	$\{Ir, Ir^{3+},\}$	$\{Ir^{3+}\}$	$\{Ir^{3+}, Ir^{4+}\}$	H
Ir^{4+}	$\{Ir, Ir^{4+}\}$	$\{Ir^{3+}, Ir^{4+}\}$	Ir^{4+}	$\{Ir^{4+}, Ir^{6+}\}$
Ir^{6+}	H	H	$\{Ir^{4+}, Ir^{6+}\}$	Ir^{6+}

According to Table 4.8, Ir^{4+} is the most common oxidation state.

5. Conclusion

This paper dealt with non-isomorphic (H_v-semigroups) hypergroups of an element with four oxidation states which has the largest number of examples among all cases for elements with four oxidation states.
For future work, it will be interesting to generalize our work to arbitrary elements with k- oxidation states.

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FUZZY SUBSETS OF THE PHENOTYPES OF F_2-OFFSPRING

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Abstract. This paper presents a connection between fuzzy sets, biological inheritance and hyperstructures in which we consider the set of phenotypes of the second generation F_2 in different types of inheritance, define fuzzy subsets of it and construct a sequence of join spaces associated to each of its types.
Keywords. Hyperstructures; fuzzy subsets; join spaces; hypergroups; automata theory.

1. Introduction

The Hyperstructure theory was introduced in 1934, at the eighth Congress of Scandinavian Mathematicians, when F. Marty [17] defined hypergroups as natural generalization of the concept of group based on the notion of hyperoperation, analyzed their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [4, 6, 7]). A hypergroup is an algebraic structure similar to a group, but the composition of two elements is a non-empty set. One of motivations for the study of hyperstructures comes from biological inheritance. In [10], M. Ghadiri and B. Davvaz used the concept of H_v-semigroup structure in the F_2-genotypes with cross operation and proved that it is an H_v-semigroup and they determined the kinds of the H_v-subsemigroups of F_2-genotypes (see also [7]). Another motivation for the study of hyperstructures comes from physical phenomenon as the nuclear fission. This motivation and the results were presented by S. Hošková, J. Chvalina and P. Račková (see [13], [14]). In [9], the

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authors provided, for the first time, a physical example of hyperstructures associated with the elementary particle physics, Leptons. They have considered this important group of the elementary particles and shown that this set along with the interactions between its members can be described by the algebraic hyperstructures. On the other hand, the concept of fuzzy sets has been introduced by L. A. Zadeh in 1965 (see [29]) as an extension of the classical notion of set, when he proposed the idea of a multi-valued logic, which extends the traditional concept of a bivalent logic, which becomes a particular case of the new theory. The fuzzy set theory is based on the principle called by L. A. Zadeh "the principle of incompatibility", that is "the closer a phenomenon is studied, the more indistinct its definition becomes". Fuzzy sets are sets whose elements have degrees of membership. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0,1]$. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.

The early theories of heredity were those of Greek scientists (Hippocrates and Aristotle); their theories were similar to Darwin's later ideas on Pangenesis. The latter states that the whole of parental organisms participate in heredity while adapting to cell theory. Much of Darwin's model was speculatively based on inheritance of tiny heredity particles that could be transmitted from parent to offspring [5]. The hypothesis was eventually replaced by Mendel's laws of inheritance where Gregor Mendel first traced patterns of certain traits in pea plants and showed that they obeyed certain statistical rules. Scientific studies of Mendelian inheritance began in 1866 with the experiments of Mendel, the founder of modern genetics [18]. Mendel worked out the mathematical rules for the inheritance of characteristics in the garden pea. The significance of his discovery was not recognized until 1900, when three botanists: Hugo de Vries, Carl Correns and Erich von Tschermak began independently conducting similar experiments with plants and arrived at conclusions similar to those of Mendel. Coming across Mendel's paper, they interpreted their results in accord with his principles and drew attention to his pioneering work. And by 1915 the basic principles of Mendelian genetics had been applied to a wide variety of organisms. Mendel discovered the principles of heredity by crossing different varieties of pea plants and analyzing the transmission pattern of traits in subsequent generations. He began by studying monohybrid crosses, those between parents that differed in a single characteristic. Mendel's approach to the study of heredity was effective for several reasons. The foremost was his choice of an experimental subject, the pea plant, Pisum sativum, which offered obvious advantages for genetic investigations. It is easy to cultivate, and Mendel had a monastery garden and a greenhouse at his disposal. Peas grow relatively rapidly, completing an entire generation in a single growing season. Mendel started with 34 varieties of peas and spent two years selecting those varieties that he would use in his experiments [20]. In [7, 10], Davvaz et al. studied the connection between weak algebraic hyperstructures and inheritance.

The aim of this paper is to investigate a new connection between fuzzy sets and biological inheritance. More precisely, we consider the set of phenotypes of F_2 under mating and define a fuzzy subset of it. Our paper is constructed as follows: after an introduction, Section 2 presents some definitions that are used throughout the paper. Section 3 presents fuzzy sets and join spaces associated to the set of phenotypes of F_2 for the cases of simple and incomplete inheritance. Finally, Section 4 presents fuzzy sets and join spaces associated to the set of phenotypes of F_2 for some examples of non Mendelian inheritance.

Throughout this paper, parents is denoted by P, first generation by F_1 and second generation by F_2.

2. Basic definitions

In this section, we present some definitions related to hyperstructures (see [1]), fuzzy sets (see [2, 3, 29]) and to biological inheritance (see [11, 12, 18]) that are used throughout the paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$, for all $x \in H$; it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$, for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$, for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup. A *canonical hypergroup* [19] is a non-empty set H endowed with a hyperoperation $\circ : H \times H \to \mathcal{P}^*(H)$, satisfying the following properties: (1) for any $x, y, z \in H, x \circ (y \circ z) = (x \circ y) \circ z$, (2) for any $x, y \in H, x \circ y = y \circ x$, (3) there exists $\imath \in H$ such that $\imath \circ x = x \circ \imath = x$, for any $x \in H$,(4) for every $x \in H$, there exists a unique element x' (or denote by x^{-1}) and we call it the inverse of x), (5) $z \in x \circ y$ implies that $y \in x' \circ z$ and $x \in z \circ y'$, that is $(H, \circ)$ is *reversible*. Two hypergroups $(H, \circ)$ and $(K, \star)$ are said to be *isomorphic hypergroups* if there exists a bijective function $f : H \to K$ such that $f(x \circ y) = f(x) \star f(y)$ for all $x, y \in H$.

Join spaces were introduced by W. Prenowitz [21, 22] and then applied by him and J. Jantosciak [23] in Euclidian as well as in non Euclidian geometry. Also, see [15, 16]. Using this notion, several branches of non Euclidian geometry were rebuilt: descriptive geometry, projective geometry and spherical geometry. Then, several important examples of join spaces have been constructed in connection with binary relations, graphs, lattices, rough sets. In order to define a join space, we need the following notation: If a, b are elements of a hypergroupoid $(H, \circ)$, then we denote $a/b = \{x \in H \mid a \in x \circ b\}$. A commutative hypergroup $(A, \star)$ is called a *join space* if for all α_1, α_2, α_3, $\alpha_4 \in A$ the following implication is true

$$\alpha_1/\alpha_2 \cap \alpha_3/\alpha_4 \neq \emptyset \Rightarrow \alpha_1 \star \alpha_4 \cap \alpha_2 \star \alpha_3 \neq \emptyset.$$

The first connection between fuzzy sets and hyperstructures was established by Corsini, when he defined a hyperoperation by means of fuzzy subsets. More precisely, let $\mu : H \to [0, 1]$ be a fuzzy subset of a nonempty set H. Define on H the hyperoperation $\star_1$, setting, for any $x, y \in H$,

$$(w') : x \star_1 y = y \star_1 x = \{z \in H : \min(\mu(x), \mu(y)) \leq \mu(z) \leq \max(\mu(x), \mu(y))\}.$$

The associated hypergroupoid $({}^1H, \star_1)$ is a join space. Also, he defined fuzzy subsets from hypergroups in the following manner: For any hypergroup $(H, \star)$, he defined a fuzzy subset $\mu : H \to [0, 1]$ of H in the following way: for $u \in H$ consider

$$(w) : \mu(u) = \frac{\sum_{(x,y)\in Q(u)} \frac{1}{|x \star y|}}{q(u)},$$

where $Q(u) = \{(a, b) \in H^2 : u \in a \star b\}$ and $q(u) = |Q(u)|$ (see [2, 3]).

Let $({}^1H, \star_1)$ be the join space obtained by applying the fuzzy subset μ as defined in (w'). By using (w) we get μ_1 and using the same procedure as in (w'), from ${}^1H$ we can obtain a membership function μ_2 and the associated join space ${}^2H$ and so on. A sequence of fuzzy sets and join spaces $(({}^iH, \star_i), \mu_i)_{i\geq 1}$ is determined in this way. If two consecutive hypergroups of the obtained sequence are isomorphic, then the sequence stops. The length of the sequence of join spaces associated with H is called the fuzzy grade of H. A hypergroupoid H has a fuzzy grade $m \in \mathbb{N}$, written as $f.g(H) = m$ if for all i, $0 \leq i < m$, the join spaces ${}^iH$ and ${}^{i+1}H$ are not isomorphic and for all $s > m$, ${}^sH$ and ${}^mH$ are isomorphic. If $f.g(H) = m$ and ${}^sH = {}^mH$, for all $s > m$, we say that the strong fuzzy grade of H; $s.f.g(H) = m$. Such construction of join spaces is important for at least two reasons: it provides examples of hypergroup structures on a given set and it gives the possibility of studying fuzzy sets in an algebraic approach. On the other hand, the construction could start either from a fuzzy subset or from a hypergroup structure on a nonempty set H.

Inheritance involves the passing of discrete units of inheritance, or genes, from parents to offspring. Gregor Mendel [18], the first who introduced the notion of inheritance explicitly in 1865, found that paired pea traits were either dominant or recessive. When pure bred parent plants (P) were cross bred, dominant traits were always seen in the progeny, whereas recessive traits were hidden until the first generation (F_1) hybrid plants were left to self pollinate. Mendel observed that in the second generation (F_2), the traits of the P generation reappeared. He concluded that traits were not blended but remained distinct in subsequent generations, which was contrary to scientific opinion at that time. Mendel didn't know about genes or discover genes, but he did speculate that there were two factors for each basic trait and that one factor was inherited from each parent. We now know that Mendel's inheritance factors are genes, or more specifically alleles (different variants of the same gene). In today's genetic language, a pure-breeding pea plant line is a homozygous (it has two identical copies of the same allele; AA) and an F_1 cross-bred pea plant is a heterozygous (it has two different alleles; AB). There are some exceptions to Mendel's principles, which have been discovered as our knowledge of genes and inheritance has increased. The principle of independent assortment doesn't apply if the genes are close together (or linked) on a chromosome. Also, alleles do not always interact in a standard dominant/recessive way (simple inheritance), particularly if they are codominant or have differences in expressivity or penetrance (incomplete inheritance). In the simple inheritance, we have two alleles (A dominant over a). The presence of the dominant allele in the genotype of an organism (AA or Aa) leads to the presence of its corresponding phenotype and its absence (aa) leads to the presence of the corresponding phenotype of the recessive trait. In the case of codominance, a cross between organisms with two different phenotypes (observed traits) produces offspring with a third phenotype that is a blending of the parental traits. For example, the cross of white and red flowers that results in the appearance of pink flowers (or white flowers with red spots) in the offspring is a good example on the codominance criteria.

Inheritance is linked to statistics in a way that we may find the probability of having a specific trait in the offspring. For example the monohybrid cross of parents with Aa genotypes in the case of simple inheritance gives offsprings having trait corresponding to A with a probability $\frac{2}{3}$ and offsprings having trait corresponding to a with a probability $\frac{1}{3}$.

3. Fuzzy sets associated to simple and incomplete inheritance

In this section, we consider hypothetical crosses of homozygous with independent number of alleles in the cases: simple inheritance, incomplete inheritance, simple and incomplete inheritance combined together. We define a fuzzy subset of the set of phenotypes of the second generation under mating ($\times$) and construct sequence of join spaces for each case.

Let H be the set of phenotypes in F_2 and define $\mu : H \longrightarrow [0,1]$ by $\mu(x) =$ probability of x for all $x \in H$. It is obvious that μ is a fuzzy subset of H.

3.1. Simple inheritance

Let A_i be the dominant allele over a_i for $i = 1, \dots, n$ and $\{A_1, \dots, A_n\}$, $\{a_1, \dots, a_n\}$ be two sets of independent alleles. We consider first results for the Monohybrid cross ($n = 1$) that differs in a single trait; a homozygous parent (A_1A_1) $\times$ a homozygous parent (a_1a_1). The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } A_1A_1 \times a_1a_1$$
$$F_1 : A_1a_1$$
$$\text{and}$$
$$F_1 \times F_1 : A_1a_1 \times A_1a_1$$
$$F_2 : \widehat{B_1} \text{ (of genotype } A_1A_1 \text{ or } A_1a_1\text{)}, \widehat{B_2} \text{ (of genotype } a_1a_1\text{)}.$$

Let $H = \{\widehat{B_1}, \widehat{B_2}\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(\widehat{B_1}) = \frac{2}{3}$ and $\mu(\widehat{B_2}) = \frac{1}{3}$.

Proposition 3.1. *Let $H = \{\widehat{B_1}, \widehat{B_2}\}$ be the set of phenotypes in F_2. By definitions of (w) and (w'), we have $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2})$ and $S.F.G(H) = 2$.*

Proof. Using (w'), we may present $({}^1H, \star_1)$ by the following table:

${}^1H$	$\widehat{B_1}$	$\widehat{B_2}$
$\widehat{B_1}$	$\{\widehat{B_1}\}$	H
$\widehat{B_2}$		$\{\widehat{B_2}\}$

Having $q(\widehat{B_1}) = q(\widehat{B_2}) = 3$ and $A(\widehat{B_1}) = A(\widehat{B_2}) = \frac{1}{1} + \frac{2}{2} = 2$ implies that $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \frac{2}{3}$.

$({}^2H, \star_2)$ can be presented by the following table:

${}^2H$	$\widehat{B_1}$	$\widehat{B_2}$
$\widehat{B_1}$	H	H
$\widehat{B_2}$		H

It is clear that $({}^2H, \star_2)$ is the total hypergroup. Therefore, $S.F.G(H) = 2$. □

We consider now results for the Dihybrid cross $(n = 2)$ that differs in two traits; a homozygous parent $(A_1A_1A_2A_2) \times$ a homozygous parent $(a_1a_1a_2a_2)$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } A_1A_1A_2A_2 \times a_1a_1a_2a_2$$
$$F_1 : A_1a_1A_2a_2$$
$$\text{and}$$
$$F_1 \times F_1 : A_1a_1A_2a_2 \times A_1a_1A_2a_2$$
$$F_2 : \widehat{B_1} \text{ (of genotype } A_1x_1A_2x_2\text{)}, \widehat{B_2} \text{ (of genotype } A_1x_1a_2a_2\text{)}, \widehat{B_3} \text{ (of genotype } a_1a_1A_2x_2\text{) and } \widehat{B_4} \text{ (of genotype } a_1a_1a_2a_2\text{)}.$$

Here, $x_i \in \{A_i, a_i\}$ for $i = 1, 2$.

Let $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}, \widehat{B_4}\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(\widehat{B_1}) = \frac{4}{9}$, $\mu(\widehat{B_2}) = \mu(\widehat{B_3}) = \frac{2}{9}$ and $\mu(\widehat{B_4}) = \frac{1}{9}$.

Proposition 3.2. *Let* $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}, \widehat{B_4}\}$. *By definitions of* (w) *and* (w'), *we have* $\mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3}) < \mu_1(\widehat{B_1}) = \mu_1(\widehat{B_4})$.

Proof. Using (w'), we may present $({}^1H, \star_1)$ by the following table:

${}^1H$	$\widehat{B_1}$	$\widehat{B_2}$	$\widehat{B_3}$	$\widehat{B_4}$
$\widehat{B_1}$	$\{\widehat{B_1}\}$	$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	H
$\widehat{B_2}$		$\{\widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_2}, \widehat{B_3}, \widehat{B_4}\}$
$\widehat{B_3}$			$\{\widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_2}, \widehat{B_3}, \widehat{B_4}\}$
$\widehat{B_4}$				$\{\widehat{B_4}\}$

Having $q(\widehat{B_1}) = q(\widehat{B_4}) = 7$, $A(\widehat{B_1}) = A(\widehat{B_4}) = \frac{1}{1} + \frac{2}{3} + \frac{2}{3} + \frac{2}{4} = \frac{17}{6}$, $q(\widehat{B_2}) = q(\widehat{B_3}) = 14$, $A(\widehat{B_2}) = A(\widehat{B_3}) = \frac{2}{3} + \frac{2}{3} + \frac{2}{4} + \frac{1}{2} + \frac{2}{2} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} = \frac{31}{6}$ implies that $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_4}) = \frac{17}{42}$ and $\mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3}) = \frac{31}{84}$. $\square$

Proposition 3.3. *Let* $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}, \widehat{B_4}\}$. *By definitions of* (w) *and* (w'), *we have* $\mu_2(\widehat{B_1}) = \mu_2(\widehat{B_2}) = \mu_2(\widehat{B_3}) = \mu_2(\widehat{B_4})$ *and* $S.F.G(H) = 3$.

Proof. We may present $({}^2H, \star_2)$ by the following table:

${}^2H$	$\widehat{B_1}$	$\widehat{B_2}$	$\widehat{B_3}$	$\widehat{B_4}$
$\widehat{B_1}$	$\{\widehat{B_1}, \widehat{B_4}\}$	H	H	$\{\widehat{B_1}, \widehat{B_4}\}$
$\widehat{B_2}$		$\{\widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_2}, \widehat{B_3}\}$	H
$\widehat{B_3}$			$\{\widehat{B_2}, \widehat{B_3}\}$	H
$\widehat{B_4}$				$\{\widehat{B_1}, \widehat{B_4}\}$

Simple computations shows that $q(\widehat{B_1}) = g(\widehat{B_2}) = q(\widehat{B_3}) = q(\widehat{B_4})$, $A(\widehat{B_1}) = A(\widehat{B_2}) = A(\widehat{B_3}) = A(\widehat{B_4})$ and thus $\mu_2(\widehat{B_1}) = \mu_2(\widehat{B_2}) = \mu_2(\widehat{B_3}) = \mu_2(\widehat{B_4})$. The latter implies that $({}^3H, \star_3)$ is the total hypergroup and hence, $S.F.G(H) = 3$. $\square$

3.2. Case of incomplete inheritance

Let B_i and $\overline{B_i}$ be codominant alleles for $i = 1, \ldots, n$ and $\{B_1, \ldots, B_n\}$, $\{\overline{B_1}, \ldots, \overline{B_n}\}$ be two sets of independent alleles. We consider first results for the Monohybrid cross ($n = 1$) that differs in a single trait; a homozygous parent (B_1B_1) $\times$ a homozygous parent $(\overline{B_1}\ \overline{B_1})$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } B_1B_1 \times \overline{B_1}\ \overline{B_1}$$
$$F_1 : B_1\overline{B_1}$$
and
$$F_1 \times F_1 : B_1\overline{B_1} \times B_1\overline{B_1}$$
$F_2 : \widehat{B_1}$ (of genotype B_1B_1), $\widehat{B_2}$ (of genotype $B_1\overline{B_1}$) and $\widehat{B_3}$ (of genotype $\overline{B_1}\ \overline{B_1}$).

Let $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(\widehat{B_1}) = \mu(\widehat{B_2}) = \mu(\widehat{B_3}) = \frac{1}{3}$.

Proposition 3.4. *Let $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$. definitions of (w) and (w'), we have $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3})$ and $S.F.G(H) = 1$.*

Proof. Using (w'), we may present $({}^1H, \star_1)$ by the following table:

${}^1H$	$\widehat{B_1}$	$\widehat{B_2}$	$\widehat{B_3}$
$\widehat{B_1}$	H	H	H
$\widehat{B_2}$		H	H
$\widehat{B_3}$			H

It is clear that $({}^1H, \star_1)$ is the total hypergroup and thus $S.F.G(H) = 1$. $\square$

We give next a generalization of Proposition 3.4 by considering the n- hybrid case of incomplete inheritance that differs in n traits; a homozygous parent $(B_1B_1 \ldots B_nB_n)$ $\times$ a homozygous parent $(\overline{B_1}\ \overline{B_1} \ldots \overline{B_n}\ \overline{B_n})$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } B_1B_1 \ldots B_nB_n \times \overline{B_1}\ \overline{B_1} \ldots \overline{B_1}\ \overline{B_1}$$
$$F_1 : B_1\overline{B_1} \ldots B_n\overline{B_n}$$
and
$$F_1 \times F_1 : B_1\overline{B_1} \ldots B_n\overline{B_n} \times B_1\overline{B_1} \ldots B_n\overline{B_n}$$
$F_2 : \widehat{B_1}$ (of genotype $B_1B_1 \ldots B_nB_n$), $\widehat{B_2}$ (of genotype $B_1B_1 \ldots B_{n-1}B_{n-1}B_n\overline{B_n}$), ..., and $\widehat{B_k}$ (of genotype $\overline{B_1}\ \overline{B_1} \ldots \overline{B_n}\ \overline{B_n}$).

The number of different phenotypes is $k = 3^n$. Let $H = \{\widehat{B_1}, \ldots, \widehat{B_k}\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(\widehat{B_1}) = \mu(\widehat{B_2}) = \ldots = \mu(\widehat{B_k}) = \frac{1}{k}$.

Theorem 3.1. *Let $H = \{\widehat{B_1}, \ldots, \widehat{B_k}\}$. By definitions of (w) and (w'), we have $\mu_1(\widehat{B_1}) = \ldots = \mu_1(\widehat{B_k})$ and $S.F.G(H) = 1$.*

Proof. Using the definition of (w'), we have $\widehat{B_i} \star_1 \widehat{B_j} = \{z \in H : \mu(z) = \frac{1}{k}\} = H$ for all $i, j \in \{1, \ldots, k\}$. Thus, $({}^1H, \star_1)$ is the total hypergroup. $\square$

3.3. Case of simple and incomplete inheritance combined together

Let A_i be the dominant allele over a_i for $i = 1, \ldots, m$ and $B_j, \overline{B_j}$ be the codominant alleles for $j = 1, \ldots, n$. This case can be given by considering the $(m+n)$- hybrid case that differs in $(m+n)$ traits; a homozygous parent $(A_1A_1 \ldots A_mA_mB_1B_1 \ldots B_nB_n)$ $\times$ a homozygous parent $(a_1a_1 \ldots a_ma_m\overline{B_1}\ \overline{B_1} \ldots \overline{B_n}\ \overline{B_n})$.

We consider first the case $m = n = 1$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } A_1A_1B_1B_1 \times a_1a_1\overline{B_1}\ \overline{B_1}$$
$$F_1 : A_1a_1B_1\overline{B_1}$$
and
$$F_1 \times F_1 : A_1a_1B_1\overline{B_1} \times A_1a_1B_1\overline{B_1}$$
$F_2 : \widehat{B_1}$ (of genotype $A_1x_1B_1B_1$), $\widehat{B_2}$ (of genotype $A_1x_1B_1\overline{B_1}$), $\widehat{B_3}$ (of genotype $A_1x_1\overline{B_1}\ \overline{B_1}$), $\widehat{B_4}$ (of genotype $a_1a_1B_1B_1$), $\widehat{B_5}$ (of genotype $a_1a_1B_1\overline{B_1}$) and $\widehat{B_6}$ (of genotype $a_1a_1\overline{B_1}\ \overline{B_1}$).

Here. $x_1 \in \{A_1, a_1\}$.

Let $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}, \widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(\widehat{B_1}) = \mu(\widehat{B_3}) = \mu(\widehat{B_2}) = \frac{2}{9}$ and $\mu(\widehat{B_4}) = \mu(\widehat{B_5}) = \mu(\widehat{B_6}) = \frac{1}{9}$.

Proposition 3.5. *Let $H = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}, \widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$. By definitions of (w) and (w'), we have $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3}) = \mu_1(\widehat{B_4}) = \mu_1(\widehat{B_5}) = \mu_1(\widehat{B_6})$ and $S.F.G(H) = 2$.*

Proof. The table below represents $({}^1H, \star_1)$:

${}^1H$	$\widehat{B_1}$	$\widehat{B_2}$	$\widehat{B_3}$	$\widehat{B_4}$	$\widehat{B_5}$	$\widehat{B_6}$
$\widehat{B_1}$	$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	H	H	H
$\widehat{B_2}$		$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	H	H	H
$\widehat{B_3}$			$\{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$	H	H	H
$\widehat{B_4}$				$\{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$	$\{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$	$\{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$
$\widehat{B_5}$					$\{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$	$\{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$
$\widehat{B_6}$						$\{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}\}$

It is easy to see that $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3}) = \mu_1(\widehat{B_4}) = \mu_1(\widehat{B_5}) = \mu_1(\widehat{B_6})$ and that $({}^2H, \star_1)$ is a total hypergroup. Therefore, $S.F.G(H) = 2$. □

We consider next the case $m = 1$ and $n \geq 1$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } A_1A_1B_1B_1 \dots B_nB_n \times a_1a_1\overline{B_1}\ \overline{B_1} \dots \overline{B_n}\ \overline{B_n}$$
$$F_1 : A_1a_1B_1\overline{B_1} \dots B_n\overline{B_n}$$
$$\text{and}$$
$$F_1 \times F_1 : A_1a_1B_1\overline{B_1} \dots B_n\overline{B_n} \times A_1a_1B_1\overline{B_1} \dots B_n\overline{B_n}.$$

Let $H = \{\widehat{B_1}, \widehat{B_2}, \dots, \widehat{B_s}\}$ be the set of phenotypes in F_2 where $s = 2k$ is the number of different phenotypes in F_2, $k = 3^n$, $\widehat{B_1}, \dots, \widehat{B_k}$ are phenotypes whose corresponding genotypes are given by $A_1x_1y_1y_1' \dots y_ny_n'$ and $\widehat{B_{k+1}}, \dots, \widehat{B_{2k}}$ are phenotypes whose corresponding genotypes are given by $a_1a_1y_1y_1' \dots y_ny_n'$. Where $x_1 \in \{A_1, a_1\}$, $\{y_i, y_i'\} \subseteq \{B_i, \overline{B_i}\}$ for $i = 1, \dots, n$. It is easy to see that $\mu(\widehat{B_1}) = \dots = \mu(\widehat{B_k}) = \frac{2}{3^{n+1}}$ and $\mu(\widehat{B_{k+1}}) = \dots = \mu(\widehat{B_{2k}}) = \frac{1}{3^{n+1}}$.

Theorem 3.2. *Let $H = \{\widehat{B_1}, \dots, \widehat{B_s}\}$. By definition of (w) and (w'), we have $\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \dots = \mu_1(\widehat{B_s})$ and $S.F.G(H) = 2$.*

Proof. Using (w'), $({}^1H, \star_1)$ may be constructed as follows:

$$\widehat{B_i} \star_1 \widehat{B_j} = \begin{cases} \{\widehat{B_1}, \dots, \widehat{B_k}\}, & \text{if } i, j \in \{1, \dots, k\}; \\ \{\widehat{B_{k+1}}, \dots, \widehat{B_{2k}}\}, & \text{if } i, j \in \{k+1, \dots, 2k\}; \\ H, & \text{otherwise.} \end{cases}$$

It is easy to see that $({}^2H, \star_2)$ is the total hypergroup. Therefore, $S.F.G(H) = 2$. $\square$

We consider next the case $m = 2$ and $n = 1$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } A_1A_1A_2A_2B_1B_1 \times a_1a_1a_2a_2\overline{B_1}\ \overline{B_1}$$
$$F_1 : A_1a_1A_2a_2B_1\overline{B_1}$$
$$\text{and}$$
$$F_1 \times F_1 : A_1a_1A_2a_2B_1\overline{B_1} \times A_1a_1A_2a_2B_1\overline{B_1}.$$

Let $H = \{\widehat{B_1}, \widehat{B_2}, \dots, \widehat{B_{12}}\}$ be the set of phenotypes in F_2 where $\widehat{B_1}, \widehat{B_2}, \widehat{B_3}$ are phenotypes whose corresponding genotypes are given by

$$A_1x_1A_2x_2y_1y_1',\ \widehat{B_4}, \dots, \widehat{B_9}$$

are phenotypes whose corresponding genotypes are given by $A_1x_1a_2a_2y_1y_1'$ or by $a_1a_1A_2x_2y_1y_1'$ and $\widehat{B_{10}}, \widehat{B_{11}}, \widehat{B_{12}}$ are phenotypes whose corresponding genotypes are given by $a_1a_1a_2a_2y_1y_1'$. Where $x_i \in \{A_i, a_i\}$ for $i = 1, 2$ and $\{y_1, y_1'\} \subseteq \{B_1, \overline{B_1}\}$. It is easy to see that $\mu(\widehat{B_1}) = \mu(\widehat{B_2}) = \mu(\widehat{B_3}) = \frac{4}{27}$, $\mu(\widehat{B_4}) = \dots = \mu(\widehat{B_9}) = \frac{2}{27}$ and $\mu(\widehat{B_{10}}) = \mu(\widehat{B_{11}}) = \mu(\widehat{B_{12}}) = \frac{1}{27}$.

