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[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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FIXED POINTS FOR TWO PAIRS OF ABSORBING MAPPINGS IN WEAK PARTIAL METRIC SPACES

Valeriu Popa and Alina-Mihaela Patriciu

Abstract. In this paper, a general fixed point theorem for two pairs of absorbing mappings in weak partial metric space, using implicit relations, has been proved.
Keywords: weak partial metric space; fixed point; pointwise absorbing mappings; implicit relation.

1. Introduction

In 1994, Matthews [13] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks and proved the Banach contraction principle in such spaces. The notion of partial metric spaces plays an important role in the constructing models in theory of computation.

Many authors studied the fixed points for mappings satisfying some contractive conditions in [1], [3], [11] and in other papers. In [11], some fixed point theorems for particular pairs of mappings are proved, generalizing some results from [1] and [3].

In 1999, Heckmann [10] introduced the notion of weak partial metric spaces, which is a generalization of partial metric spaces. Some results for mappings in weak partial metric spaces have been recently obtained by[2] and [4].

The notion of absorbing mappings have been introduced and studied in [5] - [7] as well as in other papers. Some fixed point theorems for two pairs of absorbing mappings in metric spaces have been proved in [12], [14], [15].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [16] - [18] and in other papers. Recently, the method has been used in the studies of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, Hilbert

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spaces, ultra - metric spaces, convex metric spaces, compact metric spaces, in two and three metric spaces, for single valued mappings, hybrid pairs of mappings and set-valued mappings.

Some fixed point theorems for pairs of mappings satisfying implicit relations in partial metric spaces have been proved in [8], [9], [19] - [21].

Some results for pointwise absorbing mappings satisfying implicit relations have been obtained in [15].

The purpose of this paper is to prove a general fixed point theorem for two pairs of pointwise absorbing mappings in weak partial metric spaces using an implicit relation.

2. Preliminaries

Definition 2.1. ([13]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

$(P_1) : x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$,

$(P_2) : p(x,x) \leq p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space.

If $p(x,y) = 0$, then $x = y$, but the converse does not always hold true.

Each partial metric p on X generates a T_0 - topology τ_p on X which has as base the family of open p - balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then

$$d_w(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\}$$

is a ordinary metric on X.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges with respect to τ_p to a point $x \in X$, denoted $x_n \to x$, if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n, x).$$

Remark 2.1. Let $\{x_n\}$ be a sequence in a partial metric (X,p) and $x \in X$. Then $\lim_{n\to\infty} d_w(x_n, x) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.1}$$

Definition 2.2. ([13])
a) A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
b) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x\in X$ such that

$$p(x,x)=\lim_{n,m\to\infty} p(x_n,x_m).$$

Definition 2.3. ([10]) A weak partial metric on a nonempty set X is a function $p: X\times X\to\mathbb{R}_+$ such that for all $x,y,z\in X$:
$(wP_1): x=y$ if and only if $p(x,x)=p(y,y)=p(x,y)$,
$(wP_2): p(x,y)=p(y,x)$,
$(wP_3): p(x,z)\le p(x,y)+p(y,z)-p(y,y)$.
The pair (X,p) is called a weak partial metric space.

Obviously, every partial metric space is a weak partial metric space, but the converse is not true.

For example, let $X=[0,\infty)$ and $p(x,y)=\frac{x+y}{2}$, then (X,p) is a weak partial metric space and is not a partial metric space.

Theorem 2.1. *([2]) Let (X,p) be a weak partial metric space. Then $d_w(x,y): X\times X\to\mathbb{R}_+$ is a metric on X.*

Remark 2.2. In a weak partial metric space, the convergence of sequences, Cauchy sequences and completeness are defined as in partial metric space.

Theorem 2.2. *([2]) Let (X,p) be a weak partial metric space.*
a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in metric space (X,d_w).
b) (X,p) is complete if and only if (X,d_w) is complete.

Lemma 2.1. *Let (X,p) be a weak partial metric space and $\{x_n\}$ is a sequence in X. If $\lim_{n\to\infty} x_n=x$ and $p(x,x)=0$ then*

$$\lim_{n\to\infty} p(x_n,y)=p(x,y),\forall y\in X.$$

Proof. By (wP_3),

$$p(x,y)\le p(x,x_n)+p(x_n,y),$$

hence

$$p(x,y)-p(x,x_n)\le p(x_n,y)\le p(x_n,x)+p(x,y).$$

Letting n tend to infinity we obtain

$$\lim_{n\to\infty} p(x_n,y)=p(x,y).$$

□

Remark 2.3. Remark 2.1 is still true for weak partial metric spaces.

Definition 2.4. ([6]) Let (X,d) be a metric space and f,g be self mappings on X.

1) f is called g - absorbing if there exists $R>0$ such that $d(gx,gfx)\leq Rd(fx,gx)$ for all $x\in X$.

Similarly, g is f - absorbing.

2) f is called pointwise g - absorbing if for given $x\in X$ there exists $R>0$ such that $d(gx,gfx)\leq Rd(fx,gx)$.

Similarly, g is pointwise f - absorbing.

Remark 2.4. 1) If (X,p) is a weak partial metric space we have a similar definition to Definition 2.4 with p instead d.

2) If g is the identity mapping on X, then f is trivially absorbing.

3. Implicit relations

Definition 3.1. Let $\mathfrak{F}_W$ be the set of all lower semi - continuous functions $F:\mathbb{R}_+^5\to\mathbb{R}$ satisfying the following conditions:

(F_1) : F is nonincreasing in variable t_5,

(F_2) : For all $u,v\geq 0$, there exists $h\in[0,1)$ such that

(F_{2a}) : $F(u,v,v,u,u+v)\leq 0$ and

(F_{2b}) : $F(u,v,u,v,u+v)\leq 0$,

implies $u\leq hv$.

(F_3) : $F(t,t,0,0,2t)>0,\forall t>0$.

Example 3.1. $F(t_1,...,t_5)=t_1-k\max\left\{t_2,t_3,t_4,\frac{t_5}{2}\right\}$, where $k\in[0,1)$.

(F_1) : Obviously.

(F_2) : Let $u,v\geq 0$ be and $F(u,v,v,u,u+v)=\max\left\{u,v,\frac{u+v}{2}\right\}\leq 0$. If $u>v$ then $u(1-k)\leq 0$, a contradiction. Hence $u\leq v$ which implies $u\leq hv$, where $0\leq h=k<1$.

Similarly, $F(u,v,u,v,u+v)\leq 0$ implies $u\leq hv$.

(F_3) : $F(t,t,0,0,2t)=t(1-k)>0,\forall t>0$.

The proofs for the following examples are similar to the proof of Example 3.1.

Example 3.2. $F(t_1,...,t_5)=t_1-k\max\{t_2,t_3,t_4,t_5\}$, where $k\in\left[0,\frac{1}{2}\right)$.

Example 3.3. $F(t_1,...,t_5)=t_1-k\max\left\{\frac{t_2+t_3+t_4}{3},\frac{t_5}{2}\right\}$, where $k\in[0,1)$.

Example 3.4. $F(t_1,...,t_5)=t_1^2-k\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5}{2}\right\}$, where $k\in[0,1)$.

Example 3.5. $F(t_1,...,t_5) = t_1^2 - a\max\left\{t_2^2,t_3^2,t_4^2\right\} - bt_5^2$, where $a,b \geq 0$ and $a+4b<1$.

Example 3.6. $F(t_1,...,t_5) = t_1^2 - at_2t_3 - bt_3t_4 - ct_5^2$, where $a,b,c \geq 0$ and $a+b+4c<1$.

Example 3.7. $F(t_1,...,t_5) = t_1^2 + \frac{t_1}{1+t_5} - \left(at_2^2 + bt_3^2 + ct_4^2\right)$, where $a,b,c \geq 0$ and $a+b+c<1$.

Example 3.8. $F(t_1,...,t_5) = t_1 - at_2 - bt_3 - c\max\{2t_4,t_5\}$, where $a,b,c,d \geq 0$ and $a+b+2c<1$.

Example 3.9. $F(t_1,...,t_5) = t_1 - \frac{at_3t_4}{1+t_2} - bt_2 - c(t_3+t_4) - dt_5$, where $a,b,c,d \geq 0$ and $a+b+2c+2d<1$.

Example 3.10. $F(t_1,...,t_5) = t_1^2 - t_1(at_2+bt_3+ct_4) - dt_5^2$, where $a,b,c,d \geq 0$ and $a+b+c+4d<1$.

4. Main results

Theorem 4.1. *Let (X,p) be a weak partial metric space and A,B,S and T be self mappings on X such that*
1) $T(X) \subset A(X)$ and $S(X) \subset B(X)$,
2) for all $x,y \in X$

$$F\left(\begin{array}{c} p(Sx,Ty), p(Ax,By), p(Sx,Ax), \\ p(Ty,By), p(Sx,By)+p(Ax,Ty) \end{array}\right) \leq 0. \tag{4.1}$$

If one of $A(X), B(X), S(X), T(X)$ is a closed subset of X, then
3) $\mathcal{C}(A,S) \neq \emptyset$,
4) $\mathcal{C}(B,T) \neq \emptyset$.

Moreover, if S is pointwise A - absorbing and T is pointwise B - absorbing, then A,B,S and T have a unique common fixed point z with $p(z,z)=0$.

Proof. Let x_0 be an arbitrary point of X. Since $S(X) \subset B(X)$, there exists $x_1 \in X$ such that $y_0 = Sx_0 = Bx_1$. Since $T(X) \subset A(X)$, there exists $x_2 \in X$ such that $y_1 = Tx_1 = Ax_2$. Continuing this process we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n} = Sx_{2n} = Bx_{2n+1},\ y_{2n+1} = Tx_{2n+1} = Ax_{2n+2},\ n \in \mathbb{N}. \tag{4.2}$$

First we prove that $\{y_n\}$ is a Cauchy sequence in (X,p).

By (4.1) for $x = x_{2n}$ and $y = x_{2n+1}$ we have

$$F\left(\begin{array}{c} p(Sx_{2n},Tx_{2n+1}), p(Ax_{2n},Bx_{2n+1}), p(Sx_{2n},Ax_{2n}), \\ p(Tx_{2n+1},Bx_{2n+1}), p(Sx_{2n},Bx_{2n+1})+p(Ax_{2n},Tx_{2n+1}) \end{array}\right) \leq 0.$$

By (4.2) we obtain

$$F\left(\begin{array}{c} p\left(y_{2n},y_{2n+1}\right),p\left(y_{2n-1},y_{2n}\right),p\left(y_{2n-1},y_{2n}\right), \\ p\left(y_{2n},y_{2n+1}\right),p\left(y_{2n},y_{2n}\right)+p\left(y_{2n-1},y_{2n+1}\right) \end{array}\right)\leq 0. \tag{4.3}$$

By (wP_3) we have

$$p\left(y_{2n-1},y_{2n+1}\right)\leq p\left(y_{2n-1},y_{2n}\right)+p\left(y_{2n},y_{2n+1}\right)-p\left(y_{2n},y_{2n}\right).$$

By (4.3) and (F_1) we obtain

$$F\left(\begin{array}{c} p\left(y_{2n},y_{2n+1}\right),p\left(y_{2n-1},y_{2n}\right),p\left(y_{2n-1},y_{2n}\right), \\ p\left(y_{2n},y_{2n+1}\right),p\left(y_{2n-1},y_{2n}\right)+p\left(y_{2n},y_{2n+1}\right) \end{array}\right)\leq 0.$$

By (F_{2a}) we obtain

$$p\left(y_{2n+1},y_{2n}\right)\leq hp\left(y_{2n},y_{2n-1}\right).$$

By (4.1) for $x=x_{2n}$ and $y=x_{2n-1}$ we obtain

$$F\left(\begin{array}{c} p\left(Sx_{2n},Tx_{2n-1}\right),p\left(Ax_{2n},Bx_{2n-1}\right),p\left(Sx_{2n},Ax_{2n}\right), \\ p\left(Tx_{2n-1},Bx_{2n-1}\right),p\left(Sx_{2n},Bx_{2n-1}\right)+p\left(Ax_{2n},Tx_{2n-1}\right) \end{array}\right)\leq 0.$$

By (4.1) we obtain

$$F\left(\begin{array}{c} p\left(y_{2n},y_{2n-1}\right),p\left(y_{2n-1},y_{2n-2}\right),p\left(y_{2n},y_{2n-1}\right), \\ p\left(y_{2n-1},y_{2n-2}\right),p\left(y_{2n},y_{2n-2}\right)+p\left(y_{2n-1},y_{2n-1}\right) \end{array}\right)\leq 0. \tag{4.4}$$

By (wP_3),

$$p\left(y_{2n-2},y_{2n}\right)\leq p\left(y_{2n-2},y_{2n-1}\right)+p\left(y_{2n-1},y_{2n}\right)-p\left(y_{2n-1},y_{2n-1}\right).$$

By (4.4) and (F_1) we obtain

$$F\left(\begin{array}{c} p\left(y_{2n},y_{2n-1}\right),p\left(y_{2n-1},y_{2n-2}\right),p\left(y_{2n},y_{2n-1}\right), \\ p\left(y_{2n-1},y_{2n-2}\right),p\left(y_{2n-2},y_{2n-1}\right)+p\left(y_{2n-1},y_{2n}\right) \end{array}\right)\leq 0.$$

By (F_{2b}),

$$p\left(y_{2n},y_{2n-1}\right)\leq hp\left(y_{2n-1},y_{2n-2}\right).$$

Hence,

$$p\left(y_n,y_{n+1}\right)\leq hp\left(y_{n-1},y_{n-2}\right)\leq ...\leq h^n p\left(y_0,y_1\right).$$

For $n,m\in\mathbb{N}$, $m>n$, repeating (wP_3) we obtain

$$\begin{array}{rcl} p\left(y_n,y_m\right) & \leq & p\left(y_n,y_{n+1}\right)+p\left(y_{n+1},y_{n+2}\right)+...+p\left(y_{m-1},y_m\right) \\ & \leq & h^n\left(1+h+...+h^{m-1}\right)p\left(y_0,y_1\right) \\ & \leq & \dfrac{h^n}{1-h}p\left(y_0,y_1\right). \end{array}$$

Then,

$$p(y_n, y_m) \leq \frac{h^n}{1-h} p(y_0, y_1) \to 0 \ as \ n, m \to \infty. \tag{4.5}$$

This shows that $\{y_n\}$ is a Cauchy sequence in (X, p). By Theorem 2.2 (a), $\{y_n\}$ is a Cauchy sequence in (X, d_w). Since (X, p) is complete, by Theorem 2.2 (b), (X, d_w) is a complete metric space. Since $\{y_n\}$ is Cauchy in (X, d_w), it follows that $\{y_n\}$ converges to a point z in (X, d_w). Hence,

$$\lim_{n\to\infty} d_w(y_n, z) = 0.$$

By Remark 2.3, (2.1) and (4.5) we obtain

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n,m\to\infty} p(y_n, y_m) = 0 \tag{4.6}$$

Also, by Theorem 2.2, $Sx_{2n} \to z$, $Tx_{2n+1} \to z$, $Bx_{2n+1} \to z$, $Ax_{2n+2} \to z$. Suppose that $T(X)$ is a closed subset in (X, p). Then

$$\lim_{n\to\infty} Tx_{2n+1} = z \in T(X).$$

Since $T(X) \subset A(X)$, there exists $u \in X$ such that $z = Au$.

By (4.1) for $x = u$ and $y = x_{2n+1}$ we obtain

$$F\left(\begin{array}{c} p(Su, Tx_{2n+1}), p(Au, Bx_{2n+1}), p(Su, Au), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Su, Bx_{2n+1}) + p(Au, Tx_{2n+1}) \end{array}\right) \leq 0,$$

$$F\left(\begin{array}{c} p(Su, y_{2n+1}), p(Au, y_{2n-1}), p(Su, Au), \\ p(y_{2n+1}, y_{2n}), p(Su, y_{2n}) + p(Au, y_{2n+1}) \end{array}\right) \leq 0.$$

Letting n tend to infinity, by Lemma 2.1, and (4.6) we have

$$F(p(Su, z), 0, p(Su, z), 0, p(Su, z)) \leq 0,$$

which implies by (F_{2b}) that $p(Su, z) = 0$, i.e. $z = Su$. Hence, $z = Au = Su$ and $\mathcal{C}(A, S) \neq \emptyset$.

Since $z \in S(X) \subset B(X)$, then, there exists $v \in X$ such that $z = Bv$. We prove that $Bv = Tv$.

By (4.1), for $x = u$ and $y = v$ we obtain

$$F\left(\begin{array}{c} p(Su, Tv), p(Au, Bv), p(Su, Au), \\ p(Tv, Bv), p(Su, Bv) + p(Au, Tv) \end{array}\right) \leq 0,$$

$$F(p(z, Tv), 0, 0, p(z, Tv), 0 + p(z, Tv)) \leq 0.$$

By (F_{2a}) we have $p(z,Tv)=0$, which implies $z=Tv=Bv$. Hence, $z=Au=Su=Bv=Tv$ with $p(z,z)=0$.

Moreover, if S is pointwise A - absorbing, there exists $R_1>0$ such that

$$p(Au,ASu)\leq R_1 p(Au,Su)=R_1 p(z,z)=0.$$

Hence, $z=Au=ASu=Az$ and z is a fixed point of A.

By (4.1) we have

$$F\left(\begin{array}{c} p(Sz,Tv),p(Az,Bv),p(Sz,Az), \\ p(Tv,Bv),p(Sz,Bv)+p(Az,Tv) \end{array}\right)\leq 0,$$

$$F(p(Sz,z),0,p(Sz,z),0,p(Sz,z)+p(Sz,z))\leq 0,$$

which implies by (F_{2b}) that $p(z,Sz)=0$. Hence, $z=Sz$ and z is a common fixed point of A and S.

If T is pointwise B - absorbing, then there exists $R_2>0$ such that

$$p(Bv,BTv)\leq R_2 p(Bv,Tv)=R_2 p(z,z)=0.$$

Hence, $z=Bv=BTv=Bz$ and z is a fixed point of B.

By (4.1) we have

$$F\left(\begin{array}{c} p(Su,Tz),p(Au,Bz),p(Su,Au), \\ p(Tz,Bz),p(Su,Bz)+p(Au,Tz) \end{array}\right)\leq 0,$$

$$F(p(z,Tz),0,0,p(z,Tz),0+p(z,Tz))\leq 0,$$

which implies by (F_{2a}) that $p(z,Tz)=0$. Hence, $z=Tz$ and z is a common fixed point of B and T.

Therefore, z is a common fixed point of S,T,A and B with $p(z,z)=0$.

Suppose that A,B,S and T have two common fixed points z_i, $i=1,2$ with $p(z_i,z_i)=0$.

By (4.1) we obtain

$$F\left(\begin{array}{c} p(Sz_1,Tz_2),p(Az_1,Bz_2),p(Sz_1,Az_1), \\ p(Tz_2,Bz_2),p(Sz_1,Bz_2)+p(Az_1,Tz_2) \end{array}\right)\leq 0,$$

$$F(p(z_1,z_2),p(z_1,z_2),0,0,2p(z_1,z_2))\leq 0,$$

a contradiction of (F_3) if $p(z_1,z_2)>0$. Hence, $p(z_1,z_2)=0$ which implies $z_1=z_2$. $\square$

Example 4.1. Let $X=[0,1]$ be and $p(x,y)=\frac{x+y}{2}$, which implies $d_w(x,y)=\frac{1}{2}|x-y|$. Hence, (X,p) is a complete weak partial metric space. Let the mappings $Sx=0, Ax=\frac{x}{x+2}, Bx=x, Tx=\frac{x}{3}$. Since $A(X)=\left[0,\frac{1}{3}\right], B(X)=[0,1], S(X)=\{0\}, T(X)=\left[0,\frac{1}{3}\right]$, then $T(X)\subset A(X)$, $S(X)\subset B(X)$ and $A(X), B(X)$ and $T(X)$ are closed subsets of X.

$$\begin{aligned} p(Ax,ASx) &= p\left(\frac{x}{x+2},0\right)=\frac{x}{2(x+2)}, \\ p(Ax,Sx) &= p\left(\frac{x}{x+2},0\right)=\frac{x}{2(x+2)}. \end{aligned}$$

Hence, $p(Ax,ASx)\le R_1 p(Sx,Ax)$ with $R_1\ge 1$ and S is pointwise A - absorbing.

Similarly,

$$p(Bx,BTx)=p\left(x,\frac{x}{3}\right)=\frac{\frac{x}{3}+x}{2}=\frac{2x}{3},\ p(Bx,Tx)=p\left(x,\frac{x}{3}\right)=\frac{2x}{3}.$$

Hence, $p(Bx,BTx)\le R_2 p(Bx,Tx)$ with $R_2\ge 1$ and T is pointwise B - absorbing.

On the other hand,

$$p(Sx,Ty)=\frac{Sx+Ty}{2}=\frac{0+\frac{y}{3}}{2}=\frac{y}{6},\ p(Ty,By)=\frac{\frac{y}{3}+y}{2}=\frac{2y}{3}.$$

Hence,

$$p(Sx,Ty)\le kp(Ty,By),$$

where $k\in\left[\frac{1}{4},1\right]$. Therefore,

$$p(Sx,Ty)\le k\max\{p(Ax,By),p(Sx,Ax),p(Ty,By),p(Sx,By)+p(Ax,Ty)\}$$

with $k\in\left[\frac{1}{4},1\right]$.

By Theorem 4.1 and Example 3.1, A,B,S and T have a unique common fixed point $z=0$ and $p(z,z)=0$.

If $A=B=Id$, by Theorem 4.1 and Remark 2.4 (2), we obtain

Theorem 4.2. *Let (X,p) be a weak partial metric space and S and T be self mappings on X such that for all $x,y\in X$*

$$F\begin{pmatrix} p(Sx,Ty),p(x,y),p(x,Sx), \\ p(y,Ty),p(x,By)+p(Ty,x) \end{pmatrix}\le 0. \tag{4.7}$$

for some $F\in\mathcal{F}$.

If $S(X)$ or $T(X)$ is a closed subset of X, then S and T have a unique common fixed point.

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η-RICCI SOLITONS ON KENMOTSU MANIFOLD WITH GENERALIZED SYMMETRIC METRIC CONNECTION

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Abstract. The objective of the present paper is to study the η-Ricci solitons on Kenmotsu manifold with generalized symmetric metric connection of type (α, β). Ricci and η-Ricci solitons with generalized symmetric metric connection of type (α, β) have been discussed, satisfying the conditions $\bar{R}.\bar{S} = 0$, $\bar{S}.\bar{R} = 0$, $\bar{W}_2.\bar{S} = 0$ and $\bar{S}.\bar{W}_2 = 0$.. Finally, we have constructed an example of Kenmotsu manifold with generalized symmetric metric connection of type (α, β) admitting η-Ricci solitons.
Keywords: Kenmotsu manifold; Generalized symmetric metric connection; η-Ricci soliton; Ricci soliton, Einstein manifold.

1. Introduction

A linear connection $\overline{\nabla}$ is said to be generalized symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\varphi X - u(X)\varphi Y\}, \tag{1.1}$$

for any vector fields X, Y on a manifold, where α and β are smooth functions. φ is a tensor of type $(1,1)$ and u is a $1-$form associated with a non-vanishing smooth non-null unit vector field ξ. Moreover, the connection $\overline{\nabla}$ is said to be a generalized symmetric metric connection if there is a Riemannian metric g in M such that $\overline{\nabla} g = 0$, otherwise it is non-metric.

In the equation (1.1), if $\alpha = 0$ $(\beta = 0)$, then the generalized symmetric connection is called $\beta-$ quarter-symmetric connection ($\alpha-$ semi-symmetric connection), respectively. Moreover, if we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Therefore, a generalized symmetric

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connections can be viewed as a generalization of semi-symmetric connection and quarter-symmetric connection. These two connections are important for both the geometry study and applications to physics. In [12], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors (see [1], [9], [10] , [24], [26]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab [11]. In [23], Sharfuddin and Hussian defined a semi-symmetric metric connection in an almost contact manifold, by setting

$$T(X,Y) = \eta(Y)X - \eta(X)Y.$$

In [13], [25] and [19] the authors studied the semi-symmetric metric connection and semi-symmetric non-metric connection in a Kenmotsu manifold, respectively.

In the present paper, we have defined new connection for Kenmotsu manifold, generalized symmetric metric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection.

On the other hand, a Ricci soliton is a natural generalization of an Einstein metric. In 1982, R. S. Hamilton [14] said that the Ricci solitons moved under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are sationary points of the Ricci flow:

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.2}$$

Definition 1.1. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.3}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, H. Levy [16] in Theorem 4, proved that a second order parallel symmetric non-singular tensor in real space forms is proportional to the metric tensor. Later, R. Sharma [22] initiated the study of Ricci solitons in contact Riemannian geometry. After that, Tripathi [28], Nagaraja et. al. [17] and others like C. S. Bagewadi et. al. [4] extensively studied Ricci solitons in almost contact metric manifolds. In 2009, J. T. Cho and M. Kimura [6] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. η- Ricci solitons in almost paracontact metric manifolds have been studied by A. M. Blaga et. al. [2]. A. M. Blaga and various others authors have also studied η-Ricci solitons in manifolds with different structures (see [3], [20]). It is natural and interesting to study η-Ricci solitons in almost contact metric manifolds with this new connection.

Therefore, motivated by the above studies, in this paper we will study the η-Ricci solitons in a Kenmotsu manifold with respect to a generalized symmetric metric connection. We shall consider η-Ricci solitons in the almost contact geometry, precisely, on an Kenmotsu manifold with generalized symmetric metric connection which satisfies certain curvature properties: $\bar{R}.\bar{S} = 0$, $\bar{S}.\bar{R} = 0$, $W_2.\bar{S} = 0$ and $\bar{S}.\bar{W_2} = 0$ respectively.

2. Preliminaries

A differentiable M manifold of dimension $n = 2m+1$ is called almost contact metric manifold [5], if it admits a $(1,1)$ tensor field ϕ, a contravaryant vector field ξ, a $1-$ form η and Riemannian metric g which satisfies

$$\begin{aligned}
\phi\xi &= 0, && (2.1)\\
\eta(\phi X) &= 0 && (2.2)\\
\eta(\xi) &= 1, && (2.3)\\
\phi^2(X) &= -X + \eta(X)\xi, && (2.4)\\
g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), && (2.5)\\
g(X,\xi) &= \eta(X), && (2.6)
\end{aligned}$$

for all vector fields X, Y on M. If we write $g(X, \phi Y) = \Phi(X,Y)$, then the tensor field ϕ is a anti-symmetric $(0,2)$ tensor field [5]. If an almost contact metric manifold satisfies

$$\begin{aligned}
(\nabla_X \phi)Y &= g(\phi X, Y)\xi - \eta(Y)\phi X, && (2.7)\\
\nabla_X \xi &= X - \eta(X)\xi, && (2.8)
\end{aligned}$$

then M is called a Kenmotsu manifold, where ∇ is the Levi-Civita connection of g [18].

In Kenmotsu manifolds the following relations hold [18]:

$$\begin{aligned}
(\nabla_X \eta)Y &= g(\phi X, \phi Y) && (2.9)\\
g(R(X,Y)Z,\xi) &= \eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), && (2.10)\\
R(\xi,X)Y &= \eta(Y)X - g(X,Y)\xi, && (2.11)\\
R(X,Y)\xi &= \eta(X)Y - \eta(Y)X, && (2.12)\\
R(\xi,X)\xi &= X - \eta(X)\xi, && (2.13)\\
S(X,\xi) &= -(n-1)\eta(X), && (2.14)\\
S(\phi X, \phi Y) &= S(X,Y) + (n-1)\eta(X)\eta(Y) && (2.15)
\end{aligned}$$

for any vector fields X, Y and Z, where R and S are the the curvature and Ricci the tensors of M, respectively.

A Kenmotsu manifold M is said to be generalized η Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$, where a, b and c are scalar functions such that $b \neq 0$ and $c \neq 0$. If $c = 0$ then M is called η Einstein manifold.

3. Generalized Symmetric Metric Connection in a Kenmotsu Manifold

Let $\overline{\nabla}$ be a linear connection and ∇ be a Levi-Civita connection of an almost contact metric manifold M such that

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{3.1}$$

for any vector field X and Y. Where H is a tensor of type $(1, 2)$. For $\overline{\nabla}$ to be a generalized symmetric metric connection of ∇, we have

$$H(X, Y) = \frac{1}{2}[T(X, Y) + T^{'}(X, Y) + T^{'}(Y, X)], \tag{3.2}$$

where T is the torsion tensor of $\overline{\nabla}$ and

$$g(T^{'}(X, Y), Z) = g(T(Z, X), Y). \tag{3.3}$$

From (1.1) and (3.3) we get

$$T^{'}(X, Y) = \alpha\{\eta(X)Y - g(X, Y)\xi\} + \beta\{-\eta(X)\phi Y - g(\phi X, Y)\xi\}. \tag{3.4}$$

Using (1.1), (3.2) and (3.4) we obtain

$$H(X, Y) = \alpha\{\eta(Y)X - g(X, Y)\xi\} + \beta\{-\eta(X)\phi Y\}. \tag{3.5}$$

Corollary 3.1. *For a Kenmotsu manifold, generalized symmetric metric connection $\overline{\nabla}$ is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)X - g(X, Y)\xi\} - \beta\eta(X)\phi Y. \tag{3.6}$$

If we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, generalized metric connection is reduced to a semi-symmetric metric connection and quarter-symmetric metric connection as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{3.7}$$

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.8}$$

From (3.6) we have the following proposition

Proposition 3.1. *Let M be a Kenmotsu manifold with generalized metric connection. We have the following relations:*

$$(\overline{\nabla}_X \phi)Y = (\alpha + 1)\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{3.9}$$
$$\overline{\nabla}_X \xi = (\alpha + 1)\{X - \eta(X)\xi\}, \tag{3.10}$$
$$(\overline{\nabla}_X \eta)Y = (\alpha + 1)\{g(X, Y) - \eta(Y)\eta(X)\}, \tag{3.11}$$

for any $X, Y, Z \in \Gamma(TM)$.

4. Curvature Tensor on Kenmotsu manifold with generalized symmetric metric connection

Let M be an $n-$ dimensional Kenmotsu manifold. The curvature tensor $\overline{R}$ of the generalized metric connection $\overline{\nabla}$ on M is defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \tag{4.1}$$

Using the proposition 3.1, from (3.6) and (4.1) we have

$$\begin{aligned}(4.2)\,\bar{R}(X,Y)Z &= R(X,Y)Z + \{(-\alpha^2-2\alpha)g(Y,Z)+(\alpha^2+a)\eta(Y)\eta(Z)\}X\\ &+ \{(\alpha^2+2\alpha)g(X,Z)+(-\alpha^2-\alpha)\eta(X)\eta(Z)\}Y\\ &+ \{(\alpha^2+\alpha)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\\ &+ (\beta+\alpha\beta)[g(X,\phi Z)\eta(Y)-g(Y,\phi Z)\eta(X)]\}\xi\\ &+ (\beta+\alpha\beta)\eta(Y)\eta(Z)\phi X-(\beta+\alpha\beta)\eta(X)\eta(Z)\phi Y\end{aligned}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{4.3}$$

is the curvature tensor with respect to the Levi-Civita connection ∇.

Using (2.10), (2.11), (2.12), (2.13) and (4.2) we give the following proposition:

Proposition 4.1. *Let M be an $n-$ dimensional Kenmotsu manifold with generalized symmetric metric connection of type (α,β). Then we have the following equations:*

$$\bar{R}(X,Y)\xi = (\alpha+1)\{\eta(X)Y-\eta(Y)X+\beta[\eta(Y)\phi X-\eta(X)\phi Y]\} \tag{4.4}$$

$$\bar{R}(\xi,X)Y = (\alpha+1)\{\eta(Y)X-g(X,Y)\xi+\beta[\eta(Y)\phi X-g(X,\phi Y)\xi]\}, \tag{4.5}$$

$$\bar{R}(\xi,Y)\xi = (\alpha+1)\{Y-\eta(Y)\xi-\beta\phi Y\}, \tag{4.6}$$

$$\eta(\bar{R}(X,Y)Z = (\alpha+1)\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z) \tag{4.7}$$

$$+\beta[\eta(Y)g(X,\phi Z)-\eta(X)g(Y,\phi Z)]\}$$

for any $X,Y,Z\in\Gamma(TM)$.

We know that Ricci tensor is defined by

$$\overline{S}(Y,Z) = \sum_{i=1}^{n} g(\overline{R}(e_i,Y)Z,e_i),$$

where $Y, Z \in \Gamma(TM)$, $\{e_1, e_2, ..., e_n\}$ is viewed as orthonormal frame. We can calculate the Ricci tensor with respect to generalized symmetric metric connection as follows:

$$\begin{aligned}\overline{S}(Y,Z) &= S(Y,Z) + \{(2-n)\alpha^2 + (3-2n)\alpha\}g(Y,Z) + (n-2)(\alpha^2+\alpha)\eta(Y)\eta(Z) \\ &\quad -(\beta+\alpha\beta)g(Y,\phi Z),\end{aligned} \tag{4.8}$$

where S is Ricci tensor with respect to Levi-Civita connection.

Example 4.1. We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3 : x \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Let E_1, E_2, E_3 be a linearly independent global frame on M given by

$$E_1 = x\frac{\partial}{\partial z},\ E_2 = x\frac{\partial}{\partial y},\ E_3 = -x\frac{\partial}{\partial x}. \tag{4.9}$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_1,E_3) = g(E_2,E_3) = 0, g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1,$$

Let η be the 1-form defined by $\eta(U) = g(U, E_3)$, for any $U \in TM$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2, \phi E_2 = -E_1$ and $\phi E_3 = 0$. Then, using the linearity of ϕ and g we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) an almost contact metric manifold is defined.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[E_1,E_2] = 0, \qquad [E_1,E_3] = E_1, \qquad [E_2,E_3] = E_2, \tag{4.10}$$

Using Koszul formula for the Riemannian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{E_1}E_1 = -E_3, \qquad \nabla_{E_1}E_2 = 0. \qquad \nabla_{E_1}E_3 = E_1, \\ &\nabla_{E_2}E_1 = 0, \qquad \nabla_{E_2}E_2 = -E_3, \qquad \nabla_{E_2}E_3 = 0, \\ &\nabla_{E_3}E_1 = 0, \qquad \nabla_{E_3}E_2 = 0, \qquad \nabla_{E_3}E_3 = 0. \end{aligned} \tag{4.11}$$

From the above relations, it can be easily seen that

$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, $\nabla_X\xi = X - \eta(X)\xi$, for all $E_3 = \xi$. Thus the manifold M is a Kenmotsu manifold with the structure (ϕ, ξ, η, g). for $\xi = E_3$. Therefore, the manifold M under consideration is a Kenmotsu manifold of dimension three.

5. Ricci and η-Ricci solitons on $(M, \phi, \xi, \eta, g,)$

Let $(M, \phi, \xi, \eta, g,)$ be an almost contact metric manifold. Consider the equation

$$\mathcal{L}_\xi g + 2\bar{S} + 2\lambda + 2\mu\eta \otimes \eta = 0, \tag{5.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, $\bar{S}$ is the Ricci curvature tensor field with respect to the generalized symmetric metric connection of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi$ in terms of the generalized symmetric metric connection $\bar{\nabla}$, we obtain:

$$2\bar{S}(X,Y) = -g(\bar{\nabla}_X \xi, Y) - g(X, \bar{\nabla}_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{5.2}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (4.9) is said to be an *η-Ricci soliton* on M [10]. In particular, if $\mu = 0$ then (g, ξ, λ) is called *Ricci soliton* [6] and it is called *shrinking*, *steady* or *expanding*, according as λ is negative, zero or positive respectively [6].

Here is an example of η-Ricci soliton on Kenmotsu manifold with generalized symmetric metric connection.

Example 5.1. Let $M(\phi, \xi, \eta, g)$ be the Kenmotsu manifold considered in example 4.3 .

Let $\bar{\nabla}$ be a generalized symmetric metric connection, we obtain: Using the above relations, we can calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{gathered} R(E_1,E_2)E_1 = E_2,\ R(E_1,E_2)E_2 = -E_1,\ R(E_1,E_3)E_1 = E_3 \\ R(E_1,E_3)E_3 = -E_1,\ R(E_2,E_3)E_2 = E_3,\ R(E_2,E_3)E_3 = -E_2 \end{gathered} \tag{5.3}$$

From the equations (5.3) we can easily calculate the non-vanishing components of the Ricci tensor as follows:

$$S(E_1,E_1) = -2,\ S(E_2,E_2) = -2,\ S(E_3,E_3) = -2 \tag{5.4}$$

Now, we can make similar calculations for generalized metric connection. Using (3.6) in the above equations, we get

$$\begin{array}{rrr} \overline{\nabla}_{E_1}E_1 = -(1+\alpha)E_3, & \overline{\nabla}_{E_1}E_2 = 0. & \overline{\nabla}_{E_1}E_3 = (1+\alpha)E_1, \\ \overline{\nabla}_{E_2}E_1 = 0, & \overline{\nabla}_{E_2}E_2 = -(1+\alpha)E_3, & \overline{\nabla}_{E_2}E_3 = \alpha E_2, \\ \overline{\nabla}_{E_3}E_1 = -\beta E_2, & \overline{\nabla}_{E_3}E_2 = \beta E_1, & \overline{\nabla}_{E_3}E_3 = 0. \end{array} \tag{5.5}$$

From (5.5), we can calculate the non-vanishing components of curvature tensor with respect to generalized metric connection as follows:

$$\begin{array}{rr} \overline{R}(E_1,E_2)E_1 = (1+\alpha)^2 E_2, & \overline{R}(E_1,E_2)E_2 = -(1+\alpha)^2 E_1, \\ \overline{R}(E_1,E_3)E_1 = (1+\alpha)E_3 & \overline{R}(E_1,E_3)E_3 = (1+\alpha)(\beta E_2 - E_1), \\ \overline{R}(E_2,E_3)E_2 = (1+\alpha)E_3, & \overline{R}(E_2,E_3)E_3 = -(1+\alpha)(-\beta E_1 + E_2) \\ \overline{R}(E_3,E_2)E_1 = -(1+\alpha)\beta E_3, & \overline{R}(E_3,E_1)E_2 = (1+\alpha)\beta E_3,\ . \end{array} \tag{5.6}$$

From (5.6), the non-vanishing components of the Ricci tensor are as follows:

$$\overline{S}(E_1,E_1) = -(1+\alpha)(2+\alpha), \quad \overline{S}(E_2,E_2) = -(1+\alpha)(2+\alpha),$$
$$\overline{S}(E_3,E_3) = -2(1+\alpha). \tag{5.7}$$

From (5.2) and (5.5) we get

$$2(1+\alpha)[g(e_i,e_i) - \eta(e_i)\eta(e_i)] + 2\bar{S}(e_i,e_i) + 2\lambda g(e_i,e_i) + 2\mu\eta(e_i)\eta(e_i) = 0 \tag{5.8}$$

for all $i \in \{1,2,3\}$, and we have $\lambda = (1+\alpha)^2$ (*i.e.* $\lambda > 0$) and $\mu = 1-\alpha^2$, the data (g,ξ,λ,μ) is an η-Ricci soliton on (M,ϕ,ξ,η,g). If $\alpha = -1$ which is steady and if $\alpha \neq -1$ which is expanding.

6. Parallel symmetric second order tensors and η-Ricci solitons in Kenmotsu manifolds

An important geometrical object in studying Ricci solitons is well known to be a symmetric $(0,2)$-tensor field which is parallel with respect to the generalized symmetric metric connection.

Now, let fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to generalized symmetric metric connection$\bar{\nabla}$ that is $\bar{\nabla}h = 0$. By applying Ricci identity [7]

$$\bar{\nabla}^2 h(X,Y;Z,W) - \bar{\nabla}^2 h(X,Y;Z,W) = 0, \tag{6.1}$$

we obtain the relation

$$h(\bar{R}(X,Y)Z,W) + h(Z,\bar{R}(X,Y)W) = 0. \tag{6.2}$$

Replacing $Z = W = \xi$ in (6.2) and by using (4.4) and by the symmetry of h it follows $h(\bar{R}(X,Y)\xi,\xi) = 0$ for any $X,Y \in \chi(M)$ and

$$(\alpha+1)\eta(X)h(Y,\xi) - (\alpha+1)\eta(Y)h(X,\xi) \tag{6.3}$$

$$+(\alpha+1)\eta(X)h(\xi,Y) - (\alpha+1)\eta(Y)h(\xi,X) \tag{6.4}$$

$$+\beta\eta(Y)h(\phi X,\xi) - \beta\eta(X)h(\phi Y,\xi) + \beta\eta(Y)h(\xi,\phi X) - \beta\eta(X)h(\xi,\phi Y) = 0 \tag{6.5}$$

Putting $X = \xi$ in (6.3) and by the virtue of (2.4), we obtain

$$2(\alpha+1)[h(Y,\xi) - \eta(Y)h(\xi,\xi)] - 2\beta h(\phi Y,\xi) = 0. \tag{6.6}$$

or

$$2(\alpha+1)[h(Y,\xi) - g(Y,\xi)h(\xi,\xi)] - 2\beta(\phi Y,\xi) = 0. \tag{6.7}$$

Suppose $(\alpha+1) \neq 0$, $\beta = 0$ it results

$$h(Y,\xi) - \eta(Y)h(\xi,\xi) = 0, \tag{6.8}$$

for any $Y \in \chi(M)$, equivalent to

$$h(Y,\xi) - g(Y,\xi)h(\xi,\xi) = 0, \tag{6.9}$$

for any $Y \in \chi(M)$. Differentiating the equation (6.9) covariantly with respect to the vector field $X \in \chi(M)$, we obtain

$$h(\bar{\nabla}_X Y,\xi) + h(Y,\bar{\nabla}_X \xi) = h(\xi,\xi)[g(\bar{\nabla}_X Y,\xi) + g(Y,\bar{\nabla}_X \xi)]. \tag{6.10}$$

Using (4.4) in (6.10), we obtain

$$h(X,Y) = h(\xi,\xi)g(X,Y), \tag{6.11}$$

for any $X, Y \in \chi(M)$. The above equation gives the conclusion:

Theorem 6.1. *Let $(M,\phi,\xi,\eta,g,)$ be a Kenmotsu manifold with generalized symmetric metric connection also with non-vanishing ξ-sectional curvature and endowed with a tensor field of type $(0,2)$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to $\bar{\nabla}$, then it is a constant multiple of the metric tensor g.*

On a Kenmotsu manifold with generalized symmetric metric connection using equation (3.10) and $\mathcal{L}_\xi g = 2(g - \eta \otimes \eta)$, the equation (5.2) becomes:

$$\bar{S}(X,Y) = -(\lambda + \alpha + 1)g(X,Y) + (\alpha + 1 - \mu)\eta(X)\eta(Y). \tag{6.12}$$

In particular, $X = \xi$, we obtain

$$\bar{S}(X,\xi) = -(\lambda + \mu)\eta(X). \tag{6.13}$$

In this case, the Ricci operator $\bar{Q}$ defined by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$ has the expression

$$\bar{Q}X = -(\lambda + \alpha + 1)X + (\alpha + 1 - \mu)\eta(X)\eta(X)\xi. \tag{6.14}$$

Remark that on a Kenmostu manifold with generalized symmetric metric connection, the existence of an η-Ricci soliton implies that the characteristic vector field ξ is an eigenvector of Ricci operator corresponding to the eigenvalue $-(\lambda + \mu)$.

Now we shall apply the previous results on η-Ricci solitons.

Theorem 6.2. *Let (M,ϕ,ξ,η,g) be a Kenmotsu manifold with generalized symmetric metric connection. Assume that the symmetric $(0,2)$-tensor filed $h = \mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$ is parallel with respect to the generalized symmetric metric connection associated to g. Then $(g,\xi,-\frac{1}{2}h(\xi,\xi),\mu)$ yields an η-Ricci soliton.*

Proof. Now, we can calculate

$$h(\xi,\xi) = \mathcal{L}_\xi g(\xi,\xi) + 2\bar{S}(\xi,\xi) + 2\mu\eta(\xi)\eta(\xi) = -2\lambda, \tag{6.15}$$

so $\lambda = -\frac{1}{2}h(\xi,\xi)$. From (6.11) we conclude that $h(X,Y) = -2\lambda g(X,Y)$, for any $X, Y \in \chi(M)$. Therefore $\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta = -2\lambda g$. $\square$

For $\mu = 0$ follows $\mathcal{L}_\xi g + 2S - S(\xi, \xi)g = 0$ and this gives

Corollary 6.1. *On a Kenmotsu manifold (M, ϕ, ξ, η, g) with generalized symmetric metric connection with property that the symmetric $(0, 2)$-tensor field $h = \mathcal{L}_\xi g + 2S$ is parallel with respect to generalized symmetric metric connection associated to g, the relation (5.1), for $\mu = 0$, defines a Ricci soliton.*

Conversely, we shall study the consequences of the existence of η-Ricci solitons on a Kenmotsu manifold with generalized symmetric metric connection. From (6.12), we give the conclusion:

Theorem 6.3. *If equation (4.9) defines an η-Ricci soliton on a Kenmotsu manifold (M, ϕ, ξ, η, g) with generalized symmetric metric connection, then (M, g) is quasi-Einstein.*

Recall that the manifold is called *quasi-Einstein* [8] if the Ricci curvature tensor field S is a linear combination (with real scalars λ and μ respectively, with $\mu \neq 0$) of g and the tensor product of a non-zero 1-from η satisfying $\eta = g(X, \xi)$, for ξ a unit vector field and respectively, *Einstein* [8] if S is collinear with g.

Theorem 6.4. *If (ϕ, ξ, η, g) is a Kenmotsu structure with generalized symmetric metric connection on M and (4.9) defines an η-Ricci soliton on M, then*

1. $Q \circ \phi = \phi \circ Q$
2. *Q and S are parallel along ξ.*

Proof. The first statement follows from a direct computation and for the second one, note that

$$(\bar{\nabla}_\xi Q)X = \bar{\nabla}_\xi QX - Q(\bar{\nabla}_\xi X) \tag{6.16}$$

and

$$(\bar{\nabla}_\xi S)(X, Y) = \xi(S(X, Y)) - S(\bar{\nabla}_\xi X, Y) - S(X, \bar{\nabla}_\xi Y). \tag{6.17}$$

Replacing Q and S from (6.14) and (6.13) we get the conclusion. □

A particular case arises when the manifold is ϕ-Ricci symmetric, which means that $\phi^2 \circ \nabla Q = 0$, as stated in the next theorem.

Theorem 6.5. *Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold with generalized symmetric metric connection. If M is ϕ-Ricci symmetric and (4.9) defines an η-Ricci soliton on M, then $\mu = 1$ and (M, g) is Einstein manifold [8].*

Proof. Replacing Q from (6.14) in (6.16) and applying ϕ^2 we obtain

$$(\alpha + 1 - \mu)\eta(Y)[X - \eta(X)\xi] = 0, \tag{6.18}$$

for any $X, Y \in \chi(M)$. Follows $\mu = \alpha + 1$ and $S = -(\lambda + \alpha + 1)g$. □

Remark 6.1. In particular, the existence of an η-Ricci soliton on a Kenmotsu manifold with generalized symmetric metric connection which is *Ricci symmetric* (i.e. $\bar{\nabla}S = 0$) implies that M is *Einstein* manifold. The class of Ricci symmetric manifold represents an extension of class of Einstein manifold to which the locally symmetric manifold also belong ($i.e.\ satisfying$ $\bar{\nabla}R = 0$). The condition $\bar{\nabla}S = 0$ implies $\bar{R}.\bar{S} = 0$ and the manifolds satisfying this condition are called *Ricci semi-symmetric* [7].

In what follows we shall consider η-Ricci solitons requiring for the curvature to satisfy $\bar{R}(\xi, X).\bar{S} = 0$, $\bar{S}.\bar{R}(\xi, X) = 0$, $\bar{W}_2(\xi, X).\bar{S} = 0$ and $\bar{S}.\bar{W}_2(\xi, X) = 0$ respectively, where the W_2-curvature tensor field is the curvature tensor introduced by G. P. Pokhariyal and R. S. Mishra in [21]:

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{dimM - 1}[g(X, Z)QY - g(Y, Z)QX]. \tag{6.19}$$

7. η-Ricci solitions on a Kenmotsu manifold with generalized symmetric metric connection satisfying $\bar{R}(\xi, X).\bar{S} = 0$

Now we consider a Kenmotsu manifold with with a generalized symmetric metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{S}(\bar{R}(\xi, X)Y, Z) + \bar{S}(Y, \bar{R}(\xi, X)Z) = 0, \tag{7.1}$$

for any $X, Y \in \chi(M)$.

Replacing the expression of $\bar{S}$ from (6.12) and from the symmetries of $\bar{R}$ we get

$$(\alpha + 1)(\alpha + 1 - \mu)[\eta(Y)g(X, Z) + \eta(Z)g(X, Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{7.2}$$

for any $X, Y \in \chi(M)$.
For $Z = \xi$ we have

$$(\alpha + 1)(\alpha + 1 - \mu)g(\phi X, \phi Y) = 0, \tag{7.3}$$

for any $X, Y \in \chi(M)$.

Hence we can state the following theorem:

Theorem 7.1. *If a Kenmotsu manifold with a generalized symmetric metric connection $\bar{\nabla}$, (g, ξ, λ, μ) is an η-Ricci soliton on M and it satisfies $\bar{R}(\xi, X).\bar{S} = 0$, then the manifold is an η-Einstein manifold.*

For $\mu = 0$, we deduce:

Corollary 7.1. *On a Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{R}(\xi, X).\bar{S} = 0$, there is no η-Ricci soliton with the potential vector field ξ.*

8. η-Ricci solitons on Kenmotsu manifold with generalized symmetric metric connection satisfying $\bar{S}.\bar{R}(\xi, X) = 0$

In this section, we have considered Kenmotsu manifold with a generalized symmetric metric connection $\bar{S}$ satisfying the condition

$$\bar{S}(X, \bar{R}(Y,Z)W)\xi - \bar{S}(\xi, \bar{R}(Y,Z)W)X + \bar{S}(X,Y)\bar{R}(\xi,Z)W - \tag{8.1}$$

$$-\bar{S}(\xi,Y)\bar{R}(X,Z)W + \bar{S}(X,Z)\bar{R}(Y,\xi)W - \bar{S}(\xi,Z)\bar{R}(Y,X)W + \tag{8.2}$$

$$+\bar{S}(X,W)\bar{R}(Y,Z)\xi - \bar{S}(\xi,W)\bar{R}(Y,Z)X = 0 \tag{8.3}$$

for any $X, Y, Z, W \in \chi(M)$.

Taking the inner product with ξ, the equation (8.1) becomes

$$\bar{S}(X, \bar{R}(Y,Z)W) - \bar{S}(\xi, \bar{R}(Y,Z)W)\eta(X) + \bar{S}(X,Y)\eta(\bar{R}(\xi,Z)W) - \tag{8.4}$$

$$-\bar{S}(\xi,Y)\eta(\bar{R}(X,Z)W) + \bar{S}(X,Z)\eta(\bar{R}(Y,\xi)W) - \bar{S}(\xi,Z)\eta(\bar{R}(Y,X)W) + \tag{8.5}$$

$$+\bar{S}(X,W)\eta(\bar{R}(Y,Z)\xi) - \bar{S}(\xi,W)\eta(\bar{R}(Y,Z)X) = 0 \tag{8.6}$$

for any $X, Y, Z, W \in \chi(M)$.

For $W = \xi$, using the equation (4.4), (4.5), (4.7) and (6.12) in (8.4), we get

$$(\alpha+1)(2\lambda+\mu+\alpha+1)[g(X,Y)\eta(Z)-g(X,Z)\eta(Y)+\beta g(\phi X,Y)\eta(Z) - g(\phi X,Z)\eta(Y)] \tag{8.7}$$

for any $X, Y, Z, W \in \chi(M)$.
Hence we can state the following theorem:

Theorem 8.1. *If (M, ϕ, ξ, η, g) is a Kenmotsu manifold with a generalized symmetric metric connection, (g, ξ, λ, μ) is an η-Ricci soliton on M and it satisfies $\bar{S}.\bar{R}(\xi, X) = 0$. Then*

$$(\alpha + 1)(2\lambda + \mu + \alpha + 1) = 0. \tag{8.8}$$

For $\mu = 0$ follows $\lambda = -\frac{\alpha+1}{2}$,$(\alpha \neq -1)$, therefore, we have the following corollary:

Corollary 8.1. *On a Kenmotsu manifold with a generalized symmetric metric connection, satisfying $\bar{S}.\bar{R}(\xi, X) = 0$, the Ricci soliton defined by (5.1), $\mu = 0$ is either shrinking or expanding.*

9. η-Ricci soliton on (ε)-Kenmotsu manifold with a semi-symmetric metric connection satisfying $\bar{W}_2(\xi, X).\bar{S} = 0$

The condition that must be satisfied by $\bar{S}$ is

$$\bar{S}(\bar{W}_2(\xi, X)Y, Z) + \bar{S}(Y, \bar{W}_2(\xi, X)Z) = 0, \tag{9.1}$$

for any $X, Y, Z \in \chi(M)$.

For $X = \xi$, using (4.4), (4.5), (4.7), (6.12) and (6.19) in (9.1), we get

$$\frac{(\alpha+1-\mu)(-2\mu-2\lambda+(4\alpha+4)n)}{n}\eta(Y)\eta(Z) \tag{9.2}$$

for any $X, Y, Z \in \chi(M)$. Hence, we can state the following:

Theorem 9.1. *If (M, ϕ, ξ, η, g) is an $(2n+1)$-dimensional Kenmotsu manifold with a generalized symmetric metric connection, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\bar{W}_2(\xi, X).\bar{S} = 0$, then*

$$(\alpha+1-\mu)(-2\mu-2\lambda+(4\alpha+4)n) = 0. \tag{9.3}$$

For $\mu = 0$ follows that $\lambda = \frac{(4\alpha+4)n}{2}$,$(\alpha \neq -1)$, therefore, we have the following corollary:

Corollary 9.1. *On a Kenmotsu manifold with a generalized symmetric metric connection, satisfying $\bar{W}_2(\xi, X).\bar{S} = 0$, the Ricci soliton defined by (5.1), $\mu = 0$ is either shrinking or expanding.*

10. η-Ricci soliton on Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{S}.\bar{W}_2(\xi, X) = 0$

In this section, we have considered an (ε)-Kenmotsu manifold with a semi-symmetric metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{S}(X, \bar{W}_2(Y,Z)V)\xi - \bar{S}(\xi, \bar{W}_2(Y,Z)V)X + \bar{S}(X,Y)\bar{W}_2(\xi,Z)V - \tag{10.1}$$

$$-\bar{S}(\xi,Y)\bar{W}_2(X,Z)V + \bar{S}(X,Z)\bar{W}_2(Y,\xi)V - \bar{S}(\xi,Z)\bar{W}_2(Y,X)V + \tag{10.2}$$

$$+\bar{S}(X,V)\bar{W}_2(Y,Z)\xi - \bar{S}(\xi,V)\bar{W}_2(Y,Z)X = 0, \tag{10.3}$$

for any $X, Y, Z, V \in \chi(M)$.

Taking the inner product with ξ, the equation (10.1) becomes

$$\bar{S}(X, \bar{W}_2(Y,Z)V) - \bar{S}(\xi, \bar{W}_2(Y,Z)V)\eta(X) + \bar{S}(X,Y)\eta(\bar{W}_2(\xi,Z)V) - \tag{10.4}$$

$$-\bar{S}(\xi,Y)\eta(\bar{W}_2(X,Z)V) + \bar{S}(X,Z)\eta(\bar{W}_2(Y,\xi)V) - \bar{S}(\xi,Z)\eta(\bar{W}_2(Y,X)V) + \tag{10.5}$$

$$+\bar{S}(X,V)\eta(\bar{W}_2(Y,Z)\xi) - \bar{S}(\xi,V)\eta(\bar{W}_2(Y,Z)X) = 0, \tag{10.6}$$

for any $X, Y, Z, V \in \chi(M)$.

For $X = V = \xi$, using (4.4), (4.5), (4.7), (6.12) and (6.19) in (10.4), we get

$$\{-(\alpha+1)(2\lambda+\alpha+1+\mu) + \frac{(\lambda+\alpha+1)^2 + (\lambda+\mu)^2}{2n}\}\{\eta(X)\eta(Y) - g(X,Y)\} \tag{10.7}$$

$$+\beta(\alpha+1)(2\lambda+\alpha+1+\mu)g(\phi X, Y) = 0, \tag{10.8}$$

for any $X, Y, Z \in \chi(M)$. Hence, we can state:

Theorem 10.1. *If (M, ϕ, ξ, η, g) is a $(2n+1)$-dimensional Kenmotsu manifold with generalized symmetric metric connection, (g, ξ, λ, μ) is an η-Ricci soliton on M and $\bar{S}.\bar{W}_2(\xi, X) = 0$, then*

$$-(\alpha+1)(2\lambda+\alpha+1+\mu) + \frac{(\lambda+\alpha+1)^2 + (\lambda+\mu)^2}{2n} = 0, \tag{10.9}$$

and

$$\beta(\alpha+1)(2\lambda+\alpha+1+\mu) = 0. \tag{10.10}$$

For $\mu = 0$ we get the following corollary:

Corollary 10.1. *On a Kenmotsu manifold with a generalized symmetric metric connection satisfying $\bar{S}.\bar{W}_2(\xi, X) = 0$, the Ricci soliton defined by (5.1), for $\mu = 0$, we have the following expressions:*
(i) $-(\alpha+1)(2\lambda+\alpha+1) + \frac{(\lambda+\alpha+1)^2+(\lambda)^2}{2n} = 0$ *and* $\beta(\alpha+1)(2\lambda+\alpha+1) = 0$.
(ii) *If $\alpha = -1$ or $\alpha = -2\lambda - 1$ which is steady.*

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$f-$BIHARMONIC CURVES WITH TIMELIKE NORMAL VECTOR ON LORENTZIAN SPHERE

Bilal Eftal Acet

Abstract. In this paper, we study $f-$biharmonic curves as the critical points of the $f-$bienergy functional $E_2(\psi) = \int_M f \mid \tau(\psi)^2 \mid \vartheta_g$, on a Lorentzian para-Sasakian manifold M. We give necessary and sufficient conditions for a curve such that has a timelike principal normal vector on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be an $f-$biharmonic curve. Moreover, we introduce proper $f-$biharmonic curves on the Lorentzian sphere S_1^4.
Keywords: $f-$biharmonic curves; $f-$bienergy functional; para-Sasakian manifold; Lorentzian sphere.

1. Introduction

Harmonic maps $\psi : (M, g) \to (N, h)$ between Riemannian manifolds are the critical points of the energy functional defined by

$$E(\psi) = \frac{1}{2} \int_\Omega \mid d\psi \mid^2 \vartheta_g, \tag{1.1}$$

for every compact domain $\Omega \subset M$. The Euler-Lagrange equation of the energy functional gives the harmonic equation defined by vanishing of

$$\tau(\psi) = trace \nabla d\psi, \tag{1.2}$$

where $\tau(\psi)$ is called the tension field of the map ψ.

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J.H. Sampson [7]. Biharmonic maps

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between Riemannian manifolds $\psi : (M, g) \to (N, h)$ are the critical points of the bienergy functional

$$E_2(\psi) = \frac{1}{2}\int_\Omega \mid \tau(\psi) \mid^2 \vartheta_g, \tag{1.3}$$

for any compact domain $\Omega \subset M$.

In [3], G.Y. Jiang derived the first and the second variation formulas for the bienergy, showing that the Euler-Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(\psi) &= -J^\psi(\tau(\psi)) \\ &= -\triangle\tau(\Psi) - traceR^N(d\psi, \tau(\psi))d\psi,\end{aligned}$$

where J^ψ is the Jacobi operator of ψ. The equation $\tau_2(\psi) = 0$ is called biharmonic equation. Clearly, any harmonic maps is always a biharmonic map. A biharmonic map that is not harmonic is called a proper biharmonic map.

For some recent geometric study of biharmonic maps see [14, 17, 18, 19, 24] and the references therein. Also for some recent progress on biharmonic submanifolds see [1, 2, 16, 20, 21] and for biharmonic conformal immersions and submersions see [15, 25, 27].

The concept of $f-$biharmonic maps were initiated by W.J. Lu [23]. A smooth map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is called an $f-$biharmonic map if it is a critical point of the $f-$bienergy functional defined by

$$E_{2,f}(\psi) = \frac{1}{2}\int_\Omega f \mid \tau(\psi) \mid^2 \vartheta_g, \tag{1.4}$$

for every compact domain $\Omega \subset M$.

The Euler-Lagrange equation gives the $f-$biharmonic map equation [23]

$$\begin{aligned}\tau_{2,f} &= f\tau_2(\psi) + (\triangle f)\tau(\psi) + 2\nabla^\psi_{gradf}\tau(\psi) \\ &= 0,\end{aligned}$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of ψ, respectively. Therefore, we have the following relationship among these types of maps [26]:

$$Harmonic\ \ maps \subset Biharmonic\ \ maps \subset f - Biharmonic\ \ maps. \tag{1.5}$$

From now on we will call an $f-$biharmonic map, which is neither harmonic nor biharmonic, a proper $f-$biharmonic map (see also [28]).

The study of Lorentzian almost paracontact manifold was initiated by K. Matsumoto [9]. He also introduced the notion of Lorentzian para-Sasakian manifold. In [4], I. Mihai and R. Rosca defined the same notion independently and there after many authors [5, 11, 22] studied Lorentzian para-Sasakian manifolds.

Moreover, in [17] some geometric result for spacelike and timelike curves in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold to be proper biharmonic were given. Motivated by this work, we introduced $f-$biharmonic curves on Lorentzian para-Sasakian manifold and Lorentzian sphere S^4_1.

2. Preliminaries

2.1. $f-$Biharmonic Maps

$f-$Biharmonic maps are critical points of the $f-$bienergy functional for maps $\psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_{2,f}(\psi) = \frac{1}{2}\int_{\Omega} f \mid \tau(\psi) \mid^2 \vartheta_g, \tag{2.1}$$

where Ω is a compact domain of M.

The following Theorem was proved in [23]:

Theorem 2.1. *A map $\psi : (M, g) \to (N, h)$ between Riemannian manifolds is an $f-$biharmonic map if and only if*

$$\tau_{2,f} = f\tau_2(\psi) + (\triangle f)\tau(\psi) + 2\nabla^{\psi}_{gradf}\tau(\psi) = 0, \tag{2.2}$$

where $\tau(\psi)$ and $\tau_2(\psi)$ are the tension and bitension fields of ψ, respectively. $\tau_{2,\,f}\,(\psi)$ is called the $f-$bitension field of map ψ.

A special case of $f-$biharmonic maps is $f-$biharmonic curves. We have the following.

Lemma 2.1. *[26] An arclength parametrized curve $\gamma : (a, b) \to (N^m, g)$ is an $f-$biharmonic curve with a function $f : (a, b) \to (0, \infty)$ if and only if*

$$f(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' - R^N(\gamma', \nabla^N_{\gamma'}\gamma')\gamma') + 2f'\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' + f''\nabla^N_{\gamma'}\gamma' = 0. \tag{2.3}$$

2.2. Lorentzian almost paracontact manifolds

Let M be an n-dimensional differentiable manifold with a Lorentzian metric g, i.e., g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in M$, the tensor

$$g_p : T_pM \times T_pM \to R,$$

is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_pM is the tangent space of M at the point p. Then (M, g) is called a Lorentzian manifold. A non-zero vector $X_p \in T_pM$ can be spacelike, null or timelike, if it satisfies $g_p(X_p, X_p) > 0$, $g_p(X_p, X_p) = 0$ or $g_p(X_p, X_p) < 0$, respectively.

Let M be an n-dimensional differentiable manifold equipped with a structure (φ, ξ, η), where φ is a $(1, 1)$-tensor field, ξ is a vector field, η is a 1-form on M such that [9]

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.4}$$

$$\eta(\xi) = -1. \tag{2.5}$$

The above equations imply that

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad rank(\varphi) = n - 1.$$

Then M admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y),$$

and M is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). Then we get

$$g(X, \xi) = \eta(X). \tag{2.6}$$

The manifold M endowed with a Lorentzian almost paracontact structure (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold [9, 10]. In equations (2.4) and (2.5) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by I. Sato [6].

A Lorentzian almost paracontact manifold equipped with the structure (φ, ξ, η, g) is called a Lorentzian para-Sasakian manifold [9] if

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{2.7}$$

The conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)W &= R(X,Y)W - \frac{1}{n-2}\left\{ \begin{array}{c} S(Y,W)X - S(X,W)Y \\ +g(Y,W)QX - g(X,W)QY \end{array} \right\} \\ &\quad + \frac{r}{(n-1)(n-2)}\left\{ g(Y,W)X - g(X,W)Y \right\}, \end{aligned}$$

where $S(X, Y) = g(QX, Y)$. The Lorentzian para-Sasakian manifold is called conformally flat if conformal curvature tensor vanishes i.e., $C = 0$.

The quasi-conformal curvature tensor $\hat{C}$ is defined by

$$\begin{aligned} \hat{C}(X,Y)W &= aR(X,Y)W - b\left\{ \begin{array}{c} S(Y,W)X - S(X,W)Y \\ +g(Y,W)QX - g(X,W)QY \end{array} \right\} \\ &\quad - \frac{r}{n}\left(\frac{a}{(n-1)} + 2b \right)\left\{ g(Y,W)X - g(X,W)Y \right\}, \end{aligned}$$

where a, b constants such that $ab \neq 0$. Similarly the Lorentzian para-Sasakian manifold is called quasi-conformally flat if $\hat{C} = 0$.

We know that a conformally flat and quasi-conformally flat Lorentzian para-Sasakian manifold M^n $(n > 3)$ is of constant curvature 1 and also a Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation $R(X, Y) \cdot C = 0$ holds on M [12]. For a conformally symmetric Riemannian manifold [13], we get $\nabla C = 0$. Thus for a conformally symmetric space the relation $R(X, Y) \cdot$

$C = 0$ satisfies. Hence a conformally symmetric Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere [12].

Therefore, for a conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M, we have [12]

$$R(X,Y)W = g(Y,W)X - g(X,W)Y, \tag{2.8}$$

for any vector fields $X, Y, W \in TM$.

3. $f-$Biharmonic Curves in Lorentzian Para-Sasakian Manifolds

For a Lorentzian para-Sasakian manifold M, an arbitrary curve $\gamma : I \to M$, $\gamma = \gamma(s)$ is called spacelike, timelike or lightlike (null), if all of its velocity vectors $\gamma'(s)$ are spacelike, timelike or lightlike (null), respectively. In this section, we give some conditions for a curve having timelike normal vector on a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M to be an $f-$biharmonic curve.

Theorem 3.1. *Let $\gamma : I \to M$ be a curve parametrized by arclength and M be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Asuume that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along γ such that principal normal vector N is timelike. Then γ is a proper $f-$biharmonic curve if and only if one of the following cases happens:*

i) The first curvature κ_1 of the γ solves the following ordinary differential equation,

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^4 - 4\kappa_1^2, \tag{3.1}$$

with $f = t_1\kappa_1^{-\frac{3}{2}}$ and $\kappa_2 = 0$.

ii) The first curvature κ_1 of the γ solves the following ordinary differential equation,

$$3(\kappa_1')^2 - 2\kappa_1\kappa_1'' = 4\kappa_1^4 + 4\kappa_1^4 t_3^2 - 4\kappa_1^2, \tag{3.2}$$

with $f = t_1\kappa_1^{-\frac{3}{2}}$, $\kappa_2 \neq 0$, $\kappa_3 = 0$, $\frac{\kappa_2}{\kappa_1} = t_3$.

Proof. Let γ be a curve parametrized by arclength on lying a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M and let $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along γ such that principal normal vector N is timelike.

In this case for this curve, the Frenet frame equations are given by [8]

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_1 & 0 & \kappa_2 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{3.3}$$

where T, N, B_1, B_2 are mutually orthogonal vectors and κ_1, κ_2 and κ_3 are respectively the first, the second and the third curvature of the γ.

In view of the Frenet formulas given in (3.3) and equation (2.8), we obtain

$$\nabla_T T = \kappa_1 N,$$

$$\nabla_T \nabla_T T = \kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1,$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T &= (3\kappa_1\kappa_1')T + (\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2)N \\ &\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')B_1 + (\kappa_1\kappa_2\kappa_3)B_2, \end{aligned}$$

and

$$R(T, \nabla_T T)T = -\kappa_1 N,$$

where κ_1, κ_2 and κ_3 are the first, the second and the third curvature of the γ, respectively.

Considering Theorem 2.1 and equation (2.3), we get

$$\begin{aligned} \tau_{2,f} &= f \begin{bmatrix} (3\kappa_1\kappa_1')T + (\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2 + \kappa_1 N) \\ +(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')B_1 + (\kappa_1\kappa_2\kappa_3)B_2 \end{bmatrix} \\ &\quad +2f' \left[\kappa_1^2 T + \kappa_1' N + \kappa_1\kappa_2 B_1\right] + f'' \left[\kappa_1 N\right] \\ &= 0. \end{aligned}$$

Comparing the coefficients of above equation, we obtain that γ is an $f-$biharmonic curve if and only if

$$3\kappa_1\kappa_1' + 2\kappa_1^2 \frac{f'}{f} = 0, \tag{3.4}$$

$$\kappa_1'' + \kappa_1^3 + \kappa_1\kappa_2^2 + \kappa_1 + 2\kappa_1' \frac{f'}{f} + \kappa_1 \frac{f''}{f} = 0, \tag{3.5}$$

$$2\kappa_1'\kappa_2 + \kappa_1\kappa_2' + 2\kappa_1\kappa_2 \frac{f'}{f} = 0, \tag{3.6}$$

$$\kappa_1\kappa_2\kappa_3 = 0. \tag{3.7}$$

Let κ_1 be a non zero constant. Then from (3.4) we get f is constant. So γ is biharmonic. Let κ_2 be a non zero constant. From (3.4) and (3.6) one can easily see that f is constant and γ is biharmonic.

By using (3.4) - (3.7), if $\kappa_2 = 0$, then $f-$biharmonic curve equation reduces to

$$3\kappa_1\kappa_1' + 2\kappa_1^2\frac{f'}{f} = 0, \tag{3.8}$$

$$\kappa_1'' + \kappa_1^3 + \kappa_1 + 2\kappa_1'\frac{f'}{f} + \kappa_1\frac{f''}{f} = 0. \tag{3.9}$$

Integrating the equation (3.8) we get $f = t_1\kappa_1^{-\frac{3}{2}}$ and using this result in (3.9), we arrive at (i).

