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[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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ON SOME TOPOLOGICAL PROPERTIES IN GRADUAL NORMED SPACES *

Mina Ettefagh, Farnaz Y. Azari and Sina Etemad

Abstract. In this paper, we have investigated some topological properties of sets in a given gradual normed space. We have stated gradual Hausdorff property and then, we have studied the relationship between gradual closed sets and gradual compact sets. Also, we have given a result about having the closure point for an infinite set in a gradual normed space. In the end, we have provided some illustrative examples.
Keywords: gradual normed space; Hausdorff property; gradual numbers.

1. Introduction

In 1965, Zadeh first introduced a new class of sets named *fuzzy sets* to quantify some linguistic terms and stated these terms mathematically [12]. Indeed, fuzzy sets are generalization of classical sets and also, under certain conditions, we consider a fuzzy subset as a fuzzy number. But, when we study this notion in fuzzy metric spaces, the term *fuzzy number* is used instead of fuzzy intervals.

From this point of view, fuzzy numbers are generalization of intervals, not numbers. On the other hand, some algebraic properties of numbers not hold for fuzzy numbers. These problems have been implied to avoid confusion between the researchers.

In this way, in 2006, Fortin, Dubois and Fargier introduced gradual numbers as elements of fuzzy intervals [5]. In this new structure, gradual numbers are considered as an unique generalization of real numbers which are equipped with all algebraic properties of classical real numbers [5]. Since then, gradual numbers have been applied as a strong tool for computations and optimization problems.

In [7], Kasperski et al. investigated gradual numbers and applied this notion to solving combinatorial optimization problems. Some years later, Fortin et al. [6] suggested some methods for evaluating the optimality by using gradual numbers.

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*

For more details about other applications of gradual numbers, see ([1], [2], [8], [11], [13], [14]).

Recently, Sadeqi and Azari [9] have studied some properties of gradual numbers and introduced gradual normed linear space. In [4], Ettefagh et al. investigated some properties of sequences in gradual normed spaces.

Motivated by above works, in this paper, we have investigated some topological properties of sets in a given gradual normed space. We have stated gradual Hausdorff property and then, we have studied the relationship between gradual closed sets and gradual compact sets. Also, we have given a result about having the closure point for an infinite set in a gradual normed space. Finally, in the last section, we have presented some illustrative examples.

2. Preliminaries

In this section, we recall some basic definitions and theorems on the gradual numbers and gradual normed space. For more details, see ([3], [5], [9]).

Definition 2.1. ([5]) A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}}$ from $(0,1]$ to the set of real numbers $\mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. We say that a gradual real number $\tilde{r}$ is non-negative if for each $\alpha \in (0,1]$, $A_{\tilde{r}}(\alpha) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

The gradual operations on the elements of $G(\mathbb{R})$ can be defined as follows.

Definition 2.2. ([5]) Assume that $*$ is any operation in real numbers and $\tilde{r}_1$ and $\tilde{r}_2$ are two arbitrary gradual numbers with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$, respectively. Then $\tilde{r}_1 * \tilde{r}_2$ is the gradual number with an assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ given by

$$A_{\tilde{r}_1 * \tilde{r}_2}(\alpha) = A_{\tilde{r}_1}(\alpha) * A_{\tilde{r}_2}(\alpha), \qquad (\alpha \in (0,1]).$$

Then, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$A_{\tilde{r}_1 + \tilde{r}_2}(\alpha) = A_{\tilde{r}_1}(\alpha) + A_{\tilde{r}_2}(\alpha), \qquad\qquad A_{c\tilde{r}}(\alpha) = cA_{\tilde{r}}(\alpha),$$

for each $\alpha \in (0,1]$.

For each real number $t \in \mathbb{R}$, the constant gradual number $\tilde{t}$ is defined by $A_{\tilde{t}}(\alpha) = t$ for each $\alpha \in (0,1]$. In particular, $\tilde{0}$ and $\tilde{1}$ are constant gradual numbers defined by $A_{\tilde{0}}(\alpha) = 0$ and $A_{\tilde{1}}(\alpha) = 1$, respectively. It can be easily proved that $G(\mathbb{R})$ with the gradual addition and gradual scalar multiplication is a real linear space [5].

Definition 2.3. ([7]) Let $\tilde{r}, \tilde{s} \in G(\mathbb{R})$. The partial order relation $\leq$ in $G(\mathbb{R})$ is defined by $\tilde{r} \leq \tilde{s}$ if and only if $A_{\tilde{r}}(\alpha) \leq A_{\tilde{s}}(\alpha)$ for all $\alpha \in (0,1]$.

Theorem 2.1. *([7]) Let $\tilde{r}, \tilde{s}, \tilde{t} \in G(\mathbb{R})$. We have*

(i) if $\tilde{r} \leq \tilde{s}$, then $\tilde{r} - \tilde{t} \leq \tilde{s} - \tilde{t}$;

(ii) if $\tilde{r} \leq \tilde{s}$ and $\tilde{0} \leq \tilde{t}$, then $\tilde{r} \cdot \tilde{t} \leq \tilde{s} \cdot \tilde{t}$ and $\dfrac{\tilde{r}}{\tilde{t}} \leq \dfrac{\tilde{s}}{\tilde{t}}, \tilde{t} \neq \tilde{0}$;

(iii) $\dfrac{(\tilde{r} \cdot \tilde{s})}{\tilde{t}} = \tilde{r} \cdot (\dfrac{\tilde{s}}{\tilde{t}}), \tilde{t} \neq \tilde{0}$.

Definition 2.4. ([9]) Let X be a real vector space and $x, y \in X$. The mapping $\| \cdot \|_G$ from X to $G^*(\mathbb{R})$ is called a gradual norm on X if for each $\alpha \in (0, 1]$, we have

(G1) $A_{\|x\|_G}(\alpha) = A_{\tilde{0}}(\alpha)$ iff $x = 0$;

(G2) $A_{\|kx\|_G}(\alpha) = |k| A_{\|x\|_G}(\alpha); \quad (k \in \mathbb{R})$

(G3) $A_{\|x+y\|_G}(\alpha) \leq A_{\|x\|_G}(\alpha) + A_{\|y\|_G}(\alpha)$.

Then the pair $(X, \| \cdot \|_G)$ is called a gradual normed space (GNS).

Example 2.1. ([9]) (i) Let $X = \mathbb{R}^n$. For each $\alpha \in (0, 1]$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, consider the function $\| \cdot \|_G : \mathbb{R}^n \to G^*(\mathbb{R})$ by

$$A_{\|x\|_G}(\alpha) = e^{\alpha} \sum_{i=1}^{n} |x_i|.$$

Then $\| \cdot \|_G$ is a gradual norm on $\mathbb{R}^n$ and $(\mathbb{R}^n, \| \cdot \|_G)$ is a gradual normed linear space.
(ii) Let $X = C([0, 1])$ be the space of all continuous real-valued functions on $[0, 1]$. Consider two norms on $C([0, 1])$ by $\|f\|_0 = \Big(\int_0^1 |f(t)|^2 \,\mathrm{d}t \Big)^{\frac{1}{2}}$ and $\|f\|_1 = \max\limits_{0 \leq t \leq 1} \{|f(t)|\}$. Now, the function $\| \cdot \|_G : C[0, 1] \to G^*(\mathbb{R})$ defined by

$$A_{\|f\|_G}(\lambda) = \begin{cases} \|f\|_0, & 0 < \lambda \leq \dfrac{1}{2} \\ \|f\|_1, & \dfrac{1}{2} < \lambda \leq 1 \end{cases}$$

is a gradual norm on X.

Definition 2.5. ([9]) Let X be a gradual normed space. A gradual neighborhood of $x_0 \in X$ with radius of $\epsilon > 0$ is defined by

$$x_0 + N(\alpha, \epsilon) = \{x : \ A_{\|x - x_0\|_G}(\alpha) < \epsilon\}, \qquad \alpha \in (0, 1].$$

In particular, if $x_0 = 0$, then $N(\alpha, \epsilon) = \{x : \ A_{\|x\|_G}(\alpha) < \epsilon\}$.

Lemma 2.1. *([9]) Let X be a gradual normed space and $\alpha \in (0, 1]$ and $\epsilon > 0$. We have*

(N1) $N(\alpha, \epsilon) = \epsilon N(\alpha, 1)$*;*

(N2) if $\epsilon_1 \leq \epsilon_2$*, then* $N(\alpha, \epsilon_1) \subseteq N(\alpha, \epsilon_2)$*;*

(N3) if for every $x \in X$*, the assignment function* $A_{\|x\|_G}$ *be decreasing and* $\alpha_1 \leq \alpha_2$*, then* $N(\alpha_1, \epsilon) \subseteq N(\alpha_2, \epsilon)$*.*

Definition 2.6. ([9]) Let X be a gradual normed space and $A \subseteq X$. Then

(H1) the point $x_0 \in X$ is called a closure point of A if for each $\alpha \in (0, 1]$, we have

$$(x_0 + N(\alpha, \alpha)) \cap A \neq \emptyset.$$

The set of all closure points of A is denoted by $\bar{A}$.

(H2) The point $x_0 \in X$ is called a limit point of A if for each $\alpha \in (0, 1]$, $(x_0 + N(\alpha, \alpha)) \cap A$ contains at least one point of A different from x_0 itself, or

$$(x_0 + N(\alpha, \alpha))^* \cap A \neq \emptyset,$$

where $(x_0 + N(\alpha, \alpha))^* = (x_0 + N(\alpha, \alpha)) \backslash \{x_0\}$.

(H3) the point $x_0 \in A$ is called an interior point of A if there exists $N(\alpha_0, \epsilon_0)$ such that $x_0 + N(\alpha_0, \epsilon_0) \subseteq A$. The set of all interior points of A is denoted by $IntA$.

(H4) the set A is said to be gradual closed set iff $\bar{A} = A$.

(H5) the set A is said to be gradual open set iff $IntA = A$.

The following theorems state an effective relationship between gradual open sets and gradual closed sets.

Theorem 2.2. *Let* $(X, \| \cdot \|_G)$ *be a gradual normed space and for every* $x \in X$*,* $A_{\|x\|_G}$ *be a decreasing function. If* B *is the gradual open subset of* X*, then* $X \setminus B$ *is gradual closed.*

Proof. Let B be a gradual open set and x_0 be a closure point of $X \setminus B$. Then for each $\alpha \in (0, 1]$, we have $(x_0 + N(\alpha, \alpha)) \cap (X \setminus B) \neq \emptyset$ and so $(x_0 + N(\alpha, \alpha)) \nsubseteq B$. Now, for each $\epsilon > 0$ and $\alpha \in (0, 1]$, let $\alpha_0 = \min\{\alpha, \epsilon\}$. Thus we have

$$(x_0 + N(\alpha_0, \alpha_0)) \subseteq (x_0 + N(\alpha, \epsilon))$$

and then $(x_0 + N(\alpha, \epsilon)) \nsubseteq B$. Hence $x_0 \notin IntB = B$ or $x_0 \in X \setminus B$ and we conclude that $X \setminus B$ is gradual closed. $\square$

Theorem 2.3. *Let* $(X, \| \cdot \|_G)$ *be a gradual normed space. For every subset* B *of* X*, if* $X \setminus B$ *is the gradual closed set, then* B *is gradual open set.*

Proof. Suppose that $X \setminus B$ is a gradual closed set. Then $\overline{X \setminus B} = X \setminus B$. Let $x_0 \in B$, thus $x_0 \notin X \setminus B = \overline{X \setminus B}$. Hence for some $\alpha \in (0, 1]$,

$$(x_0 + N(\alpha, \alpha)) \cap (X \setminus B) = \emptyset,$$

or $(x_0 + N(\alpha, \alpha)) \subseteq B$. We conclude that x_0 is an interior point for B and B is a gradual open set. $\square$

3. Main Results

Now, in this section, we are ready to state main results.

Theorem 3.1. *Let $(X, \|\cdot\|_G)$ be a gradual normed space and A and B be subsets of X. Then*

(i) $(\overline{A \cap B}) \subseteq \bar{A} \cap \bar{B}$ *and* $(\overline{A \cup B}) = \bar{A} \cup \bar{B}$.

(ii) if every assignment function is decreasing, we have

$$Int(A) \cup Int(B) \subseteq Int(A \cup B) \quad \text{and} \quad Int(A \cap B) = Int(A) \cap Int(B).$$

Proof. (i) Let $x \in (\overline{A \cap B})$. Then for every $\alpha \in (0, 1]$, we have

$$(x + N(\alpha, \alpha)) \cap (A \cap B) \neq \emptyset.$$

So for every $\alpha \in (0, 1]$, we have $(x + N(\alpha, \alpha)) \cap A \neq \emptyset$ and $(x + N(\alpha, \alpha)) \cap B \neq \emptyset$. This shows that $x \in \bar{A} \cap \bar{B}$.

Also, $x \in (\overline{A \cup B})$ if and only if for every $\alpha \in (0, 1]$, we have

$$(x + N(\alpha, \alpha)) \cap (A \cup B) \neq \emptyset.$$

Thus, one can write $(x + N(\alpha, \alpha)) \cap A \neq \emptyset$ or $(x + N(\alpha, \alpha)) \cap B \neq \emptyset$. This means that $x \in \bar{A} \cup \bar{B}$.

(ii) Let $x \in Int(A) \cup Int(B)$. Then $x \in Int(A)$ or $x \in Int(B)$. So for some $\alpha_1, \alpha_2 \in (0, 1]$ and $\epsilon_1, \epsilon_2 > 0$, we have

$$(x + N(\alpha_1, \epsilon_1)) \subset A \quad \text{or} \quad (x + N(\alpha_2, \epsilon_2)) \subset B.$$

Now, let $\alpha < \min\{\alpha_1, \alpha_2\}$ and $\epsilon < \min\{\epsilon_1, \epsilon_2\}$. Since every assignment function is decreasing, thus $(x + N(\alpha, \epsilon)) \subset (A \cup B)$ and we get $x \in Int(A \cup B)$.

One can similarly prove the equality $Int(A \cap B) = Int(A) \cap Int(B)$ and we omit this part of the proof. $\square$

Theorem 3.2. *Let $(X, \|\cdot\|_G)$ be a gradual normed space. Then*

(i) for any collection $\{G_\gamma\}_\gamma$ of gradual open sets, $\bigcup_\gamma G_\gamma$ is a gradual open set.

(ii) for any collection $\{F_\gamma\}_\gamma$ of gradual closed sets, $\bigcap_\gamma F_\gamma$ is a gradual closed set.

Proof. (i) Let $\{G_\gamma\}_\gamma$ be an arbitrary collection of gradual open sets and let $x \in \bigcup_\gamma G_\gamma$. Then, there is some γ_0 such that $x \in G_{\gamma_0}$. Since G_{γ_0} is a gradual open set, thus there exists $\alpha \in (0, 1]$ and $\epsilon > 0$ such that

$$(x + N(\alpha, \epsilon)) \subseteq G_{\gamma_0} \subseteq \bigcup_\gamma G_\gamma.$$

Consequently, $\bigcup_\gamma G_\gamma$ is a gradual open set.

(ii) Let $\{F_\gamma\}_\gamma$ be an arbitrary collection of gradual closed sets and put $F = \bigcap_\gamma F_\gamma$. If $x \in \bar{F}$, then for each $\alpha \in (0,1]$, we have $(x + N(\alpha,\alpha)) \cap F \neq \emptyset$. Hence for every γ, we have $(x + N(\alpha,\alpha)) \cap F_\gamma \neq \emptyset$ and so $x \in \bar{F_\gamma}$. On the other hand, since for each γ, F_γ is a gradual closed set, so $x \in \bar{F_\gamma}$ implies that $x \in F_\gamma$. Therefore $x \in \bigcap_\gamma F_\gamma = F$. This shows that $\bar{F} \subseteq F$ and so F is a gradual closed set. $\square$

Now, for the finite number of gradual open (closed) sets, we have the following theorem.

Theorem 3.3. *Let $(X, \|\cdot\|_G)$ be a gradual normed space. Hence*

(i) if every assignment function is decreasing, then for any finite collection $G_1, G_2, \cdots, G_n$ of gradual open sets, $\bigcap_{i=1}^n G_i$ is a gradual open set.

(ii) for any finite collection $F_1, F_2, \cdots, F_n$ of gradual closed sets, $\bigcup_{i=1}^n F_i$ is a gradual closed set.

Proof. (i) Let $G_1, G_2, \cdots, G_n$ be a finite collection of gradual open sets and let $x \in \bigcap_{i=1}^n G_i$ be an arbitrary element. Then for any $1 \leq i \leq n$, $x \in G_i$. But every G_i is a gradual open set, thus for each $1 \leq i \leq n$, there are $\alpha_i \in (0,1]$ and $\epsilon_i > 0$ such that $(x + N(\alpha_i, \epsilon_i)) \subseteq G_i$. Put $\alpha < \min\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\epsilon < \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. Since for each $x \in X$, the assignment function $A_{\|x\|_G}$ is decreasing, so we have $N(\alpha, \epsilon) \subseteq N(\alpha_i, \epsilon_i)$, for all $1 \leq i \leq n$. Thus

$$(x + N(\alpha, \epsilon)) \subseteq (x + N(\alpha_i, \epsilon_i)) \subseteq G_i.$$

Therefore, $(x + N(\alpha, \epsilon)) \subseteq \bigcap_{i=1}^n G_i$. Hence $\bigcap_{i=1}^n G_i$ is a gradual open set.

(ii) Since each F_i is gradual closed, thus $\bar{F_i} = F_i$. Now, suppose that $x_0 \in \overline{\bigcup_{i=1}^n F_i}$, so for each $\alpha \in (0,1]$, we have

$$(x_0 + N(\alpha,\alpha)) \cap \Big(\bigcup_{i=1}^n F_i\Big) \neq \emptyset.$$

This means that for each $\alpha \in (0,1]$, there exists $1 \leq i \leq n$ with

$$(x_0 + N(\alpha,\alpha)) \cap F_i \neq \emptyset,$$

and so $x_0 \in \bar{F_i} = F_i$. Therefore, $x_0 \in \bigcup_{i=1}^n F_i$. Hence, we conclude that $\overline{\bigcup_{i=1}^n F_i} \subseteq \bigcup_{i=1}^n F_i$ and the proof is completed. $\square$

Theorem 3.4. **(Gradual Hausdorff property)** *Let $(X, \|\cdot\|_G)$ be a gradual normed space and $x, y \in X$ with $x \neq y$. Then there exists two disjoint neighborhoods of x and y.*

Proof. Since $x \neq y$, so for each $\alpha \in (0,1]$, we have $A_{\|x-y\|_G}(\alpha) \neq A_{\tilde{0}}(\alpha)$. Thus, there is $\alpha_0 \in (0,1]$ such that $A_{\|x-y\|_G}(\alpha_0) > 0$. Put $\epsilon_0 < \frac{1}{2}A_{\|x-y\|_G}(\alpha_0)$ and we claim that $(x + N(\alpha_0, \epsilon_0)) \cap (y + N(\alpha_0, \epsilon_0)) = \emptyset$. To prove this claim, let $z \in (x + N(\alpha_0, \epsilon_0)) \cap (y + N(\alpha_0, \epsilon_0))$. Then, we have $A_{\|x-z\|_G}(\alpha_0) < \epsilon_0$ and $A_{\|y-z\|_G}(\alpha_0) < \epsilon_0$. Now, one can write

$$\begin{aligned} A_{\|x-y\|_G}(\alpha_0) &\leq A_{\|x-z\|_G}(\alpha_0) + A_{\|y-z\|_G}(\alpha_0) \\ &< 2\epsilon_0 < A_{\|x-y\|_G}(\alpha_0) \end{aligned}$$

and this is a contradiction. □

Theorem 3.5. *Let B be a subset of the gradual normed space $(X, \|\cdot\|_G)$ and for each $x \in X$, the assignment function $A_{\|x\|_G}$ be decreasing. If x_0 is a limit point of B, then every neighborhood of x_0 contains infinitely many points of B.*

Proof. Suppose that there exists $\alpha \in (o,1]$ and $\epsilon > 0$ such that the neighborhood $(x_0 + N(\alpha, \epsilon))$ of x_0 contains only finite number of elements of B, i.e. $(x_0 + N(\alpha, \epsilon)) \cap B = \{y_1, y_2, \ldots, y_n\}$. Put

$$\epsilon_0 < \min\{A_{\|x_0-y_1\|_G}(\alpha), A_{\|x_0-y_2\|_G}(\alpha), \ldots, A_{\|x_0-y_n\|_G}(\alpha)\}.$$

Since $\epsilon_0 < \epsilon$, hence $(x_0 + N(\alpha, \epsilon_0)) \subset (x_0 + N(\alpha, \epsilon))$. Now, we claim that the neighborhood $(x_0 + N(\alpha, \epsilon_0))$ contains no point of B. Indeed, if for some $i (i = 1, 2, \ldots, n)$, $y_i \in (x_0 + N(\alpha, \epsilon_0))$, then we have

$$A_{\|x_0-y_i\|_G}(\alpha) < \epsilon_0 < A_{\|x_0-y_i\|_G}(\alpha)$$

and this is a contradiction. Therefore $(x_0 + N(\alpha, \epsilon_0))^* \cap B = \emptyset$. Now, put $\alpha_0 < \min\{\epsilon_0, \alpha\}$. So, we have $\alpha_0 < \epsilon_0$ and $\alpha_0 < \alpha$. Since every assignment function is decreasing, thus

$$(x_0 + N(\alpha_0, \alpha_0)) \subset (x_0 + N(\alpha, \epsilon_0)).$$

Hence, it is followed that $(x_0 + N(\alpha_0, \alpha_0))^* \cap B = \emptyset$, which contradicts to the being limit point x_0 for B and the proof is completed. □

Now, we define the new concept "gradual compact set" in a gradual normed space.

Definition 3.1. Let $(X, \|\cdot\|_G)$ be a gradual normed space and K be an arbitrary nonempty subset of X. We say that K is a gradual compact set if for each cover $\{V_i\}_{i \in I}$ of gradual open sets for K, there exists finite number $V_i (i = 1, \cdots, n)$ such that $K \subseteq \bigcup_{i=1}^n V_i$.

Theorem 3.6. *Let $(X, \|\cdot\|_G)$ be a gradual normed space and for each $x \in X$, the assignment function $A_{\|x\|_G}$ be decreasing. Then every gradual compact set is gradual closed.*

Proof. Let K be a gradual compact subset of X. We will show that $X \setminus K$ is a gradual open set. For this, assume that $p \in (X \setminus K)$ and for each $q \in K$, consider neighborhoods $V_q = (p + N(\alpha_0, \epsilon))$ and $W_q = (q + N(\alpha_0, \epsilon))$ for p and q, respectively, where $\alpha_0 \in (0, 1]$ is a fixed number and

$$\epsilon < \frac{1}{2} A_{\|p-q\|_G}(\alpha_0).$$

For each $q \in K$, we have $V_q \cap W_q = \emptyset$; because if $z \in V_q \cap W_q$, then we get $A_{\|z-p\|_G}(\alpha_0) < \epsilon$ and $A_{\|z-q\|_G}(\alpha_0) < \epsilon$. So

$$A_{\|p-q\|_G}(\alpha_0) < 2\epsilon < A_{\|p-q\|_G}(\alpha_0)$$

which is a contradiction.

Now, the cover $\bigcup_{q \in K} W_q$ of gradual open sets for K has finite subcover as follows:

$$\exists\ q_1, q_2, \cdots, q_n \in K \quad s.t. \quad K \subseteq \bigcup_{i=1}^{n} W_{q_i}.$$

Let $V = \bigcap_{i=1}^{n} V_{q_i}$. Since V is a finite intersection of gradual open sets containing p and each assignment function is decreasing, so by Theorem 3.3 (i), V is a gradual open set containing p. Therefore $V \cap K = \emptyset$ and so $V \subseteq (X \setminus K)$. This proves that $X \setminus K$ is a gradual open set and by Theorem 2.2, K is a gradual closed set. □

Theorem 3.7. *Let $(X, \| \cdot \|_G)$ be a gradual normed space and K be a gradual compact subset of X. If F is a gradual closed subset of K, then F is gradual compact.*

Proof. Suppose that $\{V_i\}_{i \in I}$ is a cover of gradual open subsets for F. By Theorem 2.3, $X \setminus F$ is a gradual open subset and so $\{V_i\}_{i \in I} \cup (X \setminus F)$ is a cover for K. Since K is gradual compact, there exists a finite cover $\{V_i\}_{i=1}^{n} \cup (X \setminus F)$ for K. This implies that $\{V_i\}_{i=1}^{n}$ is a finite cover for F. Hence F will be a gradual compact set. □

Corollary 3.1. *Let $(X, \| \cdot \|_G)$ be a gradual normed space such that every assignment function is decreasing. Suppose that F and K are gradual closed and gradual compact subsets of X, respectively. Then $F \cap K$ is a gradual compact set.*

Proof. This is a consequence of Theorems 3.6, 3.7 and 3.2(ii). □

Finally, we give the last result about having closure point for an infinite set in a gradual normed space. This property is like to Bolzano-Weierstrass property in metric spaces [10].

Theorem 3.8. *Let $(X, \| \cdot \|_G)$ be a gradual normed space and E be an infinite subset of the gradual compact set K. Then E has the limit point in K.*

Proof. Suppose that every $x \in K$ is not a limit point of E. Then there exists $\alpha_x \in (0,1]$ such that $(x+N(\alpha_x,\alpha_x))^* \cap E = \emptyset$. Hence, if $x \in E$ is arbitrary, then $(x+N(\alpha_x,\alpha_x)) \cap E = \{x\}$. This shows that there is no finite collection of infinite cover $\{x+N(\alpha_x,\alpha_x)\}_{x\in E}$ of gradual open sets for E which covers the set E. Now, let $\{G_i\}_{i\in I}$ be an arbitrary cover of gradual open sets for $K \setminus E$. Then

$$\{x+N(\alpha_x,\alpha_x)\}_{x\in E} \cup \{G_i\}_{i\in I}$$

is a cover of gradual open sets for K which contains no finite subcover and this contradicts to the gradual compactness of K. □

4. Examples

The following examples are generalizations of the Example 2.1. Also we have some arguments about gradual interior and gradual closure points in each example.

Example 4.1. Let $(X, \|\cdot\|)$ be a real normed space. We define the function $\|\cdot\|_G : X \to G^*(\mathbb{R})$ by

$$A_{\|x\|_G}(\alpha) = f(\alpha)\|x\|; \quad (\alpha \in (0,1], x \in X)$$

where $f : (0,1] \to \mathbb{R}^+$ is a nonzero function. One can easily verify that $\|\cdot\|_G$ is a gradual norm on X. Also, if we denote the neighborhoods in $(X, \|\cdot\|)$ by

$$N_\epsilon(x) = \{a \in X : \ \|x-a\| < \epsilon\}, \quad (\epsilon > 0),$$

then in gradual normed space $(X, \|\cdot\|_G)$, for $\epsilon > 0$ and $\alpha \in (0,1]$ we have

$$N(\alpha,\epsilon) = N_{\frac{\epsilon}{f(\alpha)}}(0)$$

and for $x \in X$, $(x+N(\alpha,\epsilon)) = N_{\frac{\epsilon}{f(\alpha)}}(x)$.
Hence we conclude that $Int(A) = Int_G(A)$, where $Int(A)$ and $Int_G(A)$ denote the set of all interior points of A in $(X, \|\cdot\|)$ and $(X, \|\cdot\|_G)$, respectively.
Now, suppose that $A \subset X$ and $x \in X$ is a closure point of A in $(X, \|\cdot\|)$. Then for every $\epsilon > 0$, we have $N_\epsilon(x) \cap A \neq \emptyset$. Thus for every $\alpha \in (0,1]$ and $\epsilon = \dfrac{\alpha}{f(\alpha)}$, we can write

$$(x+N(\alpha,\alpha)) \cap A = N_{\frac{\alpha}{f(\alpha)}}(x) \cap A \neq \emptyset,$$

and we conclude that x is a closure point of A in $(X, \|\cdot\|_G)$ or $\bar{A} \subset \bar{A}^G$, in which $\bar{A}$ and $\bar{A}^G$ denote the closure of A in $(X, \|\cdot\|)$ and $(X, \|\cdot\|_G)$, respectively.

Example 4.2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on real vector space X such that they are not equivalent norms. For $x \in X$ and $\alpha \in (0,1]$, we define the function $\|\cdot\|_G : X \to G^*(\mathbb{R})$ by

$$A_{\|x\|_G}(\alpha) = \begin{cases} \|x\|_1, & 0 < \alpha \leq \dfrac{1}{2} \\ \|x\|_2, & \dfrac{1}{2} < \alpha \leq 1 \end{cases}$$

It is easy to check that $(X, \|\cdot\|_G)$ will be a gradual normed space. This example can be extended to a finite number of non-equivalent norms.
Now, for $x \in X$ and $\epsilon > 0$, let

$$_1N_\epsilon(x) = \{a \in X : \ \|x-a\|_1 < \epsilon\},$$

$$_2N_\epsilon(x) = \{a \in X : \ \|x-a\|_2 < \epsilon\}.$$

Therefore in $(X, \|\cdot\|_G)$, we can write for $\epsilon > 0$ and $\alpha \in (0,1]$,

$$N(\alpha,\epsilon) = \{x \in X : \ A_{\|x\|_G}(\alpha) < \epsilon\} = \begin{cases} {_1N_\epsilon(0)}, & 0 < \alpha \leq \frac{1}{2} \\ {_2N_\epsilon(0)}, & \frac{1}{2} < \alpha \leq 1 \end{cases},$$

and also for $x \in X$,

$$(x + N(\alpha,\epsilon)) = \begin{cases} {_1N_\epsilon(x)}, & 0 < \alpha \leq \frac{1}{2} \\ {_2N_\epsilon(x)}, & \frac{1}{2} < \alpha \leq 1 \end{cases}.$$

Then we can conclude that

$$Int_G(A) = Int_1(A) \cup Int_2(A),$$

where $Int_G(A)$, $Int_1(A)$ and $Int_2(A)$ denote the set of all interior points of A in $(X, \|\cdot\|_G)$, $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$, respectively.
Now, suppose that $x \in \bar{A}^G$. So for every $\alpha \in (0,1]$, $(x + N(\alpha,\alpha)) \cap A \neq \emptyset$; In particular, it will be true for each $\alpha \in (0, \frac{1}{2}]$. This shows that $\bar{A}^G \subset \bar{A}^1$, where $\bar{A}^1$ denote the closure of A in the space $(X, \|\cdot\|_1)$. Finally, suppose that $x \in \bar{A}^1 \cap \bar{A}^2$. Then for each $\alpha > 0$, we have $_1N_\alpha(x) \cap A \neq \emptyset$ and $_2N_\alpha(x) \cap A \neq \emptyset$. Consequently, for each $\alpha \in (0, \frac{1}{2}]$, we have

$$(x + N(\alpha,\alpha)) \cap A = {_1N_\alpha(x)} \cap A \neq \emptyset,$$

and for each $\alpha \in (\frac{1}{2}, 1]$, we have

$$(x + N(\alpha,\alpha)) \cap A = {_2N_\alpha(x)} \cap A \neq \emptyset.$$

This shows that $x \in \bar{A}^G$ and we conclude that $(\bar{A}^1 \cap \bar{A}^2) \subset \bar{A}^G$.

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ABSTRACT KOROVKIN THEOREMS VIA RELATIVE MODULAR CONVERGENCE FOR DOUBLE SEQUENCES OF LINEAR OPERATORS

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Abstract. We will obtain an abstract version of the Korovkin type approximation theorems with respect to the concept of statistical relative convergence in modular spaces for double sequences of positive linear operators. We will give an application showing that our results are stronger than classical ones. We will also study an extension to non-positive operators.
Keywords: Korovkin type approximation; modular spaces; statistical relative convergence; non-positive operators.

1. Introduction

As we know, Korovkin([15]) proved an approximation theorem via simple and easy criterion to check if a sequence of positive linear operators converges uniformly to the function to be approximated. Many researchers studied some versions of this theorem in different spaces and Bardaro and Mantellini studied this theorem on modular spaces which is the natural generalization of L_p $(p > 0)$, Orlicz, Lorentz, and Köthe spaces ([5]) and so on([7, 8]). In addition, general versions of the Korovkin theorem were studied in which a various kind of convergence methods is used, particularly statistical convergence methods([2, 3, 14, 22]). More recently, Demirci and Orhan ([9]) have introduced statistical relative uniform convergence of single sequences by using the notions of the natural density and the relative uniform convergence. Then, many researchers defined some versions of this interesting convergence method and proved Korovkin type approximation theorems for single and double sequences of linear operators in different spaces (see [11, 12, 13, 21, 23]). In [10], we studied Korovkin type theorems in modular function spaces for functions defined on a compact set I^2 where $I = [a, b]$, using the classical test set $\left\{1, x, y, x^2 + y^2\right\}$. In this paper, we will study generalized versions of the Korovkin

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type approximation theorems for a double sequence of positive linear operators $(T_{m,n})$ acting on an abstract modular function space. Then, we will give an application showing that our results are non-trivial extensions of the existing ones. Finally, we will study an extension to non-positive linear operators.

2. Preliminaries

Now we shall recall some well known notations and properties of modular spaces.

Assume G be a locally compact Hausdorff topological space endowed with a uniform structure $\mathcal{U} \subset 2^{G\times G}$ that generates the topology of G (see, [17]). Let $\mathcal{B}$ be the $\sigma-$algebra of all Borel subsets of G and $\mu : \mathcal{B} \to \mathbb{R}$ is a positive $\sigma-$finite regular measure. Let $L^0(G)$ be the space of all real valued $\mu-$measurable functions on G with identification up to sets of measure μ zero, $C(G)$ be the space of all continuous real valued functions on G, $C_b(G)$ be the space of all continuous real valued and bounded functions on G and $C_c(G)$ be the subspace of $C_b(G)$ of all functions with compact support on G. In this case, we say that a functional $\rho : L^0(G) \to [0,\infty]$ is a modular on $L^0(G)$ if it satisfies the following conditions:

(i) $\rho(f) = 0$ if and only if $f = 0$ $\mu-$almost everywhere on G,

(ii) $\rho(-f) = \rho(f)$ for every $f \in L^0(G)$,

(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ for every $f, g \in L^0(G)$ and for any $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

A modular ρ is $N-$quasi convex if there exists a constant $N \ge 1$ such that the inequality

$$\rho(\alpha f + \beta g) \le N\alpha\rho(Nf) + N\beta\rho(Ng)$$

holds for every $f, g \in L^0(G)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Note that if $N = 1$, then ρ is called convex. Furthermore, a modular ρ is $N-$quasi semiconvex if there exists a constant $N \ge 1$ such that

$$\rho(\alpha f) \le N\alpha\rho(Nf)$$

holds for every $f \in L^0(G)$ and $\alpha \in (0,1]$.

The modular space $L^\rho(G)$ generated by ρ, is given by

$$L^\rho(G) := \left\{ f \in L^0(G) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\}$$

and the space of the finite elements of $L^\rho(G)$, is given by

$$E^\rho(G) := \{ f \in L^\rho(G) : \rho(\lambda f) < \infty \text{ for all } \lambda > 0 \}.$$

Also, note that if ρ is $N-$quasi semiconvex, then the space

$$\{ f \in L^0(G) : \rho(\lambda f) < \infty \text{ for some } \lambda > 0 \}$$

coincides with $L^{\rho}(G)$.

We will need the following notions.

A modular ρ is said to be monotone if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$. A modular ρ is finite if $\chi_A \in L^{\rho}(G)$ whenever $A \in \mathcal{B}$ with $\mu(A) < \infty$. A modular ρ is strongly finite if $\chi_A \in E^{\rho}(G)$ for all $A \in \mathcal{B}$ such that $\mu(A) < \infty$ and a modular ρ is said to be absolutely continuous if there exists an $\alpha > 0$ such that: for every $f \in L^0(G)$ with $\rho(f) < \infty$, the following conditions hold:

- for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ such that $\mu(A) < \infty$ and $\rho\left(\alpha f \chi_{G \setminus A}\right) \leq \varepsilon$,
- for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho(\alpha f \chi_B) \leq \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

If a modular ρ is monotone and finite, then $C(G) \subset L^{\rho}(G)$. If ρ is monotone and strongly finite, then $C(G) \subset E^{\rho}(G)$. Also, if ρ is monotone, strongly finite and absolutely continuous, $\overline{C_c(G)} = L^{\rho}(G)$ with respect to the modular convergence (for details and properties see also [16, 18, 20]).

Now we recall the statistical relative modular and strong convergence for double sequences (see also [10]).

Definition 2.1. Let $(f_{m,n})$ be a double function sequence whose terms belong to $L^{\rho}(G)$. Then, $(f_{m,n})$ is said to be statistically *relatively modularly convergent* to a function $f \in L^{\rho}(G)$ if there exists a function $\sigma(u)$, called a scale function $\sigma \in L^0(G)$, $|\sigma(u)| \neq 0$ such that

$$st_2 - \lim_{m,n} \rho\left(\lambda_0\left(\frac{f_{m,n} - f}{\sigma}\right)\right) = 0 \quad \text{for some } \lambda_0 > 0.$$

Also, $(f_{m,n})$ is statistically relatively $F-$*norm convergent* (or, statistically relatively *strongly convergent*) to f iff

$$st_2 - \lim_{m,n} \rho\left(\lambda\left(\frac{f_{m,n} - f}{\sigma}\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in L^0(G)$, see [19].

Note that if the scale function is a non-zero constant, then statistical modular convergence is the special case of statistical relative modular convergence. Moreover, if $\sigma(u)$ is bounded, statistical relative modular convergence implies statistical modular convergence. However, if $\sigma(u)$ is unbounded, then statistical relative modular convergence does not imply statistical modular convergence.

3. Korovkin type approximation theorems

We now prove some Korovkin type theorems with respect to an abstract finite set of test functions $e_0, e_1, ..., e_k$ in the sense of the statistical relative convergence.

Let $\mathbb{T} = (T_{m,n})$ be a double sequence of positive linear operators from D into $L^0(G)$ with $C_b(G) \subset D \subset L^0(G)$. Let ρ be monotone and finite modular on $L^0(G)$ and $\sigma \in L^0(G)$ is an unbounded function satisfying $\sigma(u) \neq 0$. Assume further that the double sequence $\mathbb{T}$, together with modular ρ, satisfies the following property:

there exists a subset $X_{\mathbb{T}} \subset D \cap L^\rho(G)$ with $C_b(G) \subset X_{\mathbb{T}}$ such that the inequality

$$st_2 - \limsup_{m,n} \rho\left(\lambda\left(\frac{T_{m,n}h}{\sigma}\right)\right) \leq R\rho(\lambda h) \tag{3.1}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R.

Set $e_0(v) \equiv 1$ for all $v \in G$, let e_r, $r = 1, 2, ..., k$ and a_r, $r = 0, 1, 2, ..., k$, be functions in $C_b(G)$. Put

$$P_u(v) = \sum_{r=0}^{k} a_r(u)\, e_r(v), \ u, v \in G, \tag{3.2}$$

and suppose that $P_u(v)$, $u, v \in G$, satisfies the following properties:

$(P1)$ $P_u(u) = 0$, for all $u \in G$,

$(P2)$ for every neighbourhood $U \in \mathcal{U}$ there is a positive real number η with $P_u(v) \geq \eta$ whenever $u, v \in G$, $(u, v) \notin U$ (see for examples [4]).

In order to obtain our main theorem, we will first give the following result.

Theorem 3.1. *Let ρ be a monotone, strongly finite and $N-$quasi semiconvex modular. Suppose that e_r and a_r, $r = 0, 1, 2, ..., k$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{m,n})$ be a double sequence of positive linear operators from D into $L^0(G)$ and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)| \geq b_r > 0$ $(r = 0, 1, 2, ..., k)$. If*

$$st_2 - \lim_{m,n} \rho\left(\lambda_0\left(\frac{T_{m,n}e_r - e_r}{\sigma_r}\right)\right) = 0 \quad \textit{for some } \lambda_0 > 0, \tag{3.3}$$

$r = 0, 1, 2, ..., k$, in $L^\rho(G)$ then for every $f \in C_c(G)$

$$st_2 - \lim_{m,n} \rho\left(\gamma\left(\frac{T_{m,n}f - f}{\sigma}\right)\right) = 0 \quad \textit{for some } \gamma > 0 \tag{3.4}$$

in $L^\rho(G)$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., k\}$. If

$$st_2 - \lim_{m,n} \rho\left(\lambda\left(\frac{T_{m,n}e_r - e_r}{\sigma_r}\right)\right) = 0 \quad \textit{for every } \lambda > 0,$$

$r=0,1,2,...,k,$ in $L^{\rho}(G)$ then for every $f\in C_c(G)$

$$st_2-\lim_{m,n}\rho\left(\lambda\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)=0 \quad \text{for every } \lambda>0$$

in $L^{\rho}(G)$ where $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,k\}$.

Proof. *We first claim that, for every $f\in C_c(G)$,*

$$st_2-\lim_{m,n}\rho\left(\gamma\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)=0 \quad \text{for some } \gamma>0. \tag{3.5}$$

To see this assume that $f\in C_c(G)$. Then, since G is endowed with the uniform structure $\mathcal{U}$, f is uniformly continuous and bounded on G. By the uniform continuity of f, choose $\varepsilon\in(0,1]$, there exists a set $U\in\mathcal{U}$ such that $|f(u)-f(v)|\leq\varepsilon$ whenever $u,v\in G$, $(u,v)\in U$.

For all $u,v\in G$ let $P_u(v)$ be as in (3.2), and $\eta>0$ satisfy condition $(P2)$. Then for $u,v\in G$, $(u,v)\notin U$, we have $|f(u)-f(v)|\leq\frac{2M}{\eta}P_u(v)$ where $M:=\sup\limits_{v\in G}|f(v)|$. Therefore, in any case we get $|f(u)-f(v)|\leq\varepsilon+\frac{2M}{\eta}P_u(v)$ for all $u,v\in G$, namely,

$$-\varepsilon-\frac{2M}{\eta}P_u(v)\leq f(u)-f(v)\leq\varepsilon+\frac{2M}{\eta}P_u(v). \tag{3.6}$$

Since $T_{m,n}$ is linear and positive, by applying $T_{m,n}$ to (3.6) for every $m,n\in\mathbb{N}$ we have

$$\begin{aligned}-\varepsilon T_{m,n}(e_0;u)-\frac{2M}{\eta}T_{m,n}(P_u;u) &\leq f(u)T_{m,n}(e_0;u)-T_{m,n}(f;u)\\ &\leq \varepsilon T_{m,n}(e_0;u)+\frac{2M}{\eta}T_{m,n}(P_u;u).\end{aligned}$$

Hence

$$\begin{aligned}|T_{m,n}(f;u)-f(u)| &\leq |T_{m,n}(f;u)-f(u)T_{m,n}(e_0;u)|\\ &\quad+|f(u)|\,|T_{m,n}(e_0;u)-e_0(u)|\\ &\leq \varepsilon T_{m,n}(e_0;u)+\frac{2M}{\eta}T_{m,n}(P_u;u)\\ &\quad+M\,|T_{m,n}(e_0;u)-e_0(u)|\\ &\leq \varepsilon+(\varepsilon+M)\,|T_{m,n}(e_0;u)-e_0(u)|\\ &\quad+\frac{2M}{\eta}\sum_{r=0}^{k}a_r(u)\,|T_{m,n}(e_r;u)-e_r(u)|.\end{aligned}$$

Let $\gamma>0$. Now for each $r=0,1,2,...,k$ and $u\in G$, choose $M_0>0$ such that $|a_r(u)|\leq M_0$ and multiplying the both sides of the above inequality by $\frac{1}{|\sigma(u)|}$, the last inequality gives that

$$\gamma\left|\frac{T_{m,n}(f;u)-f(u)}{\sigma(u)}\right|\leq\frac{\gamma\varepsilon}{|\sigma(u)|}+K\gamma\sum_{r=0}^{k}\left|\frac{T_{m,n}(e_r;u)-e_r(u)}{\sigma(u)}\right|$$

where $K := \varepsilon + M + \frac{2M}{\eta} M_0$. *Now, applying the modular* ρ *to both sides of the above inequality, since* ρ *is monotone and* $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., k\}$, *we get*

$$\rho\left(\gamma\left(\frac{T_{m,n}f - f}{\sigma}\right)\right) \leq \rho\left(\frac{\gamma\varepsilon}{|\sigma|} + K\gamma \sum_{r=0}^{k} \left|\frac{T_{m,n}e_r - e_r}{\sigma_r}\right|\right).$$

Thus, we can see that

$$\rho\left(\gamma\left(\frac{T_{m,n}f - f}{\sigma}\right)\right) \leq \rho\left(\frac{(k+2)\gamma\varepsilon}{\sigma}\right) + \sum_{r=0}^{k} \rho\left((k+2) K\gamma\left(\frac{T_{m,n}e_r - e_r}{\sigma_r}\right)\right).$$

Let $\lambda_0 > 0$ *be as in the hypothesis (3.3), such* $\lambda_0 > 0$, *by hypothesis, does exist. Let* $\gamma > 0$ *be with* $(k+2) K\gamma \leq \lambda_0$ *and since* ρ *is* $N-$*quasi semiconvex and strongly finite, we have,*

$$\text{(3.7)} \quad \rho\left(\gamma\left(\frac{T_{m,n}f - f}{\sigma}\right)\right) \leq N\varepsilon\rho\left(\frac{(k+2)\gamma N}{\sigma}\right) + \sum_{r=0}^{k} \rho\left(\lambda_0\left(\frac{T_{m,n}e_r - e_r}{\sigma_r}\right)\right).$$

For a given $\varepsilon^* > 0$, *choose an* $\varepsilon \in (0, 1]$ *such that* $N\varepsilon\rho\left(\frac{(k+2)\gamma N}{\sigma}\right) < \varepsilon^*$. *Now define the following sets:*

$$\begin{aligned} S_\gamma &: = \left\{(m,n) : \rho\left(\gamma\left(\frac{T_{m,n}f - f}{\sigma}\right)\right) \geq \varepsilon^*\right\} \\ S_{\gamma,r} &: = \left\{(m,n) : \rho\left(\lambda_0\left(\frac{T_{m,n}e_r - e_r}{\sigma_r}\right)\right) \geq \frac{\varepsilon^* - N\varepsilon\rho\left(\frac{(k+2)\gamma N}{\sigma}\right)}{k+1}\right\}, \end{aligned}$$

where $r = 0, 1, 2, ..., k$. *Then, it is easy to see that* $S_\gamma \subseteq \bigcup_{r=0}^{k} S_{\gamma,r}$. *So, we can see that*

$$\delta_2(S_\gamma) \leq \sum_{r=0}^{k} \delta_2(S_{\gamma,r}).$$

Using the hypothesis (3.3), we get

$$\delta_2(S_\gamma) = 0,$$

which proves our claim (3.5).

The last part of theorem can be proved similarly to the first one. □

Now, we can give our main theorem of this paper.

Theorem 3.2. *Let* ρ *be a monotone, strongly finite, absolutely continuous and* $N-$*quasi semiconvex modular. Suppose that* e_r *and* a_r, $r = 0, 1, 2, ..., k$, *satisfy* *properties* $(P1)$ *and* $(P2)$. *Let* $\mathbb{T}=(T_{m,n})$ *be a double sequence of positive linear operators satisfying (3.1) and assume that* $\sigma_r(u)$ *is an unbounded function satisfying* $|\sigma_r(u)|\geq b_r>0$ $(r=0,1,2,...,k)$. *If*

$$st_2-\lim_{m,n}\rho\left(\lambda\left(\frac{T_{m,n}e_r-e_r}{\sigma_r}\right)\right)=0 \ \ \textit{for every } \lambda>0,$$

$r=0,1,2,...,k$ *in* $L^{\rho}(G)$, *then for every* $f\in D\cap L^{\rho}(G)$ *with* $f-C_b(G)\subset X_{\mathbb{T}}$,

$$st_2-\lim_{m,n}\rho\left(\lambda_0\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)=0 \ \ \textit{for some } \lambda_0>0$$

in $L^{\rho}(G)$ *where* $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,k\}$ *and* D, $X_{\mathbb{T}}$ *are as before.*

Proof. Let $f\in D\cap L^{\rho}(G)$ with $f-C_b(G)\subset X_{\mathbb{T}}$. It is known from [6, 18] that there exists a sequence $(g_{k,j})\subset C_c(G)$ such that $\rho(3\lambda_0^{*}f)<\infty$ and $P-\lim\limits_{k,j}\rho(3\lambda_0^{*}(g_{k,j}-f))=0$ for some $\lambda_0^{*}>0$. This means that, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ with

$$\rho(3\lambda_0^{*}(g_{k,j}-f))<\varepsilon \text{ for every } k,j\geq k_0. \tag{3.8}$$

For all $m,n\in\mathbb{N}$, *by linearity and positivity of the operators* $T_{m,n}$, *we have*

$$\begin{aligned}\lambda_0^{*}|T_{m,n}(f;u)-f(u)| &\leq \lambda_0^{*}|T_{m,n}(f-g_{k_0,k_0};u)|+\lambda_0^{*}|T_{m,n}(g_{k_0,k_0};u)-g_{k_0,k_0}(u)|\\ &\quad+\lambda_0^{*}|g_{k_0,k_0}(u)-f(u)|\end{aligned}$$

holds for every $u\in G$. *Now, applying the modular* ρ *in the last inequality and using the monotonicity of* ρ *and moreover multiplying both sides of the above inequality by* $\frac{1}{|\sigma(u)|}$, *we get*

$$\begin{aligned}&\rho\left(\lambda_0^{*}\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)\\ \leq\ &\rho\left(3\lambda_0^{*}\left(\frac{T_{m,n}(f-g_{k_0,k_0})}{\sigma}\right)\right)+\rho\left(3\lambda_0^{*}\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right)\\ &+\rho\left(3\lambda_0^{*}\left(\frac{g_{k_0,k_0}-f}{\sigma}\right)\right).\end{aligned}$$

Hence, observing $|\sigma|\geq b>0$, $(b=\max\{b_r:r=0,1,2,...,k\})$, *we can write that*

$$\begin{aligned}&\rho\left(\lambda_0^{*}\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)\\ \leq\ &\rho\left(3\lambda_0^{*}\left(\frac{T_{m,n}(f-g_{k_0,k_0})}{\sigma}\right)\right)+\rho\left(3\lambda_0^{*}\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right)\\ &+\rho\left(\frac{3\lambda_0^{*}}{b}(g_{k_0,k_0}-f)\right).\end{aligned} \tag{3.9}$$

Then using the (3.8) in (3.9), we have

$$\begin{aligned}\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right) &\leq \varepsilon+\rho\left(3\lambda_0^*\left(\frac{T_{m,n}\left(f-g_{k_0,k_0}\right)}{\sigma}\right)\right)\\ &\quad+\rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right).\end{aligned}$$

By property (3.1) and also using the facts that $g_{k_0,k_0}\in C_c(G)$ and $f-g_{k_0,k_0}\in X_{\mathbb{T}}$, we obtain

$$\begin{aligned}&st_2-\limsup_{m,n}\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)\\ &\leq \varepsilon+R\rho\left(3\lambda_0^*\left(f-g_{k_0,k_0}\right)\right)+st_2-\limsup_{m,n}\rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right)\\ &\leq \varepsilon\left(1+R\right)+st_2-\limsup_{m,n}\rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right)\end{aligned}$$

also, resulting from previous theorem,

$$\begin{aligned}0 &= st_2-\lim_{m,n}\rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right)\\ &= st_2-\limsup_{m,n}\rho\left(3\lambda_0^*\left(\frac{T_{m,n}g_{k_0,k_0}-g_{k_0,k_0}}{\sigma}\right)\right)\end{aligned}$$

which gives

$$0\leq st_2-\limsup_{m,n}\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)\leq\varepsilon\left(1+R\right).$$

From arbitrariety of $\varepsilon>0$, it follows that

$$st_2-\limsup_{m,n}\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)=0.$$

Furthermore,

$$st_2-\lim_{m,n}\rho\left(\lambda_0^*\left(\frac{T_{m,n}f-f}{\sigma}\right)\right)=0,$$

and this completes the proof. □

Remark 3.1. Note that, in Theorem 3.2, in general it is not possible to obtain statistical relative strong convergence unless the modular ρ satisfies the Δ_2-condition.

If one replaces the scale function by a nonzero constant, then the condition (3.1) is reduced to

$$st_2-\limsup_{m,n}\rho\left(\lambda\left(T_{m,n}h\right)\right)\leq R\rho\left(\lambda h\right) \tag{3.10}$$

for every $h\in X_{\mathbb{T}}$, $\lambda>0$ and for an absolute positive constant R. In this case, the next result immediately follows from our Theorem 3.2.

Corollary 3.1. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that e_r and a_r, $r = 0,1,2,...,k$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{m,n})$ be a double sequence of positive linear operators satisfying (3.10). If $(T_{m,n}e_r)$ is statistically strongly convergent to e_r, $r = 0,1,2,...,k$, in $L^\rho(G)$, then $(T_{m,n}f)$ is statistically modularly convergent to f in $L^\rho(G)$ such that f is any function belonging to $D \cap L^\rho(G)$ with $f - C_b(G) \subset X_{\mathbb{T}}$.*

If one replaces the statistical limit by the Pringsheim limit, then the condition (3.1) is reduced to

$$P - \limsup_{m,n} \rho\left(\lambda\left(\frac{T_{m,n}h}{\sigma}\right)\right) \leq R\rho(\lambda h) \tag{3.11}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the following result immediately follows from our Theorem 3.2.

Corollary 3.2. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that e_r and a_r, $r = 0,1,2,...,k$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T} = (T_{m,n})$ be a double sequence of positive linear operators satisfying (3.11) and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)| \geq b_r > 0$ $(r = 0,1,2,...,k)$. If $(T_{m,n}e_r)$ is relatively strongly convergent to e_r to the scale function σ_r, $r = 0,1,2,...,k$, in $L^\rho(G)$ then $(T_{m,n}f)$ is relatively modularly convergent to f to the scale function σ in $L^\rho(G)$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0,1,2,...,k\}$ and f is any function belonging to $D \cap L^\rho(G)$ with $f - C_b(G) \subset X_{\mathbb{T}}$.*

Now, we give an application showing that in general, our results are stronger than classical ones.

Example 3.1. Let us consider $G = [0,1]^2 \subset \mathbb{R}^2$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and convex function with $\varphi(0) = 0$, $\varphi(x) > 0$ *for any* $x > 0$ *and* $\lim_{x\to\infty} \varphi(x) = \infty$. Then, the functional ρ^φ defined by

$$\rho^\varphi(f) := \int_0^1 \int_0^1 \varphi(|f(x,y)|)\, dxdy \quad \text{for} \quad f \in L^0(G),$$

is a convex modular on $L^0(G)$ and

$$L^\varphi(G) := \left\{f \in L^0(G) : \rho^\varphi(\lambda f) < +\infty \quad \text{for some } \lambda > 0\right\}$$

is the Orlics space generated by φ.

For every $(x,y) \in G$, let $e_0(x,y) = a_3(x,y) = 1$, $e_1(x,y) = x$, $e_2(x,y) = y$, $e_3(x,y) = a_0(x,y) = x^2 + y^2$, $a_1(x,y) = -2x$, $a_2(x,y) = -2y$. For every $m,n \in \mathbb{N}$, $u_1, u_2 \in [0,1]$, let $K_{m,n}(u_1,u_2) = (m+1)(n+1)u_1^m u_2^n$ and for $f \in C(G)$ and $x,y \in [0,1]$ set

$$M_{m,n}(f;x,y) = \int_0^1 \int_0^1 K_{m,n}(u_1,u_2)\, f(u_1 x, u_2 y)\, du_1 du_2.$$

Then we get

$$\begin{aligned}&\int_0^1\int_0^1 K_{m,n}(u_1,u_2)\,du_1du_2\\ &= (m+1)\left(\int_0^1 u_1^m du_1\right)(n+1)\left(\int_0^1 u_2^n du_2\right)=1,\end{aligned}$$

and hence, $M_{m,n}(e_0;x,y)=e_0(x,y)=1$. Also, we know from [3] that

$$\begin{aligned}|M_{m,n}(e_1;x,y)-e_1(x,y)| &\le \frac{1}{m+2},\ \ |M_{m,n}(e_2;x,y)-e_2(x,y)|\le\frac{1}{n+2},\\ \left|M_{m,n}(e_1^2;x,y)-e_1^2(x,y)\right| &\le \frac{2}{m+3},\ \ \left|M_{m,n}(e_2^2;x,y)-e_2^2(x,y)\right|\le\frac{2}{n+3},\end{aligned}$$

and for each $m,n\ge 2$, $f\in L^{\varphi}(G)$ we get $\rho^{\varphi}(M_{m,n}f)\le 32\rho^{\varphi}(f)$. Moreover, $(M_{m,n})$ satisfies the condition (14) in [22] with $X_{\mathbb{M}}=L^{\varphi}(G)$ and $(M_{m,n}f)$ is modulary convergent to $f\in L^{\varphi}(G)$. Using the operators $\mathbb{M}=(M_{m,n})$, we define the double sequence of positive linear operators $\mathbb{T}=(T_{m,n})$ on $L^{\varphi}(G)$ as follows:

$$T_{m,n}(f;x,y)=(1+g_{m,n}(x,y))\,M_{m,n}(f;x,y),\ \text{ for } f\in L^{\varphi}(G),$$

$x,y\in[0,1]$ and $m,n\in\mathbb{N}$, where $g_{m,n}:G\to\mathbb{R}$ defined by

$$g_{m,n}(x,y)=\begin{cases}1, & m=k^2 \text{ and } n=l^2\\ \left(m^2+1\right)n^3xy, & (x,y)\in\left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right);m\neq k^2 \text{ and } n\neq l^2\\ 0, & (x,y)\notin\left(0,\frac{1}{m}\right)\times\left(0,\frac{1}{n}\right);\ m\neq k^2 \text{ and } n\neq l^2\end{cases}$$

$k,l=1,2,\ldots$ If $\varphi(x)=x^p$ for $1\le p<\infty$, $x\ge 0$ then $L^{\varphi}(G)=L_p(G)$ and we have for any function $f\in L^{\varphi}(G)$, $\rho^{\varphi}(f)=\|f\|_p^p$. From now on we have $p=1$.

It is clear that

$$\begin{aligned}\rho(\lambda_0(g_{m,n}-g)) &= \|\lambda_0(g_{m,n}-g)\|_1\\ &= \lambda_0\begin{cases}1, & m=k^2 \text{ and } n=l^2\\ \frac{(m^2+1)n}{4m^2}, & m\neq k^2 \text{ and } n\neq l^2\end{cases},k,l=1,2,\ldots,\end{aligned}$$

where $g=0$, then $(g_{m,n})$ does not converge statistically modularly to $g=0$. Now, we choose $\sigma_r(x,y)=\sigma(x,y)$ $(r=0,1,2,3)$ where $\sigma(x,y)=\begin{cases}\frac{1}{x^2y}, & (x,y)\in(0,1]\times(0,1]\\ 1, & \text{otherwise}\end{cases}$ *on* $L_1(G)$. Then, it can be seen that, for every $h\in L_1(G)$, $\lambda>0$ and for positive constant R_0 that

$$st_2-\limsup_{m,n}\left\|\lambda\left(\frac{T_{m,n}h}{\sigma}\right)\right\|_1\le R_0\|\lambda h\|_1.$$

Now, observe that

$$\begin{aligned}T_{m,n}(e_0;x,y)-e_0(x,y) &= g_{m,n}(x,y),\\ T_{m,n}(e_1;x,y)-e_1(x,y) &\le \frac{1+g_{m,n}(x,y)}{m+2}+g_{m,n}(x,y),\\ T_{m,n}(e_2;x,y)-e_2(x,y) &\le \frac{1+g_{m,n}(x,y)}{n+2}+g_{m,n}(x,y),\\ T_{m,n}(e_3;x,y)-e_3(x,y) &\le (1+g_{m,n}(x,y))\left(\frac{2}{m+3}+\frac{2}{n+3}\right)+2g_{m,n}(x,y).\end{aligned}$$

Hence, we can see, for any $\lambda > 0$, that

$$\begin{aligned} & \left\| \lambda \left(\frac{T_{m,n} e_0 - e_0}{\sigma} \right) \right\|_1 \\ (3.12) \qquad = & \left\| \lambda \left(\frac{g_{m,n}}{\sigma} \right) \right\|_1 = \lambda \begin{cases} 1, & m = k^2 \text{ and } n = l^2 \\ \frac{m^2+1}{12m^4}, & m \neq k^2 \text{ and } n \neq l^2 \end{cases}, k, l = 1, 2, ..., \end{aligned}$$

then, we get

$$st_2 - \lim_{m,n} \left\| \lambda \left(\frac{T_{m,n} e_0 - e_0}{\sigma} \right) \right\|_1 = 0.$$

Also, we have

$$\begin{aligned} \left\| \lambda \left(\frac{T_{m,n} e_1 - e_1}{\sigma} \right) \right\|_1 & \leq \left\| \frac{\lambda}{\sigma} \left(\frac{1 + g_{m,n}}{m+2} + g_{m,n} \right) \right\|_1 \\ & \leq \frac{\lambda}{m+2} \left\| \frac{1 + g_{m,n}}{\sigma} \right\|_1 + \left\| \lambda \left(\frac{g_{m,n}}{\sigma} \right) \right\|_1 \\ & \leq \frac{1}{m+2} \left(\frac{\lambda}{6} + \left\| \lambda \left(\frac{g_{m,n}}{\sigma} \right) \right\|_1 \right) + \left\| \lambda \left(\frac{g_{m,n}}{\sigma} \right) \right\|_1, \end{aligned}$$

from above inequality, since the equality (3.12), we have

$$st_2 - \lim_{m,n} \left\| \lambda \left(\frac{T_{m,n} e_1 - e_1}{\sigma} \right) \right\|_1 = 0.$$

Similarly, we get

$$st_2 - \lim_{m,n} \left\| \lambda \left(\frac{T_{m,n} e_2 - e_2}{\sigma} \right) \right\|_1 = 0.$$

Finally, since

$$\begin{aligned} & \left\| \lambda \left(\frac{T_{m,n} e_3 - e_3}{\sigma} \right) \right\|_1 \\ \leq & \left\| \frac{\lambda}{\sigma} \left((1 + g_{m,n}(x,y)) \left(\frac{2}{m+3} + \frac{2}{n+3} \right) + 2 g_{m,n}(x,y) \right) \right\|_1 \\ \leq & \left(\frac{2}{m+3} + \frac{2}{n+3} \right) \left(\frac{\lambda}{6} + \left\| \lambda \left(\frac{g_{m,n}}{\sigma} \right) \right\|_1 \right) + 2 \left\| \lambda \left(\frac{g_{m,n}}{\sigma} \right) \right\|_1, \end{aligned}$$

then,

$$st_2 - \lim_{m,n} \left\| \lambda \left(\frac{T_{m,n} e_3 - e_3}{\sigma} \right) \right\|_1 = 0.$$

So, our new operator $\mathbb{T} = (T_{m,n})$ satisfies all conditions of Theorem 3.2 and therefore we obtain

$$st_2 - \lim_{m,n} \left\| \lambda_0 \left(\frac{T_{m,n} f - f}{\sigma} \right) \right\|_1 = 0$$

for some $\lambda_0 > 0$, for any $f \in L_1(G)$. However, $(T_{m,n} e_0)$ is neither relatively modularly convergent *to the scale function* σ nor statistically modularly convergent. Thus $(T_{m,n})$ does not fulfil the Corollary 3.1 and 3.2.

4. An Extension to Non-Positive Operators

In this section, we relax the positivity condition of linear operators in the Korovkin theorems. In [1, 3, 4] there are some positive answers. Following this approach, we give some positive answers for statistical relative modular convergence and prove a Korovkin type approximation theorem.

Let I be a bounded interval of $\mathbb{R}$, $C^2(I)$ (resp. $C_b^2(I)$) be the space of all functions defined on I, (resp. bounded and) continuous together with their first and second derivatives, $C_+ := \left\{f \in C_b^2(I) : f \geq 0\right\}$, $C_+^2 := \left\{f \in C_b^2(I) : f'' \geq 0\right\}$.

Let e_r, $r = 1, 2, ..., k$ and a_r, $r = 0, 1, 2, ..., k$, be functions in $C_b^2(I)$, $P_u(v)$, $u, v \in I$, be as in (3.2), and suppose that $P_u(v)$ satisfies the properties $(P1)$, $(P2)$ and

$(P3)$ there is a positive real constant S_0 such that $P_u''(v) \geq S_0$ for all $u, v \in I$ (Here the second derivative is intended with respect to v).

Now we prove the following Korovkin type approximation theorem for not necessarily positive linear operators.

Theorem 4.1. *Let ρ and σ_r be as in Theorem 3.1 and e_r, a_r, $r = 0, 1, 2, ..., k$ and $P_u(v)$, $u, v \in I$, satisfies the properties $(P1)$, $(P2)$ and $(P3)$. Assume that $\mathbb{T} = (T_{m,n})$ be a double sequence of linear operators and $T_{m,n}\left(C_+ \cap C_+^2\right) \subset C_+$ for all $m, n \in \mathbb{N}$. If $T_{m,n}e_r$ is statistically relatively modularly convergent to e_r to the scale function σ_r in $L^\rho(I)$ for each $r = 0, 1, 2, ..., k$, then $T_{m,n}f$ is statistically relatively modularly convergent to f to the scale function σ in $L^\rho(I)$ for every $f \in C_b^2(I)$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., k\}$.*

If $T_{m,n}e_r$ is statistically relatively strongly convergent to e_r to the scale function σ_r, $r = 0, 1, 2, ..., k$, in $L^\rho(I)$ then $T_{m,n}f$ is statistically relatively strongly convergent to f to the scale function σ in $L^\rho(I)$ for every $f \in C_b^2(I)$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., k\}$.

Furthermore, if ρ is absolutely continuous, $\mathbb{T}$ satisfies the property (3.1) and $T_{m,n}e_r$ is statistically relatively strongly convergent to e_r to the scale function σ_r, $r = 0, 1, 2, ..., k$, in $L^\rho(I)$ then $T_{m,n}f$ is statistically relatively modularly convergent to f to the scale function σ in $L^\rho(I)$ for every $f \in D \cap L^\rho(G)$ with $f - C_b(I) \subset X_{\mathbb{T}}$ where $\sigma(u) = \max\{|\sigma_r(u)| : r = 0, 1, 2, ..., k\}$.

Proof. *Let $f \in C_b^2(I)$. Since f is uniformly continuous and bounded on I, given $\varepsilon > 0$ with $0 < \varepsilon \leq 1$, there exists a $\delta > 0$ such that $|f(u) - f(v)| \leq \varepsilon$ for all $u, v \in I$, $|u - v| \leq \delta$. Let $P_u(v)$, $u, v \in I$, be as in (3.2) and let $\eta > 0$ be associated with δ, satisfying $(P2)$. As in Theorem 3.1, for every $\beta \geq 1$ and $u, v \in I$, we have*

$$-\varepsilon - \frac{2M\beta}{\eta}P_u(v) \leq f(u) - f(v) \leq \varepsilon + \frac{2M\beta}{\eta}P_u(v) \tag{4.1}$$

where $M = \sup_{v \in I}|f(v)|$. From (4.1) it follows that

$$h_{1,\beta}(v) := \varepsilon + \frac{2M\beta}{\eta}P_u(v) + f(v) - f(u) \geq 0, \tag{4.2}$$

$$h_{2,\beta}(v) := \varepsilon + \frac{2M\beta}{\eta} P_u(v) - f(v) + f(u) \geq 0. \tag{4.3}$$

Let H_0 satisfy $(P3)$. For each $v \in I$, we get

$$h''_{1,\beta}(v) \geq \frac{2M\beta H_0}{\eta} + f''(v), \ h''_{2,\beta}(v) \geq \frac{2M\beta H_0}{\eta} - f''(v).$$

Because of f'' is bounded on I, we can choose $\beta \geq 1$ in such a way that $h''_{1,\beta}(v) \geq 0$, $h''_{2,\beta}(v) \geq 0$ for each $v \in I$. Hence $h_{1,\beta}, h_{2,\beta} \in C_+ \cap C_+^2$ and then, by hypothesis

$$T_{m,n}(h_{j,\beta}; u) \geq 0 \text{ for all } m, n \in \mathbb{N}, \ u \in I \text{ and } j = 1, 2. \tag{4.4}$$

From (4.2)-(4.4) and the linearity of $T_{m,n}$, we get

$$\begin{aligned} \varepsilon T_{m,n}(e_0; u) + \frac{2M\beta}{\eta} T_{m,n}(P_u; u) + T_{m,n}(f; u) - f(u) T_{m,n}(e_0; u) &\geq 0, \\ \varepsilon T_{m,n}(e_0; u) + \frac{2M\beta}{\eta} T_{m,n}(P_u; u) - T_{m,n}(f; u) + f(u) T_{m,n}(e_0; u) &\geq 0, \end{aligned}$$

thus,

$$\begin{aligned} -\varepsilon T_{m,n}(e_0; u) - \frac{2M\beta}{\eta} T_{m,n}(P_u; u) &\leq f(u) T_{m,n}(e_0; u) - T_{m,n}(f; u) \\ &\leq \varepsilon T_{m,n}(e_0; u) + \frac{2M\beta}{\eta} T_{m,n}(P_u; u). \end{aligned}$$

By arguing similarly as in the proof of Theorem 3.1, multiplying the inequality by $\frac{1}{|\sigma(u)|}$, using the modular ρ and for $m, n \in \mathbb{N}$, we have the assertion of the first part.

The other parts can be proved similarly as in the proofs of Theorems 3.1 and 3.2. □

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NEW INEQUALITIES ON LIPSCHITZ FUNCTIONS

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Abstract. In this study, some inequalities of Hermite Hadamard type obtained for p-convex functions are given for Lipschitz mappings. Also, some applications for special means have been given.

Keywords: Hermite Hadamard inequalities; p-convex functions; convex functions.

1. Preliminaries and fundamentals

Definition 1.1. [21] A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$.

This definition is well known in literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

One of the most important integral inequalities for convex functions is the Hermite-Hadamard inequality. The following double inequality is well known as the Hadamard inequality in literature.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality.

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Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [2, 3, 4, 11, 17, 20, 23, 24]).

Let us recall some definitions of several kinds of convex functions:

Definition 1.2. [5] Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then the function f is said to be harmonically concave.

Definition 1.3. [7] Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f\left(\left[tx^p+(1-t)\,y^p\right]^{\frac{1}{p}}\right) \leq tf(x)+(1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. If this inequality is reversed, then the function f is said to be p-concave.

According to this definition, it can be easily seen that for $p = 1$ and $p = -1$, p-convexity is reduced to ordinary convexity and harmonical convexity of functions defined on $I \subset (0,\infty)$, respectively.

Hermite-Hadamard inequalities for the p-convex function are the following:

Theorem 1.2. *[7] Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$ and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

These inequalities are sharp [7]. We refer the reader to the recent papers related to p-convexity (see [13, 14, 15, 16, 18, 19]) and references therein.

The purpose of this article is to obtain new inequalities on the right and left sides of the inequality (1.2) for Lipshitz functions.

Definition 1.4. [22] $f : I \to \mathbb{R}$ is said to satisfy the Lipschitz condition if there is a constant $M > 0$ such that

$$|f(x)-f(y)| \leq M\,|x-y|\,, \quad \forall x, y \in I.$$

Theorem 1.3. *[22] If $f : I \to \mathbb{R}$ is convex, then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Consequently, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Lemma 1.1. [1] *Let the function $f : I \subset \mathbb{R} \to \mathbb{R}$ a differentiable function on interval I, $a, b \in I$ with $a < b$ and $M = \sup_{t\in[a,b]} |f'(t)| < \infty$. Then the function f is an M-Lipschitzian functions.*

In [1], Dragomir et al. obtained new inequalities on the right and left sides of the inequality (1.1) for Lipshitz functions as follows.

Theorem 1.4. *[1] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an M-Lipschitzian mapping on I and $a, b \in I$ with $a < b$. Then we have the inequalities:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{4}(b-a)$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{3}(b-a).$$

Corollary 1.1. *[1]Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a convex and differentiable function on interval I and $a, b \in I$ with $a < b$ and $M := \sup_{t\in[a,b]} |f'(x)| < \infty$. Then we have the inequalities:*

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{M}{4}(b-a)$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{M}{3}(b-a).$$

Some results obtained in this study are reduced to the results of Theorem 1.4 and Corollary 1.1 in special cases. For more recent results connected with inequalities of the Hermite-Hadamard type on Lipschitzian functions, see [6, 8, 9, 10].

In the following part, we will give some necessary definitions and simple mathematical inequalities that will be used to achieve our main results.

Definition 1.5. [12] The beta function denoted by $\beta(m,n)$ is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

Definition 1.6. [12] The hypergeometric function denoted by ${}_2F_1(a,b;c;z)$ is defined by the integral equality

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \ \ c > b > 0, \ \ |z| < 1.$$

The following Lemmas are well known in literature. Especially, Lemma 1.3 and Lemma 1.4 are easily seen from the Lagrange mean value theorem.

Lemma 1.2. *Let $0 \leq x < y$. Then the following inequality holds for $0 < \alpha \leq 1$:*

$$\left|y^{\alpha} - x^{\alpha}\right| \leq |y - x|^{\alpha}.$$

Lemma 1.3. *Let $0 < x < y$. Then the following inequality holds for $\alpha < 0$:*

$$\left|y^{\alpha} - x^{\alpha}\right| \leq |y - x| \left(-\alpha\right) x^{\alpha - 1}.$$

Lemma 1.4. *Let $0 < x < y$. Then the following inequality holds for $\alpha \geq 1$:*

$$\left|y^{\alpha} - x^{\alpha}\right| \leq |y - x| \, \alpha y^{\alpha - 1}.$$

Let $0 < x < y$, throughout this paper we will use

$$\begin{aligned}
A &= A\left(x,y\right) = \frac{x+y}{2} \\
G &= G\left(x,y\right) = \sqrt{xy} \\
H &= H(x,y) = \frac{2xy}{x+y} \\
M_p &= M_p(x,y) = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} \\
I &= I(x,y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} \\
L &= L\left(x,y\right) = \frac{x-y}{\ln x - \ln y} \\
L_p &= L_p\left(x,y\right) = \left(\frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)}\right)^{\frac{1}{p}}, \ p \neq -1, 0
\end{aligned}$$

for the arithmetic, geometric, harmonic, power mean of the order p, identric, logarithmic and p-logarithmic mean, respectively.

2. Main results

In this section, we shall establish some Hermite-Hadamard-type inequalities for Lipschitzian functions.

Theorem 2.1. *$f : I \subset (0,\infty) \to \mathbb{R}$ be an M-Lipschitzian function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then following inequalities hold:*

a) For $p \geq 1$;

$$i) \quad \left| \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \right| \leq 2M\left|b^p-a^p\right|^{\frac{1}{p}} \frac{p^2}{(p+1)(2p+1)}$$

$$ii) \quad \left| \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \right| \leq M\left(\frac{1}{2}\right)^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}} \frac{p}{p+1}.$$

b) For $p < 0$;

$$\begin{aligned} i) \quad & \left| \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \right| \\ & \leq M\frac{b^p-a^p}{6pb^{p-1}}\left[{}_2F_1\left(1-\frac{1}{p},2;4;1-\left(\frac{b}{a}\right)^p\right)+1\right], \\ ii) \quad & \left| \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \right| \\ & \leq M\frac{b^p-a^p}{2p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\left[{}_2F_1\left(1-\frac{1}{p},1;3;1-\left(\frac{b}{a}\right)^p\right)+1\right] \end{aligned}$$

c) For $0 < p < 1$;

$$\begin{aligned} i) \quad & \left| \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \right| \leq M\frac{b^p-a^p}{6p}\left[a^{1-p}+b^{1-p}\right], \\ ii) \quad & \left| \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \right| \\ & \leq M\frac{b^p-a^p}{4p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right). \end{aligned}$$

Proof. *a)* For $p \geq 1$:

(i) Using Lemma 1.2 and taking into account that f is an M-Lipschitzian function on interval I, we have

$$\begin{aligned} & \left| tf(a)+(1-t)f(b) - f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right) \right| \\ = & \left| t\left(f(a) - f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right) + (1-t)\left(f(b) - f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\right) \right| \\ \leq & \; t\left| f(a) - f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right) \right| + (1-t)\left| f(b) - f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right) \right| \\ \leq & \; tM\left| a - \left(ta^p+(1-t)b^p\right)^{\frac{1}{p}} \right| + (1-t)M\left| b - \left(ta^p+(1-t)b^p\right)^{\frac{1}{p}} \right| \\ \leq & \; M\left[t\left| (a^p)^{\frac{1}{p}} - \left(ta^p+(1-t)b^p\right)^{\frac{1}{p}} \right| + (1-t)\left| (b^p)^{\frac{1}{p}} - \left(ta^p+(1-t)b^p\right)^{\frac{1}{p}} \right| \right] \\ = & \; M\left[t\left|(1-t)\left(b^p-a^p\right)\right|^{\frac{1}{p}} + (1-t)\left|t\left(b^p-a^p\right)\right|^{\frac{1}{p}} \right] \\ = & \; M\left[t(1-t)^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}} + (1-t)t^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}} \right]. \end{aligned}$$

So, for all $t \in [0,1]$, we can write the following inequality:

$$\begin{aligned} & \left| tf(a)+(1-t)f(b) - f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right) \right| \\ \leq & \; M\left[t(1-t)^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}} + (1-t)t^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}} \right]. \end{aligned} \tag{2.1}$$

Integrating this inequality on $[0,1]$ over t we get the inequality:

$$\begin{aligned}
(2.2) \qquad & \left|f(a)\int_0^1 tdt+f(b)\int_0^1(1-t)dt-\int_0^1 f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)dt\right| \\
\leq \;& M\left|b^p-a^p\right|^{\frac{1}{p}}\left[\int_0^1 t(1-t)^{\frac{1}{p}}dt+\int_0^1 t^{\frac{1}{p}}(1-t)dt\right].
\end{aligned}$$

Here, it is easy to see that

$$\int_0^1 t^{\frac{1}{p}}(1-t)dt=\int_0^1 t(1-t)^{\frac{1}{p}}dtdt=\frac{p^2}{(p+1)(2p+1)}, \tag{2.3}$$

$$f(a)\int_0^1 tdt+f(b)\int_0^1(1-t)dt=\frac{f(a)+f(b)}{2} \tag{2.4}$$

and

$$\int_0^1 f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)dt=\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p-1}}dx. \tag{2.5}$$

Using the inequalities (2.3), (2.4) and (2.5) in the inequality (2.2), we can derive the desired inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq 2M\left|b^p-a^p\right|^{\frac{1}{p}}\frac{p^2}{(p+1)(2p+1)}$$

(ii) Putting $t=\frac{1}{2}$ in the inequality (2.1), we have

$$\left|\frac{f(a)+f(b)}{2}-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|\leq M\left(\frac{1}{2}\right)^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}}. \tag{2.6}$$

If we replace a by $\left(ta^p+(1-t)\,b^p\right)^{\frac{1}{p}}$ and b by $\left((1-t)a^p+tb^p\right)^{\frac{1}{p}}$ in the inequality (2.6), we obtain

$$\begin{aligned}
(2.7) \qquad & \left|\frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)+f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)}{2}-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\
\leq \;& M\left(\frac{1}{2}\right)^{\frac{1}{p}}\left|(2t-1)\left(b^p-a^p\right)\right|^{\frac{1}{p}} \\
= \;& M\left(\frac{1}{2}\right)^{\frac{1}{p}}\left|2t-1\right|^{\frac{1}{p}}\left|b^p-a^p\right|^{\frac{1}{p}}
\end{aligned}$$

all $t\in[0,1]$. Now, if we integrate the inequality (2.7) on $[0,1]$ over t we can state that

$$\left|\int_0^1\frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)}{2}dt+\int_0^1\frac{f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)}{2}dt-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|$$

$$\leq M\left(\frac{1}{2}\right)^{\frac{1}{p}}|b^p-a^p|^{\frac{1}{p}}\int_0^1|2t-1|^{\frac{1}{p}}\,dt. \tag{2.8}$$

Here, it is easy to see that

$$\int_0^1 f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)dt=\int_0^1 f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)dt=\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx \tag{2.9}$$

and

$$\int_0^1|2t-1|^{\frac{1}{p}}\,dt=\frac{p}{p+1}. \tag{2.10}$$

If we put the equalities (2.9) and (2.10) in the inequality (2.8), then we have the desired inequality:

$$\left|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|\leq M\left(\frac{1}{2}\right)^{\frac{1}{p}}|b^p-a^p|^{\frac{1}{p}}\,\frac{p}{p+1}.$$

b) For $p<0$:

(i) Taking into account that f is an M-Lipschitzian function on interval I, we have

$$\begin{aligned}
&\left|tf(a)+(1-t)f(b)-f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right| \qquad (2.11)\\
= &\left|t\left(f(a)-f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right)+(1-t)\left(f(b)-f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right)\right|\\
\leq\ & t\left|f(a)-f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right|+(1-t)\left|f(b)-f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)\right|\\
\leq\ & tM\left|(a^p)^{\frac{1}{p}}-(ta^p+(1-t)\,b^p)^{\frac{1}{p}}\right|+(1-t)M\left|(b^p)^{\frac{1}{p}}-(ta^p+(1-t)\,b^p)^{\frac{1}{p}}\right|.
\end{aligned}$$

Also, using Lemma 1.3, we get

$$\begin{aligned}
&\left|(a^p)^{\frac{1}{p}}-(ta^p+(1-t)\,b^p)^{\frac{1}{p}}\right| \qquad (2.12)\\
\leq\ & |a^p-ta^p-b^p+tb^p|\left(-\frac{1}{p}\right)(ta^p+(1-t)\,b^p)^{\frac{1}{p}-1}\\
=\ & |(1-t)a^p-(1-t)b^p|\left(-\frac{1}{p}\right)(ta^p+(1-t)\,b^p)^{\frac{1}{p}-1}\\
=\ & (1-t)\frac{b^p-a^p}{p}(ta^p+(1-t)\,b^p)^{\frac{1}{p}-1}
\end{aligned}$$

and

$$\begin{aligned}
\left|(b^p)^{\frac{1}{p}}-(ta^p+(1-t)\,b^p)^{\frac{1}{p}}\right| &\leq |b^p-ta^p-b^p+tb^p|\left(-\frac{1}{p}\right)(b^p)^{\frac{1}{p}-1} \qquad (2.13)\\
&= |t\,(a^p-b^p)|\left(-\frac{1}{p}\right)(b^p)^{\frac{1}{p}-1}\\
&= t\frac{b^p-a^p}{p}b^{1-p}.
\end{aligned}$$

If we put the equalities (2.12) and (2.13) in the inequality (2.11), we obtain the following inequality:

$$
\begin{aligned}
(2.14)\qquad & \left|tf(a)+(1-t)f(b)-f\left(\left[ta^{p}+(1-t)\,b^{p}\right]^{\frac{1}{p}}\right)\right| \\
& \leq M\frac{b^{p}-a^{p}}{p}\left[\frac{t(1-t)}{\left(ta^{p}+(1-t)\,b^{p}\right)^{1-\frac{1}{p}}}+t(1-t)b^{1-p}\right].
\end{aligned}
$$

If we integrate the inequality (2.13) on $[0,1]$ over t we get

$$
\begin{aligned}
(2.15)\qquad & \left|f(a)\int_{0}^{1}tdt+f(b)\int_{0}^{1}(1-t)dt-\int_{0}^{1}f\left(\left[ta^{p}+(1-t)\,b^{p}\right]^{\frac{1}{p}}\right)dt\right| \\
& \leq M\frac{b^{p}-a^{p}}{p}\left[\int_{0}^{1}\frac{t(1-t)}{\left(ta^{p}+(1-t)\,b^{p}\right)^{1-\frac{1}{p}}}dt+b^{1-p}\int_{0}^{1}t(1-t)dt\right]
\end{aligned}
$$

for all $t\in[0,1]$. Here, we can write the following equality:

$$
(2.16)\qquad \int_{0}^{1}\frac{t(1-t)}{\left(ta^{p}+(1-t)\,b^{p}\right)^{1-\frac{1}{p}}}dt=\int_{0}^{1}\frac{t(1-t)}{b^{p-1}\left[1-t\left(1-\left(\frac{a}{b}\right)^{p}\right)\right]^{1-\frac{1}{p}}}dt
$$

Further, if we calculate the equality (2.16) using the definitions of the beta and hypergeometric functions, then we obtain

$$
\begin{aligned}
(2.17)\quad \int_{0}^{1}\frac{t(1-t)dt}{b^{p-1}\left[1-t\left(1-\left(\frac{a}{b}\right)^{p}\right)\right]^{1-\frac{1}{p}}} &= \frac{\beta\,(2,2)}{b^{p-1}}\cdot{}_{2}F_{1}\left(1-\frac{1}{p},2;4;1-\left(\frac{a}{b}\right)^{p}\right) \\
&= \frac{1}{6b^{p-1}}\cdot{}_{2}F_{1}\left(1-\frac{1}{p},2;4;1-\left(\frac{a}{b}\right)^{p}\right)
\end{aligned}
$$

where

$$
\beta(2,2)=\int_{0}^{1}t(1-t)dt=\frac{1}{6}.
$$

If we put the equalities (2.5) and (2.17) in the inequality (2.15), then we have the desired inequality as follows:

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2}-\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx\right| \\
& \leq M\frac{b^{p}-a^{p}}{6pb^{p-1}}\left[{}_{2}F_{1}\left(1-\frac{1}{p},2;4;1-\left(\frac{b}{a}\right)^{p}\right)+1\right]
\end{aligned}
$$

(ii) If we take $t=\frac{1}{2}$ and $\frac{b^{p}-a^{p}}{p}=\frac{|b^{p}-a^{p}|}{-p}$ in the inequality (2.14), we have

$$
(2.18)\quad \left|\frac{f(a)+f(b)}{2}-f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{\frac{1}{p}}\right)\right|\leq M\frac{|b^{p}-a^{p}|}{-4p}\left[\frac{1}{\left(\frac{a^{p}+b^{p}}{2}\right)^{1-\frac{1}{p}}}+b^{1-p}\right].
$$

If we replace in the inequality (2.18) a with $(ta^p+(1-t)\,b^p)^{\frac{1}{p}}$ and b with $((1-t)a^p+tb^p)^{\frac{1}{p}}$, we can write the following inequality for all $t\in[0,1]$:

$$\begin{aligned}
(2.19)\quad & \left|\frac{f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)+f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)}{2}-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\
\leq\; & M\left|\frac{(1-t)a^p+tb^p-(ta^p+(1-t)\,b^p)}{-4p}\right| \\
& \times\left[\frac{1}{\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}+\frac{1}{\left(((1-t)a^p+tb^p)^{\frac{1}{p}}\right)^{p-1}}\right] \\
=\; & M\,|2t-1|\,\frac{b^p-a^p}{4p}\left[\frac{1}{\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}+\frac{1}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}\right]
\end{aligned}$$

Integrating the last inequality (2.19) on $[0,1]$ over t we get

$$\begin{aligned}
(2.20)\quad & \left|\frac{f\left([ta^p+(1-t)\,b^p]^{\frac{1}{p}}\right)+f\left([(1-t)a^p+tb^p]^{\frac{1}{p}}\right)}{2}-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\
\leq\; & M\frac{b^p-a^p}{4p}\left[\frac{1}{\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\int_0^1|2t-1|\,dt+\int_0^1\frac{|2t-1|}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt\right]
\end{aligned}$$

where

$$(2.21)\qquad \int_0^1|2t-1|\,dt=\frac{1}{2}.$$

Now, let calculate the second integral in the inequality (2.22):

$$\begin{aligned}
(2.22)\quad & \int_0^1\frac{|2t-1|}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt \\
=\; & \int_0^{\frac{1}{2}}\frac{1-2t}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{2t-1}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
(2.23)\int_0^{\frac{1}{2}}\frac{(1-2t)\,dt}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}} &= \frac{\beta\,(2,1)}{2\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right) \\
&= \frac{1}{4\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right)
\end{aligned}$$

and

$$(2.24)\int_{\frac{1}{2}}^1\frac{2t-1}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt = \frac{1}{4\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right)$$

where

$$\beta(2,1)=\int_0^1 tdt=\frac{1}{2}.$$

Adding the equalities (2.23) and (2.24), we get

$$\int_0^1 \frac{|2t-1|}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt=\frac{1}{2\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right). \tag{2.25}$$

If we put the equalities (2.21) and (2.25) in the inequality (2.20), then we obtain the following inequality for all $t\in[0,1]$:

$$\begin{aligned}&\left|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right|\\ \leq\ & M\frac{b^p-a^p}{2p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\left[{}_2F_1\left(1-\frac{1}{p},1;3;1-\left(\frac{b}{a}\right)^p\right)+1\right].\end{aligned}$$

c) For $0<p<1$:

Using the inequality (2.11), we can write the inequality

$$\begin{aligned}&\left|tf(a)+(1-t)f(b)-f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\right|\\ \leq\ & tM\left|(a^p)^{\frac{1}{p}}-(ta^p+(1-t)b^p)^{\frac{1}{p}}\right|+(1-t)M\left|(b^p)^{\frac{1}{p}}-(ta^p+(1-t)b^p)^{\frac{1}{p}}\right|\end{aligned} \tag{2.26}$$

for all $t\in[0,1]$. Also, by using Lemma 1.4, we get

$$\begin{aligned}\left|(a^p)^{\frac{1}{p}}-(ta^p+(1-t)b^p)^{\frac{1}{p}}\right| &\leq |a^p-ta^p-b^p+tb^p|\left(\frac{1}{p}\right)(a^p)^{\frac{1}{p}-1}\\ &= (1-t)\frac{b^p-a^p}{p}a^{1-p}\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\left|(b^p)^{\frac{1}{p}}-(ta^p+(1-t)b^p)^{\frac{1}{p}}\right| &\leq |b^p-ta^p-b^p+tb^p|\left(\frac{1}{p}\right)(b^p)^{\frac{1}{p}-1}\\ &= t\frac{b^p-a^p}{p}b^{1-p}.\end{aligned} \tag{2.28}$$

From the inequalities (2.27) and (2.28), the inequality (2.26) can be wriiten as

$$\begin{aligned}&\left|tf(a)+(1-t)f(b)-f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)\right|\\ \leq\ & M\frac{b^p-a^p}{p}\left[t(1-t)a^{1-p}+t(1-t)b^{1-p}\right].\end{aligned} \tag{2.29}$$

Integrating the inequality (2.29) on $[0,1]$ over t we obtain

$$\begin{aligned}(2.30)\qquad & \left| f(a)\int_0^1 t dt + f(b)\int_0^1 (1-t)dt - \int_0^1 f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)dt\right| \\ \leq\ & M\frac{b^p-a^p}{p}\left[a^{1-p}\int_0^1 t(1-t)dt + b^{1-p}\int_0^1 t(1-t)dt\right]\end{aligned}$$

where

$$\int_0^1 t(1-t)dt = \frac{1}{6}. \tag{2.31}$$

If we put (2.4), (2.5) and (2.31) in the inequality (2.29), then we get the following inequality for all $t \in [0,1]$:

$$\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \leq M\frac{|b^p-a^p|}{6p}\left[a^{1-p}+b^{1-p}\right]. \tag{2.32}$$

Putting $t=\frac{1}{2}$ in the inequality (2.29), we have

$$\left|\frac{f(a)+f(b)}{2} - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \leq M\frac{|b^p-a^p|}{4p}\left[a^{1-p}+b^{1-p}\right]. \tag{2.33}$$

If we replace in the inequality (2.33) a with $\left(ta^p+(1-t)\,b^p\right)^{\frac{1}{p}}$ and b with $\left((1-t)a^p+tb^p\right)^{\frac{1}{p}}$, we can write the following inequality for all $t \in [0,1]$:

$$\begin{aligned}(2.34)\qquad & \left|\frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)+f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)}{2} - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\ \leq\ & M\frac{\left|\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)^p - \left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)^p\right|}{4p} \\ & \times\left[\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)^{1-p} + \left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)^{1-p}\right] \\ =\ & M\frac{b^p-a^p}{4p}\left[\frac{|2t-1|}{\left(ta^p+(1-t)b^p\right)^{1-\frac{1}{p}}} + \frac{|2t-1|}{\left((1-t)a^p+tb^p\right)^{1-\frac{1}{p}}}\right].\end{aligned}$$

Integrating the inequality (2.34) on the interval $[0,1]$ over t we have

$$\begin{aligned}& \left|\int_0^1 \frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{\frac{1}{p}}\right)}{2}dt + \int_0^1 \frac{f\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)}{2}dt - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \\ (2.35)\ \leq\ & M\frac{b^p-a^p}{4p}\left[\int_0^1 \frac{|2t-1|}{\left(ta^p+(1-t)b^p\right)^{1-\frac{1}{p}}}dt + \int_0^1 \frac{|2t-1|}{\left((1-t)a^p+tb^p\right)^{1-\frac{1}{p}}}dt\right].\end{aligned}$$

Here, using (2.22) we see that the following equalities

$$\int_0^1 \frac{|2t-1|}{(ta^p+(1-t)b^p)^{1-\frac{1}{p}}}dt = \int_0^1 \frac{|2t-1|}{((1-t)a^p+tb^p)^{1-\frac{1}{p}}}dt$$

$$= \frac{1}{2\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot {}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right) \tag{2.36}$$

hold for all $t\in[0,1]$. Finally, writting the equalities (2.9) and (2.25) in (2.35)

$$\left|\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\right| \le M\frac{b^p-a^p}{4p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot {}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right)$$

Thus, the proof of the Theorem is completed. □

Remark 2.1. If we choose $p=1$ in Theorem 2.1, then the results we obtained coincide with the Theorem 1.4.