Proposition 3.6. *Let $H = \{\widehat{B_1}, \dots, \widehat{B_{12}}\}$. By definition of (w), we have*

$$\mu_1(\widehat{B_4}) = \dots = \mu_1(\widehat{B_9}) < \mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3}) = \mu_1(\widehat{B_{10}}) = \mu_1(\widehat{B_{11}}) = \mu_1(\widehat{B_{12}}).$$

Proof. The table below represents $({}^1H, \star_1)$:

${}^1H$	$\widehat{B_1}$	$\widehat{B_2}$	$\widehat{B_3}$	$\widehat{B_4}$	$\widehat{B_5}$	$\widehat{B_6}$	$\widehat{B_7}$	$\widehat{B_8}$	$\widehat{B_9}$	$\widehat{B_{10}}$	$\widehat{B_{11}}$	$\widehat{B_{12}}$
$\widehat{B_1}$	M	M	M	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	H	H	H
$\widehat{B_2}$		M	M	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	H	H	H
$\widehat{B_3}$			M	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	$H \setminus R$	H	H	H
$\widehat{B_4}$				N	N	N	N	N	N	$H \setminus M$	$H \setminus M$	$H \setminus M$
$\widehat{B_5}$					N	N	N	N	N	$H \setminus M$	$H \setminus M$	$H \setminus M$
$\widehat{B_6}$						N	N	N	N	$H \setminus M$	$H \setminus M$	$H \setminus M$
$\widehat{B_7}$							N	N	N	$H \setminus M$	$H \setminus M$	$H \setminus M$
$\widehat{B_8}$								N	N	$H \setminus M$	$H \setminus M$	$H \setminus M$
$\widehat{B_9}$									N	$H \setminus M$	$H \setminus M$	$H \setminus M$
$\widehat{B_{10}}$										R	R	R
$\widehat{B_{11}}$											R	R
$\widehat{B_{12}}$												R

where $M = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}\}$, $R = \{\widehat{B_{10}}, \widehat{B_{11}}, \widehat{B_{12}}\}$ and $N = H \setminus (M \cup R)$.
We have that $q(\widehat{B_1}) = 63$, $A(\widehat{B_1}) = \frac{9}{3} + \frac{18}{12} + \frac{36}{9}$, $q(\widehat{B_4}) = 126$ and $A(\widehat{B_4}) = \frac{36}{6} + \frac{72}{9} + \frac{18}{12}$. Simple calculations implies that

$$\mu_1(\widehat{B_1}) = \mu_1(\widehat{B_2}) = \mu_1(\widehat{B_3}) = \mu_1(\widehat{B_{10}}) = \mu_1(\widehat{B_{11}}) = \mu_1(\widehat{B_{12}}) = \frac{17}{126}$$

and

$$\mu_1(\widehat{B_4}) = \mu_1(\widehat{B_5}) = \mu_1(\widehat{B_6}) = \mu_1(\widehat{B_7}) = \mu_1(\widehat{B_8}) = \mu_1(\widehat{B_9}) = \frac{31}{252}.$$

□

Proposition 3.7. *Let $H = \{\widehat{B_1}, \ldots, \widehat{B_{12}}\}$. By definition of (w), we have $\mu_2(\widehat{B_1}) = \ldots = \mu_2(\widehat{B_{12}})$ and $S.F.G(H) = 3$.*

Proof. We may present $({}^2H, \star_2)$ by the following table:

^2H	$\widehat{B_1}$	$\widehat{B_2}$	$\widehat{B_3}$	$\widehat{B_4}$	$\widehat{B_5}$	$\widehat{B_6}$	$\widehat{B_7}$	$\widehat{B_8}$	$\widehat{B_9}$	$\widehat{B_{10}}$	$\widehat{B_{11}}$	$\widehat{B_{12}}$
$\widehat{B_1}$	P	P	P	H	H	H	H	H	H	P	P	P
$\widehat{B_2}$		P	P	H	H	H	H	H	H	P	P	P
$\widehat{B_3}$			P	H	H	H	H	H	H	P	P	P
$\widehat{B_4}$				Q	Q	Q	Q	Q	Q	H	H	H
$\widehat{B_5}$					Q	Q	Q	Q	Q	H	H	H
$\widehat{B_6}$						Q	Q	Q	Q	H	H	H
$\widehat{B_7}$							Q	Q	Q	H	H	H
$\widehat{B_8}$								Q	Q	H	H	H
$\widehat{B_9}$									Q	H	H	H
$\widehat{B_{10}}$										P	P	P
$\widehat{B_{11}}$											P	P
$\widehat{B_{12}}$												P

where $P = \{\widehat{B_1}, \widehat{B_2}, \widehat{B_3}, \widehat{B_{10}}, \widehat{B_{11}}, \widehat{B_{12}}\}$ and $Q = \{\widehat{B_4}, \widehat{B_5}, \widehat{B_6}, \widehat{B_7}, \widehat{B_8}, \widehat{B_9}\}$. Simple computations implies that $\mu_2(\widehat{B_1}) = \ldots = \mu_2(\widehat{B_{12}}) = \frac{1}{9}$. Thus, $({}^3H, \star_3)$ is the total hypergroup and hence $S.F.G(H) = 3$. $\square$

We consider next the case $m = 2$ and $n \geq 1$. The results of this hypothetical experiment can be summarized in the following way:

$$\text{P: } A_1A_1A_2A_2B_1B_1\ldots B_nB_n \times a_1a_1a_2a_2\overline{B_1}\ \overline{B_1}\ldots\overline{B_n}\ \overline{B_n}$$
$$F_1 : A_1a_1A_2a_2B_1\overline{B_1}\ldots B_n\overline{B_n}$$
$$\text{and}$$
$$F_1 \times F_1 : A_1a_1A_2a_2B_1\overline{B_1}\ldots B_n\overline{B_n} \times A_1a_1A_2a_2B_1\overline{B_1}\ldots B_n\overline{B_n}.$$

Let $H = \{\widehat{B_1}, \widehat{B_2}, \ldots, \widehat{B_r}\}$ be the set of phenotypes in F_2 where $r = 4k$, $k = 3^n$, $\widehat{B_1}, \ldots, \widehat{B_k}$ are phenotypes whose corresponding genotypes are given by

$$A_1x_1A_2x_2y_1y_1'\ldots y_ny_n',\ \widehat{B_{k+1}}, \ldots, \widehat{B_{3k}}$$

are phenotypes whose corresponding genotypes are given by $A_1x_1a_2a_2y_1y_1'\ldots y_ny_n'$ or by $a_1a_1A_2x_2y_1y_1'\ldots y_ny_n'$ and $\widehat{B_{3k+1}}, \ldots, \widehat{B_{4k}}$ are phenotypes whose corresponding genotypes are given by $a_1a_1a_2a_2y_1y_1'\ldots y_ny_n'$. Where $x_i \in \{A_i, a_i\}$ for $i = 1, 2$ and $\{y_j, y_j'\} \subseteq \{B_j, \overline{B_j}\}$ for $j = 1, \ldots, n$. It is easy to see that $\mu(\widehat{B_1}) = \ldots = \mu(\widehat{B_k}) = \frac{4}{3^{n+2}}$, $\mu(\widehat{B_{k+1}}) = \ldots = \mu(\widehat{B_{3k}}) = \frac{2}{3^{n+2}}$ and $\mu(\widehat{B_{3k+1}}) = \ldots = \mu(\widehat{B_{4k}}) = \frac{1}{3^{n+2}}$.

Proposition 3.8. *Let $H = \{\widehat{B_1}, \ldots, \widehat{B_s}\}$ be the set of phenotypes of F_2. By definitions of (w) and (w'), we have*

$$\mu_1(\widehat{B_{k+1}}) = \ldots = \mu_1(\widehat{B_{3k}}) < \mu_1(\widehat{B_1}) = \ldots = \mu_1(\widehat{B_k}) = \mu_1(\widehat{B_{3k+1}}) = \ldots = \mu_1(\widehat{B_{4k}}).$$

Proof. The table below represents $({}^1H, \star_1)$:

${}^1H$	$\widehat{B_1}$	$\ldots$	$\widehat{B_k}$	$\widehat{B_{k+1}}$	$\ldots$	$\widehat{B_{3k}}$	$\widehat{B_{3k+1}}$	$\ldots$	$\widehat{B_{4k}}$
$\widehat{B_1}$	M_1	$\ldots$	M_1	$M_1 \cup M_2$	$\ldots$	$M_1 \cup M_2$	H	$\ldots$	H
$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\widehat{B_k}$			M_1	$M_1 \cup M_2$	$\ldots$	$M_1 \cup M_2$	H	$\ldots$	H
$\widehat{B_{k+1}}$				M_2	$\ldots$	M_2	$M_2 \cup M_3$	$\ldots$	$M_2 \cup M_3$
$\vdots$					$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\widehat{B_{3k}}$						M_2	$M_2 \cup M_3$	$\ldots$	$M_2 \cup M_3$
$\widehat{B_{3k+1}}$							M_3	$\ldots$	M_3
$\vdots$								$\ddots$	$\vdots$
$\widehat{B_{4k}}$									M_3

where $M_1 = \{\widehat{B_1}, \ldots, \widehat{B_k}\}$, $M_2 = \{\widehat{B_{k+1}}, \ldots, \widehat{B_{3k}}\}$ and $M_3 = \{\widehat{B_{3k+1}}, \ldots, \widehat{B_{4k}}\}$.
It is easy to see that

$$\mu_1(\widehat{B_1}) = \ldots = \mu_1(\widehat{B_k}) = \mu_1(\widehat{B_{3k+1}}) = \ldots = \mu_1(\widehat{B_{4k}})$$

and

$$\mu_1(\widehat{B_{k+1}}) = \ldots = \mu_1(\widehat{B_{4k}}).$$

We have that $q(\widehat{B_1}) = 7k^2$, $q(\widehat{B_{k+1}}) = 14k^2$. Simple computations shows that $A(\widehat{B_1}) = \frac{k^2}{|M_1|} + \frac{4k^2}{|M_1|+|M_2|} + \frac{2k^2}{|H|} = \frac{17k}{6}$ and $A(\widehat{B_{k+1}}) = \frac{4k^2}{|M_2|} + \frac{4k^2}{|M_1|+|M_2|} + \frac{2k^2}{|H|} + \frac{4k^2}{|M_2|+|M_3|} = \frac{31k}{6}$. We get now

$$\mu_1(\widehat{B_1}) = \ldots = \mu_1(\widehat{B_k}) = \mu_1(\widehat{B_{3k+1}}) = \ldots = \mu_1(\widehat{B_{4k}}) = \frac{17}{6k}$$

and

$$\mu_1(\widehat{B_{k+1}}) = \ldots = \mu_1(\widehat{B_{3k}}) = \frac{31}{84k}.$$

□

Proposition 3.9. *Let $H = \{\widehat{B_1}, \ldots, \widehat{B_r}\}$. By definition of (w), we have $\mu_2(\widehat{B_1}) = \ldots = \mu_2(\widehat{B_r})$ and $S.F.G(H) = 3$.*

Proof. We may present $({}^2H, \star_2)$ by the following table:

${}^2H$	$\widehat{B_1}$	$\ldots$	$\widehat{B_k}$	$\widehat{B_{k+1}}$	$\ldots$	$\widehat{B_{3k}}$	$\widehat{B_{3k+1}}$	$\ldots$	$\widehat{B_{4k}}$
$\widehat{B_1}$	$M_1 \cup M_3$	$\ldots$	$M_1 \cup M_3$	H	$\ldots$	H	$M_1 \cup M_3$	$\ldots$	$M_1 \cup M_3$
$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\widehat{B_k}$			$M_1 \cup M_3$	H	$\ldots$	H	$M_1 \cup M_3$	$\ldots$	$M_1 \cup M_3$
$\widehat{B_{k+1}}$				M_2	$\ldots$	M_2	H	$\ldots$	H
$\vdots$					$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\widehat{B_{3k}}$						M_2	H	$\ldots$	H
$\widehat{B_{3k+1}}$							$M_1 \cup M_3$	$\ldots$	$M_1 \cup M_3$
$\vdots$								$\ddots$	$\vdots$
$\widehat{B_{4k}}$									$M_1 \cup M_3$

where $M_1 = \{\widehat{B_1}, \ldots, \widehat{B_k}\}$, $M_2 = \{\widehat{B_{k+1}}, \ldots, \widehat{B_{3k}}\}$ and $M_3 = \{\widehat{B_{3k+1}}, \ldots, \widehat{B_{4k}}\}$.
We have:

$$q(\widehat{B_i}) = 12k^2 \text{ for } i = 1, \ldots, r,$$

and

$$A(x) = \begin{cases} \frac{4k^2}{|M_1|+|M_3|} + \frac{8k^2}{|H|} = 4k, & \text{for } x \in M_1 \cup M_3; \\ \frac{4k^2}{|M_2|} + \frac{8k^2}{|H|} = 4k, & \text{for } x \in M_2. \end{cases}$$

We get now that $A(x) = 4k$ for all $x \in H$. The latter implies that $\mu(x) = \frac{1}{3k}$ for all $x \in H$. Therefore, $({}^3H, \star_3)$ is the total hypergroup and $S.F.G(H) = 3$. □

4. Fuzzy sets associated to other types of inheritance

In this section, we study some examples of different types of non- Mendelian inheritance (Epistasis, Supplementary gene and Inhibitory gene), define fuzzy subsets of them and construct sequence of join spaces for each type.
The fuzzy subset μ of the set of phenotypes in F_2 of each type is defined by $\mu(x) =$ probability of x for all $x \in F_2$.

Example 4.1. *Epistasis of dominant gene in the coat color of dogs.* There are two allelomorphic pairs which may be named Aa and Bb, A and B are dominant over a and b respectively. They interact as follows: $AxBy$ and $Axbb$ have phenotype white, $aaBy$ has phenotype black and $aabb$ has phenotype brown. Here $x = A$ or a and $y = B$ or b. The results of this experiment can be summarized in the following way:

P: $AABB \otimes aabb$
$F_1 : AaBb$

and
$F_1 \otimes F_1 : AaBb \otimes AaBb$
F_2 : White, Black, Brown.

White is denoted by A_1, Black by A_2 and Brown by A_3.

Let $H = \{A_1, A_2, A_3\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(A_1) = \frac{6}{9}$, $\mu(A_2) = \frac{2}{9}$ and $\mu(A_3) = \frac{1}{9}$.

Proposition 4.1. *Let $H = \{A_1, A_2, A_3\}$. By definitions of (w) and (w'), we have $\mu_1(A_2) < \mu_1(A_1) = \mu_1(A_3)$.*

Proof. The table below represents $({}^1H, \star_1)$:

${}^1H$	A_1	A_2	A_3
A_1	$\{A_1\}$	$\{A_1, A_2\}$	H
A_2		$\{A_2\}$	$\{A_2, A_3\}$
A_3			$\{A_3\}$

We have $q(A_1) = q(A_3) = 5$, $q(A_2) = 7$, $A(A_1) = A(A_3) = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} = \frac{8}{3}$ and $A(A_2) = \frac{1}{1} + \frac{2}{2} + \frac{2}{2} + \frac{2}{3} = \frac{11}{3}$. Thus, $\mu_1(A_1) = \mu_1(A_3) = \frac{8}{15}$ and $\mu_1(A_2) = \frac{11}{21}$. □

Proposition 4.2. *Let $H = \{A_1, A_2, A_3\}$. By definition of (w), we have $\mu_2(A_1) = \mu_2(A_3) < \mu_2(A_2)$.*

Proof. The table below represents $({}^2H, \star_1)$:

${}^2H$	A_1	A_2	A_3
A_1	$\{A_1, A_3\}$	H	$\{A_1, A_3\}$
A_2		$\{A_2\}$	H
A_3			$\{A_1, A_3\}$

We have $q(A_1) = q(A_3) = 8$, $q(A_2) = 5$, $A(A_1) = A(A_3) = \frac{4}{2} + \frac{4}{3} = \frac{10}{3}$ and $A(A_2) = \frac{1}{1} + \frac{4}{3} = \frac{7}{3}$. Thus, $\mu_2(A_1) = \mu_2(A_3) = \frac{5}{12}$ and $\mu_2(A_2) = \frac{7}{15}$. □

Proposition 4.3. *Let $H = \{A_1, A_2, A_3\}$ be the set of phenotypes in F_2. Then $S.F.G(H) = 2$.*

Proof. The table below represents $({}^3H, \star_1)$:

^3H	A_1	A_2	A_3
A_1	$\{A_1, A_3\}$	H	$\{A_1, A_3\}$
A_2		$\{A_2\}$	H
A_3			$\{A_1, A_3\}$

Having $(^3H, \star_1) = (^2H, \star_1)$ implies that $S.F.G(H) = 2$. □

Example 4.2. *Supplementary gene, The anthocyanin pigmentation of flowers.* The red-type anthocyanin color of many flowers is caused by two alleles which may be termed as Aa and Bb. In the snapdragon (Antirrhinum) flower:

$AxBy$ is the genotype of magenta flower, $Axbb$ is the genotype of ivory flower and $aaBy$, $aabb$ are the genotypes of white flower where $x = A$ or a and $y = B$ or b. The results of this experiment can be summarized in the following way:

P: $AABB \otimes aabb$
$F_1 : AaBb$
and
$F_1 \otimes F_1 : AaBb \otimes AaBb$
F_2 : Magneta, Ivory, White.

Magneta is denoted by B_1, White by B_2 and Ivory by B_3.

Let $K = \{B_1, B_2, B_3\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(B_1) = \frac{4}{9}$, $\mu(B_2) = \frac{3}{9}$ and $\mu(B_3) = \frac{2}{9}$.

Theorem 4.1. *Let $K = \{B_1, B_2, B_3\}$ be the set of phenotypes in F_2. Then $S.F.G(K) = 2$.*

Proof. Since $\mu(B_3) < \mu(B_2) < \mu(B_1)$ then using (w'), we may present $(^1K, \star_1)$ as follows:

^1K	B_1	B_2	B_3
B_1	$\{B_1\}$	$\{B_1, B_2\}$	H
B_2		$\{B_2\}$	$\{B_2, B_3\}$
B_3			$\{B_3\}$

It is easy to see that $(^1H, \star_1)$ (of Proposition 4.1, Example 4.1) and $(^1K, \star_1)$ are isomorphic. This implies that $S.F.G(K) = S.F.G(H)$. Therefore, $S.F.G(K) = 2$ by Proposition 4.3. □

Example 4.3. *Inhibitory gene, Rice leaf.* In some rice variety the presence of the gene P causes its leaves to be colored deep purple. But if a gene I is present then the purple color is inhibited and the leaf becomes normal green. The I gene may be considered as epistatic over P. They interact as follows:

The genotypes $IxPy, Ixpp, iipp$ correspond to green and the genotype $iiPy$ corresponds to purple where $x = I$ or i and $y = P$ or p. The results of this experiment can be summarized in the following way:

$$\text{P: } IIPP \otimes iipp$$
$$F_1 : IiPp$$
$$\text{and}$$
$$F_1 \otimes F_1 : IiPp \otimes IiPp$$
$$F_2 : \text{Green, Purple.}$$

Green is denoted by G and Purple by P.

Let $L = \{G, P\}$ be the set of phenotypes in F_2. It is easy to see that $\mu(G) = \frac{7}{9}$ and $\mu(P) = \frac{2}{9}$.

Proposition 4.4. *Let $L = \{G, P\}$ be the set of phenotypes in F_2. Then $S.F.G(L) = 2$.*

Proof. The table below represents $({}^1L, \star_1)$:

${}^1L$	G	P
G	$\{G\}$	H
P		$\{P\}$

It is easy to see that $({}^1H, \star_1)$ (of Proposition 3.1) and $({}^1L, \star_1)$ are isomorphic. Therefore $S.F.G(L) = S.F.G(H) = 2$. $\square$

5. Conclusion

After the introduction of hyperstructures and fuzzy sets by Marty and Zadeh there have been many researches that study their importance in different fields where one of these fields is biological inheritance. This paper studied a new relationship between hyperstructures, fuzzy sets and the phenotypes of the second generation F_2. Fuzzy subsets of F_2 were defined and join spaces associated to F_2 were constructed.

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A SURVEY ON THE AUTOMORPHISM GROUPS OF THE COMMUTING GRAPHS AND POWER GRAPHS

Mahsa Mirzargar

Abstract. Let G be a finite group. The power graph $P(G)$ of a group G is the graph whose vertex set is the set of group elements where two elements are adjacent if one is a power of the other. The commuting graph $\Delta(G)$ of a group G, is the graph whose vertices are the group elements, two of them are joined if they commute. When the vertex set is $G \setminus Z(G)$, this graph is denoted by $\Gamma(G)$. Since the results based on the automorphism groups of these kinds of graphs are so sporadic, in this paper, we give a survey of all results on the automorphism groups of power graphs and commuting graphs obtained in the literature.
Keywords. Finite group; graph; vertex set; commuting graph; automorphism groups.

1. Introduction

There are many connections between graphs and groups. Generating graphs from semigroups and groups has a long history. In 1964, Bosak [6] studied a certain graph over semigroups. In [13], Zelinka studied the intersection graphs of nontrivial subgroups of finite Abelian groups. The well-known study of a directed graphs defined on the elements of a groups is the Cayley digraph [7, 22, 40]. The investigation of graphs like these is very important, because they have valuable and numerous applications presented, for example, in the books [27], [28] and [29]. The directed power graph of a group was introduced by Kelarev and Quinn [24]. The definition was formulated so that it applied to semigroups as well. Accordingly, the power graphs of semigroups were first considered in [25], [23] and [26]. It is also explained in the survey [2] that the definition given in [24] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [24] (see [2] for more detailed explanations). All of these papers used only the brief term 'power graph', even though they covered both directed and undirected power graphs. Kelarve and Quinn [23] defined another interesting classes of directed graphs, namely,

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the divisibility graphs of semigroups. Let S be a semigroup, the divisibility graph, $Div(S)$, of a semigroup S is a directed graph with vertex set S and there is an arc from u to v if and only if $u \neq v$ and $u|v$, i.e., the ideal generated by v contains u. On the other hand, the power graph, $\overrightarrow{P}(S)$, of a semigroup S is a directed graph in which the set of vertices is again S and for $a, b \in S$ there is an arc from a to b if and only if $a \neq b$ and $b = a^m$ for some positive integer m.

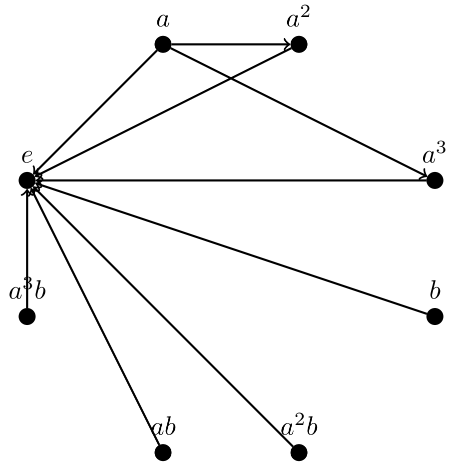

Figure 1. The directed power graph of the dihedral group D_8.

The undirected power graph $P(S)$ was also considered by Chakrabarty, Ghosh and Sen in [11]. Recall that $P(S)$ has vertex set S and two vertices $a, b \in S$ are adjacent if and only if $a \neq b$ and $< a > \subseteq < b >$ or $< b > \subseteq < a >$ (which is equivalent to saying $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer m). As a consequence, they proved that $P(G)$ is connected for any finite group G and $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m [11].

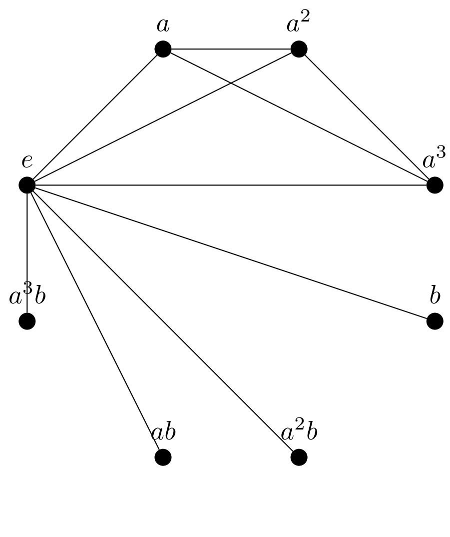

Figure 2. The undirected power graph of the dihedral group D_8.

The undirected power graphs became the main focus of study in [11] and in the subsequent papers by P. J. Cameron et al. [8, 9], which introduced the use of the brief term 'power graph' in the second meaning of an undirected power graph. For a group G, the digraph $\overrightarrow{P}(G)$ was considered in [37] as the main subject of study. The interested readers can be consulted [2, 32, 1] for more information about the power graphs. In this paper, we are also interested in the well-known commuting graphs and their automorphism groups. Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate a graph $\Gamma(G)$ with G as follows: Take $G\setminus Z(G)$ as the vertices of $\Gamma(G)$ and join two distinct vertices x and y, whenever $xy = yx$. The complement of the $\Gamma(G)$ is said to be the noncommuting graph. The noncommuting graph was first considered by Paul Erdos, when he posed the following problem in 1975 [36]: Let G be a group whose noncommuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of the noncommuting graph of G? B. H. Neumann [36] answered positively Erdos' question. We refer the readers to [3, 4, 14, 35, 31] for more details about the noncommuting graph. In [1], authors related the power graph to the commuting graph and characterize when they are equal for finite groups. A new graph pops up while considering these graphs, a graph whose vertex set consists of all group elements, in which two vertices x and y are adjacent if they generate a cyclic group. They called this graph as the enhanced power graph of G. The enhanced power graph contains the power graph and is a subgraph of the commuting graph. We consider the commuting graph with vertex set G and denoted it by $\Delta(G)$.

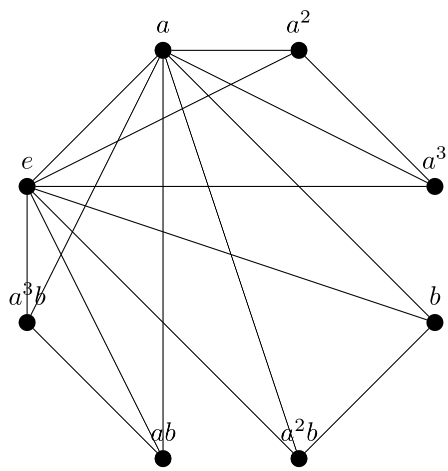

Figure 3. The commuting graph $\Delta(D_8)$.

2. Preliminaries and background information

An action of a group G on a set X is the choice, for each $g \in G$ of a permutation $\pi_g : X \to X$ such that the following two conditions hold:

1. π_e is the identity: $\pi_e(x) = x$ for each $x \in X$,
2. for every g_1, g_2 in G, $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$.

For example, any group G acts on itself ($X = G$) by left multiplication functions. A group action of G on X is said to be *faithful* if different elements of G act on X in different ways: when $g_1 \neq g_2$ in G, there is an $x \in X$ such that $g_1 \Delta x \neq g_2 \Delta x$. For any graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. Suppose $v \in V(\Gamma)$ and $V_1(\Gamma) \subseteq V(\Gamma)$, then $N(v)$ is the set of neighbours of v and $\langle V_1(\Gamma) \rangle$ is the subgraph of Γ induced by $V_1(\Gamma)$. The closed neighbourhood of a vertex x, denoted by $N[x]$, is the set of its neighbours and itself. The complement of Γ is the graph $\bar{\Gamma}$ on the same vertices such that two vertices of $\bar{\Gamma}$ are adjacent if and only if they are not adjacent in Γ. For two graphs with disjoint vertex sets V_1 and V_2 their union is the graph H in which $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2$. Define nH to be the union of n disjoint copies of G. The automorphism group of a graph Γ is that set of all permutations on $V(\Gamma)$ that fix as a set the edges $E(\Gamma)$. The set of all automorphisms of a graph Γ forms a permutation group, $Aut(\Gamma)$, acting on the object set $V(\Gamma)$. See [10] for the terminology and main results of permutation group theory. Let A and B be permutation groups acting on object sets X and Y, respectively. Define $B \wr A = \{(a, f) \mid a \in A, f : X \to B\}, (a, f)(x, y) = (ax, b_x y)$ where $f(x) = b_x$. $B \wr A$ is said to be *wreath product*. It acts on $X \times Y$ as follows: for each $a \in A$ and any sequence $b_1, b_2, \cdots, b_n$ (where $n = |X|$) in B, there is a unique permutation in $A \wr B$ written $(a; b_1, \cdots, b_n)$, and $(a; b_1, \cdots, b_n)(x_i, y_i) = (ax_i, b_i y_i)$. Suppose S_n denotes the symmetric group on $\{1, 2, \cdots, n\}$, φ is the Euler's totient function . In what follows, we describe some important results relating the automorphism groups of a graph which are crucial in this paper. Frucht [18] described if Γ is a connected graph, then $Aut(n\Gamma) \cong (Aut(\Gamma)) \wr S_n$, if no component of Γ_1 is isomorphic with a component of Γ_2, then $Aut(\Gamma_1 \cup \Gamma_2) \cong Aut(\Gamma_1) \times Aut(\Gamma_2)$ and applying the last two theorems we have the result: Let $\Gamma = n_1\Gamma_1 \cup n_2\Gamma_2 \cup \cdots \cup n_r\Gamma_r$, where n_i is the number of components of Γ isomorphic to Γ_i, then

$$Aut(\Gamma) \cong ((Aut(\Gamma_1)) \wr S_{n_1}) \times ((Aut(\Gamma_2)) \wr S_{n_2}) \times \cdots \times ((Aut(\Gamma_r)) \wr S_{n_r}).$$

An operation $\cdot$ on the set S is associative if it satisfies the following associative law: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$. A semigroup is a set S equipped with an associative binary operation $\cdot$. The set of the orders of all elements of G is denoted by $\pi_e(G)$ and is said to be the *spectrum* of G. For $n \in N$, the cyclic group of order n can be defined as the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of residues modulo n, the set $< g >= \{g^n \mid n \in \mathbb{Z}\}$ is the cyclic group generated by g in G. For a prime p, a group G is said to be an elementary abelian p-group if G is finite, abelian and every nontrivial element of G has order p. A group G is an AC-group, whenever the centralizers of non-central elements are abelian. The dihedral group D_{2n} is an example of an *AC-group.* The group G is said to be an *EPPO-group*, if all elements of G have prime power order.

3. Automorphism groups of power graphs

The first result about the automorphism groups of power graphs was obtained by P. Cameron in [8], where he explained that when the automorphism group and its graph are equal. P. Cameron proved the only finite group G for which $Aut(G) = Aut(P(G))$ is the Klein group $Z_2 \times Z_2$.