Otherwise, by use of (3.4) - (3.7), if $\kappa_1 \neq constant$ and $\kappa_2 \neq constant$ $f-$biharmonic curve the equation is equivalent to

$$f^2\kappa_1^3 = t_1^2, \tag{3.10}$$

$$(f\kappa_1)'' = -f\kappa_1(\kappa_1^2 + \kappa_2^2 + 1), \tag{3.11}$$

$$f^2\kappa_1^2\kappa_2 = t_2, \tag{3.12}$$

$$\kappa_3 = 0. \tag{3.13}$$

In view of (3.10), we find $f = t_1\kappa_1^{-\frac{3}{2}}$ and using this result in (3.11), we get $\frac{\kappa_2}{\kappa_1} = t_3$. Finally substituting these equation in (3.11), we arrive at (ii). $\square$

Proposition 3.1. *Let M be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold and $\gamma : I \to M$ be an $f-$biharmonic spacelike curve parametrized by arclength such that principal normal vector is timelike. If γ has constant geodesic curvature then γ is biharmonic.*

4. $f-$Biharmonic Curves on Lorentzian Sphere S_1^4

Suppose that M is a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Since M is locally isometric to a Lorentzian unit sphere S_1^4, we give some characterizations for $f-$biharmonic curves in S_1^4. The Lorentzian unit sphere of radius 1 can be seen as the hyperquadradic

$$S_1^4 = \{p \in \mathbb{R}_1^5 :< p, p >= 1\},$$

in a Minkowski space $\mathbb{R}_1^5$ with the metric

$$< \ , \ >: -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2.$$

Let $\gamma : I \to S_1^4$ be a curve parametrized by arclength. For an arbitrary vector field X along γ, we have

$$\nabla_T X = X' + < T, X > \gamma, \tag{4.1}$$

where ∇ is covariant derivative along γ in S_1^4.

Since S_1^4 is a Lorentzian space form of the scalar curvature 1, we have

$$R(X,Y)W =< Y, W > X - < X, W > Y,$$

for all vector fields X, Y, W in the tangent bundle of S_1^4, where R is the curvature tensor of S_1^4.

Now, we give the following:

Proposition 4.1. *Let $\gamma : I \to S_1^4$ be a non-geodesic $f-$biharmonic curve parametrized by arclength and $\{T, N, B_1, B_2\}$ be a Frenet frame along γ such that*

$$g(T,T) = g(B_1, B_1) = g(B_2, B_2) = 1, \;\; g(N,N) = -1.$$

Then, we have

$$\gamma^{(4)} - \left(\frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f}\right)\gamma'' - \left(\kappa_1^2 + \frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f} + 1\right)\gamma = 0. \tag{4.2}$$

Proof. Using (3.5) and taking the covariant derivative of the second equation in (3.3), we get

$$\begin{aligned} \nabla_T^2 N &= \nabla_T(\kappa_1 T + \kappa_2 B_1) \\ &= \kappa_1 \nabla_T T + \kappa_2 \nabla_T B_1 \\ &= (\kappa_1^2 + \kappa_2^2)N + \kappa_2\kappa_3 B_2. \end{aligned}$$

Using (3.5) in (4.3), we have

$$\nabla_T^2 N = -\left(\frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f} + 1\right)N. \tag{4.3}$$

On the other hand from (4.1), we arrive at

$$\begin{aligned} \nabla_T^2 N &= \nabla_T(N' + < T, N > \gamma) \\ &= N'' + < T, N' > \gamma \\ &= N'' + < T, \nabla_T N - < N, T > \gamma > \gamma \\ &= N'' + < T, \kappa_1 T + \kappa_2 B_1 > \gamma \\ &= N'' + \kappa_1 \gamma. \end{aligned}$$

From (4.3) and (4.4), we obtain

$$\left(\frac{\kappa_1''}{\kappa_1} + 2\frac{\kappa_1'}{\kappa_1}\frac{f'}{f} + \frac{f''}{f} + 1\right)N = N'' + \kappa_1\gamma. \tag{4.4}$$

Also in view of (4.1), we have

$$\nabla_T T = T' + < T, T > \gamma = \gamma'' + \gamma,$$

which yields

$$N = \frac{1}{\kappa_1}(\gamma'' + \gamma). \tag{4.5}$$

By use of (4.5) and (4.4), we obtain (4.2). □

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PSEUDO-PARALLEL KAEHLERIAN SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Abstract. Let $\tilde{M}^m(c)$ be a complex m-dimensional space form of holomorphic sectional curvature c and M^n be a complex n-dimensional Kaehlerian submanifold of $\tilde{M}^m(c)$. We prove that if M^n is pseudo-parallel and $Ln - \frac{1}{2}(n+2)c \geqslant 0$ then M^n is totally geodesic. Also, we study Kaehlerian submanifolds of complex space form with recurrent second fundamental form.

Keywords. Pseudo-parallel submanifolds; Kaehlerian submanifolds; recurrent second fundamental form.

1. Introduction

Among all submanifolds of an almost Hermitian manifold, there are two typical classes: one is the class of holomorphic submanifolds and the other is the class of totally real submanifolds. A submanifold M of an almost Hermitian manifold $\tilde{M}$ is called holomorphic (resp. totally real) if each tangent space of M is mapped into itself (resp. the normal space) by the almost complex structure of $\tilde{M}$. There are many results in the theory of holomorphic submanifolds.

The class of isometric immersions in a Riemannian manifold with parallel second fundamental form is very wide, as it is shown, for instance, in the classical Ferus paper [10]. Certain generalizations of these immersions have been studied, obtaining classification theorems in some cases.

Given an isometric immersion $f : M \longrightarrow \tilde{M}$, let h be the second fundamental form and $\overline{\nabla}$ the van der Waerden-Bortolotti connection of M. Then J. Deprez defined the immersion to be semi-parallel if

$$\bar{R}(X,Y) \cdot h = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]})h = 0, \tag{1.1}$$

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holds for any vector fields X, Y tangent to M. J. Deprez mainly paid attention to the case of semi-parallel immersions in real space forms (see [5] and [6]). Later, Ü. Lumiste showed that a semi-parallel submanifold is the second order envelope of the family of parallel submanifolds [13]. In the case of hypersurfaces in the sphere and the hyperbolic space, F. Dillen showed that they are flat surfaces, hypersurfaces with parallel Weingarten endomorphism or rotation hypersurfaces of certain helices [9].

In [8], the authors obtained some results in hypersurfaces in 4-dimensional space form $N^4(c)$ satisfying the curvature condition

$$\bar{R} \cdot h = LQ(g, h). \tag{1.2}$$

The submanifolds satisfying the condition (1.2) are called *pseudo-parallel* (see [1] and [2]).

In [1], Asperti et al. considered the isometric immersions $f : M \longrightarrow \tilde{M}^{n+d}(c)$ of n-dimensional Riemannian manifold into $(n+d)$-dimensional real space form $\tilde{M}^{n+d}(c)$ satisfying the curvature condition (1.2). They have shown that if f is pseudo-parallel with $H(p) = 0$ and $L_h(p) - c \geq 0$ then the point p is a geodesic point of M, i.e. the second fundamental form vanishes identically, where H is the mean curvature vector of M.

They also showed that a pseudo-parallel hypersurfaces of a space form is either quasi-umbilical or a cyclic of Dupin [2].

The study of complex hypersurfaces was initiated by Smyth [18]. He classified the complete Kaehler-Einstein manifolds which occur as hypersurfaces in complex space forms. The corresponding full local classification was given by Chern [4]. Similar classification under the weaker assumption of parallel Ricci tensor was obtained by Takahashi [19] and Nomizu and Smyth [16]. A classification of the complete Kaehler hypersurfaces of space forms which satisfy the condition $R \cdot R = 0$ and a partial classification (the case $c \neq 0$) of such hypersurfaces satisfying the condition $R \cdot S = 0$ were given by Ryan in [17]. He also classified the complex hypersurfaces of $\mathbb{C}^{n+1}$ having $R \cdot S = 0$ and constant scalar curvature.

In [7], Deprez et al. presented a new characterization of complex hyperspheres in complex projective spaces, of complex hypercylinders in complex Euclidean spaces and of complex hyperplanes in complex space forms in terms of the conditions on the tensors R, S, C and B, where B is the Bochner tensor which was introduced as a complex version of the Weyl conformal curvature tensor C of a Riemannian manifold [3]. In [23], Yaprak studied pseudosymmetry type curvature conditions on Kaehler hypersurfaces. The submanifolds in a complex space form $\widetilde{M}^m(c)$ $n \geqslant 2$,of constant holomorphic sectional curvature $4c$, parallel second fundamental form were classified by H.Naitoh in [15]. S.Maeda [14] studied semi-parallel real hypersurfaces in a complex space form $\widetilde{M}^m(c)$ for $c > 0$ and $n \geqslant 3$. In [12] Lobos and Ortega classify all connected Pseudo-parallel real hypersurfaces in a non-flat complex space form. Then, Yıldız et al. [22] studied C-totally real pseudo-parallel submanifolds in Sasakian space forms.

In the present study, we have generalized their results for the case of M^n, that is a Kaehlerian submanifold of complex space form $\tilde{M}^m(c)$ of holomorphic sectional curvature c. We will prove the following:

Theorem 1.1. *Let $\tilde{M}^m(c)$ be complex m-dimensional space form of constant holomorphic sectional curvature c and M^n be a complex n-dimensional Kaehlerian submanifold of $\tilde{M}(c)$. If M^n is pseudo-parallel and $Ln - \frac{1}{2}(n+2)c \geqslant 0$, then M^n is totally geodesic.*

Also, we study Kaehlerian submanifolds of complex space form with recurrent second fundamental form.

2. Basic Concepts

Let $\tilde{M}(c)$ be a non-flat complex space form endowed with the metric g of constant holomorphic sectional curvature c. We denote by ∇, R, S and τ the Levi-Civita connection, Riemann curvature tensor, the Ricci tensor and scalar curvature of (M, g), respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X, Y) = S(X, Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being Lie algebra of vector fields on M. We next define endomorphisms $R(X, Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.1}$$

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y, \tag{2.2}$$

respectively, where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0,2)$-tensor.

The *concircular curvature tensor* $\tilde{Z}$ in a Riemannian manifold (M^n, g) is defined by

$$\tilde{Z}(X,Y) = R(X,Y) - \frac{\tau}{n(n-1)}(X \wedge_g Y), \tag{2.3}$$

respectively, where τ is the scalar curvature of M^n.

Now, for a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field B on (M, g) we define the tensor $Q(B, T)$ by

$$\begin{aligned} Q(B,T)(X_1, ..., X_k; X, Y) &= -T((X \wedge_B Y)X_1, X_2, ..., X_k) \\ &\quad - ... - T(X_1, ..., X_{k-1}, (X \wedge_B Y)X_k), \end{aligned} \tag{2.4}$$

respectively. Putting into the above formula $T = h$ and $B = g$, we obtain the tensor $Q(g, h)$.

Let $f : M^n \longrightarrow \tilde{M}^m(c)$ be an isometric immersion of an complex n-dimensional (of real dimension $2n$) M into complex m-dimensional (of real dimension $2m$) space form $\tilde{M}^m(c)$. We denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M^n and $\tilde{M}^m(c)$, respectively. Then for vector fields X, Y which are tangent to M^n, the second fundamental form h is given by the formula $h(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$. Furthermore, for $\xi \in N(M^n)$, $A_\xi : TM \longrightarrow TM$ will denote the Weingarten operator in the ξ direction, $A_\xi X = \nabla^\perp_X \xi - \widetilde{\nabla}_X \xi$, where $\nabla^\perp$ denotes the normal connection of M. The second fundamental form h and A_ξ are related by $\widetilde{g}(h(X,Y),\xi) = g(A_\xi X, Y)$, where g is the induced metric of $\widetilde{g}$ for any vector fields X, Y tangent to M. The mean curvature vector H of M is defined to be

$$H = \frac{1}{n} Tr(h).$$

A submanifold M is said to be *minimal* if $H = 0$ identically.

The covariant derivative $\overline{\nabla} h$ of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^\perp_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.5}$$

where, $\overline{\nabla} h$ is a normal bundle valued tensor of type $(0,3)$ and is called the third fundamental form of M. The equation of Codazzi implies that $\overline{\nabla} h$ is symmetric hence

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) = (\overline{\nabla}_Z h)(X,Y). \tag{2.6}$$

Here, $\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M. If $\overline{\nabla} h = 0$, then f is called parallel, [10].

The second covariant derivative $\overline{\nabla}^2 h$ of h is defined by

$$\begin{aligned}(\overline{\nabla}^2 h)(Z,W,X,Y) &= (\overline{\nabla}_X \overline{\nabla}_Y h)(Z,W) \\ &= \nabla^\perp_X ((\overline{\nabla}_Y h)(Z,W)) - (\overline{\nabla}_Y h)(\nabla_X Z, W) \\ &\quad -(\overline{\nabla}_X h)(Z, \nabla_Y W) - (\overline{\nabla}_{\nabla_X Y} h)(Z,W).\end{aligned} \tag{2.7}$$

Then we have

$$\begin{aligned}(\overline{\nabla}_X \overline{\nabla}_Y h)(Z,W) - (\overline{\nabla}_Y \overline{\nabla}_X h)(Z,W) &= (\bar{R}(X,Y) \cdot h)(Z,W) \\ &= R^\perp(X,Y) h(Z,W) - h(R(X,Y)Z,W) \\ &\quad -h(Z, R(X,Y)W).\end{aligned} \tag{2.8}$$

where $\bar{R}$ is the curvature tensor belonging to the connection $\overline{\nabla}$.

3. Kaehlerian Submanifolds

Let $\tilde{M}$ be a Kahlerian manifold of complex dimension m (of real dimension $2m$) with almost complex structure J and with Kahlerian metric g. Let M be a complex n-dimensional analytic submanifold of $\tilde{M}$, that is, the immersion $f : M \longrightarrow \tilde{M}$ is holomorphic, i.e., $J \cdot f_* = f_* \cdot J$, where f_* is the differential of the immersion f and we denote by the same J the induced complex structure on M. Then the Riemannian metric g, which will be denoted by the same letter of $\tilde{M}$, induced on M is Hermitian. It is easy to see that the second fundamental form with this Hermitian metric g is the restriction of the second fundamental form of $\tilde{M}$ and hence is closed. This show that every complex analytic submanifold M a Kaehlerian manifold $\tilde{M}$ is also a Kaehlerian manifold with respect to the induced structure. We call such a submanifold M of a Kaehlerian manifold $\tilde{M}$ a Kaehlerian submanifold. In the other words, a Kaehlerian submanifold M of a Kaehlerian manifold $\tilde{M}$ is an invariant submanifold under the action of the complex structure J of $\tilde{M}$, i.e., $JT_x(M) \subset T_x(M)$ for every point x of M [21].

For each plane p in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined to be $K(p) = R(X,Y,X,Y) = g(R(X,Y)Y,X)$, where $\{X,Y\}$ is an orthonormal basis for p. If p is invariant by J, then $K(p)$ is called *holomorphic sectional curvature* by p. If $K(p)$ is a constant for all J-invariant planes p in $T_x(M)$ and for all points $x \in M$ is called a *space of constant holomorphic sectional curvature* or a *complex space form*. Sometimes, a complex space form is defined to be a simply connected complete Kaehlerian manifold of constant holomorphic sectional curvature defined by [21]

$$\tilde{R}(X,Y)Z = \frac{1}{4}c\{g(X,Z)Y - g(Y,Z)X + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\},$$

for any vector fields X, Y and Z on M. If this space is complete and simply connected, it is well known that it is isometric to

- a complex projective space $\mathbb{C}P^m(c)$, if $c > 0$;
- the complex Euclidean space $\mathbb{C}^m$, if $c = 0$;
- a complex hyperbolic space $\mathbb{C}H^m$, if $c < 0$.

The equations of Gauss and Ricci are

$$\begin{aligned} g(R(X,Y)Z,W) \quad &= \quad \frac{1}{4}c[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) \\ &\quad -g(JX,Z)g(JY,W) + 2g(X,JY)g(JZ,W)] \\ &\quad +g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)), \end{aligned} \tag{3.1}$$

and

$$g(R(X,Y)U,V) + g([A_V,A_U]X,Y) = \frac{1}{2}cg(X,JY)g(JU,V), \tag{3.2}$$

respectively. For an orthonormal frame field $\{e_1, e_2, ..., e_n\}$ of M, the Ricci tensor S is defined by

$$S(X,Y) = \sum_{k=1}^{n} g(R(e_k,X)Y,e_k). \tag{3.3}$$

Consequently, by the use of (3.1) the equation (3.3) turns into

$$S(X,Y) = \frac{1}{2}(n+1)cg(X,Y) - \sum_i g(h(X,e_i),h(Y,e_i)). \tag{3.4}$$

Lemma 3.1. *[21] The second fundamental form h of a Kaehlerian submanifold M satisfies*

$$h(JX,Y) = h(X,JY) = Jh(X,Y),$$

or equivalently

$$JA_V X = -A_V JX = A_{JV} X.$$

Proposition 3.1. *[21] Any Kaehlerian submanifold M is a minimal submanifold.*

Theorem 3.1. *[21] Let M^n be a Kaehlerian hypersurface of a complex space form $\tilde{M}^{n+1}(c)$. Then the following conditions are equivalent:*

(i) The Ricci tensor S of M^n is parallel;

(ii) The second fundamental form of M^n is parallel;

(iii) M is an Einstein manifold.

4. Proof of the Theorem 1.1

Let M^n be a complex n-dimensional (of real dimensional $2n$) Kaehlerian submanifold with complex structure J of a complex m-dimensional (of real dimensional $2m$) space form $\tilde{M}^m(c)$ of constant *holomorphic sectional curvature* c. Take an orthonormal basis $e_1, e_2, ..., e_{2n}$ in $T_X(M)$ such that $e_{n+t} = Je_t$ $(t = 1, ..., n)$ and an orthonormal basis $v_1, ..., v_{2p}$ for $T_X(M)^{\perp}$ such that $v_{p+s} = Jv_s$ $(s = 1, ..., p)$, where we have put $p = m - n$. Then for $1 \leq i, j \leq n$, $1 \leq \alpha \leq p$, the components of the second fundamental form h are given by

$$h^{\alpha}_{ij} = g(h(e_i, e_j), e_{\alpha}). \tag{4.1}$$

Similarly, the components of the first and the second covariant derivative of h are given by

$$h^{\alpha}_{ijk} = g((\overline{\nabla}_{e_k} h)(e_i, e_j), e_{\alpha}) = \overline{\nabla}_{e_k} h^{\alpha}_{ij}, \tag{4.2}$$

and

$$\begin{aligned} h^{\alpha}_{ijkl} &= g((\overline{\nabla}_{e_l}\overline{\nabla}_{e_k} h)(e_i, e_j), e_{\alpha}) \\ &= \overline{\nabla}_{e_l} h^{\alpha}_{ijk} \\ &= \overline{\nabla}_{e_l}\overline{\nabla}_{e_k} h^{\alpha}_{ij}, \end{aligned} \tag{4.3}$$

respectively.

If f is pseudo-parallel, then by definition, the condition

$$\bar{R}(e_l, e_k) \cdot h = L[(e_l \wedge_g e_k)]h \tag{4.4}$$

is fulfilled where

$$[(e_l \wedge_g e_k)h]\,(e_i, e_j) = -h((e_l \wedge_g e_k)e_i, e_j) - h(e_i, (e_l \wedge_g e_k)e_j) \tag{4.5}$$

for $1 \leq i, j, k, l \leq n$. Substituting (2.2) into (4.5), we get

$$\begin{aligned} [(e_l \wedge_g e_k)h](e_i, e_j) &= -g(e_k, e_i)h(e_l, e_i) + g(e_l, e_i)h(e_k, e_i) \\ &\quad -g(e_k, e_j)h(e_l, e_i) + g(e_l, e_j)h(e_k, e_i). \end{aligned} \tag{4.6}$$

By (2.8) we have

$$(\bar{R}(e_l, e_k) \cdot h)(e_i, e_j) = (\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}h)(e_i, e_j) - (\overline{\nabla}_{e_k}\overline{\nabla}_{e_l}h)(e_i, e_j). \tag{4.7}$$

Making use of (4.1), (4.3), (4.6) and (4.7), the pseudo-parallelity condition (4.4) turns into

$$h^\alpha_{ijkl} = h^\alpha_{ijlk} - L\{\delta_{ki}h^\alpha_{lj} - \delta_{li}h^\alpha_{kj} + \delta_{kj}h^\alpha_{il} - \delta_{lj}h^\alpha_{ki}\}, \tag{4.8}$$

where $g(e_i, e_j) = \delta_{ij}$ and $1 \leq i, j, k, l \leq n,\ 1 \leq \alpha \leq p$.

Recall that the Laplacian Δh^α_{ij} of h^α_{ij} is defined by

$$\Delta h^\alpha_{ij} = \sum_{i,j,k=1}^{n} h^\alpha_{ijkk}. \tag{4.9}$$

Then we obtain

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{p} h^\alpha_{ij}h^\alpha_{ijkk} + \left\|\overline{\nabla}h\right\|^2, \tag{4.10}$$

where

$$\|h\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{p}(h^\alpha_{ij})^2, \tag{4.11}$$

and

$$\left\|\overline{\nabla}h\right\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{p}(h^\alpha_{ijkk})^2, \tag{4.12}$$

are the square of the length of second and the third fundamental forms of M^n, respectively. In addition, making use of (4.1) and (4.3), we obtain

$$\begin{aligned} h^\alpha_{ij}h^\alpha_{ijkk} &= g(h(e_i, e_j), e_\alpha)g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j)g(h(e_i, e_j), e_\alpha), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), h(e_i, e_j)). \end{aligned} \tag{4.13}$$

Therefore, due to (4.13), the equation (4.10) becomes

$$\frac{1}{2}\Delta(\|h\|^2) = \sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i, e_j), h(e_i, e_j)) + \left\|\overline{\nabla}h\right\|^2. \tag{4.14}$$

Further, by the use of (4.4), (4.6) and (4.7), we get

$$
\begin{aligned}
g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}h)(e_i,e_j),h(e_i,e_j) &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_i}h)(e_k,e_j),h(e_i,e_j)) \\
&= g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_k}h)(e_j,e_k),h(e_i,e_j)) \\
&\quad -L\{g(e_i,e_j)g(h(e_k,e_k),h(e_i,e_j)) \\
&\quad -g(e_k,e_j)g(h(e_k,e_i),h(e_i,e_j)) \\
&\quad +g(e_k,e_i)g(h(e_j,e_k),h(e_i,e_j)) \\
&\quad -g(e_k,e_k)g(h(e_i,e_j),h(e_i,e_j))\}.
\end{aligned} \tag{4.15}
$$

Substituting (4.15) into (4.14), we have

$$
\begin{aligned}
\frac{1}{2}\Delta(\|h\|^2) &= \sum_{i,j,k=1}^{n}[g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}h)(e_k,e_k),h(e_i,e_j)) \\
&\quad -L\{g(e_i,e_j)g(h(e_k,e_k),h(e_i,e_j)) \\
&\quad -g(e_k,e_j)g(h(e_k,e_i),h(e_i,e_j)) \\
&\quad +g(e_k,e_i)g(h(e_j,e_k),h(e_i,e_j)) \\
&\quad -g(e_k,e_k)g(h(e_i,e_j),h(e_i,e_j))\}] + \left\|\overline{\nabla}h\right\|^2
\end{aligned} \tag{4.16}
$$

Furthermore, by definition

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)), \tag{4.17}$$

$$H^\alpha = \sum_{k=1}^{n} h_{kk}^\alpha,$$

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=1}^{p}(H^\alpha)^2,$$

and after some calculations, we get

$$
\begin{aligned}
\frac{1}{2}\Delta(\|h\|^2) &= \sum_{i,j=1}^{n}\sum_{\alpha=1}^{p} h_{ij}^\alpha(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H^\alpha) \\
&\quad -L\{n^2\|H\|^2 - n\|h\|^2\} + \left\|\overline{\nabla}h\right\|^2.
\end{aligned} \tag{4.18}
$$

Using Proposition 3.1, the equation (4.18) is reduced to

$$\frac{1}{2}\Delta(\|h\|^2) = Ln\|h\|^2 + \left\|\overline{\nabla}h\right\|^2. \tag{4.19}$$

Yano and Kon have shown in [21], that

$$\frac{1}{2}\Delta(\|h\|^2) = \left\|\overline{\nabla} h\right\|^2 - \sum_{\alpha,\beta=1}^{p}\{[Tr(A_\alpha \circ A_\beta)]^2 + \|[A_\alpha, A_\beta]\|^2 + \frac{1}{2}(n+2)c\,\|h\|^2 . \tag{4.20}$$

Hence comparing the equation (4.19) with (4.20), one can get

$$\begin{aligned} 0 &= (Ln - \frac{1}{2}(n+2)c)\,\|h\|^2 \\ &+ \sum_{\alpha,\beta=1}^{p}\left\{[Tr\,(A_\alpha \circ A_\beta)]^2 + \|[A_\alpha, A_\beta]\|^2\right\}. \end{aligned}$$

If $Ln - \frac{1}{2}(n+2)c \geq 0$ then $Tr\,(A_\alpha \circ A_\beta) = 0$. In particular, $\|A_\alpha\|^2 = Tr(A_\alpha \circ A_\alpha) = 0$, hence $h = 0$. This completes the proof of our Theorem.

Corollary 4.1. *Let $\tilde{M}^m(c)$ be complex m-dimensional space form of constant holomorphic sectional curvature c and M^n be a complex n-dimensional Kaehlerian submanifold of $\tilde{M}^m(c)$. If $\tilde{Z}(X,Y)\cdot h = 0$ and $\frac{\tau}{(n-1)} - \frac{1}{2}(n+2)c \geq 0$ then M is totally geodesic.*

We recall the following well-known:

Theorem 4.1. *([4], [16], [19]) Let M^n be a Kaehlerian hypersurface of a complex space form $\tilde{M}^{n+1}(c)$ with parallel Ricci tensor. If $c \leq 0$, then M^n is totally geodesic. If $c > 0$, then either M is totally geodesic, or an Einstein manifold $|A|^2 = nc$ and hence $\tau = n^2 c$.*

Using Theorem 3.1 and Theorem 4.1, we can easily obtain the following:

Corollary 4.2. *Let M^n be a Kaehlerian hypersurface of a complex space form $\tilde{M}^{n+1}(c)$ with parallel second fundamental form. If $c \leq 0$, then M^n is totally geodesic.*

Using Theorem 1.1, we get the following:

Corollary 4.3. *Let M^n be a complex n-dimensional Kaehlerian submanifold of $\tilde{M}(c)$ with semi-parallel. If $c \leq 0$, then M^n is totally geodesic.*

Remark 4.1. *(i) The main Theorem is generalization of Corollary* 4.2 *and Corollary* 4.3. *(ii) İf second fundamental form of M^n is parallel then it is semi-parallel. But the converse is not necessary to be parallel.*

5. Kaehlerian submanifolds of comlex space form with recurrent second fundamental form

In this section, we will consider the condition by which the second fundamental tensor A is recurrent, i.e., there exists a 1-form α such that $\bar{\nabla} A = \alpha \otimes A$. We may regard parallel condition as a special case. We know that the recurrent condition has a close relation to a holonomy group (cf,[11], [20]). Using definition of recurrent second fundamental form, we get

$$\bar{\nabla}_X \bar{\nabla}_Y A = (X\alpha(Y) + \alpha(X)\alpha(Y))A,$$

which implies that

$$\begin{aligned} \bar{R}(X,Y)\cdot A &= \bar{\nabla}_X\bar{\nabla}_Y A - \bar{\nabla}_Y\bar{\nabla}_X A - \bar{\nabla}_{[X,Y]}A \\ &= (X\alpha(Y) + \alpha(X)\alpha(Y))A - (Y\alpha(X) + \alpha(Y)\alpha(X))A \\ &\quad -\alpha([X,Y])A \\ &= (X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])A \\ &= 2d\alpha(X,Y)A. \end{aligned} \tag{5.1}$$

We now define a function on M^n by $f^2 = g(A,A)$, where the metric g is extended to the inner product between the tensor fields in the standard fashion [11]. Using the fact that $\nabla g = 0$ it follows from $f^2 = g(A,A)$ that

$$f(Y(f)) = f^2\alpha(Y). \tag{5.2}$$

So from (5.2), we have

$$Yf = f\alpha(Y) \neq 0. \tag{5.3}$$

Therefore we get

$$\{\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]}\}f = \{X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]\}f. \tag{5.4}$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M by our assumption, we obtain

$$d\alpha(X,Y) = 0, \tag{5.5}$$

that is, the 1-form α is closed. Hence, from (5.1) and (5.5), we get

$$\bar{R}(X,Y)\cdot A = 0. \tag{5.6}$$

It means that M is semi-parallel. So, by the use of Corollary 4.3, we can give the following:

Theorem 5.1. *Let $\tilde{M}^m(c)$ be complex m-dimensional space form of constant holomorphic sectional curvature c. If $c \leq 0$, there are no Kaehler submanifolds with non-trivial recurrent second fundamental form of $\tilde{M}^m(c)$.*

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CLASSIFICATION OF CONFORMAL SURFACES OF REVOLUTION IN HYPERBOLIC 3-SPACE

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Abstract. In this paper, we classify conformal surfaces of revolution in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ satisfying an equation in terms of the position vector field and the Laplace operators with respect to the first, the second and the third fundamental forms of the surface.

Keywords: hyperbolic 3-space; Laplace operators; fundamental forms of the surface.

1. Introduction

Surfaces of revolution is form of the most easily recognized class of surfaces. The use of surfaces of revolution is essential in many fields such as physics and engineering. Surfaces of revolution have been well known since ancient times as well as common objects in geometric modelling which can be found everywhere in nature, human artefacts, technical practice and also in mathematics. Furthermore, many objects from everyday life such as cans, table glasses and furniture legs are surfaces of revolution. The process of lathing wood produces surfaces of revolution by its very nature [1, 19].

The notion of finite type immersion of submanifolds of a Euclidean space has been used in classifying and characterizing the well known Riemannian submanifolds. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. A Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ . Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space. The theory of submanifolds of finite type has been studied by many geometers [7, 11].

In $\mathbb{H}^3(-c^2)$, surfaces of constant mean curvature $\mathbf{H} = c$ are particularly interesting, because they exhibit many geometric properties in common with minimal

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surfaces in $\mathbb{E}^3$. This is not a coincidence. There is a one-to-one correspondence, so-called Lawson correspondence, between surfaces of constant mean curvature $\mathbf{H}_h$ in $\mathbb{H}^3(-c^2)$ and surfaces of constant mean curvature $\mathbf{H}_e = \sqrt{\mathbf{H}_h^2 - c^2}$. Those corresponding constant mean curvature surfaces satisfy the same Gauss-Codazzi equations, so they share many geometric properties in common. Lee and Zarske constructed surfaces of revolution with constant mean curvature $\mathbf{H} = c$ and minimal surfaces of revolution in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant curvature $-c^2$. In addition, they have showed that, the limit of the surfaces of revolution with $\mathbf{H} = c$ in $\mathbb{H}^3(-c^2)$ is a catenoid, the minimal surface of revolution in Euclidean 3-space as c approaches 0 15. Lee and Martin studied spacelike and timelike surfaces of constant mean curvature in de sitter 3-space [14, 16]. Kaimakamis, Papantoniou and Petoumenos studied Lorentz invariant spacelike surfaces of constant mean curvature in anti de sitter 3-space [12].

We know that, $\mathbf{x}$ is harmonic if $\Delta\mathbf{x} = 0$ in Euclidean $3-$space. However, this is no longer true in $\mathbb{H}^3(-c^2)$ because the Laplacian equation $\Delta\mathbf{x} = 0$ is not the harmonic map equation in $\mathbb{H}^3(-c^2)$ [15].

Let $\mathbf{x} : \mathbf{M} \to \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$ induced from that of $\mathbb{E}^m$, respectively [6]. Takahashi proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta x = \lambda x$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$ are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ [18].

As an extension of Takahashi theorem, Garay studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta x = Ax, \tag{1.1}$$

where $A \in Mat\ (m, \mathbb{R})$ is an $m \times m$- diagonal matrix, and proved that such hypersurfaces are minimal ($\mathbf{H} = 0$) in $\mathbb{E}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders [10].

Related to this, Dillan, Pas and Vertraelen investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$\Delta x = Ax + B, \tag{1.2}$$

where $A \in Mat\ (3, \mathbb{R})$ is a 3×3-real matrix and $B \in \mathbb{R}^3$ [8]. In other words, each coordinate function is of 1-type in the sense of Chen [7]. For the Lorentzian version of surfaces satisfying (1.2), Alias, Ferrandez and Lucas proved that the only such surfaces are minimal surfaces and open pieces of Lorentz circular cylinders, hyperbolic cylinders, Lorentz hyperbolic cylinders, hyperbolic spaces or pseudo-spheres [2].

The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, then the Gauss map can be identified with the unit normal vector field to it. Dillen and Vertraelen studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \tag{1.3}$$

where $A \in Mat\,(3, \mathbb{R})$ [9]. Baikoussis and Vertraelen studied the helicoidal surfaces in $\mathbb{E}^3$ [3]. Choi completely classified the surfaces of revolution satisfying the condition (1.3) in the three dimensional Minkowski space $\mathbb{E}^3_1$ [5]. Bekkar, Zoubir and Senoussi classified surfaces of revolution satisfying (1.1) in the three dimensional Minkowski space [4, 17]. Kaimakamis, Papantpniou and Peteoumenos classified surfaces of revolution satisfying

$$\Delta^{III} r = Ar,$$

in the three dimensional Lorentz-Minkowski space [12]. Choi, Kim and Yoon investigated the surfaces of revolution satisfying an equation in terms of the position vector field and the 2nd-Laplacian in Minkowski 3-space [6].

The main purpose of this paper is complete the classification of conformal surfaces of revolution in $\mathbb{H}^3(-c^2)$ in terms of the position vector field and the Laplacian operators.

2. Preliminaries

Let $\mathbb{R}^{3+1}$ denote the Minkowski spacetime with rectangular coordinates x_0, x_1, x_2, x_3 and the Lorentzian metric

$$ds^2 = -\left(dx_0\right)^2 + \left(dx_1\right)^2 + \left(dx_2\right)^2 + \left(dx_3\right)^2 .$$

Hyperbolic 3-space is the hyperquadric:

$$\mathbb{H}^3(-c^2) = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^{3+1} \,\middle|\, -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -\frac{1}{c^2} \right\}$$

which has the constant sectional curvature -c^2. This is a hyperboloid of two sheets in spacetime so it is called the hyperboloid model of hyperbolic 3-space. Consider the chart

$$U = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{H}^3(-c^2) \,|\quad x_0 + x_1 > 0 \right\}$$

and define

$$t = -\frac{1}{c} \log c\left(x_0 + x_1\right),$$

$$x = \frac{x_2}{c\left(x_0 + x_1\right)},$$

$$y = \frac{x_3}{c\,(x_0 + x_1)}.$$

Then

$$ds^2 = (dt)^2 + e^{-2ct}\left\{(dx)^2 + (dy)^2\right\}.$$

$\mathbb{R}^3$ with coordinates t, x, y and the metric

$$g_c = (dt)^2 + e^{-2ct}\left\{(dx)^2 + (dy)^2\right\}$$

is called the flat model of hyperbolic 3-space . We will still denote it by $\mathbb{H}^3(-c^2)$.The flat chart model is a local chart of hyperbolic 3-space, so it is not regarded as a standard model of hyperbolic 3-space. As $c \to 0$, $\mathbb{H}^3(-c^2)$ flattens out to Euclidean 3-space $\mathbb{E}^3$ [15].

Let $\mathbb{R}^3$ be equipped with the metric

$$ds^2 = (dt)^2 + e^{-2ct}\left\{(dx)^2 + (dy)^2\right\}. \tag{2.1}$$

The space $(\mathbb{R}^3, g)$ has constant curvature $-c^2$. It is denoted by $\mathbb{H}^3(-c^2)$ and is called the pseudospherical model of hyperbolic 3-space. From the metric (2.1), one can easily see that $\mathbb{H}^3(-c^2)$ flattens out to $\mathbb{E}^3$ Euclidean 3-space as $c \to 0$ [15].

Let $\mathbf{M}$ be a domain and $\mathbf{x} : \mathbf{M} \to \mathbb{H}^3(-c^2)$ a parametric surface. The metric (2.1) induces an inner product on each tangent space $T_p\ \mathbb{H}^3(-c^2)$. This inner product can be used to define conformal surfaces in $\mathbb{H}^3(-c^2)$. $\mathbf{x} : \mathbf{M} \to \mathbb{H}^3(-c^2)$ is said to be conformal if

$$\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \quad |\mathbf{x}_u| = |\mathbf{x}_v| = e^{\frac{w}{2,}}, \tag{2.2}$$

where (u, v) is a local coordinate system in $\mathbf{M}$ and $w : \mathbf{M} \to \mathbb{R}$ is a real-valued function in $\mathbf{M}$. The induced metric on the conformal parametric surface is given by

$$ds^2_{\mathbf{x}} = e^{w}\left((du)^2 + (dv)^2\right).$$

In order to calculate the mean curvature of $\mathbf{x}$, we need to find a unit normal vector field $\mathbf{G}$ of $\mathbf{x}$. For that, we need something like cross product. $\mathbb{H}^3(-c^2)$ is not a vector space but we can define an analogue of cross product locally on each tangent space $T_p\ \mathbb{H}^3(-c^2)$. Let

$$\begin{aligned}
\mathbf{v} &= v_1\left(\frac{\partial}{\partial t}\right)_p + v_2\left(\frac{\partial}{\partial x}\right)_p + v_3\left(\frac{\partial}{\partial y}\right)_p, \\
\mathbf{w} &= w_1\left(\frac{\partial}{\partial t}\right)_p + w_2\left(\frac{\partial}{\partial x}\right)_p + w_3\left(\frac{\partial}{\partial y}\right)_p,
\end{aligned}$$

$x, y \in T_p\ \mathbb{H}^3(-c^2)$, where $\left\{\left(\frac{\partial}{\partial t}\right)_p, \left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p\right\}$ denote the canonical basis for T_p $\mathbb{H}^3(-c^2)$. The cross product is defined by

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\left(\frac{\partial}{\partial t}\right)_p$$

$$+e^{2ct}\left(v_3w_1 - v_1w_3\right)\left(\frac{\partial}{\partial x}\right)_p$$

$$+e^{2ct}\left(v_1w_2 - v_2w_1\right)\left(\frac{\partial}{\partial y}\right)_p,$$

where $p = (t; x; y) \in \mathbb{H}^3(-c^2)$. Then by a direct calculation we obtain

$$g_{11} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle\,,\ g_{22} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle\,,\ g_{12} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle\,. \tag{2.3}$$

Let $\mathbf{x} : \mathbf{M} \to \mathbb{H}^3(-c^2)$ be a parametric surface. Then on each tangent plane T_p $\mathbf{x}(\mathbf{M})$, we have

$$\left\|\mathbf{x}_u \times \mathbf{x}_v\right\|^2 = e^{4ct(u,v)}\left(g_{11}g_{22} - g_{12}^2\right), \tag{2.4}$$

where $p = (t(u; v), x(u; v), y(u; v)) \in \mathbb{H}^3(-c^2)$. If $c \to 0$, (2.3) becomes the familiar formula

$$\left\|\mathbf{x}_u \times \mathbf{x}_v\right\|^2 = g_{11}g_{22} - g_{12}^2$$

from the Euclidean case. In this case, the Gaussian curvature and the mean curvature of a parametric surface $\mathbf{x}(u, v)$ may be calculated by

$$\begin{cases} \mathbf{K} = \widetilde{\mathbf{K}} + \epsilon\frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2}, \\ \mathbf{H} = \frac{g_{22}h_{11}+g_{11}h_{22}-2g_{12}h_{12}}{2\left(g_{11}g_{22}-g_{12}^2\right)}, \end{cases} \tag{2.5}$$

where $\widetilde{\mathbf{K}}$ is the sectional curvature and $\mathbf{G}$ is unit normal vector field of $\mathbf{M}$, respectively. So the coefficient of second fundamental forms are given by

$$h_{11} = \langle \mathbf{x}_{uu}, \mathbf{G} \rangle\,,\ h_{22} = \langle \mathbf{x}_{vv}, \mathbf{G} \rangle\,,\ h_{12} = \langle \mathbf{x}_{uv}, \mathbf{G} \rangle\,.$$

Let $\mathbf{x} : \mathbf{M} \to \mathbb{H}^3(-c^2)$ be a conformal surface satisfying (2.2). The mean curvature $\mathbf{H}$ of $\mathbf{x}$ is given by

$$\mathbf{H} = \frac{1}{2}e^{-w}\left\langle \Delta\mathbf{x}, \mathbf{G} \right\rangle. \tag{2.6}$$

One can easily see that the the formulas (2.4) and (2.5) coincide for conformal surfaces in $\mathbb{H}^3(-c^2)$ [15].

Rotations about the $t-$axis are the only type of Euclidean rotations that can be considered in $\mathbb{H}^3(-c^2)$. Consider a profile curve $\alpha(u) = (u, f(u), 0)$ in the $tx-$plane. Denote $\mathbf{x}(u, v)$ as the rotation of $\alpha(u)$ about the $t-$axis through an angle v. Then,

$$\mathbf{x}(u, v) = (u, f(u)\cos v, f(u)\sin v)\,. \tag{2.7}$$

[15]. It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$, $\Delta^{\mathbf{III}}$ of the first, the second and the third fundamental forms on $\mathbf{M}$ are defined by [4, 12, 17]

$$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\left[\frac{\partial}{\partial u}\left(\frac{g_{22}\mathbf{x}_u - g_{12}\mathbf{x}_v}{\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\right) - \frac{\partial}{\partial v}\left(\frac{g_{12}\mathbf{x}_u - g_{11}\mathbf{x}_v}{\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\right)\right], \tag{2.8}$$

$$\text{(2.9)}\quad \Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{\left|h_{11}h_{22}-h_{12}^2\right|}}\left[\frac{\partial}{\partial u}\left(\frac{h_{22}\mathbf{x}_u - h_{12}\mathbf{x}_v}{\sqrt{\left|h_{11}h_{22}-h_{12}^2\right|}}\right) - \frac{\partial}{\partial v}\left(\frac{h_{12}\mathbf{x}_u - h_{11}\mathbf{x}_v}{\sqrt{\left|h_{11}h_{22}-h_{12}^2\right|}}\right)\right].$$

and

$$\text{(2.10)}\quad \Delta^{\mathbf{III}}\mathbf{x} = -\frac{1}{\left(h_{11}h_{22}-h_{12}^2\right)\sqrt{g_{11}g_{22}-g_{12}^2}}\left[\begin{array}{c} \frac{\partial}{\partial u}\left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{\left(h_{11}h_{22}-h_{12}^2\right)\sqrt{g_{11}g_{22}-g_{12}^2}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{\left(h_{11}h_{22}-h_{12}^2\right)\sqrt{g_{11}g_{22}-g_{12}^2}}\right)\end{array}\right],$$

where

$$\begin{aligned} X &= g_{11}h_{12}^2 - 2g_{12}h_{11}h_{12} + g_{22}h_{11}^2, \\ Y &= g_{11}h_{12}h_{22} - g_{12}h_{11}h_{22} + g_{22}h_{11}h_{12} - g_{12}h_{12}^2, \\ Z &= g_{22}h_{12}^2 - 2g_{12}h_{22}h_{12} + g_{11}h_{22}^2. \end{aligned}$$

3. Conformal Surfaces of Revolution Satisfying $\Delta^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x}$

In this section, we will classify conformal surfaces of revolution $\mathbb{H}^3(-c^2)$ satisfying the equation

$$\text{(3.1)}\quad \Delta^{\mathbf{I}}\mathbf{x} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$ and

$$\Delta^{\mathbf{I}}\mathbf{x}_i = \left(\Delta^{\mathbf{I}}\mathbf{x}_1, \Delta^{\mathbf{I}}\mathbf{x}_2, \Delta^{\mathbf{I}}\mathbf{x}_3\right),$$

where

$$\text{(3.2)}\quad \mathbf{x}_1 = u,\ \mathbf{x}_2 = f(u)\cos v,\ \mathbf{x}_3 = f(u)\sin v.$$

The coefficients of the first fundamental form are given by

$$\text{(3.3)}\quad \begin{cases} g_{11} = e^{-2cu}\left(e^{2cu} + f'^2(u)\right), \\ g_{22} = e^{-2cu}f^2(u), \\ g_{12} = 0. \end{cases}$$

If we require $\mathbf{x}(u, v)$ to be conformal, then

$$\text{(3.4)}\quad e^{2cu} + f'^2(u) = f^2(u).$$

The coefficients of the second fundamental form are given by

$$\text{(3.5)}\quad \begin{cases} h_{11} = -\frac{f(u)f''(u)}{\sqrt{f^2(u)\left(e^{2cu}+f'^2(u)\right)}}, \\ h_{22} = \frac{f^2(u)}{\sqrt{f^2(u)\left(e^{2cu}+f'^2(u)\right)}}, \\ h_{12} = 0. \end{cases}$$

So the Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are calculated by

$$\mathbf{K} = \frac{-c^2 f(u)\left(e^{2cu} + f'^2(u)\right)^2 - e^{4cu} f''(u)}{f(u)\left(e^{2cu} + f'^2(u)\right)^2}$$

and

$$\mathbf{H} = \frac{-f(u)f''(u) + e^{2cu} + f'^2(u)}{2e^{-2cu}\left(e^{2cu} + f'^2(u)\right)\sqrt{f^2(u)\left(e^{2cu} + f'^2(u)\right)}},$$

respectively. With the conformality condition (3.4), $\mathbf{H}$ is reduced to

$$\mathbf{H} = \frac{-f''(u) + f(u)}{2e^{-2cu} f^3(u)}. \tag{3.6}$$

Let $\mathbf{H} = c$. Then (3.6) can be written as

$$f''(u) - f(u) + 2ce^{-2cu} f^3(u) = 0. \tag{3.7}$$

The differential equation (3.7) cannot be solved analytically [15]. If $c = 0$, then (3.7) becomes

$$f''(u) - f(u) = 0.$$

Hence we can see that if $\mathbf{H} = 0$ then $c = 0$ .Thus we have:

Proposition 3.1. *Let* $\mathbf{M}$ *be surfaces of revolution given by (2.6) in* $\mathbb{H}^3(-c^2)$*. If* $c \neq 0$*, then there are no minimal conformal surfaces of revolution in* $\mathbb{H}^3(-c^2)$*. If* $c = 0$*, then there are minimal conformal surfaces of revolution in* $\mathbb{E}^3$ *if and only if* $f'' - f = 0$ *which has the general solution* $f(u) = c_1 e^u + c_2 e^{-u}$ *for some constant* c_1, c_2 *[15].*

With the conformality condition (3.4), $\mathbf{K}$ is reduced to

$$\mathbf{K} = \frac{-c^2 f^5(u) - e^{4cu} f''(u)}{f^5(u)}. \tag{3.8}$$

Let $\mathbf{K} = c$. Then (3.8) can be written as

$$cf^5(u) + c^2 f^5(u) + e^{4cu} f''(u) = 0. \tag{3.9}$$

The differential equation (3.9) cannot be solved analytically. Hence we see that if $\mathbf{K} = 0$ then $c = 0$ and $f''(u) = 0$. Thus we have:

Proposition 3.2. *Let* $\mathbf{M}$ *be surfaces of revolution given by (2.6) in* $\mathbb{H}^3(-c^2)$*. If* $c \neq 0$*, then there are no flat conformal surfaces of revolution in* $\mathbb{H}^3(-c^2)$*. If* $c = 0$ *then there are flat conformal surfaces of revolution in* $\mathbb{E}^3$ *if and only if* $f'' = 0$ *which has the general solution* $f(u) = c_1 u + c_2$ *for some constant* c_1, c_2 *[15].*

Assume that $\mathbf{K}=\mathbf{K}_0\in\mathbb{R}\backslash\{0\}$. By (3.8), we get

$$f^5(u)\mathbf{K}_0+c^2f^5(u)+e^{4cu}f''(u)=0, \tag{3.10}$$

where $f(u)\neq 0$. The differential equation (3.10) cannot be solved analytically for $c\neq 0$. If $c=0$, then the solution of (3.10) given by

$$f(u)=\pm\frac{3^{\frac{1}{6}}\sqrt{c_1^{\frac{1}{3}}\left(-1+JacobiCN\left[2.3^{\frac{1}{12}}\sqrt{c_1^{\frac{2}{3}}(u+c_2)^2(-\mathbf{K}_0)^{\frac{1}{3}}},\frac{1}{4}\left(2+\sqrt{3}\right)\right]\right)}}{\sqrt{1+\sqrt{3}}\sqrt{-2+\sqrt{3}-JacobiCN\left[2.3^{\frac{1}{12}}\sqrt{c_1^{\frac{2}{3}}\left(u+c_2\right)^2\left(-\mathbf{K}_0\right)^{\frac{1}{3}}},\frac{1}{4}\left(2+\sqrt{3}\right)\right]}}, \tag{3.11}$$

where $c_1,c_2\in\mathbb{R}$.

Theorem 3.1. *There is no conformal surface of revolution which has constant the Gaussian curvature in $\mathbb{H}^3\left(-c^2\right)$. The Gaussian curvature of conformal surface of revolution is constant, $\mathbf{K}=\mathbf{K}_0$, in $\mathbb{E}^3$ if and only if the function $f(u)$ is (3.11).*

Similar calculations are also used for the mean curvature, what we get there is not conformal on the surfaces of revolution which has the real mean curvature in $\mathbb{E}^3$. By straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (3.1), (3.2), (3.3) and (2.7) turns out to be

$$\Delta^{\mathbf{I}}\mathbf{x}_i=\begin{pmatrix}\frac{e^{2cu}\left(-f'\left(e^{2cu}+f'^2\right)+f\left(ce^{2cu}+f'f''\right)\right)}{f\left(e^{2cu}+f'^2\right)^2}, \\ \frac{e^{4cu}\cos v\left(e^{2cu}+f'^2+f\left(cf'-f''\right)\right)}{f\left(e^{2cu}+f'^2\right)^2}, \\ \frac{e^{4cu}\sin v\left(e^{2cu}+f'^2+f\left(cf'-f''\right)\right)}{f\left(e^{2cu}+f'^2\right)^2}\end{pmatrix}. \tag{3.12}$$

With the conformality condition (3.4), the equation (3.12) is reduced to

$$\Delta^{\mathbf{I}}\mathbf{x}_i=\begin{pmatrix}\frac{e^{2cu}\left(-f'f^2+f\left(ce^{2cu}+f'f''\right)\right)}{f^5}, \\ \frac{e^{4cu}\cos v\left(f^2+f\left(cf'-f''\right)\right)}{f^5}, \\ \frac{e^{4cu}\sin v\left(f^2+f\left(cf'-f''\right)\right)}{f^5}\end{pmatrix}.$$

Suppose that $\mathbf{M}$ satisfies (3.1). Then from (3.1) , we have

$$\begin{cases} a_{11}u+a_{12}f(u)\cos v+a_{13}f(u)\sin v=\frac{e^{2cu}\left(-f'f^2+f\left(ce^{2cu}+f'f''\right)\right)}{f^5}, \\ a_{21}u+a_{22}f(u)\cos v+a_{23}f(u)\sin v=\frac{e^{4cu}\cos v\left(f^2+f\left(cf'-f''\right)\right)}{f^5}, \\ a_{31}u+a_{32}f(u)\cos v+a_{32}f(u)\sin v=\frac{e^{4cu}\sin v\left(f^2+f\left(cf'-f''\right)\right)}{f^5}. \end{cases} \tag{3.13}$$

Since the functions $\cos v, \sin v$ and the constant function are linearly independent, by (3.13) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = \lambda$, $a_{22} = a_{33} = \mu$. Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

The equation (3.13) is rewritten as the following:

$$\left\{ \begin{array}{r} \lambda u = \frac{e^{2cu}\left(-f'f+ce^{2cu}+f'f''\right)}{f^4}, \\ \mu f(u) = \frac{e^{4cu}\left(f+cf'-f''\right)}{f^4}. \end{array} \right. \tag{3.14}$$

This means that $\mathbf{M}$ is at most of 2-type. From (3.14), we have

$$\left\{ \begin{array}{r} \frac{e^{4cu}}{f^4} = \frac{\lambda u}{c} - \frac{e^{2cu}f'\left(f''-f\right)}{cf^4}, \\ \frac{e^{4cu}}{f^4} = \frac{\mu f}{f+cf'-f''}. \end{array} \right. \tag{3.15}$$

Combining the first and the second equation of (3.15), we obtain

$$\frac{\lambda u}{c} - \frac{e^{2cu}f'\left(f''-f\right)}{cf^4} = \frac{\mu f}{f+cf'-f''}. \tag{3.16}$$

The equation (3.16) is reduced to

$$e^{2cu}f'\left(f-f''\right)\left(\left(-1+c\right)f-f''\right)+f^4\left(\left(\left(-1+c\right)\lambda u+c\mu\right)f-\lambda uf''\right)=0. \tag{3.17}$$

In the cases $\{c \neq 0, \lambda \neq 0, \mu = 0\}$, $\{c \neq 0, \lambda = 0, \mu \neq 0\}$, $\{c \neq 0, \lambda \neq 0, \mu \neq 0\}$, the second order nonlinear differential equation (3.17) cannot be solved analytically. For the case $\{c \neq 0, \lambda = 0, \mu = 0\}$, (3.17) can be written as

$$e^{2cu}f'\left(f-f''\right)\left(\left(1-c\right)f'-f''\right)=0. \tag{3.18}$$

The general solutions of (3.18) are given by

$$\left\{ \begin{array}{l} f(u) = c_1, \\ f(u) = c_1e^{u} + c_2e^{-u}, \\ f(u) = c_1e^{u\sqrt{c-1}} + c_2e^{-u\sqrt{c-1}}, \end{array} \right. \tag{3.19}$$

where $c_1, c_2 \in \mathbb{R}$ and $c - 1 > 0$. Substituting the solutions (3.19) into (3.14), respectively. They don' t satisfy these equations. Thus, we can give the following theorem:

Theorem 3.2. *Let* $\mathbf{M}$ *be conformal surfaces of revolution given by (2.6) and* $c \neq 0$ *in* $\mathbb{H}^3(-c^2)$. *Then there are no harmonic and non- harmonic conformal surfaces of revolution satisfying the conditions* $\Delta^{\mathbf{I}}\mathbf{x} = 0$ *and* $\Delta^{\mathbf{I}}\mathbf{x} = A\mathbf{x}$,*respectively, where* $A{\in}Mat\ (3, \mathbb{R})$.

In the cases $\{c=0,\lambda\neq 0,\mu\neq 0\}, \{c=0,\lambda\neq 0,\mu=0\}$, we cannot obtain any conformal surfaces of revolution in $\mathbb{E}^3$. In the cases $\{c=0,\lambda=0,\mu\neq 0\}$, $\{c=0,\lambda=0,\mu=0\}$, the general solutions of (3.18) are given by

$$f(u)=c_1, f(u)=c_1\cos u+c_2\sin u,$$

and

$$f(u)=c_1e^u+c_2e^{-u}, \tag{3.20}$$

respectively. The solution $f(u)=c_1$ satisfies (3.14) for $\mu=\frac{1}{c_1^4}$. Then, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u,v)=(u,c_1\cos v,c_1\sin v)\,. \tag{3.21}$$

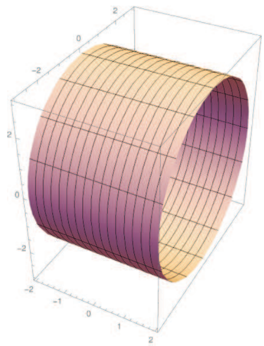

FIG. 3.1:

The solution $f(u)=c_1\cos u+c_2\sin u$ does not satisfies (3.14). But the solution (3.20) satisfies (3.14). Then, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u,v)=\left(u,\left(c_1e^u+c_2e^{-u}\right)\cos v,\left(c_1e^u+c_2e^{-u}\right)\sin v\right). \tag{3.22}$$

Thus we can give the following theorem:

Theorem 3.3. *Let $\mathbf{M}$ be conformal surfaces of revolution given by (2.6) and $c=0$ in $\mathbb{E}^3$. If $\mathbf{M}$ is harmonic or non-harmonic conformal surfaces of revolution, then it is an open part of the surfaces (3.21) or (3.22), respectively.*

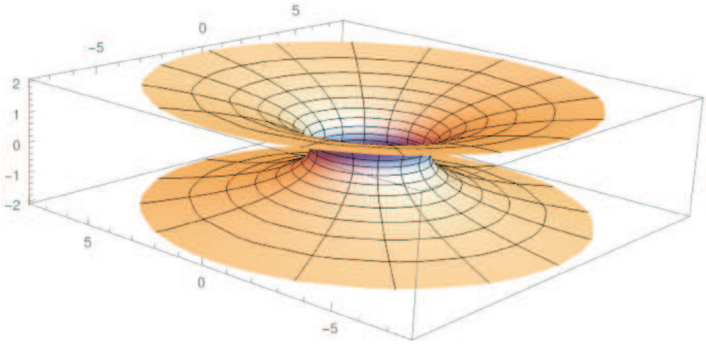

FIG. 3.2:

4. Conformal Surfaces of Revolution Satisfying $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}$

In this section, we classify conformal surfaces of revolution with non-degenerate second fundamental form in $\mathbb{H}^3(-c^2)$ satisfying the equation

$$\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$ and

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \left(\Delta^{\mathbf{II}}\mathbf{x}_1, \Delta^{\mathbf{II}}\mathbf{x}_2, \Delta^{\mathbf{II}}\mathbf{x}_3\right). \tag{4.2}$$

By straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (3.2), (3.5), (4.2) and (2.8) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \begin{pmatrix} \dfrac{\sqrt{f^2\left(e^{2cu}+f'^2\right)}\left(ff'''-f'f''\right)}{2f^2f''^2}, \\ \dfrac{\cos v\sqrt{f^2\left(e^{2cu}+f'^2\right)}\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f^2f''^2}, \\ \dfrac{\sin v\sqrt{f^2\left(e^{2cu}+f'^2\right)}\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f^2f''^2} \end{pmatrix}. \tag{4.3}$$

With the conformality condition (3.4), the equation (4.3) is reduced to

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \begin{pmatrix} \dfrac{ff'''-f'f''}{2f''^2}, \\ \dfrac{\cos v\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2}, \\ \dfrac{\sin v\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2} \end{pmatrix}.$$

Suppose that $\mathbf{M}$ satisfies (4.1). Then from (4.1) and (4.2), we have

$$(4.4)\quad \begin{cases} a_{11}u + a_{12}f(u)\cos v + a_{13}f(u)\sin v = \frac{ff'''-f'f''}{2f''^2}, \\ a_{21}u + a_{22}f(u)\cos v + a_{23}f(u)\sin v = \frac{\cos v\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2}, \\ a_{31}u + a_{32}f(u)\cos v + a_{32}f(u)\sin v = \frac{\sin v\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2}. \end{cases}$$

Since the functions $\cos v, \sin v$ and the constant function are linearly independent, by (4.4) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = \lambda$, $a_{22} = a_{33} = \mu$. Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

and the equation (4.4) is rewritten as follows:

$$(4.5)\quad \begin{cases} \lambda u = \frac{ff'''-f'f''}{2f''^2}, \\ \mu f(u)\cos v = \frac{\cos v\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2}, \\ \mu f(u)\sin v = \frac{\sin v\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2}. \end{cases}$$

From (4.5), we obtain

$$(4.6)\quad \begin{cases} \lambda u = \frac{ff'''-f'f''}{2f''^2}, \\ \mu f(u) = \frac{\left(ff'f'''-f'^2f''-2ff''^2\right)}{2f''^2}. \end{cases}$$

This means that $\mathbf{M}$ is at most of 2-type. From (4.6), we have

$$(4.7)\quad \begin{cases} \frac{ff'''-f'f''}{2f''^2} = \lambda u, \\ \frac{ff'''-f'f''}{2f''^2} = \frac{f(\mu-1)}{f'}. \end{cases}$$

Combining the first and the second equation of (4.7), we obtain

$$(4.8)\quad \lambda u f'(u) + f(u)(1-\mu) = 0.$$

If we solve the ordinary differential equation, we get

$$(4.9)\quad f(u) = c_1 u^{\frac{\mu-1}{\lambda}},$$

where $\lambda \neq 0$, $\mu \neq 0$, $c_1 \in \mathbb{R}$. If we apply the solution (4.9) into the first and the second line of the equation (4.6), we can easily see that it does not satisfies these equations. If we choose

$$\lambda = \frac{(\mu-1)^2}{\mu-2}, \mu \neq 1, \mu \neq 2,$$

then the equation (4.8) is reduced to

$$\left(\frac{(\mu-1)^2}{\mu-2}\right) uf'(u) + f(u)(1-\mu) = 0. \tag{4.10}$$

Its general solution is given by

$$f(u) = c_1 u^{\frac{\mu-2}{\mu-1}}. \tag{4.11}$$

The solution (4.11) provides the system (4.6). Thus the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \frac{(\mu-1)^2}{\mu-2} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Then, the parametrization of $\mathbf{M}$ is given by

$$\mathbf{x}(u,v) = \left(u, c_1 u^{\frac{\mu-2}{\mu-1}} \cos v, c_1 u^{\frac{\mu-2}{\mu-1}} \sin v\right). \tag{4.12}$$

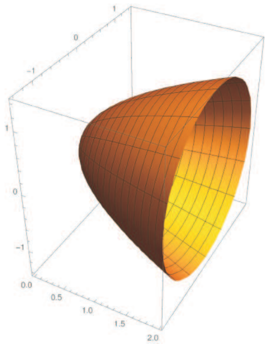

FIG. 4.1:

Let $\mu = 0$, then from (4.10), we obtain

$$f(u) - \frac{uf'(u)}{2} = 0.$$

Its general solution is

$$f(u) = c_1 u^2. \tag{4.13}$$

The solution (4.13) does not satisfies (4.7). Thus we can give the following theorems:

Definition 4.1. A surface in $\mathbb{H}^3(-c^2)$ is said to be **II**-harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 4.1. *Let* $\mathbf{M}$ *be conformal surfaces of revolution given by (2.6) and* $c \neq 0$ *in* $\mathbb{H}^3(-c^2)$. *Then there are no* **II**-*harmonic conformal surfaces of revolution satisfying the condition* $\Delta^{\mathbf{II}}\mathbf{x} = 0$.

Theorem 4.2. *Let* $\mathbf{M}$ *be conformal surfaces of revolution given by (2.6) and* $c \neq 0$ *in* $\mathbb{H}^3(-c^2)$. *If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x} = A\mathbf{x}$, *where* $A \in Mat(3, \mathbb{R})$, *then it is an open part of the surface (4.12).*

5. Conformal Surfaces of Revolution Satisfying $\Delta^{\mathbf{III}}\mathbf{x} = \mathbf{A}\mathbf{x}$

In this section, we will classify conformal surfaces of revolution with non-degenerate second fundamental form in $\mathbb{H}^3(-c^2)$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{III}}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{5.1}$$

where $\mathbf{A} = (\mathbf{a}_{ij}) \in Mat(3, R)$ and

$$\Delta^{\mathbf{III}}\mathbf{x}_i = \left(\Delta^{\mathbf{III}}\mathbf{x}_1, \Delta^{\mathbf{III}}\mathbf{x}_2, \Delta^{\mathbf{III}}\mathbf{x}_3\right). \tag{5.2}$$

By straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (3.2), (3.3), (3.5), (5.2) and (2.9) turns out to be

$$\Delta^{\mathbf{III}}\mathbf{x}_i = \begin{pmatrix} \dfrac{e^{4cu}\left(f''' - cf''\right) + e^{2cu} f'\left(f''' f' - f''^2\right)}{f^2 f''^3}, \\ \dfrac{e^{2cu}\cos v\left(-e^{2cu}\left(f''^2 + f'\left(cf'' - f'''\right)\right) + f'^2\left(f''' f' - 2f''^2\right)\right)}{f^2 f''^3}, \\ \dfrac{e^{2cu}\sin v\left(-e^{2cu}\left(f''^2 + f'\left(cf'' - f'''\right)\right) + f'^2\left(f''' f' - 2f''^2\right)\right)}{f^2 f''^3} \end{pmatrix}. \tag{5.3}$$

Suppose that $\mathbf{M}$ satisfies (5.1). Then from (5.1) and (5.2), we have

$$a_{11}u + a_{12}f(u)\cos v + a_{13}f(u)\sin v = A(u)e^{2cu} \tag{5.4}$$

$$a_{21}u + a_{22}f(u)\cos v + a_{23}f(u)\sin v = B(u)e^{2cu}\cos v$$

$$a_{31}u + a_{32}f(u)\cos v + a_{33}f(u)\sin v = B(u)e^{2cu}\sin v,$$

where

$$\begin{aligned} A(u) &= \left(\frac{e^{2cu}\left(f''' - cf''\right) + f'\left(f''' f' - f''^2\right)}{f^2 f''^3}\right), \\ B(u) &= \frac{\left(-e^{2cu}\left(f''^2 + f'\left(cf'' - f'''\right)\right) + f'^2\left(f''' f' - 2f''^2\right)\right)}{f^2 f''^3}. \end{aligned}$$

Since the functions $\cos v, \sin v$ and the constant function are linearly independent, by (5.4) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = \lambda, a_{22} = a_{33} = \mu$.

Consequently the matrix $\mathbf{A}$ satisfies

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

and (5.4) is rewritten as the following:

$$\lambda u = \frac{e^{4cu}\left(f''' - cf''\right) + e^{2cu} f' \left(f''' f' - f''^2\right)}{f^2 f''^3}, \tag{5.5}$$

$$\mu f(u) = \frac{e^{2cu}\left(-e^{2cu}\left(f''^2 + f'\left(cf'' - f'''\right)\right) + f'^2\left(f''' f' - 2f''^2\right)\right)}{f^2 f''^3}.$$

Combining the first and the second equation of (5.5), we obtain

$$e^{4cu} + e^{2cu} f'^2 + \mu f^3 f'' - u\lambda f^2 f' f'' = 0. \tag{5.6}$$

In the cases $\{c \neq 0, \lambda \neq 0, \mu = 0\}, \{c \neq 0, \lambda = 0, \mu \neq 0\}, \{c \neq 0, \lambda \neq 0, \mu \neq 0\}$ and $\{c = 0, \lambda \neq 0, \mu = 0\}, \{c = 0, \lambda = 0, \mu \neq 0\}, \{c = 0, \lambda \neq 0, \mu \neq 0\}$, we can not obtain any conformal surfaces of revolution in $\mathbb{H}^3(-c^2)$ and $\mathbb{E}^3$, respectively. Because the second order nonlinear differential equation (5.6) cannot be solved analytically. We will discuss two cases according to constant c.

Case 1: Let $c \neq 0, \lambda = 0, \mu = 0$, from (5.6), we obtain

$$e^{4cu} + e^{2cu} f'^2 = 0. \tag{5.7}$$

Its general solution is

$$f(u) = c_1 \pm i\frac{e^{cu}}{c}, \tag{5.8}$$

Case 2: Let $c = 0, \lambda = 0, \mu = 0$, from (5.6), we obtain

$$1 + f'^2 = 0. \tag{5.9}$$

The general solution of the equation (5.9) is given by

$$f(u) = c_1 \pm iu. \tag{5.10}$$

Since the solutions (5.8) and (5.10) are complex, it is a contradiction. Thus we can give the following theorem:

Definition 5.1. A surface in $\mathbb{H}^3(-c^2)$ is said to be **III**-harmonic if it satisfies the condition $\Delta^{\mathbf{III}}\mathbf{x} = \mathbf{0}$.

Theorem 5.1. *Let* $\mathbf{M}$ *be conformal surfaces of revolution given by (2.6) and* $c \neq 0$ *in* $\mathbb{H}^3(-c^2)$*. Then there are no* **III***-harmonic and non* **III***-harmonic conformal surfaces of revolution satisfying the conditions* $\Delta^{\mathbf{III}}\mathbf{x} = 0$ *and* $\Delta^{\mathbf{III}}\mathbf{x} = A\mathbf{x}$*, where* $A \in Mat\ (3, \mathbb{R})$*, respectively.*

Theorem 5.2. *Let* $\mathbf{M}$ *be conformal surfaces of revolution given by (2.6) and* $c = 0$ *in* $\mathbb{E}^3$*. Then there are no* **III***-harmonic and non* **III***-harmonic conformal surfaces of revolution satisfying the conditions* $\Delta^{\mathbf{III}}\mathbf{x} = 0$ *and* $\Delta^{\mathbf{III}}\mathbf{x} = A\mathbf{x}$*, where* $A \in Mat$ $(3, \mathbb{R})$*, respectively.*

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INEQUALITIES FOR GRADIENT EINSTEIN AND RICCI SOLITONS

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Abstract. This short note concerns with two inequalities in the geometry of gradient Einstein solitons (g, f, λ) on a smooth manifold M. These inequalities provide some relationships between the curvature of the Riemannian metric g and the behavior of the scalar field f through two quadratic equations satisfied by the scalar λ. The similarity with gradient Ricci solitons and a slight generalization involving a g-symmetric endomorphism A are provided.
Keywords: gradient Einstein solitons; smooth manifold; Riemannian metric; g-symmetric endomorphism.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold endowed with a smooth function $f \in C^\infty(M)$; an excellent textbook in Riemannian geometry is [6]. The scalar field f yields *the Hessian endomorphism*: $h_f : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $h_f(X) = \nabla_X \nabla f$, where ∇ is the Levi-Civita connection of g. Then we know the symmetry of *the Hessian tensor field* of f: $H_f(X,Y) := g(h_f(X), Y)$, namely $H_f(X,Y) = H_f(Y,X)$. What follows is the existence of a g-orthonormal frame field $E = \{E_i\}_{i=1,\dots,n} \subset \mathfrak{X}(M)$ and the existence of the eigenvalues $\lambda = \{\lambda_i\}_{i=1,\dots,n} \subset C^\infty(M)$:

$$h_f(E_i) = \lambda_i E_i. \tag{1.1}$$

Hence we express all the geometric objects related to f in terms of the pair (E, λ) which we call *the spectral data* of f:

$$\nabla f = \sum_{i=1}^n E_i(f) E_i, \quad \|\nabla f\|_g^2 = \sum_{i=1}^n [E_i(f)]^2, \quad h_f(X) = \sum_{i=1}^n (\lambda_i X^i) E_i, \tag{1.2}$$

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for $X = \sum_{i=1}^{n} X^i E_i$. Also *the Hessian* and *the Laplacian* of f are:

$$H_f(X,Y) = \sum_{i=1}^{n} \lambda_i (X^i Y^i), \quad \Delta f := Tr_g H_f = \sum_{i=1}^{n} \lambda_i. \tag{1.3}$$

Let us remark that if ∇f does not have zeros and E_1 is exactly its unit vector field i.e. $E_1 = \frac{\nabla f}{\|\nabla f\|_g}$, then ∇f is a *geodesic vector field*: $\nabla_{\nabla f} \nabla f = \lambda_1 \nabla f$ which means that the flow of ∇f consists in geodesics of g.