If we choose $p=-1$ in Theorem 2.1, then we can also give the following corollary.

Corollary 2.1. *$f: I\subset(0,\infty)\to\mathbb{R}$ be an M-Lipschitzian function on the interval I of real numbers and $a,b\in I$ with $a<b$. Then, following inequality it holds that:*

i) $$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le M\frac{b-a}{6ab^3}\left[{}_2F_1\left(2,2;4;1-\frac{a}{b}\right)+1\right],$$

ii) $$\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\right| \le M\frac{b-a}{2ab\left(\frac{a+b}{2ab}\right)^2}\left[{}_2F_1\left(2,1;3;1-\frac{a}{b}\right)+1\right].$$

By using Theorem 1.2, Lemma 1.1 and Theorem 2.1, we can state the following corollary:

Corollary 2.2. *Let the function $f: I\subset(0,\infty)\to\mathbb{R}$ a differentiable p-convex function on interval I, $a,b\in I$ with $a<b$ and $M=\sup_{t\in[a,b]}|f'(t)|<\infty$. Then:*

For $p\ge 1$;

$$\begin{aligned} i)\ 0 &\le \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \le 2M\left(b^p-a^p\right)^{\frac{1}{p}}\frac{p^2}{(p+1)(2p+1)}, \\ ii)\ 0 &\le \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx - f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \le M\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(b^p-a^p\right)^{\frac{1}{p}}\frac{p}{p+1}, \end{aligned}$$

For $p<0$;

$$\begin{aligned} i)\ 0 &\leq \frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \\ &\leq M\frac{b^p-a^p}{6pb^{p-1}}\left[{}_2F_1\left(1-\frac{1}{p},2;4;1-\left(\frac{b}{a}\right)^p\right)+1\right], \end{aligned}$$

$$\begin{aligned} ii)\ 0 &\leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\ &\leq M\frac{b^p-a^p}{2p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\left[{}_2F_1\left(1-\frac{1}{p},1;3;1-\left(\frac{b}{a}\right)^p\right)+1\right]. \end{aligned}$$

For $0<p<1$;

$$\begin{aligned} i)\ 0 &\leq \frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq M\frac{|b^p-a^p|}{3p}A\left(a^{1-p},b^{1-p}\right), \\ ii)\ 0 &\leq \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx-f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\ &\leq M\frac{b^p-a^p}{4p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right). \end{aligned}$$

Remark 2.2. If we choose $p=1$ in Corollary 2.2, then the results we obtained coincide with the Corollary 1.1.

Proposition 2.1. *For $p\geq 1$ and $0<a<b$, the inequalities*

$$\begin{aligned} i)\quad 0 &\leq H^{-1}\left(a^p,b^p\right)-L^{-1}\left(a^p,b^p\right)\leq \frac{\left(b^p-a^p\right)^{\frac{1}{p}}}{a^{p+1}}\frac{2p^3}{(p+1)(2p+1)} \\ ii)\quad 0 &\leq L^{-1}\left(a^p,b^p\right)-A^{-1}\left(a^p,b^p\right)\leq \frac{\left(b^p-a^p\right)^{\frac{1}{p}}}{a^{p+1}}\left(\frac{1}{2}\right)^{\frac{1}{p}}\frac{p^2}{p+1} \end{aligned}$$

hold.

Proof. The function $f:(0,\infty)\to\mathbb{R},\ f(x)=x^{-p}\ (p\geq 1)$ is a differentiable p-convex function. By using Corollary 2.2 for $p\geq 1$:

(i) Since $|f'(x)|=\left|-px^{-p-1}\right|=px^{-p-1}$, *we obtain*

$$M=\sup_{x\in[a,b]}|f'(x)|=\sup_{x\in[a,b]}px^{-p-1}=\frac{p}{a^{p+1}},$$

$$\frac{f(a)+f(b)}{2}=\frac{a^{-p}+b^{-p}}{2}=\frac{1}{2}\left(\frac{1}{a^p}+\frac{1}{b^p}\right)=\frac{1}{2}\left(\frac{a^p+b^p}{(ab)^p}\right)$$

and

$$\int_a^b \frac{x^{-p}}{x^{1-p}}dx = \int_a^b \frac{1}{x}dx = \ln b - \ln a.$$

Hence we have

$$\frac{a^p+b^p}{2\,(ab)^p} - \frac{p(\ln b - \ln a)}{b^p-a^p} \leq \frac{(b^p-a^p)^{\frac{1}{p}}}{a^{p+1}} \frac{2p^3}{(p+1)(2p+1)}.$$

(ii) It is easy to see that

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) = \left(\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}\right)^{-p} = \left(\frac{a^p+b^p}{2}\right)^{-1} = \frac{2}{a^p+b^p},$$

$$\frac{p(\ln b - \ln a)}{b^p-a^p} - \frac{2}{a^p+b^p} \leq \frac{(b^p-a^p)^{\frac{1}{p}}}{a^{p+1}} \left(\frac{1}{2}\right)^{\frac{1}{p}} \frac{p^2}{p+1}.$$

□

Proposition 2.2. *For $p \leq -1$ and $0 < a < b$, the following inequalities hold:*

$$\begin{aligned} i)\ 0 &\leq H^{-1}(a^p,b^p) - L^{-1}(a^p,b^p) \\ &\leq \frac{|b^p-a^p|}{6a^{p+1}b^{p-1}}\left[{}_2F_1\left(1-\frac{1}{p},2;4;1-\left(\frac{b}{a}\right)^p\right)+1\right], \end{aligned}$$

$$\begin{aligned} ii)\ 0 &\leq L^{-1}(a^p,b^p) - A^{-1}(a^p,b^p) \\ &\leq \frac{|b^p-a^p|}{2a^{p+1}\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}}\left[{}_2F_1\left(1-\frac{1}{p},1;3;1-\left(\frac{b}{a}\right)^p\right)+1\right]. \end{aligned}$$

Proof. The function $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^{-p}$ $(p\leq -1)$ be a differentiable p-convex function. By using Corollary 2.2 for $p<0$, we obtain desired inequalities. □

Proposition 2.3. *For $0 < p < 1$ and $0 < a < b$, the following inequalities hold:*

$$i) \qquad 0 \leq A(a,b) - L_{p-1}^{1-p}(a,b)\,L_p^p(a,b) \leq \frac{b^p-a^p}{3p}A\left(a^{1-p},b^{1-p}\right)$$

$$ii) \qquad 0 \leq L_{p-1}^{1-p}(a,b)\,L_p^p(a,b) - M_p(a,b) \leq \frac{b^p-a^p}{4p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right).$$

Proof. The function $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$ $(0 < p < 1)$ be a differentiable p-convex function. By using the Corollary 2.2 for $p < 1$, we can write

$$\begin{aligned} \frac{f(a)+f(b)}{2} &= \frac{a+b}{2}, \\ \int_a^b \frac{x}{x^{1-p}}dx &= \frac{b^{p+1}-a^{p+1}}{p+1}, \\ M &= \sup_{x\in[a,b]} |f'(x)| = 1. \end{aligned}$$

So, we obtain followings for Corollary 2.2 (i)-(ii) for $0 < p < 1$, respectively:

$$\left|\frac{a+b}{2} - \frac{p}{b^p-a^p}\frac{b^{p+1}-a^{p+1}}{p+1}\right| = \frac{a+b}{2} - \frac{p}{b^p-a^p}\frac{b^{p+1}-a^{p+1}}{p+1} \leq \frac{b^p-a^p}{3p}A\left(a^{1-p}, b^{1-p}\right)$$

and

$$\frac{p}{b^p-a^p}\frac{b^{p+1}-a^{p+1}}{p+1} - \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \leq \frac{b^p-a^p}{4p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 3; \frac{a^p-b^p}{a^p+b^p}\right).$$

□

3. Conclusion

In this paper, using the definition of M-Lipschitzian function and some simple mathematical inequalities, we obtained new inequalities related to the right and left sides of the inequality (1.2) for Lipshitz functions. Some results obtained in this study are reduced to the results obtained in [1] in special cases.

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SOME STATISTICAL CONVERGENCE TYPES OF ORDER α FOR DOUBLE SET SEQUENCES

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Abstract. In this study, we have introduced the concepts of Wijsman statistical convergence of order α, Hausdorff statistical convergence of order α and Wijsman strongly p-Cesàro summability of order α for double set sequences. Also, we have investigated some properties of these concepts and examined the relationships among them.
Keywords: Statistical convergence; Cesàro summability; Double sequence; Order α; Wijsman convergence; Hausdorff convergence; Set sequences.

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [30] and Fast [11], and later reintroduced by Schoenberg [28] independently. Moreover, many rearchers have studied this concept until recently (see, [5, 6, 12, 14, 15, 25, 31, 34]). The order of statistical convergence of a single sequence of numbers was given by Gadjiev and Orhan [13]. Then, the concepts of statistical convergence of order α and strongly p-Cesàro summability of order α were studied by Çolak [8] and Çolak and Bektaş [9].

In [24], Pringsheim introduced the concept of convergence for double sequences. Recently, Mursaleen and Edely [19] have extended this concept to statistical convergence. More developments on double sequences can be found in [4,7,16–18,20]. Very recently, the concepts of statistical convergence of order α and strongly p-Cesàro summability of order α for double sequences have been studied by Savaş [26] and Çolak and Altın [10].

The concepts of convergence for number sequences were transferred to the concepts of convergence for set sequences by many authors. In this study, the concepts of Wijsman convergence and Hausdorff convergence which are two of these transfers are considered (see, [1–3, 35]). Nuray and Rhoades [21] extended the concept of

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Wijsman convergence and Hausdorff convergence to statistical convergence for set sequences and gave some basic theorems. Very recently, the concept of Wijsman $\mathcal{I}$-statistical convergence of order α have been studied by Savaş [27] and Şengül and Et [32].

Nuray et al. [22] introduced the concepts of Wijsman convergence and Wijsman strongly p-Cesàro summability for double set sequences. Then, the concepts of Hausdorff convergence for double set sequences was studied by Sever et al. [29]. Also, the concepts of Wijsman statistical convergence and Hausdorff statistical convergence were studied by Nuray et al. [23] and Talo et al. [33], respectively.

In this study, we shall introduce the concepts of Wijsman statistical convergence of order α, Hausdorff statistical convergence of order α and Wijsman strongly p-Cesàro summability of order α for double set sequences. Also, we shall investigate some properties of these concepts and examine the relationships among them.

2. Definitions and Notations

Firstly, we recall the basic concepts that need for a good understanding of our study (see, [1–3, 19, 22–24, 29, 33, 35]).

A double sequence (x_{ij}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j > N_\varepsilon$.

A double sequence (x_{ij}) is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\Big|\big\{(i,j) :\ i \le n, j \le m,\ |x_{ij} - L| \ge \varepsilon\big\}\Big| = 0.$$

Let X be any non-empty set. The function $f : \mathbb{N} \to P(X)$ is defined by $f(i) = U_i \in P(X)$ for each $i \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{U_i\} = (U_1, U_2, ...)$, which is the range's elements of f, is said to be set sequences.

Let (X, d) be a metric space. For any point $x \in X$ and any non-empty subset U of X, the distance from x to U is defined by

$$\rho(x, U) = \inf_{u\in U} d(x, u).$$

Throughout the study, we will take (X, d) be a metric space and U, U_{ij} be any non-empty closed subsets of X.

A double sequence $\{U_{ij}\}$ is said to be Wijsman convergent to U if for each $x \in X$,

$$\lim_{i,j\to\infty} \rho(x, U_{ij}) = \rho(x, U).$$

A double sequence $\{U_{ij}\}$ is said to be Hausdorff convergent to U if for each $x \in X$,

$$\lim_{i,j\to\infty} \sup_{x\in X} |\rho(x, U_{ij}) - \rho(x, U)| = 0.$$

A double sequence $\{U_{ij}\}$ is said to be Wijsman statistical convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\Big|\big\{(i,j):\ i \le n, j \le m,\ |\rho(x,U_{ij}) - \rho(x,U)| \ge \varepsilon\big\}\Big| = 0.$$

The class of all Wijsman statistical convergent sequences is simply denoted by $W(S_2)$.

A double sequence $\{U_{ij}\}$ is said to be Hausdorff statistical convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\Big|\big\{(i,j):\ i \le n, j \le m,\ \sup_{x\in X} |\rho(x,U_{ij}) - \rho(x,U)| \ge \varepsilon\big\}\Big| = 0.$$

The class of all Hausdorff statistical convergent sequences is simply denoted by $H(S_2)$.

A double sequence $\{U_{ij}\}$ is said to be Wijsman Cesàro summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} \rho(x,U_{ij}) = \rho(x,U).$$

Let $0 < p < \infty$. A double sequence $\{U_{ij}\}$ is said to be Wijsman strongly p-Cesàro summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} |\rho(x,U_{ij}) - \rho(x,U)|^p = 0.$$

The class of all Wijsman strongly p-Cesàro summable sequences is simply denoted by $W[C_2]^p$.

From now on, in short, we shall use $\rho_x(U)$ and $\rho_x(U_{ij})$ instead of $\rho(x,U)$ and $\rho(x,U_{ij})$, respectively.

3. Main Results

In this section, we shall introduce the concepts of Wijsman statistical convergence of order α, Hausdorff statistical convergence of order α and Wijsman strongly p-Cesàro summability of order α for double set sequences. Also, we shall investigate some properties of these concepts and examine the relationships among them.

Definition 3.1. Let $0 < \alpha \le 1$. A double sequence $\{U_{ij}\}$ is Wijsman statistically convergent of order α to U or $W(S_2^\alpha)$-convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha}\Big|\big\{(i,j):\ i \le m, j \le n,\ |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\big\}\Big| = 0.$$

In this case, we write $U_{ij} \longrightarrow_{W(S_2^\alpha)} U$ or $U_{ij} \longrightarrow U\big(W(S_2^\alpha)\big)$.

The class of all $W(S_2^\alpha)$-convergent sequences will be simply denoted by $W(S_2^\alpha)$.

Example 3.1. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:

$$U_{ij} := \begin{cases} \{(x,y) \in \mathbb{R}^2 : (x-i)^2 + (y-j)^2 = 1\} & , \text{ if } i \text{ and } j \text{ are square integers} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

Then, the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent of order α to the set $U = \{(0,0)\}$.

Remark 3.1. For $\alpha = 1$, the concept of $W(S_2^\alpha)$-convergence coincides with the concept of Wijsman statistical convergence for double set sequences in [23].

Theorem 3.1. *If $0 < \alpha \leq \beta \leq 1$, then $W(S_2^\alpha) \subseteq W(S_2^\beta)$.*

Proof. Let $0 < \alpha \leq \beta \leq 1$ and suppose that $U_{ij} \longrightarrow_{W(S_2^\alpha)} U$. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\frac{1}{(mn)^\beta}\Big|\big\{(i,j) :\ i \leq m, j \leq n,\ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big|$$
$$\leq \frac{1}{(mn)^\alpha}\Big|\big\{(i,j) :\ i \leq m, j \leq n,\ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big|.$$

Hence, by our assumption, we get $U_{ij} \longrightarrow_{W(S_2^\beta)} U$. Consequently, $W(S_2^\alpha) \subseteq W(S_2^\beta)$. $\square$

If we take $\beta = 1$ in Theorem 3.1, then we obtain the following corollary.

Corollary 3.1. *If a double sequence $\{U_{ij}\}$ is Wijsman statistically convergent of order α to U for some $0 < \alpha \leq 1$, then the double sequence is Wijsman statistically convergent to U, i.e., $W(S_2^\alpha) \subseteq W(S_2)$.*

Definition 3.2. Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is Wijsman Cesàro summable of order α to U or $W(C_2^\alpha)$-summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} \rho_x(U_{ij}) = \rho_x(U).$$

In this case, we write $U_{ij} \longrightarrow_{W(C_2^\alpha)} U$ or $U_{ij} \longrightarrow U\big(W(C_2^\alpha)\big)$.

The class of all $W(C_2^\alpha)$-summable sequences will be simply denoted by $W(C_2^\alpha)$.

Definition 3.3. Let $0 < \alpha \leq 1$ and $0 < p < \infty$. A double sequence $\{U_{ij}\}$ is Wijsman strongly p-Cesàro summable of order α to U or $W[C_2^\alpha]^p$-summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p = 0.$$

In this case, we write $U_{ij} \longrightarrow_{W[C_2^\alpha]^p} U$ or $U_{ij} \longrightarrow U\big(W[C_2^\alpha]^p\big)$. If $p=1$, then a double sequence $\{U_{ij}\}$ is simply said to be Wijsman strongly Cesàro summable of order α to U and we write $U_{ij} \longrightarrow_{W[C_2^\alpha]} U$ or $U_{ij} \longrightarrow U\big(W[C_2^\alpha]\big)$.

The class of all $W[C_2^\alpha]^p$-summable sequences will be simply denoted by $W[C_2^\alpha]^p$.

Example 3.2. Let $X=\mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:

$$U_{ij} := \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : x^2+(y-1)^2 = \frac{1}{ij}\right\} & , \text{ if } i \text{ and } j \text{ are square integers,} \\ \{(1,0)\} & , \text{ otherwise.} \end{cases}$$

Then, the double sequence $\{U_{ij}\}$ is Wijsman strongly Cesàro summable of order α to the set $U=\{(1,0)\}$.

Remark 3.2. For $\alpha=1$, the concepts of $W(C_2^\alpha)$-summability and $W[C_2^\alpha]^p$-summability coincides with the concepts of Wijsman Cesàro summability and Wijsman strongly p-Cesàro summability for double set sequences in [22], respectively.

Theorem 3.2. *If $0<\alpha\le\beta\le 1$, then $W[C_2^\alpha]^p \subseteq W[C_2^\beta]^p$.*

Proof. Let $0<\alpha\le\beta\le1$ and suppose that $U_{ij} \longrightarrow_{W[C_2^\alpha]^p} U$. For each $x\in X$, we have

$$\frac{1}{(mn)^\beta}\sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij})-\rho_x(U)|^p \le \frac{1}{(mn)^\alpha}\sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij})-\rho_x(U)|^p.$$

Hence, by our assumption, we get $U_{ij} \longrightarrow_{W[C_2^\beta]^p} U$. Consequently, $W[C_2^\alpha]^p \subseteq W[C_2^\beta]^p$. $\square$

If we take $\beta=1$ in Theorem 3.2, then we obtain the following corollary.

Corollary 3.2. *If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-Cesàro summable of order α to U for some $0<\alpha\le1$, then the double sequence is Wijsman strongly p-Cesàro summable to U, i.e., $W[C_2^\alpha]^p \subseteq W[C_2]^p$.*

Now, without proof, we shall state a theorem that gives a relation between $W[C_2^\alpha]^p$ and $W[C_2^\alpha]^q$, where $0<\alpha\le1$ and $0<p<q<\infty$.

Theorem 3.3. *Let $0<\alpha\le1$ and $0<p<q<\infty$. Then, $W[C_2^\alpha]^q \subset W[C_2^\alpha]^p$.*

Theorem 3.4. *Let $0<\alpha\le\beta\le1$ and $0<p<\infty$. If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-Cesàro summable of order α to U, then the double sequence is Wijsman statistically convergent of order β to U.*

Proof. Let $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$ and assume that the double sequence $\{U_{ij}\}$ is Wijsman strongly p-Cesàro summable of order α to U. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\begin{aligned}
\sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p &= \sum_{i,j=1,1;\ |\rho_x(U_{ij})-\rho_x(U)|\geq\varepsilon}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p \\
&\quad + \sum_{i,j=1,1;\ |\rho_x(U_{ij})-\rho_x(U)|<\varepsilon}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p \\
&\geq \sum_{i,j=1,1;\ |\rho_x(U_{ij})-\rho_x(U)|\geq\varepsilon}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p \\
&\geq \varepsilon^p \left|\left\{(i,j):\ i \leq m, j \leq n,\ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\right\}\right|
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)|^p \\
&\qquad \geq \frac{\varepsilon^p}{(mn)^\alpha} \left|\left\{(i,j):\ i \leq m, j \leq n,\ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\right\}\right| \\
&\qquad \geq \frac{\varepsilon^p}{(mn)^\beta} \left|\left\{(i,j):\ i \leq m, j \leq n,\ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Hence, by our assumption, we get the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent of order β to U. $\square$

If we take $\beta = \alpha$ in Theorem 3.4, then we obtain the following corollary.

Corollary 3.3. *Let $0 < \alpha \leq 1$ and $0 < p < \infty$. If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-Cesàro summable of order α to U, then the double sequence is Wijsman statistically convergent of order α to U.*

Definition 3.4. Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is Hausdorff statistically convergent of order α to U or $H(S_2^\alpha)$-convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \left|\left\{(i,j):\ i \leq m, j \leq n,\ \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $U_{ij} \longrightarrow_{H(S_2^\alpha)} U$ or $U_{ij} \longrightarrow U\big(H(S_2^\alpha)\big)$.

The class of all $H(S_2^\alpha)$-convergent sequences will be denoted by simply $H(S_2^\alpha)$.

Remark 3.3. For $\alpha = 1$, the concept of $H(S_2^\alpha)$-convergence coincides with the concept of Hausdorff statistical convergence for double set sequences in [33].

Theorem 3.5. *If* $0 < \alpha \le \beta \le 1$, *then* $H(S_2^\alpha) \subseteq H(S_2^\beta)$.

Proof. Let $0 < \alpha \le \beta \le 1$ and suppose that $U_{ij} \longrightarrow_{H(S_2^\alpha)} U$. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\frac{1}{(mn)^\beta}\left|\left\{(i,j): \ i \le m, j \le n, \ \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|$$
$$\le \frac{1}{(mn)^\alpha}\left|\left\{(i,j): \ i \le m, j \le n, \ \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|.$$

Hence, by our assumption, we get $U_{ij} \longrightarrow_{H(S_2^\beta)} U$. Consequently, $H(S_2^\alpha) \subseteq H(S_2^\beta)$. □

If we take $\beta = 1$ in Theorem 3.5, then we obtain the following corollary.

Corollary 3.4. *If a double sequence* $\{U_{ij}\}$ *is Hausdorff statistically convergent of order* α *to* U *for some* $0 < \alpha \le 1$, *then the double sequence is Hausdorff statistically convergent to* U, *i.e.,* $H(S_2^\alpha) \subseteq H(S_2)$.

Theorem 3.6. *Let* $0 < \alpha \le \beta \le 1$. *If a double sequence* $\{U_{ij}\}$ *is Hausdorff statistically convergent of order* α *to* U, *then the double sequence is Wijsman statistically convergent of order* β *to* U.

Proof. Let $0 < \alpha \le \beta \le 1$ and assume that $U_{ij} \longrightarrow_{H(S_2^\alpha)} U$. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\frac{1}{(mn)^\beta}\left|\left\{(i,j): \ i \le m, j \le n, \ |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|$$
$$\le \frac{1}{(mn)^\beta}\left|\left\{(i,j): \ i \le m, j \le n, \ \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|$$
$$\le \frac{1}{(mn)^\alpha}\left|\left\{(i,j): \ i \le m, j \le n, \ \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|.$$

Hence, by our assumption, we achieve the desired result. □

If we take $\beta = \alpha$ in Theorem 3.6, then we obtain the following corollary.

Corollary 3.5. *Let* $0 < \alpha \le 1$. *If a double sequence* $\{U_{ij}\}$ *is Hausdorff statistically convergent of order* α *to* U, *then the double sequence is Wijsman statistically convergent of order* α *to* U.

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ON SURFACES CONSTRUCTED BY EVOLUTION ACCORDING TO QUASI FRAME

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Abstract. The present paper presents evolutions of spherical indicatrix of a space curve according to the quasi-frame. Then, some geometric properties of these surfaces constructed by evolutions have been obtained. At the end, illustrative examples of the spherical images of a space curve have been presented.
Keywords: space curve; spherical images; quasi-frame.

1. Introduction

The curves obtained with the help of a given space curve have been studied by many researchers. Bertrand curve pairs, involute-evolute curve pairs and spherical images of a space curve can be given as examples of these curves, [7]. For example, Korpinar [4] investigated the surfaces constructed by the binormal spherical image of a space curve. They derived the time evolution equations for the Frenet frame of binormal spherical image as a curve occurring on the sphere and gave some geometric properties of these surfaces such as fundamental forms and curvatures. The spherical image of the curve moving with time occurs on a sphere. In [6], time evolution equations of a space curve given with the quasi frame are obtained.

In this paper, we have found relations between the motion of curves and the motion of their spherical image. We have obtained the Frenet elements of the spherical images of the curve given with the quasi frame. Then we have derived some geometric properties of the surfaces constructed by the evolution of the spherical images of a space curve. At the end, we have given the illustrative examples of the spherical images of a space curve.

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2. Preliminaries

In this section, we present the Frenet frame and the quasi frame along a space curve which are given by Soliman in [6]. Also, we give some geometric properties for these frames.

Let $r = r(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The Frenet frame of r consists of the vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ which are given by

$$\begin{aligned} \mathbf{T} &= r'(s), \\ \mathbf{N} &= \frac{r''(s)}{\|r''(s)\|}, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N}, \end{aligned}$$

where $\mathbf{T}$ is the tangent vector, $\mathbf{N}$ is the normal vector and $\mathbf{B}$ is the binormal vector of the curve r.

The curvature κ and the torsion τ are given by

$$\begin{aligned} \kappa &= \|r''(s)\|, \\ \tau &= \frac{\det(r', r'', r''')}{\|r''(s)\|^2}. \end{aligned}$$

The quasi frame of a space curve $r = r(s)$ which is parameterized with arc-length consists of the vectors $\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q$. They are given by

$$\begin{aligned} \mathbf{T}_q &= \mathbf{T}, \\ \mathbf{N}_q &= \frac{\mathbf{T} \times \overrightarrow{\mathbf{k}}}{\left\|\mathbf{T} \times \overrightarrow{\mathbf{k}}\right\|}, \\ \mathbf{B}_q &= \mathbf{T} \times \mathbf{N}_q, \end{aligned}$$

where $\overrightarrow{\mathbf{k}}$ is the projection vector which can be chosen as $\overrightarrow{\mathbf{k}} = (1,0,0)$ or $\overrightarrow{\mathbf{k}} = (0,1,0)$ or $\overrightarrow{\mathbf{k}} = (0,0,1)$. In this paper, we choose the projection vector $\overrightarrow{\mathbf{k}} = (0,0,1)$. $\mathbf{N}_q$ and $\mathbf{B}_q$ are called the quasi normal vector and the quasi binormal vector, respectively.

Let θ be the angle between the normal $\mathbf{N}$ and the quasi normal $\mathbf{N}_q$. The quasi formulas are given by, [1],

$$\frac{\partial}{\partial s}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix},$$

where k_i are called the quasi curvatures $(1 \le i \le 3)$ which are given by

$$\begin{aligned} k_1 &= \kappa\cos\theta = \left\langle \mathbf{T}_q', \mathbf{N}_q \right\rangle, \\ k_2 &= -\kappa\sin\theta = \left\langle \mathbf{T}_q', \mathbf{B}_q \right\rangle, \\ k_3 &= \theta' + \tau = -\left\langle \mathbf{N}_q, \mathbf{B}_q' \right\rangle. \end{aligned}$$

3. The Spherical Images of a Space Curve

In this section, we give the representation of the Frenet frame, curvature and torsion for spherical images of the curve in terms of the quasi frame and curvatures of the curve.

Given a space curve r parameterized with arc-length in $\mathbb{R}^3$. Let $\mathbf{T}$ be the unit tangent vector of r. When we take $\overrightarrow{PQ} = \mathbf{T}$; while the moving point P is drawing the curve r, the moving point Q draws a curve on the unit sphere. This curve is called the spherical image of the tangent to the curve r. The spherical image of the normal and the binormal to the curve are defined similarly. Now we give these concepts according to the quasi frame of the curve.

Definition 3.1. Let $r = r(s)$ be a space curve parameterized with arc-length in $\mathbb{R}^3$. The following space curves lie on a unit sphere

$$\begin{aligned} r_1(s) &= \mathbf{T}_q(s), \\ r_2(s) &= \mathbf{N}_q(s), \\ r_3(s) &= \mathbf{B}_q(s) \end{aligned}$$

and they are called the spherical image of the tangent, the quasi normal and the quasi binormal to the curve, respectively.

3.1. Spherical Image of $\mathbf{T}_q$

Let $\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\}$ be the quasi frame of the curve $r = r(s)$ parameterized with arc-length and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame of the curve $r_1(s) = \mathbf{T}_q(s)$. The quasi curvatures of the curve r are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve r_1 are denoted by κ and τ, respectively.

Theorem 3.1. *The Frenet elements of r_1 can be given in terms of the quasi elements of r as follows:*

$$\begin{aligned} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{k_1}{\sqrt{k_1^2+k_2^2}} & \frac{k_2}{\sqrt{k_1^2+k_2^2}} \\ \frac{A_1}{\sqrt{U_1}} & \frac{B_1}{\sqrt{U_1}} & \frac{C_1}{\sqrt{U_1}} \\ \frac{K_1}{\sqrt{V_1}} & \frac{L_1}{\sqrt{V_1}} & \frac{M_1}{\sqrt{V_1}} \end{bmatrix} \begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} \\ \kappa &= (1 + \frac{K_1}{(k_1^2+k_2^2)^3})^{\frac{1}{2}}, \\ \tau &= \frac{W_1}{V_1}, \end{aligned}$$

where

$$\begin{aligned}
A_1 &= -\left(k_1^2+k_2^2\right)^2, \\
B_1 &= k_1'k_2^2 - k_1k_2k_2' - k_1^2k_2k_3 - k_2^3k_3, \\
C_1 &= k_1^3k_3 + k_1^2k_2' + k_1k_2^2k_3 - k_1k_1'k_2, \\
K_1 &= k_1^2k_3 + k_2^2k_3 + k_1k_2' - k_1'k_2, \\
L_1 &= -k_2\left(k_1^2+k_2^2\right), \\
M_1 &= k_1\left(k_1^2+k_2^2\right), \\
U_1 &= \left(k_1^2+k_2^2\right)^4 + \left(k_1^2+k_2^2\right)\left(k_1^2k_3+k_2^2k_3+k_1k_2'-k_1'k_2\right)^2, \\
V_1 &= \left(k_1^2+k_2^2\right)^3 + \left(k_1^2k_3+k_2^2k_3+k_1k_2'-k_1'k_2\right)^2, \\
W_1 &= 3\left(k_1\left(k_1'\right)^2k_2 + k_1'k_2'k_2^2 - k_1^2k_1'k_2' - k_1k_2\left(k_2'\right)^2\right) \\
&\quad + \left(k_1^2+k_2^2\right)\left(k_1k_2''+k_1^2k_3'+k_2^2k_3'-k_1''k_2-k_1k_1'k_3-k_2k_2'k_3\right).
\end{aligned}$$

Proof. By simple calculations one can easily get the first, the second and the third derivatives of r_1 as follows:

$$\begin{aligned}
r_1'(s) &= k_1\mathbf{N}_q + k_2\mathbf{B}_q, \\
r_1''(s) &= -\left(k_1^2+k_2^2\right)\mathbf{T}_q + (k_1'-k_2k_3)\mathbf{N}_q + (k_1k_3+k_2')\mathbf{B}_q, \\
r_1'''(s) &= -3(k_1k_1'+k_2k_2')\mathbf{T}_q + (k_1''-2k_2'k_3-k_2k_3'-k_1k_3^2-k_1k_2^2-k_1^3)\mathbf{N}_q \\
&\quad +(k_2''+2k_1'k_3+k_1k_3'-k_2k_3^2-k_2k_1^2-k_2^3)\mathbf{B}_q.
\end{aligned}$$

Then, it is easy to compute the following:

$$\begin{aligned}
\|r_1'\| &= \sqrt{k_1^2+k_2^2}, \\
r_1'\times r_1'' &= K_1\mathbf{T}_q + L_1\mathbf{N}_q + M_1\mathbf{B}_q, \\
\|r_1'\times r_1''\| &= \sqrt{V_1}, \\
\det(r_1', r_1'', r_1''') &= \langle r_1'\times r_1'', r_1'''\rangle = W_1.
\end{aligned}$$

Using the Frenet formulas, one can easily obtain the Frenet elements of r_1 in terms of the quasi elements of r as indicated in the theorem. □

3.2. Spherical Image of $\mathbf{N}_q$

Let $\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\}$ be the quasi frame of the curve $r = r(s)$ parameterized with arc-length and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame of the curve $r_2(s) = \mathbf{N}_q(s)$. The quasi curvatures of the curve r are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve r_2 are denoted by κ and τ, respectively.

Theorem 3.2. *The Frenet elements of r_2 can be given in terms of the quasi ele-*

ments of r as follows:

$$
\begin{aligned}
\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} &= \begin{bmatrix} \frac{-k_1}{\sqrt{k_1^2+k_3^2}} & 0 & \frac{k_3}{\sqrt{k_1^2+k_3^2}} \\ \frac{A_2}{\sqrt{U_2}} & \frac{B_2}{\sqrt{U_2}} & \frac{C_2}{\sqrt{U_2}} \\ \frac{K_2}{\sqrt{V_2}} & \frac{L_2}{\sqrt{V_2}} & \frac{M_2}{\sqrt{V_2}} \end{bmatrix} \begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} \\
\kappa &= (1+\frac{K_2}{\left(k_1^2+k_3^2\right)^3})^{\frac{1}{2}}, \\
\tau &= \frac{W_2}{V_2},
\end{aligned}
$$

where

$$
\begin{aligned}
A_2 &= k_1 k_3 k_3' - k_1^2 k_2 k_3 - k_1' k_3^2 - k_2 k_3^3, \\
B_2 &= -k_1^2 \left(k_1^2 + k_3^2\right), \\
C_2 &= k_1^2 k_3' - k_1 k_1' k_3 - k_1^3 k_2 - k_1 k_2 k_3^2 -, \\
K_2 &= k_3 \left(k_1^2 + k_3^2\right), \\
L_2 &= k_1 k_3' - k_1' k_3 - k_1^2 k_2 - k_2 k_3^2, \\
M_2 &= k_1 \left(k_1^2 + k_3^2\right), \\
U_2 &= \left(k_1^2 + k_3^2\right)^4 + \left(k_1^2 + k_3^2\right) \left(k_1 k_3' - k_1' k_3 - k_1^2 k_2 - k_2 k_3^2\right)^2, \\
V_2 &= \left(k_1^2 + k_3^2\right)^3 + \left(k_1 k_3' - k_1' k_3 - k_1^2 k_2 - k_2 k_3^2\right)^2, \\
W_2 &= 3\left(k_1 \left(k_1'\right)^2 k_3 + k_1' k_3' k_3^2 - k_1^2 k_3' k_3' - k_1 k_3 \left(k_3'\right)^2\right) \\
&\quad + \left(k_1^2 + k_3^2\right) \left(k_1 k_3'' - k_1'' k_3 - k_2' k_3^2 - k_1^2 k_2' + k_2 k_3' k_3 + k_1 k_1' k_2\right).
\end{aligned}
$$

Proof. The calculations can be made similar to the proof of the first theorem. □

3.3. Spherical Image of $\mathbf{B}_q$

Let $\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\}$ be the quasi frame of the curve $r = r(s)$ parameterized with arc-length and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame of the curve $r_3(s) = \mathbf{B}_q(s)$. The quasi curvatures of the curve r are denoted by k_1, k_2, k_3 and the curvature and the torsion of the curve r_3 are denoted by κ and τ, respectively.

Theorem 3.3. *The Frenet elements of r_3 can be given in terms of the quasi elements of r as follows:*

$$
\begin{aligned}
\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} &= \begin{bmatrix} \frac{-k_2}{\sqrt{k_2^2+k_3^2}} & 0 & \frac{-k_3}{\sqrt{k_2^2+k_3^2}} \\ \frac{A_3}{\sqrt{U_3}} & \frac{B_3}{\sqrt{U_3}} & \frac{C_3}{\sqrt{U_3}} \\ \frac{K_3}{\sqrt{V_3}} & \frac{L_3}{\sqrt{V_3}} & \frac{M_3}{\sqrt{V_3}} \end{bmatrix} \begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} \\
\kappa &= (1+\frac{K_3}{\left(k_2^2+k_3^2\right)^3})^{\frac{1}{2}}, \\
\tau &= \frac{W_3}{V_3},
\end{aligned}
$$

where

$$\begin{aligned}
A_3 &= k_2k_3k_3' + k_1k_2^2k_3 + k_1k_3^3 - k_2'k_3^2, \\
B_3 &= k_2k_2'k_3 - k_1k_2k_3^2 - k_1k_2^3 - k_3'k_2^2, \\
C_3 &= -\left(k_2^2 + k_3^2\right)^2, \\
K_3 &= k_3\left(k_2^2 + k_3^2\right), \\
L_3 &= -k_2\left(k_2^2 + k_3^2\right), \\
M_3 &= k_1^2k_2 + k_1k_3^2 + k_2k_3' - k_2'k_3, \\
U_3 &= \left(k_2^2 + k_3^2\right)^4 + \left(k_2^2 + k_3^2\right)\left(k_2k_3' - k_2'k_3 + k_1k_2^2 + k_1k_3^2\right)^2, \\
V_3 &= \left(k_2^2 + k_3^2\right)^3 + \left(k_2k_3' - k_2'k_3 + k_1k_2^2 + k_1k_3^2\right)^2, \\
W_3 &= 3\left(k_2\left(k_2'\right)^2k_3 + k_2'k_3'k_3^2 - k_2^2k_2'k_3' - k_2k_3\left(k_3'\right)^2\right) \\
&\quad + \left(k_2^2 + k_3^2\right)\left(k_1'k_2^2 + k_1'k_3^2 + k_3''k_2 - k_3k_2'' - k_1k_2k_2' - k_1k_3k_3'\right).
\end{aligned}$$

Proof. The calculations can be made similar to the proof of the first theorem. □

An evolving curve can be thought as a collection of curves parameterized by time. This means that each curve in the collection has a space parameter s and a time parameter t, [3]. The following definitions can be given according to quasi frame in $\mathbb{R}^3$ considering references [6] and [7].

$$\frac{\partial}{\partial s}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix}, \tag{3.1}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & \nu \\ -\mu & -\nu & 0 \end{bmatrix}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix}. \tag{3.2}$$

Applying the compatibility condition

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix} \mathbf{T}_q \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix},$$

in the light of the equations (3.1) and (3.2) one can easily get

$$\begin{bmatrix} 0 & \left(\frac{\partial k_1}{\partial t} - \nu k_2 + \mu k_3 - \frac{\partial \lambda}{\partial s}\right) & \left(\frac{\partial k_2}{\partial t} + \nu k_1 - \lambda k_3 + \frac{\partial \mu}{\partial s}\right) \\ -\left(\frac{\partial k_1}{\partial t} - \nu k_2 + \mu k_3 - \frac{\partial \lambda}{\partial s}\right) & 0 & \left(\frac{\partial k_3}{\partial t} - \mu k_1 + \lambda k_2 - \frac{\partial \nu}{\partial s}\right) \\ -\left(\frac{\partial k_2}{\partial t} + \nu k_1 - \lambda k_3 + \frac{\partial \mu}{\partial s}\right) & -\left(\frac{\partial k_3}{\partial t} - \mu k_1 + \lambda k_2 - \frac{\partial \nu}{\partial s}\right) & 0 \end{bmatrix} = 0_{3\times 3}.$$

Thus, the compatibility condition becomes

$$\begin{aligned}\frac{\partial k_1}{\partial t} &= \nu k_2 - \mu k_3 + \frac{\partial \lambda}{\partial s},\\ \frac{\partial k_2}{\partial t} &= \lambda k_1 - \nu k_3 - \frac{\partial \mu}{\partial s},\\ \frac{\partial k_3}{\partial t} &= \mu k_1 - \lambda k_2 + \frac{\partial \nu}{\partial s}.\end{aligned}$$

4. Surfaces Constructed by the Evolution of the Spherical Images of a Space Curve

In this section, we study the surfaces constructed by the evolution of the spherical image of the tangent, spherical image of the quasi normal and spherical image of the quasi binormal to the curve.

4.1. Surfaces Constructed Using the Spherical Image of the Tangent

The equation of surfaces constructed by the spherical image of the tangent is given by

$$\Psi = \mathbf{T}_q(s,t).$$

Theorem 4.1. *Under the assumption $\mu k_1 - \lambda k_2 > 0$, the Gaussian curvature K_1, the mean curvature H_1 and the principal curvatures k_{11} and k_{21} of Ψ are given by*

$$K_1 = 1,\ H_1 = -1,\ k_{11} = -1,\ k_{21} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned}\Psi_s &= k_1\mathbf{N}_q + k_2\mathbf{B}_q,\\ \Psi_t &= \lambda\mathbf{N}_q + \mu\mathbf{B}_q,\end{aligned} \tag{4.1}$$

where the lower indices show partial differentiation. Then the unit normal to Ψ is given by

$$\mathbf{N}_\Psi = \frac{\Psi_s \times \Psi_t}{\|\Psi_s \times \Psi_t\|} = \mathbf{T}_q.$$

Using the equations (3.1), (3.2) and (4.1), the second order derivatives are calculated and given by

$$\begin{aligned}\Psi_{ss} &= -\left(k_1^2 + k_2^2\right)\mathbf{T}_q + \left((k_1)_s - k_2k_3\right)\mathbf{N}_q + \left((k_2)_s + k_1k_3\right)\mathbf{B}_q,\\ \Psi_{tt} &= -\left(\lambda^2 + \mu^2\right)\mathbf{T}_q + \left(\lambda_t - \mu\nu\right)\mathbf{N}_q + (\mu_t + \lambda\nu)\mathbf{B}_q,\\ \Psi_{st} &= -(\lambda k_1 + \mu k_2)\mathbf{T}_q + \left(\lambda_s - \mu k_3\right)\mathbf{N}_q + (\lambda k_3 - \mu_s)\mathbf{B}_q.\end{aligned}$$

The components of the first fundamental form $g_{ij}, (1 \le i,j \le 2)$ are obtained as follows:

$$\begin{aligned}g_{11} &= \langle \Psi_s, \Psi_s\rangle = k_1^2 + k_2^2,\\ g_{12} &= \langle \Psi_s, \Psi_t\rangle = \lambda k_1 + \mu k_2,\\ g_{22} &= \langle \Psi_t, \Psi_t\rangle = \lambda^2 + \mu^2.\end{aligned}$$

The components of the second fundamental form $l_{ij},(1 \leq i,j \leq 2)$ are obtained as follows:

$$\begin{aligned} l_{11} &= \langle \Psi_{ss}, \mathbf{N}_\Psi \rangle = -\left(k_1^2 + k_2^2\right), \\ l_{12} &= \langle \Psi_{st}, \mathbf{N}_\Psi \rangle = -\left(\lambda k_1 + \mu k_2\right), \\ l_{22} &= \langle \Psi_{tt}, \mathbf{N}_\Psi \rangle = -\left(\lambda^2 + \mu^2\right). \end{aligned}$$

Thus, we get the following equalities:

$$\begin{aligned} K_1 &= \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = 1, \\ H_1 &= \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2\left(g_{11}g_{22} - g_{12}^2\right)} = -1, \\ k_{11} &= H_1 + \sqrt{H_1^2 - K_1} = -1, \\ k_{21} &= H_1 - \sqrt{H_1^2 - K_1} = -1. \end{aligned}$$

□

4.2. Surfaces Constructed Using the Spherical Image of the Quasi Normal

The equation of surfaces constructed by the spherical image of the quasi normal is given by

$$\phi = \mathbf{N}_q(s,t).$$

Theorem 4.2. *Under the assumption $\nu k_1 - \lambda k_3 > 0$, the Gaussian curvature K_2, the mean curvature H_2 and the principal curvatures k_{12} and k_{22} of ϕ are given by*

$$K_2 = 1,\ H_2 = -1,\ k_{12} = -1,\ k_{22} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned} \phi_s &= -k_1\mathbf{T}_q + k_3\mathbf{B}_q, \\ \phi_t &= -\lambda\mathbf{T}_q + \nu\mathbf{B}_q, \end{aligned} \tag{4.2}$$

where the lower indices show partial differentiation. Then the unit normal to ϕ is given by

$$\mathbf{N}_\phi = \frac{\phi_s \times \phi_t}{\|\phi_s \times \phi_t\|} = \mathbf{N}_q.$$

Using the equations (3.1), (3.2) and (4.2), the second order derivatives are calculated and given by

$$\begin{aligned} \phi_{ss} &= -\left((k_1)_s + k_2k_3\right)\mathbf{T}_q - \left(k_1^2 + k_3^2\right)\mathbf{N}_q + \left((k_3)_s - k_1k_2\right)\mathbf{B}_q, \\ \phi_{tt} &= -\left(\lambda_t + \mu\nu\right)\mathbf{T}_q - \left(\lambda^2 + \nu^2\right)\mathbf{N}_q + (\nu_t - \lambda\mu)\mathbf{B}_q, \\ \phi_{st} &= -\left(\lambda_s + \nu k_2\right)\mathbf{T}_q - \left(\lambda k_1 + \nu k_3\right)\mathbf{N}_q + (\nu_s - \lambda k_2)\mathbf{B}_q. \end{aligned}$$

The components of the first fundamental form $g_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned} g_{11} &= \langle \phi_s, \phi_s \rangle = k_1^2 + k_3^2, \\ g_{12} &= \langle \phi_s, \phi_t \rangle = \lambda k_1 + \nu k_3, \\ g_{22} &= \langle \phi_t, \phi_t \rangle = \lambda^2 + \nu^2. \end{aligned}$$

The components of the second fundamental form $l_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned} l_{11} &= \langle \phi_{ss}, \mathbf{N}_\phi \rangle = -\left(k_1^2 + k_3^2\right), \\ l_{12} &= \langle \phi_{st}, \mathbf{N}_\phi \rangle = -\left(\lambda k_1 + \nu k_3\right), \\ l_{22} &= \langle \phi_{tt}, \mathbf{N}_\phi \rangle = -\left(\lambda^2 + \nu^2\right). \end{aligned}$$

Thus, we get the following equalities:

$$\begin{aligned} K_2 &= \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2} = 1, \\ H_2 &= \frac{l_{11} g_{22} - 2 l_{12} g_{12} + l_{22} g_{11}}{2\left(g_{11} g_{22} - g_{12}^2\right)} = -1, \\ k_{12} &= H_2 + \sqrt{H_2^2 - K_2} = -1, \\ k_{22} &= H_2 - \sqrt{H_2^2 - K_2} = -1. \end{aligned}$$

□

4.3. Surfaces Constructed Using the Spherical Image of the Quasi Binormal

The equation of surfaces constructed by the spherical image of the quasi binormal is given by

$$\varphi = \mathbf{B}_q(s, t).$$

Theorem 4.3. *Under the assumption $\nu k_2 - \mu k_3 > 0$, the Gaussian curvature K_3, the mean curvature H_3 and the principal curvatures k_{13} and k_{23} of φ are given by*

$$K_3 = 1, \ H_3 = -1, \ k_{13} = -1, \ k_{23} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned} \varphi_s &= -k_2 \mathbf{T}_q - k_3 \mathbf{N}_q, \\ \varphi_t &= -\mu \mathbf{T}_q - \nu \mathbf{N}_q, \end{aligned} \tag{4.3}$$

where the lower indices show partial differentiation. Then the unit normal to φ is given by

$$\mathbf{N}_{\varphi} = \frac{\varphi_s \times \varphi_t}{\|\varphi_s \times \varphi_t\|} = \mathbf{B}_q.$$

Using the equations (3.1), (3.2) and (4.3), the second order derivatives are calculated and given by

$$\begin{aligned}
\varphi_{ss} &= \left(k_1 k_3 - (k_2)_s\right) \mathbf{T}_q - \left((k_3)_s + k_1 k_2\right) \mathbf{N}_q - \left(k_2^2 + k_3^2\right) \mathbf{B}_q, \\
\varphi_{tt} &= \left(\lambda\nu - \mu_t\right) \mathbf{T}_q - (\nu_t + \lambda\mu)\mathbf{N}_q - \left(\mu^2 + \nu^2\right) \mathbf{B}_q, \\
\varphi_{st} &= \left(\mu_s + \nu k_1\right) \mathbf{T}_q - (\nu_s - \mu k_1)\mathbf{N}_q - (\mu k_2 + \nu k_3)\mathbf{B}_q.
\end{aligned}$$

The components of the first fundamental form $g_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned}
g_{11} &= \langle \varphi_s, \varphi_s \rangle = k_2^2 + k_3^2, \\
g_{12} &= \langle \varphi_s, \varphi_t \rangle = \mu k_2 + \nu k_3, \\
g_{22} &= \langle \varphi_t, \varphi_t \rangle = \mu^2 + \nu^2.
\end{aligned}$$

The components of the second fundamental form $l_{ij}, (1 \leq i, j \leq 2)$ are obtained as follows:

$$\begin{aligned}
l_{11} &= \langle \varphi_{ss}, \mathbf{N}_{\varphi} \rangle = -\left(k_2^2 + k_3^2\right), \\
l_{12} &= \langle \varphi_{st}, \mathbf{N}_{\varphi} \rangle = -\left(\mu k_2 + \nu k_3\right), \\
l_{22} &= \langle \varphi_{tt}, \mathbf{N}_{\varphi} \rangle = -\left(\mu^2 + \nu^2\right).
\end{aligned}$$

Thus, we get the following equalities:

$$\begin{aligned}
K_3 &= \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2} = 1, \\
H_3 &= \frac{l_{11} g_{22} - 2 l_{12} g_{12} + l_{22} g_{11}}{2\left(g_{11} g_{22} - g_{12}^2\right)} = -1, \\
k_{13} &= H_3 + \sqrt{H_3^2 - K_3} = -1, \\
k_{23} &= H_3 - \sqrt{H_3^2 - K_3} = -1.
\end{aligned}$$

□

5. Examples

We give two illustrative examples to the spherical images of a regular space curve according to quasi frame.

Example 5.1. Let us consider the space curve α which is defined by

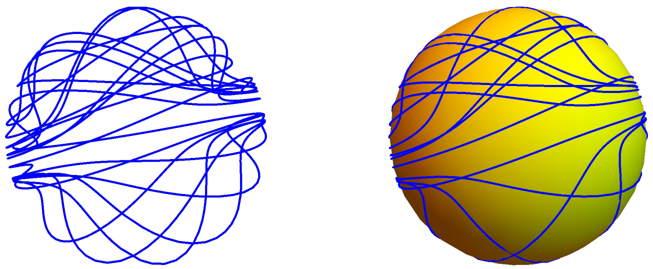
$$\begin{aligned} \alpha &: \mathbb{R} \longrightarrow \mathbb{R}^3, \\ \alpha(t) &= ((2+\cos t+\sin t)\sin t\cos(\sin(10t)), \\ &\quad (2+\cos t+\sin t)\sin t\sin(\sin(10t)), \\ &\quad (2+\cos t+\sin t)\cos t). \end{aligned}$$

Calculating the first derivative of α, one can easily see that

$$\left\|\alpha'(t)\right\| \neq 0$$

for all $t \in \mathbb{R}$. So we can say that α is a regular space curve. In the light of the quasi formulas, one can easily obtain the quasi frame $\left\{\mathbf{T}_q, \mathbf{N}_q, \mathbf{B}_q\right\}$ of α. The graphics of the curve α and its spherical images are given below.

FIG. 5.1: The curve α



(a) (b)

FIG. 5.2: The spherical image of tangent of curve α

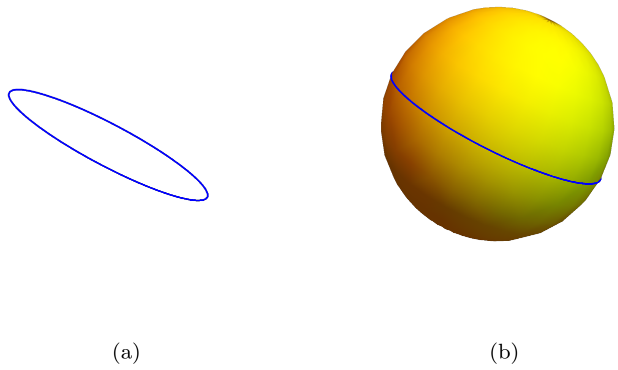


FIG. 5.3: The spherical image of the quasi-normal of curve α

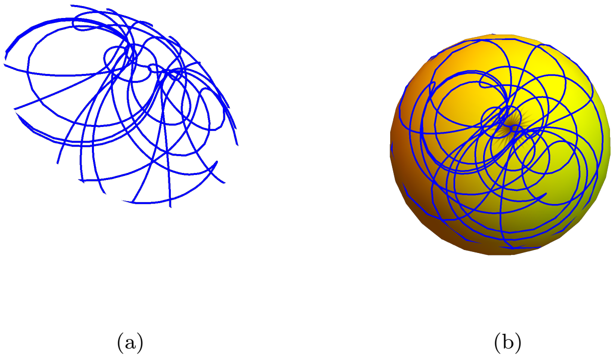


FIG. 5.4: The spherical image of the quasi-binormal of curve α

Example 5.2. Let us consider the space curve β which is defined in [5] by

$$\begin{aligned} \beta &: \mathbb{R} \longrightarrow \mathbb{R}^3, \\ \beta(t) &= (-\frac{18}{5}\sin(-\frac{t}{4}) + \frac{2}{45}\sin(\frac{9t}{4}), -\frac{18}{5}\cos(-\frac{t}{4}) + \frac{2}{45}\cos(\frac{9t}{4}), \frac{3}{5}\cos t). \end{aligned}$$

Calculating the first derivative of β, one can easily see that

$$\left\|\beta'(t)\right\| \neq 0$$

for all $t \in \mathbb{R}$. So we can say that β is a regular space curve. In the light of the quasi formulas, one can easily obtain the quasi frame $\{\overline{\mathbf{T}}_q, \overline{\mathbf{N}}_q, \overline{\mathbf{B}}_q\}$ of β. The graphics of the curve β and its spherical images are given below.

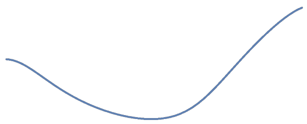
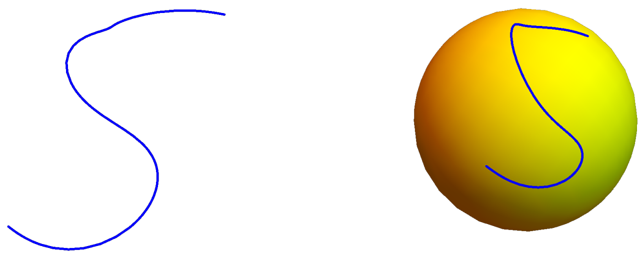
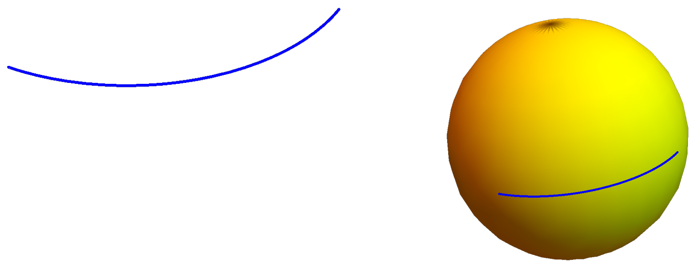


FIG. 5.5: The curve β



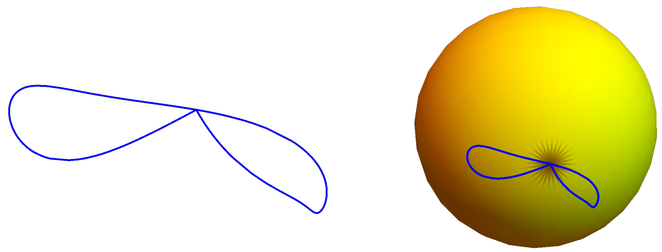
(a) (b)

FIG. 5.6: The spherical image of tangent of curve β



(a) (b)

Fig. 5.7: The spherical image of the quasi-normal of curve β



(a) (b)

Fig. 5.8: The spherical image of the quasi-binormal of curve β

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PARTIAL $b_v(s)$, PARTIAL v-GENERALIZED AND $b_v(\theta)$ METRIC SPACES AND RELATED FIXED POINT THEOREMS

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Abstract. In this paper, we have introduced three new generalized metric spaces called partial $b_v(s)$, partial v-generalized and $b_v(\theta)$ metric spaces which extend $b_v(s)$ metric space, b-metric space, rectangular metric space, v-generalized metric space, partial metric space, partial b-metric space, partial rectangular b-metric space and so on. We have proved some famous theorems such as Banach, Kannan and Reich fixed point theorems in these spaces. Also, we have given some numerical examples to support our definitions. Our results generalize several corresponding results in literature.

Keywords: partial $b_v(s)$ metric space; $b_v(\theta)$ metric space; generalized metric spaces; fixed point theorems; weakly contractive mappings.

1. Introduction and Preliminaries

Metric space was introduced by Maurice Fréchet [1] in 1906. Since a metric induces topological properties, it has very large application area in mathematics, especially in fixed point theory. Generalizing of notions is in the nature of mathematics. So, after the notion of metric space, many different type generalized metric spaces were introduced by many researchers. In 1989, Bakhtin introduced the notion of b-metric spaces by adding a multiplier to triangle ineuality. In 1994, Matthews [2] introduced the notion of partial metric spaces. In this kind of spaces, self-distance of any point needs not to be zero. This space is used in the study of denotational semantics of dataflow network. In 2000, Branciari [9] introduced rectangular metric space by adding four points instead of three points in triangle inequality. These three spaces are the basis of other generalized metric spaces. After all these spaces, v-generalized metric space [9], rectangular b-metric spaces [3], $b_v(s)$ metric space [10], partial b-metric space [4] and partial rectangular b-metric space [5] were introduced in recent years. Below, we shall give the definitions of some generalized metric spaces.

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Definition 1.1. [8] Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ a function. (E, ρ) is called b-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w, v \in E$:

1. $\rho(u, w) = 0$ iff $u = w$;
2. $\rho(u, w) = \rho(w, u)$;
3. $\rho(u, w) \leq s[\rho(u, v) + \rho(v, w)]$.

Clearly a b-metric space with $s = 1$ is exactly a usual metric space.

Definition 1.2. [2] Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ a mapping. (E, ρ) is called partial metric space if following conditions hold for all $u, w, v \in E$:

1. $u = w$ iff $\rho(u, u) = \rho(u, w) = \rho(w, w)$;
2. $\rho(u, u) \leq \rho(u, w)$;
3. $\rho(u, w) = \rho(w, u)$;
4. $\rho(u, w) \leq \rho(u, v) + \rho(v, w) - \rho(v, v)$.

It is clear that every metric space is also a partial metric spaces.

Definition 1.3. [9] Let E be a nonempty set and let $\rho : E \times E \to [0, \infty)$ be a mapping. (E, ρ) is called a rectangular metric space if following conditions hold for all $u, w \in E$ and for all distinct points $c, d \in E \setminus \{u, w\}$:

1. $\rho(u, w) = 0$ iff $u = w$;
2. $\rho(u, w) = \rho(w, u)$;
3. $\rho(u, w) \leq \rho(u, c) + \rho(c, d) + \rho(d, w)$.

Definition 1.4. [4] Let E be a nonempty set and mapping $\rho : E \times E \to [0, \infty)$ a mapping. (E, ρ) is called partial b-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w, v \in E$:

1. $u = w$ iff $\rho(u, u) = \rho(u, w) = \rho(w, w)$;
2. $\rho(u, u) \leq \rho(u, w)$;
3. $\rho(u, w) = \rho(w, u)$;
4. $\rho(u, w) \leq s[\rho(u, v) + \rho(v, w)] - \rho(v, v)$.

Remark 1.1. [4] It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact does not hold.

In 2017, Mitrovic and Radenovic introduced following generalized metric space which is referred to as $b_v(s)$ metric space. Under the suitable assumptions, this kind of spaces can be reduced to the other spaces.

Definition 1.5. [10] Let E be a nonempty set, $\rho : E \times E \to [0, \infty)$ a mapping and $v \in \mathbb{N}$. Then (E, ρ) is said to be a $b_v(s)$ metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w \in E$ and for all distinct points $z_1, z_2, \ldots, z_v \in E \setminus \{u, w\}$:

1. $\rho(u, w) = 0$ iff $u = w$;
2. $\rho(u, w) = \rho(w, u)$;
3. $\rho(u, w) \leq s[\rho(u, z_1) + \rho(z_1, z_2) + \cdots + \rho(z_v, w)]$.

This metric space can be reduced to v-generalized metric space by taking $s = 1$, rectangular metric space by taking $v = 2$ and $s = 1$, rectangular b-metric space by taking $v = 2$, b-metric space by taking $v = 1$ and usual metric space by taking $v = s = 1$.

2. Main Results

In this part, motivated and inspired by mentioned studies, we introduce $b_v(\theta)$ (or extended $b_v(s)$) metric space and partial $b_v(s)$ metric space. Also we give some fixed point theorems in these spaces.

First we introduce partial $b_v(s)$ metric space and give some properties of it.

2.1. Partial $b_v(s)$ Metric Spaces

Definition 2.1. Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ be a mapping and $v \in \mathbb{N}$. Then (E, ρ) is said to be a partial $b_v(s)$ metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w, z_1, z_2, \ldots, z_v \in E$:

1. $u = w \Leftrightarrow \rho(u, u) = \rho(u, w) = \rho(w, w)$;
2. $\rho(u, u) \leq \rho(u, w)$;
3. $\rho(u, w) = \rho(w, u)$;
4. $\rho(u, w) \leq s[\rho(u, z_1) + \rho(z_1, z_2) + \ldots + \rho(z_{v-1}, z_v) + \rho(z_v, w)] - \sum_{i=1}^{v} \rho(z_i, z_i)$.

It is easy to see that every $b_v(s)$ metric space is a partial $b_v(s)$ metric space. However, the converse is not true in general.

Remark 2.1. In Definition 2.1;

1. if we take $v = 2$, then we derive partial rectangular b-metric space.
2. if we take $v = 1$, then we derive partial b-metric space.
3. if we take $v = s = 1$, then we derive partial metric space.

Remark 2.2. Let (E, ρ) be a partial $b_v(s)$ metric space, if $\rho(u, w) = 0$, for $u, w \in E$, then $u = w$.

Proof. Let $\rho(u, w) = 0$ for $u, w \in E$. From the second condition of partial $b_v(s)$ metric space, since $\rho(u, u) \leq \rho(u, w) = 0$, we have $\rho(u, u) = 0$. Similarly, we have $\rho(w, w) = 0$. So, we get $\rho(u, w) = \rho(u, u) = \rho(w, w) = 0$. It follows from the first condition that $u = w$. $\square$

Proposition 2.1. *Let E be a nonempty set such that d_1 is a partial metric and d_2 is a $b_v(s)$ metric on E. Then (E, ρ) is a partial $b_v(s)$ metric space where $\rho : E \times E \to [0, \infty)$ is a mapping defined by $\rho(u, w) = d_1(u, w) + d_2(u, w)$ for all $u, w \in E$.*

Proof. Let (E, d_1) be a partial metric space and (E, d_2) be a $b_v(s)$ metric space.Then it is clear that first three conditions of the partial $b_v(s)$ metric space are satisfied for the function ρ. Let $u, w, z_1, z_2, \ldots, z_v \in E$ be arbitrary points. Then, we have

$$\begin{aligned}
\rho(u, w) &= d_1(u, w) + d_2(u, w) \\
&\leq d_1(u, z_1) + d_1(z_1, z_2) + \ldots + d_1(z_v, w) - \sum_{i=1}^{v} d_1(z_i, z_i) \\
&\quad +s\left[d_2(u, z_1) + d_2(z_1, z_2) + \ldots + d_2(z_v, w)\right] \\
&\leq s\Bigg[d_1(u, z_1) + d_1(z_1, z_2) + \ldots + d_1(z_v, w) - \sum_{i=1}^{v} d_1(z_i, z_i) \\
&\quad +d_2(u, z_1) + d_2(z_1, z_2) + \ldots + d_2(z_v, w)\Bigg] \\
&= s\left[\rho(u, z_1) + \rho(z_1, z_2) + \ldots + \rho(z_v, w) - \sum_{i=1}^{v} \rho(z_i, z_i)\right] \\
&\leq s\left[\rho(u, z_1) + \rho(z_1, z_2) + \ldots + \rho(z_v, w)\right] - \sum_{i=1}^{v} \rho(z_i, z_i).
\end{aligned}$$

So, (E, ρ) is a partial $b_v(s)$ metric space. $\square$

Example 2.1. Let $E = \mathbb{R}^+$. Define mapping $d_1 : E \times E \to [0, \infty)$ by

$$d_1(u, w) = \begin{cases} 0, & \text{if } u = w \\ 2, & \text{if } u \text{ or } w \notin \{1, 2\} \text{ and } u \neq w \\ 25, & \text{if } u, w \in \{1, 2\} \text{ and } u \neq w \end{cases}$$

for all $u, w \in \mathbb{R}^+$. Then, it is obvious that (E, d_1) is a $b_v(s)$ metric space with $v = 9$ and $s \geq \frac{5}{4}$. On the other hand, let $p > 1$, $d_2 : E \times E \to \mathbb{R}^+$ be a mapping defined by $d_2(u, w) = [\max\{u, w\}]^p$ for all $u, w \in E$. Then, it is easy to see that (E, d_2) is a partial metric space. So, we have from Proposition 2.1 that (E, ρ) is a partial $b_v\,(s)$ metric space where $\rho(u, w) = d_1(u, w) + d_2(u, w)$ for all $u, w \in E$.

Now, we give definitions of convergent sequence, Cauchy sequence and complete partial $b_v(s)$ metric space in following way.

Definition 2.2. Let (E, ρ) be a partial $b_v(s)$ metric space and let $\{u_n\}$ be any sequence in E and $u \in E$. Then:

1. The sequence $\{u_n\}$ is said to be convergent and converges to u, if $\lim_{n\to\infty} \rho(u_n, u) = \rho(u, u)$.
2. The sequence $\{u_n\}$ is said to be Cauchy sequence in (E, ρ) if $\lim_{n,m\to\infty} \rho(u_n, u_m)$ exists and is finite.
3. (E, ρ) is said to be a complete partial $b_v(s)$ metric space if for every Cauchy sequence $\{u_n\}$ in E there exists $u \in E$ such that

$$\lim_{n,m\to\infty} \rho(u_n, u_m) = \lim_{n\to\infty} \rho(u_n, u) = \rho(u, u).$$

Note that the limit of a convergent sequence may not be unique in a partial $b_v(s)$ metric space.

Now we give an analogue of Banach contraction principle. Our proof is very different from the original proof of Banach contraction principle in usual metric space.

Theorem 2.1. *Let (E, ρ) be a complete partial $b_v(s)$ metric space and $S : E \to E$ be a contraction mapping, i.e., S satisfies*

$$\rho(Su, Sw) \leq \lambda \rho(u, w) \tag{2.1}$$

for all $u, w \in E$, where $\lambda \in [0, 1)$. Then S has a unique fixed point $b \in S$ and $\rho(b, b) = 0$.

Proof. Let $G = S^{n_0}$ and define a sequence $\{u_n\}$ by $Gu_n = u_{n+1}$ for all $n \in \mathbb{N}$ and arbitrary point $u_0 \in E$. Since $\lambda \in [0, 1)$ and $\lim_{n\to\infty} \lambda^n = 0$, there exists a natural number n_0 such that $\lambda^{n_0} < \frac{\varepsilon}{4s}$ for given $0 < \varepsilon < 1$. Then, for all $u, w \in E$ we get

$$\rho(Gu, Gw) = \rho(S^{n_0}u, S^{n_0}w) \leq \lambda^{n_0} \rho(u, w). \tag{2.2}$$

So, we have

$$\rho(u_{k+1}, u_k) = \rho(Gu_k, Gu_{k-1}) \leq \lambda^{n_0} \rho(u_k, u_{k-1}) \leq \lambda^{kn_0} \rho(u_1, u_0) \to 0,$$

as $k \to \infty$. Hence, there exists a $l \in \mathbb{N}$ such that

$$\rho(u_{l+1}, u_l) < \frac{\varepsilon}{4s}.$$

Now, let

$$B_\rho[u_l, \varepsilon/2] := \left\{ w \in E : \rho(u_l, w) \leq \frac{\varepsilon}{2} + \rho(u_l, u_l) \right\}.$$

We need to prove that G maps the set $B_\rho[u_l, \varepsilon/2]$ into itself. Since $u_l \in B_\rho[u_l, \varepsilon/2]$, it is a nonempty set. Let z be an arbitrary point in $B_\rho[u_l, \varepsilon/2]$. Then, using (2.2) we get

$$\begin{aligned}
\rho(Gz, u_l) &\leq s\left[\rho(Gz, Gu_l) + \rho(Gu_l + Gu_{l+1}) + \ldots + \rho(Gu_{l+v-2}, Gu_{l+v-1})\right. \\
&\quad \left. + \rho(Gu_{l+v-1}, u_l)\right] - \sum_{i=0}^{v-1} \rho(Gu_{l+i}, Gu_{l+i}) \\
&\leq s\left[\rho(Gz, Gu_l) + \rho(Gu_l + Gu_{l+1}) + \ldots + \right. \\
&\quad \left. + \rho(Gu_{l+v-2}, Gu_{l+v-1}) + \rho(Gu_{l+v-1}, u_l)\right] \\
&\leq s[\lambda^{n_0}(\frac{\varepsilon}{2} + \rho(u_l, u_l)) + \rho(u_{l+1}, u_{l+2}) + \ldots + \\
&\quad + \rho(u_{l+v-1}, u_{l+v}) + \rho(u_{l+v}, u_l) \\
&\leq s\Big\{\lambda^{n_0}(\frac{\varepsilon}{2} + \rho(u_l, u_l)) + \rho(u_{l+1}, u_{l+2}) + \ldots + \\
&\quad + \rho(u_{l+v-1}, u_{l+v}) + s[\rho(u_l, u_{l+1}) + \rho(u_{l+1}, u_{l+2}) + \ldots \\
&\quad + \rho(u_{l+v-1}, u_{l+v}) + \rho(u_{l+v}, u_{l+v})] - \sum_{i=1}^{v} \rho(u_{l+i}, u_{l+i})\Big\} \\
&\leq s\Big\{\lambda^{n_0}(\frac{\varepsilon}{2} + \rho(u_l, u_l)) + (s+1)\,\rho(u_l, u_{l+1}) + (s+1)\rho(u_{l+1}, u_{l+2}) + \\
&\quad (s+1)\rho(u_{l+2}, u_{l+3}) + \ldots + (s+1)\rho(u_{l+v-1}, u_{l+v}) + s\rho(u_{l+v}, u_{l+v})\Big\} \\
&\leq s\Big\{\lambda^{n_0}(\frac{\varepsilon}{2} + \rho(u_l, u_l)) + (s+1)\rho(u_l, u_{l+1}) + (s+1)\rho(u_{l+1}, u_{l+2}) + \\
&\quad (s+1)\rho(u_{l+2}, u_{l+3}) + \ldots + (s+1)\rho(u_{l+v-1}, u_{l+v}) + s\lambda^{vn_0}\rho(u_l, u_l)\Big\} \\
&= \rho(u_l, u_l)\left[s\lambda^{n_0} + s^2\lambda^{vn_0}\right] + s\lambda^{n_0}\frac{\varepsilon}{2} + s^2\rho(u_l, u_{l+1}) + \\
&\quad s(s+1)\left[\rho(u_{l+1}, u_{l+2}) + \ldots + \rho(u_{l+v-1}, u_{l+v})\right].
\end{aligned}$$

Since $\lambda^{n_0} < \frac{\varepsilon}{4s}$ and $\rho(u_l, u_{l+1}) \leq \frac{\varepsilon}{4v(s^2+s)}$, we have

$$\begin{aligned}
\rho(Gz, u_l) &\leq \rho(u_l, u_l)\left[s\frac{\varepsilon}{4s} + s^2\frac{\varepsilon^v}{(4s)^v}\right] + s\frac{\varepsilon}{4s}\frac{\varepsilon}{2} + \\
&\quad (s^2 + s)\left[\rho(u_l, u_{l+1}) + \rho(u_{l+1}, u_{l+2}) + \ldots + \rho(u_{l+v-1}, u_{l+v})\right] \\
&\leq \rho(u_l, u_l) + \frac{\varepsilon}{4} + (s^2 + s)v\frac{\varepsilon}{4v(s^2 + s)} \\
&= \frac{\varepsilon}{2} + \rho(u_l, u_l).
\end{aligned}$$

So, $Gz \in B_\rho[u_l, \varepsilon/2]$. Therefore, G maps $B_\rho[u_l, \varepsilon/2]$ into itself. Since $u_l \in B_\rho[u_l, \varepsilon/2]$ and $Gu_l \in B_\rho[u_l, \varepsilon/2]$, we obtain that $G^n u_l \in B_\rho[u_l, \varepsilon/2]$ for all $n \in \mathbb{N}$, that is, $u_m \in B_\rho[u_l, \varepsilon/2]$ for all $m \geq l$. On the other hand, from definition of partial $b_v\,(s)$ metric space, since $\rho(u_l, u_l) \leq \rho(u_l, u_{l+1}) < \frac{\varepsilon}{4v(s^2+s)} < \frac{\varepsilon}{2}$, we have

$$\rho(u_n, u_m) < \frac{\varepsilon}{2} + \rho(u_l, u_l) < \varepsilon$$

for all $n, m > l$. This means that the sequence $\{u_n\}$ is a Cauchy sequence. Completeness of E implies that there exists $b \in E$ such that

$$\lim_{n\to\infty} \rho(u_n, b) = \lim_{n,m\to\infty} \rho(u_n, u_m) = \rho(b, b) = 0. \tag{2.3}$$

Now, we need to show that, b is a fixed point of S. For any $n \in \mathbb{N}$ we get

$$\begin{aligned}
\rho(b, Sb) &\leq s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \right.\\
&\quad \left.\rho(u_{n+v}, Sb)\right] - \sum_{i=1}^{v} \rho(u_{n+i}, u_{n+i})\\
&\leq s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \rho(u_{n+v}, Sb)\right]\\
&\leq s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \rho(Su_{n+v-1}, Sb)\right]\\
&\leq s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \lambda\rho(u_{n+v-1}, b)\right].
\end{aligned}$$

So, it follows from (2.3) that $\rho(b, Sb) = 0$. So, b is a fixed point of S.

Now, we show that S has a unique fixed point. Let $a, b \in E$ be two distinct fixed points of S, that is, $Sa = a$, $Sb = b$. Then, contractivity of mapping S implies that

$$\rho(a, b) = \rho(Sa, Sb) \leq \lambda\rho(a, b) < \rho(a, b),$$

which is a contradiction. So, it folllows that $\rho(a, b) = 0$, that is, $a = b$. Moreover, for a fixed point a, let assume that $\rho(a, a) > 0$. Then we get $\rho(a, a) = \rho(Sa, Sa) \leq \lambda\rho(a, a) < \rho(a, a)$ which is a contradiction. So, we have $\rho(a, a) = 0$. □

Now, we prove an analogue of Kannan fixed point theorem.