In 2013, Doostabadi, Erfanian and Jafarzadeh asserted that the full automorphism group of the power graph of the cyclic group Z_n is isomorphic to the direct product of some symmetry groups.

Conjecture 3.1. [16] *For every positive integer n,*

$$Aut(P(Z_n)) \cong S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1,n\}} S_{\varphi(d)}$$

where $D(n)$ is the set of positive divisors of n, and φ is the Euler's totient function.

In fact, if n is a prime power, then $P(Z_n)$ is a complete graph by [11] which implies that $Aut(P(Z_n) \cong S_n$. Hence, the conjecture does not hold if $n = p^m$ for any prime p and integer $m > 2$. In [17], proved that this conjecture holds for the remaining case. Feng, Ma and Wang [17], describe the full automorphism group of the power (di)graph of an arbitrary finite group. As an application, this conjecture is valid if n is not a prime power. Denote by $C(G)$ the set of all cyclic subgroups of G. For $C \in C(G)$, let $[C]$ denote the set of all generators of C. Write

$$C(G) = \{C_1, \cdots C_k\} \text{ and } [C_i] = \{[C_i]_1, \cdots [C_i]_{s_i}\}.$$

Define $\mathbf{P}(G)$ as the set of permutations σ on $C(G)$ preserving order, inclusion and noninclusion, i.e., $|C_i^\sigma| = |C_i|$ for each $i \in \{1, \cdots, k\}$ and $C_i \subseteq C_j$ if and only if $C_i^\sigma \subseteq C_j^\sigma$. Note that $\mathbf{P}(G)$ is a permutation group on C(G). This group induces the faithful action on the set G:

$$G \times \mathbf{P}(G) \longrightarrow G, \qquad ([C_i]_j, \sigma) \longmapsto [C_i^\sigma]_j. \tag{3.1}$$

For $\Omega \subseteq G$, let S_Ω denote the symmetric group on Ω. Since G is the disjoint union of $[C_1], \cdots, [C_k]$, we get the faithful group action on the set G:

$$G \times \prod_{i=1}^{k} S_{[C_i]} \longrightarrow G, \quad ([C_i]_j, (\xi_1, \cdots, \xi_k)) \longmapsto ([C_i]_j)^{\xi_i}. \tag{3.2}$$

By using the above-mentioned symbols we have:

Theorem 3.1. [17] *Let G be a finite group. Then*

$$Aut(\overrightarrow{P}(G)) = (\prod_{i=1}^{k} S_{[C_i]}) \times \boldsymbol{P}(G),$$

where $\boldsymbol{P}(G)$ and $\prod_{i=1}^{k} S_{[C_i]}$ act on G as in (3.1) and (3.2), respectively.

In the power graph $P(G)$, for $x, y \in G$, define $x \equiv y$ if $N[x] = N[y]$. Observe that $\equiv$ is an equivalence relation. Let $\bar{x}$ denote the equivalence class containing x. Write

$$\mathcal{U}(G) = \{\bar{x} | x \in G\} = \{\bar{u_1}, \cdots, \bar{u_l}\}.$$

Since G is the disjoint union of $u_1, \cdots, u_l$, the following is a faithful group action on the set G:

$$G \times \prod_{i=1}^{l} S_{\bar{u}_i} \longrightarrow G, \quad (x, (\tau_1, \tau_2, \cdots, \tau_l)) \longmapsto x^{\tau_i}, \quad where \ \ x \in \bar{u}_i. \tag{3.3}$$

Similar to the last theorem, for the automorphism groups of undirected power graphs we have:

Theorem 3.2. [17] *Let G be a finite group. Then*

$$Aut(P(G)) = (\prod_{i=1}^{l} S_{\bar{u}_i)} \times \boldsymbol{P}(G),$$

where $\boldsymbol{P}(G)$ and $\prod_{i=1}^{l} S_{\bar{u}_i}$ act on G as in (3.1) and (3.3), respectively.

By combining Theorems 3.1 and 3.2, the authors in [17], obtained that $Aut(P(G)) = Aut(\overrightarrow{P}(G))$ if and only if $x = [x]$ for each $x \in G$. Indeed, this result demonstrates relationship between power graphs and directed power graphs.

A graph Γ is said to be a *subgraph* of another graph Δ (or Δ is a supergraph of Γ), if $V(\Gamma) \subset V(\Delta)$ and $E(\Gamma) \subset E(\Delta)$. Hamzeh and Ashrafi [19] defined the main supergraph $\mathcal{S}(G)$ of $P(G)$ with the vertex set G and two elements $x, y \in G$ are adjacent if and only if $o(x)|o(y)$ or $o(y)|o(x)$ and proved that there is not a group G, such that $Aut(\mathcal{S}(G)) = Aut(G)$. In what follows, $\Omega_{a_i}(G) = |\{y|o(y) = a_i\}|$. Authors in [19] also define the graph Δ with vertex set $V(\delta) = \pi_e(G)$ and two vertices a_i and a_j are adjacent if and only if $a_i|a_j$ or $a_j|a_i$.

Theorem 3.3. [19] *Let G be a finite group with spectrum $\pi_e(G) = \{a_1, \cdots, a_k\}$ and choose a representative set $\{t_1, t_2, \cdots, t_k\}$, where for each i, $1 \leq i \leq k, ti \in K_{\Omega_{a_i}}(G)$. Then,*

1. *If $deg(t_i)$'s are distinct then $Aut(\mathcal{S}(G)) = S_{\Omega_{a_1}}(G) \times \cdots \times S_{\Omega_{a_k}}(G)$.*

2. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, any two distinct vertices of $K_{\Omega_{a_{i_1}}}(G), \cdots, K_{\Omega_{a_{i_r}}}(G)$ are adjacent and $N_\Delta[a_{i_1}] = \cdots = N_\Delta[a_{i_r}]$ then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}(G)+\cdots+\Omega_{a_{i_r}}(G)}$.*

3. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, all vertices of $K_{\Omega_{a_{i_1}}}(G), \cdots, K_{\Omega_{a_{i_r}}}(G)$ are adjacent and $N_\Delta[a_{i_l}]$'s are distinct then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G)$.*

4. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, $N_\Delta[a_{i_1}] = \cdots = N_\Delta[a_{i_r}]$ and for each two $m, n, 1 \leq m, n \leq r$, $K_{\Omega_{a_{i_m}}}(G)$ and $K_{\Omega_{a_{i_n}}}(G)$ are disjoint then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}}(G) \wr S_r$.*

5. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$,$N_\Delta[a_{i_l}]$'s are distinct and for each $m, n, 1 \leq m, n \leq r$, $K_{\Omega_{a_{i_m}}}(G)$ and $K_{\Omega_{a_{i_n}}}(G)$ are disjoint then $Aut(\mathcal{S}(G))$ has a subgroup isomorphic to $S_{\Omega_{a_{i_1}}}(G) \times \cdots \times S_{\Omega_{a_{i_r}}}(G)$.*

6. *$Aut(\mathcal{S}(G)) = A_1 \times \cdots \times A_q$, where $A_i, 1 \leq i \leq q$, are subgroups appeared in Cases (2–5).*

In [[20], Theorem 2.8], it is proved that if G is an EPPO-group of order $p_1^{n_1} \cdots p_k^{n_k}$ and $V_i = \{1 \neq g \in G \mid o(g)|p_i^{ni}\}$ then $\mathcal{S}(G) = K_1 + (\bigcup_{i=1}^k K_{|Vi|})$. The authors applied the structure of $\mathcal{S}(G)$ to determine its automorphism.

Theorem 3.4. [19] *Let G be a finite group and $e_1, \cdots, e_t$ are distinct values of $|V_1|, \cdots, |V_k|$. Define $B_i = |\{|V_j| \mid |V_j| = e_i\}|$. Then,*

$$Aut(\mathcal{S}(G)) = (S_{|V_1|} \wr S_{B_1}) \times \cdots \times (S_{|V_k|} \wr S_{B_k}).$$

Suppose G is a finite group and $C(G) = \{C_1, \cdots, C_k\}$ is the set of all cyclic subgroups of G. Define L_G to be the graph with vertex set $C(G)$ in which two cyclic subgroups C_i and C_j are adjacent if one is contained in the other or there is a cyclic subgroup C_k such that $C_i \subseteq C_k$ and $C_j \subseteq C_k$. It is clear that the subgraphs of $P(G)$ induced by a cyclic subgroup are complete. So, $P(G) = W_G[K_{b_1}, K_{b_2}, \cdots, K_{b_k}]$ with $b_i = \varphi(|C_i|)$.

Theorem 3.5. [19] *Let G be a finite group with $C(G) = \{C_1, \cdots, C_k\}$ and choose a representative set $\{t_1, t_2, \cdots, t_k\}$, where for each i, $1 \leq i \leq k, ti \in K_{b_i}$. Then,*

1. *If $deg(t_i)$'s are distinct then $Aut(P(G)) = S_{b_1} \times \cdots \times S_{b_k}$.*

2. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, any two distinct vertices of $K_{b_{i_1}}, \cdots, K_{b_{i_r}}$ are adjacent and $N_{W_G}[C_{i_1}] = \cdots = N_{W_G}[C_{i_r}]$ then $Aut(P(G))$ has a subgroup isomorphic to $S_{b_{a_{i_1}}+\cdots+b_{a_{i_r}}}$.*

3. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, all vertices of $K_{b_{i_1}}, \cdots, K_{b_{i_r}}$ are adjacent and $N_{W_G}[C_{i_l}]$'s are distinct then $Aut(P(G))$ has a subgroup isomorphic to $S_{b_{i_1}} \times \cdots \times S_{b_{i_r}}$.*

4. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, $N_{W_G}[C_{i_1}] = \cdots = N_{W_G}[C_{i_r}]$ and for each two $m, n, 1 \leq m, n \leq r$, $K_{b_{i_m}}$ and $K_{b_{i_n}}$ are disjoint then $Aut(P(G))$ has a subgroup isomorphic to $S_{b_{i_1}} \wr S_r$.*

5. *If $deg(t_{i_1}) = \cdots = deg(t_{i_r})$, $N_{W_G}[C_{i_l}]$'s are distinct and for each $m, n, 1 \leq m, n \leq r$, $K_{b_{i_m}}$ and $K_{b_{i_n}}$ are disjoint then $Aut(P(G))$ has a subgroup isomorphic to $S_{b_{i_1}} \times \cdots \times S_{b_{i_r}}$.*

6. *$Aut(P(G)) = A_1 \times \cdots \times A_q$, where $A_i, 1 \leq i \leq q$, are subgroups appeared in Cases (2–5).*

3.1. Examples

In this section, we present $Aut(P(G))$ and $Aut(\overrightarrow{P}(G))$ for some families of finite groups such as $Z_n, Z_n^p, D_{2n}, Q_{4n}, U_{6n}, V_{8n}$ and so on. These results obtained in several papers in different ways. In [5], the authors used the graph structure from [30] and computed the automorphism groups of $P(G)$ for the above groups. In [17], the authors by using Theorem 3.1 and Theorem 3.2, computed the automorphism groups of $P(G)$ and $\overrightarrow{P}(G)$ for these groups. In [19], authors obtained these results from Theorem 3.3.

Example 3.1. [17] If n be a positive integer then,

$$
\begin{aligned}
Aut(\overrightarrow{P}(Z_n)) &\cong \prod_{d \in D(n)} S_{\varphi(d)}, \\
Aut(P(Z_n)) &\cong \begin{cases} S_n & n \textit{ is a prime power} \\ S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1,n\}} S_{\varphi(d)} & \textit{otherwise} \end{cases},
\end{aligned}
$$

and if $n \geq 2$ then,

$$Aut(P(Z_p^n)) = Aut(\overrightarrow{P}(Z_p^n) \cong S_{p-1} \wr S_m,$$

where $m = \frac{p^n-1}{p-1}$ and Z_p^n denote the elementary abelian $p-$group.

In the [21, 15], the dihedral group D_{2n}, semi-dihedral group SD_{2^n}, generalized quaternion group of Q_{4n}, semidihedral groups SD_{8n} are defined by the following presentations:

$$
\begin{aligned}
D_{2n} &= \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\
SD_{2^n} &= \langle a, b \mid a^{2^n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\
Q_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\
U_{6n} &= \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle, \\
V_{8n} &= \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle.
\end{aligned}
$$

Now, we are ready to state next example.

Example 3.2. [17] For $n \geq 3$,

$$Aut(\overrightarrow{P}(D_{2n})) \cong \prod_{d\in D(n)} S_{\varphi(d)} \times S_n,$$

$$Aut(P(D_{2n})) \cong \begin{cases} S_{n-1} \times S_n, & n \text{ is a prime power} \\ S_n \times \prod_{d\in D(n)} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

and let $n \geq 3$ then,

$$Aut(\overrightarrow{P}(Q_{4n})) \cong \prod_{d\in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n),$$

$$Aut(P(Q_{4n})) \cong \begin{cases} S_2 \times S_{2n-2} \times (S_2 \wr S_n), & n \text{ is a power of } 2 \\ \prod_{d\in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n) & \text{otherwise} \end{cases}.$$

Example 3.3. [5] If k is nonnegative integer and satisfies $n = 3^k t$ for some positive integer t such that $3 \not| t$ then,

$$Aut(P(U_{6n})) \cong \begin{cases} \prod_{d|3n} S_{\varphi(d)} \times \prod_{d|2n, d\not| n} S_{\varphi(d)} \wr S_3 & k=0 \\ \prod_{d|2n, d\not| n} S_{\varphi(d)} \wr S_3 \times \prod_{d|n} S_{\varphi(d)} \times \prod_{d|n, d\not| t} S_{\varphi(d)} \wr S_3 & k=1 \\ \prod_{d|2n, d\not| n} S_{\varphi(d)} \wr S_3 \times \prod_{d|n} S_{\varphi(d)} \times \prod_{d|3t, d\not| t} S_{\varphi(d)} \wr S_3 & \\ \times \prod_{d|n, d\not| 3t} S_{\varphi(d)} \wr S_2 & k \geq 2 \end{cases},$$

if $n = 2^k t$ for a nonnegative k and some positive odd integer t then,

$$Aut(P(V_{8n})) \cong \begin{cases} S_{2n} \times S_2 \wr S_n \times \prod_{d|2n, d\not| n} S_{\varphi(d)} \wr S_2 \times \prod_{d|2n} S_{\varphi(d)} & k=0 \\ S_{2n+1} \times S_2 \wr S_n \times \prod_{l=1}^{k-1} S_{2^l}^2 \times S_{2^k} \wr S_2 & t=1, k\geq 1 \\ S_{2n} \times S_2 \wr S_n \times \prod_{d|t} S_{\varphi(d)}^4 \times \prod_{s=2}^{k} \prod_{d|2^s t, d\not| 2^{s-1} t} S_{\varphi(d)}^2 & \\ \times \prod_{d|2^{k+1} t, d\not| 2^k t} S_{\varphi(d)} \wr S_2 & t>1, k\geq 1 \end{cases},$$

also,

$$Aut(P(SD_{8n})) \cong \begin{cases} S_{4n-2} \times S_{2n} \times (S_2 \wr S_n), & n \text{ is a power of } 2 \\ \prod_{d|4n} S_{\varphi(d)} \times S_{2n} \times (S_2 \wr S_n) & \text{otherwise} \end{cases}.$$

The smallest sporadic group is the first Mathieu group M_{11}, it has order 7920. There are many presentations for the group M_{11}, we give two of its known presentation, [39].

$$\begin{aligned} M_{11} &\cong < a, b, c | a^{11} = b^5 = c^4 - (ac)^3 = 1, b^4ab = a^4, c^3bc = b^2 >, \\ &\cong < a, b, c, d | a^2 = b^2 = c^2 = d^2 = (ab)^5 = (bc)^3 = (bd)^4 = (cd)^3 = (abdbd)^3 = 1 > . \end{aligned}$$

The paper by Around (1960) increased the interest to finite simple groups, as Janko in Australia found (1965) the first new sporadic group J_1 a century later after Mathieu's. It turns out that J_1 had order 175560. A presentation for J_1 in terms of its standard generators is given below [12]:

$$J_1 \cong < a, b | a^2 = b^3 = (ab)^7 = (ab(abab^{-1})^3)^5 = (ab(abab^{-1})^6 abab(ab^{-1})^2)^2 = 1 > .$$

The automorphism groups of M_{11} and J_1 are determined as follows:

Example 3.4. [5] Let M_{11} be the first Mathieu group and J_1 be the first Janko group, then,

$$\begin{aligned} Aut(P(M_{11})) &\cong (S_{10} \wr S_{144}) \times (S_4 \wr S_{396}) \times (S_2 \wr S_{55}) \times ((S_6 \wr S_3) \times (S_2 \wr S_4) \times S_2) \wr S_{165}, \\ Aut(P(J_1)) &\cong (S_{10} \wr S_{596}) \times (S_6 \wr S_{4180}) \times (S_{18} \wr S_{1540}) \\ &\times ((S_2 \times S_8 \times S_4 \times (S_4 \wr S_3) \times (S_2 \wr S_5)) \wr S_2) \wr S_{1463}. \end{aligned}$$

Moreover, in [30] the automorphism groups of $P(Z_{pq})$, $P(Z_{pqr})$ and $P(Z_{p^2q^2})$ are calculated as follows:

$$\begin{aligned} Aut(P(Z_{pq})) &\cong S_{\varphi(pq)+1} \times S_{p-1} \times S_{q-1}, \\ Aut(P(Z_{pqr})) &\cong S_{\varphi(pqr)} \times S_{p-1} \times S_{q-1} \times S_{r-1} \times S_{\varphi(pq)} \times S_{\varphi(pr)} \times S_{\varphi(qr)}, \\ Aut(P(Z_{p^2q^2})) &\cong S_{\varphi(p^2q^2)+1} \times S_{p-1} \times S_{\varphi(p^2)} \times S_{q-1} \times S_{\varphi(q^2)} \times S_{\varphi(pq)} \times S_{\varphi(pq^2)} \times S_{\varphi(p^2q)}. \end{aligned}$$

As we mentioned in above Theorem 3.4 is playing a main role in finding automorphism group of power graphs. In [19], the authors obtained the following results from Theorem 3.3.

Example 3.5. [19] If n is odd, then

$$Aut(\mathcal{S}(D_{2n})) = \begin{cases} S_{n-1} \times S_n & n \text{ is a prime power} \\ S_n \times \prod_{d|n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

and if n is even then

$$Aut(\mathcal{S}(D_{2n})) = \begin{cases} S_{2n} & n \text{ is a power of } 2 \\ S_{\varphi(n)+1} \times S_{n+1} \prod_{\{1,n,2\} \neq d|n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

if n is odd, then

$$Aut(\mathcal{S}(T_{4n})) = S_{2n} \times \prod_{d|2n} S_{\varphi(d)},$$

and if n is even then

$$Aut(\mathcal{S}(T_{4n})) = \begin{cases} S_{4n} & n \text{ is a power of } 2 \\ S_{\varphi(2n)+1} \times S_{2n+2} \prod_{\{1,2n,4\} \neq d|2n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

for arbitrary n,

$$Aut(\mathcal{S}(SD_{8n})) = \begin{cases} S_{8n} & n \text{ is a power of } 2 \\ S_{\varphi(4n)+1} \times S_{2n+1} \times S_{2n+2} \prod_{\{1,4n,2,4\} \neq d|4n} S_{\varphi(d)} & \text{otherwise} \end{cases},$$

if $n = 2^k$ then $Aut(\mathcal{S}(V_{8n})) \cong S_{8n}$, and if n is an odd prime then $Aut(\mathcal{S}(V_{8n})) = S_{2n+3} \times S_{2n} \times S_{3\varphi(n)} \times \prod_{\{1,2n,2\} \neq d|2n} S_{\varphi(d)}$.

4. Automorphism groups of commuting graphs

The commuting graphs $\Delta(G)$ and $\Gamma(G)$ of a group G are defined in the introduction. The following theorem established the relation between $Aut(G)$, $Aut(\Delta(G))$ and $Aut(\Gamma(G))$.

Theorem 4.1. [33] *Let G be a finite group, then*

1. $Aut(G) = Aut(\Delta(G))$ *if and only if* $|G| = 1$.
2. $Aut(\Delta(G)) \cong Aut(\Gamma(G)) \times S_{Z(G)}$.

Mirzargar, Pach and Ashrafi studied the subgroups of $Aut(\Delta(G))$ in [33, 34]. The first subgroups are $Aut(\Gamma(G))$ and $Aut(G)$, then they added some automorphisms of graph to $Aut(G)$ and constructed bigger subgroups. Define two permutations $\Phi_{x,y}, \phi : G \to G$ as follows: $\Phi_{x,y}$ fixed each element $a \in G \setminus \{x, y\}$ and maps x into y and vice-versa; and, the permutation ϕ is defined by $x \to x^{-1}$ for each element $x \in G$. They also defined $Aut^*(G) = \langle Aut(G), \phi\rangle$ and considered to the equality of the subgroups and the main group.

Theorem 4.2. [33] $Aut^*(G) = Aut(\Delta(G))$ *if and only if* $G \cong S_3$.

Let the cosets $Z(G)x_1, Z(G)x_2, \cdots , Z(G)x_{m-1}$ of the group $G/Z(G)$ and define a new graph $\Delta^u(G)$ with $V(\Delta^u(G)) = \{x_0 = 1, x_1, \cdots , x_{m-1}\}$ and $E(\Delta^u(G)) = \{x_ix_j|x_ix_j = x_jx_i, 0 \leq i < j \leq m - 1\}$. Notice when $|Z(G)| = 1$ then $\Delta(G) \cong \Delta^u(G)$. It is clear that every two elements in one of these cosets commute. Hence we have a complete graph in any of these cosets. On the other hand, if there exists $x_i \in Z(G)x_i, x_j \in Z(G)x_j$ satisfying $x_ix_j = x_jx_i$, then for every $y_i \in Z(G)x_i, y_j \in Z(G)x_j$ we have $y_iy_j = y_jy_i$. Finally, the set of all $\phi \in Aut(\Delta(G))$ such that for $a, b \in G$ if $ab^{-1} \in Z(G)$, then $\phi(a)\phi(b)^{-1} \in Z(G)$ is denoted by T. These notations are applied in [33] to prove two following theorems.

Theorem 4.3. [33] *Let G be a group. Then,*

1. $Aut(\Delta^u(G))$ *is a subgroup of* $Aut(\Delta(G))$. *Moreover,* $Aut(\Delta^u(G)) = Aut(\Delta(G))$ *if and only if* $|Z(G)| = 1$.
2. *If G is not centerless then T is a subgroup of $Aut(\Delta(G))$, and $Aut(\Delta(G)) = T$ if and only if for each pair a, b of elements of G with $C_G(a) = C_G(b)$, we have $ab^{-1} \in Z(G)$.*

Theorem 4.4. [33] *Let $|Z(G)| \geq 2$, where G be a nonabelian group. If $T = Aut(\Delta(G))$ then $G/Z(G)$ is an elementary abelian $2-$group.*

For a finite group G define a labelled graph $\Delta^v(G)$ as follows. For $a, b \in G$ let $a \sim b$ if $C_G(a) = C_G(b)$. Clearly, $\sim$ is an equivalence relation, the equivalence class of $a \in G$ is $A(a) = \{x | C_G(x) = C_G(a)\}$. Let us denote the equivalence classes by $A_1, \ldots, A_k$, these are the vertices of $\Delta^v(G)$. Two vertices A_i and A_j are connected if and only if $a_i a_j = a_j a_i$, for some $a_i \in A_i, a_j \in A_j$. At first, we note that if there exists $a_i \in A_i, a_j \in A_j$ satisfying $a_i a_j = a_j a_i$, then for every $b_i \in A_i, b_j \in A_j$ we have $a_j \in C_G(a_i) = C_G(b_i)$. So, $b_i \in C_G(a_j) = C_G(b_j)$ implies that $b_i b_j = b_j b_i$. Each equivalence class is the union of some sets of the form $tZ(G)$, hence there exists a positive integers c_i such that $|A_i| = c_i|Z(G)|$. Let $\alpha(A_i) = c_i$ be the label of the vertex A_i in $\Delta^v(G)$. One can see $\phi : V(\Delta^v(G)) \to V(\Delta^v(G))$ is an automorphism of the labelled graph $\Delta^v(G)$ if ϕ is a bijection, it preserves the edges (and the non-edges) and it preserves the labels. The automorphism group formed by these automorphisms is denoted by $Aut(\Delta^v(G))$. Define $S_{A_i} = \{f_\sigma \mid \sigma \in S_{|A_i|}, \forall x \in A_i, f_\sigma(x) = \sigma(x), \forall x \notin A_i, f_\sigma(x) = x\}$, $1 \leq i \leq k$. Clearly, S_{A_i} is a subgroup of $Aut(\Delta(G))$. The connection between $Aut(\Delta(G))$ and $Aut(\Delta^v(G))$ is described by the following theorem:

Theorem 4.5. [33] *There is a subgroup A of $Aut(\Delta(G))$ such that $A \cong Aut(\Delta^v(G))$ and $Aut(\Delta(G)) = \langle S_{A_1}, \cdots, S_{A_k}\rangle \times A$.*

In [38], Rocke proved that the following are equivalent:

1. G has abelian centralizers;
2. If $xy = yx$, then $C_G(x) = C_G(y)$ whenever $x, y \notin Z(G)$;
3. If $xy = yx$ and $xz = zx$, then $yz = zy$ whenever $x \notin Z(G)$;
4. If U and B are subgroups of G and $Z(G) < C_G(U) \leq C_G(B) < G$ then $C_G(U) = C_G(B)$.

Therefore, the intersection of two proper element centralizers of an AC-group is the center of G. If G is an AC-group, then $\Delta(G)$ is a union of some complete graphs with all vertices adjacent to the elements of $Z(G)$. So, $\Delta(G)$ is $n_1(C_G(x_1) \setminus Z(G)) \cup n_2(C_G(x_2) \setminus Z(G)) \cup \cdots \cup (n_r C_G(x_r) \setminus Z(G))$ and also every element of $Z(G)$ is adjacent to all elements of G, such that for each $i, 1 \leq i \leq r$, we have n_i isomorphic components with complete graph of size $|C_G(x_i) \setminus Z(G)|$. In [33], the authors proved that if G is an AC-group with the above notations then,

$$\begin{aligned} Aut(\Delta(G)) &\cong ((S_{|C_G(x_1)|-|Z(G)|}) \wr S_{n_1}) \times ((S_{|C_G(x_2)|-|Z(G)|}) \wr S_{n_2}) \times \cdots \\ &\times ((S_{|C_G(x_n)|-|Z(G)|}) \wr S_{n_r}) \times S_{Z(G)}. \end{aligned}$$

Finally, from [33], $|Aut(\Delta(G))|$ can not be a prime power or a square-free number. Moreover, $|Aut(\Delta(G))| = 1$ if and only if G is trivial, $Aut(\Gamma(G))$ is abelian if and only if G is a group of order 1 or 2. Also if $|G| > 2$ then $Aut(\Delta(G))$ is a nonabelian group.

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DOUBLE EXPONENTIAL EULER–SINC COLLOCATION METHOD FOR A TIME–FRACTIONAL CONVECTION–DIFFUSION EQUATION

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Abstract. In this research, a new version of Sinc–collocation method incorporated with a Double Exponential (DE) transformation is implemented for a class of convectiondiffusion equations that involve time fractional derivative in the Caputo sense. Our approach uses the DE Sinc functions in space and the Euler polynomials in time, respectively. The problem is reduced to the solution of a system of linear algebraic equations. A comparison between the proposed approximated solution and numerical/exact/available solution reveals the reliability and significant advantages of our newly proposed method.
Keywords. Time-fractional convection–diffusion equation; Shifteted Legendre polynomials; Euler-Sinc collocation; Caputo fractional derivative; Double exponential.

1. Introduction

We focus on a time–fractional convection–diffusion equation with variable coefficients of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + a(x)\frac{\partial u(x,t)}{\partial x} + b(x)\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),\ 0 < x < 1,\ 0 < t \leqslant 1, \tag{1.1}$$

with initial condition

$$u(x,0) = g(x),\ \ 0 < x < 1, \tag{1.2}$$

and boundary conditions

$$u(0,t) = h(t),\ \ u(1,t) = k(t),\ \ 0 < t \leqslant 1, \tag{1.3}$$

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where $a(x), b(x) \neq 0$ are continuous functions, $\alpha \in (0, 1]$ and the oprator $\frac{\partial^\alpha}{\partial t^\alpha}$ (or D_t^α) is defined in the Caputo sense as follows:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}. \tag{1.4}$$

We recall that the Caputo fractional derivative of the power function satisfies (1.5)

$$D_t^\alpha t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{p-\alpha}, & p \in \mathbb{N}_0 \text{ and } p \geqslant \lceil \alpha \rceil \text{ or } p \notin \mathbb{N} \text{ and } p > \lfloor \alpha \rfloor, \\ 0, & p \in \mathbb{N}_0 \text{ and } p < \lceil \alpha \rceil. \end{cases}$$

where $\lceil \alpha \rceil$ and $\lfloor \alpha \rfloor$ denote the ceil and floor of α, respectively.
The problem (1.1)–(1.3) is a class of time–fractional diffusion/wave equations which appears frequently e.g., in earthquake modeling, non-Markov Processes and in the mathematical modelling of the earth surface transport [6, 4]. In recent years, due to its extensive engineering applications, much attention has been focused on providing an effective numerical method to solve it.
Izadkhah and Saberi-Najafi [2] expanded the required approximate solution as the elements of the Gegenbauer polynomials in time and Lagrange polynomials in space and by using a global collocation, reduced the problem to a system of linear algebraic equations. Other numerical/analytic methods, such as standard Sinc–Legendre collocation, finite difference, finite element, ADM, HAM and VIM have also been developed to solve time–fractional diffusion/wave equations and have been fully addressed in [6, 2].
The numerical methods based on Sinc approximations have been studied extensively during the last three decades. Recently, these approaches have been used for solution of fractional ordinary/partial differential equations and usually give a result with high accuracy even for problems with an algebraic singularity at the end point. Many of these methods have been found very effective and reliable and under some conditions, have convergency of order $\mathscr{O}\left(\exp\left(\frac{-cN}{\ln(N)}\right)\right)$, where $c > 0$ is a constant and $N \in \mathbb{N}$ depends on number of mesh points. For more historical remarks and technical details about Sinc numerical methods see [7, 8] and the references therein. In the present paper, we apply the Euler–Sinc collocation method coupled with Double Exponential (DE) transformation for solving equations (1.1)–(1.3). Our method consists of reducing the solution to a set of algebraic equations by expanding $u(x,t)$ as a combination of modified Sinc functions (in space) and Euler polynomials (in time) with a special boundary treatment. The numerical experiments are implemented in Maple 15 programming. The programs are executed on a Notebook System with 2.0 GHz Intel Core 2 Duo processor with 2 GB 533 MHz DDR2 SDRAM.