2. Results

Assume now that the triple $(g, f, \lambda \in \mathbb{R})$ is a *gradient Einstein soliton* on M, [2, p. 67]:

$$H_f + Ric + \left(\lambda - \frac{R}{2}\right) g = 0, \tag{2.1}$$

where Ric is the Ricci tensor field of g and R is the scalar curvature. Einstein solitons generate self-similar solutions of the Einstein flow (1.1) of [2] and are more rigid than the well-known Ricci solitons. By considering the Ricci endomorphism $Q \in \mathcal{T}^1_1(M)$ provided by:

$$Ric(X,Y) = g(QX, Y), \tag{2.2}$$

we can express (2.1) as:

$$h_f + Q + \left(\lambda - \frac{R}{2}\right) I = 0 \tag{2.3}$$

with I the Kronecker endomorphism. From (2.3) we get that Q is also of diagonal form with respect to the frame E:

$$Q(X) = -\sum_{i=1}^{n} \left(\lambda_i + \lambda - \frac{R}{2}\right) X^i E_i, \quad \|Q\|_g^2 = \sum_{i=1}^{n} \left(\lambda_i + \lambda - \frac{R}{2}\right)^2. \tag{2.4}$$

By developing the second formula above we derive:

$$\|Ric\|_g^2 = \sum_{i=1}^{n} \lambda_i^2 + (2\lambda - R) \sum_{i=1}^{n} \lambda_i + n\left(\lambda^2 - \lambda R + \frac{R^2}{4}\right) =$$

$$= \|H_f\|_g^2 + (2\lambda - R)\Delta f + n\left(\lambda^2 - \lambda R + \frac{R^2}{4}\right). \tag{2.5}$$

Hence the scalar λ is a solution of the quadratic equation:

$$n\lambda^2 + 2\left(\Delta f - \frac{nR}{2}\right)\lambda + \left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f\right) = 0 \tag{2.6}$$

which means the non-negativity:

$$0 \leq \Delta' := \left(\Delta f - \frac{nR}{2}\right)^2 - n\left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f\right). \tag{2.7}$$

It follows a lower boundary of the geometry of g in terms of f:

$$\|Ric\|_g^2 \geq \|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2. \tag{2.8}$$

An "exotic" consequence is provided by the case of strict inequality in (2.7), more precisely, it follows that the data (g, f, λ) is doubled by $\left(g, f, \frac{2\Delta f}{n} - R - \lambda = -\frac{2}{n}R - \lambda\right)$.

Example 1 i) (*Gaussian soliton*) We have $(M = \mathbb{R}^n, g_{can})$ and $f(x) = -\frac{\lambda}{2}\|x\|^2$. It results $h_f = -\lambda I_n$ and $\Delta f = -n\lambda$. Since $\|H_f\|^2 = n\lambda^2$, the left hand side of (2.6) is:

$$n\lambda^2 + 2\left(\Delta f - \frac{nR}{2}\right)\lambda + \left(\|H_f\|_g^2 - \|Ric\|_g^2 + \frac{nR^2}{4} - R\Delta f\right) = n\lambda^2 + 2(-n\lambda)\lambda + n\lambda^2$$

which is exactly zero. Also: $\Delta' = (n\lambda)^2 - n(n\lambda^2 - 0) = 0$ which means the uniqueness of λ and the equality case in (2.8): $0 = n\lambda^2 - \frac{(n\lambda)^2}{n}$.
ii) A generalization of the previous example is provided on a Ricci-flat manifold by a smooth function f satisfying a generalization of Hessian structures:

$$H_f = -\lambda g. \tag{2.9}$$

Then $\Delta f = -n\lambda$ and $\|H_f\|^2 = n\lambda^2$ exactly as for the Gaussian soliton. Using Lemma 4.1. of [3, p. 1540] it results form (2.9) that ∇f is a particular *concircular vector field*: $h_f = -\lambda I$; hence $\lambda_1 = ... = \lambda_n = -\lambda$ is the spectral part of the spectral data of f. If ∇f is without zeros it follows from Theorem 3.1. of [3, p. 1539] that (M, g) is locally a warped product manifold with a 1-dimensional basis: $(M, g) = (I \subseteq \mathbb{R}, g_{can}) \times_\varphi (F^{n-1}, g_F)$. In fact, $\nabla f = \varphi(s)\frac{\partial}{\partial s}$ with $\varphi'(s) = -\lambda$ which means an affine warping function, $\varphi(s) = -\lambda s + C$. □

A new quadratic equation, similar to (2.6), follows from:

$$\Delta f + \left(1 - \frac{n}{2}\right)R + n\lambda = 0 \tag{2.10}$$

obtained by tracing (2.1). Hence the companion equation of (2.6) is:

$$n\lambda^2 + 2\left(1 - \frac{n}{2}\right)R\lambda + \left(\|Ric\|_g^2 - \|H_f\|_g^2 + \frac{n-4}{4}R^2\right) = 0. \tag{2.11}$$

The new inequality is then:

$$0 \leq \Delta' := \left(1 - \frac{n}{2}\right)^2 R^2 - n\left(\|Ric\|_g^2 - \|H_f\|_g^2 + \frac{n-4}{4}R^2\right) \tag{2.12}$$

and it results a lower boundary of the behavior of f in terms of the geometry of g:

$$\|H_f\|_g^2 \geq \|Ric\|_g^2 - \frac{R^2}{n} = \frac{1}{n}\sum_{i \neq j}(\lambda_i - \lambda_j)^2. \tag{2.13}$$

We remark that (2.8) and (2.13) can be unified in the double inequality:

$$\|H_f\|_g^2 - \frac{1}{n}(\Delta f)^2 \leq \|Ric\|_g^2 \leq \|H_f\|_g^2 + \frac{R^2}{n} \tag{2.14}$$

and the simultaneous equalities for $n \geq 3$ hold if and only if $R = \Delta f = 0 = \lambda$ and $H_f = -Ric$; hence f is a harmonic map on a steady gradient Einstein soliton. The vanishing of the right-hand side of (2.13) means that g is an Einstein metric; other interesting aspects concerning the functional $F_g := \frac{R^2}{\|Ric\|_g^2}$ on the space of non-flat metrics appear in [5]. This raises the first future problem to study the similar functional $F_f^g := \frac{(\Delta f)^2}{\|H_f\|_g^2}$ on the space of smooth functions which are not *linear on* M after the name from [6, p. 283]. Remark that for the Hessian structures (2.9) we have a constant and maximal $F_f^g = n$.

Example 1 revisited i) (*Gaussian soliton*) The inequality (2.13) becomes $n\lambda^2 \geq 0$.
ii) Again, (2.13) means $n\lambda^2 \geq 0$.
iii) (*relationship with gradient Ricci solitons*) If $R = 0$, then the gradient Einstein soliton becomes a gradient Ricci soliton and we remark that (2.14) is exactly the double inequality (20) of [4, p. 3339]. The explication of this fact is provided by the following remark. □

Remark An unified proof of the double inequality (2.14) is provided by the following relation satisfied by an Einstein soliton, which is a direct consequence of the equations (2.5) and (2.10):

$$n\left(\|H_f\|_g^2 - \|Ric\|_g^2\right) = (\Delta f)^2 - R^2 \tag{2.15}$$

and it is important to point out that this equation does not involves the scalar λ. In other words, (2.15) is a universal formula of the gradient Einstein solitons. With $\lambda \to \lambda + \frac{R}{2}$ we get that (2.15) holds also for gradient Ricci solitons and hence we obtain the similarity between gradient Ricci and Einstein solitons with respect to (2.14). □

Returning to (2.3) we remark that the Ricci endomorphism Q commutes with h_f for an Einstein or Ricci gradient soliton. It results the commuting property also for the Einstein endomorphism:

$$Einst_g := Q - \frac{R}{n}I \tag{2.16}$$

which is the trace-free part of Q. We will assume now that the data $(g, f, \lambda, \mu \in \mathbb{R})$ satisfies:

$$h_f + Q + \lambda I + \mu Einst_g = 0. \tag{2.17}$$

The corresponding relation in terms of Ricci endomorphism is:

$$h_f + (1+\mu)Q + \left(\lambda - \frac{\mu R}{n}\right)I = 0 \tag{2.18}$$

or, for $\mu \neq -1$:

$$h_{\frac{f}{1+\mu}} + Q + \left(\frac{\lambda}{1+\mu} - \frac{\mu R}{n(1+\mu)}\right) I = 0. \tag{2.19}$$

This last equation is an example of ρ-Einstein soliton as is introduced in Definition 1.1 of [2, p. 67] with $\rho = \frac{\mu}{n(1+\mu)}$ and (f, λ) of [2] replaced by $\frac{1}{1+\mu}(f, \lambda)$.

Hence we naturally arrive to the following slight generalization of all the above considerations. Fix a g-symmetric endomorphism $A \in \mathcal{T}^1_1(M)$ which is also diagonal with respect to the frame E:

$$A(E_i) = \rho_i E_i, \quad \rho_i \in C^\infty(M). \tag{2.20}$$

Hence A and h_f commutes: $A \circ h_f = h_f \circ A$. We introduce:

Definition The data $(g, f, \lambda, \mu \in \mathbb{R})$ is an A-*Ricci gradient soliton* if:

$$h_f + Q + \lambda I + \mu A = 0. \tag{2.21}$$

We get that A commutes also with Q and the corresponding generalization of (2.15) is:

$$n\left[\|H_f\|_g^2 - \|Ric\|_g^2 + \mu^2\|A\|_g^2 + 2\mu Tr_g(h_f \circ A)\right] = (\Delta f + \mu Tr_g A)^2 - R^2 \tag{2.22}$$

yielding the double inequality:

$$\|H_f\|_g^2 - \frac{1}{n}(\Delta f + \mu Tr_g A)^2 + \mu^2\|A\|_g^2 + 2\mu Tr_g(h_f \circ A) \leq \|Ric\|_g^2 \leq$$

$$\leq \|H_f\|_g^2 + \frac{R^2}{n} + \mu^2\|A\|_g^2 + 2\mu Tr_g(h_f \circ A). \tag{2.23}$$

There is another problem: to find remarkable endomorphisms commuting with a given h_f. We will finish this note with an example.

Example 2 Suppose that (M, g) is a hypersurface in (N^{n+1}, g) and let $A = S$ be the shape endomorphism of M commuting with h_f for the fixed scalar field $f \in C^\infty(M)$. If $(g, f, \lambda, \mu \in \mathbb{R})$ is a *shape-Ricci gradient soliton* on M i.e. (2.21) holds for S, then denoting by H the mean curvature of M, we get:

$$\|H_f\|_g^2 - \frac{1}{n}(\Delta f + \mu H)^2 + \mu^2\|S\|_g^2 + 2\mu Tr_g(h_f \circ S) \leq \|Ric\|_g^2 \leq$$

$$\leq \|H_f\|_g^2 + \frac{R^2}{n} + \mu^2\|S\|_g^2 + 2\mu Tr_g(h_f \circ S). \tag{2.24}$$

We point out that immersions of (almost) Ricci solitons into another Riemannian manifold are studied in [1]. $\square$

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LEGENDRE CURVES ON 3-DIMENSIONAL f-KENMOTSU MANIFOLDS ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In the present paper, biharmonic Legendre curves with respect to Schouten-Van Kampen connection have been studied on three-dimensional f-Kenmotsu manifolds. Locally ϕ-symmetric Legendre curves on three-dimensional f-Kenmotsu manifolds with respect to Schouten-Van Kampen Connection have been introduced. Also, slant curves have been studied on three-dimensional f-Kenmotsu manifolds with respect to Schouten-Van Kampen connection. Finally, we have constructed an example of a Legendre curve in a 3-dimensional f-Kenmotsu manifold.
Keywords: Legendre curves; f-Kenmotsu manifold; Locally ϕ-symmetric Legendre curves; Schouten-Van Kampen connection; Slant curve.

1. Introduction

In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [2]. Originally, the notion of Legendre curve was defined for curves in a contact three-manifolds with the help of a contact form. This notion of Legendre curves can be also extended to almost contact manifolds [22]. Curves satisfying the properties of Legendre curves in almost contact metric manifolds are known as almost contact curves [5]. In [16], A. Sarkar, S. K. Hui and M. Sen have studied Legendre curves on three dimensional trans-Saskian manifold. J. Welyzcko [22], studied Legendre curves on a three-dimensional trans-Sasakian manifolds with respect to Levi-Civita connections. In [5], the authors have introduced a 1-parameter family of linear connections on three-dimensional almost contact metric manifolds to study biharmonic curves on almost contact manifolds. The author has studied some curves on three-dimensional

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trans-Sasakian manifolds with semi-symmetric metric connection [14]. The author of the present paper has also studied some curves on α-Sasakian manifolds with indefinite metric [15]. The Schouten-Van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [3],[6], [17]. Solov'es investigated hyperdistributions in Riemannian manifolds using Schouten-Van Kampen connection [18]. Then Olszak studied the Schouten-Van Kampen connection to an almost contact metric structure [13]. He characterized some classes of almost contact metric manifolds with Schouten-Van Kampen connection and found certain curvature properties of this connection of these manifolds. In the present paper, we are interested to study biharmonic Legendre curves with respect to Schouten-Van Kampen connection on a three-dimensional f-Kenmotsu manifold. We also introduce locally ϕ-symmetric almost contact curves with respect to Schouten-Van Kampen connections on a three-dimensional f-Kenmotsu manifold.

The present paper is organized as follows: After the introduction, we have given some required preliminaries in Section 2. In Section 3, we have studied biharmonic Legendre curves with respect to Schoute-Van Kampen connection. In Section 4, we have considered locally ϕ-symmetric Legendre curves on three-dimensional f-Kenmotsu manifolds with respect to Schouten-Van Kampen connection. In Section 5, we have studied slant curves with respect to Schouten-Van Kampen connection. In the last section, we have constructed an example of Legendre curve in a three-dimensional f-Kenmotsu manifold.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [1]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$. An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called f-Kenmotsu manifold if this may be expressed by the condition [11]

$$(\nabla_X \phi)Y = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.5}$$

where $f \in C^{\infty}(M)$ such that $df \wedge \eta = 0$ and ∇ is Levi-Civita connection on M. If $f = \alpha$=constant$\neq 0$, then the manifold is an $\alpha-$Kenmotsu manifold [7]. 1-Kenmotsu manifold is a Kenmotsu [8]. If f=0, then the manifold is cosymplectic [7]. An f-Kenmotsu manifold is said to be to be regular if $f^2 + f' \neq 0$, where $f' = \xi(f)$.

For an f-Kenmotsu manifold it follows that

$$\nabla_X \xi = f\{X - \eta(X)\xi\}, \tag{2.6}$$

Then using (2.6), we have

$$(\nabla_X \eta)Y = f(g(X,Y) - \eta(X)\eta(Y)). \tag{2.7}$$

The condition $df \wedge \eta = 0$ holds if dim M$\geqslant$ 5. This does not hold in general if dim M=3 [12]. In a 3-dimensional f-Kenmotsu manifold M, we have [12]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')\{g(Y,Z)X - g(X,Z)Y\} \\ &- (\frac{r}{2} + 3f^2 + 3f')\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{2.8}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.9}$$

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi, \tag{2.10}$$

where R denotes the curvature tensor, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold M.

The Schouten-Van Kampen connections [9], [13] $\tilde{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{2.11}$$

for all vector fields X, Y on M. With the help of (2.6) and (2.7), the above equation takes the form

$$\tilde{\nabla}_X Y = \nabla_X Y + f\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.12}$$

for an f-Kenmotsu manifold. So, we obtain the following Proposition by using (2.1)-(2.6) and (2.10):

Proposition 2.1. The Schouten-Van Kampen connections $\tilde{\nabla}$ on f-Kenmotsu manifold, we have the following properties $\tilde{\nabla} g = 0$, $\tilde{\nabla}\eta = 0$ and $\tilde{\nabla}\xi = 0$. Also from (2.10), the tensor $\tilde{T}$ of the Schouten-Van Kampen connection is [21]

$$\tilde{T}(X,Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2d\eta(X,Y)\xi, \tag{2.13}$$

for all fields X, Y on M.

Let $\tilde{\nabla}_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to Schouten-Van Kampen connection on M. We shall say that γ is a Frenet curve with respect to Schouten-Van Kampen connection if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\tilde{\nabla}_t t = 0$ (geodesic).

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(= \dot{\gamma})$, n and a non-negative function $\tilde{k}$ (curvature) along γ such that $\tilde{\nabla}_t t = \tilde{k}n$, $\tilde{\nabla}_t n = -\tilde{k}t$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(= \dot{\gamma})$, n, b and two non-negative functions $\tilde{k}$(curvature) and $\tilde{\tau}$(torsion) along γ such that

$$\tilde{\nabla}_t t = \tilde{k}n, \tag{2.14}$$

$$\tilde{\nabla}_t n = -\tilde{k}t + \tilde{\tau}b, \tag{2.15}$$

$$\tilde{\nabla}_t b = -\tilde{\tau}n. \tag{2.16}$$

The vector fields t and n along γ in the above equations are related by $n = \phi t$ and hence $b = \xi$ along γ. With respect to Schouten-Van Kampen connection, a Frenet curve of osculating order 3 for which $\tilde{k}$ is a positive constant and $\tilde{\tau} = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if $\tilde{k}$ and $\tilde{\tau}$ both are positive constants and the curve is called a generalized helix with respect to Schouten-Van Kampen connection if $\frac{\tilde{k}}{\tilde{\tau}}$ is a constant.

A Frenet curve γ in an almost contact metric manifold is said to be almost contact curve if it is an integral curve of the distribution $\mathcal{D} = \ker\eta$. Formally, it is also said that a Frenet curve γ in an almost contact metric manifold is an almost contact curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For more details we refer [2], [4], [10], [22]. It is to be mentioned that in the paper [5], curves satisfying the above properties on almost contact manifolds have been termed as almost contact curve while Welyczko [22] has termed such curves on almost contact manifolds as Legendre curves. Henceforth, by Legendre curves on almost contact manifolds we shall mean almost contact curves.

A Frenet curve is called a slant curve if it makes a constant angle with Reeb vector field ξ. If a curve γ on an almost contact metric manifold is a slant curve then $\eta(\dot{\gamma}) = \cos\theta$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$ where θ is a constant and is called slant angle. In particular if the angle is $\frac{\pi}{2}$, the curve becomes a Legendre curve.

3. Biharmonic almost contact curves with respect to Schouten-Van Kampen connection

In this section we study biharmonic almost contact curves on a three-dimensional f-Kenmotsu manifold with respect to Schouten-Van Kampen connection.

Definition 3.1. *A Legendre curve γ on a three-dimensional f-Kenmotsu manifold is called biharmonic with respect to Schouten-Van Kampen connection if it* *satisfies the equation [5]*

$$\tilde{\nabla}_t^3 t + \tilde{\nabla}_t \tilde{T}(\tilde{\nabla}_t t, t) + \tilde{R}(\tilde{\nabla}_t t, t)t = 0, \tag{3.1}$$

where $\tilde{T}$ is torsion of the Schouten-Van Kampen connection and $\dot{\gamma} = t$ is tangent vector field of the curve.

Let us consider a Legendre curve γ with respect to Schouten-Van Kampen connection on a three-dimensional f-Kenmotsu manifold. We take $\{t, \phi t, \xi\}$ as right handed system when $\phi t = -n$, $\phi n = t$.

Let $\tilde{R}$ and R be the Riemannian curvature tensor with respect to Schouten-Van Kampen connection and Levi-civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [9]

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + f^2\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f'\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{3.2}$$

For a Legendre curve $\eta(t) = 0, \eta(n) = 0$ because we have considered Frenet frame as $\{t, \phi t, \xi\}$ as a right handed system when $\phi t = -n, \phi n = t$. Using this facts in (3.2) and considering (2.8), we get

$$\tilde{R}(n,t)t = (\frac{r}{2} + 3f^2 + 2f')n. \tag{3.3}$$

By Frenet formula (2.15), (2.16), (2.17) and (2.18) we get

$$\tilde{\nabla}_t^3 t = -3\tilde{k}\tilde{k}'t + (\tilde{k}'' - \tilde{k}^3 - \tilde{\kappa}\tilde{\tau}^3)n + (2\tilde{\tau}\tilde{\kappa}' + \tilde{\kappa}\tilde{\tau}')b, \tag{3.4}$$

where $n = -\phi t, b = \xi$. In view of (2.13)

$$\tilde{\nabla}_t \tilde{T}(\tilde{\nabla}_t t, t) = 0. \tag{3.5}$$

By virtue of (3.3), (3.4) and (3.5) we get

$$\begin{aligned} \tilde{\nabla}_t^3 t + \tilde{k}\tilde{R}(n,t)t &= -3\tilde{k}\tilde{k}'t + \{\tilde{k}'' - \tilde{k}^3 - \tilde{\kappa}\tilde{\tau}^2 + \tilde{k}(\frac{r}{2} + 3f^2 + 2f')\}n \\ &+ (2\tilde{\tau}\tilde{\kappa}' + \tilde{\kappa}\tilde{\tau}')b. \end{aligned} \tag{3.6}$$

By virtue of (3.1) and observing the components of the right hand side of (3.6), we get $\tilde{\kappa}$ and $\tilde{\tau}$ are constant such that $\tilde{\kappa}^2 + \tilde{\tau}^2 = \frac{r}{2} + 3f^2 + 2f'$. Hence we can state the following theorem

Theorem 3.1. Let γ be a non-geodesic Legendre curve with respect to Schouten-Van Kampen connection on three-dimensional f-Kenmotsu manifold. Then the Legendre curve is a helix with respect to the Schouten-Van Kampen connection such that $\tilde{\kappa}^2 + \tilde{\tau}^2 = \frac{r}{2} + 3f^2 + 2f'$. The converse statement is true if the torsion tensor is constant along γ.

4. Locally ϕ-symmetric Legendre curves with respect to Schouten-Van Kampen connection

The notion of locally ϕ-symmetric manifolds was introduced by T. Takahashi [19] in the context of Sasakian geometry. Since every smooth curve is a one-dimensional differentiable manifold, we may apply the concept of local ϕ-symmetry on a smooth curve. In [14], locally ϕ-symmetric Legendre curves have been studied.

Definition 4.1. *With respect to Schouten-Van Kampen connection a three-dimensional f-Kenmotsu manifold will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z=0, \tag{4.1}$$

for all vector fields X,Y,Z,W orthogonal to ξ.

Definition 4.2. *With respect to Schoutn-Van Kampen connection a Legendre curve γ on a three-dimensional f-Kenmotsu manifold will be called locally ϕ-symmetric if it satisfies*

$$\phi^2(\tilde{\nabla}_t\tilde{R})(\tilde{\nabla}_t t,t)t=0, \tag{4.2}$$

where $t=\dot{\gamma}$. Here we shall establish the following:

Theorem 4.1. A necessary and sufficient condition for a non-geodesic Legendre curve on a three-dimensional f-Kenmotsu manifold with constant structure function to be locally $\phi-$symmetric with respect to the Schouten-Van Kampen connection is $r=-6f^2$, where r is the scalar curvature of the manifold with respect to Levi-Civita connection.

By definition of covariant differentiation of the Riemannian curvature tensor R of type $(1,3)$ we obtain

$$(\tilde{\nabla}_t\tilde{R})(\tilde{\nabla}_t t,t)t=\tilde{\nabla}_t\tilde{R}(\tilde{\nabla}_t t,t)t-\tilde{R}(\tilde{\nabla}_t^2 t,t)t-\tilde{R}(\tilde{\nabla}_t t,\tilde{\nabla}_t t)t-\tilde{R}(\tilde{\nabla}_t t,t)\tilde{\nabla}_t t.$$

Using Serret-Frenet formula, from the above equation we get

$$\begin{aligned}(\tilde{\nabla}_t\tilde{R})(\tilde{\nabla}_t t,t)t &= \tilde{\nabla}_t\tilde{R}(\tilde{\nabla}_t t,t)t+\tilde{k}^2\tilde{R}(t,t)t-\tilde{k}\tilde{\tau}\tilde{R}(b,t)t\\ &- \tilde{k}'\tilde{R}(n,t)t-\tilde{k}^2\tilde{R}(n,n)t-\tilde{k}^2\tilde{R}(n,t)n.\end{aligned} \tag{4.3}$$

After some straightforward calculation, the above equation together with (3.3) and (2.18) we have

$$(\tilde{\nabla}_t\tilde{R})(\tilde{\nabla}_t t,t)t=\tilde{k}(\frac{r}{2}+3f^2+2f')'n+\tilde{k}\tilde{\tau}(\frac{r}{2}+3f^2+2f')\xi. \tag{4.4}$$

Applying ϕ^2 in both sides of the above equation, we have

$$\phi^2(\tilde{\nabla}_t\tilde{R})(\tilde{\nabla}_t t,t)t=-\tilde{k}(\frac{r}{2}+3f^2+2f')'n-\tilde{k}\tilde{\tau}(\frac{r}{2}+3f^2+2f')\xi. \tag{4.5}$$

By virtue of (4.2) and the above relation, the theorem follows.

5. Slant curves in a three-dimensional f-Kenmotsu manifold with respect to Schouten-Van Kampen connection

Definition 5.1. A unit speed curve γ in an almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be slant if its tangent vector field makes constant angle θ with ξ i.e., $\eta(\dot{\gamma}) = cos\theta$ is constant along γ. By definition, slant curves with constant angle $\frac{\pi}{2}$ are called Legendre curves or almost contact curves. Slant curves in 3-dimensional Kenmotsu manifolds with respect to semi-symmetric metric connection have been studied by W. Tang, P. Majhi, P. Zhao and U.C. De [20]. In this section, we are interested to study slant curves on 3-dimensional f-Kenmotsu manifolds with respect to Schouten-Van Kampen connection. Let us consider a unit speed curve γ on a f-Kenmotsu manifold, by virtue of (2.12) we get

$$\tilde{\nabla}_t t = \nabla_t t + f(\xi - \eta(t)t), \tag{5.1}$$

where ∇ is Levi-Civita connection. Now if γ is a Legendre curve in M and {t,n,b} the Frenet frame along γ, then the tangent vector field t can be defined by t=$\dot{\gamma}$. Then from (5.1) we get

$$\tilde{\nabla}_t t = \nabla_t t + f\xi. \tag{5.2}$$

Then from above equation we have:

Proposition 5.1. The curvature vector field $\nabla_t t$ coincides with the $\tilde{\nabla}_t t$ if and only if the manifold is cosymplectic.

Let γ be a non-geodesic Frenet curve in three-dimensional f-Kenmotsu manifold with Schouten-Van Kampen connection.

Differentiating the equation $g(t, \xi) = \cos\theta$ with respect to t along γ we get

$$\tilde{\nabla}_t g(t, \xi) - g(\tilde{\nabla}_t t, \xi) - g(t, \tilde{\nabla}_t \xi) = 0. \tag{5.3}$$

Using (2.12) in the above equation we get,

$$\tilde{\kappa}\eta(n) = -\sin\theta.\theta', \tag{5.4}$$

where $\{t, n, b\}$ is Frenet frame with $t = \dot{\gamma}$. If γ is slant curve, then above equation reduces to

$$\tilde{\kappa}\eta(n) = 0. \tag{5.5}$$

Therefore, from the above equation we can state the following proposition:

Proposition 5.2. A non-geodesic curve γ in a three-dimensional f-Kenmotsu manifold with Schouten-Van Kampen connection is slant if and only if it satisfies $\eta(n) = 0$.

Hence the reeb vector field ξ can be written as follows $\xi = \cos\theta t \mp \sin\theta b$. This means that the reeb vector field is in the plane spanned by t and b, namely $g(\xi, n) = 0$. On the other hand, with respect to an adapted local orthonormal frame fields X, ϕX,

ξ of M such that $\eta(X) = 0$ we have the following equalities of the Frenet vector fields t, n, b for some function $\lambda(s)$,

$$t = \sin\theta\{\cos\lambda X + \sin\lambda\phi X\} + \cos\theta\xi,$$

$$n = -\sin\lambda X + \cos\lambda\phi X,$$

$$b = \mp\cos\theta\cos\lambda X \mp \cos\theta\sin\lambda\phi X \pm co\sec\theta\xi.$$

Differentiating the equation $g(\xi, n) = 0$ along the slant curve γ of M, it follows that

$$g(\tilde{\nabla}_t n, \xi) + g(n, \tilde{\nabla}_t \xi) = 0,$$

using (2.6) and (2.12) we get

$$\kappa\cos\theta \pm \tau\sin\theta = 0.$$

Hence we can state the following theorem:
Theorem 5.1. If a non-geodesic curve of a three-dimensional f-Kenmotsu manifold with respect to Schouten-Van Kampen connection is a slant curve, then $\frac{\kappa}{\tau}$ =constant.

6. An example of a three-dimensional f-Kenmotsu manifold with respect to Schouten-Van Kampen connection

In this section, we would like to construct an example of a three-dimensional f-Kenmotsu manifold with respect to Schouten-Van Kampen connection.

We considered a three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = z^2\frac{\partial}{\partial x}, \qquad e_2 = z^2\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -\frac{2}{z}e_2, \qquad [e_1, e_3] = -\frac{2}{z}e_1.$$

By Koszul formula

$$\begin{array}{lll} \nabla_{e_1} e_3 = -\frac{2}{z} e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -\frac{2}{z} e_3, \\ \nabla_{e_2} e_3 = -\frac{2}{z} e_2, & \nabla_{e_2} e_2 = -\frac{2}{z} e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From above we see that the manifold satisfies $\nabla_X \xi = f(X - \eta(X)\xi)$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence the manifold is a f-Kenmotsu manifold. Now the Schouten-Van Kampen connection on the manifold we have

$$\begin{array}{lll} \tilde{\nabla}_{e_1} e_3 = (-\frac{2}{z} - r) e_1, & \tilde{\nabla}_{e_1} e_2 = 0, & \tilde{\nabla}_{e_1} e_1 = \frac{2}{z}(e_3 - \xi), \\ \tilde{\nabla}_{e_2} e_3 = (-\frac{2}{z} - f) e_2, & \tilde{\nabla}_{e_2} e_2 = \frac{2}{z}(e_3 - \xi), & \tilde{\nabla}_{e_2} e_1 = 0, \\ \tilde{\nabla}_{e_3} e_3 = -f(e_3 - \xi), & \tilde{\nabla}_{e_3} e_2 = 0, & \tilde{\nabla}_{e_3} e_1 = 0. \end{array}$$

From above we see that $\tilde{\nabla}_{e_i} e_j = 0, (0 \le i, j \le 3)$ for $\xi = e_3$ and $f = -\frac{2}{z}$. Hence the manifold is f-Kenmotsu manifold with respect to Schouten-Van Kampen connection.

Example 6.1. Consider a curve $\gamma : I \to M$ defined by $\gamma(s) = (\sqrt{\frac{2}{3}} s, \sqrt{\frac{1}{3}} s, 1)$. Hence $\dot{\gamma}_1 = \sqrt{\frac{2}{3}}$, where $\dot{\gamma}_2 = \sqrt{\frac{1}{3}}$ and $\dot{\gamma}_3 = 0$, $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$. Now

$$\eta(\dot{\gamma}) = g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, e_3) = 0.$$

$$\begin{aligned} g(\dot{\gamma}, \dot{\gamma}) &= g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, \dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3) \\ &= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2 \\ &= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 \\ &= \frac{2}{3} + \frac{1}{3} \\ &= 1. \end{aligned}$$

Hence the curve is Legendre curve.

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NEW FIXED POINT RESULTS FOR T-CONTRACTIVE MAPPING WITH C-DISTANCE IN CONE METRIC SPACES

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Abstract. In this article, we generalize and improve the results of Fadail et al.[Z. M. Fadail and S. M. Abusalim, Int. Jour. of Math. Anal., Vol. 11, No. 8(2017), pp. 397-405.] and Dubey et al.[AnilKumar Dubey and Urmila Mishra, Non. Func. Anal. Appl., Vol. 22, No. 2(2017), pp 275-286.] under the concept of a c-distance in cone metric spaces. We prove the existence and uniqueness of the fixed point for T-contractive type mapping under the concept of c-distance in cone metric spaces.
Keywords: Fixed point; T-contractive mapping; Cone metric space; c-distance.

1. Introduction

In 2007, Huang and Zhang[12] first introduced the concept of cone metric spaces and they established and proved the existence of fixed point theorems which is an extension of the Banach contraction mapping principle in to the cone metric spaces. Recently, Cho et al.[3] introduced the concept of c-distance in a cone metric spaces and proved some fixed point results in ordered cone metric spaces. Afterward, many authors have generalized and studied fixed point theorems under c-distance in cone metric spaces (see [1, 7, 8, 9, 10, 11, 14, 15, 16]). In 2009, Beiranvand et al.[2] introduced new classes of contractive functions and established the Banach principle. Since then, fixed point theorems for T-contraction mapping on cone metric spaces have been appeared, see for instance [4, 5, 6] and [11].

The purpose of this paper is to extend and generalize some results on c-distance in cone metric spaces. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is $intP \neq \phi$. Also, in this paper we assume $\mathbb{R}$ as a set of real numbers and $\mathbb{N}$ as a set of natural numbers.

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2. Preliminaries

Definition 2.1. ([12]) Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that:

(1) P is a non-empty, closed and $P \neq \{\theta\}$;

(2) If a, b are non-negative real numbers and $x, y \in P$ then $ax + by \in P$;

(3) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

Definition 2.2. ([12]) A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, \theta \preceq x \preceq y$ implies $||x|| \leq K||y||$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a Banach space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 2.3. ([12]) Let X be a non empty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(i) If $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.1. Let $E = \mathbb{R}^2$, and $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$ where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. It is easy to see that d is a cone metric, and hence (X, d) becomes a cone metric space over (E, P). Also, we have P is a solid and normal cone where the normal constant $K = 1$.

Definition 2.4. ([12]) Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$:

(1) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.

(2) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) if every Cauchy sequence in X is convergent in X then (X, d) is called a complete cone metric space.

The following Lemma is useful to prove our results.

Lemma 2.1. *([13])*

(1) If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(2) If $c \in intP$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next, we give the notion of c-distance on a cone metric space (X, d) of Cho et al. in [3].

Definition 2.5. ([3]) Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$;

(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q_3) for each $x \in X$ and $n \geq 1$ if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.2. ([3]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$, $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ is defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

The following Lemma is very important to prove our results.

Lemma 2.2. ([3]) *Let (X, d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following hold:*

(1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.

(2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.

(3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.1. ([3])

(1) $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Next definition taken from [2]:

Definition 2.6. Let (X, d) be a cone metric space, P a solid cone and $T : X \to X$. Then

(a) T is said to be continuous if $\lim_{n\to\infty} x_n = x^*$ implies that $lim_{n\to\infty} Tx_n = Tx^*$, for all $\{x_n\}$ in X;

(b) T is said to be sequentially convergent if we have, for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent, then $\{x_n\}$ is also convergent;

(c) T is said to be subsequentially convergent if we have, for every sequence $\{x_n\}$ that $\{Tx_n\}$ is convergent, implies $\{x_n\}$ has a convergent subsequence.

Now, we give our main results in this paper.

3. Main Results

Theorem 3.1. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be an one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition, suppose that there exists mapping $k, l : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \le k(x), l(fx) \le l(x)$, *for all* $x \in X$;

(b) $(k + 2l)(x) < 1$ *for all* $x \in X$;

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx, Ty) + l(x)[q(Tfx, Ty) + q(Tfy, Tx)]$

for all $x, y \in X$. Then the map f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$, then $q(Tu, Tu) = \theta$.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, ...x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &= q(Tfx_{n-1}, Tfx_n) \\ &\preceq k(x_{n-1})q(Tx_{n-1}, Tx_n) + l(x_{n-1})[q(Tfx_{n-1}, Tx_n) \\ &\quad +q(Tfx_n, Tx_{n-1})] \\ &= k(fx_{n-2})q(Tx_{n-1}, Tx_n) + l(fx_{n-2})[q(Tx_n, Tx_n) \\ &\quad +q(Tx_{n+1}, Tx_{n-1})] \\ &\preceq k(x_{n-2})q(Tx_{n-1}, Tx_n) + l(x_{n-2})[q(Tx_{n-1}, Tx_n) \\ &\quad +q(Tx_n, Tx_{n+1})], \end{aligned}$$

continuing in this manner, we can get

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\preceq k(x_0)q(Tx_{n-1}, Tx_n) + l(x_0)q(Tx_{n-1}, Tx_n) \\ &\quad + l(x_0)q(Tx_n, Tx_{n+1}) \end{aligned}$$

and hence

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &\preceq \frac{k(x_0) + l(x_0)}{1 - l(x_0)} q(Tx_{n-1}, Tx_n) \\ &= hq(Tx_{n-1}, Tx_n) \\ &\preceq h^2 q(Tx_{n-2}, Tx_{n-1}) \\ &\preceq h^n q(Tx_0, Tx_1), \end{aligned}$$

where $h = \frac{k(x_0)+l(x_0)}{1-l(x_0)} < 1$. Note that,

$$q(Tfx_{n-1}, Tfx_n) = q(Tx_n, Tx_{n+1}) \preceq hq(Tx_{n-1}, Tx_n). \tag{3.1}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(Tx_n, Tx_m) &\preceq q(Tx_n, Tx_{n+1}) + q(Tx_{n+1}, Tx_{n+2}) \\ &\quad + ... + q(Tx_{m-1}, Tx_m) \\ &\preceq (h^n + h^{n+1} + ... + h^{m-1}) q(Tx_0, Tx_1) \\ &\preceq \frac{h^n}{1-h} q(Tx_0, Tx_1) \to \theta \quad as \quad n \to \infty. \end{aligned}$$

Thus, Lemma 2.2(3) shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $Tx_n \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So, there are $x^* \in X$ and $\{x_{n_i}\}$ such that $x_{n_i} \to x^*$ as $i \to \infty$. Since T is continuous, we obtain $\lim Tx_{n_i} \to Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_n, Tx^*) \preceq \frac{k^n}{1-k} q(Tx_0, Tx_1). \tag{3.2}$$

Now by using (3.1), we have

$$\begin{aligned} q(Tx_n, Tfx^*) &= q(Tfx_{n-1}, Tfx^*) \\ &\preceq hq(Tx_{n-1}, Tx^*) \\ &\preceq h\frac{k^{n-1}}{1-k} q(Tx_0, Tx_1) \end{aligned}$$

$$= \frac{h^n}{1-h} q(Tx_0, Tx_1). \tag{3.3}$$

By Lemma 2.2(1), (3.2) and (3.3), we have $Tx^* = Tfx^*$. Since T is one to one, then $x^* = fx^*$. Thus, x^* is fixed point of f. Suppose that $u = fu$, then we have

$$\begin{aligned} q(Tu, Tu) &= q(Tfu, Tfu) \\ &\preceq k(u)q(Tu, Tu) + l(u)[q(Tfu, Tu) + q(Tfu, Tu)] \\ &= k(u)q(Tu, Tu) + l(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &\preceq (k + 2l)(x_0)q(Tu, Tu). \end{aligned}$$

Since $(k+2l)(x_0) < 1$, Lemma 2.1(1) shows that $q(Tu, Tu) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned} q(Tx^*, Ty^*) &= q(Tfx^*, Tfy^*) \\ &\preceq k(x^*)q(Tx^*, Ty^*) + l(x^*)[q(Tfx^*, Ty^*) + q(Tfy^*, Tx^*)] \\ &= k(x^*)q(Tx^*, Ty^*) + l(x^*)[q(Tx^*, Ty^*) + q(Ty^*, Tx^*)] \\ &= (k + 2l)(x^*)q(Tx^*, Ty^*). \end{aligned}$$

Since $(k+2l)(x^*) < 1$, Lemma 2.1(1) shows that $q(Tx^*, Ty^*) = \theta$. Also we have $q(Tx^*, Tx^*) = \theta$. Thus Lemma 2.2(1), $Tx^* = Ty^*$. Since T is one to one, then $x^* = fx^*$. Therefore, the fixed point is unique $\square$

Corollary 3.1. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition, suppose that there exists mapping $k, l : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \le k(x), l(fx) \le l(x)$, *for all* $x \in X$;

(b) $(k + 2l)(x) < 1$ *for all* $x \in X$;

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx, Ty) + l(x)[q(Tfx, Tx) + q(Tfy, Ty)]$

for all $x, y \in X$. Then the map f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$, then $q(Tu, Tu) = \theta$.

Theorem 3.2. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be an one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition suppose that there exists mapping $k, l, r : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \le k(x), l(fx) \le l(x), r(fx) \le r(x)$ *for all* $x \in X$;

(b) $(k+2l+2r)(x)<1$ *for all* $x\in X$;

(c) $q(Tfx,Tfy)\preceq k(x)q(Tx,Ty)+l(x)[q(Tfy,Tx)+q(Tfx,Ty)]$
$+r(x)[q(Tfx,Tx)+q(Tfy,Ty)]$

for all $x,y\in X$. *Then the map* f *has a unique fixed point* $x^*\in X$ *and for any* $x\in X$, *iterative sequence* $\{fx_n\}$ *converges to the fixed point. If* $u=fu$, *then* $q(Tu,Tu)=\theta$.

Proof. Choose $x_0\in X$. Set $x_1=fx_0, x_2=fx_1=f^2x_0, ...x_{n+1}=fx_n=f^{n+1}x_0$. Then we have

$$\begin{aligned} q(Tx_n,Tx_{n+1}) &= q(Tfx_{n-1},Tfx_n)\\ &\preceq k(x_{n-1})q(Tx_{n-1},Tx_n)+l(x_{n-1})[q(Tfx_n,Tx_{n-1})\\ &\quad +q(Tfx_{n-1},Tx_n)]+r(x_{n-1})[q(Tfx_{n-1},Tx_{n-1})\\ &\quad +q(Tfx_n,Tx_n)]\\ &= k(fx_{n-2})q(Tx_{n-1},Tx_n)+l(fx_{n-2})[q(Tx_{n+1},Tx_{n-1})\\ &\quad +q(Tx_n,Tx_n)]+r(fx_{n-2})[q(Tx_n,Tx_{n-1})+q(Tx_{n+1},Tx_n)]\\ &\preceq k(x_{n-2})q(Tx_{n-1},Tx_n)+l(x_{n-2})[q(Tx_{n-1},Tx_n)\\ &\quad +q(Tx_n,Tx_{n+1})]+r(x_{n-2})[q(Tx_{n-1},Tx_n)+q(Tx_n,Tx_{n+1})], \end{aligned}$$

continuing in this manner, we can get

$$\begin{aligned} q(Tx_n,Tx_{n+1}) &\preceq (k(x_0)+l(x_0)+r(x_0))q(Tx_{n-1},Tx_n)+(l(x_0)\\ &\quad +r(x_0))q(Tx_n,Tx_{n+1}) \end{aligned}$$

and hence

$$\begin{aligned} q(Tx_n,Tx_{n+1}) &\preceq \frac{k(x_0)+l(x_0)+r(x_0)}{1-l(x_0)-r(x_0)}q(Tx_{n-1},Tx_n)\\ &= hq(Tx_{n-1},Tx_n)\\ &\preceq h^2q(Tx_{n-2},Tx_{n-1})\\ &\preceq h^nq(Tx_0,Tx_1), \end{aligned}$$

where $h=\frac{k(x_0)+l(x_0)+r(x_0)}{1-l(x_0)-r(x_0)}<1$. Note that,

$$q(Tfx_{n-1},Tfx_n)=q(Tx_n,Tx_{n+1})\preceq hq(Tx_{n-1},Tx_n). \tag{3.4}$$

Let $m>n\geq 1$. Then it follows that

$$\begin{aligned} q(Tx_n,Tx_m) &\preceq q(Tx_n,Tx_{n+1})+q(Tx_{n+1},Tx_{n+2})....+q(Tx_{m-1},Tx_m)\\ &\preceq (h^n+h^{n+1}+....+h^{m-1})q(Tx_0,Tx_1)\\ &\preceq \frac{h^n}{1-h}q(Tx_0,Tx_1)\to\theta \quad as \quad n\to\infty. \end{aligned}$$

Thus, Lemma 2.2(3) shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $Tx_n \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So there are $x^* \in X$ and $\{x_{n_i}\}$ such that $x_{n_i} \to x^*$ as $i \to \infty$. Since T is continuous, we obtain $\lim Tx_{n_i} = Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_n, Tx^*) \preceq \frac{k^n}{1-k} q(Tx_0, Tx_1). \tag{3.5}$$

Now by using (3.4), we have

$$\begin{aligned} q(Tx_n, Tfx^*) &= q(Tfx_{n-1}, Tfx^*) \\ &\preceq hq(Tfx_{n-1}, Tx^*) \\ &\preceq h\frac{k^{n-1}}{1-k} q(Tx_0, Tx_1) \\ &= \frac{h^n}{1-h} q(Tx_0, Tx_1). \end{aligned} \tag{3.6}$$

By Lemma 2.2(1), (3.5) and (3.6), we have $Tx^* = Tfx^*$. Since T is one to one, then $x^* = fx^*$. Thus, x^* is a fixed point of f. Suppose that $u = fu$, then we have

$$\begin{aligned} q(Tu, Tu) &= q(Tfu, Tfu) \\ &\preceq k(u)q(Tu, Tu) + l(u)[q(Tfu, Tu) + q(Tfu, Tu)] \\ &\quad +r(u)[q(Tfu, Tu) + q(Tfu, Tu)] \\ &= k(u)q(Tu, Tu) + l(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &\quad +r(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &\preceq (k + 2l + 2r)(x_0)q(Tu, Tu). \end{aligned}$$

Since $(k + 2l + 2r)(x_0) < 1$, Lemma 2.1(1) shows that $q(Tu, Tu) = \theta$. Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned} q(Tx^*, Ty^*) &= q(Tfx^*, Tfy^*) \\ &\preceq k(x^*)q(Tx^*, Tx^*) + l(x^*)[q(Tfy^*, Tx^*) + q(Tfx^*, Ty^*)] \\ &\quad +r(x^*)[q(Tfx^*, Tx^*) + q(Tfy^*, Ty^*)] \\ &= k(x^*)q(Tx^*, Ty^*) + l(x^*)[q(Ty^*, Tx^*) + q(Tx^*, Ty^*)] \\ &\quad +r(x^*)[q(Tx^*, Tx^*) + q(Ty^*, Ty^*)] \\ &= (k + 2l)(x^*)q(Tx^*, Ty^*) \\ &\preceq (k + 2l + 2r)(x^*)q(Tx^*, Ty^*). \end{aligned}$$

Since $(k + 2l + 2r)(x^*) < 1$, Lemma 2.1(1) shows that $q(Tx^*, Ty^*) = \theta$. Also we have, $q(Tx^*, Tx^*) = \theta$. Thus, by Lemma 2.2(1), $Tx^* = Ty^*$. Since T is one to one, then $x^* = fx^*$. Therefore, the fixed point is unique. $\square$

Theorem 3.3. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. Let $T : X \to X$ be an one to one, continuous function and subsequentially convergent and $f : X \to X$ be a mapping. In addition suppose that there exists mapping $k, r, l, t : X \to [0, 1)$ such that the following conditions hold:*

(a) $k(fx) \leq k(x), r(fx) \leq r(x), l(fx) \leq l(x), t(fx) \leq t(x)$ *for all* $x \in X$*;*

(b) $(k + r + l + 2t)(x) < 1$ *for all* $x \in X$*;*

(c) $q(Tfx, Tfy) \preceq k(x)q(Tx, Ty) + r(x)q(Tfx, Tx) + l(x)q(Tfy, Ty) + t(x)[q(Tfx, Ty) + q(Tfy, Tx)]$

for all $x, y \in X$. Then map f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$, then $q(Tu, Tu) = \theta$.

Proof. The proof of this theorem is same as Theorem 3.1. □

Now we give an example which illustrates our Theorems 3.1.

Example 3.1. Let $E = \mathbb{R}$ and $P = \{x \in E, x \geq 0\}$, let $X = [0, 1]$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|e^t$ where $e^t \in E$. Then (X, d) is complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = ye^t$ for all $x, y \in X$. Then q is a c-distance on X. Define the mapping $T, f : X \to X$ by $fx = \frac{x^2}{4}$ and $T(x) = x^4$ for all $x \in X$. Take mapping $k, l : X \to [0, 1)$ by $k(x) = \frac{x+1}{4}$ and $l(x) = \frac{x}{8}$, for all $x \in X$. Observe that

(i) $k(fx) = k\left(\frac{x^2}{4}\right) = \left(\frac{\frac{x^2}{4}+1}{4}\right) = \frac{1}{4}\left(\frac{x^2}{4} + 1\right) \leq (\frac{1}{4})(x + 1) = k(x)$ for all $x \in X$.

(ii) $l(fx) = l(\frac{x^2}{4}) = \left(\frac{\frac{x^2}{4}}{8}\right) = \frac{1}{8}\left(\frac{x^2}{4}\right) \leq \frac{1}{8}(x) = l(x)$, for all $x \in X$.

(iii) $(k + 2l)(x) = \frac{x+1}{4} + \frac{x}{4} = \frac{1}{4}(2x + 1) < 1$, for all $x \in X$.

Now, we have

$$\begin{aligned} q(Tfx, Tfy) &= Tfye^t \\ &= \frac{y^8}{256}e^t \\ &\preceq \left(\frac{y+1}{4}\right)y^4e^t \\ &= k(x)q(Tx, Ty) \\ &\preceq k(x)q(Tx, Ty) + l(x)[q(Tfx, Ty) + q(Tfy, Tx)]. \end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied. Hence f has a unique fixed point $x = 0$ with $q(0, 0) = \theta$.

Competing Interests

The authors declare that they have no competing interests.

Authors' contribution
All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

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FIXED POINT THEOREMS FOR SUBSEQUENTIALLY MULTI-VALUED F_δ-CONTRACTIONS IN METRIC SPACES *

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Abstract. The aim of this paper is to prove common fixed point theorems for multi-valued contraction of Wordowski type, by using the concept of subsequential continuity in the setting of set valued context contractions with compatibility. We have also given an example and an application to integral inclusions of Fredholm type to support our results.

keywords: Subsequentially continuous; δ-compatible; F-contraction; Hardy Rogers contraction; integral inclusion.

1. Introduction

The multi-valued fixed point theory has many different applications, for example in integral or differential inclusions, economics, optimization, etc. The contraction principle due to Banach has been generalized in different directions and one of such generalizations is connected to Nadler [12], where he used the Hausdorff metric to prove the existence of a fixed point of multi valued mapping in metric space. Later, many authors have obtained some results in non linear analysis concerning the multivalued fixed point theory and its applications using two types of distances. One is the Hausdorff distance and another is the δ-distance which was defined by Fisher [8]. Although δ-distance is not a metric like the Hausdorff distance, it shares most of the properties of a metric and some results on δ-distance can be found in [1, 2, 3]. In this paper, we have used a Ćirić type F-contraction and Hardy-Rogers type F-contraction inequality introduced by Minak et al.[11](independently by Wardowski and Dung [17] as F-weak contraction and Cosentino and Vetro [7] respectively, using δ-distance to establish the existence of a strict coincidence and a common strict fixed point of a weakly compatible hybrid pair of maps which are

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strongly tangential. However, it is worth mentioning that the idea of F-contraction was initiated by Wardowski [16], and later, it became generalized by several authors in different directions. The examples are Minak et al. [11], Wardowski and Dung [17], Cosentino and Vetro [7].

2. Preliminaries

Let (X, d) be a metric space, $B(X)$ is the set of all non-empty bounded subsets of X. For all $A, B \in B(X)$, we define the two functions:$D, \delta : B(X) \times B(X) \to \mathbb{R}_+$ such that

$$D(A, B) = \inf\{d(a, b); a \in A, b \in B\},$$

$$\delta(A, B) = \sup\{d(a, b); a \in A, b \in B\}.$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$ and $D(A, B) = D(a, B)$, also if $B = \{b\}$ is a singleton we write

$$\delta(A, B) = D(A, B) = d(a, b).$$

It is clear that δ satisfies the following properties:

$$\delta(A, B) = \delta(B, A) \geq 0,$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B),$$

$$\delta(A, A) = \text{diam}A,$$

$$\delta(A, B) = 0 \ \text{ implies } \ A = B = \{a\},$$

for all $A, B, C \in B(X)$.
Notice that for all $a \in A$ and $b \in B$ we have

$$D(A, B) \leq d(a, b) \leq \delta(A, B),$$

where $A, B \in B(X)$.

Definition 2.1. [14] Two mappings $S : X \to B(X)$ and $f : X \to X$ are to be weakly commuting on X if $fSx \in B(X)$ and for all $x \in X$:

$$\delta(Sfx, fSx) \leq \max\{\delta(fx, Sx), diam(fSx)\}.$$

Definition 2.2. [10] A hybrid pair of mappings (f, S) of a metric space (X, d) is δ-compatible if

$$\lim_{n\to\infty} \delta(Sfx_n, fSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $fSx_n \in B(X)$, $\lim\limits_{n\to\infty} Sx_n = \{z\}$, and $\lim\limits_{n\to\infty} fx_n = z$, for some $z \in X$.

Definition 2.3. [13] The pair of self mappings (f, g) on a metric space(X, d) is said to be reciprocally continuous if

$$\lim_{n\to\infty} fgx_n = ft$$

and

$$\lim_{n\to\infty} gfx_n = gt,$$

where $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t in X.

Later, Singh and Mishra [15] generalized the concept of reciprocal continuity to the setting of single and set-valued maps as follows.

Definition 2.4. [15] Two maps $f : X \to X$ and $S : X \to B(X)$ are reciprocally continuous on X (resp. at $t \in X$) if and only if $fSx \in B(X)$ for each $x \in X$ (resp. $fSt \in B(X)$) and

$$\lim_{n\to\infty} fSx_n = fM, \ \lim_{n\to} Sfx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = M \in B(X)$, $\lim_{n\to\infty} fx_n = t \in M$

In 2009, Bouhadjera and Godet Thobie [5] introduced the concept of subcompatibility and subsequential continuity as follows:
Two self-mappings f and g on a metric space (X, d) are said to be subcompatible if there exists a sequence $\{x_n\}$ such that:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \ \text{ and } \ \lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

for some $t \in X$.
The pair (f, g) of self mappings is said to be subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} fgx_n = fz$, $\lim_{n\to\infty} gfx_n = gz$.

Definition 2.5. [4] Let $f : X \to X$ and $S : X \to CB(X)$ two single and set-valued mappings respectively, the hybrid pair (f, S) is to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = M \in CB(X) \quad \text{and} \quad \lim_{n\to\infty} fx_n = z \in M,$$

for some $z \in X$ and $\lim_{n\to\infty} fSx_n = fM$, $\lim_{n\to\infty} Sfx_n = Sz$.

Notice that continuity or reciprocal continuity implies subsequential continuity, but the converse may be not.

Example 2.1. Let $X=[0,1]$ and d the euclidian metric, we define f,S by

$$fx=\begin{cases} 1-x, & 0\leq x\leq 1\\ \frac{1}{4}, & \frac{1}{2}<x\leq 1\end{cases} \qquad Sx=\begin{cases} [0,x], & 0\leq x\leq 1\\ [x-\frac{1}{2},x], & \frac{1}{2}<x\leq 1\end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n\geq 1$ we have: $x_n=\frac{1}{2}-\frac{1}{n}$, clearly that $\lim\limits_{n\to\infty} fx_n=\dfrac{1}{2}\in[0,\dfrac{1}{2}]$ and $\lim\limits_{n\to\infty} Sx_n=[0,\dfrac{1}{2}]\in B(X)$, also we have:

$$\lim_{n\to\infty} fSx_n=\lim_{n\to\infty}[\frac{1}{2}+\frac{1}{n},1]=[\frac{1}{2},1]=f([0,\frac{1}{2}]),$$

and

$$\lim_{n\to\infty} Sfx_n=\lim_{n\to\infty}[\frac{1}{n},\frac{1}{2}+\frac{1}{n}]=[0,\frac{1}{2}]=S(\frac{1}{2}),$$

then (f,S) is subsequentially continuous.
On the other hand, consider a sequence $\{y_n\}$ which defined for all $n\geq 1$ by: $y_n=1+\frac{1}{n}$, we have

$$\lim_{n\to\infty} fx_n=\frac{1}{2}\in[0,1], \quad \text{and} \quad \lim_{n\to\infty} Sx_n=[0,1]\in B(X),$$

however

$$\lim_{n\to\infty} fSx_n=\lim_{n\to\infty} f([\frac{1}{n},1+\frac{1}{n}])\neq f([0,1]),$$

then f and S are never reciprocally continuous.

Let $\mathcal{F}$ be the set of all functions $F:(0,+\infty)\to\mathbb{R}$ satisfying the following conditions:

(F_1) : F is strictly increasing,

(F_2) : for each sequence $\{\alpha_n\}$ in X, $\lim\limits_{n\to\infty}\alpha_n=0$ if and only if $\lim\limits_{n\to\infty}F(\alpha_n)=-\infty$,

(F_3) : there exists $k\in(0,1)$ satisfying $\lim\limits_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

Example 2.2. Let $F_i:(0,+\infty)\to\mathbb{R},\ i\in\{1,2,3\}$, defined by

1. $F_1(t)=\ln t$,
2. $F_2(t)=t+\ln t$,
3. $F_3(t)=-\frac{1}{\sqrt{t}}$.

Then $F_i\in\mathcal{F}$, for each $i\in\{1,2,3\}$.

Definition 2.6. [16] Let (X,d) be a metric space and $T:X\to X$ be a mapping. For $F\in\mathcal{F}$, we say T is F-contraction, if there exists $\tau>0$ such that for $x,y\in X$, $d(Tx,Ty)>0$ implies $\tau+F(d(Tx,Ty))\leq F(d(x,y))$.

Definition 2.7. [7] A self mapping T on a metric space (X,d) is a Hardy- Rogers type F-contraction if there exists $F\in\mathcal{F}$ and $\tau\in\mathbb{R}^+$ such that $d(Tx,Ty)>0$ implies that

$$F(d(Tx,Ty))\leq F(\alpha d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty)+\lambda d(x,Ty)+\mu d(y,Tx)),$$

for all $x,y\in X$, where, $\alpha+\beta+\gamma+2\lambda=1$, $\gamma\neq 1$, $\mu\geq 0$.

Definition 2.8. [11] A self mapping T on a metric space (X,d) is a Ćirić type F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that $d(Tx,Ty) > 0$ implies that

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y)),$$

$\forall x,y \in X$. where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty)+d(y,Tx))\}$

Notice that every F-contraction is a Ćirić type F-contraction or Hardy-Rogers type F-contraction but the reverse implication does not hold.

Definition 2.9. [7] A self mapping T on a metric space (X,d) is a Hardy- Rogers type F-contraction if there exists $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that $d(Tx,Ty) > 0$ implies that

$$F(d(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \lambda d(x,Ty) + Ld(y,Tx)),$$

for all $x,y \in X$, where, $\alpha + \beta + \gamma + 2\lambda = 1$, $\gamma \neq 1$ and $L \geq 0$.

Definition 2.10. [11] A self mapping T on a metric space (X,d) is a Ćirić type F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that $d(Tx,Ty) > 0$ implies that

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y)),$$

$\forall x,y \in X$, where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty)+d(y,Tx))\}$.

Notice that every F-contraction is a Ćirić type F-contraction or Hardy-Rogers type F-contraction but the reverse implication does not hold.

Definition 2.11. [1] Let (X,d) be a metric space and $T : X \to B(X)$. we say that T is a generalized multivalued F-contraction, if there exists τ such that

$$\tau + F(\delta(Tx,Ty)) \leq F(M(x,y),$$

for all $x,y \in X$ with $\min\{d(x,y), \delta(Tx,Ty)\} > 0$, where $F \in \mathcal{F}$ and $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}$.

3. Main results

Theorem 3.1. *Let $f,g : X \to X$ be single valued mappings and $S,T : X \to B(X)$ be multi-valued mappings of metric space (X,d). If the two pairs (f,S) and (g,T) are subsequentially continuous and δ-compatible. Then pairs (f,S) and (g,T) have a strict coincidence point. Moreover, f,g,S and T have a common strict fixed point provided there exists $\tau > 0$ such that for all x,y in X we have:*

$$\delta(Sx,Ty) > 0 \text{ implies } \tau + F(\delta(Sx,Ty)) \leq F(R(x,y)), \tag{3.1}$$

where $F \in \mathcal{F}$ and

$$R(x,y) = \max\{d(fx,gy), D(fx,Sx), D(gy,Ty), \frac{1}{2}[D(fx,Ty) + D(gy,Sx)]\}$$

Proof. Since (f, S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = M \in B(X), \quad \lim_{n\to\infty} fx_n = z \in M.$$

$$\lim_{n\to\infty} fSx_n = fM, \quad \lim_{n\to\infty} Sfx_n = Sz,$$

Also, the pair (f, S) is δ-compatible implies that

$$\lim_{n\to\infty} \delta(fSx_n, Sfx_n) = \delta(fM, Sz) = 0,$$

which gives that $fM = Sz = \{fz\}$, and so z is a coincidence point of f and S. Similarly, for the pair (g, T) there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ty_n = N \in B(X) \quad \text{and} \quad \lim_{n\to\infty} gy_n = t \in N$$

and

$$\lim_{n\to\infty} gTy_n = gN, \quad \lim_{n\to\infty} Tgy_n = Tt.$$

The pair (g, T) is δ-compatible, implies that

$$\lim_{n\to\infty} \delta(gTy_n, Tgy_n) = \delta(gN, Tt) = 0.$$

Then $gN = Tt$ and Tt is a singleton, i.e, $Tt = \{gt\}$ and t is strict coincidence point of g and T.
Now, we claim $fz = gt$, if not by using (3.1), $\delta(Sz, Tt) > 0$, if not $d(fz, gt) \leq \delta(Sz, Tt) = 0$, which a contradiction. So we have:

$$\tau + F(\delta(Sz, Tt)) \leq F(R(z, t)).$$

Since $Sz = \{fz\}$ and $Tt = \{gt\}$, then

$$D(fz, Sz) = D(gt, Tt) = 0,$$

$$D(fz, Tt) = d(fz, gt)$$

and $D(gt, Sz) = d(fz, gt)$. Hence

$$\begin{aligned} R(z,t) &= \max\{d(fz, gt), D(fz, Sz), D(gt, Tt), \frac{1}{2}(D(fz, Tt) + D(gt, Sz))\} \\ &= d(fz, gt). \end{aligned}$$

Subsisting in (3.1) we get

$$\tau + F(\delta(Sz, Tt)) \leq F(d(fz, gt)).$$

This yields

$$F(\delta(Sz, Tt)) < \tau + F(\delta(Sz, Tt)) \leq F(d(fz, gt)) = F(\delta(Sz, Tt)),$$

F is a strictly increasing function, implies that

$$\delta(Sz, Tt)) < \delta(Sz, Tt),$$

which is a contradiction. Then $fz = gt$ and so $Sz = Tt$,
Now we claim $z = fz$, if not by taking $x = x_n$ and $y = t$ in (3.1), $\delta(Sx_n, Tt) > 0$, otherwise letting $n \to \infty$, we get

$$d(z, fz) = d(z, gt) \leq \delta(M, Tt) = 0,$$

which contradicts that $z \neq fz$, and so we have

$$\tau + F(\delta(Sx_n, Tt)) \leq F(\max\{d(fx_n, gt), D(fx_n, Sx_n),$$

$$D(gt, Tt), \frac{1}{2}(D(fx_n, Tt) + D(gt, Sx_n))\}).$$

Letting $n \to \infty$, we get:

$$F(d(z, fz) < \tau + F(\delta(M, Tt) \leq F(d(z, fz)),$$

which is a contradiction. Hence z is a fixed point for f and S.
We will show $z = t$, if not by taking $x = x_n$ and $y = y_n$ in (3.1), $\delta(Sx_n, Ty_n) > 0$, if not letting $n \to \infty$, we obtain

$$d(z, t) \leq \delta(M, N) = 0,$$

which is a contradiction, so we have:

$$\tau + F(\delta(Sx_n, Ty_n)) \leq F(\max\{d(fx_n, gy_n), D(fx_n, Sx_n),$$

$$D(gy_n, Ty_n), \frac{1}{2}(D(fx_n, Ty_n) + D(gy_n, Sx_n))\}).$$

Letting $n \to \infty$, we get

$$F(d(z, t) < \tau + F(\delta(M, N) \leq F(d(z, t)),$$

which is a contradiction. Hence $z = t$ and consequently z is a common fixed point for f, g, S and T.
For the uniqueness, suppose there is another fixed point w and using (3.1), $\delta(Sz, Tw) > 0$, if not $d(z, t) \leq \delta(Sz, Tt) = 0$, which is a contradiction, then we have:

$$d(z, w) < \tau + F(\delta(Sz, Tw)) \leq F(d(z, w)),$$

which is a contradiction. Then z is unique. $\square$

If $f = g$ and $S = T$ we obtain the following corollary:

Corollary 3.1. *Let $f : X \to X$ be a single valued mapping and $S : X \to B(X)$ be a multi-valued mapping of metric space (X, d). Suppose that the pair (f, S) is subsequentially continuous, as well is δ-compatible and there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all x, y in X we have:*

$$\delta(Sx, Sy) > 0 \ \ implies \ \ \tau + F(\delta(Sx, Sy)) \leq F(M(x, y)),$$

where $F \in \mathcal{F}$ and

$$R(x, y) = \max\{d(fx, fy), D(fx, Sx), D(fy, Sy), \frac{1}{2}[D(fx, Sy) + D(fy, Sx)]\}.$$

Therefore, f and T have a strict common fixed point.

If S and T are single valued maps, we get the following corollary:

Corollary 3.2. *Let (X, d) be a metric space and let $f, g, S, T : X \to X$ be self mappings if the hybrid pair (f, S) is subsequentially continuous as well as compatible. Then f and S have a coincidence point. Moreover, f and S have a common fixed point provided there exists $\tau > 0$ such that for all x, y in X we have:*

$$d(Sx, Ty) > 0 \ \ implies \ \ \tau + F(d(Tx, Ty)) \leq F(R(x, y)),$$

where $F \in \mathcal{F}$ and

$$R(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\}.$$

Now we shall state and prove our second main result using Hardy-Rogers type F-contractions [7] to establish strict coincidence and common strict fixed point of two hybrid pairs of self maps.

Theorem 3.2. *Let $f, g : X \to X$ be single valued mappings and $S, T : X \to B(X)$ be multi-valued mappings of metric space (X, d) such that the pairs (f, S) and (g, T) are subsequentially continuous as well as δ-compatible. Then, the pairs (f, S) and (g, T) have a strict coincidence point. Moreover, f, g, S and T have a common strict fixed point provided there exists $\tau > 0$ such that for all x, y in X we have: $\delta(Sx, Ty) > 0$ implies*

$$\tau + F(\delta(Sx, Ty)) \leq F\{\alpha d(fx, gy) + \beta d(fx, Sx)$$
$$+\gamma d(gy, Ty) + \lambda d(fx, Ty) + L d(gy, Sx)\}, \tag{3.2}$$

for all $x, y \in X$ with $\delta(Sx, Ty) > 0$, where $F \in \mathcal{F}$, $\alpha + \beta + \gamma + \lambda + L < 1$ and $L \geq 0$.