Theorem 2.2. *Let (E, ρ) be a complete partial $b_v(s)$ metric space and $S : E \to E$ a mapping satisfying the following condition:*

$$\rho(Su, Sy) \leq \lambda\left[\rho(u, Su) + \rho(w, Sw)\right] \tag{2.4}$$

for all $u, w \in E$, where $\lambda \in [0, \frac{1}{2})$, $\lambda \neq \frac{1}{s}$. Then S has a unique fixed point $b \in E$ and $\rho(b, b) = 0$.

Proof. .First we show the existence of fixed points of S. Let define a sequence $\{u_n\}$ by $u_n = S^n u_0$ for all $n \in \mathbb{N}$ and an arbitrary point $u_0 \in E$ and $\sigma_n = \rho(u_n, u_{n+1})$. If $\sigma_n = 0$, then for at least one n, u_n is a fixed point of S. So, let assume that $\sigma_n > 0$ for all $n \geq 0$. Since S is a Kannan mapping, it follows from (2.4) that

$$\begin{aligned}
\sigma_n &= \rho(u_n, u_{n+1}) = \rho(Su_{n-1}, Su_n)\\
&\leq \lambda\left[\rho(u_{n-1}, Su_{n-1}) + \rho(u_n, Su_n)\right]\\
&= \lambda\left[\rho(u_{n-1}, u_n) + \rho(u_n, u_{n+1})\right]\\
&= \lambda\left[\sigma_{n-1} + \sigma_n\right].
\end{aligned}$$

Therefore, we get $\sigma_n \leq \frac{\lambda}{1-\lambda}\sigma_{n-1}$. On repeating this process we obtain

$$\sigma_n \leq \left(\frac{\lambda}{1-\lambda}\right)^n \sigma_0.$$

From hypothesis, since $\lambda \in [0, \frac{1}{2})$, we have

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \rho(u_n, u_{n+1}) = 0. \tag{2.5}$$

So, for every $\varepsilon > 0$, there exists a natural number n_0 such that $\sigma_n < \varepsilon/2$ and $\sigma_m < \varepsilon/2$ for all $n, m \geq n_0$. From (2.5), we have

$$\begin{aligned}
\rho(u_n, u_m) &= \rho(Su_{n-1}, Su_{m-1}) \\
&\leq \lambda\left[\rho(u_{n-1}, Su_{n-1}) + \rho(u_{m-1}, Su_{m-1})\right] \\
&= \lambda\left[\rho(u_{n-1}, u_n) + \rho(u_{m-1}, u_m)\right] \\
&= \lambda\left[\sigma_{n-1} + \sigma_{m-1}\right] \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for $n, m > n_0$. Hence, $\{u_n\}$ is Cauchy sequence in E and $\lim_{n,m\to\infty} \rho(u_n, u_m) = 0$. It follows from the completeness of E that there exists $b \in E$ such that

$$\lim_{n\to\infty} \rho(u_n, b) = \lim_{n,m\to\infty} \rho(u_n, u_m) = \rho(b, b) = 0.$$

Now,we show that b is a fixed point of S. From definition of Kannan mappings and partial $b_v\,(s)$ metric space, we have

$$\begin{aligned}
\rho(b, Sb) \leq\ & s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) +\right. \\
& \left.\rho(u_{n+v}, Sb)\right] - \sum_{i=1}^{v} \rho(u_{n+i}, u_{n+i}) \\
\leq\ & s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \rho(u_{n+v}, Sb)\right] \\
\leq\ & s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \rho(Su_{n+v-1}, Sb)\right] \\
\leq\ & s\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) +\right. \\
& \left.\lambda\left\{\rho(u_{n+v-1}, Su_{n+v-1}) + \rho(b, Sb)\right\}\right].
\end{aligned}$$

So, it follows from the last inequality that

$$\begin{aligned}
\rho(b, Sb) \leq\ & \frac{s}{(1 - s\lambda)}\left[\rho(b, u_{n+1}) + \rho(u_{n+1}, u_{n+2})\right. \\
& \left.+ \ldots + \rho(u_{n+v-1}, u_{n+v}) + \lambda\rho(u_{n+v-1}, Su_{n+v-1})\right].
\end{aligned}$$

Since $\lambda \neq \frac{1}{s}$ and $\{u_n\}$ is a Cauchy and convergent sequence, we have $\rho(b, Sb) = 0$, so $Sb = b$. It means that b is a fixed point of S. Now we show the uniqueness of fixed point. But first, we need to show that if $b \in E$ is a fixed point of S, then $\rho(b, b) = 0$. Let assume to the contrary that $\rho(b, b) > 0$. Then, from (2.4) we have

$$\rho(b, b) = \rho(Sb, Sb) \leq \lambda\left[\rho(b, Sb) + \rho(b, Sb)\right] = 2\lambda\rho(b, b) < \rho(b, b),$$

which is a contradiction. So, assumption is wrong, namely, $\rho(b,b)=0$. Now, we can show that S has a unique fixed point. Suppose $a,b\in E$ be two distinct fixed points of S. Then we have $\rho(b,b)=\rho(a,a)=0$, and it follows from (2.4) that

$$\begin{aligned}\rho(b,a) &= \rho(Sb,Sa)\le\lambda\left[\rho(b,Sb)+\rho(a,Sa)\right]\\ &= \lambda\left[\rho(b,b)+\rho(a,a)\right]=0\end{aligned}$$

Therefore, we have $\rho(b,a)=0$ and so $b=a$. Thus S has a unique fixed point. This completes the proof. □

Theorem 2.3. *Let (E,ρ) be a complete partial $b_v(s)$ metric space and $S:E\to E$ a mapping satisfying:*

$$\rho(Su,Sw)\le\lambda\max\left\{\rho(u,w),\rho(u,Su),\rho(w,Sw)\right\} \tag{2.6}$$

for all $u,w\in E$ and $\lambda\in\left[0,\frac{1}{s}\right)$. Then, S has a unique fixed point $b\in E$ and $\rho(b,b)=0$.

Proof. We begin with the fixed points of S. Let $u_0\in E$ be an arbitrary initial point and let $\{u_n\}$ be a sequence defined by $u_{n+1}=Su_n$ for all n. If $u_n=u_{n+1}$ for at least one natural number n, then it is clear that this point is a fixed point of S. So, let assume that $u_{n+1}\neq u_n$ for all n. Now, it follows from (2.6) that

$$\begin{aligned}\rho(u_{n+1},u_n) &= \rho(Su_n,Su_{n-1})\\ &\le \lambda\max\left\{\rho(u_n,u_{n-1}),\rho(u_n,Su_n),\rho(u_{n-1},Su_{n-1})\right\}\\ &= \lambda\max\left\{\rho(u_n,u_{n-1}),\rho(u_n,u_{n+1}),\rho(u_{n-1},u_n)\right\}\\ &= \lambda\max\left\{\rho(u_n,u_{n-1}),\rho(u_n,u_{n+1})\right\}.\end{aligned}$$

Set $L=\max\left\{\rho(u_n,u_{n-1}),\rho(u_n,u_{n+1})\right\}$. There exists two cases. If $L=\rho(u_n,u_{n+1})$, then we get $\rho(u_{n+1},u_n)\le\lambda\rho(u_{n+1},u_n)<\rho(u_{n+1},u_n)$ which is a contradiction. So, we must have $L=\rho(u_n,u_{n-1})$ and then we have

$$\rho(u_{n+1},u_n)\le\lambda\rho(u_n,u_{n-1}).$$

By repeating this process, we obtain

$$\rho(u_{n+1},u_n)\le\lambda^n\rho(u_1,u_0) \tag{2.7}$$

for all n. On the other hand, since $\lambda^n\to 0$ for $n\to\infty$, there exists a natural number n_0 such that $0<\lambda^{n_0}s<1$. For $m,n\in\mathbb{N}$ with $m>n$, by using inequality (2.7), we obtain

$$\begin{aligned}\rho(u_n,u_m) \le\ & s\left[\rho(u_n,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-3},u_{n+v-2})\right.\\ &\left.+\rho(u_{n+v-2},u_{n+n_0})+\rho(u_{n+n_0},u_{m+n_0})+\rho(u_{m+n_0},u_m)\right]\\ &-\sum_{i=1}^{v-2}\rho(u_{n+i},u_{n+i})-\rho(u_{n+n_0},u_{n+n_0})-\rho(u_{m+n_0},u_{m+n_0})\end{aligned}$$

$$\begin{aligned}
&\leq s\left[\rho(u_n,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-3},u_{n+v-2})\right.\\
&\quad\left.+\rho(u_{n+v-2},u_{n+n_0})+\rho(u_{n+n_0},u_{m+n_0})+\rho(u_{m+n_0},u_m)\right]\\
&\leq s\left(\lambda^n+\lambda^{n+1}+\cdots+\lambda^{n+v-3}\right)\rho(u_0,u_1)\\
&\quad+s\lambda^n\rho(u_{v-2},u_{n_0})+s\lambda^{n_0}\rho(u_n,u_m)+s\lambda^m\rho(u_{n_0},u_0).
\end{aligned}$$

So, we get

$$\begin{aligned}
(1-s\lambda^{n_0})\,\rho(u_n,u_m) \;\leq\;& s\left(\lambda^n+\lambda^{n+1}+\cdots+\lambda^{n+v-3}\right)\rho(u_0,u_1)\\
&+s\lambda^n\rho(u_{v-2},u_{n_0})+s\lambda^m\rho(u_{n_0},u_0).
\end{aligned}$$

By taking limit from both side, we have

$$\lim_{n,m\to\infty}\rho(u_n,u_m)=0$$

Therefore, $\{u_n\}$ is a Cauchy sequence in E. By completing E, there exists $b\in E$ such that

$$\lim_{n\to\infty}\rho(u_n,b)=\lim_{n,m\to\infty}\rho(u_n,u_m)=\rho(b,b)=0. \tag{2.8}$$

Now, we show that b is a fixed point of S. From definition of partial $b_v\,(s)$ metric space and inequality (2.6), we have

$$\begin{aligned}
\rho(b,Sb) \;\leq\;& s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\right.\\
&\left.\rho(u_{n+v},Sb)\right]-\sum_{i=1}^{v}\rho(u_{n+i},u_{n+i})\\
\leq\;& s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\rho(u_{n+v},Sb)\right]\\
\leq\;& s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\rho(Su_{n+v-1},Sb)\right]\\
\leq\;& s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\right.\\
&\left.\lambda\max\left\{\rho(u_{n+v-1},b),\rho(u_{n+v-1},u_{n+v}),\rho(b,Sb)\right\}\right].
\end{aligned}$$

Set $F=\max\left\{\rho(u_{n+v-1},b),\rho(u_{n+v-1},u_{n+v}),\rho(b,Sb)\right\}$. There exists three cases:

1. If $F=\rho(u_{n+v-1},b)$, then we get

$$\rho(b,Sb)\leq s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\lambda\rho(u_{n+v-1},b)\right].$$

So, it follows from (2.8) that $\rho(b,Sb)=0$.

2. If $F=\rho(u_{n+v-1},u_{n+v})$, then we get

$$\rho(b,Sb)\leq s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+(1+\lambda)\,\rho(u_{n+v-1},u_{n+v})\right].$$

Again by using (2.8), we obtain that $\rho(b,Sb)=0$.

3. If $F=\rho(b,Sb)$ then we get

$$(1-s\lambda)\,\rho(b,Sb)\leq s\left[\rho(b,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})\right].$$

Since $\lambda \in \left[0, \frac{1}{s}\right)$, we obtain that $\rho(b, Sb) = 0$, that is, $Sb = b$. Thus, b is a fixed poit of S.

Now we show the uniqueness of fixed point of S. Suppose on the contrary that a and b are two distinct fixed points of S and $\rho(a,b) > 0$. It follows from (2.6) that

$$\begin{aligned} \rho(a,b) &= \rho(Sa,Sb) \leq \lambda \max\{\rho(a,b), \rho(a,Sa), \rho(b,Sb)\} \\ &= \lambda \max\{\rho(a,b), \rho(a,a), \rho(b,b)\} \\ &= \lambda\rho(a,b) < \rho(a,b), \end{aligned}$$

which is a cotradiction. Therefore, we must have $\rho(a,b) = 0$ and so $a = b$. Hence, S has a unique fixed point. □

In definition 2.1, if we take $s = 1$, then we derive following definition of partial v-generalized metric space.

Definition 2.3. Let E be a nonempty set and $\rho : E \times E \to [0,\infty)$ be a mapping and $v \in \mathbb{N}$. Then (E,ρ) is said to be a partial v-generalized metric space if following conditions hold for all $u, w, z_1, z_2, \ldots, z_v \in E$:

1. $u = w \Leftrightarrow \rho(u,u) = \rho(u,w) = \rho(w,w)$;
2. $\rho(u,u) \leq \rho(u,w)$;
3. $\rho(u,w) = \rho(w,u)$;
4. $\rho(u,w) \leq \rho(u,z_1) + \rho(z_1,z_2) + \ldots + \rho(z_{v-1},z_v) + \rho(z_v,y) - \sum_{i=1}^{v} \rho(z_i,z_i)$.

In Theorems 2.1,2.2 and 2.3, if take $s = 1$, then we derive following fixed point theorems in partial v-generalized metric space.

Corollary 2.1. *Let (E,ρ) be a complete partial v-generalized metric space and $S : E \to E$ be a contraction mapping, i.e., S satisfies*

$$\rho(Su, Sw) \leq \lambda\rho(u,w)$$

for all $u, w \in E$, where $\lambda \in [0,1)$. Then S has a unique fixed point $b \in S$ and $\rho(b,b) = 0$.

Corollary 2.2. *Let (E,ρ) be a complete partial v-generalized metric space and $S : E \to E$ a mapping satisfying the following condition:*

$$\rho(Su, Sy) \leq \lambda\left[\rho(u,Su) + \rho(w,Sw)\right]$$

for all $u, w \in E$, where $\lambda \in [0, \frac{1}{2})$. Then S has a unique fixed point $b \in E$ and $\rho(b,b) = 0$.

Corollary 2.3. *Let (E,ρ) be a complete partial v-generalized metric space and $S : E \to E$ a mapping satisfying:*

$$\rho(Su, Sw) \leq \lambda \max\{\rho(u,w), \rho(u,Su), \rho(w,Sw)\}$$

for all $u, w \in E$ and $\lambda \in [0,1)$. Then, S has a unique fixed point $b \in E$ and $\rho(b,b) = 0$.

2.2. $b_v(\theta)$ Metric Spaces

In 2017, Kamran et al. introduced following generalized metric space which they called extended b-metric space.

Definition 2.4. [6] Let E be a nonempty set and let $\theta : E \times E \to [1, \infty)$ be a function. A function $\rho_\theta : E \times E \to [0, \infty)$ is called an extended b-metric if for all $u, v, w \in E$ it satisfies:

1. $\rho_\theta(u, w) = 0$ iff $u = w$;
2. $\rho_\theta(u, w) = \rho(w, u)$;
3. $\rho_\theta(u, w) \leq \theta(u, w)\left[\rho_\theta(u, v) + \rho_\theta(v, w)\right]$.
   The pair (E, ρ_θ) is called an extended b-metric space.

It is clear that if $\theta(u, w) = s$ for all $u, w \in E$, then we obtain b-metric space.

From this point of view, we introduce following generalized metric space called as $b_v(\theta)$ (or extended $b_v(s)$) metric space.

Definition 2.5. Let E be a nonempty set, $\theta : E \times E \to [1, \infty)$ a function and $v \in \mathbb{N}$. Then $\rho_\theta : E \times E \to [0, \infty)$ is called $b_v(\theta)$ metric if for all $u, z_1, z_2, ..., z_v, w \in E$, each of them different from each other, it satisfies

1. $\rho_\theta(u, w) = 0$ iff $u = w$;
2. $\rho_\theta(u, w) = \rho_\theta(w, u)$;
3. $\rho_\theta(u, w) \leq \theta(u, w)\left[\rho_\theta(u, z_1) + \rho_\theta(z_1, z_2) + \cdots + \rho_\theta(z_v, w)\right]$.
   The pair (E, ρ_θ) is called $b_v(\theta)$ metric space.

Remark 2.3. It is clear that if for all $u, w \in E$

1. $\theta(u, w) = s$, then we obtain $b_v(s)$ metric space,
2. $v = 1$, then we obtain extended b-metric space,
3. $\theta(u, w) = s$ and $v = 1$, then we obtain b-metric space,
4. $\theta(u, w) = s$ and $v = 2$, then we obtain rectangular b-metric space,
5. $\theta(u, w) = 1$ and $v = 2$, then we obtain rectangular metric space,
6. $\theta(u, w) = 1$, then we obtain v-generalized metric space,
7. $\theta(u, w) = 1$ and $v = 1$, then we obtain usual metric space.

Example 2.2. Let $E = \mathbb{N}$. Define mappings $\theta : \mathbb{N} \times \mathbb{N} \to [1, \infty)$ and $\rho_\theta : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by $\theta(u, w) = 3 + u + w$ and

$$\rho_\theta(u, w) = \begin{cases} 6, & \text{if } u, w \in \{1, 2\} \text{ and } u \neq w \\ 1, & \text{if } u \text{ or } w \notin \{1, 2\} \text{ and } u \neq w \\ 0, & \text{if } u = w \end{cases}$$

for all $u, w \in \mathbb{N}$. Then, it is easy to see that (E, ρ_θ) is a $b_v(\theta)$ metric space with $v = 5$.

Definitions of Cauchy sequence, convergence and completeness can be easily extended to the case of $b_v(\theta)$ metric space by the following way.

Definition 2.6. Let (E, ρ_θ) be a $b_v(\theta)$ metric space, $\{u_n\}$ a sequence in E and $u \in E$. Then,

a) $\{u_n\}$ is said to converge to u in (E, ρ_θ) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho_\theta(u_n, u) < \varepsilon$ for all $n \geq n_0$ and this convergence is denoted by $u_n \to u$.

b) $\{u_n\}$ is said to be Cauchy sequence in (E, ρ_θ) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\rho_\theta(u_n, u_{n+p}) < \varepsilon$ for all $n \geq n_0$ and $p > 0$.

c) (E, ρ_θ) is said to be complete if every Cauchy sequence in E is convergent in E.

Now, we are in the position to prove fixed point theorems in $b_v(\theta)$ metric spaces. But first, we prove following lemmas which we need in the proof of main theorems.

Lemma 2.1. *Let (E, ρ_θ) be a $b_v(\theta)$ metric space, $S : E \to E$ a mapping and $\{u_n\}$ a sequence in E defined by $u_{n+1} = Su_n = S^n u_0$ such that $u_n \neq u_{n+1}$. Suppose that $c \in [0, 1)$ such that*

$$\rho_\theta(u_{n+1}, u_n) \leq c\rho_\theta(u_n, u_{n-1})$$

for all $n \in \mathbb{N}$. Then $u_n \neq u_m$ for all distinct $n, m \in \mathbb{N}$.

Proof. Since the proof is very similar with the proof of Lemma 1.11 of [10], we omit it. □

Lemma 2.2. *Let (E, ρ_θ) be a $b_v(\theta)$ metric space with a bounded function θ and $\{u_n\}$ a sequence in E defined by $u_{n+1} = Su_n = S^n u_0$ such that $u_n \neq u_m$ for all $n, m \in \mathbb{N}$. Assume that there exist $c \in [0, 1)$ and $k_1, k_2 \in \mathbb{R}^+ \cup \{0\}$ such that*

$$\rho_\theta(u_m, u_n) \leq c\rho_\theta(u_{m-1}, u_{n-1}) + k_1 c^m + k_2 c^m \tag{2.9}$$

for all $n, m \in \mathbb{N}$. Then $\{u_n\}$ is a Cauchy sequence in E.

Proof. It is easy to see that $\{u_n\}$ is Cauchy if $c = 0$. So, we should assume that $c \neq 0$. Since function $\theta(u, w)$ is bounded, there exists a number $n_0 \in \mathbb{N}$ such that

$$0 < c^{n_0}\theta(u, w) < 1 \tag{2.10}$$

for all $u, w \in E$. From hypothesis of lemma, we can write

$$\begin{aligned}
\rho_\theta(u_{n+1}, u_n) &\leq c\rho_\theta(u_n, u_{n-1}) + k_1 c^{n+1} + k_2 c^n \\
&\leq c\left(c\rho_\theta(u_{n-1}, u_{n-2}) + k_1 c^n + k_2 c^{n-1}\right) + k_1 c^{n+1} + k_2 c^n \\
&= c^2\rho_\theta(u_{n-1}, u_{n-2}) + 2\left(k_1 c^{n+1} + k_2 c^n\right) \\
&\;\;\vdots \\
&\leq c^n\rho_\theta(u_1, u_0) + n\left(k_1 c^{n+1} + k_2 c^n\right).
\end{aligned}$$

Similarly, for all $k \geq 1$, we can write

$$\rho_\theta\left(u_{m+k}, u_{n+k}\right) \leq c^k \rho_\theta\left(u_m, u_n\right) + k\left(k_1 c^{m+k} + k_2 c^{n+k}\right).$$

If $v \geq 2$, then from the definition of $b_v(\theta)$ metric space, we get

$$\begin{aligned}
\rho_\theta\left(u_n, u_m\right) \leq\ & \theta\left(u_n, u_m\right)\left[\rho_\theta\left(u_n, u_{n+1}\right) + \rho_\theta\left(u_{n+1}, u_{n+2}\right)\right. \\
& + \cdots + \rho_\theta\left(u_{n+v-3}, u_{n+v-2}\right) + \rho_\theta\left(u_{n+v-2}, u_{n+n_0}\right) \\
& \left. + \rho_\theta\left(u_{n+n_0}, u_{m+n_0}\right) + \rho_\theta\left(u_{m+n_0}, u_m\right)\right].
\end{aligned}$$

Then, we have

$$\begin{aligned}
\rho_\theta\left(u_n, u_m\right) \leq\ & \theta\left(u_n, u_m\right)\left[\left(c^n + c^{n+1} + \cdots + c^{n+v-3}\right)\rho_\theta\left(u_0, u_1\right)\right. \\
& + \left(k_1 c + k_2\right)\left(n c^n + (n+1) c^{n+1} + \cdots + (n+v-3) c^{n+v-3}\right) \\
& + c^n \rho_\theta\left(u_{v-2}, u_{n_0}\right) + n c^n\left(k_1 c^{v-2} + k_2 c^{n_0}\right) \\
& + c^{n_0} \rho_\theta\left(u_n, u_m\right) + n_0 c^{n_0}\left(k_1 c^n + k_2 c^m\right) \\
& \left. + c^m \rho_\theta\left(u_{n_0}, u_0\right) + m c^m\left(k_1 c^{n_0} + k_2\right)\right].
\end{aligned}$$

So, we obtain

$$\begin{aligned}
& \rho_\theta\left(u_n, u_m\right)\left(1 - c^{n_0}\theta\left(u_n, u_m\right)\right) \leq \\
& \leq \theta\left(u_n, u_m\right)\left[\left(c^n + c^{n+1} + \cdots + c^{n+v-3}\right)\rho_\theta\left(u_0, u_1\right)\right. \\
& + \left(k_1 c + k_2\right)\left(n c^n + (n+1) c^{n+1} + \cdots + (n+v-3) c^{n+v-3}\right) \\
& + c^n \rho_\theta\left(u_{v-2}, u_{n_0}\right) + n c^n\left(k_1 c^{v-2} + k_2 c^{n_0}\right) + n_0 c^{n_0}\left(k_1 c^n + k_2 c^m\right) \\
& \left. + c^m \rho_\theta\left(u_{n_0}, u_0\right) + m c^m\left(k_1 c^{n_0} + k_2\right)\right].
\end{aligned}$$

Since $\lim_{n\to\infty} n c^n = 0$ and $1 - c^{n_0}\theta\left(u_n, u_m\right) > 0$, using (2.9), we have $\rho_\theta\left(u_n, u_m\right) \to 0$ as $n, m \to \infty$. This means that $\{u_n\}$ is a Cauchy sequence. Since $b_v(s)$ metric space is a $b_{2v}\left(s^2\right)$ metric space, if $v = 1$, then $\{u_n\}$ is Cauchy. $\square$

Now we can give Banach fixed point theorem in complete $b_v(\theta)$ metric space.

Theorem 2.4. *Let (E, ρ_θ) be a complete $b_v(\theta)$ metric space with a bounded function θ and $S : E \to E$ a contraction mapping, i.e., there exists a constant $c \in [0, 1)$ such that*

$$\rho_\theta(Su, Sw) \leq c\rho_\theta(u, w) \tag{2.11}$$

for all $u, w \in E$. Then S has a unique fixed point.

Proof. Let $u_0 \in E$ be an arbitrary initial point and let $\{u_n\}$ be a sequence defined by $u_{n+1} = Su_n = S^{n+1}u_0$ and $u_n \neq u_{n+1}$ for all $n \geq 0$. It follows from Lemma 2.1 that $u_n \neq u_m$ for all $n, m \in \mathbb{N}$. Since S is a contraction mapping, we can write

$$\rho_\theta\left(u_n, u_m\right) = \rho_\theta\left(Su_{n-1}, Su_{m-1}\right) \leq c\rho_\theta\left(u_{n-1}, u_{m-1}\right).$$

From Lemma 2.2, we have $\{u_n\}$ is a Cauchy sequence. So, it follows from completeness of E that there exists an element $u \in E$ such that $u_n \to u$. Now, we show that $u \in FixS$, i.e., $u = Su$.

$$\begin{aligned}
\rho_\theta(u, Su) &\leq \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2})\right. \\
&\quad \left. + \cdots + \rho_\theta(u_{n+v-1}, u_{n+v}) + \rho_\theta(u_{n+v}, Su)\right] \\
&= \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2})\right. \\
&\quad \left. + \cdots + \rho_\theta(u_{n+v-1}, u_{n+v}) + \rho_\theta(Su_{n+v-1}, Su)\right] \\
&\leq \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2})\right. \\
&\quad \left. + \cdots + \rho_\theta(u_{n+v-1}, u_{n+v}) + c\rho_\theta(u_{n+v-1}, u)\right].
\end{aligned}$$

Since θ is a bounded function and $\{u_n\}$ is Cauchy with $u_n \to u$, we have $\rho_\theta(u, Su) = 0$. This means that $u \in FixS$. Next, we need to show that u is a unique fixed point. To the contrary, let assume that there exists another fixed point w. Since

$$\rho_\theta(u, w) = \rho_\theta(Su, Sw) \leq c\rho_\theta(u, w) < \rho_\theta(u, w),$$

we get $u = w$ that is u is the unique fixed point of S. □

Remark 2.4. In Theorem 2.4,

1. if we take the constant $v = 1$ and the function $\theta(u, w) = 1$ for all $u, w \in E$, then we derive classical Banach fixed point theorem in usual metric spaces.
2. if we take $\theta(u, w) = s$ for all $u, w \in E$ where $s \geq 1$, then we derive Theorem 2.1 of [10] in $b_v(s)$ metric spaces.
3. if $v = 1$ and $\theta(u, w) = s$ for all $u, w \in E$, then we derive Theorem 2.1 of [13] in b-metric spaces.
4. if $v = 2$ and $\theta(u, w) = s$ for all $u, w \in E$, then we derive Theorem 2.1 of [14] and so main theorem of [3] in rectangular b-metric spaces.
5. if $\theta(u, w) = 1$ for all $u, w \in E$, then we derive main result of Branciari [9] in v-generalized metric spaces.

In literature, there exist various type of contraction mappings. Weakly contractive mapping is one of this type of contractions which generalize usual contractions. A mapping $S : E \to E$ is called weakly contractive if there exists a continuous and nondecreasing function $\psi(t)$ defined from $\mathbb{R}^+ \cup \{0\}$ onto itself such that $\psi(0) = 0, \psi(t) \to \infty$ as $t \to \infty$ and for all $u, w \in E$

$$\rho_\theta(Su, Sw) \leq \rho_\theta(u, w) - \psi(\rho_\theta(u, w)). \tag{2.12}$$

Now, we generalize Banach fixed point theorem for weakly contractive mappings in $b_v(\theta)$ metric space.

Theorem 2.5. *Let E be a complete $b_v(\theta)$ metric space and S a weakly contractive mapping on E. Then S has a unique fixed point.*

Proof. Let $u_0 \in E$ be an arbitrary initial point. Define sequence $\{u_n\}$ by $u_1 = Su_0$, $u_2 = Su_1 = S^2u_0, \ldots, u_{n+1} = Su_n = S^n u_0$. If $u_n = u_{n+1}$ for all $n \in \mathbb{N}$ where $\mathbb{N}$ is the set of positive integer, then proof is trivial. So, let assume that $u_n \neq u_{n+1}$ for all n. Moreover, the case that $u_n \neq u_m$ for all different n and m can be easily proved. From (2.12), we can write

$$\begin{aligned} \rho_\theta\left(u_{n+1}, u_{n+p+1}\right) &= \rho_\theta\left(Su_n, Su_{n+p}\right) \\ &\leq \rho_\theta\left(u_n, u_{n+p}\right) - \psi\left(\rho_\theta\left(u_n, u_{n+p}\right)\right) \end{aligned}$$

for all $n, p \in \mathbb{N}$. Let $\alpha_n = \rho_\theta\left(u_n, u_{n+p}\right)$. Since ψ is nondecreasing, we have

$$\alpha_{n+1} \leq \alpha_n - \psi\left(\alpha_n\right) \leq \alpha_n. \tag{2.13}$$

Thus, the sequence $\{\alpha_n\}$ has a limit $\alpha \geq 0$. Now we should show that $\alpha = 0$. Assume to the contrary that $\alpha > 0$. Using (2.13), we have

$$\psi\left(\alpha_n\right) \geq \psi\left(\alpha\right) > 0.$$

So, we get

$$\alpha_{n+1} \leq \alpha_n - \psi\left(\alpha\right).$$

Hence, we obtain $\alpha_{N+m} \leq \alpha_m - N\psi\left(\alpha\right)$ which is a contradiction for large enough N. This proves that $\alpha = 0$. This means that $\{u_n\}$ is Cauchy. Completeness of E implies that there exists a point $u \in E$ such that $u_n \to u$. Now, we show that u is a fixed point of S. Using (2.12) and definition of ρ_θ, we get

$$\begin{aligned} \rho_\theta\left(u, Su\right) &\leq \theta\left(u, Su\right)\left[\rho_\theta\left(u, u_{n+1}\right) + \rho_\theta\left(u_{n+1}, u_{n+2}\right)\right. \\ &\quad \left. + \ldots + \rho_\theta\left(u_{n+v-1}, u_{n+v}\right) + \rho_\theta\left(u_{n+v}, Su\right)\right] \\ &= \theta\left(u, Su\right)\left[\rho_\theta\left(u, u_{n+1}\right) + \rho_\theta\left(u_{n+1}, u_{n+2}\right)\right. \\ &\quad \left. + \ldots + \rho_\theta\left(u_{n+v-1}, u_{n+v}\right) + \rho_\theta\left(Su_{n+v-1}, Su\right)\right] \\ &\leq \theta\left(u, Su\right)\left[\rho_\theta\left(u, u_{n+1}\right) + \rho_\theta\left(u_{n+1}, u_{n+2}\right) + \ldots + \right. \\ &\quad \left. \rho_\theta\left(u_{n+v-1}, u_{n+v}\right) + \rho_\theta\left(u_{n+v-1}, u\right) - \psi\left(\rho_\theta\left(u_{n+v-1}, u\right)\right)\right]. \end{aligned}$$

Since $\rho_\theta\left(u_n, u_{n+p}\right) \to 0$ and $u_n \to u$ as $n \to \infty$ and $\psi\left(0\right) = 0$, we have u is a fixed point of S.

To prove the uniqueness of fixed point, we can assume that there exist one more fixed point w. Since S is a weakly contractive mapping, we have

$$\rho_\theta\left(u, w\right) = \rho_\theta\left(Su, Sw\right) \leq \rho_\theta\left(u, w\right) - \psi\left(\rho_\theta\left(u, w\right)\right) < \rho_\theta\left(u, w\right).$$

So $u = w$ This finishes the proof. $\square$

Remark 2.5. In Theorem 2.5,

1. if we take the constant $v = 1$, the function $\theta\left(u, w\right) = 1$ for all $u, w \in E$ and $\psi(t) = ct$, then we derive classical Banach fixed point theorem.

2. if we take $\psi(t) = ct$ and $\theta(u, w) = s$ where $s \in [1, \infty)$, then we derive Theorem 2.1 of [10]
3. if $v = 1$, $\theta(u, w) = s$ and $\psi(t) = ct$, then we derive Theorem 2.1 of [13].
4. if $v = 2$, $\theta(u, w) = s$ and $\psi(t) = ct$, then we derive Theorem 2.1 of [14] and so main theorem of [3].
5. if $v = 1$ and $\theta(u, w) = s$, then we derive main theorem of [12].

Now, we give Reich fixed point theorem.

Theorem 2.6. *Let (E, ρ_θ) be a complete $b_v(\theta)$ metric space with a bounded function θ and $S : E \to E$ a mapping satisfying:*

$$\rho_\theta(Su, Sw) \leq \alpha\rho_\theta(u, w) + \beta\rho_\theta(u, Su) + \gamma\rho_\theta(w, Sw) \tag{2.14}$$

for all $u, w \in E$ where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma < 1$ and $\Gamma_1 < \frac{1}{\Gamma_2}$ where $\Gamma_1 = \min\{\beta, \gamma\}$ and $\Gamma_2 = \max\{\theta(u, Su), \theta(Su, u)\}$. Then S has a unique fixed point. Moreover, sequence $\{u_n\}$ defined by $u_n = Su_{n-1}$ converges strongly to the unique fixed point of S.

Proof. Let $\{u_n\}$ be a sequence defined by $u_{n+1} = Su_n = S^{n+1}u_0$ where $u_0 \in E$ is an arbitrary initial point. If $u_n = u_{n+1}$ for all $n \in \mathbb{N}$, it is easy to see that u_0 is a fixed point of S. Now, we assume that $u_n \neq u_{n+1}$ for all n. From (2.14) and definition of $\{u_n\}$, we have

$$\begin{aligned}
\rho_\theta(u_{n+1}, u_n) &= \rho_\theta(Su_n, Su_{n-1}) \\
&\leq \alpha\rho_\theta(u_n, u_{n-1}) + \beta\rho_\theta(u_n, Su_n) + \gamma\rho_\theta(u_{n-1}, Su_{n-1}) \\
&= \alpha\rho_\theta(u_n, u_{n-1}) + \beta\rho_\theta(u_n, u_{n+1}) + \gamma\rho_\theta(u_{n-1}, u_n).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\rho_\theta(u_{n+1}, u_n) &\leq \frac{\alpha+\gamma}{1-\beta}\rho_\theta(u_n, u_{n-1}) \\
&\leq \left(\frac{\alpha+\gamma}{1-\beta}\right)^n \rho_\theta(u_1, u_0).
\end{aligned}$$

Since $\alpha + \beta + \gamma < 1$, then it is clear that $0 \leq \frac{\alpha+\gamma}{1-\beta} < 1$. So, we obtain

$$\lim_{n\to\infty} \rho_\theta(u_{n+1}, u_n) = 0. \tag{2.15}$$

Also, since we assume that $u_n \neq u_{n+1}$ for all n and $\rho_\theta(u_{n+1}, u_n) \leq \frac{\alpha+\gamma}{1-\beta}\rho_\theta(u_n, u_{n-1})$, then it follows from Lemma 2.1 that $u_n \neq u_m$ for all $n, m \in \mathbb{N}$. So, we have

$$\begin{aligned}
\rho_\theta(u_n, u_m) &= \rho_\theta(Su_{n-1}, Su_{m-1}) \\
&\leq \alpha\rho_\theta(u_{n-1}, u_{m-1}) + \beta\rho_\theta(u_{n-1}, Su_{n-1}) + \gamma\beta\rho_\theta(u_{m-1}, Su_{m-1}) \\
&= \alpha\rho_\theta(u_{n-1}, u_{m-1}) + \beta\rho_\theta(u_{n-1}, u_n) + \gamma\beta\rho_\theta(u_{m-1}, u_m) \\
&\leq \alpha\rho_\theta(u_{n-1}, u_{m-1}) + \left(\beta\left(\frac{\alpha+\gamma}{1-\beta}\right)^{n-1} + \gamma\left(\frac{\alpha+\gamma}{1-\beta}\right)^{m-1}\right)\rho_\theta(u_1, u_0).
\end{aligned}$$

It follows from Lemma 2.2 that $\{u_n\}$ is a Cauchy sequence. So, from the completeness of E, we obtain that there exists a point $u \in E$ such that $u_n \to u$. Now, we show that u is a fixed point of S, i.e., $\rho_\theta(u, Su) = 0$. Since

$$\begin{aligned}
\rho_\theta(u, Su) &\leq \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2}) + \cdots + \right.\\
&\quad \left. + \rho_\theta(u_{n+v-1}, u_{n+v}) + \rho_\theta(u_{n+v}, Su)\right]\\
&\leq \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2}) + \cdots + \right.\\
&\quad \left. + \rho_\theta(u_{n+v-1}, u_{n+v}) + \rho_\theta(Su_{n+v-1}, Su)\right]\\
&\leq \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2}) + \cdots + \right.\\
&\quad + \rho_\theta(u_{n+v-1}, u_{n+v}) + \\
&\quad \left. + \alpha\rho_\theta(u_{n+v-1}, u) + \beta\rho_\theta(u_{n+v-1}, u_{n+v}) + \gamma\rho_\theta(u, Su)\right],
\end{aligned}$$

we have

$$\begin{aligned}
(1 - \gamma\theta(u, Su))\,\rho_\theta(u, Su) &\leq \theta(u, Su)\left[\rho_\theta(u, u_{n+1}) + \rho_\theta(u_{n+1}, u_{n+2}) + \cdots \right.\\
&\quad \left. + \rho_\theta(u_{n+v-1}, u_{n+v}) + \alpha\rho_\theta(u_{n+v-1}, u) + \beta\rho_\theta(u_{n+v-1}, u_{n+v})\right].
\end{aligned}$$

Since $\Gamma_1 < \frac{1}{\Gamma_2}$, we get $(1 - \gamma\theta(u, Su)) \in [0, 1)$. So, it follows from (2.15) and convergence of $\{u_n\}$ that $\rho_\theta(u, Su) = 0$. This means that u is a fixed point of S. Now, we need to show that u is a unique fixed point. Let assume that there exists another fixed point v. Then, we have

$$\begin{aligned}
\rho_\theta(u, v) &= \rho_\theta(Su, Sv) \leq \alpha\rho_\theta(u, v) + \beta\rho_\theta(u, Su) + \delta\rho_\theta(v, Sv)\\
&= \alpha\rho_\theta(u, v).
\end{aligned}$$

Since $\alpha < 1$, we obtain that $\rho_\theta(u, v) = 0$, i.e., u is the unique fixed point of S. $\square$

Remark 2.6. In Theorem 2.6, if we take $\theta(u, w) = s$ for all $u, w \in E$ where $s \geq 1$, then we derive Theorem 2.4 of [10].

In Reich fixed point theorem, if we get $\alpha = 0$, then we obtain following generalized Kannan fixed point theorem in $b_v(\theta)$ metric spaces.

Theorem 2.7. *Let E be a complete $b_v(\theta)$ metric space and S a mapping on E satisfying:*

$$\rho_\theta(Su, Sw) \leq \beta\rho_\theta(u, Su) + \gamma\rho_\theta(w, Sw)$$

for all $u, w \in E$ where β and γ are nonnegative constants with $\beta + \gamma < 1$ and $\Gamma_1 < \frac{1}{\Gamma_2}$ where $\Gamma_1 = \min\{\beta, \gamma\}$ and $\Gamma_2 = \max\{\theta(u, Su), \theta(Su, u)\}$. Then S has a unique fixed point.

Remark 2.7. In Theorem 2.7,

1. if $v = 1$ and $\theta(u, w) = 1$ for all $u, w \in E$ where $s \geq 1$, then we obtain Kannan fixed point theorem [7] in complete usual metric spaces.
2. if $v = 2$ and $\theta(u, w) = s$ for all $u, w \in E$ where $s \geq 1$, then we derive Theorem 2.4 of [3].
3. if $v = 2$ and $\theta(u, w) = 1$ for all $u, w \in E$ where $s \geq 1$, then we obtain main theorem of [15] without the assumption of orbitally completeness of the space and the main theorem of [11].

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k-TYPE SLANT HELICES FOR SYMPLECTIC CURVE IN 4-DIMENSIONAL SYMPLECTIC SPACE

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Abstract. In this study, we have expressed the notion of k-type slant helix in 4-symplectic space. Also, we have generated some differential equations for k-type slant helix of symplectic regular curves.

Keywords: Symplectic curve; k-Type slant helix.

1. Introduction

The helix concept is an important area for differential geometers due to its numerous applications in many areas from physics to engineering. So, many authors are interested in helices to study in Euclidean 3-space and Euclidean 4-space. In [7, 6, 9], the authors gave new characterizations for an helix. The notion of a slant helix belongs to Izumiya and Takeuchi [4]. They consider the principle normal vector field of the curve instead of tangent vector field and they defined a new kind of helix which is called slant helix. Recently, some studies have been done to extend the definitions of helix and slant helix to Minkowski space (see [1, 2, 3]) and other frames [8].

2. Preliminaries

Let us give a brief related to symplectic space. One can found a brief account of the symplectic space in [10, 5]. The symplectic space $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ is the vector space $\mathbb{R}^4$ endowed with the standard symplectic form Ω, given in global Darboux coordinates by $\Omega = \sum\limits_{i=1}^{2} dx_i \wedge dy_i$. Each tangent space is endowed with symplectic inner product defined in canonical basis by

$$\begin{aligned} \langle u, v\rangle &= \Omega(u, v) \\ &= x_1\eta_1 + x_2\eta_2 - y_1\xi_1 - y_2\xi_2 \end{aligned}$$

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where $u=(x_1,x_2,y_1,y_2)$ and $v=(\xi_1,\xi_2,\eta_1,\eta_2)$.

A symplectic frame is a smooth section of the bundle of linear frames over $\mathbb{R}^4$ which assigns to every point $z\in\mathbb{R}^4$ an ordered basis of tangent vectors $a_1,a_2,a_3,a_4$with the property that

$$\langle a_i,a_j\rangle=\langle a_{2+i},a_{2+j}\rangle=0,\quad 1\leq i,j\leq 2, \tag{2.1}$$

$$\langle a_i,a_{2+j}\rangle=0,\qquad 1\leq i\neq j\leq 2, \tag{2.2}$$

$$\langle a_i,a_{2+i}\rangle=1,\quad 1\leq i\leq 2.$$

Let $z(t):\mathbb{R}\to\mathbb{R}^4$ denotes a local parametrized curve. In our notation, we allow z to be defined on an open interval of $\mathbb{R}$. As it is customary in classical mechanics, we use the notation $\dot{z}$ to denote differentiation with respect to the parameter t

$$\dot{z}=\frac{dz}{dt}.$$

Definition 2.1. A curve $z\,(t)$ is said to be symplectic regular if it satisfies the following non-degeneracy condition

$$\langle \dot{z}\ ,\ \ddot{z}\rangle\neq 0\ ,\quad for\ all\ t\in\mathbb{R}. \tag{2.3}$$

Definition 2.2. Let $t_0\in\mathbb{R}$, then the symplectic arc length s of a symplectic regular curve starting at t_0 is defined by

$$s\,(t)=\int\limits_{t_0}^{t}\langle \dot{z}\ ,\ \dddot{z}\rangle^{1/3}dt,\qquad for\ t\geqslant t_0. \tag{2.4}$$

Taking the extrerior differential of (2.4) we obtain the symplectic arc length element as

$$ds=\langle \dot{z}\ ,\ \dddot{z}\rangle^{1/3}\,dt.$$

Dually, the arc length derivative operator is

$$D=\frac{d}{ds}=\langle \dot{z}\ ,\ \dddot{z}\rangle^{-1/3}\,\frac{d}{dt}. \tag{2.5}$$

In the following, primes are used to denote differentiation with respect to the symplectic arc length derivative operator (2.5)

$$z^{'}=\frac{dz}{ds}.$$

Definition 2.3. A symplectic regular curve is parametrized by symplectic arc length if

$$\langle \dot{z},\dddot{z}\rangle=1,\quad for\ all\ t\in\mathbb{R}.$$

Let $z(s)$ be a symplectic regular curve in $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$. In this case there exist only one Frenet frame $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ for which $z(s)$ is a symplectic regular curve with Frenet equations

$$\begin{aligned} &a_1'(s) = a_3(s), \quad a_2'(s) = H_2(s)a_4(s), \\ &a_3'(s) = k_1(s)a_1(s) + a_2(s), \quad a_3'(s) = a_1(s) + k_2(s)a_2(s), \end{aligned} \tag{2.6}$$

where $H_2(s) = constant(\neq 0)$ [10].

In [2], the authors introduced the k-type slant helix in Minkowski 4-dimensional space E_1^4. Now, we extend the concept of slant helix for symplectic regular curve as follows:

Definition 2.4. Let z be a symplectic regular curve with the Frenet frame $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$. We say that z is a k-type slant helix if there exists a (non-zero) constant vector field $U \in R^4$ such that

$$\langle a_{k+1}(s), U\rangle = const.$$

for $0 \leq k \leq 3$ where U is an axis of the curve.

In particular, 0-type slant helices are general helices and 1-type slant helices are slant helices.

3. k-Type Slant Helices

Theorem 3.1. *Let z be a symplectic regular curve in $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$. Then z is 0-type slant helix(or general helix) if and only if*

$$\frac{k_1(s)}{k_2(s)} = const. \tag{3.1}$$

Morever, z is also a k-type slant helix, for $k \in \{1, 2, 3\}$.

Proof. Assume that z is a 0-type slant helix. Then for a constant vector field U, we have $\langle a_1(s), U\rangle = c$ is constant. Differentiating this equation and using Frenet equations, we obtain $\langle a_3(s), U\rangle = 0$. So U is orthogonal to $a_3(s)$ and we can decompose U as differentiating (3.1) and using Frenet equations, one arrives to

$$\begin{aligned} ck_1(s) + U_4(s) &= 0, \\ U_4'(s) &= 0, \\ c + U_4k_2(s) &= 0. \end{aligned} \tag{3.2}$$

Thus U_4 is constant. By (3.1) and (3.2) can easily obtained. Converse of proof is obvious. □

Theorem 3.2. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$, *where* $k_1(s) \neq const(\neq 0)$, $k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const = c_0$. *If z is a* 0*-slant helix, then*

$$k_1^{''}(s) - c_0 k_1(s) k_2(s) + c_0 = 0. \tag{3.3}$$

Proof. Assume that z is a 0-type slant helix. Then for a constant vector field U, we have

$$\langle a_1(s), U \rangle = c. \tag{3.4}$$

Differentiating this equation and using Frenet equations, we obtain

$$\langle a_3(s), U \rangle = 0. \tag{3.5}$$

Taking the derivative of equation (3.5) with respect to s, we have

$$\langle a_2(s), U \rangle = -c k_1(s). \tag{3.6}$$

Now, if we differentiate (3.6) and use the Frenet frame, we get

$$\langle a_4(s), U \rangle = \frac{-c}{H_2(s)} k_1'(s). \tag{3.7}$$

Hence, we differentiate (3.7) for the last time. Taking into account of hypotesis of the Theorem and the Frenet frame, we obtain (3.3). □

Corollary 3.1. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$. *If z is a* 0*-type slant helix, then we have following diffrential equation*

$$k_2^{''}(s) - c_0 k_2^2(s) - 1 = 0, \tag{3.8}$$

where $c_0 = H_2(s) = const(\neq 0)$ *and* $k_1(s) \neq const(\neq 0)$ *and* $k_2(s) \neq const(\neq 0)$.

Proof. From (3.3) and (3.1) we obtain (3.8). □

Corollary 3.2. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$. *a) If z is a 0-type slant helix with* $k_1(s) = const(\neq 0)$, *then we have*

$$k_2(s) = \frac{1}{k_1(s)}, \tag{3.9}$$

b) If z is a 0-type slant helix with $k_2(s) = const(\neq 0)$, *then we have*

$$H_2(s) = -\frac{1}{k_2^2(s)}. \tag{3.10}$$

Similarly, we can give the following conclusions:

Theorem 3.3. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with* $k_1(s) \neq const(\neq 0), k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const(\neq 0)$. *If z is a 1-type slant helix, then*

$$k_2^{''}(s) - k_1(s)k_2(s) + 1 = 0. \tag{3.11}$$

Corollary 3.3. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with non-zero constant* $k_1(s), k_2(s), H_2(s)$. *If z is a 1-type slant helix, then*

$$k_1(s) = \frac{1}{k_2(s)}. \tag{3.12}$$

Morever z is also a k-type slant helix, for $k \in \{2, 3\}$. In this case, we have the following:

Theorem 3.4. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with* $k_1(s) \neq const(\neq 0), k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const(\neq 0)$. *If z is a 2-type slant helix, then we get*

$$(k_1^{''}(s) + H_2(s) - H_2(s)k_1(s)k_2(s) \langle a_1(s), U \rangle = -2k_1'(s)c, \tag{3.13}$$

where c is a constant.

Theorem 3.5. *Let z be a symplectic regular curve in* $\mathrm{Si}\,m = (\mathbb{R}^4, \Omega)$ *with Frenet frame* $\{a_1(s), a_2(s), a_3(s), a_4(s)\}$ *with* $k_1(s) \neq const(\neq 0), k_2(s) \neq const(\neq 0)$ *and* $H_2(s) = const(\neq 0)$. *If z is a 3-type slant helix, then we have*

$$(k_2^{''}(s) - k_1(s)k_2(s) + 1) \langle a_2(s), U \rangle = -2ck_2'(s)H_2(s), \tag{3.14}$$

where c is a constant.

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EXISTENCE AND BLOW UP FOR A NONLINEAR VISCOELASTIC HYPERBOLIC PROBLEM WITH VARIABLE EXPONENTS *

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Abstract. Our aim in this paper is to establish the weak existence theorem and find under suitable assumptions sufficient conditions on m, p and the initial data for which the blow up takes place for the following boundary value problem:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u.$$

This paper extends some of the results obtained by the authors and it is focused on new results which are consequence of the presence of variable exponents.
Keywords: Variable exponents; weak solutions; blow up.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded Lipschitz domain and $0 < T < \infty$. We consider the following initial boundary value problem:

$$(1.1) \qquad \begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds \\ +|u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u, \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in S_T, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

where $Q_T = \Omega \times (0,T]$ and S_T denote the lateral boundary of the cylinder Q_T. It is assumed throughout the paper that the exponents $m(x)$ and $p(x)$ are continuous in Ω with logarithmic module of continuity:

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(1.2) $$1 < m^- = ess \inf_{x\in\Omega} m(x) \leq m(x) \leq m^+ = ess \sup_{x\in\Omega} m(x) < \infty,$$

(1.3) $$1 < p^- = ess \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ = ess \sup_{x\in\Omega} p(x) < \infty,$$

(1.4) $$\forall z, \xi \in \Omega, |z - \xi| < 1, |m(x) - m(\xi)| + |p(z) - p(\xi)| \leq \omega(|z - \xi|),$$

where

(1.5) $$\lim_{\tau\to 0^+} \sup \omega(\tau) ln\frac{1}{\tau} = C < +\infty.$$

Remark 1.1. We use the standard Lebesgue space $L^p(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar product and norms. We will use the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \leq s \leq 2n/(n-2)$ if $n \geq 3$ or $s \geq 2$ if $n = 1, 2$. The generic embedding constant, denoted by C_* is given by

(1.6) $$\|u\|_s \leq C_* \|\nabla u\|_2.$$

And we also assume that

(H_1) : ρ is a constant that satisfies

$$0 < \rho \leq \frac{2}{n-2} \text{ if } n \geq 3 \quad \text{and} \quad 0 < \rho \quad \text{if} \quad n = 1, 2.$$

(H_2) : $g : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded $\mathcal{C}^1$ function satisfying

$$g(0) > 0, \; 1 - \int_0^\infty g(s) ds = l > 0.$$

(H_3) : There exists $\xi > 0$ such that

$$g'(t) < -\xi(t) g(t), t \geq 0.$$

If m, p are constants, there have been many results about the existence and blow-up properties of the solutions, we refer the readers to the bibliography given in [5]-[25]. In recent years, a great attention has been focused on the study of mathematical models of electro-rheological fluids. These models include hyperbolic, parabolic or elliptic equations which are nonlinear with respect to gradient of the thought solution and with variable exponents of nonlinearity see ([3]-[12]-[15]-[23]-[24]) and the references therein. It should be mentioned that questions of existence, uniqueness and regularity of weak solutions for parabolic and elliptic equations have been studied by many authors under various conditions on the data and by different methods-(see [[1],[2]] and the further references therein).

To the best of our knowledge, there are only a few works about viscoelastic hyperbolic equations with variable exponents of nonlinearity. In [4] the authors investigated the finite time blow-up of solutions for viscoelastic hyperbolic equations, and in [5] the authors discussed only the viscoelastic hyperbolic problem with constant exponents. Motivated by the works of [[5],[4]], we shall study the existence and energy decay of the solutions to problem (1.1) and state some properties to the

solutions.
The present paper is organized as follows. In Section 2, we introduce the function spaces of Orlicz-Sobolev type and a brief description of their main properties, give the definition of the weak solution to the problem and prove the existence of weak solutions for problem (1.1) with Galerkin's method. In the last sections, we finally prove the desired results.

2. Existence of weak solutions

In this section, the existence of weak solutions is studied. Firstly, we introduce some Banach spaces
$L^{p(x)}(\Omega) = \left\{ u(x) : u \text{ is measurable in } \Omega, A_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}dx < \infty \right\},$
with the following Luxembourg-type norm

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0, A_{p(.)}(u/\lambda) \leq 1 \right\}.$$

We, next, define the variable-exponent Lebesgue Sobolev space $W^{1,p(.)}(\Omega)$ as follows:

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \quad \text{such that} \quad \nabla u \quad \text{exists and} \quad |\nabla u| \in L^{p(.)}(\Omega) \right\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,p(.)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}$. Furthermore, we set $W_0^{1,p(.)}(\Omega)$ to be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined in a different way for the variable exponent case. However, both definitions are equivalent (see [10]). The dual of $W_0^{1,p(.)}(\Omega)$ is defined as $W^{-1,p'(.)}(\Omega)$; in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Lemma 2.1. *([3]) For $u \in L^{p(x)}(\Omega)$, the following relations hold:*

1. $\|u\|_{p(.)} < 1 (= 1; > 1) \Leftrightarrow A_{p(.)}(u) < 1 (= 1; > 1)$;
2. $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p^+} \leq A_{p(.)}(u) \leq \|u\|_{p(.)}^{p^-}$;
   $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p^+} \geq A_{p(.)}(u) \geq \|u\|_{p(.)}^{p^-}$;
3. $\|u\|_{p(.)} \to 0 \Leftrightarrow A_{p(.)}(u) \to 0; \|u\|_{p(.)} \to \infty \Leftrightarrow A_{p(.)}(u) \to \infty$.

Lemma 2.2. *([26]) For $u \in W_0^{1,p(.)}(\Omega)$, if p satisfies condition* (1.2)*, the $p(.)$-Poincaré's inequality*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)},$$

holds, where the positive constant C depends on p and Ω.

Remark 2.1. Note that the following inequality

$$\int_\Omega |u|^{p(x)}dx \le C\int_\Omega |\nabla u|^{p(x)}dx,$$

does not in general hold.

Lemma 2.3. *([10]). Let Ω be an open domain (that may be unbounded) in $\mathbb{R}^n$ with cone property. If $p(x) : \bar{\Omega} \to \mathbb{R}$ is Lipschitz continuous function satisfying $1 < p^- \le p^+ < \frac{n}{k}$ and $r(x) : \bar{\Omega} \to \mathbb{R}$ is measurable and satisfies*

$$p(x) \le r(x) \le p^*(x) = \frac{np(x)}{n - kp(x)} \quad a.e\ x \in \bar{\Omega},$$

then there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$.

The main theorem in this section is the following:

Theorem 2.1. *Let $u_0, u_1 \in H_0^1(\Omega)$ be given. Assume that the exponents $m(x)$ and $p(x)$ satisfy conditions (1.2)-(1.4). Then the problem (1.1) has at least one weak solution $u : \Omega \times (0,\infty) \to \mathbb{R}$ in the class*

$$u \in L^\infty(0,\infty; H_0^1(\Omega)),\ u' \in L^\infty(0,\infty; H_0^1(\Omega)),\ u'' \in L^\infty(0,\infty; H_0^1(\Omega)).$$

And one of the following conditions holds:

$$(A_1)\ 2 < p^- < p^+ < \max\left\{n, \frac{np^-}{n-p^-}\right\}, \quad 2 < m^- < m^+ < p^-;$$

$$(A_2)\ \max\left\{1, \frac{2n}{n+2}\right\} < p^- < p^+ < 2, \quad 1 < m^- < m^+ < \frac{3p^- - 2}{p^-}.$$

Proof. Let us take for $\{w_j\}_{j=1}^\infty$ the orthogonal basis of $H_0^1(\Omega)$ such that

$$-\Delta w_j = \lambda_j w_j, x \in \Omega,\ w_j = 0,\ x \in \partial\Omega.$$

We denote by $V_k = span\ \{w_i, ..., w_k\}$ the subspace generated by the first k vectors of the basis $\{w\}_{j=1}^\infty$. By normalization, we have $\|w_j\|_2 = 1$. Let us define the operator:

$$\begin{aligned} < Lu, \phi > &= \int_\Omega \Big[|u_t|^\rho u_{tt}\phi + \nabla u \nabla\phi + \nabla u_{tt}\nabla\phi - \int_0^t g(t-s)\nabla u\nabla\phi ds \\ &\quad + |u_t|^{m(x)-2}u_t\phi - \alpha|u|^{p(x)-2}u\phi\Big]dx, \quad \phi \in V_k. \end{aligned}$$

For any given integer k, we consider the approximate solution $u_k = \sum_{i=1}^{k} c_i^k(t) w_i$, which satisfies

$$
\left\{
\begin{array}{ll}
< Lu_k, w_i >= 0 & i = 1, 2, ...k, \\
u_k(0) = u_{0k}, & u_{kt}(0) = u_{1k},
\end{array}
\right.
\tag{2.1}
$$

where

$$
u_{0k} = \sum_{i=1}^{k} (u_0, w_i) w_i, u_{1k} = \sum_{i=1}^{k} (u_1, w_i) w_i \text{ and } u_{0k} \to u_0, u_{1k} \to u_1 \ in \ H_0^1(\Omega).
$$

Here we denote by (.,.) the inner product in $\mathbb{L}^2(\Omega)$.

Problem (1.1) generates the system of k ordinary differential equations

$$
\left\{
\begin{array}{l}
\left| \sum_{i=1}^{k} (c_i^k(t))', w_i \right|^{\rho} (\sum_{i=1}^{k} c_i^k(t), w_i)'' = -\lambda_i c_i^k(t) + \lambda_i \int_0^t g(t-s) c_i^k(s) ds \\
+ \left| (\sum_{i=1}^{k} (c_i^k(t))', w_i) \right|^{m(x)-2} (\sum_{i=1}^{k} (c_i^k(t))', w_i) \\
-\alpha \left| (\sum_{i=1}^{k} c_i^k(t), w_i) \right|^{p(x)-2} (\sum_{i=1}^{k} c_i^k(t), w_i), \\
c_i^k(0) = (u_0, w_i), \ (c_i^k(0))' = (u_1, w_i), \quad i = 1, 2, ...k.
\end{array}
\right.
\tag{2.2}
$$

By the standard theory of the ODE system, we infer that the problem (2.2) admits a unique solution $c_i^k(t)$ in $[0, t_k]$, where $t_k > 0$. Then we can obtain an approximate solution $u_k(t)$ for (1.1) in V_k, over $[0, t_k]$. This solution can be extended to $[0, T]$, for any given $T > 0$, by the estimate below. Multiplying (2.1) by $(c_i^k(t))'$ and summing with respect to i we arrive at the relation

$$
\begin{aligned}
0 &= \tfrac{d}{dt} \left(\tfrac{1}{\rho+2} \|u_k'\|_{\rho+2}^{\rho+2} + \tfrac{1}{2} \|\nabla u_k\|_2^2 + \tfrac{1}{2} \|\nabla u_k'\|_2^2 \right) + \int_\Omega |u_k'|^{m(x)} dx \\
&- \tfrac{d}{dt} \left(\int_0^t g(t-s) \int_\Omega (\nabla u_k(s) \nabla u_k'(t) dx ds) \right) - \alpha \tfrac{d}{dt} \left(\int_\Omega \frac{1}{p(x)} |u_k|^{p(x)} dx \right).
\end{aligned}
\tag{2.3}
$$

Multiplying (2.1) by $(c_i^k(t))'$, integrating over Q_T, using integration by part and Green formula, one obtains

$$
\begin{aligned}
&- \int_0^t g(t-s) \int_\Omega (\nabla u_k(s), \nabla u_k'(t)) dx ds = \frac{1}{2} \frac{d}{dt} (g \diamond \nabla u_k)(t) \\
&- \tfrac{1}{2} (g' \diamond \nabla u_k)(t) - \tfrac{1}{2} \tfrac{d}{dt} \int_0^t g(s) ds \|\nabla u_k\|_2^2 + \frac{1}{2} g(t) \|\nabla u_k\|_2^2,
\end{aligned}
\tag{2.4}
$$

here
$$(\varphi \diamond \nabla\psi)(t) = \int_0^t \varphi(t-s)\|\nabla\psi(t) - \nabla\psi(s)\|_2^2 ds.$$
Combining (2.3)-(2.4) and $(H_2) - (H_3)$, we get

$$\begin{aligned} &\frac{d}{dt}\Bigg(\frac{1}{\rho+2}\|u_k'\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}(g\diamond \nabla u_k)(t) \\ &+\frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_k\|_2^2 - \alpha\int_\Omega \frac{1}{p(x)}|u_k|^{p(x)}dx\Bigg) \\ &= \frac{1}{2}(g\diamond\nabla u_k)(t) - \frac{1}{2}g(t)\|\nabla u_k\|_2^2 - \int_\Omega |u_k'|^{m(x)}dx. \end{aligned} \tag{2.5}$$

Integrating (2.5) over $(0,t)$, and using the assumptions (1.2)-(1.4), it is easy to verify that

$$\begin{aligned} \tfrac{1}{\rho+2}\|u_k'\|_{\rho+2}^{\rho+2} \quad &+\frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g\diamond\nabla u_k)(t) \\ &-\alpha\tfrac{1}{p(x)}|u_k|^{p(x)} \le C_1, \end{aligned}$$

where C_1 is a positive constant depending only on $\|u_0\|_{H_0^1}, \|u_1\|_{H_0^1}$.
According to the Lemma 2.1, we also have

$$\begin{aligned} \tfrac{1}{\rho+2}\|u_k'\|_{\rho+2}^{\rho+2} + \tfrac{1}{2}\|\nabla u_k'\|_2^2 \quad &+\frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_k\|_2^2 + \frac{1}{2}(g\diamond\nabla u_k)(t) \\ &- \max\left\{\alpha\tfrac{1}{p^-}\|u_k\|_{p(x)}^{p^-}, \alpha\tfrac{1}{p^-}\|u_k\|_{p(x)}^{p^+}\right\} \le C_1. \end{aligned} \tag{2.6}$$

In view of $(H_1) - (H_2) - (H_3)$ and $(A_1) - (A_2)$, we get

$$\|u_k'\|_{\rho+2}^{\rho+2} + \|\nabla u_k'\|_2^2 + (g\diamond\nabla u_k)(t) \le C_2, \tag{2.7}$$

where C_2 is positive constant depending only on $\|u_0\|_{H_0^1}, \|u_1\|_{H_0^1}, l, p^-, p^+$. It follows from (2.7) that

$$u_k \quad \text{is uniformly bounded in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)). \tag{2.8}$$

$$u_k' \quad \text{is uniformly bounded in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)). \tag{2.9}$$

Next, multiplying (1.1) by $(c_i^k(t))''$ and then summing with respect to i, we get the following

$$\begin{aligned} &\int_\Omega |u_k'|^\rho |u_k''|_2^2 dx + \|\nabla u_k''\|_2^2 + \frac{d}{dt}\left(\frac{1}{m(x)}|u_k'|^{m(x)}\right) = -\int_\Omega \nabla u_k \nabla u_k'' dx \\ &+\int_\Omega g(t-s)\int_\Omega \nabla u_k(s)\nabla u_k'' dx ds + \alpha\int_\Omega |u_k|^{p(x)-2}u_k u_k'' dx. \end{aligned} \tag{2.10}$$

Note that we have the estimates for $\varepsilon > 0$

$$\int_\Omega |u_k'|^\rho |u_k''|^2 dx \le C_\varepsilon \||u_k'|^\rho\|_2^2 + \frac{1}{4\varepsilon}\|u_k''\|_2^2, \tag{2.11}$$

$$\left| - \int_\Omega \nabla u_k \nabla u_k'' dx \right| \le \varepsilon \|\nabla u_k''\|_2^2 + \frac{1}{4\epsilon} \|\nabla u_k\|_2^2, \tag{2.12}$$

$$\begin{aligned}
&\left| - \int_0^t g(t-s) \int_\Omega \nabla u_k(s) \nabla u_k''(t) dx ds \right| \\
&\le \tfrac{1}{4\varepsilon} \int_\Omega \left(\int_0^t g(t-s) \nabla u_k(s) ds \right)^2 dx + \varepsilon \|\nabla u_k''\|_2^2 \\
&\le \varepsilon \|\nabla u_k''\|_2^2 + \tfrac{1}{4\varepsilon} \int_0^t g(s) ds \int_0^t g(t-s) \int_\Omega |\nabla u_k(s)|^2 dx ds \\
&\le \varepsilon \|\nabla u_k''\|_2^2 + \tfrac{(1-l)g(0)}{4\varepsilon} \int_0^t \|\nabla u_k(s)\|_2^2 ds,
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\alpha \| |u_k|^{p(x)-2} u_k u_k'' \| &\le \alpha\varepsilon \|u_k''\|_2^2 + \tfrac{\alpha}{4\varepsilon} \| |u_k|^{p(x)-2} u_k \|_2^2 \\
&\le \alpha\varepsilon \|u_k''\|_2^2 + \tfrac{\alpha}{4\varepsilon} \int_\Omega (|u_k|^{p(x)-2} u_k)^2 dx.
\end{aligned} \tag{2.14}$$

From Lemma 2.2, we have

$$\|u_k''\|_2^2 \le C^2 \|\nabla u_k''\|_2^2, \tag{2.15}$$

and

$$\begin{aligned}
&\int_\Omega (|u_k'|^{p(x)-2} u_k)^2 dx = \int_\Omega |u_k|^{2(p(x)-1)} u_k dx \\
&\le \max\left\{ \int_\Omega |u_k|^{2(p^- -1)} dx, \int_\Omega |u_k|^{2(p^+ -1)} dx \right\} \\
&\le \max\left\{ C^{*\frac{1}{2(p^- -1)}} \|\nabla u_k\|^{\frac{2}{2(p^- -1)}}, C^{*\frac{1}{2p^+ -1}} \|\nabla u_k'\|^{\frac{2}{2p^+ -1}} \right\},
\end{aligned} \tag{2.16}$$

where C, C^* are embedding constants. Taking into account (2.10)-(2.16), we obtain

$$\begin{aligned}
&C_\varepsilon \int_\Omega |\nabla u_t|^{2\rho} dx + \frac{1}{4\varepsilon} \int_\Omega |u_k''|^2 dx + (1 - 2\varepsilon - \alpha\varepsilon C) \|\nabla u_k''\|_2^2 \\
&+ \tfrac{d}{dt} \left(\tfrac{1}{m(x)} |u_k'|^{m(x)} \right) \le \tfrac{1}{4\varepsilon} \|\nabla u_k\|_2^2 + \tfrac{(1-l)g(0)}{4\varepsilon} \int_0^t \|\nabla u_k(s)\|_2^2 ds \\
&+ \max\left\{ C^{*\frac{1}{2(p^- -1)}} \|\nabla u_k\|^{\frac{1}{p^- -1}}, C^{*\frac{1}{2p^+ -1}} \|\nabla u_k\|^{\frac{1}{p^+ -1}} \right\}.
\end{aligned} \tag{2.17}$$

Integrating (2.17) over $(0, t)$ and using (2.7), Lemma 2.3 we get

$$\begin{aligned}
&C_\varepsilon T C_2^{2\rho} + \tfrac{1}{4\varepsilon} \int_0^t \|u_k''\|^2 dx + (1 - 2\varepsilon - \alpha\varepsilon C) \int_\Omega \|\nabla u_k''\|_2^2 ds \\
&+ \int_\Omega \frac{1}{m(x)} |u_k'|^{m(x)} dx \le \frac{1}{4\varepsilon} (C_3 + (1-l) g(0) T) + C_4,
\end{aligned} \tag{2.18}$$

where C_4 is a positive constant depending only on $\|u_1\|_{H_0^1}$. Taking α,ε small enough in (2.18), we obtain the estimate

$$\frac{1}{4\varepsilon} \int_0^t \|u_k''\|^2 ds + \int_\Omega \frac{1}{m(x)} |u_k'|^{m(x)} dx \le C_5. \tag{2.19}$$

Hence according to the Lemma 2.1, we have that

$$\frac{1}{4\varepsilon}\int_0^t \|u_k''\|^2 ds + \min\left\{\frac{1}{m^+}\|u_k'\|_{m(x)}^{m^-}, \frac{1}{m^+}\|u_k'\|_{m(x)}^{m^+}\right\} \le C_5, \tag{2.20}$$

where C_5 is a positive constant depending only on $\|u_0\|_{H_0^1}$, $\|u_1\|_{H_0^1}$, l, $g(0)$, T. From estimate (2.20), we get

$$u_k'' \text{ is uniformly bounded in } \mathbb{L}^2(0,T;H_0^1(\Omega)). \tag{2.21}$$

By (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad L^\infty(0,T;H_0^1(\Omega)), \tag{2.22}$$

by (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and a function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.23}$$

$$u_i \rightharpoonup u \quad \text{weakly in} \quad \mathbb{L}^{p^-}(0,T;W^{1,p(x)}(\Omega)), \tag{2.24}$$

where C_5 is a positive constant depending only on $\|u_0\|_{H_0^1}$, $\|u_1\|_{H_0^1}$, l, $g(0)$, T. From estimate (2.20), we get

$$u_k'' \text{ is uniformly bounded in } \mathbb{L}^2(0,T;H_0^1(\Omega)). \tag{2.25}$$

By (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.26}$$

by (2.7)-(2.9) and (2.25), we infer that there exists a subsequence u_i of u_k and function u such that

$$u_i \rightharpoonup u \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.27}$$

$$u_i \rightharpoonup u \quad \text{weakly in} \quad \mathbb{L}^{p^-}(0,T;W^{1,p(x)}(\Omega)), \tag{2.28}$$

$$u_i' \rightharpoonup u' \quad \text{weakly star in} \quad \mathbb{L}^\infty(0,T;H_0^1(\Omega)), \tag{2.29}$$

$$u_i'' \rightharpoonup u'' \quad \text{weakly in} \quad \mathbb{L}^2(0,T;H_0^1(\Omega)). \tag{2.30}$$

Next, we will deal with the nonlinear term. From the Aubin-Lions theorem, see ([20], pp.57-58], it follows from (2.29) and (2.30) that there exists a subsequence of u_i, still represented by the same notation, such that

$u_i' \to u'$ strongly in $\mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$, which implies that $u_i' \to u'$ almost everywhere in $\Omega \times (0,T)$. Hence, by (2.27) – (2.30), we have

$$|u_i'|^\rho u_i'' \to |u'|^\rho u'' \quad \text{weakly in} \quad \Omega \times (0,T), \tag{2.31}$$

$$|u_i|^{p(x)-2}u_i \rightharpoonup |u|^{p(x)-2}u \quad \text{weakly in} \quad \Omega \times (0,T), \tag{2.32}$$

$$|u_i'|^{m(x)-2}u_i' \to |u'|^{m(x)-2}u' \quad \text{almost everywhere in} \quad \Omega \times (0,T). \tag{2.33}$$

Multiplying (2.2) by $\phi(t) \in C(0,T)$(which $C(0,T)$ is space of C^∞ function with compact support in $(0,T)$) and integrating the obtained result over $(0,T)$, we obtain that

$$< Lu_k, w_i\phi(t) >= 0, \quad i = 1, 2, ..., k. \tag{2.34}$$

Note that $\{w_i\}_{i=1}^\infty$ is basis of $H_0^1(\Omega)$. Convergence (2.27)-(2.33) is sufficient to pass to the limit in (2.34) in order to get

$$|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u \ \text{ in } \ \mathbb{L}^2(0,T;H_0^1(\Omega)),$$

for arbitrary $T > 0$. In view of (2.27) – (2.30) and Lemma 3.3.17 in [?], we derive that

$$u_k(0) \rightharpoonup u(0) \text{ weakly in } H_0^1(\Omega), \ u_k'(0) \rightharpoonup u'(0) \text{ weakly in } H_0^1(\Omega).$$

Hence, we get $u(0) = u_0$, $u_1(0) = u_1$. Then, we conclude the proof of the Theorem 2.1. $\square$

3. Blow up

In this section, we shall prove our main result concerning the blow-up of solutions to Theorem 2.1. For this task, we define

$$\begin{aligned} E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ &+ \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p, \end{aligned} \tag{3.1}$$

where

$$(g \circ v)(t) = \int_0^\infty g(s)ds < \frac{\frac{p}{2} - 1}{\frac{p}{2} - 1 + \frac{1}{2p}} \tag{3.2}$$

Lemma 3.1. *([22]) The modified energy functional satisfies the solution of (1.1)*

$$E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \|u_t\|_m^m \leq \frac{1}{2}(g' \circ \nabla u)(t). \tag{3.3}$$

Theorem 3.1. *Suppose that*

$$\max\{m, p\} \leq \frac{2(n-1)}{n-2}, \quad n \geq 3, \tag{3.4}$$

holds. Assume further that $u_0, u_1 \in H_0^1(\Omega)$ and $E(0) < 0$. Then the solution of theorem 2.1 blows up in finite time

$$T^* \leq \frac{C(1-\alpha)}{\epsilon\gamma\alpha L^{\frac{\alpha}{1-\alpha}}(0)}$$

Lemma 3.2. *Suppose that (3.4) holds. Then there exists a positive constant $C > 1$ depending on Ω only such that for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$, we have*

$$\|u\|_p^s \leq C\left(\|\nabla u\|_2^2 + \|u\|_p^p\right). \tag{3.5}$$

Proof. 1. If $\|u\|_p \leq 1$ then $\|u\|_p^s \leq \|u\|_p^2 \leq C\|\nabla u\|_2^2$ by Sobolev embedding theorems.

2. If $\|u\|_p > 1$ then $\|u\|_p^s \leq \|u\|_p^p$. Therefore (3.5) follows.

□

We set

$$H(t) = -E(t).$$

We use, throughout this paper, C to denote a generic positive constant depending on Ω only. As a result of (3.1) and (3.5) we have

Corollary 3.1. *Let the assumptions of the lemma 3.2 hold. Then we have the following for all* $t \in [0, T)$,

$$\|u\|_p^s \leq C\left(-H(t) - \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 - (g \circ \nabla u)(t) + \|u\|_p^p\right). \tag{3.6}$$

Proof. (Theorem 3.1) By multiplying equation (1.1) by $-u_t$ and integrating over Ω we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{-\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{p}\int_\Omega |u|^p dx\right\}\\ &+\int_0^t g(t-s)\int_\Omega \nabla u_t(t).\nabla u(s)dxds = \int_\Omega |u_t|^m dx,\end{aligned} \tag{3.7}$$

for any regular solution. This result can be extended to weak solutions by density argument. But

$$
\begin{aligned}
&\int_0^t g(t-s)\int_\Omega \nabla u_t(t).\nabla u(\tau)dxds\\
&= \int_0^t \int_\Omega \nabla u_t(t).|\nabla u(s)-\nabla u(t)|dxds + \int_0^t g(t-s)\int_\Omega \nabla u_t(t).\nabla u(t)dxds\\
&= -\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds\\
&+ \int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx\right)ds\\
&= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds\right]\\
&+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega |\nabla u(t)|^2 dxds\right]\\
&+\frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2 dxds.
\end{aligned} \tag{3.8}
$$

We then insert (3.8) in (3.7) to get

$$
\begin{aligned}
&\frac{d}{dt}\left\{-\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{p}\int_\Omega |u|^p dx\right\}\\
&-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxd\tau\right]\\
&+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)\int_\Omega |\nabla u(t)|^2 dxds\right]\\
&= \int_\Omega |u_t|^m dx - \frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds + \frac{1}{2}g(t)\|\nabla u(t)\|^2.
\end{aligned} \tag{3.9}
$$

By using the definition of $H(t)$ the estimate (3.9) becomes

$$
\begin{aligned}
H'(t) &= \int_\Omega |u_t|^m dx - \frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds\\
&+\frac{1}{2}g(t)\|\nabla u(t)\|^2 \geq 0.
\end{aligned} \tag{3.10}
$$

Consequently, we have

$$
0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p. \tag{3.11}
$$

We define

$$
L(t) = H^{1-\alpha}(t) + \frac{\epsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \epsilon\int_\Omega \nabla u_t.\nabla u dx, \tag{3.12}
$$

where ϵ small to be chosen later and

$$
0 < \alpha \leq \frac{p-m}{m-1}
$$

By taking derivative of (3.12) and using (1.1) we obtain

$$
\begin{aligned}
L'(t) &= -\frac{1}{2}(1-\alpha)H^{-\alpha}(t)\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds\\
&+(1-\alpha)H^{-\alpha}(t)\left\{\int_\Omega |u_t|^m dx+\frac{1}{2}g(t)\|\nabla u(t)\|^2\right\}\\
&+\frac{\epsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}-\epsilon\|\nabla u\|_2^2+\epsilon\|\nabla u_t\|_2^2+\epsilon\int_0^t g(t-s)\int_\Omega \nabla u(s).\nabla u(t)ds\\
&-\epsilon\int_\Omega |u_t|^{m-2}u_t u dx+\epsilon\|u\|_p^p.
\end{aligned}\tag{3.13}
$$

We then exploit Young's inequality, and use (3.1) to substitute for $\int_\Omega |u(x,t)|^p dx$ hence (3.13) becomes

$$
\begin{aligned}
L'(t) &\geq (1-\alpha)H^{1-\alpha}(t)\|u_t\|_m^m+\frac{\epsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}\\
&-\epsilon\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\epsilon\|\nabla u_t\|_2^2-\epsilon\eta(g\circ\nabla u)\\
&-\frac{\epsilon}{4\eta}\int_0^t g(s)ds\|\nabla u\|_2^2-\epsilon\int_\Omega |u_t|^{m-2}u_t u dx+\epsilon\frac{p}{2}(g\circ\nabla u)\\
&+\epsilon\left(pH(t)+\frac{p}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{p}{2}\|\nabla u_t\|_2^2+\frac{p}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right)\\
&\geq (1-\alpha)H^{1-\alpha}(t)\|u_t\|_m^m+\epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2}+\epsilon pH(t)\\
&+\epsilon\left(\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t g(s)ds\right)\|\nabla u\|_2^2\\
&+\epsilon\left(\frac{p}{2}-\eta\right)(g\circ\nabla u)+\epsilon\left(\frac{p}{2}+1\right)\|\nabla u_t\|_2^2,
\end{aligned}\tag{3.14}
$$

for some number η with $0<\eta<\frac{p}{2}$. By recalling (3.2), the estimate (3.14) is reduced to

$$
\begin{aligned}
L'(t) &\geq (1-\alpha)H^{-\alpha}(t)\|u_t\|_m^m+\epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2}\\
&+\epsilon pH(t)+\epsilon a_1(g\circ\nabla u)+\epsilon a_2\|\nabla u\|_2^2+\epsilon a_3\|\nabla u_t\|_2^2-\epsilon\int_\Omega |u_t|^{m-2}u_t u dx,
\end{aligned}\tag{3.15}
$$

where

$$
a_1=\frac{p}{2}-\eta>0,\quad a_2=\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t g(s)ds>0,\quad a_3=\frac{p}{2}+1>0.
$$

To estimate the last term of (3.15), we use again Young's inequality

$$
XY\leq\frac{\delta^r}{r}X^r+\frac{\delta^{-q}}{q}Y^q,\quad X,Y\geq 0,\quad for\quad all\quad \delta>0,\quad \frac{1}{r}+\frac{1}{q}=1,
$$

with $r = m$ and $q = \frac{m}{(m-1)}$. So we have

$$\int_\Omega |u_t|^{m-1}|u|dx \geq \frac{\delta^m}{m}\|u\|_m^m + \frac{m-1}{m}\delta^{\frac{-m}{(m-1)}}\|u_t\|_m^m,$$

which yields, by substitution in (3.15), for all $\delta > 0$

$$\begin{aligned}L'(t) \geq &\left[(1-\alpha)H^{-\alpha}(t) - \frac{m-1}{m}\delta^{\frac{-m}{m-1}}\right]\|u_t\|_m^m + \epsilon a_3\|\nabla u_t\|_2^2 - \epsilon\frac{\delta^m}{m}\|u\|_m^m \\ &+\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon pH(t) + \epsilon a_1(g \circ \nabla u) + \epsilon a_2\|\nabla u\|_2^2.\end{aligned} \tag{3.16}$$

The inequality (3.16) remains valid even if δ is time dependant since the integral is taken over the x variable. Therefore by taking δ so that $\delta^{\frac{-m}{m-1}} = kH^{-\alpha}(t)$, for large k to be specified later, and substituting in (3.16) we arrive at

$$\begin{aligned}L'(t) \geq &\left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right]H^{-\alpha}(t)\|u_t\|_m^m + \epsilon\left(\frac{1}{\rho+1}\right)\|u_t\|_{\rho+2}^{\rho+2} \\ &+\epsilon a_1(g \circ \nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 + \epsilon\left(\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} \\ &+\epsilon\left[pH(t) - \frac{k^{1-m}}{m}H^{\alpha(m-1)}(t)\|u\|_m^m\right].\end{aligned} \tag{3.17}$$

By exploiting (3.11) and inequality $\|u\|_m^m \leq C\|u\|_p^m$, we obtain

$$H^{\alpha(m-1)}(t)\|u\|_m^m \leq \left(\frac{1}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)},$$

therefore, from (3.17), one obtains

$$\begin{aligned}L'(t) \geq &\left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right]H^{-\alpha}(t)\|u_t\|_m^m \\ &+\epsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon a_1(g \circ \nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 \\ &+\epsilon\left[pH(t) - \frac{k^{1-m}}{m}\left(\frac{1}{p}\right)^{\alpha(m-1)} C\|u\|_p^{m+\alpha p(m-1)}\right].\end{aligned} \tag{3.18}$$

At this stage, we use Corollary 3.1 for $s = m+\alpha(m-1) \le p$, to deduce from (3.18)

$$
\begin{aligned}
L'(t) &\ge \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
&+\epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon a_1(g\circ\nabla u) + \epsilon a_2\|\nabla u\|_2^2 + \epsilon a_3\|\nabla u_t\|_2^2 \\
&+\epsilon\Bigg[pH(t) - C_1 k^{1-m}\Big\{ - H(t) - \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 \\
&-(g\circ\nabla u)(t) + \|u\|_p^p\Big\}\Bigg] \qquad (3.19)\\
&\ge \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
&+\epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2} + C_1k^{1-m}\right)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\epsilon\left(a_1 + C_1k^{1-m}\right)(g\circ\nabla u) + \epsilon\left(a_2 + C_1k^{1-m}\right)\|\nabla u\|_2^2 \\
&+\epsilon\left(a_3 - C_1k^{1-m}\right)\|\nabla u_t\|_2^2 + \epsilon\left(p + C_1k^{1-m}\right)H(t) - \epsilon C_1k^{1-m}\|u\|_p^p,
\end{aligned}
$$

where $C_1 = \left(\frac{1}{p}\right)^{\alpha(m-1)} C/m$. By noting that

$$
H(t) \ge -\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \frac{1}{2}\|\nabla u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}(g\circ\nabla u) + \frac{1}{p}\|u\|_p^p,
$$

and writing $p = 2a_4 + (p-2a_4)$, where $a_4 = \min\{a_1, a_2, a_3\}$, the estimate (3.19) yields

$$
\begin{aligned}
L'(t) &\ge \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
&+\epsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2} + C_1k^{1-m} - a_4\right)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon\left(\frac{2a_4}{p} - C_1k^{1-m}\right)\|u\|_p^p \qquad (3.20)\\
&+\epsilon\left(a_1 + C_1k^{1-m} - a_4\right)(g\circ\nabla u) + \epsilon\left(a_2 + C_1k^{1-m} - a_4\right)\|\nabla u\|_2^2 \\
&+\epsilon\left(a_3 - C_1k^{1-m} - a_4\right)\|\nabla u_t\|_2^2 + \epsilon\left(p + C_1k^{1-m} - 2a_4\right)H(t).
\end{aligned}
$$

We choose k large enough so that (3.20) becomes

$$
\begin{aligned}
L'(t) &\ge \left[(1-\alpha) - \frac{m-1}{m}\epsilon k\right] H^{-\alpha}(t)\|u_t\|_m^m \\
&+\epsilon\gamma\left[H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \|u\|_p^p\right],
\end{aligned} \qquad (3.21)
$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\|u_t\|_2^2$, $\|u\|_p^p$, and $(g\circ\nabla u)(t)$ in (3.21). Once k is fixed (hence γ), we pick ϵ small enough so that

$$
(1-\alpha) - \frac{\epsilon k(m-1)}{m} \ge 0,
$$

and

$$
L(0) = H^{1-\alpha}(0) + \frac{\epsilon}{\rho+1}\int_\Omega |u_t|^\rho u_1 u_0 dx + \epsilon\int_\Omega \nabla u_1.\nabla u_0 dx > 0.
$$

Therefore (3.21) takes the form

$$L'(t) \geq \epsilon\gamma \left[H(t) + \|u_t\|_{\rho 2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t) + \|u\|_p^p \right]. \tag{3.22}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad for \quad all \quad t \geq 0.$$

We now estimate

$$\left| \int_\Omega |u_t|^\rho u_t u dx \right| \leq \|u_t\|_{\rho+2}^{\rho+1} \|u\|_{\rho+2} \leq C \|u_t\|_{\rho+2}^{\rho+2} \|u\|_p,$$

we have

$$\left| \int_\Omega |u_t|^\rho u_t u dx \right|^{\frac{1}{1-\alpha}} \leq C \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\alpha}} \|u\|_p^{\frac{1}{1-\alpha}} \leq C \left(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\alpha}\mu} + \|u\|_p^{\frac{\theta}{1-\alpha}} \right).$$

Where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Choose $\mu = \frac{(1-\alpha)(\rho+2)}{\rho+1} (> 1)$, then

$$\frac{\theta}{1-\alpha} = \frac{\rho+2}{(1-\alpha)(\rho+2) - (\rho+1)} < p.$$

Using Corollary 3.1, we obtain for all $t \geq 0$

$$\left| \int_\Omega |u_t|^\rho u_t u dx \right|^{\frac{1}{1-\alpha}} \leq C \left[-H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 - (g \circ \nabla u)(t) + \|u\|_p^p \right].$$

Therefore,

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \frac{\epsilon}{\rho+1} \int_\Omega |u_t|^\rho u_t u dx + \epsilon \int_\Omega \nabla u_t . \nabla u dx \right)^{\frac{1}{1-\alpha}} \\ &\leq C \left[\|u_t\|_{\rho+2}^{\rho+2} + H(t) + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^{\frac{2}{1-2\alpha}} + \|u\|_p^p \right], \quad \forall t \geq 0. \end{aligned} \tag{3.23}$$

Noting that

$$\|\nabla u\|_2^{\frac{2}{1-2\alpha}} \leq C^{\frac{1}{1-2\alpha}} \leq \frac{C^{\frac{1}{1-2\alpha}}}{H(0)} H(t), \tag{3.24}$$

it follows from (3.23) and (3.24) that

$$L^{\frac{1}{1-\alpha}}(t) \leq C \left[\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|u\|_p^p \right], \quad \forall t \geq 0. \tag{3.25}$$

Combining (3.22) and (3.25), we arrive at

$$L'(t) \geq \frac{\epsilon\gamma}{C} L^{\frac{1}{1-\alpha}}(t), \quad \forall t \geq 0. \tag{3.26}$$

A simple integration of (3.26) over $(0,t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{\frac{-\alpha}{1-\alpha}}(0) - \epsilon\gamma t\alpha/|C(1-\alpha)|} \tag{3.27}$$

This shows that $L(t)$ blows up in finite time.

$$T^* \leq \frac{C(1-\alpha)}{\epsilon\gamma\alpha L^{\frac{\alpha}{1-\alpha}}(0)} \tag{3.28}$$

Summarizing, the proof is completed. $\square$

4. Asymptotic Behavior

In this section, we investigate the asymptotic behavior of the problem (1.1). We define

$$G(t) = ME(t) + \epsilon\psi(t) + \chi(t), \tag{4.1}$$

where ϵ and M are positive constants which shall be determined later, and

$$\psi(t) = \frac{1}{\rho+1}\xi(t)\int_\Omega |u_t|^\rho u_t u dx + \xi(t)\int_\Omega \nabla u_t.\nabla u dx, \tag{4.2}$$

$$\chi(t) = \xi(t)\int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g(t-s)\left[u(t)-u(s)\right] dsdx. \tag{4.3}$$

Theorem 4.1. *Let $(u_0,u_1) \in H_0^1(\Omega)\times H_0^1(\Omega)$ be given. Assume that $(H_1)-(H_3)$ and (3.4) hold. Then for each $t_0 > 0$, there exists two positive constants K and κ such that the solution of (1.1) satisfies*

$$E(t) \leq Ke^{-\kappa\int_0^t \xi(s)ds}, \quad t \geq t_0. \tag{4.4}$$

For our purposes, we need:

Theorem 4.2. *([22]) Suppose that $(H_1)-(H_3)$ and (3.4) hold. If $u_0,u_1 \in H_0^1(\Omega)$ and*

$$\frac{C_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{4.5}$$

where C_* is the best Poincare's constant. Then the solution of the problem (1.1) is global in time and satisfies

$$l\|\nabla u(t)\| + \|\nabla u_t(t)\| \leq \frac{2p}{p-2}E(0). \tag{4.6}$$

The proof of the theorem 4.2 is detailed in [22].