2. Basic Definitions and Theorems of the Method

In this section, we introduce some basic definitions and derive preliminary results for developing our method.

2.1. Sinc Functions

The sine cardinal or Sinc function on $\mathbb{C}$ defined by

$$\mathrm{Sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

For step size $h > 0$, and any integer k, the translated Sinc function on uniform meshes is denoted $S(k,h)(z)$ and defined by

$$S(k,h)(z) = \mathrm{Sinc}\left(\frac{z-kh}{h}\right).$$

For target equation (1.1) on spatial interval $0 < x < 1$, we employ a conformal map

$$w = \phi(x) = \ln\left(\frac{1}{\pi}\ln\left(\frac{x}{1-x}\right) + \sqrt{1+\left(\frac{1}{\pi}\ln\left(\frac{x}{1-x}\right)\right)^2}\right),$$

with inverse

$$x = \psi(w) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh w\right), \tag{2.1}$$

as the DE transformation, on a subinterval $\Gamma = (0,1) = \psi(\mathbb{R})$ with $\phi(0) = -\infty$ and $\phi(1) = \infty$. Now we have

$$f(x) \simeq \sum_{k=-N}^{N} f(x_k) S_k(x),$$

as a method of interpolation where, $x_k = \psi(kh) \in (0,1)$ and $S_k(x) = S(k,h) \circ \phi(x)$ are defined as Sinc grid points and the translated Sinc basic functions, respectively. Also, the pth order derivative of $S_j(x)$ with respect ot ϕ at the node x_k is denoted by $\delta_{j,k}^{(p)}$ and computed by the following relation

$$\delta_{j,k}^{(p)} = \frac{d^p}{d\phi^p}\left[S(j,h)\circ\phi(x)\right]\Big|_{x=x_k} = \frac{\pi^p}{h^p}\begin{cases} \lim\limits_{t\to 0}\dfrac{d^p}{dt^p}\left(\dfrac{\sin t}{t}\right), & j = k \\ \dfrac{d^p}{dt^p}\left(\dfrac{\sin t}{t}\right)\Big|_{t=\pi(k-j)}, & j \neq k \end{cases}, \quad p = 0,1,\ldots \tag{2.2}$$

Leibniz's rule for higher-order derivatives of products provides the following recurrence formulas of (2.2) [5]:

$$\delta_{j,k}^{(2r)} = \frac{1}{h^{2r}}\begin{cases} \frac{(-1)^r\pi^{2r}}{2r+1}, & j = k, \\ \frac{(-1)^{k-j}(2r)!}{(k-j)^{2r}}\sum\limits_{s=0}^{r-1}\frac{(-1)^{s+1}\pi^{2s}}{(2s+1)!}(k-j)^{2s}, & j \neq k, \end{cases}$$

$$\delta_{j,k}^{(2r+1)} = \frac{1}{h^{2r+1}} \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}(2r+1)!}{(k-j)^{2r+1}} \sum\limits_{s=0}^{r} \frac{(-1)^s \pi^{2s}}{(2s+1)!}(k-j)^{2s}, & j \neq k, \end{cases}$$

with $r = 0, 1, 2, \ldots$.
Hence for $p = 0, 1, 2$ these quantities are as following

$$\delta_{j,k}^{(0)} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.3}$$

$$\delta_{j,k}^{(1)} = \frac{1}{h} \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{2.4}$$

$$\delta_{j,k}^{(2)} = \frac{1}{h^2} \begin{cases} -\frac{\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k, \end{cases} \tag{2.5}$$

Definition 2.1. [8] A function f is said to decay double exponentially with respect to ψ, if there exist positive constants α and β such that

$$|f(\psi(\xi))| \leq \alpha \exp\left(-\beta \exp(|\xi|)\right), \text{ for all } \xi \in \mathbb{R}.$$

Moreover, under some conditions on f, $f(\psi(\xi))$ decays double exponentially with respect to ψ.

If the function f is double exponentially decreasing, then the interpolation formula of it over [0, 1] takes the form

$$\sum_{j=-N}^{N} f(x_j) S_j(x)$$

and under some restrictions on f, it is shown by both theoretical analysis and numerical experiments that the approximation error on $x \in [0, 1]$ can be estimated by

$$\left\| f(x) - \sum_{j=-N}^{N} f(x_j) S_j(x) \right\|_\infty \leqslant C \exp\left[\frac{-\pi d N}{\ln(\pi d N/\beta)}\right] \tag{2.6}$$

where h is taken as

$$h = \frac{\ln(\pi d N/\beta)}{N}, \tag{2.7}$$

and C is a constant independent of f and N [8].

2.2. Euler Functions

We end this section by introducing the classical Euler polynomials $E_n(t)$ and deriving some of their features. The classical Euler polynomials denoted by $E_n(t)$, are usually defined by means of the following generating function:

$$\frac{2e^{tx}}{e^x+1} = \sum_{n=0}^{\infty} E_n(t)\frac{x^n}{n!}, \qquad |x| < \pi. \tag{2.8}$$

It is possible to get explicit expressions for the Euler polynomials as following [3]:

$$E_n(t) = \frac{1}{n+1}\sum_{k=0}^{n}\left(2-2^{n+2-k}\right)\binom{n+1}{k}B_{n+1-k}\,t^k, \qquad n = 0,\,1,\,2,\,3,\,\ldots \tag{2.9}$$

where B_k's are the Bernoulli numbers.

Lemma 2.1. *Let $\alpha > 0$. Then the fractional derivative of $E_j(t)$ of order α is*

$$D_t^{\alpha}E_j(t) = \sum_{k=\lceil\alpha\rceil}^{j} e_{j,k}t^{k-\alpha}, \qquad j = 0,\,1,\,2,\,3,\,\ldots \tag{2.10}$$

where $e_{j,k} = \frac{1}{j+1}\left(2-2^{j+2-k}\right)\binom{j+1}{k}B_{j+1-k}\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}$.

Proof. The proof is straightforward and deduces from (1.5) and (2.9). □

3. Euler–Sinc Collocation Method

In order to discretize equations (1.1)–(1.3) by using Euler–Sinc collocation approach, we approximate $u(x,t)$ as

$$u_{m,n}(x,t) = \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{i,j}S_i(x)E_j(t) + B(x,t). \tag{3.1}$$

Here, $S_k(x)$'s convege to zero as x tends to 0 or 1. Hence we can pick a nice function satisfying the boundary conditions (1.3), say $B(x,t) = (1-x)h(t) + xk(t)$. The $(2m+1)(n+1)$ unknown expansion coefficients $\{c_{i,j}\}$ in (3.1) are determined by substituting $u_{m,n}(x,t)$ into equations (1.1)–(1.2) and evaluating the results at the collocation points $x_k = \psi(kh)$, $k = -m,\,\ldots,\,m$, and t_l, $l = 1,\,\ldots,\,n$ (as the n first roots of the shifteted Legendre polynomial $P_{n+1}(t)$ in $[0,\,1]$).

Lemma 3.1. *If the assumed approximate solution of the initial–boundary value problem (1.1)–(1.3) is (3.1), then the discrete Euler–Sinc collocation system for the* *determination of the unknown coefficients* $\{c_{i,j}\}$ *is given by the following* $(2m+1)\times(n+1)$ *equations*

$$\sum_{j=0}^{n}\sum_{r=\lceil\alpha\rceil}^{j} c_{k,j}e_{j,r}t_l^{r-\alpha}+a\left(x_k\right)\sum_{i=-m}^{m}\sum_{j=0}^{n}c_{i,j}\lambda_{i,k}E_j\left(t_l\right)$$

$$+\,b\left(x_k\right)\sum_{i=-m}^{m}\sum_{j=0}^{n}c_{i,j}\mu_{i,k}E_j\left(t_l\right)=F\left(x_k,t_l\right),\tag{3.2}$$

$$\sum_{j=0}^{n}c_{k,j}E_j(0)=g\left(x_k\right),\quad k=-m,\,-m+1,\,\ldots,m,\quad l=0,\,1,\,\ldots,n,$$

where,

$$\lambda_{i,k}=\delta_{i,k}^{(1)}\phi'\left(x_k\right),\qquad \mu_{i,k}=\delta_{i,k}^{(2)}\left(\phi'\left(x_k\right)\right)^2+\delta_{i,k}^{(1)}\phi''\left(x_k\right),\tag{3.3}$$

and

$$F\left(x_k,t_l\right)=f\left(x_k,t_l\right)-D_t^{\alpha}B\left(x_k,t_l\right)-\frac{\partial B\left(x_k,t_l\right)}{\partial x}-\frac{\partial^2 B\left(x_k,t_l\right)}{\partial x^2}.\tag{3.4}$$

Proof. The process of proof is similar to the proof of lemma 1 in [6] and left to the reader. □

Finally, the linear system (3.2) for the unknown coefficients $\{c_{i,j}\ :\ i=-m,\,-m+1,\,\ldots,m,\ j=0,\,1,\,\ldots,n\}$ can be solved by using `fsolve` command in MAPLE.

4. Illustrative Examples

In this section we show numerical results of the Euler–Sinc collocation method. In order to verify the performance and reliability of the proposed method, three examples are examined in this section. In all examples, we heuristically choose $d=\frac{\pi}{6}$ and $\beta=\frac{\pi}{2}$ which leads to $h=\frac{1}{m}\ln\left(\frac{m\pi}{3}\right)$. In the presence of exact solutions, we also define maximum absolute error $e_{m,n}$ as

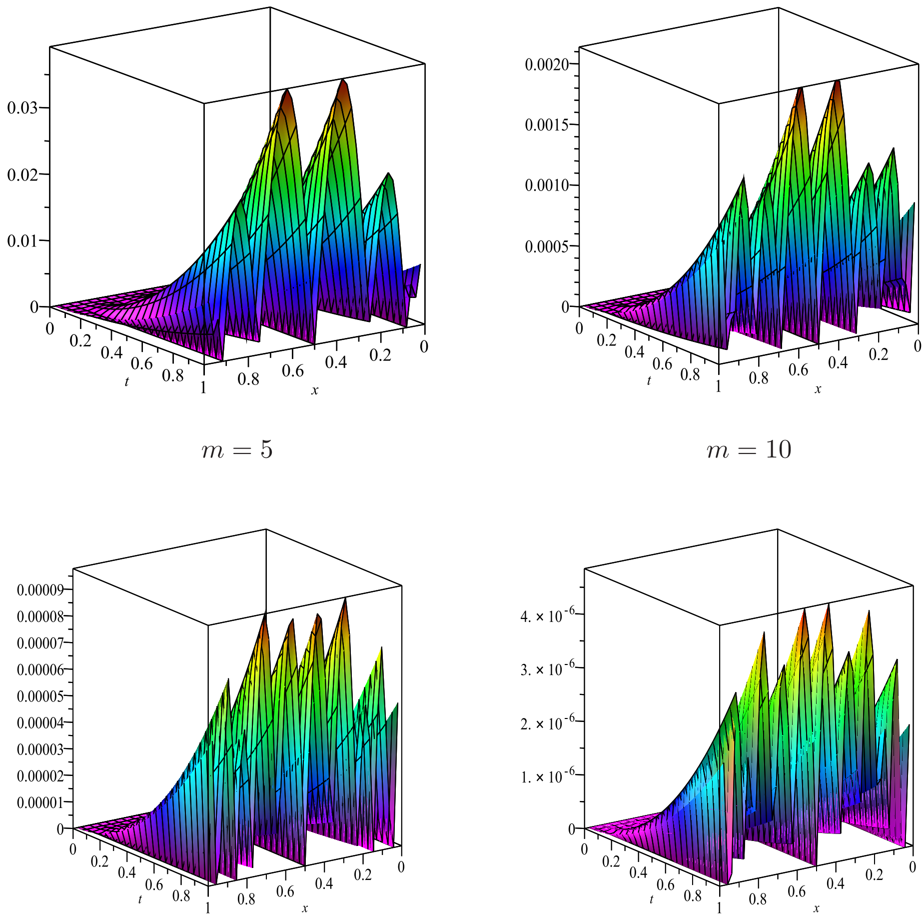
$$e_{m,n}:=\max\left\{|u(x,t)-u_{m,n}(x,t)|:\ \ 0\leqslant x\leqslant 1,\ 0<t\leqslant 1\right\}$$

.

Example 4.1. [6] Consider the following time–fractional diffusion equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}-\frac{\partial^2u(x,t)}{\partial x^2}=f(x,t),\quad 0<x<1,\ 0<t\leqslant 1,\ \ 0<\alpha\leqslant 1\tag{4.1}$$

with $h(t)=0,\ k(t)=0,\ f(x,t)=\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\sin(2\pi x)+4\pi^2t^2\sin(2\pi x)$ and $u(x,0)=0$. This problem has the exact solution $u(x,t)=t^2\sin(2\pi x)$.



$m = 5$ $m = 10$

$m = 15$ $m = 20$

FIG. 4.1: The convergence of the sequence $e_{m,3}$ for different values of m.

In Figure 4.1, as a convergence criterion of $u_{m,n}(x,t)$ to $u(x,t)$, we illustrate the error function $e_{m,n}$ for $n = 3$ and $m = 5$, 10, 15, 20. This figure illustrates that the error function $e_{m,3}$ gets smaller and smaller values by increasing spatial resolution m.

Example 4.2. [6, 2] Consider the problem (1.1)–(1.3) of order $0 < \alpha < 1$ with $a(x) = x$, $b(x) = 1$, $f(x,t) = 2t^\alpha + 2x^2 + 2$, $g(x) = x^2$, $h(t) = \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$, $k(t) = 1 + \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$ and the exact solution $u(x,t) = x^2 + \frac{2\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$.

Table 4.1: Comparison of absolute error $e_{m,n}$ for $\alpha = 0.5$, $n = 3$ and $t = \frac{1}{2}$, for Example 4.2.

x	Method of [1] for $m = 64$	Method of [6] for $m = 25$ and $n = 7$	Present method $m = 5$	$m = 10$
0.1	1.210×10^{-03}	6.462×10^{-06}	7.816×10^{-05}	5.648×10^{-07}
0.2	1.259×10^{-03}	1.578×10^{-05}	1.100×10^{-04}	5.198×10^{-07}
0.3	1.865×10^{-03}	2.272×10^{-05}	9.353×10^{-05}	4.891×10^{-07}
0.4	7.412×10^{-03}	2.674×10^{-05}	8.441×10^{-05}	5.835×10^{-07}
0.5	1.000×10^{-06}	2.759×10^{-05}	8.440×10^{-05}	5.951×10^{-07}
0.6	7.460×10^{-03}	2.534×10^{-05}	8.604×10^{-05}	5.736×10^{-07}
0.7	1.724×10^{-03}	2.035×10^{-05}	9.540×10^{-05}	4.833×10^{-07}
0.8	4.990×10^{-03}	1.320×10^{-05}	1.101×10^{-04}	5.227×10^{-07}
0.9	1.678×10^{-02}	4.653×10^{-06}	7.636×10^{-05}	5.591×10^{-07}

Taking $\alpha = 0.5$ and temporal resolution $n = 3$, in Table 4.1, we compare our method for moderate values of spatial resolution, say $m = 5$ and $m = 10$, together with the results obtained by using the wavelet method [1] for $m = 64$ and Sinc-Legendre collocation method [6] with $n = 7$ and $m = 25$. The results of this Table show that our computations are in good agreement with those obtained by the existing methods and a slight increase in m significantly improves our numerical results.

5. Conclusion

The present work exhibits the reliability of the Euler–Sinc method to solve a Caputo time fractional convection-diffusion equation that arises frequently in the mathematical modeling of real-world physical problems such as earthquake modeling, traffic flow model and financial option pricing problems. Approximated results are in close agreement with numerical/exact solutions and the results of the previous section reveal that our scheme can be used to obtain accurate numerical solutions of problem (1.1)-(1.3) with very little computational effort.

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COUNTING THE NUMBER OF SUBGROUPS AND NORMAL SUBGROUPS OF THE GROUP U_{2np}, p IS AN ODD PRIME

Haider Baqer Shelash

Abstract. The aim of this paper is to compute the number of subgroups and normal subgroups of the group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e, aba^{-1} = b^{-1}\rangle$, where p is an odd prime. Suppose $n = 2^r \prod_{1\le i\le s} p_i^{\alpha_i}$ in which p_i's are distinct odd primes, α_i's are positive integers and $t = \prod_{1\le i\le s} p_i^{\alpha_i}$. It is proved that the number of subgroups is $2\tau(2n) + (p-1)\left(\tau(\frac{n}{p}) + \tau(\frac{n}{2^r})\right)$, when $p \mid n$ and $2\tau(2n) + (p-1)\,[\tau(t)]$, otherwise. It will be also proved that this group has $\tau(2n) + \tau(n)$ normal subgroups.
Keywords. group; subgroup; dihedral group; finite group.

1. Introduction

Cavior [1] proved that the number of subgroups of a dihedral group of order $2n$ can by computed by $\tau(n) + \sigma(n)$. After publishing this work Calhoun [2] computed the number of subgroups in certain finite groups. For more information on this problem, we encourage the readers to consult the interesting book of Tărnăuceanu [6].

Following Darafsheh and Yaghoobian [3], we define:

$$U_{2nm} = \langle a, b \mid a^{2n} = b^m = e \mid aba^{-1} = b^{-1}\rangle.$$

This group has order $2nm$ and can be written as the semi-direct product of two cyclic groups that one of them is of order m and another one has order $2n$. Set $n = 2^r \prod_{1\le i\le s} p_i^{\alpha_i}$, where p_i's are distinct odd prime numbers and α_i's are positive integers. Shelash [4], introduced an algorithm for computing all subgroups and normal subgroups of a finite group. Shelash and Ashrafi [5] applied this algorithm to compute the number of minimal and maximal subgroups of certain finite groups.

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Here, we apply this algorithm to obtain the number of subgroups and normal subgroups of the group U_{2np}, where p is an odd prime.

The order table of U_{2np} is defined as the matrix $A = [a_{ij}]$ with $a_{ij} = 2^{i-1}c_{j-1}$, $1 \leq i \leq \tau(2^{r+1})$ and $1 \leq j \leq \tau(\prod_{1\leq i\leq s} p_i^{\alpha_i})$, where c_j is an odd divisor of $|U_{2np}|$ and the function $\tau(n)$ is defined as the number of positive divisors of n. For simplicity of our argument, we assume that $c_0 < c_1 < \cdots < c_{\alpha-1}$, where $\alpha = \tau(\prod_{1\leq i\leq s} p_i^{\alpha_i})$. For example if $|G| = 60$, then the order table of G is as follows:

a_{ij}	1	2	2^2
$c_0 = 1$	1	2	4
$c_1 = 3$	3	6	12
$c_2 = 5$	5	10	20
$c_3 = 15$	15	30	60

Throughout this paper our notations are standard and can be taken from the standard books on group theory. The function $\sigma(n)$ is defined as the summation of all divisors of n. Furthermore, the number of subgroups and normal subgroups of a group G are denoted by $Sub(G)$ and $NSub(G)$, respectively. Our calculations are done with the aid of GAP [7].

2. Main Results

The group $U_{2np} = \langle a, b \mid a^{2n} = b^p = e \mid aba^{-1} = b^{-1}\rangle$ is a finite group of order $2np$, where p is an odd prime. Suppose $n = 2^r \prod_{1\leq i\leq s} p_i^{\alpha_i}$ in which p_i's are distinct odd primes and α_i's are positive integers. For simplicity of our argument, we assume that $t = \prod_{1\leq i\leq s} p_i^{\alpha_i}$. If $p = p_k \mid n$ then the order of U_{2np} is equal to $2^{r+1}p_1^{\alpha_1}\cdots p_k^{\alpha_{k+1}}\cdots p_s^{\alpha_s}$, otherwise it is $2^{r+1}p\prod_{1\leqslant i\leq s} p_i^{\alpha_i}$.

Lemma 2.1. *The following hold:*

1. *If q is even then $a^q b^w = b^w a^q$;*
2. *If q is odd then $a^q b^w = b^{-w} a^q$.*

Proof. By presentation of the group U_{2np}, we have $aba^{-1} = b^{-1}$ and so if q is even then $a^q b = ba^q$. Furthermore, if q is odd then $a^q b = b^{-1}a^q$. Choose positive integer w. Then $a^q b^w = ba^q b^{w-1}$. If q is even number, thus $a^q b^w = b^w a^q$. If q is odd number then $a^q b^w = b^{-1}a^q b^{w-1}$, then $a^q b^w = b^{-w}a^q$. □

Proposition 2.1. *Let $n = 2^r t$, $t = \prod_{1\leq i\leq s} p_i^{\alpha_i}$ and $m = p$ be an odd prime number. Then the structure description of the group U_{2np} is $C_t \times (C_p : C_{2^{r+1}})$.*

Proof. Suppose $\Phi = \langle a^{2^{r+1}}\rangle$, $\Psi = \langle b\rangle$ and $\Omega = \langle a^t\rangle$ are subgroups of U_{2np}. By Lemma 2.1, one can see that $g\Phi g^{-1} = g\langle a^{2^{r+1}}\rangle g^{-1} = \langle a^{2^{r+1}}\rangle = \Phi$, for all $g \in U_{2np}$.

Thus $\Phi \trianglelefteq U_{2np}$. Define $(\Psi : \Omega) = \langle b, a^t\rangle$. If i is odd then,

$$\begin{aligned} a^ib^j(\Psi : \Omega)b^{-j}a^{-i} &= a^ib^j\langle b, a^t\rangle b^{-j}a^{-i} \\ &= \langle a^ib^jbb^{-j}a^{-i}, a^ib^ja^tb^{-j}a^{-i}\rangle \\ &= \langle b, a^tb^{2j}\rangle \\ &= (\Psi : \Omega), \end{aligned}$$

and if i is an even number,

$$\begin{aligned} a^ib^j(\Psi : \Omega)b^{-j}a^{-i} &= a^ib^j\langle b, a^t\rangle b^{-j}a^{-i} \\ &= \langle a^ib^jbb^{-j}a^{-i}, a^ib^ja^tb^{-j}a^{-i}\rangle \\ &= \langle b, a^tb^{2}\rangle \\ &= (\Psi : \Omega). \end{aligned}$$

Hence $(\Psi : \Omega)$ is a normal subgroup of U_{2np}. On the other hand, $\langle a^{2^{r+1}}\rangle \cap \langle b, a^t\rangle = e$ and $\frac{|\langle a^{2^{r+1}}\rangle| \times |\langle b, a^t\rangle|}{|\langle a^{2^{r+1}}\rangle \cap \langle b, a^t\rangle} = 2np$, which completes our argument. $\square$

Lemma 2.2. *The group U_{2np} has the following types of subgroup:*

1. *The cyclic subgroups $\langle a^i\rangle$ of order $\frac{2n}{i}$, where $i \mid 2n$;*
2. *The subgroups $\langle a^i, b\rangle$ of order $\frac{2np}{i}$, where $i \mid 2n$;*
3. *The cyclic subgroups $\langle a^ib^j\rangle$, where $i \mid 2n$, $2p^k \nmid i$ and $j = 1, \cdots, p-1$.*

Proof. Set $H = \langle a^i\rangle$ and $K = \langle b\rangle$, $i \mid 2n$. By presentation of U_{2np}, K is normal and so $HK = \langle a^i, b\rangle$ has order $\frac{2np}{i}$. The result now follows from Lemma 2.1. $\square$

Proposition 2.2. *Let $n = 2^r \prod_{1\leq i\leq s} p_i^{\alpha_i}$ be a positive integer and p be an odd prime number. The following hold:*

1. *There is at most one subgroup of order k such that $2 \mid k, 2^{r+1} \nmid k$ and $p \nmid k$;*
2. *If $p \mid n$, then there exists one subgroup of order k such that $p^{\alpha_i+1} \mid k$;*
3. *There exists p subgroups of order k when $p \nmid k$ and $2^{r+1} \mid k$;*
4. *There exists $\sigma(p)$ subgroups of order k when $p \mid k$ and $p^{\alpha_{i+1}} \nmid k$.*

Proof. Our main proof will consider the following parts:

1. Suppose $p \nmid 2^hv$, $1 \leq h \leq r$, and $v \mid n$. Then $\langle a^{\frac{2^{r+1-h}t}{v}}\rangle$ is a cyclic group of order 2^hv and the order of subgroups $\langle a^{\frac{2^{r+1-h}m}{v}}b\rangle$ and $\langle a^{\frac{2^{r+1-h}m}{v}}, b\rangle$ are not 2^hv. We now apply Lemma 2.2 to get the result.

2. Suppose $2^{r+1} \mid k$. Since $\frac{t}{v}$ is an odd number, by Lemma 2.1 $\langle a^{\frac{t}{v}} b^j \rangle$ are cyclic subgroups of order $2^{r+1}v$, $1 \leq j \leq p$.

3. Consider the subgroups $\langle a^{\frac{2n}{2^h p}} \rangle$ and $\langle a^{\frac{2n}{2^h p}}, b \rangle$, where $1 \leq h \leq r+1$. Since there are $p-1$ subgroups of type $\langle a^{\frac{2n}{2^h p}} b^j \rangle$, $1 \leq j \leq p-1$, the number of all subgroups of order k is equal to $\sigma(p)$

Hence the result. $\square$

Theorem 2.1. *Let p be an odd prime and $n = 2^r \prod_{1 \leq i \leq s} p_i^{\alpha_i}$, where p_i's are distinct odd primes, α_i's are positive integers and $t = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$. Then the number of all subgroups of the group U_{2np} is given by the following:*

1. *If $p \mid n$ then $Sub(U_{2np}) = 2\tau(2n) + (p-1)\left[\tau(\frac{n}{p}) + \tau(\frac{n}{2^r})\right]$.*

2. *If $p \nmid n$ then $Sub(U_{2np}) = 2\tau(2n) + (p-1)\left[\tau(t)\right]$.*

Proof. By presentation of the group U_{2np}, it has $\tau(2n)$ subgroups contained in $\langle a \rangle$. Since $\langle b \rangle$ is a normal subgroup, the group U_{2np} has $\tau(2n)$ subgroups of the form $H\langle b \rangle$ such that H is a subgroup of $\langle a \rangle$. We now assume that $p \mid n$. By Lemma 2.2, it is enough to count the number of subgroups in the form $\langle a^i b^j \rangle$, where $i \mid 2n$, $2p^\alpha \nmid i$ and $1 \leq j \leq p-1$. Note that $2n$ has exactly $\tau(\frac{2n}{2^{r+1}}) = \tau(\frac{n}{2^r})$ odd divisors and the number of all divisors of $2n$ such that $2p \mid i$ and $2p^\alpha \nmid i$ is equal to $\tau(\frac{2n}{2p}) = \tau(\frac{n}{p})$. So the group U_{2np} has exactly $(p-1)[\tau(\frac{n}{p}) + \tau(\frac{n}{2^r})]$ subgroups, when $p \mid n$. If $p \nmid n$, then the number of subgroups of type $\langle a^i b^j \rangle$ is equal to $(p-1)\tau(\frac{n}{2^r}) = (p-1)\tau(t)$. $\square$

We are now ready to count the number of normal subgroups of the group U_{2np}.

Lemma 2.3. *The normal subgroup of the group U_{2np} has one of the following forms:*

1. *All cyclic subgroups $\langle a^i \rangle$ such that $2 \mid i \mid 2n$;*

2. *All subgroups $\langle a^i, b \rangle$, when $i \mid 2n$.*

Proof. The first part follows from Lemma 2.1. We apply the presentation of U_{2np} to prove that $\langle a^k, b \rangle$ is normal, when $k \mid 2n$. Choose the element $a^i b^j$ in U_{2np}. Then we have four cases for the subgroup $a^i b^j \langle a^k, b \rangle b^{-j} a^{-i}$ as follows:

1. *k and i are even numbers.* In this case $\langle a^i b^j a^k b^{-j} a^{-i}, a^i b^j b b^{-j} a^{-i} \rangle = \langle a^k, b \rangle$, as desired.

2. *k is even and i is odd.* Then, $\langle a^i b^j a^k b^{-j} a^{-i}, a^i b^j b b^{-j} a^{-i} \rangle = \langle a^k, b \rangle$ which proves our claim.

3. *k and i are odd numbers.* This shows that $\langle a^ib^ja^kb^{-j}a^{-i}, a^ib^jbb^{-j}a^{-i}\rangle = \langle a^kb^{2j}, b\rangle = \langle a^k, b\rangle$.

4. *k is even and i is odd.* In this case, $\langle a^ib^ja^kb^{-j}a^{-i}, a^ib^jbb^{-j}a^{-i}\rangle = \langle a^kb^{-2j}, b\rangle = \langle a^k, b\rangle$.

Note that a^k and a^kb^j has the same order, when k is odd number. □

Choose $a^i \in U_{2np}$, where i is an odd number. Then $a^i\langle a^ib^j\rangle a^{-i} = \langle a^ia^ib^ja^{-i}\rangle = \langle a^ib^{-j}\rangle$. Since $\langle a^ib^{-j}\rangle \neq \langle a^ib^j\rangle$, all subgroups $\langle a^ib^j\rangle$, $1 \leq j \leq p$ and $i \mid 2n$, are not normal in U_{2np}.