Proof. As in proof of Theorem 3.1, (f, S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} Sx_n = M \in B(X), \quad \lim_{n \to \infty} fx_n = z \in M$$

and

$$\lim_{n\to\infty} fSx_n = fM, \quad \lim_{n\to\infty} Sfx_n = Sz,$$

again, the pair (f, S) is δ-compatible we get

$$\lim_{n\to\infty} \delta(fSx_n, Sfx_n) = \delta(fM, Sz) = 0,$$

which implies that $fM = Sz = \{fz\}$, and so z is a coincidence point of f and S. Similarly, for g and T there is a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ty_n = N \in B(X) \quad \text{and} \quad \lim_{n\to\infty} gy_n = t \in N$$

and

$$\lim_{n\to\infty} gTy_n = gN, \quad \lim_{n\to\infty} Tgy_n = Tt.$$

The pair (g, T) is δ-compatible, implies that

$$\lim_{n\to\infty} \delta(gTy_n, Tgy_n) = \delta(gN, Tt) = 0.$$

then $gN = Tt$ and Tt is a singleton,i.e, $Tt = \{gt\}$ and t is a strict coincidence point of B and T.
We show $fz = gt$, if not so $\delta(Sz, Tt) > 0$, by using (3.2) we get

$$\begin{aligned} F(\delta(Sz, Tt)) &< \tau + F(\delta(Sz, Tt)) \\ &\leq F((\alpha + \lambda + L)d(fz, gt)) \\ &\leq F(d(fz, gt)) = F(\delta(Sz, Tt)). \end{aligned}$$

Since F is increasing, we get

$$\delta(Sz, Tt) < \delta(Sz, Tt),$$

which is a contradiction. Hence $fz = gt$.
Now we claim $z = fz$, if not by taking $x = x_n$ and $y = t$ in (3.2), $\delta(Sx_n, Tt) > 0$, otherwise letting $n \to \infty$, we get

$$d(z, fz) = d(z, gt) \leq \delta(M, Tt) = 0,$$

which is a contradiction. Then using (3.2) we get

$$\begin{aligned} \tau + F(\delta(Sx_n, Tt)) \leq F\{&(\alpha d(fx_n, gt) + \beta d(fx_n, Sx_n) \\ &+\gamma D(gt, Tt) + \lambda d(fx_n, Tt) + LDd(gt, Sx_n)\}. \end{aligned}$$

Taking $n \to \infty$, we get

$$\tau + F(\delta(M, Tt)) \leq F((\alpha + \lambda + L)d(z, fz))$$

then

$$F(d(z, fz)) < \tau + F(\delta(M, Tt)) \leq F(d(z, fz)),$$

which is a contradiction. Hence $z = fz$.
We will show $z = t$, if not by taking $x = x_n$ and $y = y_n$ in (3.2), $\delta(Sx_n, Ty_n) > 0$, if not letting $n \to \infty$, we get:

$$d(z,t) \leq \delta(M,N) = 0,$$

which is a contradiction, using (3.2 we get:

$$\tau + F(\delta(Sx_n, Ty_n)) \leq F(\alpha d(fx_n, gy_n) + \beta D(fx_n, Sx_n)$$

$$+\gamma D(gy_n, Ty_n) + \lambda D(fx_n, Ty_n) + L(gy_n, Sx_n))).$$

Letting $n \to \infty$, we get

$$F(d(z,t) < \tau + F(\delta(M,N) \leq F((\alpha + \lambda + L)d(z,t))$$

$$\leq F(d(z,t)),$$

which is a contradiction. Hence $z = t$ and consequently z is a common fixed point for f, g, S and T.
For the uniqueness, suppose there is another fixed point w and using (3.2) we get:

$$d(z,w) < \tau + F(\delta(Sz, Tw)) \leq F(\alpha d(z,w) + \beta d(z, Sz)$$

$$+\gamma d(w, Tw) + \lambda d(z, Tw) + Ld(w, Sz))$$

$$\leq F((\alpha + \lambda + L)d(z,w))$$

$$\leq F(d(z,w)),$$

which is a contradiction. Then z is unique. □

Example 3.1. Let $X = [0,4]$, $d(x,y) = |x-y|$ and f, g, S and T defined by

$$fx = gx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 1, & 2 < x \leq 4 \end{cases} \qquad Tx = Sx = \begin{cases} \{2\}, & 0 \leq x \leq 2 \\ [\frac{3}{2}, 2], & 2 < x \leq 4 \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geq 1$ such that $x_n = 2 - \frac{1}{n}$, it is clear that

$$\lim_{n\to\infty} fx_n = 2 \in \{2\}$$

and

$$\lim_{n\to\infty} Sx_n = \{2\},$$

which implies that the pair (f, S) is subsequentially continuous. On other hand, we have

$$\lim n \to \infty \delta(fSx_n, Sfx_n) = \delta(\{2\}, \{2\}) = 0,$$

so (f, S) is δ-compatible.
For the inequality (3.1), we discuss the following cases:

1. For $x, y \in [0,1]$, we have: $\delta(Sx, Sy) = 0$, so (3.1) is satisfied for all x, y.

2. For $x \in [0,1]$ and $y \in (1,5]$, we have:

$$\delta(Sx, Sy) = \frac{1}{2} \leq e^{-\frac{1}{2}} \leq e^{-\frac{1}{2}} D(fy, Sy).$$

3. For $x, y \in (1,5]$ we have

$$\delta(Sx, Ty) = \frac{1}{2} \leq e^{-\frac{1}{2}} = e^{-\frac{1}{2}} D(fx, Sx).$$

4. For $x \in (1,5]$ and $y \in [0,1]$ we have

$$\delta(Sx, Sy) = \frac{1}{2} \leq e^{-\frac{1}{2}} = e^{-\frac{1}{2}} D(fx, Sx).$$

Then f and S satisfy(3.1), therefore 2 is the unique common strict fixed point of f and S.

4. Application to integral inclusions

In this subsection, we shall apply the obtained results to assert the existence of solution for a system of integral inclusions.
Let us consider the following integral inclusion systems.

$$x_i(t) \in f(t) + \int_0^1 K_i(t, s, x_i(s))ds, i = 1, 2 \tag{4.1}$$

where f is a continuous function on $[0,1]$, i,e., $f \in C([0,1])$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to CB(\mathbb{R})$ is a set valued function.
Clearly $X = C([0,1])$ with convergence uniform metrics $d_\infty(x, y) = \sup_{x \in X} |x(t) - y(t)|$ is a complete metric space. Assume that

1. the function $K_i : (t, s) \mapsto K(t, s, x_1(s))$ is continuous on $[0,1] \times (0,1]$ for all $x \in C((0,1])$.
2. For all $x_i \in X$ and $k_i \in K_i$ $(i = 1, 2)$, there exists a function $\varphi : [0,1] \times [0,1] \to [0, +\infty)$ such that

$$|k_1(t, s, x_1(s)) - k_2(t, s, x_2(s)) \leq \varphi(t, s)|x_1 - x_2|.$$

3. There exists $\tau > 0$ such that

$$\sup_{t \in [0,1]} \int_0^1 \varphi(t, s)ds \leq e^{-\tau}.$$

4. There exist two sequences $\{x_n\}, \{y_n\}$ and two elements x, y in X such that

$$\lim_{n \to \infty} Sx_n = M \in B(X),$$

$$\lim_{n \to \infty} x_n = x \in M$$

and

$$\lim_{n\to\infty} Ty_n = N \in B(X),$$
$$\lim_{n\to\infty} y_n = y \in N.$$

Theorem 4.1. *Under the assumptions* (1) − (4) *the system of integral inclusions (4.1) has a solution in* $C((0,1]) \times C([0,1])$.

Proof. Define two set valued mapping:

$$Sx_1(t) = \{z \in X, z(t) \in f(t) + \int_0^1 K_1(t,s,x_1(s))ds\},$$

$$Tx_2(t) = \{z \in X, z(t) \in f(t) + \int_0^1 K_2(t,s,x_2(s))ds\}.$$

The system (4.1) has a solution if and only if S and T have a common fixed point. Denote I_X the identity operator on X.
From condition (4), the two pairs (I_X, S) and $(I_{X,T})$ are subsequentially continuous as well as δ-compatible.

For the contractive condition (3.1), let $x_1, x_2 \in C([0,1])$ and $z_1 \in Sx_1$, then there exists $k_1 \in K_1$ such that

$$z_1(t) = \int_0^1 k_1(s,t)ds,$$

for $z_2 \in f(t) + \int_0^1 K_2(t,s,x_2(t))ds$, i.e., $z_2(t) = f(t) + \int_0^1 k_2(t,s)ds$, we have

$$|z_1 - z_2| \leq \int_0 |k_1(t,s) - k_2(t,s)|ds$$
$$\leq \int_0^1 |x_1 - x_2|\varphi(t,s)ds.$$

Since $K_i, i = 1,2$ are bounded, so we have

$$\sup_{z_i \in X} |z_1 - z_2 \leq \|x_1 - x_2\|_\infty \int_0^1 \varphi(t,s)ds,$$

which implies that

$$\delta(Sx_1, Tx_2) \leq e^{-\tau} d(x_1, x_2)$$
$$\leq e^{-\tau} \max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{1}{2}(d(x_1,Tx_2) + d(x_2,Sx_1))\}.$$

taking logarithm of two sides we get

$$\ln(\delta(Sx_1,Tx_2)) \leq -\tau + \ln\Big(\max\{d(x_1,x_2), d(x_1,Sx_1), d(x_2,Tx_2), \frac{1}{2}(d(x_1,Tx_2)+d(x_2,Sx_1))\}\Big).$$

Hence all hypotheses of Theorem 3.1 satisfied with $F(t) = \ln t$ and $f = g = I_X$, therefore the system (4.1) has a solution. □

Conclusion. We have established common fixed point theorems for two hybrid pairs contraction of Wordowski type using δ-distance without exploiting the notion of continuity or reciprocal continuity, weak reciprocal continuity. Since F-contraction is a proper generalization of ordinary contraction, our results generalize, extend and improve the results of Wordowski [16] and others existing in literature, for instance Acar et al. [1], Ćirić [6], Cosentino et al. [7], Hardy Rogers [9] and Minak et al.[11] without using the completeness of space or subspace, and the containment requirement of range space.

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Δ^m-STATISTICAL CONVERGENCE OF ORDER $\tilde{\alpha}$ FOR DOUBLE SEQUENCES OF FUNCTIONS

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Abstract. In this paper, we have introduced and examined the concepts of Δ^m-pointwise and Δ^m-uniform statistical convergence of order $\tilde{\alpha}$ for double sequences of real valued functions. Also, we have given the concept of Δ^m-statistically Cauchy sequence for double sequences of real valued functions and proven that it is equivalent to Δ^m-pointwise statistical convergence of order $\tilde{\alpha}$ for double sequences of real valued functions. Some relations between $S^2_{\tilde{\alpha}}(\Delta^m, f)$-statistical convergence and strong $\left[w^2_p\right]_{\tilde{\alpha}}(\Delta^m, f)-$summability have also been given.
Keywords. Statistical convergence; Cauchy sequence; summability.

1. Introduction

The idea of statistical convergence was given by Zygmund [27]] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [25] and Fast [11] and later reintroduced by Schoenberg [23] independently. Many mathematicians have studied various properties of statistical convergence and applications of this concept in different areas such as Fourier analysis, Ergodic theory, number theory, measure theory, Trigonometric series, Turnpike theory and Banach spaces Cınar *et al.* [1], Colak [2], Colak and Altın [3], Connor [4], Et *et al.* ([7],[8],[9],[10]), Fridy [12], Işık [16], Mohiuddine *et al.* [18], Móricz [19], Mursaleen [21], Et and Şengül [24], Tripathy and Sarma [26] and many authors have examined the relationship between statistical convergence with sequences spaces and summability theory.

Pointwise and uniform statistical convergence of sequences of real valued functions were defined by Gökhan *et al.* ([13], [14], [15]) and independently by Duman and Orhan [5]. The aim of the present paper is to introduce and examine the concepts of Δ^m-pointwise and Δ^m-uniform statistical convergence of order $\tilde{\alpha}$ for

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double sequences of real valued functions. In Section 2 we give a brief overview of statistical convergence of order $\tilde{\alpha}$ and strong $p-$Cesàro summability of double sequences of functions. In Section 3 we give the concepts of Δ^m-pointwise and Δ^m-uniform statistical convergence of order $\tilde{\alpha}$ and the concept of Δ^m-statistically Cauchy sequence for sequences of real valued functions.

2. Definition and Preliminaries

A double sequence $x = (x_{jk})$ is said to be convergent in the Pringsheim [22] sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case, we write $P - \lim x = L$.

A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ is bounded if there exists a positive real number M such that $|x_{jk}| < M$ for all j and k, that is, $\|x\| = \sup_{j,k \geq 0} |x_{jk}| < \infty$. Although every convergent single sequence is bounded, a convergent double sequence need not be bounded.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K(m,n) = \{(j,k) : j \leq m, k \leq n\}$. The double natural density of K is defined by

$$\delta^2(K) = P - \lim_{m,n} \frac{1}{mn} |K(m,n)|, \qquad \text{if the limit exists.}$$

A double sequence $x = (x_{jk})$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$ the set $\{(j,k) : j \leq m, k \leq n : |x_{jk} - L| \geq \varepsilon\}$ has double natural density zero [21].

A convergent double sequence is statistically convergent but the converse is not true in general. Also, a statistically convergent double sequence need not be bounded.

Throughout the paper, we have taken $s, t, u, v \in (0,1]$ and written $\tilde{\alpha}$ instead of (s,t) and $\tilde{\beta}$ instead of (u,v). We have defined

$$\begin{array}{rcl}
\tilde{\alpha} & \preceq & \tilde{\beta} \Leftrightarrow s \leq u \text{ and } t \leq v \\
\tilde{\alpha} & \prec & \tilde{\beta} \Leftrightarrow s < u \text{ and } t < v \\
\tilde{\alpha} & \cong & \tilde{\beta} \Leftrightarrow s = u \text{ and } t = v \\
\tilde{\alpha} & \in & (0,1] \Leftrightarrow s, t \in (0,1] \\
\tilde{\beta} & \in & (0,1] \Leftrightarrow u, v \in (0,1] \\
\tilde{\alpha} & \cong & 1 \text{ in case } s = t = 1 \\
\tilde{\beta} & \cong & 1 \text{ in case } u = v = 1 \\
\tilde{\alpha} & \succ & 1 \text{ in case } s > 1 \text{ and } t > 1
\end{array}$$

Let $\tilde{\alpha} \in (0,1]$ be given. The $\tilde{\alpha}-$double density of a subset K of $\mathbb{N} \times \mathbb{N}$ was defined by Çolak and Altın as follows [3]

$$\delta_{\tilde{\alpha}}^2 (K) = P - \lim_{n,m} \frac{1}{n^s m^t} |K(n,m)|, \text{ if the limit exists.}$$

$\delta^2 (K^c) = 1 - \delta^2 (K)$ holds, but $\delta_{\tilde{\alpha}}^2 (K^c) = 1 - \delta_{\tilde{\alpha}}^2 (K)$ does not hold for $0 < \widetilde{\alpha} < 1$ in general.

A double sequence $x = (x_{jk})$ is said to be statistically convergent order $\widetilde{\alpha}$ to the number L if for each $\varepsilon > 0$, the set

$$\{(j,k) : j \leq n, k \leq m : |x_{jk} - L| \geq \varepsilon\}$$

has double natural density zero, i.e.

$$|x_{jk} - L| < \varepsilon \quad a.a. (j,k) (\widetilde{\alpha}).$$

A double sequence $x = (x_{jk})$ is said to be strongly Cesàro summable to a number L if

$$P - \lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} |x_{jk} - L| = 0.$$

The idea of difference sequences defined by Kızmaz [17] and the notion was generalized by Et and Çolak [6] such as

$$\Delta^m (X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. Recently difference sequence spaces have been studied in ([7], [8], [20]).

For a double sequence $x = (x_{jk})$ we have generalized difference sequences as follows:

$$\Delta^m x_{jk} = \sum_{v_1=0}^{m} \sum_{v_2=0}^{m} (-1)^{v_1+v_2} \binom{m}{v_1} \binom{m}{v_2} x_{j+v_1 k+v_2}$$

where $\Delta x_{jk} = x_{jk} - x_{jk+1} - x_{j+1k} + x_{j+1k+1}$ for all $j,k \in \mathbb{N}$.

3. Main Result

In this section, we have given the relations between Δ^m-pointwise statistical convergence of order $\tilde{\alpha}$ and Δ^m-pointwise statistical convergence of order $\tilde{\beta}$ and the relations between strong Δ_p^m-pointwise Cesàro summability of order $\tilde{\alpha}$ and strong Δ_p^m-pointwise Cesàro summability of order $\tilde{\beta}$ and the relations between strong Δ_p^m-pointwise Cesàro summability of order $\tilde{\alpha}$ and Δ^m-pointwise statistical convergence of order $\tilde{\beta}$ for double sequences of functions, where $\tilde{\alpha} \preceq \tilde{\beta}$.

Definition 3.1. Let $\tilde{\alpha} \in (0,1]$ be given. A double sequence of functions $\{f_{jk}\}$ is said to be Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$ (or $S^2_{\tilde{\alpha}}(\Delta^m, f)-$summable) to the function f on a set A if for every $\varepsilon > 0$ and for every $x \in A$

$$\lim_{n,m} \frac{1}{n^s m^t} \left|\{(j,k) : j \leq n, k \leq m : |\Delta^m f_{jk}(x) - f(x)| \geq \varepsilon \}\right| = 0$$

i.e. for every $x \in A$,

$$|\Delta^m f_{jk}(x) - f(x)| < \varepsilon \qquad a.a.\,(j,k) \ \ (\tilde{\alpha}).$$

In this case, we write $S^2_{\tilde{\alpha}} - \lim \Delta^m f_{jk}(x) = f(x)$ on A. The function f is said to be double Δ^m-statistical limit of order $\tilde{\alpha}$ of the sequence $\{f_{jk}\}$ (or Pringsheim Δ^m-statistical limit of order $\tilde{\alpha}$). The set of all Δ^m-pointwise statistically convergent sequences of functions order $\tilde{\alpha}$ will be denoted by $S^2_{\tilde{\alpha}}(\Delta^m, f)$.

For $\tilde{\alpha} \in (0,1]$, Δ^m-pointwise statistical convergence of order $\tilde{\alpha}$ is well defined, but is not well defined for $\tilde{\alpha} \succ 1$. For this, a sequence of functions have been defined $\{f_{jk}\}$ by

$$f_{jk}(x) = \begin{cases} 1 & j+k = 2n \\ x^{j+k} & j+k \neq 2n \end{cases} \qquad n = 1,2,3..., x \in \left[0, \tfrac{1}{2}\right].$$

Then we calculate $\Delta f_{jk}(x)$ as follows;

$$\Delta f_{jk}(x) = \begin{cases} 2 - 2x^{j+k+1} & j+k = 2n \\ x^{j+k} + x^{j+k+2} - 2 & j+k \neq 2n \end{cases} \qquad n = 1,2,3..., x \in \left[0, \tfrac{1}{2}\right].$$

Then for every $x \in A$, both

$$\lim_{n,m\to\infty} \frac{1}{n^s m^t} \left|\left\{(j,k) : j \leq n, k \leq m : \left|\Delta f_{jk}(x) - \left(2 - 2x^{j+k+1}\right)\right| \geq \varepsilon\right\}\right|$$

$$\leq \lim_{n,m\to\infty} \frac{\left(\frac{n}{2}+1\right)\left(\frac{m}{2}+1\right)}{n^s m^t} = 0$$

and

$$\lim_{n,m\to\infty} \frac{1}{n^s m^t} \left|\left\{(j,k) : j \leq n, k \leq m : \left|\Delta f_{jk}(x) - \left(x^{j+k} + x^{j+k+2} - 2\right)\right| \geq \varepsilon\right\}\right|$$

$$\leq \lim_{n,m\to\infty} \frac{\left(\frac{n}{2}+1\right)\left(\frac{m}{2}+1\right)}{n^s m^t} = 0$$

for $\tilde{\alpha} \succ 1$, hence $S^2_{\tilde{\alpha}} - \lim \Delta f_{jk}(x) = 2$ and $S^2_{\tilde{\alpha}} - \lim \Delta f_{jk}(x) = -2$ which is impossible.

Theorem 3.1. *Let $\tilde{\alpha} \in (0,1]$, $\{f_{jk}\}$ and $\{g_{jk}\}$ be two double sequences of real valued functions defined on a set A.*

(i) If $S_{\tilde{\alpha}}^2 - \lim \Delta^m f_{jk}(x) = f(x)$ and $c \in \mathbb{R}$, then $S_{\tilde{\alpha}}^2 - \lim c\Delta^m f_{jk}(x) = cf(x)$,

(ii) If $S_{\tilde{\alpha}}^2 - \lim \Delta^m f_{jk}(x) = f(x)$ and $S_{\tilde{\alpha}}^2 - \lim \Delta^m g_{jk}(x) = g(x)$, then $S_{\tilde{\alpha}}^2 - \lim(\Delta^m f_{jk}(x) + \Delta^m g_{jk}(x)) = f(x) + g(x)$.

Proof. Omitted. □

It is easy to see that every Δ^m-pointwise convergent sequences of function is Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$, but the converse does not hold. To see this, a sequence $\{f_{jk}\}$ is defined by

$$f_{jk}(x) = \begin{cases} 1 & j,k = n^2 \\ \frac{jkx}{2+j^2k^2x^2} & j,k \neq n^2 \end{cases}.$$

Then we calculate $\Delta f_{jk}(x)$ as follows:

$$\Delta f_{jk}(x) = \begin{cases} 1 - \frac{j(k+1)x}{2+j^2(k+1)^2x^2} - \frac{(j+1)kx}{2+(j+1)^2k^2x^2} + \frac{(j+1)(k+1)x}{2+(j+1)^2(k+1)^2x^2} & j,k = n^2 \\ \frac{jkx}{2+j^2k^2x^2} - \frac{j(k+1)x}{2+j^2(k+1)^2x^2} - \frac{(j+1)kx}{2+(j+1)^2k^2x^2} - 1 & j,k = n^2 - 1 \\ \frac{jkx}{2+j^2k^2x^2} - \frac{j(k+1)x}{2+j^2(k+1)^2x^2} - \frac{(j+1)kx}{2+(j+1)^2k^2x^2} + \frac{(j+1)(k+1)x}{2+(j+1)^2(k+1)^2x^2} & j,k \neq n^2 \end{cases}$$

The sequence $\{f_{jk}\}$ is $\Delta-$pointwise statistically convergent of order $\tilde{\alpha}$ with $S_{\tilde{\alpha}}^2 - \lim \Delta f_{jk}(x) = 0$ for $\tilde{\alpha} \succ \frac{1}{2}$, but it is not $\Delta-$pointwise convergent.

Theorem 3.2. *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ be given such that $\tilde{\alpha} \preceq \tilde{\beta}$, then $S_{\tilde{\alpha}}^2(\Delta^m, f) \subseteq S_{\tilde{\beta}}^2(\Delta^m, f)$ and the inclusion is strict.*

Proof. The inclusion part of the proof is easy. To show that the inclusion is strict, a double sequence $\{f_{jk}\}$ is defined by

$$f_{jk}(x) = \begin{cases} 1 & j,k = n^2 \\ \frac{j^2k^2x}{1+j^3k^3x^2} & j,k \neq n^2 \end{cases}.$$

So we have

$$\Delta f_{jk}(x) = \begin{cases} 1 - \frac{j^2(k+1)^2x}{1+j^3(k+1)^3x^2} - \frac{(j+1)^2k^2x}{1+(j+1)^3k^3x^2} + \frac{(j+1)^2(k+1)^2x}{1+(j+1)^3(k+1)^3x^2} & j,k = n^2 \\ \frac{j^2k^2x}{1+j^3k^3x^2} - \frac{j^2(k+1)^2x}{1+j^3(k+1)^3x^2} - \frac{(j+1)^2k^2x}{1+(j+1)^3k^3x^2} - 1 & j,k = n^2 - 1 \\ \frac{j^2k^2x}{1+j^3k^3x^2} - \frac{j^2(k+1)^2x}{1+j^3(k+1)^3x^2} - \frac{(j+1)^2k^2x}{1+(j+1)^3k^3x^2} + \frac{(j+1)^2(k+1)^2x}{1+(j+1)^3(k+1)^3x^2} & j,k \neq n^2 \end{cases}.$$

Then $S_{\tilde{\beta}}^{2}-\lim \Delta f_{jk}(x)=0$ for $\tilde{\beta}\in\left(\frac{1}{2},1\right]$, but $f\notin x\in S_{\tilde{\alpha}}^{2}(\Delta,f)$ for $\tilde{\alpha}\in\left(0,\frac{1}{2}\right]$. □

Corollary 3.1. *If a double sequence of functions $\{f_{jk}\}$ is $\Delta^{m}-$pointwise statistically convergent of order $\tilde{\alpha}$ to the function f, then it is $\Delta^{m}-$pointwise statistically convergent to the function f.*

Definition 3.2. Let $\tilde{\alpha}\in(0,1]$. The sequence $\{f_{jk}\}$ is a $\Delta^{m}-$pointwise statistically Cauchy sequence of order $\tilde{\alpha}$, provided that for every $\varepsilon>0$ there are two numbers $N\,(=N(\varepsilon))\,,M\,(=M(\varepsilon))$ such that

$$|\Delta^{m}f_{jk}(x)-\Delta^{m}f_{N,M}(x)|<\varepsilon \quad a.a.\,(j,k)\ \ (\tilde{\alpha})\ \text{ and for each } x\in A$$

i.e.

$$\lim_{n,m\to\infty}\frac{1}{n^{s}m^{t}}\left|\{(j,k):j\leq n,k\leq m:|\Delta^{m}f_{jk}(x)-\Delta^{m}f_{N,M}(x)|\geq\varepsilon\}\right|=0$$

for each $x\in A$.

Using the same technique in proof of [1][Thorem3.4], we obtain the proof of the following theorem.

Theorem 3.3. *Let $\{f_{jk}\}$ be a double sequence of functions defined on a set A. The following statements are equivalent:*

(i) $\{f_{jk}\}$ is $\Delta^{m}-$pointwise statistically convergent of order $\tilde{\alpha}$ to $f(x)$ on A;

(ii) $\{f_{jk}\}$ is $\Delta^{m}-$pointwise statistically Cauchy sequence of order $\tilde{\alpha}$ on A;

(iii) $\{f_{jk}\}$ is a double sequence of functions for which there is a $\Delta^{m}-$pointwise convergent sequence of functions $\{g_{jk}\}$ such that $\Delta^{m}f_{jk}(x)=\Delta^{m}g_{jk}(x)$ a.a. (j,k) $(\tilde{\alpha})$ for every $x\in A$.

Definition 3.3. Let $\tilde{\alpha}\in(0,1]$ and p be a positive real number. A double sequence of functions $\{f_{jk}\}$ is said to be strongly $\Delta_{p}^{m}-$pointwise Cesàro summable of order $\tilde{\alpha}$ (or $\left[w_{p}^{2}\right]_{\tilde{\alpha}}(\Delta^{m},f)-$summable) if there is a function f such that

$$\lim_{n,m\to\infty}\frac{1}{n^{s}m^{t}}\sum_{j=1}^{n}\sum_{k=1}^{m}|\Delta^{m}f_{jk}(x)-f(x)|^{p}=0.$$

In this case, we write $\left[w_{p}^{2}\right]_{\tilde{\alpha}}-\lim \Delta^{m}f_{jk}(x)=f(x)$ on A. The set of all strongly $\Delta_{p}^{m}-$Cesàro summable double sequences of functions of order $\tilde{\alpha}$ will be denoted by $\left[w_{p}^{2}\right]_{\tilde{\alpha}}(\Delta^{m},f)$.

Theorem 3.4. *Let p be a positive real number and $\tilde{\alpha},\tilde{\beta}\in(0,1]$ such that $\tilde{\alpha}\preceq\tilde{\beta}$. Then $\left[w_{p}^{2}\right]_{\tilde{\alpha}}(\Delta^{m},f)\subseteq\left[w_{p}^{2}\right]_{\tilde{\beta}}(\Delta^{m},f)$ and the inclusion is strict for some $\tilde{\alpha}=(s,t)$ and $\tilde{\beta}=(u,v)$ such that $\tilde{\alpha}\prec\tilde{\beta}$.*

Proof. The inclusion part of the proof is easy. To show that the inclusion is strict define a double sequence $\{f_{jk}\}$ by

$$f_{jk}(x) = \begin{cases} \frac{jkx}{1+jkx} & j,k = n^2 \\ 0 & j,k \neq n^2 \end{cases} \quad x \in [1,2].$$

Then we calculate $\Delta f_{jk}(x)$ as follows

$$\Delta f_{jk}(x) = \begin{cases} \frac{jkx}{1+jkx} & j,k = n^2 \\ \frac{-(j+1)(k+1)x}{1+(j+1)(k+1)x} & j,k = n^2-1 \\ 0 & j,k \neq n^2 \end{cases}.$$

Therefore we get

$$\frac{1}{n^s m^t}\sum_{j=1}^{n}\sum_{k=1}^{m}|\Delta f_{jk}(x) - f(x)|^p \leq \frac{2\sqrt{n}\sqrt{m}}{n^s m^t} = \frac{1}{n^{s-\frac{1}{2}}m^{t-\frac{1}{2}}} \to 0 \ as \ n,m \to \infty$$

and so the sequence $\{f_{jk}\}$ is strongly Δ_p-pointwise Cesàro summable of order $\tilde{\alpha}$, for $\tilde{\alpha}, \tilde{\beta} \in \left(\frac{1}{2}, 1\right]$, but since

$$\frac{1}{n^s m^t}\sum_{j=1}^{n}\sum_{k=1}^{m}|\Delta f_{jk}(x) - f(x)|^p \geq \frac{2\sqrt{n}\sqrt{m}}{2n^s m^t} \to \infty \quad as \ n,m \to \infty.$$

the sequence $\{f_{jk}\}$ is not strongly Δ_p-pointwise Cesàro summable of order $\tilde{\alpha}$, for $\tilde{\alpha}, \tilde{\beta} \in \left(0, \frac{1}{2}\right]$. □

Corollary 3.2. *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ and p be a positive real number. Then*

(*i*) *if $\tilde{\alpha} \cong \tilde{\beta}$, then $\left[w_p^2\right]_{\tilde{\alpha}}(\Delta^m, f) = \left[w_p^2\right]_{\tilde{\beta}}(\Delta^m, f)$,*

(*ii*) *$\left[w_p^2\right]_{\tilde{\alpha}}(\Delta^m, f) \subseteq \left[w_p^2\right](\Delta^m, f)$ for each $\tilde{\alpha} \in (0,1]$ and $0 < p < \infty$.*

Definition 3.4. A double sequence of functions $\{f_{jk}\}$ is said to be $\Delta^m_{(C,1,1)}-$pointwise statistically summable of order $\tilde{\alpha}$ (or $(C,1,1)_{S^2_{\tilde{\alpha}}}$ $-$summable) to the function f if for every $\varepsilon > 0$ and $x \in A$, the set $K_\epsilon(\sigma_{mn})$ has double natural density zero. In this case we write $(C,1,1)_{S^2_{\tilde{\alpha}}} - \lim \Delta^m f_{jk} = f$, where

$$K_\epsilon(\sigma_{mn}) = \{(j,k) : j \leq n, k \leq m : |\sigma_{mn}(\Delta^m, f) - f(x)| \geq \varepsilon\}$$

and

$$\sigma_{mn}(\Delta^m, f) = \frac{1}{(n+1)(m+1)}\sum_{j=0}^{n}\sum_{k=0}^{m}\Delta^m f_{jk}(x).$$

Theorem 3.5. *If a double sequence of functions $\{f_{jk}\}$ is bounded and $S^2_{\tilde{\alpha}}(\Delta^m, f)-$summable to f then it is statistically $(C,1,1)_{S^2_{\tilde{\alpha}}}-$summable to f, but the converse does not hold.*

Proof. Let $\{f_{jk}\}$ be bounded and $S^2_{\tilde{\alpha}}(\Delta^m, f)-$summable to f, we can write $\sup\limits_{j,k}|\Delta^m f_{jk} - f| = M$ and K_ϵ has double natural density zero, where

$$K_\epsilon = \{(j,k) : j \le n, k \le m : |\Delta^m f_{jk}(x) - f(x)| \ge \varepsilon\}.$$

Then

$$\begin{aligned}
&\left|\frac{1}{(n+1)(m+1)}\sum_{j=0}^{n}\sum_{k=0}^{m}\Delta^m f_{jk}(x) - f(x)\right| \le \frac{1}{(n+1)(m+1)}\sum_{j=0}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| \\
= {} & \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\in K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| + \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\notin K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| \\
\le {} & \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\in K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| + \frac{1}{(n+1)(m+1)}\sum_{\substack{j=0 \\ (j,k)\in K(\varepsilon)}}^{n}\sum_{k=0}^{m}|\Delta^m f_{jk}(x) - f(x)| \\
= {} & \frac{1}{(n+1)(m+1)}M|K(\varepsilon)| + \varepsilon \to 0 \ as \ n, m \to \infty
\end{aligned}$$

which implies that $P - \lim \sigma_{mn}(\Delta^m, f) = f$.

For the converse if we define $f_{jk}(x) = (-1)^{j+k} x$, $x \in (0,1)$ then we get $\Delta f_{jk}(x) = 4(-1)^{j+k} x$. The sequence of functions $\{f_{jk}\}$ is statistically $(C,1,1)_{S^2_{\tilde{\alpha}}}-$summable of order $\tilde{\alpha}$ to 0 but neither bounded nor statistically convergent. □

Definition 3.5. Let $\tilde{\alpha}$ be any real number such that $\tilde{\alpha} \in (0,1]$. A double sequence of functions $\{f_{jk}\}$ is said to be Δ^m-uniformly statistically convergent of order $\tilde{\alpha}$ to the function f on a set A if, for every $\varepsilon > 0$

$$P-\lim_{n,m\to\infty}\frac{1}{n^s m^t}|\{(j,k) : j \le n, k \le m : |\Delta^m f_{jk}(x) - f(x)| \ge \varepsilon \text{ for all } x \in A\}| = 0.$$

i.e., for all $x \in A$,

$$|\Delta^m f_{jk}(x) - f(x)| < \varepsilon \ a.a.(j,k)(\tilde{\alpha})$$

In this case we write

$$S^2_{\tilde{\alpha}} - \lim \Delta^m f_{jk}(x) = f(x) \text{ uniformly on } A \text{ or } S^2_{\tilde{\alpha},u} - \lim \Delta^m f_{jk}(x) = f(x) \text{ on } A.$$

The set of all Δ^m-uniformly statistically convergent sequences of order $\tilde{\alpha}$ will be denoted by $S^2_{\tilde{\alpha},u}(\Delta^m, f)$.

We can give this definition as follows:

f_{jk}, Δ^m-uniformly statistically of order $\tilde{\alpha}$ converges to $f \iff$ for all $\varepsilon > 0$, $\exists$K$\subset \mathbb{N}\times\mathbb{N}$, $\delta^2_{\tilde{\alpha}}(K) = 1$ and exists $(n_0, m_0) \in K$, $n_0 = n_0(\varepsilon)$,$m_0 = m_0(\varepsilon) \ni$,$\forall j > n_0$, $k > m_0$ and$(j,k)\in$ K and $\forall \in A$, $|\Delta^m f_{jk}(x) - f(x)| < \varepsilon$.

Theorem 3.6. *Let f and f_{jk} (for all $j,k\in\mathbb{N}$) be continuous functions on $A=[a,b]\subset\mathbb{R}$ and $\tilde{\alpha}\in(0,1]$. Then $S_{\tilde{\alpha}}^2-\lim\Delta^m f_{jk}(x)=f(x)$ uniformly on A if and only if $S_{\tilde{\alpha}}^2-\lim c_{j,k}=0$, where $c_{j,k}=\max\limits_{x\in A}|\Delta^m f_{jk}(x)-f(x)|$.*

Proof. Omitted. □

It follows from (3.2) that, if $\lim\Delta^m f_{jk}(x)=f(x)$ uniformly on A, then $S_{\tilde{\alpha}}^2-\lim\Delta^m f_{jk}(x)=f(x)$ uniformly on A. But the converse is not true, for this consider a sequence defined by

$$f_{jk}(x)=\begin{cases} 3 & j=m^2,k=n^2 \\ \frac{j}{1+j^2x^2} & otherwise \end{cases}\quad j,k=1,2,3...,x\in[0,1].$$

So we have

$$\Delta f_{j,k}(x)=\begin{cases} 3-\frac{j}{1+j^2x^2} & j=m^2,k=n^2 \\ -3+\frac{(j+1)}{1+(j+1)^2x^2} & j=m^2-1,k=n^2 \\ -\frac{(j+1)}{1+(j+1)^2x^2}+3 & j=m^2-1,k=n^2-1 \\ \frac{j}{1+j^2x^2}-3 & j=m^2,k=n^2-1 \\ 0 & otherwise \end{cases}.$$

Then $\{f_{jk}\}$ is $\Delta-$uniformly statistically convergent sequences of order $\tilde{\alpha}$ to $f(x)=0$ on $[0,1]$ for $\tilde{\alpha}\in\left[\frac{1}{2},1\right]$ since $S_{\tilde{\alpha}}^2-\lim c_{j,k}=0$, where

$$c_{jk}=\max_{x\in[0,1]}|\Delta f_{jk}(x)-0|=\begin{cases} 3-\frac{1}{2\sqrt{j}} & j=m^2,k=n^2 \\ 0 & otherwise \end{cases},$$

but $\{f_{jk}\}$ is not $\Delta-$uniformly convergent on $[0,1]$ since $\lim\limits_{k\to\infty}c_{j,k}$ does not exist.

It can be shown that if a sequence $\{f_{jk}\}$ is Δ^m-uniformly statistically convergent of order $\tilde{\alpha}$, then it is Δ^m-pointwise statistically convergent of order $\tilde{\alpha}$, but the converse does not hold. For this consider a sequence defined by

$$f_{jk}(x)=\begin{cases} 1 & j=m^2,k=n^2 \\ \frac{j^2x}{1+j^3x^2} & otherwise \end{cases}\quad j,k=1,2,3...,\ k\in\mathbb{N},x\in[0,1].$$

then we have

$$\Delta f_{j,k}(x) = \begin{cases} 1 - \frac{j^2 x}{1+j^3x^2} & j = m^2, k = n^2 \\ -1 + \frac{(j+1)^2 x}{1+(j+1)^3x^2} & j = m^2 - 1, k = n^2 \\ -\frac{j^2 x}{1+j^3x^2} + 1 & j = m^2 - 1, k = n^2 - 1 \\ \frac{(j+1)^2 x}{1+(j+1)^3x^2} - 1 & j = m^2, k = n^2 - 1 \\ 0 & otherwise \end{cases}$$

The sequence $\{f_{jk}\}$ is $\Delta-$pointwise statistically convergent of order $\tilde{\alpha}$ to $f(x) = 0$ on $[0,1]$ but $\{f_{jk}\}$ is not $\Delta-$uniformly statistically convergent of order $\tilde{\alpha}$ to $f(x) = 0$ on $[0,1]$ by Theorem 3.13, because

$$c_{jk} = \max x \in [0,1] \left|\Delta f_{jk}(x) - 0\right| = \begin{cases} 1 - \frac{\sqrt{j}}{2} & j = m^2, k = n^2 \\ -1 + \frac{\sqrt{j+1}}{2} & j = m^2 - 1, k = n^2 \\ -\frac{\sqrt{j}}{2} + 1 & j = m^2 - 1, k = n^2 - 1 \\ \frac{\sqrt{j+1}}{2} - 1 & j = m^2, k = n^2 - 1 \\ 0 & otherwise \end{cases}$$

and $S^2_{\tilde{\alpha}} - \lim c_{j,k}$ does not exist.

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THE $D_p^q(\Delta^{+r})$-STATISTICAL CONVERGENCE

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Abstract. Let $p(n)$ and $q(n)$ be nondecreasing sequences of positive integers such that $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$ hold. Firstly, in this paper $D_p^q(\Delta^{+r})$-statistical convergence of $x = (x_n)$ where Δ^{+r} is r-th difference of the sequence $x = (x_n)$ for any $r \in \mathbb{Z}^+$ has been defined whereas the results are given under some restrictions on the sequence $p(n)$ and $q(n)$. Secondly, it has been determined that the sets of sequences A and B of the form $[D_p^q]_\alpha^0$ satisfy $A \subset [D_p^q]_0(\Delta^{+r}) \subset B$ and the sets C and D of the form $[D_p^q]_\alpha$ satisfy $C \leq [D_p^q]_\infty(\Delta^{+r}) \leq D$.
Keywords: $D_p^q(\Delta^{+r})$-statistical convergence; summability methods; Deferred Cesaro mean; sequence space.

1. Introduction and main definitions

One of the main problems of the analysis is to determine the set of convergent sequences of the space with considered method. Over the years, this problem has been examined by taking into consideration different summability methods. In recent years, this kind of works have been gained further momentum especially by using the concept of natural density in positive integers.

The concept of statistical convergence was introduced by [16] and [9] independently in the same year. The notion was associated with summability theory by [2] ,[10], [12, 13] and many others.

In this study, the results from [3] were extended and some new results were obtained using Deferred Cesaro mean defined by [1] in as follows:

$$(D_p^q x)_n := \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \tag{1.1}$$

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where $(p(n))$ and $(q(n))$ are sequences of nondecreasing integers satisfying

$$p(n) < q(n) \quad \text{and} \quad \lim_{n\to\infty} q(n) = \infty \ . \tag{1.2}$$

The set of all real valued sequences will be denoted by s and U^+ is denoted by

$$U^+ := \{(u_n) \in s : \ u_n > 0, \ \text{for all } n \in \mathbb{N}\}.$$

For any $\alpha = (\alpha_n) \in U^+$ a new set of sequences can be defined as follows:

$$\alpha \circledast E := \left\{ x \in s : \left(\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} \right) \in E \right\}$$

where E is any sequence space. So, we get

$$\alpha \circledast E := \begin{cases} [D_p^q]_\alpha^0 & if \quad E = c_0, \\ [D_p^q]_\alpha^c & if \quad E = c, \\ [D_p^q]_\alpha & if \quad E = l_\infty, \\ [D_p^q]_\alpha^t & if \quad E = l_p. \end{cases}$$

where

$$[D_p^q]_\alpha^0 := \left\{ x \in s : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} = 0 \right\},$$

$$[D_p^q]_\alpha := \left\{ x \in s : \sup_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} < \infty \right\},$$

$$[D_p^q]_\alpha^t := \left\{ x \in s : \sum_{n=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} \right|^t < \infty \right\}, 1 < t < \infty$$

and

$$[D_p^q]_\alpha^c := \left\{ x \in s : \exists L \in \mathbb{R} \text{ such that} \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} = L \right\}.$$

The idea of difference sequence space was defined by [11] and it was generalized by [6]. Later on, [7] improved this idea by considering any sequence space X as follows

$$\Delta^{+r}(X) := \{x = (x_k) : (\Delta^{+r} x_k) \in X\}$$

where $r \in \mathbb{N}$, $\Delta^0 x := (x_k)$, $\Delta^+ x_k := x_k - x_{k+1}$ and $\Delta^{+r} x_k := \sum_{j=0}^{r} (-1)^j \binom{r}{j} x_{k+j}$.

If $x \in \Delta^{+r}(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^{+r} x_k$ and

$$x_k = \sum_{j=1}^{k-r} (-1)^r \binom{k-j-1}{r-1} y_j = \sum_{j=1}^{k} (-1)^r \binom{k+r-j-1}{r-1} y_{j-r},$$

where $y_{1-r} = y_{2-r} = ... = y_0 = 0$ for sufficiently large k, for instance $k > 2m$ (see more info in [4], [8]).
We can define following sets of sequences for any $r \geq 1$ as:

$$[D_p^q]_0(\Delta^{+r}) := \left\{ x \in s : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k| = 0 \right\},$$

$$[D_p^q]_\infty(\Delta^{+r}) := \left\{ x \in s : \sup_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k| < \infty \right\},$$

$$[D_p^q]_t(\Delta^{+r}) := \left\{ x \in s : \sum_{n=1}^{\infty} \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k| \right|^t < \infty \right\}, 1 < t < \infty$$

and

$$[D_p^q]_c(\Delta^{+r}) := \left\{ x \in s : \exists L \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |\Delta^{+r} x_k - L| = 0 \right\}.$$

In the case when $q_n = n$ and $p_n = 0$, we will denote the previous sets by $[S]_0(\Delta^{+r})$, $[S]_\infty(\Delta^{+r})$, $[S]_t(\Delta^{+r})$ and $[S]_c(\Delta^{+r})$, respectively.

Now, let us define $[D_p^q]_\alpha(\Delta^{+r})$ -statistical convergence of sequence for any $r \geq 1$:

Definition 1.1. A sequence $x = (x_n)$ is said $[D_p^q]_\alpha(\Delta^{+r})$- statistical convergent to zero if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| = 0 \tag{1.3}$$

holds. It is denoted by $x_k \to 0([D_p^q S]_\alpha(\Delta^{+r}))$.
The set of $[D_p^q]_\alpha(\Delta^{+r})$-statistical convergent sequence is also denoted by $[D_p^q S]_\alpha(\Delta^{+r})$.
Remark 1. It is clear that for any positive integer r, if

(i) $q(n) = n$ and $p(n) = 0$, then (1.3) coincides with convergence of $\Delta^{+r} x_k$.

(ii) $q(n) = n$ and $p(n) = n - 1$, then (1.3) coincides with $s_\alpha(\Delta^{+r})$, where

$$s_\alpha(\Delta^{+r}) := \{x \in s : \sup_k |\frac{\Delta^{+r} x_k}{\alpha_k}| < \infty\}.$$

(iii) $q(n) = \lambda_n$ and $p(n) = 0$ where λ_n is a strictly increasing sequence of natural numbers such that $\lim_{n\to\infty} \lambda_n = \infty$, then (1.3) coincides with λ-statistical convergence of sequences which is given by [15].

(iv) $q(n) = n$ and $p(n) = n - \lambda_n$ where (λ_n) is a nondecrasing sequence of natural numbers such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ holds then (1.3) coincides with the $\lambda^{+r}(\mu)$-statistical convergence defined by [3] and with the definition of λ^m-statistcal convergence defined by [5].

(v) $q(n) = k_n$ and $p(n) = k_{n-1}$, where (k_n) is a lacunary sequence of nonnegative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$ then $[D_p^q]_\alpha(\Delta^{+r})$- statistical convergence coincides with Δ^r- lacunary statistical convergence defined by [17].

2. Main results

2.1 Comparasion of $[D_p^q]_\alpha^0(\Delta^{+r})$ and $[D_p^q S]_\alpha(\Delta^{+r})$ when $r \geq 1$.

Theorem 2.1. *Let $r \geq 1$ be an integer. Then,*

(a) $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q S]_\alpha(\Delta^{+r})$ holds and this inclusion is proper,

(b) if $x \in l_\infty^\alpha(\Delta^{+r})$ then $[D_p^q S]_\alpha(\Delta^{+r}) \subset [D_p^q]_\alpha^0(\Delta^{+r})$, where $l_\infty^\alpha(\Delta^{+r}) := \{x \in s : \sup_k |\frac{\Delta^{+r} x_k}{\alpha_k}| < \infty\}$.

Proof. (a) Let us assume that $x \in [D_p^q]_\alpha^0(\Delta^{+r})$. So, for any $\epsilon > 0$, the following inequality

$$\begin{aligned}
\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| &= \\
&= \frac{1}{q(n)-p(n)} \left[\sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon}}^{q(n)} \right] \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \\
&\geq \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \\
&\geq \epsilon . \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} 1 \\
&\geq \epsilon . \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| > \epsilon \right\} \right| \geq 0
\end{aligned}$$

holds. Since $x \in [D_p^q S]_\alpha(\Delta^{+r})$, then desired result is obtained.
The following example shows that this inclusion is proper. To see this, let a sequence $x = (x_n)$ as follows:

$$\frac{\Delta^{+r}x_k}{\alpha_k} := \begin{cases} k, & q(n) - [|\sqrt{q(n)}|] + 1 < k \leq q(n), \\ 0, & \text{otherwise.} \end{cases}$$

If we consider the method $[D_p^q]_\alpha^0(\Delta^{+r})$ for the sequence $p(n)$ satisfying

$$0 < p(n) \leq q(n) - [|\sqrt{q(n)}|] + 1,$$

then, for an arbitrary $\epsilon > 0$ we have

$$\frac{1}{q(n)-p(n)} \left| \left\{ q(n) - [|\sqrt{q(n)}|] + 1 < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon \right\} \right| = \frac{[|\sqrt{q(n)}|]}{q(n)-p(n)} \to 0$$

when $n \to \infty$. This calculation shows that $x \in [D_p^q S]_\alpha(\Delta^{+r})$.
But, it is clear that the sequence

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right|$$

is not convergent to zero. That is, $x \notin [D_p^q]_\alpha^0(\Delta^{+r})$. (b) Let $x \in l_\infty^\alpha(\Delta^{+r})$. Then, there exists $M > 0$ such that $\left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \leq M$ holds for all k. Then, for any $\epsilon > 0$, the following inequality

$$\begin{aligned}
\frac{1}{q(n)-p(n)} & \sum_{k=p(n)+1}^{q(n)} \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| = \\
&= \frac{1}{q(n)-p(n)} \left[\sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon}}^{q(n)} \right] \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \\
&\leq \frac{1}{q(n)-p(n)} \left[M. \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon}}^{q(n)} 1 + \epsilon. \sum_{\substack{k=p(n)+1 \\ \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon}}^{q(n)} 1 \right] \\
&\leq \frac{M}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon \right\} \right| \\
&+ \frac{\epsilon}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| < \epsilon \right\} \right| \\
&\leq \frac{M}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \geq \epsilon \right\} \right| + \epsilon
\end{aligned}$$

holds. By taking limit when $n \to \infty$ in above inequality we obtain $x \in [D_p^q S]_\alpha(\Delta^{+r})$ because of $x \in [D_p^q]_\alpha^0(\Delta^{+r})$. So, proof is complated. $\square$

2.2. Comparasion of $S_\alpha(\Delta^{+r})$ and $[D_p^q S]_\alpha(\Delta^{+r})$ when $r \geq 1$.

Let us denote the set of sequences $x = (x_n)$ by $S_\alpha(\Delta^{+r})$ for any fixed $\alpha \in U^+$ such that

$$\left(\frac{\Delta^{+r}x_k}{\alpha_k}\right) \in S.$$

In this section, the set of sequences $S_\alpha(\Delta^{+r})$ and $[D_p^q S]_\alpha(\Delta^{+r})$ will be compared under some restriction on $p(n)$ and $q(n)$.

Theorem 2.2. *$S_\alpha(\Delta^{+r}) \subseteq [D_p^q S]_\alpha(\Delta^{+r})$ if and only if*

$$\liminf_{n\to\infty} \frac{q(n)-p(n)}{q(n)} > 0. \tag{2.1}$$

Proof. Let $x \in S_\alpha(\Delta^{+r})$ be an arbitrary sequence such that

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right| = 0,$$

holds for every $\epsilon > 0$. Since the sequence $q(n)$ satisfies $\lim_{n\to\infty} q(n) = \infty$, then

$$\left\{\frac{1}{q(n)}\left|\left\{k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right|\right\}_{n\in\mathbb{N}}$$

is also convergent to zero because of (see in [12], Theorem 2.2.1). Hence, by a simple calculation we have the following inequality

$$\frac{1}{q(n)}\left|\left\{k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right| \ge \frac{1}{q(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right|$$

$$\ge \frac{q(n)-p(n)}{q(n)}.\frac{1}{q(n)-p(n)}\left|\left\{p(n) < k \le q(n) : \left|\frac{\Delta^{+r}x_k}{\alpha_k}\right| \ge \epsilon\right\}\right|.$$

Taking in to consider (2.1), if we take limit when $n \to \infty$ in the above inequality then,

$$x \in [D_p^q S]_\alpha(\Delta^{+r}).$$

Conversely, assume that

$$\liminf_{n\to\infty} \frac{q(n)-p(n)}{q(n)} = 0$$

holds. Now, let us choose a subsequence $(n(j))_{j\ge 1}$ such that $\frac{q(n_{(j)})-p(n_{(j)})}{q(n_{(j)})} < \frac{1}{j}$ holds for all $i \in \mathbb{N}$. Let a sequence $x = (x_n)$ such that

$$\frac{\Delta^{+r}x_k}{\alpha_k} := \begin{cases} 1, & p(n_{(j)}) + 1 < k \le q(n_{(j)}), \\ 0 & \text{otherwise}, \end{cases}$$

holds. Then, $x \in [S]_0(\Delta^{+r})$ and hence by Theorem 1 (a), we have $x \in S_\alpha(\Delta^{+r})$. But $x \notin [D_p^q]_\alpha^0(\Delta^{+r})$, and therefore by Theorem 1 (b), we have $x \notin ([D_p^q S]_\alpha(\Delta^{+r}))$. □

Corollary 2.1. *Let $\{q(n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence with $q(n) < n$ for all $n \in \mathbb{N}$ and $\left\{\frac{n}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ be a bounded sequence. Then, $S_\alpha(\Delta^{+r}) \subset [D_p^q S]_\alpha(\Delta^{+r})$ for all $r \ge 1$.*

Theorem 2.3. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then, $[D_p^n S]_\alpha(\Delta^{+r}) \subset S_\alpha(\Delta^{+r})$ holds for all $r \ge 1$.*

Proof. Let us assume that $x \in [D_p^n S]_\alpha(\Delta^{+r})$. We shall apply the technique which was suggested by [1] and was also used in [10].
Let

$$p(n) = n^{(1)} > p(n^{(1)}) = n^{(2)} > p(n^{(2)}) = n^{(3)} > ...,$$

and we may write the set $\left\{k \leq n : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$ as

$$= \left\{k \leq n^{(1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(1)} < k \leq n : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$$

and the set $\left\{1 < k \leq n^{(1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$ as

$$= \left\{k \leq n^{(2)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(2)} < k \leq n^{(1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\},$$

and the set $\left\{k \leq n^{(2)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$ as

$$= \left\{k \leq n^{(3)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(3)} < k \leq n^{(2)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}.$$

If we continue this operation consecutively, after the final step we have

$$\left\{k \leq n^{(h-1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$$

$$= \left\{k \leq n^{(h)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\} \cup \left\{n^{(h)} < k \leq n^{(h-1)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}$$

for a certain positive integer $h > 0$ depending on n such that $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$.
By combining all the equalities obtained above we have

$$\frac{1}{n}\left|\left\{k \leq n : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}\right|$$

$$= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n} . \frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \leq n^{(m)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}\right|.$$

As a result of this equality it can be said that statistical convergence of the sequence $(\frac{\Delta^{+r} x_k}{\alpha_k})$ is a linear combination of the following sequence

$$\left\{\frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \leq n^{(m)} : \left|\frac{\Delta^{+r} x_k}{\alpha_k}\right| \geq \epsilon\right\}\right|\right\}_{m \in \mathbb{N}}.$$

Now, let us consider a matrix $A = (a_{nm})$ as

$$a_{n,m} := \begin{cases} \frac{n^{(m)} - n^{(m+1)}}{n}, & m = 0, 1, 2, ..., h, \\ 0, & \text{otherwise}, \end{cases}$$

It is clear that, where $n^{(0)} = n$.

The matrix $A = (a_{n,m})$ satisfied the Silverman Toeplitz Theorem (see in [14]). So, we have

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| = 0$$

because of

$$\frac{1}{n^{(m)} - n^{(m+1)}} \left| \left\{ n^{(m+1)} < k \leq n^{(m)} : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| \to 0,$$

when $n \to \infty$. This completes the proof. □

2.3. Comparasion of $[D_p^q S]_\alpha(\Delta^{+r})$ and $[D_r^s S]_\alpha(\Delta^{+r})$ for all $r \geq 1$.

In this section, the sequence spaces $[D_p^q S]_\alpha(\Delta^{+r})$ and $[D_r^s S]_\alpha(\Delta^{+r})$ will be compared under which for all $n \in \mathbb{N}$ in addition to (1.2),

$$p(n) \leq r(n) < s(n) \leq q(n) \tag{2.2}$$

holds.

Theorem 2.4. *Let $r(n)$ and $s(n)$ be sequences of positive natural numbers satisfying (2.2) in addition to (1.2) such that the sets*

$$\{k : p(n) < k \leq r(n)\} \quad \text{and} \quad \{k : s(n) < k \leq q(n)\}$$

are finite for all $n \in \mathbb{N}$. Then, $[D_r^s S]_\alpha(\Delta^{+r}) \subset [D_p^q S]_\alpha(\Delta^{+r})$ holds.

Proof. Let us consider a sequence $x = (x_n)$ such that $x \in [D_r^s S]_\alpha(\Delta^{+r})$. For an arbitrary $\epsilon > 0$ the equality

$$\left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} = \left\{ p(n) < k \leq r(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\}$$

$$\cup \left\{ r(n) < k \leq s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \cup \left\{ s(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\}$$

holds. So, the following inequality

$$\begin{aligned}
& \frac{1}{q(n) - p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| \\
\leq \; & \frac{1}{s(n) - r(n)} \left| \left\{ p(n) < k \leq r(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| \\
+ \; & \frac{1}{s(n) - r(n)} \left| \left\{ r(n) < k \leq s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| \\
+ \; & \frac{1}{s(n) - r(n)} \left| \left\{ s(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right|
\end{aligned}$$

holds. By taking limit of each side in the above inequality when $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| = 0.$$

This implies that $x \in [D_p^q S]_\alpha(\Delta^{+r})$. So, the proof is completed. $\square$

Theorem 2.5. *Let $p = p(n)$, $q = q(n)$ and $r = r(n)$, $s = s(n)$ be sequences of positive natural numbers satisfying* (1.2) *and* (2.2) *such that*

$$\liminf_{n\to\infty} \frac{s(n)-r(n)}{q(n)-p(n)} > 0$$

holds. Then, $[D_p^q S]_\alpha(\Delta^{+r}) \subset [D_r^s S]_\alpha(\Delta^{+r})$ holds.

Proof. It is easy to see from (2.2) and (1.2) that the following inclusion

$$\left\{ r(n) < k \leq s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\}$$
$$\subset \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\}$$

and the following inequality

$$\left| \left\{ r(n) < k \leq s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right|$$
$$\leq \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right|$$

hold. So, the last inequality gives that

$$\frac{s(n)-r(n)}{q(n)-p(n)} . \frac{1}{s(n)-r(n)} \left| \left\{ r(n) < k \leq s(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right|$$
$$\leq \frac{1}{q(n)-p(n)} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^{+r} x_k}{\alpha_k} \right| \geq \epsilon \right\} \right| .$$

Therefore, by taking the limit of each side in above inequality when $n \to \infty$, a desired result is obtained. $\square$

2.4. Some properties of the set $D_{p,q}^{\sim}$. Now, define the set $D_{p,q}^{\sim}$ of sequence $\alpha \in U^+$ satisfying the condition

$$sup_n \left(\frac{1}{\alpha_{q(n)}} \sum_{k=p(n)+1}^{q(n)} \alpha_k \right) < \infty.$$

Let Δ be the well known operator defined by $\Delta x_n = x_n - x_{n-1}$ for all n, with $x_0 = 0$.

Lemma 2.1. *Let $\alpha \in U^+$. The following statements are equivalent:*

(i) $\alpha \in D^{\sim}_{p,q}$ *and* $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$,

(ii) the operator Δ *is bijective from* $[D_p^q]_\alpha$ *to itself,*

(iii) the operator Δ *is bijective from* $[D_p^q]^0_\alpha$ *to itself.*

Proof. Firstly, let us show that (i) implies (ii). Let $x = (x_n)$ and $y = (y_n) \in [D_p^q]_\alpha$ be arbitrary sequences and assume that $\Delta x = \Delta y$. It means that

$$(x_1, x_2 - x_1, ..., x_n - x_{n-1}, ...) = (y_1, y_2 - y_1, ..., y_n - y_{n-1}, ...)$$

holds. From this assumption we have, $x_n - x_{n-1} = y_n - y_{n-1}$ for all $n \geq 1$. This calculation gives that $x_n = y_n$ holds for all $n \in \mathbb{N}$. Hence, Δ is an injective function from $[D_p^q]_\alpha$ to itself.

Now, let $y \in [D_p^q]_\alpha$ be an arbitrary sequence. We must find a sequence $x \in [D_p^q]_\alpha$ such that $\Delta x_n = y_n$ holds. That is,

$$(y_1, y_2, ..., y_n, ...) = (x_1, x_2 - x_1, ..., x_n - x_{n-1}, ...)$$

holds. Therefore, the sequence $x = (x_n)$ must be as $x_n := \sum_{k=1}^{n} y_k$, for all $n \in \mathbb{N}$ Now, let us check that $x \in [D_p^q]_\alpha$. By using the method in [1], we have

$$\begin{aligned}
\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| &= \left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\
&+ \left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\
&= \left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{y_1}{\alpha_1}\right| + \left|\frac{y_1+y_2}{\alpha_2}\right| + ... + \left|\frac{y_1+y_2+...+y_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\
&+ \left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{y_1}{\alpha_1}\right| + \left|\frac{y_1+y_2}{\alpha_2}\right| + ... + \left|\frac{y_1+y_2+...+y_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\
&= \frac{1}{q(n)-p(n)}\left(\left|\frac{y_{p(n)+1}}{\alpha_{p(n)+1}}\right| + \left|\frac{y_{p(n)+1}+y_{p(n)+2}}{\alpha_{p(n)+2}}\right| + ... + \left|\frac{y_{p(n)+1}+...+y_{q(n)}}{\alpha_{q(n)}}\right|\right)
\end{aligned}$$

Also, with a simple calculation, the following inequality

$$\left|\frac{y_{p(n)+1}+y_{p(n)+2}}{\alpha_{p(n)+2}}\right| \leq \left|\frac{y_{p(n)+1}}{\alpha_{p(n)+1}}\right| \frac{\alpha_{p(n)+1}}{\alpha_{p(n)+2}} + \left|\frac{y_{p(n)+2}}{\alpha_{p(n)+2}}\right|$$

and

$$\left|\frac{y_{p(n)+1}+...+y_{q(n)}}{\alpha_{q(n)}}\right| \leq \left|\frac{y_{p(n)+1}}{\alpha_{q(n)}}\right| + \left|\frac{y_{p(n)+2}}{\alpha_{q(n)}}\right| + ... + \left|\frac{y_{q(n)}}{\alpha_{q(n)}}\right|$$

$$\leq \left|\frac{y_{p(n)+1}}{\alpha_{p(n)+1}}\right| \frac{\alpha_{p(n)+1}}{\alpha_{q(n)}} + \left|\frac{y_{p(n)+2}}{\alpha_{p(n)+2}}\right| \frac{\alpha_{p(n)+2}}{\alpha_{q(n)}} + ... + \left|\frac{y_{q(n)}}{\alpha_{q(n)}}\right| \frac{\alpha_{q(n)}}{\alpha_{q(n)}}$$

$$\leq K. \left(\frac{\alpha_{p(n)+1} + \alpha_{p(n)+2} + ... + \alpha_{q(n)}}{\alpha_{q(n)}}\right)$$

holds for a positive K. Consequently, we conclude that $x \in [D_p^q]_\alpha$ for $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$. Similarly, it can be proved that $\Delta : [D_p^q]_\alpha^0 \to [D_p^q]_\alpha^0$ is a bijective function. □

It can easily be deduced that if $\alpha \in D_{p,q}^{\sim}$, then for any given integer $r \geq 1$ the operator Δ^r is a bijective function from $[D_p^q]_\alpha$ to itself. So, $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$. It is the same for the operator Δ considered as an operator from $[D_p^q]_\alpha$ to itself.

Lemma 2.2. *Let $r \geq 1$ be an integer and $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$. The following statements are equivalent:*

(i) $\alpha \in D_{p,q}^{\sim}$,

(ii) $[D_p^q]_\alpha(\Delta) = [D_p^q]_\alpha$,

(iii) $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$.

Proof. First show that (i) implies (ii). Let $x \in [D_p^q]_\alpha$ be an arbitrary sequence. Then, the following inequality

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k - x_{k-1}}{\alpha_k}\right| \leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left(\left|\frac{x_k}{\alpha_k}\right| + \left|\frac{x_{k-1}}{\alpha_k}\right|\right)$$

$$\leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| + \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_{k-1}}{\alpha_{k-1}}\right| . \left|\frac{\alpha_{k-1}}{\alpha_k}\right|$$

holds. It gives that

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k - x_{k-1}}{\alpha_k}\right| < \infty.$$

So, $x \in [D_p^q]_\alpha(\Delta)$. Conversely, let $x \in [D_p^q]_\alpha(\Delta)$. This implies that $b := (\Delta x) \in [D_p^q]_\alpha$ for every n. So, we have

$$x_n = u + \sum_{k=1}^{n} b_k$$

for $u \in \mathbb{C}$ (see in [3] Lemma 2.2). Then, $b = (b_n) \in [D_p^q]_\alpha$.
So, if we take $u = 0$, then we obtain

$$\begin{aligned}
\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_k}\right| &= \left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\
&+ \left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{x_1}{\alpha_1}\right| + \left|\frac{x_2}{\alpha_2}\right| + ... + \left|\frac{x_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\
&= \left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{b_1}{\alpha_1}\right| + \left|\frac{b_1+b_2}{\alpha_2}\right| + ... + \left|\frac{b_1+b_2+...+b_{p(n)}}{\alpha_{p(n)}}\right|\right)\right] \\
&+ \left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{b_1}{\alpha_1}\right| + \left|\frac{b_1+b_2}{\alpha_2}\right| + ... + \left|\frac{b_1+b_2+...+b_{q(n)}}{\alpha_{q(n)}}\right|\right)\right] \\
&= \frac{1}{q(n)-p(n)}\left(\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right| + \left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+2}}\right| + ... + \left|\frac{b_{p(n)+1}+...+b_{q(n)}}{\alpha_{q(n)}}\right|\right).
\end{aligned}$$

So, from Lemma 1, we have $[D_p^q]_\alpha(\Delta) = [D_p^q]_\alpha$.

Now, let us show that (ii) implies (iii). Hence, Δ is bijective function and so does the composition Δ^r. In that case $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$. We obtain that

$$[D_p^q]_\alpha = [D_p^q]_\alpha(\Delta) = ... = [D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha(\Delta^{r+1}).$$

On the contrary, let $[D_p^q]_\alpha(\Delta^r) = [D_p^q]_\alpha$. Therefore, (iii) must imply (i) to achieve this equality. $\square$

Lemma 2.3. *Let $\alpha, \beta \in U^+$. Then, $[D_p^q]_\alpha^0 = [D_p^q]_\beta^0$ if and only if there exists $M_1, M_2 > 0$ such that for all $n \in \mathbb{N}$,*

$$M_1 \le \frac{\alpha_n}{\beta_n} \le M_2 \tag{2.3}$$

holds.

Proof. It is easy to see from (2.3) that

$$M_1 \frac{x_k}{\alpha_k} \le \frac{x_k}{\beta_k} \le M_2 \frac{x_k}{\alpha_k}$$

holds. Also, this inequality implies that

$$M_1 \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k} \le \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\beta_k} \le \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_k}{\alpha_k}$$

holds. Then, if we take limit in the above inequality when $n \to \infty$, a desired implication will be obtained. $\square$

Theorem 2.6. *Let $\alpha \in U^+$ and $r \ge 1$ be arbitrary integer. Then, the following statements are true:*

(i) $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ *if and only if* $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$.

(ii) the following statements are equivalent:

(a) $\alpha \in D^{\sim}_{p,q}$ *and* $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$,

(b) $[D_p^q]^0_\alpha(\Delta^+) = [D_p^q]^0_{(\alpha_{n-1})}$

(c) $[D_p^q]^0_\alpha(\Delta^+) \subset [D_p^q]^0_{(\alpha_{n-1})}$.

(iii) (a) $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ *if and only if, for any given integer* $r \geq 1$,

$$[D_p^q]^0_\alpha \subset [D_p^q]^0_\alpha(\Delta^+) \subset ... \subset [D_p^q]^0_\alpha(\Delta^{+r});$$

(b) If $\alpha \in D^{\sim}_{p,q}$ *and* $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$, *then* $[D_p^q]^0_\alpha = [D_p^q]^0_\alpha(\Delta^+)$.

Proof. (i) Assume that $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ and let $x \in [D_p^q]^0_{(\alpha_{n-1})}$ be an arbitrary sequence. Then,

$$\begin{aligned} \frac{1}{q(n)-p(n)} \quad & \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k - x_{k+1}}{\alpha_k}\right| \\ \leq \quad & \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left(\left|\frac{x_k}{\alpha_k}\right| \cdot \left|\frac{\alpha_{k-1}}{\alpha_{k-1}}\right| + \left|\frac{x_{k+1}}{\alpha_k}\right| \right) \\ \leq \quad & \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_{k-1}}\right| \cdot \left|\frac{\alpha_{k-1}}{\alpha_k}\right| \\ + \quad & \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_{k+1}}{\alpha_k}\right|. \end{aligned}$$

under assumptions, the above inequality implies that $x \in [D_p^q]^0_\alpha(\Delta^+)$ when $n \to \infty$. This gives $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$.

Conversely, assume that $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$. Therefore, it is clear that $\alpha \in D^{\sim}_{p,q}$ and $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$.
(ii) Let us show that (a) implies (b). Now $\alpha \in D^{\sim}_{p,q}$ implies that $(\alpha_{p(n)-1}/\alpha_{p(n)}) \in l_\infty$ and by (i), $[D_p^q]^0_{(\alpha_{n-1})} \subset [D_p^q]^0_\alpha(\Delta^+)$.

Conversely, $x \in [D_p^q]^0_\alpha(\Delta^+)$ implies that $b = \Delta^+ x \in [D_p^q]^0_\alpha$. So, for every n, we have $x_n = -\sum_{k=1}^{n-1} b_k$ for $x_1 = 0$ (see in [3], Lemma 2.2).
Then, $b = (b_n) \in [D_p^q]^0_\alpha$. Since $b = \Delta^+ x$, then $(b_1, b_2, ..., b_n...) = (x_1 - x_2, x_2 - x_3, ..., x_{n-1} - x_n, ...)$, for all $n \in \mathbb{N}$. Therefore,

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left|\frac{x_k}{\alpha_{k-1}}\right| == \left[-\frac{p(n)}{q(n)-p(n)}\right] \left[\frac{1}{p(n)} \left(\left|\frac{x_2}{\alpha_1}\right| + \left|\frac{x_3}{\alpha_2}\right| + ... + \left|\frac{x_{p(n)}}{\alpha_{p(n)-1}}\right| \right)\right]$$

$$+\left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{x_2}{\alpha_1}\right|+\left|\frac{x_3}{\alpha_2}\right|+...+\left|\frac{x_{q(n)}}{\alpha_{q(n)-1}}\right|\right)\right]$$

$$=\left[-\frac{p(n)}{q(n)-p(n)}\right]\left[\frac{1}{p(n)}\left(\left|\frac{-b_1}{\alpha_1}\right|+\left|\frac{-(b_1+b_2)}{\alpha_2}\right|+...+\left|\frac{-(b_1+b_2+...+b_{p(n)-1})}{\alpha_{p(n)-1}}\right|\right)\right]$$

$$+\left[\frac{q(n)}{q(n)-p(n)}\right]\left[\frac{1}{q(n)}\left(\left|\frac{-b_1}{\alpha_1}\right|+\left|\frac{-(b_1+b_2)}{\alpha_2}\right|+...+\left|\frac{-(b_1+b_2+...+b_{q(n)-1})}{\alpha_{q(n)-1}}\right|\right)\right]$$

$$=\frac{1}{q(n)-p(n)}\left(\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right|+\left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+2}}\right|+...+\left|\frac{b_{p(n)+1}+...+b_{q(n)-1}}{\alpha_{q(n)-1}}\right|\right)$$

holds. From here, the following inequalities

$$\left|\frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+2}}\right|\leq\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right|\frac{\alpha_{p(n)+1}}{\alpha_{p(n)+2}}+\left|\frac{b_{p(n)+2}}{\alpha_{p(n)+2}}\right|$$

and

$$\left|\frac{b_{p(n)+1}+b_{p(n)+2}+...+b_{q(n)-1}}{\alpha_{q(n)-1}}\right|\leq\left|\frac{b_{p(n)+1}}{\alpha_{q(n)-1}}\right|+\left|\frac{b_{p(n)+2}}{\alpha_{q(n)-1}}\right|+...+\left|\frac{b_{q(n)-1}}{\alpha_{q(n)-1}}\right|$$

$$\leq\left|\frac{b_{p(n)+1}}{\alpha_{p(n)+1}}\right|\frac{\alpha_{p(n)+1}}{\alpha_{q(n)-1}}+\left|\frac{b_{p(n)+2}}{\alpha_{p(n)+2}}\right|\frac{\alpha_{p(n)+2}}{\alpha_{q(n)-1}}+...+\left|\frac{b_{q(n)-1}}{\alpha_{q(n)-1}}\right|\frac{\alpha_{q(n)-1}}{\alpha_{q(n)-1}}$$

$$\leq K.\left(\frac{\alpha_{p(n)+1}+\alpha_{p(n)+2}+...+\alpha_{q(n)-1}}{\alpha_{q(n)-1}}\right)$$

hold for any $K>0$. Then, $x\in[D_p^q]^0_{(\alpha_{n-1})}$ and we conclude that $[D_p^q]^0_\alpha(\Delta^+)\subset[D_p^q]^0_{(\alpha_{n-1})}$.