Lemma 4.1. *Let $u \in L^\infty(0,T;H_0^1(\Omega))$ be the solution of (1.1), then we have*

$$\int_\Omega \left(\int_0^t g(t-s)\,[u(t)-u(s)]\,ds\right)^{\rho+2} dx \le C_*^{\rho+2}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} \times (g\circ\nabla u)(t). \tag{4.7}$$

Proof. Here, we point out that

$$\int_0^t g(t-s)\,[u(t)-u(s)]\,ds = \int_0^t [g(t-s)]^{\frac{\rho+1}{\rho+2}}\,[g(t-s)]^{\frac{1}{\rho+2}}\,[u(t)-u(s)]\,ds,$$

then by using Hölder's inequality, we get

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^{\rho+2} dx \le \\
&\le \left(\int_0^\infty g(s)ds\right)^{\rho+1} \int_0^t g(t-s)\int_\Omega |u(t)-u(s)|^{\rho+2}dxds \\
&\le C_*^{\rho+2}(1-l)^{\rho+1}\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^{\rho+2}ds \\
&\le C_*^{\rho+2}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}(g\circ\nabla u)(t).
\end{aligned}$$

This ends the proof. □

Lemma 4.2. *For $\epsilon > 0$ small enough while $M > 0$ is large enough, the relation*

$$\alpha_1 G(t) \le E(t) \le \alpha_2 G(t),$$

holds for two positive constants α_1 and α_2.

Proof. By using Young's inequality, the Sobolev embedding theorem, (1.6),(4.6) and Lemma 4.1, we can derive that

$$\begin{aligned}
\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx\right| &\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|u\|_{\rho+2}^{\rho+2} \\
&\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\|\nabla u\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)[u(t)-u(s)]dsdx\right| \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^{\rho+2}dx \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}(g\circ\nabla u)(t).
\end{aligned}$$

It follows that

$$\begin{aligned}
G(t) &\leq ME(t) + \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)\xi(t)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]\xi(t)\|\nabla u\|_2^2 + \frac{\epsilon+1}{2}\xi(t)\|\nabla u_t\|_2^2 \\
&+\left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]\xi(t)(g\circ\nabla u)(t) \\
&\leq ME(t) + \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)N\|u_t\|_{\rho+2}^{\rho+2} \\
&+\epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]N\|\nabla u\|_2^2 + \frac{\epsilon+1}{2}N\|\nabla u_t\|_2^2 \\
&+\left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]N(g\circ\nabla u)(t) \leq \frac{1}{\alpha_1}E(t),
\end{aligned}$$

and

$$\begin{aligned}
G(t) &\geq ME(t) - \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)\xi(t)\|u_t\|_{\rho+2}^{\rho+2} \\
&-\epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]\xi(t)\|\nabla u\|_2^2 - \frac{\epsilon+1}{2}\xi(t)\|\nabla u_t\|_2^2 \\
&-\left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]\xi(t)(g\circ\nabla u)(t) \\
&\geq \left[\frac{M}{\rho+2} - \left(\frac{1}{\rho+1} + \frac{\epsilon}{\rho+2}\right)N\right]\|u_t\|_{\rho+2}^{\rho+2} + \left(\frac{M}{2} - \frac{\epsilon+1}{2}N\right)\|\nabla u_t\|_2^2 \\
&+\left\{\frac{M}{2}l - \epsilon\left[\frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{2pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}} + \frac{1}{2}\right]N\right\} - \frac{M}{p}\|u\|_p^p \\
&+\left\{\frac{M}{2} - \left[\frac{1-l}{2} + \frac{C_*^{\rho+2}}{(\rho+1)(\rho+2)}(1-l)^{\rho+1}\left(\frac{4pE(0)}{(p-2)l}\right)^{\frac{\rho}{2}}\right]N\right\}(g\circ\nabla u)(t) \geq \frac{1}{\alpha_2}E(t).
\end{aligned}$$

For $\epsilon > 0$ small enough while $M > 0$ is large enough. This completes the proof. □

Lemma 4.3. *Under the assumptions $(H_1) - (H_3)$ and (4.6), the functional*

$$\psi(t) = \frac{1}{\rho+1}\xi(t)\int_\Omega |u_t|^\rho u_t u dx + \xi(t)\int_\Omega \nabla u_t.\nabla u dx,$$

satisfies the solutions of (1.1),

$$\begin{aligned}
\psi'(t) &\leq -\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right]\xi(t)\|\nabla u\|_2^2 \\
&+\left\{1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^{\rho}\right]\right\}\xi(t)\|\nabla u_t\|_2^2 \\
&+\frac{1-l}{2l}\xi(t)(g\circ\nabla u)(t) + \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \xi(t)\|u\|_p^p.
\end{aligned} \tag{4.8}$$

Proof. By using the equation of (1.1), we easily see that

$$
\begin{aligned}
\psi'(t) &= \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \xi(t)\int_\Omega |u_t|^\rho u_{tt} u dx + \xi(t)\|\nabla u_t\|_2^2 \\
&+\xi(t)\int_\Omega \nabla u.\nabla u_{tt} dx + \frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t u dx + \xi'(t)\int_\Omega \nabla u_t.\nabla u dx \\
&= \frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \xi(t)\|\nabla u_t\|_2^2 - \xi(t)\|\nabla u\|_2^2 \\
&+\xi(t)\int_\Omega \nabla u.\int_0^t g(t-s)\nabla u(s) ds dx - \xi(t)\int_\Omega |u_t|^{m-2}u_t u dx + \xi(t)\|u\|_p^p \\
&+\frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t u dx + \xi'(t)\int_\Omega \nabla u_t.\nabla u dx.
\end{aligned} \tag{4.9}
$$

Now we estimate

$$
\begin{aligned}
\int_\Omega \nabla u(t).\int_0^t g(t-s)\nabla u(s) ds dx &\le \frac{1}{2}\|\nabla u\|_2^2 \\
+\frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|) ds\right)^2 dx.
\end{aligned} \tag{4.10}
$$

We use Young's inequality and the fact that

$$\int_0^t g(s)ds \le \int_0^\infty g(s)ds = 1-l,$$

it follows from (4.10) for $\eta = \frac{l}{1-l} > 0$ that

$$
\begin{aligned}
&\int_\Omega \nabla u(t).\int_0^t g(t-s)\nabla u(s) ds dx \le \frac{1}{2}(1+\eta)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)| ds\right)^2 dx \\
&+\frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)| ds\right)^2 dx + \frac{1}{2}\|\nabla u\|_2^2 \\
&\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(1+\eta)(1-l)^2\|\nabla u(t)\|_2^2 + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t) \\
&\le \frac{2-l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2l}(1-l)(g\circ\nabla u)(t),
\end{aligned}
$$

and

$$\int_\Omega |u_t|^\rho u_t u dx \le \frac{1}{4\delta}\|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \delta C_*^2\|\nabla u\|_2^2, \tag{4.11}$$

for any $\delta > 0$. In view of (4.6) and the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega), \quad for \quad 0 < \rho \le \frac{2}{n-2} \quad if \quad n \ge 3 \quad and \quad \rho > 0 \quad if \quad n = 1,2,$$

we get

$$\|u_t\|_{2(\rho+1)}^{2(\rho+1)} \le C_*^{2(\rho+1)}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2. \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\begin{aligned}\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx &\leq \frac{1}{4\delta}\frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2 \\ &+\frac{1}{\rho+1}\delta C_*^2\|\nabla u\|_2^2,\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}&\left|\int_\Omega |u_t|^{m-2}u_t u dx\right| \leq \delta\int_\Omega |u|^2dx+\frac{1}{4\delta}\int_\Omega |u_t|^{2m-2}dx \\ &\leq \delta C_*^2\|\nabla u\|_2^2+\tfrac{1}{4\delta}C_*^{2m-2}\|\nabla u_t\|_{2m-2}^{2m-2} \\ &\leq \delta C_*^2\|\nabla u\|_2^2+\tfrac{1}{4\delta}C_*^{2m-2}\left((\tfrac{2pE(0)}{p-2}\right)^{\frac{m-2}{2}}\|\nabla u_t\|_2^2 \\ &\leq \delta C_*^2\|\nabla u\|_2^2+\tfrac{A}{4\delta}\|\nabla u_t\|_2^2.\end{aligned} \tag{4.14}$$

Also

$$\int_\Omega \nabla u_t.\nabla u dx \leq \frac{1}{4\delta}\|\nabla u_t\|_2^2+\delta\|\nabla u\|_2^2. \tag{4.15}$$

By combining (4.9),(4.10),(4.13),(4.14) and (4.15), we deduce easily the estimate (4.8). This completes our proof. □

Lemma 4.4. *Under the assumptions* $(H_1)-(H_3)$, *the functional*

$$\chi(t)=\xi(t)\int_\Omega\left(\Delta u_t-\frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g(t-s)[u(t)-u(s)]dsdx,$$

satisfies, along solutions of (1.1) and for $\delta>0$

$$\begin{aligned}&\chi'(t)\leq \delta_1\left[1+2(1-l)^2+C_*^{2p-2}\left(\tfrac{2pE(0)}{(p-2)l}\right)^{p-2}\right]\xi(t)\|\nabla u\|_2^2 \\ &\left[\frac{(\rho+2)k}{4\delta_1(\rho+1)}+2\delta_1+\frac{1}{2\delta_1}+\frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g\circ\nabla u)(t) \\ &-\frac{g(0)}{4\delta_1}\left(1+\frac{C_*^2}{\rho+1}\right)\xi(t)(g'\circ\nabla u)(t)-\frac{1}{\rho+1}\xi(t)\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2} \\ &+\left\{(k+1)\delta_1\left[1+\frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right]+\delta_1\tilde{A}-\int_0^t g(s)ds\right\}\xi(t)\|\nabla u_t\|_2^2.\end{aligned} \tag{4.16}$$

Proof. Applying (1.1), the computation yields

$$\begin{aligned}\chi'(t)&=\xi(t)\int_\Omega(\Delta u_{tt}-|u_t|^\rho u_{tt})\int_0^t g(t-s)[u(t)-u(s)]dsdx \\ &+\xi(t)\int_\Omega\left(\Delta u_t-\frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g'(t-s)[u(t)-u(s)]dsdx \\ &+\xi(t)\int_0^t g(s)ds\int_\Omega\left(\Delta u_t-\frac{|u_t|^\rho u_t}{\rho+1}\right)u_t dx \\ &+\xi'(t)\int_\Omega \Delta u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx \\ &-\frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx.\end{aligned}$$

By integrating the parts, it follows that

$$
\begin{aligned}
\chi'(t) &= \xi(t)\int_\Omega \nabla u(t).\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx \\
&-\xi(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)ds.\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx \\
&-\xi(t)\int_0^t g(s)ds\|\nabla u_t\|_2^2-\xi(t)\int_\Omega \nabla u_t.\int_0^t g'(t-s)[\nabla u(t)-\nabla u(s)]dsdx \\
&-\frac{1}{\rho+1}\xi(t)\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx \\
&-\frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2}\int_0^t g(s)ds \\
&+\xi(t)\int_\Omega |u_t|^{m-2}u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx \\
&-\xi(t)\int_\Omega |u|^{p-2}u\int_0^t g(t-s)[u(t)-u(s)]dsdx \\
&-\xi'(t)\int_\Omega \nabla u_t.\int_0^t g(t-s)[\nabla u(t)-u(s)]dsdx \\
&-\frac{1}{\rho+1}\xi'(t)\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx.
\end{aligned}
\tag{4.17}
$$

In fact, by exploiting Young's inequality, we get that for any $\delta_1>0$

$$
\begin{aligned}
\int_\Omega \nabla u(t).\left(\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right)dx &\le \delta_1\|\nabla u\|_2^2 \\
&+\frac{1}{4\delta_1}(1-l)(g\circ\nabla u)(t),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\int_\Omega\int_0^t g(t-s)\nabla u(s)ds.\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx \\
&\le \delta_1\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|ds\right)^2dx \\
&+\frac{1}{4\delta_1}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2dx \\
&\le \left(2\delta_1+\frac{1}{4\delta_1}\right)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2dx \\
&\le \left(2\delta_1+\frac{1}{4\delta_1}\right)(1-l)(g\circ\nabla u)(t)+2\delta_1(1-l)^2\|\nabla u\|_2^2+2\delta_1(1-l)^2\|\nabla u\|_2^2,
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
\int_\Omega \nabla u_t.\int_0^t g'(t-s)[\nabla u(t)-\nabla u(s)]dsdx &\le \delta_1\|\nabla u_t\|_2^2 \\
&+\frac{g(0)}{4\delta_1}(-g'\circ\nabla u)(t).
\end{aligned}
\tag{4.20}
$$

Also

$$\begin{aligned}(4.21)\qquad &\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)[u(t)-u(s)]dsdx\\ &\le \delta_1 \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho \|\nabla u_t\|_2^2 + \frac{g(0)}{4\delta_1(\rho+1)} C_*^2(-g'\circ\nabla u)(t).\end{aligned}$$

Similarly, we get

$$\begin{aligned}(4.22)\qquad &\left|\int_\Omega |u_t|^{m-2}u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\ &\le \delta_1\|u_t\|_{2m-2}^{2m-2} + \frac{1}{4\delta_1}\int_\Omega \left(\int_0^t g(t-s)|u(t)-u(s)|ds\right)^2 dx\\ &\le \delta_1 C_*^{2m-2}\|\nabla u_t\|_{2m-2}^{2m-2} + \frac{1}{4\delta_1}\int_0^t g(s)ds\int_0^t g(t-s)\int_\Omega |u(t)-u(s)|^2 dsdx\\ &\le \delta_1\tilde{A}\|\nabla u_t\|_2^2 + \frac{1}{4\delta_1}(1-l)C_*^2(g\circ\nabla u)(t),\end{aligned}$$

$$\begin{aligned}(4.23)\qquad &-\int_\Omega |u|^{p-2}u\int_0^t g(t-s)[u(t)-u(s)]dsdx\\ &\le \frac{1}{4\delta_1}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^2 dx\\ &+\delta_1\|u\|_{2p-2}^{2p-2} \le \delta_1\|u\|_{2p-2}^{2p-2} + \frac{C_*^2(1-l)}{4\delta_1}(g\circ\nabla u)(t)\\ &\le \delta_1 C_*^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}\|\nabla u\|_2^2 + \frac{C_*^2(1-l)}{4\delta_1}(g\circ\nabla u)(t),\end{aligned}$$

and

$$\begin{aligned}(4.24)\qquad \int_\Omega \nabla u_t.\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx \quad &\le \delta_1\|\nabla u_t\|_2^2\\ &+\frac{1}{4\delta_1}(1-l)(g\circ\nabla u)(t).\end{aligned}$$

We estimate

$$\begin{aligned}(4.25)\qquad &\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx\\ &\le \frac{1}{\rho+1}\delta_1\|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{4\delta_1(\rho+1)}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^2 dx\\ &\le \delta_1\frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\|\nabla u_t\|_2^2 + \frac{1-l}{4\delta_1(\rho+1)}(g\circ\nabla u)(t).\end{aligned}$$

Combining the estimates (4.18)-(4.25) and (4.17), the assertion of the lemma 4.4 is established. □

Proof. (Theorem 4.1). Since g is positive, we have that, for any $t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \quad t \geq t_0.$$

By using (4.1),(4.8),(4.16) and Lemma 4.1, a series of computations yields, for $t \geq t_0$,

$$\begin{aligned}
G'(t) \leq{}& \frac{M}{2}(g' \circ \nabla u)(t) - \epsilon\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right]\xi(t)\|\nabla u\|_2^2 \\
&+\epsilon\frac{1-l}{2l}\xi(t)(g \circ \nabla u)(t) + \epsilon\frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} + \epsilon\xi(t)\|u\|_p^p \\
&+\epsilon\left[1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^{\rho}\right]\right]\xi(t)\|\nabla u_t\|_2^2 \\
&+\delta_1\left[1 + 2(1-l)^2 + C_*^{2p-2}\left(\tfrac{2pE(0)}{(p-2)l}\right)^{p-2}\right]\xi(t)\|\nabla u\|_2^2 \\
&+\left[\frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g \circ \nabla u)(t) \\
&-\frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)\xi(t)(g' \circ \nabla u)(t) - \frac{1}{\rho+1}\xi(t)\left(\int_0^t g(s)ds\right)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\left\{(k+1)\delta_1\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^{\rho}\right] + \delta_1\tilde{A} - \int_0^t g(s)ds\right\}\xi(t)\|\nabla u_t\|_2^2 \\
\leq{}& -\left\{\left[g_0 - \epsilon\left(1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left(1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^{\rho}\right)\right)\right]\right. \\
&\left.-(k+1)\delta_1\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^{\rho} + \delta_1\tilde{A}\right]\right\}\xi(t)\|u_t\|_2^2 \\
&-\left\{\epsilon\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right]\right. \\
&\left.-\delta_1\left[1 + 2(1-l)^2 + C_*^{2p-2}\left(\tfrac{2pE(0)}{(p-2)l}\right)^{p-2}\right]\right\}\xi(t)\|\nabla u\|_2^2 + \epsilon\xi(t)\|u\|_p^p \\
&-\left[\frac{M}{2} - \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)N\right](-g' \circ \nabla u)(t) - (g_0 - \epsilon)\frac{1}{\rho+1}\xi(t)\|u_t\|_{\rho+2}^{\rho+2} \\
&+\left[\frac{\epsilon}{2l} + \frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g \circ \nabla u)(t).
\end{aligned} \tag{4.26}$$

At this point, we choose $\delta > 0$ so small that

$$\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right) > \frac{l}{4}.$$

Hence δ is fixed, we choose $\epsilon > 0$ small enough so that Lemma 4.2 holds and that

$$\epsilon < \frac{g_0}{2\left[1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left(1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^{\rho}\right)\right]}$$

Once δ and ϵ are fixed, we choose a positive constant δ_1 satisfying

$$\delta_1 < \min\{\xi_1, \xi_2\},$$

where

$$\xi_1 = \frac{g_0}{2(k+1)\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1 \tilde{A}\right]},$$

and

$$\xi_2 = \frac{g_0}{2(k+1)\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1 \tilde{A}\right]}.$$

and be such that

$$\begin{aligned} &g_0 - \epsilon\left[1 + \frac{A}{4\delta} + \frac{k}{4\delta}\left(1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho\right)\right] \\ &-(k+1)\delta_1\left[1 + \frac{C_*^{2(\rho+1)}}{\rho+1}\left(\frac{2pE(0)}{p-2}\right)^\rho + \delta_1\tilde{A}\right] > 0, \end{aligned}$$

also

$$\epsilon\left[\frac{l}{2} - \delta C_*^2 - k\delta\left(1 + \frac{C_*^2}{\rho+1}\right)\right] - \delta_1\left[1 + 2(1-l)^2 + C_*^{2p-2}\left(\frac{2pE(0)}{(p-2)l}\right)^{p-2}\right] > 0.$$

We then pick M sufficiently large so that Lemma 4.2 holds and that

$$\left[\frac{M}{2} - \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)N\right] - \left[\frac{\epsilon}{2l} + \frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l) > 0.$$

Hence, using (H_3), we get

$$\begin{aligned} &\left[\frac{M}{2} - \frac{g(0)}{4\delta_1}\left(1 + \frac{C_*^2}{\rho+1}\right)N\right](-g' \circ \nabla u)(t) \\ &-\left[\frac{\epsilon}{2l} + \frac{(\rho+2)k}{4\delta_1(\rho+1)} + 2\delta_1 + \frac{1}{2\delta_1} + \frac{C_*^2}{2\delta_1}\right](1-l)\xi(t)(g \circ \nabla u)(t) \geq k_3\xi(t)(g \circ \nabla u)(t). \end{aligned}$$

By using Lemma 4.2 and (4.26), we arrive $\forall t \geq t_0$ at

$$G'(t) \leq -\beta_1\xi(t)E(t) \leq \alpha_1\beta_1\xi(t)G(t), \tag{4.27}$$

for some positive constant β_1. A simple integration of (4.27) leads to

$$G(t) \leq G(t_0)e^{-\alpha_1\beta_1\int_{t_0}^t \xi(s)ds}, \quad \forall t \geq t_0. \tag{4.28}$$

Thus, from Lemma 4.2 and (4.28), we get

$$E(t) \leq \alpha_2 G(t_0)e^{-\alpha_1\beta_1\int_{t_0}^t \xi(s)ds} = Ke^{-\kappa\int_{t_0}^t \xi(s)ds}, \quad \forall t \geq t_0. \tag{4.29}$$

This completes our proof. $\square$

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ON S-CURVATURE OF A HOMOGENEOUS FINSLER SPACE WITH RANDERS CHANGED SQUARE METRIC *

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Abstract. The study of curvature properties of homogeneous Finsler spaces with (α, β)-metrics is one of the central problems in Riemann-Finsler geometry. In the present paper, the existence of invariant vector fields on a homogeneous Finsler space with Randers changed square metric has been proved. Further, an explicit formula for S-curvature of Randers changed square metric has been established. Finally, using the formula of S-curvature, the mean Berwald curvature of afore said (α, β)-metric has been calculated.

Keywords: Homogeneous Finsler space, square metric, Randers change, invariant vector field, S-curvature, mean Berwald curvature.

1. Introduction

According to S. S. Chern [6], Finsler geometry is just Riemannian geometry without quadratic restriction. Finsler geometry is an interesting and active area of research for both pure and applied reasons [2, 1, 13, 16]. In 1972, M. Matsumoto [17] introduced the concept of $(\alpha, \beta)-$metrics which are the generalizations of Randers metric introduced by G. Randers [20]. Z. Shen [25] introduced the notion of S-curvature, a non-Riemannian quantity, for a comparison theorem in Finsler geometry. It is non-Riemannian in the sense that any Riemannian manifold has vanishing S-curvature. One special class of Finsler spaces is homogeneous and symmetric Finsler spaces. It is an active area of research these days. Many authors [8, 12, 15, 21, 23, 30] have worked in this area. The main aim of this paper is to establish an explicit formula for S-curvature of a homogeneous Finsler space with Randers change of square metric. The importance of S-curvature in Riemann-Finsler geometry can be seen in several papers (e.g., [26, 27]).

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The simplest non-Riemannian metrics are the Randers metrics given by $F = \alpha + \beta$ with $\|\beta\|_\alpha < 1$, where α is a Riemannian metric and β is a 1-form. Besides Randers metrics, other interesting kind of non-Riemannian metrics are square metrics. Berwald's metric, constructed by Berwald [4] in 1929 as

$$F = \frac{\left(\sqrt{(1-|x|^2)\,|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\right)^2}{\left(1-|x|^2\right)^2 \sqrt{(1-|x|^2)\,|y|^2 + \langle x, y\rangle^2}}$$

is a classical example of square metric. Berwald's metric can be rewritten as follows:

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{1.1}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)\,|y|^2 + \langle x, y\rangle^2}}{\left(1-|x|^2\right)^2},$$

and

$$\beta = \frac{\langle x, y\rangle}{\left(1-|x|^2\right)^2}.$$

An (α, β)-metric expressed in the form (1.1) is called square metric [28]. Just as Randers metrics, square metrics play an important role in Finsler geometry. The importance of square metric can be seen in papers [28, 29, 31]). Square metrics can also be expressed in the form [31]

$$F = \frac{\left(\sqrt{(1-b^2)\,\alpha^2 + \beta^2} + \beta\right)^2}{\left(1-b^2\right)^2 \sqrt{(1-b^2)\,\alpha^2 + \beta^2}},$$

where $b := \|\beta_x\|_\alpha$ is the length of β.
In this case, $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, where $\phi = \phi(b^2, s)$ is a smooth function, is called general (α, β)-metric. If $\phi = \phi(s)$ is independent of b^2, then F is called an (α, β)-metric. An interesting fact is that if $\alpha = |y|$, and $\beta = \langle x, y\rangle$, then $F = |y|\phi\left(|x|^2, \frac{\langle x,y\rangle}{|y|}\right)$ becomes spherically symmetric metric.

If $F(\alpha, \beta)$ is a Finsler metric, then $F(\alpha, \beta) \longrightarrow \bar{F}(\alpha, \beta)$ is called a Randers change if

$$\bar{F}(\alpha, \beta) = F(\alpha, \beta) + \beta. \tag{1.2}$$

Above change of a Finsler metric has been introduced by M. Matsumoto [18], and it was named as "Randers change" by M. Hashiguchi and Y. Ichijyō [14]. In the current paper, we deal with Randers changed square metrics

$$F = \frac{(\alpha+\beta)^2}{\alpha} + \beta = \alpha\phi(s), \text{where} \;\; \phi(s) = 1 + s^2 + 3s.$$

The paper is organized as follows:
In section 2, we discuss some basic definitions and results to be used in consequent sections. The existence of invariant vector fields on homogeneous Finsler spaces with Randers changed square metric has been proved in section 3 (see Theorem 3.1). Further, in section 4, we have established an explicit formula for S-curvature of afore said metric (see Theorem 4.2). Finally, in section 5, the mean Berwald curvature of this metrics has been calculated (see Theorem 5.1).

2. Preliminaries

First, we discuss some basic definitions and results required to study aforesaid spaces. We refer [3, 7, 9] for notations and further details.

Definition 2.1. An n-dimensional real vector space V is said to be a **Minkowski space** if there exists a real valued function $F : V \longrightarrow [0, \infty)$, called Minkowski norm, satisfying the following conditions:

- F is smooth on $V \backslash \{0\}$,
- F is positively homogeneous, i.e., $F(\lambda v) = \lambda F(v), \quad \forall\ \lambda > 0$,
- For any basis $\{u_1,\ u_2,\ ...,\ u_n\}$ of V and $y = y^i u_i \in V$, the Hessian matrix $\left(g_{ij}\right) = \left(\frac{1}{2} F^2_{y^i y^j}\right)$ is positive-definite at every point of $V \backslash \{0\}$.

Definition 2.2. Let M be a connected smooth manifold. If there exists a function $F : TM \longrightarrow [0, \infty)$ such that F is smooth on the slit tangent bundle $TM \backslash \{0\}$ and the restriction of F to any $T_xM,\ x \in M$, is a Minkowski norm, then M is called a Finsler space and F is called a Finsler metric.

An (α, β)-metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and a one-form $\beta = b_i(x) y^i$ on M and is of the form $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$, where ϕ is a smooth function on M. Basically, (α, β)-metrics are the generalization of Randers metrics. Many authors [12, 15, 21, 22, 24, 30] have worked on (α, β)-metrics. Let us recall Shen's lemma [7] which provides necessary and sufficient condition for an (α, β)-metric to be a Finsler metric.

Lemma 2.1. *Let $F = \alpha\phi(s),\ s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F is a Finsler metric if and only if the following conditions are satisfied:*

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0, \quad \forall \quad |s| \leq b < b_0.$$

Before defining homogeneous Finsler spaces, we shall discuss some basic concepts below.

Definition 2.3. Let G be a smooth manifold having the structure of an abstract group. G is called a Lie group, if the maps $i : G \longrightarrow G$ and $\mu : G \times G \longrightarrow G$ defined as $i(g) = g^{-1}$, and $\mu(g, h) = gh$ respectively, are smooth.

Let G be a Lie group and M, a smooth manifold. Then a smooth map $f : G \times M \longrightarrow M$ satisfying

$$f(g_2, f(g_1, x)) = f(g_2 g_1, x), \quad \text{for all} \quad g_1, g_2 \in G, \quad \text{and} \quad x \in M$$

is called a smooth action of G on M.

Definition 2.4. Let M be a smooth manifold and G, a Lie group. If G acts smoothly on M, then G is called a **Lie transformation group** of M.

The following theorem gives us a differentiable structure on the coset space of a Lie group.

Theorem 2.1. *Let G be a Lie group and H, its closed subgroup. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns G/H into a smooth manifold such that G is a Lie transformation group of G/H.*

Definition 2.5. Let (M, F) be a connected Finsler space and $I(M, F)$ the group of isometries of (M, F). If the action of $I(M, F)$ is transitive on M, then (M, F) is said to be a **homogeneous Finsler space**.

Let G be a Lie group acting transitively on a smooth manifold M. Then for $a \in M$, the isotropy subgroup G_a of G is a closed subgroup and by theorem 2.1, G is a Lie transformation group of G/G_a. Further, G/G_a is diffeomorphic to M.

Theorem 2.2. *[9] Let (M, F) be a Finsler space. Then $G = I(M, F)$, the group of isometries of M is a Lie transformation group of M. Let $a \in M$ and $I_a(M, F)$ be the isotropy subgroup of $I(M, F)$ at a. Then $I_a(M, F)$ is compact.*

Let (M, F) be a homogeneous Finsler space, i.e., $G = I(M, F)$ acts transitively on M. For $a \in M$, let $H = I_a(M, F)$ be a closed isotropy subgroup of G which is compact. Then H is a Lie group itself being a closed subgroup of G. Write M as the quotient space G/H.

Definition 2.6. [19] Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, where $\mathfrak{k}$ is a subspace of $\mathfrak{g}$ such that $\mathrm{Ad}(h)(\mathfrak{k}) \subset \mathfrak{k}\ \forall\, h \in H$, is called a reductive decomposition of $\mathfrak{g}$, and if such decomposition exists, then $(G/H, F)$ is called reductive homogeneous space.

Therefore, we can write, any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric F is viewed as G invariant Finsler metric on M.

Definition 2.7. A one-parameter subgroup of a Lie group G is a homomorphism $\psi : \mathbb{R} \longrightarrow G$, such that $\psi(0) = e$, where e is the identity of G.

Recall [9] the following result which gives us the existence of one-parameter subgroup of a Lie group.

Theorem 2.3. *Let G be a Lie group having Lie algebra $\mathfrak{g}$. Then for any $Y \in \mathfrak{g}$, there exists a unique one-parameter subgroup ψ such that $\dot{\psi}(0) = Y_e$, where e is the identity element of G.*

Definition 2.8. Let G be a Lie group with identity element e and $\mathfrak{g}$ its Lie algebra. The exponential map $\exp : \mathfrak{g} \longrightarrow G$ is defined by

$$\exp(tY) = \psi(t), \quad \forall\, t \in \mathbb{R},$$

where $\psi : \mathbb{R} \longrightarrow G$ is unique one-parameter subgroup of G with $\dot{\psi}(0) = Y_e$.

In case of reductive homogeneous manifold, we can identify the tangent space $T_H(G/H)$ of G/H at the origin $eH = H$ with $\mathfrak{k}$ through the map

$$Y \longmapsto \frac{d}{dt} exp(tX)H|_{t=0}, \quad Y \in \mathfrak{k},$$

since M is identified with G/H and Lie algebra of any Lie group G is viewed as T_eG.

3. Invariant Vector Field

For a homogeneous Finsler space with Randers changed square metric $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$, in Theorem 3.1, we prove the existence of invariant vector field corresponding to 1-form β. For this, first we prove following lemmas:

Lemma 3.1. *Let (M, α) be a Riemannian space and $\beta = b_i y^i$, a 1-form with $\|\beta\| = \sqrt{b_i b^i} < 1$. Then the Randers changed square Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$, consists of a Riemannian metric α along with a smooth vector field X on M with $\alpha\left(X|_x\right) < 1$, $\forall\, x \in M$, i.e.,*

$$F\left(x, y\right) = \frac{\left(\alpha\left(x, y\right) + \langle X|_x, y\rangle\right)^2}{\alpha\left(x, y\right)} + \langle X|_x, y\rangle\,, \quad x \in M, \quad y \in T_xM,$$

where $\langle\ ,\ \rangle$ is the inner product induced by the Riemannian metric α.

Proof. We know that the restriction of a Riemannian metric to a tangent space is an inner product. Therefore, the bilinear form $\langle u, v\rangle = a_{ij}u^i v^j, \quad u, v \in T_xM$ is an inner product on T_xM for $x \in M$, and this inner product induces an inner product on T_x^*M, the cotangent space of M at x which gives us $\left\langle dx^i, dx^j\right\rangle = a^{ij}$. A linear isomorphism exists between T_x^*M and T_xM, which can be defined by using this inner product. It follows that the 1-form β corresponds to a smooth vector field X on M, which can be written as

$$X|_x = b^i \frac{\partial}{\partial x^i}, \quad \text{where} \quad b^i = a^{ij} b_j.$$

Then, for $y \in T_xM$, we have

$$\langle X|_x, y\rangle = \left\langle b^i \frac{\partial}{\partial x^i} \ , \ y^j \frac{\partial}{\partial x^j} \right\rangle = b^i y^j a_{ij} = b_j y^j = \beta(y).$$

Also, we have

$$\alpha^2(x, y) = a_{ij} y^i y^j,$$

which implies

$$\alpha^2\left(X|_x\right) = a_{ij} b^i b^j = \|\beta\|^2 < 1,$$

i. e.,

$$\alpha\left(X|_x\right) < 1.$$

This completes the proof. $\square$

Lemma 3.2. *Let (M, F) be a Finsler space with Randers changed square Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$. Let $I(M, F)$ be the group of isometries of (M, F) and $I(M, \alpha)$ be that of Riemannian space (M, α). Then $I(M, F)$ is a closed subgroup of $I(M, \alpha)$.*

Proof. Let $x \in M$ and $\phi : (M, F) \longrightarrow (M, F)$ be an isometry. Therefore, we have

$$F(x, y) = F(\phi(x), d\phi_x(y)), \quad \forall y \in T_xM.$$

By Lemma 3.1, we get

$$\begin{aligned}&\frac{(\alpha\left(x, y\right) + \langle X|_x, y\rangle)^2}{\alpha\left(x, y\right)} + \langle X|_x, y\rangle = \\ &= \frac{\left(\alpha\left(\phi(x), d\phi_x(y)\right) + \left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle\right)^2}{\alpha\left(\phi(x), d\phi_x(y)\right)} + \left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle,\end{aligned}$$

which gives us

$$\begin{aligned}&\alpha\left(\phi(x), d\phi_x(y)\right)\alpha^2\left(x, y\right) + \alpha\left(\phi(x), d\phi_x(y)\right)\langle X|_x, y\rangle^2 \\ &+3\alpha\left(\phi(x), d\phi_x(y)\right)\alpha\left(x, y\right)\langle X|_x, y\rangle \\ &= \alpha\left(x, y\right)\alpha^2\left(\phi(x), d\phi_x(y)\right) + \alpha\left(x, y\right)\left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle^2 \\ &+3\alpha\left(x, y\right)\alpha\left(\phi(x), d\phi_x(y)\right)\left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle \qquad (3.1)\end{aligned}$$

Replacing y by $-y$ in equation (3.1), we get

$$\begin{aligned}
&\alpha\left(\phi(x), d\phi_x(y)\right)\alpha^2(x,y)+\alpha\left(\phi(x), d\phi_x(y)\right)\langle X|_x, y\rangle^2\\
&-3\alpha\left(\phi(x), d\phi_x(y)\right)\alpha(x,y)\langle X|_x, y\rangle\\
&=\alpha(x,y)\,\alpha^2\left(\phi(x), d\phi_x(y)\right)+\alpha(x,y)\left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle^2\\
&-3\alpha(x,y)\,\alpha\left(\phi(x), d\phi_x(y)\right)\left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle
\end{aligned} \tag{3.2}$$

Subtracting equation (3.2) from equation (3.1), we get

$$\alpha\left(\phi(x), d\phi_x(y)\right)\alpha(x,y)\langle X|_x, y\rangle=\alpha(x,y)\,\alpha\left(\phi(x), d\phi_x(y)\right)\left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle,$$

which implies

$$\langle X|_x, y\rangle=\left\langle X|_{\phi(x)}, d\phi_x(y)\right\rangle. \tag{3.3}$$

Adding equations (3.1) and (3.2) and using equation (3.3), we get

$$\begin{aligned}
&\alpha\left(\phi(x), d\phi_x(y)\right)\alpha^2(x,y)+\alpha\left(\phi(x), d\phi_x(y)\right)\langle X|_x, y\rangle^2\\
&=\alpha(x,y)\,\alpha^2\left(\phi(x), d\phi_x(y)\right)+\alpha(x,y)\langle X|_x, y\rangle^2,
\end{aligned}$$

which leads to

$$\alpha(x,y)=\alpha\left(\phi(x), d\phi_x(y)\right). \tag{3.4}$$

Therefore ϕ is an isometry with respect to the Riemannian metric α and $d\phi_x\left(X|_x\right)=X|_{\phi(x)}$. Thus $I(M,F)$ is a closed subgroup of $I(M,\alpha)$. $\square$

From Lemma (3.2), we conclude that if (M,F) is a homogeneous Finsler space with Randers change of square metric $F=\frac{(\alpha+\beta)^2}{\alpha}+\beta$, then the Riemannian space (M,α) is homogeneous. Further, M can be written as a coset space G/H, where $G=I(M,F)$ is a Lie transformation group of M and H, the compact isotropy subgroup $I_a(M,F)$ of $I(M,F)$ at some point $a\in M$ [10]. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. If $\mathfrak{g}$ can be written as a direct sum of subspaces $\mathfrak{h}$ and $\mathfrak{k}$ of $\mathfrak{g}$ such that $\mathrm{Ad}(h)\mathfrak{k}\subset\mathfrak{k}\ \ \forall\, h\in H$, then from definition 2.6, $(G/H,F)$ is a reductive homogeneous space.

Therefore, we can write, any homogeneous Finsler space as a coset space of a connected Lie group with an invariant Finsler metric. Here, the Finsler metric F is viewed as G invariant Finsler metric on M.

Theorem 3.1. *Let $F=\frac{(\alpha+\beta)^2}{\alpha}+\beta$ be a G-invariant Randers changed square metric on G/H. Then α is a G-invariant Riemannian metric and the vector field X corresponding to the 1-form β is also G-invariant.*

Proof. Since F is a G-invariant metric on G/H, we have

$$F(y) = F(\mathrm{Ad}(h)\,y), \quad \forall\, h \in H, \quad y \in \mathfrak{k}.$$

By Lemma 3.1, we get

$$\frac{(\alpha(y) + \langle X, y\rangle)^2}{\alpha(y)} + \langle X, y\rangle = \frac{(\alpha(\mathrm{Ad}(h)\,y) + \langle X, \mathrm{Ad}(h)\,y\rangle)^2}{\alpha(\mathrm{Ad}(h)\,y)} + \langle X, \mathrm{Ad}(h)\,y\rangle.$$

After simplification, we get

$$\begin{aligned} &\alpha(\mathrm{Ad}(h)\,y)\,\alpha^2(y) + \alpha(\mathrm{Ad}(h)\,y)\,\langle X, y\rangle^2 + 3\alpha(\mathrm{Ad}(h)\,y)\,\alpha(y)\,\langle X, y\rangle \\ (3.5) = {}&\alpha(y)\,\alpha^2(\mathrm{Ad}(h)\,y) + \alpha(y)\,\langle X, \mathrm{Ad}(h)\,y\rangle^2 + 3\alpha(y)\,\alpha(\mathrm{Ad}(h)\,y)\,\langle X, \mathrm{Ad}(h)\,y\rangle. \end{aligned}$$

Replacing y by $-y$ in equation (3.5), we get

$$\begin{aligned} &\alpha(\mathrm{Ad}(h)\,y)\,\alpha^2(y) + \alpha(\mathrm{Ad}(h)\,y)\,\langle X, y\rangle^2 - 3\alpha(\mathrm{Ad}(h)\,y)\,\alpha(y)\,\langle X, y\rangle \\ (3.6) = {}&\alpha(y)\,\alpha^2(\mathrm{Ad}(h)\,y) + \alpha(y)\,\langle X, \mathrm{Ad}(h)\,y\rangle^2 - 3\alpha(y)\,\alpha(\mathrm{Ad}(h)\,y)\,\langle X, \mathrm{Ad}(h)\,y\rangle. \end{aligned}$$

Subtracting equation (3.6) from equation (3.5), we get

$$\alpha(\mathrm{Ad}(h)\,y)\,\alpha(y)\,\langle X, y\rangle = \alpha(y)\,\alpha(\mathrm{Ad}(h)\,y)\,\langle X, \mathrm{Ad}(h)\,y\rangle,$$

which gives us

$$\langle X, y\rangle = \langle X, \mathrm{Ad}(h)\,y\rangle. \tag{3.7}$$

Adding equations (3.5) and (3.6) and using equation (3.7), we get

$$\alpha(\mathrm{Ad}(h)\,y)\,\alpha^2(y) + \alpha(\mathrm{Ad}(h)\,y)\,\langle X, y\rangle^2 = \alpha(y)\,\alpha^2(\mathrm{Ad}(h)\,y) + \alpha(y)\,\langle X, y\rangle^2$$

which leads to

$$\alpha(y) = \alpha(\mathrm{Ad}(h)\,y). \tag{3.8}$$

Therefore, α is a G-invariant Riemannian metric and $\mathrm{Ad}(h)\,X = X$, which proves that X is also G-invariant. □

The following theorem gives us a complete description of invariant vector fields.

Theorem 3.2. *[11] There exists a bijection between the set of invariant vector fields on G/H and the subspace*

$$V = \{Y \in \mathfrak{k} : Ad(h)\,Y = Y,\ \forall\, h \in H\}.$$

4. S-curvature of homogeneous Finsler space with

Now, we discuss S-curvature, a quantity used to measure the rate of change of the volume form of a Finsler space along geodesics. Let V be an n-dimensional real vector space having a basis $\{\alpha_i\}$ and F be a Minkowski norm on V. Let $Vol\ B$ to be the volume of a subset B of $\mathbb{R}^n$, and B^n be the open unit ball. The function $\tau = \tau(y)$ defined as

$$\tau(y) = \ln\left(\frac{\sqrt{det(g_{ij}(y))}}{\sigma_F}\right), \quad y \in V - \{0\},$$

where

$$\sigma_F = \frac{Vol\,(B^n)}{Vol\,\{(y^i) \in \mathbb{R}^n : F\,(y^i\alpha_i) < 1\}},$$

is called the distortion of (V, F).
For a Finsler space (M, F), $\tau = \tau(x, y)$ is the distortion of Minkowski norm F_x on T_xM, $x \in M$. Let γ be a geodesic with $\gamma(0) = x$, $\dot{\gamma}(0) = y$, where $y \in T_xM$, then S-curvature denoted as $S(x, y)$ is the rate of change of distortion along the geodesic γ, i.e.,

$$S(x, y) = \frac{d}{dt}\left\{\tau\Big(\gamma(t), \dot{\gamma}(t)\Big)\right\}\Bigg|_{t=0}.$$

Here, it is to be noted that $S(x, y)$ is positively homogeneous of degree one, i.e., for $\lambda > 0$, we have $S(x, \lambda y) = \lambda S(x, y)$.

S-curvature of a Finsler space is related to a volume form. There are two important volume forms in Finsler geometry: the Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$ and the Holmes-Thompson volume form $dV_{HT} = \sigma_{HT}(x)dx$ defined respectively as

$$\sigma_{BH}(x) = \frac{Vol\,(B^n)}{Vol A},$$

and

$$\sigma_{HT}(x) = \frac{1}{Vol\,(B^n)}\int_A det\,(g_{ij})\,dy,$$

where $A = \left\{\left(y^i\right) \in \mathbb{R}^n \ : \ F\left(x, y^i\frac{\partial}{\partial x^i}\right) < 1\right\}$.

If the Finsler metric F is replaced by a Riemannian metric, then both the volume forms reduce to a single Riemannian volume form $dV_{HT} = dV_{BH} = \sqrt{det\,(g_{ij}(x))}dx$.

Next, for the function

$$T(s) = \phi\,(\phi - s\phi')^{n-2}\left\{(\phi - s\phi') + \left(b^2 - s^2\right)\phi''\right\},$$

the volume form $dV = dV_{BH}$ or dV_{HT} is given by $dV = f(b)dV_\alpha$, where

$$f(b) = \begin{cases} \dfrac{\int_0^\pi \sin^{n-2} t \ dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(b\cos t)^n} \ dt}, & \text{if } \ dV = dV_{BH} \\ \dfrac{\int_0^\pi \left(\sin^{n-2} t\right) T(b\cos t) \ dt}{\int_0^\pi \sin^{n-2} t \ dt}, & \text{if } \ dV = dV_{HT} \end{cases},$$

and $dV_\alpha = \sqrt{det\,(a_{ij})}dx$ is the Riemannian volume form of α.

The formula for S-curvature of an (α, β)-metric, in local co-ordinate system, introduced by Cheng and Shen [5], is as follows:

$$S = \left(2\psi - \frac{f'(b)}{bf(b)}\right)(r_0 + s_o) - \frac{\Phi}{2\alpha\Delta^2}\left(r_{00} - 2\alpha Q s_0\right), \tag{4.1}$$

where

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Delta &= 1 + sQ + \left(b^2 - s^2\right) Q', \\ \psi &= \frac{Q'}{2\Delta}, \\ \Phi &= \left(sQ' - Q\right)\left(n\Delta + 1 + sQ\right) - \left(b^2 - s^2\right)\left(1 + sQ\right) Q'', \\ r_{ij} &= \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \ r_j = b^i r_{ij}, \ r_0 = r_i y^i, \ r_{00} = r_{ij} y^i y^j, \\ s_{ij} &= \frac{1}{2}\left(b_{i|j} - b_{j|i}\right), \ s_j = b^i s_{ij}, \ s_0 = s_i y^i. \end{aligned}$$

It is well known [5] that if the Riemannian length b is constant, then $r_0 + s_0 = 0$. Therefore, in this case, the equation (4.1) takes the form

$$S = -\frac{\Phi}{2\alpha\Delta^2}\left(r_{00} - 2\alpha Q s_0\right). \tag{4.2}$$

After Shen's work on S-curvature, Cheng and Shen [5] characterized Finsler metrics with isotropic S-curvature in 2009. In the same year, Deng [8] gave an explicit formula for S-curvature of homogeneous Randers spaces and he proved that a homogeneous Randers space having almost isotropic S-curvature has vanishing S-curvature. Later in 2010, Deng and Wang [12] gave a formula for S-curvature of homogeneous (α, β)-metrics. They also derived a formula for mean Berwald curvature E_{ij} of Randers metric. Recently, Shanker and Kaur [22] have proved that there is a mistake in the formula of S-curvature given in [12], and they have given the correct version of the formula for S-curvature of homogeneous (α, β)-metrics. Further, some progress has been done in the study of S-curvature of homogeneous Finsler spaces (see [15, 30] for detail).

Definition 4.1. Let (M, F) be an n-dimensional Finsler space. If there exists a smooth function $c(x)$ on M and a closed 1-form ω such that

$$S(x, y) = (n+1)\Big(c(x)F(y) + \omega(y)\Big), \quad x \in M, \ y \in T_x(M),$$

then (M, F) is said to have almost isotropic S-curvature. In addition, if ω is zero, then (M, F) is said to have isotropic S-curvature.
Also, if ω is zero and $c(x)$ is constant, then we say, (M, F) has constant S-curvature.

With above notations, let us recall the following theorem:

Theorem 4.1. *[22] Let $F = \alpha\phi(s)$ be a G-invariant (α, β)-metric on the reductive homogeneous Finsler space G/H with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$. Then the S-curvature is given by*

$$S(H, y) = \frac{\Phi}{2\alpha\Delta^2}\left(\left\langle [v, y]_{\mathfrak{k}}, y\right\rangle + \alpha Q\left\langle [v, y]_{\mathfrak{k}}, v\right\rangle\right), \tag{4.3}$$

where $v \in \mathfrak{k}$ corresponds to the 1-form β and $\mathfrak{k}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Now, we establish a formula for S-curvature of a homogeneous Finsler space with Randers changed square metric.

Theorem 4.2. *Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, and $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ be a G-invariant Randers changed square metric on G/H. Then the S-curvature is given by*

(4.4)
$$S(H, y) = \left[\frac{\begin{array}{c}-12s^5n + (-27n+9)s^4 + (8nb^2 + 4n - 4b^2 + 16)s^3 \\ + (18nb^2 + 18n - 18b^2 + 18)s^2 + -12b^2 s - 3 - 6b^2 - 6nb^2 - 3n\end{array}}{2\left(-3s^2 + 1 + 2b^2\right)\left(1 - 2s^2 - 3s^4 + 3s - 9s^3 + 2b^2 + 2b^2 s^2 + 6b^2 s\right)}\right] \times \left(\frac{2s+3}{1-s^2}\left\langle [v, y]_{\mathfrak{k}}, v\right\rangle + \frac{1}{\alpha}\left\langle [v, y]_{\mathfrak{k}}, y\right\rangle\right),$$

where $v \in \mathfrak{k}$ corresponds to the 1-form β and $\mathfrak{k}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. For Randers changed square metric

$$F = \alpha\phi(s), \text{where} \ \ \phi(s) = 1 + s^2 + 3s,$$

the entities written in the equation (4.1) take the values as follows:

$$\begin{aligned}
Q &= \frac{\phi'}{\phi - s\phi'} = \frac{2s+3}{1-s^2},\\
Q' &= \frac{2s^2+6s+2}{(1-s^2)^2},\\
Q'' &= \frac{4s^3+18s^2+12s+6}{(1-s^2)^3},\\
\Delta &= 1+sQ+\left(b^2-s^2\right)Q'\\
&= 1+s\left(\frac{2s+3}{1-s^2}\right)+\left(b^2-s^2\right)\frac{2s^2+6s+2}{(1-s^2)^2}\\
&= \frac{-3s^4-9s^3+(2b^2-2)s^2+(6b^2+3)s+2b^2+1}{(1-s^2)^2},
\end{aligned}$$

$$\begin{aligned}
\Phi &= (sQ'-Q)(1+n\Delta+sQ)+\left(s^2-b^2\right)(1+sQ)Q''\\
&= \left(\frac{2s^3+6s^2+2s}{(1-s^2)^2}-\frac{2s+3}{1-s^2}\right)\times\\
&\quad \left\{1+\frac{-3ns^4-9ns^3+(2nb^2-2n)s^2+(6nb^2+3n)s+2nb^2+n}{(1-s^2)^2}+\frac{2s^2+3s}{1-s^2}\right\}\\
&\quad +\left(s^2-b^2\right)\left\{1+\frac{2s^2+3s}{1-s^2}\right\}\left(\frac{4s^3+18s^2+12s+6}{(1-s^2)^3}\right)\\
&= \frac{1}{(1-s^2)^4}\Big\{-(12n+4)s^7-(63n+21)s^6+(8nb^2-89n-27)s^5\\
&\quad +(42nb^2+3n+15)s^4+(62nb^2+58n+40)s^3+(12nb^2+15n+9)s^2\\
&\quad -(18nb^2+9n+9)s-(6nb^2+3n+3)\Big\}+\frac{1}{(1-s^2)^4}\Big\{4s^7+30s^6+(70-4b^2)s^5\\
&\quad +(60-30b^2)s^4+(30-70b^2)s^3+(6-60b^2)s^2-30b^2s-6b^2\Big\}.\\
&= \frac{1}{(1-s^2)^4}\Big\{-12ns^7+(9-63n)s^6+(8nb^2-4b^2-89n+43)s^5\\
&\quad +(42nb^2-30b^2+3n+75)s^4+(62nb^2-70b^2+58n+70)s^3\\
&\quad +(12nb^2-60b^2+15n+15)s^2-(18nb^2+30b^2+9n+9)s\\
&\quad -(6nb^2+6b^2+3n+3)\Big\}
\end{aligned}$$

After substituting these values in equation (4.3), we get the formula 4.4 for S-curvature of homogeneous Finsler space with Randers changed square metric. □

Corollary 4.1. *Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, and $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ be a G-invariant Randers* *changed square metric on G/H. Then $(G/H, F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. Converse part is obvious. For necessary part, suppose G/H has isotropic S-curvature, then

$$S(x,y) = (n+1)c(x)F(y), \quad x \in G/H, \ y \in T_x(G/H).$$

Taking $x = H$ and $y = v$ in the equation (4.4), we get $c(H) = 0$.
Consequently $S(H,y) = 0 \ \forall \ y \in T_H(G/H)$.
Since F is a homogeneous metric, we have $S = 0$ everywhere.
□

5. Mean Berwald Curvature

There is another quantity [7] associated with S-curvature called Mean Berwald curvature.
Let $E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} S(x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right)(x,y)$, where G^m are spray coefficients. Then $\mathcal{E} := E_{ij} dx^i \otimes dx^j$ is a tensor on $TM\backslash\{0\}$, which we call E tensor. E tensor can also be viewed as a family of symmetric forms $E_y : T_xM \times T_xM \longrightarrow \mathbb{R}$ defined as

$$E_y(u,v) = E_{ij}(x,y)u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}|_x, \quad v = v^i \frac{\partial}{\partial x^i}|_x \ \in T_xM$. Then the collection $\{E_y : y \in TM\backslash\{0\}\}$ is called E-curvature or mean Berwald curvature.
In this section, we calculate the mean Berwald curvature of a homogeneous Finsler space with the aforesaid metrics. To calculate it, we need the following:
At the origin, $a_{ij} = \delta^i_j$,
therefore $y_i = a_{ij}y^j = \delta^i_j y^j = y^i$,

$$\begin{aligned}
\alpha_{y^i} &= \frac{y_i}{\alpha}, \\
\beta_{y^i} &= b_i, \\
s_{y^i} &= \frac{\partial}{\partial y^i}\left(\frac{\beta}{\alpha}\right) = \frac{b_i\alpha - sy_i}{\alpha^2}, \\
s_{y^i y^j} &= \frac{\partial}{\partial y^j}\left(\frac{b_i\alpha - sy_i}{\alpha^2}\right) \\
&= \frac{\alpha^2\left\{b_i\frac{y_j}{\alpha} - \left(\frac{b_j\alpha - sy_j}{\alpha^2}\right)y_i - s\delta^i_j\right\} - (b_i\alpha - sy_i)\,2\alpha\frac{y_j}{\alpha}}{\alpha^4} \\
&= \frac{-(b_iy_j + b_jy_i)\,\alpha + 3sy_iy_j - \alpha^2 s\delta^i_j}{\alpha^4},
\end{aligned}$$

Assuming

$$\frac{\begin{array}{l}-12s^5n + (-27n+9)s^4 + (8nb^2 + 4n - 4b^2 + 16)s^3 \\ \quad + (18nb^2 + 18n - 18b^2 + 18)s^2 + -12b^2s - 3 - 6b^2 - 6nb^2 - 3n\end{array}}{2\left(-3s^2 + 1 + 2b^2\right)\left(1 - 2s^2 - 3s^4 + 3s - 9s^3 + 2b^2 + 2b^2s^2 + 6b^2s\right)} = B$$

in the equation (4.4), we find

$$\begin{aligned}
\frac{\partial B}{\partial y^j} &= \frac{1}{2}\left(-3s^2+1+2b^2\right)^{-1}\left(-3s^4-9s^3+\left(2b^2-2\right)s^2+\left(3+6b^2\right)s+1+2b^2\right)^{-2}\times\\
&\Big\{-36ns^8+(-162n+54)\,s^7+\left(-207n-36b^2+225\right)s^6\\
&+\left(-252b^2+90n-36nb^2+522\right)s^5\\
&+\left(-488b^2+631-80nb^2+199n-8b^4+16nb^4\right)s^4\\
&+\left(-30n+186+96nb^4-408b^2-120nb^2-48b^4\right)s^3\\
&+\left(-228b^2+156nb^4-108b^4-33-60nb^2-69n\right)s^2\\
&+\left(96nb^4+6n+6-48b^4-12b^2+60nb^2\right)s\\
&+9+24b^2+36nb^4+12b^4+36nb^2+9n\Big\}s_{y^j},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 B}{\partial y^i \partial y^j} &= \frac{1}{2}\frac{\partial}{\partial y^i}\Big[\left(-3s^2+1+2b^2\right)^{-1}\left(-3s^4-9s^3+\left(2b^2-2\right)s^2+\left(3+6b^2\right)s+1+2b^2\right)^{-2}\times\\
&\Big\{-36ns^8\,(-162n+54)\,s^7+\left(-207n-36b^2+225\right)s^6\\
&+\left(-252b^2+90n-36nb^2+522\right)s^5\\
&+\left(-488b^2+631-80nb^2+199n-8b^4+16nb^4\right)s^4\\
&+\left(-30n+186+96nb^4-408b^2-120nb^2-48b^4\right)s^3\\
&+\left(-228b^2+156nb^4-108b^4-33-60nb^2-69n\right)s^2\\
&+\left(96nb^4+6n+6-48b^4-12b^2+60nb^2\right)s\\
&+9+24b^2+36nb^4+12b^4+36nb^2+9n\Big\}\Big]s_{y^j}\\
&+\frac{1}{2}\left(-3s^2+1+2b^2\right)^{-1}\left(-3s^4-9s^3+\left(2b^2-2\right)s^2+\left(3+6b^2\right)s+1+2b^2\right)^{-2}\times\\
&\Big\{-36ns^8+(-162n+54)\,s^7+\left(-207n-36b^2+225\right)s^6\\
&+\left(-252b^2+90n-36nb^2+522\right)s^5\\
&+\left(-488b^2+631-80nb^2+199n-8b^4+16nb^4\right)s^4\\
&+\left(-30n+186+96nb^4-408b^2-120nb^2-48b^4\right)s^3\\
&+\left(-228b^2+156nb^4-108b^4-33-60nb^2-69n\right)s^2\\
&+\left(96nb^4+6n+6-48b^4-12b^2+60nb^2\right)s\\
&+9+24b^2+36nb^4+12b^4+36nb^2+9n\Big\}s_{y^iy^j}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \left(-3s^2+1+2b^2\right)^{-1}\left(-3s^4-9s^3+\left(2b^2-2\right)s^2+\left(3+6b^2\right)s+1+2b^2\right)^{-3}\times \\
& \Big\{-108s^{11}n+(243-729n)s^{10}+\left(1593-216b^2-144nb^2-1935n\right)s^9 \\
& +\left(5940-2052b^2-1404nb^2-1512n\right)s^8 \\
& +\left(-144b^4-6570b^2+144nb^4+1440n-4338nb^2+13356\right)s^7 \\
& +\left(15894-10188b^2+1260nb^4+1638n-1332b^4-6300nb^2\right)s^6 \\
& +\left(8706-4884b^4-4254nb^2-1122n-8574b^2+3756nb^4\right)s^5 \\
& +\left(3132-7560b^4-1080n+5400nb^4-3834b^2+54nb^2\right)s^4 \\
& +\left(2634nb^2+3960nb^4+1700-4680b^4+40b^6+40nb^6+332n+402b^2\right)s^3 \\
& +\left(1476nb^4+1368nb^2+720b^2+315n-1116b^4+639\right)s^2 \\
& +\left(-90nb^2-15n-162b^2-180nb^4+21-468b^4-120nb^6-120b^6\right)s \\
& -24-24n-120nb^6-120b^6-126b^2-216nb^4-126nb^2-216b^4\Big\}s_{y^i}s_{y^j} \\
+ \quad & \frac{1}{2}\left(-3s^2+1+2b^2\right)^{-1}\left(-3s^4-9s^3+\left(2b^2-2\right)s^2+\left(3+6b^2\right)s+1+2b^2\right)^{-2}\times \\
& \Big\{-36ns^8+(-162n+54)s^7+\left(-207n-36b^2+225\right)s^6 \\
& +\left(-252b^2+90n-36nb^2+522\right)s^5 \\
& +\left(-488b^2+631-80nb^2+199n-8b^4+16nb^4\right)s^4 \\
& +\left(-30n+186+96nb^4-408b^2-120nb^2-48b^4\right)s^3 \\
& +\left(-228b^2+156nb^4-108b^4-33-60nb^2-69n\right)s^2 \\
& +\left(96nb^4+6n+6-48b^4-12b^2+60nb^2\right)s \\
& +9+24b^2+36nb^4+12b^4+36nb^2+9n\Big\}s_{y^iy^j}.
\end{aligned}
$$

Theorem 5.1. *Let G/H be a reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{k}$, and $F=\frac{(\alpha+\beta)^2}{\alpha}+\beta$ be a G-invariant Randers changed square metric on G/H. Then the mean Berwald curvature of the homogeneous Finsler space with Randers changed square metric is given by*

$$
\begin{aligned}
E_{ij}(H,y) \quad = \quad & \frac{1}{2}\Bigg[\left(\frac{1}{\alpha}\frac{\partial^2 B}{\partial y^i\partial y^j}-\frac{y_i}{\alpha^3}\frac{\partial B}{\partial y^j}-\frac{y_j}{\alpha^3}\frac{\partial B}{\partial y^i}-\frac{B}{\alpha^3}\delta_i^j+\frac{3B}{\alpha^5}y_iy_j\right)\langle [v,y]_{\mathfrak{k}},y\rangle \\
& +\left(\frac{1}{\alpha}\frac{\partial B}{\partial y^j}-\frac{B}{\alpha^3}y_j\right)\Big(\langle [v,v_i]_{\mathfrak{k}},y\rangle+\langle [v,y]_{\mathfrak{k}},v_i\rangle\Big) \\
& +\left(\frac{1}{\alpha}\frac{\partial B}{\partial y^i}-\frac{B}{\alpha^3}y_i\right)\Big(\langle [v,v_j]_{\mathfrak{k}},y\rangle+\langle [v,y]_{\mathfrak{k}},v_j\rangle\Big)
\end{aligned}
$$

(5.1)

$$
\begin{aligned}
&+\frac{B}{\alpha}\Big(\left\langle [v,v_j]_{\mathfrak{k}},v_i\right\rangle + \left\langle [v,v_i]_{\mathfrak{k}},v_j\right\rangle \Big)\\
&+\Bigg\{ \frac{2s+3}{1-s^2}\frac{\partial^2 B}{\partial y^i \partial y^j} + \frac{2s^2+6s+2}{(1-s^2)^2}\; s_{y^i}\;\frac{\partial B}{\partial y^j} + \frac{2s^2+6s+2}{(1-s^2)^2}\; s_{y^j}\;\frac{\partial B}{\partial y^i}\\
&+\frac{(4s^3+18s^2+12s+6)B}{(1-s^2)^3} s_{y^i}s_{y^j} + \frac{(2s^2+6s+2)B}{(1-s^2)^2} s_{y^iy^j}\Bigg\} \left\langle [v,y]_{\mathfrak{k}},v\right\rangle\\
&+\left\{ \frac{2s+3}{1-s^2}\frac{\partial B}{\partial y^j} + \frac{(2s^2+6s+2)B}{(1-s^2)^2} s_{y^j}\right\} \left\langle [v,v_i]_{\mathfrak{k}},v\right\rangle\\
&+\left\{ \frac{2s+3}{1-s^2}\frac{\partial B}{\partial y^i} + \frac{(2s^2+6s+2)B}{(1-s^2)^2} s_{y^i}\right\} \left\langle [v,v_j]_{\mathfrak{k}},v\right\rangle \Bigg],
\end{aligned}
$$

where $v \in \mathfrak{k}$ corresponds to the 1-form β and $\mathfrak{k}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. From the equation (4.4), we can write S- curvature at the origin as follows

$$S(H,y) = \phi_2 + \psi_2,$$

where

$$\phi_2 = \frac{B}{\alpha}\left\langle [v,y]_{\mathfrak{k}},y\right\rangle \quad \text{and} \quad \psi_2 = \frac{2s+3}{1-s^2}\; B\; \left\langle [v,y]_{\mathfrak{k}},v\right\rangle.$$

Therefore, mean Berwald curvature is

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial y^i\partial y^j} = \frac{1}{2}\left(\frac{\partial^2 \phi_2}{\partial y^i\partial y^j} + \frac{\partial^2 \psi_2}{\partial y^i\partial y^j}\right), \tag{5.2}$$

where $\frac{\partial^2 \phi_2}{\partial y^i\partial y^j}$ and $\frac{\partial^2 \psi_2}{\partial y^i\partial y^j}$ are calculated as follows:

$$
\begin{aligned}
\frac{\partial \phi_2}{\partial y^j} &= \frac{\partial}{\partial y^j}\left(\frac{B}{\alpha}\left\langle [v,y]_{\mathfrak{k}},y\right\rangle\right)\\
&= \left(\frac{1}{\alpha}\frac{\partial B}{\partial y^j} - \frac{B}{\alpha^2}\frac{y_j}{\alpha}\right)\left\langle [v,y]_{\mathfrak{k}},y\right\rangle + \frac{B}{\alpha}\Big(\left\langle [v,v_j]_{\mathfrak{k}},y\right\rangle + \left\langle [v,y]_{\mathfrak{k}},v_j\right\rangle \Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \phi_2}{\partial y^i\partial y^j} &= \frac{\partial}{\partial y^i}\left\{\left(\frac{1}{\alpha}\frac{\partial B}{\partial y^j} - \frac{By_j}{\alpha^3}\right)\left\langle [v,y]_{\mathfrak{k}},y\right\rangle + \frac{B}{\alpha}\Big(\left\langle [v,v_j]_{\mathfrak{k}},y\right\rangle + \left\langle [v,y]_{\mathfrak{k}},v_j\right\rangle \Big)\right\}\\
&= \left(\frac{1}{\alpha}\frac{\partial^2 B}{\partial y^i\partial y^j} - \frac{y_i}{\alpha^3}\frac{\partial B}{\partial y^j} - \frac{y_j}{\alpha^3}\frac{\partial B}{\partial y^i} - \frac{B}{\alpha^3}\delta^j_i + \frac{3B}{\alpha^5}y_iy_j\right)\left\langle [v,y]_{\mathfrak{k}},y\right\rangle\\
&\quad+\left(\frac{1}{\alpha}\frac{\partial B}{\partial y^j} - \frac{B}{\alpha^3}y_j\right)\Big(\left\langle [v,v_i]_{\mathfrak{k}},y\right\rangle + \left\langle [v,y]_{\mathfrak{k}},v_i\right\rangle \Big)\\
&\quad+\left(\frac{1}{\alpha}\frac{\partial B}{\partial y^i} - \frac{B}{\alpha^3}y_i\right)\Big(\left\langle [v,v_j]_{\mathfrak{k}},y\right\rangle + \left\langle [v,y]_{\mathfrak{k}},v_j\right\rangle \Big)\\
&\quad+\frac{B}{\alpha}\Big(\left\langle [v,v_j]_{\mathfrak{k}},v_i\right\rangle + \left\langle [v,v_i]_{\mathfrak{k}},v_j\right\rangle \Big),
\end{aligned}
$$

and

$$\begin{aligned}\frac{\partial\psi_2}{\partial y^j} &= \frac{\partial}{\partial y^j}\left(\frac{(2s+3)B}{1-s^2}\left\langle [v,y]_{\mathfrak{k}},v\right\rangle\right)\\ &= \left\{\frac{2s+3}{1-s^2}\frac{\partial B}{\partial y^j}+\frac{(2s^2+6s+2)B}{(1-s^2)^2}s_{y^j}\right\}\left\langle [v,y]_{\mathfrak{k}},v\right\rangle+\frac{(2s+3)B}{1-s^2}\left\langle [v,v_j]_{\mathfrak{k}},v\right\rangle,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\psi_2}{\partial y^i\partial y^j} &= \frac{\partial}{\partial y^i}\Bigg[\left\{\frac{2s+3}{1-s^2}\frac{\partial B}{\partial y^j}+\frac{(2s^2+6s+2)B}{(1-s^2)^2}s_{y^j}\right\}\left\langle [v,y]_{\mathfrak{k}},v\right\rangle\\ &\quad+\frac{(2s+3)B}{1-s^2}\left\langle [v,v_j]_{\mathfrak{k}},v\right\rangle\Bigg]\\ &= \left\{\frac{2s+3}{1-s^2}\frac{\partial^2 B}{\partial y^i\partial y^j}+\frac{2s^2+6s+2}{(1-s^2)^2}s_{y^i}\frac{\partial B}{\partial y^j}+\frac{2s^2+6s+2}{(1-s^2)^2}s_{y^j}\frac{\partial B}{\partial y^i}\right.\\ &\quad\left.+\frac{(4s^3+18s^2+12s+6)B}{(1-s^2)^3}s_{y^i}s_{y^j}+\frac{(2s^2+6s+2)B}{(1-s^2)^2}s_{y^iy^j}\right\}\left\langle [v,y]_{\mathfrak{k}},v\right\rangle\\ &\quad+\left\{\frac{2s+3}{1-s^2}\frac{\partial B}{\partial y^j}+\frac{(2s^2+6s+2)B}{(1-s^2)^2}s_{y^j}\right\}\left\langle [v,v_i]_{\mathfrak{k}},v\right\rangle\\ &\quad+\left\{\frac{2s+3}{1-s^2}\frac{\partial B}{\partial y^i}+\frac{(2s^2+6s+2)B}{(1-s^2)^2}s_{y^i}\right\}\left\langle [v,v_j]_{\mathfrak{k}},v\right\rangle.\end{aligned}$$

Substituting all above values in the equation (5.2), we get the formula (5.1). □

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SOLVABILITY FOR A CLASS OF NONLINEAR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN OPERATOR IN BANACH SPACES

Choukri Derbazi

Abstract. This paper is devoted to the existence of solutions for certain classes of nonlinear differential equations involving the Caputo-Hadamard fractional-order with p-Laplacian operator in Banach spaces. The arguments are based on Mönch's fixed point theorem combined with the technique of measures of noncompactness. An example is also presented to illustrate the effectiveness of the main results.

Keywords: Banach spaces; differential equations; Caputo-Hadamard fractional-order; Laplacian operator; Mönch's fixed point theorem.

1. Introduction

Fractional calculus is a branch of mathematical analysis that deals with the derivatives and integrals of arbitrary (non-integer) order. In fact fractional calculus has developed into an important field of research during the last few decades in view of its widespread applications in a variety of disciplines such as physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, financial mathematics, economics, etc. (see [28, 32, 33, 37, 38, 42, 44]).

Fixed point theory is an important tool in Nonlinear Analysis, in particular, in obtaining existence results for a variety of mathematical problems. Although there are many methods (such as Banach contraction principle, Schauder's fixed point theorem, and Krasnoselskii's fixed point theorem, etc.) to analyze, under suitable conditions, the existence and uniqueness of solution of various problems with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions, and periodic boundary conditions for fractional differential equations,

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for more details see for instance [5, 13, 18, 19, 20, 21, 26, 29, 43]. We focus here on the so-called measures of noncompactness, see for instance [1, 2, 4, 12, 14, 39]. A key result in the cited works to prove the existence of solutions is the celebrated Mönch fixed point theorem, based on such measures. We also refer the readers to the recent book [11], where several applications of the measure of noncompactness can be found.

A p-Laplacian differential equation was first introduced by Leibenson [31] when he studied the turbulent flow in a porous medium. Since then, fractional differential equations and the differential equation with a p-Laplacian operator are widely applied in different fields of physics and natural phenomena, for example, non-Newtonian mechanics, fluid mechanics, viscoelasticity mechanics, combustion theory, mathematical biology, the theory of partial differential equations. Hence, there have been many published papers that are devoted to the existence of solutions of boundary value problems for the p-Laplacian operator equations, see [9, 22, 23, 34, 35, 41] and their references. On the other hand, it has been noticed that most of the above-mentioned work on the topic is based on Riemann–Liouville or Caputo fractional derivatives. In 1892, Hadamard [27] introduced another fractional derivative, which differs from the above-mentioned ones because its definition involves the logarithmic function of arbitrary exponent and named the Hadamard derivative. For some developments on the f of the Hadamard fractional differential equations, we can refer to [15, 17, 25, 43]. Very recently Jarad et al [30] have modified the Hadamard fractional derivative into a more suitable one having physical interpretable initial conditions similar to the singles in the Caputo setting and called it Caputo–Hadamard fractional derivative. Details and properties of the modified derivative can be found in [30]. To the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo–Hadamard fractional differential equations [2, 7, 8, 16]. There are no contributions, as far as we know, concerning the Caputo–Hadamard fractional differential equations with p-Laplacian operator in Banach spaces. As a result, the goal of this paper is to enrich this academic area. So, in this paper, we mainly study the following problem of Caputo–Hadamard fractional differential equation with p-Laplacian operator of the form

$$ {}^{C}_{H}\mathcal{D}^{\beta}_{1}\Big(\phi_p\Big[{}^{C}_{H}\mathcal{D}^{\alpha}_{1}u(t)\Big]\Big) = f(t,u(t)), \;\; 1<\alpha\leq 2 \;\, t\in J:=[1,T], \tag{1.1} $$

supplemented with boundary conditions

$$ \begin{array}{rcl} a_1u(1)+b_1{}^{C}_{H}\mathcal{D}^{\gamma}_{1}u(1) & = & \lambda_1 \; {}^{H}\mathcal{I}^{\sigma_1}_{1}u(\eta_1), \; 0<\gamma\leq 1 \\ a_2u(T)+b_2{}^{C}_{H}\mathcal{D}^{\gamma}_{1}u(T) & = & \lambda_2 \; {}^{H}\mathcal{I}^{\sigma_2}_{1}u(\eta_2), \; 1<\eta_1,\eta_2<T \\ {}^{C}_{H}\mathcal{D}^{\alpha}_{1}u(1) & = & 0, \end{array} \tag{1.2} $$

where ${}^{C}_{H}\mathcal{D}^{\mu}_{1}$ is the Caputo–Hadamard fractional derivative order $\mu\in\{\alpha,\beta,\gamma\}$ such that $1<\alpha\leq 2, 0<\beta,\gamma\leq 1, {}^{H}\mathcal{I}^{\theta}_{1}$ is the Hadamard fractional integral of order $\theta>0, \theta\in\{\sigma_1,\sigma_2\}$ and $f:[1,T]\times E\longrightarrow E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$. $a_i, b_i, \lambda_i, i = 1, 2$ are suitably chosen real constants.

This paper is organized as follows. The second section provides the definitions and preliminary results to be used in this paper. The existence results, which rely on Mönch's fixed point theorem have been presented in Section 3. An example illustrating the obtained results is presented in Section 4, and the paper concludes with some conclusions in Section 5.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Let $C(J, E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm

$$\|u\|_\infty = \sup\{\|u(t)\|, t \in J\}.$$

By $L^1(J)$ we denote the space of Bochner-integrable functions $u : J \to E$, with the norm

$$\|u\|_1 = \int_1^T \|x(t)\|\mathrm{dt}.$$

Next, we define the Kuratowski measure of noncompactness and give some of its important properties.

Definition 2.1. [10] Let E be a Banach space, Ω_E the bounded subsets of E The Kuratowski measure of noncompactness is the map $\kappa : \Omega_E \to [0, \infty)$ defined by

$$\kappa(B) = \inf\{\varepsilon > 0 : B \subset \cup_{j=1}^m B_j,\ B \in \Omega_E \textit{ and } \operatorname{diam}(B_j) \leq \varepsilon\}.$$

Properties 2.1. *The Kuratowski measure of noncompactness satisfies some properties. For more details see [10].*

(1) $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$,

(2) $\kappa(A) = 0 \Leftrightarrow A$ *is relatively compact,*

(3) $\kappa(A) = \kappa(\overline{A}) = \kappa(conv(A))$, *where* $\overline{A}$ *and* $convA$ *represent the closure and the convex hull of* A, *respectively,*

(4) $\kappa(A + B) \leq \kappa(A) + \kappa(B)$,

(5) $\kappa(\lambda A) = |\lambda|\kappa(A), \lambda \in \mathbb{R}$.

Now, we give some results and properties from the theory of of fractional calculus. We begin by defining Hadamard fractional integrals and derivatives. In what follows,

Definition 2.2. [33] The Hadamard fractional integral of order $\alpha > 0$, for a function $u \in L^1(J)$, is defined as

$$\left({}^{H}\mathcal{I}_1^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{\mathrm{d}s}{s}, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} \mathrm{d}t, \quad \alpha > 0.$$

Set

$$\delta = t\frac{\mathrm{d}}{\mathrm{d}t}, \quad \alpha > 0,\ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of α. Define the space

$$AC_\delta^n[1,T] := \left\{u : [1,T] \longrightarrow \mathbb{R} : \delta^{n-1} u(t) \in AC([1,T])\right\}.$$

Definition 2.3. [33] The Hadamard fractional derivative of order $\alpha > 0$ applied to the function $u \in AC_\delta^n[1,T]$ is defined as

$$\left({}^{H}\mathcal{D}_1^{\alpha} u\right)(t) = \delta^n \left({}^{H}\mathcal{I}_1^{n-\alpha} u\right)(t).$$

Definition 2.4. [30, 33] The Caputo–Hadamard fractional derivative of order $\alpha > 0$ applied to the function $u \in AC_\delta^n[1,T]$ is defined as

$$\left({}_{H}^{C}\mathcal{D}_1^{\alpha} u\right)(t) = \left({}^{H}\mathcal{I}_1^{n-\alpha} \delta^n u\right)(t).$$

Lemmas of the following type are rather standard in the study of fractional differential equations.

Lemma 2.1. [30, 33] *Let $\alpha > 0, r > 0, n = [\alpha] + 1$, and $a > 0$, then the following relations hold*

- $\left({}^{H}\mathcal{I}_1^{\alpha} \left(\log \frac{s}{a}\right)^{r-1}\right)(t) = \frac{\Gamma(r)}{\Gamma(\alpha+r)} \left(\log \frac{t}{a}\right)^{\alpha+r-1}$,
- $$\begin{aligned} \left({}_{H}^{C}\mathcal{D}_1^{\alpha} \left(\log \frac{s}{a}\right)^{r-1}\right)(t) &= \frac{\Gamma(r)}{\Gamma(r-\alpha)} \left(\log \frac{t}{a}\right)^{r-\alpha-1}, \quad (r > n), \\ \left({}_{H}^{C}\mathcal{D}_1^{\alpha} \left(\log \frac{s}{a}\right)^{r-1}\right)(t) &= 0, \quad r \in \{0, \ldots, n-1\}. \end{aligned}$$

Lemma 2.2. [24, 33] *Let $\alpha > \beta > 0$, and $u \in AC_\delta^n[1,T]$. Then we have:*

- ${}^{H}\mathcal{I}_1^{\alpha}\, {}^{H}\mathcal{I}_1^{\beta} u(t) = {}^{H}\mathcal{I}_1^{\alpha+\beta} u(t)$,

- ${}^{C}_{H}\mathcal{D}^{\alpha}_{1}\, {}^{H}\mathcal{I}^{\alpha}_{1} u(t) = u(t)$,
- ${}^{C}_{H}\mathcal{D}^{\beta}_{1}\, {}^{H}\mathcal{I}^{\alpha}_{1} u(t) = {}^{H}\mathcal{I}^{\alpha-\beta}_{1} u(t)$.