Theorem 2.2. *The number of normal subgroups in the group U_{2np} is given by $NSub(U_{2np}) = \tau(2n) + \tau(n)$.*

Proof. Let p be an odd prime and $n = 2^r \prod_{1\leq i\leq s} p_i^{\alpha_i}$, where p_i's are distinct odd primes, α_i's are positive integers and $t = \prod_{1\leq i\leq s} p_i^{\alpha_i}$. To prove the theorem, we apply Lemma 2.3. We now that each subgroup of type $\langle a^i\rangle$, i is even, is normal. Since

$$\begin{aligned} \tau(2^{r+1}t) - \tau(t) &= \\ \tau(2^{r+1}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) &= (r+2)\tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) - \tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) \\ &= (r+1)\tau(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) \\ &= \tau(2^rp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) \\ &= \tau(n), \end{aligned}$$

$\tau(2^{r+1}t)$ is the number all divisors of $2n$ and $\tau(t)$ is the number of odd divisors of $2n$, $\tau(2^{r+1}t) - \tau(t) = \tau(2^rt) = \tau(n)$ is the number of even divisors of $2n$. On the other hand, the number of all normal subgroups of type $\langle a^i, b\rangle$, $i \mid 2n$, is equal to $\tau(2n)$. Therefore, $NSub(U_{2np}) = \tau(2n) + \tau(n)$. □

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INDEPENDENCE AND PI POLYNOMIALS FOR FEW STRINGS

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Abstract. If s_k is the number of independent sets of cardinality k in a graph G, then $I(G;x) = s_0 + s_1x + \ldots + s_\alpha x^\alpha$ is the independence polynomial of G [Gutman, I. and Harary, F., Generalizations of the matching polynomial, Utilitas Mathematica 24 (1983) 97-106] , where $\alpha = \alpha(G)$ is the size of a maximum independent set. Also the PI polynomial of a molecular graph G is defined as $A + \sum x^{|E(G)|-N(e)}$, where $N(e)$ is the number of edges parallel to e, $A = |V(G)|(|V(G)|+1)/2 - |E(G)|$ and summation goes over all edges of G. In [T. Došlić, A. Loghman and L. Badakhshian, Computing Topological Indices by Pulling a Few Strings, MATCH Commun. Math. Comput. Chem. 67 (2012) 173-190], several topological indices for all graphs consisting of at most three strings are computed. In this paper we compute the PI and independence polynomials for graphs containing one, two and three strings.
Keywords. independent sets; molecular graph; independence polynomials.

1. Introduction

Let G be a simple molecular graph without directed and multiple edges and without loops, the vertex and edge-sets of which are represented by $V(G)$ and $E(G)$, respectively. The set of neighbors of the vertex v will be denoted by $N_G(v)$, if there is no confusion we will simply write $N(v)$ instead of $N_G(v)$. The closed neighborhood of the vertex v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of the vertex v will be denoted by $d(v) = |N_G(v)|$. For $S \subset V(G)$ the graph $G - S$ denotes the subgraph of G induced by the vertices $V(G) \setminus S$. If $e \in E(G)$ then $G - e$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$.

An edge set X is called independent if there is no vertex in common between any two edges in X. Also if this set has r elements we call r-edge set to be independent. Matching polynomial of graph G is defined by the sum of $(-1)^r q(G,r)x^{n-2r}$ [10].

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In which $q(G,r)$ is the number of r-edge independent set of G. A vertex set Y is called independent if there is no edge between any two vertices in Y and if this set has r elements we call r-vertex independent set. Let s_k be the number of r-vertex independent set of cardinality k in a graph G. The polynomial

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + \cdots + s_{\alpha(G)} x^{\alpha(G)}$$

is called the independence polynomial of G, (Gutman and Harary,[5]). We have $s_0 = 1$, $s_1 = |V(G)|$, the number of vertices of G and $s_2 = |E(\overline{G})|$, the number of edges of the complement of G. The following results are easily obtained,(see [2] and [5, 6]). The join of two disjoint graphs G and H is the graph $G+H$ such that $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{v_1v_2 : v_1 \in V(G), v_2 \in V(H)\}$.

Theorem 1.1. If G and H be two vertex-disjoint graphs. Then:

a). $I(G \cup H; x) = I(G;x)I(H;x)$

b). $I(G + H; x) = I(G;x) + I(H;x) - 1$

In [2], Arocha showed that $I(P_n;x) = F_{n+1}(x)$, where $F_n(x)$, $n \geq 0$, are the so-called Fibonacci polynomials. Hoede and Li [6] obtained the following recursive formula for the independence polynomial of a graph.

Theorem 1.2. For any vertex v of a graph G, we have:

$$I(G;x) = I(G - v; x) + xI(G - N_G[v]; x)$$

In [1], Ashrafi, Manoochehrian and Yousefi-Azari defined a new polynomial and they named the Padmakar-Ivan polynomial. They abbreviated this new polynomial as $PI(G,x)$, for a molecular graph G and investigate some of the elementary properties of this polynomial. PI polynomial of G is defined as follow :

$$PI(G,x) = \sum_{f \in E(G)} x^{|E(G)|-N(f)} + \frac{|V(G)|(|V(G)|+1)}{2} - |E(G)|$$

Where $N(f)$ is the number of edges parallel to f. (See survey article [8] and [11] for details)

A thread in a graph G is any maximal connected subgraph induced by a set of vertices of degree 2 in G. A string in G is a subgraph induced by a thread and the vertices adjacent to it. A graph G consists of s strings if it can be represented as a union of s strings so that any two strings have at most two vertices in common [4]. In the extreme case $s = 1$, G is either a path or a cycle, and this, together with the number of vertices, gives us complete information on G. In general, the smaller s, the more information on G is packed into its string decomposition.

The first attempt on a systematic investigation of topological indices of graphs consisting of a few strings was made in a paper by Lukovits [9]. Also in [4], Došlić and co-authors computed explicit formulas for the values of several topological indices (the eccentric connectivity index, the reverse Wiener index, the geometric-arithmetic index, two connectivity indices and two Zagreb indices) for all graphs consisting of at most three strings. The ten classes of graphs considered in paper [4] are shown in Fig 3.1.

Throughout the paper we will consider G_1 denotes a path of length k (or P_{k+1}), G_4 denotes two cycles of length k and m spliced in one vertex, and G_5 denotes three paths of lengths k, m and n spliced together in one of their respective end vertices. Further, whenever referring to the strings of the same type, we assume that the lengths increase with the lexicographic order of the corresponding notational parameter. For example, we assume $k \leq m$ in G_4, G_6 and $k \leq m \leq n$ in G_5, G_8 and G_{10}. Similarly, we take $m \leq n$ in G_7 and G_9, but do not make any assumptions about the relationship of either of them with k. Also, sometimes it will be necessary to refer to the values of the string lengths. In such cases, we put the lengths in the superscripts in the alphabetic order. For example, $G_5^{1,1,n}$ denotes a graph of type (5) whose two path-like strings have length 1. Similarly, $G_9^{1,m,n}$ denotes a graph of type (9) whose path-like string is trivial. Our notation is standard and mainly taken from [3, 7].

2. Independence and PI Polynomials for few Strings

In this section we compute independence and Pi polynomials of graphs G_i, $1 \leq i \leq 10$.

Theorem 2.1. The independence polynomial of P_n, $n \geq 3$, can be obtained from the following equality:

$$I(P_n;x) = \frac{1}{\sqrt{1+4x}}((\frac{1+\sqrt{1+4x}}{2})^{n+2} + (\frac{1-\sqrt{1+4x}}{2})^{n+2})$$

Proof. It is easy to see that $I(P_0;x) = 1$ and $I(P_1;x) = 1+x$. By Theorem 1.2, The independence polynomial of P_n, $n \geq 3$, satisfies the following equation:

$$I(P_n;x) = I(P_{n-1};x) + xI(P_{n-2};x)$$

In order to solve this recurrence we use a characteristic equation. The characteristic equation corresponding to the above recurrence is:

$$\lambda^2 - \lambda - x = 0$$

This gives $\lambda = \frac{1\pm s}{2}$, where $s = \sqrt{1+4x}$. Then we have:

$$I(P_n;x) = c_1(\frac{1+s}{2})^n + c_2(\frac{1-s}{2})^n$$

By $I(P_0;x)=1$ and $I(P_1;x)=1+x$ we can compute c_1 and c_2 are as follows:

$$\begin{aligned} c_1 &= \frac{1+2x+s}{2s}, \\ c_2 &= \frac{-1-2x+s}{2s} \end{aligned}$$

And so

$$\begin{aligned} I(P_n;x) &= (\frac{1+2x+s}{2s})(\frac{1+s}{2})^n + (\frac{-1-2x+s}{2s})(\frac{1-s}{2})^n \\ &= (\frac{1}{2^{n+1}s})((2+4x+2s)(1+s)^n - (2+4x-2s)(1-s)^n) \\ &= (\frac{1}{2^{n+1}s})((1+s)^2(1+s)^n - (1-s)^2(1-s)^n) \\ &= \frac{1}{s}[(\frac{1+s}{2})^{n+2} - (\frac{1-s}{2})^{n+2}]. \end{aligned}$$

□

We know $G_1 = P_{k+1}$ and $G_2 = C_k$ then we have:

Corollary 2.1. The independence polynomial of G_1, $k \geq 2$, is as follows:

$$I(G_1;x) = \frac{1}{s}[(\frac{1+s}{2})^{k+3} - (\frac{1-s}{2})^{k+3}]$$

Where $s = \sqrt{1+4x}$.

Corollary 2.2. The independence polynomial of G_2, $k \geq 3$, is as follows:

$$I(G_2;x) = \frac{1}{2^k}[(1+s)^k + (1-s)^k$$

Where $s = \sqrt{1+4x}$.

Proof. By Theorem 1.2, the independence polynomial of C_k satisfies the following equation:

$$I(C_k;x) = I(P_{k-1};x) + xI(P_{k-3};x)$$

and by Theorem 2.1, we have:

$$\begin{aligned} I(C_k;x) &= I(P_{k-1};x) + xI(P_{k-3};x) \\ &= \frac{1}{s}[(\frac{1+s}{2})^{k+1} - (\frac{1-s}{2})^{k+1}] + \frac{x}{s}[(\frac{1+s}{2})^{k-1} - (\frac{1-s}{2})^{k-1}] \\ &= (\frac{1}{2^{k+1}s})[((1+s)^{k+1} - (1-s)^{k+1}) + 4x((1+s)^{k-1} - (1-s)^{k-1})] \\ &= (\frac{1}{2^{k+1}s})[(1+s)^{k+1} - (1-s)^{k+1} + (s^2-1)((1+s)^{k-1} - (1-s)^{k-1})] \\ &= (\frac{1}{2^{k+1}s})[2s(1+s)^k + 2s(1-s)^k] = \frac{1}{2^k}[(1+s)^k + (1-s)^k]. \end{aligned}$$

□

Figure 3.2, shows some selected points in these graphs. Apply Theorem 1.1 and Theorem 1.2, for selected vertex u, to compute this polynomial for G_i, $3 \leq i \leq 10$.

Corollary 2.3. The independence polynomials of G_i, $3 \leq i \leq 10$, are as follows:

$$\begin{aligned}
I(G_3^{k,m};x) &= \frac{(Z^{k+2}-M^{k+2})(Z^{m+2}-M^{m+2})+x(Z^{k+1}-M^{k+1})(Z^m-M^m)}{s^2}\\
I(G_4^{k,m};x) &= \frac{(Z^{k+2}-M^{k+2})(Z^{m+2}-M^{m+2})+x(Z^{k}-M^{k})(Z^m-M^m)}{s^2}\\
I(G_5^{k,m,n};x) &= \frac{(Z^{k+2}-M^{k+2})(Z^{m+2}-M^{m+2})(Z^{n+2}-M^{n+2})}{s^3}\\
&+ \frac{x(Z^{k+1}-M^{k+1})(Z^{m+1}-M^{m+1})(Z^{n+1}-M^{n+1})}{s^3}\\
I(G_6^{k,m,n};x) &= \frac{(Z^{k+2}-M^{k+2})(Z^{m+2}-M^{m+2})(Z^{n+2}-M^{n+2})}{s^3}\\
&+ \frac{x(Z^{k+1}-M^{k+1})(Z^{m+1}-M^{m+1})(Z^{n}-M^{n})}{s^3}\\
I(G_7^{k,m,n};x) &= \frac{(Z^{n+2}-M^{n+2})(Z^{k+1}-M^{k+1})(Z^{m+2}-M^{m+2})}{s^3}\\
&+ \frac{x(Z^{n+2}-M^{n+2})(Z^{k}-M^{k})(Z^{m}-M^{m})}{s^3}\\
&+ \frac{x(Z^{n}-M^{n})(Z^{k}-M^{k})(Z^{m+2}-M^{m+2})}{s^3}\\
&+ \frac{x^2(Z^{n}-M^{n})(Z^{k-1}-M^{k-1})(Z^{m}-M^{m})}{s^3}\\
I(G_8^{k,m,n};x) &= \frac{(Z^{n+1}-M^{n+1})(Z^{k+1}-M^{k+1})(Z^{m+1}-M^{m+1})}{s^3}\\
&+ \frac{2x(Z^{n}-M^{n})(Z^{k}-M^{k})(Z^{m}-M^{m})}{s^3}\\
&+ \frac{x^2(Z^{n-1}-M^{n-1})(Z^{k-1}-M^{k-1})(Z^{m-1}-M^{m-1})}{s^3}\\
I(G_9^{k,m,n};x) &= \frac{(Z^{n+2}-M^{n+2})(Z^{k+2}-M^{k+2})(Z^{m+2}-M^{m+2})}{s^3}\\
&+ \frac{x(Z^{n}-M^{n})(Z^{k+1}-M^{k+1})(Z^{m}-M^{m})}{s^3}\\
I(G_{10}^{k,m,n};x) &= \frac{(Z^{n+2}-M^{n+2})(Z^{k+2}-M^{k+2})(Z^{m+2}-M^{m+2})}{s^3}\\
&+ \frac{x(Z^{n}-M^{n})(Z^{k}-M^{k})(Z^{m}-M^{m})}{s^3}
\end{aligned}$$

Where $s=\sqrt{1+4x}$, $Z=\frac{1+s}{2}$ *and* $M=\frac{1-s}{2}$.

Now, we are ready to compute the PI polynomial of graphs G_i, $1 \leq i \leq 10$. For computing the PI polynomial of G_i, it is enough to calculate $N(e)$, for every $e \in$

$E(G_i)$. To calculate $N(e)$, we consider two cases that e is edge of paths or edge of cycles. We start by quoting known results for paths and cycles.

Lemma 2.1. Let $G_2 = C_k$ and $G_1 = P_{k+1}$ be a cycle and a path of length k. Then we have:

$$\begin{aligned} PI(G_1,x) &= k(x^{k-1}-1)+C(k+2,2), \\ PI(G_2,x) &= \begin{cases} kx^{k-2}+C(k,2) & k \text{ is even} \\ kx^{k-1}+C(k,2) & k \text{ is odd} \end{cases}. \end{aligned}$$

The results for two-parameter graphs depend on the parity of the cycle length(s).

Theorem 2.2. The PI polynomial of G_3 and G_4 are computed as follows:

$$\begin{aligned} PI(G_3,x) &= \begin{cases} kx^{k+m-1}+mx^{k+m-2}+C(k+m,2) & m \text{ is even} \\ (k+m-1)x^{k+m-1}+x^{m-1}+C(k+m,2) & m \text{ is odd} \end{cases}, \\ PI(G_4,x) &= \begin{cases} (k+m)x^{k+m-2}+T & k,m \text{ are even} \\ mx^{k+m-2}+(k-1)x^{k+m-1}+x^{k-1}+T & k \text{ is odd}, m \text{ is even} \\ kx^{k+m-2}+(m-1)x^{k+m-1}+x^{m-1}+T & m \text{ is odd}, k \text{ is even} \\ (k+m-2)x^{k+m-1}+x^{m-1}+x^{k-1}+T & k,m \text{ are odd} \end{cases}. \end{aligned}$$

Where $T = \frac{(k+m)(k+m-1)}{2}$.

Proof. Consider G_3 to compute its PI polynomial. It is clear that $|E(G_3)| = |V(G_3)| = k+m$. From Figures 3.1, one can see that there are two types of edges of G_3. If $e \in E(P_{k+1})$ then $N(e) = 1$ and if $e \in E(C_m)$ then $N(e) = 1$ or 2 (m is odd or even). Now, by according to definition PI polynomial we have:

$$\begin{aligned} PI(G_3,x) &= \sum_{f\in E(G_3)} x^{|E(G_3)|-N(f)} + \frac{|V(G_3)|(|V(G_3)|+1)}{2} - |E(G_3)| \\ &= \sum_{f\in E(P_{k+1})} x^{|E(G_3)|-N(f)} + \sum_{f\in E(C_m)} x^{|E(G_3)|-N(f)} + C(k+m,2) \\ &= kx^{k+m-1} + \begin{cases} mx^{k+m-2}+C(k+m,2) & m \text{ is even} \\ (m-1)x^{k+m-1}+x^{m-1}+C(k+m,2) & m \text{ is odd} \end{cases} \\ &= \begin{cases} kx^{k+m-1}+mx^{k+m-2}+C(k+m,2) & m \text{ is even} \\ (k+m-1)x^{k+m-1}+x^{m-1}+C(k+m,2) & m \text{ is odd} \end{cases}. \end{aligned}$$

To compute $PI(G_4,x)$, we consider four separate cases as follow:

$$\begin{aligned} PI(G_4,x) &= \sum_{f\in E(G_4)} x^{|E(G_4)|-N(f)} + \frac{|V(G_4)|(|V(G_4)|+1)}{2} - |E(G_4)| \\ &= \sum_{f\in E(C_k)} x^{|E(G_3)|-N(f)} + \sum_{f\in E(C_m)} x^{|E(G_3)|-N(f)} + \frac{(k+m)(k+m-1)}{2} \end{aligned}$$

$$= \begin{cases} (k+m)x^{k+m-2}+T & k, m \text{ are even} \\ mx^{k+m-2}+(k-1)x^{k+m-1}+x^{k-1}+T & k \text{ is odd}, m \text{ is even} \\ kx^{k+m-2}+(m-1)x^{k+m-1}+x^{m-1}+T & m \text{ is odd}, k \text{ is even} \\ (k+m-2)x^{k+m-1}+x^{m-1}+x^{k-1}+T & k, m \text{ are odd} \end{cases}.$$

This completes the proof. □

Finally, we compute the PI polynomial of the graphs with three strings. Using a similar argument as above, we have:

Theorem 2.3. The PI polynomial of G_i, $5 \leq i \leq 10$, are computed as follows:

$$\begin{aligned} PI(G_5,x) &= tx^{k+m+n-1}+C(t+2,2)-t \\ PI(G_6,x) &= \begin{cases} (m+k)x^{t-1}+nx^{t-2}+C(t,2) & n \text{ is even} \\ (m+k)x^{t-1}+(n-1)x^{t-1}+x^{n-1}+C(t,2) & n \text{ is odd} \end{cases}, \\ PI(G_7,x) &= PI(G_9,x) = \\ &= \begin{cases} (n+m)x^{t-2}+s & n, m \text{ are even} \\ mx^{t-2}+(n-1)x^{t-1}+x^{n-1}+s & n \text{ is odd}, m \text{ is even} \\ nx^{t-2}+(m-1)x^{t-1}+x^{m-1}+s & m \text{ is odd}, n \text{ is even} \\ (m+n-2)x^{t-1}+x^{n-1}+x^{m-1}+s & n, m \text{ are odd} \end{cases}. \end{aligned}$$

If k is even then:

$$PI(G_8,x) = \begin{cases} tx^{t-3}+w & n, m \text{ are even} \\ (m+k)x^{t-2}+nx^{t-1}+w & n \text{ is odd}, m \text{ is even} \\ (n+k)x^{t-2}+mx^{t-1}+w & m \text{ is odd}, n \text{ is even} \\ (m+n)x^{t-2}+kx^{t-1}+w & n, m \text{ are odd} \end{cases},$$

$$PI(G_{10},x) = \begin{cases} tx^{t-2}+w & n, m \text{ are even} \\ (m+k)x^{t-2}+(n-1)x^{t-1}+x^{n-1}+w & n \text{ is odd}, m \text{ is even} \\ (n+k)x^{t-2}+(m-1)x^{t-1}+x^{m-1}+w & m \text{ is odd}, n \text{ is even} \\ kx^{t-2}+(m+n-2)x^{t-1}+x^{n-1}+x^{m-1}+w & n, m \text{ are odd} \end{cases}.$$

If k is odd then:

$$PI(G_8,x) = \begin{cases} tx^{t-3}+w & n, m \text{ are odd} \\ (m+k)x^{t-2}+nx^{t-1}+w & m \text{ is odd}, n \text{ is even} \\ (n+k)x^{t-2}+mx^{t-1}+w & n \text{ is odd}, m \text{ is even} \\ (m+n)x^{t-2}+kx^{t-1}+w & n, m \text{ are even} \end{cases},$$

$$PI(G_{10},x) = \begin{cases} (m+n)x^{t-2}+(k-1)x^{t-1}+x^{k-1}+w & n, m \text{ are even} \\ mx^{t-2}+(n+k-2)x^{t-1}+x^{n-1}+x^{k-1}+w & n \text{ is odd}, m \text{ is even} \\ nx^{t-2}+(m+k-2)x^{t-1}+x^{m-1}+x^{k-1}w & m \text{ is odd}, n \text{ is even} \\ (t-3)x^{t-1}+x^{n-1}+x^{k-1}+x^{m-1}+w & n, m \text{ are odd} \end{cases}.$$

Where $t = k+m+n$, $s = kx^{t-1}+\frac{t(t-3)}{2}$ and $w = \frac{t^2-5t+2}{2}$.

3. Tables and Figures

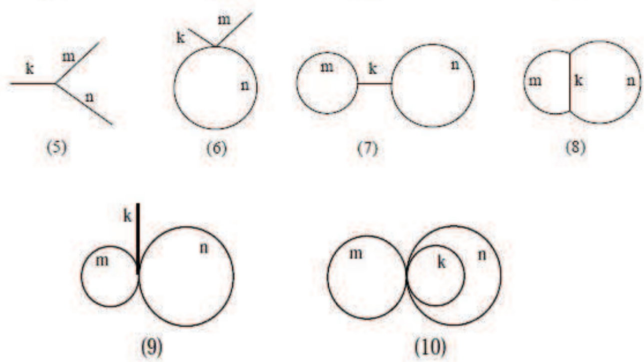

FIG. 3.1: Graphs from [4], including at most three strings

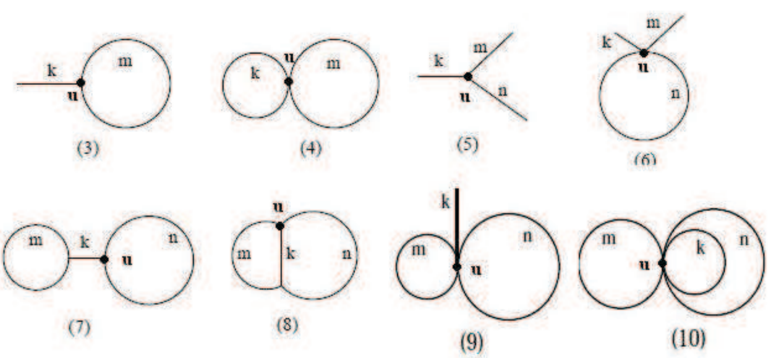

FIG. 3.2: Graphs G_i, $3 \leq i \leq 10$, with vertex u

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ON A DESIGN FROM PRIMITIVE REPRESENTATIONS OF THE FINITE SIMPLE GROUPS

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Abstract. In this paper we present a design construction from primitive permutation representations of a finite simple group G. The group G acts primitively on the points and transitively on the blocks of the design. T he construction has this property that with some conditions we can obtain t-design for $t \geq 2$. We examine our design for fourteen sporadic simple groups. As a result we found a 2-(176,5,4) design with full automorphism group M_{22}.
Keywords. primitive permutation; group; finite simple group; automorphism group.

1. Introduction

Designs are interesting combinatorial objects. They have important applications in coding theory and information theory. Constructing combinatorial designs by using finite permutation groups is a well-studied subject. Well-known methods to construct 1-design from primitive permutation representations of finite simple groups were introduced by Moori and Key [10, 11]. Also in [5] a generalization of the construction in [10] was described. Here we present a design construction from primitive representations of a finite simple group. The groups we consider are primitive on the points and transitive on the blocks of constructed designs. In some conditions we can construct t-designs for $t \geq 2$. We employ this method on some simple groups and calculate full automorphism groups of constructed designs.

Sporadic simple groups are interesting family of the finite simple groups. Designs that are invariant under sporadic groups or have full automorphism group equal to sporadic groups are very interesting. Some of these designs were presented in [2, 8, 10, 12, 13]. Here we consider fourteen sporadic groups for our purpose. For these sporadic groups we obtained some designs for which the full automorphism groups are the same as the sporadic group or double cover of that. These constructed

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designs from sporadic groups are usually 1-design or 2-design and in some cases groups act primitively on the points and the blocks of the designs.

In Section 2, first we present some preliminary definitions and lemmas which will be used in the proof of our main results. In Theorem 2.1 we give our design construction from primitive permutation representations of a finite simple group. Then some properties of these design are considered. One advantage of our design in comparison with the designs presented in [5, 10, 11] is that we can determine some conditions to construct t-design for $t \geq 2$. In Proposition 2.3 these conditions are determined. Applying this result on some finite simple groups we found some 2-designs from 1-transitive actions of these simple groups. In Section 3 we describ constructed designs from fourteen sporadic simple groups. Especially we make use of large sporadic simple groups Co_3 and Fi_{23}. These groups are full automorphism group of the constructed designs which act primitively on the points and the blocks. In Section 4 we present a 2-(176,5,4) design from Mathieu group M_{22}. With the best of knowledge, this design is new and group M_{22} acts primitively on the points and transitively on the blocks of this design. The full automorphism group of this design is isomorphic to M_{22}.

2. Design Construction

In this paper all groups are assumed to be finite. Our notations are standard and for design are from [3] and for group theory and character theory are from [6, 9]. For the name and structure of finite simple groups we use the Atlas notation [4]. All computations were done with GAP [14] and Magma [1]. All programs are accessible from the author upon request.

Let t, λ, v and k be integers such that $1 \leq t \leq k \leq v$ and $\lambda > 0$. Let X be an v-set. A t-(v, k, λ) design is a pair $D = (X, B)$ such that B is a collection of k-subsets of X called blocks such that every t-subset of X appears in exactly λ blocks. A design is called simple if it has no repeated blocks. All designs in this paper are simple. The design D is called symmetric if the number of points and the blocks are equal.

An automorphism of D is a permutation f on X such that $f(b) \in B$ for each $b \in B$. A group whose elements are automorphism of D is called an automorphism group of D. We use $Aut(D)$ to denote the full automorphism group of D.

Let G be a finite permutation group acting on a set X. The orbit of $x \in X$ is defined as $O(x) = \{x^g | g \in G\}$ and the stabilizer subgroup of x is $G_x = \{g \in G | x^g = x\}$. It is well-known that $|G| = |O(x)|.|G_x|$. For $g \in G$, the conjugacy class of g is $cl(g) = \{a^{-1}ga | a \in G\}$. It is well-known that $|G| = |cl(g)|.|C_G(g)|$ such that $C_G(g) = \{a \in G | ag = ga\}$ is centralizer subgroup of g in G.

Let G be a finite group and H be a subgroup of G. Assume that Ω is the set of all conjugates of H in G. Let $\chi_H = \chi(G|H)$ be the permutation character corresponding to the action of G on Ω. For $g \in G$ if $cl(g) \cap H = \emptyset$ then $\chi_H(g) = 0$.

In what follows we present some lemmas that are used in constructing design.

Lemma 2.1. *[6, Corollary 1.5A] Let G be a group acting transitively on a set Ω with at least two points. Then action of G on Ω is primitive if and only if for each $x \in \Omega$, G_x is a maximal subgroup of G.*

Lemma 2.2. *[7, Corollary 3.1.3] Let G be a finite group and H a subgroup of G containing a fixed element x. Then the number h of conjugates of H in G containing x is given by*

$$h = [N_G(H) : H]^{-1} \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, ..., x_m$ are representatives of H-conjugacy classes that fuse to the G-class $cl(x)$.

Lemma 2.3. *[9, Corollary 5.14] Let G be a finite group and H be a subgroup of G. Let Ω be the set of all conjugates of H in G. Then for all $g \in G$, $\chi_H(g)$ is equal to the number of points in Ω fixed by g.*

Let G be a finite simple group and M be a maximal subgroup of G. Let Ω be the set of all conjugates of M in G. In the action of G on Ω, the stabilizer subgroup of each point of Ω is conjugate to M. For $g \in G$ let $cl(g) \cap M \neq \emptyset$. Then by Lemma 2.2 each $g \in G$ is in $r = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_M(g_i)|}$ conjugates of M, where $g_1, ..., g_m$ are representatives of the M-conjugacy classes that fuse to the G-class $cl(g)$. Then by Lemma 2.3, $\chi_M(g) = r$. Let nX be a conjugacy class of the elements of order n such that $nX \cap M \neq \emptyset$. Consider $g \in nX$. Then g is contained in r conjugates of M. We define $B_{g,M} = \{M_1, ..., M_r\}$ such that M_i for $i \in \{1, 2, ..., r\}$, is a conjugate of M for which $g \in M_i$. Also set $S_{g,M} = \cap_{i=1}^{r} M_i$. Clearly for $g, g' \in nX$ if the subgroups $S_{g,M}$ and $S_{g',M}$ are conjugate, then $|S_{g,M} \cap nX| = |S_{g',M} \cap nX|$.