So, $[D_p^q]^0_\alpha(\Delta^+)=[D_p^q]^0_{(\alpha_{n-1})}$ and we have shown that (a) implies (b). Consequantly, conclude that (b) implies (c).

(iii) (a) Let $(\alpha_{p(n)}/\alpha_{p(n)-1})\in l_\infty$. Since $[D_p^q]^0_\alpha\subset[D_p^q]^0_\alpha(\Delta^+)$, then for all $x\in[D_p^q]^0_\alpha$ we have

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|\frac{x_k-x_{k+1}}{\alpha_k}\right|$$

$$\leq\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left(\left|\frac{x_k}{\alpha_k}\right|+\left|\frac{x_{k+1}}{\alpha_k}\right|.\left|\frac{\alpha_{k+1}}{\alpha_{k+1}}\right|\right)$$

$$\leq\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|\frac{x_k}{\alpha_k}\right|+\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\left|\frac{x_{k+1}}{\alpha_{k+1}}\right|.\left|\frac{\alpha_{k+1}}{\alpha_k}\right|.$$

under assumption, this inequality implies that $[D_p^q]^0_\alpha\subset[D_p^q]^0_\alpha(\Delta^+)$ when $n\to\infty$.

Now, from the mathematical induction method for any given integer $r\geq1$ and $x\in[D_p^q]^0_\alpha(\Delta^{+r})$; then $\Delta^{+r}x\in[D_p^q]^0_\alpha$ and with $[D_p^q]^0_\alpha\subset[D_p^q]^0_\alpha(\Delta^+)$ holds because of

$\Delta^{+r}x \in [D_p^q]_\alpha^0(\Delta^+)$ and $x \in [D_p^q]_\alpha^0(\Delta^{+(r+1)})$.
So, we have $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q]_\alpha^0(\Delta^{+(r+1)})$.
(b) Now, $\alpha \in D_{p,q}^{\sim}$ implies that $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ and by (iii) (a), we have $[D_p^q]_\alpha^0 \subset [D_p^q]_\alpha^0(\Delta^+)$.

Conversely, let $x \in [D_p^q]_\alpha^0(\Delta^+)$ implies that $b = \Delta^+ x \in [D_p^q]_\alpha^0$ and for every n, we have $x_n = -\sum_{k=1}^{n-1} b_k$ for $x_1 = 0$ Theorem 6 (ii). Then $b \in [D_p^q]_\alpha^0$.

$x \in [D_p^q]_\alpha^0(\Delta^+)$ for $x \in [D_p^q]_\alpha^0$ when shown similarly with (ii). Therefore, the conditions $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ are equivalent to $[D_p^q]_\alpha^0 = [D_p^q]_\alpha^0(\Delta^+)$. $\square$

Corollary 2.2. *Let $r \geq 1$ be an integer and assume that $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$. Then, $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q]_\alpha^0$ implies that $[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_\alpha^0$.*

Proof. By Theorem 3 (iii) (a), the condition $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$ implies that $[D_p^q]_\alpha^0 \subset [D_p^q]_\alpha^0(\Delta^{+r})$. Since $[D_p^q]_\alpha^0(\Delta^{+r}) \subset [D_p^q]_\alpha^0$ then,

$$[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_\alpha^0$$

holds. $\square$

Remark 2. In Theorem 3, the conditions $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)}/\alpha_{p(n)-1})_n \in l_\infty$ are equivalent to $[D_p^q]_\alpha^0(\Delta^+) = [D_p^q]_{(\alpha_{n-1})}^0 = [D_p^q]_\alpha^0$.

Proof. If $\alpha \in D_{p,q}^{\sim}$ and $(\alpha_{p(n)}/\alpha_{p(n)-1}) \in l_\infty$, then there are $K_1, K_2 > 0$ such that

$$K_1 \leq \frac{\alpha_{p(n)}}{\alpha_{p(n)-1}} \leq K_2$$

holds for all $n \in \mathbb{N}$. Then by Lemma 3, we have $[D_p^q]_{(\alpha_{n-1})}^0 = [D_p^q]_\alpha^0$. By Theorem 6 (ii), we conclude that the condition $\alpha \in D_{p,q}^{\sim}$ implies that $[D_p^q]_\alpha^0(\Delta^+) = [D_p^q]_{(\alpha_{n-1})}^0 = [D_p^q]_\alpha^0$. $\square$

Corollary 2.3. *Let $\alpha \in U^+$ and $r \geq 1$ be an integer. Then, the condition $\alpha \in D_{p,q}^{\sim}$ implies $[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_{(\alpha_{n-r})}^0$.*

Proof. The condition $\alpha \in D_{p,q}^{\sim}$ implies by Theorem 6 (ii) $[D_p^q]_\alpha^0(\Delta^+) = [D_p^q]_{(\alpha_{n-1})}^0$. Now let $r \geq 1$ be an integer and assume that

$$[D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_{(\alpha_{n-r})}^0.$$

Then, $x \in [D_p^q]_\alpha^0(\Delta^{+(r+1)})$ if and only if $(\Delta^{+(r+1)})x \in [D_p^q]_\alpha^0$, which in turn is

$$\Delta^+ x \in [D_p^q]_\alpha^0(\Delta^{+r}) = [D_p^q]_{(\alpha_{n-r})}^0.$$

So, $[D_p^q]_\alpha^0(\Delta^{+(r+1)}) = [D_p^q]_{(\alpha_{n-r})}^0(\Delta^+)$ since $\alpha \in D_{p,q}^{\sim}$, then $(\alpha_{n-r}) \in D_{p,q}^{\sim}$ and

$$[D_p^q]_{(\alpha_{n-r})}^0(\Delta^+) = [D_p^q]_\alpha^0(\Delta^{+(r+1)}) = [D_p^q]_{(\alpha_{n-(r+1)})}^0.$$

This shows (i). $\square$

Theorem 2.7. *Let $\alpha \in D_{p,q}^{\sim}$ and $r \geq 1$ be an integer. Then, $(\alpha_{p(n)}) \in l_\infty$ implies that*

$$[D_p^q]_\alpha^0 \subset [D_p^q]_0(\Delta^{+r}) \quad \textit{and} \quad [D_p^q]_\alpha \subset [D_p^q]_\infty(\Delta^{+r})$$

holds.

Proof. Let $(\alpha_{p(n)}) \in l_\infty$ and $x \in [D_p^q]_\alpha$. Then, the following inequality

$$\begin{aligned}\frac{1}{q(n)-p(n)} \qquad \sum_{k=p(n)+1}^{q(n)} |x_k - x_{k+1}| &\leq \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k| \\ &+ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_{k+1}| \\ &\leq \frac{1}{q(n)-p(n)} \left[\sum_{k=p(n)+1}^{q(n)} \left| \frac{x_k}{\alpha_k} \right| .\alpha_k + \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_{k+1}}{\alpha_{k+1}} \right| .\alpha_{k+1} \right]\end{aligned}$$

holds. Hence, $x \in [D_p^q]_\infty(\Delta^+)$, because of $[D_p^q]_\alpha \subset [D_p^q]_\infty(\Delta^{+r})$. Similarly $[D_p^q]_\alpha^0 \subset [D_p^q]_0(\Delta^{+r})$ is satisfied. $\square$

Theorem 2.8. *Let $\alpha \in U^+$ and $r \geq 1$ be an integer. Assume that $(1/\alpha_{p(n)}) \in l_\infty$. Then, we have*

$$[D_p^q]_0(\Delta^{+r}) \subset [D_p^q]_\alpha^0 \quad \textit{and} \quad [D_p^q]_\infty(\Delta^{+r}) \subset [D_p^q]_\alpha.$$

Proof. Assume that $\alpha \in D_{p,q}^{\sim}$ and let $(1/\alpha_{p(n)}) \in l_\infty$. Let $x \in [D_p^q]_0(\Delta^+)$ implies that $b = \Delta^+ x \in [D_p^q]_0$ and for every n, we have from Theorem 6 that $x_n = -\sum_{k=1}^{n-1} b_k$.

$$\begin{aligned}&\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_k}{\alpha_k} \right| = \left[-\frac{p(n)}{q(n)-p(n)} \right] \left[\frac{1}{p(n)} \left(\left| \frac{x_1}{\alpha_1} \right| + \left| \frac{x_2}{\alpha_2} \right| + ... + \left| \frac{x_{p(n)}}{\alpha_{p(n)}} \right| \right) \right] \\ &+ \left[\frac{q(n)}{q(n)-p(n)} \right] \left[\frac{1}{q(n)} \left(\left| \frac{x_1}{\alpha_1} \right| + \left| \frac{x_2}{\alpha_2} \right| + ... + \left| \frac{x_{q(n)}}{\alpha_{q(n)}} \right| \right) \right] \\ &= \left[-\frac{p(n)}{q(n)-p(n)} \right] \left[\frac{1}{p(n)} \left(\left| \frac{-b_1}{\alpha_2} \right| + \left| \frac{-(b_1+b_2)}{\alpha_3} \right| + ... + \left| \frac{-(b_1+b_2+...+b_{p(n)})}{\alpha_{p(n)}} \right| \right) \right] \\ &+ \left[\frac{q(n)}{q(n)-p(n)} \right] \left[\frac{1}{q(n)} \left(\left| \frac{-b_1}{\alpha_2} \right| + \left| \frac{-(b_1+b_2)}{\alpha_3} \right| + ... + \left| \frac{-(b_1+b_2+...+b_{q(n)})}{\alpha_{q(n)}} \right| \right) \right] \\ &= \frac{1}{q(n)-p(n)} \left(\left| \frac{b_{p(n)+1}}{\alpha_{p(n)+2}} \right| + \left| \frac{b_{p(n)+1}+b_{p(n)+2}}{\alpha_{p(n)+3}} \right| + ... + \left| \frac{b_{p(n)+1}+...+b_{q(n)-1}}{\alpha_{q(n)}} \right| \right) \\ &= \frac{M}{q(n)-p(n)} \left[\left| b_{p(n)+1} \right| + \left| b_{p(n)+1} + b_{p(n)+2} \right| + ... + \left| b_{p(n)+1} + ... + b_{q(n)-1} \right| \right]\end{aligned}$$

the inequality is provided. So, $x \in [D_p^q]_\alpha^0$. $\square$

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ON BÄCKLUND TRANSFORMATIONS WITH SPLIT QUATERNIONS

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Abstract. The present paper deals with the introduction of Bäcklund transformations with split quaternions in Minkowski space. Firstly, we have briefly summarized the basic concepts of split quaternion theory and Bishop frames of non-null curves in Minkowski space. Then, for Bäcklund transformations defined with each case of non-null curves, we have given the relationships between Bäcklund transformations and split quaternions. Some special propositions for transformations constructed with split quaternions have also been presented . At the end, the results obtained with the mathematical model have been evaluated.

Keywords: Minkowski space; quaternions; Bäcklund transformations; Bishop frames.

1. Introduction

Bäcklund transformations give a correlation between PDE and their solution. In other words, one can estimate Bäcklund transformations generating a PDEs's solution if we know a solution of PDE. There exists a class of Bäcklund transformations which are called auto-Bäcklund transformations, when the connected PDEs are the same. To generate new solutions on the integrable theories the Bäcklund transformations have been widely used. These transformations help to connect diffucult PDE to simpler one that has easier solution. In the case of solutions, these transformations are highly effective in generating multi-solutions from the familiar solutions. By applying Bäcklund transformations to trivial solution one can generate a non-trivial case [16]. Due to the aforementioned features, numerous studies have been carried out on Bäcklund transformations from past to present. For example, Weiss studied the B*ä*cklund transformations on focal surfaces in [14], Sen gave *darb*oux Bäcklund transformation of nonlinear optical waves in [12], B*ä*ck obtained Bäcklund transformations for minimal surfaces in [3].

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Hamilton introduced a new algebra of the real quaternions. Additionally, he wanted to replicate what Gauss did with complex numbers and the Euclidean plane over the real numbers. So, Hamilton obtained the 4-dimensional real division algebra

$$\mathcal{H}_{\mathbb{R}} = \left\{ \mu_1 + \mu_2\delta_1 + \mu_3\delta_2 + \mu_4\delta_3 \mid \ \delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_1\delta_2\delta_3 = -1 \right\}, \tag{1.1}$$

where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$. The key of Hamilton's work is the interpretation of the multiplication of the imaginary units as the wedge product of the canonical basis $\left\{ \vec{i}, \vec{j}, \vec{k} \right\}$ of the 3-dimensional Euclidean space, [4].

In next years, Cokle found new examples in [5]: coquaternions, tessarines and cotessarines. The first ones are precisely the split quaternions. Then, a split quaternion μ is a linear combination of the form

$$\mathcal{P}_{\mathbb{R}} = \left\{ \mu = \mu_1 + \mu_2\delta_1 + \mu_3\delta_2 + \mu_4\delta_3 \mid -\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_1\delta_2\delta_3 = 1 \right\}, \tag{1.2}$$

where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$, [4]. In recent papers, split quaternions have been widely used . For example, Aslan and Kocakuşaklı studied canal surfaces with split quaternions in [1],[9]. Aslan gave quaternionic shape operator in [2], and Tunçer studied circular surfaces with split quaternions.

There are many studies on surfaces with quaternions and split quaternions considering the aforementioned articles, but construction of the Bäcklund transformation with split quaternions has not been studied until now. Because classical Bäcklund transformations mainly focus on the transformation of surfaces, the relationship between the results obtained with this study and the theory of surfaces can be investigated. Therefore, we will explain the relationship between Bäcklund Transformations of non-null curves and a timelike split quaternion having a timelike vector part, a timelike split quaternion having a spacelike vector part and a spacelike split quaternion.

2. Preliminaries

Let us recall some known concepts of split quaternion theory and Bishop frame given by [11] and [8].

Assume that $r = (r_1, r_2, r_3)$ and $s = (s_1, s_2, s_3)$ are two vectors in Minkowski 3-space. Then, Lorentzian inner product and vector product of these curves are defined by

$$\langle r, s \rangle_L = -r_1s_1 + r_2s_2 + r_3s_3$$

and

$$r\Lambda_L s = (r_3s_2 - r_2s_3, r_1s_3 - r_3s_1, r_1s_2 - r_2s_1).$$

For a vector $r \in \mathbb{E}_1^3, r$ is called

i) a spacelike vector if $\langle r, r \rangle_L > 0$ or $r = 0$,

ii) a timelike vector if $\langle r, r \rangle_L < 0$, *iii)* a null vector if $\langle r, r\rangle_L = 0$ for $r \neq 0$, [10].

Let $\mathcal{P}_{\mathbb{R}}$ denote a four dimensional vector space over a field $\mathbb{R}$ whose characteristics is greater than 2, [6]:

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, \omega_1, \omega_2, \omega_3\}$ satisfying the equalities $-\omega_1^2 = \omega_2^2 = \omega_3^2 = 1$ and

$$\omega_1 * \omega_2 = \omega_3, \omega_2 * \omega_3 = -\omega_1,\ \omega_3 * \omega_1 = \omega_2.$$

Furthermore, $S_w = w_1$ and $\vec{V}_w = w_2\omega_1 + w_3\omega_2 + w_4\omega_3$ are scalar and vector parts of a real split quaternion $w = w_1 + w_2\omega_1 + w_3\omega_2 + w_4\omega_3$, respectively. Let $w = (w_1, w_2, w_3, w_4)$ and $q = (q_1, q_2, q_3, q_4)$ be two split quaternions. Then, the split quaternion product of the split quaternions w and q is defined as

$$w * q = w_1 q_1 + \left\langle \vec{V}_w, \vec{V}_q \right\rangle_L + w_1\vec{V}_q + q_1\vec{V}_w + \vec{V}_w \Lambda_L \vec{V}_q,$$

where $\langle , \rangle_L$ and Λ_L are Lorentzian inner product and vector product respectively. Also, a split quaternion $w = (w_1, w_2, w_3, w_4)$ is expressed as

i) w is a spacelike if $-I_w = -w_1^2 - w_2^2 + w_3^2 + w_4^2 < 0$,

ii) w is a timelike if $-I_w = -w_1^2 - w_2^2 + w_3^2 + w_4^2 > 0$,

iii) w is a lightlike(null) quaternion if $-I_w = -w_1^2 - w_2^2 + w_3^2 + w_4^2 = 0$. The norm of $w = (w_1, w_2, w_3, w_4)$ is defined as

$$N_w = \sqrt{|w_1^2 + w_2^2 - w_3^2 - w_4^2|}.$$

If $N_w = 1$ then w is called unit split quaternion and $w_0 = w/N_w$ is a unit split quaternion for $N_w \neq 0$.

Also, each spacelike unit split quaternion $w = (w_1, w_2, w_3, w_4)$ is expressed as

$$w = N_w \left(\sinh\varphi + \varepsilon \cosh\varphi\right), \tag{2.1}$$

where

$$\sinh\varphi = \frac{w_1}{N_w},\ \cosh\varphi = \frac{\sqrt{-w_2^2 + w_3^2 + w_4^2}}{N_w}$$

and

$$\varepsilon = \frac{w_2\omega_1 + w_3\omega_2 + w_4\omega_3}{\sqrt{-w_2^2 + w_3^2 + w_4^2}}.$$

Each timelike split quaternion having a spacelike vector part $w = (w_1, w_2, w_3, w_4)$ is expressed as

$$w = N_w \left(\cosh\varphi + \varepsilon \sinh\varphi\right), \tag{2.2}$$

where

$$\cosh\varphi = \frac{w_1}{N_w},\ \sinh\varphi = \frac{\sqrt{-w_2^2 + w_3^2 + w_4^2}}{N_w}$$

and

$$\varepsilon = \frac{w_2\omega_1 + w_3\omega_2 + w_4\omega_3}{\sqrt{-w_2^2 + w_3^2 + w_4^2}}.$$

Finally, each timelike split quaternion having a timelike vector part $w = (w_1, w_2, w_3, w_4)$ is expressed, as

$$w = N_w (\cos\varphi + \varepsilon \sin\varphi), \tag{2.3}$$

where

$$\cos\varphi = \frac{w_1}{N_w}, \ \sin\varphi = \frac{\sqrt{w_2^2 - w_3^2 - w_4^2}}{N_w}$$

and

$$\varepsilon = \frac{w_2\omega_1 + w_3\omega_2 + w_4\omega_3}{\sqrt{w_2^2 - w_3^2 - w_4^2}}.$$

On the other hand, assume that $\alpha : s \to \alpha(s)$, which is parameterized by arc-length parameter s, is a spatial curve in Minkowski 3-space. The relation between Bishop frame and Frenet frame of a curve α according to the arc-length parameter is governed by the relations:

$$\begin{aligned} \mathcal{T} &= \mathcal{T}, \\ \mathcal{N} &= \cos\theta\mathcal{N}_1 - \sin\theta\mathcal{N}_2, \\ \mathcal{B} &= \sin\theta\mathcal{N}_1 + \cos\theta\mathcal{N}_2. \end{aligned} \tag{2.4}$$

Also, $\kappa_1(s) = \kappa(s)\cos\theta(s)$ and $\kappa_2(s) = \tau(s)\sin\theta(s)$ are called Bishop curvatures.

The Bishop equations can be given as below, if the curve α is a timelike curve:

$$\begin{aligned} \mathcal{T}' &= \kappa_1\mathcal{N}_1 + \kappa_2\mathcal{N}_2, \\ \mathcal{N}_1' &= \kappa_1\mathcal{T}, \\ \mathcal{N}_2' &= \kappa_2\mathcal{T}, \end{aligned} \tag{2.5}$$

where

$$\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}, \ \theta = \arctan\left(\frac{\kappa_2}{\kappa_1}\right), \ \tau = \frac{d\theta}{ds}, \tag{2.6}$$

$$\langle\mathcal{T},\mathcal{T}\rangle_L = -1, \ \langle\mathcal{N}_1,\mathcal{N}_1\rangle_L = 1, \ \langle\mathcal{N}_2,\mathcal{N}_2\rangle_L = 1.$$

The Bishop equations can be given as below, if the curve α is a spacelike curve with a timelike principal normal:

$$\begin{aligned} \mathcal{T}' &= \kappa_1\mathcal{N}_1 - \kappa_2\mathcal{N}_2, \\ \mathcal{N}_1' &= \kappa_1\mathcal{T}, \\ \mathcal{N}_2' &= \kappa_2\mathcal{T}, \end{aligned} \tag{2.7}$$

where

$$\kappa = \sqrt{|\kappa_2^2 - \kappa_1^2|},\ \theta = \arg\tanh\left(\frac{\kappa_2}{\kappa_1}\right),\ \tau = \frac{d\theta}{ds}, \tag{2.8}$$
$$\langle \mathcal{T}, \mathcal{T} \rangle_L = 1,\ \langle \mathcal{N}_1, \mathcal{N}_1 \rangle_L = -1,\ \langle \mathcal{N}_2, \mathcal{N}_2 \rangle_L = 1.$$

The Bishop equations can be given as below, if the curve α is a spacelike curve with a timelike principal normal:

$$\begin{aligned} \mathcal{T}' &= \kappa_1 \mathcal{N}_1 - \kappa_2 \mathcal{N}_2, \\ \mathcal{N}_1' &= -\kappa_1 \mathcal{T}, \\ \mathcal{N}_2' &= -\kappa_2 \mathcal{T}, \end{aligned} \tag{2.9}$$

where

$$\kappa = \sqrt{|\kappa_1^2 - \kappa_2^2|},\ \theta = \arg\tanh\left(\frac{\kappa_2}{\kappa_1}\right),\ \tau = -\frac{d\theta}{ds}, \tag{2.10}$$
$$\langle \mathcal{T}, \mathcal{T} \rangle_L = 1,\ \langle \mathcal{N}_1, \mathcal{N}_1 \rangle_L = 1,\ \langle \mathcal{N}_2, \mathcal{N}_2 \rangle_L = -1.$$

3. Results And Discussion

Let us start by assuming $q : I \to R \subset \mathcal{P}_{\mathbb{R}}$ is a unit split quaternion with arc-length parameter s, [7]. Then, a split quaternion q can be written as

$$\begin{aligned} q:\ I &\to \mathcal{P}_{\mathbb{R}} \\ s &\to q(s) = \sum_{i=1}^{4} q_i(s)\,\omega_i; \quad 1 \le i \le 4,\ \omega_1 = 1. \end{aligned} \tag{3.1}$$

3.1. The Construction of Bäcklund Transformations of Timelike Curve

Let us recall Bäcklund transformations of a timelike curve introduced by Karacan in [8] assume that α is a timelike curve and $\{\mathcal{T}, \mathcal{N}_1, \mathcal{N}_2\}$ and $\{\kappa_1, \kappa_2\}$ are its Bishop frame and Bishop curvatures, respectively. Then, the Bäcklund transformation of the curve α is expressed as

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}\left(\cosh\gamma\mathcal{T} + \sinh\gamma\mathcal{N}_1\right), \tag{3.2}$$

where

$$\mathcal{K} = \kappa_2 \tan\frac{\phi}{2} \tag{3.3}$$

and

$$\frac{d\gamma}{ds} = \kappa_2^{\beta}\cosh\gamma\tan\frac{\phi}{2} - \kappa_1. \tag{3.4}$$

Theorem 3.1. *Let β be Bäcklund transformations of a timelike curve given by equation (3.3) in $\mathbb{E}_1^3$ and q be a spacelike split quaternion defined as*

$$q = \sinh\gamma + \cosh\gamma\mathcal{N}_1. \tag{3.5}$$

The Bäcklund transformations of a timelike curve with a spacelike split quaternion can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\frac{1}{2}(q * \mathcal{N}_2 - \mathcal{N}_2 * q) + \mathcal{N}_1 * q - \cosh\gamma]. \tag{3.6}$$

Proof. Let us assume that q is a spacelike split quaternion given in the form $q = \sinh\gamma + \cosh\gamma\mathcal{N}_1$. As a direct result of equation (3.3), the equality

$$\mathcal{N}_1 * q = \sinh\gamma * \mathcal{N}_1 + \cosh\gamma \tag{3.7}$$

holds, which directs us to the equality

$$\sinh\gamma * \mathcal{N}_1 = \mathcal{N}_1 * q - \cosh\gamma. \tag{3.8}$$

By multiplying both sides of the equation (3.5) with $\mathcal{N}_2$ from the left side, we reach

$$\mathcal{N}_2 * q = \sinh\gamma * \mathcal{N}_2 - \cosh\gamma\mathcal{T}. \tag{3.9}$$

In a similar way, the equation

$$q * \mathcal{N}_2 = \sinh\gamma * \mathcal{N}_2 + \cosh\gamma\mathcal{T} \tag{3.10}$$

can be obtained analoguously, hence both the equations of (3.9) and (3.10) allow us to write

$$\cosh\gamma\mathcal{T} = \frac{1}{2}(q * \mathcal{N}_2 - \mathcal{N}_2 * q). \tag{3.11}$$

As a consequence of the equations (3.8), (3.11), we immediately have the equation

$$\cosh\gamma\mathcal{T} + \sinh\gamma\mathcal{N}_1 = \frac{1}{2}(q * \mathcal{N}_2 - \mathcal{N}_2 * q) + \mathcal{N}_1 * q - \cosh\gamma, \tag{3.12}$$

which states the fact that Bäcklund transformations β can be calculated as

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\frac{1}{2}(q * \mathcal{N}_2 - \mathcal{N}_2 * q) + \mathcal{N}_1 * q - \cosh\gamma]. \tag{3.13}$$

□

The above equalities direct us to the following theorem.

Theorem 3.2. *Let β be Bäcklund transformations of a timelike curve given by equation (3.2) in $\mathbb{E}_1^3$ and q be a spacelike split quaternion defined as*

$$q = \sinh\gamma + \cosh\gamma\mathcal{N}_2. \tag{3.14}$$

The Bäcklund transformations of a timelike curve with a spacelike split quaternion can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\mathcal{N}_1 * q]. \tag{3.15}$$

Proof. Assume that q is a spacelike split quaternion defined as $q = \sinh\gamma+\cosh\gamma\mathcal{N}_2$. If we multiply both sides of the equation (3.14) with $\mathcal{N}_1$ from the left side, then

$$\mathcal{N}_1 * q = \cosh\gamma\mathcal{T} + \sinh\gamma\mathcal{N}_1. \tag{3.16}$$

In this case, the Bäcklund transformations β can be calculated as

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\mathcal{N}_1 * q]. \tag{3.17}$$

□

The proof of the next theorem is similar to Theorem 1.

Theorem 3.3. *Let β be Bäcklund transformations of a timelike curve given by equation (3.2) in $\mathbb{E}_1^3$ and q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{N}_2. \tag{3.18}$$

The Bäcklund transformations of a timelike curve with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\mathcal{T} * q]. \tag{3.19}$$

Proposition 3.1. *Let q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{N}_2 \tag{3.20}$$

and u, v be any two vectors, where the angle between them is γ, and β be Bäcklund transformations given by equation (3.19) in $\mathbb{E}_1^3$. Then, the Bäcklund transformations of a timelike curve with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\mathcal{T} * v * u^{-1}], \tag{3.21}$$

where they satisfy the following conditions:
i) if u and v are timelike vectors, then u and v are perpendicular to $\mathcal{N}_2$,
ii) if u and v are spacelike vectors, then $|\langle u, v\rangle| > 1$ and u and v are perpendicular to$\mathcal{N}_2$.

Theorem 3.4. *Let β be Bäcklund transformations of a timelike curve given by the equation (3.2) in $\mathbb{E}_1^3$ and q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{N}_1. \tag{3.22}$$

The Bäcklund transformations of a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\frac{1}{2}(\mathcal{T} * q + q * \mathcal{T}) + q - \cosh\gamma]. \tag{3.23}$$

Proof. Let us assume that q is a timelike split quaternion having a spacelike vector part given in the form $\cosh\gamma + \sinh\gamma\mathcal{N}_1$. As a direct result of the equation (3.22), the equality

$$\sinh\gamma\mathcal{N}_1 = q - \cosh\gamma \tag{3.24}$$

holds, which directs us to the equality. By multiplying both sides of the equation (3.22) with $\mathcal{T}$ from the left side, we reach

$$\mathcal{T} * q = \cosh\mathcal{T} - \sinh\gamma\mathcal{N}_2. \tag{3.25}$$

In a similar way, the equation

$$q * \mathcal{T} = \cosh\mathcal{T} + \sinh\gamma\mathcal{N}_2 \tag{3.26}$$

can be obtained analoguously, hence both the equations of (3.25) and (3.26) allow us to write

$$\cosh\gamma\mathcal{T} = \frac{1}{2}(\mathcal{T} * q + q * \mathcal{T}). \tag{3.27}$$

As a consequence of the equations (3.24), (3.27), we immediately have the equation

$$\cosh\gamma\mathcal{T} + \sinh\gamma\mathcal{N}_1 = \frac{1}{2}(\mathcal{T} * q + q * \mathcal{T}) + q - \cosh\gamma. \tag{3.28}$$

In this case, the Bäcklund transformations with a timelike split quaternion having a spacelike vector part can be calculated as

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\frac{1}{2}(\mathcal{T} * q + q * \mathcal{T}) + q - \cosh\gamma]. \tag{3.29}$$

□

Proposition 3.2. *Let q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{N}_1 \tag{3.30}$$

and u, v be any two vectors, where the angle between them is γ, and β be Bäcklund transformations given by the equation (3.23) in $\mathbb{E}_1^3$. Then, the Bäcklund transformations of a timelike curve with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 + \mathcal{K}^2}[\frac{1}{2}(\mathcal{T} * v * u^{-1} + v * u^{-1} * \mathcal{T}) + v * u^{-1} - \cosh\gamma], \tag{3.31}$$

where they satisfy the following conditions;
i) if u and v are timelike vectors, then u and v are perpendicular to $\mathcal{N}_1$,
ii) if u and v are spacelike vectors, then $|\langle u, v\rangle| > 1$ and u and v are perpendicular to $\mathcal{N}_1$.

In the light of the above theorems, the following theorems will be given without proof.

3.2. The Construction of Bäcklund Transformations of Spacelike Curve with Spacelike Principal Normal

Let us recall Bäcklund transformations of a spacelike curve with spacelike principal normal introduced by Karacan in [8]. Therefore, we assume that α is a spacelike curve with a spacelike principal normal and $\{\mathcal{T}, \mathcal{N}_1, \mathcal{N}_2\}$ and $\{\kappa_1, \kappa_2\}$ are its Bishop frame and Bishop curvatures, respectively. Then, the Bäcklund transformation of the curve α is expressed as

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 - \mathcal{K}^2}\left(\cos\gamma\mathcal{T} + \sin\gamma\mathcal{N}_1\right), \tag{3.32}$$

where

$$\mathcal{K} = \kappa_2 \tan\frac{\phi}{2} \tag{3.33}$$

and

$$\frac{d\gamma}{ds} = -\kappa_2\cos\gamma\tanh\frac{\phi}{2} - \kappa_1^\beta. \tag{3.34}$$

Theorem 3.5. *Let β be Bäcklund transformations of a spacelike curve with spacelike principal normal given by the equation (3.32) in $\mathbb{E}_1^3$ and q be a timelike split quaternion having a timelike vector part defined as*

$$q = \cos\gamma + \sin\gamma\mathcal{N}_2. \tag{3.35}$$

The Bäcklund transformations of a spacelike curve having spacelike principal normal with a timelike split quaternion having a timelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 - \mathcal{K}^2}[\mathcal{T} * q]. \tag{3.36}$$

Proposition 3.3. *Let q be a timelike split quaternion having a timelike vector part defined as*

$$q = \cos\gamma + \sin\gamma\mathcal{N}_2 \tag{3.37}$$

and u, v be any two spacelike vectors,when the angle between them is γ, and β be Bäcklund transformations given by equation (3.36) in $\mathbb{E}_1^3$. Then, Bäcklund transformations of a spacelike curve having a spacelike principal normal with a timelike split quaternion having a timelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\tanh\gamma}{\kappa_2^2 - \mathcal{K}^2}[\mathcal{T} * u * v], \tag{3.38}$$

where u and v are perpendicular to $\mathcal{N}_2$.

3.3. The Construction of Bäcklund Transformations of Spacelike Curve with Spacelike Binormal

Let us recall Bäcklund transformations of a spacelike curve with spacelike binormal introduced by Karacan in [8]. Hence, we assume that α is a spacelike curve with spacelike binormal and $\{\mathcal{T}, \mathcal{N}_1, \mathcal{N}_2\}$ and $\{\kappa_1, \kappa_2\}$ are its Bishop frame and Bishop curvatures, respectively. Then, the Bäcklund transformation of the curve α is expressed as

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}\left(\cosh\gamma\mathcal{T} + \sinh\gamma\mathcal{N}_1\right), \tag{3.39}$$

where

$$\mathcal{K} = \kappa_2 \tanh\frac{\phi}{2} \tag{3.40}$$

and

$$\frac{d\gamma}{ds} = -\kappa_2\cosh\gamma\tanh\frac{\phi}{2} - \kappa_1^\beta. \tag{3.41}$$

Theorem 3.6. *Let β be Bäcklund transformations of a spacelike curve with spacelike binormal given by equation (3.41) in $\mathbb{E}_1^3$ and q be a spacelike split quaternion defined as*

$$q = \sinh\gamma + \cosh\gamma\mathcal{N}_2. \tag{3.42}$$

The Bäcklund transformations of a spacelike curve with spacelike binormal with a spacelike split quaternion can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}[q * \mathcal{N}_1]. \tag{3.43}$$

Theorem 3.7. *Let β be Bäcklund transformations of a spacelike curve with a spacelike binormal given by the equation (3.39) in $\mathbb{E}_1^3$ and q be a spacelike split quaternion defined as*

$$q = \sinh\gamma + \cosh\gamma\mathcal{T}. \tag{3.44}$$

The Bäcklund transformations of a spacelike curve having a spacelike binormal with a spacelike split quaternion can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}[q - \sinh\gamma + \frac{1}{2}(\mathcal{N}_1 * q + q * \mathcal{N}_1)]. \tag{3.45}$$

Theorem 3.8. *Let β be Bäcklund transformations of a spacelike curve with spacelike binormal given by equation (3.39) in $\mathbb{E}_1^3$ and q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{T}. \tag{3.46}$$

The Bäcklund transformations of a spacelike curve having a spacelike binormal with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}[q * \mathcal{T} - \sinh\gamma + \frac{1}{2}(\mathcal{N}_2 * q - q * \mathcal{N}_2)]. \tag{3.47}$$

Proposition 3.4. *Let q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{T} \tag{3.48}$$

and u, v be any two vectors, when the angle between them is γ, and β be Bäcklund transformations given by the equation (3.47) in $\mathbb{E}_1^3$. Then, Bäcklund transformations of a spacelike curve having a spacelike binormal with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}[v * u^{-1} * \mathcal{T} - \sinh\gamma + \frac{1}{2}(\mathcal{N}_2 * v * u^{-1} - v * u^{-1} * \mathcal{N}_2)], \tag{3.49}$$

where they satisfy the following conditions:
i) if u and v are timelike vectors, then u and v are perpendicular to $\mathcal{T}$,
ii) if u and v are spacelike vectors, then $|\langle u, v\rangle| > 1$ and u and v are perpendicular to $\mathcal{T}$.

Theorem 3.9. *Let β be Bäcklund transformations of a spacelike curve with a spacelike binormal given by the equation (3.39) in $\mathbb{E}_1^3$ and q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{N}_2. \tag{3.50}$$

The Bäcklund transformations of a spacelike curve having a spacelike binormal with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}[\mathcal{T} * q]. \tag{3.51}$$

Proposition 3.5. *Let q be a timelike split quaternion having a spacelike vector part defined as*

$$q = \cosh\gamma + \sinh\gamma\mathcal{N}_2. \tag{3.52}$$

and u, v be any two vectors, when the angle between them is γ, and β be Bäcklund transformations given by the equation (3.51) in $\mathbb{E}_1^3$. Then, Bäcklund transformations of a spacelike curve having a spacelike binormal with a timelike split quaternion having a spacelike vector part can be stated by

$$\beta = \alpha + \frac{2\mathcal{K}\sinh\gamma}{\kappa_2^2 - \mathcal{K}^2}[\mathcal{T} * v * u^{-1}], \tag{3.53}$$

where they satisfy the following conditions;
i) if u and v are timelike vectors, then u and v are perpendicular to $\mathcal{N}_2$,
ii) if u and v are spacelike vectors, then $|\langle u, v\rangle| > 1$ and u and v are perpendicular to $\mathcal{N}_2$.

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NEW APPROACHES ON DUAL SPACE

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Abstract. In this paper, we have explained how to define the basic concepts of differential geometry on Dual space. To support this, dual tangent vectors that have $\overline{p}$ as dual point of application have been defined. Then, the dual analytic functions defined by Dimentberg have been examined in detail, and by using the derivative of the these functions, dual directional derivatives and dual tangent maps have been introduced.
Keywords: Dual space; dual tangent vectors; dual analytic functions; tangent maps.

1. Introduction and Basic Concepts

Sir Isaac Newton invented calculus in about 1665. The solution to the problem which he was interested in was too difficult for mathematics used in that time. For this reason, he found a new approach to mathematics. Also, he tried to compute the velocity of n object at any instant. Nowadays, many scientists tend to calculate the rate at which satellite's position changes according to time. A comparison of the change in one quantity to the simultaneous change in a second quantity is known as a rate of change. If both changes emerge in the course of an infinitely short period of time, the rate is called instantaneous. Then, the derivatives are important to the solution of the problems in calculus. Calculus has application fields in physics and engineering [1].

Dual numbers were defined by W. K. Clifford [3] (1845-1879) as a tool of his geometrical studies. Their first applications were given by Kotelnikov [9] and Study [13]. Dual variable functions were introduced by Dimentberg [4]. He investigated the analytic conditions of these functions, and by means of conditions, he described the derivative concept of these functions. In 1999, by using these dual analytic functions, Brodsky *et al.* [2] showed that the derivatives of products of two dual analytic functions with respect to dual variables are equal to moment-product derivative.

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In recent years, dual numbers have been widely used in kinematics, dynamics, mechanism design, and field and group theories ([5], [6], [7], [8] and [12]). For example in kinematics, constraint manifolds of spatial mechanisms are explained using dual numbers system [10]. The aim of this study is to calculate the derivative of dual analytic functions with respect to dual vectors, by expanding the definition of the derivative in dual analytic function. After then, by using this derivative concept and dual analytic functions, the authors showed how to define vector fields and tangent maps on Dual space. These concepts will give us a new perspective in Dual space.

This paper is organized in the following way: In section II, the dual analytic functions defined by Dimentberg are introduced, and by using these functions, the partial derivatives of the functions $\overline{f} : D^n \to D$ are calculated.

In section III, dual tangent vectors are introduced, and the derivative of $\overline{f}$ with respect to dual tangent vectors is computed. For $1 \leq i \leq n$, it is shown that partial derivatives calculated in the second part is the derivative of $\overline{f}$ with respect to vectors e_i, where $e_i = (\delta_{i1}, ..., \delta_{in})$. Here, δ_{ij} is the Kronecker delta (0 if $i \neq j$, 1 if $i = j$).

In section IV, dual vector fields are introduced, and in the last section, the dual tangent map that sends the dual tangent vectors at dual point $\overline{p}$ to the dual tangent vectors at dual point $\overline{f}\,(\overline{p})$ is defined.

Now, we recall a brief summary of the theory of dual numbers and the fundamental concepts of Differential Geometry.

Let the set $\mathbb{R} \times \mathbb{R}$ be shown as D. On the set $D = \{\overline{x} = (x, x^*) \mid x, x^* \in \mathbb{R}\}$, two operators and equality are defined as follows.

$$\begin{aligned} \overline{x} \oplus \overline{y} &= (x + y, x^* + y^*), \\ \overline{x} \odot \overline{y} &= (xy, x^* y + x y^*), \\ \overline{x} &= \overline{y} \Longleftrightarrow x = y,\ x^* = y^*. \end{aligned}$$

The set D is called the dual numbers system and $(x, x^*) \in D$ is called a dual number. The dual numbers $(1, 0) = 1$ and $(0, 1) = \varepsilon$ are called the unit element of multiplication operation in D, and dual unit which satisfies the condition that $\varepsilon^2 = 0$, respectively. Also, the dual number $\overline{x} = (x, x^*)$ can be written as $\overline{x} = x + \varepsilon x^*$, and the set of all dual numbers is shown by

$$D = \left\{ \overline{x} = x + \varepsilon x^* \mid x, x^* \in \mathbb{R},\ \varepsilon^2 = 0 \right\}.$$

The set of

$$D^3 = \{\overline{v} = (\overline{v}_1, \overline{v}_2, \overline{v}_3) \mid \overline{v}_i \in D,\ 1 \leq i \leq 3\}$$

gives all triples of dual numbers. The element of D^3 is called as dual vectors and a dual vector can be written in the following form

$$\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*,$$

where $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^3$. The addition and multiplication operations on D^3 are as below:

$$\begin{aligned}\overline{v}+\overline{w} &= \overrightarrow{v}+\overrightarrow{w}+\varepsilon\left(\overrightarrow{v}^*+\overrightarrow{w}^*\right),\\ \overline{\lambda}\overline{v} &= \lambda\overrightarrow{v}+\varepsilon\left(\lambda\overrightarrow{v}^*+\lambda^*\overrightarrow{v}\right),\end{aligned}$$

where $\overline{v}=\overrightarrow{v}+\varepsilon\overrightarrow{v}^*, \overline{w}=\overrightarrow{w}+\varepsilon\overrightarrow{w}^*\in D^3$ and $\overline{\lambda}=\lambda+\varepsilon\lambda^*\in D$. The set D^3 is a module over the ring D, and is called D-module or dual space.

The set of dual vectors on D^n is represented by

$$D^n=\left\{\overline{v}=\left(\overline{v}_1,...,\overline{v}_n\right) \mid \overline{v}_i\in D,\ 1\le i\le n\right\}.$$

These vectors can be given in the form

$$\overline{v}=\overrightarrow{v}+\varepsilon\overrightarrow{v}^*,$$

where $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^n$. On this set, the addition and multiplication are given as follows

$$\begin{aligned}\overline{v}+\overline{w} &= \overrightarrow{v}+\overrightarrow{w}+\varepsilon\left(\overrightarrow{v}^*+\overrightarrow{w}^*\right),\\ \overline{\lambda}\overline{v} &= \lambda\overrightarrow{v}+\varepsilon\left(\lambda\overrightarrow{v}^*+\lambda^*\overrightarrow{v}\right).\end{aligned}$$

The set D^n is a module over the ring D. On the other hand, since $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^n$, we can write the equalities below

$$\overrightarrow{v}=v_1e_1+...+v_ne_n,$$

and

$$\overrightarrow{v}^*=v_1^*e_1+...+v_n^*e_n,$$

where $e_i=\left(\delta_{i1},...,\delta_{in}\right)$ for $1\le i\le n$. Thus, we have

$$\begin{aligned}\overline{v} &= \overrightarrow{v}+\varepsilon\overrightarrow{v}^*\\ &= v_1e_1+...+v_ne_n+\varepsilon\left(v_1^*e_1+...+v_n^*e_n\right)\\ &= \left(v_1+\varepsilon v_1^*\right)e_1+...+\left(v_n+\varepsilon v_n^*\right)e_n\\ &= \overline{v}_1e_1+...+\overline{v}_ne_n.\end{aligned}$$

For $1\le i\le n$, let $x_i:\mathbb{R}^n\to\mathbb{R}$ be the function that sends each point $p=\left(p_1,...,p_n\right)$ to its ith coordinate p_i. Then $x_1,...,x_n$ are the natural coordinate functions of $\mathbb{R}^n$. On the set

$$T_p\mathbb{R}^n=\{p\}\times\mathbb{R}^n=\left\{\left(p,\overrightarrow{v}\right) \mid \overrightarrow{v}\in\mathbb{R}^n\right\},$$

addition and scalar product operators are defined as follows, respectively.

$+:T_p\mathbb{R}^n\times T_p\mathbb{R}^n\to T_p\mathbb{R}^n$, for $\left(p,\overrightarrow{v}\right),\ \left(p,\overrightarrow{w}\right)$ defined as

$$\left(p,\overrightarrow{v}\right)+\left(p,\overrightarrow{w}\right)=\left(p,\overrightarrow{v}+\overrightarrow{w}\right).$$

$\cdot : \mathbb{R} \times T_p\mathbb{R}^n \to T_p\mathbb{R}^n$, for $\lambda \in \mathbb{R}$ and $(p, \overrightarrow{v})$ defined as

$$\lambda (p, \overrightarrow{v}) = (p, \lambda \overrightarrow{v}).$$

In this case, the set $(T_p\mathbb{R}^n, +, (\mathbb{R}, +, \cdot), \cdot)$ is a vector space otherwise known as a tangent space. The element $\overrightarrow{v}_p = (p, \overrightarrow{v})$ is called a tangent vector to $\mathbb{R}^n$ at p.

A real-valued function of f on $\mathbb{R}^n$ is differentiable proved all mixed partial derivatives of f exist and are continuous.

For $1 \leq j \leq n$, if the functions $f_j : \mathbb{R}^n \to \mathbb{R}$ are differentiable, then the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

Let f be a differentiable real-valued function on $\mathbb{R}^n$. Gradient of the function f is defined as

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right).$$

Definition 1.1. Let f be a differentiable real-valued function on $\mathbb{R}^n$ and $\overrightarrow{v}_p$ be a tangent vector to $\mathbb{R}^n$. Then, the number

$$\overrightarrow{v}_p [f] = \frac{d}{dt} f (p + t\overrightarrow{v}) \mid_{t=0}$$

is called the derivative of f with respect to $\overrightarrow{v}_p$.

A vector field is a function that assigns to each point p of $\mathbb{R}^n$ a tangent vector $\overrightarrow{v}_p$ to $\mathbb{R}^n$ at p.

Definition 1.2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. For every $p \in \mathbb{R}^n$, the function $f_{*p} : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ is defined as follows:

$$f_{*p} (\overrightarrow{v}_p) = (\overrightarrow{v}_p [f_1], ..., \overrightarrow{v}_p [f_m]) \mid_{f(p)}.$$

This function is called tangent map of f.

For the vectors $\overrightarrow{v} = (v_1, ..., v_n)$ and $\overrightarrow{w} = (w_1, ..., w_n)$, the inner product on $\mathbb{R}^n$ is given by

$$\overrightarrow{v} \cdot \overrightarrow{w} = v_1 w_1 + ... + v_n w_n.$$

For more details, we refer the readers to [11].

2. Derivative of Dual Analytic Functions

Let $\overline{x} = x + \varepsilon x^*$ be a dual number. A dual variable function $\overline{f} : D \to D$ is defined as follows:

$$\overline{f} (\overline{x}) = f (x, x^*) + \varepsilon f^o (x, x^*),$$

where f and f^o are real functions with two real variables x and x^*. Dimentberg comprehensively investigated the properties of dual functions. He showed that the analytic conditions of dual functions are

$$\frac{\partial f}{\partial x^*} = 0 \text{ and } \frac{\partial f^o}{\partial x^*} = \frac{\partial f}{\partial x}. \tag{2.1}$$

From the above first condition, the function f is a function which has only variable x, i.e.,

$$f(x, x^*) = f(x)$$

and the second implies that the function f^o is as below expression

$$f^o(x, x^*) = x^* \frac{\partial f}{\partial x} + \widetilde{f}(x),$$

where $\widetilde{f}(x)$ is a certain function of x. General notation of dual analytic function is given by following equality

$$\overline{f}(\overline{x}) = \overline{f}(x + \varepsilon x^*) = f(x) + \varepsilon\left(x^* \frac{df}{dx} + \widetilde{f}(x)\right). \tag{2.2}$$

For $x^* = 0$, the function must be written in the form

$$\overline{f}(\overline{x}) = \overline{f}(x + \varepsilon x^*) = f(x) + \varepsilon \widetilde{f}(x).$$

The derivative of the dual analytic function $\overline{f}$ is defined by

$$\begin{aligned} \frac{d\overline{f}}{d\overline{x}} &= \frac{df}{dx} + \varepsilon \frac{d}{dx}\left(x^* \frac{df}{dx} + \widetilde{f}(x)\right) \\ &= \frac{df}{dx} + \varepsilon\left(x^* \frac{d^2 f}{dx^2} + \frac{d\widetilde{f}}{dx}\right). \end{aligned} \tag{2.3}$$

It is seen that the derivative of the function $\overline{f}$ with respect to dual variable $\overline{x}$ is equal to the derivative with respect to real variable x [4]. Now, we shall study dual analytic functions $\overline{f} : D^n \to D$, i.e.,

$$\overline{f}(\overline{x}) = \overline{f}(x + \varepsilon x^*) = f(x_1, ..., x_n, x_1^*, ..., x_n^*) + \varepsilon f^o(x_1, ..., x_n, x_1^*, ..., x_n^*),$$

where $x = (x_1, ..., x_n)$ and $x^* = (x_1^*, ..., x_n^*)$. Using the above equalities (2.1), the analytic conditions of this function can be given

$$\frac{\partial f}{\partial x_i^*} = 0 \text{ and } \frac{\partial f^o}{\partial x_i^*} = \frac{\partial f}{\partial x_i}, \qquad (1 \leq i \leq n).$$

In that case, general expression of the dual analytic functions is defined as follows:

$$\overline{f}(\overline{x}) = f(x_1, ..., x_n) + \varepsilon\left(\sum_{i=1}^{n} x_i^* \frac{\partial f}{\partial x_i} + \widetilde{f}(x_1, ..., x_n)\right).$$

If the equality (2.2) is used, then the partial derivatives of these dual analytic functions are given by

$$\frac{\partial \overline{f}}{\partial \overline{x}_j} = \frac{\partial f}{\partial x_j} + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial \widetilde{f}}{\partial x_j} \right),$$

where $1 \leq j \leq n$. Similarly, the partial derivatives of the function $\overline{f}$ according to dual variables $\overline{x}_j$ are reduced to the partial derivatives according to real variables x_j. For the general dual functions $\overline{f} : D^n \to D^m$, if the functions

$$\overline{f}_k : D^n \to D, \quad (1 \leq k \leq m)$$

are dual analytic functions, then the dual function $\overline{f}$ is a dual analytic function, and the set of the dual analytic functions is shown by

$$C(D^n, D^m) = \left\{ \overline{f} \mid \overline{f} : D^n \to D^m \text{ is a dual analytic function} \right\}.$$

For the dual-valued analytic functions on D^n, the following equalities can be defined

$$\begin{aligned} \left(\overline{f} + \overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x}) + \overline{g}(\overline{x}) \\ &= f(x) + g(x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \left(\frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i} \right) + \widetilde{f}(x) + \widetilde{g}(x) \right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \left(\overline{\lambda f}\right)(\overline{x}) &= \overline{\lambda f}(\overline{x}) \\ &= \lambda f(x) + \varepsilon \left(\lambda \left(\sum_{i=1}^{n} x_i^* \frac{\partial f}{\partial x_i} \right) + \lambda^* f(x) + \lambda \widetilde{f}(x) \right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \left(\overline{f} \cdot \overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x}) \cdot \overline{g}(\overline{x}) \\ &= f(x)\, g(x) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \left(\frac{\partial (fg)}{\partial x_i} \right) + f(x)\, \widetilde{g}(x) + g(x)\, \widetilde{f}(x) \right), \end{aligned} \tag{2.6}$$

where $\overline{x} = x + \varepsilon x^* = (x_1, ..., x_n) + \varepsilon (x_1^*, ..., x_n^*)$. It is clear that the above equations are the dual analytic functions.

Let $\overline{p} = p + \varepsilon p^*$ be a dual point of D^n, and $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*$ be a dual vector to D^n. The equation of dual straight line is given by

$$\begin{aligned} \overline{\alpha}\left(\overline{t}\right) &= p + t\overrightarrow{v} + \varepsilon \left(t^* \overrightarrow{v} + p^* + t\overrightarrow{v}^*\right) \\ &= \alpha(t) + \varepsilon \left(t^* \alpha'(t) + \widetilde{\alpha}(t)\right). \end{aligned} \tag{2.7}$$

It is seen that the equality (2.7) is a dual analytic function.

Definition 2.1. Let $x_1, ..., x_n, x_1^*, ..., x_n^*$ be coordinate functions of $\mathbb{R}^{2n}$. For $1 \leq i \leq n$, these functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$ are given as follows:

$$x_i(\widetilde{p}) = p_i, \quad x_i^*(\widetilde{p}) = p_i^*,$$

where $\widetilde{p} = (p_1, ..., p_n, p_1^*, ..., p_n^*)$ is a point of $\mathbb{R}^{2n}$. In this case, dual coordinate functions $\overline{x}_i : D^n \to D$ are defined by

$$\begin{aligned} \overline{x}_i(\overline{p}) &= x_i(\widetilde{p}) + \varepsilon x_i^*(\widetilde{p}) \\ &= p_i + \varepsilon p_i^* \\ &= \overline{p}_i, \end{aligned}$$

where $\overline{p} = (p_1 + \varepsilon p_1^*, ..., p_n + \varepsilon p_n^*) = (p_1, ..., p_n) + \varepsilon (p_1^*, ..., p_n^*) = p + \varepsilon p^*$ is a point of D^n.

The above definition shows how to implement the dual point in the dual analytic functions. For example, for the dual-valued analytic functions on D^n, the following equalities can be written

$$\begin{aligned} \overline{f}(\overline{p}) &= f(\widetilde{p}) + \varepsilon \left(\sum_{i=1}^{n} p_i^* \frac{\partial f}{\partial x_i}(\widetilde{p}) + \widetilde{f}(\widetilde{p}) \right) \\ &= f(\widetilde{p}) + \varepsilon f^o(\widetilde{p}), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \overline{f}}{\partial \overline{x}_j}(\overline{p}) &= \frac{\partial f}{\partial x_j}(\widetilde{p}) + \varepsilon \left(\sum_{i=1}^{n} x_i^* \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial \widetilde{f}}{\partial x_j} \right)(\widetilde{p}) \\ &= \frac{\partial f}{\partial x_j}(\widetilde{p}) + \varepsilon \left(\sum_{i=1}^{n} p_i^* \frac{\partial^2 f}{\partial x_i \partial x_j}(\widetilde{p}) + \frac{\partial \widetilde{f}}{\partial x_j}(\widetilde{p}) \right) \\ &= \frac{\partial f}{\partial x_j}(\widetilde{p}) + \varepsilon \frac{\partial f^o}{\partial x_j}(\widetilde{p}). \end{aligned}$$

Definition 2.2. Let $\overline{f}$ and $\overline{g}$ be dual-valued analytic functions on D. Composition of the dual analytic functions $\overline{f}$ and $\overline{g}$ is determined by

$$\begin{aligned} \overline{f} \circ \overline{g} &: D \to D \\ (\overline{f} \circ \overline{g})(\overline{x}) &= \overline{f}(\overline{g}(\overline{x})), \end{aligned} \tag{2.8}$$

where

$$\overline{f}(\overline{g}(\overline{x})) = (f \circ g)(x) + \varepsilon \left(x^* (f \circ g)'(x) + \widetilde{g}(x)(f' \circ g)(x) + \left(\widetilde{f} \circ g \right)(x) \right). \tag{2.9}$$

If $(f \circ g)(x) = h(x)$ and $\widetilde{g}(x)(f' \circ g)(x) + \left(\widetilde{f} \circ g\right)(x) = \widetilde{h}(x)$ are taken,

$$\left(\overline{f} \circ \overline{g}\right)(\overline{x}) = h(x) + \varepsilon\left(x^* h'(x) + \widetilde{h}(x)\right)$$

can be written. This formula demonstrates that the dual function $\overline{f} \circ \overline{g}$ is a dual analytic function. The explanation on how to calculate the derivative of this function is given in the following theorem.

Theorem 2.1. *Let $\overline{f}$ and $\overline{g}$ be dual-valued analytic functions on D. The derivative of the dual analytic composite function is given by*

$$\frac{d}{d\overline{x}}\left(\overline{f} \circ \overline{g}\right)(\overline{x}) = \frac{d\overline{g}}{d\overline{x}}(\overline{x})\,\frac{d\overline{f}}{d\overline{x}}(\overline{g}(\overline{x})).$$

Proof. Since $\overline{f}$ and $\overline{g}$ are the dual analytic functions,

$$\overline{f}(\overline{x}) = f(x) + \varepsilon\left(x^* f'(x) + \widetilde{f}(x)\right)$$

and

$$\overline{g}(\overline{x}) = g(x) + \varepsilon\left(x^* g'(x) + \widetilde{g}(x)\right)$$

can be written. We know that the derivative of the dual analytic functions are attained by the following equalities:

$$\frac{d\overline{f}}{d\overline{x}} = f'(x) + \varepsilon\left(x^* f''(x) + \widetilde{f}'(x)\right)$$

and

$$\frac{d\overline{g}}{d\overline{x}} = g'(x) + \varepsilon\left(x^* g''(x) + \widetilde{g}'(x)\right).$$

Moreover, since the dual function $\overline{f} \circ \overline{g}$ is the dual analytic function, by using defined derivative of the dual analytic functions, the below equality is obtained

$$\begin{aligned}
\frac{d}{d\overline{x}}\left(\overline{f} \circ \overline{g}\right)(\overline{x}) &= \frac{d}{dx}(f \circ g)(x) \\
&\quad + \varepsilon \frac{d}{dx}\left(x^* (f \circ g)'(x) + \widetilde{g}(x)(f' \circ g)(x) + \left(\widetilde{f} \circ g\right)(x)\right) \\
&= \left(g'(x) + \varepsilon\left(x^* g''(x)\right) + \widetilde{g}'(x)\right) \\
&\quad \cdot \left(f'(g(x)) + \varepsilon\left(\left(x^* g'(x) + \widetilde{g}(x)\right)\right) f''(g(x)) + \widetilde{f}'(g(x))\right) \\
&= \frac{d\overline{g}}{d\overline{x}}(\overline{x})\,\frac{d\overline{f}}{d\overline{x}}(\overline{g}(\overline{x})).
\end{aligned}$$

□

3. Directional Derivatives on Dual Space

Let $\overline{p} = p + \varepsilon p^*$ be a dual point of D^n and $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*$ be dual vector to D^n. A dual tangent vector that has $\overline{p}$ as point of application is given as following equality

$$\overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon \overrightarrow{v}^*_{\widetilde{p}},$$

where $\widetilde{p}$ is the point of $\mathbb{R}^{2n}$. The set of all the dual tangent vectors is shown by

$$T_{\overline{p}}D^n = \left\{ \overline{v}_{\overline{p}} \mid \overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon \overrightarrow{v}^*_{\widetilde{p}},\ \widetilde{p} \in \mathbb{R}^{2n};\ \overrightarrow{v}_{\widetilde{p}}, \overrightarrow{v}^*_{\widetilde{p}} \in T_{\widetilde{p}}\mathbb{R}^n \right\}.$$

Since the tangent vectors of $T_{\widetilde{p}}\mathbb{R}^n$ are written in the form $\overrightarrow{v}_{\widetilde{p}} = (\widetilde{p}, \overrightarrow{v})$, the dual tangent vectors can be determined by

$$\overline{v}_{\overline{p}} = (\overline{p}, \overline{v}) = (\widetilde{p}, \overrightarrow{v}) + \varepsilon (\widetilde{p}, \overrightarrow{v}^*).$$

On the set $T_{\overline{p}}D^n$, we can define the following operations:

$+ : T_{\overline{p}}D^n \times T_{\overline{p}}D^n \to T_{\overline{p}}D^n$, for $(\overline{p}, \overline{v})$, $(\overline{p}, \overline{w})$ defined as

$$(\overline{p}, \overline{v}) + (\overline{p}, \overline{w}) = (\overline{p}, \overline{v} + \overline{w}) = (\widetilde{p}, \overrightarrow{v} + \overrightarrow{w}) + \varepsilon (\widetilde{p}, \overrightarrow{v}^* + \overrightarrow{w}^*).$$

$\cdot : D \times T_{\overline{p}}D^n \to T_{\overline{p}}D^n$, for $\overline{\lambda}$, $(\overline{p}, \overline{v})$ defined as

$$\overline{\lambda} \cdot (\overline{p}, \overline{v}) = \left(\overline{p}, \overline{\lambda v}\right) = (\widetilde{p}, \lambda \overrightarrow{v}) + \varepsilon (\widetilde{p}, \lambda^* \overrightarrow{v} + \lambda \overrightarrow{v}^*).$$

Taken into account the above operations, the set $\{T_{\overline{p}}D^n, +, (D, \oplus, \odot), \cdot\}$ is a D-module and is called a dual tangent space. Besides, since $\overrightarrow{v}$ and $\overrightarrow{v}^*$ are the vectors of $\mathbb{R}^n$, $\overline{v}_{\overline{p}} = (\overline{p}, \overline{v})$ can be written in the form

$$\begin{aligned} (\overline{p}, \overline{v}) &= (\widetilde{p}, \overrightarrow{v}) + \varepsilon (\widetilde{p}, \overrightarrow{v}^*) \\ &= (\widetilde{p}, v_1 e_1 + ... + v_n e_n) + \varepsilon (\widetilde{p}, v_1^* e_1 + ... + v_n^* e_n) \\ &= v_1 (\widetilde{p}, e_1) + ... + v_n (\widetilde{p}, e_n) + \varepsilon (v_1^* (\widetilde{p}, e_1) + ... + v_n^* (\widetilde{p}, e_n)) \\ &= (v_1 + \varepsilon v_1^*) e_{1\widetilde{p}} + ... + (v_n + \varepsilon v_n^*) e_{n\widetilde{p}} \\ &= \overline{v}_1 e_{1\overline{p}} + ... + \overline{v}_n e_{n\overline{p}}, \end{aligned} \tag{3.1}$$

where $e_{i\overline{p}} = e_{i\widetilde{p}} + \varepsilon 0_{\widetilde{p}} = e_{i\widetilde{p}}$, for $1 \leq i \leq n$. On the other hand, let us assume that

$$\sum_{i=1}^{n} \overline{\lambda}_i e_{i\overline{p}} = 0_{\overline{p}}, \tag{3.2}$$

where $\overline{\lambda}_i = \lambda_i + \varepsilon \lambda_i^*$ is a dual number, for $1 \leq i \leq n$. Expanding the equality (3.2), we obtain the following equalities:

$$\lambda_1 e_{1\widetilde{p}} + ... + \lambda_n e_{n\widetilde{p}} + \varepsilon (\lambda_1^* e_{1\widetilde{p}} + ... + \lambda_n^* e_{n\widetilde{p}}) = 0_{\widetilde{p}} + \varepsilon 0_{\widetilde{p}}$$

and

$$(\widetilde{p}, \lambda_1 e_1 + ... + \lambda_n e_n) + \varepsilon (\widetilde{p}, \lambda_1^* e_1 + ... + \lambda_n^* e_n) = (\widetilde{p}, 0) + \varepsilon (\widetilde{p}, 0).$$

If the equality property of dual numbers is used, the second formula implies that

$$\lambda_1 e_1 + ... + \lambda_n e_n = 0$$

and

$$\lambda_1^* e_1 + ... + \lambda_n^* e_n = 0.$$

Since the set $\{e_1, ..., e_n\}$ is linear independent, we have

$$\lambda_1 = ... = \lambda_n = 0$$

and

$$\lambda_1^* = ... = \lambda_n^* = 0.$$

If we consider the equations (3.1) and (3.2), it is seen that

$$T_{\overline{p}} D^n = Sp \{e_{1\overline{p}}, ..., e_{n\overline{p}}\}.$$

Consequently, each element of $T_{\overline{p}} D^n$ can be written as a linear combination of element of the set $\{e_{1\overline{p}}, ..., e_{n\overline{p}}\}$, and this set is known as a standard base of $T_{\overline{p}} D^n$.

Definition 3.1. Let $\overline{f}$ be a dual-valued analytic function on D^n and $\overline{v}_{\overline{p}}$ be a dual tangent vector to D^n. The dual number

$$\frac{d}{d\overline{t}} \overline{f} \left(\overline{p} + \overline{t}\overline{v}\right) |_{\overline{t}=0} \tag{3.3}$$

is called a derivative of $\overline{f}$ with respect to $\overline{v}_{\overline{p}}$ and is denoted by

$$\overline{v}_{\overline{p}} \left[\overline{f}\right] = \frac{d}{d\overline{t}} \overline{f} \left(\overline{p} + \overline{t}\overline{v}\right) |_{\overline{t}=0} .$$

For example, we calculate $\overline{v}_{\overline{p}} \left[\overline{f}\right]$ for the dual analytic function $\overline{f} = x_1^2 + x_2 x_3 + \varepsilon (2x_1 x_1^* + x_2^* x_3 + x_3^* x_2)$ with $\overline{p} = (1, 0, -1) + \varepsilon (-1, 2, 1)$ and $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^* = (1, 5, 3) + \varepsilon (-1, 0 - 1)$. Then

$$\overline{p} + \overline{t}\overline{v} = (1 + t, 5t, -1 + 3t) + \varepsilon (-t + t^* - 1, 5t^* + 2, -t + 3t^* + 1)$$

is computed. Because of

$$\overline{f} = x_1^2 + x_2 x_3 + \varepsilon (2x_1 x_1^* + x_2^* x_3 + x_3^* x_2),$$

we have

$$\overline{f} \left(\overline{p} + \overline{t}\overline{v}\right) = 16t^2 - 3t + 1 + \varepsilon \left((32t - 3) t^* - 7t^2 + 7t - 4\right).$$

Now, the derivative of the function $\overline{f}$ according to $\overline{t}$ is calculated as below:

$$\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)=32t-3+\varepsilon\left(32t^{*}-14t+7\right).$$

Then, we obtain $\overline{v}_{\overline{p}}\left[\overline{f}\right]=-3+7\varepsilon$ at $\overline{t}=t+\varepsilon t^{*}=0+\varepsilon 0$.

This definition appears to be the same as the directional derivatives defined in Euclidean space. However, both definition are different. The following theorem shows how to calculate $\overline{v}_{\overline{p}}\left[\overline{f}\right]$, by using the partial derivatives of the dual analytic function $\overline{f}$ at point $\overline{p}$ on dual space D^n.

Theorem 3.1. *Let $\overline{f}=f+\varepsilon f^{o}$ be a dual-valued analytic function on D^n and $\overline{v}_{\overline{p}}=\overrightarrow{v}_{\widetilde{p}}+\varepsilon\overrightarrow{v}^{*}_{\widetilde{p}}$ be a dual tangent vector to D^n. Then, the dual directional derivatives are*

$$\overline{v}_{\overline{p}}\left[\overline{f}\right]=\left(\nabla f\right)_{\left(\widetilde{p}\right)}\cdot\overrightarrow{v}+\varepsilon\left(\left(\nabla f^{o}\right)_{\left(\widetilde{p}\right)}\cdot\overrightarrow{v}+\left(\nabla f\right)_{\left(\widetilde{p}\right)}\cdot\overrightarrow{v}^{*}\right),$$

where

$$\left(\nabla f\right)_{\left(\widetilde{p}\right)}=\left(\frac{\partial f}{\partial x_1}\left(\widetilde{p}\right),...,\frac{\partial f}{\partial x_n}\left(\widetilde{p}\right)\right)$$

and

$$\begin{aligned}\left(\nabla f^{o}\right)_{\left(\widetilde{p}\right)} &= \left(\frac{\partial f^{o}}{\partial x_1}\left(\widetilde{p}\right),...,\frac{\partial f^{o}}{\partial x_n}\left(\widetilde{p}\right)\right)\\ &= \left(\sum_{i=1}^{n}p_i^{*}\frac{\partial^2 f}{\partial x_i\partial x_1}\left(\widetilde{p}\right)+\frac{\partial\widetilde{f}}{\partial x_1}\left(\widetilde{p}\right),...,\sum_{i=1}^{n}p_i^{*}\frac{\partial^2 f}{\partial x_i\partial x_n}\left(\widetilde{p}\right)+\frac{\partial\widetilde{f}}{\partial x_n}\left(\widetilde{p}\right)\right).\end{aligned}$$

Proof. As it is has already been known, the directional derivatives are defined by

$$\overline{v}_{\overline{p}}\left[\overline{f}\right]=\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)|_{\overline{t}=0}\,.$$

Since $\overline{x}_1,...,\overline{x}_n$ are the dual coordinate functions of D^n, we can write

$$\begin{aligned}\overline{p}+\overline{t}\overline{v} &= \left(\overline{p}_1+\overline{t}\overline{v}_1,...,\overline{p}_n+\overline{t}\overline{v}_n\right)\\ &= \left(\overline{x}_1\left(\overline{p}+\overline{t}\overline{v}\right),...,\overline{x}_n\left(\overline{p}+\overline{t}\overline{v}\right)\right),\end{aligned}$$

where the expressions $\overline{x}_i\left(\overline{p}+\overline{t}\overline{v}\right)$ can be written in the form

$$\overline{x}_i\left(\overline{p}+\overline{t}\overline{v}\right)=p_i+tv_i+\varepsilon\left(t^{*}v_i+p_i^{*}+tv_i^{*}\right)$$

for $1\leq i\leq n$. Since these functions are the dual analytic functions, the derivative of these functions is as in the following equality

$$\begin{aligned}\frac{d\overline{x}_i\left(\overline{p}+\overline{t}\overline{v}\right)}{d\overline{t}} &= \frac{d}{dt}\left(p_i+tv_i\right)+\varepsilon\frac{d}{dt}\left(\left(t^{*}v_i+p_i^{*}+tv_i^{*}\right)\right)\\ &= v_i+\varepsilon v_i^{*}. \qquad (3.4)\end{aligned}$$

Due to

$$\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)=\overline{f}\left(\overline{x}_1\left(\overline{p}+\overline{t}\overline{v}\right),...,\overline{x}_n\left(\overline{p}+\overline{t}\overline{v}\right)\right),$$

the derivative of dual analytic composite functions is

$$\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)=\frac{\partial\overline{f}}{\partial\overline{x}_1}\mid_{\overline{p}+\overline{t}\overline{v}}\frac{d\overline{x}_1}{d\overline{t}}\mid_{\overline{t}}+...+\frac{\partial\overline{f}}{\partial\overline{x}_n}\mid_{\overline{p}+\overline{t}\overline{v}}\frac{d\overline{x}_n}{d\overline{t}}\mid_{\overline{t}}.$$

In this case, for $\overline{t}=0$, we find

$$\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)\mid_{\overline{t}=0}=\frac{\partial\overline{f}}{\partial\overline{x}_1}\mid_{\overline{p}}\frac{d\overline{x}_1}{d\overline{t}}\mid_{\overline{t}=0}+...+\frac{\partial\overline{f}}{\partial\overline{x}_n}\mid_{\overline{p}}\frac{d\overline{x}_n}{d\overline{t}}\mid_{\overline{t}=0}. \tag{3.5}$$

Since $\overline{f}$ is the dual-valued analytic function on D^n, the partial derivatives of this function are

$$\frac{\partial\overline{f}}{\partial\overline{x}_j}=\frac{\partial f}{\partial x_j}+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2 f}{\partial x_i\partial x_j}+\frac{\partial\widetilde{f}}{\partial x_j}\right)\quad(1\leq j\leq n).$$

For $\overline{p}\in D^n$, we can express

$$\begin{aligned}\frac{\partial\overline{f}}{\partial\overline{x}_j}\left(\overline{p}\right) &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right)+\varepsilon\left(\sum_{i=1}^{n}x_i^*\frac{\partial^2 f}{\partial x_i\partial x_j}+\frac{\partial\widetilde{f}}{\partial x_j}\right)\left(\widetilde{p}\right)\\ &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right)+\varepsilon\left(\sum_{i=1}^{n}p_i^*\frac{\partial^2 f}{\partial x_i\partial x_j}\left(\widetilde{p}\right)+\frac{\partial\widetilde{f}}{\partial x_j}\left(\widetilde{p}\right)\right)\\ &= \frac{\partial f}{\partial x_j}\left(\widetilde{p}\right)+\varepsilon\frac{\partial f^o}{\partial x_j}\left(\widetilde{p}\right). \qquad (3.6)\end{aligned}$$

By substituting (3.6) and (3.4) into (3.5), we have

$$\begin{aligned}\frac{d}{d\overline{t}}\overline{f}\left(\overline{p}+\overline{t}\overline{v}\right)\mid_{\overline{t}=0} &= \frac{\partial f}{\partial x_1}\left(\widetilde{p}\right)v_1+...+\frac{\partial f}{\partial x_n}\left(\widetilde{p}\right)v_n\\ &+\varepsilon\left(\frac{\partial f^o}{\partial x_1}\left(\widetilde{p}\right)v_1+...+\frac{\partial f^o}{\partial x_n}\left(\widetilde{p}\right)v_n+\frac{\partial f}{\partial x_1}\left(\widetilde{p}\right)v_1^*+...+\frac{\partial f}{\partial x_n}\left(\widetilde{p}\right)v_n^*\right). \qquad (3.7)\end{aligned}$$

Thus, the dual directional derivatives are obtained from (3.7) as

$$\overline{v}_{\overline{p}}\left[\overline{f}\right]=\left(\nabla f\right)_{\left(\widetilde{p}\right)}\cdot\overrightarrow{v}+\varepsilon\left(\left(\nabla f^o\right)_{\left(\widetilde{p}\right)}\cdot\overrightarrow{v}+\left(\nabla f\right)_{\left(\widetilde{p}\right)}\cdot\overrightarrow{v}^*\right).$$

□

Using this theorem, we recalculate $\overline{v}_{\overline{p}}\left[\overline{f}\right]$ for the example above. Due to

$$\overline{f}=f+\varepsilon f^o=x_1^2+x_2x_3+\varepsilon\left(2x_1x_1^*+x_2^*x_3+x_3^*x_2\right),$$

we get

$$f = x_1^2 + x_2 x_3 \text{ and } f^o = 2x_1 x_1^* + x_2^* x_3 + x_3^* x_2.$$

At the point $\widetilde{p}$, since

$$x_1(\widetilde{p}) = 1,\ x_2(\widetilde{p}) = 0,\ x_3(\widetilde{p}) = -1,\ x_1^*(\widetilde{p}) = -1,\ x_2^*(\widetilde{p}) = 2,\ x_3^*(\widetilde{p}) = 1,$$

the following equalities are obtained

$$(\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v} = -3,\ (\nabla f^o)_{(\widetilde{p})} \cdot \overrightarrow{v} = 9,\ \text{and}\ (\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v}^* = -2.$$

By the theorem

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = -3 + \varepsilon(9-2) = -3 + 7\varepsilon$$

as before.