Lemma 2.3. [30, 33] *Let $\alpha \geq 0$, and $n = [\alpha] + 1$. If $u \in AC^{n}_{\delta}[1,T]$, then the Caputo–Hadamard fractional differential equation*

$$\left({}^{C}_{H}\mathcal{D}^{\alpha}_{1} u\right)(t) = 0,$$

has a solution:

$$u(t) = \sum_{j=0}^{n-1} c_j \left(\log t\right)^j ,$$

and the following formula holds:

$${}^{H}\mathcal{I}^{\alpha}_{1}\left({}^{C}_{H}\mathcal{D}^{\alpha}_{1} u(t)\right) = u(t) + \sum_{j=0}^{n-1} c_j \left(\log t\right)^j ,$$

where $c_j \in \mathbb{R}, j = 0, 1, 2, \ldots, n-1$.

Remark 2.1. Note that for an abstract function $u : J \longrightarrow E$, the integrals which appear in the previous definitions are taken in Bochner's sense. (see, for instance, [40]).

In the sequel we will make use of the following fixed point theorem.

Theorem 2.2. (*Mönch's fixed point theorem [36]*). *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let $\mathcal{N}$ be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}\mathcal{N}(V), \, or \; V = \mathcal{N}(V) \cup \{0\} \Rightarrow \kappa(V) = 0, \tag{2.1}$$

holds for every subset $V \subset D$, then $\mathcal{N}$ has a fixed point.

Lemma 2.4. [6] *Let $H \subset C(J,E)$ be a bounded and equicontinuous subset. Then the function $t \to \kappa(H(t))$ is continuous on J, and*

$$\kappa_C(H) = \max_{t \in J} \kappa(H(t)),$$

and

$$\kappa\Big(\int_J u(s)\mathrm{ds}\Big) \leq \int_J \kappa(H(s))\mathrm{ds},$$

where $H(s) = \{u(s) : u \in H, s \in J\}$, and κ_C is the Kuratowski measure of noncompactness defined on the bounded sets of $C(J,E)$.

Definition 2.5. [45] A function $f : [1,T] \times E \longrightarrow E$ is said to satisfy the Carathéodory conditions, if the following hold

- $f(t,u)$ is measurable with respect to t for $u \in E$,
- $f(t,u)$ is continuous with respect to $u \in E$ for a.e. $t \in J$.

3. Main Results

Before starting and proving our main result, we have introduced the following auxiliary lemma.

Lemma 3.1. *For a given $h \in C(J, \mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$ {}^{C}_{H}\mathcal{D}^{\alpha}_{1} u(t) = h(t),\ 1 < \alpha \leq 2\ \ t \in J := [1, T], \tag{3.1}$$

supplemented with boundary conditions

$$\begin{aligned} a_1 u(1) + b_1 {}^{C}_{H}\mathcal{D}^{\gamma}_{1} u(1) &= \lambda_1\ {}^{H}\mathcal{I}^{\sigma_1}_{1} u(\eta_1),\ 0 < \gamma \leq 1 \\ a_2 u(T) + b_2 {}^{C}_{H}\mathcal{D}^{\gamma}_{1} u(T) &= \lambda_2\ {}^{H}\mathcal{I}^{\sigma_2}_{1} u(\eta_2),\ 1 < \eta_1, \eta_2 < T, \end{aligned} \tag{3.2}$$

is given by

$$\begin{aligned} u(t) &= {}^{H}\mathcal{I}^{\alpha}_{1} h(t) + \mu_1(t) {}^{H}\mathcal{I}^{\sigma_1+\alpha}_{1} h(\eta_1) + \mu_2(t) \Big(\lambda_2 {}^{H}\mathcal{I}^{\sigma_2+\alpha}_{1} h(\eta_2) \\ &- \left(a_2 {}^{H}\mathcal{I}^{\alpha}_{1} h(T) + b_2 {}^{H}\mathcal{I}^{\alpha-\gamma}_{1} h(T) \right) \Big), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mu_1(t) &= \lambda_1(\Delta_1 - \Delta_2 \log t), \quad \mu_2(t) = \lambda_1 \Delta_3 + \Delta_4 \log t, \\ \Delta_1 &= \frac{1}{\Delta}\left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2+2)} \right); \\ \Delta_2 &= \frac{1}{\Delta}\left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2+1)} \right) \\ \Delta_3 &= \frac{\lambda_1 (\log \eta_1)^{\sigma_1+1}}{\Delta \Gamma(\sigma_1+2)}; \\ \Delta_4 &= \frac{1}{\Delta}\left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1+1)} \right) \\ \Delta &= \left(a_1 - \frac{\lambda_1 (\log \eta_1)^{\sigma_1}}{\Gamma(\sigma_1+1)} \right) \left(a_2 \log T + \frac{b_2 (\log T)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{\lambda_2 (\log \eta_2)^{\sigma_2+1}}{\Gamma(\sigma_2+2)} \right) \\ &+ \frac{\lambda_1 (\log \eta_1)^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \left(a_2 \log T - \frac{\lambda_2 (\log \eta_2)^{\sigma_2}}{\Gamma(\sigma_2+1)} \right) \neq 0. \end{aligned} \tag{3.4}$$

Proof. By applying Lemma 2.3, we may reduce (3.1) to an equivalent integral equation

$$ u(t) = {}^{H}\mathcal{I}^{\alpha}_{1} h(t) + k_0 + k_1 \log(t), \quad k_0, k_1 \in \mathbb{R}. \tag{3.5}$$

Applying the boundary conditions (3.2) in (3.5) we may obtain

$$\begin{aligned} {}^{H}\mathcal{I}^{\sigma_i}_{1} u(\eta_i) &= {}^{H}\mathcal{I}^{\sigma_i+\alpha}_{1} h(\eta_i) + k_0 \frac{(\log \eta_i)^{\sigma_i}}{\Gamma(\sigma_i+1)} + k_1 \frac{(\log \eta_i)^{\sigma_i+1}}{\Gamma(\sigma_i+2)},\ i = 1, 2. \\ {}^{C}_{H}\mathcal{D}^{\gamma}_{1} u(T) &= {}^{H}\mathcal{I}^{\alpha-\gamma}_{1} h(T) + k_1 \frac{\Gamma(2)}{\Gamma(2-\gamma)} (\log T)^{1-\gamma}. \end{aligned}$$

After collecting the similar terms in one part, we have the following equations:

$$\begin{aligned}\left(a_1-\frac{\lambda_1\left(\log\eta_1\right)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)k_0 &- \frac{\lambda_1\left(\log\eta_1\right)^{\sigma_1+1}}{\Gamma(\sigma_1+2)}k_1=\lambda_1{}^{H}\mathcal{I}_1^{\sigma_1+\alpha}h(\eta_1),\\ \left(a_2\log T-\frac{\lambda_2\left(\log\eta_2\right)^{\sigma_2}}{\Gamma(\sigma_2+1)}\right)k_0 &+ \left(a_2\log T+\frac{b_2\left(\log T\right)^{1-\gamma}}{\Gamma(2-\gamma)}-\frac{\lambda_2\left(\log\eta_2\right)^{\sigma_2+1}}{\Gamma(\sigma_2+2)}\right)k_1\\ &= \lambda_2{}^{H}\mathcal{I}_1^{\sigma_2+\alpha}h(\eta_2)-a_2{}^{H}\mathcal{I}_1^{\alpha}h(T)-b_2{}^{H}\mathcal{I}_1^{\alpha-\gamma}h(T).\end{aligned}$$

Therefore, we get

$$\begin{aligned}k_0 &= \frac{\lambda_1}{\Delta}\left(a_2\log T+\frac{b_2\left(\log T\right)^{1-\gamma}}{\Gamma(2-\gamma)}-\frac{\lambda_2\left(\log\eta_2\right)^{\sigma_2+1}}{\Gamma(\sigma_2+2)}\right){}^{H}\mathcal{I}_1^{\alpha+\sigma_1}h(\eta_1)\\ &+ \frac{\lambda_1\left(\log\eta_1\right)^{\sigma_1+1}}{\Delta\Gamma(\sigma_1+2)}(\lambda_2{}^{H}\mathcal{I}_1^{\sigma_2+\alpha}h(\eta_2)-(a_2{}^{H}\mathcal{I}_1^{\alpha}h(T)+b_2{}^{H}\mathcal{I}_1^{\alpha-\gamma}h(T))),\\ k_1 &= \frac{\lambda_1}{\Delta}\left(a_1-\frac{\lambda_1\left(\log\eta_1\right)^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)(\lambda_2 I_{0^+}^{\sigma_2+\alpha}h(\eta_2)-(a_2{}^{H}\mathcal{I}_1^{\alpha}h(T)+b_2{}^{H}\mathcal{I}_1^{\alpha-\gamma}h(1))),\\ &- \frac{\lambda_1}{\Delta}\left(a_2\log T-\frac{\lambda_2\left(\log\eta_2\right)^{\sigma_2}}{\Gamma(\sigma_2+1)}\right)I_{0^+}^{\sigma_1+\alpha}h(\eta_1).\end{aligned}$$

Substituting the value of k_0, k_1 in (3.5) we get (3.3), which completes the proof. □

Lemma 3.2. *Let $1<\alpha\le 2, 0<\beta\le 1$. Then the boundary value problem of the fractional differential equation*

$$\text{(3.6)}\qquad {}^{C}_{H}\mathcal{D}_1^{\beta}\left(\phi_p\left[{}^{C}_{H}\mathcal{D}_1^{\alpha}u(t)\right]\right)=f(t,u(t)),\ \ 1<\alpha\le 2\ \ t\in J:=[1,T],$$

supplemented with boundary conditions

$$\begin{aligned}a_1u(1)+b_1{}^{C}_{H}\mathcal{D}_1^{\gamma}u(1) &= \lambda_1\ {}^{H}\mathcal{I}_1^{\sigma_1}u(\eta_1),\ 0<\gamma\le 1\\ a_2u(T)+b_2{}^{C}_{H}\mathcal{D}_1^{\gamma}u(T) &= \lambda_2\ {}^{H}\mathcal{I}_1^{\sigma_2}u(\eta_2),\ 1<\eta_1,\eta_2<T\\ \text{(3.7)}\qquad {}^{C}_{H}\mathcal{D}_1^{\alpha}u(1) &= 0,\end{aligned}$$

has a unique solution

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\ &+ \frac{\mu_1(t)}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\ &+ \frac{\lambda_2\mu_2(t)}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\end{aligned}$$

$$
\begin{aligned}
&- \frac{a_2\mu_2(t)}{\Gamma(\alpha)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&- \frac{b_2\mu_2(t)}{\Gamma(\alpha-\gamma)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}. \qquad (3.8)
\end{aligned}
$$

Proof. From Lemma 2.3 and the boundary value problem (3.6)–(3.7), we have

$$
\begin{aligned}
{}^H\mathcal{I}_1^{\beta}\, {}^C_H\mathcal{D}_1^{\beta}\left(\phi_p\left[{}^C_H\mathcal{D}_1^{\alpha}u(t)\right]\right) &= {}^H\mathcal{I}_1^{\beta} f(t,u(t))\\
&= \phi_p\left[{}^C_H\mathcal{D}_1^{\alpha}u(t)\right] + d_0, \quad d_0\in\mathbb{R},
\end{aligned}
$$

that is

$$
\phi_p\left[{}^C_H\mathcal{D}_1^{\alpha}u(t)\right] = {}^H\mathcal{I}_1^{\beta} f(t,u(t)) - d_0, \quad d_0\in\mathbb{R},
$$

By ${}^C_H\mathcal{D}_1^{\alpha}u(1)=0$, we have $d_0=0$. So,

$$
{}^C_H\mathcal{D}_1^{\alpha}u(t) = \phi_q\left[{}^H\mathcal{I}_1^{\beta} f(t,u(t))\right].
$$

Thus, the boundary value problem (3.6)–(3.7) is equivalent to the following problem:

$$
{}^C_H\mathcal{D}_1^{\alpha}u(t) = \phi_q\left[{}^H\mathcal{I}_1^{\beta} f(t,u(t))\right], \ t\in J:=[1,T],
$$

$$
\begin{aligned}
a_1u(1) + b_1\,{}^C_H\mathcal{D}_1^{\gamma}u(1) &= \lambda_1\, {}^H\mathcal{I}_1^{\sigma_1}u(\eta_1), \ 0<\gamma\le 1\\
a_2u(T) + b_2\,{}^C_H\mathcal{D}_1^{\gamma}u(T) &= \lambda_2\, {}^H\mathcal{I}_1^{\sigma_2}u(\eta_2), \ 1<\eta_1,\eta_2<T.
\end{aligned}
$$

Lemma 3.1 implies that boundary value problem (3.6)–(3.7) has a unique solution,

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{\mu_1(t)}{\Gamma(\sigma_1+\alpha)} \int_1^{\eta_1} \left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{\lambda_2\mu_2(t)}{\Gamma(\sigma_2+\alpha)} \int_1^{\eta_2} \left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&- \frac{a_2\mu_2(t)}{\Gamma(\alpha)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&- \frac{b_2\mu_2(t)}{\Gamma(\alpha-\gamma)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q\left(\int_1^s \frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
$$

This completes the proof. □

In the following, for computational convenience we put

$$\begin{aligned}\mathcal{M}_f &= \frac{\|p_f\|\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}}\left\{\frac{(\log T)^{\alpha+\beta(q-1)}(1+|a_2|\tilde{\mu_2})}{\Gamma(\alpha+\beta(q-1)+1)}+\frac{\tilde{\mu_1}(\log\eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{\Gamma(\sigma_1+\alpha+\beta(q-1)+1)}\right.\\ &+ \left.\tilde{\mu_2}\left[\frac{|\lambda_2|(\log\eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)}+\frac{|b_2|(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)}\right]\right\}, \end{aligned} \tag{3.9}$$

where $\tilde{\mu_1}=|\lambda_1|(|\Delta_1|+|\Delta_2|\log T)$, $\tilde{\mu_2}=|\lambda_1\Delta_3|+|\Delta_4|\log T$,

$$\begin{aligned}\mathcal{L}_f &= \frac{\|p_f\|\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}}\left\{\frac{(\log T)^{\alpha+\beta(q-1)-1}}{\Gamma(\alpha+\beta(q-1))}+\frac{|\lambda_1\Delta_2|(\log\eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{T\Gamma(\sigma_1+\alpha+\beta(q-1)+1)}\right.\\ &+ \frac{|\Delta_4|}{T}\left[\frac{|\lambda_2|(\log\eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)}+\frac{|a_2|(\log T)^{\alpha+\beta(q-1)}}{\Gamma(\alpha+\beta(q-1)+1)}\right.\\ &+ \left.\left.\frac{|b_2|(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)}\right]\right\} \end{aligned} \tag{3.10}$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1.1)–(1.2)

Theorem 3.1. *Assume that the following hypotheses hold:*

(H1) The function $f:[1,T]\times E\longrightarrow E$ satisfies Carathéodory conditions

(H2) There exisst $p_f\in L^\infty(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that

$$\|f(t,u)\| \leq \phi_p(p_f(t)\psi(\|u\|)) \text{ for a.e. } t\in J \text{ and each } u\in E.$$

(H3) For each bounded set $D\subset E$, and each $t\in J$, the following inequality holds

$$\kappa(f(t,D)) \leq \phi_p(p_f(t)\kappa(D)).$$

If

$$\mathcal{M}_f<1, \tag{3.11}$$

then the problem (1.1)–(1.2) has at least one solution defined on J.

Proof. Consider the operator $\mathcal{N}:C(J,E)\longrightarrow C(J,E)$ defined by:

$$\mathcal{N}u(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}$$

$$
\begin{aligned}
&+ \frac{\mu_1(t)}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{\lambda_2\mu_2(t)}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&- \frac{a_2\mu_2(t)}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&- \frac{b_2\mu_2(t)}{\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
$$

From Lemma 3.2, the fixed points of the operator $\mathcal{N}$ are solution of the problem (1.1)–(1.2).

Let $R>0$, such that

$$R \geq \mathcal{M}_f\psi(R),$$

and consider the ball

$$B_R = \{w \in C(J,E) : \|w\|_\infty \leq R\}.$$

We shall show that the operator $\mathcal{N}$ satisfies all the assumptions of Theorem 2.2. Take $u\in B_R, t\in J$ we have

$$
\begin{aligned}
&\|\mathcal{N}u(t)\|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\mu_1(t)|}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\lambda_2\mu_2(t)|}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|a_2\mu_2(t)|}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|b_2\mu_2(t)|}{\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u(\tau))\|\frac{d\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
$$

Using hypothese (H2) we get

$$
\begin{aligned}
&\|\mathcal{N}u(t)\|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p\left(p_f(\tau)\psi(\|u(\tau)\|)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{|\mu_1(t)|}{\Gamma(\sigma_1+\alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p \left(p_f(\tau)\psi(\|u(\tau)\|)\right) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|\lambda_2\mu_2(t)|}{\Gamma(\sigma_2+\alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p \left(p_f(\tau)\psi(\|u(\tau)\|)\right) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|a_2\mu_2(t)|}{\Gamma(\alpha)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p \left(p_f(\tau)\psi(\|u(\tau)\|)\right) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|b_2\mu_2(t)|}{\Gamma(\alpha-\gamma)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \phi_p \left(p_f(\tau)\psi(\|u(\tau)\|)\right) \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&\leq \quad \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|\mu_1(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_1+\alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|\lambda_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_2+\alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|a_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \mathrm{d}\tau \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|b_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha-\gamma)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\int_1^s \frac{\left(\log \frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)} \frac{\mathrm{d}\tau}{\tau} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&= \quad \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left(\frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|\mu_1(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_1+\alpha)} \int_1^{\eta_1} \left(\log \frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1} \phi_q \left(\frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|\lambda_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_2+\alpha)} \int_1^{\eta_2} \left(\log \frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1} \phi_q \left(\frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|a_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-1} \phi_q \left(\frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&+ \quad \frac{|b_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha-\gamma)} \int_1^{T} \left(\log \frac{T}{s}\right)^{\alpha-\gamma-1} \phi_q \left(\frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \right) \frac{\mathrm{ds}}{\mathrm{s}} \\
&= \quad \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \right)^{q-1} \frac{\mathrm{ds}}{\mathrm{s}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\mu_1(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\left(\frac{(\log s)^\beta}{\Gamma(\beta+1)}\right)^{q-1}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\lambda_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\left(\frac{(\log s)^\beta}{\Gamma(\beta+1)}\right)^{q-1}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|a_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\left(\frac{(\log s)^\beta}{\Gamma(\beta+1)}\right)^{q-1}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|b_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\left(\frac{(\log s)^\beta}{\Gamma(\beta+1)}\right)^{q-1}\frac{\mathrm{ds}}{\mathrm{s}}\\
&= \frac{\|p_f\|\psi(\|u\|)}{\Gamma(\beta+1)^{q-1}\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\beta(q-1)}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\mu_1(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\beta+1)^{q-1}\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}(\log s)^{\beta(q-1)}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\lambda_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\beta+1)^{q-1}\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}(\log s)^{\beta(q-1)}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|a_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\beta+1)^{q-1}\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}(\log s)^{\beta(q-1)}\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|b_2\mu_2(t)|\|p_f\|\psi(\|u\|)}{\Gamma(\beta+1)^{q-1}\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}(\log s)^{\beta(q-1)}\frac{\mathrm{ds}}{\mathrm{s}}\\
&= \frac{\|p_f\|\psi(\|u\|)\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}}\left\{\frac{(\log T)^{\alpha+\beta(q-1)}(1+|a_2|\tilde{\mu_2})}{\Gamma(\alpha+\beta(q-1)+1)}\right.\\
&+ \frac{\tilde{\mu_1}(\log\eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{\Gamma(\sigma_1+\alpha+\beta(q-1)+1)}+\tilde{\mu_2}\left[\frac{|\lambda_2|(\log\eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)}\right.\\
&+ \left.\left.\frac{|b_2|(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)}\right]\right\}\\
&\le \mathcal{M}_f\psi(R)\le R.
\end{aligned}
$$

Thus

$$\|\mathcal{N}u\|\le R.$$

This proves that $\mathcal{N}$ transforms the ball B_R into itself. Furthermore for any $u\in B_R$ and $t\in J$, we have

$$\|(\mathcal{N}u)'(t)\|$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha-1)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-2}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p\left(p_f(\tau)\psi(\|u(\tau)\|)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\lambda_1\Delta_2|}{T\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{T\Gamma(\beta)}\phi_p\left(p_f(\tau)\psi(\|u(\tau)\|)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_2\Delta_4|}{T\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p\left(p_f(\tau)\psi(\|u(\tau)\|)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|a_2\Delta_4|}{T\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p\left(p_f(\tau)\psi(\|u(\tau)\|)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|b_2\Delta_4|}{T\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p\left(p_f(\tau)\psi(\|u(\tau)\|)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
$$

Some computations give

$$
\begin{aligned}
\|(\mathcal{N}u)'(t)\| &\leq \frac{\|p_f\|\psi(\|u\|)\Gamma(\beta(q-1)+1)}{\Gamma(\beta+1)^{q-1}}\left\{\frac{(\log T)^{\alpha+\beta(q-1)-1}}{\Gamma(\alpha+\beta(q-1))}\right.\\
&+ \frac{|\lambda_1\Delta_2|\,(\log\eta_1)^{\sigma_1+\alpha+\beta(q-1)}}{T\Gamma(\sigma_1+\alpha+\beta(q-1)+1)}+\frac{|\Delta_4|}{T}\left[\frac{|\lambda_2|\,(\log\eta_2)^{\sigma_2+\alpha+\beta(q-1)}}{\Gamma(\sigma_2+\alpha+\beta(q-1)+1)}\right.\\
&+ \left.\left.\frac{|a_2|\,(\log T)^{\alpha+\beta(q-1)}}{\Gamma(\alpha+\beta(q-1)+1)}+\frac{|b_2|\,(\log T)^{\alpha-\gamma+\beta(q-1)}}{\Gamma(\alpha-\gamma+\beta(q-1)+1)}\right]\right\}\\
&= \mathcal{L}_f\psi(\|u\|),
\end{aligned}
$$

where $\mathcal{L}_f$ is given by (3.10). We shall show that the operator $\mathcal{N}: B_R \longrightarrow B_R$ satisfies all the assumptions of Theorem 2.2.

For clarity, we will divide the remain of the proof into several steps.

Step 1: $\mathcal{N}: B_R \longrightarrow B_R$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t\in J$,we have

$$
\|(\mathcal{N}u_n)(t)-(\mathcal{N}u)(t)\|
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u_n(\tau)-f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\mu_1(t)|}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u_n(\tau)-f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\lambda_2\mu_2(t)|}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u_n(\tau)-f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|a_2\mu_2(t)|}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u_n(\tau)-f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|b_2\mu_2(t)|}{\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\|f(\tau,u_n(\tau)-f(\tau,u(\tau))\|\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}.
\end{aligned}
\tag{3.12}
$$

Since $u_n \to u$ as $n \to \infty$ and $f, \phi_q(\cdot)$ are continuous, then by the Lebesgue dominated convergence theorem, equation (3.12) implies

$$\|\mathcal{N}u_n - \mathcal{N}u\| \to 0, \quad \text{as} \quad n \to \infty.$$

Step 2: $\mathcal{N}(B_R)$ is bounded and equicontinuous. Since $\mathcal{N}(B_R) \subset B_R$ and B_R is bounded, then $\mathcal{N}(B_R)$ is bounded. Next, let $t_1, t_2 \in J, t_1 < t_2, u \in B_R$. Thus, we have

$$\|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \leq \int_{t_1}^{t_2} \|(\mathcal{N}u)'(t)\|\mathrm{dt} \leq \mathcal{L}_f \psi(R)|t_2 - t_1|,$$

where $\mathcal{L}_f$ is given by (3.10). As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero.

Step 3: The implication (2.1) holds. Now let V be a subset of B_R such that $V \subset \overline{\mathcal{N}(V)} \cup \{0\}$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \kappa(V(t))$ is continuous on J. By assumption (H3), and the properties of the measure κ we have for each $t \in J$.

$$v(t) \leq \kappa(\overline{\mathcal{N}(V)} \cup \{0\}) \leq \kappa(\mathcal{N}(V)(t))$$

$$\begin{aligned}
&\leq \kappa\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\right)\\
&+ \kappa\left(\frac{\mu_1(t)}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\right)\\
&+ \kappa\left(\frac{\lambda_2\mu_2(t)}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\right)\\
&+ \kappa\left(\frac{a_2\mu_2(t)}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\right)\\
&+ \kappa\left(\frac{b_2\mu_2(t)}{\Gamma(\alpha-\gamma)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}f(\tau,u(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\right)\\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}\kappa\left(f(\tau,V(\tau))\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\tilde{\mu}_1|}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}\kappa\left(f(\tau,V(\tau))\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|\lambda_2\tilde{\mu}_2|}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}\kappa\left(f(\tau,V(\tau))\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \frac{|a_2\tilde{\mu}_2|}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s \frac{(\log\frac{s}{\tau})^{\beta-1}}{\Gamma(\beta)}\kappa\left(f(\tau,V(\tau))\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{|b_2\tilde{\mu}_2|}{\Gamma(\alpha-\gamma)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\kappa\left(f(\tau,V(\tau))\right)\frac{d\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&\leq \quad \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p(p_f(\tau)v(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \quad \frac{|\tilde{\mu}_1|}{\Gamma(\sigma_1+\alpha)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p(p_f(\tau)v(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \quad \frac{|\lambda_2\tilde{\mu}_2|}{\Gamma(\sigma_2+\alpha)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\sigma_2+\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p(p_f(\tau)v(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \quad \frac{|a_2\tilde{\mu}_2|}{\Gamma(\alpha)}\int_1^{T}\left(\log\frac{T}{s}\right)^{\alpha-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p(p_f(\tau)v(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&+ \quad \frac{|b_2\tilde{\mu}_2|}{\Gamma(\alpha-\gamma)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-\gamma-1}\phi_q\left(\int_1^s\frac{\left(\log\frac{s}{\tau}\right)^{\beta-1}}{\Gamma(\beta)}\phi_p(p_f(\tau)v(\tau))\frac{d\tau}{\tau}\right)\frac{\mathrm{ds}}{\mathrm{s}}\\
&\leq \quad \mathcal{M}_f\|v\|_\infty,
\end{aligned}
$$

where $\mathcal{M}_f$ is given by (3.9). Which gives

$$
\|v\|_\infty \quad \leq \quad \mathcal{M}_f\|v\|_\infty.
$$

This means that

$$
\|v\|_\infty\left(1-\mathcal{M}_f\right) \quad \leq \quad 0.
$$

From (3.11), we get $\|v\|_\infty = 0$, that is, $v(t) = \kappa(V(t)) = 0$, for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in B_R. Applying Theorem 2.2 we conclude that $\mathcal{N}$ has a fixed point which is a solution of the problem (1.1)–(1.2). $\square$

4. An Example

In this section, we give an example to illustrate the usefulness of our main result. Let

$$
E = c_0 = \left\{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0\ (n \to \infty)\right\},
$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$
\|u\|_\infty = \sup_{n\geq 1}|u_n|.
$$

Example 4.1. Consider the following fractional problem posed in c_0 :

$$
{}^{C}_{H}\mathcal{D}_1^{\frac{1}{2}}\left(\phi_p\left[{}^{C}_{H}\mathcal{D}_1^{\frac{7}{4}}u(t)\right]\right) = f(t,u(t)),\ t \in J := [1,e], \tag{4.1}
$$

supplemented with boundary conditions

$$
\begin{aligned}
u(1) + {}^{C}_{H}\mathcal{D}_1^{\frac{1}{2}} u(1) &= {}^{H}\mathcal{I}_1^{\frac{5}{2}} u(\frac{5}{4}),\ 0 < \gamma \leq 1 \\
u(e) + {}^{C}_{H}\mathcal{D}_1^{\frac{1}{2}} u(e) &= {}^{H}\mathcal{I}_1^{\frac{7}{2}} u(\frac{3}{2}),\ 1 < \eta_1, \eta_2 < T \\
{}^{C}_{H}\mathcal{D}_1^{\frac{7}{4}} u(1) &= (0, 0, \ldots, 0, \ldots).
\end{aligned} \tag{4.2}
$$

Note that, this problem is a particular case of BVP (1.1)–(1.2), where

$$
\begin{aligned}
\alpha &= \frac{7}{4}, \beta = \gamma = \frac{1}{2}, T = e \\
a_1 &= b_1 = a_2 = b_2 = \lambda_1 = \lambda_2 = 1, \\
\sigma_1 &= \frac{5}{2}, \sigma_2 = \frac{7}{2}, \eta_1 = \frac{5}{4}, \eta_2 = \frac{3}{2}, \\
p &= 2, q = 2,
\end{aligned}
$$

and $f : J \times c_0 \longrightarrow c_0$ given by

$$
f(t, u) = \left\{ \phi_p \left(\frac{1}{(t^2+2)^2} \left(\frac{1}{n^2} + \sin|u_n| \right) \right) \right\}_{n \geq 1}, \quad \text{for } t \in J, u = \{u_n\}_{n\geq 1} \in c_0.
$$

It is clear that condition (H1) holds, and as

$$
\begin{aligned}
\|f(t,u)\| &= \left\| \phi_p \left(\frac{1}{(t^2+2)^2} \left(\frac{1}{n^2} + \sin|u_n| \right) \right) \right\| \\
&\leq \phi_p \left(\frac{1}{(t^2+2)^2} (1 + \|u\|) \right) \\
&= \phi_p \left(p_f(t) \psi(\|u\|) \right).
\end{aligned}
$$

Therefore, the assumption (H2) of the Theorem 3.1 is satisfied with $p_f(t) = \frac{1}{(t^2+2)^2}, t \in J$, and $\psi(u) = 1 + u,\ u \in [0, \infty)$. On the other hand, for any bounded set $D \subset c_0$, we have

$$
\kappa(f(t, D)) \leq \frac{1}{(t^2+2)^2} \kappa(D), \text{ for each } t \in J.
$$

Hence (H3) is satisfied. We shall check that condition (3.11) is satisfied. Using the Matlab program, we can find

$$
\mathcal{M}_f = 0.4228 < 1.
$$

and

$$
(1 + R)\mathcal{M}_f < R,
$$

thus

$$
R > \frac{\mathcal{M}_f}{1 - \mathcal{M}_f} = 0.7326,
$$

Then R can be chosen as $R = 1 \geq 0.7326$. Consequently, Theorem 3.1 implies that the problem (4.1)–(4.2) has at least one solution $u \in C(J, c_0)$.

5. Conclusions

We have proved the existence of solutions for Caputo–Hadamard fractional differential equations with p–Laplacian operator in a given Banach space. The problem is issued by applying Mönch's fixed point theorem combined with the technique of measures noncompactness. We also provide an example to make our results clear.

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$\eta-$RICCI SOLITONS IN LORENTZIAN $\alpha-$SASAKIAN MANIFOLDS

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Abstract. In the present paper, we have studied η-Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds satisfying certain curvature conditions. The existence of $\eta-$Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold has been proved by a concrete example.
Keywords: $\eta-$Ricci solitons; Lorentzian $\alpha-$Sasakian manifolds; projective curvature tensor.

1. Introduction

In 1985, J. A. Oubina [14] defined and studied a new class of almost contact metric manifolds known as trans-Sasakian manifolds, which includes α-Sasakian, β-Kenmotsu and cosymplectic structures. In 2005, A. Yildiz and C. Murathan [5] studied conformally flat and quasi-conformally flat Lorentzian $\alpha-$Sasakian manifolds. Lorentzian $\alpha-$Sasakian manifolds have been studied by many authors such as [1, 3, 6]. Recently, U. C. De and P. Majhi have studied $\phi-$Weyl semisymmetric and $\phi-$projectively semisymmetric generalized Sasakian space-forms and obtained some intersesting results [21].

In 1982, R. S. Hamilton [20] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. G. Perelman [12, 13] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group

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of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [17, 18]

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively. Ricci solitons in the context of general relativity have been studied by M. Ali and Z. Ahsan [16].

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by J. T. Cho and M. Kimura [15]. They have studied Ricci solitons of real hypersurfaces in a non-flat complex space form and they defined η-Ricci soliton, which satisfies the equation

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{1.2}$$

where λ and μ are real number. In particular, if $\mu = 0$, then the notion η-Ricci soliton (g, V, λ, μ) is reduced to the notion of Ricci soliton (g, V, λ). Recenty, $\eta-$Ricci solitons have been studied by various authors such as A. Singh and S. Kishor [4], A. M. Blaga [9], D. G. Prakasha and B. S. Hadimani [11], S. Ghosh [19] and many others.

The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian $\alpha-$Sasakian manifolds. In Section 3, we discuss η-Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds. Section 4 is devoted to study η-Ricci solitons in $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifolds. In Section 5, we study η-parallel $\phi-$tensor Lorentzian α-Sasakian manifolds admitting $\eta-$Ricci solitons. $\eta-$Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor have been studied in Section 6. In Section 7, we study η-Ricci solitons in recurrent Lorentzian α-Sasakian manifolds. Finally, we construct an example of 3-dimensional Lorentzian $\alpha-$Sasakian manifold which admits an $\eta-$Ricci soliton.

2. Preliminaries

A differentiable manifold of dimension n is called a Lorentzian $\alpha-$Sasakian manifold if it admits a $(1, 1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy [5]

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$g(X, \xi) = \eta(X) \tag{2.5}$$

for all vector fields X, Y on M.
Also Lorentzian $\alpha-$Sasakian manifolds satisfy

$$\nabla_X \xi = -\alpha\phi X, \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y), \tag{2.7}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and $\alpha \in \mathbb{R}$.

Furthermore, on a Lorentzian $\alpha-$Sasakian manifold M, the following relations hold [5, 6]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.8}$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)\xi = \alpha^2[X + \eta(X)\xi], \tag{2.11}$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X), \quad S(\xi,\xi) = -(n-1)\alpha^2, \tag{2.12}$$

$$Q\xi = (n-1)\alpha^2\xi, \tag{2.13}$$

$$(\nabla_X \phi)Y = \alpha g(X,Y)\xi - \alpha\eta(Y)X \tag{2.14}$$

for any vector fields X, Y and Z on M.

Definition 2.1. A Lorentzian $\alpha-$Sasakian manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor S is of the form [7]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y),$$

where a, b and c are smooth functions on M. If $c = 0$, $b = c = 0$ and $b = 0$, then the manifold is said to be an $\eta-$Einstein, Einstein and a special type of generalized η-Einstein manifold, respectively.

Definition 2.2. The projective curvature tensor C in an $n-$dimensional Lorentzian α-Sasakian manifold M is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{2.15}$$

where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

3. η-Ricci solitions in Lorentzian $\alpha-$Sasakian manifolds

Suppose that a Lorentzian $\alpha-$Sasakian manifold admits an η-Ricci soliton (g, ξ, λ, μ). Then (1.2) holds and thus we have

$$(\pounds_{\xi} g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.1}$$

In a Lorentzian $\alpha-$Sasakian manifold, we find

$$(\pounds_{\xi} g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = -2\alpha g(X, \phi Y). \tag{3.2}$$

Combining (3.1) and (3.2), it follows that

$$S(X, Y) = -\lambda g(X, Y) + \alpha g(\phi X, Y) - \mu\eta(X)\eta(Y). \tag{3.3}$$

It yields

$$QX = -\lambda X + \alpha\phi X - \mu\eta(X)\xi. \tag{3.4}$$

By taking $Y = \xi$ in (3.3) and using (2.1), (2.3) and (2.5), we get

$$S(X, \xi) = (\mu - \lambda)\eta(X). \tag{3.5}$$

Thus from (2.12) and (3.5), we obtain

$$\mu - \lambda = (n - 1)\alpha^2. \tag{3.6}$$

Hence in view of (3.3) and (3.6), we can state the following theorem:

Theorem 3.1. *If (g, ξ, λ, μ) is an η-Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold, then the manifold is a generalized η-Einstein manifold of the form* (3.3) *and $\mu - \lambda = (n - 1)\alpha^2$.*

In particular, if we take $\mu = 0$ in (3.3) and (3.6), then we obtain

$$S(X, Y) = -\lambda g(X, Y) + \alpha g(\phi X, Y), \tag{3.7}$$

$$\lambda = -(n - 1)\alpha^2, \tag{3.8}$$

respectively. Thus we have

Corollary 3.1. *If (g, ξ, λ) is a Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold, then the manifold is a special type of genralized η-Einstein manifold and its Ricci solition is always shrinking.*

Now, let (g, V, λ, μ) be a Ricci soliton in a Lorentzian $\alpha-$Sasakian manifold such that V is pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function. Then (1.2) holds and we have

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(X, \nabla_Y \xi) + (Yb)\eta(X)$$

$$+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0$$

which in view of (2.6) takes the form

$$-2b\alpha g(\phi X,Y)+(Xb)\eta(Y)+(Yb)\eta(X) \tag{3.9}$$

$$+2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)=0.$$

Putting $Y=\xi$ in (3.9) and using (2.1), (2.3), (2.5) and (2.12), we find

$$-(Xb)+[(\xi b)+2(n-1)\alpha^2+2\lambda-2\mu]\eta(X)=0. \tag{3.10}$$

Again putting $X=\xi$ in (3.10) and using (2.1), we get

$$(\xi b)+(n-1)\alpha^2+\lambda-\mu=0. \tag{3.11}$$

Combining the equations (3.10) and (3.11), it follows that

$$db=[(n-1)\alpha^2+\lambda-\mu]\eta. \tag{3.12}$$

Now applying d on (3.12), we get

$$[(n-1)\alpha^2+\lambda-\mu]\eta=0 \quad \Rightarrow \quad \mu-\lambda=(n-1)\alpha^2, \quad d\eta\neq 0. \tag{3.13}$$

Thus from (3.12) and (3.13), we obtain $db=0$, i.e., b is a constant. Therefore, (3.9) takes form

$$S(X,Y)=-\lambda g(X,Y)+b\alpha g(\phi X,Y)-\mu\eta(X)\eta(Y). \tag{3.14}$$

Hence in view of (3.13) and (3.14), we can state the following theorem:

Theorem 3.2. *If (g,ξ,λ,μ) is an η-Ricci soliton in an $n-$dimensional Lorentzian $\alpha-$Sasakian manifold, such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is a generalized η-Einstein manifold of the form* (3.14) *and $\mu-\lambda=(n-1)\alpha^2$.*

4. η-Ricci solitions in ϕ-projectively semisymmetric Lorentzian $\alpha-$Sasakian manifolds

Definition 4.1. A Lorentzian $\alpha-$Sasakian manifold is said to be $\phi-$projectively semisymmetric if [20]

$$P(X,Y)\cdot\phi=0$$

for all X,Y on M.

Let M be an n-dimensional $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifold admits η-Ricci soliton. Therefore $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0 \tag{4.1}$$

for any vector fields $X, Y, Z \in \chi(M)$. From (2.15), it follows that

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y], \tag{4.2}$$

$$\phi P(X,Y)Z = \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y]. \tag{4.3}$$

Combining the equations (4.1), (4.2) and (4.3), we have

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{n-1}[S(Y,\phi Z)X - S(X,\phi Z)Y] \\ + \frac{1}{n-1}[S(Y,Z)\phi X - S(X,Z)\phi Y] = 0 \end{aligned} \tag{4.4}$$

which by taking $Y = \xi$ and using (2.3), (2.9) and (2.12) is reduced to

$$S(X,\phi Z) = (n-1)\alpha^2 g(X,\phi Z). \tag{4.5}$$

In view of (3.3), (4.5) takes the form

$$[\lambda + (n-1)\alpha^2]g(X,\phi Z) - \alpha g(\phi X,\phi Z) = 0. \tag{4.6}$$

By replacing X by ϕX in (4.6) and using (2.2), we get

$$[\lambda + (n-1)\alpha^2]g(\phi X,\phi Z) - \alpha g(X,\phi Z) = 0. \tag{4.7}$$

By adding (4.6) and (4.7), we obtain

$$[\lambda + (n-1)\alpha^2 - \alpha](g(\phi X,\phi Z) + g(X,\phi Z)) = 0$$

from which it follows that $\lambda = -(n-1)\alpha^2 + \alpha$ and hence from (3.6), we get $\mu = \alpha$. Thus we can state the following theorem:

Theorem 4.1. *If (g,ξ,λ,μ) is an η-Ricci soliton in an n-dimensional $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifold, then $\lambda = -(n-1)\alpha^2 + \alpha$ and $\mu = \alpha$.*

Now from the relations (3.3), (3.6) and (4.7), we obtain

$$S(X,Y) = (n-1)\alpha^2 g(X,Y). \tag{4.8}$$

Thus we have

Corollary 4.1. *An n-dimensional $\phi-$projectively semisymmetric Lorentzian $\alpha-$Sasakian manifold admitting an η-Ricci soliton (g,ξ,λ,μ) is an Einstein manifold.*

5. η-parallel $\phi-$tensor Lorentzian α-Sasakian manifolds admitting $\eta-$Ricci solitons

In this section, we study the $\eta-$parallel $\phi-$tensor in Lorentzian α-Sasakian manifolds. If the $(1,1)$ tensor ϕ is $\eta-$parallel, then we have [10]

$$g((\nabla_X \phi)Y, Z) = 0 \tag{5.1}$$

for all $X, Y, Z \in \chi(M)$. From (2.14) and (5.1), we get

$$g(X,Y)\eta(Z) - \eta(Y)g(X,Z) = 0, \quad \text{where} \quad \alpha \neq 0. \tag{5.2}$$

Putting $Z = \xi$ in (5.2), we find

$$g(X,Y) = -\eta(X)\eta(Y)$$

which by replacing Y by QY and using (2.12) yields

$$S(X,Y) = -\alpha^2(n-1)\eta(X)\eta(Y). \tag{5.3}$$

From (3.3) and (5.3), it follows that

$$\lambda g(X,Y) - \alpha g(\phi X, Y) + (\mu - (n-1)\alpha^2)\eta(X)\eta(Y) = 0$$

which by replacing Y by ϕY becomes

$$\lambda g(X, \phi Y) - \alpha g(\phi X, \phi Y) = 0. \tag{5.4}$$

Now by replacing X by ϕX in (5.4) and using (2.2), we find

$$\lambda g(\phi X, \phi Y) - \alpha g(X, \phi Y) = 0. \tag{5.5}$$

By adding (5.4) and (5.5), we obtain $\lambda = \alpha$ and hence from (3.6) we get $\mu = \alpha + (n-1)\alpha^2$. Thus we have the following theorem:

Theorem 5.1. *If (g, ξ, λ, μ) is an η-Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and if the tensor ϕ is $\eta-$parallel , then $\lambda = \alpha$ and $\mu = \alpha + (n-1)\alpha^2$.*

Now from the relations (3.3), (3.6) and (5.5), we obtain

$$S(X,Y) = -(n-1)\alpha^2\eta(X)\eta(Y). \tag{5.6}$$

Thus we have

Corollary 5.1. *If (g, ξ, λ, μ) is an η-Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and if the tensor ϕ is $\eta-$parallel, then the manifold is a special type of $\eta-$Einstein manifold.*

6. $\eta-$Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor

In this section, we consider $\eta-$Ricci solitons in Lorentzian $\alpha-$Sasakian manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. A. Gray [2] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor.

Definition 6.1. A Lorentzian $\alpha-$Sasakian manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$$

for all $X, Y, Z \in \chi(M)$,

Taking covariant derivative of (3.3) and making use of (2.7) and (2.14), we find

$$(\nabla_X S)(Y,Z) = \alpha^2[g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] \tag{6.1}$$

$$+\alpha\mu[g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)].$$

If the Ricci tensor S is of Codazzi type, then

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{6.2}$$

In view of (6.1), (6.2) takes the form

$$\alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \alpha\mu[g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)] = 0$$

which by putting $X = \xi$ and using (2.1), (2.3)-(2.5) gives

$$\alpha g(\phi Y,\phi Z) - \mu g(\phi Y,Z) = 0, \qquad \alpha \neq 0. \tag{6.3}$$

Now by replacing Z by ϕZ in (6.3) and using (2.2), we find

$$\alpha g(\phi Y,Z) - \mu g(\phi Y,\phi Z) = 0. \tag{6.4}$$

By adding (6.3) and (6.4), we obtain $\mu = \alpha$ and hence from (3.6) we get $\lambda = \alpha - (n-1)\alpha^2$. Thus we have the following:

Theorem 6.1. *Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and if the manifold has Ricci tensor of Codazzi type, then $\lambda = \alpha - (n-1)\alpha^2$ and $\mu = \alpha$.*

Definition 6.2. A Lorentzian $\alpha-$Sasakian manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{6.5}$$

for all $X, Y, Z \in \chi(M)$.

Let (g, ξ, λ, μ) be an $\eta-$Ricci soliton in an n-dimensional Lorentzian $\alpha-$Sasakian manifold and the manifold has cyclic parallel Ricci tensor, then (6.5) holds. Taking covariant derivative of (3.3) and making use of (2.7) and (2.14), we find

$$(\nabla_X S)(Y,Z) = \alpha^2[g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] \\ +\alpha\mu[g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y)]. \tag{6.6}$$

Similarly, we have

$$(\nabla_Y S)(Z,X) = \alpha^2[g(Y,Z)\eta(X) - g(Y,X)\eta(Z)] \\ +\alpha\mu[g(\phi Y,Z)\eta(X) + g(\phi Y,X)\eta(Z)], \tag{6.7}$$

and

$$(\nabla_Z S)(X,Y) = \alpha^2[g(Z,X)\eta(Y) - g(Z,Y)\eta(X)] \\ +\alpha\mu[g(\phi Z,X)\eta(Y) + g(\phi Z,Y)\eta(X)]. \tag{6.8}$$

By using (6.6)-(6.8) in (6.5), we obtain

$$\alpha\mu[g(\phi X,Y)\eta(Z) + g(\phi X,Z)\eta(Y) + g(\phi Y,Z)\eta(X)] = 0$$

which by taking $Z = \xi$ reduces to

$$\alpha\mu g(\phi X,Y) = 0. \tag{6.9}$$

Since the manifold under consideration is non-cosymplectic and $g(\phi X,Y) \neq 0$, in general, therefore (6.9) yields $\mu = 0$. Therefore the $\eta-$Ricci soliton becomes Ricci soliton. Thus we have the following:

Theorem 6.2. *An $\eta-$Ricci soliton in a non-cosymplectic Lorentzian $\alpha-$Sasakian manifold whose Ricci tensor is of Codazzi-type becomes a Ricci soliton.*

7. η-Ricci solitons on recurrent Lorentzian α-Sasakian manifolds

Definition 7.1. An n-dimensional Lorentzian α-Sasakian manifold is said to be recurrent if there exists a non-zero 1-form A such that [8]

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W \tag{7.1}$$

for all vector fields X, Y, Z and W on M. If the 1-form A vanishes, then the manifold reduces to a symmetric manifold.

Assume that M is a recurrent Lorentzian α-Sasakian manifold. Therefore the curvature tensor of the manifold satisfies (7.1). By a suitable contraction of (7.1), we get

$$(\nabla_X S)(Z,W) = A(X)S(Z,W). \tag{7.2}$$

This implies that

$$\nabla_X S(Z,W) - S(\nabla_X Z,W) - S(Z,\nabla_X W) = A(X)S(Z,W) \tag{7.3}$$

which by taking $W = \xi$ and using (2.6) and (2.12) yields

$$S(Z,\phi X) = (n-1)\alpha^2 g(\phi X,Z) + (n-1)\alpha A(X)\eta(Z), \qquad \alpha \neq 0. \tag{7.4}$$

In view of (3.3), (7.4) takes the form

$$\alpha g(X,Z) + \alpha\eta(X)\eta(Z) = [\lambda + (n-1)\alpha^2]g(\phi X,Z) + (n-1)\alpha A(X)\eta(Z). \tag{7.5}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (7.5), we have

$$\alpha g(X,Z) = [\lambda + (n-1)\alpha^2]g(\phi X,Z) + (n-2)\alpha\eta(X)\eta(Z). \tag{7.6}$$

By replacing Z by ϕZ in (7.6), we get

$$\alpha g(X,\phi Z) = [\lambda + (n-1)\alpha^2]g(\phi X,\phi Z) \tag{7.7}$$

which by replacing X by ϕX and using (2.2), becomes

$$\alpha g(\phi X,\phi Z) = [\lambda + (n-1)\alpha^2]g(X,\phi Z). \tag{7.8}$$

By adding (7.7) and (7.8), we obtain $\lambda = -(n-1)\alpha^2 - \alpha$ and hence from (3.6) we get $\mu = -\alpha$. Thus we can state the following:

Theorem 7.1. *If (g,ξ,λ,μ) is an $\eta-$Ricci soliton in an n-dimensional recurrent Lorentzian α-Sasakian manifold, then $\lambda = -(n-1)\alpha^2 - \alpha$ and $\mu = -\alpha$.*

Now from the relations (3.3), (3.6) and (7.7), we obtain

$$S(X,Y) = (n-1)\alpha^2 g(X,Y). \tag{7.9}$$

Thus we have

Corollary 7.1. *An n-dimensional recurrent Lorentzian α-Sasakian manifold admitting an η-Ricci soliton (g,ξ,λ,μ) is an Einstein manifold.*

Example. We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3 : z > 0\}$, where (x,y,z) are the standard coordinates of R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = e^{-z}\frac{\partial}{\partial y}, \quad e_2 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \alpha\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point p of M. Let g be the Lorentzian like (semi-Riemannian) metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi, \xi) = -1, \quad \phi^2 X = X + \eta(X)\xi \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M.
Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1, \quad [e_2, e_3] = \alpha e_2.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = \alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \alpha e_1, \quad \nabla_{e_2} e_1 = 0, \tag{7.10}$$

$$\nabla_{e_2} e_2 = \alpha e_3, \quad \nabla_{e_2} e_3 = \alpha e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Also, one can easily verify that

$$\nabla_X \xi = -\alpha\phi X \quad \text{and} \quad (\nabla_X \phi)Y = \alpha g(X, Y)\xi - \alpha\eta(Y)X.$$

Therefore, the manifold is a Lorentzian α-Sasakian manifold. From the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1, e_2)e_1 = -\alpha^2 e_2, \quad R(e_1, e_3)e_1 = -\alpha^2 e_3, \quad R(e_2, e_3)e_1 = 0, \tag{7.11}$$

$$R(e_1, e_2)e_2 = \alpha^2 e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = -\alpha^2 e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = -\alpha^2 e_1, \quad R(e_2, e_3)e_3 = -\alpha^2 e_2$$

from which it is clear that

$$R(X, Y)Z = \alpha^2[g(Y, Z)X - g(X, Z)Y]. \tag{7.12}$$

Hence the manifold (M, ϕ, ξ, g) is a Lorentzian α-Sasakian manifold of constant curvature α^2. With the help of the above results we get the components of Ricci tensor and scalar curvature as follows:

$$S(e_1, e_1) = S(e_2, e_2) = 2\alpha^2, \quad S(e_3, e_3) = -2\alpha^2, \tag{7.13}$$

Therefore, $r = \sum_{i=1}^{3} \epsilon_i S(e_i, e_i) = 6\alpha^2$, where $\epsilon_i = g(e_i, e_i)$. From the equation (3.3) and (7.13), we obtain $\lambda = \alpha(1 - 2\alpha)$ and $\mu = \alpha$. Thus the data (g, ξ, λ, μ) for $\lambda = \alpha(1 - 2\alpha)$ and $\mu = \alpha$ defines an $\eta-$Ricci soliton on (M, ϕ, ξ, η, g).

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LEFT INVARIANT (α, β)-METRICS ON 4-DIMENSIONAL LIE GROUPS

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Abstract. Let G be a 4-dimensional Lie group with an invariant para-hypercomplex structure and let $F = \beta + a\alpha + \beta^2/\alpha$ be a left invariant (α, β)-metric, where α is a Riemannian metric and β is a 1-form on G, and a is a real number. We prove that the flag curvature of F with parallel 1-form β is non-positive, except in Case 2, in which F admits both negative and positive flag curvature. Then, we determine all geodesic vectors of (G, F).
Keywords: para-hypercomplex structure; (α, β)-metric; Riemannian metric; flag curvature.

1. Introduction

Hypercomplex and para-hypercomplex structures are interesting and practical structures in differential geometry [13]. These structures have been used in theoretical physics and HKT-geometry, intensively [11]. According to V. V. Cortés and C. Mayer studies, the para-hypercomplex structures emerged as target manifold of hypermultiplets in Euclidean theories with rigid $N = 2$ supersymmetry [9]. M. L. Barberis classified the invariant hypercomplex structures on a simply-connected 4-dimensional real Lie group [3, 5]. In [6], N. Blažić and S. Vukmirović classified 4-dimensional Lie algebras admitting a para-hypercomplex structure.

Finsler geometry has many applications in mechanics, physics and biology [1]. Among Finsler metrics, (α, β)-metrics, which were first introduced by M. Matsumoto, are the important ones [16].
In [20] the third author introduced a new class of (α, β)-metrics given by $F = \beta + a\alpha + \beta^2/\alpha$ where $a \in (\frac{1}{4}, \infty)$ and studied the locally dually flatness for this type of metrics [21]. One of the key quantities in Riemannian geometry is the sectional curvature. In Finsler geometry, we have the notion of flag curvature as a natural extension of the notion of the sectional curvature [2].

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In the present study, we consider the left invariant 4-dimensional para-hypercomplex Lie groups and construct some Berwaldian left invariant (α, β)-metrics of type $F = \beta + a\alpha + \beta^2/\alpha$ on them. We get a formula for the flag curvature of F and prove that F is non- positive flag curvature except one case, consequently, F is not of constant Ricci curvature.

Let (G, α) be a Lie group G furnished by a left invariant Riemannian metric α. There is a natural kind of geodesics of (G, α) which are closely related to the algebraic ingredient of G. More precisely, we are interested in those geodesics which are in the form $\gamma(t) = exp(tX)$ for some tangent vector X in the Lie algebra of G, i.e., $\mathfrak{g} := T_eG$. In other words, those geodesics which are orbits of one parameter subgroups of G. In this case, X is called a geodesic vector. This notion was extended to Finsler geometry by Latifi [14]. Here, we also obtain all geodesic vectors of the invariant (α, β)-metric $F = \beta + a\alpha + \beta^2/\alpha$.

2. Preliminaries

Let us recall some known facts about para-hypercomplex structures and Finsler spaces. Let M be a smooth manifold and $\{J_i\}_{i=1,2,3}$ be a family of fiberwise endomorphisms of TM such that

$$J_1^2 = -Id_{TM}, \tag{2.1}$$

$$J_2^2 = Id_{TM}, \qquad J_2 \neq \pm Id_{TM}, \tag{2.2}$$

$$J_1J_2 = -J_2J_1 = J_3, \tag{2.3}$$

and

$$N_i = 0 \qquad i = 1, 2, 3, \tag{2.4}$$

where N_i is the Nijenhuis tensor corresponding to J_i defined as follows:

$$N_1(X, Y) = [J_1X, J_1Y] - J_1\big([X, J_1Y] + [J_1X, Y]\big) - [X, Y],$$

and

$$N_i(X, Y) = [J_iX, J_iY] - J_i\big([X, J_iY] + [J_iX, Y]\big) + [X, Y], \qquad i = 2, 3,$$

for all vector fields X, Y on M. A para-hypercomplex structure on a smooth manifold M is a triple $\{J_i\}_{i=1,2,3}$ such that J_1 is a complex structure and J_i, $i = 2, 3$, are two non-trivial integrable product structures on M satisfying (2.3).

Definition 2.1. A para-hypercomplex structure $\{J_i\}_{i=1,2,3}$ on a Lie group G is said to be left invariant if for any $a \in G$ the following diagram is commutative:

$$\begin{tikzcd} TG \arrow[r, "TL_a"] \arrow[d, "J_i"'] & TG \arrow[d, "J_i"] \\ TG \arrow[r, "TL_a"'] & TG \end{tikzcd}$$

That is

$$J_i = TL_a \circ J_i \circ TL_{a^{-1}}, \qquad i = 1, 2, 3$$

where $L_a : G \to G$ given by $L_a(x) = ax$ is the left translation along a and TL_a is its derivation.

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of the tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0} = g_{ij}(x, y)u^i v^j,$$
$$u = u^i \frac{\partial}{\partial x^i}, v = v^j \frac{\partial}{\partial x^j} \in T_xM,$$

where $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is called the fundamental tensor of F.

An important class of Finsler metrics is the class of (α, β)-metrics which was first introduced by M. Matsumoto in 1992 [16]. An (α, β)-metric on a manifold M is a Finsler metric with the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where $\alpha(x, y) = \sqrt{g_{ij}(x)y^i y^j}$, $\beta(x, y) = b_i(x)y^i$ is a Riemannian metric and a 1-form on the manifold M, respectively and $\phi : (-b_0, b_0) \to \mathbb{R}^+$ is a C^∞ function satisfying

$$\phi(s) - s\phi'(s) > 0, \ \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \tag{2.5}$$

for all $|s| \le b < b_0$ in which $b := \|\beta\|$ denotes the norm of β with respect to α (see [17], [26] and [28]).

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il}\Big\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\Big\}y^j y^k \tag{2.6}$$

is called the associated spray to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a *geodesic* in M. For Riemannian metrics, $G^i(x, y)$ are quadratic with respect to y. For a general Finsler metric F, we define the Berwald curvature of F by

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{2.7}$$

A Finsler metric is called Berwald metric if its Berwald curvature vanishes [18].

A Finsler metric F on a Lie group G is called left invariant if for all $a \in G$ and $Y \in T_aG$

$$F(a, Y) = F(e, (L_{a^{-1}})_{*a}Y). \tag{2.8}$$

One of the main quantities in Finsler geometry is the flag curvature which is defined as follows:

$$K(P,Y)=\frac{g_Y(R(U,Y)Y,U)}{g_Y(Y,Y).g_Y(U,U)-g_Y^2(Y,U)}, \tag{2.9}$$

where $P=span\{U,Y\}$ is a 2-plane in T_xM, $R(U,Y)Y=\nabla_U\nabla_Y Y-\nabla_Y\nabla_U Y-\nabla_{[U,Y]}Y$and ∇ is the Chern connection induced by F (for more details, see [4, 25]).

In [6], N. Blažić and S. Vukmirović classified 4-dimensional Lie algebras admitting left invariant para-hypercomplex structures. H. R. Salimi Moghaddam obtained some curvature properties of left invariant Riemannian metrics on such Lie groups [23]. In each case, let G_i be the connected 4-dimensional Lie group corresponding to the considered Lie algebra $\mathfrak{g}_i$ and $\langle,\rangle$ is an inner product on $\mathfrak{g}_i$ such that $\{X,Y,Z,W\}$ is an orthonormal basis for $\mathfrak{g}_i$. Additionally, we use g for the left invariant Riemannian metric on G_i induced by $\langle,\rangle$ and use ∇ for its Levi-Civita connection. Let us denote the Riemannian curvature tensor of g by R. Furthermore, suppose that $U=aX+bY+cZ+dW$and $V=\tilde{a}X+\tilde{b}Y+\tilde{c}Z+\tilde{d}W$ are any two independent vectors in $\mathfrak{g}_i$.
Now, we list all five classes of 4-dimensional Lie algebras admitting an invariant para-hypercomplex structure and non-zero parallel vector fields. These classes of Lie algebras were first introduced in [6].

Case 1. [6] Let $\mathfrak{g}_1$ be the Lie algebra spanned by the basis $\{X,Y,Z,W\}$ with the following Lie algebra structure:

$$[X,Y]=Y, \qquad [X,W]=W. \tag{2.10}$$

Hence, using Koszula's formula, we have

Table 2.1: Taken from [23]

	X	Y	Z	W
∇_X	0	0	0	0
∇_Y	$-Y$	X	0	0
∇_Z	0	0	0	0
∇_W	$-W$	0	0	X

Therefore, for U and V we have

$$R(V,U)U=(a\tilde{b}-b\tilde{a})(bX-aY)+(a\tilde{d}-d\tilde{a})(dX-aW)+(b\tilde{d}-d\tilde{b})(dY-bW).$$

Case 2. [6] The Lie algebra of Case 2 has the following Lie bracket:

$$[X,Y]=Z. \tag{2.11}$$

Therefore

Table 2.2: Taken from [23]

	X	Y	Z	W
∇_X	0	$\frac{1}{2}Z$	$-\frac{1}{2}Y$	0
∇_Y	$-\frac{1}{2}Z$	0	$\frac{1}{2}X$	0
∇_Z	$-\frac{1}{2}Y$	$\frac{1}{2}X$	0	0
∇_W	0	0	0	0

Hence for U and V we have

$$R(V,U)U = \frac{3}{4}(\tilde{a}b - b\tilde{a})(bX - aY) + \frac{1}{4}(a\tilde{c} - c\tilde{a})(aZ - cX) + \frac{1}{4}(b\tilde{c} - c\tilde{b})(bZ - cY).$$

Case 3. [6] The Lie algebra structure of g_3 is in the following form:

$$[X,Y] = X. \tag{2.12}$$

Hence,

Table 2.3: Taken from [23]

	X	Y	Z	W
∇_X	$-Y$	X	0	0
∇_Y	0	0	0	0
∇_Z	0	0	0	0
∇_W	0	0	0	0

as a result, for U and V we have

$$R(V,U)U = (a\tilde{b} - b\tilde{a})(bX - aY).$$

Case 4. [6] In the Lie algebra structure of Case 4, there are two real parameters λ and η. This Lie algebra has the following structure:

$$[X,Z] = X, \qquad [X,W] = Y, \qquad [Y,Z] = Y, \qquad [Y,W] = \lambda X + \eta\beta Y, \qquad \lambda,\eta \in \mathbb{R},$$

thus

Table 2.4: Taken from [23]

	X	Y	Z	W
∇_X	$-Z$	$\frac{-(1+\lambda)}{2}W$	X	$\frac{1+\lambda}{2}Y$
∇_Y	$\frac{-(1+\lambda)}{2}W$	$-(Z+\eta W)$	Y	$\frac{(1+\lambda)}{2}X+\eta Y$
∇_Z	0	0	0	0
∇_W	$\frac{\lambda-1}{2}Y$	$\frac{1-\lambda}{2}X$	0	0

Therefore, we have

$$
\begin{aligned}
R(V,U)U &= -\Big\{(a\tilde{b}-b\tilde{a})\Big(b\frac{(1+\lambda)^2-4}{4}X+a\frac{4-(1+\lambda)^2}{4}Y\Big)\\
&+ (a\tilde{c}-c\tilde{a})\Big(aZ+b\frac{1+\lambda}{2}W-cX-d\frac{1+\lambda}{2}Y\Big)+(a\tilde{d}-d\tilde{a})\\
&\times \Big(a\frac{-\lambda^2+2\lambda+3}{4}W+b\frac{1+\lambda}{2}Z+b\eta W-c\frac{1+\lambda}{2}Y\\
&+ d\frac{(1+\lambda)(\lambda-3)}{4}X-d\eta Y\Big)\\
&+ (b\tilde{c}-c\tilde{b})\Big(a\frac{1+\lambda}{2}W+bZ+b\eta W-cY-d\frac{1+\lambda}{2}X-d\beta Y\Big)\\
&+ (b\tilde{d}-d\tilde{b})\Big(a\frac{1+\lambda}{2}Z+a\eta W+b\eta Z+b\frac{3\lambda^2+4\eta^2+2\lambda-1}{4}W\\
&- c\frac{1+\lambda}{2}X-c\eta Y-d\frac{3\lambda^2+4\eta^2+2\lambda-1}{4}Y-\eta dX\Big)\Big\}.
\end{aligned}\tag{2.13}
$$

Case 5. [6] The last Lie algebra is $\mathfrak{g}_5$ with the following Lie algebra structure:

$$[X,Y]=W,\qquad [X,W]=-Y,\qquad [Y,W]=-X.\tag{2.14}$$

Thus

Table 2.5: Taken from [23]

	X	Y	Z	W
∇_X	0	$\frac{3}{2}W$	X	$-\frac{3}{2}Y$
∇_Y	$\frac{1}{2}W$	$-W$	0	$-\frac{1}{2}X$
∇_Z	0	0	0	0
∇_W	$-\frac{1}{2}Y$	$\frac{1}{2}X$	0	0

Thus for U and V we have

$$R(V,U)U = -\frac{1}{4}\{(a\tilde{b}-b\tilde{a})(bX-aY)+(a\tilde{d}-d\tilde{a})(dX-aW)+7(b\tilde{d}-d\tilde{b})(-dY+bW)\}.$$

3. Flag curvature of $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$

Let us give a formula for the fundamental tensor of invariant (α,β)-metrics of type $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$, where $a\in(\frac{1}{4},\infty)$ and α is a left invariant Riemannian metric on a 4-dimensional Lie group G. We consider a left invariant vector field B on G and we let β be the 1-form associated to B with respect to α, that is, for any $x\in G$ and $y\in T_xG$, $\beta_x(y)=\alpha_x(B(x),y)$. Moreover, in the reminder of this section, we require B to be parallel with respect to α, i.e., $\nabla_B B=0$, where ∇ is the Levi-Civita connection of α. It is known that in this case, the Chern connection of F coincides to the Levi-Civita connection of α, hence F is a Berwald metric [1].

For any non- zero tangent vector $Y\in T_xM$, denote the fundamental tensors of F and α by g_Y and g, respectively. By definition, we have

$$(3.1)\qquad g_Y(U,V):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(Y+sU+tV)\Big]|_{s,t=0},\quad U,V\in T_xM.$$

$$(3.2)\qquad g(U,V):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[\alpha^2(Y+sU+tV)\Big]|_{s,t=0},\quad U,V\in T_xM.$$

It is easy to see that

$$\begin{aligned}
g_Y(U,V) \;&=\; \frac{4g(X,Y)^4g(U,Y)g(V,Y)}{g(X,Y)^3}+\frac{3g(X,Y)^3g(U,Y)g(V,Y)}{g(Y,Y)^{\frac{5}{2}}}\\
&-\; \frac{g(X,Y)^3}{g(Y,Y)^2}\Big(g(X,Y)g(U,V)+4g(X,U)g(V,Y)+4g(X,V)g(U,Y)\Big)\\
&-\; \frac{g(X,Y)}{g(Y,Y)^{\frac{3}{2}}}\Big(ag(U,Y)g(V,Y)+g(X,Y)^2g(U,V)+3g(X,Y)g(X,U)g(V,Y)\\
&+\; 3g(X,Y)g(X,V)g(U,Y)\Big)+\frac{6}{g(Y,Y)}\Big(g(X,Y)^2g(X,U)g(X,V)\Big)\\
&+\; \frac{1}{\sqrt{g(Y,Y)}}\Big(ag(X,Y)g(U,V)+6g(X,Y)g(X,U)g(X,V)+ag(X,U)g(V,Y)\\
(3.3)&+\; {}^ag(X,V)g(U,Y)\Big)+a^2g(U,V)+g(X,U)g(X,V)+2ag(X,U)g(X,V).
\end{aligned}$$

Remark 3.1. We know that $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$ is an (α,β)-metric with $\phi(s)=s^2+s+a$, i.e., $F=\alpha\phi(\frac{\beta}{\alpha})$. By applying the formula obtained by Z. Shen [7], we can also get the formula of $g_Y(U,V)$. Indeed, we have

$$\begin{aligned} g_Y(U,V) &= \phi^2(s)g(U,V)+\phi(s)\phi^{'}(s)\Big(-sg(U,V)+g(X,U)\frac{g(V,Y)}{\sqrt{g(Y,Y)}} \\ &+ g(X,V)\frac{g(U,Y)}{\sqrt{g(Y,Y)}}-s\frac{g(U,V)g(V,Y)}{g(Y,Y)}\Big) \\ &+ \Big(\phi(s)\phi^{''}(s)+(\phi^{'}(s))^2\Big)\Big(g(X,U)g(X,V)-sg(X,U)\frac{g(V,Y)}{\sqrt{g(Y,Y)}} \\ &- sg(X,V)\frac{g(U,Y)}{\sqrt{g(Y,Y)}}+s^2\frac{g(U,V)g(V,Y)}{g(Y,Y)}\Big), \end{aligned} \tag{3.4}$$

where $s=\frac{g(X,Y)}{\sqrt{g(Y,Y)}}$. It is easy to see that 3.3 and 3.4 coincide.

Let G be a 4-dimensional Lie group admitting an invariant para-hypercomplex structure. As mentioned above, all such Lie groups are classified in [6] [23]. Now we consider the cases 1-5 discussed in [23] and give the explicit formula for their flag curvature in each case. Let $Y:=U$ in (3.4), in all cases.

Case 1. Here, the only left invariant and parallel vector field with respect to α is given by $B=qZ$ with $\frac{1}{4}<|q|<\infty$. Note that here $s=\frac{g(qZ,U)}{\sqrt{g(U,U)}}=cq$, where we have used $g(U,U)=1$. In this case, it follows from (3.4)

$$\begin{aligned} g_U(R(V,U)U,V) &= -\big((\phi(s))^2-s\phi(s)\phi^{'}(s)\big)\Big((a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+(b\tilde{d}-d\tilde{b})^2\Big) \\ g_U(U,U) &= (\phi(s))^2 \\ g_U(V,V) &= \big(\phi(s)\phi^{''}(s)+(\phi^{'}(s))^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi^{'}(s) \\ g_U(U,V) &= \phi(s)\phi^{'}(s)(\tilde{c}q). \end{aligned}$$

Let $P=span\{U,V\}$. In [5], Latifi gives a formula for the flag curvature of a left invariant (α,β)-metric. Using this formula, we get the following

$$K(P,U)=\frac{-\big((\phi(s))^2-s\phi(s)\phi^{'}(s)\big)\big\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+(b\tilde{d}-d\tilde{b})^2\big\}}{(\phi(s))^2\big\{\big(\phi(s)\phi''(s)+(\phi'(s))^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi'(s)\big\}-\big(\phi(s)\phi'(s)(\tilde{c}q)\big)^2}.$$

Hence, $K(P,U)\leqslant 0$. It means that (G,F) has non-positive flag curvature.

Remark 3.2. In [12], L. Huang proved that a left invariant Finsler metric F on a Lie group G admits a direction in which the flag curvature is non-negative, provided $dim[\mathfrak{g},\mathfrak{g}]\leq dim\mathfrak{g}-2$. Thus, Case 1 shows that we can not replace non-negative with positive in Huang's theorem.

Case 2. We see that the only left invariant and parallel vector field with respect to α is given by $X = qW$ with $\frac{1}{4} < |q| < \infty$. Thus $s = \frac{g(qW,U)}{\sqrt{g(U,U)}} = qd$. A similar argument as in the Case 1 yields

$$\begin{aligned} g_(RV, U)U, V) &= \big((\phi(s))^2 - \phi(s)\phi^{'}(s)\big)\big\{-\frac{3}{4}(a\tilde{b} - b\tilde{a})^2 + \frac{1}{4}(a\tilde{c} - c\tilde{a})^2 + \frac{1}{4}(b\tilde{c} - c\tilde{b})^2\big\} \\ g_U(U, U) &= (\phi(s))^2 \\ g_U(V, V) &= \big(\phi(s)\phi^{''}(s) + (\phi^{'}(s))^2\big)(\tilde{d}q)^2 + (\phi(s))^2 - dq\phi(s)\phi^{'}(s) \\ g_U(U, V) &= \phi(s)\phi^{'}(s)(\tilde{d}q). \end{aligned}$$

We obtain the flag curvature as follows:

$$K(P, U) = \frac{\big((\phi(s))^2 - \phi(s)\phi^{'}(s)\big)\big\{-\frac{3}{4}(a\tilde{b} - b\tilde{a})^2 + \frac{1}{4}(a\tilde{c} - c\tilde{a})^2 + \frac{1}{4}(b\tilde{c} - c\tilde{b})^2\big\}}{\phi^2(s)\big\{\big(\phi(s))^2 - dq\phi(s)\phi'(s) + (\phi(s)\phi''(s) + (\phi'(s))^2\big)(\tilde{d}q)^2\big\} - \big(\phi(s)\phi'(s)(\tilde{d}q)\big)^2}.$$

Unlike Case 1, in this case (G, F) admits both positive and negative flag curvature.

Case 3. According to [23], (G_3, g) admits a parallel left invariant vector field $X = q_1Z + q_2W$ such that $\frac{1}{4} < |q_1^2 + q_2^2| < \infty$. As in the previous cases, we get $s = cq_1 + dq_2$.

$$\begin{aligned} g_U(RV, U)U, V) &= -\big(\phi(s))^2 - s\phi(s)\phi^{'}(s)\big)\big(a\tilde{b} - b\tilde{a}\big)^2 \\ g_U(U, U) &= (\phi(s))^2 \\ g_U(V, V) &= \big(\phi(s)\phi^{''}(s) + (\phi^{'}(s))^2\big)(\tilde{c}q_1 + \tilde{d}q_2) + (\phi(s))^2 - \big(cq_1 + dq_2\big)\phi(s)\phi^{'}(s) \\ g_U(U, V) &= \phi(s)\phi^{'}(s)\big(\tilde{c}q_1 + \tilde{d}q_2\big) \end{aligned}$$

Therefore, the flag curvature of F is as follows:

$$K(P, U) = \frac{-\big\{\phi^2(s) - s\phi(s)\phi^{'}(s)\big\}(a\tilde{b} - b\tilde{a})^2}{\Psi}, \tag{3.5}$$

where
$\Psi := \phi^2\{(\phi\phi^{''} + \phi'^2)(\tilde{c}q_1 + \tilde{d}q_2) + \phi^2 - (cq_1 + dq_2)\phi\phi^{'}\} - (\phi\phi^{'}(\tilde{c}q_1 + \tilde{d}q_2))^2 \geqslant 0.$

Case 4. In [23], it has been shown that vector fields which are parallel to (G_4, g), are of the form $X = qW$ such that $\frac{1}{4} < |q| < \infty$. Thus $s = dq$ and we have:

$$\begin{aligned} g_U(RV, U)U, V) &= -((\phi(s))^2 - s\phi(s)\phi^{'}(s))((a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2) \\ g_U(U, U) &= (\phi(s))^2 \\ g_U(V, V) &= \big(\phi(s)\phi^{''}(s) + (\phi^{'}(s))^2\big)\big(\tilde{d}q)^2 + (\phi(s))^2 - dq\phi(s)\phi^{'}(s)\big) \\ g_U(U, V) &= \phi(s)\phi^{'}(s)(\tilde{d}q). \end{aligned}$$

We have the flag curvature of F as follows:

$$K(P,U)=\frac{\big(-(\phi(s))^2-dq\phi(s)\phi^{'}(s)\big)\{(a\tilde{c}-c\tilde{a})^2+(b\tilde{c}-c\tilde{b})^2\}}{(\phi(s))^2\{\big(\phi(s)\phi''(s)+\phi'(s)^2\big)(\tilde{d}q)^2+\phi^2(s)-dq\phi(s)\phi'(s)\}-(\phi(s)\phi'(s)(\tilde{d}q))^2},$$

which are always non-positive.

Case 5. In [23], it has been shown that the parallel left invariant vector fields are of the form $X=qZ$ such that $\frac{1}{4}<|q|<\infty$. Thus $s=cq$ and we get:

$$\begin{aligned}
g_U(RV,U)U,V) &= -\frac{1}{4}\big((\phi(s))^2-cq\phi(s)\phi^{'}(s)\big)\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+7(b\tilde{d}-d\tilde{b})^2\}\\
g_U(U,U) &= (\phi(s))^2\\
g_U(V,V) &= \big(\phi(s)\phi^{''}(s)+(\phi^{'}(s))^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi^{'}(s)\\
g_U(U,V) &= \phi(s)\phi^{'}(s)(\tilde{c}q),
\end{aligned}$$

Moreover, the flag curvature is given by the following:

$$K(P,U)=\frac{-\frac{1}{4}\big((\phi(s))^2-cq\phi(s)\phi^{'}(s)\big)\{(a\tilde{b}-b\tilde{a})^2+(a\tilde{d}-d\tilde{a})^2+7(b\tilde{d}-d\tilde{b})^2\}}{(\phi(s))^2\big\{\big(\phi(s)\phi''(s)+\phi'(s)^2\big)(\tilde{c}q)^2+(\phi(s))^2-cq\phi(s)\phi'(s)\big\}-\big(\phi(s)\phi'(s)(\tilde{c}q)\big)^2}.$$

which are always non-positive.

Sumarizing the above results, we get the following.

Theorem 3.1. *In all above cases, except for the Case 2, the flag curvature of F is non-positive. Moreover, in Case 2, (G,F) admits both positive and negative flag curvature.*

Remark 3.3. In [10], S. Deng proved that if a G-invariant Randers metric $F=\alpha+\beta$ on a homogeneous manifold $\frac{G}{H}$, which is Douglas type, has negative flag curvature, then the sectional curvature of α is negative. Case 5 shows that this fact is no longer true for (α,β)-metric of type $F=\beta+a\alpha+\frac{\beta^2}{\alpha}$.

4. Geodesic vectors

In this section, we discuss the geodesic vectors of a left invariant Finsler metric $F=\beta+\frac{a\alpha^2+\beta^2}{\alpha}$ on a 4-dimensional Lie group G admitting an invariant para-hypercomplex structure. We still asume that β is parallel with respect to α. Let us recall the definition of geodesic vectors.

Definition 4.1. Let F be a left invariant Finsler metric on a Lie group G. A non-zero tangent vector $B \in T_eG$ is said to be a geodesic vector of F, if the 1-parameter subgroup $t \longrightarrow exp(tB)$, $t \in \mathbb{R}_+$ is a geodesic of F.

To find all geodesic vectors of a left invariant Finsler metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ on a 4-dimensional Lie group G admitting an invariant para-hypercomplex structure, we need the following propositions.

Proposition 4.1. *(see [14]) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant Finsler metric on G. Then a non-zero vector $B \in \mathfrak{g}$ is a geodesic vector of F if and only if for every $Z \in \mathfrak{g}$*

$$g_B([B, Z], B) = 0, \tag{4.1}$$

Proposition 4.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant (α, β)- Berwald Finsler metric on G. Then a non-zero vector $B \in \mathfrak{g}$ is a geodesic vector of F if and only if it is a geodesic vector of α.*

Now, we find all geodesic vectors in each case of all five classes given in [23], while they equipped with Left invariant Finsler metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$. Using Proposition 4.1 and 4.2, we obtain all geodesic vectors of $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ as follows.

Theorem 4.1. *The geodesic vectors of left invariant finsler metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ are given by the following*

	geodesic vectors
Case 1	$\{aX + cZ \mid a, c \in \mathbb{R}\}$
Case 2	$\{aX + bY + cZ + dW \mid bc = ac = 0\}$
Case 3	$\{bY + cZ + dW \mid b, c, d \in \mathbb{R}\}$
Case 4	$\{aX + cZ + dW \mid ac = ad\lambda = 0\}$
Case 5	$\{aX + bY + cZ + dW \mid ad = ab = 0\}$

Now, we obtain a relation between the geodesic vectors of a general (α, β)-metric F and a Riemannian metric g.

Theorem 4.2. *Let G be a Lie group and $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β)-metric arising from a Riemannian metric g and a left invariant vector filed B, i.e., $\alpha(x, y) = \sqrt{g_x(y, y)}$ and $\beta(x, y) = \alpha_x(B, y)$ Suppose that $Y \in \mathfrak{g}$ is a unit vector for which $g(B, [Y, Z]) = 0$ for all $Z \in \mathfrak{g}$. Then Y is a geodesic vector of (M, F) if and only if Y is a geodesic vector of (M, g).*

Proof. Using (3.3) and taking into account $g(B, [Y, Z]) = 0$ for all $Z \in \mathfrak{g}$, we have

$$g_Y(Y, [Y, Z]) = \Big(\phi^2(s) - \phi(s)\phi'(s)\frac{g(B,Y)}{\sqrt{g(Y,Y)}}\Big)g(Y, [Y, Z]), \tag{4.2}$$

Let Y be a geodesic vector of g. Replacing (4.1) into (4.2) and using $g(B,[Y,Z]) = 0$, we have Y is a geodesic vector of (M,F).
Conversely, let Y be a unit geodesic vector of (M,F). We have

$$\Big(\phi^2(s) - \phi(s)\phi'(s)g(X,Y)\Big)g(Y,[Y,Z]) = 0, \tag{4.3}$$

This completes the proof. □

Theorem 4.3. *Let (G,F) be a connected Lie group and $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ be a left- invariant Finsler metric of Berwald type on G defined by the Riemannian metric α and the vector field B. Then (G,F) is complete.*

Proof. Since F is of the Berwald type then (G,F) and (G,α) have the same connection also $\nabla B = 0$ where ∇ is Riemannian connection of α. On the other hand (G,α) is a Lie group and hence a complete space. As (G,F) and (G,α) have the same geodesics. We show that these geodesics have constant Finsler speed. Let $\sigma(t)$, $-\infty < t < \infty$ be a geodesic for F , we have

$$F(\sigma(t),\dot\sigma(t)) = g_{\sigma(t)}(B,\dot\sigma(t)) + a\sqrt{g_{\sigma(t)}(\dot\sigma(t)),\dot\sigma(t))} + \frac{g^2_{\sigma(t)}(B,\dot\sigma(t))}{\sqrt{g_{\sigma(t)}(\dot\sigma(t)),\dot\sigma(t))}}$$

Since $g_{\sigma(t)}(\dot\sigma(t),\dot\sigma(t))$ is constant, it is enough to show that $g_{\sigma(t)}(B,\dot\sigma(t))$ is also constant. we have

$$\frac{d}{dt}(g_{\sigma(t)}(B,\dot\sigma(t)) = g_{\sigma(t)}(\nabla_{\dot\sigma(t)}B,\dot\sigma(t)) + g_{\sigma(t)}(B,\nabla_{\dot\sigma(t)}\dot\sigma(t)) = 0 \tag{4.4}$$

Then this yields that these geodesics have constant Finsler speed. □
The following Proposition can be found in [14].

Proposition 4.3. *Let (M,F) be a forward geodesically complete Finsler manifold. If X is a vector field such that $F(X)$ is bounded, then X is a forward complete vector field.*

Using Proposition 4.3, we get the following.

Theorem 4.4. *Let (G,F) be a connected Lie group and $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ be a left- invariant Finsler metric of Berwald type. Then the vector field B is complete.*

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TWO NOTABLE CLASSES OF PROJECTIVE VECTOR FIELDS

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Abstract. Here, we find some necessary conditions for a projective vector field on a Randers metric to preserve the non-Riemannian quantities Ξ and H. They are known in the contexts as the C-projective and H-projective vector fields. We find all projective vector fields of the Funk type metrics on the Euclidean unit ball $\mathbb{B}^n(1)$.
Keywords: projective vector field; Randers metric; Funk type metrics; Euclidean unit ball.

1. Introduction

Beltrami [3] introduced the first examples of projective transformations. The projective Finsler geometry is much complicated than the projective Riemannian geometry. This complexity may impresses several projective properties which used to be proved in projective Riemannian geometry, for example Beltrami's theorem in Riemannian geometry states that the projective (i.e., locally projectively flat). Riemannian metrics are exactly those with constant sectional curvature, while this fails generally within Finsler geometry. This may even affect on the projective algebra (i.e. the lie algebra of the projective vector fields) $proj(M, F)$ and its subalgebras. To trace this, we may refer to the subalgebras of $proj(M, F)$ [7, 8]. The special projective algebra $sproj(M, F)$ consists of the projective vector fields preserving the Berwald curvature. In [11], it is proved that given any special projective vector field X on a Randers space $(M, F = \alpha + \beta)$ with the navigation data (h, W), either F is isotropic S-curvature or X is a conformal vector field for the Riemannian metric h. This result supports a Lichnerowicz-Obata type theorem for the special projective vector fields, see [4, 11]. It is guessed that some other Lichnerowicz-Obata type theorems may be established for our two notable projective subalgebras, namely, the C-projective and the H-projective algebras. We would like to examine another subalgebras of $proj(M, F)$ namely, the C-projective algebra and the H-invariant projective algebra $cproj(M, F)$ and $hproj(M, F)$, respectively. The former is shown here to be characterized by preserving the Ξ-curvature and the latter is defined by

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preserving the **H**-curvature. In [9], C-projective vector fields are studied. We prove the following infinitesimally stated results:

Theorem 1.1. *Let us suppose that $(M, F = \alpha+\beta)$ is a Randers space of dimension $n \geq 2$ and X is a C-projective vector field of (M, F). Then, at least one of the statements of (1.1) and (1.2) is held:*

$$(1.1)\qquad \begin{cases} 1.\ \ F\ is\ of\ isotropic\ S\text{-}curvature. \\ or \\ 2.\ \ b^2\alpha^2(\mathcal{L}_{\hat{X}}(\alpha^2-\beta^2) - (\beta^2\mathcal{L}_{\hat{X}}\alpha^2 - \alpha^2\mathcal{L}_{\hat{X}}\beta^2) = \eta(\alpha^2-\beta^2), \end{cases}$$

where η is a polynomial of degree two on TM.

$$(1.2)\qquad \begin{cases} 3.\ \ \mathcal{L}_{\hat{X}}\alpha^2 = \lambda\alpha^2. \\ or \\ 4.\ \ (8e_{00}\beta s_0 - 4e_{00}^2 + e_{00;0}\beta) = \eta\alpha^2, \end{cases}$$

where $\lambda \in C^\infty(M)$ and $\eta(y)$ is a polynomials of degree two.