Lemma 2.4. *Let G be a finite simple group and M be a maximal subgroup of G. Let nX be a conjugacy class of the elements of order n such that $nX \cap M \neq \emptyset$. Then for each $g, g' \in nX$ the following hold:*

1. *$B_{g,M} = B_{g',M}$ if and only if $S_{g,M} \cap nX = S_{g',M} \cap nX$,*
2. *if $S_{g,M} \cap S_{g',M} \cap nX \neq \emptyset$ then $S_{g,M} \cap nX = S_{g',M} \cap nX$,*
3. *$\frac{|nX|}{|S_{g,M} \cap nX|}$ is a positive integer,*
4. *$\frac{|M \cap nX|}{|S_{g,M} \cap nX|}$ is a positive integer.*

Proof. (1) Let $B_{g,M} = B_{g',M}$. Then obviously $S_{g,M} \cap nX = S_{g',M} \cap nX$. Now suppose $S_{g,M} \cap nX = S_{g',M} \cap nX$. Then $g, g' \in S_{g,M}$ and $g, g' \in S_{g',M}$. If $B_{g,M} \neq B_{g',M}$ without lose of generally, there exists M', a conjugate of M such that $M' \in B_{g,M}$ but $M' \notin B_{g',M}$. Now since $g' \in S_{g,M}$ then $g' \in M'$ and $M' \in B_{g',M}$ which is a contradiction. (2) Suppose $x \in S_{g,M} \cap S_{g',M} \cap nX$. So $x \in S_{g,M}$

and $x \in S_{g',M}$. Then we have $B_{x,M} = B_{g,M} = B_{g',M}$ and by (1) the result is achieved. (3) By (2) each element of nX is contained in a unique $S_{g,M} \cap nX$. Then $nX = \cup_{i=1}^{k}(S_{g_i,M} \cap nX)$, which is a disjoint union of the elements of class nX for some $S_{g_i,M}$ for $i \in \{1, 2, ..., k\}$. Therefore $k = \frac{|nX|}{|S_{g,M} \cap nX|}$ is positive integer. (4) According to the proof of (3) we have $nX = \cup_{i=1}^{k}(S_{g_i,M} \cap nX)$. For each $i \in \{1, 2, ..., k\}$, $S_{g_i,M} \cap M \cap nX = S_{g_i,M} \cap nX$ or $S_{g_i,M} \cap M \cap nX = \emptyset$. Therefore $nX \cap M = \cup_{j=1}^{h}(S_{g_j,M} \cap nX)$ such that for each $j \in \{1, 2, ..., h\}$ we have $S_{g_j,M} \cap M \cap nX = S_{g_j,M} \cap nX$ hence $h = \frac{|M \cap nX|}{S_{g,M} \cap nX}$ is positive integer. □

Now we are ready to present a design construction from primitive permutation representations of a finite simple group.

Theorem 2.1. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. Set $B = \{B_{x,M} | x \in nX\}$. Then $D = (\Omega, B)$ is a,*

$$1 - ([G : M], \chi_M(g), \frac{|M \cap nX|}{|S_{g,M} \cap nX|})$$

design which has $\frac{|nX|}{|S_{g,M} \cap nX|}$ blocks. Also G is an automorphism group of D which acts primitively on the points and transitively on the blocks of D.

Proof. In the action of G on Ω, the stabilizer of each point of Ω is conjugate to M. Since $B_{g,M}$ is the collection of all conjugates of M that contain g, by Lemma 2.3 the length of a block is $\chi_M(g)$. Clearly $|S_{g,M} \cap nX| \geq 1$. If $|S_{g,M} \cap nX| = 1$ then by Lemma 2.4(1) for every $g', g'' \in nX$ we have $B_{g',M} \neq B_{g'',M}$ and so $|B| = |nX|$. If $|S_{g,M} \cap nX| = m > 1$ then by Lemma 2.4(1) for m elements $g_1, ..., g_m$ of $S_{g,M} \cap nX$ we have $B_{g_1,M} = B_{g_2,M} = ... = B_{g_m,M}$. So according to the proof of Lemma 2.4(3), $nX = \cup_{i=1}^{k}(S_{g_i,M} \cap nX)$ for $i \in \{1, 2, ..., k\}$, which is a disjoint union of the elements of $S_{g_i,M} \cap nX$. Then for each $S_{g_i,M} \cap nX$ there exists a unique block. Hence the number of different blocks is equal to $\frac{|nX|}{|S_{g,M} \cap nX|}$ which by Lemma 2.4(3) is a positive integer. By the proof of Lemma 2.4(4) $M \cap nX = \cup_{j=1}^{h}(S_{g_j,M} \cap nX)$ for $j \in \{1, 2, ..., h\}$, which is a disjoint union of the elements of $S_{g_j,M} \cap nX$. Then for each $j \in \{1, 2, ..., h\}$, subgroup M is in a unique block. Then each point appears in exactly $\frac{|M \cap nX|}{|S_{g,M} \cap nX|}$ blocks which by Lemma 2.4(4) is a positive integer. The group G acts on the points and the blocks with conjugation. Since M is maximal then by Lemma 2.1, G acts primitively on the points. Each block is corresponding to an element of nX and since G is transitive on nX then is transitive on the blocks. □

We denote the constructed design in Theorem 2.1 by $D(G, M, nX)$. The design $D(G, M, nX)$ is not necessary symmetric and the action of G on the blocks is not necessary primitive. In the following propositions we consider some conditions to achieve these properties.

Proposition 2.1. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. If $C_G(g)$ is maximal subgroup then the action of G on the blocks of $D(G, M, nX)$ is primitive.*

Proof. Since $C_G(g)$ is maximal, by Lemma 2.1 the action of G on right cosets of $C_G(g)$ is primitive. Each block is corresponding to an element of nX and respectively to a right coset of $C_G(g)$. Then the action of G on the blocks of $D(G, M, nX)$ is primitive. $\square$

Let $g \in nX$ and consider $D(G, M, nX)$. According to the definition, for every $x \in C_G(g)$ we have $B_{g,M} = B_{g^x,M} = (B_{g,M})^x$. Therefore $C_G(g)$ is a subgroup of stabilizer of the block $B_{g,M}$.

Proposition 2.2. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. Then the following hold,*

1. *if $C_G(g)$ is conjugate to M and $|S_{g,M} \cap nX| = 1$ then $D(G, M, nX)$ is symmetric and G acts on the blocks primitively,*
2. *if $|S_{g,M} \cap nX|.|C_G(g)| = |M|$ then $D(G, M, nX)$ is symmetric.*

Proof. (1) The number of points and blocks in $D(G, M, nX)$ are $\frac{|G|}{|M|}$ and $\frac{|nX|}{|S_{g,M} \cap nX|} = \frac{|G|}{|S_{g,M} \cap nX|.|C_G(g)|}$, respectively. Since $C_G(g)$ is conjugate to M then $|C_G(g)| = |M|$ and the number of points and blocks are equal. Also by Proposition 2.1 the action of G on blocks is primitive. (2) In this case clearly the number of points and blocks are equal and $D(G, M, nX)$ is symmetric. $\square$

In Table 2.1, 2.2 and 2.3 the columns from left are: number of row, the considered finite simple group G, maximal subgroup M of G, a conjugacy class of G, properties of the constructed design from G, number of the blocks of the design, full automorphism group of the design and symmetric property of the design.

Example 2.1. In Table 2.1 $D(G, M, nX)$ for some finite simple groups was constructed. By [4] in $L_2(11)$ centralizer of an element of class $2A$ is maximal subgroup D_{12}. The design $D(L_2(11), D_{12}, 2A)$ in row 3 of the table is a 1-(55,7,7) design with 55 blocks then is symmetric and $L_2(11)$ acts on the blocks primitively. This design is an example such that satisfies Lemma 2.2(1). In $G_2(3)$ for maximal subgroup $M = (3^{1+2} \times 3^2) : 2S_4$, consider $D(G_2(3), M, 3A)$. Let $g \in 3A$, in this case $|S_{g,M} \cap 3A| = 2$ and $|C_{G_2(3)}(g)| = \frac{|M|}{2}$ then $D(G_2(3), M, 3A)$ in row 2 of the table is a 1-(364,13,13) symmetric design. This is an example that satisfies Lemma 2.2(2).

Always t-designs with $t \geq 2$ are interesting. In two following propositions we consider some conditions to construct t-designs for $t \geq 2$.

Table 2.1: $D(G,M,nX)$ for some finite simple groups

No.	G	M	nX	$D(G,M,nX)$	No. Blocks	$Aut(D)$	Symm
1	A_9	$L_2(8):3$	$2B$	1-(120,8,9)	135	A_9	NO
2	$G_2(3)$	$(3^{1+2}\times 3^2):2S_4$	$3A$	1-(364,13,13)	364	$G_2(3)$	YES
3	$L_2(11)$	D_{12}	$2A$	1-(55,7,7)	55	$L_2(11):2$	YES
4	$S_6(2)$	$2.[2^6]:(S_3\times S_3)$	$2B$	1-(315,43,43)	315	$S_6(2)$	YES
5	$U_5(2)$	$(2^{1+6}\times 3^{1+2}):2A_4$	$2A$	1-(165,37,37)	165	$U_5(2)$	YES
6	$U_3(3)$	$4^2:S_3$	$3B$	1-(63,3,16)	336	$S_6(2)$	NO
7	$O_8^+(2)$	$(3\times U_4(2)):2$	$6A$	1-(1120,8,360)	50400	$O_8^+(2)$	NO
8	$O_8^+(2)$	$(3\times U_4(2)):2$	$3A$	1-(1120,40,40)	1120	$O_8^+(3):4$	YES

Proposition 2.3. *Let G be a finite simple group, M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M\cap nX\neq\emptyset$ and $g\in nX$. Let $m\in\{1,2,...,\chi_M(g)\}$. If intersection of every m different conjugates of M has $f\geq 1$ elements of the class nX then $D(G,M,nX)$ is an $m-([G:M],\chi_M(g),\frac{f}{|S_{g,M}\cap nX|})$ design. Also G is an automorphism group of $D(G,M,nX)$ such that acts primitively on the points and transitively on the blocks of $D(G,M,nX)$.*

Proof. Consider an m-set of different conjugates of M, set S intersection of these subgroups. By the proof of Lemma 2.4(4) we have this partition $S\cap nX=\cup_{j=1}^{h}(S_{g_j,M}\cap nX)$. Therefore for each $j\in\{1,2,...,h\}$ these m conjugates of M are in a unique block. Then every m conjugates of M appears in exactly $\frac{|S\cap nX|}{|S_{g,M}\cap nX|}=\frac{f}{|S_{g,M}\cap nX|}$ blocks and result is concluded. $\square$

Example 2.2. In Table 2.2 we consider some finite simple groups in their 1-transitively action such that satisfy conditions of Proposition 2.3.

Table 2.2: Constructed $D(G,M,nX)$ from Proposition 2.3

No.	G	M	nX	$D(G,M,nX)$	No. Blocks	$Aut(D)$	Symm
1	$S_4(3)$	$3^{1+2}:2A_4$	3A	2-(40,13,4)	40	$L_4(3):2$	YES
2	$S_4(4)$	$2^6:(3\times A_5)$	2A	2-(85,21,5)	85	$L_4(4):2$	YES
3	$S_4(5)$	$5^{1+2}:4A_5$	5A	2-(156,31,6)	156	$L_4(5):4$	YES
4	$L_2(8)$	D_{18}	2A	2-(28,4,1)	63	$L_2(8):3$	NO
5	$L_2(16)$	D_{34}	2A	2-(120,8,1)	255	$L_2(16):4$	NO

Corollary 2.1. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M\cap nX\neq\emptyset$ and $g\in nX$. For $t>1$ let the action of G on Ω be t-transitive. For $m\in\{1,2,...,t\}$ consider an m-set of different conjugates of M and set S intersection of these subgroups. If $S\neq\langle 1\rangle$ and $|S\cap nX|=k\geq 1$ then $D(G,M,nX)$ is an $m-([G:M],\chi_M(g),\frac{k}{|S_{g,M}\cap nX|})$ design*

and G is an automorphism group of $D(G, M, nX)$ that acts t-transitively on the points and transitively on the blocks.

Proof. Since G is t-transitive on Ω then intersection of any m-set of different conjugates of M is conjugate to S and result is concluded by Proposition 2.3. $\square$

Example 2.3. In Table 2.3 we consider some finite simple groups in their 2-transitively action such that satisfy conditions of Corollary 2.1.

Table 2.3: Constructed $D(G, M, nX)$ from Corollary 2.1

No.	G	M	nX	$D(G, M, nX)$	No. Blocks	$Aut(D)$	Symm
1	$L_2(11)$	A_5	2A	2-(11,3,3)	55	$L_2(11)$	NO
2	$L_3(3)$	$3^2 : 2S_4$	2A	2-(13,5,15)	117	$L_3(3)$	NO
3	$L_3(3)$	$3^2 : 2S_4$	3A	2-(13,4,1)	13	$L_3(3)$	YES
4	$L_3(5)$	$5^2 : GL_2(5)$	2A	2-(31,7,35)	775	$L_3(5)$	NO
5	$L_3(5)$	$5^2 : GL_2(5)$	5A	2-(31,6,1)	31	$L_3(5)$	YES
6	$S_6(2)$	$U_4(2) : 2$	2A	2-(28,16,20)	63	$S_6(2)$	NO

3. Constructed Designs from Sporadic Groups

In this section we construct some designs from fourteen sporadic simple groups. For each considered sporadic group, we present one or two designs that their full automorphism groups are as the same sporadic group. These results are presented in Table 3.1. For information on the sporadic simple groups and their maximal subgroups we use Atlas [4].

In Table 3.1 the columns from left are: number of row, group G, maximal subgroup M of G, a conjugacy class of G, properties of the constructed design from G, number of the blocks of the design, full automorphism group of the design and symmetric property of the design.

For instance we study properties of the designs in row 15 and 18 of Table 3.1.

The Conway group Co_3 has order $495766656000 = 2^{10}.3^7.5^3.7.11.23$. The group Co_3 has forty two conjugacy classes of elements and fourteen conjugacy classes of maximal subgroups. The centralizer subgroup of an elements of class $2A$ in Co_3 is maximal subgroup isomorphic to $2.S_6(2)$. The group $McL : 2$ is maximal subgroup of index 276 and Co_3 acts 2-transitive on conjugates of $McL : 2$. We consider design $D = D(Co_3, McL : 2, 2A)$. Intersection of every two different maximal subgroups conjugate to $McL : 2$ have 2835 elements of class $2A$ therefore by Corollary 2.1 D is a 2-(276,36,2835) design. The design D has 170775 blocks and since centralizer of an element of the class $2A$ is maximal hence by Proposition 2.1 group Co_3 acts primitively on the blocks of this design. Also Co_3 acts 2-transitively on the points of D. The full automorphism group of D is Co_3.

Table 3.1: Designs constructed from some sporadic groups

No.	G	M	nX	$D(M_{11}, M, nX)$	No. Blocks	$Aut(D)$	Symm
1	M_{11}	$L_2(11)$	2A	3-(12,4,3)	165	M_{11}	NO
2	M_{11}	$2.S_4$	2A	1-(165,13,13)	165	M_{11}	YES
3	M_{12}	$M_9 : S_3$	3A	1-(220,4,4)	220	M_{12}	YES
4	M_{22}	A_7	4B	1-(176,4,315)	13860	M_{22}	NO
5	M_{23}	M_{22}	3A	4-(23,5,16)	28336	M_{23}	NO
6	M_{23}	$L_3(4) : 2$	2A	1-(253,29,435)	3795	M_{23}	NO
7	J_1	$2 \times A_5$	2A	1-(1463,31,31)	1463	J_1	YES
8	J_1	$D_6 \times D_{10}$	2A	1-(2926,46,23)	1463	J_1	NO
9	J_3	$L_2(19)$	2A	1-(14688,96,171)	26163	J_3	NO
10	HS	$U_3(5) : 2$	2B	2-(176,12,66)	15400	HS	NO
11	McL	M_{22}	2A	1-(2025,105,1155)	22275	McL	NO
12	McL	$2^4 : A_7$	2A	1-(22275,435,435)	22275	McL	YES
13	He	$2^6 : 3.S_6$	$2A$	1-(29155,651,558)	24990	He	NO
14	Suz	$U_5(2)$	3A	1-(32760,252,176)	22880	$Suz : 2$	NO
15	Co_3	$McL : 2$	$2A$	2-(276,36,2835)	170775	Co_3	NO
16	Fi_{22}	$O_7(3)$	$2A$	1-(14080,1408,351)	3510	Fi_{22}	NO
17	Co_2	$U_6(2) : 2$	$2A$	1-(2300,284,7029)	56925	Co_2	NO
18	Fi_{23}	$2.Fi_{22}$	$2A$	1-(31671,3511,3511)	31671	Fi_{23}	YES

The Fischer sporadic simple group Fi_{23} has order $4089470473293004800 = 2^{18}.3^{13}.5^2.7.11.13.17.23$. The group Fi_{23} has ninety eight conjugacy classes of elements and fourteen conjugacy classes of maximal subgroups. The group $2.Fi_{22}$ is maximal subgroup of index 31671 and also is centralizer subgroup of an element of the class $2A$. By Proposition 2.2(1) $D(Fi_{23}, 2.Fi_{22}, 2A)$ is a symmetric $1-(31671, 3511, 3511)$ design. The group Fi_{23} acts primitively on the points and the blocks of $D(Fi_{23}, 2.Fi_{22}, 2A)$. The full automorphism group of $D(Fi_{23}, 2.Fi_{22}, 2A)$ is isomorphic to Fi_{23}.

4. A 2-design invariant under M_{22}

The Mathieu sporadic group M_{22} has order $443520 = 2^7.3^2.5.7.11$. The group M_{22} has twelve conjugacy classes of elements and eight conjugacy classes of maximal subgroups [4].

Consider permutation representation of M_{22} on 176 conjugates of maximal subgroup A_7. The group M_{22} has one conjugacy class of elements of order 3. The centralizer of an element of the class $3A$ is isomorphic to $3 \times A_4$. Each elements of class $3A$ is contained in 5 conjugates of maximal subgroup A_7.

Proposition 4.1. *The $D(M_{22}, A_7, 3A)$ is a 2-(176,5,4) design with full automorphism group isomorphic to M_{22}.*

Proof. Let M_1 and M_2 be two different maximal subgroups isomorphic to A_7. Then $M_1 \cap M_2$ is isomorphic to 3^2:4 or S_4. Both subgroups 3^2:4 and S_4 have 8 elements of order 3. Also for each $g \in 3A$, S_{g,A_7} is subgroup of order 3. Hence by Proposition

2.3 $D(M_{22}, A_7, 3A)$ is a 2-(176,5,4) design. The automorphism group is calculated by GAP [14]. $\square$

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FAULT-TOLERANT METRIC DIMENSION OF CIRCULANT GRAPHS

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Abstract. A set W of vertices in a graph G is called a resolving set for G if for every pair of distinct vertices u and v of G, there exists a vertex $w \in W$ such that the distance between u and w is different from the distance between v and w. The cardinality of a minimum resolving set is called the metric dimension of G, denoted by $\beta(G)$. A resolving set W for G is fault-tolerant if $W \setminus \{w\}$ is also a resolving set, for each w in W. The fault-tolerant metric dimension of G is the size of a smallest fault-tolerant resolving set for G, denoted by $\beta'(G)$. In this paper, we study the fault-tolerant metric dimension of a family of circulant graphs $X_{n,3}$ with connection set $C = \{1, \frac{n}{2}, n-1\}$, when n is even and circulant graphs $X_{n,4}$ with connection set $C = \{\pm 1, \pm 2\}$.
Keywords. Circulant graphs; resolving set; fault-tolerant metric dimension.

1. Introduction

The metric dimension problem was introduced independently by Slater [15] and Harary and Melter [8]. The metric dimension arises in many diverse areas, including telecommunications [3], connected joints in graphs and chemistry [4], the robot navigation [12] and geographical routing protocols [13], etc.

For a connected graph G with vertex set $V(G)$ and edge set $E(G)$, the *distance* between two vertices u and v in $V(G)$ is the number of edges in a shortest path connecting them, and is denoted by $d(u, v)$. Consider an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$. For each $v \in V(G)$ the *code* of v with respect to W is $(d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, denoted by $c_W(v)$. The set W is called a *resolving set* for G, if all vertices of G have distinct codes. The minimum cardinality of a resolving set of G is called the *metric dimension* of G and is denoted by $\beta(G)$. A resolving set of order $\beta(G)$ is called a *metric basis* of G [2].
Elements of bases were referred to as sensors in an application given in [5]. If one

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of the sensors does not work properly, we will not have enough information to deal with the intruder (fire, thief etc). In order to overcome this kind of problems, concept of fault-tolerant metric dimension was presented in [9]. Fault-tolerant resolving sets provide correct information even when one of the sensors is not working. A resolving set W of a graph G is *fault-tolerant* if $W \setminus \{w\}$ is also a resolving set, for each w in W. The *fault-tolerant metric dimension* of G is the minimum cardinality of a fault-tolerant resolving set, denoted by $\beta'(G)$. A fault-tolerant resolving set of order $\beta'(G)$ is called a *fault-tolerant metric basis*.
The *circulant* graph is a graph with vertex set $\mathbb{Z}_n$, an additive group of integers modulo n, and two vertices labeled i and j are adjacent if and only if $i-j\,(mod\ n) \in C$, where $C \subset \mathbb{Z}_n$, which is called *connection set*, has the property that $C = -C$ and $0 \notin C$. The circulant graph is denoted by $X_{n,\Delta}$ where $\Delta = |C|$.
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijective mapping f from V_1 to V_2 such that $uv \in E_1$ if and only if $f(u)f(v) \in E_2$. An *automorphism* of a graph is an isomorphism from the graph to itself. The set of all automorphisms of a graph, G, forms a group, denoted by $Aut(G)$. It is well known that, if G is a circulant graph, then $\mathbb{Z}_n$ is a subgroup of $Aut(G)$ [7].
In this paper, we consider a family of circulant graphs $X_{n,3}$ with connection set $C = \{1, \frac{n}{2}, n-1\}$, when n is even and prove that the fault-tolerant metric dimension of this family of graphs is independent of choice of n by showing that $\beta'(X_{n,3}) = 4$, for all $n \geq 4$ and $n \equiv 0\,(mod\ 4)$, in Theorem 2.2, and $\beta'(X_{n,3}) \leq 6$, for all $n \geq 10$ and $n \equiv 2\,(mod\ 4)$, in Theorem 2.4. We also consider a family of circulant graphs $X_{n,4}$ with connection set $C = \{\pm 1, \pm 2\}$ and prove that the fault-tolerant metric dimension of this family of graphs is independent of choice of n by showing that $\beta'(X_{n,4}) = 4$, for all $n \geq 10$ and $n \equiv 2(mod 4)$, in Theorem 3.1.

2. Fault-Tolerant Metric Dimension Of Circulant Graphs $X_{n,3}$

Salman et al. [14] characterized the metric dimension for family of circulant graphs $X_{n,3}$ with connection set $C = \{1, \frac{n}{2}, n-1\}$ for even n. Now we obtain the fault-tolerant metric dimension of this family of graphs.

Theorem 2.1. *[14, Theorem 2.2] Let n be an integer and $n \equiv 0\,(mod\ 4)$. If $k = \frac{n}{4}$, then for any $1 \leq i \leq k$ the set $W = \{v_i, v_{i+1}, v_{i+2k}\}$ is a resolving set and hence $\beta(X_{n,3}) = 3$.*

The following lemma, gave a new family of resolving set of $X_{n,3}$ of size 3, where $n \equiv 0\,(mod\ 4)$.

Lemma 2.1. *Let n be an integer, $n \equiv 0\,(mod\ 4)$ and $k = \frac{n}{4}$. Then the set $W = \{v_i, v_{i+1}, v_{i+2k+1}\}$ is a resolving set of $X_{n,3}$, for any $1 \leq i \leq k$.*

Proof. Let $W = \{v_i, v_{i+1}, v_{i+2k+1}\}$ for fixed i; $0 \leq i \leq k$ where $k = \frac{n}{4}$. We compute the codes of all $v \in V(X_{n,3}) \setminus W$. We have

$$c_W(v_{i+k}) = (k, k-1, k), c_W(v_{i+k+1}) = (k, k, k), c_W(v_{i+3k}) = (k, k, k-1),$$

$$c_W(v_{i+3k+1}) = (k-1, k, k), c_W(v_{i+2k}) = (1, 2, 1).$$

The codes of other vertices are listed in Table 1. By a simple computing these codes are distinct and hence W is a resolving set of $X_{n,3}$.

Table 1

Shortest paths between	v_i	v_{i+1}	v_{i+2k+1}
$v_{i+j+1}(1 \leq j \leq k-2)$	$j+1$	j	$j+1$
$v_{i+k+j}(2 \leq j \leq k-1)$	$k-j+1$	$k-j+2$	$k-j+1$
$v_{i+2k+j}(2 \leq j \leq k-1)$	j	$j-1$	$j-1$
$v_{i+3k+j}(2 \leq j \leq k-1)$	$k-j$	$k-j+1$	$k-j+2$

□

Theorem 2.2. *For all $n \geq 4$ and $n \equiv 0\,(mod\ 4)$, $\beta'(X_{n,3}) = 4$.*

Proof. From the definition of fault-tolerant metric dimension it can be seen that $\beta'(G) \geq \beta(G)+1$ [11]. This implies that $\beta'(X_{n,3}) \geq 4$ since $\beta(X_{n,3}) = 3$ by Theorem 2.1.

Now for the lower bound, Let $W' = \{v_i, v_{i+1}, v_{i+2k}, v_{i+2k+1}\}$ for fixed i; $0 \leq i \leq k$ where $k = \frac{n}{4}$. We will show that for each $x \in W'$, the set $W' \setminus \{x\}$ is a resolving set for $X_{n,3}$. At first note that $\mathbb{Z}_n$ is subgroup of $Aut(X_{n,3})$ and if $f = (v_o, v_1, \cdots, v_{n-1})$ is a cycle of order n, then $\mathbb{Z}_n \cong < f >$. In addition $f^j(v_i) = v_{i+j}$. Now we consider the following cases:

Case 1. Suppose that $x = v_i$. We have

$$f^j(\{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}) = \{v_{j+i+1}, v_{j+i+2k}, v_{j+i+2k+1}\}$$

. If $j = 2k$, then

$$f^j(\{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}) = \{v_{i+2k+1}, v_i, v_{i+1}\},$$

and

$$f^j(\{v_{i+2k+1}, v_i, v_{i+1}\}) = \{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}.$$

By Lemma 2.1, $\{v_i, v_{i+1}, v_{i+2k+1}\}$ is a resolving set for $X_{n,3}$ and since automorphisms of graphs preserves the properties of the graph, we conclude that $W' \setminus \{x\} = \{v_{i+1}, v_{i+2k}, v_{i+2k+1}\}$ is a resolving set for $X_{n,3}$.

Case 2. Let $x = v_{i+1}$. We have

$$f^j(\{v_i, v_{i+1}, v_{i+2k}\}) = \{v_{j+i}, v_{j+i+1}, v_{j+i+2k}\}.$$

If $j = 2k$, then

$$f^j(\{v_i, v_{i+1}, v_{i+2k}\}) = \{v_{i+2k}, v_{i+2k+1}, v_i\}$$

. Hence By the same argument of Case 1, the set $W' \setminus \{x\} = \{v_{i+2k}, v_{i+2k+1}, v_i\}$ is a resolving set for $X_{n,3}$.

Case 3. If $x = v_{i+2k}$, then according to the Lemma 2.1, $W' \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Case 4. If $x = v_{i+2k+1}$, then according to the Theorem 2.1, $W' \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Therefore, W' is the fault-tolerant resolving set for this family of graphs. Thus $\beta'(X_{n,3}) \leq 4$, for all $n \geq 10$ and $n \equiv 2\,(mod\ 4)$. This completes the proof. $\square$

Now we study the fault-tolerant metric dimension of $X_{n,3}$ in the case $n \equiv 2\,(mod\ 4)$.

Theorem 2.3. *[14, Theorem 2.5] Let $n \geq 6$ be an integer and $n \equiv 2\,(mod\ 4)$. If $k = \frac{n-2}{4}$, then $W = \{v_i, v_{i+1}, v_{i+2k}, v_{i+2k+1}\}$ is a resolving set for $X_{n,3}$ for any $1 \leq i \leq k$. In addition $\beta(X_{n,3}) = 4$.*

In the following lemma we gave some resolving sets of size 3 for $X_{n,3}$.

Lemma 2.2. *Let $n \geq 10$ be an integer and $n \equiv 2\,(mod\ 4)$. For $k = \frac{n-2}{4}$ and any $1 \leq i \leq k$ the following sets are resolving sets of size 4 of $X_{n,3}$,*
i) $W_1 = \{v_i, v_{i+1}, v_{i+2k+1}, v_{i+2k+2}\}$;
ii) $W_2 = \{v_{i+1}, v_{i+2k}, v_{i+2k+2}, v_{i+4k+1}\}$;
iii)$W_3 = \{v_i, v_{i+2k+1}, v_{i+2k+2}, v_{i+4k+1}\}$.

Proof. Suppose that $k = \frac{n-2}{4}$ and $W = \{v_i, v_{i+1}, v_{i+2k+1}, v_{i+2k+2}\}$ where $0 \leq i \leq k$. We compute $c_{W_1}(v)$ for $v \in V(X_{n,3}) \setminus W_1$. We have

$$c_{W_1}(v_{i+k}) = (k, k-1, k+1, k), c_{W_1}(v_{i+k+1}) = (k+1, k, k, k+1),$$

$$c_{W_1}(v_{i+2k}) = (2, 3, 1, 2), c_{W_1}(v_{i+3k+2}) = (k, k+1, k+1, k).$$

The codes of other vertices respect to W_1, are shown in Table 2. It is not difficult to see that all codes are distinct and hence W_1 is a resolving set of $X_{n,3}$.