Throughout this paper, we will use the following notations:

$$(\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v} = \overrightarrow{v}_{\widetilde{p}}[f],\quad (\nabla f^o)_{(\widetilde{p})} \cdot \overrightarrow{v} = \overrightarrow{v}_{\widetilde{p}}[f^o],\quad (\nabla f)_{(\widetilde{p})} \cdot \overrightarrow{v}^* = \overrightarrow{v}^*_{\widetilde{p}}[f].$$

In this case, dual directional derivatives are shown by

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = \overrightarrow{v}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o] + \overrightarrow{v}^*_{\widetilde{p}}[f]\right).$$

Thus, the following theorem can be given.

Theorem 3.2. *Let $\overline{f} = f + \varepsilon f^o$ and $\overline{g} = g + \varepsilon g^o$ be dual-valued analytic functions on D^n and $\overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon \overrightarrow{v}^*_{\widetilde{p}}$ be dual tangent vector to D^n. Then*

(1) $\overline{v}_{\overline{p}}\left[\overline{f} + \overline{g}\right] = \overline{v}_{\overline{p}}\left[\overline{f}\right] + \overline{v}_{\overline{p}}\left[\overline{g}\right].$

(2) $\overline{v}_{\overline{p}}\left[\overline{f}\overline{g}\right] = \overline{v}_{\overline{p}}\left[\overline{f}\right]\overline{g}(\overline{p}) + \overline{f}(p)\overline{v}_{\overline{p}}\left[\overline{g}\right].$

Proof. (1) From the above theorem, we know that

$$\overline{v}_{\overline{p}}\left[\overline{f}\right] = \overrightarrow{v}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o] + \overrightarrow{v}^*_{\widetilde{p}}[f]\right).$$

In that case, we have

$$\begin{aligned}
\overline{v}_{\overline{p}}\left[\overline{f} + \overline{g}\right] &= \overrightarrow{v}_{\widetilde{p}}[f+g] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o + g^o] + \overrightarrow{v}^*_{\widetilde{p}}[f+g]\right) \\
&= \overrightarrow{v}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o] + \overrightarrow{v}^*_{\widetilde{p}}[f]\right) + \overrightarrow{v}_{\widetilde{p}}[g] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[g^o] + \overrightarrow{v}^*_{\widetilde{p}}[g]\right) \\
&= \overline{v}_{\overline{p}}\left[\overline{f}\right] + \overline{v}_{\overline{p}}\left[\overline{g}\right].
\end{aligned}$$

(2) From (2.6), the function

$$\begin{aligned}
\left(\overline{f} \cdot \overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x}) \cdot \overline{g}(\overline{x}) \\
&= f(x)\,g(x) + \varepsilon\left(\sum_{i=1}^{n} x_i^*\left(\frac{\partial(fg)}{\partial x_i}\right) + f(x)\,\widetilde{g}(x) + g(x)\,\widetilde{f}(x)\right)
\end{aligned}$$

is a dual analytic function. If this dual analytic function is shown as below

$$\begin{aligned} \left(\overline{f}\overline{g}\right)(\overline{x}) &= \overline{f}(\overline{x})\,\overline{g}(\overline{x}) \\ &= fg+\varepsilon\left(fg^o+gf^o\right), \end{aligned}$$

the following equality is obtained

$$\begin{aligned} \overline{v}_{\overline{p}}\left[\overline{f}\overline{g}\right] &= \overrightarrow{v}_{\widetilde{p}}[fg]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[fg^o+gf^o]+\overrightarrow{v}^*_{\widetilde{p}}[fg]\right) \\ &= \overrightarrow{v}_{\widetilde{p}}[f]\,g(\widetilde{p})+f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g] \\ &\quad +\varepsilon\begin{pmatrix} \overrightarrow{v}_{\widetilde{p}}[f]\,g^o(\widetilde{p})+f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g^o]+\overrightarrow{v}_{\widetilde{p}}[f^o]\,g(\widetilde{p}) \\ +f^o(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g]+g(\widetilde{p})\,\overrightarrow{v}^*_{\widetilde{p}}[f]+f(\widetilde{p})\,\overrightarrow{v}^*_{\widetilde{p}}[g] \end{pmatrix} \\ &= \overrightarrow{v}_{\widetilde{p}}[f]\,g(\widetilde{p})+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f]\,g^o(\widetilde{p})+\overrightarrow{v}_{\widetilde{p}}[f^o]\,g(\widetilde{p})+\overrightarrow{v}^*_{\widetilde{p}}[f]\,g(\widetilde{p})\right) \\ &\quad +f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g]+\varepsilon\left(f^o(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g]+f(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}}[g^o]+f(\widetilde{p})\,\overrightarrow{v}^*_{\widetilde{p}}[g]\right) \\ &= \left(\overrightarrow{v}_{\widetilde{p}}[f]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o]+\overrightarrow{v}^*_{\widetilde{p}}[f]\right)\right)\left(g(\widetilde{p})+\varepsilon g^o(\widetilde{p})\right) \\ &\quad +\left(f(\widetilde{p})+\varepsilon f^o(\widetilde{p})\right)\left(\overrightarrow{v}_{\widetilde{p}}[g]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[g^o]+\overrightarrow{v}^*_{\widetilde{p}}[g]\right)\right) \\ &= \overline{v}_{\overline{p}}\left[\overline{f}\right]\overline{g}(\overline{p})+\overline{f}(p)\,\overline{v}_{\overline{p}}[\overline{g}]\,. \end{aligned}$$

The equalities (1) and (2) show that the dual directional derivatives satisfy linear and Leibniz rules. □

Definition 3.2. Let $\overline{f}=f+\varepsilon f^o$ be dual-valued analytic function on D^n and $\overline{v}_{\overline{p}}=\overrightarrow{v}_{\widetilde{p}}+\varepsilon\overrightarrow{v}^*_{\widetilde{p}}$ be dual tangent vector to D^n. The expression

$$\overline{v}_{\overline{p}}:C\left(D^n,D\right)\to D$$

$$\overline{v}_{\overline{p}}\left(\overline{f}\right)=\overline{v}_{\overline{p}}\left[\overline{f}\right]=\overrightarrow{v}_{\widetilde{p}}[f]+\varepsilon\left(\overrightarrow{v}_{\widetilde{p}}[f^o]+\overrightarrow{v}^*_{\widetilde{p}}[f]\right)$$

can be defined as an operator.

In section II, we showed that each element of $T_{\overline{p}}D^n$ can be written as a linear combination of element of the set $\{e_{1\overline{p}},...,e_{n\overline{p}}\}$. In this case, for $1\leq i\leq n$, the below equality

$$\begin{aligned} e_{i\overline{p}}\left[\overline{f}\right] &= e_{i\widetilde{p}}[f]+\varepsilon e_{i\widetilde{p}}[f^o] \\ &= \frac{\partial f}{\partial x_i}(\widetilde{p})+\varepsilon\frac{\partial f^o}{\partial x_i}(\widetilde{p}) \\ &= \frac{\partial f}{\partial x_i}(\widetilde{p})+\varepsilon\left(\sum_{j=1}^n p^*_j\frac{\partial^2 f}{\partial x_j\partial x_i}(\widetilde{p})+\frac{\partial\widetilde{f}}{\partial x_i}(\widetilde{p})\right) \qquad (3.8) \\ &= \frac{\partial f}{\partial x_i}(\widetilde{p})+\varepsilon\left(\sum_{j=1}^n x^*_j\frac{\partial^2 f}{\partial x_j\partial x_i}+\frac{\partial\widetilde{f}}{\partial x_i}\right)(\widetilde{p}) \\ &= \frac{\partial\overline{f}}{\partial\overline{x}_i}(\overline{p}) \end{aligned}$$

can be written. For every $\overline{p} \in D^n$, since the above equality is correct, we get the following equality:

$$e_i\left[\overline{f}\right] = e_i\left[f\right] + \varepsilon e_i\left[f^o\right] = \frac{\partial f}{\partial x_i} + \varepsilon \frac{\partial f^o}{\partial x_i} = \frac{\partial \overline{f}}{\partial \overline{x}_i}. \tag{3.9}$$

The equality (3.9) is shown that the partial derivatives of the dual analytic function $\overline{f}$ according to dual variables $\overline{x}_i$ are equal to the derivative of $\overline{f}$ with respect to vectors e_i.

Definition 3.3. Let $\overline{f} = f + \varepsilon f^o$ be dual-valued analytic function on D^n. Differential of $\overline{f}$ is shown as $d\overline{f}$ and is defined as the following equality

$$d\overline{f}\left(\overline{v}_{\overline{p}}\right) = \overline{v}_{\overline{p}}\left[\overline{f}\right] = \overrightarrow{v}_{\widetilde{p}}\left[f\right] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}\left[f^o\right] + \overrightarrow{v}^{*}_{\widetilde{p}}\left[f\right]\right).$$

If the above definition is considered, since the dual identity function is defined as

$$\overline{I}\left(\overline{x}\right) = \overline{x} = x + \varepsilon x^* = x + \varepsilon\left(x' x^* + 0\left(x\right)\right),$$

$\overline{I}\left(\overline{x}\right)$ is the dual analytic function and, for $1 \leq i \leq n$,

$$\begin{aligned} d\overline{x}_i\left(\overline{v}_{\overline{p}}\right) &= \overline{v}_{\overline{p}}\left[\overline{x}_i\right] = \overrightarrow{v}_{\widetilde{p}}\left[x_i\right] + \varepsilon\left(\overrightarrow{v}_{\widetilde{p}}\left[x_i^*\right] + \overrightarrow{v}^{*}_{\widetilde{p}}\left[x_i\right]\right) \\ &= v_i + \varepsilon v_i^* \end{aligned}$$

is calculated. In this case, it is seen that

$$d\overline{x}_i\left(\overline{v}_{\overline{p}}\right) = \overline{v}_i.$$

On the other hand, assuming that $x_1^*, ..., x_n^*$ are not dependent on $x_1, ..., x_n$, the following equality can be written

$$\begin{aligned} d\overline{x}_i &= dx_i + \varepsilon dx_i^* \\ &= dx_i\left(1 + \varepsilon \frac{dx_i^*}{dx_i}\right) \\ &= dx_i \text{ [4]}. \end{aligned}$$

In this case, $d\overline{x}_i\left(\overline{v}_{\overline{p}}\right)$ can be rewritten as follows

$$\begin{aligned} d\overline{x}_i\left(\overline{v}_{\overline{p}}\right) &= dx_i\left(\overrightarrow{v}_{\widetilde{p}}\right) + \varepsilon dx_i\left(\overrightarrow{v}^{*}_{\widetilde{p}}\right) \\ &= v_i + \varepsilon v_i^*. \end{aligned}$$

Thus, $d\overline{x}_i$ is the ith coordinate functions of the dual vector $v + \varepsilon v^*$ while $\overline{x}_i$ is the ith coordinate functions of the dual point $\overline{p} = p + \varepsilon p^*$.

Let us consider that $g_{ij} = e_i \cdot e_j$, where $1 \leq i, j \leq n$. In this case, the dual inner product on D^n is shown by

$$G = g_{ij} d\overline{x}_i d\overline{x}_j.$$

For the dual vectors $\overline{v} = \overrightarrow{v} + \varepsilon \overrightarrow{v}^*$, $\overline{w} = \overrightarrow{w} + \varepsilon \overrightarrow{w}^* \in D^3$, the dual inner product is

$$\begin{aligned} G(\overline{v}, \overline{w}) &= g_{ij} d\overline{x}_i(\overline{v}) d\overline{x}_j(\overline{w}) \\ &= d\overline{x}_1(\overline{v}) d\overline{x}_1(\overline{w}) + d\overline{x}_2(\overline{v}) d\overline{x}_2(\overline{w}) + d\overline{x}_3(\overline{v}) d\overline{x}_3(\overline{w}) \\ &= (dx_1(\overrightarrow{v}) + \varepsilon dx_1(\overrightarrow{v}^*))(dx_1(\overrightarrow{w}) + \varepsilon dx_1(\overrightarrow{w}^*)) \\ &\quad + (dx_2(\overrightarrow{v}) + \varepsilon dx_2(\overrightarrow{v}^*))(dx_2(\overrightarrow{w}) + \varepsilon dx_2(\overrightarrow{w}^*)) \\ &\quad + (dx_3(\overrightarrow{v}) + \varepsilon dx_3(\overrightarrow{v}^*))(dx_3(\overrightarrow{w}) + \varepsilon dx_3(\overrightarrow{w}^*)) \\ &= \overrightarrow{v} \cdot \overrightarrow{w} + \varepsilon(\overrightarrow{v} \cdot \overrightarrow{w}^* + \overrightarrow{v}^* \cdot \overrightarrow{w}). \end{aligned}$$

This inner product shows how to define inner product studied on D^3 in many articles.

4. Vector Fields on Dual Space

A dual vector field is a dual function that assigns to each dual point $\overline{p} = p + \varepsilon p^* \in D^n$ a dual tangent vector $\overline{X}_{\overline{p}} = \overrightarrow{X}_{\widetilde{p}} + \varepsilon \overrightarrow{X}^*_{\widetilde{p}}$ to D^n, i.e., for every $\overline{p} = p + \varepsilon p^* \in D^n$, the dual vector field is defined as below expression

$$\begin{aligned} \overline{X} &: D^n \to TD^n \\ \overline{X}(\overline{p}) &= \overline{X}_{\overline{p}} = \overrightarrow{X}_{\widetilde{p}} + \varepsilon \overrightarrow{X}^*_{\widetilde{p}}, \end{aligned}$$

where $\overline{X} = X + \varepsilon X^*$. For $1 \leq i \leq n$, let $\overline{a}_i = a_i + \varepsilon a_i^o$ be dual analytic function. In this case,

$$\overline{X} = (a_1, ..., a_n) + \varepsilon(a_1^o, ..., a_1^o) \tag{4.1}$$

is a dual vector field on D^n. For each point $\overline{p}$ of D^n, the equality (4.1) is given in the form

$$\overline{X}(\overline{p}) = (a_1(\widetilde{p}), ..., a_n(\widetilde{p})) + \varepsilon(a_1^o(\widetilde{p}), ..., a_n^o(\widetilde{p})).$$

Here, since $\overline{a}_i = a_i + \varepsilon a_i^o$ is the dual-valued analytic function on D^n, it can be written as follows

$$\begin{aligned} \overline{a}_i(\overline{x}) &= a_i(x_1, ..., x_n) + \varepsilon\left(\sum_{j=1}^n x_j^* \frac{\partial a_i}{\partial x_j} + \widetilde{a}_i(x_1, ..., x_n)\right) \\ &= a_i + \varepsilon a_i^o \end{aligned}$$

and, for every $\overline{p} = p + \varepsilon p^* \in D^n$, we have

$$\overline{a}_i(\overline{p}) = a_i(\widetilde{p}) + \varepsilon a_i^o(\widetilde{p}).$$

The set of dual vector fields is given as follows:

$$\chi(D^n) = \left\{\overline{X} \mid \overline{X} : D^n \to TD^n, \ \overline{X}(\overline{p}) = \overline{X}_{\overline{p}} = \overrightarrow{X}_{\widetilde{p}} + \varepsilon \overrightarrow{X}^*_{\widetilde{p}}\right\}.$$

On this set, we are able to define the following operators which make $\chi(D^n)$ a module called D-module. Axioms are as follows:

$$\left(\overline{X}+\overline{Y}\right)(\overline{p})=\overline{X}(\overline{p})+\overline{Y}(\overline{p})=\overrightarrow{X}_{\widetilde{p}}+\overrightarrow{Y}_{\widetilde{p}}+\varepsilon\left(\overrightarrow{X}^*_{\widetilde{p}}+\overrightarrow{Y}^*_{\widetilde{p}}\right)$$

and

$$\left(\overline{\lambda}\cdot\overline{X}\right)(\overline{p})=\overline{\lambda X}_{\overline{p}}=\lambda\overrightarrow{X}_{\widetilde{p}}+\varepsilon\left(\lambda^*\overrightarrow{X}_{\widetilde{p}}+\lambda\overrightarrow{X}^*_{\widetilde{p}}\right),$$

where $\overline{X}=\overrightarrow{X}+\varepsilon\overrightarrow{X}^*$ and $\overline{Y}=\overrightarrow{Y}+\varepsilon\overrightarrow{Y}^*$ are the dual vector fields and $\overline{\lambda}=\lambda+\varepsilon\lambda^*$ is the dual number.

Let $\overline{f}=f+\varepsilon f^o$ be a dual-valued analytic function on D^n. The function

$$\overline{X}\left[\overline{f}\right]=X[f]+\varepsilon\left(X[f^o]+X^*[f]\right) \tag{4.2}$$

is called the derivative of $\overline{f}$ with respect to the dual vector field $\overline{X}$.

Expanding the equality (4.2), we have

$$\overline{X}\left[\overline{f}\right]=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial x_i}a_i+\varepsilon\left(\begin{array}{c}\sum_{j=1}^{n}x_j^*\left(\frac{\partial f}{\partial x_i}\frac{\partial a_i}{\partial x_j}+a_i\frac{\partial^2 f}{\partial x_j\partial x_i}\right)\\+\frac{\partial f}{\partial x_i}\widetilde{a}_i+a_i\frac{\partial\widetilde{f}}{\partial x_i}\end{array}\right)\right]. \tag{4.3}$$

It is clear that the equality (4.3) is a dual analytic function on D^n. In this case, the dual vector field $\overline{X}:C(D^n,D)\to C(D^n,D)$ is able to be defined as follows:

$$\overline{X}\left(\overline{f}\right)=\overline{X}\left[\overline{f}\right]. \tag{4.4}$$

For every $\overline{p}\in D^n$, if the equalities (4.2) and (4.3) are used, the following expressions are obtained, respectively,

$$\left(\overline{X}\left[\overline{f}\right]\right)(\overline{p})=\overline{X}_{\overline{p}}\left[\overline{f}\right]=\overrightarrow{X}_{\widetilde{p}}[f]+\varepsilon\left(\overrightarrow{X}_{\widetilde{p}}[f^o]+\overrightarrow{X}^*_{\widetilde{p}}[f]\right)$$

and

$$\overline{X}_{\overline{p}}\left[\overline{f}\right]=\sum_{i=1}^{n}\left[\frac{\partial f}{\partial x_i}(\widetilde{p})\,a_i(\widetilde{p})+\varepsilon\left(\begin{array}{c}\sum_{j=1}^{n}p_j^*\left(\frac{\partial f}{\partial x_i}(\widetilde{p})\frac{\partial a_i}{\partial x_j}(\widetilde{p})+a_i(\widetilde{p})\frac{\partial^2 f}{\partial x_i\partial x_j}(\widetilde{p})\right)\\+\frac{\partial f}{\partial x_i}(\widetilde{p})\,\widetilde{a}_i(\widetilde{p})+a_i(\widetilde{p})\frac{\partial\widetilde{f}}{\partial x_i}(\widetilde{p})\end{array}\right)\right].$$

Corollary 4.1. *If $\overline{X}=\overrightarrow{X}+\varepsilon\overrightarrow{X}^*$ is a dual vector field on D^n and $\overline{f}=f+\varepsilon f^o$ and $\overline{g}=g+\varepsilon g^o$ are dual-valued analytic functions on D^n, then*

(1) $\overline{X}\left[\overline{f}+\overline{g}\right]=\overline{X}\left[\overline{f}\right]+\overline{X}\left[\overline{g}\right]$.

(2) $\overline{X}\left[\overline{\lambda f}\right]=\overline{\lambda X}\left[\overline{f}\right]$, *for all dual numbers* $\overline{\lambda}=\lambda+\varepsilon\lambda^*$.

(3) $\overline{X}\left[\overline{fg}\right]=\overline{X}\left[\overline{f}\right]\overline{g}+\overline{fX}\left[\overline{g}\right]$.

Proof. For $\overline{p} = p + \varepsilon p^* \in D^n$, in the section III, the equalities (1) and (3) were calculated in detail. Now, we know that

$$\left(\overline{X}\left[\overline{\lambda f}\right]\right)(\overline{p}) = \overline{X}_{\overline{p}}\left[\overline{\lambda f}\right].$$

In this case, we have

$$\begin{aligned}
\left(\overline{X}\left[\overline{\lambda f}\right]\right)(\overline{p}) &= \overline{X}_{\overline{p}}\left[\overline{\lambda f}\right] = \overrightarrow{X}_{\widetilde{p}}[\lambda f] + \varepsilon\left(\overrightarrow{X}_{\widetilde{p}}[\lambda^* f + \lambda f^o] + \overrightarrow{X}^*_{\widetilde{p}}[\lambda f]\right) \\
&= \lambda \overrightarrow{X}_{\widetilde{p}}[f] + \varepsilon\left(\lambda^* \overrightarrow{X}_{\widetilde{p}}[f] + \lambda \overrightarrow{X}_{\widetilde{p}}[f^o] + \lambda \overrightarrow{X}^*_{\widetilde{p}}[f]\right) \\
&= (\lambda + \varepsilon\lambda^*)\left(\overrightarrow{X}_{\widetilde{p}}[f] + \varepsilon\left(\overrightarrow{X}_{\widetilde{p}}[f^o] + \overrightarrow{X}^*_{\widetilde{p}}[f]\right)\right) \qquad (4.5) \\
&= \overline{\lambda X}_{\overline{p}}\left[\overline{f}\right] \\
&= \left(\overline{\lambda X}\left[\overline{f}\right]\right)(\overline{p}).
\end{aligned}$$

For every $\overline{p} = p + \varepsilon p^* \in D^n$, since the equality (4.5) is correct,

$$\overline{X}\left[\overline{\lambda f}\right] = \overline{\lambda X}\left[\overline{f}\right]$$

is obtained. □

5. Tangent Maps on Dual Space

Let $\overline{f} \in C(D^n, D^m)$ be a dual analytic function. For every $\overline{p} = p + \varepsilon p^* \in D^n$, the dual function

$$\overline{f}_{*\overline{p}} : T_{\overline{p}} D^n \to T_{\overline{f}(\overline{p})} D^m$$

is called as dual tangent map of $\overline{f}$ at dual point $\overline{p}$, and is defined by

$$\begin{aligned}
\overline{f}_{*\overline{p}}(\overline{v}_{\overline{p}}) &= \left((v_{\widetilde{p}}[f_1], ..., v_{\widetilde{p}}[f_m]) + \varepsilon\left(v_{\widetilde{p}}[f_1^o] + v^*_{\widetilde{p}}[f_1], ..., v_{\widetilde{p}}[f_m^o] + v^*_{\widetilde{p}}[f_m]\right)\right)|_{q+\varepsilon q^*} \\
&= f_{*\widetilde{p}}(v_{\widetilde{p}}) + \varepsilon\left(f^o_{*\widetilde{p}}(v_{\widetilde{p}}) + f_{*\widetilde{p}}\left(v^*_{\widetilde{p}}\right)\right) \qquad (5.1) \\
&= w_{\widetilde{q}} + \varepsilon w^*_{\widetilde{q}},
\end{aligned}$$

where $q + \varepsilon q^* = f(\widetilde{p}) + \varepsilon f^o(\widetilde{p})$ is the dual point of D^n. It is seen from the above formula that $\overline{f}_{*\overline{p}}$ sends dual tangent vectors at $\overline{p} = p + \varepsilon p^*$ to dual tangent vectors at $\overline{f}(\overline{p}) = f(\widetilde{p}) + \varepsilon f^o(\widetilde{p})$. On the other hand, the function $\overline{f}_* : \chi(D^n) \to \chi(D^m)$ is named as dual tangent map of $\overline{f}$ and is given as

$$\begin{aligned}
\overline{f}_*\left(\overline{X}\right) &= \left(\overline{X}\left[\overline{f}_1\right], ..., \overline{X}\left[\overline{f}_m\right]\right) \\
&= f_*\left(\overrightarrow{X}\right) + \varepsilon\left(f^o_*\left(\overrightarrow{X}\right) + f_*\left(\overrightarrow{X}^*\right)\right).
\end{aligned}$$

Theorem 5.1. *If the function $\overline{f} : D^n \to D^m$ is a dual analytic function, then the dual tangent map $\overline{f}_{*\overline{p}} : T_{\overline{p}} D^n \to T_{\overline{f}(\overline{p})} D^m$ is a linear transformation.*

Proof. Let $\overline{v}_{\overline{p}}$ and $\overline{w}_{\overline{p}}$ be dual tangent vectors and $\overline{\lambda} = \lambda + \varepsilon\lambda^*$ be dual number. We must show that

(1) $\overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}} + \overline{w}_{\overline{p}}\right) = \overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right) + \overline{f}_{*\overline{p}}\left(\overline{w}_{\overline{p}}\right)$

(2) $\overline{f}_{*\overline{p}}\left(\overline{\lambda}\overline{v}_{\overline{p}}\right) = \overline{\lambda f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right).$

Since the dual tangent vectors are shown as $\overline{v}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \varepsilon\overrightarrow{v}^*_{\widetilde{p}}$ and $\overline{w}_{\overline{p}} = \overrightarrow{w}_{\widetilde{p}} + \varepsilon\overrightarrow{w}^*_{\widetilde{p}}$, the addition of these vectors is

$$\overline{v}_{\overline{p}} + \overline{w}_{\overline{p}} = \overrightarrow{v}_{\widetilde{p}} + \overrightarrow{w}_{\widetilde{p}} + \varepsilon\left(\overrightarrow{v}^*_{\widetilde{p}} + \overrightarrow{w}^*_{\widetilde{p}}\right).$$

Considering the equality (5.1), we get

$$\begin{aligned}
\overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}} + \overline{w}_{\overline{p}}\right) &= f_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}} + \overrightarrow{w}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}} + \overrightarrow{w}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}} + \overrightarrow{w}^*_{\widetilde{p}}\right)\right) \\
&= f_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}}\right)\right) \\
&\quad + f_{*\widetilde{p}}\left(\overrightarrow{w}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{w}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{w}^*_{\widetilde{p}}\right)\right) \\
&= \overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right) + \overline{f}_{*\overline{p}}\left(\overline{w}_{\overline{p}}\right).
\end{aligned}$$

On the other hand, the multiplication of dual tangent vector with dual number is

$$\overline{\lambda}\overline{v}_{\overline{p}} = \lambda\overrightarrow{v}_{\widetilde{p}} + \varepsilon\left(\lambda^*\overrightarrow{v}_{\widetilde{p}} + \lambda\overrightarrow{v}^*_{\widetilde{p}}\right).$$

In this case, we have

$$\begin{aligned}
\overline{f}_{*\overline{p}}\left(\overline{\lambda}\overline{v}_{\overline{p}}\right) &= \left(\left(\lambda\overrightarrow{v}_{\widetilde{p}}\right)[f_1], ..., \left(\lambda\overrightarrow{v}_{\widetilde{p}}\right)[f_m]\right) \\
&\quad + \varepsilon\left(\left(\left(\lambda^*\overrightarrow{v}_{\widetilde{p}}\right)[f_1]\right)\left(\lambda\overrightarrow{v}^*_{\widetilde{p}}\right)[f_1] +, ..., \left(\left(\lambda^*\overrightarrow{v}_{\widetilde{p}}\right)[f_m]\right)\left(\lambda\overrightarrow{v}^*_{\widetilde{p}}\right)[f_m]\right).
\end{aligned}$$

When the above mentioned equality is taken into consideration, it is easily seen that

$$\begin{aligned}
\overline{f}_{*\overline{p}}\left(\overline{\lambda}\overline{v}_{\overline{p}}\right) &= \left(\lambda + \varepsilon\lambda^*\right)\left(f_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + \varepsilon\left(f^o_{*\widetilde{p}}\left(\overrightarrow{v}_{\widetilde{p}}\right) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}}\right)\right)\right) \\
&= \overline{\lambda f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right).
\end{aligned}$$

The equalities (1) and (2) show that the map $\overline{f}_{*\overline{p}}$ is a linear transformation. □

According to given bases, each linear transformation corresponds to a matrix. Now, let's find the matrix which is called dual Jacobian corresponding to this linear transformation. Let us consider that the bases of $T_{\overline{p}}D^n$ and $T_{\overline{q}}D^n$ are defined as follows:

$$\left\{e_{1\overline{p}}, ..., e_{n\overline{p}}\right\} \text{ and } \left\{e_{1\overline{q}}, ..., e_{m\overline{q}}\right\},$$

respectively, where $\overline{q} = f\left(\widetilde{p}\right) + \varepsilon f^o\left(\widetilde{p}\right)$. Thus, for $1 \leq j \leq n$, the following expression can be written:

$$\overline{f}_{*\overline{p}}\left(e_{j\overline{p}}\right) = \left(\left(e_{j\widetilde{p}}[f_1], ..., e_{j\widetilde{p}}[f_m]\right) + \varepsilon\left(e_{j\widetilde{p}}[f^o_1], ..., e_{j\widetilde{p}}[f^o_m]\right)\right)|_{q+\varepsilon q^*}$$

$$
\begin{aligned}
&= \left(\frac{\partial f_1}{\partial x_j}(\widetilde{p})\, e_{1\widetilde{q}} + ... + \frac{\partial f_m}{\partial x_j}(\widetilde{p})\, e_{m\widetilde{q}} + \varepsilon\left(\frac{\partial f_1^o}{\partial x_j}(\widetilde{p})\, e_{1\widetilde{q}} + ... + \frac{\partial f_m^o}{\partial x_j}(\widetilde{p})\, e_{m\widetilde{q}}\right)\right) \\
&= \left(\frac{\partial f_1}{\partial x_j}(\widetilde{p}), ..., \frac{\partial f_m}{\partial x_j}(\widetilde{p})\right) + \varepsilon\left(\frac{\partial f_1^o}{\partial x_j}(\widetilde{p}), ..., \frac{\partial f_m^o}{\partial x_j}(\widetilde{p})\right) \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_j}(\widetilde{p}) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f_m}{\partial x_j}(\widetilde{p}) \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{\partial f_1^o}{\partial x_j}(\widetilde{p}) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f_m^o}{\partial x_j}(\widetilde{p}) \end{bmatrix},
\end{aligned}
$$

where $\widetilde{q} = (q_1, ..., q_n, q_1^*, ..., q_n^*)$ is the point of $\mathbb{R}^{2n}$. Thus, the dual Jacobian matrix is shown by $\overline{J}\left(\overline{f}\right)(\overline{p})$ and is defined as follow equality

$$
\begin{aligned}
\overline{J}\left(\overline{f}\right)(\overline{p}) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\widetilde{p}) & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_n}(\widetilde{p}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_m}{\partial x_1}(\widetilde{p}) & \cdot & \cdot & \cdot & \frac{\partial f_m}{\partial x_n}(\widetilde{p}) \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{\partial f_1^o}{\partial x_1}(\widetilde{p}) & \cdot & \cdot & \cdot & \frac{\partial f_1^o}{\partial x_n}(\widetilde{p}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_m^o}{\partial x_1}(\widetilde{p}) & \cdot & \cdot & \cdot & \frac{\partial f_m^o}{\partial x_n}(\widetilde{p}) \end{bmatrix} \\
&= J(f)(\widetilde{p}) + \varepsilon J(f^o)(\widetilde{p}),
\end{aligned}
$$

where

$$
\frac{\partial f_k^o}{\partial x_j}(\widetilde{p}) = \sum_{i=1}^{n} p_i^* \frac{\partial^2 f_k}{\partial x_i \partial x_j}(\widetilde{p}) + \frac{\partial f_k}{\partial x_j}(\widetilde{p})
$$

for $1 \leq k \leq m$ and $1 \leq j \leq n$. In this case, the dual analytic tangent map $\overline{f}_{*\overline{p}} : T_{\overline{p}}D^n \to T_{\overline{f}(\overline{p})}D^n$ can be rewritten as follows

$$
\begin{aligned}
\overline{f}_{*\overline{p}}(\overline{v}_{\overline{p}}) &= \left(f_{*\widetilde{p}}(\overrightarrow{v}_{\widetilde{p}}) + \varepsilon\left(f^o_{*\widetilde{p}}(\overrightarrow{v}_{\widetilde{p}}) + f_{*\widetilde{p}}\left(\overrightarrow{v}^*_{\widetilde{p}}\right)\right)\right) \\
&= J(f)(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}} + \varepsilon\left(J(f^o)(\widetilde{p})\,\overrightarrow{v}_{\widetilde{p}} + J(f)(\widetilde{p})\,\overrightarrow{v}^*_{\widetilde{p}}\right). \qquad (5.2)
\end{aligned}
$$

Example 5.1. Let

$$
\begin{aligned}
\overline{f} &: D^2 \to D^3, \\
\overline{f}(\overline{x}) &= (\cos x_1, \sin x_1, x_2) + \varepsilon(-x_1^* \sin x_1 + \cos x_1, x_1^* \cos x_1 + \sin x_1, x_2^*)
\end{aligned}
$$

be a dual analytic function with $\overline{p} = \left(\frac{\pi}{4}, 0\right) + \varepsilon\left(1, \frac{\pi}{2}\right)$ and $\overline{v}_{\overline{p}} = (2, -3)_{\widetilde{p}} + \varepsilon(1, 2)_{\widetilde{p}}$. The dual tangent map of $\overline{f}$ is obtained from (5.1) as

$$
\begin{aligned}
\overline{f}_{*\overline{p}}(\overline{v}_{\overline{p}}) &= \begin{pmatrix} (v_{\widetilde{p}}[f_1], v_{\widetilde{p}}[f_2], v_{\widetilde{p}}[f_3]) \\ +\varepsilon\left(v_{\widetilde{p}}[f_1^o] + v^*_{\widetilde{p}}[f_1], v_{\widetilde{p}}[f_2^o] + v^*_{\widetilde{p}}[f_2], v_{\widetilde{p}}[f_3^o] + v^*_{\widetilde{p}}[f_3]\right) \end{pmatrix}_{|_{q+\varepsilon q^*}} \\
&= \left(-\sqrt{2}, \sqrt{2}, -3\right)_{\widetilde{q}} + \varepsilon\left(-\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\right)_{\widetilde{q}},
\end{aligned}
$$

where $q+\varepsilon q^{*}=f\left(\widetilde{p}\right)+\varepsilon f^{o}\left(\widetilde{p}\right)=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)+\varepsilon\left(0,\sqrt{2},\frac{\pi}{2}\right)$ is a dual point of D^{3}. On the other hand, if we use the dual Jacobian matrix of $\overline{f}$ at dual point $\overline{p}$, we have

$$\begin{aligned}\overline{J}\left(\overline{f}\right)\left(\overline{p}\right) &= J\left(f\right)\left(\widetilde{p}\right)+\varepsilon J\left(f^{o}\right)\left(\widetilde{p}\right)\\ &= \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0\\ \frac{\sqrt{2}}{2} & 0\\ 0 & 1\end{bmatrix}+\varepsilon\begin{bmatrix} -\sqrt{2} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Considering the equality (5.2) , we get the following equality

$$\begin{aligned}\overline{f}_{*\overline{p}}\left(\overline{v}_{\overline{p}}\right) &= J\left(f\right)\left(\widetilde{p}\right)\overrightarrow{v}_{\widetilde{p}}+\varepsilon\left(J\left(f^{o}\right)\left(\widetilde{p}\right)\overrightarrow{v}_{\widetilde{p}}+J\left(f\right)\left(\widetilde{p}\right)\overrightarrow{v}^{*}_{\widetilde{p}}\right)\\ &= \begin{bmatrix} -\sqrt{2}\\ \sqrt{2}\\ -3\end{bmatrix}+\varepsilon\begin{bmatrix} -\frac{5\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\\ 2\end{bmatrix}\\ &= \left(-\sqrt{2},\sqrt{2},-3\right)_{\widetilde{q}}+\varepsilon\left(-\frac{5\sqrt{2}}{2},\frac{\sqrt{2}}{2},2\right)_{\widetilde{q}}.\end{aligned}$$

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$\mathcal{I}$-LOCALIZED SEQUENCES IN METRIC SPACES

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Abstract. In this paper, we have introduced $\mathcal{I}$-localized and $\mathcal{I}^*$-localized sequences in metric spaces and investigated some basics properties of the $\mathcal{I}$-localized sequences related with $\mathcal{I}$-Cauchy sequences. Also, we have obtained some necessary and sufficient conditions for the $\mathcal{I}$-localized sequences to be an $\mathcal{I}$-Cauchy sequences. We have also defined uniformly $\mathcal{I}$-localized sequences on metric spaces and its relation with $\mathcal{I}$-Cauchy sequences have been obtained.
Keywords: $\mathcal{I}$-Cauchy sequences; $\mathcal{I}$-localized sequences; $\mathcal{I}^*$-localized sequences.

1. Introduction and preliminaries

The localized sequences defined in [7] can be understood as a typical generalization of a Cauchy sequence in metric spaces. Using the properties of localized sequences and the locator of a sequence, some interesting results related to closure operators in metric spaces have been obtained in [7]. If X is a metric space with a metric $d(\cdot,\cdot)$ and (x_n) is a sequence of points in X, we call the sequence (x_n) to be localized in some subset $M \subset X$ if the number sequence $\alpha_n = d(x_n, x)$ converges for all $x \in M$. The maximal subset on which (x_n) is a localized sequence is called the locator of (x_n). In addition, if (x_n) is localized on X, then it becomes localized everywhere. If the locator of a sequence (x_n) contains all elements of this sequence, except of a finite number of elements, then (x_n) is called localized in itself. For the above notations and further properties of the localized sequences we refer to [7]. It is important to remark that, every Cauchy sequence in X is localized everywhere. It is also an interesting fact that if $A : X \to X$ is a mapping with the condition $d(Ax, Ay) \leq d(x, y)$ for all $x, y \in X$, then for every $x \in X$ the sequence $(A^n x)$ is localized at every fixed point of the mapping A. This means that fixed points of the mapping A is contained in the locator of the sequence $(A^n x)$. Motivating the above-mentioned facts the authors of the present study have recently introduced

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the notations of a statistically localized sequence and the statistically locator of a sequence [9] where important properties of statistically localized sequences have been investigated.

In the present paper, the main purpose is to generalize the concept of statistically localized sequence using the notation of ideal $\mathcal{I}$ of subset of the set $\mathbb{N}$ of positive integers. Note that the $\mathcal{I}$-convergence of sequences of real numbers and $\mathcal{I}$-convergence of sequences in metric spaces were also defined and investigated (see [4, 5]).

Recall that for a non-empty set X, the family $\mathcal{I} \subset 2^X$ is an ideal if and only if for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we have $B \in \mathcal{I}$ (see [6]). Additionally, a non-empty family of set $\mathcal{F} \subset 2^X$ is a filter on X if and only if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $B \supset A$ we have $B \in \mathcal{F}$. In addition, an ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. Then, $\mathcal{I} \subset 2^X$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$ (see [4, 5]).

Let (X, d) be a fixed metric space and $\mathcal{I}$ denotes a non-trivial ideal of subsets of $\mathbb{N}$.

Definition 1.1. ([4]) A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $\xi \in X$ and it will be denoted as $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$ if and only if

$$A(\varepsilon) = \{n \in \mathbb{N} : d(x_n, \xi) \geq \varepsilon\} \in \mathcal{I}$$

for any $\varepsilon > 0$.

Definition 1.2. ([8]) A sequence $(x_n)_{n\in\mathbb{N}}$ of element of X is said to be $\mathcal{I}$-Cauchy sequence if and only if there is $n_0 \in N$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : d(x_n, x_{n_0}) \geq \varepsilon\} \in \mathcal{I} \text{ for each } \varepsilon > 0.$$

Note that the notations of $\mathcal{I}^*$-convergent and $\mathcal{I}^*$-Cauchy sequences are also related to $\mathcal{I}$-convergence.

Definition 1.3. ([4]) A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}^*$-convergent to $\xi \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that $\lim_{k\to\infty} d(x_{m_k}, \xi) = 0$ and $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$.

Definition 1.4. ([8]) A sequence (x_n) of elements of X is said to be $\mathcal{I}^*$-Cauchy sequence if and only if there is a set $M = \{m_1 < m_2 < ... < m_k\}$ such that

$$\lim_{k,p\to\infty} d\left(x_{m_k}, x_{m_p}\right) = 0.$$

Note that $\mathcal{I}^*$-convergent and $\mathcal{I}^*$-Cauchy sequences imply $\mathcal{I}$-convergent and $\mathcal{I}$-Cauchy sequences, respectively. Moreover, if $\mathcal{I}$ is an ideal with property (AP) (see [4]), then $\mathcal{I}$ and $\mathcal{I}^*$-convergence coincide. Also, in this case $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequences are the same (see [8]). More property and fact about ideal convergence and statistical convergence are contained, for instance, in Gürdal et al. [2], Nuray et al. [10], Savaş and Gürdal [11, 12], Şahiner et al. [13] and Yegül and Dündar [14].

In this paper, we have defined $\mathcal{I}$-localized and $\mathcal{I}^*$-localized sequences in metric spaces and investigated some basics properties of $\mathcal{I}$-localized sequences related with $\mathcal{I}$-Cauchy sequences.

2. $\mathcal{I}$ and $\mathcal{I}^*$-localized sequences

Let (X, d) is a metric space and $\mathcal{I}$ is a non-trivial ideal of subsets of $\mathbb{N}$.

Definition 2.1. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-localized in the subset $M \subset X$ if and only if for each $x \in M$, $\mathcal{I}\text{-}\lim_{n\to\infty} d(x_n, x)$ exists, i.e. the real number sequence $\alpha_n = d(x_n, x)$ is $\mathcal{I}$-convergent.

(b) the maximal set on which a sequence (x_n) is $\mathcal{I}$-localized we call the $\mathcal{I}$-locator of (x_n) and we denote this set as $loc_{\mathcal{I}}(x_n)$.

(c) A sequence (x_n) is called $\mathcal{I}$-localized everywhere if $\mathcal{I}$-locator of (x_n) coincides with X, i.e. $loc_{\mathcal{I}}(x_n) = X$.

(d) A sequence (x_n) is called $\mathcal{I}$-localized in itself if

$$\{n \in \mathbb{N} : x_n \notin loc_{\mathcal{I}}(x_n)\} \subset \mathcal{I}.$$

From this definition, we immediately have that if (x_n) is an $\mathcal{I}$-Cauchy sequence then, it is $\mathcal{I}$-localized everywhere. Indeed, since

$$|d(x_n, x) - d(x_{n_0}, x)| \leqslant d(x_n, x_{n_0})$$

we have

$$\{n \in \mathbb{N} : |d(x_n, x) - d(x_{n_0}, x)| \geqslant \varepsilon\} \subset \{n \in \mathbb{N} : d(x_n, x_{n_0}) \geqslant \varepsilon\}$$

which indicates that the sequence is $\mathcal{I}$-localized if it is $\mathcal{I}$-Cauchy sequence.

We also have that, every $\mathcal{I}$-convergent sequence is $\mathcal{I}$-localized. Note that, if $\mathcal{I}$ is an admissible ideal then

(i) every localized sequence in X is $\mathcal{I}$-localized sequence in X.

(ii) if additionally X is a vector space, then the sum of two $\mathcal{I}$-localized sequences is $\mathcal{I}$-localized and also multiplication of $\mathcal{I}$-localized sequence to a constant is also $\mathcal{I}$-localized.

Remark 2.1. If (x_n) and (y_n) are two $\mathcal{I}$-localized number sequences, then $(x_n y_n)$, $\left(\frac{x_n}{y_n}\right)$, $y_n \neq 0$, are also $\mathcal{I}$-localized sequences.

$\mathcal{I}^*$-localized sequences could also be defined in metric spaces. Then we call the sequence (x_n) to be $\mathcal{I}^*$-localized in a metric space X if and only if the number sequence $d(x_n, x)$ is $\mathcal{I}^*$-convergent for each $x \in X$. Therefore, every $\mathcal{I}^*$-convergent or $\mathcal{I}^*$-Cauchy sequence in a metric space X is $\mathcal{I}^*$-localized in X.

It is known that for admissible ideal $\mathcal{I}^*$-Cauchy criteria and $\mathcal{I}^*$-convergence, implies that $\mathcal{I}$-Cauchy criteria and $\mathcal{I}$-convergence, respectively. Moreover, for the admissible ideal with the property (AP) the notions of $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequences; $\mathcal{I}$-convergent and $\mathcal{I}^*$-convergent sequences coincide.

Lemma 2.1. *Let $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$ and X is a metric space. If a sequence $(x_n) \subset X$ is $\mathcal{I}^*$-localized on the set $M \subset X$, then (x_n) is $\mathcal{I}$-localized on the set M and $loc_{\mathcal{I}_*}(x_n) \subset loc_{\mathcal{I}}(x_n)$.*

Proof. If (x_n) is $\mathcal{I}^*$-localized on M, then there exist a set $H \in \mathcal{I}$ such that for $H^C = \mathbb{N} \backslash H = \{k_1 < k_2 < ... < k_j\}$ we have

$$\lim_{j \to \infty} d(x_j, x)$$

for each $x \in M$. Then, the sequence $d(x_n, x)$ is an $\mathcal{I}^*$-Cauchy sequence which implies the $d(x_n, x)$ is an $\mathcal{I}$-Cauchy sequences (see [8]). Therefore; the number sequence $d(x_n, x)$ is $\mathcal{I}$-convergent, i.e. (x_n) is $\mathcal{I}$-localized on the set M. $\square$

Lemma 2.2. *Suppose (X, d) is a metric space, then*

(i) if X has no limit point, then $\mathcal{I}$ and $\mathcal{I}^*$-localized sequences are the same in X and $\mathcal{I}_{loc}(x_n) = \mathcal{I}^*_{loc}(x_n)$ for any $(x_n) \in X$.

(ii) if X has a limit point ξ, then there is an admissible ideal $\mathcal{I}$ for which there exists an $\mathcal{I}$-localized sequence $(y_n) \subset X$ such that (y_n) is not $\mathcal{I}^*$-localized.

Proof. (i) If X has no any limit point, then the notions $\mathcal{I}$ and $\mathcal{I}^*$-convergence coincide in X (see [4]). Therefore, if (x_n) is $\mathcal{I}$-localized then it is $\mathcal{I}^*$-localized also and by the Lemma 2.1, we have $\mathcal{I}_{loc}(x_n) = \mathcal{I}^*_{loc}(x_n)$.

(ii) Let ξ is a limit point of X. Then there exists a sequence (x_n) such that $\lim_{n \to \infty} d(x_n, \xi) = 0$. Let $D = \bigcup_{j=1}^{\infty} D_j$, $D_j = \left\{\alpha^{j-1}(2s-1) : s \in \mathbb{N}\right\}$ $(j = 1, 2, ...)$ is a decomposition of integers and $\mathcal{E}$ is an ideal of sets $A \subset \mathbb{N}$ such that each A intersects only a finite member of D_j (see [4]). Let us define the sequence $y_n = x_j$ for $n \in D_j$. Then, from Theorem 3.1 in [4], we have that the sequence (y_n) is $\mathcal{I}$-localized in X. This means that for each $x \in X$ we have $\mathcal{I}\text{-}\lim_{n \to \infty} d(y_n, x) = \alpha(x)$. Let us define $d(x_n, x) = \alpha_n(x)$. It is easy to show that $\mathcal{I}\text{-}\lim_{n \to \infty} \alpha_n(x) = \alpha(x)$. If for some $x \in X$ we have $\mathcal{I}^*\text{-}\lim_{n \to \infty} \alpha_n(x) = \alpha(x)$, then there is $H \in \mathcal{E}$ such that for $M = \{m_1 < m_2 < ... < m_k < ...\} = \mathbb{N} \backslash H$ we have $\lim_{k \to \infty} \alpha_{m_k}(x) = \alpha(x)$. According to the definition of $\mathcal{E}$, the integer $\ell \in \mathbb{N}$ such that $H \subset \Delta_1 \cup ... \cup \Delta_\ell$ could be found yet, then $\Delta_{\ell+1} \subset \mathbb{N} \backslash H = M$. Therefore; for many infinite k's, we have $\alpha_{m_k}(x) \to \alpha_{\ell+1}(x)$. This contradicts that $\alpha_{m_k}(x) \to \alpha(x)$. Hence, the sequence (y_n) is $\mathcal{I}$-localized but not $\mathcal{I}^*$-localized. This proves Lemma 2.2. $\square$

We recall that a necessary and sufficient condition of equivalency of $\mathcal{I}$ and $\mathcal{I}^*$-convergent, also $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy condition, is so called (AP) properties of an admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ (see [4]) which means that for every family $\{A_i\}_{i\in\mathbb{N}}$ with $A_i \cap A_j = \emptyset \ (i \neq j)$, $A_i \in \mathcal{I} \ (i \in \mathbb{N})$, there is a family $\{B_i\}_{i\in\mathbb{N}}$ such that $(A_j \backslash B_i) \cup (B_j \backslash A_j)$ for all $j \in \mathbb{N}$ and a limit set $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. From the Theorem 3.2 in [4], if $\mathcal{I}$ is an admissible ideal with the property (AP) and (x_n) is $\mathcal{I}$-localized sequence, then $\mathcal{I}^*$-$\lim_{n\to\infty} d(x_m, x)$ exists for each x. This means that, for any admissible ideal with the property (AP) $\mathcal{I}$-localized sequence is also $\mathcal{I}^*$-localized. In contrast, if every $\mathcal{I}$-localized sequence is also $\mathcal{I}^*$-localized and the metric space (X, d) has at least one limit point, then $\mathcal{I}$ has the property (AP). Indeed, if ξ is a limit point of X then there exists a sequence (x_n) such that $d(x_n, \xi) = 0$. Then, putting $y_n = x_j$ for $n \in A_j$, where $\{A_n\}_{n\in\mathbb{N}}$ is a disjoint family of non-empty sets from $\mathcal{I}$, we have $\mathcal{I}$-$\lim_{n\to\infty} y_n = \xi$. Hence, (y_n) is $\mathcal{I}$-localized sequence. By the assumption of (y_n) is $\mathcal{I}^*$-localized, $\mathcal{I}^*$-$\lim_{n\to\infty} d(y_n, x)$ exists for each x. So, there is a set $B(x) \in \mathcal{I}$ such that if $M(x) = \mathbb{N} \backslash B(x) = \{m_1(x) < m_2(x) < \ldots\}$, then the limit

$$\lim_{k\to\infty} d(y_{m_k}, x) = \alpha(x)$$

exist. In addition, for each fixed x it could be proved that $\mathcal{I}$ has the property (AP) as in [4].

The investigation of other properties of the $\mathcal{I}$-localized sequences have been given in the following section.

3. Basic properties of ideal localized sequences

Proposition 3.1. *Every $\mathcal{I}$-localized sequence is $\mathcal{I}$-bounded.*

Proof. Let (x_n) is $\mathcal{I}$-localized. Then, the number sequence $d(x_n, x)$ is $\mathcal{I}$-convergent for some $x \in X$. This means that $\{n \in \mathbb{N} : d(x_n, x) > K\} \in \mathcal{I}$ for some $K > 0$. Consequently, the sequence (x_n) is $\mathcal{I}$-bounded. $\square$

Proposition 3.2. *Let $\mathcal{I}$ is an admissible ideal with the property (AP) and $L = loc_{\mathcal{I}}(x_n)$. Also, a point $z \in X$ be such that for any $\varepsilon > 0$ there exists $x \in L$ satisfying*

$$\{n \in \mathbb{N} : |d(x, x_n) - d(z, x_n)| \geqslant \varepsilon\} \in \mathcal{I}. \tag{1}$$

Then $z \in L$.

Proof. It is enough to show that the number sequence $\alpha_n = d(x_n, z)$ is an $\mathcal{I}$-Cauchy sequence. Let $\varepsilon > 0$ and $x \in L = loc_{\mathcal{I}}(x_n)$ is a point with the property (1). By adopting the (AP) property of $\mathcal{I}$, we have

$$|d(x, x_{k_n}) - d(z, x_{k_n})| \to 0 \text{ as } n \to \infty,$$

and

$$|d(x_{k_n}, x) - d(x_{k_m}, x)| \to 0 \text{ as } m, n \to \infty$$

where $K = \{k_1 < k_2 < ... < k_n < ...\} \in \mathcal{F}(\mathcal{I})$. Hence for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$(2) \qquad |d(x, x_{k_n}) - d(z, x_{k_n})| < \frac{\varepsilon}{3}$$

and

$$(3) \qquad |d(x, x_{k_n}) - d(x, x_{k_m})| < \frac{\varepsilon}{3}$$

for all $n \geq n_0$, $m \geq m_0$.

Now, combining (2) and (3) together with the following estimation

$$|d(z, x_{k_n}) - d(z, x_{k_m})|$$

$$\leq |d(z, x_{k_n}) - d(x, x_{k_n})| + |d(x, x_{k_n}) - d(x, x_{k_m})| + |d(x, x_{k_m}) - d(z, x_{k_m})|$$

we obtain

$$|d(z, x_{k_n}) - d(z, x_{k_m})| < \varepsilon$$

for all $n \geq n_0$, $m \geq n_0$, which gives

$$|d(z, x_{k_n}) - d(z, x_{k_m})| \to 0 \text{ as } m, n \to \infty$$

for $K = (k_n) \subset N$ and $K \in \mathcal{F}(\mathcal{I})$. This result implies that $d(x_n, z)$ is an $\mathcal{I}$-Cauchy sequence which finalizes the proof. $\square$

Proposition 3.3. *The $\mathcal{I}$-locator of any sequence is a closed subset of the metric space X.*

Proof. Let $z \in \overline{loc_{\mathcal{I}}}(x_n)$. Then, for any $\varepsilon > 0$ the ball $B(z, \varepsilon)$ will contain a point $x \in loc_{\mathcal{I}}(x_n)$. Therefore;

$$\{n \in \mathbb{N} : |d(x, x_n) - d(z, x_n)| \geqslant \varepsilon\} \in \mathcal{I}$$

for any $\varepsilon > 0$, since for each $n \in \mathbb{N}$

$$|d(x, x_n) - d(z, x_n)| \leq d(z, x) < \varepsilon.$$

Consequently, the hypothesis of Proposition 3.2 is satisfied. Then $z \in loc_{\mathcal{I}}(x_n)$, i.e. $loc_{\mathcal{I}}(x_n)$ is closed. $\square$

Recall that the point z is an $\mathcal{I}$-limit point of the sequence $(x_n) \in X$ if there is a set

$$K = \{k_1 < k_2 < ... < k_n\} \subset \mathbb{N}$$

such that $K \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = z$. A point ξ is said to be an $\mathcal{I}$-cluster point of the sequence (x_n) if for each $\varepsilon > 0$

$$\{n \in \mathbb{N} : d(x_n, \xi) < \varepsilon\} \notin \mathcal{I}$$

(see [4, 5]). In addition, if $K = \{k_1 < k_2 < ...\} \in \mathcal{I}$, then the subsequence (x_{k_n}) of the sequence (x_n) is called $\mathcal{I}$-thin subsequence of the sequence (x_n). If $M = \{m_1 < m_2 < ...\} \notin \mathcal{I}$, then the sequence $x_M = (x_m)$ is called $\mathcal{I}$-nonthin subsequence of (x_n).

Since $|d(x_n, y) - d(z, y)| \leq d(x_n, z)$, the following propositions could be given.

Proposition 3.4. *If $z \in X$ is an $\mathcal{I}$-limit point (an $\mathcal{I}$-cluster point) of a sequence $(x_n) \in X$, then for each $y \in X$ the number $d(z, y)$ is an $\mathcal{I}$-limit point (an $\mathcal{I}$-cluster point) of the sequence $\{d(x_n, y)\}$.*

Proposition 3.5. *All $\mathcal{I}$-limit points ($\mathcal{I}$-cluster points) of the $\mathcal{I}$-localized sequence (x_n) have the same distance from each point x of the locator $loc_{\mathcal{I}}(x_n)$.*

Proof. If z_1 and z_2 are two $\mathcal{I}$-limit points of the sequence (x_n), then the numbers $d(z_1, x)$ and $d(z_2, x)$ are $\mathcal{I}$-limit points of the $\mathcal{I}$-convergent sequence $d(x, x_n)$. Consequently, $d(z_1, x) = d(z_2, x)$. □

Proposition 3.6. *$loc_{\mathcal{I}}(x_n)$ does not contain more than one $\mathcal{I}$-limit ($\mathcal{I}$-cluster) point of the sequence (x_n). Particularly, everywhere the localized sequence has not more than one $\mathcal{I}$-limit ($\mathcal{I}$-cluster) point.*

Proof. If $x, y \in loc_{\mathcal{I}}(x_n)$ are two $\mathcal{I}$-limit points of the sequence (x_n), then by the Proposition 3.5, $d(x, x) = d(x, y)$. But $d(x, x) = 0$. This implies that, $d(x, y) = 0$ for $x \neq y$ which is a well-known contradiction. □

Proposition 3.7. *If the sequence (x_n) has an $\mathcal{I}$-limit point $z \in loc_{\mathcal{I}}(x_n)$, then $\mathcal{I}$-$\lim_{n\to\infty} x_n = z$.*

Proof. The sequence $\{d(x_n, z)\}$ is $\mathcal{I}$-convergent and some $\mathcal{I}$-nonthin subsequence of this sequence converges to zero. Then, (x_n) is $\mathcal{I}$-convergent to z. □

Definition 3.1. For the given $\mathcal{I}$-localized sequence (x_n), with the $\mathcal{I}$-locator $L = loc_{\mathcal{I}}(x_n)$, the number

$$\sigma = \inf_{x\in L} \left(\mathcal{I}\text{-} \lim_{n\to\infty} d(x, x_n)\right)$$

is called the $\mathcal{I}$-barrier of (x_n).

Theorem 3.1. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ is an ideal with (AP) property. Then, an $\mathcal{I}$-localized sequence is $\mathcal{I}$-Cauchy sequence if and only if $\sigma = 0$.*

Proof. Let (x_n) is an $\mathcal{I}$-Cauchy sequence in a metric space X. Then, there is a set $K = \{k_1 < k_2 < ... < k_n\} \subset \mathbb{N}$ such that $K \in \mathcal{F}(\mathcal{I})$ and $\lim_{n,m\to\infty} d(x_{k_n}, x_{k_m}) = 0$. Consequently, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d\left(x_{k_n}, x_{k_{n_0}}\right) < \varepsilon \text{ for all } n \geq n_0.$$

Since (x_n) is an $\mathcal{I}$-localized sequence, $\mathcal{I}$-$\lim_{n\to\infty} d\left(x_{k_n}, x_{k_{n_0}}\right)$ exist and we get $\mathcal{I}$-$\lim_{n\to\infty} d\left(x_{k_n}, x_{k_{n_0}}\right) \leq \varepsilon$. Hence, $\sigma \leq \varepsilon$. Because $\varepsilon > 0$, we also get $\sigma = 0$.

Let now assume the converse by taking $\sigma = 0$. Then, for each $\varepsilon > 0$ there is an $x \in loc_{\mathcal{I}}(x_n)$ such that $d(x) = \mathcal{I}$-$\lim_{n\to\infty} d(x, x_n) < \frac{\varepsilon}{2}$. In this case

$$\left\{n \in \mathbb{N} : |d(x) - d(x, x_n)| \geq \frac{\varepsilon}{2} - d(x)\right\} \in \mathcal{I}$$

which implies $\left\{n \in \mathbb{N} : d(x, x_n) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}$. Therefore, $\mathcal{I}$-$\lim_{n\to\infty} d(x, x_n) = 0$, i.e. (x_n) is an $\mathcal{I}$-Cauchy sequence. □

Remark 3.1. We have previously acquired from the proof of Theorem 3.1 that, if $\sigma = 0$ then an ideal $\mathcal{I}$ will not have (AP) properties. In other words, if an $\mathcal{I}$-barrier of a localized sequence is equal to zero, then it is an $\mathcal{I}$-Cauchy sequence.

Theorem 3.2. *If the sequence (x_n) is $\mathcal{I}$-localized in itself and (x_n) contains an $\mathcal{I}$-nonthin Cauchy subsequence, then (x_n) will be an $\mathcal{I}$-Cauchy sequence itself.*

Proof. Let (y_n) is an $\mathcal{I}$-nonthin Cauchy subsequence. It could be assumed that all members of (y_n) belong to the $loc_{\mathcal{I}}(x_n)$. Since (y_n) is a Cauchy sequence, by Theorem 3.1, $\inf_{y_n} \lim_{m\to\infty} d(y_m, y_n) = 0$. On the other hand, since (x_n) is $\mathcal{I}$-localized in itself then,

$$\mathcal{I}\text{-}\lim_{m\to\infty} d(x_m, y_n) = \mathcal{I}\text{-}\lim_{m\to\infty} d(y_m, y_n) = 0.$$

This implies that, the $\mathcal{I}$-barrier of (x_n) is equal to zero: $\sigma = 0$. Then, by using the Remark 3.1, we figure out that (x_n) is an $\mathcal{I}$-Cauchy sequence. □

Let $a \in X$, $r > 0$ and $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal. Recall that the sequence (x_n) in a metric space (X, d) is called $\mathcal{I}$-bounded if there is a subset $K = \{k_1 < k_2 < ... < k_n \subset ...\} \subset \mathbb{N}$ such that $K \in \mathcal{F}(\mathcal{I})$ and $\left(x_{k_n}\right) \subset B(a, r)$, where $B(a, r)$ is the ball with center at the point a and radius r. Clearly, $\left(x_{k_n}\right)$ is a bounded sequence in X sequence and it has a localized in itself subsequence (see [7]). As a result, the following assertion also becomes true.

Proposition 3.8. *Every $\mathcal{I}$-bounded sequence in a metric space has an $\mathcal{I}$-localized in itself subsequence.*

Theorem 3.3. *If every $\mathcal{I}$-localized in itself sequence of X is an $\mathcal{I}$-Cauchy sequence, then every bounded set in X is totally bounded.*

Proof. Let (x_n) is an $\mathcal{I}$-localized in itself sequence of metric space (X, d), but the assertion is not true. Then, there is a bounded subset $M \subset X$ such that $d(x_n, x_m) > \varepsilon$ $(n \neq m)$ for some $\varepsilon > 0$ and for some sequence $(x_n) \subset M$. Since (x_n) is bounded by Proposition 3.8, (x_n) has an $\mathcal{I}$-localized in itself sequence (x'_n). Since $d(x'_n, x'_m) > \varepsilon$ for any $n \neq m$, the subsequence is not an Cauchy sequence. This contradicts our assumption. □

Definition 3.2. We indicate an infinite subset $M \subset X$ is thick relatively to a nonempty subset $Y \subset X$ if for each $\varepsilon > 0$ there is the *a* point $y \in Y$ such that the ball $B(y, \varepsilon)$ has infinitely many points of M. In particular, if the set M is thick relatively to its subset $Y \subset M$ then M is called thick in itself.

Proposition 3.9. *If every bounded infinite set of X is thick in itself, then every $\mathcal{I}$-localized in itself sequence of X is an $\mathcal{I}$-Cauchy sequence.*

Proof. Let the assumption is satisfied and (x_n) is an $\mathcal{I}$-localized in itself sequence of X. Then, (x_n) is an $\mathcal{I}$-bounded sequence in X. This implies that there is an infinite set M of points of (x_n) such that M is a bounded subset of X. By this assumption, M is thick in itself. Then, for every $\varepsilon > 0$ we can choose $x_k \in M$ such that the ball $B(x_k, \varepsilon)$ contains infinitely many points of $x'_1, ..., x'_n, ...$ of X. For the sequence (x'_n) the sequence $d(x'_n, x_k)_{n=1}^{\infty}$ is $\mathcal{I}$-convergent and

$$\mathcal{I} - \lim_{n \to n_0} d(x'_n, x_K) \leq \varepsilon.$$

Hence, the $\mathcal{I}$-barrier of (x_n) is equal to zero so (x_n) is an $\mathcal{I}$-Cauchy sequence. □

Since every bounded subset of X is totally bounded if and only if every bounded infinite subset of X is thick in itself (see [7]), the equality of Theorem 3.3 and Proposition 3.9 have been figure out. Moreover, from Theorem 3.2 and Theorem 3.1 it has been figured out that in Theorem 3.3 the assumption could be replaced equivalently with the assumption regarding every $\mathcal{I}$-bounded sequence to have an $\mathcal{I}$-Cauchy subsequence.

Proposition 3.10. *If every $\mathcal{I}$-bounded sequence has an $\mathcal{I}$-Cauchy subsequence, then every bounded subset in X is totally bounded.*

Note that in separable metric spaces, Proposition 3.10 can be weakened.

Proposition 3.11. *If every $\mathcal{I}$-bounded sequence of separable metric space X has an $\mathcal{I}$-localized subsequence in everywhere, then every bounded subset of X could be said that totally bounded.*

Definition 3.3. A sequence (x_n) in metric space (X, d) is called uniformly $\mathcal{I}$-localized on a subset $M \subset X$ if the sequence $\{d(x, x_n)\}$ is uniformly $\mathcal{I}$-convergent for all $x \in M$.

The following proposition is proved analogously to Proposition 3.2.

Proposition 3.12. *Let sequence (x_n) be uniformly $\mathcal{I}$-localized on the set $M \subset X$ and $z \in Y$ is such that for every $\varepsilon > 0$ there is $y \in M$ for which*

$$\{n \in \mathbb{N} : |d(z, x_n) - d(y, x_n)| \geqslant \varepsilon\} \in \mathcal{I}$$

is satisfied. Then, $z \in loc_{\mathcal{I}}(x_n)$ and (x_n) is uniformly $\mathcal{I}$-localized on the set of such points z.

Remark 3.2. The function $d_M(x, z) = \sup\limits_{y \in M} |d(x, y) - d(y, z)|$, $x, z \in Y$ is called a pseudo metric on the set Y.

It is easy to prove by standard techniques (see [1, 3]) that a sequence (x_n) is uniformly $\mathcal{I}$-localized on the set $M \subset X$ if and only if (x_n) is an $\mathcal{I}$-Cauchy sequence with respect to the metric $d_M(x, z)$.

Theorem 3.4. *Every uniformly $\mathcal{I}$-localized in itself sequence is $\mathcal{I}$-Cauchy sequence.*

Proof. Let $\{n \in \mathbb{N} : x_n \in M\} \in \mathcal{F}(\mathcal{I})$ the uniformly $\mathcal{I}$-localized in itself sequence (x_n). From the definition of d_M we get that if at most one of the points x, z belongs to M, then $d_M(x, z) = d(x, z)$. In particular, there is $n_0 \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d_M(x_n, x_{n_0}) = d(x_n, x_{n_0})\} \in \mathcal{F}(\mathcal{I})$. Now the assertion is obtained from Proposition 3.2. $\square$

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TAUBERIAN CONDITIONS FOR q-CESÀRO INTEGRABILITY

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Abstract. Given a q-integrable function f on $[0, \infty)$, we define $s(x) = \int_0^x f(t)d_qt$ and $\sigma(s(x)) = \frac{1}{x}\int_0^x s(t)d_qt$ for $x > 0$. It is known that if $\lim_{x\to\infty} s(x)$ exists and is equal to A, then $\lim_{x\to\infty} \sigma(s(x)) = A$. But the converse of this implication is not true in general. Our goal is to obtain Tauberian conditions imposed on the general control modulo of $s(x)$ under which the converse implication holds. These conditions generalize some previously obtained Tauberian conditions.
Keywords: q-integrable function; Tauberian conditions; q-derivative; q-integrals; quantum calculus.