In the following, we investigate a bigger class of C-projective vector fields, namely, the vector fields which preserve H-curvature.

Theorem 1.2. *Let us suppose that $(M, F = \alpha + \beta)$ be a compact Randers space of dimension $n \geq 2$ and X is a H-projective vector field of (M, F). Then, at least one of the statements of (1.3) and (1.4) is held:*

$$(1.3)\qquad \begin{cases} 1.\ \ F\ is\ of\ isotropic\ S\text{-}curvature. \\ or \\ 2.\ \ (\alpha^2 b^4 + \beta^2)\mathcal{L}_{\hat{X}}(\alpha^2-\beta^2) + 2b^2(\alpha^2\mathcal{L}_{\hat{X}}\beta^2 - \beta^2\mathcal{L}_{\hat{X}}\alpha^2) \\ \qquad\qquad = \eta(x,y)(\alpha^2-\beta^2), \end{cases}$$

where η is a polynomial of degree two on TM and

$$(1.4)\qquad \begin{cases} 3.\ \ \mathcal{L}_{\hat{X}}\alpha^2 = \lambda\alpha^2. \\ or \\ 4.\ \ -192e_{00}\beta s_0 + 96e_{00}^2 - 19e_{00;0}\beta = \lambda\alpha^2, \end{cases}$$

where $\lambda \in C^\infty(M)$ and $\eta(y)$ is a polynomials of degree two.

Here, we study a bigger class of projective transformation, namely, C-projective transformation of Randers space. C-projective algebra, i. e, the algebra of C-projective vector fields on an n-dimensional Finsler space is a sub-algebra of projective algebra, and its dimension is $n(n+2)$ at most. Let $F = \alpha + \beta$ be a Randers space. We find the conditions for a vector field to be a C-projective vector field.

2. Preliminaries

Let M be a smooth and connected manifold of dimension $n \geq 2$. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x\in M} T_xM$. We will denote the elements of TM by (x, y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM_0 \to M$ is given by $\pi(x, y) := x$. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) the Hessian of F^2 with elements $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^iy^j}$ is positive-definite matrix on TM_0. The pair (M, F) is then called a *Finsler space*. Throughout this paper, we denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a 1-form by $\beta = b_i(x)y^i$. A globally defined spray $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(x, y)$ are local functions on TM_0 satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\lambda > 0$ and given by

$$G^i = \frac{1}{4}g^{ik}\{y^h F^2_{x^hy^k} - F^2_{x^k}\}. \tag{2.1}$$

Assume the following conventions:

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$$

The local functions G^i_j are coefficients of a connection in the pullback bundle $\pi^*TM \to M$ which is called the Berwald connection denoted by D. Recall that for instance, the derivatives of a vector field X and a 2-covariant tensor $T = T_{ij}dx^i \otimes dx^j$ are given by:

$$\begin{aligned} X_{i|j} &= \frac{\delta X_i}{\delta x^j} - X_r G^r_{ij} \\ T_{ij|k} &= \frac{\delta T_{ij}}{\delta x^k} - T_{rj}G^r_{ik} - T_{ir}G^r_{jk} \end{aligned} \tag{2.2}$$

where $\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - G^i_k\frac{\partial}{\partial y^i}$. Given a Finsler metric F on an n-dimensional manifold M, *the Busemann-Hausdorff volume form* $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(y^i\frac{\partial}{\partial x^i}|_x) < 1\}}.$$

Define $\underline{g} = \det(g_{ij}(x, y))$ and $\tau(x, y) := \ln\frac{\sqrt{g}}{\sigma_F(x)}$. Given a vector $y \in T_xM$, let $\gamma(t), -\epsilon < t < \epsilon$, denote the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = y$. The function $S(x, y) := \frac{d}{dt}[\tau(\gamma(t), \dot\gamma(t))]_{|t=0}$ is called the S-curvature with respect to Busemann-Hausdorff volume form. A Finsler space is said to be *of isotropic S-curvature* if there is a function $c = c(x)$ defined on M such that $S = (n + 1)c(x)F$. It is called a Finsler space *of constant S-curvature* when c is constant. Let (M, α) be a Riemannian space and $\beta = b_i(x)y^i$ be a 1-form defined on M such that $\|\beta\|_x :=$

$\sup_{y\in T_xM}\beta(y)/\alpha(y)<1$. The Finsler metric $F=\alpha+\beta$ is called a Randers metric on the manifold M. Denote the geodesic spray coefficients of α and F by G^i_α and G^i, respectively. The Levi-Civita covariant derivative of α is denoted by ∇. Define $\nabla_j b_i$ by $(\nabla_j b_i)\theta^j := db_i - b_j\theta_i^{\ j}$, where $\theta^i := dx^i$ and $\theta_i^{\ j} := \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms and ∇ denotes its associated covariant derivation of α. Let us put

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(\nabla_j b_i + \nabla_i b_j),\ s_{ij} := \frac{1}{2}(\nabla_j b_i - \nabla_i b_j),\\ s^i_{\ j} &:= a^{ih}s_{hj},\ s_j := b_i s^i_{\ j},\ e_{ij} := r_{ij} + b_i s_j + b_j s_i. \end{aligned}$$

Then G^i are given by

$$G^i = G^i_\alpha + \Big(\frac{e_{00}}{2F} - s_0\Big)y^i + \alpha s^i_{\ 0}, \tag{2.3}$$

where $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$, $s^i_{\ 0} := s^i_{\ j}y^j$ and G^i_α denote the geodesic coefficients of α.
Given any Randers space $(M, F=\alpha+\beta)$, the S-curvature takes the following form:

$$\mathbf{S} = (n+1)\{\frac{e_{00}}{2F} - s_0 - \rho_0\} \tag{2.4}$$

where $\rho = \ln(\sqrt{1-||\beta||^2_\alpha})$ and $\rho_0 = \rho_{x^i}y^i$. Akbar-Zadeh in [2] studied another non-Riemannian quantity H-curvature, which is invariant by special projective and C-projective algebras [15]. At every point $x\in M$, $\Xi_y = \Xi_i(y)dx^i$ and $H = H_{ij}dx^i\otimes dx^j$ are defined as follows:

$$\Xi_i = y^m S_{.i|m} - S_{|i} \tag{2.5}$$

$$H_{ij} = \frac{1}{2}S_{.i.j|m}y^m = \frac{1}{4}(\Xi_{i.j} + \Xi_{j.i}) \tag{2.6}$$

where "." and "|" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Berwald connection. The quantity Ξ has been introduced by Zhongmin Shen using the S-curvature, cf. [14, 16]. The above quantities do not depend on the choice of connection for performing horizontal derivatives and can be derived for the Finsler metric itself.

The notion of Riemann curvature for Riemann metrics is extended to Finsler metrics. For a vector $y\in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by

$$R_y(u) := R^i_k(y)u^k\frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j\partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := \{R_y\}_{y\in TM_0}$ is called the Riemann curvature. We define the Ricci curvature as the trace of R_y, i.e., $Ric(x,y) := trace(R_y) = (n-1)R^m_m(y)$.

Every vector field X on a manifold M induces naturally an infinitesimal coordinate transformations on TM given by $(x^i, y^i) \to (\overline{x}^i, \overline{y}^i)$, given by

$$\overline{x}^i = x^i + V^i dt, \quad \overline{y}^i = y^i + y^k \frac{\partial V^i}{\partial x^k} dt.$$

Using this coordinates transformation, we may consider the notion of the complete lift $\hat{X}$ of V to a vector field on TM_0 given by

$$\hat{X} = X^i \frac{\partial}{\partial x^i} + y^k \frac{\partial X^i}{\partial x^k} \frac{\partial}{\partial y^i}$$

It is a notable remark on the Lie derivative computations that, $\mathcal{L}_{\hat{X}} y^i = 0$ and the differential operators $\mathcal{L}_{\hat{X}}$, $\frac{\partial}{\partial x^i}$, exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute. The vector field $\hat{X}$ is called a projective vector field, if there is a function P (called the projective factor) on TM_0 such that $\mathcal{L}_{\hat{X}} G^i = P y^i$, cf. [1]. In this case, given any appropriate t, the local flow $\{\phi_t\}$ associated with X is a projective transformation. A projective vector field is said to be affine if $P = 0$. It is well-known that every Killing vector field is affine. Recall that, given any projective vector field X on a Riemannian space, the projective factor $P = P(x, y)$ is linear with respect to y and also it is the natural lift of a closed 1-form on M to a function on TM_0, while in the Finslerian setting these issues are non-Riemannian features. Consider the following conventional definitions of a projective vector field X; X is said to be (cf. [10])

1. special if $\mathcal{L}_{\hat{X}} E = 0$, or equivalently, $P(x, y) = P_i(x) y^i$.
2. C-projective if $P_{i|j} = P_{j|i}$.
3. H-invariant if $\mathcal{L}_{\hat{X}} H = 0$, equivalently, $P_{jk|l} = P_{jl|k}$.

The projective factor P for a projective vector field X on a Riemannian manifold is simultaneously a special and a C-projective vector field. The following theorem provides the equivalence conditions of C-projective vector fields [13]:

Theorem 2.1. *[13] Let $(M, F = \alpha + \beta)$ be a Randers space and V is a projective vector field $V \in \chi(M)$. The following statements are equivalent:*

1. V is C-projective,

2. $\mathcal{L}_V \Xi = 0$,

3. $\mathcal{L}_V \Sigma = 0$.

where $\Sigma = \Sigma_{ij} dx^i \otimes dx^j$ is defined as follows:

$$\Sigma_{ij} = \frac{1}{n+1}(S_{.i|j} - S_{.j|i})$$

Let $\hat{V}$ be a projective vector field. We have the following identities for arbitrary tensors $T_{i|j}$ and $T_{ij|k}$ [1]:

$$\begin{aligned} \mathcal{L}_{\hat{X}} T_{i|j} &= (\mathcal{L}_{\hat{X}} T_i)_{|j} - (\mathcal{L}_{\hat{X}} G^r_i) T_{r.j} - (\mathcal{L}_{\hat{X}} G^r_{ij}) T_r \\ \mathcal{L}_{\hat{X}} T_{ij|k} &= (\mathcal{L}_{\hat{X}} T_{ij})_{|k} - (\mathcal{L}_{\hat{X}} G^r_k) T_{ij.r} - (\mathcal{L}_{\hat{X}} G^r_i) T_{rj} - (\mathcal{L}_{\hat{X}} G^r_j) T_{ir} \end{aligned} \tag{2.7}$$

3. Projective vector field v.s. Ξ-curvature

Let $F = \alpha + \beta$ be a Randers metric. Due to (2.4), Ξ-curvature is directly obtained as follows:

$$\Xi_i = \frac{e_{i0|0}}{F} - \frac{e_{00|0}u_i}{2F^2} - \frac{e_{00|i}}{2F} - \Theta_i \tag{3.1}$$

where Θ_i is $\Theta_i := s_{i|0} - s_{0|i}$.

Using spray coefficients of Randers metric (2.3) and the general formula of horizontal derivatives (2.2) one can obtain the following:

$$\begin{aligned}
e_{00|0} &= e_{00;0} - \frac{4e_{00}^2}{F} + 8e_{00}s_0 - 4\alpha(q_{00} + \beta t_0 + s_0^2)\\
e_{00|i} &= e_{00;i} - \frac{5e_{00}e_{i0}}{F} + 2e_{i0}s_0 + \frac{2e_{00}^2u_i}{F^2} + 4e_{00}s_i\\
&\quad -2\alpha(q_{i0} + \beta t_i + s_0s_i) - 2\alpha_i(q_{00} + \beta t_0 + s_0^2)\\
e_{i0|0} &= e_{i0;0} - \frac{7e_{i0}e_{00}}{2F} + 3e_{i0}s_0 + \frac{e_{00}^2u_i}{F^2} + 2e_{00}s_i\\
&\quad -\alpha_i(q_{00} + \beta t_0 + s_0^2) - \alpha(q_{i0} + \beta t_i + s_0s_i)\\
e_{ij|0} &= e_{ij;0} - 2e_{ij}(\frac{e_{00}}{2F} - s_0) - e_{i0}(\frac{e_{j0}}{F} - \frac{e_{00}u_j}{2F^2} - s_j)\\
&\quad -e_{j0}(\frac{e_{i0}}{F} - \frac{e_{00}u_i}{2F^2} - s_i)\\
&\quad -\alpha_i(q_{j0} + b_jt_0 + s_0s_j) - \alpha_j(q_{i0} + b_it_0 + s_0s_i)\\
&\quad -\alpha(q_{ij} + q_{ji} + b_it_j + b_jt_i + 2s_is_j)
\end{aligned} \tag{3.2}$$

where ";" is the covariant derivatives with respect to α.

By substituting (3.2) in (3.1), we obtain:

$$\begin{aligned}
\Xi_i &= \frac{1}{2F^3}\big[(-e_{00;i} + 2e_{i0;0} - 4e_{i0}s_0)F^2 - u_i(-4(\beta t_0 + s_0^2 + q_{00})\alpha + 8s_0e_{00} + e_{00;0})F\\
&+ 4e_{00}^2u_i + 2e_{i0}e_{00}F + 4F^2\alpha(b_it_0 + s_is_0 + q_{0i}) - 2A_{i0}F^3\big]
\end{aligned}$$

where $u_i = \frac{y_i}{F}$.

Proof of Theorem 1.1 Let us suppose X be a C-projective vector field. Then by Theorem 2.1, we have $\mathcal{L}_{\hat{X}}\Xi_i = 0$. Suppose $\mathcal{L}_{\hat{X}}\alpha^2 = t_{00}$. It is easy to see that every C-projective vector field is a projective vector field and every projective vector field of $F = \alpha + \beta$, is a projective vector field of α (see [12]), then there is a projective factor $\eta = \eta(x, y)$ which is linear with respect to y. By using the Lie identity of (2.7) we obtain:

$$\begin{aligned}
\mathcal{L}_{\hat{X}}e_{i0;0} &= (\mathcal{L}_{\hat{X}}e_{i0})_{;0} - 2e_{00}\eta_i - 3e_{i0}\eta\\
\mathcal{L}_{\hat{X}}e_{00;i} &= (\mathcal{L}_{\hat{X}}e_{00})_{;0} - 4e_{00}\eta_i - 2e_{i0}\eta\\
\mathcal{L}_{\hat{X}}e_{00;0} &= (\mathcal{L}_{\hat{X}}e_{00})_{;0} - 8e_{00}\eta
\end{aligned} \tag{3.3}$$

Then by substituting (3.3) and using the Maple program, we obtain the following:

$$
\begin{array}{rcl}
\mathcal{L}_{\hat{X}}\Xi_i &=& \dfrac{3\Xi_i\mathcal{L}_{\hat{X}}F}{F} - \\
&-& \dfrac{1}{2F^3}(-8u_ie_{00}\mathcal{L}_{\hat{X}}e_{00} - 4F^2s_0\mathcal{L}_{\hat{X}}e_{i0} - 4F^2e_{i0}\mathcal{L}_{\hat{X}}s_0 + 2Fe_{i0}\mathcal{L}_{\hat{X}}e_{00} \\
&+& 2Fe_{00}\mathcal{L}_{\hat{X}}e_{i0} + 6F^2\mathcal{L}_{\hat{X}}FA_{i0} + \\
&+& (-4F\alpha\mathcal{L}_{\hat{X}}u_i - 4u_i\alpha\mathcal{L}_{\hat{X}}F - 4Fu_i\dfrac{t_{00}}{2\alpha})(\beta t_0 + s_0^2 + q_{00}) \\
&+& 4F^2\alpha(\mathcal{L}_{\hat{X}}b_it_0 + \mathcal{L}_{\hat{X}}b_it_0 + \mathcal{L}_{\hat{X}}s_0s_i + s_0\mathcal{L}_{\hat{X}}s_i + \mathcal{L}_{\hat{X}}q_{i0}) + 2e_{i0}e_{00}\mathcal{L}_{\hat{X}}F \\
&-& 8F\mathcal{L}_{\hat{X}}Fs_0e_{i0} + s_0\mathcal{L}_{\hat{X}}Fu_ie_{00} + 8Fs_0e_{00}\mathcal{L}_{\hat{X}}u_i + \\
&+& 8Fu_ie_{00}\mathcal{L}_{\hat{X}}s_0 + 8Fu_is_0\mathcal{L}_{\hat{X}}e_{00} \\
&-& 4Fu_i\alpha(\mathcal{L}_{\hat{X}}\beta t_0 + 2s_0\mathcal{L}_{\hat{X}}s_0 + \mathcal{L}_{\hat{X}}t_0\beta + \mathcal{L}_{\hat{X}}q_{00}) + \\
&+& 2F^3\mathcal{L}_{\hat{X}}A_{i0} + 2F\mathcal{L}_{\hat{X}}Fe_{00;i} \\
&-& 4F\mathcal{L}_{\hat{X}}Fe_{i0;0} + Fu_i((\mathcal{L}_{\hat{X}}e_{00})_{;0}) - 8\eta e_{00} + e_{00;0}F\mathcal{L}_{\hat{X}}u_i + e_{00;0}u_i\mathcal{L}_{\hat{X}}F \\
&+& (4(2F\mathcal{L}_{\hat{X}}F\alpha + F^2\dfrac{t_{00}}{2\alpha}))(b_it_0 + s_0s_i + q_{i0}) + \\
&+& F^2(\mathcal{L}_{\hat{X}}e_{00})_{;i} - 2\eta_ie_{00} - 3\eta e_{i0}) \\
&-& 2F^2(\mathcal{L}_{\hat{X}}e_{i0})_{;0} - 4\eta_ie_{00} - 2\eta e_{i0}) - 4e_{00}^2\mathcal{L}_{\hat{X}}u_i) \\
&=& -2F^4(Rat_i + \alpha Irrat_i)
\end{array}
$$

where $Rat_i = A_0 + A_2\alpha^2 + A_4\alpha^4 + A_6\alpha^6$ and $Irrat_i = A_7\alpha^6 + A_5\alpha^4 + A_3\alpha^2 + A_1$ and the terms $A_0, \cdots, A_6$ are respectively given in Appendix 1.

$$
\begin{array}{rcl}
A_0 &=& 8e_{00}t_{00}\beta^2s_0y_i - 4e_{00}^2t_{00}y_i\beta + e_{00;0}t_{00}\beta^2y_i \\
A_1 &=& -2t_{00}(8e_{00}\beta s_0y_i - 4e_{00}^2y_i + e_{00;0}\beta y_i)
\end{array}
$$

The equation (3.4) is equivalent to $Rat_i = 0$ and $Irrat_i = 0$, $(i = 1, ..., n)$. The system of equations $Rat_i = 0$ and $Irrat_i = 0$ is itself equivalent to the system of equations $Rat_i - \beta Irrat_i = 0$ and $Irrat_i = 0$. By using Maple, we obtain the followings:

$$
\begin{array}{l}
Rat_i - \beta Irrat_i = \\
= (\alpha^2 - \beta^2)\big\{\big[- 32\mathcal{L}_{\hat{X}}\beta b_it_0 - 40\mathcal{L}_{\hat{X}}\beta s_0s_i - 24\mathcal{L}_{\hat{X}}b_i\beta t_0 + 8\mathcal{L}_{\hat{X}}b_is_0^2 \\
-16\mathcal{L}_{\hat{X}}s_0\beta s_i + 16\mathcal{L}_{\hat{X}}s_0b_is_0 - 16\mathcal{L}_{\hat{X}}s_i\beta s_0 + 8\mathcal{L}_{\hat{X}}t_0\beta b_i - 24a_{i0}\mathcal{L}_{\hat{X}}\beta \\
+8e_{i0}\mathcal{L}_{\hat{X}}s_0 - 8e_{i0}\eta - 12\mathcal{L}_{\hat{X}}a_{i0}\beta + 8\mathcal{L}_{\hat{X}}e_{i0}s_0 - 24\mathcal{L}_{\hat{X}}\beta q_{0i} - 16\mathcal{L}_{\hat{X}}\beta q_{i0} \\
+8\mathcal{L}_{\hat{X}}b_iq_{00} + 8\mathcal{L}_{\hat{X}}q_{00}b_i - 16\mathcal{L}_{\hat{X}}q_{i0}\beta - 2\mathcal{L}_{\hat{X}}e_{00;i} + 4\mathcal{L}_{\hat{X}}e_{i0;0}\big]\alpha^4 \\
+\big[- 24a_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta^2 - 64e_{00}\mathcal{L}_{\hat{X}}\beta b_is_0 - 16e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_0 - 16e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_i \\
+16e_{00}\beta b_i\eta + 40e_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta s_0 + 8e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 - 8e_{i0}\beta^2\eta - 4\mathcal{L}_{\hat{X}}a_{i0}\beta^3
\end{array}
$$

$$
\begin{aligned}
& -16\mathcal{L}_{\hat{X}}e_{00}\beta b_i s_0 + 8\mathcal{L}_{\hat{X}}e_{i0}\beta^2 s_0 + 40(\mathcal{L}_{\hat{X}}\beta)\beta t_0 y_i + 32\mathcal{L}_{\hat{X}}\beta s_0^2 y_i \\
& +16\mathcal{L}_{\hat{X}}s_0\beta s_0 y_i + 8\mathcal{L}_{\hat{X}}t_0\beta^2 y_i + 8\mathcal{L}_{\hat{X}}y_i\beta^2 t_0 + 8\mathcal{L}_{\hat{X}}y_i\beta s_0^2 - 8t_{00}\beta b_i t_0 \\
- \quad & 28t_{00}\beta s_0 s_i + 20t_{00}b_i s_0^2 - 24a_{i0}t_{00}\beta + 8e_{00}^2\mathcal{L}_{\hat{X}}b_i - 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta \\
& +16e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i - 4e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta - 16e_{00}\mathcal{L}_{\hat{X}}s_0 y_i - 16e_{00}\mathcal{L}_{\hat{X}}y_i s_0 \\
& +16e_{00}\eta y_i - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i - 2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta - 10e_{00;i}(\mathcal{L}_{\hat{X}}\beta)\beta - 4e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta \\
& +20e_{i0}t_{00}s_0 + 20e_{i0;0}(\mathcal{L}_{\hat{X}}\beta)\beta - 16\mathcal{L}_{\hat{X}}e_{00}s_0 y_i - 2\mathcal{L}_{\hat{X}}e_{00;0}\beta b_i - 2\mathcal{L}_{\hat{X}}e_{00;i}\beta^2 \\
& +4\mathcal{L}_{\hat{X}}e_{i0;0}\beta^2 + 32\mathcal{L}_{\hat{X}}\beta q_{00}y_i + 8\mathcal{L}_{\hat{X}}q_{00}\beta y_i + 8\mathcal{L}_{\hat{X}}y_i\beta q_{00} - 12t_{00}\beta q_{0i} \\
& -16t_{00}\beta q_{i0} + 20t_{00}b_i q_{00} - 2e_{00;0}\mathcal{L}_{\hat{X}}y_i - 5e_{00;i}t_{00} + 10e_{i0;0}t_{00} - 2\mathcal{L}_{\hat{X}}e_{00;0}y_i]\alpha^2 \\
- \quad & 24e_{00}^2\mathcal{L}_{\hat{X}}(\beta)\beta b_i + 24e_{00}^2\mathcal{L}_{\hat{X}}\beta y_i + 12e_{00}^2 t_{00}b_i - 24e_{00}t_{00}s_0 y_i - 3e_{00;0}t_{00}y_i\} \\
+ \quad & 12e_{00}^2\{[(-2\beta b_i + 2y_i)\mathcal{L}_{\hat{X}}\beta + b_i t_{00}]\alpha^2 - y_i t_{00}\beta\}
\end{aligned}
$$

By the above equation, for any point $x \in M$, the irreducible polynomial $\alpha^2 - \beta^2$ divides e_{00} or $[(-2\beta b_i + 2y_i)\mathcal{L}_{\hat{X}}\beta + b_i t_{00}]\alpha^2 - y_i t_{00}\beta$. In the first case, for a function $c \in C^\infty(M)$, $e_{00} = 2c(x)(\alpha^2 - \beta^2)$ which means that F is of isotropic S-curvature. In the second case, we have

$$
[(-2\beta b_i + 2y_i)\mathcal{L}_{\hat{X}}\beta + b_i t_{00}]\alpha^2 - y_i t_{00}\beta = \eta_i(x,y)(\alpha^2 - \beta^2) \tag{3.4}
$$

where η_i are polynomials of degree two. By contracting above equation with b^i, we obtain

$$
b^2(\mathcal{L}_{\hat{X}}(\alpha^2 - \beta^2) - (\beta^2\mathcal{L}_{\hat{X}}\alpha^2 - \alpha^2\mathcal{L}_{\hat{X}}\beta^2) = \eta(x,y)(\alpha^2 - \beta^2)
$$

where $\eta = \eta_i(x,y)b^i$ is a polynomial of degree two on TM.

By $Irrat_i = 0$, we have $\mathcal{L}_{\hat{X}}\alpha^2 = \sigma(x)\alpha^2$ which means that $\hat{X}$ is a conformal vector field of α, or α^2 divides $(8e_{00}\beta s_0 y_i - 4e_{00}^2 y_i + e_{00;0}\beta y_i)$. By contracting it with b^i, we obtain

$$
(8e_{00}\beta s_0 - 4e_{00}^2 + e_{00;0}\beta) = \lambda(x,y)\alpha^2
$$

where $\lambda(x,y)$ is a polynomial of degree two. □

Example 3.1. *Let $M = \mathbb{R}^2$ and $F(x,y,u,v) = \sqrt{u^2+v^2} + au + bv$, where $a, b \in \mathbb{R}$ and $a^2 + b^2 < 1$. Since F is Berwaldian, $S = 0$ and $\Xi = 0$, i.e., every projective vector field is a C-projective vector field. Let $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Then by a dircet calculation, we have $\mathcal{L}_{\hat{X}}\alpha^2 = 2\alpha^2$ and $\mathcal{L}_{\hat{X}}\beta = \beta$. If we substitute them in (3.4) we obtain:*

$$
[(-2b_i\beta + 2y_i)\beta + 2b_i\alpha^2]\alpha^2 - 2y_i\alpha^2\beta = \eta_i(x,y)(\alpha^2 - \beta^2) \tag{3.5}
$$

By contracting (4.4) with y^i we have $\eta(x,y) = 2\beta\alpha^2$. In this case all of the conditions Theorem 1.1 is held.

Here we make an example which satiesfies condition 1 and 3 of Theorem 1.1.

Example 3.2. *Let $M = \mathbb{B}^2(1)$ open ball on $\mathbb{R}^2$ and let F be Funk metric as:*

$$
\theta = \frac{\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2}}{1 - |x|^2} + \frac{<x,y>}{1 - |x|^2} \tag{3.6}
$$

By [6] one can see that Funk metric is of constant S-curvature $S = \frac{1}{2}\theta$. It is easy to see that the vector field $V = (x^i < a, x > -a^i)\frac{\partial}{\partial x^i}$ is a Killing vector field of $\alpha := \frac{\sqrt{|y|^2-|x|^2|y|^2+<x,y>^2}}{1-|x|^2}$. By direct calculation we have

$$\mathcal{L}_{\hat{V}}(\beta) = - < a, y > \tag{3.7}$$

where $\beta := \frac{<x,y>}{1-|x|^2}$. So conditions 1 and 3 of Theorem 1.1 hold. Here, we examine condition 2 and 4. Since $s_j = 0$ and F is of constant S-curvature, we have $r_{00} = \alpha^2 - \beta^2$. By taking the covariant derivative with respect to α, we have

$$r_{00;0} = -2\beta(\alpha^2 - \beta^2). \tag{3.8}$$

If we substitute (3.8) in condition 4, we have

$$\frac{|y|^2}{1-|x|^2}\{2|y|^2(1-|x|^2)+ < x, y >^2 \} = (|y|^2(1-|x|^2)+ < x, y >^2)\lambda(x, y)$$

but $\lambda(x, y)$ should be a polynomial of degree 2, which is a contradiction. Now we consider condition 2. Since $\mathcal{L}_{\hat{V}}\alpha^2 = 0$, condition 2 reduces to

$$\alpha^2(1-b^2)\mathcal{L}_{\hat{V}}\beta^2 = \eta(x, y)(\alpha^2 - \beta^2)$$

Then by substituting $b^2 = |x|^2$ and $\mathcal{L}_{\hat{V}}\beta^2$ we have

$$-2 < a, y >< x, y > \{|y|^2 + \frac{< x, y >^2}{1-|y|^2}\} = |y|^2\eta_V(x, y) \tag{3.9}$$

where η_V is a polynomial of degree two. If we write the extended form of (3.9) we have

$$\begin{aligned}(a_1y_1 + a_2y_2) \qquad & (x_1y_1 + x_2y_2)\{y_1^2 + y_2^2 + \frac{x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2}{1-|x|^2}\} \\ & = (y_1^2 + y_2^2)(a_{11}(x)y_1^2 + 2a_{12}(x)y_1y_2 + a_{22}y_2^2)\end{aligned}$$

By comparing left and right side of above equation we have $a_1x_1x_2^2+a_2x_1^2x_2 = 0$, i.e. $a = 0$ which is contradition. Therefore, there is not any $\eta(x, y)$ to applies condition 2.

Here we make an example which satisfies condition 2 and 3 of Theorem 1.1.

Example 3.3. *Let α be the Bergman metric on $D = \{x \in \mathbb{R}^{2N} : |x| < 1\}$ and $f = \frac{1}{2}ln(1-|x|^2)$ be the potential of α, and J be the complex structure. B. Chen and L. Zhao [5] proved Randers metric*

$$F_\epsilon(x, y) = \sqrt{\alpha(y, y)} + df(\epsilon y - Jy), \qquad \epsilon \neq 0$$

is of scalar flag curvature and neither projectively flat nor of isotropic S-curvature. That is the condition 1 of Theorem 1.1 is not held. Let $\hat{V}$ be a Killing vector field of F, i. e. condition 2 and 3 are established automatically. By [5] we have

$$\begin{aligned} &b_k = -J^i_k f_i, \quad s_k = -f_k, \quad a_{kj} = -\tfrac{1}{2}(f_{si}J^s_k J^i_j + f_{jk}), \quad r_{00} + 2s_0\beta = 0 \\ &e_{00}(\epsilon) = 2\epsilon\beta(\beta + \epsilon f_0), \quad e_{00;0}(\epsilon) = 4\epsilon\beta f_0(\beta + \epsilon f_0 + f_0) \end{aligned} \tag{3.10}$$

By substituting (3.10) in condition 4 Theorem 1.1, we find out that the term

$$4\epsilon\beta^2(-3f_0(\beta + \epsilon f_0) + f_0^2 - 4\epsilon(\beta + \epsilon f_0)^2) \tag{3.11}$$

should be a multiple of α^2, which is impossible. Thus condition 4 is not true.

4. Projective vector fields vs. H-projective

Let $F = \alpha + \beta$ be a Randers metric. Due to (2.6), H-curvature can be directly obtained as follows:

$$4H_{ij} = \frac{2e_{ij|0}}{F} - \frac{2e_{i0|0}u_j}{F^2} - \frac{2e_{j0|0}u_i}{F^2} - \frac{e_{00|0}F_{ij}}{F^2} + \frac{2e_{00|0}u_iu_j}{F^3} \tag{4.1}$$

By substituting (3.2) in (4.1) we obtain:

Proof of Theorem 1.2 Let us suppose X be a H-invariant vector field, i. e. $\mathcal{L}_{\hat{X}}H_{ij} = 0$ Suppose $\mathcal{L}_{\hat{X}}\alpha^2 = t_{00}$, then by using Maple program and using equations (4.1), (3.2) and (3.3) we obtain the following:

$$H_{ij} = -2F^4(Rat_{ij} + \alpha Irrat_{ij}) \tag{4.2}$$

where $Rat_{ij} = A_0 + A_2\alpha^2 + \cdots + A_{10}\alpha^{10}$ and $Irrat_{ij} = A_9\alpha^8 + \cdots + A_1$ and the terms $A_0, \cdots, A_{10}$ are respectively given in Appendix 2.

$$A_1 = 3t_{00}\beta y_i y_j(-40e_{00}\beta s_0 + 16e_{00}^2 - 5e_{00;0}\beta) \tag{4.3}$$

The equation (4.2) is equivalent to $Rat_{ij} = 0$ and $Irrat_{ij} = 0$. The system of equations $Rat_{ij} = 0$ and $Irrat_{ij} = 0$ is itself equivalent to the system of equations $Rat_{ij} - \beta Irrat_{ij} = 0$ and $Irrat_{ij} = 0$. By using Maple we obtain the followings:

$$\begin{aligned} &Rat_{ij} - \beta Irrat_{ij} = \\ &= (\alpha^2 - \beta^2)\{[-2(\mathcal{L}_{\hat{X}}b_it_j + \mathcal{L}_{\hat{X}}b_jt_i + 2\mathcal{L}_{\hat{X}}s_is_j + 2\mathcal{L}_{\hat{X}}s_js_i + \mathcal{L}_{\hat{X}}t_ib_j \\ &+\mathcal{L}_{\hat{X}}t_jb_i + \mathcal{L}_{\hat{X}}q_{ij} + \mathcal{L}_{\hat{X}}q_{ji})]\alpha^8 + [12\mathcal{L}_{\hat{X}}\beta\beta b_it_j + 12\mathcal{L}_{\hat{X}}\beta\beta b_jt_i + 40\mathcal{L}_{\hat{X}}\beta\beta s_is_j \\ &-72\mathcal{L}_{\hat{X}}\beta b_ib_jt_0 - 44\mathcal{L}_{\hat{X}}\beta b_is_0s_j - 44\mathcal{L}_{\hat{X}}\beta b_js_0s_i - 10e_{i0}\mathcal{L}_{\hat{X}}\beta s_j - 20e_{ij}\mathcal{L}_{\hat{X}}\beta s_0 \\ &-10e_{j0}\mathcal{L}_{\hat{X}}\beta s_i + 20\mathcal{L}_{\hat{X}}\beta\beta q_{ij} + 20\mathcal{L}_{\hat{X}}\beta\beta q_{ji} - 24\mathcal{L}_{\hat{X}}\beta b_iq_{0j} - 12\mathcal{L}_{\hat{X}}\beta b_iq_{j0} \\ &-28\mathcal{L}_{\hat{X}}\beta b_jq_{0i} - 12\mathcal{L}_{\hat{X}}\beta b_jq_{i0} + 5t_{00}b_it_j + 5t_{00}b_jt_i + 10t_{00}s_is_j - 12\mathcal{L}_{\hat{X}}s_i\beta^2s_j \\ &-12\mathcal{L}_{\hat{X}}s_j\beta^2s_i - 6\mathcal{L}_{\hat{X}}t_i\beta^2b_j - 2\mathcal{L}_{\hat{X}}t_j\beta^2b_i + 4\mathcal{L}_{\hat{X}}t_j\beta^2b_j \end{aligned}$$

$$
\begin{aligned}
&+4a_{ij}\mathcal{L}_{\hat{X}}\beta t_0 + 8a_{ij}\mathcal{L}_{\hat{X}}s_0 s_0\\
&+4a_{ij}\mathcal{L}_{\hat{X}}t_0\beta - 4e_{00}\mathcal{L}_{\hat{X}}b_i s_j - 4e_{00}\mathcal{L}_{\hat{X}}b_j s_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}s_i b_j - 4e_{00}\mathcal{L}_{\hat{X}}s_j b_i + 4e_{00}b_i\eta_j\\
&+4e_{00}b_j\eta_i - 6e_{i0}\mathcal{L}_{\hat{X}}b_j s_0 - 6e_{i0}\mathcal{L}_{\hat{X}}s_0 b_j + 6e_{i0}\mathcal{L}_{\hat{X}}s_j\beta + 2\mathcal{L}_{\hat{X}}t_j\beta y_i + 2\mathcal{L}_{\hat{X}}t_j\beta y_j\\
&+2\mathcal{L}_{\hat{X}}y_i\beta t_j - 2\mathcal{L}_{\hat{X}}y_i b_j t_0 + 4\mathcal{L}_{\hat{X}}y_i s_0 s_j + 4\mathcal{L}_{\hat{X}}y_i s_j t_0\\
&-6e_{i0}\beta\eta_j + 6e_{i0}b_j\eta + 12e_{ij}\mathcal{L}_{\hat{X}}s_0\beta\\
&-12e_{ij}\beta\eta - 6e_{j0}\mathcal{L}_{\hat{X}}b_i s_0 - 6e_{j0}\mathcal{L}_{\hat{X}}s_0 b_i + 6e_{j0}\mathcal{L}_{\hat{X}}s_i\beta\\
&-6e_{j0}\beta\eta_i + 6e_{j0}b_i\eta + 4\mathcal{L}_{\hat{X}}a_{ij}\beta t_0\\
&-4\mathcal{L}_{\hat{X}}e_{00}b_i s_j - 4\mathcal{L}_{\hat{X}}e_{00}b_j s_i + 6\mathcal{L}_{\hat{X}}e_{i0}\beta s_j + 4\mathcal{L}_{\hat{X}}a_{ij}\beta t_0 + 12e_{ij}\mathcal{L}_{\hat{X}}s_0\beta\\
&-6\mathcal{L}_{\hat{X}}e_{i0}b_j s_0 + 12\mathcal{L}_{\hat{X}}e_{ij}\beta s_0 + 6\mathcal{L}_{\hat{X}}e_{j0}\beta s_i\\
&-6\mathcal{L}_{\hat{X}}e_{j0}b_i s_0 + 2\mathcal{L}_{\hat{X}}\beta t_i y_j - 2\mathcal{L}_{\hat{X}}b_i\beta^2 t_j\\
&-8\mathcal{L}_{\hat{X}}b_i b_j s_0^2 - 2\mathcal{L}_{\hat{X}}b_j\beta^2 t_i - 8\mathcal{L}_{\hat{X}}b_j b_i s_0^2 + 2\mathcal{L}_{\hat{X}}\beta t_j y_i + 8\mathcal{L}_{\hat{X}}b_i\beta q_{0j} + 4\mathcal{L}_{\hat{X}}b_i\beta q_{j0}\\
&-8\mathcal{L}_{\hat{X}}b_i b_j q_{00} - 2\mathcal{L}_{\hat{X}}b_i t_0 y_j + 8\mathcal{L}_{\hat{X}}b_j\beta q_{0i}\\
&+4\mathcal{L}_{\hat{X}}b_j\beta q_{i0} - 8\mathcal{L}_{\hat{X}}b_j b_i q_{00} - 2\mathcal{L}_{\hat{X}}b_j t_0 y_i\\
&-8\mathcal{L}_{\hat{X}}q_{00}b_i b_j + 8\mathcal{L}_{\hat{X}}q_{0i}\beta b_j + 8\mathcal{L}_{\hat{X}}q_{0j}\beta b_i + 4\mathcal{L}_{\hat{X}}q_{i0}\beta b_j + 4\mathcal{L}_{\hat{X}}q_{j0}\beta b_i + 4\mathcal{L}_{\hat{X}}s_i s_0 y_j\\
&+4\mathcal{L}_{\hat{X}}s_i t_0 y_j + 4\mathcal{L}_{\hat{X}}s_j s_0 y_i + 4\mathcal{L}_{\hat{X}}s_j t_0 y_i - 2\mathcal{L}_{\hat{X}}t_0 b_i y_j - 2\mathcal{L}_{\hat{X}}t_0 b_j y_i + 4\mathcal{L}_{\hat{X}}t_0 s_i y_j\\
&+4\mathcal{L}_{\hat{X}}t_0 s_j y_i + 2\mathcal{L}_{\hat{X}}y_j\beta t_i - 2\mathcal{L}_{\hat{X}}y_j b_i t_0 + 4\mathcal{L}_{\hat{X}}y_j s_0 s_i + 4\mathcal{L}_{\hat{X}}y_j s_i t_0 - 2(\mathcal{L}_{\hat{X}}e_{j0})_{;0}b_i\\
&+4\mathcal{L}_{\hat{X}}q_{0i}y_j + 4\mathcal{L}_{\hat{X}}q_{0j}y_i + 4\mathcal{L}_{\hat{X}}y_i q_{0j} + 4\mathcal{L}_{\hat{X}}y_j q_{0i}\\
&+4\mathcal{L}_{\hat{X}}a_{ij}s_0^2 - 6\mathcal{L}_{\hat{X}}q_{ij}\beta^2 - 10e_{ij;0}\mathcal{L}_{\hat{X}}\beta\\
&+5t_{00}q_{ij} + 5t_{00}q_{ji} - 6\mathcal{L}_{\hat{X}}q_{ji}\beta^2 + 4a_{ij}\mathcal{L}_{\hat{X}}q_{00}\\
&-2e_{00}\mathcal{L}_{\hat{X}}e_{ij} - 2e_{i0;0}\mathcal{L}_{\hat{X}}b_j - 2e_{ij}\mathcal{L}_{\hat{X}}e_{00}\\
&-2e_{j0;0}\mathcal{L}_{\hat{X}}b_i + 4\mathcal{L}_{\hat{X}}a_{ij}q_{00} - 2\mathcal{L}_{\hat{X}}e_{i0;0}b_j\\
&+6\mathcal{L}_{\hat{X}}e_{ij;0}\beta + 4\mathcal{L}_{\hat{X}}s_0\beta b_j s_i - 16\mathcal{L}_{\hat{X}}s_0 b_i b_j s_0\\
&+12\mathcal{L}_{\hat{X}}s_i\beta b_j s_0 + 8\mathcal{L}_{\hat{X}}s_i\beta b_j t_0 + 12\mathcal{L}_{\hat{X}}s_j\beta b_i s_0\\
&+8\mathcal{L}_{\hat{X}}\beta b_i s_j t_0 + 8\mathcal{L}_{\hat{X}}\beta b_j s_i t_0 - 8\mathcal{L}_{\hat{X}}b_i\beta b_j t_0\\
&+12\mathcal{L}_{\hat{X}}b_i\beta s_0 s_j + 8\mathcal{L}_{\hat{X}}b_i\beta s_j t_0 - 8\mathcal{L}_{\hat{X}}b_j\beta b_i t_0\\
&+12\mathcal{L}_{\hat{X}}b_j\beta s_0 s_i + 8\mathcal{L}_{\hat{X}}b_j\beta s_i t_0 + 4\mathcal{L}_{\hat{X}}s_0\beta b_i s_j\\
&+8\mathcal{L}_{\hat{X}}s_j\beta b_i t_0 - 8\mathcal{L}_{\hat{X}}t_0\beta b_i b_j + 8\mathcal{L}_{\hat{X}}t_0\beta b_i s_j\\
&+8\mathcal{L}_{\hat{X}}t_0\beta b_j s_i - 4\mathcal{L}_{\hat{X}}e_{i0}e_{j0} - 4\mathcal{L}_{\hat{X}}e_{j0}e_{i0}\big]\alpha^6\\
&+\big[12e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta b_j - 26e_{ij}t_{00}\beta s_0 + 36e_{j0}\mathcal{L}_{\hat{X}}\beta s_0 y_i\\
&-13e_{j0}t_{00}\beta s_i + 21e_{j0}t_{00}b_i s_0 - 20a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 t_0\\
&-24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta s_0^2 - 10e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 s_j - 20e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0\\
&-10e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2 s_i - 10\mathcal{L}_{\hat{X}}\beta\beta^2 t_i y_j\\
&-10\mathcal{L}_{\hat{X}}\beta\beta^2 t_j y_i + 44\mathcal{L}_{\hat{X}}\beta b_i s_0^2 y_j + 44\mathcal{L}_{\hat{X}}\beta b_j s_0^2 y_i\\
&+12e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta b_i - 28\mathcal{L}_{\hat{X}}\beta\beta q_{0i}y_j - 24\mathcal{L}_{\hat{X}}\beta\beta q_{0j}y_i\\
&+8\mathcal{L}_{\hat{X}}\beta\beta q_{i0}y_j + 8\mathcal{L}_{\hat{X}}\beta\beta q_{j0}y_i + 44\mathcal{L}_{\hat{X}}\beta b_i q_{00}y_j
\end{aligned}
$$

$$
\begin{aligned}
&+44\mathcal{L}_{\hat{X}}\beta b_j q_{00} y_i - 12t_{00}\beta b_i q_{0j} - 6t_{00}\beta b_i q_{j0}\\
&-18t_{00}\beta b_j q_{0i} - 6t_{00}\beta b_j q_{i0} - 7t_{00}\beta t_i y_j - 7t_{00}\beta t_j y_i\\
&+28t_{00} b_i b_j q_{00} - 13t_{00} b_i t_0 y_j - 13t_{00} b_j t_0 y_i\\
&-22t_{00} s_0 s_i y_j - 22t_{00} s_0 s_j y_i - t_{00}\beta^2 b_i t_j - t_{00}\beta^2 b_j t_i\\
&+10t_{00}\beta^2 s_i s_j + 28t_{00} b_i b_j s_0^2 + 48a_{ij} e_{00}\mathcal{L}_{\hat{X}}\beta s_0\\
&-24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta q_{00} - 14a_{ij} t_{00}\beta t_0 - 56e_{00} e_{i0}\mathcal{L}_{\hat{X}}\beta b_j\\
&+12e_{00} e_{ij}\mathcal{L}_{\hat{X}}\beta\beta - 56e_{00} e_{j0}\mathcal{L}_{\hat{X}}\beta b_i + 24e_{00}\mathcal{L}_{\hat{X}}\beta s_i y_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta s_j y_i + 14e_{00} t_{00} b_i s_j + 14e_{00} t_{00} b_j s_i\\
&-14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i b_j + 24e_{i0} e_{j0}\mathcal{L}_{\hat{X}}\beta\beta + 36e_{i0}\mathcal{L}_{\hat{X}}\beta s_0 y_j\\
&-13e_{i0} t_{00}\beta s_j + 21e_{i0} t_{00} b_j s_0 + 28\mathcal{L}_{\hat{X}}\beta\beta b_i t_0 y_j\\
&+28\mathcal{L}_{\hat{X}}\beta\beta b_j t_0 y_i + 12e_{j0;0}\mathcal{L}_{\hat{X}}\beta y_i + 7e_{j0;0} t_{00} b_i\\
&-18t_{00} q_{0i} y_j - 14t_{00} q_{0j} y_i - 14a_{ij} t_{00} s_0^2 + 4a_{ij}\mathcal{L}_{\hat{X}} t_0\beta^3\\
&-10e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta^2 + 5t_{00}\beta^2 q_{ij} + 5t_{00}\beta^2 q_{ji}\\
&+4a_{ij} e_{00;0}\mathcal{L}_{\hat{X}}\beta - 14a_{ij} t_{00} q_{00} + 7e_{00} e_{ij} t_{00}\\
&+14e_{i0} e_{j0} t_{00} + 12e_{i0;0}\mathcal{L}_{\hat{X}}\beta y_j + 7e_{i0;0} t_{00} b_j - 13e_{ij;0} t_{00}\beta\\
&+2\mathcal{L}_{\hat{X}} t_j\beta^3 y_i + 2\mathcal{L}_{\hat{X}} t_j\beta^3 y_j + 4\mathcal{L}_{\hat{X}} a_{ij}\beta^3 t_0 + 4\mathcal{L}_{\hat{X}} a_{ij}\beta^2 s_0^2\\
&+4a_{ij}\mathcal{L}_{\hat{X}} q_{00}\beta^2 - 12e_{00}^2\mathcal{L}_{\hat{X}} b_i b_j - 12e_{00}^2\mathcal{L}_{\hat{X}} b_j b_i\\
&-2e_{00}\mathcal{L}_{\hat{X}} e_{ij}\beta^2 - 4e_{i0}\mathcal{L}_{\hat{X}} e_{j0}\beta^2 - 2e_{i0;0}\mathcal{L}_{\hat{X}} b_j\beta^2\\
&+8e_{00} e_{j0}\mathcal{L}_{\hat{X}} y_i + 8e_{00}\mathcal{L}_{\hat{X}} e_{i0} y_j + 8e_{00}\mathcal{L}_{\hat{X}} e_{j0} y_i\\
&-2e_{00;0}\mathcal{L}_{\hat{X}} a_{ij}\beta + 2e_{00;0}\mathcal{L}_{\hat{X}} b_i y_j + 2e_{00;0}\mathcal{L}_{\hat{X}} b_j y_i\\
&+2e_{00;0}\mathcal{L}_{\hat{X}} y_i b_j + 2e_{00;0}\mathcal{L}_{\hat{X}} y_j b_i + 8e_{i0}\mathcal{L}_{\hat{X}} e_{00} y_j\\
&-2e_{ij}\mathcal{L}_{\hat{X}} e_{00}\beta^2 - 4e_{j0}\mathcal{L}_{\hat{X}} e_{i0}\beta^2 - 2e_{j0;0}\mathcal{L}_{\hat{X}} b_i\beta^2\\
&+4\mathcal{L}_{\hat{X}} a_{ij}\beta^2 q_{00} - 2\mathcal{L}_{\hat{X}} e_{i0;0}\beta^2 b_j - 2\mathcal{L}_{\hat{X}} e_{j0;0}\beta^2 b_i\\
&+4\mathcal{L}_{\hat{X}} q_{0i}\beta^2 y_j + 4\mathcal{L}_{\hat{X}} q_{0j}\beta^2 y_i - 4\mathcal{L}_{\hat{X}} q_{i0}\beta^2 y_j - 4\mathcal{L}_{\hat{X}} q_{j0}\beta^2 y_i\\
&+2\mathcal{L}_{\hat{X}} y_i\beta^3 t_j + 2\mathcal{L}_{\hat{X}} y_j\beta^3 t_i - 4e_{i0;0}\mathcal{L}_{\hat{X}} y_j\beta\\
&-4e_{j0;0}\mathcal{L}_{\hat{X}} y_i\beta + 8e_{j0}\mathcal{L}_{\hat{X}} e_{00} y_i + 2\mathcal{L}_{\hat{X}} e_{00;0} b_i y_j\\
&+2(\mathcal{L}_{\hat{X}} e_{00})_{;0} b_j y_i - 4\mathcal{L}_{\hat{X}} e_{i0;0}\beta y_j - 4\mathcal{L}_{\hat{X}} e_{j0;0}\beta y_i\\
&-8\mathcal{L}_{\hat{X}} q_{00} y_i y_j - 8\mathcal{L}_{\hat{X}} y_i q_{00} y_j - 8\mathcal{L}_{\hat{X}} y_j q_{00} y_i + 4\mathcal{L}_{\hat{X}} y_i\beta^2 q_{0j}\\
&-4\mathcal{L}_{\hat{X}} y_i\beta^2 q_{j0} - 8\mathcal{L}_{\hat{X}} y_i s_0^2 y_j + 4\mathcal{L}_{\hat{X}} y_j\beta^2 q_{0i}\\
&-4\mathcal{L}_{\hat{X}} y_j\beta^2 q_{i0} - 8\mathcal{L}_{\hat{X}} y_j s_0^2 y_i + 8a_{ij} e_{00}\mathcal{L}_{\hat{X}} e_{00}\\
&-2a_{ij}\mathcal{L}_{\hat{X}} e_{00;0}\beta + 8e_{00} e_{i0}\mathcal{L}_{\hat{X}} y_j + 16e_{00}\mathcal{L}_{\hat{X}} b_i\beta b_j s_0\\
&+16e_{00}\mathcal{L}_{\hat{X}} b_j\beta b_i s_0 + 16e_{00}\mathcal{L}_{\hat{X}} s_0\beta b_i b_j - 16e_{00}\beta b_i b_j\eta\\
&+12t_{00}\beta b_i s_j t_0 + 12t_{00}\beta b_j s_i t_0 + 16\mathcal{L}_{\hat{X}} e_{00}\beta b_i b_j s_0\\
&+8\mathcal{L}_{\hat{X}}\beta\beta s_i t_0 y_j + 8\mathcal{L}_{\hat{X}}\beta\beta s_j t_0 y_i - 8\mathcal{L}_{\hat{X}} s_0\beta b_i s_0 y_j\\
&-8\mathcal{L}_{\hat{X}} s_0\beta b_j s_0 y_i + 36e_{i0}\mathcal{L}_{\hat{X}}\beta\beta b_j s_0 + 36e_{j0}\mathcal{L}_{\hat{X}}\beta\beta b_i s_0\\
&-24\mathcal{L}_{\hat{X}}\beta\beta s_0 s_i y_j - 24\mathcal{L}_{\hat{X}}\beta\beta s_0 s_j y_i - 20t_{00}\beta b_i b_j t_0 - 30t_{00}\beta b_i s_0 s_j
\end{aligned}
$$

$$
\begin{aligned}
&-30t_{00}\beta b_j s_0 s_i + 24e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i s_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_j s_i - 112e_{00}\mathcal{L}_{\hat{X}}\beta b_i b_j s_0 + 4e_{00}^2\mathcal{L}_{\hat{X}}a_{ij}\\
&+8a_{ij}\mathcal{L}_{\hat{X}}s_0\beta^2 s_0 - 4e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2 s_j - 4e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2 s_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}s_i\beta^2 b_j - 4e_{00}\mathcal{L}_{\hat{X}}s_j\beta^2 b_i + 4e_{00}\beta^2 b_i\eta_j\\
&+4e_{00}\beta^2 b_j\eta_i - 10\mathcal{L}_{\hat{X}}b_j\beta^2 t_0 y_i - 4\mathcal{L}_{\hat{X}}b_j\beta s_0^2 y_i\\
&-4\mathcal{L}_{\hat{X}}s_0\beta^2 s_i y_j - 4\mathcal{L}_{\hat{X}}s_0\beta^2 s_j y_i - 6e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^2 s_0\\
&-6e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 b_j + 6e_{i0}\beta^2 b_j\eta - 6e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^2 s_0\\
&-6e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^2 b_i + 6e_{j0}\beta^2 b_i\eta - 4\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_i s_j\\
&-4\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_j s_i - 6\mathcal{L}_{\hat{X}}e_{i0}\beta^2 b_j s_0 - 6\mathcal{L}_{\hat{X}}e_{j0}\beta^2 b_i s_0\\
&+4\mathcal{L}_{\hat{X}}s_i\beta^2 t_0 y_j + 4\mathcal{L}_{\hat{X}}s_j\beta^2 t_0 y_i - 10\mathcal{L}_{\hat{X}}t_0\beta^2 b_i y_j\\
&-10\mathcal{L}_{\hat{X}}t_0\beta^2 b_j y_i + 4\mathcal{L}_{\hat{X}}t_0\beta^2 s_i y_j + 4\mathcal{L}_{\hat{X}}t_0\beta^2 s_j y_i\\
&-10\mathcal{L}_{\hat{X}}y_i\beta^2 b_j t_0 + 4\mathcal{L}_{\hat{X}}y_i\beta^2 s_j t_0 - 4\mathcal{L}_{\hat{X}}y_i\beta b_j s_0^2\\
&-10\mathcal{L}_{\hat{X}}y_j\beta^2 b_i t_0 + 4\mathcal{L}_{\hat{X}}y_j\beta^2 s_i t_0 - 10\mathcal{L}_{\hat{X}}b_i\beta^2 t_0 y_j\\
&-4\mathcal{L}_{\hat{X}}b_i\beta s_0^2 y_j + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta b_j + 8e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta b_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}b_i s_0 y_j + 16e_{00}\mathcal{L}_{\hat{X}}b_j s_0 y_i + 16e_{00}\mathcal{L}_{\hat{X}}s_0 b_i y_j\\
&+16e_{00}\mathcal{L}_{\hat{X}}s_0 b_j y_i - 8e_{00}\mathcal{L}_{\hat{X}}s_i\beta y_j - 8e_{00}\mathcal{L}_{\hat{X}}s_j\beta y_i\\
&-8e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_j + 16e_{00}\mathcal{L}_{\hat{X}}y_i b_j s_0 - 8e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}y_j b_i s_0 + 8e_{00}\beta\eta_i y_j + 8e_{00}\beta\eta_j y_i - 16e_{00}b_i\eta y_j\\
&-16e_{00}b_j\eta y_i + 2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta b_j - 4\mathcal{L}_{\hat{X}}y_j\beta b_i s_0^2\\
&-8a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta + 16a_{ij}e_{00}\beta\eta - 16a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta s_0\\
&+8e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j\beta + 2(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta b_i b_j\\
&-12\mathcal{L}_{\hat{X}}e_{i0}\beta s_0 y_j - 12\mathcal{L}_{\hat{X}}e_{j0}\beta s_0 y_i - 8\mathcal{L}_{\hat{X}}\beta t_0 y_i y_j\\
&-4\mathcal{L}_{\hat{X}}b_i\beta q_{00}y_j - 4\mathcal{L}_{\hat{X}}b_j\beta q_{00}y_i\\
&-4\mathcal{L}_{\hat{X}}q_{00}\beta b_i y_j - 4\mathcal{L}_{\hat{X}}q_{00}\beta b_j y_i - 16\mathcal{L}_{\hat{X}}s_0 s_0 y_i y_j\\
&-8\mathcal{L}_{\hat{X}}t_0\beta y_i y_j + 2e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta b_i\\
&+8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta b_j - 12e_{i0}\mathcal{L}_{\hat{X}}s_0\beta y_j - 12e_{i0}\mathcal{L}_{\hat{X}}y_j\beta s_0\\
&+12e_{i0}\beta\eta y_j + 8e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta b_i\\
&-12e_{j0}\mathcal{L}_{\hat{X}}s_0\beta y_i - 12e_{j0}\mathcal{L}_{\hat{X}}y_i\beta s_0 + 12e_{j0}\beta\eta y_i\\
&+8e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\beta + 16\mathcal{L}_{\hat{X}}e_{00}b_j s_0 y_i\\
&-16e_{00}\mathcal{L}_{\hat{X}}a_{ij}\beta s_0 - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i b_j - 4\mathcal{L}_{\hat{X}}y_i\beta b_j q_{00}\\
&-8\mathcal{L}_{\hat{X}}y_i\beta t_0 y_j - 4\mathcal{L}_{\hat{X}}y_j\beta b_i q_{00}\\
&+6t_{00}s_i t_0 y_j + 6t_{00}s_j t_0 y_i - 8\mathcal{L}_{\hat{X}}e_{00}\beta s_i y_j\\
&-8\mathcal{L}_{\hat{X}}e_{00}\beta s_j y_i + 16\mathcal{L}_{\hat{X}}e_{00}b_i s_0 y_j\\
&-2e_{j0}\beta^3\eta_i + 2\mathcal{L}_{\hat{X}}e_{i0}\beta^3 s_j + 4\mathcal{L}_{\hat{X}}e_{ij}\beta^3 s_0\\
&+2\mathcal{L}_{\hat{X}}e_{j0}\beta^3 s_i + 2e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^3 - 2e_{i0}\beta^3\eta_j\\
&+4e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^3 - 4e_{ij}\beta^3\eta + 2e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^3 + 2\mathcal{L}_{\hat{X}}e_{ij;0}\beta^3\big]\alpha^4
\end{aligned}
$$

$$
\begin{aligned}
&+\big[-e_{i0}t_{00}\beta^3 s_j - 2e_{ij}t_{00}\beta^3 s_0 - 9e_{00;0}t_{00}b_j y_i\\
&+10e_{i0;0}t_{00}\beta y_j + 10e_{j0;0}t_{00}\beta y_i - t_{00}\beta^3 t_i y_j\\
&-t_{00}\beta^3 t_j y_i - 2a_{ij}t_{00}\beta^2 q_{00} + 96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i y_j\\
&+96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_j y_i + 54e_{00}^2 t_{00}b_i b_j + e_{00}e_{ij}t_{00}\beta^2\\
&+2e_{i0}e_{j0}t_{00}\beta^2 + e_{i0;0}t_{00}\beta^2 b_j + e_{j0;0}t_{00}\beta^2 b_i\\
&-e_{j0}t_{00}\beta^3 s_i - 2a_{ij}t_{00}\beta^3 t_0 - 2a_{ij}t_{00}\beta^2 s_0^2\\
&-96\beta e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i b_j - e_{ij}\mathcal{L}_{\hat{X}}\mathcal{L}_{\hat{X}}0t_{00}\beta^3\\
&-18a_{ij}e_{00}^2 t_{00} - 16e_{00}^2\mathcal{L}_{\hat{X}}y_i y_j - 16e_{00}^2\mathcal{L}_{\hat{X}}y_j y_i\\
&+36t_{00}\beta t_0 y_i y_j - 6t_{00}\beta b_j q_{00}y_i + t_{00}\beta^2 b_i t_0 y_j\\
&+t_{00}\beta^2 b_j t_0 y_i - 6t_{00}\beta b_i s_0^2 y_j - 6t_{00}\beta b_j s_0^2 y_i\\
&+40a_{ij}e_{00}t_{00}\beta s_0 - 4e_{00}e_{i0}t_{00}\beta b_j - 4e_{00}e_{j0}t_{00}\beta b_i + 2e_{00}t_{00}\beta^2 b_i s_j\\
&+2e_{00}t_{00}\beta^2 b_j s_i\\
&+3e_{i0}t_{00}\beta^2 b_j s_0 + 3e_{j0}t_{00}\beta^2 b_i s_0 - 160e_{00}\mathcal{L}_{\hat{X}}\beta s_0 y_i y_j\\
&+20e_{00}t_{00}\beta s_i y_j + 20e_{00}t_{00}\beta s_j y_i\\
&-72e_{00}t_{00}b_i s_0 y_j - 72e_{00}t_{00}b_j s_0 y_i - e_{00;0}t_{00}\beta b_i b_j\\
&+30e_{i0}t_{00}\beta s_0 y_j + 30e_{j0}t_{00}\beta s_0 y_i\\
&-6t_{00}\beta b_i q_{00}y_j + 4t_{00}\beta^2 s_0 s_i y_j + 4t_{00}\beta^2 s_0 s_j y_i\\
&+32e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_0 y_i - 32e_{00}\beta\eta y_i y_j\\
&+2t_{00}\beta^2 s_i t_0 y_j + 2t_{00}\beta^2 s_j t_0 y_i + 32\mathcal{L}_{\hat{X}}e_{00}\beta s_0 y_i y_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}s_0\beta y_i y_j - 18e_{00;0}\mathcal{L}_{\hat{X}}\beta y_i y_j\\
&+32e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_0 y_j - 4t_{00}\beta^2 q_{0i}y_j - 2t_{00}\beta^2 q_{0j}y_i\\
&+10t_{00}\beta^2 q_{i0}y_j + 10t_{00}\beta^2 q_{j0}y_i + 3a_{ij}e_{00;0}t_{00}\beta\\
&-36e_{00}e_{i0}t_{00}y_j - 36e_{00}e_{j0}t_{00}y_i\\
&-9e_{00;0}t_{00}b_i y_j - 32e_{00}\mathcal{L}_{\hat{X}}e_{00}y_i y_j + 4e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta y_j\\
&+4e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta y_i + 4\mathcal{L}_{\hat{X}}e_{00;0}\beta y_i y_j\\
&+36t_{00}q_{00}y_i y_j + 36t_{00}s_0^2 y_i y_j - 8e_{00}t_{00}\beta b_i b_j s_0\big]\alpha^2\\
&-96e_{00}^2\mathcal{L}_{\hat{X}}\beta\beta^3 b_i b_j + (48(b_i b_j t_{00}\\
&+2\mathcal{L}_{\hat{X}}\beta(b_i y_j + b_j y_i)))e_{00}^2\beta^2 + (((-48b_i y_j - 48b_j y_i)t_{00} - 96y_i y_j\mathcal{L}_{\hat{X}}\beta)e_{00}^2\\
&-80e_{00}t_{00}s_0 y_i y_j - 8e_{00;0}t_{00}y_i y_j)\beta + 88e_{00}^2 t_{00}y_i y_j\big\}\\
&+48e_{00}^2\big\{\big[(-2\beta b_i b_j + 2b_i y_j + 2b_j y_i)\mathcal{L}_{\hat{X}}\beta + b_i b_j t_{00}\big]\alpha^2 - 2y_i y_j\beta\mathcal{L}_{\hat{X}}\beta\\
&-\big[(b_i y_j + b_j y_i)\beta - y_i y_j\big]t_{00}\big\}\alpha^2
\end{aligned}
$$

By above equation, for any point $x \in M$, the irreducible polynomial $(\alpha^2 - \beta^2)$ divides e_{00}, or irreducible polynomial $(\alpha^2 - \beta^2)$ divides last equation. In the first case, for a function $c \in C^\infty(M)$, $e_{00} = 2c(x)(\alpha^2 - \beta^2)$ which means that F is of isotropic S-curvature. In the second case, the irreducible polynomial $(\alpha^2 - \beta^2)$

divides:

$$[(-2\beta b_i b_j + 2b_i y_j + 2b_j y_i)\mathcal{L}_{\hat{X}}\beta + b_i b_j t_{00}]\alpha^2$$
$$-2y_i y_j \beta\mathcal{L}_{\hat{X}}\beta - [(b_i y_j + b_j y_i)\beta - y_i y_j]t_{00}$$

By contracting previous equation with $b^i b^j$ we obtain

$$(\alpha^2 b^4 + \beta^2)\mathcal{L}_{\hat{X}}(\alpha^2 - \beta^2) + 2b^2(\alpha^2\mathcal{L}_{\hat{X}}\beta^2 - \beta^2\mathcal{L}_{\hat{X}}\alpha^2) = \eta(x, y)(\alpha^2 - \beta^2)$$

where $\eta(x, y)$ is a quadratic form.

By $Irrat = 0$, then α^2 divides A_1, where

$$A_1 = \frac{1}{2}\beta t_{00} y_i y_j (-192 e_{00}\beta s_0 + 96 e_{00}^2 - 19 e_{00;0}\beta)$$

In this case, $\mathcal{L}_{\hat{X}}\alpha^2 = \sigma(x)\alpha^2$ which means that $\hat{X}$ is a Killing vector field of α, or α^2 divides $-192e_{00}\beta s_0 + 96e_{00}^2 - 19e_{00;0}\beta$. We have

$$-192 e_{00}\beta s_0 + 96 e_{00}^2 - 19 e_{00;0}\beta = \lambda(x, y)\alpha^2$$

where $\lambda(x, y)$ is a quadratic form.

□

Example 4.1. *Let $M = \mathbb{R}^2$ and $F(x, y, u, v) = \sqrt{u^2 + v^2} + au + bv$, as it defined in Example 4.1. Then by the same argument, one can see the condition 1, 2 and 3 of Theorem 1.2 are held. If we substitute $\mathcal{L}_{\hat{X}}\alpha^2 = 2\alpha^2$ and $\mathcal{L}_{\hat{X}}\beta = \beta$ in 4 we have:*

$$[(\alpha^2 b^4 + \beta^2)(2\alpha^2 - \beta^2) = \eta^2(x, y)(\alpha^2 - \beta^2) \tag{4.4}$$

which means that $\eta(x, y) = 2(\alpha^2 b^4 + \beta^2)$. In this case, all of the conditions Theorem 2.1 hold.

Here we give an example which satisfies condition 1 and 3 of Theorem 2.1.

Example 4.2. *Let (θ, M) be the Funk metric as Example 3.2, i. e. condition 1 and 3 of Theorem 1.2 are held. If we substitute (3.8) in condition 4, we have*

$$\frac{|y|^2}{1 - |x|^2}\{48|y|^2(1 - |x|^2) + 19 < x, y >^2\} = (|y|^2(1 - |x|^2) + < x, y >^2)\lambda(x, y)$$

but $\lambda(x, y)$ should be a polynomial of degree 2, which is a contradiction. Now we consider condition 2. Since $\mathcal{L}_{\hat{V}}\alpha^2 = 0$, condition 2 is reduced to

$$(2b^2\alpha^2 - b^4\alpha^2 - beta^2)\mathcal{L}_{\hat{V}}\beta^2 = \eta(x, y)(\alpha^2 - \beta^2)$$

Then by substituting $b^2 = |x|^2$ and $\mathcal{L}_{\hat{V}}\beta^2$ we have

$$\begin{aligned}\{(\frac{|x|^2|y|^2}{1 - |x|^2} &+ \frac{|x|^2 < x, y >^2}{(1 - |x|^2)^2})(2 - |x|^2) - \frac{< x, y >^2}{(1 - |x|^2)^2}\} \\ &\times(-2 < x, y >< a, y >) = |y|^2\eta(x, y)\end{aligned} \tag{4.5}$$

where η is a polynomial of degree two. If we compute the extended form of (4.5) we have

$$a_1x_1(x_1^2 - 3x_2^2 + a_2x_2(x_2^2 - 3x_1^2) = 0$$

which means that $a = 0$, which is a contradiction. Therefore, there is not any $\eta(x, y)$ to applies condition 2.

Here we make an example which satisfies condition 2 and 3 of Theorem 1.2.

Example 4.3. *We consider Theorem 1.2 for Bergman metric. As we see in Example 3.3, conditions 2 and 3 are established automatically for a Killing vector field of F and the condition 1 of Theorem 1.1 do not hold. Then by substituting (3.10) in condition 4, we find out that the term*

$$\epsilon\beta^2(77f_0(\beta + \epsilon f_0) - 19f_0 + 96\epsilon(\beta + \epsilon f_0)^2) \tag{4.6}$$

a should be a multiple of α^2, which is impossible. Thus condition 4, does not exist.

5. C-projective vector fields of Funk type Finsler metrics

Let M be a bounded convex in $\mathbb{R}^n$ and θ be the Funk type Finsler metric on M, i.e.,

$$\theta_{x^k} = \theta\theta_{y^k} \tag{5.1}$$

The spray of a Funk type Finsler metric is given by $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ where $G^i = \frac{1}{2}\theta y^i$. Thus, every Funk type Finsler metric is locally projectively flat. It is easy to see that every Funk type Finsler metric is also of constant flag curvature $K = -\frac{1}{4}$. We are going to characterize all projective vector fields of Funk type Finsler metrics.

Theorem 5.1. *Let $V = V^i\frac{\partial}{\partial x^i}$ be a vector field on M. Then the complete lift of V, i.e., $\hat{V}$ is a projective vector field of (M, θ) if and only if there is a 1-form $\eta := \eta_t(x)y^t$ on M which satiesties*

$$2\frac{\partial^2 V^i}{\partial x^t \partial x^s} = \eta_t(x){\delta^i}_s + \eta_s(x){\delta^i}_t \tag{5.2}$$

Proof. Suppose that $\hat{V} = V^i\frac{\partial}{\partial x^i} + y^k\frac{\partial V^i}{\partial x^k}\frac{\partial}{\partial y^i}$ is a projective vector field of (M, θ). Then by [1], $\hat{V}$ is a projective vector field if and only if there exsits a function $P(x, y)$ satisfying

$$[\hat{V}, G] = Py^i. \tag{5.3}$$

By a direct calculation, we have

$$[\hat{V}, G] = (Qy^i - A^i)\frac{\partial}{\partial y^i}, \tag{5.4}$$

where $Q = (V^j\theta + y^k\frac{\partial V^j}{\partial x^k})\theta_j$ and $A^i = y^jy^k\frac{\partial^2 V^i}{\partial x^k\partial x^j}$. Comparing (5.3) with (5.4), we get

$$Qy^i - A^i = Py^i \tag{5.5}$$

By differentiating (5.5) with respect to y^s we have

$$2y^k\frac{\partial^2 V^i}{\partial x^k\partial x^s} = (P_s - Q_s)y^i + (P-Q)\delta^i{}_s \tag{5.6}$$

Again by differentiating (5.6) with respect to y^t we have

$$2\frac{\partial^2 V^i}{\partial x^t\partial x^s} = (P_{st} - Q_{st})y^i + (P_s - Q_s)\delta^i{}_t + (P_t - Q_t)\delta^i{}_s \tag{5.7}$$

By differentiating (5.7) with respect to y^r we have

$$(P_{rst} - Q_{rst})y^i + (P_{st} - Q_{st})\delta^i{}_r + (P_{sr} - Q_{sr})\delta^i{}_t + (P_{tr} - Q_{tr})\delta^i{}_s = 0 \tag{5.8}$$

Let $i = r$ in (5.8), then we have $(n+1)(P_{st} - Q_{st}) = 0$, which means that

$$P = Q + \eta_j(x)y^j, \tag{5.9}$$

for some 1-form η on M. If we put (5.9) into (5.5), we have

$$y^jy^k\frac{\partial^2 V^i}{\partial x^k\partial x^j} = \eta_j(x)y^jy^i. \tag{5.10}$$

By differentiating (5.10) with respect to y^s we have

$$2y^k\frac{\partial^2 V^i}{\partial x^k\partial x^s} = \eta_s(x)y^i + \eta_j(x)y^j\delta^i{}_s \tag{5.11}$$

Again by differentiating (5.11) with respect to y^t we have

$$2\frac{\partial^2 V^i}{\partial x^t\partial x^s} = \eta_s(x)\delta^i{}_t + \eta_t(x)\delta^i{}_s \tag{5.12}$$

Thus if $\hat{V}$ is a projective vector field of the Funk type metric θ, then there is a 1-form $\eta_t(x)$ which satisfies (5.12).