Table 2

Shortest paths between	v_i	v_{i+1}	v_{i+2k+1}	v_{i+2k+2}
$v_{i+j+1}(1 \leq j \leq k-2)$	$j+1$	j	$j+2$	$j+1$
$v_{i+k+j}(2 \leq j \leq k-1$	$k-j+2$	$k-j+3$	$k-j+1$	$k-j+2$
$v_{i+2k+j+1}(2 \leq j \leq k)$	$j+1$	j	j	$j-1$
$v_{i+3k+j+1}(2 \leq j \leq k)$	$k-j+1$	$k-j+2$	$k-j+2$	$k-j+3$

For W_2 we have,

$$c_{W_2}(v_{i+k}) = (k-1, k, k, k+1), c_{W_2}(v_{i+k+1}) = (k, k-1, k+1, k),$$

Table 3

Shortest paths between	v_{i+1}	v_{i+2k}	v_{i+2k+2}	v_{i+4k+1}
$v_{i+j+1}(1 \leq j \leq k-2)$	j	$j+3$	$j+1$	$j+2$
$v_{i+k+j}(2 \leq j \leq k-1$	$k-j+3$	$k-j$	$k-j+2$	$k-j+1$
$v_{i+2k+j+1}(2 \leq j \leq k)$	j	$j+1$	$j-1$	j
$v_{i+3k+j+1}(2 \leq j \leq k-1)$	$k-j+2$	$k-j+1$	$k-j+3$	$k-j$

$$c_{W_2}(v_{i+2k}) = (2,1,1,2), c_{W_2}(v_{i+3k+2}) = (k+1,k,k,k-1),$$

and the codes of other vertices are listed Table 3. These codes are distinct and we conclude that W_2 is a resolving set for $X_{n,3}$.

Similarly for W_3, we have

$$c_{W_3}(v_1) = (1,2,1,2), c_{W_3}(v_{i+k}) = (k,k+1,k,k+1), c_{W_3}(v_{i+k+1}) = (k+1,k,k+1,k),$$

$$c_{W_3}(v_{i+2k}) = (2,1,2,1), c_{W_3}(v_{i+3k+2}) = (k,k+1,k,k-1),$$

and for other vertices, the codes are listed in Table 4. By these codes, we conclude that W_3 is a resolving set for $X_{n,3}$.

Table 4

Shortest paths between	v_i	v_{i+2k+1}	v_{i+2k+2}	v_{i+4k+1}
$v_{i+j+1}(1 \leq j \leq k-2)$	$j+1$	$j+2$	$j+1$	$j+2$
$v_{i+k+j}(2 \leq j \leq k-1)$	$k-j+2$	$k-j+1$	$k-j+2$	$k-j+1$
$v_{i+2k+j+1}(2 \leq j \leq k)$	$j+1$	j	$j-1$	j
$v_{i+3k+j+1}(2 \leq j \leq k-1)$	$k-j+1$	$k-j+2$	$k-j+3$	$k-j$

□

Theorem 2.4. *For all $n \geq 6$ and $n \equiv 2\,(mod\ 4)$, $\beta'(X_{n,3}) \leq 6$.*

Proof. For $n = 6$, $X_{6,3} \simeq K_{3,3}$. This implies that $\beta'(X_{6,3}) = 6$ since $\beta'(K_{m,n}) = m+n$ [6, Proposition 1].
Now suppose that $n \geq 10$. Let $k = \frac{n-2}{4}$. For a fixed i, where $0 \leq i \leq k$, consider the set

$$W = \{v_i, v_{i+1}, v_{i+2k}, v_{i+2k+1}, v_{i+2k+2}, v_{i+4k+1}\}.$$

Since W contains the set W_1 of Theorem 2.3 (i), so W is a resolving set for $X_{n,3}$. Now we will show that for each $x \in W'$, the set $W' \setminus \{x\}$ is a resolving set for $X_{n,3}$. We have the following cases:

Case 1. If $x \in \{v_i, v_{i+2k+1}\}$, then $W \setminus \{x\}$ contains a set W_2 listed in Lemma 2.2 (ii). Thus $W \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Case 2. If $x = v_{i+1}$, then $W \setminus \{x\}$ contains a set W_3 listed in Lemma 2.2 (iii). So $W \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Case 3. If $x = v_{i+2k}$, then $W \setminus \{x\}$ contains a set W listed in Lemma 2.2 (i). Hence $W \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Case 4. If $x \in \{v_{i+2k+2}, v_{i+4k+1}\}$, then $W \setminus \{x\}$ contains a set W' listed in Theorem 2.3. Hence $W \setminus \{x\}$ is a resolving set for $X_{n,3}$.

Therefore, W is the fault-tolerant resolving set for this family of graphs. Thus $\beta'(X_{n,3}) \leq 6$, for all $n \geq 10$ and $n \equiv 2\,(mod\ 4)$. □

3. Fault-Tolerant Metric Dimension Of Circulant Graphs $X_{n,4}$

In this section consider $X_{n,4}$ with connection set $C = \{\pm 1, \pm 2\}$. In [1], Borchert and Gosselin showed that $dim(X_{n,4}) = 4$ if $n = 1\,(mod\ 4)$ and $dim(X_{n,4}) = 3$ otherwise. Now we study the fault-tolerant metric dimension of this family of graphs in the case $n \equiv 2\,(mod\ 4)$.

In the following lemma we obtain some resolving sets for $X_{n,4}$.

Lemma 3.1. *Let $n \geq 10$ and $n \equiv 2\,(mod\ 4)$. For $k = \frac{n-2}{4}$ and any $1 \leq i \leq k$, the following sets are resolving sets for $X_{n,4}$,*
i)$W_1 = \{v_i, v_{i+1}, v_{i+2}\}$;
ii)$W_2 = \{v_i, v_{i+1}, v_{i+3}\}$;
iii)$W_3 = \{v_i, v_{i+2}, v_{i+3}\}$.

Proof. The set W_1 is a resolving set by [10, Theorem 5]. For the parts (ii) and (iii), we prove that the sets W_2 and W_3 are resolving sets of $X_{n,4}$ for $i = 0$. The remaining cases, obtained by this fact that $\mathbb{Z}_n$ is a subgraph of $Aut(X_{n,4})$. By a simple computing we can obtain the codes of vertices respect to W_2 and W_3. These codes listed in Table 5 and Table 6. Clearly these codes are distinct and hence the sets W_2 and W_3 are resolving sets.

Table 5

Shortest paths between	v_0	v_1	v_3
v_2	1	1	1
v_j $(4 \leq j \leq \frac{n}{2})$	$\lceil \frac{j}{2} \rceil$	$\lceil \frac{j-1}{2} \rceil$	$\lceil \frac{j-3}{2} \rceil$
$v_{\frac{n}{2}+1}$	$\lceil \frac{n-2}{4} \rceil$	$\lceil \frac{n}{4} \rceil$	$\lceil \frac{n-4}{4} \rceil$
$v_{\frac{n}{2}+2}$	$\lfloor \frac{n-2}{4} \rfloor$	$\lceil \frac{n-2}{4} \rceil$	$\lceil \frac{n-2}{4} \rceil$
$v_{\frac{n}{2}+3}$	$\lceil \frac{n-6}{4} \rceil$	$\lceil \frac{n-4}{4} \rceil$	$\lceil \frac{n}{4} \rceil$
v_j $(\frac{n}{2}+4 \leq j \leq n-1)$	$\lceil \frac{n-j}{2} \rceil$	$\lfloor \frac{n-j+2}{2} \rfloor$	$\lfloor \frac{n-j+4}{2} \rfloor$

□

Theorem 3.1. *For $n \geq 10$ and $n \equiv 2\,(mod\ 4)$, $\beta'(X_{n,4}) = 4$.*

Table 6

Shortest paths between	v_0	v_2	v_3
v_1	1	1	1
v_j $(4 \leq j \leq \frac{n}{2})$	$\lceil \frac{j}{2} \rceil$	$\lceil \frac{j-2}{2} \rceil$	$\lceil \frac{j-3}{2} \rceil$
$v_{\frac{n}{2}+1}$	$\lceil \frac{n-2}{4} \rceil$	$\lfloor \frac{n}{4} \rfloor$	$\lfloor \frac{n-4}{4} \rfloor$
$v_{\frac{n}{2}+2}$	$\lfloor \frac{n-2}{4} \rfloor$	$\lceil \frac{n}{4} \rceil$	$\lceil \frac{n-2}{4} \rceil$
$v_{\frac{n}{2}+3}$	$\lceil \frac{n-6}{4} \rceil$	$\lceil \frac{n-4}{4} \rceil$	$\lceil \frac{n}{4} \rceil$
v_j $(\frac{n}{2}+4 \leq j \leq n-1)$	$\lceil \frac{n-j}{2} \rceil$	$\lfloor \frac{n-j+2}{2} \rfloor$	$\lfloor \frac{n-j+4}{2} \rfloor$

Proof. From the definition of fault-tolerant metric dimension it can be seen that $\beta'(G) \geq \beta(G) + 1$ [11]. This implies that $\beta'(X_{n,4}) \geq 4$ since $\beta(X_{n,4}) = 3$ [1].
Now for the lower bound, consider the set $W' = \{v_1, v_2, v_3, v_4\}$. Since W contains the set W_1 listed in Theorem 3.1, so W is a resolving set for $X_{n,4}$. Now we will show that for each $x \in W$, the set $W \setminus \{x\}$ is a resolving set for $X_{n,4}$. If $x \in \{v_1, v_4\}$, then $W \setminus \{x\}$ is a resolving set by setting $i = 1$ and $i = 2$ in part (i) of Lemma 3.1. If $x = v_2$, then by setting $i = 1$ in part (iii) of Lemma 3.1, we conclude that $W \setminus \{x\}$ is a resolving set. Finally $\{v_1, v_2, v_4\}$ is a resolving set of $X_{n,4}$ by Lemma 3.1 (ii). Therefore $\beta'(X_{n,4}) = 4$

□

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DOMINATION, TOTAL DOMINATION AND OPEN PACKING OF THE CORCOR DOMAIN OF GRAPHENE

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Abstract. A dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one vertex in D. A dominating set D is a total dominating set, if every vertex in V is adjacent to at least one vertex in D. The set P is said to be an open packing set if no two vertices of P have a common neighbor in G. In this paper, we obtain domination number, total domination number and open packing number of the molecular graph of a new type of graphene named CorCor that is a 2-dimensional carbon network.

Keywords. Graph; vertex; dominating set; domination number; total domination number.

1. Introduction

Throughout this paper, all graphs are assumed to be simple connected, undirected with $n \geq 1$ vertices and m edges. Let $G = (V, E)$ be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. By the neighborhood of a vertex v of G, we mean the set $N_G(v) = N(v) = \{u \in V : uv \in E\}$. The closed neighborhood of vertex v is $N_G[v] = N(v) \cup v$. For $S \subseteq V$, the neighborhood of S is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$.

A set P of vertices of G is an open packing of G, if the open neighborhoods of the vertices of P are pairwise disjoint in G. The open packing number of G, denoted by $\rho^0(G)$, is the maximum cardinality among all open packings of G.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - D$ has a neighbor in D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A subset $D \subseteq V(G)$ is a total dominating

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set, abbreviated TDS, of G if every vertex of G has a neighbor in D. The total domination number of G, denoted by $\gamma_t(G)$ and introduced by Cockayne, Dawes, and Hedetniemi [2],[4].

Open packing is the natural dual object of total dominating sets, and so $\rho^0(G) \leq \gamma_t(G)$ holds for every graph G, [5]. Moreover, if there exists a total dominating set D of G and D is also an open packing, then $\rho^0(G) = \gamma_t(G) = |D|$.

Molecules arranging themselves into predictable patterns on silicon chips could lead to microprocessors with much smaller circuit elements. Mathematically, assembling in predictable patterns is equivalent to packing in graphs. An H-packing of a graph G is a set of vertex disjoint subgraphs of G, each of which is isomorphic to a fixed graph H. From the optimization point of view, maximum H-packing problem is to find the maximum number of vertex disjoint copies of H in G called the packing number denoted by $\lambda(G, H)$. For our convenience $\lambda(G, H)$ is sometimes represented as λ. An H-packing in G is called perfect if it covers all vertices of G. If H is the complete graph K_2, the maximum H-packing problem becomes the familiar maximum matching problem. Structures realized by arrangements of regular hexagons in the plane are of interest in the chemistry of benzenoid hydrocarbons, where perfect matchings correspond to kekule structures and feature in the calculation of molecular energies associated with benzenoid hydrocarbon molecules. H-Packing, is of practical interest in the areas of scheduling, wireless sensor tracking, wiring-board design, code optimization and many others. A benzenoid system is a geometric collection of congruent regular hexagons arranged in the plane, so that two hexagons are either disjoint or have a common edge. It follows from the conditions of regularity and congruence that benzenoid systems are subsets (with 1-connected interior) of a regular tiling of the plane by hexagonal tiles. Benzenoid systems are of considerable importance in theoretical chemistry because they are the natural graph representation of benzenoid hydrocarbons. In each benzenoid system as a graph, we assign vertices of hexagons as the vertices of the graph, and the sides of hexagons as the edges of the graph. Benzenoid graph is simple, plane, and bipartite. A vertex of a hexagonal system belongs to, atmost, three hexagons. A vertex shared by three hexagons is called an internal vertex, other vertices, it called external vertex.

Domination and its variations in graphs have attracted considerable attention, including a few chemically relevant applications, [6], [7] and [9]. A new 2-dimensional carbon network (benzenoid system), named Coronene of coronene (CorCor for short) was introduced by M. V. Diudea [3], [8]. CorCor is a benzene of benzenes, a domain of graphene. In this paper, we obtain domination number, total domination number and open packing number of the molecular graph of CorCor.

2. Domination Number of CorCor

In this paper we obtain domination number, total domination number and open packing for a new 2-dimensional carbon network, named CorCor.

The graphs $G[n] = CorCor[n]$ with n layers (CorCor of dimension n), have been shown in Figure 1. For $n = 1$, we have one coronene, for $n = 2$, this coronene is surrounded by 6 other coronenes, for $n = 3$, graph $G[3]$ is obtained from $G[2]$ surrounded by 12 coronenes and in general $G[n]$ is obtained from $G[n-1]$ surrounded by $6(n-1)$ coronenes. Number of coronenes in $G[n]$ are $3n^2 - 3n + 1$.

Theorem 2.1. ([7], Theorem 2.5) In a graph G, if there exists a perfect H-packing when $H \cong K_{1,\Delta(G)}$, then $\gamma(G) = \lambda$, where $\Delta(G)$ and λ are the maximum degree and the packing number of G respectively.

By Theorem 2.1, packing number of CorCor[n] is equal with domination number because it is clear that $\gamma(K_{1,\Delta(G)}) = 1$. In a perfect H-packing when $H \cong K_{1,\Delta(G)}$, all the vertices are dominated by exactly one vertex. Hence, the packing number and the domination number of G are same. In other words $\gamma(G[n]) = \lambda$.

Theorem 2.2. Let $G[n]$ be a CorCor of dimension n. Then

$$\gamma(G[n]) = \begin{cases} \frac{3}{2}(2n + \lceil \frac{n}{3} \rceil - 2)^2 + N - 34 & if\ \lceil \frac{n}{3} \rceil\ is\ even \\ \frac{3}{2}(2n + \lceil \frac{n}{3} \rceil - 3)(2n + \lceil \frac{n}{3} \rceil - 1) + N - 34 & if\ \lceil \frac{n}{3} \rceil\ is\ odd \end{cases} .$$

where $N = (n - \lceil \frac{n}{3} \rceil + 3)(n - \lceil \frac{n}{3} \rceil + 4)$.

Proof. Let D be any minimum dominating set of $G[n]$. For computing $\gamma(G[n])$, it is enough to calculate $|D|$.

For $n = 1$, it is easy to see $\gamma(G[1]) = 6$. We determine vertices in dominating set $G[2]$ and $G[3]$ in Figure 3.2, also we can dominate $G[3]$ with six hexagonal that it has been shown in Figure 3.2. The number of vertices of dominating set, respectively from interior of the hexagonal dominating to the outside as follows:

$$2, 6, 6, 12, 12, 18.$$

So $\gamma(G[3]) = 2 + 6 + 6 + 12 + 12 + 18 + 24 = 80$.

For $n > 3$, we have two cases.

Case 1. If $\lceil \frac{n}{3} \rceil$ is even. Then the number of dominating vertices in a hexagonal dominating (see Figure 3.2, $G[3]$) as follows:

$$2, 6, 6, 12, 12, \cdots, 6(n-1), 6(n-1), 6n, 6(n - \lceil \frac{n}{3} \rceil + 3), 6(n - \lceil \frac{n}{3} \rceil + 2), \cdots, 30.$$

Where:

$$\underbrace{2, 6, 6, 12, 12, \cdots, 6(n-1), 6(n-1), 6n,}_{(2n + \lceil \frac{n}{3} \rceil - 2)}$$

$$\underbrace{6(n - \lceil \frac{n}{3} \rceil + 3), 6(n - \lceil \frac{n}{3} \rceil + 2), \cdots, 30\,.}_{(n - \lceil \frac{n}{3} \rceil - 1)}$$

Also we have 24 vertices of dominating set are located out of hexagonal dominating. For obtaining $\gamma(G[n])$, it is that to sum the numbers.

Case 2. If $\lceil \frac{n}{3} \rceil$ is odd. Then the number of dominating vertices in a hexagonal dominating (see Figure 3.2, $G[3]$) as follows:

$$2, 6, 6, 12, 12, \cdots, 6(n-1), 6(n-1), 6n, 6n, 6(n - \lceil \frac{n}{3} \rceil + 3), 6(n - \lceil \frac{n}{3} \rceil + 2), \cdots, 30.$$

where

$$\underbrace{2, 6, 6, 12, 12, \cdots, 6(n-1), 6(n-1), 6n, 6n,}_{(2n+\lceil \frac{n}{3} \rceil - 2)}$$
$$\underbrace{6(n - \lceil \frac{n}{3} \rceil + 3), 6(n - \lceil \frac{n}{3} \rceil + 2), \cdots, 30}_{(n-\lceil \frac{n}{3} \rceil - 1)}.$$

Similarly Case 1, we may to calculate the. $\gamma(G[n])$. The bold vertices in Figure 3.2 are in dominating set. $\square$

Theorem 2.3. Let $G[n]$ be a CorCor of dimension n. Then

$$\rho^0(G[n]) \geq 12n^2 - 12n + 4$$

This inequality is sharp.

Proof. We consider G be a CorCor of dimension n, we can see in Figure 3.3, number of vertices of open packing $G[2]$ is shown, $\rho^0(G[2]) = 29$. As we mentioned earlier, number of coronenes in $G[n]$ are $3n^2 - 3n + 1$ and every coronenes can have four points from set of open packing, so

$$\rho^0(G[n]) \geq 4(3n^2 - 3n + 1)$$

Also $\rho^0(G) \leq \gamma_t(G)$ holds for every graph G, according to the before theorem, $\rho^0(G)$ have upper bound.

For $n = 3$, can see in Figure 3.3, $\rho^0(G[3]) = 76$, so inequality is sharp. $\square$

In the next result the total domination of CorCor[n] is studied, but at first, we notice that the molecular graph of $G[n]$ has exactly $42n^2 - 24n + 6$ vertices and $63n^2 - 45n + 12$ edges. The molecular graph $G[n]$ is constructed from $6n - 3$ rows of hexagons. For example, the graph $G[3]$ has exactly 15 rows of hexagons and the number of hexagons in each row is according to the following sequence:

$$2, 5, 9, 10, 11, 12, 12, 11, 12, 12, 11, 10, 9, 5, 2.$$

The $(3n-1)^{th}$ row of $G[n]$ is called the central row of $G[n]$. This row has exactly $2(3\lceil \frac{n}{3} \rceil + 2(n - \lceil \frac{n}{3} \rceil)) - 3 = 4n + 2\lceil \frac{n}{3} \rceil - 3$ hexagons. The central hexagon of $G[n]$ is surrounded by six hexagons. If we replace each hexagon by a vertex and connect such vertices according to the adjacency of hexagons, then we will find a new hexagon containing the central hexagon of $G[n]$ hexagon containing the last one and so on, see Figure 3.1. The hexagons constructed from this algorithm are called the big hexagons.

Theorem 2.4. Let $G[n]$ be a CorCor of dimension n. Then

$$4(3n^2 - 3n + 1) < \gamma_t(G[n]) \leq \frac{|V(G[n])|}{3} + 6n - 2.$$

Proof. If G is a CorCor of dimension 1, then $G[1]$ consists of just a single coronene, and it is easy to see that $\gamma_t(G[1]) = 12$. Now we consider the case of dimension at least two. In the CorCor, any zigzag line not containing vertical edges is called a zigzag horizontal line, [1]. The zigzag horizontal lines of CorCor are denoted by $L_j, 1 \leq j \leq 6n-2$, Figure 3.4. We have

$$|L_1| = |L_{(6n-2)}|$$

and

$$\begin{aligned} L_1 &= \{v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}\}, \\ L_{(6n-2)} &= \{v_{(6n-2,1)}, v_{(6n-2,2)}, v_{(6n-2,3)}, v_{(6n-2,4)}, v_{(6n-2,5)}\}. \end{aligned}$$

Also

$$|L_2| = |L_{(6n-3)}|$$

And so on.

Let T be any minimum total dominating set of $G[n]$. For computing $\gamma_t(G)$, it is enough to calculate $|T|$.

$$\begin{aligned} |T| = 2|\{&v_{1,3}, v_{1,4}, v_{2,1}, v_{2,2}, v_{2,7}, v_{2,8}, v_{2,9}, v_{3,1}, v_{3,2}, v_{3,7}, v_{3,8}, v_{3,13}, v_{3,14} \\ &, v_{4,2}, v_{4,5}, v_{4,6}, v_{4,11}, v_{4,12}, v_{4,15}, v_{5,2}, v_{5,3}, v_{5,8}, v_{5,9}, v_{5,14}, v_{5,15}, v_{6,4}\}|. \end{aligned}$$

Then $|T| \leq \frac{|V(G[n])|}{3} + 6n - 2 = 14n^2 - 2n$. Since $\rho^0(G) \leq \gamma_t(G)$ holds for every graph $G[5]$. So that the inequality is hold. Also in Figure 3.5 is shown vertices of total dominating set of $G[2]$. $\square$

3. Tables and Figures

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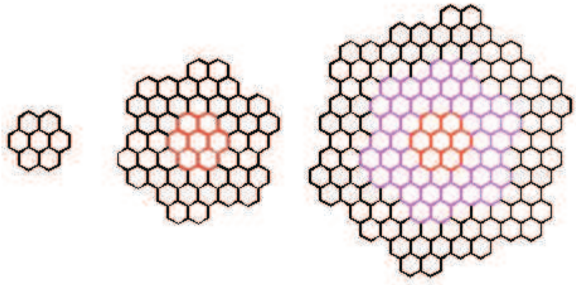

FIG. 3.1: G[n]=CorCor[n] for n=1,2,3.

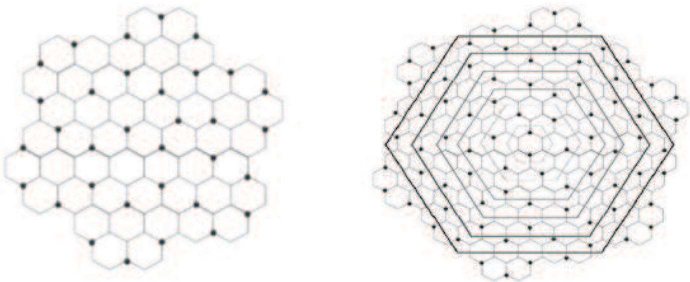

FIG. 3.2: Domination number of G[2] and Hexagonal dominating set of G[3].

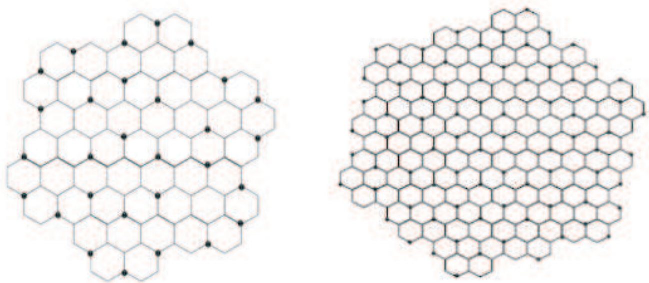

FIG. 3.3: Open packing of G[2] and Open packing of G[3].

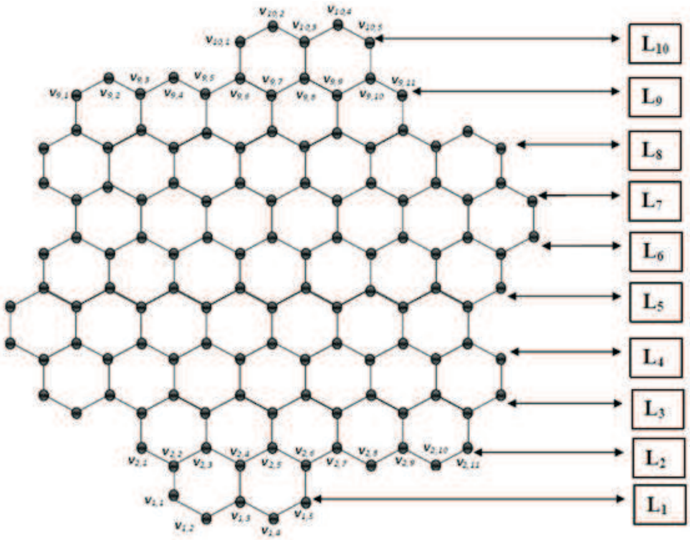

FIG. 3.4: The vertices and level in CorCor[2].

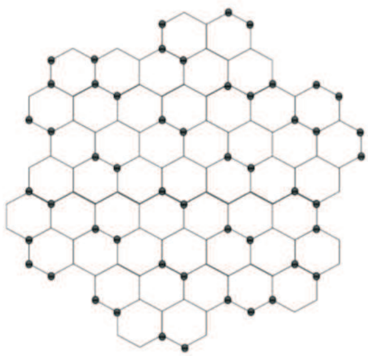

FIG. 3.5: Total domination number of CorCor[2].

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THE WEIGHT HIERARCHY OF HADAMARD CODES

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Abstract. The support of an (n, M, d) binary code C over the set $\mathbf{A} = \{0, 1\}$ is the set of all coordinate positions i, such that at least two codewords of C have distinct entry in coordinate i. If C is a code of size M, then r-th generalized Hamming weight, $d_r(C)$, $1 \leq r \leq 1 + log_2(M-1)$, of C is defined as the minimum of the cardinalities of the supports of all subset of C of cardinality $2^{r-1}+1$. The sequence $(d_1(C), d_2(C), \ldots, d_k(C))$ is called the Hamming weight hierarchy (HWH) of C. In this paper we obtain HWH for $(2^k-1, 2^k, 2^{k-1})$ binary Hadamard code corresponding to Sylvester Hadamard matrix H_{2^k} and we show that

$$d_r = 2^{k-r}(2^r - 1).$$

Also we compute the HWH of $(4n-1, 4n, 2n)$ Hadamard codes for $2 \leq n \leq 8$.
Keywords. Binary code; Hamming weight; Hadamard codes.

1. introduction

Let $\mathbf{A} = \{0, 1\}$. For positive integer n, every non-empty subset, C, of $\mathbf{A}^n$ is called a *binary code* of length n. The *Hamming distance* of two vectors X, Y is defined the number of the coordinates that they differ and is denoted by $d(X, Y)$. The Hamming distance of C is denoted by $d = d(C)$ and defined as

$$\min_{X \neq Y \in C} d(X, Y).$$

A binary code C of length n, size M and distance d is called (n, M, d) binary code. The *support* of an (n, M, d) binary code C over the set $\mathbf{A} = \{0, 1\}$ is the set of all coordinate positions i, such that at least two codewords have distinct entry in coordinate i and is denoted by $supp(C)$. The *rth generalized Hamming weight* (GHW), $d_r(C)$, $1 \leq r \leq 1 + log_2(M-1)$, of C is defined as follows

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$$d_r = d_r(C) = \min\{\|D\| : D \subset C, |D| = 2^{r-1} + 1\},$$

where $\|D\| = |supp(D)|$. The sequence $(d_1(C), d_2(C), \dots, d_k(C))$ is called the *Hamming weight hierarchy* (HWH) of C.

For the first time, the generalized Hamming weights (GHW) were introduced by V. K. Wei in [17] for linear codes. In [17] , the basic properties of GHW are studied and the weight hierarchy for Hamming code, Reed-Solomon codes, binary Reed-Muller code, etc are determined. This concept is a generalization of minimum Hamming weight of a code. It is not difficult to see that $d_1(C) = d(C)$. The concept of GHW were extended for various version of codes, such as non-linear code and codes over rings, for example see [2, 3]. Study of this notion was motivated by applications in cryptography. It is a well-known fact that the sequence of generalized Hamming weights is strictly increasing, that is,

$$d_1(C) < d_2(C) < \cdots < d_k(C) = n.$$

Among non-linear codes, Hadamard codes are the most useful codes in engineering, coding theory and mathematics. First we mention the definition of Hadamard matrices and specific version of Hadamard codes which are not linear. Then we obtain GHW for these codes.

A square matrix H of order n with elements in $\{1, -1\}$ is called a *Hadamard matrix* when $HH^t = I_n$, in which I_n denotes the identity matrix. We will denote by H_n the Hadamard matrix of order n. For clarifying we bring some examples:

$$H_1 = \begin{pmatrix} 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{1.1}$$

It is known (Paley, 1933) that if Hadamard matrices of order n exist, then $n = 1, 2$ or $n = 4s$, where s is a positive integer. Note that changing the sign of elements in a row or column can not affect the orthogonality. Hence a Hadamard matrix can always be reduced to the standard form in which the initial row and column contain only $+1$.