1. Introduction

The first formulae of what we now call quantum calculus or q-calculus were introduced by Euler in the 18th century. Many notable results were obtained in the 19th century. In the early 20th century, Jackson defined the notions of q-derivative [9] and definite q-integral [10]. Also, he was the first to develop q-calculus in a systematic way. Following Jackson's papers, q-calculus has received an increasing attention of many researchers due to its vast applications in mathematics and physics.

We wiill now give some concepts of the q-calculus necessary for the understanding of this work. We follow the terminology and notations from the book of Kac and Cheung [11]. In what follows, q is a real number satisfying $0 < q < 1$.

The q-derivative $D_qf(x)$ of an arbitrary function $f(x)$ is defined by

$$D_qf(x) = \frac{f(x) - f(qx)}{x - qx}, \text{ if } x \neq 0,$$

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where $D_q f(0) = f'(0)$ provided $f'(0)$ exists. If $f(x)$ is differentiable, then $D_q f(x)$ tends to $f'(x)$ as q tends to 1.
Notice that the q-derivative satisfies the following q-analogue of Leibniz rule

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x).$$

The q-integrals from 0 to a and from 0 to ∞ are given by

$$\int_0^a f(x)d_q x = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_0^\infty f(x)d_q x = (1-q)a\sum_{n=-\infty}^{\infty} f(q^n)q^n$$

provided the sums converge absolutely. On a general interval $[a,b]$, the q-integral is defined by

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x.$$

The q-integral and the q-derivative are related by the fundamental theorem of quantum calculus as follows:

If $F(x)$ is an anti q-derivative of $f(x)$ and $F(x)$ is continuous at $x=0$, then

$$\int_a^b f(x)d_q x = F(b) - F(a), \quad 0 \le a < b \le \infty.$$

In addition, we have

$$D_q\left(\int\limits_0^x f(t)d_q t\right) = f(x).$$

A function $f(x)$ is said to be q-integrable on $\mathbb{R}_+ := [0,\infty)$ if the series $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ converges absolutely. We denote the set of all functions that are q-integrable on $\mathbb{R}_+$ by $L_q^1(\mathbb{R}_{q,+})$, where

$$\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}.$$

One may consult the recent books [2, 1] for further results and several applications of q-calculus.

Throughout this paper we assume that $f(x)$ is q-integrable on $\mathbb{R}_+$ and $s(x) = \int_0^x f(t)d_q t$. The symbol $s(x) = o(1)$ means that $\lim_{x\to\infty} s(x) = 0$. The q-Cesàro mean of $s(x)$ are defined by

$$\sigma(x) = \sigma(s(x)) = \frac{1}{x}\int_0^x s(t)d_q t.$$

The integral $\int_0^\infty f(t)d_qt$ is said to be q-Cesàro integrable (or $(C_q, 1)$ integrable) to a finite A, in symbols: $s(x) \to A(C_q, 1)$, if

$$\lim_{x\to\infty} \sigma(x) = A. \tag{1.1}$$

If the q-integral

$$\int_0^\infty f(t)d_qt = A \tag{1.2}$$

exists, then the limit (1.1) also exists [6]. That is, q-Cesàro integrability method is regular. The converse is not necessarily true (see [15], Example 1). Adding some suitable condition to (1.1), which is called a Tauberian condition, may imply (1.2). Any theorem which states that the convergence of the q-integral follows from its q-Cesàro integrability and some Tauberian condition is called a Tauberian theorem.

The difference between $s(x)$ and its q-Cesàro mean is given by the identity [6]

$$s(x) - \sigma(x) = qv(x), \tag{1.3}$$

where $v(x) = \frac{1}{x}\int_0^x tf(t)d_qt$. The identity (1.3) will be used in the various steps of proofs.

For each integer, $m \geq 0$, $\sigma_m(x)$ and $v_m(x)$ are defined by

$$\sigma_m(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x \sigma_{m-1}(t)d_qt & , m \geq 1 \\ s(x) & , m = 0 \end{cases}$$

and

$$v_m(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x v_{m-1}(t)d_qt & , m \geq 1 \\ v(x) & , m = 0 \end{cases}$$

The relationship between $\sigma_m(x)$ and $v_m(x)$ can be easily obtained by (1.3) as follows:

$$\sigma_m(x) - \sigma_{m+1}(x) = qv_m(x). \tag{1.4}$$

The classical control modulo of $s(x) = \int_0^x f(t)d_qt$ is denoted by

$$\omega_0(x) = xD_q(s(x)) = xf(x),$$

and the general control modulo of integer order $m \geq 1$ of $s(x)$ is defined by

$$\omega_{m-1}(x) - \sigma(\omega_{m-1}(x)) = q\omega_m(x).$$

Note that the concepts of classical and general control modulo were first introduced by Çanak and Totur [3] for the integrals in standard calculus.

A function $f(x)$ is said to satisfy the property (P) (see [7]), if for all $\epsilon > 0$ there exists $K > 0$ such that

$$|f(x) - f(qx)| < \epsilon$$

for all $x > K$.

Recently, Fitouhi and Brahim [7], Çanak et al. [6] and Totur et al. [15] have determined Tauberian conditions using this property. Moreover, Çanak et al. [6] showed that if $s(x)$ satisfies the property $(\mathcal{P})$, its q-Cesàro mean $\sigma(x)$ then also satisfies the property $(\mathcal{P})$.

Slowly oscillating real-valued functions were introduced by Schmidt [14]. A function $f(x)$ is said to be slowly oscillating, if for every $\varepsilon > 0$ there exists $K > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x > y > K$ and $x/y \to 1$. Slow oscillation condition were used in a number of Tauberian theorems for the Cesàro integrability [4, 5], logarithmic integrability [12, 16] and weighted mean integrability [13, 17] in standard calculus. Consider that, as q tends to 1, the property $(\mathcal{P})$ corresponds to slow oscillation of a function.

The following theorems are the q-analogues of classical Tauberian theorems due to the Hardy [8] and Schmidt [14], respectively.

Theorem 1.1. ([7]) *If $s(x)$ is q-Cesàro integrable to A and*

$$\omega_0(x) = o(1), \tag{1.5}$$

then $\int_0^\infty f(t)d_qt = A$.

Theorem 1.2. ([6],[7]) *If $s(x)$ is q-Cesàro integrable to A and satisfies the property $(\mathcal{P})$, then $\int_0^\infty f(t)d_qt = A$.*

The purpose of this study is to generalize the above theorems by imposing Tauberian conditions on the general control modulo of integer order $m \geq 1$.

2. Main Results

In this paper, we shall prove the following Tauberian theorems.

Theorem 2.1. *If $s(x)$ is q-Cesàro integrable to A and*

$$\omega_m(x) = o(1) \tag{2.1}$$

for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.

Remark 2.1. It follows from the definition of the general control modulo that condition (1.5) implies the condition (2.1).

Theorem 2.2. *If $s(x)$ is q-Cesàro integrable to A and $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$ for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.*

Remark 2.2. Let the function $s(x)$ satisfy the property $(\mathcal{P})$, then so does the function $\sigma(\omega_m(x))$ for any non-negative integer m.

Remark 2.3. For the case $m = 0$ in Theorem 2.2, we observe that $v(x)$ satisfies the property $(\mathcal{P})$ which means that it is a Tauberian condition for the q-Cesàro integrability [6].

3. Auxiliary Results

In this section we state and prove some lemmas which are needed for the brevity of proofs of our main results.

Lemma 3.1. *For every integer $m \geq 1$,*

$$xD_q(\sigma_m(x)) = v_{m-1}(x). \tag{3.1}$$

Proof. Taking the q-derivative of $\sigma_m(x)$ gives

$$\begin{aligned} D_q(\sigma_m(x)) &= D_q\left(\frac{1}{x}\int\limits_0^x \sigma_{m-1}(t)d_qt\right) \\ &= \frac{1}{qx}\sigma_{m-1}(x) - \frac{1}{qx^2}\int\limits_0^x \sigma_{m-1}(t)d_qt \\ &= \frac{1}{qx}\left(\sigma_{m-1}(x) - \sigma_m(x)\right). \end{aligned}$$

Hence, applying the identity (1.3) to $\sigma_{m-1}(x)$, we get $D_q(\sigma_m(x)) = \dfrac{v_{m-1}(x)}{x}$, which completes the proof. □

Lemma 3.2. *For every integer $m \geq 1$,*

- **(i)** $xf(x) - v(x) = qxD_q(v(x))$
- **(ii)** $v_{m-1}(x) - v_m(x) = qxD_q(v_m(x))$.

Proof. (i) Taking the q-derivative and then multiplying both sides of identity (1.3) by x, we get

$$xD_q(s(x)) - xD_q(\sigma(x)) = qxD_q(v(x)).$$

It follows from Lemma 3.1 that

$$xf(x) - v(x) = qxD_q(v(x)).$$

(ii) Taking the $q-$derivative of both sides of (1.4), we have

$$D_q(\sigma_m(x)) - D_q(\sigma_{m+1}(x)) = qD_q(v_m(x)). \tag{3.2}$$

Then, multiplying (3.2) by x yields

$$xD_q(\sigma_m(x)) - xD_q(\sigma_{m+1}(x)) = qxD_q(v_m(x)).$$

Using Lemma 3.1, we prove that

$$v_{m-1}(x) - v_m(x) = qxD_q(v_m(x)).$$

□

Lemma 3.3. *For every integer $m \geq 1$,*

$$\sigma(xD_q(v_{m-1}(x))) = xD_q(v_m(x)). \tag{3.3}$$

Proof. Taking Cesàro means of both sides of the identity in Lemma 3.2 (ii), we find

$$\begin{aligned} \sigma(xD_q(v_{m-1}(x))) &= q^{-1}[\sigma(v_{m-2}(x)) - \sigma(v_{m-1}(x))] \\ &= q^{-1}\left(v_{m-1}(x) - v_m(x)\right) \\ &= xD_q(v_m(x)). \end{aligned}$$

□

For a function $f(x)$, we define

$$(xD_q)_m(f(x)) = (xD_q)_{m-1}(xD_q(f(x))) = xD_q((xD_q)_{m-1}(f(x))),$$

where $(xD_q)_0(f(x)) = f(x)$ and $(xD_q)_1(f(x)) = xD_q(f(x))$.

Lemma 3.4. *For every integer $m \geq 1$,*

$$\omega_m(x) = (xD_q)_m(v_{m-1}(x)). \tag{3.4}$$

Proof. We prove the assertion by using mathematical induction. From the definition of the general control modulo for $m = 1$ and Lemma 3.2 (i), we get

$$\omega_1(x) = q^{-1}(\omega_0(x) - \sigma(\omega_0(x))) = q^{-1}(xf(x) - v(x)) = xD_q(v(x)).$$

Assume the assertion holds for some positive integer $m = k$. That is, assume that

$$\omega_k(x) = (xD_q)_k(v_{k-1}(x)). \tag{3.5}$$

We show that the assertion is true for $m = k + 1$. That is,

$$\omega_{k+1}(x) = (xD_q)_{k+1}(v_k(x)).$$

By definition of the general control modulo for $m = k + 1$, we have

$$\omega_{k+1}(x) = q^{-1}(\omega_k(x) - \sigma(\omega_k(x))).$$

Considering Lemma 3.2 (ii) and Lemma 3.3 together with (3.5), we obtain

$$\begin{aligned} \omega_{k+1}(x) &= q^{-1}[(xD_q)_k(v_{k-1}(x)) - (xD_q)_k(v_k(x))] \\ &= q^{-1}(xD_q)_k(v_{k-1}(x) - v_k(x)) \\ &= (xD_q)_{k+1}(v_k(x)). \end{aligned}$$

Therefore, we conclude that Lemma 3.4 is true for each integer $m \geq 1$. □

Lemma 3.5. *If $s(x)$ is q-Cesàro integrable to some finite number A, then for each non-negative integer m, $\sigma(\omega_m(x))$ is q-Cesàro integrable to 0.*

Proof. If $s(x) \to A(C_q, 1)$, then it is known that $\sigma(x) \to A(C_q, 1)$. Thus, it follows from the identity (1.3) that $v(x) = \sigma(\omega_0(x)) \to 0(C_q, 1)$. Replacing $s(x)$ with $v(x)$ in (1.3), we write

$$v(x) - v_1(x) = qxD_q(v_1(x)) = q\sigma(\omega_1(x)). \tag{3.6}$$

Then, (3.6) implies $\sigma(\omega_1(x)) \to 0(C_q, 1)$. Now, applying (1.3) to $xD_q(v_1(x))$, we get

$$xD_q(v_1(x)) - xD_q(v_2(x)) = q(xD_q)_2 v_2(x) = q\sigma(\omega_2(x)). \tag{3.7}$$

Hence from (3.7), $\sigma(\omega_2(x)) \to 0(C_q, 1)$. Continuing in the same manner, we obtain $\sigma(\omega_m(x)) \to 0(C_q, 1)$ for each non-negative integer m.

□

Lemma 3.6. *For every non-negative integer m and k,*

$$\sigma_k(\omega_m(x)) = \omega_m(\sigma_k(x)). \tag{3.8}$$

Proof. Using Lemma 3.4 and Lemma 3.3 respectively, it follows

$$\begin{aligned} \sigma_k(\omega_m(x)) &= \sigma_k((xD_q)_m v_{m-1}(x)) \\ &= (xD_q)_{m+1}\sigma_{m+k}(x). \end{aligned} \tag{3.9}$$

On the other hand, taking Lemma 3.4 and Lemma 3.1 into account we find

$$\begin{aligned} \omega_m(\sigma_k(x)) &= (xD_q)_m v_{m-1}(\sigma_k(x)) \\ &= (xD_q)_{m+1}(\sigma_{m+k}(x)). \end{aligned} \tag{3.10}$$

Therefore, the proof is completed from the equality of (3.9) and (3.10). □

The following lemma shows a different representation of the difference $s(x) - \sigma(x)$.

Lemma 3.7. *For any function $s(x)$ defined on $(0, \infty)$, we have the identity*

$$s(x) - \sigma(x) = \frac{q}{1-q}(\sigma(x) - \sigma(qx)), \tag{3.11}$$

where $\sigma(qx) = \dfrac{1}{qx}\displaystyle\int_0^{qx} s(t)d_qt.$

Proof. By the definition of the q-integral, we may write

$$\begin{aligned} \int_0^{qx} s(t)d_qt &= (1-q)qx\sum_{n=0}^{\infty} s(xq^{n+1})q^n \\ &= (1-q)x\sum_{n=1}^{\infty} s(xq^n)q^n \\ &= (1-q)x\left(\sum_{n=0}^{\infty} s(xq^n)q^n - s(x)\right) \\ &= \int_0^x s(t)d_qt - (1-q)xs(x). \end{aligned}$$

Dividing the both sides of the last equality by qx, we get

$$\frac{q}{1-q}(\sigma(x) - \sigma(qx)) = s(x) - \sigma(x).$$

□

It is clear from Lemma 3.7 that, even if $\sigma(x)$ is convergent, $\sigma(x)$ and $\sigma(qx)$ do not tend to same value when $s(x)$ is not convergent.

4. Proofs

In this section, we give proofs of our main results.

4.1. Proof of Theorem 2.1

From the hypothesis we have

$$\omega_m(x) = xD_q\sigma(\omega_{m-1}(x)) = o(1), \tag{4.1}$$

for some integer $m \geq 1$. On the other hand, from Lemma 3.5, $\sigma(\omega_{m-1}(x)) \to 0(C_q, 1)$. Hence, applying Theorem 1.1 to $\sigma(\omega_{m-1}(x))$ we obtain

$$\sigma(\omega_{m-1}(x)) = o(1). \tag{4.2}$$

Considering (4.1) and (4.2) together with the identity

$$\omega_{m-1}(x) - \sigma(\omega_{m-1}(x)) = q\omega_m(x),$$

we get

$$\omega_{m-1}(x) = xD_q\sigma(\omega_{m-2}(x)) = o(1). \tag{4.3}$$

By Lemma 3.5, we also have $\sigma(\omega_{m-2}(x)) \to 0(C_q, 1)$. Now, applying Theorem 1.1 to $\sigma(\omega_{m-2}(x))$ we obtain

$$\sigma(\omega_{m-2}(x)) = o(1). \tag{4.4}$$

From (4.3), (4.4) and the identity

$$\omega_{m-2}(x) - \sigma(\omega_{m-2}(x)) = q\omega_{m-1}(x),$$

we find

$$\omega_{m-2}(x) = xD_q\sigma(\omega_{m-3}(x)) = o(1). \tag{4.5}$$

Taking (4.1), (4.3) and (4.5) into account and proceeding likewise, we observe that $\omega_0(x) = o(1)$. Therefore, the proof follows from Theorem 1.1. □

4.2. Proof of Theorem 2.2

Considering Lemma 3.7 we may construct the identity

$$\sigma(\omega_m(x)) - \sigma_2(\omega_m(x)) = \frac{q}{1-q}[\sigma_2(\omega_m(x)) - \sigma_2(\omega_m(qx))].$$

Since $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$, its $q-$Cesàro mean $\sigma_2(\omega_m(x))$ also satisfies the property $(\mathcal{P})$. Let $\epsilon > 0$ be given. Then, there exists $K > 0$ such that

$$-\epsilon < \sigma_2(\omega_m(x)) - \sigma(\omega_m(x)) < \epsilon \tag{4.6}$$

for every $x > K$. By (4.6), we write

$$\sigma(\omega_m(x)) - \epsilon < \sigma_2(\omega_m(x)) < \sigma(\omega_m(x)) + \epsilon. \tag{4.7}$$

Since $s(x) \to A(C_q, 1)$, we have by using Lemma 3.5 that $\lim\limits_{x\to\infty} \sigma_2(\omega_m(x)) = 0$. Thus, it follows from (4.7)

$$-\epsilon < \liminf_{x\to\infty} \sigma(\omega_m(x)) < \limsup_{x\to\infty} \sigma(\omega_m(x)) < \epsilon,$$

which is equivalent to

$$\lim_{x\to\infty} \sigma(\omega_m(x)) = 0. \tag{4.8}$$

It yields from the equality

$$\begin{aligned}
\sigma(\omega_m(x)) &= \sigma((xD_q)_m v_{m-1}(x)) \\
&= xD_q(xD_q)_{m-1} v_m(x) \\
&= xD_q \sigma_2(\omega_{m-1}(x)),
\end{aligned}$$

that $xD_q\sigma_2(\omega_{m-1}(x)) = o(1)$. Also, by Lemma 3.5, $\sigma(\omega_{m-1}(x)) \to 0(C_q, 1)$. Further, regularity of q-Cesàro integrability implies $\sigma_2(\omega_{m-1}(x)) \to 0(C_q, 1)$. Then, if we apply Theorem 1.1 to $\sigma_2(\omega_{m-1}(x))$ we obtain

$$\lim_{x\to\infty} \sigma_2(\omega_{m-1}(x)) = 0. \tag{4.9}$$

From the definition of the general control modulo, it is easy to see

$$\sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = q\sigma(\omega_m(x)). \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we reach

$$\lim_{x\to\infty} \sigma(\omega_{m-1}(x)) = 0. \tag{4.11}$$

Now, since

$$\begin{aligned}
\sigma(\omega_{m-1}(x)) &= \sigma((xD_q)_{m-1} v_{m-2}(x)) \\
&= xD_q(xD_q)_{m-2} v_{m-1}(x) \\
&= xD_q \sigma_2(\omega_{m-2}(x)),
\end{aligned}$$

we find $xD_q\sigma_2(\omega_{m-2}(x)) = o(1)$. Besides, we have $\sigma_2(\omega_{m-2}(x)) \to 0(C_q, 1)$ from Lemma 3.5 and the regularity of q-Cesàro integrability. Now, applying Theorem 1.1 to $\sigma_2(\omega_{m-2}(x))$ we get

$$\lim_{x\to\infty} \sigma_2(\omega_{m-2}(x)) = 0. \tag{4.12}$$

Considering (4.11), (4.12) and the identity

$$\sigma(\omega_{m-2}(x)) - \sigma_2(\omega_{m-2}(x)) = q\sigma(\omega_{m-1}(x)), \tag{4.13}$$

we have

$$\lim_{x\to\infty} \sigma(\omega_{m-2}(x)) = 0. \tag{4.14}$$

In the light of (4.8), (4.11) and (4.14), continuing in the same fashion we conclude

$$\lim_{x\to\infty} \sigma(\omega_0(x)) = \lim_{x\to\infty} v(x) = 0.$$

Therefore, since $s(x) \to A(C_q, 1)$, we obtain via (1.3) that $\lim\limits_{x\to\infty} s(x) = A$. □

5. Extensions

In this section, we will present the q-Hölder or (H_q, k) integrability method which is an obvious generalization of the q-Cesàro integrability. Later, we extend our main results to this method.

If

$$\lim_{x\to\infty} \sigma_k(x) = A,$$

then $\int_0^\infty f(t)d_qt$ is said to be integrable by the q-Hölder method of order $k \in \mathbb{N}_0$ (shortly, (H_q, k) integrable) to A, and this fact is denoted by $s(x) \to A(H_q, k)$. In particular, the method $(H_q, 0)$ indicates the convergence in the ordinary sense and the method $(H_q, 1)$ is equivalent to $(C_q, 1)$. The (H_q, k) methods are regular for any k and are compatible for all k. The power of the method increases with increasing k: The (H_q, k) integrability implies (H_q, k') integrability for any $k' > k$.

Theorem 5.1. *Let $s(x) \to A(H_q, k+1)$. If*

$$\omega_m(x) = o(1) \tag{5.1}$$

for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.

Proof. By (5.1) and the regularity of the $(C_q, 1)$ method, we obtain $\sigma_k(\omega_m(x)) = o(1)$ for each integer $k \geq 0$. Then, from Lemma 3.6 it is clear that

$$\omega_m(\sigma_k(x)) = o(1) \quad \text{for each } k \in \mathbb{N}_0. \tag{5.2}$$

Besides, from the assumption since $\sigma_k(x) \to A(C_q, 1)$, Theorem 2.1 implies

$$\lim_{x\to\infty} \sigma_k(x) = A$$

which is also equivalent to $\sigma_{k-1}(x) \to A(C_q, 1)$. From (5.2), we know that $\omega_m(\sigma_{k-1}(x)) = o(1)$. Now, applying Theorem 2.1 to $\sigma_{k-1}(x)$ yields

$$\lim_{x\to\infty} \sigma_{k-1}(x) = A$$

which is also equivalent to $\sigma_{k-2}(x) \to A(C_q, 1)$. Repeating the same steps k-times we conclude

$$\lim_{x\to\infty} \sigma_0(x) = \int_0^\infty f(t) d_q t = A.$$

□

Theorem 5.2. *Let $s(x) \to A(H_q, k+1)$. If $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$ for some integer $m \geq 0$, then $\int_0^\infty f(t) d_q t = A$.*

Proof. If $\sigma(\omega_m(x))$ satisfies the property $(\mathcal{P})$, then so does $\sigma_k(\omega_m(x))$ for every non-negative integer k. From Lemma 3.6, since

$$\sigma_k(\omega_m(x)) = \omega_m(\sigma_k(x))$$

we find that $\sigma(\omega_m(\sigma_k(x)))$ also satisfies $(\mathcal{P})$ for all $k \in \mathbb{N}_0$. Considering the hypothesis $\sigma_k(x) \to A(C_q, 1)$ and Theorem 2.2 we obtain

$$s(x) \to A(H_q, k)$$

which requires $\sigma_{k-1}(x) \to A(C_q, 1)$. Moreover, since $\sigma(\omega_m(\sigma_{k-1}(x)))$ satisfies $(\mathcal{P})$, we get

$$s(x) \to A(H_q, k-1)$$

which requires $\sigma_{k-2}(x) \to A(C_q, 1)$. Applying the same reasoning k-times we reach that

$$s(x) \to A(H_q, 0)$$

which means $\lim_{x\to\infty} s(x) = A$. □

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SOME NEW TYPES OF CONTINUITY IN ASYMMETRIC METRIC SPACES

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Abstract. Using the notions of forward and backward arithmetic convergence in asymmetric metric spaces, we have defined arithmetic ff-continuity and arithmetic fb-continuity and prove some interesting results. Moreover, we have introduced the concepts of forward and backward arithmetic compactness and obtained the related results in the setting of asymmetric metric space.

Keywords: asymmetric metric spaces; forward and backward arithmetic compactness; forward and backward arithmetic convergence; arithmetic ff-continuity.

1. Introduction and Preliminaries

In 1931, Wilson [18] first introduced asymmetric metric spaces as quasi-metric spaces, and afterwards they were studied by many other authors (see [1, 14, 15, 16]). An asymmetric metric space is a generalization of a metric space but the symmetry axiom is eliminated in the definition of metric spaces. We can come up with some troubles in several classical statements of symmetric analysis without the symmetry property in the definition of such spaces. In asymmetric metric spaces, some notions such as convergence, completeness and compactness are different from the metric case. There are two notions for each of them, namely forward and backward ones, since we have two topologies which are the forward topology and the backward topology in the same space (see [13]). Collins and Zimmer [10] studied these notions in the asymmetric context.

An example that asymmetric metrics are common in real life is taxicab geometry topology including one-way streets, where a path from point A to point B contains a different set of streets than a path from B to A. Also, the examples of the latest applications of asymmetric metric spaces in the field of pure and applied

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mathematics and material science are as in [8]. In [9], Cobzas gave the basic results on asymmetric normed spaces.

Ruckle [17] introduced the notion of*arithmetic convergence* as a sequence $x = (x_k)$ defined on $\mathbb{N}$, and it is said to be arithmetic convergent if for each $\varepsilon > 0$ there is an integer n such that for every integer m we have $|x_m - x_{<m,n>}| < \varepsilon$. Here and henceforth, $< m, n >$ denotes the greatest common divisor of m and n. Çakalli [4] gave another definition of arithmetic convergence of a sequence (x_k) as a sequence $x = (x_k)$ is said to be arithmetically convergent if for each $\varepsilon > 0$ there is an integer n_0 such that $|x_m - x_{<m,n>}| < \varepsilon$ for every integers m, n satisfying $< m, n >\geq n_0$. Throughout the article, we follow the definition given by Çakalli in his corrigendum to the paper [4]. For more details on arithmetic convergence and arithmetic continuity, we refer to [4, 19, 20, 21, 22, 23]. For different types of continuity and b- metric spaces, we refer to [2, 3, 5, 6, 7, 11, 12].

In this article, we will first introduce the concepts of forward and backward arithmetic convergence and using these notions we will define forward and backward arithmetic continuity in asymmetric metric spaces and establish some interesting results. In the last section, we will introduce forward and backward arithmetic compactness and obtain related results.

2. Asymmetric Metric Spaces

Let us recall some definitions and results on asymmetric metric spaces which were given in [10].

Definition 2.1. A function $d : X \times X \to R$ is an asymmetric metric and (X, d) is an asymmetric metric space if

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$ for every $x, y \in X$.

(ii) $d(x, z) \leq d(x, y) + d(y, z)$; for every $x, y, z \in X$.

Definition 2.2. The *forward topology* τ_+ induced by d is the topology generated by the forward open balls $B^+(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for $x \in X; \varepsilon > 0$.

Likewise, the *backward topology* τ_- induced by d is the topology generated by the backward open balls $B^-(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$ for $x \in X; \varepsilon > 0$.

Definition 2.3. A set $S \subset X$ is *forward bounded* (resp. *backward bounded*), if there exists $x \in X$ and $\varepsilon > 0$ such that $S \subset B^+(x, \varepsilon)$ (resp. $S \subset B^-(x, \varepsilon)$).

Definition 2.4. A sequence (x_n) is said to be *forward convergent* to $x \in X$ (*backward convergent* to $x \in X$) if and only if

$$\lim_{n\to\infty} d(x, x_n) = 0 \ (\lim_{n\to\infty} d(x_n, x) = 0)$$

and is denoted by $x_n \xrightarrow{f} x$ ($x_n \xrightarrow{b} x$).

Definition 2.5. A sequence (x_n) in an asymmetric metric space (X, d) is *forward Cauchy (backward Cauchy)* if for each $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for $k \geqslant n \geqslant N$; $d(x_n, x_k) < \varepsilon$ $(d(x_k, x_n) < \varepsilon)$ holds.

Definition 2.6. [Sequential definition of continuity] Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. A function $f : X \to Y$ is $ff-$*continuous* at $x \in X$ if and only if whenever $x_n \xrightarrow{f} x$ in (X, d_X) we have $f(x_n) \xrightarrow{f} f(x)$ in (Y, d_Y).

The statement holds analogously for the other types. Note that the forward uniform continuity is same as the backward uniform continuity.

Definition 2.7. A set $S \subset X$ is

(i) *forward compact* if every open cover of S in the forward topology has a finite subcover.

(ii) *forward relatively compact* if $\overline{S}$ is forward compact, where $\overline{S}$ denotes the closure of S in the forward topology.

(iii) *forward sequentially compact* if every sequence in X contains a forward convergent subsequence.

(iv) *forward complete* if every forward Cauchy sequence is forward convergent.

Definition 2.8. Let (f_n) be a function sequence and f be a function from X to Y. We say that the sequence (f_n) is *forward convergent uniformly (backward convergent uniformly)* to limit f if for every $\varepsilon > 0$ there exists a positive number N such that for all $x \in X$ and all $n \geq N$ we have $d(f(x), f_n(x)) < \varepsilon$ $(d(f_n(x), f(x)) < \varepsilon)$.

3. Arithmetic Continuity in Asymmetric Metric Spaces

In this section, we introduce the concepts of forward and backward arithmetic convergence and forward and backward arithmetic continuity in asymmetric metric spaces and prove some results using these notions.

Definition 3.1. A sequence $x = (x_k)$ is called forward arithmetic convergent (resp. backward arithmetic convergent) in an asymmetric metric space (X, d) if for each $\varepsilon > 0$ there is an integer N such that $d(x_{<m,n>}, x_m) < \varepsilon$ (resp. $d\,(x_m, x_{<m,n>}) < \varepsilon$), for every integers m, n satisfying $<m, n> \geq N$. We shall denote it by writing $x_m \xrightarrow{af} x_{<m,n>}$ (resp. $x_m \xrightarrow{ab} x_{<m,n>}$).

Definition 3.2. Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces. A function $f : X \to Y$ is arithmetic $ff-$*continuous* (respectively arithmetic $fb-$*continuous*), iff it transforms forward arithmetic convergent sequence in (X, d_X) to forward arithmetic convergent sequence (respectively backward arithmetic convergent sequence) in (Y, d_Y).

Theorem 3.1. *Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. If $f : X \to Y$ is uniformly continuous then it is arithmetic ff-continuous.*

Proof. Let $f : X \to Y$ be uniformly continuous and (x_n) be any forward arithmetic convergence sequence in X. Since f is uniformly continuous, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ with $d_X(x, y) < \delta$, $d_Y(f(x), f(y)) < \varepsilon$. Again, the sequence (x_n) is forward arithmetic convergent in X, hence for the same $\delta > 0$ there exists a positive integer m_0 such that for all integers m, n satisfying $< m, n >\geq 0$,

$$\begin{aligned} d_X(x_{<n,m>}, x_n) < \delta \text{ for each } n \quad & \Rightarrow \quad d_Y(f(x_{<n,m>}), f(x_n)) < \varepsilon \text{ for each } n \\ & \Rightarrow \quad \text{the sequence } (f(x_n)) \text{ is forward arithmetic convergent} \\ & \Rightarrow \quad \text{the function } f \text{ is arithmetic } \textit{ff-continuous}. \end{aligned}$$

This completes the proof. □

Definition 3.3. A sequence of functions (f_n) from an asymmetric metric space (X, d_X) to an asymmetric metric space (Y, d_Y) is said to be forward arithmetic convergent (resp. backward arithmetic convergent) if for any $\varepsilon > 0$ and $\forall\ x \in X$ there exists a positive integer m_0 such that for all integers m, n satisfying $< m, n >\geq 0$,

$$d_Y(f_{<n,m>}(x), f_n(x)) < \varepsilon \ (\textit{resp. } d_Y(f_n(x), f_{<n,m>}(x)) < \varepsilon).$$

Theorem 3.2. *If (f_n) be a sequence of forward arithmetic convergent functions from an asymmetric metric space (X, d_X) to an asymmetric metric space (Y, d_Y) and x_o is a point in X such that*

$$\lim_{x \to x_o} f_n(x) = y_n, \ \ n = 1, 2, 3 \ldots$$

then (y_n) is also forward arithmetic convergent.

Proof. Since the sequence (f_n) is forward arithmetic convergent, therefore, for $\varepsilon > 0$ and a positive integer m_0 such that for all integers m, n satisfying $< m, n >\geq 0$

$$d_Y(f_{<n,m>}(x), f_n(x)) < \varepsilon \ \forall x \in X.$$

Keeping n, m fixed and letting $x \to x_o$,

$$d_Y(y_{<n,m>}, y_n) < \varepsilon.$$

Hence, the sequence (y_n) is forward arithmetic convergent. □

Remark 3.1. *The same result can be written for backward arithmetic convergence.*

Theorem 3.3. *If (f_n) is a sequence of arithmetic $ff-$continuous functions from asymmetric metric space (X, d_X) to asymmetric metric space (Y, d_Y) with forward convergence equivalent to backward convergence in Y and (f_n) forward converges uniformly to a function f, then f is arithmetic ff-continuous.*

Proof. Let $\varepsilon > 0$ and (x_n) be any forward arithmetic convergent sequence in X. Since $f_n \xrightarrow{f} f$ uniformly, we can choose $N_1 \in \mathbb{N}$ so that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $n \geq N_1$ and $x \in X$. Now, in particular, $f_n(x_{<n,m>}) \xrightarrow{f} f(x_{<n,m>})$ and so $f_n(x_{<n,m>}) \xrightarrow{b} f(x_{<n,m>})$. Thus, we can find $N_2 \in \mathbb{N}$ so that $d_Y(f_n(x_{<n,m>}), f(x_{<n,m>})) < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Further, (f_n) is given to be a sequence of arithmetic ff-continuous functions. In particular, f_N is arithmetic ff-continuous function, and thus arithmetic fb-continuous by equivalence of forward and backward convergence in Y. So there exists an integer n_0, greater than N and $\delta > 0$ such that

$$d_Y(f_N(x_n), f_N(x_{<n,m>})) < \frac{\varepsilon}{3} \text{ for } d_X(x_{<n,m>}, x_n) < \delta,$$

for all integers m, n satisfying $< m, n > \geq n_0$. Consequently, whenever $d_X(x_{<n,m>}, x_n) < \delta$ and $< m, n > \geq n_0$, we have

$$\begin{aligned} d_Y(f(x_n), f(x_{<n,m>})) &\leq d_Y(f(x_n), f_N(x_n)) + d_Y(f_N(x_n), f_N(x_{<n,m>})) \\ &\quad + d_Y(f_N(x_{<n,m>}), f(x_{<n,m>})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore f is arithmetic fb-continuous and by equivalence of convergence it is also arithmetic ff-continuous. □

Theorem 3.4. *Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces. Then the set of all arithmetic ff-continuous functions from X to Y, with forward convergence equivalent to backward convergence in Y, is a closed subset of all continuous functions from X to Y i.e. $\mathbb{A}^{ff}(X, Y) = \overline{\mathbb{A}^{ff}(X, Y)}$ where $\mathbb{A}^{ff}(X, Y)$ is the set of all arithmetic ff-continuous functions from X to Y and $\overline{\mathbb{A}^{ff}(X, Y)}$ denotes the closure of $\mathbb{A}^{ff}(X, Y)$.*

Proof. Let $f \in \overline{\mathbb{A}^{ff}(X, Y)}$. Then there exists a sequence of points in $\mathbb{A}^{ff}(X, Y)$ such that $f_n \xrightarrow{f} f$ as $n \to \infty$. Let $\varepsilon > 0$ and (x_n) be any forward arithmetic convergent sequence in X. Since $f_n \xrightarrow{f} f$ uniformly, we can choose $N_1 \in \mathbb{N}$ so that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $n \geq N_1$ and $x \in X$. In particular, $f_n(x_{<n,m>}) \xrightarrow{f} f(x_{<n,m>})$ and so $f_n(x_{<n,m>}) \xrightarrow{b} f(x_{<n,m>})$. Therefore, we can find $N_2 \in \mathbb{N}$ so that $d_Y(f_n(x_{<n,m>}), f(x_{<n,m>})) < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Assume that $N = \max\{N_1, N_2\}$. Moreover, (f_n) is given to be a sequence of arithmetic ff-continuous functions.

In particular, f_N is arithmetic ff-continuous function, and thus arithmetic fb-continuous by equivalence of forward and backward convergence in Y. So there exists an integer n_0 greater than N and $\delta > 0$ such that

$$d_Y(f_N(x_n), f_N(x_{<n,m>})) < \frac{\varepsilon}{3} \text{ for } d_X(x_{<n,m>}, x_n) < \delta$$

for all integers m, n satisfying $< m, n >\geq n_0$. Thus, whenever $d_X(x_{<n,m>}, x_n) < \delta$ and $< m, n >\geq n_0$,, we have

$$\begin{aligned} d_Y(f(x_n), f(x_{<n,m>})) &\leq d_Y(f(x_n), f_N(x_n)) + d_Y(f_N(x_n), f_N(x_{<n,m>})) \\ &\quad + d_Y(f_N(x_{<n,m>}), f(x_{<n,m>})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore f is arithmetic fb-continuous and by equivalence of convergence it is also arithmetic ff-continuous. So $f \in \mathbb{A}^{ff}(X, Y)$. This completes the prove of the theorem. □

In [4], Çakalli introduced the notion of (cAC)-continuity as follows: a function f is said to be (cAC)-*continuous* (or $f \in (cAC)$) if f transforms convergent sequences to arithmetic convergent sequences. We define this notion in the sense of arithmetic forward (or backward) convergence as follows:

A function f from asymmetric metric space X to asymmetric metric space Y is said to be *forward* (cAC)-*continuous* if it transforms forward convergent sequences in X to forward arithmetic convergent sequences in Y, i.e. (x_n) is forward convergent in X implies $f(x_n)$ is forward arithmetic convergent in Y.

Theorem 3.5. *Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces, with forward convergence equivalent to backward convergence in Y. If (f_n) is a sequence of forward (cAC)-continuous functions from X to Y and (f_n) forward converges uniformly to a function f, then f is forward (cAC)-continuous.*

Proof. Let $\varepsilon > 0$ be given and (x_k) be any forward convergent sequence in X. Since f_n forward converges uniformly to f, there exists a positive integer N_1 such that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $x \in X$ and $n \geq N_1$. In particular, $f_n(x_n) \xrightarrow{f} f(x_n)$ and so $f_n(x_n) \xrightarrow{b} f(x_n)$. Thus we can find $N_2 \in \mathbb{N}$ so that $d_Y(f_n(x_n), f(x_n)) < \frac{\varepsilon}{3}$ for all $n \geq N_2$. Assume that $N = \max\{N_1, N_2\}$. By hypothesis, f_n is forward (cAC)-continuous. In particular f_N is forward (cAC)-continuous, so there exists an integer n_0, greater than N such that $d_Y(f_N(x_{<m,n>}), f_N(x_n)) < \frac{\varepsilon}{3}$ for all $x \in X$ and for all integers m, n satisfying $< m, n >\geq n_0$. Thus, it follows that

$$\begin{aligned} d_Y(f(x_{<m,n>}), f(x_n)) &\leq d_Y(f(x_{<m,n>}), f_N(x_{<m,n>})) \\ &\quad + d_Y(f_N(x_{<m,n>}), f_N(x_n)) + d_Y(f_N(x_n), f(x_n)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This establishes the result. □

Theorem 3.6. *Let (X, d_X) and (Y, d_Y) be two asymmetric metric spaces. Then the set of all forward (cAC)-continuous functions from X to Y, with forward convergence equivalent to backward convergence in Y, is a closed subset of the set of all continuous functions from X to Y.*

Proof. The result immediately follows from the previous theorem. □

4. Compactness in Asymmetric Metric Spaces

We will first introduce forward arithmetic compactness and backward arithmetic compactness in the setting of asymmetric metric space as follows:

Definition 4.1. A subset A of an asymmetric metric space (X, d_X) is said to be

(i) *forward arithmetic compact* if every sequence in A has forward arithmetic convergent subsequence.

(ii) *backward arithmetic compact* if every sequence in A has backward arithmetic convergent subsequence.

Theorem 4.1. *An arithmetic ff-continuous image of an forward arithmetic compact subset of an asymmetric metric space (X, d) is forward arithmetic compact.*

Proof. Let (X, d_X) and (Y, d_Y) be asymmetric metric spaces. Let $f : X \to Y$ be an arithmetic ff-continuous function and $A \subset X$ be forward arithmetic compact. Let (y_n) be a sequence in $f(A)$. Then we can write $y_n = f(x_n)$ where $x_n \in X$ for each $n \in \mathbb{N}$. Since A is forward arithmetic compact, there exists an forward arithmetic convergent subsequence (x_{n_k}) of (x_n). Again, it is given that f is arithmetic ff-continuous, this implies that $f(x_{n_k})$ is forward arithmetic convergent subsequence of $f(x_n)$. Hence, $f(A)$ is forward arithmetic compact. □

Theorem 4.2. *An arithmetic fb-continuous image of a backward arithmetic compact subset of an asymmetric metric space (X, d) is backward arithmetic compact.*

Proof. The proof is the same as in the previous theorem. □

Theorem 4.3. *Any closed subset of a forward arithmetic compact subset of an asymmetric metric space (X, d) is forward arithmetic compact.*

Proof. Let A be any forward arithmetic compact subset of X and B be a closed subset of A. Let $x = (x_n)$ be any sequence of points in B. Then $x = (x_n)$ is a sequence of points in A. Since A is forward arithmetic compact, there exists an forward arithmetic convergent subsequence (x_{n_k}) of the sequence x. Since B is closed, so any sequence $x = (x_n)$ of points in B has forward arithmetic convergent subsequence in B. Hence the result. □

Theorem 4.4. *Any closed subset of a backward arithmetic compact subset of an asymmetric metric space (X, d) is backward arithmetic compact.*

Proof. The proof is the same as in the previous theorem. $\square$

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$f-$LACUNARY STATISTICAL CONVERGENCE AND STRONG $f-$LACUNARY SUMMABILITY OF ORDER α OF DOUBLE SEQUENCES

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Abstract. The main objective of this article is to introduce the concepts of $f-$lacunary statistical convergence of order α and strong $f-$lacunary summability of order α of double sequences and give some inclusion relations between these concepts.

Keywords: $f-$lacunary statistical convergence; strong $f-$lacunary summability; sequence spaces.

1. Introduction

In 1951, Steinhaus [41] and Fast [19] introduced the concept of statistical convergence while later in 1959, Schoenberg [40] reintroduced it independently. Bhardwaj and Dhawan [4], Caserta et al. [5], Connor [6], Çakallı [11], Çınar et al. [12], Çolak [13], Et et al. ([15],[17]), Fridy [21], Işık [27], Salat [39], Di Maio and Kočinac [14], Mursaleen et al. ([31],[30],[32]), Belen and Mohiuddine [3] and many authors investigated the arguments related to this notion.

A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f-$density of a subset $E\subset\mathbb{N}$ for any unbounded modulus f by

$$d^f(E)=\lim_{n\to\infty}\frac{f\left(\left|\{k\leq n:k\in E\}\right|\right)}{f(n)}, \textit{if the limit exists}$$

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and defined $f-$statistical convergence for any unbounded modulus f by

$$d^{f}\left(\{k\in\mathbb{N}:|x_{k}-\ell|\geq\varepsilon\}\right)=0$$

i.e.

$$\lim_{n\to\infty}\frac{1}{f\left(n\right)}f\left(\left|\{k\leq n:|x_{k}-\ell|\geq\varepsilon\}\right|\right)=0,$$

and we write it as $S^{f}-\lim x_{k}=\ell$ or $x_{k}\to\ell\left(S^{f}\right)$. Every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence does not need to be $f-$statistically convergent for every unbounded modulus f.

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_{r})$ of non-negative integers such that $k_{0}=0$ and $h_{r}=(k_{r}-k_{r-1})\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_{r}=(k_{r-1},k_{r}]$ and the ratio $\frac{k_{r}}{k_{r-1}}$ will be abbreviated by q_{r}, and $q_{1}=k_{1}$ for convenience.

In [22], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (x_{k}) of real numbers is called lacunary statistically convergent to a real number ℓ, if

$$\lim_{r\to\infty}\frac{1}{h_{r}}\left|\{k\in I_{r}:|x_{k}-\ell|\geq\varepsilon\}\right|=0$$

for every positive real number ε.

Lacunary sequence spaces were studied in ([7],[8],[9],[10],[18],[20],[22],[23],[25],[26],[28],[36],[43]).

A double sequence $x=(x_{j,k})_{j,k=0}^{\infty}$ has Pringsheim limit ℓ provided that given for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{j,k}-\ell|<\varepsilon$ whenever $j,k>N$. In this case, we write $P-\lim x=\ell$ (see Pringsheim [38]).

Let $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K\left(m,n\right)=\left\{(j,k):j\leq m,k\leq n\right\}$. The double natural density of K is defined by

$$\delta_{2}\left(K\right)=P-\lim_{m,n}\frac{1}{mn}\left|K\left(m,n\right)\right|,\text{ \textit{if the limit exists.}}$$

A double sequence $x=\left(x_{jk}\right)_{j,k\in\mathbb{N}}$ is said to be statistically convergent to a number ℓ if for every $\varepsilon>0$ the set $\left\{(j,k):j\leq m,k\leq n:|x_{jk}-\ell|\geq\varepsilon\right\}$ has double natural density zero (see Mursaleen and Edely [31]).

In [35], Patterson and Savaş introduced the concept of double lacunary sequence in the sense that double sequence $\theta''=\{(k_{r},l_{s})\}$ is called double lacunary sequence, if there exists two increasing sequences of integers such that

$$k_{0}=0,h_{r}=k_{r}-k_{r-1}\to\infty\ \textit{as}\ r\to\infty$$

and

$$l_{0}=0,\overline{h}_{s}=l_{s}-l_{s-1}\to\infty\ \textit{as}\ s\to\infty.$$

where $k_{r,s}=k_rl_s$, $h_{r,s}=h_r\overline{h}_s$ and the following intervals are determined by θ'', $I_r=\{(k):k_{r-1}<k\leq k_r\}$, $I_s=\{(l):l_{s-1}<l\leq l_s\}$, $I_{r,s}=\{(k,l):k_{r-1}<k\leq k_r \text{ and } l_{s-1}<l\leq l_s\}$, $q_r=\frac{k_r}{k_{r-1}}$, $\overline{q}_s=\frac{l_s}{l_{s-1}}$ and $q_{r,s}=q_r\overline{q}_s$.

The double number sequence x is $S_{\theta''}-$convergent to ℓ provided that for every $\varepsilon>0$,

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:|x_{k,l}-\ell|\geq\varepsilon\}\right|)=0.$$

In this case, we write $S_{\theta''}-\lim x_{k,l}=\ell$ or $x_{k,l}\to\ell\,(S_{\theta''})$ (see [35]).

The notion of a modulus was given by Nakano [33]. Maddox [29] used a modulus function to construct some sequence spaces. Afterwards, different sequence spaces defined by modulus have been studied by Altın and Et [2], Et et al. [16], Işık [27], Gaur and Mursaleen [24], Nuray and Savaş [34], Pehlivan and Fisher [37], Şengül [42] and many others.

2. Main Results

In this section, we will introduce the concepts of $f-$lacunary statistical convergence of order α and strong $f-$lacunary summability of order α of double sequences, where f is an unbounded modulus and also give some results related to these concepts.

Definition 2.1. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence and α be a real number such that $0<\alpha\leq 1$. We say that the double sequence $x=(x_{k,l})$ is $f-$lacunary statistically convergent of order α, if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{[f\,(h_{r,s})]^{\alpha}}f\left(\left|\{(k,l)\in I_{r,s}:|x_{k,l}-\ell|\geq\varepsilon\}\right|\right)=0.$$

This space will be denoted by $S_{\theta''}^{f,\alpha}$. In this case, we write $S_{\theta''}^{f,\alpha}-\lim x_{k,l}=\ell$ or $x_{k,l}\to\ell\left(S_{\theta''}^{f,\alpha}\right)$. In the special case $\theta''=\{(2^r,2^s)\}$, we shall write $S''^{f,\alpha}$ instead of $S_{\theta''}^{f,\alpha}$.

Definition 2.2. Let f be a modulus function, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and α be a positive real number. We say that the double sequence $x=(x_{k,l})$ is strongly $w^{\alpha}\left[\theta'',f,p\right]-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{[h_{r,s}]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left[f\left(|x_{k,l}-\ell|\right)\right]^{p_k}=0.$$

In this case we write $w^{\alpha}\left[\theta^{''},f,p\right]-\lim x_{k,l}=\ell$. The set of all strongly $w^{\alpha}\left[\theta^{''},f,p\right]-$ summable sequences will be denoted by $w^{\alpha}\left[\theta^{''},f,p\right]$. If we take $p_k=1$ for all $k\in\mathbb{N}$, we write $w^{\alpha}\left[\theta^{''},f\right]$ instead of $w^{\alpha}\left[\theta^{''},f,p\right]$.

Definition 2.3. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and α be a positive real number. We say that the double sequence $x=(x_{k,l})$ is strongly $w_{\theta''}^{f,\alpha}(p)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k}=0.$$

In the present case, we write $w_{\theta''}^{f,\alpha}(p)-\lim x_{k,l}=\ell$. The set of all strongly $w_{\theta''}^{f,\alpha}(p)-$ summable sequences will be denoted by $w_{\theta''}^{f,\alpha}(p)$. In case of $p_k=p$ for all $k\in\mathbb{N}$ we write $w_{\theta''}^{f,\alpha}[p]$ instead of $w_{\theta''}^{f,\alpha}(p)$.

Definition 2.4. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and α be a positive real number. We say that the double sequence $x=(x_{k,l})$ is strongly $w_{\theta'',f}^{\alpha}(p)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left|x_{k,l}-\ell\right|^{p_k}=0.$$

In the present case, we write $w_{\theta'',f}^{\alpha}(p)-\lim x_{k,l}=\ell$. The set of all strongly $w_{\theta'',f}^{\alpha}(p)-$summable sequences will be denoted by $w_{\theta'',f}^{\alpha}(p)$. In case of $p_k=p$ for all $k\in\mathbb{N}$ we write $w_{\theta'',f}^{\alpha}[p]$ instead of $w_{\theta'',f}^{\alpha}(p)$.

The proof of each of the following results is fairly straightforward, so we choose to state these results without proof, where we shall assume that the sequence $p=(p_k)$ is bounded and $0<h=\inf_k p_k\le p_k\le\sup_k p_k=H<\infty$.

Theorem 2.1. *The space $w_{\theta''}^{f,\alpha}(p)$ is paranormed by*

$$g(x)=\sup_{r,s}\left\{\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|x_{k,l}\right|\right)\right]^{p_k}\right\}^{\frac{1}{M}}$$

where, $M=\max(1,H)$.

Proposition 2.1. ([37]) *Let f be a modulus and $0<\delta<1$. Then for each $\|u\|\ge\delta$, we have $f(\|u\|)\le 2f(1)\delta^{-1}\|u\|$.*

Theorem 2.2. *Let f be an unbounded modulus, α be a real number such that $0<\alpha\leq 1$ and $p>1$. If $\lim_{u\to\infty}\inf\frac{f(u)}{u}>0$, then $w_{\theta''}^{f,\alpha}[p]=w_{\theta'',f}^{\alpha}[p]$.*

Proof. Let $p>1$ be a positive real number and $x\in w_{\theta''}^{f,\alpha}[p]$. If $\lim_{u\to\infty}\inf\frac{f(u)}{u}>0$ then there exists a number $c>0$ such that $f(u)>cu$ for $u>0$. Clearly

$$\begin{aligned}\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}[f(|x_{k,l}-\ell|)]^{p} &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}[c\,|x_{k,l}-\ell|]^{p}\\ &= \frac{c^{p}}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-\ell|^{p},\end{aligned}$$

and therefore $w_{\theta''}^{f,\alpha}[p]\subset w_{\theta'',f}^{\alpha}[p]$.

Now let $x\in w_{\theta'',f}^{\alpha}[p]$. Then we have

$$\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-\ell|^{p}\to 0 \ \textit{as}\ r,s\to\infty.$$

Let $0<\delta<1$. We can write

$$\begin{aligned}\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-\ell|^{p} &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{\substack{(k,l)\in I_{r,s}\\ |x_{k,l}-\ell|\geq\delta}}|x_{k,l}-\ell|^{p}\\ &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{\substack{(k,l)\in I_{r,s}\\ |x_{k,l}-\ell|\geq\delta}}\left[\frac{f(|x_{k,l}-\ell|)}{2f(1)\delta^{-1}}\right]^{p}\\ &\geq \frac{1}{[f(h_{r,s})]^{\alpha}}\frac{\delta^{p}}{2^{p}f(1)^{p}}\sum_{(k,l)\in I_{r,s}}[f(|x_{k,l}-\ell|)]^{p}\end{aligned}$$

by Proposition 2.1. Therefore $x\in w_{\theta''}^{f,\alpha}[p]$.

If $\lim_{u\to\infty}\inf\frac{f(u)}{u}=0$, the equality $w_{\theta''}^{f,\alpha}[p]=w_{\theta'',f}^{\alpha}[p]$ cannot be hold as shown in the following example:

Let $f(x)=2\sqrt{x}$ and define a double sequence $x=(x_{k,l})$ by

$$x_{k,l}=\begin{cases}\sqrt[3]{h_{r,s}}, & \textit{if } k=k_r \textit{ and } l=l_s\\ 0, & \textit{otherwise}\end{cases}\qquad r,s=1,2,....$$

For $\ell=0$, $\alpha=\frac{3}{4}$ and $p=\frac{6}{5}$, we have

$$\frac{1}{[f(h_{r,s})]^{\alpha}}\sum_{(k,l)\in I_{r,s}}[f(|x_{k,l}|)]^{p}=\frac{\left(2\,[h_{r,s}]^{\frac{1}{6}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r,s}}\right)^{\frac{3}{4}}}=\frac{\left(2\left(h_r\overline{h_s}\right)^{\frac{1}{6}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_r\overline{\overline{h_s}}}\right)^{\frac{3}{4}}}\to 0 \ \textit{as}\ r,s\to\infty$$

hence $x \in w_{\theta''}^{f,\alpha}[p]$, but

$$\frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} |x_{k,l}|^{p} = \frac{\left(\sqrt[3]{h_{r,s}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r,s}}\right)^{\frac{3}{4}}} \to \infty \ as \ r,s \to \infty$$

and so $x \notin w_{\theta'',f}^{\alpha}[p]$. $\square$

Maddox [29] showed that the existence of an unbounded modulus f for which there is a positive constant c such that $f(xy) \geq cf(x) f(y)$, for all $x \geq 0$, $y \geq 0$.

Theorem 2.3. *Let f be an unbounded modulus and α be a positive real number. If $\lim_{u\to\infty} \frac{[f(u)]^{\alpha}}{u^{\alpha}} > 0$, then $w^{\alpha}\left[\theta'', f\right] \subset S_{\theta''}^{f,\alpha}$.*

Proof. Let $x \in w^{\alpha}[\theta'', f]$ and $\lim_{u\to\infty} \frac{f(u)^{\alpha}}{u^{\alpha}} > 0$. For $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{[h_{r,s}]^{\alpha}} \sum_{(k,l)\in I_{r,s}} f(|x_{k,l}-\ell|) &\geq \frac{1}{[h_{r,s}]^{\alpha}} f\left(\sum_{(k,l)\in I_{r,s}} |x_{k,l}-\ell|\right) \geq \frac{1}{[h_{r,s}]^{\alpha}} f\left(\sum_{\substack{(k,l)\in I_{r,s} \\ |x_{k,l}-\ell|\geq\varepsilon}} |x_{k,l}-\ell|\right) \\
&\geq \frac{1}{[h_{r,s}]^{\alpha}} f\left(|\{(k,l)\in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\}|\,\varepsilon\right) \\
&\geq \frac{c}{[h_{r,s}]^{\alpha}} f\left(|\{(k,l)\in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\}|\right) f(\varepsilon) \\
&= \frac{c}{[h_{r,s}]^{\alpha}} \frac{f\left(|\{(k,l)\in I_{r,s} : |x_{k,l}-\ell| \geq \varepsilon\}|\right)}{[f(h_{r,s})]^{\alpha}} [f(h_{r,s})]^{\alpha} f(\varepsilon).
\end{aligned}$$

Therefore, $w^{\alpha}\left[\theta'', f\right] - \lim x_{k,l} = \ell$ implies $S_{\theta''}^{f,\alpha} - \lim x_{k,l} = \ell$. $\square$

Theorem 2.4. *Let α_1, α_2 be two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$, f be an unbounded modulus function and let $\theta'' = \{(k_r, l_s)\}$ be a double lacunary sequence, then we have $w_{\theta''}^{f,\alpha_1}(p) \subset S_{\theta''}^{f,\alpha_2}$.*

Proof. Let $x \in w_{\theta''}^{f,\alpha_1}(p)$ and $\varepsilon > 0$ be given and $\sum_1$, $\sum_2$ denote the sums over $(k,l) \in I_{r,s}$, $|x_{k,l}-\ell| \geq \varepsilon$ and $(k,l) \in I_{r,s}$, $|x_{k,l}-\ell| < \varepsilon$ respectively. Since

$f\left(h_{r,s}\right)^{\alpha_1} \leq f\left(h_{r,s}\right)^{\alpha_2}$ for each r and s, we may write

$$\begin{aligned}
& \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_1}} \sum_{(k,l)\in I_{r,s}} \left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k} \\
= & \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_1}} \left[\sum\nolimits_1 \left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k} + \sum\nolimits_2 \left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k}\right] \\
\geq & \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[\sum\nolimits_1 \left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k} + \sum\nolimits_2 \left[f\left(\left|x_{k,l}-\ell\right|\right)\right]^{p_k}\right] \\
\geq & \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[\sum\nolimits_1 \left[f\left(\varepsilon\right)\right]^{p_k}\right] \\
\geq & \frac{1}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[f\left(\sum\nolimits_1 \left[\varepsilon\right]^{p_k}\right)\right] \\
\geq & \frac{1}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} \left[f\left(\sum\nolimits_1 \min(\left[\varepsilon\right]^{h}, \left[\varepsilon\right]^{H})\right)\right] \\
\geq & \frac{1}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} f\left(\left|\left\{(k,l)\in I_{r,s} : \left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right| \left[\min(\left[\varepsilon\right]^{h}, \left[\varepsilon\right]^{H})\right]\right) \\
\geq & \frac{c}{H.\left[f\left(h_{r,s}\right)\right]^{\alpha_2}} f\left(\left|\left\{(k,l)\in I_{r,s} : \left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right) f\left(\left[\min(\left[\varepsilon\right]^{h}, \left[\varepsilon\right]^{H})\right]\right).
\end{aligned}$$

Hence $x \in S_{\theta''}^{f,\alpha_2}$. □

Theorem 2.5. *Let* $\theta'' = \{(k_r, l_s)\}$ *be a double lacunary sequence and* α *be a fixed real number such that* $0 < \alpha \leq 1$. *If* $\liminf_r q_r > 1$, $\liminf_s q_s > 1$ *and* $\lim_{u\to\infty} \frac{[f(u)]^{\alpha}}{u^{\alpha}} > 0$, *then* $S''^{f,\alpha} \subset S_{\theta''}^{f,\alpha}$.

Proof. Suppose first that $\liminf_r q_r > 1$ and $\liminf_s q_s > 1$; then there exists $a, b > 0$ such that $q_r \geq 1 + a$ and $q_s \geq 1 + b$ for sufficiently large r and s, which implies that

$$\frac{h_r}{k_r} \geq \frac{a}{1+a} \Longrightarrow \left(\frac{h_r}{k_r}\right)^{\alpha} \geq \left(\frac{a}{1+a}\right)^{\alpha}$$

and

$$\frac{\overline{h}_s}{l_s} \geq \frac{b}{1+b} \Longrightarrow \left(\frac{\overline{h}_s}{l_s}\right)^{\alpha} \geq \left(\frac{b}{1+b}\right)^{\alpha}.$$

If $S''^{f,\alpha} - \lim x_{k,l} = \ell$, then for every $\varepsilon > 0$ and for sufficiently large r and s, we have

$$
\begin{aligned}
& \frac{1}{\left[f\left(k_r l_s\right)\right]^\alpha} f\left(\left|\left\{k \leq k_r, l \leq l_s:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right) \\
\geq & \frac{1}{\left[f\left(k_r l_s\right)\right]^\alpha} f\left(\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right) \\
= & \frac{\left[f\left(h_{r,s}\right)\right]^\alpha}{\left[f\left(k_r l_s\right)\right]^\alpha} \frac{1}{\left[f\left(h_{r,s}\right)\right]^\alpha} f\left(\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right) \\
= & \frac{\left[f\left(h_{r,s}\right)\right]^\alpha}{\left[h_{r,s}\right]^\alpha} \frac{k_r^\alpha}{\left[f\left(k_r l_s\right)\right]^\alpha} \frac{\left[h_{r,s}\right]^\alpha}{k_r^\alpha} \frac{f\left(\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right)}{\left[f\left(h_{r,s}\right)\right]^\alpha} \\
= & \frac{\left[f\left(h_{r,s}\right)\right]^\alpha}{\left[h_{r,s}\right]^\alpha} \frac{k_r^\alpha l_s^\alpha}{\left[f\left(k_r l_s\right)\right]^\alpha} \frac{h_r^\alpha \overline{h}_s^\alpha}{k_r^\alpha l_s^\alpha} \frac{f\left(\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right)}{\left[f\left(h_{r,s}\right)\right]^\alpha} \\
\geq & \frac{\left[f\left(h_{r,s}\right)\right]^\alpha}{\left[h_{r,s}\right]^\alpha} \frac{\left(k_r l_s\right)^\alpha}{\left[f\left(k_r l_s\right)\right]^\alpha}\left(\frac{a}{1+a}\right)^\alpha\left(\frac{b}{1+b}\right)^\alpha \frac{f\left(\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right|\right)}{\left[f\left(h_{r,s}\right)\right]^\alpha} .
\end{aligned}
$$

This proves the sufficiency. □

Theorem 2.6. *Let f be an unbounded modulus, $\theta=(k_r)$ and $\theta'=(l_s)$ be two lacunary sequences, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence and $0<\alpha\leq 1$. If $S_{f,\theta}^\alpha-\lim x_k=\ell$ and $S_{f,\theta'}^\alpha-\lim x_l=\ell$, then $S_{f,\theta''}^\alpha-\lim x_{k,l}=\ell$.*

Proof. Suppose $S_{f,\theta}^\alpha-\lim x_k=\ell$ and $S_{f,\theta'}^\alpha-\lim x_l=\ell$. Then for $\varepsilon>0$ we can write

$$
\lim_r \frac{1}{\left[f\left(h_r\right)\right]^\alpha}\left|\left\{k \in I_r:\left|x_k-\ell\right| \geq \varepsilon\right\}\right|=0
$$

and

$$
\lim_s \frac{1}{\left[f\left(\overline{h}_s\right)\right]^\alpha}\left|\left\{l \in I_s:\left|x_l-\ell\right| \geq \varepsilon\right\}\right|=0 .
$$

So we have

$$
\begin{aligned}
& \frac{1}{\left[f\left(h_{r,s}\right)\right]^\alpha}\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right| \\
\leq & \frac{1}{\left[c f\left(h_r\right) f\left(\overline{h}_s\right)\right]^\alpha}\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right| \\
\leq & \frac{1}{c^\alpha\left[f\left(h_r\right)\right]^\alpha\left[f\left(\overline{h}_s\right)\right]^\alpha}\left|\left\{(k,l) \in I_{r,s}:\left|x_{k,l}-\ell\right| \geq \varepsilon\right\}\right| \\
\leq & \left[\frac{1}{\left[f\left(h_r\right)\right]^\alpha}\left|\left\{k \in I_r:\left|x_k-\ell\right| \geq \varepsilon\right\}\right|\right]\left[\frac{1}{\left[f\left(\overline{h}_s\right)\right]^\alpha}\left|\left\{l \in I_s:\left|x_l-\ell\right| \geq \varepsilon\right\}\right|\right] .
\end{aligned}
$$

Hence $S_{f,\theta''}^\alpha-\lim x_{k,l}=\ell$. □

Theorem 2.7. *Let f be an unbounded modulus. If $\lim p_k > 0$, then $w_{\theta''}^{f,\alpha}(p) - \lim x_{k,l} = \ell$ uniquely.*

Proof. Let $\lim p_k = s > 0$. Assume that $w_{\theta''}^{f,\alpha}(p) - \lim x_{k,l} = \ell_1$ and $w_{\theta''}^{f,\alpha}(p) - \lim x_{k,l} = \ell_2$. Then

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} = 0,$$

and

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} = 0.$$

By definition of f, we have

$$\begin{aligned} & \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|\ell_1 - \ell_2|)]^{p_k} \\ \leq\ & \frac{D}{[f(h_{r,s})]^{\alpha}} \left(\sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} + \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} \right) \\ =\ & \frac{D}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_1|)]^{p_k} + \frac{D}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l} - \ell_2|)]^{p_k} \end{aligned}$$

where $\sup_k p_k = H$ and $D = \max\left(1, 2^{H-1}\right)$. Hence

$$\lim_{r,s} \frac{1}{[f(h_{r,s})]^{\alpha}} \sum_{(k,l)\in I_{r,s}} [f(|\ell_1 - \ell_2|)]^{p_k} = 0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $\ell_1 - \ell_2 = 0$. Thus the limit is unique. $\square$

Theorem 2.8. *Let $\theta_1'' = \{(k_r, l_s)\}$ and $\theta_2'' = \{(s_r, t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$ and α_1, α_2 two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If*

$$\lim_{r,s\to\infty} \inf \frac{[f(h_{r,s})]^{\alpha_1}}{[f(\ell_{r,s})]^{\alpha_2}} > 0 \tag{2.1}$$

then $w_{\theta_2''}^{f,\alpha_2}(p) \subset w_{\theta_1''}^{f,\alpha_1}(p)$, where $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r$ and $l_{s-1} < l \leq l_s\}$, $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h}_s$ and $J_{r,s} = \{(s,t) : s_{r-1} < s \leq s_r$ and $t_{s-1} < l \leq t_s\}$, $s_{r,s} = s_r t_s$, $\ell_{r,s} = \ell_r \overline{\ell}_s$.

Proof. Let $x \in w_{\theta_2''}^{f,\alpha_2}(p)$. We can write

$$\begin{aligned}\frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in J_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} &= \frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in J_{r,s}-I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} \\ &+ \frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} \\ &\geq \frac{1}{[f(\ell_{r,s})]^{\alpha_2}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k} \\ &\geq \frac{[f(h_{r,s})]^{\alpha_1}}{[f(\ell_{r,s})]^{\alpha_2}} \frac{1}{[f(h_{r,s})]^{\alpha_1}} \sum_{(k,l)\in I_{r,s}} [f(|x_{k,l}-\ell|)]^{p_k}.\end{aligned}$$

Thus if $x \in w_{\theta_2''}^{f,\alpha_2}(p)$, then $x \in w_{\theta_1''}^{f,\alpha_1}(p)$. $\square$

From Theorem 2.8. we have the following results.

Corollary 2.1. *Let $\theta_1'' = \{(k_r, l_s)\}$ and $\theta_2'' = \{(s_r, t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$ and α_1, α_2 two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If (2.1) holds then*

(*i*) $w_{\theta_2''}^{f,\alpha}(p) \subset w_{\theta_1''}^{f,\alpha}(p)$, *if* $\alpha_1 = \alpha_2 = \alpha$,

(*ii*) $w_{\theta_2''}^{f}(p) \subset w_{\theta_1''}^{f,\alpha_1}(p)$, *if* $\alpha_2 = 1$,

(*iii*) $w_{\theta_2''}^{f}(p) \subset w_{\theta_1''}^{f}(p)$, *if* $\alpha_1 = \alpha_2 = 1$.

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ON STAR COLORING OF DEGREE SPLITTING OF COMB PRODUCT GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring in which every path on four vertices in G is not bi-colored. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G. Let $G=(V,E)$ be a graph with $V=S_1\cup S_2\cup S_3\cup\ldots\cup S_t\cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T=V(G)-\bigcup_{i=1}^{t}S_i$. The degree splitting graph $DS(G)$ is obtained by adding vertices $w_1,w_2,\ldots w_t$ and joining w_i to each vertex of S_i for $1\le i\le t$. The comb product between two graphs G and H, denoted by $G\triangleright H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i^{th} copy of H at the vertex o to the i^{th} vertex of G. In this paper, we give the exact value of star chromatic number of degree splitting of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph.

Keywords: Star coloring; degree splitting graph; comb product

1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph in [4, 5, 10]. The concept of star chromatic number was introduced by Branko Grunbaum in 1973. A star coloring [1, 8, 9] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraph formed by the vertices of any two colors has connected components that are star graph. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

Guillaume Fertin et al. [8] determined the star chromatic number of trees, cycles, complete bipartite graphs, outer planar graphs and 2-dimensional grids. They also

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investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, graphs with bounded treewidth and cubic graphs and planar graphs with high - girth.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. Coleman et al. [6] proved that star coloring remains NP-hard problem even on bipartite graphs.

For a given graph $G = (V(G), E(G))$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \ldots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. The degree splitting graph [11, 12] of G, denoted by $DS(G)$, is obtained by adding vertices $w_1, w_2, \ldots w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.

Comb product is also same as the hierarchical product graphs was first introduced by Barriére et al. [3] in 2009. Also, the exact value of metric dimension of hierarchical product graphs was obtained by Tavakoli et al. in [14]. Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H, denoted by $G \triangleright H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i^{th} copy of H at the vertex o to the i^{th} vertex of G. By the definition of comb product, we can say that $V(G \triangleright H) = \{(a, u) \mid a \in V(G), u \in V(H)\}$ and $(a, u)(b, v) \in E(G \triangleright H)$ whenever $a = b$ and $uv \in E(H)$, or $ab \in E(G)$ and $u = v = o$. Ridho Alfarisi et al. [2] determined the partition dimension of comb product of path and complete graph and in [7] they also determined the star partition dimension of comb product of cycle and complete graph. Saputro et al. showed the metric dimension of comb product of the connected graphs G and H in [13].