The converse is trivial. $\square$

Theorem 5.2. *Let θ be a Funk type metric on a bounded convex domain M in $\mathbb{R}^n$. Then $V = V^i\frac{\partial}{\partial x^i}$ is a projective vector field of θ if and only if V^i's are given by*

$$V^i = x^i < a, x > +Q^i_jx^j + c^i, \tag{5.13}$$

where $a, c = (c^1, \cdots, c^n) \in \mathbb{R}^n$ are two fixed vectors, (Q^i_j) is a fixed $n\times n$ real matrix and $< . >$ is the Euclidean inner product on $\mathbb{R}^n$. 1

$$[\hat{V}, G] = PY$$

where

$$\begin{aligned} P \quad &= -2 < a, y > -\theta_{x^k}(x^k < a, x > +Q^k_jx^j + c^k) \\ &\quad -\theta_{y^k}(y^k < a, x > +x^k < a, y > +Q^k_jy^j). \end{aligned}$$

Proof. By [12] the maximum degree of projective vector fields is $n(n+2)$, thus the projective vector field of Funk type metrics are exactly as (5.13) $\square$

Appendix 1

$$
\begin{aligned}
A_0 &= 8e_{00}t_{00}\beta^2 s_0 y_i - 4e_{00}^2 t_{00} y_i \beta + e_{00;0} t_{00}\beta^2 y_i \\
A_1 &= -2t_{00}(8e_{00}\beta s_0 y_i - 4e_{00}^2 y_i + e_{00;0}\beta y_i) \\
A_2 &= -4t_{00}\beta^3 s_0 s_i + 4t_{00}\beta^2 b_i s_0^2 - 12A_{i0}t_{00}\beta^3 \\
&\quad -64e_{00}(\mathcal{L}_{\hat{X}}\beta)\beta s_0 y_i - 16e_{00}\mathcal{L}_{\hat{X}} s_0 \beta^2 y_i \\
&\quad -16e_{00}\mathcal{L}_{\hat{X}} y_i \beta^2 s_0 - 32e_{00}t_{00}\beta b_i s_0 + 16e_{00}\beta^2 \eta y_i \\
&\quad +20e_{i0}t_{00}\beta^2 s_0 - 16\mathcal{L}_{\hat{X}} e_{00}\beta^2 s_0 y_i \\
&\quad -4t_{00}\beta^3 q_{i0} + 4t_{00}\beta^2 b_i q_{00} + 16t_{00}\beta^2 t_0 y_i \\
&\quad +16t_{00}\beta s0^2 y_i + 24e_{00}^2\mathcal{L}_{\hat{X}}\beta y_i + 8e_{00}^2\mathcal{L}_{\hat{X}} y_i \beta \\
&\quad +12e_{00}^2 t_{00} b_i - 8e_{00}e_{i0}t_{00}\beta + 16e_{00}\mathcal{L}_{\hat{X}} e_{00}\beta y_i \\
&\quad -24e_{00}t_{00}s_0 y_i - 8e_{00;0}(\mathcal{L}_{\hat{X}}\beta)\beta y_i - 2e_{00;0}\mathcal{L}_{\hat{X}} y_i \beta^2 \\
&\quad -4e_{00;0}t_{00}\beta b_i - 5e_{00;i}t_{00}\beta^2 + 10e_{i0;0}t_{00}\beta^2 \\
&\quad -2\mathcal{L}_{\hat{X}} e_{00;0}\beta^2 y_i + 16t_{00}\beta q_{00} y_i - 3e_{00;0}t_{00} y_i \\
A_3 &= 8\mathcal{L}_{\hat{X}} y_i \beta^3 t_0 + 8\mathcal{L}_{\hat{X}} y_i \beta^2 s_0^2 + 8\mathcal{L}_{\hat{X}} y_i \beta^2 q_{00} \\
&\quad +16t_{00}s_0^2 y_i + 16t_{00}q_{00}y_i - 2e_{00;0}\mathcal{L}_{\hat{X}} b_i \beta^2 \\
&\quad -8e_{00}e_{i0}t_{00} + 20e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 - 24A_{i0}\mathcal{L}_{\hat{X}}\beta\beta^3 \\
&\quad -36A_{i0}t_{00}\beta^2 - 10e_{00;i}\mathcal{L}_{\hat{X}}\beta\beta^2 - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta y_i \\
&\quad +20e_{i0;0}t_{00}\beta - 10e_{00;i}t_{00}\beta - 4\mathcal{L}_{\hat{X}} e_{00;0} y_i \beta \\
&\quad -2e_{i0}\beta^3\eta + 24e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i - 12t_{00}\beta^2 q_{0i} \\
&\quad -4e_{00;0}t_{00}b_i + 16e_{00}\mathcal{L}_{\hat{X}} e_{00} y_i - 2\mathcal{L}_{\hat{X}} e_{00;0}\beta^2 b_i \\
&\quad -20t_{00}\beta^2 q_{i0} + 8e_{i0}\mathcal{L}_{\hat{X}} s_0 \beta^3 + 8\mathcal{L}_{\hat{X}} e_{i0}\beta^3 s_0 \\
&\quad -4e_{00}\mathcal{L}_{\hat{X}} e_{i0}\beta^2 - 4e_{i0}\mathcal{L}_{\hat{X}} e_{00}\beta^2 + 8\mathcal{L}_{\hat{X}} t_0 \beta^3 y_i \\
&\quad +8\mathcal{L}_{\hat{X}} q_{00}\beta^2 y_i - 4e_{00;0}\mathcal{L}_{\hat{X}} y_i \beta + 8e_{00}^2\mathcal{L}_{\hat{X}} b_i \beta \\
&\quad -12e_{00}\beta^3\eta_i - 4\mathcal{L}_{\hat{X}} A_{i0}\beta^4 - 2\mathcal{L}_{\hat{X}} e_{00;i}\beta^3 \\
&\quad +4\mathcal{L}_{\hat{X}} e_{i0;0}\beta^3 + 8e_{00}^2\mathcal{L}_{\hat{X}} y_i - 64e_{00}\mathcal{L}_{\hat{X}}\beta b_i s_0 \beta \\
&\quad +16\mathcal{L}_{\hat{X}} s_0 \beta^2 s_0 y_i - 64e_{00}\mathcal{L}_{\hat{X}}\beta s_0 y_i - 32e_{00}t_{00}b_i s_0 + 40e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 \\
&\quad +40e_{i0}t_{00}\beta s_0 - 8t_{00}\beta^2 b_i t_0 \\
&\quad +24t_{00}\beta b_i s_0^2 + 24t_{00}\beta b_i q_{00} + 16t_{00}\beta t_0 y_i + 32\mathcal{L}_{\hat{X}}\beta s_0^2 y_i \beta \\
&\quad +32\mathcal{L}_{\hat{X}}\beta q_{00} y_i \beta + 16e_{00}\beta^2 b_i \eta \\
&\quad +32e_{00}\beta\eta y_i - 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta\beta - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i \beta \\
&\quad +16e_{00}\mathcal{L}_{\hat{X}} e_{00} b_i \beta - 32t_{00}\beta^2 s_0 si \\
&\quad -16e_{00}\mathcal{L}_{\hat{X}} b_i \beta^2 s_0 - 32e_{00}\mathcal{L}_{\hat{X}} y_i \beta s_0 - 16e_{00}\mathcal{L}_{\hat{X}} s_0 \beta^2 b_i \\
&\quad -32e_{00}\mathcal{L}_{\hat{X}} s_0 \beta y_i s_0 - 16e_{00}\mathcal{L}_{\hat{X}} y
\end{aligned}
$$

$$
\begin{aligned}
A_4 \quad = \quad & -8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta - 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta + 32\mathcal{L}_{\hat{X}}\beta s_0^2 y_i \\
& +8\mathcal{L}_{\hat{X}}b_i\beta^2 q_{00} + 16\mathcal{L}_{\hat{X}}y_i\beta^2 t_0 + 20t_{00}b_i s_0^2 \\
& +32\mathcal{L}_{\hat{X}}\beta q_{00}y_i + 20t_{00}b_i q_{00} - 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta \\
& -72A_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 - 8e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i \\
& +(40e_{i0;0}\mathcal{L}_{\hat{X}}\beta)\beta - 36A_{i0}t_{00}\beta - 20e_{00;i}(\mathcal{L}_{\hat{X}}\beta)\beta \\
& -4\mathcal{L}_{\hat{X}}e_{00;0}b_i\beta - 28t_{00}q_{i0}\beta - 8\mathcal{L}_{\hat{X}}s_0\beta^3 s_i \\
& -36e_{00}\beta^2\eta_i - 8\mathcal{L}_{\hat{X}}b_i\beta^3 t_0 - 6e_{i0}\beta^2\eta \\
& +8\mathcal{L}_{\hat{X}}b_i\beta^2 s_0^2 - 24(\mathcal{L}_{\hat{X}}\beta)\beta^2 q_{0i} - 24t_{00}\beta q_{0i} \\
& +16e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i + 16e_{00}\eta y_i - (16\mathcal{L}_{\hat{X}}\beta)\beta^2 q_{i0} \\
& -16e_{00}\mathcal{L}_{\hat{X}}y_i s_0 + 24e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 \\
& +24\mathcal{L}_{\hat{X}}e_{i0}\beta^2 s_0 - 16e_{00}\mathcal{L}_{\hat{X}}s_0 y_i - 16\mathcal{L}_{\hat{X}}e_{00}s_0 y_i \\
& +8\mathcal{L}_{\hat{X}}t_0\beta^3 b_i + 8\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_i + 16\mathcal{L}_{\hat{X}}t_0\beta^2 y_i \\
& +20e_{i0}t_{00}s_0 + 16\mathcal{L}_{\hat{X}}y_i s_0^2\beta + 16\mathcal{L}_{\hat{X}}y_i q_{00}\beta \\
& -4e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta + 16\mathcal{L}_{\hat{X}}q_{00}y_i\beta - 8\mathcal{L}_{\hat{X}}s_i\beta^3 s_0 \\
& -8\mathcal{L}_{\hat{X}}q_{i0}\beta^3 - 16\mathcal{L}_{\hat{X}}A_{i0}\beta^3 - 2e_{00;0}\mathcal{L}_{\hat{X}}y_i + 8e_{00}^2\mathcal{L}_{\hat{X}}b_i \\
& -5e_{00;i}t_{00} - 2\mathcal{L}_{\hat{X}}e_{00;0}y_i - 6\mathcal{L}_{\hat{X}}e_{00;i}\beta^2 \\
& +12\mathcal{L}_{\hat{X}}e_{i0;0}\beta^2 + 10e_{i0;0}t_{00} + 32\mathcal{L}_{\hat{X}}s_0\beta s_0 y_i \\
& -64e_{00}\mathcal{L}_{\hat{X}}\beta b_i s_0 + 80e_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta s_0 - 32t_{00}\beta b_i t_0 \\
& +32e_{00}\beta b_i\eta + 32\mathcal{L}_{\hat{X}}\beta b_i s_0^2\beta + 32\mathcal{L}_{\hat{X}}\beta b_i q_{00}\beta \\
& -40(\mathcal{L}_{\hat{X}}\beta)\beta^2 s_0 s_i - 52t_{00}\beta s_0 s_i - 32e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_0 \\
& -32e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_i - 32\mathcal{L}_{\hat{X}}e_{00}\beta b_i s_0 \\
& +16\mathcal{L}_{\hat{X}}s_0\beta^2 b_i s_0 + 48(\mathcal{L}_{\hat{X}}\beta)\beta t_0 y_i \\
A_5 \quad = \quad & -36e_{00}\eta_i\beta - 6e_{i0}\eta\beta + 8\mathcal{L}_{\hat{X}}y_i\beta t_0 - \\
& 72A_{i0}(\mathcal{L}_{\hat{X}}\beta)\beta - 32\mathcal{L}_{\hat{X}}\beta q_{i0}\beta + 24e_{i0}\mathcal{L}_{\hat{X}}s_0\beta \\
& +24\mathcal{L}_{\hat{X}}e_{i0}s_0\beta + 16\mathcal{L}_{\hat{X}}b_i s_0^2\beta - 24\mathcal{L}_{\hat{X}}s_i\beta^2 s_0 \\
& -24\mathcal{L}_{\hat{X}}s_0\beta^2 s_i + 32\mathcal{L}_{\hat{X}}\beta b_i s_0^2 - 16e_{00}\mathcal{L}_{\hat{X}}s_0 b_i \\
& +32\mathcal{L}_{\hat{X}}\beta b_i q_{00} - 48(\mathcal{L}_{\hat{X}}\beta)\beta q_{0i} + 16e_{00}b_i\eta \\
& -24t_{00}b_i t_0 - 24t_{00}s_0 s_i - 16e_{00}\mathcal{L}_{\hat{X}}b_i s_0 \\
& -16\mathcal{L}_{\hat{X}}e_{00}b_i s_0 + 16\mathcal{L}_{\hat{X}}t_0\beta^2 b_i + 8\mathcal{L}_{\hat{X}}\beta t_0 y_i \\
& +16(\mathcal{L}_{\hat{X}}s_0)s_0 y_i + 8\mathcal{L}_{\hat{X}}t_0\beta y_i + 40e_{i0}\mathcal{L}_{\hat{X}}\beta s_0 \\
& -32\mathcal{L}_{\hat{X}}b_i\beta^2 t_0 + 16\mathcal{L}_{\hat{X}}b_i q_{00}\beta + 16\mathcal{L}_{\hat{X}}q_{00}b_i\beta \\
& -12t_{00}q_{0i} - 4e_{00}\mathcal{L}_{\hat{X}}e_{i0} - 4e_{i0}\mathcal{L}_{\hat{X}}e_{00} \\
& -24\mathcal{L}_{\hat{X}}q_{i0}\beta^2 + 8\mathcal{L}_{\hat{X}}y_i s_0^2 + 8\mathcal{L}_{\hat{X}}y_i q_{00} - 2e_{00;0}\mathcal{L}_{\hat{X}}b_i \\
& +8\mathcal{L}_{\hat{X}}q_{00}y_i - 10e_{00;i}\mathcal{L}_{\hat{X}}\beta \\
& -2\mathcal{L}_{\hat{X}}e_{00;0}b_i - 12t_{00}q_{i0} - 6\mathcal{L}_{\hat{X}}e_{00;i}\beta + 12\mathcal{L}_{\hat{X}}e_{i0;0}\beta \\
& -24\mathcal{L}_{\hat{X}}A_{i0}\beta^2 + 20e_{i0;0}\mathcal{L}_{\hat{X}}\beta - 12A_{i0}t_{00}
\end{aligned}
$$

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\begin{aligned}
& -80\mathcal{L}_{\hat{X}}\beta)\beta s_0 s_i - 32\mathcal{L}_{\hat{X}}\beta b_i t_0\beta + 32\mathcal{L}_{\hat{X}} s_0\beta b_i s_0 \\
A_6 \;=\; & -32\mathcal{L}_{\hat{X}}\beta b_i t_0 - 40\mathcal{L}_{\hat{X}}\beta s_0 s_i - 40\mathcal{L}_{\hat{X}} b_i\beta t_0 \\
& +8\mathcal{L}_{\hat{X}} b_i s_0^2 - 24\mathcal{L}_{\hat{X}} s_0\beta s_i + 16\mathcal{L}_{\hat{X}} s_0 b_i s_0 \\
& -24\mathcal{L}_{\hat{X}} s_i\beta s_0 + 8\mathcal{L}_{\hat{X}} t_0\beta b_i - 24A_{i0}\mathcal{L}_{\hat{X}}\beta \\
& -12e_{00}\eta_i + 8e_{i0}\mathcal{L}_{\hat{X}} s_0 - 2e_{i0}\eta - 16\mathcal{L}_{\hat{X}} A_{i0}\beta \\
& +8\mathcal{L}_{\hat{X}} e_{i0}s_0 - 24\mathcal{L}_{\hat{X}}\beta q_{0i} - 16\mathcal{L}_{\hat{X}}\beta q_{i0} + 8\mathcal{L}_{\hat{X}} b_i q_{00} + 8\mathcal{L}_{\hat{X}} q_{00} b_i \\
& -24\mathcal{L}_{\hat{X}} q_{i0}\beta - 2\mathcal{L}_{\hat{X}} e_{00;i} + 4\mathcal{L}_{\hat{X}} e_{i0;0} \\
A_7 \;=\; & -16\mathcal{L}_{\hat{X}} b_i t_0 - 8\mathcal{L}_{\hat{X}} s_0 s_i - 8\mathcal{L}_{\hat{X}} s_i s_0 - 4\mathcal{L}_{\hat{X}} A_{i0} - 8\mathcal{L}_{\hat{X}} q_{i0}
\end{aligned}
$$

Appendix 2

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\begin{aligned}
A_0 \;=\; & \frac{1}{2} t_{00}\beta^2 y_i y_j(-32e_{00}\beta s_0 + 16e_{00}^2 - 3e_{00;0}\beta) \\
A_1 \;=\; & \frac{1}{2}\beta t_{00} y_i y_j(-192e_{00}\beta s_0 + 96e_{00}^2 - 19e_{00;0}\beta) \\
A_2 \;=\; & 8a_{ij}e_{00}t_{00}\beta^3 s_0 + 28e_{00}^2\mathcal{L}_{\hat{X}}\beta\beta y_i y_j - 4e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta^2 y_i y_j \\
& +2t_{00}\beta^4 s_0 s_i y_j + 2t_{00}\beta^4 s_0 s_j y_i \\
& -2t_{00}\beta^3 b_i s_0^2 y_j - 2t_{00}\beta^3 b_j s_0^2 y_i + 6e_{j0}t_{00}\beta^3 s_0 y_i \\
& +4e_{00}t_{00}\beta^3 s_i y_j + 4e_{00}t_{00}\beta^3 s_j y_i + 6e_{i0}t_{00}\beta^3 s_0 y_j \\
& +8e_{00}\mathcal{L}_{\hat{X}} y_j\beta^3 s_0 y_i + 12e_{00}^2 t_{00} b_i y_j\beta + 12e_{00}^2 t_{00} b_j y_i\beta \\
& -8e_{00}\beta^3\eta y_i y_j - 2t_{00}\beta^3 b_i q_{00} y_j \\
& -2t_{00}\beta^3 b_j q_{00} y_i - 8e_{00}e_{i0}t_{00}\beta^2 y_j - 8e_{00}e_{j0}t_{00}\beta^2 y_i \\
& -2e_{00;0}t_{00}\beta^2 b_i y_j - 2e_{00;0}t_{00}\beta^2 b_j y_i \\
& +8\mathcal{L}_{\hat{X}} e_{00}\beta^3 s_0 y_i y_j - 8e_{00}\mathcal{L}_{\hat{X}} e_{00}\beta^2 y_i y_j \\
& +4e_{00}\mathcal{L}_{\hat{X}} s_0\beta^3 y_i y_j + 8e_{00}\mathcal{L}_{\hat{X}} y_i\beta^3 s_0 y_j - 4a_{ij}e_{00}^2 t_{00}\beta^2 \\
& +(1/2)a_{ij}e_{00;0}t_{00}\beta^3 + 2e_{i0;0}t_{00}\beta^3 y_j + 2e_{j0;0}t_{00}\beta^3 y_i \\
& +2t_{00}\beta^4 q_{i0} y_j + 2t_{00}\beta^4 q_{j0} y_i - 4e_{00}^2\mathcal{L}_{\hat{X}} y_i\beta^2 y_j \\
& -4e_{00}^2\mathcal{L}_{\hat{X}} y_j\beta^2 y_i + e_{00;0}\mathcal{L}_{\hat{X}} y_i\beta^3 y_j + e_{00;0}\mathcal{L}_{\hat{X}} y_j\beta^3 y_i \\
& +(\mathcal{L}_{\hat{X}} e_{00})_{;0}\beta^3 y_i y_j + 8t_{00}\beta^3 t0 y_i y_j \\
& +8t_{00}\beta^2 q_{00} y_i y_j + 8t_{00}\beta^2 s_0^2 y_i y_j + 88e_{00}^2 t_{00} y_i y_j \\
& -16e_{00}t_{00}\beta^2 b_i s_0 y_j - 16e_{00}t_{00}\beta^2 b_j s_0 y_i \\
& -200e_{00}t_{00}\beta s_0 y_i y_j - 48e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 y_i y_j - (43/2e_{00;0})t_{00}\beta y_i y_j \\
A_3 \;=\; & -e_{ij;0}t_{00}\beta^4 - 8t_{00}\beta^2 b_i q_{00} y_j - 8t_{00}\beta^2 b_j q_{00} y_i + 6t_{00}\beta^3 s_0 s_i y_j \\
& +6t_{00}\beta^3 s_0 s_j y_i - 8t_{00}\beta^2 b_i s_0^2 y_j \\
& -8t_{00}\beta^2 b_j s_0^2 y_i + 48a_{ij}e_{00}t_{00}\beta^2 s_0 - 4e_{00}e_{i0}t_{00}\beta^2 b_j \\
& -4e_{00}e_{j0}t_{00}\beta^2 b_i + 44t_{00}\beta^2 t_0 y_i y_j \\
& +44t_{00}\beta s_0^2 y_i y_j - 44e_{00}e_{i0}t_{00}\beta y_j - 44e_{00}e_{j0}t_{00}\beta y_i \\
& -22e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta y_i y_j - 11e_{00;0}t_{00}\beta b_i y_j
\end{aligned}
$$

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\begin{aligned}
& -11e_{00;0}t_{00}\beta b_j y_i + 44t_{00}\beta q_{00}y_i y_j + 24e_{00}t_{00}\beta^2 s_i y_j \\
& +24e_{00}t_{00}\beta^2 s_j y_i + 36e_{i0}t_{00}\beta^2 s_0 y_j \\
& +36e_{j0}t_{00}\beta^2 s_0 y_i + t_{00}\beta^3 b_i t_0 y_j + t_{00}\beta^3 b_j t_0 y_i \\
& -e_{00;0}t_{00}\beta^2 b_i b_j + 2e_{00}t_{00}\beta^3 b_i s_j + 2e_{00}t_{00}\beta^3 b_j s_i \\
& +3e_{i0}t_{00}\beta^3 b_j s_0 + 3e_{j0}t_{00}\beta^3 b_i s_0 + 40e_{00}\mathcal{L}_{\hat{X}}y_j\beta^2 s_0 y_i \\
& -40e_{00}\beta^2 etay_i y_j + 2t_{00}\beta^3 s_i t_0 y_j \\
& +2t_{00}\beta^3 s_j t_0 y_i + 6e_{00}^2 t_{00}b_i b_j\beta + 40\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_0 y_i y_j \\
& -40e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta y_i y_j + 28e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2 y_i y_j \\
& +40e_{00}\mathcal{L}_{\hat{X}}y_i\beta^2 s_0 y_j - 2t_{00}\beta^3 q_{0j}y_i - e_{j0}t_{00}\beta^4 s_i \\
& +2e_{i0}e_{j0}t_{00}\beta^3 + e_{00}e_{ij}t_{00}\beta^3 - e_{i0}t_{00}\beta^4 s_0 \\
& -2e_{ij}t_{00}\beta^4 s_0 - t_{00}\beta^4 t_i y_j - t_{00}\beta^4 t_j y_i + e_{i0;0}t_{00}\beta^3 b_j \\
& +e_{j0;0}t_{00}\beta^3 b_i - 4t_{00}\beta^3 q_{0i}y_j - 2a_{ij}t_{00}\beta^3 q_{00} \\
& -2a_{ij}t_{00}\beta^4 t_0 - 2a_{ij}t_{00}\beta^3 s_0^2 + 12t_{00}\beta^3 q_{i0}y_j \\
& +12t_{00}\beta^3 q_{j0}y_i - 22a_{ij}e_{00}^2 t_{00}\beta + 124e_{00}^2\mathcal{L}_{\hat{X}}\beta y_i y_j \\
& +60e_{00}^2 t_{00}b_i y_j + 60e_{00}^2 t_{00}b_j y_i - 20e_{00}^2\mathcal{L}_{\hat{X}}y_i\beta y_j \\
& -20e_{00}^2\mathcal{L}_{\hat{X}}y_j\beta y_i + 5e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta^2 y_j \\
& +5e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta^2 y_i + 5(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta^2 y_i y_j \\
& +(7/2a_{ij})e_{00;0}t_{00}\beta^2 - 120e_{00}t_{00}s_0 y_i y_j \\
& -8e_{00}t_{00}\beta^2 b_i b_j s_0 - 208e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_0 y_i y_j - 88e_{00}t_{00}\beta b_i s_0 y_j \\
& -88e_{00}t_{00}\beta b_j s_0 y_i + 12e_{i0;0}t_{00}\beta^2 y_j \\
& +12e_{j0;0}t_{00}\beta^2 y_i - (27/2e_{00;0})t_{00}y_i y_j \\
A_4 \quad = \quad & -9e_{ij;0}t_{00}\beta^3 - 18a_{ij}e_{00}^2 t_{00} - 2\mathcal{L}_{\hat{X}}q_{i0}\beta^4 y_j \\
& -2\mathcal{L}_{\hat{X}}q_{j0}\beta^4 y_i - 2e_{i0;0}\mathcal{L}_{\hat{X}}y_j\beta^3 - 2e_{j0;0}\mathcal{L}_{\hat{X}}y_i\beta^3 \\
& -2(\mathcal{L}_{\hat{X}}e_{i0})_{;0}\beta^3 y_j - 2\mathcal{L}_{\hat{X}}e_{j0;0}\beta^3 y_i - 16e_{00}^2\mathcal{L}_{\hat{X}}y_i y_j \\
& -16e_{00}^2\mathcal{L}_{\hat{X}}y_j y_i - 2\mathcal{L}_{\hat{X}}y_i\beta^4 q_{j0} - 2\mathcal{L}_{\hat{X}}y_j\beta^4 q_{i0} \\
& -a_{ij}(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta^3 + 4e_{00}^2\mathcal{L}_{\hat{X}}a_{ij}\beta^2 - e_{00;0}\mathcal{L}_{\hat{X}}a_{ij}\beta^3 \\
& +17t_{00}\beta^2 b_i t_0 y_j + 17t_{00}\beta^2 b_j t_0 y_i + 18t_{00}\beta b_i s_0^2 y_j \\
& +18t_{00}\beta b_j s_0^2 y_i + 72a_{ij}e_{00}t_{00}\beta s_0 - 56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta\beta y_j \\
& -36e_{00}e_{i0}t_{00}\beta b_j - 56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta y_i \\
& -36e_{00}e_{j0}t_{00}\beta b_i + 18e_{00}t_{00}\beta^2 b_i s_j + 18e_{00}t_{00}\beta^2 b_j s_i \\
& +36e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 y_j + 27e_{i0}t_{00}\beta^2 b_j s_0 \\
& +36e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0 y_i + 27e_{j0}t_{00}\beta^2 b_i s_0 - 12\mathcal{L}_{\hat{X}}\beta\beta^2 b_i q_{00}y_j \\
& -12\mathcal{L}_{\hat{X}}\beta\beta^2 b_j q_{00}y_i + 48\mathcal{L}_{\hat{X}}\beta\beta^2 t_0 y_i y_j \\
& +56\mathcal{L}_{\hat{X}}\beta\beta s_0^2 y_i y_j - 160e_{00}\mathcal{L}_{\hat{X}}\beta s_0 y_i y_j + 36e_{00}t_{00}\beta s_i y_j \\
& +36e_{00}t_{00}\beta s_j y_i - 72e_{00}t_{00}b_i s_0 y_j \\
& -72e_{00}t_{00}b_j s_0 y_i - 14e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta b_i y_j - 14e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta b_j y_i
\end{aligned}
$$

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\begin{aligned}
&-9e_{00;0}t_{00}\beta b_ib_j+54e_{i0}t_{00}\beta s_0y_j\\
&+54e_{j0}t_{00}\beta s_0y_i+56\mathcal{L}_{\hat{X}}\beta\beta q_{00}y_iy_j+18t_{00}\beta b_iq_{00}y_j\\
&+18t_{00}\beta b_jq_{00}y_i+4\mathcal{L}_{\hat{X}}s_0\beta^3b_js_0y_i\\
&-16t_{00}\beta^2s_0s_iy_j-16t_{00}\beta^2s_0s_jy_i-8t_{00}\beta^3b_ib_jt_0+4t_{00}\beta^3b_is_jt_0\\
&+4t_{00}\beta^3b_js_it_0-4t_{00}\beta^3b_is_0s_j\\
&-4t_{00}\beta^3b_js_0s_i+4\mathcal{L}_{\hat{X}}s_0\beta^3b_is_0y_j+56e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_0y_i\\
&-56e_{00}\beta\eta y_iy_j+16\mathcal{L}_{\hat{X}}e_{00}\beta^2b_is_0y_j\\
&+16\mathcal{L}_{\hat{X}}e_{00}\beta^2b_js_0y_i-16\mathcal{L}_{\hat{X}}s_0\beta^2s_0y_iy_j+10t_{00}\beta^2s_it_0y_j\\
&+10t_{00}\beta^2s_jt_0y_i+16e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2s_0y_j\\
&+16e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2s_0y_i+16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2b_iy_j+16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2b_jy_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}y_i\beta^2b_js_0+16e_{00}\mathcal{L}_{\hat{X}}y_j\beta^2b_is_0\\
&-16e_{00}\beta^2b_i\eta y_j-16e_{00}\beta^2b_j\eta y_i+10\mathcal{L}_{\hat{X}}\beta\beta^3s_0s_iy_j\\
&+10\mathcal{L}_{\hat{X}}\beta\beta^3s_0s_jy_i-12\mathcal{L}_{\hat{X}}\beta\beta^2b_is_0^2y_j\\
&-12\mathcal{L}_{\hat{X}}\beta\beta^2b_js_0^2y_i+48a_{ij}e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2s_0+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2s_iy_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2s_jy_i+56\mathcal{L}_{\hat{X}}e_{00}\beta s_0y_iy_j\\
&-24e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta b_iy_j-24e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta b_jy_i+44e_{00}\mathcal{L}_{\hat{X}}s_0\beta y_iy_j\\
&+56e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_0y_j-2t_{00}\beta^3b_jq_{0i}\\
&-26t_{00}\beta^2q_{0i}y_j-18t_{00}\beta^2q_{0j}y_i+18t_{00}\beta^2q_{i0}y_j\\
&+18t_{00}\beta^2q_{j0}y_i-36e_{00}e_{i0}t_{00}y_j-36e_{00}e_{j0}t_{00}y_i\\
&-2\mathcal{L}_{\hat{X}}s_0\beta^4s_iy_j-2\mathcal{L}_{\hat{X}}s_0\beta^4s_jy_i-2\mathcal{L}_{\hat{X}}y_i\beta^4s_0s_j\\
&+2\mathcal{L}_{\hat{X}}y_i\beta^3b_js_0^2+2\mathcal{L}_{\hat{X}}b_i\beta^3s_0^2y_j+2\mathcal{L}_{\hat{X}}b_j\beta^3s_0^2y_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}y_i\beta^3s_j-4e_{00}\mathcal{L}_{\hat{X}}y_j\beta^3s_i+4e_{00}\beta^3\eta_iy_j\\
&+4e_{00}\beta^3\eta_jy_i-2\mathcal{L}_{\hat{X}}y_j\beta^4s_0s_i+2\mathcal{L}_{\hat{X}}y_j\beta^3b_is_0^2\\
&-4a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta^3+8a_{ij}e_{00}\beta^3\eta-8a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^3s_0\\
&-2\mathcal{L}_{\hat{X}}s_i\beta^4s_0y_j-2\mathcal{L}_{\hat{X}}s_j\beta^4s_0y_i+2\mathcal{L}_{\hat{X}}q_{00}\beta^3b_iy_j\\
&+2\mathcal{L}_{\hat{X}}q_{00}\beta^3b_jy_i-8\mathcal{L}_{\hat{X}}t_0\beta^3y_iy_j-6e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^3y_j\\
&-6e_{i0}\mathcal{L}_{\hat{X}}y_j\beta^3s_0+6e_{i0}\beta^3\eta y_j-6e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^3y_i\\
&-6e_{j0}\mathcal{L}_{\hat{X}}y_i\beta^3s_0+6e_{j0}\beta^3\eta y_i-8e_{00}\mathcal{L}_{\hat{X}}a_{ij}\beta^3s_0\\
&-4e_{00}\mathcal{L}_{\hat{X}}s_i\beta^3y_j-4e_{00}\mathcal{L}_{\hat{X}}s_j\beta^3y_i+8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta^2y_j\\
&+8e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta^2y_i+2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta^2y_j+2e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta^2y_i\\
&+2e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta^2b_j+2e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta^2b_i\\
&+2\mathcal{L}_{\hat{X}}y_i\beta^3b_jq_{00}-8\mathcal{L}_{\hat{X}}y_i\beta^3t_0y_j-8\mathcal{L}_{\hat{X}}y_i\beta^2s_0^2y_j2\mathcal{L}_{\hat{X}}y_j\beta^3b_iq_{00}\\
&-8\mathcal{L}_{\hat{X}}y_j\beta^3t_0y_i-8\mathcal{L}_{\hat{X}}y_j\beta^2s_0^2y_i\\
&+8a_{ij}e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta^2-4\mathcal{L}_{\hat{X}}e_{00}\beta^3s_iy_j-4\mathcal{L}_{\hat{X}}e_{00}\beta^3s_jy_i\\
&-6\mathcal{L}_{\hat{X}}e_{i0}\beta^3s_0y_j-6\mathcal{L}_{\hat{X}}e_{j0}\beta^3s_0y_i\\
&+2\mathcal{L}_{\hat{X}}b_i\beta^3q_{00}y_j+2\mathcal{L}_{\hat{X}}b_j\beta^3q_{00}y_i-32e_{00}\mathcal{L}_{\hat{X}}e_{00}y_iy_j
\end{aligned}
$$

$$
\begin{aligned}
& +7e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta y_j + 7e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta y_i \\
& +7\mathcal{L}_{\hat{X}}e_{00;0}\beta y_i y_j + 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 y_j + 8e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 y_i \\
& +2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_i y_j + 2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_j y_i \\
& -8\mathcal{L}_{\hat{X}}q_{00}\beta^2 y_i y_j - 8\mathcal{L}_{\hat{X}}y_i\beta^2 q_{00}y_j - 8\mathcal{L}_{\hat{X}}y_j\beta^2 q_{00}y_i \\
& -12e_{00}^2\mathcal{L}_{\hat{X}}b_i\beta y_j - 12e_{00}^2\mathcal{L}_{\hat{X}}b_j\beta y_i \\
& -12e_{00}^2\mathcal{L}_{\hat{X}}y_i\beta b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}y_j\beta b_i + 8e_{00}e_{i0}\mathcal{L}_{\hat{X}}y_j\beta^2 \\
& +8e_{00}e_{j0}\mathcal{L}_{\hat{X}}y_i\beta^2 + (11/2a_{ij})e_{00;0}t_{00}\beta \\
& +36t_{00}\beta t_0 y_i y_j + 36t_{00}q_{00}y_i y_j + 36t_{00}s_0^2 y_i y_j - 112e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i s_0 y_j \\
& -112e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_j s_0 y_i \\
& -72e_{00}t_{00}\beta b_i b_j s_0 - 9e_{i0}t_{00}\beta^3 s_j - 18e_{ij}t_{00}\beta^3 s_0 \\
& +10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{i0}y_j + 10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{j0}y_i - 18a_{ij}t_{00}\beta^3 t_0 \\
& -18a_{ij}t_{00}\beta^2 s_0^2 - 18e_{00;0}\mathcal{L}_{\hat{X}}\beta y_i y_j - 9e_{00;0}t_{00}b_i y_j \\
& -9e_{00;0}t_{00}b_j y_i + 18e_{i0;0}t_{00}\beta y_j + 18e_{j0;0}t_{00}\beta y_i \\
& -9t_{00}\beta^3 tiy_j - 9t_{00}\beta^3 tjy_i - 28a_{ij}e_{00}^2\mathcal{L}_{\hat{X}}\beta\beta \\
& +4a_{ij}e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta^2 - 18a_{ij}t_{00}\beta^2 q_{00} + 96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i y_j \\
& +96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_j y_i + 54e_{00}^2 t_{00}b_i b_j + 9e_{00}e_{ij}t_{00}\beta^2 \\
& +18e_{i0}e_{j0}t_{00}\beta^2 + 12e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 y_j + 9e_{i0;0}t_{00}\beta^2 b_j \\
& +12e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 y_i + 9e_{j0;0}t_{00}\beta^2 b_i - 9e_{j0}t_{00}\beta^3 s_i \\
A_5 \;=\;& 4a_{ij}\mathcal{L}_{\hat{X}}t_0\beta^4 - 10e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta^3 + 5t_{00}\beta^3 q_{ij} \\
& 5t_{00}\beta^3 q_{ji} - 28a_{ij}e_{00}^2\mathcal{L}_{\hat{X}}\beta - 21e_{ij;0}t_{00}\beta^2 \\
& +8e_{i0;0}t_{00}y_j + 8e_{j0;0}t_{00}y_i - 2e_{i0}\beta^4\eta_j + 4e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^4 \\
& -4e_{ij}\beta^4 eta + 2e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^4 - 2e_{j0}\beta^4\eta_i \\
& +4\mathcal{L}_{\hat{X}}a_{ij}\beta^4 t_0 + 4\mathcal{L}_{\hat{X}}a_{ij}\beta^3 s_0^2 + 2\mathcal{L}_{\hat{X}}e_{i0}\beta^4 s_j \\
& +4\mathcal{L}_{\hat{X}}e_{ij}\beta^4 s_0 + 2\mathcal{L}_{\hat{X}}e_{j0}\beta^4 s_i - 2e_{00}\mathcal{L}_{\hat{X}}e_{ij}\beta^3 \\
& -4e_{i0}\mathcal{L}_{\hat{X}}e_{j0}\beta^3 + 2\mathcal{L}_{\hat{X}}t_j\beta^4 y_i + 2\mathcal{L}_{\hat{X}}t_j\beta^4 y_j \\
& +2e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^4 - 2e_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^3 - 4e_{j0}\mathcal{L}_{\hat{X}}e_{i0}\beta^3 \\
& -2e_{j0;0}\mathcal{L}_{\hat{X}}b_i\beta^3 + 4\mathcal{L}_{\hat{X}}a_{ij}\beta^3 q_{00} - 2\mathcal{L}_{\hat{X}}e_{i0;0}\beta^3 b_j \\
& -2\mathcal{L}_{\hat{X}}e_{j0;0}\beta^3 b_i + 4\mathcal{L}_{\hat{X}}q_{0i}\beta^3 y_j + 4\mathcal{L}_{\hat{X}}q_{0j}\beta^3 y_i \\
& +2\mathcal{L}_{\hat{X}}y_i\beta^4 t_j + 2\mathcal{L}_{\hat{X}}y_j\beta^4 t_i + 4a_{ij}\mathcal{L}_{\hat{X}}q_{00}\beta^3 \\
& -6e_{i0;0}\mathcal{L}_{\hat{X}}y_j\beta^2 - 6e_{j0;0}\mathcal{L}_{\hat{X}}y_i\beta^2 - 6(\mathcal{L}_{\hat{X}}e_{i0})_{;0}\beta^2 y_j \\
& -6(\mathcal{L}_{\hat{X}}e_{j0})_{;0}\beta^2 y_i + 3e_{00;0}\mathcal{L}_{\hat{X}}y_i y_j + 3e_{00;0}\mathcal{L}_{\hat{X}}y_j y_i \\
& +3\mathcal{L}_{\hat{X}}e_{00;0}y_i y_j + 4\mathcal{L}_{\hat{X}}y_i\beta^3 q_{0j} + 4\mathcal{L}_{\hat{X}}y_j\beta^3 q_{0i} \\
& -3a_{ij}\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 + 8e_{00}^2\mathcal{L}_{\hat{X}}a_{ij}\beta - 12e_{00}^2\mathcal{L}_{\hat{X}}b_i y_j \\
& -12e_{00}^2\mathcal{L}_{\hat{X}}b_j y_i - 12e_{00}^2\mathcal{L}_{\hat{X}}y_i b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}y_j b_i \\
& -3e_{00;0}\mathcal{L}_{\hat{X}}a_{ij}\beta^2 - 2e_{i0;0}\mathcal{L}_{\hat{X}}b_j\beta^3 - 14e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta b_i b_j \\
& +72e_{i0}\mathcal{L}_{\hat{X}}\beta\beta s_0 y_j + 45e_{i0}t_{00}\beta b_j s_0
\end{aligned}
$$

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\begin{aligned}
&+72e_{j0}\mathcal{L}_{\hat{X}}\beta\beta s_0y_i+45e_{j0}t_{00}\beta b_is_0+32\mathcal{L}_{\hat{X}}\beta\beta b_iq_{00}y_j\\
&+32\mathcal{L}_{\hat{X}}\beta\beta b_jq_{00}y_i+40\mathcal{L}_{\hat{X}}\beta\beta t_0y_iy_j\\
&+28t_{00}\beta b_ib_jq_{00}+3t_{00}\beta b_it_0y_j+3t_{00}\beta b_jt_0y_i\\
&-42t_{00}\beta s_0s_iy_j-42t_{00}\beta s_0s_jy_i+16e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2b_js_0\\
&+16e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2b_is_0+16e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2b_ib_j-16e_{00}\beta^2b_ib_j\eta\\
&+16t_{00}\beta^2b_is_jt_0+16t_{00}\beta^2b_js_it_0\\
&-24e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta b_ib_j+32e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_0y_j+16\mathcal{L}_{\hat{X}}e_{00}\beta^2b_ib_js_0\\
&-14\mathcal{L}_{\hat{X}}\beta\beta^2s_0s_iy_j-14\mathcal{L}_{\hat{X}}\beta\beta^2s_0s_jy_i\\
&+8\mathcal{L}_{\hat{X}}\beta\beta^2s_it_0y_j+8\mathcal{L}_{\hat{X}}\beta\beta^2s_jt_0y_i-4\mathcal{L}_{\hat{X}}s_0\beta^2b_is_0y_j\\
&-4\mathcal{L}_{\hat{X}}s_0\beta^2b_js_0y_i+32\mathcal{L}_{\hat{X}}e_{00}\beta b_is_0y_j\\
&+32\mathcal{L}_{\hat{X}}e_{00}\beta b_js_0y_i-28t_{00}\beta^2b_ib_jt_0-32\mathcal{L}_{\hat{X}}s_0\beta s_0y_iy_j\\
&+14t_{00}\beta s_it_0y_j+14t_{00}\beta s_jt_0y_i\\
&+32e_{00}\mathcal{L}_{\hat{X}}b_j\beta s_0y_i+32e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_iy_j+32e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_jy_i\\
&+32e_{00}\mathcal{L}_{\hat{X}}y_i\beta b_js_0+32e_{00}\mathcal{L}_{\hat{X}}y_j\beta b_is_0\\
&-32e_{00}\beta b_i\eta y_j-32e_{00}\beta b_j\eta y_i+36e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2b_js_0\\
&+36e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2b_is_0+28\mathcal{L}_{\hat{X}}\beta\beta^2b_it_0y_j\\
&+28\mathcal{L}_{\hat{X}}\beta\beta^2b_jt_0y_i+32\mathcal{L}_{\hat{X}}\beta\beta b_is_0^2y_j+32\mathcal{L}_{\hat{X}}\beta\beta b_js_0^2y_i\\
&-34t_{00}\beta^2b_is_0s_j-34t_{00}\beta^2b_js_0s_i\\
&+28t_{00}\beta b_ib_js_0^2+96a_{ij}e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_0-56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta\beta b_j\\
&-56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta b_i+48e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_iy_j\\
&+48e_{00}\mathcal{L}_{\hat{X}}\beta\beta s_jy_i-112e_{00}\mathcal{L}_{\hat{X}}\beta b_is_0y_j-112e_{00}\mathcal{L}_{\hat{X}}\beta b_js_0y_i\\
&+30e_{00}t_{00}\beta b_is_j+30e_{00}t_{00}\beta b_js_i\\
&-64e_{00}t_{00}b_ib_js_0+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2b_is_j+24e_{00}\mathcal{L}_{\hat{X}}\beta\beta^2b_js_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}b_j\beta^3s_i-4e_{00}\mathcal{L}_{\hat{X}}s_i\beta^3b_j\\
&-4e_{00}\mathcal{L}_{\hat{X}}s_j\beta^3b_i+4e_{00}\beta^3b_i\eta_j+4e_{00}\beta^3b_j\eta_i-t_{00}\beta^3b_it_j\\
&-t_{00}\beta^3b_jt_i+8t_{00}\beta q_{j0}y_i+24t_{00}b_iq_{00}y_j\\
&+24t_{00}b_jq_{00}y_i-6e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^3s_0-6e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^3b_j\\
&+6e_{i0}\beta^3b_j\eta-6e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^3s_0-6e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^3b_i\\
&+6e_{j0}\beta^3b_ieta-4\mathcal{L}_{\hat{X}}e_{00}\beta^3b_is_j-4\mathcal{L}_{\hat{X}}e_{00}\beta^3b_js_i\\
&-6\mathcal{L}_{\hat{X}}e_{i0}\beta^3b_js_0-6\mathcal{L}_{\hat{X}}e_{j0}\beta^3b_is_0+8a_{ij}\mathcal{L}_{\hat{X}}s_0\beta^3s_0\\
&-4e_{00}\mathcal{L}_{\hat{X}}b_i\beta^3s_j+4\mathcal{L}_{\hat{X}}t_0\beta^3s_iy_j+4\mathcal{L}_{\hat{X}}t_0\beta^3s_jy_i\\
&+4\mathcal{L}_{\hat{X}}y_i\beta^3s_jt_0-2\mathcal{L}_{\hat{X}}y_i\beta^2b_js_0^2-2\mathcal{L}_{\hat{X}}b_i\beta^2s_0^2y_j\\
&-2\mathcal{L}_{\hat{X}}b_j\beta^2s_0^2y_i-12e_{00}\mathcal{L}_{\hat{X}}y_i\beta^2s_j-12e_{00}\mathcal{L}_{\hat{X}}y_j\beta^2s_i\\
&+12e_{00}\beta^2\eta_iy_j+12e_{00}\beta^2\eta_jy_i+2e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta^2b_j\\
&+2e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta^2b_i+4\mathcal{L}_{\hat{X}}y_j\beta^3s_it_0-2\mathcal{L}_{\hat{X}}y_j\beta^2b_is_0^2\\
&-12a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta^2+24a_{ij}e_{00}\beta^2\eta
\end{aligned}
$$

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\begin{aligned}
&-24a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_0 - 12e_{00}^2\mathcal{L}_{\hat{X}}b_i\beta b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}b_j\beta b_i\\
&+8e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^2 + 4\mathcal{L}_{\hat{X}}s_i\beta^3 t_0 y_j\\
&+4\mathcal{L}_{\hat{X}}s_j\beta^3 t_0 y_i - 2\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_i y_j - 2\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_j y_i\\
&-16\mathcal{L}_{\hat{X}}t_0\beta^2 y_i y_j + 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_j - 18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 y_j\\
&-18e_{i0}\mathcal{L}_{\hat{X}}y_j\beta^2 s_0 + 18e_{i0}\beta^2\eta y_j + 8e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_i\\
&-18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^2 y_i - 18e_{j0}\mathcal{L}_{\hat{X}}y_i\beta^2 s_0 + 18e_{j0}\beta^2\eta y_i\\
&+8e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^2 - 24e_{00}\mathcal{L}_{\hat{X}}a_{ij}\beta^2 s_0 + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta^2 b_j\\
&+8e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta^2 b_i - 12e_{00}\mathcal{L}_{\hat{X}}s_i\beta^2 y_j\\
&-12e_{00}\mathcal{L}_{\hat{X}}s_j\beta^2 y_i + 16e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta y_i + 20e_{00}\mathcal{L}_{\hat{X}}s_0 y_i y_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}y_i s_0 y_j + 24e_{00}\mathcal{L}_{\hat{X}}y_j s_0 y_i\\
&-24e_{00}\eta y_i y_j + 4e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta y_j + 4e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta y_i\\
&+4e_{00;0}\mathcal{L}_{\hat{X}}y_i\beta b_j + 4e_{00;0}\mathcal{L}_{\hat{X}}y_j\beta b_i - 2\mathcal{L}_{\hat{X}}y_i\beta^2 b_j q_{00}\\
&-16\mathcal{L}_{\hat{X}}y_i\beta^2 t_0 y_j - 16\mathcal{L}_{\hat{X}}y_i\beta s_0^2 y_j - 2\mathcal{L}_{\hat{X}}y_j\beta^2 b_i q_{00}\\
&-16\mathcal{L}_{\hat{X}}y_j\beta^2 t_0 y_i - 16\mathcal{L}_{\hat{X}}y_j\beta s_0^2 y_i + 16a_{ij}e_{00}\mathcal{L}_{\hat{X}}e_{00}\beta\\
&-12\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_i y_j - 12\mathcal{L}_{\hat{X}}e_{00}\beta^2 s_j y_i + 2\mathcal{L}_{\hat{X}}e_{00;0}\beta^2 b_i b_j\\
&-18\mathcal{L}_{\hat{X}}e_{i0}\beta^2 s_0 y_j - 18\mathcal{L}_{\hat{X}}e_{j0}\beta^2 s_0 y_i\\
&-2\mathcal{L}_{\hat{X}}b_i\beta^2 q_{00}y_j - 2\mathcal{L}_{\hat{X}}b_j\beta^2 q_{00}y_i + 16e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta y_j\\
&+16e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta y_i + 24\mathcal{L}_{\hat{X}}e_{00}s_0 y_i y_j + 4(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta b_i y_j\\
&+4(\mathcal{L}_{\hat{X}}e_{00})_{;0}\beta b_j y_i - 16\mathcal{L}_{\hat{X}}q_{00}\beta y_i y_j - 16\mathcal{L}_{\hat{X}}y_i\beta q_{00}y_j\\
&-16\mathcal{L}_{\hat{X}}y_j\beta q_{00}y_i + 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}y_j\beta\\
&+16e_{00}e_{j0}\mathcal{L}_{\hat{X}}y_i\beta - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_i y_j - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_j y_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta y_j + (5/2a_{ij})e_{00;0}t_{00}\\
&+2\mathcal{L}_{\hat{X}}e_{ij;0}\beta^4 - 10\mathcal{L}_{\hat{X}}t_0\beta^3 b_i y_j - 10\mathcal{L}_{\hat{X}}t_0\beta^3 b_j y_i\\
&-10\mathcal{L}_{\hat{X}}y_i\beta^3 b_j t_0 - 2\mathcal{L}_{\hat{X}}y_i\beta^3 s_0 s_j - 10\mathcal{L}_{\hat{X}}y_j\beta^3 b_i t_0\\
&-2\mathcal{L}_{\hat{X}}y_j\beta^3 s_0 s_i - 10\mathcal{L}_{\hat{X}}b_i\beta^3 t_0 y_j - 10\mathcal{L}_{\hat{X}}b_j\beta^3 t_0 y_i\\
&-6\mathcal{L}_{\hat{X}}s_0\beta^3 s_i y_j - 6\mathcal{L}_{\hat{X}}s_0\beta^3 s_j y_i - 2\mathcal{L}_{\hat{X}}s_i\beta^3 s_0 y_j\\
&-2\mathcal{L}_{\hat{X}}s_j\beta^3 s_0 y_i - 6\mathcal{L}_{\hat{X}}q_{i0}\beta^3 y_j - 6\mathcal{L}_{\hat{X}}q_{j0}\beta^3 y_i\\
&-6\mathcal{L}_{\hat{X}}y_i\beta^3 q_{j0} - 6\mathcal{L}_{\hat{X}}y_j\beta^3 q_{i0} - 112e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i b_j s_0\\
&-21e_{i0}t_{00}\beta^2 s_j + 12e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 b_j - 42e_{ij}t_{00}\beta^2 s_0\\
&-20a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^3 t_0 - 24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 s_0^2 - 10e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^3 s_j\\
&-20e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^3 s_0 - 10e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^3 s_i - 10\mathcal{L}_{\hat{X}}\beta\beta^3 t_i y_j\\
&-10\mathcal{L}_{\hat{X}}\beta\beta^3 t_j y_i - 28\mathcal{L}_{\hat{X}}\beta\beta^2 q_{0i}y_j - 24\mathcal{L}_{\hat{X}}\beta\beta^2 q_{0j}y_i\\
&+18\mathcal{L}_{\hat{X}}\beta\beta^2 q_{i0}y_j + 18\mathcal{L}_{\hat{X}}\beta\beta^2 q_{j0}y_i + 56\mathcal{L}_{\hat{X}}\beta s_0^2 y_i y_j\\
&-12t_{00}\beta^2 b_i q_{0j} - 6t_{00}\beta^2 b_i q_{j0} - 20t_{00}\beta^2 b_j q_{0i}\\
&-6t_{00}\beta^2 b_j q_{i0} - 15t_{00}\beta^2 t_i y_j - 15t_{00}\beta^2 t_j y_i + 24t_{00}b_i s_0^2 y_j\\
&+24t_{00}b_j s_0^2 y_i + 10t_{00}\beta^3 s_i s_j
\end{aligned}
$$

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\begin{aligned}
& -30a_{ij}t_{00}\beta^2 t_0 - 30a_{ij}t_{00}\beta s_0^2 + 96e_{00}^2\mathcal{L}_{\hat{X}}\beta b_i b_j \\
& +12e_{00}e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 + 24e_{i0}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2 + 32a_{ij}e_{00}t_{00}s_0 \\
& +8a_{ij}e_{00;0}\mathcal{L}_{\hat{X}}\beta\beta - 30a_{ij}t_{00}\beta q_{00} - 56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta y_j \\
& -32e_{00}e_{i0}t_{00}b_j + 15e_{00}e_{ij}t_{00}\beta - 56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta y_i \\
& -32e_{00}e_{j0}t_{00}b_i + 16e_{00}t_{00}s_i y_j + 16e_{00}t_{00}s_j y_i - 14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_i y_j \\
& -14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_j y_i - 8e_{00;0}t_{00}b_i b_j \\
& +30e_{i0}e_{j0}t_{00}\beta + 24e_{i0}t_{00}s_0 y_j + 24e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta y_j \\
& +15e_{i0;0}t_{00}\beta b_j + 24e_{j0}t_{00}s_0 y_i + 24e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta y_i \\
& +15e_{j0;0}t_{00}\beta b_i + 56\mathcal{L}_{\hat{X}}\beta q_{00}y_i y_j - 40t_{00}\beta q_{0i}y_j \\
& -30t_{00}\beta q_{0j}y_i + 8t_{00}\beta q_{i0}y_j - 21e_{j0}t_{00}\beta^2 s_i \\
& +12e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta^2 b_i - 24a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2 q_{00} \\
A_6 \;=\;& 10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{ij} + 10\mathcal{L}_{\hat{X}}\beta\beta^3 q_{ji} - 14a_{ij}t_{00}s_0^2 \\
& -2\mathcal{L}_{\hat{X}}t_i\beta^4 b_j + 2\mathcal{L}_{\hat{X}}t_j\beta^4 b_j + 12a_{ij}\mathcal{L}_{\hat{X}}t_0\beta^3 \\
& -30e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta^2 + 15t_{00}\beta^2 q_{ij} + 15t_{00}\beta^2 q_{ji} \\
& +4a_{ij}e_{00;0}\mathcal{L}_{\hat{X}}\beta - 14a_{ij}t_{00}q_{00} + 7e_{00}e_{ij}t_{00} + 14e_{i0}e_{j0}t_{00} \\
& +12e_{i0;0}\mathcal{L}_{\hat{X}}\beta y_j + 7e_{i0;0}t_{00}b_j - 19e_{ij;0}t_{00}\beta \\
& +12e_{j0;0}\mathcal{L}_{\hat{X}}\beta y_i + 7e_{j0;0}t_{00}b_i - 18t_{00}q_{0i}y_j - 14t_{00}q_{0j}y_i \\
& +12a_{ij}\beta^3 t_0 + 12a_{ij}\beta^2 s_0^2 - 4\mathcal{L}_{\hat{X}}s_i\beta^4 s_j - 4\mathcal{L}_{\hat{X}}s_j\beta^4 s_i \\
& -6e_{00}\mathcal{L}_{\hat{X}}e_{ij}\beta^2 - 12e_{i0}\mathcal{L}_{\hat{X}}e_{j0}\beta^2 - 6e_{i0;0}\mathcal{L}_{\hat{X}}b_j\beta^2 \\
& +4\mathcal{L}_{\hat{X}}b_i\beta^3 q_{0j} + 2\mathcal{L}_{\hat{X}}b_i\beta^3 q_{j0} + 4\mathcal{L}_{\hat{X}}b_j\beta^3 q_{0i} \\
& +2\mathcal{L}_{\hat{X}}b_j\beta^3 q_{i0} + 4\mathcal{L}_{\hat{X}}q_{0i}\beta^3 b_j + 4\mathcal{L}_{\hat{X}}q_{0j}\beta^3 b_i + 2\mathcal{L}_{\hat{X}}q_{i0}\beta^3 b_j \\
& +2\mathcal{L}_{\hat{X}}q_{j0}\beta^3 b_i + 6\mathcal{L}_{\hat{X}}t_j\beta^3 y_i + 6\mathcal{L}_{\hat{X}}t_j\beta^3 y_j \\
& -6e_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta^2 - 12e_{j0}\mathcal{L}_{\hat{X}}e_{i0}\beta^2 - 6e_{j0;0}\mathcal{L}_{\hat{X}}b_i\beta^2 \\
& +12a_{ij}\beta^2 q_{00} - 6\mathcal{L}_{\hat{X}}e_{i0;0}\beta^2 b_j - 6\mathcal{L}_{\hat{X}}e_{j0;0}\beta^2 b_i \\
& +12\mathcal{L}_{\hat{X}}q_{0i}\beta^2 y_j + 12\mathcal{L}_{\hat{X}}q_{0j}\beta^2 y_i - 6\mathcal{L}_{\hat{X}}q_{i0}\beta^2 y_j \\
& -6\mathcal{L}_{\hat{X}}q_{j0}\beta^2 y_i + 6\mathcal{L}_{\hat{X}}y_i\beta^3 t_j + 6\mathcal{L}_{\hat{X}}y_j\beta^3 t_i \\
& +12a_{ij}\mathcal{L}_{\hat{X}}q_{00}\beta^2 - 12e_{00}^2\mathcal{L}_{\hat{X}}b_i b_j - 12e_{00}^2\mathcal{L}_{\hat{X}}b_j b_i \\
& -6e_{i0;0}\mathcal{L}_{\hat{X}}y_j\beta - 6e_{j0;0}\mathcal{L}_{\hat{X}}y_i\beta + 2\mathcal{L}_{\hat{X}}e_{00;0}b_i y_j \\
& +2\mathcal{L}_{\hat{X}}e_{00;0}b_j y_i - 6\mathcal{L}_{\hat{X}}e_{i0;0}\beta y_j - 6\mathcal{L}_{\hat{X}}e_{j0;0}\beta y_i \\
& -8\mathcal{L}_{\hat{X}}q_{00}y_i y_j - 8\mathcal{L}_{\hat{X}}y_i q_{00}y_j - 8\mathcal{L}_{\hat{X}}y_j q_{00}y_i \\
& +12\mathcal{L}_{\hat{X}}y_i\beta^2 q_{0j} - 6\mathcal{L}_{\hat{X}}y_i\beta^2 q_{j0} - 8\mathcal{L}_{\hat{X}}y_i s_0^2 y_j \\
& +12\mathcal{L}_{\hat{X}}y_j\beta^2 q_{0i} - 6\mathcal{L}_{\hat{X}}y_j\beta^2 q_{i0} - 8\mathcal{L}_{\hat{X}}y_j s_0^2 y_i \\
& +8a_{ij}e_{00}\mathcal{L}_{\hat{X}}e_{00} - 3a_{ij}\mathcal{L}_{\hat{X}}e_{00;0}\beta + 8e_{00}e_{i0}\mathcal{L}_{\hat{X}}y_j \\
& +8e_{00}e_{j0}\mathcal{L}_{\hat{X}}y_i + 8e_{00}\mathcal{L}_{\hat{X}}e_{i0}y_j + 8e_{00}\mathcal{L}_{\hat{X}}e_{j0}y_i \\
& -3e_{00;0}a_{ij}\beta + 2e_{00;0}\mathcal{L}_{\hat{X}}b_i y_j + 2e_{00;0}\mathcal{L}_{\hat{X}}b_j y_i \\
& +2e_{00;0}\mathcal{L}_{\hat{X}}y_i b_j + 2e_{00;0}\mathcal{L}_{\hat{X}}y_j b_i + 8e_{i0}\mathcal{L}_{\hat{X}}e_{00}y_j
\end{aligned}
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\begin{aligned}
&+8e_{j0}\mathcal{L}_{\hat{X}}e_{00}y_i + 8\mathcal{L}_{\hat{X}}\beta\beta^2 b_j s_i t_0 - 16\mathcal{L}_{\hat{X}}s_0\beta^2 b_i b_j s_0\\
&+32e_{00}\mathcal{L}_{\hat{X}}b_i\beta b_j s_0 + 32e_{00}\mathcal{L}_{\hat{X}}b_j\beta b_i s_0 + 32e_{00}\mathcal{L}_{\hat{X}}s_0\beta b_i b_j\\
&-32e_{00}\beta b_i b_j\eta + 20t_{00}\beta b_i s_j t_0 + 20t_{00}\beta b_j s_i t_0\\
&+32\mathcal{L}_{\hat{X}}e_{00}\beta b_i b_j s_0 + 16\mathcal{L}_{\hat{X}}\beta\beta s_i t_0 y_j + 16\mathcal{L}_{\hat{X}}\beta\beta s_j t_0 y_i\\
&-20\mathcal{L}_{\hat{X}}s_0\beta b_i s_0 y_j - 20\mathcal{L}_{\hat{X}}s_0\beta b_j s_0 y_i\\
&-16\mathcal{L}_{\hat{X}}\beta\beta^2 b_i b_j t_0 + 8\mathcal{L}_{\hat{X}}\beta\beta^2 b_i s_j t_0 + 72e_{j0}\mathcal{L}_{\hat{X}}\beta\beta b_i s_0\\
&+56\mathcal{L}_{\hat{X}}\beta\beta b_i b_j q_{00} - 58\mathcal{L}_{\hat{X}}\beta\beta s_0 s_i y_j\\
&-58\mathcal{L}_{\hat{X}}\beta\beta s_0 s_j y_i - 60t_{00}\beta b_i b_j t_0 - 56t_{00}\beta b_i s_0 s_j\\
&-56t_{00}\beta b_j s_0 s_i - 44\mathcal{L}_{\hat{X}}\beta\beta^2 b_i s_0 s_j\\
&-44\mathcal{L}_{\hat{X}}\beta\beta^2 b_j s_0 s_i + 56\mathcal{L}_{\hat{X}}\beta\beta b_i b_j s_0^2 + 48e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_i s_j\\
&+48e_{00}\mathcal{L}_{\hat{X}}\beta\beta b_j s_i - 112e_{00}\mathcal{L}_{\hat{X}}\beta b_i b_j s_0\\
&+72e_{i0}\mathcal{L}_{\hat{X}}\beta\beta b_j s_0 - 12e_{00}\mathcal{L}_{\hat{X}}s_i\beta^2 b_j - 12e_{00}\mathcal{L}_{\hat{X}}s_j\beta^2 b_i\\
&+12e_{00}\beta^2 b_i\eta_j + 12e_{00}\beta^2 b_j\eta_i - 18e_{i0}\mathcal{L}_{\hat{X}}b_j\beta^2 s_0\\
&-18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta^2 b_j + 18e_{i0}\beta^2 b_j\eta - 18e_{j0}\mathcal{L}_{\hat{X}}b_i\beta^2 s_0\\
&-18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta^2 b_i + 18e_{j0}\beta^2 b_i eta - 12\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_i s_j\\
&-12\mathcal{L}_{\hat{X}}e_{00}\beta^2 b_j s_i - 18\mathcal{L}_{\hat{X}}e_{i0}\beta^2 b_j s_0 - 18\mathcal{L}_{\hat{X}}e_{j0}\beta^2 b_i s_0\\
&+6\mathcal{L}_{\hat{X}}s_j\beta^3 b_i s_0 + 4\mathcal{L}_{\hat{X}}s_j\beta^3 b_i t_0\\
&-8\mathcal{L}_{\hat{X}}t_0\beta^3 b_i b_j + 4\mathcal{L}_{\hat{X}}t_0\beta^3 b_i s_j + 4\mathcal{L}_{\hat{X}}t_0\beta^3 b_j s_i\\
&+24a_{ij}\mathcal{L}_{\hat{X}}s_0\beta^2 s_0 - 12e_{00}\mathcal{L}_{\hat{X}}b_i\beta^2 s_j - 12e_{00}\mathcal{L}_{\hat{X}}b_j\beta^2 s_i\\
&-22\mathcal{L}_{\hat{X}}t_0\beta^2 b_j y_i + 12\mathcal{L}_{\hat{X}}t_0\beta^2 s_i y_j + 12\mathcal{L}_{\hat{X}}t_0\beta^2 s_j y_i\\
&-22\mathcal{L}_{\hat{X}}y_i\beta^2 b_j t_0 + 6\mathcal{L}_{\hat{X}}y_i\beta^2 s_0 s_j + 12\mathcal{L}_{\hat{X}}y_i\beta^2 s_j t_0\\
&-10\mathcal{L}_{\hat{X}}y_i\beta b_j s_0^2 - 22\mathcal{L}_{\hat{X}}y_j\beta^2 b_i t_0 + 6\mathcal{L}_{\hat{X}}y_j\beta^2 s_0 s_i\\
&+12\mathcal{L}_{\hat{X}}y_j\beta^2 s_i t_0 - 10\mathcal{L}_{\hat{X}}y_j\beta b_i s_0^2 - 8\mathcal{L}_{\hat{X}}b_i\beta^2 b_j q_{00}\\
&-22\mathcal{L}_{\hat{X}}b_i\beta^2 t_0 y_j - 10\mathcal{L}_{\hat{X}}b_i\beta s_0^2 y_j - 8\mathcal{L}_{\hat{X}}b_j\beta^2 b_i q_{00}\\
&-22\mathcal{L}_{\hat{X}}b_j\beta^2 t_0 y_i - 10\mathcal{L}_{\hat{X}}b_j\beta s_0^2 y_i - 8\mathcal{L}_{\hat{X}}q_{00}\beta^2 b_i b_j\\
&-6\mathcal{L}_{\hat{X}}s_0\beta^2 s_i y_j - 6\mathcal{L}_{\hat{X}}s_0\beta^2 s_j y_i - 12e_{00}\mathcal{L}_{\hat{X}}y_j\beta s_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}y_j b_i s_0 + 12e_{00}\beta\eta_i y_j + 12e_{00}\beta\eta_j y_i\\
&-16e_{00}b_i\eta y_j - 16e_{00}b_j\eta y_i + 4e_{00;0}\mathcal{L}_{\hat{X}}b_i\beta b_j\\
&+4e_{00;0}\mathcal{L}_{\hat{X}}b_j\beta b_i - 12a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0\beta + 24a_{ij}e_{00}\beta\eta\\
&-24a_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta s_0 + 16e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j\beta + 6\mathcal{L}_{\hat{X}}s_i\beta^2 s_0 y_j\\
&+12\mathcal{L}_{\hat{X}}s_i\beta^2 t_0 y_j + 6\mathcal{L}_{\hat{X}}s_j\beta^2 s_0 y_i + 12\mathcal{L}_{\hat{X}}s_j\beta^2 t_0 y_i\\
&-22\mathcal{L}_{\hat{X}}t_0\beta^2 b_i y_j - 16\mathcal{L}_{\hat{X}}s_0 s_0 y_i y_j - 8\mathcal{L}_{\hat{X}}t_0\beta y_i y_j\\
&-10\mathcal{L}_{\hat{X}}y_i\beta b_j q_{00} - 8\mathcal{L}_{\hat{X}}y_i\beta t_0 y_j + 16e_{i0}\mathcal{L}_{\hat{X}}e_{00}\beta b_j\\
&-18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta y_j - 18e_{i0}\mathcal{L}_{\hat{X}}y_j\beta s_0 + 18e_{i0}\beta\eta y_j\\
&+16e_{j0}\mathcal{L}_{\hat{X}}e_{00}\beta b_i - 18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta y_i - 18e_{j0}\mathcal{L}_{\hat{X}}y_i\beta s_0\\
&+18e_{j0}\beta\eta y_i - 12\mathcal{L}_{\hat{X}}e_{00}\beta s_i y_j - 12\mathcal{L}_{\hat{X}}e_{00}\beta s_j yi
\end{aligned}
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\begin{aligned}
& +16e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\beta - 24e_{00}a_{ij}\beta s_0 - 24e_{00}\mathcal{L}_{\hat{X}}e_{00}b_ib_j \\
& +16e_{00}\mathcal{L}_{\hat{X}}e_{i0}\beta b_j + 16e_{00}\mathcal{L}_{\hat{X}}e_{j0}\beta b_i + 16e_{00}\mathcal{L}_{\hat{X}}b_is_0y_j \\
& +16e_{00}\mathcal{L}_{\hat{X}}b_js_0y_i + 16e_{00}\mathcal{L}_{\hat{X}}s_0b_iy_j \\
& +16e_{00}\mathcal{L}_{\hat{X}}s_0b_jy_i - 12e_{00}\mathcal{L}_{\hat{X}}s_i\beta y_j - 12e_{00}\mathcal{L}_{\hat{X}}s_j\beta y_i \\
& -12e_{00}\mathcal{L}_{\hat{X}}y_i\beta s_j + 16e_{00}\mathcal{L}_{\hat{X}}y_ib_js_0 - 10\mathcal{L}_{\hat{X}}y_j\beta b_iq_{00} \\
& -8\mathcal{L}_{\hat{X}}y_j\beta t_0y_i + 6t_{00}s_it_0y_j + 6t_{00}s_jt_0y_i \\
& +16\mathcal{L}_{\hat{X}}e_{00}b_is_0y_j + 16\mathcal{L}_{\hat{X}}e_{00}b_js_0y_i + 4\mathcal{L}_{\hat{X}}e_{00;0}\beta b_ib_j \\
& -18\mathcal{L}_{\hat{X}}e_{i0}\beta s_0y_j - 18\mathcal{L}_{\hat{X}}e_{j0}\beta s_0y_i - 8\mathcal{L}_{\hat{X}}\beta t_0y_iy_j \\
& -10\mathcal{L}_{\hat{X}}b_i\beta q_{00}y_j - 10\mathcal{L}_{\hat{X}}b_j\beta q_{00}y_i - 10\mathcal{L}_{\hat{X}}q_{00}\beta b_iy_j \\
& -10\mathcal{L}_{\hat{X}}q_{00}\beta b_jy_i + 2\mathcal{L}_{\hat{X}}s_0\beta^3b_is_j + 2\mathcal{L}_{\hat{X}}s_0\beta^3b_js_i \\
& +6\mathcal{L}_{\hat{X}}s_i\beta^3b_js_0 + 4\mathcal{L}_{\hat{X}}s_i\beta^3b_jt_0 - 8\mathcal{L}_{\hat{X}}b_i\beta^3b_jt_0 \\
& +6\mathcal{L}_{\hat{X}}b_i\beta^3s_0s_j + 4\mathcal{L}_{\hat{X}}b_i\beta^3s_jt_0 - 8\mathcal{L}_{\hat{X}}b_i\beta^2b_js_0^2 \\
& -8\mathcal{L}_{\hat{X}}b_j\beta^3b_it_0 + 6\mathcal{L}_{\hat{X}}b_j\beta^3s_0s_i + 4\mathcal{L}_{\hat{X}}b_j\beta^3s_it_0 \\
& -8\mathcal{L}_{\hat{X}}b_j\beta^2b_is_0^2 - 2\mathcal{L}_{\hat{X}}q_{ij}\beta^4 - 2\mathcal{L}_{\hat{X}}q_{ji}\beta^4 \\
& +4e_{00}^2\mathcal{L}_{\hat{X}}a_{ij} + 8e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^3 - 8e_{i0}\beta^3\eta_j + 16e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^3 \\
& -16e_{ij}\beta^3\eta + 8e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^3 - 8e_{j0}\beta^3\eta_i + 8\mathcal{L}_{\hat{X}}e_{i0}\beta^3s_j \\
& +16\mathcal{L}_{\hat{X}}e_{ij}\beta^3s_0 + 8\mathcal{L}_{\hat{X}}e_{j0}\beta^3s_i + 8\mathcal{L}_{\hat{X}}e_{ij;0}\beta^3 \\
& +21e_{i0}t_{00}b_js_0 + 24e_{i0;0}\mathcal{L}_{\hat{X}}\beta\beta b_j - 38e_{ij}t_{00}\beta s_0 \\
& +36e_{j0}\mathcal{L}_{\hat{X}}\beta s_0y_i - 19e_{j0}t_{00}\beta s_i + 21e_{j0}t_{00}b_is_0 \\
& +20\mathcal{L}_{\hat{X}}\beta\beta^3s_is_j - 36a_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2t_0 - 48a_{ij}\mathcal{L}_{\hat{X}}\beta\beta s_0^2 \\
& -30e_{i0}\mathcal{L}_{\hat{X}}\beta\beta^2s_j - 60e_{ij}\mathcal{L}_{\hat{X}}\beta\beta^2s_0 - 30e_{j0}\mathcal{L}_{\hat{X}}\beta\beta^2s_i \\
- & \quad 24\mathcal{L}_{\hat{X}}\beta\beta^2b_iq_{0j} - 12\mathcal{L}_{\hat{X}}\beta\beta^2b_iq_{j0} - 28\mathcal{L}_{\hat{X}}\beta\beta^2b_jq_{0i} \\
& -12\mathcal{L}_{\hat{X}}\beta\beta^2b_jq_{i0} - 18\mathcal{L}_{\hat{X}}\beta\beta^2t_iy_j - 18\mathcal{L}_{\hat{X}}\beta\beta^2t_jy_i \\
& +44\mathcal{L}_{\hat{X}}\beta b_is_0^2y_j + 44\mathcal{L}_{\hat{X}}\beta b_js_0^2y_i + 44\mathcal{L}_{\hat{X}}\beta b_iq_{00}y_j \\
& +44\mathcal{L}_{\hat{X}}\beta b_jq_{00}y_i - 24t_{00}\beta b_iq_{0j} - 12t_{00}\beta b_iq_{j0} \\
& -34t_{00}\beta b_jq_{0i} - 12t_{00}\beta b_jq_{i0} - 7t_{00}\beta t_iy_j - 7t_{00}\beta t_jy_i \\
& +28t_{00}b_ib_jq_{00} - 13t_{00}b_it_0y_j - 13t_{00}b_jt_0y_i \\
- & \quad 22t_{00}s_0s_iy_j - 22t_{00}s_0s_jy_i + 3t_{00}\beta^2b_it_j + 3t_{00}\beta^2b_jt_i \\
& +30t_{00}\beta^2s_is_j + 28t_{00}b_ib_js_0^2 + 48a_{ij}e_{00}\mathcal{L}_{\hat{X}}\beta s_0 \\
- & \quad 48a_{ij}\mathcal{L}_{\hat{X}}\beta\beta q_{00} - 14a_{ij}t_{00}\beta t_0 - 56e_{00}e_{i0}\mathcal{L}_{\hat{X}}\beta b_j \\
& +24e_{00}e_{ij}\mathcal{L}_{\hat{X}}\beta\beta - 56e_{00}e_{j0}\mathcal{L}_{\hat{X}}\beta b_i + 24e_{00}\mathcal{L}_{\hat{X}}\beta s_iy_j \\
& +24e_{00}\mathcal{L}_{\hat{X}}\beta s_jy_i + 14e_{00}t_{00}b_is_j + 14e_{00}t_{00}b_js_i \\
& -14e_{00;0}\mathcal{L}_{\hat{X}}\beta b_ib_j + 48e_{i0}e_{j0}\mathcal{L}_{\hat{X}}\beta\beta + 36e_{i0}\mathcal{L}_{\hat{X}}\beta s_0y_j \\
& -19e_{i0}t_{00}\beta s_j + 24e_{j0;0}\mathcal{L}_{\hat{X}}\beta\beta b_i - 56\mathcal{L}_{\hat{X}}\beta\beta q_{0i}y_j \\
& -48\mathcal{L}_{\hat{X}}\beta\beta q_{0j}y_i + 6\mathcal{L}_{\hat{X}}\beta\beta q_{i0}y_j + 6\mathcal{L}_{\hat{X}}\beta\beta q_{j0}y_i \\
A_7 = & \quad 30\mathcal{L}_{\hat{X}}\beta\beta^2q_{ij} + 30\mathcal{L}_{\hat{X}}\beta\beta^2q_{ji} - 24a_{ij}\mathcal{L}_{\hat{X}}\beta q_{00}
\end{aligned}
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\begin{aligned}
&+12e_{00}e_{ij}\mathcal{L}_{\hat{X}}\beta+24e_{i0}e_{j0}\mathcal{L}_{\hat{X}}\beta+6\mathcal{L}_{\hat{X}}t_j\beta^3b_j\\
&+12a_{ij}\mathcal{L}_{\hat{X}}t_0\beta^2+12e_{i0}\mathcal{L}_{\hat{X}}s_j\beta^2-6e_{i0}t_{00}s_j\\
&+12e_{i0;0}\mathcal{L}_{\hat{X}}\beta b_j-12e_{ij}t_{00}s_0-30e_{ij;0}\mathcal{L}_{\hat{X}}\beta\beta-6e_{j0}t_{00}s_i\\
&+12e_{j0;0}\mathcal{L}_{\hat{X}}\beta b_i-28\mathcal{L}_{\hat{X}}\beta q_{0i}y_j-24\mathcal{L}_{\hat{X}}\beta q_{0j}y_i\\
&+15t_{00}\beta q_{ij}+15t_{00}\beta q_{ji}-12t_{00}b_iq_{0j}-6t_{00}b_iq_{j0}\\
&-16t_{00}b_jq_{0i}-6t_{00}b_jq_{i0}-24a_{ij}\mathcal{L}_{\hat{X}}\beta s_0^2-12e_{i0}\beta^2\eta_j\\
&+24e_{ij}\mathcal{L}_{\hat{X}}s_0\beta^2-24e_{ij}\beta^2\eta+12e_{j0}\mathcal{L}_{\hat{X}}s_i\beta^2\\
&-12e_{j0}\beta^2\eta_i+12\mathcal{L}_{\hat{X}}a_{ij}\beta^2t_0+12\mathcal{L}_{\hat{X}}a_{ij}\beta s_0^2\\
&+12\mathcal{L}_{\hat{X}}e_{i0}\beta^2s_j+24\mathcal{L}_{\hat{X}}e_{ij}\beta^2s_0+12\mathcal{L}_{\hat{X}}e_{j0}\beta^2s_i\\
&+8e_{00}\mathcal{L}_{\hat{X}}e_{j0}b_i-4e_{00}\mathcal{L}_{\hat{X}}s_iy_j-4e_{00}\mathcal{L}_{\hat{X}}s_jy_i\\
&-4e_{00}\mathcal{L}_{\hat{X}}y_is_j-4e_{00}\mathcal{L}_{\hat{X}}y_js_i+4e_{00}\eta_iy_j+4e_{00}\eta_jy_i\\
&+2e_{00;0}\mathcal{L}_{\hat{X}}b_ib_j+2e_{00;0}\mathcal{L}_{\hat{X}}b_jb_i+8e_{i0}\mathcal{L}_{\hat{X}}e_{00}b_j\\
&-12e_{i0}\mathcal{L}_{\hat{X}}e_{j0}\beta-6e_{i0}\mathcal{L}_{\hat{X}}s_0y_j-6e_{i0}\mathcal{L}_{\hat{X}}y_js_0\\
&+6e_{i0}\eta y_j-6e_{i0;0}\mathcal{L}_{\hat{X}}b_j\beta+12\mathcal{L}_{\hat{X}}b_i\beta^2q_{0j}+6\mathcal{L}_{\hat{X}}b_i\beta^2q_{j0}\\
&-6\mathcal{L}_{\hat{X}}b_is_0^2y_j+12\mathcal{L}_{\hat{X}}b_j\beta^2q_{0i}+6\mathcal{L}_{\hat{X}}b_j\beta^2q_{i0}\\
&-6\mathcal{L}_{\hat{X}}b_js_0^2y_i+12\mathcal{L}_{\hat{X}}q_{0i}\beta^2b_j+12\mathcal{L}_{\hat{X}}q_{0j}\beta^2b_i\\
&+6\mathcal{L}_{\hat{X}}q_{i0}\beta^2b_j+6\mathcal{L}_{\hat{X}}q_{j0}\beta^2b_i+6\mathcal{L}_{\hat{X}}t_j\beta^2y_i+6\mathcal{L}_{\hat{X}}t_j\beta^2y_j\\
&+8e_{j0}\mathcal{L}_{\hat{X}}e_{00}b_i-12e_{j0}\mathcal{L}_{\hat{X}}e_{i0}\beta-6e_{j0}\mathcal{L}_{\hat{X}}s_0y_i-6e_{j0}\mathcal{L}_{\hat{X}}y_is_0\\
&+6e_{j0}\eta y_i-6e_{j0;0}\mathcal{L}_{\hat{X}}b_i\beta+12\mathcal{L}_{\hat{X}}a_{ij}\beta q_{00}\\
&-4\mathcal{L}_{\hat{X}}e_{00}s_iy_j-4\mathcal{L}_{\hat{X}}e_{00}s_jy_i+2\mathcal{L}_{\hat{X}}e_{00;0}b_ib_j\\
&-6\mathcal{L}_{\hat{X}}e_{i0}s_0y_j-6\mathcal{L}_{\hat{X}}e_{i0;0}\beta b_j-6\mathcal{L}_{\hat{X}}e_{j0}s_0y_i\\
&-6\mathcal{L}_{\hat{X}}e_{j0;0}\beta b_i-2\mathcal{L}_{\hat{X}}\beta q_{i0}y_j-2\mathcal{L}_{\hat{X}}\beta q_{j0}y_i-6\mathcal{L}_{\hat{X}}b_iq_{00}y_j\\
&-6\mathcal{L}_{\hat{X}}b_jq_{00}y_i-6\mathcal{L}_{\hat{X}}q_{00}b_iy_j-6\mathcal{L}_{\hat{X}}q_{00}b_jy_i\\
&+12\mathcal{L}_{\hat{X}}q_{0i}\beta y_j+12\mathcal{L}_{\hat{X}}q_{0j}\beta y_i-2\mathcal{L}_{\hat{X}}q_{i0}\beta y_j-2\mathcal{L}_{\hat{X}}q_{j0}\beta y_i\\
&+6\mathcal{L}_{\hat{X}}y_i\beta^2t_j-6\mathcal{L}_{\hat{X}}y_ib_js_0^2+6\mathcal{L}_{\hat{X}}y_j\beta^2t_i\\
&-6\mathcal{L}_{\hat{X}}y_jb_is_0^2-4a_{ij}e_{00}\mathcal{L}_{\hat{X}}s_0+8a_{ij}e_{00}\eta-8a_{ij}\mathcal{L}_{\hat{X}}e_{00}s_0\\
&+12a_{ij}\mathcal{L}_{\hat{X}}q_{00}\beta+8e_{00}e_{i0}\mathcal{L}_{\hat{X}}b_j+8e_{00}e_{j0}\mathcal{L}_{\hat{X}}b_i\\
&-8e_{00}\mathcal{L}_{\hat{X}}a_{ij}s_0+8e_{00}\mathcal{L}_{\hat{X}}e_{i0}b_j-6e_{00}\mathcal{L}_{\hat{X}}e_{ij}\beta\\
&+12\mathcal{L}_{\hat{X}}y_i\beta q_{0j}-2\mathcal{L}_{\hat{X}}y_i\beta q_{j0}-6\mathcal{L}_{\hat{X}}y_ib_jq_{00}+12\mathcal{L}_{\hat{X}}y_j\beta q_{0i}\\
&-2\mathcal{L}_{\hat{X}}y_j\beta q_{i0}-6\mathcal{L}_{\hat{X}}y_jb_iq_{00}-6e_{ij}\mathcal{L}_{\hat{X}}e_{00}\beta\\
&-32\mathcal{L}_{\hat{X}}s_0\beta b_ib_js_0+16\mathcal{L}_{\hat{X}}\beta\beta b_is_jt_0+16\mathcal{L}_{\hat{X}}\beta\beta b_js_it_0\\
&-88\mathcal{L}_{\hat{X}}\beta\beta b_ib_jt_0-88\mathcal{L}_{\hat{X}}\beta\beta b_is_0s_j-88\mathcal{L}_{\hat{X}}\beta\beta b_js_0s_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}s_0b_ib_j-12e_{00}\mathcal{L}_{\hat{X}}s_i\beta b_j\\
&-12e_{00}\mathcal{L}_{\hat{X}}s_j\beta b_i+12e_{00}\beta b_i\eta_j+12e_{00}\beta b_j\eta_i-16e_{00}b_ib_j\eta\\
&+10\mathcal{L}_{\hat{X}}s_i\beta s_0y_j-18e_{i0}\mathcal{L}_{\hat{X}}b_j\beta s_0\\
&-18e_{i0}\mathcal{L}_{\hat{X}}s_0\beta b_j+18e_{i0}\beta b_j\eta-18e_{j0}\mathcal{L}_{\hat{X}}b_i\beta s_0
\end{aligned}
$$

$$
\begin{aligned}
&-18e_{j0}\mathcal{L}_{\hat{X}}s_0\beta b_i+18e_{j0}\beta b_i\eta-12\mathcal{L}_{\hat{X}}e_{00}\beta b_is_j\\
&-12\mathcal{L}_{\hat{X}}e_{00}\beta b_js_i+16\mathcal{L}_{\hat{X}}e_{00}b_ib_js_0-18\mathcal{L}_{\hat{X}}e_{i0}\beta b_js_0\\
&-18\mathcal{L}_{\hat{X}}e_{j0}\beta b_is_0+12\mathcal{L}_{\hat{X}}s_j\beta^2b_it_0-16\mathcal{L}_{\hat{X}}t_0\beta^2b_ib_j\\
&+12\mathcal{L}_{\hat{X}}t_0\beta^2b_is_j+12\mathcal{L}_{\hat{X}}t_0\beta^2b_js_i+24a_{ij}\mathcal{L}_{\hat{X}}s_0\beta s_0\\
&-12e_{00}\mathcal{L}_{\hat{X}}b_i\beta s_j+16e_{00}\mathcal{L}_{\hat{X}}b_ib_js_0-12e_{00}\mathcal{L}_{\hat{X}}b_j\beta s_i\\
&+16e_{00}\mathcal{L}_{\hat{X}}b_jb_is_0-14\mathcal{L}_{\hat{X}}y_i\beta b_jt_0+10\mathcal{L}_{\hat{X}}y_i\beta s_0s_j\\
&+12\mathcal{L}_{\hat{X}}y_i\beta s_jt_0-14\mathcal{L}_{\hat{X}}y_j\beta b_it_0+10\mathcal{L}_{\hat{X}}y_j\beta s_0s_i\\
&+12\mathcal{L}_{\hat{X}}y_j\beta s_it_0+8\mathcal{L}_{\hat{X}}\beta s_it_0y_j+8\mathcal{L}_{\hat{X}}\beta s_jt_0y_i\\
&-16\mathcal{L}_{\hat{X}}b_i\beta b_jq_{00}-14\mathcal{L}_{\hat{X}}b_i\beta t_0y_j-16\mathcal{L}_{\hat{X}}b_j\beta b_iq_{00}\\
&-14\mathcal{L}_{\hat{X}}b_j\beta t_0y_i-16\mathcal{L}_{\hat{X}}q_{00}\beta b_ib_j\\
&-2\mathcal{L}_{\hat{X}}s_0\beta s_iy_j-2\mathcal{L}_{\hat{X}}s_0\beta s_jy_i-12\mathcal{L}_{\hat{X}}s_0b_is_0y_j-12\mathcal{L}_{\hat{X}}s_0b_js_0y_i\\
&+8t_{00}b_is_jt_0+8t_{00}b_js_it_0+12\mathcal{L}_{\hat{X}}s_i\beta t_0y_j\\
&+10\mathcal{L}_{\hat{X}}s_j\beta s_0y_i+12\mathcal{L}_{\hat{X}}s_j\beta t_0y_i-14\mathcal{L}_{\hat{X}}t_0\beta(b_iy_j+b_jy_i)\\
&+12\mathcal{L}_{\hat{X}}t_0\beta s_iy_j+12\mathcal{L}_{\hat{X}}t_0\beta s_jy_i+6\mathcal{L}_{\hat{X}}s_0\beta^2b_is_j\\
&+6\mathcal{L}_{\hat{X}}s_0\beta^2b_js_i+18\mathcal{L}_{\hat{X}}s_i\beta^2b_js_0+12\mathcal{L}_{\hat{X}}s_i\beta^2b_jt_0\\
&+18\mathcal{L}_{\hat{X}}s_j\beta^2b_is_0-16\mathcal{L}_{\hat{X}}b_i\beta^2b_jt_0+18\mathcal{L}_{\hat{X}}b_i\beta^2s_0s_j\\
&+12\mathcal{L}_{\hat{X}}b_i\beta^2s_jt_0-16\mathcal{L}_{\hat{X}}b_i\beta b_js_0^2-16\mathcal{L}_{\hat{X}}b_j\beta^2b_it_0\\
&+18\mathcal{L}_{\hat{X}}b_j\beta^2s_0s_i+12\mathcal{L}_{\hat{X}}b_j\beta^2s_it_0-16\mathcal{L}_{\hat{X}}b_j\beta b_is_0^2\\
&-6e_{ij;0}t_{00}+12(\mathcal{L}_{\hat{X}}e_{ij})_{;0}\beta^2-a_{ij}(\mathcal{L}_{\hat{X}}e_{00})_{;0}\\
&-e_{00;0}\mathcal{L}_{\hat{X}}a_{ij}-2e_{i0;0}\mathcal{L}_{\hat{X}}y_j-2e_{j0;0}\mathcal{L}_{\hat{X}}y_i-2(\mathcal{L}_{\hat{X}}e_{i0})_{;0}y_j\\
&-2(\mathcal{L}_{\hat{X}}e_{j0})_{;0}y_i-2\mathcal{L}_{\hat{X}}b_i\beta^3t_j-2\mathcal{L}_{\hat{X}}b_j\beta^3t_i\\
&-16\mathcal{L}_{\hat{X}}s_i\beta^3s_j-16\mathcal{L}_{\hat{X}}s_j\beta^3s_i-8\mathcal{L}_{\hat{X}}t_i\beta^3b_j-2\mathcal{L}_{\hat{X}}t_j\beta^3b_i\\
&-8\mathcal{L}_{\hat{X}}q_{ij}\beta^3-8\mathcal{L}_{\hat{X}}q_{ji}\beta^3-26t_{00}b_is_0s_j\\
&+12\mathcal{L}_{\hat{X}}\beta\beta^2b_it_j+12\mathcal{L}_{\hat{X}}\beta\beta^2b_jt_i+60\mathcal{L}_{\hat{X}}\beta\beta^2s_is_j\\
&+56\mathcal{L}_{\hat{X}}\beta b_ib_js_0^2-12a_{ij}\mathcal{L}_{\hat{X}}\beta\beta t_0+24e_{00}\mathcal{L}_{\hat{X}}\beta b_is_j\\
&+24e_{00}\mathcal{L}_{\hat{X}}\beta b_js_i-30e_{i0}\mathcal{L}_{\hat{X}}\beta\beta s_j+36e_{i0}\mathcal{L}_{\hat{X}}\beta b_js_0\\
&-60e_{ij}\mathcal{L}_{\hat{X}}\beta\beta s_0-30e_{j0}\mathcal{L}_{\hat{X}}\beta\beta s_i+36e_{j0}\mathcal{L}_{\hat{X}}\beta b_is_0\\
&-48\mathcal{L}_{\hat{X}}\beta\beta b_iq_{0j}-24\mathcal{L}_{\hat{X}}\beta\beta b_iq_{j0}-56\mathcal{L}_{\hat{X}}\beta\beta b_jq_{0i}\\
&-24\mathcal{L}_{\hat{X}}\beta\beta b_jq_{i0}-6\mathcal{L}_{\hat{X}}\beta\beta t_iy_j-6\mathcal{L}_{\hat{X}}\beta\beta t_jy_i\\
&+56\mathcal{L}_{\hat{X}}\beta b_ib_jq_{00}-28\mathcal{L}_{\hat{X}}\beta b_it_0y_j-28\mathcal{L}_{\hat{X}}\beta b_jt_0y_i-34\mathcal{L}_{\hat{X}}\beta s_0s_iy_j\\
&-34\mathcal{L}_{\hat{X}}\beta s_0s_jy_i+9t_{00}\beta b_it_j\\
&+9t_{00}\beta b_jt_i+30t_{00}\beta s_is_j-40t_{00}b_ib_jt_0-26t_{00}b_js_0s_i
\end{aligned}
$$

$$
\begin{aligned}
A_8 \;=\;&-20e_{ij}\mathcal{L}_{\hat{X}}\beta s_0-10e_{j0}\mathcal{L}_{\hat{X}}\beta s_i+30\mathcal{L}_{\hat{X}}\beta\beta q_{ij}\\
&+30\mathcal{L}_{\hat{X}}\beta\beta q_{ji}-24\mathcal{L}_{\hat{X}}\beta b_iq_{0j}-12\mathcal{L}_{\hat{X}}\beta b_iq_{j0}\\
&-28\mathcal{L}_{\hat{X}}\beta b_jq_{0i}-12\mathcal{L}_{\hat{X}}\beta b_jq_{i0}+5t_{00}b_it_j+5t_{00}b_jt_i\\
&+10t_{00}s_is_j-12\mathcal{L}_{\hat{X}}t_i\beta^2b_j-6\mathcal{L}_{\hat{X}}t_j\beta^2b_i
\end{aligned}
$$

$$
\begin{aligned}
& +6\mathcal{L}_{\hat{X}}t_j\beta^2 b_j+4a_{ij}\mathcal{L}_{\hat{X}}\beta t_0+8a_{ij}\mathcal{L}_{\hat{X}}s_0s_0+4a_{ij}\mathcal{L}_{\hat{X}}t_0\beta-4e_{00;0}\mathcal{L}_{\hat{X}}b_is_j \\
& -4e_{00;0}\mathcal{L}_{\hat{X}}b_js_i-4e_{00;0}\mathcal{L}_{\hat{X}}s_ib_j \\
& -4e_{00;0}\mathcal{L}_{\hat{X}}s_jb_i+4e_{00;0}b_i\eta_j+4e_{00;0}b_j\eta_i-6e_{i0}\mathcal{L}_{\hat{X}}b_js_0 \\
& -6e_{i0}\mathcal{L}_{\hat{X}}s_0b_j+8e_{i0}\mathcal{L}_{\hat{X}}s_j\beta-10e_{i0}\mathcal{L}_{\hat{X}}\beta s_j \\
& -8e_{i0}\beta\eta_j+6e_{i0}b_j\eta+16e_{ij}\mathcal{L}_{\hat{X}}s_0\beta-16e_{ij}\beta\eta \\
& -6e_{j0}\mathcal{L}_{\hat{X}}b_is_0-6e_{j0}\mathcal{L}_{\hat{X}}s_0b_i+8e_{j0}\mathcal{L}_{\hat{X}}s_i\beta-8e_{j0}\beta\eta_i \\
& +6e_{j0}b_i\eta+4\mathcal{L}_{\hat{X}}a_{ij}\beta t_0-4\mathcal{L}_{\hat{X}}e_{00;0}b_is_j \\
& -4\mathcal{L}_{\hat{X}}e_{00;0}b_js_i+8\mathcal{L}_{\hat{X}}e_{i0}\beta s_j-6\mathcal{L}_{\hat{X}}e_{i0}b_js_0+16\mathcal{L}_{\hat{X}}e_{ij}\beta s_0 \\
& +8\mathcal{L}_{\hat{X}}e_{j0}\beta s_i-6\mathcal{L}_{\hat{X}}e_{j0}b_is_0-6\mathcal{L}_{\hat{X}}b_i\beta^2t_j-8\mathcal{L}_{\hat{X}}b_ib_js_0^2 \\
& -6\mathcal{L}_{\hat{X}}b_j\beta^2t_i-8\mathcal{L}_{\hat{X}}b_jb_is_0^2-24\mathcal{L}_{\hat{X}}s_i\beta^2s_j \\
& -24\mathcal{L}_{\hat{X}}s_j\beta^2s_i+2\mathcal{L}_{\hat{X}}\beta t_iy_j+2\mathcal{L}_{\hat{X}}\beta t_jy_i+12\mathcal{L}_{\hat{X}}b_i\beta q_{0j} \\
& +6\mathcal{L}_{\hat{X}}b_i\beta q_{j0}-8\mathcal{L}_{\hat{X}}b_ib_jq_{00}-2\mathcal{L}_{\hat{X}}b_it_0y_j \\
& +12\mathcal{L}_{\hat{X}}b_j\beta q_{0i}+6\mathcal{L}_{\hat{X}}b_j\beta q_{i0}-8\mathcal{L}_{\hat{X}}b_jb_iq_{00}-2\mathcal{L}_{\hat{X}}b_jt_0y_i \\
& -8\mathcal{L}_{\hat{X}}q_{00}b_ib_j+12\mathcal{L}_{\hat{X}}q_{0i}\beta b_j+12\mathcal{L}_{\hat{X}}q_{0j}\beta b_i \\
& +6\mathcal{L}_{\hat{X}}q_{i0}\beta b_j+6\mathcal{L}_{\hat{X}}q_{j0}\beta b_i+4\mathcal{L}_{\hat{X}}s_is_0y_j+4\mathcal{L}_{\hat{X}}s_it_0y_j \\
& +4\mathcal{L}_{\hat{X}}s_js_0y_i+4\mathcal{L}_{\hat{X}}s_jt_0y_i-2\mathcal{L}_{\hat{X}}t_0b_iy_j \\
& -2\mathcal{L}_{\hat{X}}t_0b_jy_i+4\mathcal{L}_{\hat{X}}t_0s_iy_j+4\mathcal{L}_{\hat{X}}t_0s_jy_i+2\mathcal{L}_{\hat{X}}t_j\beta y_i \\
& +2\mathcal{L}_{\hat{X}}t_j\beta y_j+2\mathcal{L}_{\hat{X}}y_i\beta t_j-2\mathcal{L}_{\hat{X}}y_ib_jt_0 \\
& +4\mathcal{L}_{\hat{X}}y_is_0s_j+4\mathcal{L}_{\hat{X}}y_is_jt_0+2\mathcal{L}_{\hat{X}}y_j\beta t_i-2\mathcal{L}_{\hat{X}}y_jb_it_0 \\
& +4\mathcal{L}_{\hat{X}}y_js_0s_i+4\mathcal{L}_{\hat{X}}y_js_it_0-8\mathcal{L}_{\hat{X}}t_0\beta b_ib_j \\
& +12\mathcal{L}_{\hat{X}}t_0\beta b_is_j+12\mathcal{L}_{\hat{X}}t_0\beta b_js_i+6\mathcal{L}_{\hat{X}}s_0\beta b_is_j \\
& +6\mathcal{L}_{\hat{X}}s_0\beta b_js_i-16\mathcal{L}_{\hat{X}}s_0b_ib_js_0+18\mathcal{L}_{\hat{X}}s_i\beta b_js_0 \\
& +12\mathcal{L}_{\hat{X}}s_i\beta b_jt_0+18\mathcal{L}_{\hat{X}}s_j\beta b_is_0+12\mathcal{L}_{\hat{X}}s_j\beta b_it_0 \\
& +8\mathcal{L}_{\hat{X}}\beta b_is_jt_0+8\mathcal{L}_{\hat{X}}\beta b_js_it_0-8\mathcal{L}_{\hat{X}}b_i\beta b_jt_0 \\
& +18\mathcal{L}_{\hat{X}}b_i\beta s_0s_j+12\mathcal{L}_{\hat{X}}b_i\beta s_jt_0-8\mathcal{L}_{\hat{X}}b_j\beta b_it_0 \\
& +18\mathcal{L}_{\hat{X}}b_j\beta s_0s_i+12\mathcal{L}_{\hat{X}}b_j\beta s_it_0+4\mathcal{L}_{\hat{X}}a_{ij}s_0^2 \\
& -12\mathcal{L}_{\hat{X}}q_{ij}\beta^2-12\mathcal{L}_{\hat{X}}q_{ji}\beta^2+4a_{ij}\mathcal{L}_{\hat{X}}q_{00}-2e_{00;0}\mathcal{L}_{\hat{X}}e_{ij} \\
& -2e_{i0;0}\mathcal{L}_{\hat{X}}b_j-2e_{ij}\mathcal{L}_{\hat{X}}e_{00;0}-2e_{j0;0}\mathcal{L}_{\hat{X}}b_i \\
& +4\mathcal{L}_{\hat{X}}a_{ij}q_{00}-2\mathcal{L}_{\hat{X}}e_{i0;0}b_j-10e_{ij;0}\mathcal{L}_{\hat{X}}\beta+5t_{00}q_{ij}+5t_{00}q_{ji} \\
& +8\mathcal{L}_{\hat{X}}e_{ij;0}\beta-2\mathcal{L}_{\hat{X}}e_{j0;0}b_i+4\mathcal{L}_{\hat{X}}q_{0i}y_j \\
& +4\mathcal{L}_{\hat{X}}q_{0j}y_i+4\mathcal{L}_{\hat{X}}y_iq_{0j}+4\mathcal{L}_{\hat{X}}y_jq_{0i} \\
& +24\mathcal{L}_{\hat{X}}\beta\beta b_it_j+24\mathcal{L}_{\hat{X}}\beta\beta b_jt_i+60\mathcal{L}_{\hat{X}}\beta\beta s_is_j \\
& -72\mathcal{L}_{\hat{X}}\beta b_ib_jt_0-44\mathcal{L}_{\hat{X}}\beta b_is_0s_j-44\mathcal{L}_{\hat{X}}\beta b_js_0s_i \\
& -4\mathcal{L}_{\hat{X}}e_{i0}e_{j0}-4\mathcal{L}_{\hat{X}}e_{j0}e_{i0} \\
A_9 \;=\; & 4\mathcal{L}_{\hat{X}}t_0b_js_i-8\mathcal{L}_{\hat{X}}t_i\beta b_j-6\mathcal{L}_{\hat{X}}t_j\beta b_i+2\mathcal{L}_{\hat{X}}t_j\beta b_j \\
& -6\mathcal{L}_{\hat{X}}b_i\beta t_j+12\mathcal{L}_{\hat{X}}\beta b_it_j+12\mathcal{L}_{\hat{X}}\beta b_jt_i
\end{aligned}
$$

$$
\begin{aligned}
& +20\mathcal{L}_{\hat{X}}\beta s_i s_j + 6\mathcal{L}_{\hat{X}} b_i s_0 s_j + 4\mathcal{L}_{\hat{X}} b_i s_j t_0 - 6\mathcal{L}_{\hat{X}} b_j \beta t_i \\
& +6\mathcal{L}_{\hat{X}} b_j s_0 s_i + 4\mathcal{L}_{\hat{X}} b_j s_i t_0 + 2\mathcal{L}_{\hat{X}} s_0 b_i s_j \\
& +2\mathcal{L}_{\hat{X}} s_0 b_j s_i - 16\mathcal{L}_{\hat{X}} s_i \beta s_j + 6\mathcal{L}_{\hat{X}} s_i b_j s_0 + 4\mathcal{L}_{\hat{X}} s_i b_j t_0 \\
& -16\mathcal{L}_{\hat{X}} s_j \beta s_i + 6\mathcal{L}_{\hat{X}} s_j b_i s_0 + 4\mathcal{L}_{\hat{X}} s_j b_i t_0 \\
& +4\mathcal{L}_{\hat{X}} t_0 b_i s_j + 2\mathcal{L}_{\hat{X}} e_{j0} s_i + 2e_{j0}\mathcal{L}_{\hat{X}} s_i + 4\mathcal{L}_{\hat{X}} e_{ij} s_0 \\
& +4e_{ij}\mathcal{L}_{\hat{X}} s_0 + 2\mathcal{L}_{\hat{X}} e_{i0} s_j + 2e_{i0}\mathcal{L}_{\hat{X}} s_j - 4e_{ij}\eta \\
& -2e_{j0}\eta_i + 4\mathcal{L}_{\hat{X}} b_i q_{0j} + 2\mathcal{L}_{\hat{X}} b_i q_{j0} + 4\mathcal{L}_{\hat{X}} b_j q_{0i} \\
& +2\mathcal{L}_{\hat{X}} b_j q_{i0} + 4\mathcal{L}_{\hat{X}} q_{0i} b_j + 4\mathcal{L}_{\hat{X}} q_{0j} b_i + 2\mathcal{L}_{\hat{X}} q_{i0} b_j \\
& -8\mathcal{L}_{\hat{X}} q_{ij}\beta + 2\mathcal{L}_{\hat{X}} q_{j0} b_i - 8\mathcal{L}_{\hat{X}} qji\beta + 10\mathcal{L}_{\hat{X}}\beta q_{ij} \\
& +10\mathcal{L}_{\hat{X}}\beta q_{ji} - 2e_{i0}\eta_j + 2\mathcal{L}_{\hat{X}} e_{ij;0} \\
A_{10} \quad = \quad & -2(\mathcal{L}_{\hat{X}} b_i t_j + \mathcal{L}_{\hat{X}} b_j t_i + 2\mathcal{L}_{\hat{X}} s_i s_j + 2\mathcal{L}_{\hat{X}} s_j s_i \\
& +\mathcal{L}_{\hat{X}} t_i b_j + \mathcal{L}_{\hat{X}} t_j b_i + \mathcal{L}_{\hat{X}} q_{ij} + \mathcal{L}_{\hat{X}} q_{ji}).
\end{aligned}
$$

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TAUBERIAN THEOREMS FOR THE WEIGHTED MEAN METHOD OF SUMMABILITY OF INTEGRALS

Fırat Özsaraç and İbrahim Çanak

Abstract. Let q be a positive weight function on $\mathbf{R}_+ := [0,\infty)$ which is integrable in Lebesgue's sense over every finite interval $(0,x)$ for $0 < x < \infty$, in symbol: $q \in L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \to \infty$ as $x \to \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \ \ (x > 0, m = 1, 2, ...),$$

provided that $Q(x) > 0$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m-th iteration of weighted mean method determined by the function $q(x)$, or for short, $(\overline{N}, q, m)$ integrable to a finite number L if

$$\lim_{x\to\infty} \tau_q^{(m)}(x) = L.$$

In this case, we write $s(x) \to L(\overline{N}, q, m)$.