The *Kronecker product* or *tensor product* of matrices A and B is defined as follows

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \tag{1.2}$$

Lemma 1.1. *(Sylvester) Let H_1 and H_2 be Hadamard matrices of orders h_1 and h_2, then the Kronecker product of H_1 and H_2 is an Hadamard matrix of order h_1h_2.*

Lemma 1.2. *(Sylvester (1867)) There is an Hadamard matrix of order 2^t for all non-negative t.*

The matrices of order 2^t constructed using Sylvester's construction are usually referred to as *Sylvester-Hadamard matrices.* The Sylvester-Hadamard matrices are associated with discrete orthogonal functions called Walsh functions [15].

A (v, k, λ) *design*, is a pair $(\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v elements, called *points* and $\mathcal{B}$ is a collection of distinct subsets of $\mathcal{P}$ of size k, called *blocks*, such that every pair is contained in precisely λ blocks. The number of blocks in $\mathcal{B}$ is denoted by b and *Fisher's inequality* state that $b \geq v$. If $b = v$, the (v, k, λ) design is called *symmetric*. Symmetric designs have interesting properties. One of them is that every two distinct blocks intersect in exactly λ points. Another properties is that every point appears in exactly k blocks. For a (v, k, λ) design $D = (\mathcal{P}, \mathcal{B})$, consider $\overline{\mathcal{B}} = \{\mathcal{P} \backslash B : B \in \mathcal{B}\}$. It is not difficult to see that $\overline{D} = (\mathcal{P}, \overline{\mathcal{B}}$ is a $(v, v-n, v-2k+\lambda)$ design. Let $\mathcal{P} = \{x_1, x_2, \ldots, x_v\}$ and $\mathcal{B} = \{B_, B_2, \cdots, B_v\}$. The *Incidence matrix* of a (v, k, λ) design, $(\mathcal{P}, \mathcal{B})$, is the $v \times b$ matrix M whose entries m_{ij} are defined as $m_{ij} = 1$ if $x_i \in B_j$ and $m_{ij} = 0$ if $x_i \notin B_j$. In the next section we compute the weight hierarchy of some families of code arising from Hadamard matrix by properties of symmetric designs.

2. Main Result

We firstly recall Levenshtein's method [13] for constructing optimal error correcting codes from suitable Hadamard matrices. Starting from a normalized (i.e. the first row and column formed all of $1'$s) Hadamard matrix H of order $4n$, some codes (which are termed Hadamard codes) may be constructed (see [14], for instance). More concretely, consider the matrix A_{4n} related to H_{4n}, which consists in replacing the $+1'$s by $0'$s and the $-1'$s by $1'$s. Since the rows of H_{4n} are orthogonal, any two rows of A_{4n} agree in $2n$ places and differ in $2n$ places and so have Hamming distance $2n$ apart. In these circumstances, one may construct an $(4n-1, 4n, 2n)$ code, C_{4n}, consisting of the rows of A_{4n} with the first column deleted. This code called *Hadamard code.* Also if we deleted the first row and column of A_{4n}, then the remaining matrix is the incidence matrix of a $(4n-1, 2n, n-1)$ symmetric design, which called *Hadamard design.* For further information about Hadamard matrices and Hadamard design reader can see [1].

Theorem 2.1. *Suppose that H_n is a Hadamard matrix and C_n is the Hadamard code corresponding to H_n. Then $d_2(C_n) = \frac{3n}{4}$.*

Proof. Since $d_2(C_n) = min\{\|D\|; D \subseteq C_n, |D| = 3\}$, therefore there are the rows r_1, r_2, r_3 in A_n such that $\|r_1, r_2, r_3\| = d_2(C_n)$. Let $D = \{\{r_1, r_2, r_3\}\}$. Hence the elements of D may have the following cases:

$$\underbrace{\begin{matrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{1}\end{matrix}}_{a-tuple} \quad \underbrace{\begin{matrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{-1}\end{matrix}}_{b-tuple} \quad \underbrace{\begin{matrix}\mathbf{1}\\ \mathbf{-1}\\ \mathbf{1}\end{matrix}}_{c-tuple} \quad \underbrace{\begin{matrix}\mathbf{1}\\ \mathbf{-1}\\ \mathbf{-1}\end{matrix}}_{d-tuple}$$

Note that $\mathbf{1}$ denotes the $m-$ tuple vector of 1, in which $m \leq n$.

Using the orthogonality of the distinct rows and $H_n H_n^t = I_n$, we have the following equations:

$$a+b+c+d = n-1, a+b-c-d = -1, a-b+c-d = -1, a-b-c+d = -1.$$

The solution of this system of equations is $b = c = d = \frac{n}{4}$. So, we have

$$\|D)\| = b+c+d = \frac{3n}{4}.$$

□

Theorem 2.2. *Suppose that H_{2^k} is the Sylvester Hadamard matrix. If C_{2^k} is the Hadamard code of order 2^k corresponding to H_{2^k}, then*

$$d_r(C_{2^k}) = 2^{k-r}(2^r - 1).$$

Proof. Let C_{2^k} be the code associated with the Sylvester Hadamard matrix H_{2^k}. The proof is by induction on k. It is true for $k = 1$. Suppose that the relation is true for k. In the other words, suppose that the weight hierarchy of C_{2^k} is $d_r = 2^{k-r}(2^r - 1)$. We know that

$$d_r = min\{\|D\|; D \subset C_{2^k}, |D| = 2^{r-1} + 1\}.$$

Therefore there are $2^{r-1} + 1$ rows in C_{2^k}, say them $h_1, h_2, ..., h_{2^{r-1}+1}$, such that

$$supp(h_1, h_2, ..., h_{2^{r-1}+1}) = 2^{k-r}(2^r - 1).$$

We know that

$$H_{2^{k+1}} = \begin{pmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{pmatrix}.$$

Now by using the construction of $H_{2^{k+1}}$, the support of $h_1, h_2, ..., h_{2^{r-1}+1}$ in $C_{2^{k+1}}$ is equal to $2.2^{k-r}(2^r - 1)$. □

Two Hadamard matrices are called *equivalent* if one is obtained from the other by a sequence of permutations and negations of rows and columns. The equivalent classes of Hadamard matrices of small orders have been determined by several authors. It is well known that order up to 12, there is a unique Hadamard matrix. For orders

$16, 20, 24, 28$ and 32 there are $5, 3, 60, 487$ and 3710027 inequivalent Hadamard matrices, respectively [4, 5, 6, 7, 8, 9, 10, 11, 12, 16].
It is obvious that, if H and H' are two equivalent Hadamard matrices of order $4n$ and C and C' are two $(4n-1, 4n, 2n)$ nonlinear codes correspondence to H and H', with $r-th$ generalized Hamming weight d_r and d'_r, respectively, then $d_r = d'_r$. In the following theorem, we prove that if C_n is a code from H_n, then $d_r(C_n)$ is independent of choice of H_n for $8 \leq n \leq 32$.

Theorem 2.3. *Let C_{4n} be a $(4n-1, 4n, 2n)$ Hadamard code. If $k = 1 + [log_2(4n-1)]$, then $d_k = 4n-1$ and $d_{k-1} = 4n-2$.*

Proof. Let D_{4n} be the $(4n-1, 2n-1, n-1)$ Hadamard design, corresponding to C_{4n}. If $d_k \leq 4n-2$, then there exist a coordinate, i, such that all code words are equal to 1 or all code words are equal to 0 in position i. Hence there exists an element of $x \in \mathcal{P}$ such that x belong to every blocks of D_{4n} (or $\overline{D_{4n}}$), which is impossible. Hence $d_k = 4n-1$. If $d_{k-1} \leq 4n-3$, then there are a subset D Of C_{4n} of size $2^{k-2}+1$ and two coordinates j_1 and j_2, such that all code word of D are agree in these coordinates. If $j_1 = j_2 = 1$, then every code word of D indicated a block of D_{4n}, and hence there exists a pair, which appear in $|D|$ blocks, which is impossible. If $j_1 = j_2 = 0$, then there exists a pair, which appear in at least $|D|-1$ blocks of $\overline{D_{4n}}$, which is impossible. If $j_1 = 1$ and $j_2 = 0$, then there exists a pair $\{x, y\}$ such that there are at least $|D|$ blocks $B_1, B_2, \cdots, B_{|D|}$ of D_{4n}, which $x \in B_i$ and $y \notin B_i$ for $1 \leq i \leq |D|$. But the number of blocks , B, such that $x \in B$ and $y \notin B$ is equal to n and we get a contradiction. Hence $d_{k-1} = 4n-2$. □

Theorem 2.4. *Suppose that $n = 4k$ and $2 \leq k \leq 8$. If C_{4n} and C'_{4n} are two $(4n-1, 4n, 2n)$ binary Hadamard codes, corresponding to two Hadamard matrices H_{4n} and H'_{4n}, respectively, then $d_r(C_{4n}) = d_r(C'_{4n})$.*

Proof. For $n \in \{2, 3\}$, the result is obvious, since H_8 and H_{12} are unique. Suppose that C_{16} is the $(15, 16, 8)$ binary Hadamard, constructed by a Hadamard matrix of order 16. By Theorem 2.2, $d_2(C_{16}) = 12$. Since generalized Hamming weights is strictly increasing, then $d_3(C_{16}) = 13, d_4(C_{16}) = 14, d_5(C_{16}) = 15$. Now consider the code C_{20} constructed from H_{20}. The code C_{20} is a $(19, 20, 10)$ binary code. Theorem 2.2 implies that $d_2(C_{20}) = 15$ and by Theorem 2.3 we have $d_4(C_{20}) = 18$ and $d_5(C_{20}) = 19$. If $d_3(C_{20}) = 16$, then there are 5 code words, which agree in 3 coordinates. But all 3 inequivalent Hadamard matrices have not this property. The same argument works for other cases. □

In the following table we give the generalized Hamming weights of Hadamard matrices of order up to 32.

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Hadamard Matrix	Code	Design	d_1	d_2	d_3	d_4	d_5
H_8	$(7,8,4)$	$(7,3,1)$	4	6	7	-	-
H_{12}	$(11,12,6)$	$(11,5,2)$	6	9	10	11	-
H_{16}	$(15,16,8)$	$(15,7,3)$	8	12	13	14	15
H_{20}	$(19,20,10)$	$(19,9,4)$	10	15	17	18	19
H_{24}	$(23,24,12)$	$(23,11,5)$	12	18	21	22	23
H_{28}	$(27,28,14)$	$(27,13,6)$	14	21	25	26	27
H_{32}	$(31,32,16)$	$(31,15,7)$	16	24	28	30	31

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FAMILY OF ELLIPTIC CURVES $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$ *

Mehrdad Khazali and Hassan Daghigh

Abstract. In this paper we show that for any two primes p and q greater than 5, the elliptic curve $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$ has rank at least 2. We will also provide two independent points on $E_{(p,q)}$. Then we will show that, conjecturally, the family $\{E_{(p,q)}\}$ contains an infinite subfamily of rank three elliptic curves.
Keywords. Elliptic curves; Abelian group; group homomorphism.

1. Introduction

Let E be an elliptic curve over $\mathbb{Q}$ and $E(\mathbb{Q})$ be its Mordell-Weil group over $\mathbb{Q}$ which is a finitely generated Abelian group. The rank of the free part of $E(\mathbb{Q})$ as a $\mathbb{Z}$ -module is called the rank of E over $\mathbb{Q}$. There has been a lot of research to compute the rank of the families of elliptic curves. Despite these attempts, there is no efficient algorithm for finding the rank of elliptic curves. So finding special forms of elliptic curves whose structure is known is very interesting. Many authors [7, 8, 9, 10, 11, 12, 16, 5, 6] have considered different families of elliptic curves and have computed their rank and integral points.
In this paper, we study elliptic curves of the form $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$ over $\mathbb{Q}$, where p and q are primes greater than 5. We show that the torsion group of these curves is trivial, and also find at least two independent points on these curves, which means that $E_{(p,q)}$ has rank at least 2.

2. feature of points in $E(\mathbb{Q})$

In this section, we consider the structure of the group of rational points on the family of the following elliptic curves

$$E = E_{(p,q)} : y^2 = x^3 - p^2x + q^2, \tag{2..1}$$

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where p and q are primes greater than 5. We will show that E has no torsion points and has rank at least 2. To show that $E(\mathbb{Q})$ has no nontrivial torsion point we need the following lemma.

Lemma 2.1. *Let E be an elliptic curve with integer coefficients. Suppose that E has good reduction modulo the prime r, and E_r is the reduction modulo r. The map*

$$E(\mathbb{Q})_{Tor} \longrightarrow E_r(\mathbb{F}_r)$$

is an injective group homomorphism.

Proof. This is a direct corollary of [14]. □

Theorem 2.2. *Let p and q be prime numbers greater than 5. The torsion part of E is trivial.*

Proof. We have $\Delta_E = -4p^6 + 27q^4$. It is easy to see that $3 \nmid \Delta_E$ and $5 \nmid \Delta_E$, therefore E has good reductions modulo 3 and 5. Let E_3 and E_5 be reductions of E modulo 3 and 5 respectively. By direct computation, we see that $|E_3(\mathbb{F}_3)| = 7$ and we have

$$|E_5(\mathbb{F}_5)| = \begin{cases} 8 & p^2 \equiv 1 \pmod 5. \\ 9 & p^2 \equiv 4 \pmod 5. \end{cases}$$

Now using Lemma 2.1, we see that $|E_{tors}(\mathbb{Q})|$ divides 7, and also 8 or 9, which means that $|E_{tors}(\mathbb{Q})| = 1$. This means that $E_{tors}(\mathbb{Q}) = \{\mathcal{O}\}$. □

By the Mordell-Weil theorem, $E(\mathbb{Q})$ is a finitely generated abelian group. Hence

$$E(\mathbb{Q}) = E(\mathbb{Q})_{Tor} \oplus \mathbb{Z}^r,$$

where r is the rank of $E(\mathbb{Q})$. In fact, (2..1) shows that in our case

$$E(\mathbb{Q}) \cong \mathbb{Z}^r.$$

Using this we have

$$E(\mathbb{Q})/2E(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

Therefore $E(\mathbb{Q})/2E(\mathbb{Q})$ determines r. We record the above as the following proposition:

Proposition 2.3. *Let E be an elliptic curve on $\mathbb{Q}$ such that $E(\mathbb{Q})$ has no torsion point. Then*

$$E(\mathbb{Q})/2E(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r,$$

where r is the rank of $E(\mathbb{Q})$.

To study our elliptic curve $E_{(p,q)}$, we start by the following lemmas about the features of the points in $E(\mathbb{Q})$.

Proposition 2.4. *Assume that $P = (x, y) \in E(\mathbb{Z})$. Then*

(i) y is odd.

(ii) $x \not\equiv 2, 4, 6 \pmod 8$.

Proof. The proof of (i) and (ii) are straightforward. $\square$

Lemma 2.5. *Let $P = (x, y)$ be a point in $E(\mathbb{Q})$. Then, $x = \frac{u}{s^2}$ and $y = \frac{u'}{s^3}$, where $s, u, u' \in \mathbb{Z}$ and $gcd(u, s) = gcd(u', s) = 1$.*

Proof. See [17]. $\square$

Lemma 2.6. *For every point $P = (x, y) \in E(\mathbb{Q})$, we have*

$$x(2P) = \frac{x^4 + 2p^2x^2 + p^4 - 8q^2x}{4y^2}. \tag{2..2}$$

Proof. The proof is straightforward. $\square$

In the following proposition, we see some features of elements of $2E(\mathbb{Q})$.

Proposition 2.7. *Let $P = (\frac{u}{s^2}, \frac{u'}{s^3})$ and $Q = (\frac{w}{t^2}, \frac{w'}{t^3}) \in E(\mathbb{Q})$, where $u, u', s, w, w', t \in \mathbb{Z}$ and $gcd(uu', s) = gcd(ww', t) = 1$. If $P = 2Q$ then:*

(i) $t|s$.

(ii) If s is odd, then w and t are odd.

(iii) u is odd and $u \not\equiv s^2 \pmod 4$.

Proof. Suppose that $P = 2Q$, since $x(2Q) = x(P) = \frac{u}{s^2}$, using Lemma 2.6 and the fact that Q is on E we have

$$4ut^2(w^3 - p^2wt^4 + q^2t^6) = s^2[(w^2 + p^2t^4)^2 - 8q^2wt^6]. \tag{2..3}$$

Since $gcd(t, w) = 1$, from the above equality we see that $t|s$. This proves (i). Considering (2..3) modulo 8, proves part (ii). For part (iii), it suffices to consider (2..3) modulo 16. $\square$

3. group structure of $E(\mathbb{Q})$

In this section, using previous results, we will find two independent points in $E(\mathbb{Q})$, which proves that $r \geq 2$. Fix two prime numbers $p > 5$ and $q > 5$ and let E be the elliptic curve $E : y^2 = x^3 - p^2x + q^2$ over $\mathbb{Q}$. Consider the points $P_1 = (0, q)$, $P_2 = (-p, q)$ and $P_3 = (p, q)$ in $E(\mathbb{Q})$. We will show that P_1 and P_2 are independent.

Lemma 3.1. *None of the points P_1, P_2 and P_3 belong to $2E(\mathbb{Q})$.*

Proof. By part (iii) in Proposition 2.7, P_1 and $P_2 \notin 2E(\mathbb{Q})$. We prove the lemma for P_3, other parts is similar. Suppose that there exist $Q = (\frac{w}{t^2}, \frac{w'}{t^3}) \in E(\mathbb{Q})$, such that $P_3 = 2Q$. By Proposition 2.7(i), $t|1$, so $t = \pm 1$ and by Lemma 2.6, we have

$$4p(w^3 - p^2w + q^2) = [(w^2 + p^2)^2 - 8q^2w]. \tag{3..1}$$

Which is equivalent to

$$(w - p)^4 - 4p^2(w - p)^2 - 8q^2(w - p) - 12pq^2 + 4p^4 = 0. \tag{3..2}$$

By Proposition 2.4, we know w is odd, let $w - p = 2s$. We have

$$(2s^2 - p^2)^2 = 4q^2s + 3pq^2. \tag{3..3}$$

Assume that $n = 2s^2 - p^2$ hence $q|n$ so n is odd. Therefore we have

$$s = \frac{n^2 - 3q^2p}{4q^2}.$$

Hence

$$n + p^2 = 2s^2 = \frac{(n^2 - 3pq^2)^2}{8q^4},$$

and therefore

$$n^4 - 6q^2pn^2 - 8q^4n + p^2q^4 = 0. \tag{3..4}$$

On the other hand

$$p = \frac{n^2 - 4q^2s}{3q^2}.$$

So

$$n = 2s^2 - \frac{n^4 + 16q^4s^2 - 8n^2q^2s}{9q^4},$$

and therefore

$$9nq^4 + n^4 - 8n^2q^2s = 2s^2q^4. \tag{3..5}$$

From this we have $n|2s^2q^4$. Since $gcd(n, 2s^2) = 1$, we have $n|q^4$. This is impossible since the product of roots of equation is p^2q^4. In fact, we have no integer roots for the equation (3..4). Therefore, we must reject the assumption that there exist $Q \in E(\mathbb{Q})$, such that $P_3 = 2Q$. Hence $P_3 \neq 2Q$. □

Theorem 3.2. *Let* $\overline{P}_i = P_i + 2E(\mathbb{Q})$, $i = 1, 2, 3$, *be elements in* $E(\mathbb{Q})/2E(\mathbb{Q})$. *The set* $H = \{\overline{O}, P_1, \overline{P_2}, \overline{P_3}\}$ *is a subgroup of* $E(\mathbb{Q})/2E(\mathbb{Q})$ *of order* 4, *so* $4 || E(\mathbb{Q})/2E(\mathbb{Q})|$ *and hence* $|E(\mathbb{Q})/2E(\mathbb{Q})| \geq 4$.

Proof. By Lemma 3.1, we know that $\bar{P}_1,\ \bar{P}_2,\ \bar{P}_3 \neq \bar{\mathcal{O}}$. On the other hand, it is easy to see that $-P_3 = P_1 + P_2$ and H is closed. This shows that H is a subgroup of $E(\mathbb{Q})/2E(\mathbb{Q})$. To prove the theorem, we consider the following cases:

1. Suppose that $\bar{P}_1 = \bar{P}_2$ then, $\bar{P}_3 = 2\bar{P}_1 = \bar{\mathcal{O}}$, which is a contradiction according to Lemma 3.1.
2. Suppose that $\bar{P}_1 = \bar{P}_3$ then, $\bar{P}_2 = \bar{\mathcal{O}}$, which is a contradiction according to Lemma 3.1.
3. Suppose that $\bar{P}_2 = \bar{P}_3$ then, $\bar{P}_1 = \bar{\mathcal{O}}$, which is a contradiction according to Lemma 3.1.

Therefore, these four classes are distinct classes of $E(\mathbb{Q})/2E(\mathbb{Q})$, so $|H| = 4$. □

We have shown that $|E(\mathbb{Q})/2E(\mathbb{Q})| \geq 4$, which implies that $rank(E) \geq 2$, by Proposition 2.3. In fact, we have the following theorem.

Theorem 3.3. *The point* P_1 *and* P_2 *are independent rational points in* $E(\mathbb{Q})$ *and so* $rank(E(\mathbb{Q})) \geq 2$.

Proof. Assume on the contrary that two rational points P_1 and P_2 are dependent. Then there exist $m, n \in \mathbb{Z}$,not both zero, such that $mP_1 + nP_2 = \mathcal{O}$. Without loss of generality, let $m \in \mathbb{N}$ be the smallest among all. We have four cases,

1. Assume that m is even and n is odd then, $\bar{\mathcal{O}} = \bar{P}_2$, which contradicts Theorem 3.2.
2. Assume that m is odd and n is even then, $\bar{\mathcal{O}} = \bar{P}_1$, which is a contradiction according to Theorem 3.2.
3. Assume that m is odd and n is odd then, $\bar{\mathcal{O}} = \bar{P}_3$, which contradicts Theorem 3.2.
4. Assume that $m = 2t$ and $n = 2t'$, both are even then $2(tP_1 + t'P_2) = \mathcal{O}$. Now Theorem 2.2 implies that $(tP_1 + t'P_2) = \mathcal{O}$. This contradicts the minimality of m.

This completes the proof. □

Example 3.4. *By Theorem* 3.3 *the points* $P_1 = (0, 7)$ *and* $P_2 = (-7, 7)$ *are independent points on the elliptic curve* $E = E_{(7,7)} : y^2 = x^3 - 7^2x + 7^2$. *The computer algebra system Sage [15] suggests that the rank of* $E = E_{(7,7)}$ *is in fact, 2 and the points* $P_1 = (0, 7)$ *and* $P_2 = (-7, 7)$ *generate* $E = E_{(7,7)}$.

4. A family of rank 3 elliptic curves

Already, we have identified an infinite family of rank two elliptic curves. In this section, we find a subfamily of rank three elliptic curves in this family. We will show that under a famous conjecture this subfamily has infinitely many members. Suppose that p and q satisfy $p^2+q^2-1=b^2$ for an integer, then the point $P_4=(-1,b)$ is a point on $E_{(p,q)}$. We will show that P_1, P_2 and P_4 are independent. We need the following:

Lemma 4.1. *Let p and q are prime numbers greater than 5. If there exists $b\in\mathbb{Z}$ such that $p^2+q^2-1=b^2$, then the point $P_4=(-1,b)\in E(\mathbb{Q})$, satisfies the followings:*

(i) $P_4\notin 2E(\mathbb{Q})$.

(ii) $P_5=P_4+P_1\notin 2E(\mathbb{Q})$.

(iii) $P_6=P_4+P_2\notin 2E(\mathbb{Q})$.

(iv) $P_7=P_4+P_3\notin 2E(\mathbb{Q})$.

Proof. First, we will show that $P_4\notin 2E(\mathbb{Q})$. Assume on the contrary that there exists $Q=(\frac{w}{t^2},\frac{w'}{t^3})\in E(\mathbb{Q})$, such that $P_4=2Q$. Then, $-1=x(2Q)$. By Proposition 2.7 we can set $t=1$. Now by Lemma 2.6 We have

$$4(w^3-p^2w+q^2)+(w^2+p^2)^2=8q^2w.$$

From this and Proposition 2.4 $1\le w\in\mathbb{Z}$. We rewrite the above formula as a quadratic equation in p^2. Then we have

$$p^4+2p^2(w^2-2w)+(w^4+4w^3-8wq^2+4q^2)=0. \tag{4..1}$$

The above equation has integer solutions if and only if

$$\Delta_{p^2}=16(-2w^3+w^2+2wq^2-q^2)$$

is the square of an integer. Now from this we have

$$-2w^3+w^2+2wq^2-q^2=m^2$$

for some integer m. Hence w satisfies the equation

$$-2w^3+w^2+2wq^2-q^2-m^2=0.$$

The sum of the roots of this equation is $\frac{1}{2}$. This impossible since $1\le w\in\mathbb{Z}$. This prove (i).
(ii) Let $P_5=2Q$ we have $(q-b)^2+1\equiv 1 \pmod 4$. This contradicts Proposition 2.7(iii). The proofs of (iii) and (iv) are similar to that of (ii). $\square$

Lemma 4.2. *Let* $\overline{P}_i = P_i + 2E(\mathbb{Q})$, $1 \leq i \leq 7$. *The set*

$$H = \{\overline{O},\ \bar{P}_1,\ \bar{P}_2,\ \bar{P}_3,\ \bar{P}_4,\ \bar{P}_5,\ \bar{P}_6,\ \bar{P}_7\}$$

is a subgroup of $E(\mathbb{Q})/2E(\mathbb{Q})$ *of order 8.*

Proof. The fact that the 8 elements in H are distinct and H is closed under addition is easy to prove using Theorem 3.2 and Lemma 4.1. □

The Lemma 4.2 and Proposition 2.3 show that the rank of $E(\mathbb{Q})$ is at least 3. In fact, we have the following result.

Theorem 4.3. *The points* P_1, P_2 *and* P_4, *are independent rational points on* $E(\mathbb{Q})$ *and therefore the rank of* $E(\mathbb{Q})$ *is at least three.*

Proof. Assume on the contrary that two rational points P_1, P_2 and P_4 are dependent. Then there exist $m, n, s \in \mathbb{Z}$,not both zero, such that $mP_1 + nP_2 + sP_4 = \mathcal{O}$.First we note that since P_1, P_2 are independent by Theorem 3.3 we have $s \neq 0$. Without loss of generality, let $s \in \mathbb{N}$ be the smallest among all. We have eight cases. When s is odd we have the following four cases,

1. Assume that m is even and n is odd. Then, $\bar{\mathcal{O}} = \bar{P}_6$, which contradicts Theorem 4.2.

2. Assume that m is odd and n is even. Then, $\bar{\mathcal{O}} = \bar{P}_5$, which is a contradiction according to Theorem 4.2.

3. Assume that m is odd and n is odd. Then, $\bar{\mathcal{O}} = \bar{P}_7$, which contradicts Theorem 4.2.

4. Assume that m is even and n is even. Then, $\bar{\mathcal{O}} = \bar{P}_4$, which contradicts Theorem 4.2.

If s is even we have four cases,

1. Assume that m is even and n is odd. Then, $\bar{\mathcal{O}} = \bar{P}_2$, which contradicts Theorem 4.2.

2. Assume that m is odd and n is even. Then, $\bar{\mathcal{O}} = \bar{P}_1$, which is a contradiction according to Theorem 4.2.

3. Assume that m is odd and n is odd. Then, $\bar{\mathcal{O}} = \bar{P}_3$, which contradicts Theorem 4.2.

4. Assume that $m = 2t$, $n = 2t'$ and, $s = 2t''$ both are even then

$$2(tP_1 + t'P_2 + t''P_4) = \mathcal{O}.$$

Now Theorem 2.2 implies that $(tP_1 + t'P_2 + t''P_4) = \mathcal{O}$. This contradicts the minimality of s.

This completes the proof. $\square$

Example 4.4. *By Theorem 4.2 the points* $P_1 = (0, 11)$, $P_2 = (-7, 11)$, $P_4 = (-1, 13)$ *are independent points on the elliptic curves* $E = E_{(7,11)} : y^2 = x^3 - 7^2x + 11^2$. *The computer algebra system Sage [15] suggests that the rank of* $E = E_{(7,11)}$ *is in fact, 3 and the points* $P_1 = (0, 11)$, $P_2 = (-7, 11)$, $P_4 = (-1, 13)$ *generate* $E = E_{(7,11)}$.

Here we investigate the number of primes p and q, for which p^2+q^2-1 is square. For this, we recall the Schinzel and Sierpinski [13] conjecture.

Conjecture 4.5. *Let* $f_1(x), f_2(x), \ldots, f_m(x) \in \mathbb{Z}[x]$ *be irreducible polynomials with positive leading coefficients. Assume that there exists no integer* $n > 1$ *dividing* $f_1(k), f_2(k), \ldots, f_m(k)$ *for all integers* k. *Then there exist infinitely many positive integers* l *such that each of the numbers* $f_1(l), f_2(l), \ldots, f_m(l)$ *is prime.*

Proposition 4.6. *There are infinitely many prime* p *and* q *for which* $p^2 + q^2 - 1$ *is a square.*

Proof. Consider $f_1(x) = 2x + 1$, $f_2(x) = x^2 + x - 1$ and $f_3(x) = x^2 + x + 1$. We have

$$f_1(x)^2 + f_2(x)^2 - 1 = f_3(x)^2.$$

On the other hand if there exist integers k and n such that $n|f_2(k), f_3(k)$, then $n|f_2(k) - f_3(k) = 2$, thus $n \nmid f_1(k)$.
So Conjecture 4.5 implies that there exist infinitely many k, such that $p = f_1(k)$, $q = f_2(k)$ and $b = f_3(k)$ are prime, which completes the proof. $\square$

Corollary 4.7. *Assuming the above conjecture, there is an infinite family elliptic curves of the form* $E = E_{(p,q)}$ *of rank at least 3.*

Proof. This follows from Theorem 4.3. $\square$

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