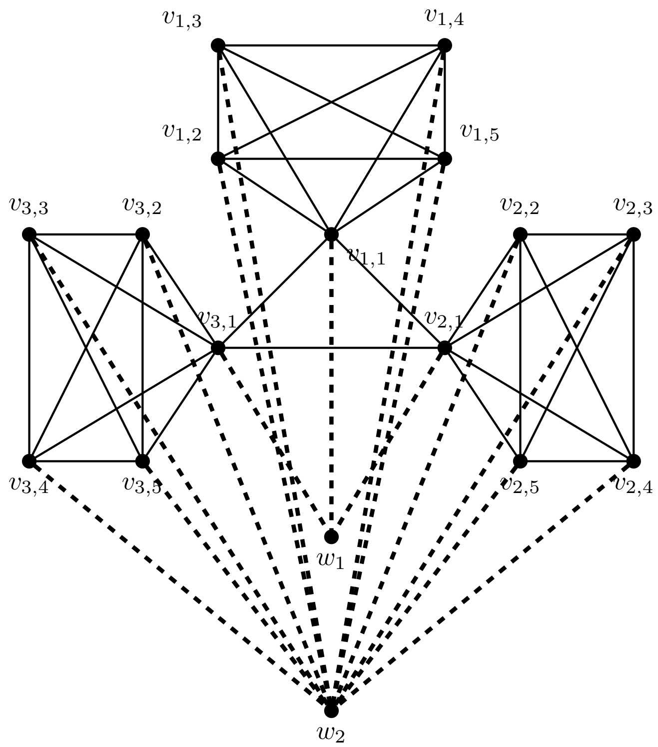

Figure 1: $DS(K_3 \triangleright K_5)$

In this paper, we have given the exact value of star chromatic number of degree splitting graph of comb product of complete graph with complete graph, complete graph with path, complete graph with cycle, complete graph with star graph, cycle with complete graph, path with complete graph and cycle with path graph denoted by $DS(K_m \triangleright K_n)$, $DS(K_m \triangleright P_n)$, $DS(K_m \triangleright C_n)$, $DS(K_m \triangleright K_{1,n})$, $DS(C_m \triangleright K_n)$, $DS(P_m \triangleright K_n)$ and $DS(C_m \triangleright P_n)$ respectively.

In order to prove our results, we shall make use of the following theorem by Guillaume et al. [8].

Theorem 1.1. *[8] If C_n is a cycle with $n \geq 3$ vertices, then*

$$\chi_s(C_n) = \begin{cases} 4, & \textit{when} \quad n = 5 \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. The proof of the theorem can be found in [8]. □

2. Main Results

In the following subsections, we will find the star chromatic number of degree splitting graph of comb product of complete with complete graph, complete with path, complete with cycle, complete with star graph, comb product of cycle with complete, path with complete and cycle with path graph denoted by $DS\left(K_m \triangleright K_n\right)$, $DS\left(K_m \triangleright C_n\right)$, $DS\left(K_m \triangleright K_{1,n}\right)$, $DS\left(K_m \triangleright P_n\right)$, $DS\left(C_m \triangleright K_n\right)$, $DS\left(P_m \triangleright K_n\right)$ and $DS\left(C_m \triangleright P_n\right)$ respectively. Figure 1 shows an example of degree splitting of comb product $\left(K_3 \triangleright K_5\right)$.

2.1. Star Coloring of Degree Splitting of $\left(K_m \triangleright K_n\right)$

The comb product between K_m and K_n, denoted by $K_m \triangleright K_n$ has vertex set

$$V\left(K_m \triangleright K_n\right) = \left\{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\right\}$$

and edge set

$$\begin{aligned} E\left(K_m \triangleright K_n\right) &= \left\{v_{i,1}v_{i+k,1} : 1 \leq i \leq m, 1 \leq k \leq m-i\right\} \\ &\bigcup \left\{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n-j\right\}. \end{aligned}$$

Thus

$$\left|V\left(K_m \triangleright K_n\right)\right| = mn$$

and

$$\left|E\left(K_m \triangleright K_n\right)\right| = \frac{mn(n-1) + m(m-1)}{2}.$$

Theorem 2.1. *Let K_m and K_n be two complete graphs of order $m, n \geq 3$ and $m \leq n$, then*

$$\chi_s\left(DS\left(K_m \triangleright K_n\right)\right) = m + n.$$

Proof. We have,

$$V\left(K_m \triangleright K_n\right) = \left\{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\right\} = S_1 \cup S_2$$

where

$$S_1 = \left\{v_{i,1} : 1 \leq i \leq m\right\}$$

and

$$S_2 = \left\{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\right\}.$$

To obtain $DS\left(K_m \triangleright K_n\right)$ from $K_m \triangleright K_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus, we get $V\left(DS\left(K_m \triangleright K_n\right)\right) = V\left(K_m \triangleright K_n\right) \cup \left\{w_1, w_2\right\}$. First we find the upper bound for $\chi_s\left(DS\left(K_m \triangleright K_n\right)\right)$.

Clearly, $m + n$ colors are needed at least to star color $DS\left(K_m \triangleright K_n\right)$. We now distinguish n as three cases: For every $1 \leq i \leq m$, Case(i): When $n \equiv 3(\text{mod } 3)$.

$$\sigma(v_{i,3k-2}) = i + j - 1, for \;\; 1 \le k \le \frac{n}{3}$$

$$\sigma(v_{i,3k-1}) = i + j - 1, for \;\; 1 \le k \le \frac{n}{3}$$

$$\sigma(v_{i,3k}) = i + j - 1, for \;\; 1 \le k \le \frac{n}{3}$$

and

$$\sigma(w_1) = \sigma(w_2) = m + n$$

Case(ii): When $n \equiv 1(\text{mod } 3)$.

$$\sigma(v_{i,3k-2}) = i + j - 1, for \;\; 1 \le k \le \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1, for \;\; 1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor$$

$$\sigma(v_{i,3k}) = i + j - 1, for \;\; 1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = m + n$$

Case(iii): When $n \equiv 2(\text{mod } 3)$.

$$\sigma(v_{i,3k-2}) = i + j - 1, for \;\; 1 \le k \le \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1, for \;\; 1 \le k \le \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k}) = i + j - 1, for \;\; 1 \le k \le \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = m + n.$$

Thus, the upper bound for the star coloring of $(DS(K_m \rhd K_n)) \le m + n$.

Now, we prove the lower bound for $\chi_s(DS(K_m \rhd K_n))$.

Suppose $\chi_s(DS(K_m \rhd K_n)) < m + n$. Let $\chi_s(DS(K_m \rhd K_n)) = m + n - 1$, then there exists a bicolored path P_4. Since $\{v_{1,i}\}$ induce a clique of order n (say K_n). If we assign the same n colors to the second copy of K_n, then we get a path on four vertices between these clique which is bicolored, a contradiction for proper star coloring. Thus, $\chi_s(DS(K_m \rhd K_n)) = m+n-1$ color is impossible. Therefore, the lower bound of $\chi_s(DS(K_m \rhd K_n)) \ge m + n$. Thus we get the lower and upper bound of $\chi_s(DS(K_m \rhd K_n))$. Hence $\chi_s(DS(K_m \rhd K_n)) = m + n$. This completes the proof of the theorem. $\square$

2.2. Star Coloring of Degree Splitting of $(K_m \triangleright C_n)$

A graph $K_m \triangleright C_n$ has vertex set

$$V(K_m \triangleright C_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} |E(K_m \triangleright C_n)| &= \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m, 1 \leq k \leq m-i\} \\ &\bigcup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \bigcup \{v_{m,1}v_{1,1}\}. \end{aligned}$$

Thus

$$|V(K_m \triangleright C_n)| = mn$$

and

$$|E(K_m \triangleright C_n)| = \frac{m(m-1)+2mn}{2}.$$

Theorem 2.2. *Let K_m and C_n be two connected graphs of order $m \geq 4$ and $n \geq 5$, then*

$$\chi_s(DS(K_m \triangleright C_n)) = m+1.$$

Proof. We have

$$V(K_m \triangleright C_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\}$$

and

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.$$

To obtain $DS(K_m \triangleright C_n)$ from $K_m \triangleright C_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus we get

$$V(DS(K_m \triangleright C_n)) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \bigcup \{w_1, w_2\}.$$

We first prove the lower bound for the star chromatic number of degree splitting of comb product of complete graph with cycle. For this, we show that any coloring with m colors will give us at least one bicolored path of length 4. Since each $\{v_{i,1} : 1 \leq i \leq m\}$ is adjacent to w_1, it gives a complete graph of order $m+1$. Thus, no coloring that uses m colors can be a star coloring. Therefore, the lower bound of star chromatic number is $\chi_s(DS(K_m \triangleright C_n)) \geq m+1$.

Now, we prove the upper bound for the star chromatic number of degree splitting of $(K_m \triangleright C_n)$. Since the complete graph has the chromatic number m. We assign the m colors to the mn vertices of the graph $K_m \triangleright C_n$ alternatively and we assign $\sigma(w_1) = \sigma(w_2) = m+1$. Thus the upper bound of the $\chi_s(DS(K_m \triangleright C_n)) \leq m+1$.

Thus we get the lower and upper bound of the $\chi_s(DS(K_m \triangleright C_n))$. Therefore,

$$\chi_s(DS(K_m \triangleright C_n)) = m+1.$$

This concludes the proof of the theorem. $\square$

2.3. Star Coloring of Degree Splitting of $(K_m \triangleright P_n)$

A graph $K_m \triangleright P_n$ has vertex set

$$V(K_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E(K_m \triangleright P_n) &= \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m, 1 \leq k \leq m-i\} \\ &\quad \bigcup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}. \end{aligned}$$

Thus

$$|V(K_m \triangleright P_n)| = mn$$

and

$$|E(K_m \triangleright P_n)| = \frac{m(m-1) + 2m(n-1)}{2}.$$

Theorem 2.3. *Let K_m be a complete graph of order $m \geq 3$ and P_n be a path graph of order $n \geq 3$ then,*

$$\chi_s(DS(K_m \triangleright P_n)) = m + 1.$$

Proof. We have

$$V(K_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\},$$

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n-1\}$$

and

$$S_3 = \{v_{i,n} : 1 \leq i \leq m\}.$$

To obtain $DS(K_m \triangleright P_n)$ from $K_m \triangleright P_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3 respectively. Thus, $V(DS(K_m \triangleright P_n)) = V(K_m \triangleright P_n) \cup \{w_1, w_2, w_3\}$. Now, we assign the following coloring pattern:

For every $1 \leq i \leq m$

For $n \equiv 1 (\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

$$\sigma(v_{i,3k}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1.$$

For $n \equiv 2(\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1.$$

For $n \equiv 3(\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k}) = i + j - 1(\text{mod } m), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = m + 1.$$

Thus the upper bound of star coloring of degree splitting of $(K_m \triangleright P_n) \leq m + 1$.

Now, we prove the lower bound of $\chi_s(DS(K_m \triangleright P_n))$. Suppose the lower bound of the

$$\chi_s(DS(K_m \triangleright P_n)) < m + 1.$$

That is

$$\chi_s(DS(K_m \triangleright P_n)) = m.$$

We must assign m colors for $\{v_{i,1}, 1 \leq i \leq m\}$ for proper star coloring. Since each $\{v_{i,1}\}$ is adjacent to w_1, it gives a complete graph of order $m + 1$. Therefore $\chi_s(DS(K_m \triangleright P_n))$ with m colors is impossible. Therefore $\chi_s(DS(K_m \triangleright P_n)) \geq m + 1$. Hence, $\chi_s((DS(K_m \triangleright P_n)) = m + 1$. This concludes the proof of the theorem. $\square$

2.4. Star Coloring of Degree Splitting of $(K_m \triangleright K_{1,n})$

A graph $K_m \triangleright K_{1,n}$ has a vertex set

$$V(K_m \triangleright K_{1,n}) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E(K_m \triangleright K_{1,n}) &= \{v_{i,1}v_{i+k,1} : 1 \leq i \leq m-1, 1 \leq k \leq m-i\} \\ & \bigcup \{v_{i,1}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

Thus

$$|V(K_m \triangleright K_{1,n})| = mn$$

and

$$|E(K_m \triangleright K_{1,n})| = \frac{m(m-1)+2mn}{2}.$$

Theorem 2.4. *Let K_m be a complete graph of order m, $(m \geq 3)$ and $K_{1,n}$ be a star graph with n + 1 vertices $(n \geq 2)$ then*

$$\chi_s(DS(K_m \triangleright K_{1,n})) = m+1.$$

Proof. We have

$$V(K_m \triangleright K_{1,n}) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2.$$

To obtain $DS(K_m \triangleright K_{1,n})$ from $(K_m \triangleright K_{1,n})$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively, where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\}$$

and

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.$$

Thus we get

$$V(DS(K_m \triangleright K_{1,n})) = V(K_m \triangleright K_{1,n}) \cup \{w_1, w_2\}.$$

Now we assign the coloring pattern as follows:
For every $1 \leq i \leq m$, assign i to $\sigma(v_{i,1})$, and
For $2 \leq j \leq n$ assign

$$\sigma(v_{i,j}) = \begin{cases} 2 \; if \; i \equiv 1 (\text{mod } 3) \\ 3 \; if \; i \equiv 2 (\text{mod } 3) \\ 1 \; if \; i \equiv 3 (\text{mod } 3) \end{cases}$$

alternatively

and

$$\sigma(w_1) = \sigma(w_2) = m + 1.$$

Thus $\chi_s((DS(K_m \triangleright K_{1,n})) \geq m + 1$.

Now, we prove the lower bound of $\chi_s(DS(K_m \triangleright K_{1,n}))$. Suppose the lower bound of the $\chi_s(DS(K_m \triangleright K_{1,n})) < m + 1$. That is $\chi_s(DS(K_m \triangleright K_{1,n})) = m$. We must assign m colors for $\{v_{i,1} : 1 \leq i \leq m\}$ for proper star coloring. Since each $\{v_{i,1}\}$ is adjacent to w_1, it gives a complete graph of order $m + 1$. Therefore $\chi_s(DS(K_m \triangleright K_{1,n}))$ with m color is impossible. Therefore $\chi_s(DS(K_m \triangleright K_{1,n})) \geq m + 1$. Thus we get the lower and upper bound of $\chi_s(DS(K_m \triangleright K_{1,n}))$. Hence, $\chi_s(DS(K_m \triangleright K_{1,n})) = m + 1$. It concludes the proof of the theorem. $\square$

2.5. Star Coloring of Degree Splitting of $(C_m \triangleright K_n)$

A graph $C_m \triangleright K_n$ has vertex set

$$V(C_m \triangleright K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E(C_m \triangleright K_n) &= \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m-1\} \\ &\bigcup \{v_{m,1}v_{1,1}\} \bigcup \{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n-j\}. \end{aligned}$$

Thus

$$|V(C_m \triangleright K_n)| = mn$$

and

$$|E(C_m \triangleright K_n)| = \frac{mn^2 - mn + 2m}{2}.$$

Theorem 2.5. *Let C_m and K_n be two connected graphs of order $m > n$ and $m > 3, n \geq 3$, then*

$$\chi_s(DS(C_m \triangleright K_n)) = m + 1.$$

Proof. We have

$$V(C_m \triangleright K_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} = S_1 \cup S_2$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\}$$

and

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.$$

To obtain $DS(C_m \triangleright K_n)$ from $C_m \triangleright K_n$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus we get $V(DS(C_m \triangleright K_n)) = V(C_m \triangleright K_n) \cup \{w_1, w_2\}$. First we find the upper bound for $\chi_s(DS(C_m \triangleright K_n))$.

We define the coloring pattern as follows:

For every $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\sigma\left(v_{i,j}\right) = i + j - 1(\text{mod } m)$$

and also

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = m + 1.$$

Thus the upper bound for star chromatic number of $(DS\left(C_m \triangleright K_n\right)) \leq\ m + 1$.

Now, we prove the lower bound for $\chi_s((DS\left(C_m \triangleright K_n\right))$.

Suppose the lower bound of $\chi_s((DS\left(C_m \triangleright K_n\right)) < m+1$. Let $\chi_s((DS\left(C_m \triangleright K_n\right)) = m$, then there exist a bicolored path P_4. Since $\{v_{1,i}\}$ induce a clique of order n. If we assign the same n colors to the second copy of the clique, then we get a path on four vertices between these cliques which is bicolored, a contradiction for proper star coloring. Thus we obtain $\chi_s((DS\left(C_m \triangleright K_n\right)) = m$ color is impossible. It concludes that the lower bound is $\chi_s((DS\left(C_m \triangleright K_n\right)) \geq m + 1$. Therefore, $\chi_s((DS\left(C_m \triangleright K_n\right)) = m + 1$. Hence the proof of the theorem. □

2.6. Star Coloring of Degree Splitting of $(P_m \triangleright K_n)$

A graph $P_m \triangleright K_n$ has a vertex set

$$V\left(P_m \triangleright K_n\right) = \left\{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\right\}$$

and edge set

$$\begin{aligned} E\left(P_m \triangleright K_n\right) &= \left\{v_{i,1}v_{i+1,1} : 1 \leq i \leq m - 1\right\} \\ &\bigcup \left\{v_{i,j}v_{i,j+k} : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq n - j\right\}. \end{aligned}$$

Thus

$$\left|V\left(P_m \triangleright K_n\right)\right| = mn$$

and

$$\left|E\left(P_m \triangleright K_n\right)\right| = \frac{mn(n-1) + 2(m-1)}{2}.$$

Theorem 2.6. *Let P_m be a path graph of order $m \geq 4$ and K_n be a complete graph with $n \geq 2$, then*

$$\chi_s\left(DS\left(P_m \triangleright K_n\right)\right) = n + 2.$$

Proof. We have

$$V\left(P_m \triangleright K_n\right) = \left\{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\right\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \left\{v_{1,1}, v_{m,1}\right\},$$

$$S_2 = \left\{v_{i,1} : 2 \leq i \leq m - 1\right\}$$

and

$$S_3 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\}.$$

To obtain $DS(P_m \triangleright K_n)$ from $P_m \triangleright K_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3 respectively. Thus we get $V(DS(P_m \triangleright K_n)) = \{v_{i,j} : 1 \leq i \leq m; 1 \leq j \leq n\} \cup \{w_1, w_2, w_3\}$. First we find the upper bound for $\chi_s(DS(P_m \triangleright K_n))$. For every $1 \leq i \leq m$,

For $n \equiv 1 (\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$$

$$\sigma(v_{i,3k}) \ \ = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2.$$

For $n \equiv 2 (\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma(v_{i,3k}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2.$$

For $n \equiv 3 (\text{mod } 3)$

$$\sigma(v_{i,3k-2}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k-1}) = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma(v_{i,3k}) \ \ = i + j - 1(\text{mod } n+1), \ for \ \ 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = n + 2.$$

Thus $\chi_s(DS(P_m \triangleright K_n) \leq n + 2$.

Now, we prove that $\chi_s(DS\,(P_m \triangleright K_n) \geq n+2$. Suppose $\chi_s(DS\,(P_m \triangleright K_n) < n+2$. That is $\chi_s(DS\,(P_m \triangleright K_n) = n+1$. Since $\{v_{1,i}\}$ induce a clique of order n. If we assign the same n colors to the second copy of the clique, then we get a path on four vertices between these cliques which is bicolored, a contradiction for proper star coloring. Thus $\chi_s(DS\,(P_m \triangleright K_n) \geq n+1$. Therefore, $\chi_s(DS\,(P_m \triangleright K_n) = n+2$. Hence, there is a another proof to the theorem. □

2.7. Star Coloring of Degree Splitting of $(C_m \triangleright P_n)$

A graph $C_m \triangleright P_n$ has vertex set

$$V\,(C_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E\,(C_m \triangleright P_n) &= \{v_{i,1}v_{i+1,1} : 1 \leq i \leq m-1\} \\ &\bigcup \{v_{m,1}v_{1,1}\} \bigcup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}. \end{aligned}$$

Thus

$$|V\,(C_m \triangleright P_n)| = mn$$

and

$$|E\,(C_m \triangleright P_n)| = m + m(n-1).$$

Theorem 2.7. *Let C_m be a cycle of length $m \geq 3$ and P_n be a path of length $n \geq 3$ then,*

$$\chi_s\,(DS\,(C_m \triangleright P_n)) = \begin{cases} 4, if\ m = 3k, k \geq 1 \\ 5, otherwise \end{cases}.$$

Proof. We have

$$V\,(C_m \triangleright P_n) = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{v_{i,1} : 1 \leq i \leq m\},$$

$$S_2 = \{v_{i,j} : 1 \leq i \leq m, 2 \leq j \leq n-1\}$$

and

$$S_3 = \{v_{i,n} : 1 \leq i \leq m\}.$$

To obtain $DS\,(C_m \triangleright P_n)$ from $C_m \triangleright P_n$, we add three vertices w_1, w_2 and w_3 corresponding to S_1, S_2 and S_3 respectively. Thus we get $V\,(DS\,(C_m \triangleright P_n)) = V\,(C_m \triangleright P_n) \cup \{w_1, w_2, w_3\}$. First we find the upper bound for $\chi_s\,(DS\,(C_m \triangleright P_n))$.

The star chromatic number is defined as follows:

Case(i):

If $m = 3k$, $k \geq 1$

For $n \equiv 1(\text{mod } 3)$

$$\sigma\left(v_{i,3k-2}\right) = i(\text{mod } 3),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i+1(\text{mod } 3),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

$$\sigma\left(v_{i,3k}\right) = i+2(\text{mod } 3),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = \sigma\left(w_3\right) = 4.$$

For $n \equiv 2(\text{mod } 3)$

$$\sigma\left(v_{i,3k-2}\right) = i(\text{mod } 3),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k-1}\right) = i+1(\text{mod } 3),\ for\ \ 1 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$$

$$\sigma\left(v_{i,3k}\right) = i+2(\text{mod } 3),\ for\ \ 1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$$

and

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = \sigma\left(w_3\right) = 4.$$

For $n \equiv 3(\text{mod } 3)$

$$\sigma\left(v_{i,3k-2}\right) = i(\text{mod } 3),\ for\ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma\left(v_{i,3k-1}\right) = i+1(\text{mod } 3)\ for\ \ 1 \leq k \leq \frac{n}{3}$$

$$\sigma\left(v_{i,3k}\right) = i+2(\text{mod } 3),\ for\ \ 1 \leq k \leq \frac{n}{3}$$

and

$$\sigma\left(w_1\right) = \sigma\left(w_2\right) = \sigma\left(w_3\right) = 4.$$

Thus, $\chi_s\left(DS\left(C_m \triangleright P_n\right)\right) = 4$ if $m = 3k,\ k \geq 1$.

Case(ii)(a): When $m = 3k+1,\ k \in N$.

We color the $3k$ vertices of C_m by $\sigma\left(v_{i,1}\right) = i(\text{ mod } 3)$ and we assign the remains of one uncolored vertex by 4.

Also, we assign

$$\sigma\left(v_{i,j}\right) = i+j-1(\text{mod } 3).$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 5.$$

Case(ii)(b): When $m = 3(k-1)+2$, $k \in N$, here $m = 5$ is not included. That is $m = 3(k-1)+5$, $k \geq 2$.

We color the $3(k-1)$ vertices of C_m by 1, 2 and 3 and for the remaining five vertices assign the color 4, 1, 2, 3, 4.

Also, we assign

$$\sigma(v_{i,j}) = i + j - 1 (\text{mod } 3).$$

and

$$\sigma(w_1) = \sigma(w_2) = \sigma(w_3) = 5.$$

Thus, $\chi_s(DS(C_m \triangleright P_n)) = 5$.

When $m = 5$, then $\chi_s(DS(C_m \triangleright P_n)) = 5$. Hence, the theorem is proved. □

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HYPERBOLIC TYPE SOLUTIONS FOR THE COUPLE BOITI-LEON-PEMPINELLI SYSTEM

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Abstract. In this paper, the $(1/G')$-expansion method is proposed to construct hyperbolic type solutions of the nonlinear evolution equations. To asses the applicability and effectiveness of the method, two cases of the coupled Boiti-Leon-Pempinelli (CBLP) system have been investigated in this study. It is shown that, with the help of symbolic computation, the $(1/G')$-expansion method provides a powerful and straightforward mathematical tool for solving nonlinear partial differential equations.
Keywords: nonlinear evolution equations; partial differential equations; symbolic computation.

1. Introduction

Nonlinear evolution equations usually used to describe the nonlinear phenomena of waves in plasma physics, ocean engineering, quantum mechanics, fluid dynamics, solid state physics, hydrodynamics and many other branches of sciences and engineering. These types of equations have been used to describe the liquid flow containing gas bubbles, the propagation of waves, fluid flow in elastic oceans, rivers, tubes, lakes as well as a gravity waves in a smaller domain and Spatio-temporal rescaling of the nonlinear wave motion.
There are several approaches for finding solutions of nonlinear partial differential equations which have been developed and employed successfully. Some of these are a new sub equation method [1], homotopy analysis method [2, 3], homotopy-Pade method [4], homotopy perturbation method [5, 6], (G'/G)-expansion method [7, 8], modified variational iteration algorithm-I [9, 10, 11], sub equation method [12], Variational iteration method with an auxiliary parameter [13, 14, 15, 16], sumudu transform approach [17], $(1/G')$-expansion method [18, 19], variational iteration method [20, 21], auto-Bäcklund transformation method [22], Clarkson–Kruskal direct method [23], Bernoulli sub-equation function technique [24], decomposition

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method [25, 26, 27, 28], modified variational iteration algorithm-II [29, 30, 31], first integral method [32], homogeneous balance method [33], modified Kudryashov technique [34], residual power series approach [35], collocation method [36], extended rational SGEEM [37], sine-Gordon expansion method [38, 39] and many more [40, 41, 42, 43].

Consider the following coupled Boiti-Leon-Pempinelli System [44]

$$\begin{aligned} u_{ty} &= \left(u^2 - u_x\right)_{xy} + 2v_{xxx}, \\ v_t &= v_{xx} + 2uv_x. \end{aligned} \tag{1.1}$$

There have been numerous studies about the analytical treatment of CBLP System. In some of the studies, new traveling wave solutions of CBLP System have been attained utilizing the generalized (G'/G)-expansion method [44], while the analytic solutions of CBLP System have been obtained in [45].

In current work, we will construct the exact solutions of the CBLP System employing $(1/G')$-expansion method.

The remaining portion of this paper is as follows: In section 2, $(1/G')$-expansion method is elaborated, in section 3, $(1/G')$-expansion method's applications are discussed and utilized to obtain hyperbolic type solutions of the CBLP System, applicability and reliability of the proposed techniques are shown through 3D, contour and 2D graphics. The conclusion is discussed in the last section.

2. Description of the Method

Consider a general form of the following nonlinear PDE,

$$\sigma\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, ...\right) = 0. \tag{2.1}$$

Here, let $u = u(\xi) = u(x, y, t)$, $\quad \xi = x + y - ct$, $\quad c \neq 0$, where c is a constant and the speed of the wave. We can convert it into the following nODE for $u(\xi)$

$$\tau\left(u, -cu', u', u', u'', ...\right) = 0. \tag{2.2}$$

The solution of Eq. (2.2) is assumed to have the form

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \left(\frac{1}{G'}\right)^i, \tag{2.3}$$

whereas $a_i, \quad i = 0, 1, ..., n$ are nonzero constants, $G = G(\xi)$ provides the following second order IODE

$$G'' + \lambda G' + \mu = 0, \tag{2.4}$$

where μ and λ are constants to be determined after,

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + B\cosh[\xi\lambda] - B\sinh[\xi\lambda]}, \tag{2.5}$$

where B is integral constant. If the desired derivatives of the Eq. (2.3) are calculated and substituting in the Eq. (2.2), a polynomial with the argument $(1/G')$ is attained. An algebraic equation system is created by equalizing the coefficients of this polynomial to zero. The equation are solved using a package program and put into place in the default Eq. (2.2) solution function. Lastly, the solutions of Eq. (1.1) are found.

3. Solutions of CBLP System

The traveling wave transmutation $\xi = x + y - ct$, allows us to convert Eq. (1.1) into an ODE for $u = u(\xi)$

$$-cu'' = \left(u^2 - u'\right)'' + 2v, \tag{3.1}$$

$$-cv' = v'' + 2uv', \tag{3.2}$$

here by integrating twice the Eq. (3.1), we attain

$$v' = \frac{1}{2}u' - \frac{1}{2}cu - \frac{1}{2}u^2. \tag{3.3}$$

According to ξ in Eq. (3.3) and considering zero constants for integration, we attain

$$v = \frac{1}{2}u - \frac{1}{2}\int \left(cu + u^2\right) d\xi. \tag{3.4}$$

Replacing Eq. (3.3) into the Eq. (3.2),

$$u'' - 2u^3 - 3cu^2 - c^2u = 0. \tag{3.5}$$

In Eq. (3.5), we get balancing term $n = 1$ and in Eq.(2.3), the following situation is obtained:

$$u(\xi) = a_0 + a_1\left(\frac{1}{G'}\right), \quad a_1 \neq 0. \tag{3.6}$$

Replacing Eq. (3.6) into Eq. (3.5) and the coefficients of the algebraic Eq. (1.1) are equal to zero, can find the following algebraic equation systems

$$\begin{aligned}
&Const &&: -c^2a_0 - 3ca_0^2 - 2a_0^3 = 0,\\
&\left(\tfrac{1}{G'[\xi]}\right)^1 &&: -c^2a_1 + \lambda^2 a_1 - 6ca_0a_1 - 6a_0^2a_1 = 0,\\
&\left(\tfrac{1}{G'[\xi]}\right)^2 &&: 3\lambda\mu a_1 - 3ca_1^2 - 6a_0a_1^2 = 0,\\
&\left(\tfrac{1}{G'[\xi]}\right)^3 &&: 2\mu^2a_1 - 2a_1^3 = 0.
\end{aligned} \tag{3.7}$$

Case1:

$$a_0 = 0,\ a_1 = -\mu,\ c = -\lambda, \tag{3.8}$$

replacing Eq.(3.8) into the Eq.(3.6) and the following hyperbolic type solutions is obtained for Eq. (1.1):

$$u_1(x,y,t) = -\frac{\mu}{-\frac{\mu}{\lambda} + B\cosh\left[\lambda\left(t\lambda + x + y\right)\right] - B\sinh\left[\lambda\left(t\lambda + x + y\right)\right]}, \tag{3.9}$$

$$\begin{aligned} v_1(x,y,t) = \frac{1}{4\mu}(\lambda\mu &\left(\lambda(t\lambda+x+y) + 2\log\left[\begin{array}{l} (-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right] \\ +(B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right] \end{array}\right]\right) \\ &-\lambda\mu(\lambda(t\lambda+x+y) + 2\log\left[\begin{array}{l} (-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right] \\ +(B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right] \end{array}\right] \\ &-\frac{4B\lambda\mu\sinh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right]}{(A\lambda-\mu)\left(\begin{array}{l} (-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right] \\ +(B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(t\lambda+x+y)\right] \end{array}\right)})) \\ &-\frac{\mu}{2\left(-\frac{\mu}{\lambda}+B\cosh[\lambda(t\lambda+x+y)]-B\sinh[\lambda(t\lambda+x+y)]\right)}. \end{aligned} \tag{3.10}$$

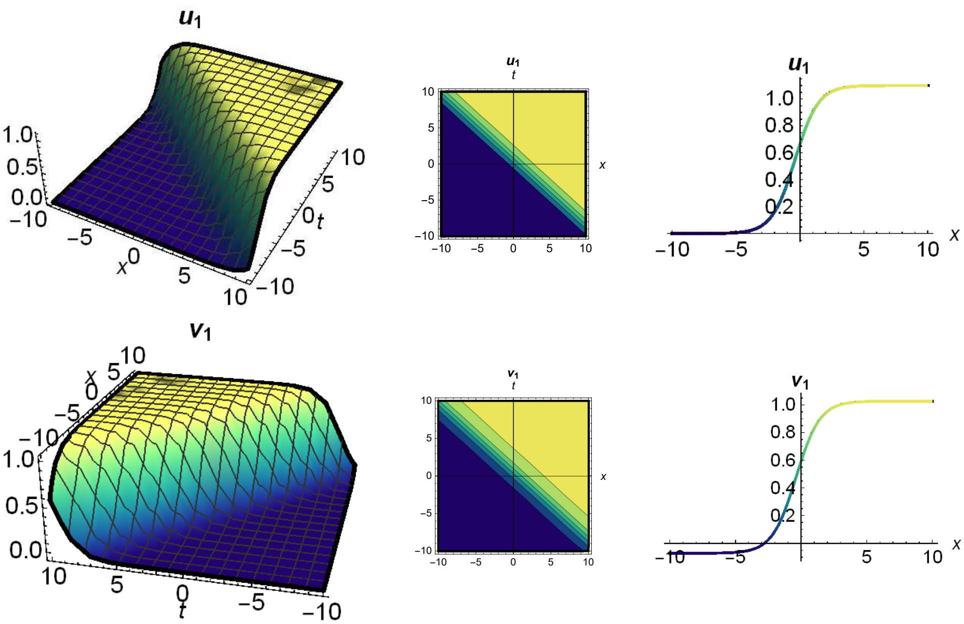

FIG. 3.1: 3D, contour and 2D graphs respectively for $B = 0.6,\ \mu = -0.1,\ y = 1,\ \lambda = 1.1$ values of Eqs. (3.9) and (3.10).

Case 2:

$$a_0 = -\lambda,\ a_1 = -\mu,\ c = \lambda, \tag{3.11}$$

replacing values Eq. (3.11) into Eq. (3.6) and the following hyperbolic type solutions are obtained for Eq. (1.1):

$$u_2(x,y,t) = -\lambda - \frac{\mu}{-\frac{\mu}{\lambda} + B\cosh\left[\lambda\left(x - t\lambda + y\right)\right] - B\sinh\left[\lambda\left(x - t\lambda + y\right)\right]}, \tag{3.12}$$

$$
\begin{aligned}
&v_2(x,y,t) = \tfrac{1}{2}(\lambda^2(-x-y+t\lambda)\\
&+\tfrac{1}{\mu}\lambda\left(\lambda(x-t\lambda+y)\mu+\mu\left(\lambda(x-t\lambda+y)+2\log\begin{bmatrix}(-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(x+y-t\lambda)\right]+\\ (B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(x+y-t\lambda)\right]\end{bmatrix}\right)\right)\\
&\quad+\tfrac{1}{2\mu}(-\lambda\mu\left(\lambda(x-t\lambda+y)+2\log\begin{bmatrix}(-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]\\ +(B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]\end{bmatrix}\right)\\
&\quad-\lambda\mu(\lambda(x-t\lambda+y)+2\log\begin{bmatrix}(-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]\\ +(B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]\end{bmatrix}\\
&\quad-\frac{4B\lambda\mu\sinh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]}{(B\lambda-\mu)\begin{pmatrix}(-B\lambda+\mu)\cosh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]\\ +(B\lambda+\mu)\sinh\left[\frac{1}{2}\lambda(x-t\lambda+y)\right]\end{pmatrix}})))\\
&\quad+\tfrac{1}{2}\left(-\lambda-\frac{\mu}{-\frac{\mu}{\lambda}+B\cosh[\lambda(x-t\lambda+y)]-B\sinh[\lambda(x-t\lambda+y)]}\right).
\end{aligned}
$$

(3.13)

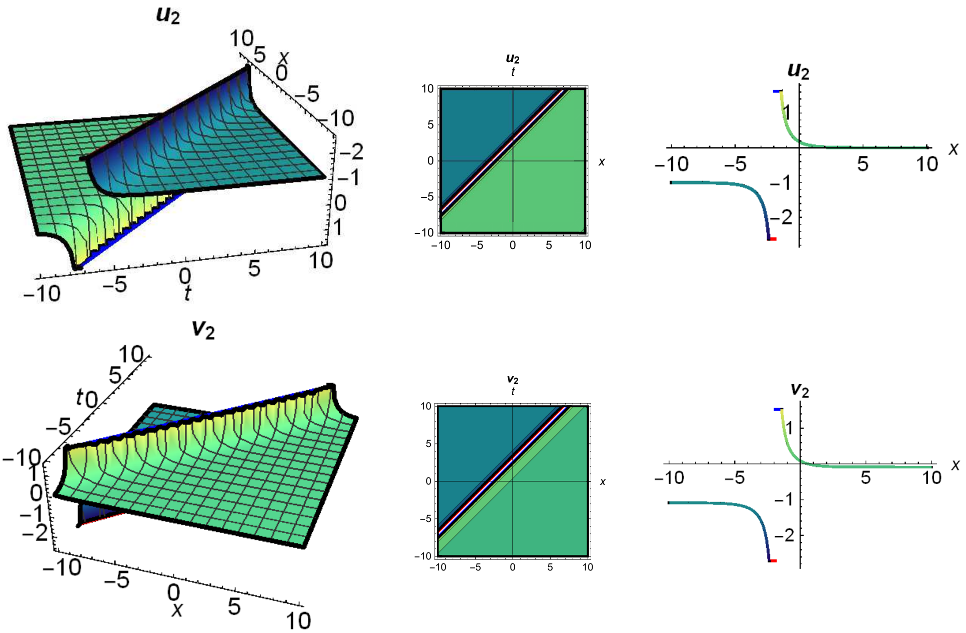

FIG. 3.2: 3D, contour and 2D graphs respectively for $B = 0.6$, $\mu = -0.1$, $y = 1$, $\lambda = 1.1$ values of Eqs. (3.12) and (3.13).

4. Conclusion

In this work, we have achieved hyperbolic type exact solutions of the CBLP System with the help of $(1/G')$-expansion method. Computer technology utilized in the construction of 3D, 2D and contour graphics of the obtained solutions. The CBLP System, which plays an important role in mathematical physics,has been investigated analytically for the effectiveness and reliability of the proposed method.

Furthermore, the applied method is an effective, powerful method and can be used to establish new exact solutions of many other nonlinear partial differential equations arising in applied sciences and engineering.

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NEW UPPER BOUND ON THE LARGEST LAPLACIAN EIGENVALUE OF GRAPHS

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Abstract. Let $G = (V, E)$ be a simple, undirected graph with maximum and minimum degree Δ and δ respectively, and let A be the adjacency matrix and Q be the Laplacian matrix of G. In the past decades, the Laplacian spectrum has received much more attention, since it has been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning. In this paper, we will compute lower and upper bounds for the largest Laplacian eigenvalue which is related to a given maximum and minimum degree and a given number of vertices and edges. We will also compare our results in this paper with other published results.

Keywords: Laplacian matrix; Laplacian spectrum; Laplacian eigenvalue; adjacency matrix.

1. Introduction

Let $G = (V, E)$ be a simple graph (i.e. finite, undirected graph without loops or multiple edges) on vertex set $V = \{v_1, ..., v_n\}$ and edge set $E = \{e_1, ..., e_m\}$ (so $n =\mid V(G) \mid$ is its order, and $m =\mid E(G) \mid$ is its size). For $v_i \in V(G)$, the degree of v_i, written by $d(v_i)$ or d_i , is the number of edges incident with v. Let $\Delta = \max\ \{d_i : v_i \in V(G)\}$ and $\delta = \min\ \{d_i : v_i \in V(G)\}$. Spectral graph theory [1, 2, 3] studies properties of graphs using the spectrum of related matrices. The most studied matrix associated with G appears to be the adjacency matrix $A = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. Another much studied matrix is the Laplacian matrix, defined by $Q(G) = D(G) - A(G)$, where $D(G) = diag(d_1, d_2, \ldots, d_n)$ (see [4, 5, 6]). Notice also that $Q = C^T C$, where C is the matrix whose rows are indexed by the edges of G and whose columns are indexed by its vertices, in which each row corresponding to the edge $e = \{u, v\}$,

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$(u < v)$, has a (1) in the column corresponding to u, a (-1) in that corresponding to v and 0 in every other place. Therefore Q is a symmetric, positive semi-definite matrix.

For an $n \times n$ real symmetric matrix M, its eigenvalues are real numbers. The eigenvalues (or spectrum) of $A(G)$ and $Q(G)$ which are real eigenvalues, are called A-eigenvalues (or A-spectrum) Q-eigenvalues (or Q-spectrum) respectively. These eigenvalues will be denoted by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant ... \geqslant \lambda_n(G)$ and $\mu = \mu_1(G) \geqslant \mu_2(G) \geqslant ... \geqslant \mu_n(G) = 0$ respectively.

2. Application

Applications of eigenvalue methods in combinatorics, graph theory and in combinatorial optimization have a long history. For example, eigenvalue bounds on the chromatic number were formulated by Wilf [7] and Homan [8] at the end of the sixties. Historically, the next applications related to combinatorial optimization, according to Fiedler [9] and Donath and Hoffman [10] in 1973, concerned the area of graph partition. A very important use of eigenvalues is the Lovász notion of the theta-function from 1979 [11]. Using it, he solved the long standing Shannon capacity problem for the 5-cycle. The theta-function provides the only known way to compute the chromatic number of perfect graphs in polynomial time.

The next important result was the use of eigenvalues in the construction of superconcentrators and expanders by Alon and Milman [12] in 1985. Their work motivated the study of eigenvalues of random regular graphs. Eigenvalues of random 01-matrices had already been studied by F. Juhász, who analyzed the behavior of the theta-function on random graphs, and introduced eigenvalues in clustering [13]. Isoperimetric properties of graphs and their eigenvalues play a crucial role in the design of several randomized algorithms. These applications are based on the so-called rapidly mixing Markov chains. The most important discoveries in this area include random polynomial time algorithms for approximating the volume of a convex body (cf., e.g., [14, 15, 16]), polynomial time algorithms for approximate counting (e.g., approximating the permanent or counting the number of perfect matchings, see [17] for additional information), etc. Isoperimetric properties and related expansion properties of graphs are the basis for various other applications, ranging from the fast convergence of Markov chains, efficient approximation algorithms, randomized or derandomized algorithms, complexity lower bounds, and building efficient communication networks and networks for parallel computation.

There are several known results that relate μ and to various structural properties of the graph G. In particular, there is a correspondence between μ and the expansion properties of G. Expander graphs have been widely used in Computer Science, in areas ranging from parallel computation to complexity theory and cryptography. See, e.g. [18]. In view of this correspondence, it is interesting to study the maximum possible value of μ for a graph with a given maximum and minimum degree and a given number of vertices and edges.

3. Main Results

There are some known results for upper bounds of μ. Research on the bound involving eigenvalues of A, Q has attracted much attention [19, 20]. In 1985, Anderson and Morley gave an upper bound for largest Laplasian graph eigenvalue in [21]. In 1997, Li and Zhang [22] improved researches of Anderson and Morley. In 1998, Merris [23] showed an upper bound of μ. In 1998, Li and Zhang [24] improved the researches of Merris. In 2000, Rojo et al. [25] obtained an always-nontrivial bound. In 2002, Pan [26] improved researches of Li and Zhang. In 2003, Das [27] improved the bound of Merris. In 2010, Dongmei Zhu gave a new upper bound in [28]. In the following part, we will compute lower and upper bounds for the largest Laplacian eigenvalue of G which is related with given a maximum and minimum degree and a given number of vertices and edges. We have also compared our results with other relevant results.

Theorem 1: Let G be a graph with n vertices and m edges. Then,

$$\mu \geqslant \frac{2m}{n-1} - \sqrt{\frac{n-2}{n-1}(\sum_{i=1}^{n}(d_i)^2 + 2m) - \frac{4m^2}{n-1} + \frac{4m^2}{(n-1)^2}}. \tag{3.1}$$

Proof. Let $\mu = \mu_1(G) \geqslant \mu_2(G) \geqslant ... \geqslant \mu_n(G) = 0$ be eigenvalues of Laplacian matrix of G. We know that

$$2m = \sum_{i=1}^{n} \mu_i, \tag{3.2}$$

and

$$\sum_{i=1}^{n} d_i^2 + 2m = \sum_{i=1}^{n} \mu_i^2. \tag{3.3}$$

Applying (3.2) and using the Cauchy-Schwarz inequality one can obtain

$$2m - \mu = \sum_{i=2}^{n-1} \mu_i^2 \leqslant \sqrt{n-2}\sqrt{\sum_{i=2}^{n-1} \mu_i^2}.$$

Raising both sides to power two and using (3.3), we obtain

$$\begin{aligned}(2m-\mu)^2 &\leqslant (n-2)\left(\sum_{i=1}^{n}\mu_i^2-\mu^2\right)\\ &= (n-2)\left(\sum_{i=1}^{n}d_i^2+2m-\mu^2\right).\end{aligned}$$

Thus

$$4m^2+\mu^2-4m\mu \leqslant (n-2)\left(\sum_{i=1}^{n}d_i^2+2m-\mu^2\right).$$

Therefore

$$(n-1)\mu^2-4m\mu \leqslant (n-2)\left(\sum_{i=1}^{n}d_i^2+2m\right)-4m^2.$$

Consequently,

$$\mu^2-(\tfrac{4m}{n-1})\mu \leqslant \tfrac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\tfrac{4m^2}{n-1}.$$

As a result, we have

$$(\mu-\tfrac{2m}{n-1})^2 \leqslant \tfrac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\tfrac{4m^2}{n-1}+\tfrac{4m^2}{(n-1)^2}.$$

Hence

$$\mu-\tfrac{2m}{n-1} \geqslant -\sqrt{\tfrac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\tfrac{4m^2}{n-1}+\tfrac{4m^2}{(n-1)^2}}.$$

Finally,

$$\mu \geqslant \frac{2m}{n-1}-\sqrt{\frac{n-2}{n-1}\left(\sum_{i=1}^{n}d_i^2+2m\right)-\frac{4m^2}{n-1}+\frac{4m^2}{(n-1)^2}}$$

we complete the proof. □

Theorem 2: Let G be a simple graph with n vertices and m edges, and Δ , δ be the maximum and minimum degree of G respectively. Then we have

$$\mu \leqslant \sqrt{2m-(n-1)-\delta^2+\left(\frac{2\Delta-1}{2}\right)^2}+\left(\frac{2\Delta-1}{2}\right). \tag{3.4}$$

Proof. Let $X=(x_1,x_2,\ldots,x_n)^T$ be the eigenvector of $Q(G)$ and $\|X\|^2=1$ corresponding to $\mu(G)$. Let Q_i denote the ith row of Q. Let $X(i)$ denote the vector obtained from X by replacing x_j with 0 if v_i is not adjacent to v_j and replacing x_j with $(-x_j)$ if v_i is adjacent to v_j. Since

$$Q(G)X = \mu(G)X,$$

and

$$Q_iX(i) = \sum_{a_{ij}=1} x_j,$$

it follows that

$$d_ix_i - \mu x_i = Q_iX(i).$$

Both sides of the above equation are brought to power two, which leads to

$$d_i^2x_i^2 + \mu^2x_i^2 - 2\mu d_ix_i^2 = \|Q_iX(i)\|^2.$$

On the other hand, by the Lagrange identity we have

$$\|Q_iX(i)\|^2 = \|Q_i\|^2\|X(i)\|^2 - d_i^2\left(\sum_{a_{ij}=1} x_j^2\right) - \sum_{\substack{1\leq k<j\leq n \\ a_{ij}=a_{ik}=1}} (x_j - x_k)^2.$$

We also have

$$\|Q_i\|^2\|X(i)\|^2 = (d_i^2 + d_i)\left(\sum_{a_{ij}=1} x_j^2\right).$$

By summing over i and using Raleigh's relation we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\left(\sum_{\substack{1\leqslant k<j\leqslant n \\ a_{ij}=a_{ik}=1}} (x_j - x_k)^2\right) &\geqslant \sum_{\substack{1\leqslant k<j\leqslant n \\ a_{jk}=1}} (x_j - x_k)^2 \\
&= \sum_{j=1}^{n} d_jx_j^2 - 2\sum_{a_{jk}=1} x_jx_k \geqslant \mu.
\end{aligned}$$

Note that we have three inequalities, (3.5),(3.6) and (3.7), as below:

$$\begin{aligned}
\sum_{i=1}^{n} d_i\left(\sum_{a_{ij}=1} x_j^2\right) &= \sum_{i=1}^{n} d_i\left(\sum_{i=1}^{n} x_i^2 - \sum_{a_{ij}=0} x_j^2\right) \\
&= \sum_{i=1}^{n} d_i - \left(\sum_{i=1}^{n} d_ix_i^2 + \sum_{i=1}^{n} d_i(\sum_{\substack{a_{ij}=0 \\ i\neq j}} x_j^2)\right) \\
&\leqslant \sum_{i=1}^{n} d_i - \left(\sum_{i=1}^{n} d_ix_i^2 + \sum_{i=1}^{n}(n-1-d_i)x_i^2\right) \\
&= \sum_{i=1}^{n} d_i - (n-1),
\end{aligned} \tag{3.5}$$

$$\sum_{i=1}^{n} d_i^2 x_i^2 \geqslant \delta^2 \sum_{i=1}^{n} x_i^2 = \delta^2, \tag{3.6}$$

and

$$\sum_{i=1}^{n} 2\mu d_i x_i^2 \leqslant 2\mu\Delta \sum_{i=1}^{n} x_i^2 = 2\mu\Delta. \tag{3.7}$$

Then, by using the above inequalities it is possible to verify

$$\mu^2 \leqslant \sum_{i=1}^{n} d_i - (n-1) - \delta^2 + 2\Delta\mu - \mu$$

and further

$$\mu^2 + \mu - 2\mu\Delta \leqslant \sum_{i=1}^{n} d_i - (n-1) - \delta^2.$$

Thus

$$\left(\mu - \left(\frac{2\Delta - 1}{2}\right)^2\right) \leqslant \sum_{i=1}^{n} d_i - (n-1) - \delta^2 + \left(\frac{2\Delta - 1}{2}\right)^2.$$

Hence,

$$\mu \leqslant \sqrt{\sum_{i=1}^{n} d_i - (n-1) - \delta^2 + \left(\frac{2\Delta - 1}{2}\right)^2} + \left(\frac{2\Delta - 1}{2}\right)^2.$$

Finally,

$$\mu \leqslant \sqrt{2m - (n-1) - \delta^2 + \left(\tfrac{2\Delta-1}{2}\right)^2} + \left(\tfrac{2\Delta-1}{2}\right). \ \square$$

Remark 1: For circle graph, the upper bound in (3.4) occurs if $n \geqslant 7$. The upper bound in (3.4) is equal when $G = C_7$ be a circle graph with 7 vertices.

Remark 2: The upper bound in (3.4) and [28, 29] are comparable. For instance, let $G = K_n$ be a complete graph with n vertices. Then the upper bound of Laplacian matrix $G = K_n$ in (3.4) is $2\Delta - 1$ and the upper bound of Laplacian matrix $G = K_n$ in [28, 29] is 2Δ.

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ON THE SIGNED MATCHINGS OF GRAPHS

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Abstract. For a graph G and any $v \in V(G)$, $E_G(v)$ is the set of all edges incident with v. A function $f : E(G) \to \{-1, 1\}$ is called a *signed matching* of G if $\sum_{e \in E_G(v)} f(e) \leq 1$ for every $v \in V(G)$. The weight of a signed matching f, is defined by $w(f) = \sum_{e \in E(G))} f(e)$. The signed matching number of G, denoted by $\beta_1'(G)$, is the maximum $w(f)$ where the maximum is taken over all signed matchings over G. In this paper, we have obtained the signed matching number of some families of graphs and studied the signed matching number of subdivision and the edge deletion of edges of a graph.

Keywords: signed matching; signed matching number; bipartite graphs.

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ and size $|E|$ of G is denoted by $n(G)$ and $m(G)$, respectively.

Let $G = (V, E)$ be a graph. For $u \in V$, $E_G(v)$=$\{uv \in E | u \in V\}$ are called the *edge-neighborhood* of v in G. For simplicity $E_G(v)$ is denoted by E(v). The *degree* of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |E(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbour is called a *support vertex.* A graph, G, is called r-regular graph if $\deg_G(v) = r$ for every $v \in V(G)$. For a nonempty subset $X \subseteq E$ the *edge induced subgraph* of G, induced by X, denote by $\langle X \rangle$, is a subgraph with edge set X and a vertex v belong to $\langle X \rangle$ if v is incident with at least one edge in X. A *k-partite graph* is a graph which its vertex set can be partitioned into k sets $V_1, V_2, \cdots, V_k$ such that every edge of the graph has an end point in V_i and an end point in V_j for some $1 \leq i \neq j \leq k$. A *complete k-partite graph* is a k-partite graph that every vertex of each partite set is adjacent to all vertices of the other partite sets. We denote the complete k-partite graph

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by $K_{n_1,n_2,\cdots,n_k}$, where $|V_i| = n_i$ for $1 \leq i \leq k$. In the case $k = 2$, the k-partite and complete k-partite graph are called *bipartite* and *complete bipartite* graphs. We denote by P_n, C_n, K_n and $\overline{K_n}$, the path, the cycle, complete graph and the empty graph of order n, respectively. A *double star* $DS_{a,b}$ is a graph containing exactly two non-leaf vertices which one is adjacent to a leaves and the other is adjacent to b leaves. These two non-leaf of double star are called *centers* of double star. For a graph $G = (V, E)$ and $e = uv \in E$, a *subdivision* of G respect to e, denote by $S(G)$, is a graph obtained from G by deleting the edge e and add new vertex x and new edges xu and xv. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint vertex sets. A graph $G = (V, E)$ is the *join graph* of G_1 and G_2, if $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. If G is the join graph of G_1 and G_2, we shall write $G = G_1 + G_2$. The graphs $W_n = C_n + K_1, F_n = P_n + K_1$, and $Fr_n = nK_2 + K_1$ are called *wheel, fan* and *friendship* graphs, respectively. For all graph-theoretic terminology not defined here, the reader is referred to [2].

Let $f : E(G) \to \{-1, 1\}$ be a function. For every vertex v, we define $f_G(v) = \sum_{e\in E_G(v)} f(e)$. A function $f : E(G) \to \{-1, 1\}$ is called a *signed matching* of G if $f_G(v) \leq 1$ for every $v \in V(G)$. The weight of a signed matching f is defined by $w(f) = f(E(G)) = \sum_{e\in E(G)} f(e)$. The *signed matching number* of G is $\beta_1'(G)$ =max $w(f)$, where the maximum is taken over all signed matchings. It seems natural to define $\beta_1'(\overline{K_n}) = 0$ for all totally disconnected graphs $\overline{K_n}$. A signed matching f on G, with $w(f) = \beta_1'(G)$ is called a β_1'- signed matching.

The concept of signed matching is defined by Wang [4], and further studied in, for example [3, 5, 6]. In [4], it is shown that a maximum signed matching can be found in strongly polynomial time. In addition, the exact value of $\beta_1'(G)$ for paths, cycles, complete graphs and complete bipartite graphs were found [4].

In this paper, we have studied the signed matchings of subdivision and edge deletion of a graph. Also, we have studied the signed matchings of join of graphs.

2. Main Results

In this section, we first stated some of the results which would be useful in the remaining part of the paper. The following proposition provides a relation between $|E(G)|$ and $\beta_1'(G)$.

Proposition 2.1. *For any graph* $G = (V(G), E(G))$*, we have* $\beta_1'(G) \equiv |E(G)| \ (mod\, 2)$.

Proof. Let f be a β_1'-signed matching on G. Suppose that P and M are the numbers of positive and negative edges respect to f, respectively. Hence

$$P + M = |E(G)|, P - M = \beta_1'(G).$$

Therefore, $\beta_1'(G) - |E(G)| = -2M$ and we conclude that $\beta_1'(G) \equiv |E(G)| \ (mod\, 2)$. □

In [4], $\beta_1'(G)$ for Eulerian graphs is given as follows.

Theorem 2.1. *[4] Let G be a Eulerian graph of order n and size m. Then*

$$\beta_1'(G) = \frac{1}{2}((-1)^m - 1).$$

Corollary 2.1. *[4]Let n be a natural number. Then*

$$\beta_1'(C_n) = \begin{cases} -1, & \text{if } n \neq 2k, \\ 0, & \text{if } n = 2k. \end{cases}$$

For non-Eulerian graph, the following theorem was given in [4]. Here we give an alternative proof for this theorem.

Theorem 2.2. *Let G be a graph of order n with $2k(k \geq 1)$ odd vertices. Then*

$$0 \leq \beta_1'(G) \leq k.$$

Proof. Let $f : E(G) \longrightarrow \{1, -1\}$ be a β_1'-signed matching of G. Hence $f_G(v) \leq 0$ for any even vertex v and $f_G(v) \leq 1$ for any odd vertex v. Therefore

$$2\beta_1'(G) = 2\sum_{e \in E} f(e) = \sum_{v \in V} f_G(v)) \leq 2k,$$

and hence $\beta_1'(G) \leq k$.
For the lower bound, note that, the edges of G can be partitioned to subsets $E_1, E_2, \cdots E_k$, such that for each i, the induced subgraph $\langle E_i \rangle$ is a trail connected odd vertices and at most one of these trails has odd length (see Theorem 5.3 of [2]). If we label the edges of each E_i alternately by 1 and -1, we can find a signed matching with positive weight. Hence $\beta_1'(G) \geq 0$. □

As a straight result of Theorems 2.1 and 2.2, we have the following corollary.

Corollary 2.2. *Let G be a graph. Hence $\beta_1'(G) = -1$ if and only if G is a Eulerian graph of odd size.*

Theorem 2.3. *[4] Let m and n be two natural numbers. Then*

$$\beta_1'(K_{m,n}) = \begin{cases} 0 & \text{if } mn \equiv 0 \ (mod\ 2), \\ min\{m, n\} & \text{if } mn \equiv 1 \ (mod\ 2). \end{cases}$$

Theorem 2.4. *Let m, n, p be positive integers. Then*

$$\beta_1'(K_{m,n,p}) = \begin{cases} 0 & \text{if } m \equiv n \equiv p \equiv 0 \ (mod\ 2), \\ -1 & \text{if } m \equiv n \equiv p \equiv 1 \ (mod\ 2), \\ 0 & \text{if } m \equiv n \equiv 0 \ (mod\ 2), p \equiv 1 \ (mod\ 2), \\ min\{m, n\} & \text{if } m \equiv n \equiv 1 \ (mod\ 2), p \equiv 0 (mod\ 2) \end{cases}$$

Proof. If $m \equiv n \equiv p$ $(mod\ 2)$, then each vertex of $K_{m,n,p}$ has even degree and hence $K_{m,n,p}$ is an Eulerian graph. Therefore, the first and the second parts of theorem are obtained by Theorem 2.1. Now suppose that $V_1 = \{v_1, v_2, \ldots, v_m\}, V_2 = \{u_1, u_2, \ldots, u_n\}$ and $V_3 = \{w_1, w_2, \ldots w_p\}$ are three parts of $K_{m,n,p}$ of sizes m, n and p, respectively. Let $f : E(G) \longrightarrow \{1, -1\}$ be a signed matching of $K_{m,n,p}$. At first consider the case $m \equiv n \equiv 0$ $(mod\ 2)$ and $p \equiv 1$ $(mod\ 2)$. Hence every vertex of V_3 has even degree. Therefore $f_{K_{m,n,p}}(v) \le 0$ for any $v \in V_3$. On the other hand $K_{m,n,p} \cong K_{m+n,p} \cup K_{m,n}$ and hence

$$w(f) = \sum_{v \in V_3} f_{K_{m,n,p}}(v) + \sum_{v \in V_2} f_{K_{m,n}}(v).$$

Note that For any $v \in V_2$, the degree of v in $K_{m,n}$ is even and hence $f_{K_{m,n}}(v) \le 0$. Therefore $w(f) \le 0$. Hence $\beta_1'(K_{m,n,p}) \le 0$. Now consider the function $g : E(K_{m,n,p}) \longrightarrow \{1, -1\}$ as follows:

$$g(u_i v_j) = (-1)^{i+j},\ g(u_i w_j) = (-1)^{i+j},\ g(w_i v_j) = (-1)^{i+j}.$$

It is not difficult to see that g is a signed matching and $w(g) = 0$. Therefore,, in this case $\beta_1'(K_{m,n,p}) = 0$.
Now suppose that $m \equiv n \equiv 1$ $(mod\ 2)$ and $p \equiv 0$ $(mod\ 2)$. Again, every vertex of V_3 has even degree. Therefore $f_{K_{m,n,p}}(v) \le 0$ for any $v \in V_3$. By the same argument as above we have

$$w(f) = \sum_{v \in V_3} f_{K_{m,n,p}}(v) + f(E(K_{m,n})) \le f(E(K_{m,n})).$$

But $f(E(K_{m,n})) \le min\{m, n\}$ by Theorem 2.3. Hence $\beta_1'(K_{m,n,p}) \le min\{m, n\}$ By the same argument as above $\beta_1'(K_{m,n,p}) = min\{m, n\}$. $\square$

Theorem 2.5. *Suppose that a and b are two integers. Then*

$$\beta_1'(DS_{a,b}) = \begin{cases} 3 & \text{if } a \equiv b \equiv 0\ (mod\ 2) \\ 1 & \text{if } a \equiv b \equiv 1\ (mod\ 2) \\ 2 & \text{if } a \equiv 1\ (mod\ 2), b \equiv 0\ (mod\ 2) \end{cases}$$

Proof. Let u and v be centers of double star $DS_{a,b}$ of degrees $a + 1$ and $b + 1$. Suppose that $f : E(DS_{a,b}) \longrightarrow \{1, -1\}$ is a signed matching set. Hence $w(f) = f_{DS_{a,b}}(u) + f_{DS_{a,b}}(v) - f(uv)$.
If $a \equiv b \equiv 1$ $(mod\ 2)$, then deg(u) and deg(v) are even. Therefore, it follows that $f_{DS_{a,b}}(u), f_{DS_{a,b}}(v) \le 0$. We conclude $w(f) \le -f(uv) \le 1$. Hence $\beta_1'(DS_{a,b}) \le 1$. Now consider $g : E(DS_{a,b}) \longrightarrow \{1, -1\}$ such that $g(e) = 1$ for $\frac{a+1}{2}$ edges of $E_{DS_{a,b}}(v) \setminus \{uv\}$ and $\frac{b+1}{2}$ edges of $E_{DS_{a,b}}(u) \setminus \{uv\}$ and $g(e) = -1$ for the remaining edges of $E_{DS_{a,b}}(v) \cup E_{DS_{a,b}}(u)$. Clearly g is a signed matching and $w(g) = 1$. Hence $\beta_1'(DS_{a,b}) \ge 1$ and we conclude $\beta_1'(DS_{a,b}) = 1$.

If $a \equiv b \equiv 0$ $(mod\ 2)$, then deg(u) and deg(v) are odd. Therefore, it follows $f_{DS_{a,b}}(v)), f_{DS_{a,b}}(u) \le 1$. We conclude that $w(f) \le 2 - f(uv) \le 3$. Hence

$\beta'_1(DS_{a,b}) \leq 3$. Now consider $g : E(DS_{a,b}) \longrightarrow \{1,-1\}$ such that $g(e) = 1$ for $\frac{a+2}{2}$ edges of $E_{DS_{a,b}}(v) \setminus \{uv\}$ and $\frac{b+2}{2}$ edges of $E_{DS_{a,b}}(u) \setminus \{uv\}$ and $g(e) = -1$ for the remaining edges of $E(v) \cup E(u)$. Again g is a signed matching with $w(g) = 3$ and we conclude that $\beta'_1(DS_{a,b}) = 3$.
For the last case, suppose that $a \equiv 0 \ (mod\ 2)$ and $b \equiv 1 (mod\ 2)$. By the same argument as above, we conclude that $\beta'_1(S_{a,b}) = 2$. $\square$

Theorem 2.6. *Let n be an integer. Then*

$$\beta'_1(W_n) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor & \text{if } n \equiv 0, 3 \ (mod\ 4), \\ \lfloor \frac{n+1}{2} \rfloor - 1 & \text{if } n \equiv 1, 2 \ (mod\ 4). \end{cases}$$

Proof. Suppose that $E(W_n) = \{v_i v_{i+1}, uv_i : 1 \leq i \leq n\}$, where indices computing in module n. Note that the vertex u has degree equal to n, and other vertices have degree 3. If $n \equiv 0 \ (mod\ 4)$, then W_n has n vertices of odd degree. Hence $\beta(W_n) \leq \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$ by Theorem 2.2. Now define $f : E(W_n) \longrightarrow \{1,-1\}$ by

$$f(uv_{4i+1}) = f(uv_{4i+2}) = f(v_{4i+2}v_{4i+3}) = f(v_{4i+3}v_{4i+4}) = f(v_{4i+4}v_{4i+5}) = 1$$

for $0 \leq i \leq \frac{n}{4} - 1$ and $f(e) = -1$ for other edges of W_n. Clearly f is a signed matching with $w(f) = \frac{n}{2} = \lfloor \frac{n+1}{2} \rfloor$. So $\beta'_1(W_n) \geq \lfloor \frac{n+1}{2} \rfloor$. Hence $\beta'_1(W_n) = \lfloor \frac{n+1}{2} \rfloor$. The case $n \equiv 3 \ (mod\ 4)$ is obtained by a similar argument as the above.
Now suppose that $n \equiv 2 \ (mod\ 4)$. Hence $\beta'_1(W_n) \leq \frac{n}{2}$ by Theorem 2.2. But $\beta'_1(W_n) \neq \frac{n}{2}$ by Proposition 2.1 and therefore $\beta'_1(W_n) \leq \frac{n}{2} - 1$. Now define $f : E(W_n) \longrightarrow \{1,-1\}$ by

$$f(uv_{n-1}) = f(v_1 v_n) = 1,$$

$$f(uv_{4i+1}) = f(uv_{4i+2}) = f(v_{4i+2}v_{4i+3}) = f(v_{4i+3}v_{4i+4}) = f(v_{4i+4}v_{4i+5}) = 1$$

for $0 \leq i \leq \frac{n-6}{4}$ and $f(e) = -1$ for other edges of W_n. Clearly f is a signed matching with $w(f) = \frac{n}{2} - 1 = \lfloor \frac{n+1}{2} \rfloor - 1$. So $\beta'_1(W_n) \geq \lfloor \frac{n+1}{2} \rfloor - 1$. $\square$

Theorem 2.7. *Let n be an integer. Then*

$$\beta'_1(F_n) = \begin{cases} \lfloor \frac{n-1}{2} \rfloor - 1 & \text{if } n \equiv 0, 3 \ (mod\ 4), \\ \lfloor \frac{n-1}{2} \rfloor & \text{if } n \equiv 1, 2 \ (mod\ 4). \end{cases}$$

Proof. The result follows by a similar argument as the proof of Theorem 2.6. $\square$

Theorem 2.8. *Let n be an integer. Then*

$$\beta'_1(Fr_n) = \begin{cases} 0 & \text{if } n \equiv 0 \ (mod\ 2) \\ -1 & \text{if } n \equiv 1 \ (mod\ 2) \end{cases}$$

Proof. Since the graph Fr_n is an Eulerian graph, the result follows from Theorem 2.1. $\square$

Theorem 2.9. *Let G be a graph and e be an edge of G. If $S(G)$ is the subdivision of G by edge e, then*

$$\beta_1'(G) - 1 \leq \beta_1'(S(G)) \leq \beta_1'(G) + 1.$$

In addition these bounds are sharp.

Proof. Suppose that $e = uv$ and $S(G) = G \setminus \{e\} \cup \{xu, xv\}$, where x is the new vertex. Let f be a β_1'-signed matching of G. If $f(e) = 1$ (or $f(e) = -1$), then define $g : E(S(G)) \longrightarrow \{1, -1\}$ by $g(xu) = 1$(or $g(xu) = -1$), $g(xv) = -1$ and $g(w) = f(w)$ for other edges of $S(G)$. Clearly g is a signed matching on $S(G)$ and $w(g) = \beta_1'(G) - 1$. Hence $\beta_1'(G) - 1 \leq \beta_1'(S(G))$.
Now suppose that f is a β_1'-signed matching of $S(G)$. Define signed matching g on G by $g(e) = -1$ and $g(w) = f(w)$ for other edges of G. we conclude that $\beta_1'(S(G)) \leq \beta_1'(G) + 1$.
For any positive integer n, we have $S(C_n) = C_{n+1}$. If n is even, then $\beta_1'(C_n) = 0$ and $\beta_1'(C_{n+1}) = -1$ by Corollary 2.1 and the lower bound is occurred. If n is odd, then $\beta_1'(C_n) = -1$ and $\beta_1'(C_{n+1}) = 0$ by Corollary 2.1 and the we obtain the upper bound. □

Theorem 2.10. *Let G be a graph. Then*

$$\beta_1'(G) - 3 \leq \beta_1'(G - e) \leq \beta_1'(G) + 1.$$

In addition these bounds are sharp.

Proof. Suppose that $e = uv$. Let f be a β_1'-signed matching of G. If $f(e) = 1$, then define $g : E(G - e) \longrightarrow \{1, -1\}$ by $g(x) = f(x)$ for any edge x of $G - e$. Clearly g is a signed matching on $G - e$ and $w(g) = \beta_1'(G) - 1$. Hence $\beta_1'(G) - 1 \leq \beta_1'(G - e)$. If $f(e) = -1$, change the label of two edges e_1 and e_2 (which are adjacent to u and v in $G - e$, respectively) from 1 to -1. Hence we have a signed matching on $G - e$ of weight $\beta_1'(G) - 3$ and hence $\beta_1'(G) - 3 \leq \beta_1'(G - e)$.
Now suppose that f is a β_1'-signed matching of $G - e$. Define signed matching g on G by $g(e) = -1$ and $g(w) = f(w)$ for other edges of G. We conclude that $\beta_1'(G - e) \leq \beta_1'(G) + 1$.
Suppose that n is an even integer. We have $\beta_1'(DS_{n,n}) = 3$ by Theorem2.5. If x, y are centers of double star and $e = xy$, then $DS_{n,n} - e = 2K_{1,n}$ and we have $\beta_1'(DS_{n,n} - e) = 0$ by Theorem 2.3. Hence the lower bound is obtained. If n is even and m is odd, then $\beta_1'(K_{1,n} \cup K_{1,m}) = 1$ by Theorem 2.3. But $K_{1,n} \cup K_{1,m} + e = DS_{m,n}$, where e joint two stars $K_{1,m}$ and $K_{1,n}$. Hence $\beta_1'(DS_{m.n}) = 2$ and upper bound is occurred. □

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