It is known that if the limit $\lim_{x\to\infty} s(x) = L$ exists, then $\lim_{x\to\infty} \tau_q^{(m)}(x) = L$ also exists. However, the converse of this implication is not always true. Some suitable conditions together with the existence of the limit $\lim_{x\to\infty} \tau_q^{(m)}(x)$, which is so called Tauberian conditions, may imply convergence of $\lim_{x\to\infty} s(x)$.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for $(\overline{N}, q, m)$ summable integrals of real- or complex-valued functions have been obtained. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability $(\overline{N}, q)$ have been extended and generalized.

Keywords: Tauberian conditions; weight function; summable integrals; finite interval.

1. Introduction

Let q be a positive weight function on $\mathbf{R}_+ := [0,\infty)$ which is integrable in Lebesgue's sense over every finite interval $(0,x)$ for $0 < x < \infty$, in symbol: $q \in$

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$L^1_{loc}(\mathbf{R}_+)$ such that $Q(x) := \int_0^x q(t)dt \neq 0$ for each $x > 0$, $Q(0) = 0$ and $Q(x) \to \infty$ as $x \to \infty$. Given a real or complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$, we define $s(x) := \int_0^x f(t)dt$ and

$$\tau_q^{(0)}(x) := s(x), \tau_q^{(m)}(x) := \frac{1}{Q(x)} \int_0^x \tau_q^{(m-1)}(t)q(t)dt \ \ (x > 0, m = 1, 2, ...),$$

provided that $Q(x) > 0$.

For each integer $m \geq 0$, we define $v_q^{(m)}(x)$ by

$$v_q^{(m)}(x) = \begin{cases} \frac{Q(x)}{q(x)} f(x) & , m = 0 \\ \frac{1}{Q(x)} \int_0^x f(t)Q(t)dt & , m = 1 \\ \frac{1}{Q(x)} \int_0^x v_q^{(m-1)}(t)q(t)dt & , m \geq 2. \end{cases}$$

The identity

$$\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) = v_q^{(m)}(x) \tag{1.1}$$

is known as the weighted Kronecker identity for the weighted mean method of summability.

It is clear from (1.1) that

$$\frac{Q(x)}{q(x)} \frac{d}{dx} \tau_q^{(m)}(x) = v_q^{(m)}(x)$$

for each integer $m \geq 0$ (see [14]). Here, we call $v_q^{(m)}(x)$ the generator of $\tau_q^{(m-1)}(x)$ for each integer $m \geq 1$.

We say that $\int_0^\infty f(x)dx$ is summable to L by the m-th iteration of weighted mean method determined by the function $q(x)$, or for short, $(\overline{N}, q, m)$ summable to a finite number L if

$$\lim_{x \to \infty} \tau_q^{(m)}(x) = L. \tag{1.2}$$

It is obvious that $(\overline{N}, q, m)$ summability reduces to the ordinary convergence for $m = 0$ and $(\overline{N}, q, 1)$ is the $(\overline{N}, q)$ method. If $q(x) = 1$ on $\mathbf{R}_+$, then $(\overline{N}, q, m)$ method is the Hölder method of order m and $(\overline{N}, q, 1)$ method is the Cesàro summability method $(C, 1)$.

It is well known that condition $Q(x) \to \infty$ as $x \to \infty$ is a necessary and sufficient condition that the existence of the integral

$$\int_0^\infty s(x)dx = L \tag{1.3}$$

implies (1.2). That is, the $(\overline{N}, q, m)$ summability method is regular, where m is a nonnegative integer. However, the converse of this implication is not always true. Notice that some suitable condition on $s(x)$ together with (1.2) may imply (1.3).

Such a condition is called a Tauberian condition and resulting theorem is said to be a Tauberian theorem.

Móricz [8] and Fekete and Móricz [6] obtained one-sided and two-sided Tauberian conditions for the weighted mean method $(\overline{N}, q)$ of integrals. Following these works, Totur and Okur [13] proved one-sided boundedness of $v_q^{(0)}(x)$ is a Tauberian condition for the weighted mean method of summability $(\overline{N}, q)$ of integrals. From the fact that condition $v_q^{(0)}(x) \geq -C$ implies slow decreasing of $s(x)$, Totur and Okur [13] generalized their first result and proved that slow decrease of $s(x)$ is also a Tauberian condition for $(\overline{N}, q)$ method. For a detailed study and some interesting results related to Tauberian theorems for the weighted mean method of summability, we refer the reader to Borwein and Kratz [1], Çanak and Totur [2], Çanak and Totur [3], Çanak and Totur [4], Özsaraç and Çanak [9], Sezer and Çanak [10], Tietz and Zeller [11] and Totur and Çanak [12], etc.

In this paper, one- and two-sided Tauberian conditions in terms of the generating function and its generalizations for summable integrals by m-th iteration of weighted means of real- or complex-valued functions have been obtained, respectively. Some classical type Tauberian theorems given for Cesàro summability $(C, 1)$ and weighted mean method of summability $(\overline{N}, q)$ have been extended and generalized.

2. Main results

For the main results of the paper, we need the following definitions and notations.

Definition 2.1. ([7]) A positive function Q is called regularly varying of index $\alpha > 0$ if

$$\lim_{x\to\infty} \frac{Q(\rho x)}{Q(x)} = \rho^{\alpha}, \quad \rho > 0. \tag{2.1}$$

It easily follows from Definition 2.1 that for all $\rho > 1$ and sufficiently large x,

$$\frac{\rho^{\alpha}}{2(\rho^{\alpha}-1)} \leq \frac{Q(\rho x)}{Q(\rho x) - Q(x)} \leq \frac{3\rho^{\alpha}}{2(\rho^{\alpha}-1)} \tag{2.2}$$

and for all $0 < \rho < 1$ and sufficiently large x,

$$\frac{\rho^{\alpha}}{2(1-\rho^{\alpha})} \leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \leq \frac{3\rho^{\alpha}}{2(1-\rho^{\alpha})}. \tag{2.3}$$

We note that if (2.1) holds, then the following equivalent conditions are clearly satisfied (see [5]):

$$\liminf_{x\to\infty} \frac{Q(x)}{Q(\rho x)} < 1, \text{ for every } \rho > 1 \tag{2.4}$$

and

$$\liminf_{x\to\infty} \frac{Q(\rho x)}{Q(x)} < 1, \text{ for every } 0 < \rho < 1. \tag{2.5}$$

First, we consider real-valued functions and prove the following Tauberian theorems.

Theorem 2.1. *Let (2.1) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $v_q^{(m-1)}(x)$ is one-sided bounded, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 2.1. *([13]) Let (2.1) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and $v_q^{(0)}(x)$ is one-sided bounded, then $s(x)$ converges to L.*

Theorem 2.2. *Let (2.4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 2.2. *([13]) Let (2.4) be satisfied. If a real-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and slowly decreasing, then $s(x)$ converges to L.*

A real-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly decreasing if

$$\lim_{\rho \to 1^+} \liminf_{x \to \infty} \min_{x \le t \le \rho x} \left(s(t) - s(x)\right) \ge 0. \tag{2.6}$$

Note that condition (2.6) can be equivalently reformulated as follows:

$$\lim_{\rho \to 1^-} \liminf_{x \to \infty} \min_{\rho x \le t \le x} \left(s(x) - s(t)\right) \ge 0. \tag{2.7}$$

Second, we consider complex-valued functions and prove the following Tauberian theorems.

Theorem 2.3. *Let (2.1) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $v_q^{(m-1)}(x)$ is bounded, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 2.3. *Let (2.1) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and $v_q^{(0)}(x)$ is bounded, then $s(x)$ converges to L.*

Theorem 2.4. *Let (2.4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating, then $s(x)$ is $\left(\overline{N}, q, m-1\right)$ summable to L.*

Corollary 2.4. *Let (2.4) be satisfied. If a complex-valued function $f \in L^1_{loc}(\mathbf{R}_+)$ is such that its integral function $s(x)$ is $\left(\overline{N}, q, 1\right)$ summable to L and slowly oscillating, then $s(x)$ converges to L.*

A complex-valued function $s(x)$ defined on $\mathbf{R}_+$ is said to be slowly oscillating if

$$\lim_{\rho\to 1^+}\limsup_{x\to\infty}\max_{x\le t\le \rho x}|s(t)-s(x)|=0. \tag{2.8}$$

Note that condition (2.8) can be equivalently reformulated as follows:

$$\lim_{\rho\to 1^-}\limsup_{x\to\infty}\max_{\rho x\le t\le x}|s(x)-s(t)|=0. \tag{2.9}$$

3. An auxiliary result

The following two representations of $s(x)-\tau_q^{(1)}(x)$ will be needed in the proofs of our main results.

Lemma 3.1. *([13])*
(i) For $\rho>1$ and sufficiently large x,

$$\begin{aligned} s(x)-\tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(1)}(\rho x)-\tau_q^{(1)}(x)\right) \\ &- \frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}(s(t)-s(x))\,q(t)\,dt. \end{aligned}$$

(ii) For $0<\rho<1$ and sufficiently large x,

$$\begin{aligned} s(x)-\tau_q^{(1)}(x) &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(1)}(x)-\tau_q^{(1)}(\rho x)\right) \\ &+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}(s(x)-s(t))\,q(t)\,dt. \end{aligned}$$

4. Proofs of main results

Proof of Theorem 2.1 Suppose that $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L and $v_q^{(m-1)}(x)$ is one-sided bounded. By Lemma 3.1 (i), we have

$$\begin{aligned} \tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right) \\ &- \frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left(\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right)q(t)\,dt \\ &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right) \\ &- \frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left(\int_x^{t}\frac{d}{dz}\tau_q^{(m-1)}(z)\,dz\right)q(t)\,dt. \end{aligned}$$

Since $v_q^{(m-1)}(x)$ is one-sided bounded, we get

$$\begin{aligned}
\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &\leq \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)\\
&+\frac{C}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left(\int_x^t \frac{q(z)}{Q(z)}dz\right)q(t)\,dt\\
&=\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)\\
&+\frac{C}{Q(\rho x)-Q(x)}\int_x^{\rho x} q(t)\log\frac{Q(t)}{Q(x)}dt\\
&=\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)+C\log\frac{Q(\rho x)}{Q(x)}. \qquad (4.1)
\end{aligned}$$

By (2.2) and $\left(\overline{N},q,m\right)$ summability of $s(x)$, we have

$$\lim_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)=0. \qquad (4.2)$$

Taking (4.2) into account in (4.1), we obtain

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\leq\limsup_{x\to\infty}\left(C\log\frac{Q(\rho x)}{Q(x)}\right)=C\log\rho^{\alpha}.$$

Letting $\rho\to 1^+$ in the last inequality, we have

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\leq 0. \qquad (4.3)$$

Similarly, from Lemma 3.1 (ii), we have

$$\begin{aligned}
\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\
&+\frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right)q(t)\,dt\\
&=\frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\
&+\frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left(\int_t^x \frac{d}{dz}\tau_q^{(m-1)}(z)\,dz\right)q(t)\,dt.
\end{aligned}$$

Since $v_q^{(m-1)}(x)$ is one-sided bounded, we get

$$\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) \geq \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)$$

$$\begin{aligned}
& - \frac{C}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} \left(\int_{t}^{x} \frac{q(z)}{Q(z)} dz \right) q(t)\, dt \\
& = \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) \\
& - \frac{C}{Q(x)-Q(\rho x)} \int_{\rho x}^{x} q(t) \log \frac{Q(x)}{Q(t)} dt \\
& = \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) - C \log \frac{Q(x)}{Q(\rho x)}. \qquad (4.4)
\end{aligned}$$

By (2.3) and $(\overline{N}, q, m)$ summability of $s(x)$, we obtain

$$\lim_{x\to\infty} \frac{Q(\rho x)}{Q(x)-Q(\rho x)} \left(\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x) \right) = 0. \qquad (4.5)$$

Taking (4.5) into account in (4.4), we obtain

$$\limsup_{x\to\infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq - \liminf_{x\to\infty} \left(C \log \frac{Q(x)}{Q(\rho x)} \right) = -C \log \rho^{\alpha}.$$

Letting $\rho \to 1^-$ in the last inequality, we have

$$\limsup_{x\to\infty} \left(\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) \right) \geq 0. \qquad (4.6)$$

Combining (4.3) and (4.6), we obtain $s(x)$ is $(\overline{N}, q, m-1)$ summable to L. $\square$

Proof of Theorem 2.2 Suppose that $s(x)$ is $(\overline{N}, q, m)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly decreasing. By Lemma 3.1 (i), we have

$$\begin{aligned}
\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x) & = \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
& - \frac{1}{Q(\rho x)-Q(x)} \int_{x}^{\rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) q(t)\, dt \\
& \leq \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
& - \frac{1}{Q(\rho x)-Q(x)} \int_{x}^{\rho x} q(t) \min_{x\leq t\leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right) dt \\
& = \frac{Q(\rho x)}{Q(\rho x)-Q(x)} \left(\tau_q^{(m)}(\rho x) - \tau_q^{(m)}(x) \right) \\
& - \min_{x\leq t\leq \rho x} \left(\tau_q^{(m-1)}(t) - \tau_q^{(m-1)}(x) \right). \qquad (4.7)
\end{aligned}$$

Taking the lim sup of both sides of (4.7), we get

$$\begin{aligned}\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right) &\le \limsup_{x\to\infty}\tfrac{Q(\rho x)}{Q(\rho x)-Q(x)}\left(\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right)\\ &-\liminf_{x\to\infty}\min_{x\le t\le\rho x}\left(\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right). \end{aligned}\tag{4.8}$$

By (2.4), we have

$$0<\limsup_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}=1+\left(\liminf_{x\to\infty}\frac{Q(\rho x)}{Q(x)}-1\right)^{-1}<\infty.$$

Since $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L, the first term on the right-hand side vanishes in (4.8). From this, we obtain

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\le-\liminf_{x\to\infty}\min_{x\le t\le\rho x}\left(\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right).$$

Taking the limit of (4.8) as $\rho\to 1^+$, we have

$$\limsup_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\le 0.\tag{4.9}$$

Similarly, by Lemma 3.1 (ii), we have

$$\begin{aligned}\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x) &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\ &+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right)q(t)\,dt\\ &\ge \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\ &+ \frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}q(t)\min_{\rho x\le t\le x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right)dt\\ &= \frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\ &+ \min_{\rho x\le t\le x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right). \end{aligned}\tag{4.10}$$

From (4.10), we get

$$\begin{aligned}\liminf_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right) &\ge \liminf_{x\to\infty}\tfrac{Q(\rho x)}{Q(x)-Q(\rho x)}\left(\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right)\\ &+\liminf_{x\to\infty}\min_{\rho x\le t\le x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right). \end{aligned}\tag{4.11}$$

By (2.4), we have

$$0<\liminf_{x\to\infty}\frac{Q(\rho x)}{Q(x)-Q(\rho x)}=\left(\limsup_{x\to\infty}\frac{Q(x)}{Q(\rho x)}-1\right)^{-1}<\infty.$$

Since $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L, the first term on the right-hand side vanishes in (4.11). From this, we obtain

$$\liminf_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\geq\liminf_{x\to\infty}\min_{\rho x\leq t\leq x}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right).$$

Taking the limit of (4.11) as $\rho\to 1^-$, we have

$$\liminf_{x\to\infty}\left(\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right)\geq 0. \tag{4.12}$$

Combining (4.9) and (4.12), we obtain $s(x)$ is $\left(\overline{N},q,m-1\right)$ summable to L. □

Proof of Theorem 2.3 Suppose that $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L and $v_q^{(m-1)}(x)$ is bounded. By Lemma 3.1 (i), we have

$$\begin{aligned}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &\leq \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|q(t)\,dt\\ &=\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left|\int_x^t\frac{d}{dz}\tau_q^{(m-1)}(z)\,dz\right|q(t)\,dt.\end{aligned}$$

Since $v_q^{(m-1)}(x)$ is bounded, we get

$$\begin{aligned}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &\leq \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\frac{C}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left|\int_x^t\frac{q(z)}{Q(z)}dz\right|q(t)\,dt\\ &=\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\frac{C}{Q(\rho x)-Q(x)}\int_x^{\rho x}q(t)\log\frac{Q(t)}{Q(x)}dt\\ &\leq \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|+C\log\frac{Q(\rho x)}{Q(x)}. \qquad (4.13)\end{aligned}$$

By (2.2) and $(\overline{N},q,m)$ summability of $s(x)$, we have

$$\lim_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|=0.$$

Taking the lim sup of both sides of (4.13) gives

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le\limsup_{x\to\infty}\left(C\log\frac{P(\rho x)}{Q(x)}\right)=C\log\rho^{\alpha}.$$

Letting $\rho\to 1^{+}$ in last inequality, we have

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le 0. \tag{4.14}$$

Similarly, from Lemma 3.1 (ii), we have

$$\begin{aligned}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &\le \frac{P(\rho x)}{Q(x)-Q(\rho x)}\left|\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right|\\ &+\frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m-1)}(t)\right|q(t)\,dt\\ &=\frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left|\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right|\\ &+\frac{1}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left|\int_{t}^{x}\frac{d}{dz}\tau_q^{(m-1)}(z)\,dz\right|q(t)\,dt.\end{aligned}$$

Since $v_q^{(m-1)}(x)$ is bounded, we get

$$\begin{aligned}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &\le \frac{P(\rho x)}{Q(x)-Q(\rho x)}\left|\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right|\\ &+\frac{C}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}\left|\int_{t}^{x}\frac{p(z)}{P(z)}dz\right|q(t)\,dt\\ &=\frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left|\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right|\\ &+\frac{C}{Q(x)-Q(\rho x)}\int_{\rho x}^{x}q(t)\log\frac{Q(x)}{Q(t)}dt\\ &\le\frac{Q(\rho x)}{Q(x)-Q(\rho x)}\left|\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right|+C\log\frac{Q(x)}{Q(\rho x)}.\end{aligned} \tag{4.15}$$

By (2.3) and $(\overline{N},p,m)$ summability of $s(x)$, we have

$$\lim_{x\to\infty}\frac{Q(\rho x)}{P(x)-Q(\rho x)}\left|\tau_q^{(m)}(x)-\tau_q^{(m)}(\rho x)\right|=0.$$

From (4.15), we get

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le\limsup_{x\to\infty}\left(C\log\frac{Q(x)}{Q(\rho x)}\right)=C\log\rho^{\alpha}.$$

Letting $\rho\to 1^-$ in last inequality, we have

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le 0. \tag{4.16}$$

From either (4.14) or (4.16), we conclude $s(x)$ is $\left(\overline{N},q,m-1\right)$ summable to L. □

Proof of Theorem 2.4 Suppose that $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L and $\tau_q^{(m-1)}(x)$ is slowly oscillating. By Lemma 3.1 (i), we have

$$\begin{aligned}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &= \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+ \frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|q(t)\,dt\\ &\le \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+ \frac{1}{Q(\rho x)-Q(x)}\int_x^{\rho x}q(t)\max_{x\le t\le\rho x}\left(\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|\right)dt\\ &\le \frac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+ \max_{x\le t\le\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|. \qquad (4.17)\end{aligned}$$

From (4.17), we get

$$\begin{aligned}\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right| &\le \limsup_{x\to\infty}\tfrac{Q(\rho x)}{Q(\rho x)-Q(x)}\left|\tau_q^{(m)}(\rho x)-\tau_q^{(m)}(x)\right|\\ &+\limsup_{x\to\infty}\max\nolimits_{x\le t\le\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|. \qquad (4.18)\end{aligned}$$

By (2.4), we have

$$0<\limsup_{x\to\infty}\frac{Q(\rho x)}{Q(\rho x)-Q(x)}=1+\left(\liminf_{x\to\infty}\frac{Q(\rho x)}{Q(x)}-1\right)^{-1}<\infty.$$

Since $s(x)$ is $\left(\overline{N},q,m\right)$ summable to L, the first term on the right side vanishes in (4.18). From this, we obtain

$$\limsup_{x\to\infty}\left|\tau_q^{(m-1)}(x)-\tau_q^{(m)}(x)\right|\le\limsup_{x\to\infty}\max_{x\le t\le\rho x}\left|\tau_q^{(m-1)}(t)-\tau_q^{(m-1)}(x)\right|.$$

Taking the limit of (4.18) as $\rho \to 1^{+}$, we have

$$\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| \leq 0. \tag{4.19}$$

Similarly, by Lemma 3.1 (ii), we have

$$\begin{aligned}
\left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| &= \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right| q(t)\, dt \\
&\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \frac{1}{Q(x) - Q(\rho x)} \int_{\rho x}^{x} q(t) \max_{\rho x \leq t \leq x} \left(\left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|\right) dt \\
&\leq \frac{Q(\rho x)}{Q(x) - Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \max_{\rho x \leq t \leq x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|. \qquad (4.20)
\end{aligned}$$

From (4.20), we get

$$\begin{aligned}
\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| &\leq \limsup_{x\to\infty} \tfrac{Q(\rho x)}{Q(x)-Q(\rho x)} \left|\tau_q^{(m)}(x) - \tau_q^{(m)}(\rho x)\right| \\
&+ \limsup_{x\to\infty} \max_{\rho x \leq t \leq x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|. \qquad (4.21)
\end{aligned}$$

By (2.4), we have

$$0 < \liminf_{x\to\infty} \frac{Q(\rho x)}{Q(x) - Q(\rho x)} = \left(\limsup_{x\to\infty} \frac{Q(x)}{Q(\rho x)} - 1\right)^{-1} < \infty.$$

Since $s(x)$ is $(\overline{N}, q, m)$ summable to L, the first term on the right-hand side vanishes in (4.21). From this, we obtain

$$\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| \leq \limsup_{x\to\infty} \max_{\rho x \leq t \leq x} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m-1)}(t)\right|.$$

Taking the limit of (4.21) as $\rho \to 1^{-}$, we have

$$\limsup_{x\to\infty} \left|\tau_q^{(m-1)}(x) - \tau_q^{(m)}(x)\right| \leq 0. \tag{4.22}$$

From either (4.19) or (4.22), we conclude $s(x)$ is $(\overline{N}, q, m-1)$ summable to L. $\square$

5. Conclusion

In this paper, we introduce Tauberian conditions in terms of the generator and its generalizations for summable integrals by m-th iteration of weighted means of real- or complex-valued functions, respectively. Tauberian conditions for summable double integrals by m-th iteration of weighted means of real- or complex-valued functions will be illustrated in a forthcoming work.

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SPACELIKE TRANSLATION SURFACES IN MINKOWSKI 4−SPACE $\mathbb{E}_1^4$

Sezgin Büyükkütük and Günay Öztürk

Abstract. In the present paper, we consider spacelike translation surfaces in 4-dimensional Minkowski space. We characterize such surfaces in terms of their Gaussian curvature and mean curvature functions. We classify flat and minimal spacelike translation surfaces in $\mathbb{E}_1^4$.

Keywords: spacelike translation surfaces; Minkowski space; Gaussian curvature.

1. Introduction

A surface's geometry consists of some properties like area, distance, angle and curvature. The most important of these is curvature, which reveals the structural differences between surfaces. Flat and minimal surfaces with zero Gaussian and zero mean curvature have major significance in geometry. Especially, a minimal surface is a surface that locally minimizes its area. In addition to the planes, catenoids and helicoids, the appearance of minimal surfaces can also be observed in nature: in the structures that animals build, in various plants and animal anatomies, etc. In history, some mathematicians such as Riemann, Schwarz, Scherk, Weierstrass and Enneper made major advances on minimal surfaces (see, [18]). During 1960s, the pioneering work of Osserman influenced the majority of modern theories of minimal surfaces in three dimensional spaces [17]. Minimal surfaces have also been the subject of today's work (see, [16]).

A special surface: Translation surface which is known as double curved in differential geometry are base for roofing structures. The construction and design of freeform glass roofing structures are generally created with the help of curved (formed) glass panes or planar triangular glass facets. Especially, double curved surface are made up of quadrilateral, that is four sided, facets. They lead to economic advantages compared to triangular glass facets. Because of these advantages, translation surfaces are used to construct free form glass roofing structures [11]. Also,

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due to geometric property of these surfaces, they are used for Teichmüller theory in physics (see, [9]).

Translation surfaces can be parameterized locally as $\phi(u,v) = (u, v, f(u)+g(v))$. In [4], Baikoussis and Verstraelen (1992) investigated the Gauss map of translation surfaces with in $3-$space. Particularly, in [19], H. Scherk introduced the special translation surface named Scherk surface which is the only non flat minimal. Then, these type of surfaces have been studied in Euclidean spaces by many geometers with different perspectives (see, [1, 2, 7, 20]). Also, in [3], the authors characterized the translation surfaces in the $3-$dimensional Lorentz-Minkowski space.

In the present study, we consider a spacelike translation surface in Minkowski 4-space. We define the surface which locally can be written as a monge patch

$$\phi(u,v) = (u, v, f_1(u) + g_1(v), f_2(u) + g_2(v)),$$

for some differentiable functions, $f_i(u), g_i(v)$, $i = 1, 2$. We characterize such surfaces in terms of their Gaussian curvature and mean curvature functions and give the conditions for such surfaces to become flat and minimal.

2. Basic Concepts

The Minkowski $4-$space denoted by $\mathbb{E}_1^4$ is the space given by the Lorentzian inner product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

Let $S : \phi = \phi(u,v) : (u,v) \in D$ ($D \subset \mathbb{E}^2$) be a spacelike surface in $\mathbb{E}_1^4$, then $\langle , \rangle$ induces a Riemannian metric on S. Thus, at each point p of a spacelike surface S, the following decomposition is available:

$$\mathbb{E}_1^4 = T_p^{\perp}S \oplus T_pS,$$

where the restriction of the metric $\langle , \rangle$ onto the normal space $T_p^{\perp}S$ and T_pS have the signatures (1,1) and (2,0), respectively.

$\widetilde{\nabla}$ and ∇ indicate the Levi-Civita connections on $\mathbb{E}_1^4$ and S. Suppose X and Y be vector fields tangent to M and ξ be a normal vector field. The formulas of Weingarten and Gauss decompose the vector fields $\widetilde{\nabla}_X\xi$ and $\widetilde{\nabla}_XY$ into normal and tangent components:

$$\begin{aligned}
\widetilde{\nabla}_X\xi &= -A_\xi X + D_X\xi, \\
\widetilde{\nabla}_XY &= \nabla_XY + h(X,Y),
\end{aligned}$$

where h, D, and A_ξ are the second fundamental form, the normal connection and the shape operator, respectively [6].

The mean curvature vector field H of S can be calculated by $H = \frac{1}{2}trh$, i.e. given a local orthonormal frame $\{X, Y\}$ of the tangent bundle, $H = \frac{1}{2}\left((h(X,X) + h\left(Y,Y\right)\right).$

Let $S : \phi = \phi(u,v) : (u,v) \in D$ ($D \subset \mathbb{E}^2$) be a local parametrization on a spacelike surface in $\mathbb{E}_1^4$. The tangent space at an arbitrary point $p = \phi(u,v)$ of S is $T_pS = span\{\phi_u, \phi_v\}$, where $\langle \phi_u, \phi_u \rangle > 0$, $\langle \phi_v, \phi_v \rangle > 0$. The standard indications $E = \langle \phi_u, \phi_u \rangle$, $F = \langle \phi_u, \phi_v \rangle$, $G = \langle \phi_v, \phi_v \rangle$ are used for the coefficients of the first fundamental form

$$I(\lambda,\mu) = E\lambda^2 + 2F\lambda\mu + G\mu^2,\ \lambda,\mu \in IR. \tag{2.1}$$

[12] Since $I(\lambda,\mu)$ is positive definite, we set $W = \sqrt{EG - F^2}$. We choose a normal frame field $\{\xi_1, \xi_2\}$ such that $\langle \xi_1, \xi_1 \rangle = -1$, $\langle \xi_2, \xi_2 \rangle = 1$, and the quadruple $\{\phi_u, \phi_v, \xi_1, \xi_2\}$ is positively oriented in $\mathbb{E}_1^4$. Then we have the following derivative formulas:

$$\begin{aligned} \widetilde{\nabla}_{\phi_u}\phi_u &= \phi_{uu} = \Gamma_{11}^1\phi_u + \Gamma_{11}^2\phi_v - c_{11}^1\xi_1 + c_{11}^2\xi_2, \\ \widetilde{\nabla}_{\phi_u}\phi_v &= \phi_{uv} = \Gamma_{12}^1\phi_u + \Gamma_{12}^2\phi_v - c_{12}^1\xi_1 + c_{12}^2\xi_2, \\ \widetilde{\nabla}_{\phi_v}\phi_v &= \phi_{vv} = \Gamma_{22}^1\phi_u + \Gamma_{22}^2\phi_v - c_{22}^1\xi_1 + c_{22}^2\xi_2, \end{aligned} \tag{2.2}$$

where Γ_{ij}^k and c_{ij}^k, $(i,j,k = 1,2)$ denote Cristoffel symbols and coefficients of second fundamental form, respectively. Then, these coefficients are given by

$$\begin{aligned} c_{11}^1 &= \langle \phi_{uu}, \xi_1 \rangle, \quad c_{12}^1 = \langle \phi_{uv}, \xi_1 \rangle, \quad c_{22}^1 = \langle \phi_{vv}, \xi_1 \rangle, \\ c_{11}^2 &= \langle \phi_{uu}, \xi_2 \rangle, \quad c_{12}^2 = \langle \phi_{uv}, \xi_2 \rangle, \quad c_{22}^1 = \langle \phi_{vv}, \xi_2 \rangle. \end{aligned} \tag{2.3}$$

[13] h represents the second fundamental tensor of the surface S, then

$$\begin{aligned} h(\phi_u, \phi_u) &= -c_{11}^1\xi_1 + c_{11}^2\xi_2, \\ h(\phi_u, \phi_v) &= -c_{12}^1\xi_1 + c_{12}^2\xi_2, \\ h(\phi_v, \phi_v) &= -c_{22}^1\xi_1 + c_{22}^2\xi_2. \end{aligned} \tag{2.4}$$

The second fundamental tensor can be written as

$$h(X,Y) = -\langle A_{\xi_1}(X), Y \rangle \xi_1 + \langle A_{\xi_2}(X), Y \rangle \xi_2. \tag{2.5}$$

[15] The component of H along a given normal connection N_k, denoted by H_k, is called the expansion along ξ_k, i.e., $H_k = \langle H, \xi_k \rangle = \frac{tr(A_{\xi_k})}{2}$ and we obtain

$$H_k = \frac{c_{11}^k G - 2c_{12}^k F + c_{22}^k E}{2(EG - F^2)}. \tag{2.6}$$

With regard to the normal basis the mean curvature vector field H becomes

$$H = -H_1\xi_1 + H_2\xi_2. \tag{2.7}$$

The norm of the mean curvature vector $\left\| \vec{H} \right\|$ is called the mean curvature of S. If mean curvature vector of a surface is zero, then it is called minimal.

Gaussian curvature of a regular patch $\phi(u,v)$ can be expressed in terms of the coefficients of the first and second fundamental forms as

$$K = \frac{-\det(A_{\xi_1}) + \det(A_{\xi_2})}{W^2} = \frac{-c_{11}^1 c_{22}^1 + c_{11}^2 c_{22}^2 + \left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2}{EG - F^2}. \tag{2.8}$$

A surface S is said to be flat if its Gauss curvature vanishes. [5].

3. Spacelike Translation Surface in $\mathbb{E}_1^3$

The translation surface S determined by curves $\alpha, \beta : (a,b) \to \mathbb{E}_1^3$ is the patch

$$S : \phi(u,v) = \alpha(u) + \beta(v). \tag{3.1}$$

It is the surface formed by moving α parallel to itself in such a way that a point of the curve moves along β [8].

A surface that can be generated from two space curves by translating either one of them parallel to itself in such a way that each of its points describes a curve that is a translation of the other curve. For the spacelike surface S, both of the generator curves $\alpha(u)$, $\beta(v)$ are spacelike. These curves are defined by the parameterizations

$$\begin{aligned} \alpha(u) &= (u, 0, f(u)), \\ \beta(v) &= (0, v, g(v)), \end{aligned}$$

where $f(u)$ and $g(v)$ are smooth functions. Thus, the representation of the surface is

$$\phi(u,v) = (u, v, f(u) + g(v)). \tag{3.2}$$

The natural frame $\{\phi_u, \phi_v\}$ is given by

$$\begin{aligned} \phi_u &= (1, 0, f'(u)), \\ \phi_v &= (0, 1, g'(v)). \end{aligned}$$

Then it follows that the unit normal vector ξ is given by

$$\xi = \frac{1}{\sqrt{1 - f'^2 + g'^2}} (f', -g', 1).$$

The curvatures of the surface in Minkowski 3−space are given by

$$K = -\frac{f''g''}{\left(f'^2 + g'^2 - 1\right)^2}$$

and

$$H = \frac{\left(1 - f'^2\right) g'' + \left(1 - g'^2\right) f''}{2\left(f'^2 + g'^2 - 1\right)^{\frac{3}{2}}}.$$

Theorem 3.1. *[10] A translation surface parameterized by (3.2) in Minkowski* $3-$*space has constant Gaussian curvature if and only if it is (a part) of a plane or a generalized cylinder and thus, is a flat surface.*

Theorem 3.2. *[10] A spacelike translation surface in Minkowski* $3-$*space parameterized by (3.2) has mean curvature zero if and only if it is (a part of) either a spacelike plane or the surface of Scherk of the first kind which is parameterized by*

$$\phi(u,v)=\left(u,v,\frac{1}{a}\ln\left|\frac{\cosh(av)}{\sinh(au)}\right|\right) \quad \textit{with} \quad \tanh^2(au)+\tanh^2(av)<1 \textit{ and } a\in\mathbb{R}_0.$$

4. Spacelike Translation Surface in $\mathbb{E}_1^4$

Definition 4.1. A surface can be determined by the curves $\alpha,\beta:(a,b)\to\mathbb{E}_1^4$ is the patch

$$\begin{aligned}\phi &: \mathbb{E}^2\to\mathbb{E}_1^4\\ \phi(u,v) &= \alpha(u)+\beta(v).\end{aligned}$$

If the generating curves $\alpha(u)$ and $\beta(v)$ are space curves has the parameterizations

$$\begin{aligned}\alpha(u) &= (u,0,f_1(u),f_2(u)),\\ \beta(v) &= (0,v,g_1(v),g_2(v)),\end{aligned}$$

then this surface is still called translation surface in $\mathbb{E}_1^4$. Thus, the translation surface is defined by the patch

$$\phi(u,v)=(u,v,f_1(u)+g_1(v),f_2(u)+g_2(v)). \tag{4.1}$$

Let the surface S be spacelike, then both of the generator curves $\alpha(u)$, $\beta(v)$ are spacelike. The first partial derivatives of $\phi(u,v)$ are given by

$$\begin{aligned}\phi_u &= (1,0,f_1^{'}(u),f_2^{'}(u)),\\ \phi_v &= (0,1,g_1^{'}(v),g_2^{'}(v)).\end{aligned} \tag{4.2}$$

Hence, the coefficients of the first fundamental form of the surface as we can find

$$\begin{aligned}E &= \langle\phi_u,\phi_u\rangle=-1+(f_1^{'})^2+(f_2^{'})^2,\\ F &= \langle\phi_u,\phi_v\rangle=f_1^{'}g_1^{'}+f_2^{'}g_2^{'},\\ G &= \langle\phi_v,\phi_v\rangle=1+(g_1^{'})^2+(g_2^{'})^2,\end{aligned} \tag{4.3}$$

where $\langle\ ,\ \rangle$ is Lorentzian inner product in $\mathbb{E}_1^4$. Since the first fundamental form is positive definite, we set $W=\sqrt{EG-F^2}$.

The second partial derivatives of $\phi(u,v)$ are expressed as

$$
\begin{aligned}
\phi_{uu} &= (0,0,f_1''(u),f_2''(u)),\\
\phi_{uv} &= (0,0,0,0),\\
\phi_{vv} &= (0,0,g_1''(v),g_2''(v)).
\end{aligned} \tag{4.4}
$$

It follows that chosen normal frame field $\{\xi_1,\xi_2\}$

$$
\begin{aligned}
\xi_1 &= \frac{1}{\sqrt{|A|}}(f_1'(u),-g_1'(v),1,0),\\
\xi_2 &= \frac{1}{\sqrt{AD}}(Af_1'(u)-Bf_2'(u),Bg_1'(v)-Ag_2'(v),-B,A),
\end{aligned} \tag{4.5}
$$

where

$$
\begin{aligned}
A &= 1-(f_1')^2+(g_1')^2,\\
B &= -f_1'f_2'+g_1'g_2',\\
C &= 1-(f_3')^2+(g_3')^2,\\
D &= AC-B^2,
\end{aligned}
$$

and by the use of (4.4) and (4.5), the functions c_{ij}^k, $(i,j,k=1,2)$ are given by

$$
\begin{aligned}
c_{11}^1 &= \frac{f_1''}{\sqrt{|A|}},\quad c_{22}^1=\frac{g_1''}{\sqrt{|A|}},\\
c_{12}^1 &= c_{12}^2=0,\\
c_{11}^2 &= \frac{Af_2''-Bf_1''}{\sqrt{AD}},\\
c_{22}^2 &= \frac{Ag_2''-Bg_1''}{\sqrt{AD}}.
\end{aligned} \tag{4.6}
$$

Using Gram-Schmidt orthonormalization method for the spacelike vector fields ϕ_u and ϕ_v, we get orthonormal tangent vectors

$$
\begin{aligned}
X &= \frac{\phi_u}{\sqrt{E}},\\
Y &= \frac{\sqrt{E}}{W}\left(\phi_v-\frac{F}{E}\phi_u\right).
\end{aligned} \tag{4.7}
$$

By the use of (2.3), (2.4), (2.5) and (4.7), the shape operator matrices can be written as

$$(4.8)\quad \begin{aligned} A_{\xi_1} &= \frac{1}{E\sqrt{|A|}}\begin{bmatrix} f_1'' & \frac{-f_1''F}{W} \\ \frac{-f_1''F}{W} & \frac{g_1''E^2+f_1''F^2}{W^2} \end{bmatrix}, \\ A_{\xi_2} &= \frac{1}{E\sqrt{|AD|}}\begin{bmatrix} Af_2''-Bf_1'' & \frac{-(Af_2''-Bf_1'')F}{W} \\ \frac{-(Af_2''-Bf_1'')F}{W} & \frac{\left(Ag_2''-Bg_1''\right)E^2+(Af_2''-Bf_1'')F^2}{W^2} \end{bmatrix}. \end{aligned}$$

Theorem 4.1. *Let S be a spacelike translation surface in $\mathbb{E}_1^4$. Then Gaussian curvature of S is given by*

$$(4.9)\qquad K=\frac{f_1''g_1''C-(f_1''g_2''+g_1''f_2'')B+f_2''g_2''A}{W^2D}.$$

Proof. By the use of the equations (2.8) and (4.6), we get the result. □

Theorem 4.2. *Let S be a spacelike translation surface parameterized by (4.1). Then S is a flat surface if and only if it is (a part) of a plane or a generalized cylinder given by*

$$(4.10)\qquad \phi(u,v)=(u,0,f_1(u)+a_1,f_2(u)+a_2)+v(0,1,b_1,b_2)$$

or

$$(4.11)\qquad \phi(u,v)=(0,v,g_1(v)+c_1,g_2(v)+c_2)+u(1,0,d_1,d_2),$$

where a_i,b_i,c_i,d_i $(i=1,2)$ are real constants.

Proof. Let S be a spacelike translation surface parameterized by (4.1). If the Gaussian curvature of the surface is zero, then we get $f_i'=0$, $g_i'=0$, or $f_i''=0$, or $g_i''=0$ $(i=1,2)$. For the first case we obtain a plane. For the second and third case, we get generalized cylinders with the parameterizations (4.10) and (4.11), respectively. This completes the proof. □

Theorem 4.3. *Let S be a spacelike translation surface with the parametrization (4.1) in $\mathbb{E}_1^4$. Then the mean curvature vector field is given by*

$$(4.12)\qquad \vec{H}=-\frac{f_1''G+g_1''E}{2\sqrt{|A|}W^2}\xi_1+\frac{G(f_2''A-f_1''B)+E(g_2''A-g_1''B)}{2\sqrt{AD}W^2}\xi_2.$$

Proof. By the use of the equations (2.6), (2.7) and (4.6), we obtain the desired result. □

Proposition 4.1. *Let S be a spacelike translation surface with the parametrization (4.1) in $\mathbb{E}_1^4$. Then S is a minimal surface if and only if*

$$-\frac{f_i^{''}}{E} = \frac{g_i^{''}}{G} = c_i \ , \quad i = 1, 2 \tag{4.13}$$

where c_i, $(i = 1, 2)$ are real constants.

Proof. Let S be a spacelike translation surface with the parametrization (4.1). If S is minimal, then the mean curvature vector field is zero, namely the components of $\vec{H}$ are zero. From the equation (4.12), we get the result. $\square$

Theorem 4.4. *Let S be a spacelike translation surface in $\mathbb{E}_1^4$ with the parametrization (4.1). Then S is a minimal surface if and only if either S is a plane or the functions $f_i(u)$, $g_i(v)$ are defined by*

$$f_i(u) = \frac{c_i}{c_1^2 + c_2^2}\left(\ln\left(\cos\sqrt{d}u\right) - bu\right) + k_i u, \ d > 0 \qquad i = 1, 2$$

or

$$f_i(u) = -\frac{c_i}{c_1^2 + c_2^2}\left(\ln\left(\cosh\sqrt{|d|}u\right) + bu\right) + k_i u, \ d < 0 \qquad i = 1, 2$$

and

$$g_i(v) = -\frac{c_i}{c_1^2 + c_2^2}\left(\ln\left(\cos\sqrt{d'}v\right) + b'v\right) + l_2 v, \qquad i = 1, 2$$

where d' positive and b, b', c_i, d, k_i, l_i are real constants.

Proof. Let S be a translation surface in $\mathbb{E}_1^4$ which satisfies the equation (4.13). Then

$$-\frac{f_i''(u)}{-1 + (f_1'(u))^2 + (f_2'(u))^2} = \frac{g_i''(v)}{1 + (g_1'(u))^2 + (g_2'(u))^2} = c_i.$$

We know that the variables u and v are independent. Hence, left and right side of the equation must be constant. Thus, we have

$$\begin{aligned} f_i''(u) &= -c_i\left(-1 + (f_1'(u))^2 + (f_2'(u))^2\right), \\ g_i''(v) &= c_i\left(1 + (g_1'(u))^2 + (g_2'(u))^2\right). \end{aligned} \tag{4.14}$$

Suppose $c_i = 0$, $i = 1, 2$, then we obtain $f_i(u) = a_i u + b_i$ and $g_i(v) = c_i v + d_i$. As a result of this, M is a plane in Minkowski 4−space. Furthermore, we assume that $c_1 \neq 0$, by dividing the equations (4.14) by the same equations for $i = 1$, we get

$$\frac{f_i''(u)}{f_1''(u)} = \frac{g_i''(v)}{g_1''(v)} = \frac{c_i}{c_1}, \quad i = 1, 2.$$

Therefore

$$\begin{aligned} f_i''(u) &= \frac{c_i}{c_1} f_1''(u), \\ g_i''(v) &= \frac{c_i}{c_1} g_1''(v), \end{aligned}$$

and then

$$\begin{aligned} f_i'(u) &= \frac{c_i}{c_1} f_1'(u) + k_i \\ g_i'(v) &= \frac{c_i}{c_1} g_1'(v) + l_i \end{aligned} \tag{4.15}$$

where k_i and l_i are real constants for $i = 1, 2$ with $k_1 = l_1 = 0$. Consider the equations (3.2) for $i = 1$:

$$\begin{aligned} f_1''(u) &= -c_1 \left(-1 + (f_1'(u))^2 + (f_2'(u))^2 \right), \\ g_1''(v) &= c_1 \left(1 + (g_1'(u))^2 + (g_2'(u))^2 \right) \end{aligned}$$

and substitute (4.15) into these equations. We have

$$\begin{aligned} f_1''(u) &= -\frac{c_1^2 + c_2^2}{c_1} (f_1')^2 - 2c_2 k_2 f_1' - c_1 \left(k_2^2 - 1 \right), \\ g_1''(v) &= \frac{c_1^2 + c_2^2}{c_1} (g_1')^2 + 2c_2 l_2 g_1' + c_1 \left(l_2^2 + 1 \right). \end{aligned}$$

Then taking

$$\begin{aligned} f_1'(u) &= p, \quad g_1'(v) = q, \quad a = \frac{c_1^2 + c_2^2}{c_1}, \quad b = c_2 k_2, \\ c &= c_1 \left(k_2^2 - 1 \right), \quad b' = c_2 l_2, \quad c' = c_1 \left(l_2^2 + 1 \right), \end{aligned}$$

we obtain the differential equations

$$\begin{aligned} \frac{dp}{du} &= -(ap^2 + 2bp + c), \\ \frac{dq}{dv} &= (aq^2 + 2b'q + c'), \end{aligned}$$

or we can write

$$\begin{aligned} \frac{dp}{du} &= -\frac{1}{a} \left[(ap + b)^2 + ac - b^2 \right], \\ \frac{dq}{dv} &= \frac{1}{a} \left[(aq + b')^2 + ac' - b'^2 \right]. \end{aligned}$$

Put $d = ac - b^2$ and $d' = ac' - b'^2$, then

$$\begin{aligned} d &= c_1^2 \left(k_2^2 - 1 \right) - c_2^2, \\ d' &= c_2^2 \left(l_2^2 + 1 \right) + c_2^2, \end{aligned}$$

where d is constant and d' is positive constant. Assume both of them are positive, we get

$$\begin{aligned} p &= f_1'(u) = \frac{-b-\sqrt{d}\tan\left(\sqrt{d}u\right)}{a}, \\ q &= g_1'(v) = \frac{-b'+\sqrt{d'}\tan\left(\sqrt{d'}v\right)}{a}. \end{aligned}$$

If d is negative, then

$$p = f_1'(u) = \frac{-b+\sqrt{|d|}\tanh\left(\sqrt{|d|}u\right)}{a}.$$

Using these result and equality (4.15), the other functions are

$$\begin{aligned} f_2'(u) &= -\frac{c_2}{c_1^2+c_2^2}\left(\sqrt{d}\tan\left(\sqrt{d}u\right)+b\right)+k_2u,\ d>0, \\ g_2'(v) &= \frac{c_2}{c_1^2+c_2^2}\left(\sqrt{d'}\tan\left(\sqrt{d'}v\right)-b'\right)+l_2v,\ d'>0, \end{aligned}$$

or

$$f_2'(u) = \frac{c_2}{c_1^2+c_2^2}\left(\sqrt{|d|}\tanh\left(\sqrt{|d|}u\right)-b\right)+k_2u,\ d<0.$$

Consequently, we have all the solutions

$$f_i(u) = \frac{c_i}{c_1^2+c_2^2}\left(\ln\left(\cos\sqrt{d}u\right)-bu\right)+k_iu,\ d>0$$

or

$$f_i(u) = -\frac{c_i}{c_1^2+c_2^2}\left(\ln\left(\cosh\sqrt{|d|}u\right)+bu\right)+k_iu,\ d<0$$

and

$$g_i(v) = -\frac{c_i}{c_1^2+c_2^2}\left(\ln\left(\cos\sqrt{d'}v\right)+b'v\right)+l_2v,\ d'>0$$

for $i=1,2$ $\square$

Example 4.1. The surface given by the parametrization

$$\phi(u,v) = \left(u, v, -3u+2v-\ln\left(\cosh 2u\cos 3v\right), -2u+v-2\ln(\cosh 2u\cos 3v)\right) \tag{4.16}$$

is minimal in Minkowski 4−space.

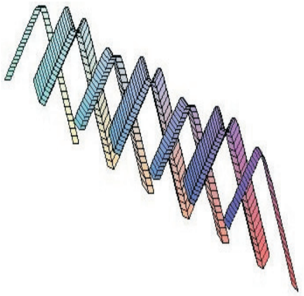

Fig. 4.1: 3D model obtained by the projection of Spacelike Minimal Surface (4.16)

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ON $\mathcal{I}_2$-CONVERGENCE AND $\mathcal{I}_2$-CAUCHY DOUBLE SEQUENCES OF FUNCTIONS IN 2-NORMED SPACES

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Abstract. In this study, firstly, we studied some properties of $\mathcal{I}_2$-convergence. Then, we introduced $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy sequence of double sequences of functions in 2-normed space. Also, we investigated the relationships between them for double sequences of functions in 2-normed spaces.

Keywords: $\mathcal{I}_2$-Convergence, $\mathcal{I}_2$-Cauchy, Double sequences of Functions, 2-normed Spaces.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [15] and Schoenberg [36]. Gökhan et al. [20] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [28] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$ [15, 16]. Gezer and Karakuş [19] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [5] investigated $\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions. Das et al. [7] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Dündar and Altay [8, 10] studied the concepts of pointwise and uniformly $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [13] investigated some results of $\mathcal{I}_2$-convergence of double sequences of functions. Also, a lot of development has been made about double sequences of functions (see [9, 11, 14, 30, 34, 40–42]).

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The concept of 2-normed spaces was initially introduced by Gähler [17, 18] in the 1960's. Statistical convergence and statistical Cauchy sequence of functions in 2-normed space were studied by Yegül and Dündar [43]. Yegül and Dündar [44] introduced concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Also, Yegül and Dündar [45] introduced concepts of $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions in 2-normed space. Recently, Arslan and Dündar [1,2] inroduced $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy sequences of functions in 2-normed spaces. Futhermore, there has been a lot of development in this area (see [3, 4, 6, 26, 27, 29, 31–33, 37–39]).

2. Definitions and Notations

Now, we recall the concept of density, statistical convergence, 2-normed space and some fundamental definitions and notations (See [1, 2, 7, 12, 16, 18–25, 28, 31, 35, 43–45]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$.

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula $\|x, y\| = |x_1y_2 - x_2y_1|$; $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be convergent to f if $f_n(x) \to f(x)(\|.,.\|_Y)$ for each $x \in X$. We write $f_n \to f(\|.,.\|_Y)$. This can be expressed by the formula $(\forall y \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), y\| < \varepsilon$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, **(ii)** For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, **(iii)** For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin \mathcal{F}$, **(ii)** For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, **(iii)** For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

$\mathcal{I}$ is nontrivial ideal in $\mathbb{N}$ if and only if $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N}\backslash A)\}$ is a filter in $\mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

The sequence of functions $\{f_n\}$ is said to be $\mathcal{I}$-convergent (pointwise) to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$ $A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}$ or $\mathcal{I} - \lim_{n\to\infty} \|f_n(x) - f(x), z\|_Y = 0$, for each $x \in X$. This can be expressed by the formula $(\forall z \in Y)$ $(\forall \varepsilon > 0)$ $(\exists M \in \mathcal{I})$ $(\forall n_0 \in \mathbb{N}\backslash M)$ $(\forall x \in X)(\forall n \geq n_0)$ $\|f_n(x) - f(x), z\| \leq \varepsilon$. In this case, we write $f_n \to_{\mathcal{I}} f(\|.,.\|_Y)$.

The sequence of functions $\{f_n\}$ is said to be $\mathcal{I}^*$-convergent (pointwise sense) to f, if there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., $\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$ $\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|$ and we write $\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ or $f_n \to_{\mathcal{I}^*} f(\|.,.\|_Y)$.

The sequence of functions $\{f_n\}$ is said to be $\mathcal{I}$-Cauchy sequence, if for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, x) \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\| \geq \varepsilon\} \in \mathcal{I}$, for each nonzero $z \in Y$.

The sequence of functions $\{f_n\}$ is said to be $\mathcal{I}^*$-Cauchy sequence, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, such that the subsequence $\{f_M\} = \{f_{m_k}\}$ is a Cauchy sequence, i.e., $\lim_{k,p\to\infty} \|f_{m_k}(x) - f_{m_p}(x), z\| = 0$, for each $x \in X$ and each nonzero $z \in Y$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Throughout the paper, we let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$, $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{h_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be three double sequences of functions, f, g and k be three functions from X to Y.

A double sequence $\{f_{mn}\}$ is said to be convergent (pointwise) to f if, for each point $x \in X$ and for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x, \varepsilon)$ such that for all $m, n \geq k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \longrightarrow f(\|.,.\|_Y)$.

The double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-convergent (pointwise sense) to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in X$. This can be expressed by the formula

$$(\forall z \in Y)\ (\forall x \in X)\ (\forall \varepsilon > 0)\ (\exists H \in \mathcal{I}_2)\ (\forall (m,n) \notin H)\ \|f_{mn}(x) - f(x), z\| < \varepsilon.$$

In this case, we write $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, *or* $f_{mn} \to_{\mathcal{I}_2} f(\|.,.\|_Y)$.

The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathcal{I}_2^*$-convergent (pointwise) to f, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ $(H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2)$ such that for each $x \in X$, each nonzero $z \in Y$ and all $(m,n) \in M$ $\lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ and we write $\mathcal{I}_2^* - \lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ or $f_{mn} \to_{\mathcal{I}_2^*} f(\|.,.\|_Y)$.

Lemma 2.1. [45] *For each $x \in X$ and nonzero $z \in Y$,*

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 2.2. [45] *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property $(AP2)$. For each $x \in X$ and nonzero $z \in Y$,*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Lemma 2.3. [11] *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $\{P_i\}_{i=1}^{\infty} \in F(\mathcal{I}_2)$ for each i, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ with the property $(AP2)$. Then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P \backslash P_i$ is finite for all i.*

Lemma 2.4. [45] *For each $x \in X$ and each nonzero $z \in Y$, If*
$\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ *and* $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|$, *then*

(i) $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|$,

(ii) $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|c.f_{mn}(x), z\| = \|c.f(x), z\|$, $c \in \mathbb{R}$,

(iii) $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|f_{mn}(x).g_{mn}(x), z\| = \|f(x).g(x), z\|$.

3. Main Results

In this study, firstly, we studied some properties of $\mathcal{I}_2$-convergence. Then, we introduced $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy sequence of double sequences of functions in 2-normed space. Also, were investigated relationships between them for double sequences of functions in 2-normed spaces.

Theorem 3.1. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property $(AP2)$. Then, for each $x \in X$ and each nonzero $z \in Y$, following conditions are equivalent*

(i) $\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$

(ii) *There exists $\{g_{mn}(x)\}$ and $\{h_{mn}(x)\}$ be two sequences of functions from X to Y such that*

$$f_{mn}(x) = g_{mn}(x)+h_{mn}(x),\ \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\| \text{ and } supp\{h_{mn}(x)\} \in \mathcal{I}_2,$$

where supp $h_{mn}(x) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\}$.

Proof. **(i)** $\Rightarrow$ **(ii)**: $\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. Then, by Lemma 2.2 there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ $(i.e., H = \mathbb{N}\times\mathbb{N}\setminus M \in \mathcal{I}_2)$ such that for each $x \in X$, each nonzero $z \in Y$ and all $(m,n) \in M$

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Let us define the sequence $\{g_{mn}(x)\}$ by

$$g_{mn}(x) = \begin{cases} f_{mn}(x), & (m,n) \in M, \\ f(x), & (m,n) \in \mathbb{N}\times\mathbb{N}\setminus M. \end{cases} \tag{3.1}$$

It is clear that $\{g_{mn}(x)\}$ is a double sequence of functions and $\lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Besides let

$$h_{mn}(x) = f_{mn}(x) - g_{mn}(x),\ (m,n) \in \mathbb{N}\times\mathbb{N} \tag{3.2}$$

for each $x \in X$. Since

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) \neq g_{mn}(x)\} \subset \mathbb{N}\times\mathbb{N}\setminus M \in \mathcal{I}_2,$$

for each $x \in X$, so we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2.$$

It follows that supp $h_{mn}(x) \in \mathcal{I}_2$ and by (3.1) and (3.2) we get $f_{mn}(x) = g_{mn}(x) + h_{mn}(x)$, for each $x \in X$.

(ii)$\Rightarrow$**(i)**: Assume that there exist two sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ such that

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x),\ \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\| \\ \text{and } supph_{mn}(x) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2 \tag{3.3}$$

for each $x \in X$ and each nonzero $z \in Y$. We show that $\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Let

$$M = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) = 0\} = \mathbb{N}\times\mathbb{N}\setminus supph_{mn}(x). \tag{3.4}$$

Since *supp* $h_{mn}(x) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2$, then from (3.3) and (3.4), we have $M \in \mathcal{F}(\mathcal{I}_2)$ and $f_{mn}(x) = g_{mn}(x)$ for $(m,n) \in M$. Hence, we conclude that exists a set $M \in \mathcal{F}(\mathcal{I}_2)$, (*i.e.*, $H = \mathbb{N}\times\mathbb{N} \setminus M \in \mathcal{I}_2$) such that $\lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ and so

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for $(m,n) \in M$, each $x \in X$ and each nonzero $z \in Y$. By Lemma 2.2 it follows that

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. This completes the proof. $\square$

Corollary 3.1. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property $(AP2)$. Then, $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ if and only if there exist $\{g_{mn}\}$ and $\{h_{mn}\}$ be two sequences of functions from X to Y such that*

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x),\ \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} \|h_{mn}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

Proof. Let $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ and $\{g_{mn}(x)\}$ is sequence defined by (3.1). Consider the sequence

$$h_{mn}(x) = f_{mn}(x) - g_{mn}(x), \quad (m,n) \in \mathbb{N}\times\mathbb{N} \tag{3.5}$$

for each $x \in X$. Then, we have

$$\lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$$

and since $\mathcal{I}_2$ is a strongly admissible ideal so

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. By Lemma 2.4 and by (3.5) we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|h_{mn}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$. Now, let

$$f_{mn}(x) = g_{mn}(x) + h_{mn}(x),$$

where

$$\lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} \|h_{mn}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$. Since $\mathcal{I}_2$ is a strongly admissible ideal so

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|f(x), z\|$$

and by Lemma 2.4 we get

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. $\square$

Remark 3.1. In Theorem 3.1 if **(ii)** is satisfied then the admissible ideal $\mathcal{I}_2$ need not have the property $(AP2)$. Since for each $x \in X$ and each nonzero $z \in Y$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|h_{mn}(x), z\| \geq \varepsilon\} \subset \{(m,n) \in \mathbb{N}\times\mathbb{N} : h_{mn}(x) \neq 0\} \in \mathcal{I}_2,$$

for each $\varepsilon > 0$, then

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|h_{mn}(x), z\| = 0.$$

Hence, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$.

Definition 3.1. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-Cauchy sequence, if for every $\forall\varepsilon > 0$ and each $x \in X$ there exist $s = s(\varepsilon, x)$, $t = t(\varepsilon, x) \in \mathbb{N}$ such that

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each nonzero $z \in Y$.

Theorem 3.2. *If $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent if and only if it is $\mathcal{I}_2$-Cauchy sequence in 2-normed spaces.*

Proof. Assume that $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent to f. Then, for $\varepsilon > 0$

$$A\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2,$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$A^c\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

for each $x \in X$ and each nonzero $z \in Y$ and thus $A^c\left(\frac{\varepsilon}{2}, z\right)$ is non-empty. So we can select a positive integers k, l such that $(k,l) \notin A\left(\frac{\varepsilon}{2}, z\right)$ and $\|f_{kl}(x) - f(x), z\| < \frac{\varepsilon}{2}$. Now, we define the set

$$B(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{kl}(x), z\| \geq \varepsilon\},$$

for each $x \in X$ and each nonzero $z \in Y$, such that we show that $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$. Let $(m, n) \in B(\varepsilon, z)$, then we have

$$\begin{aligned}\varepsilon \leq \|f_{mn}(x) - f_{kl}(x), z\| &\leq \|f_{mn}(x) - f(x), z\| + \|f_{kl}(x) - f(x), z\| \\ &< \|f_{mn}(x) - f(x), z\| + \frac{\varepsilon}{2},\end{aligned}$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that $\frac{\varepsilon}{2} < \|f_{mn}(x) - f(x), z\|$ and so, $(m, n) \in A(\frac{\varepsilon}{2}, z)$. Hence, we have $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$ and so $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence.

Conversely, assume that $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence. We prove that $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent. Let (ε_{pq}) be a strictly decreasing sequence of number converging to zero since $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence, there exist two strictly increasing sequences (k_p) and (l_q) of positive integers such that

$$A(\varepsilon_{pq}, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_p l_q}(x), z\| \geq \varepsilon_{pq}\} \in \mathcal{I}_2, (p, q = 1, 2, ...),$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$(3.6) \qquad \emptyset \neq \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_p l_q}(x), z\| < \varepsilon_{pq}\} \in \mathcal{F}(\mathcal{I}_2),$$

$(p, q = 1, 2, ...)$, for each $x \in X$ and each nonzero $z \in Y$. Let p, q, s and t be four positive integers such that $p \neq q$ and $s \neq t$. By (3.6), both the sets

$$C(\varepsilon_{pq}, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_p l_q}(x), z\| < \varepsilon_{pq}\}$$

and

$$D(\varepsilon_{st}, z) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{k_s l_t}(x), z\| < \varepsilon_{st}\}$$

are non empty sets in $\mathcal{F}(\mathcal{I}_2)$, for each $x \in X$ and each nonzero $z \in Y$. Since $\mathcal{F}(\mathcal{I}_2)$ is a filter on $\mathbb{N} \times \mathbb{N}$, so

$$\emptyset \neq C(\varepsilon_{pq}, z) \cap D(\varepsilon_{st}, z) \in \mathcal{F}(\mathcal{I}_2).$$

Therefore, for each pair (p, q) and (s, t) of positive integers with $p \neq q$ and $s \neq t$, we can select a pair $(m_{(p,q),(s,t)}, n_{(p,q),(s,t)}) \in \mathbb{N} \times \mathbb{N}$ such that

$$\|f_{m_{pqst} n_{pqst}}(x) - f_{k_p l_q}(x), z\| < \varepsilon_{pq} \text{ and } \|f_{m_{pqst} n_{pqst}}(x) - f_{k_s l_t}(x), z\| < \varepsilon_{st},$$

for each $x \in X$ and each nonzero $z \in Y$. It follows that

$$\begin{aligned}\|f_{k_p l_q}(x) - f_{k_s l_t}(x), z\| &\leq \|f_{m_{pqst} n_{pqst}}(x) - f_{k_p l_q}(x), z\| \\ &\quad + \|f_{m_{pqst} n_{pqst}}(x) - f_{k_s l_t}(x), z\| \\ &\leq \varepsilon_{pq} + \varepsilon_{st} \to 0,\end{aligned}$$

as $p, q, s, t \to \infty$. This implies that $\{f_{k_p l_q}\}$ $(p, q = 1, 2, ...)$ is a Cauchy sequence and therefore it satisfies the Cauchy convergence criterion. Thus, the sequence $\{f_{k_p l_q}\}$ converges to a limit f (say) i,e.,

$$\lim_{p,q \to \infty} \|f_{k_p l_q}, z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. Also, we have $\varepsilon_{pq} \to 0$ as $p, q \to \infty$, so for each $\varepsilon > 0$ we can choose positive integers p_0, q_0 such that

$$\varepsilon_{p_0 q_0} < \frac{\varepsilon}{2} \text{ and } \|f_{k_p l_q} - f(x), z\| < \frac{\varepsilon}{2}, \ (for\ p > p_0 \text{ and } q > q_0). \tag{3.7}$$

Now, we define the set

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\},$$

for each $x \in X$ and each nonzero $z \in Y$. We prove that $A(\varepsilon, z) \subset A(\varepsilon_{p_0 q_0}, z)$. Let $(m,n) \in A(\varepsilon, z)$, then by second half of (3.7) we have

$$\begin{aligned}
\varepsilon \leq \|f_{mn}(x) - f(x), z\| &\leq \|f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x), z\| + \|f_{k_{p_0} l_{q_0}}(x) - f(x), z\| \\
&\leq \|f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x), z\| + \frac{\varepsilon}{2},
\end{aligned}$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\frac{\varepsilon}{2} < \|f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x), z\|$$

and therefore by first half of (3.7)

$$\varepsilon_{p_0 q_0} < \|f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. Thus, we have $(m,n) \in A(\varepsilon_{p_0 q_0}, z)$ and therefore $A(\varepsilon, z) \subset A(\varepsilon_{p_0 q_0}, z)$. Since $A(\varepsilon_{p_0 q_0}, z) \in \mathcal{I}_2$ so $A(\varepsilon, z) \in \mathcal{I}_2$ by property of ideal. Hence $\{f_{k_p l_q}\}$ is $\mathcal{I}_2$-convergent. □

Definition 3.2. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2^*$- Cauchy sequence, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (*i.e.*, $H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$) and for every $\varepsilon > 0$ and each $x \in X$, $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for all $(m,n), (s,t) \in M$ and each $z \in Y$

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

whenever $m, n, s, t > k_0$. In this case, we write

$$\lim_{m,n,s,t \to \infty} \|f_{mn}(x) - f_{st}(x), z\| = 0.$$

Theorem 3.3. *If double sequence of functions $\{f_{mn}\}$ is a $\mathcal{I}_2^*$-Cauchy sequence, then it is $\mathcal{I}_2$-Cauchy sequence in 2-normed spaces.*

Proof. Let $\{f_{mn}\}$ is a $\mathcal{I}_2^*$-Cauchy sequence in 2-normed spaces. Then, by definition there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (*i.e.*, $H = \mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$) and for every $\varepsilon > 0$ and each $x \in X$, $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for all $(m,n), (s,t) \in M$ and each $z \in Y$

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

whenever $m, n, s, t > k_0$. Then, for each $x \in X$ and nonzero each $z \in Y$ we have

$$\begin{array}{rcl} A(\varepsilon, z) & = & \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \\ & \subset & H \cup [M \cap ((\{1,2,3,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,...,(k_0-1)\}))] \end{array}$$

Since $\mathcal{I}_2$ is an admissible ideal, then

$$H \cup [M \cap ((\{1,2,3,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,...,(k_0-1)\}))] \in \mathcal{I}_2.$$

Therefore, we have $A(\varepsilon, z) \in \mathcal{I}_2$ i.e., $\{f_{mn}\}$ is a $\mathcal{I}_2$-Cauchy sequence. □

Theorem 3.4. *If $\mathcal{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x) - f(x), z\| = 0$, then $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence in 2-normed spaces.*

Proof. By assumption there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (*i.e.*, $H = \mathbb{N} \times \mathbb{N}\ M \in \mathcal{I}_2$) such that $\lim_{m,n\to\infty} \|f_{mn}(x) - f(x), z\| = 0$ for each $x \in X$ and each $z \in Y$. It shows that for each $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for each $x \in X$, each $z \in Y$

$$\|f_{mn}(x) - f(x), z\| < \frac{\varepsilon}{2}$$

for all $(m, n) \in M$ and $m, n > k_0$. Since for each $\varepsilon > 0$,

$$\begin{array}{rcl} \|f_{mn}(x) - f_{st}(x), z\| & \leq & \|f_{mn}(x) - f(x), z\| + \|f_{st}(x) - f(x), z\| \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{array}$$

for each $x \in X$, each $z \in Y$ and $m, n, s, t \geq k_0$ we have

$$\lim_{m,n,s,t\to\infty} \|f_{mn}(x) - f_{st}(x), z\| = 0,$$

i.e., $\{f_{mn}\}$ is a $\mathcal{I}_2^*$-Cauchy sequence. Then, by Theorem 3.3 $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence. □

Theorem 3.5. *Let $\mathcal{I}_2$ be an admissible ideal with property $(AP2)$ and a double sequence of functions $\{f_{mn}\}$. Then, the concepts $\mathcal{I}_2$-Cauchy double sequence and $\mathcal{I}_2^*$-Cauchy double sequence of functions coincide in 2-normed spaces.*

Proof. By Theorem 3.3 $\mathcal{I}_2^*$-Cauchy sequence implies $\mathcal{I}_2$-Cauchy sequence (in this case $\mathcal{I}_2$ need not to have $(AP2)$ condition).

Now, it is sufficient to prove that a double sequence $\{f_{mn}\}$ is a $\mathcal{I}_2^*$-Cauchy double sequence under assumption that $\{f_{mn}\}$ is a $\mathcal{I}_2$-Cauchy double sequence. Let $\{f_{mn}\}$ is a $\mathcal{I}_2$-Cauchy double sequence. Then, for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, z), t = t(\varepsilon, z) \in \mathbb{N}$ such that

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{st}(x), z\| \geq \varepsilon\} \in \mathcal{I}_2$$

for each nonzero $z \in Y$. Let

$$P_i = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|f_{mn}(x) - f_{s_i t_i}(x), z\| < \frac{1}{i}\right\},\ (i = 1, 2, ...),$$

where $s = s(\frac{1}{i}),\ t = t(\frac{1}{i})$. It is clear that

$$P_i \in \mathcal{F}(\mathcal{I}_2),\ (i = 1, 2, ...).$$

Since $\mathcal{I}_2$ has $(AP2)$ property then by Lemma 2.3 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in F(\mathcal{I}_2)$ and $P \backslash P_i$ is finite for all i. Now we show that

$$\lim_{m,n,s,t \to \infty} \|f_{mn}(x) - f_{st}(x), z\| = 0$$

for each $x \in X$, $(m,n), (s,t) \in P$ and each nonzero $z \in Y$. Let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$, if $(m,n), (s,t) \in P$ then $P \backslash P_i$ is a finite set, so there exists $k = k(j)$ such that $(m,n), (s,t) \in P_j$ for all $m, n, s, t > k(j)$. Therefore, for each $x \in X$

$$\|f_{mn}(x) - f_{s_j t_j}(x), z\| < \frac{1}{j} \text{ and } \|f_{st}(x) - f_{s_j t_j}(x), z\| < \frac{1}{j},$$

for each nonzero $z \in Y$ and all $m, n, s, t > k(j)$. Hence, for each $x \in X$ it follows that

$$\begin{aligned} \|f_{mn}(x) - f_{st}(x), z\| &\leq \|f_{mn}(x) - f_{s_j t_j}(x), z\| + \|f_{st}(x) - f_{s_j t_j}(x), z\| \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon \end{aligned}$$

for all $m, n, s, t > k(j)$ and each nonzero $z \in Y$. Therefore, for any $\varepsilon > 0$ and each $x \in X$ there exists $k = k(\varepsilon, x)$ such that for $m, n, s, t > k$ and $(m,n), (s,t) \in P \in \mathcal{F}(\mathcal{I}_2)$

$$\|f_{mn}(x) - f_{st}(x), z\| < \varepsilon,$$

for each nonzero $z \in Y$ and so, the sequence $\{f_{mn}\}$ is a $\mathcal{I}_2^*$-Cauchy sequence in 2-normed space. $\square$

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REFINEMENTS AND REVERSES OF HöLDER-MCCARTHY OPERATOR INEQUALITY VIA A CARTWRIGHT-FIELD RESULT

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Abstract. By the use of a classical result of Cartwright and Field, in this paper we have obtained new refinements and reverses of Hölder-McCarthy operator inequality in the case of $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, has also been performed.
Keywords: Hölder-McCarthy operator inequality; selfadjoint operator; Hilbert space; nonnegative operator.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, namely $\langle Ax, x \rangle \geq 0$ for any $x \in H$. We write this as $A \geq 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [16] proved that

$$\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle, \; p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x \rangle \leq \langle Ax, x \rangle^p, \; p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator on H with

$$mI \leq A \leq MI, \tag{1.3}$$

where I is the *identity operator* and m, M are real numbers with $m < M$.

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In [7, Theorem 3] Fujii et al. obtained the following interesting ratio inequality that provides a reverse of the *Hölder-McCarthy inequality* (1.1) for an operator A that satisfies the condition (1.3) with $m > 0$

$$\langle A^p x, x\rangle \le \left\{ \frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}} \right\}^p \langle Ax, x\rangle^p , \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where $q = p/(p-1)$, $p > 1$.

If A satisfies the condition (1.3) with $m \ge 0$, then we also have the additive reverse of (1.1) that has been obtained by the author in 2008, see [4]

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le \tfrac{1}{2} p (M-m) \left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2 \right]^{1/2} \\ &\le \tfrac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right) \end{aligned}$$

and

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le \tfrac{1}{2} p \left(M^{p-1} - m^{p-1}\right) \left[\left\|Ax\right\|^2 - \left\langle Ax, x\right\rangle^2 \right]^{1/2} \\ &\le \tfrac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1}\right) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

We also have the alternative upper bounds [4]

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le \tfrac{1}{4} p \tfrac{(M-m)\left(M^{p-1} - m^{p-1}\right)}{M^{p/2} m^{p/2}} \langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle, \text{ (for } m > 0), \\ &\le p \tfrac{1}{4} (M-m) \left(M^{p-1} - m^{p-1}\right) \left(\tfrac{M}{m}\right)^{p/2}, \text{ (for } m > 0) \end{aligned}$$

and

$$\begin{aligned} \langle A^p x, x\rangle - \langle Ax, x\rangle^p &\le p \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) \left[\langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle\right]^{\frac{1}{2}} \\ &\le p \left(\sqrt{M} - \sqrt{m}\right) \left(M^{(p-1)/2} - m^{(p-1)/2}\right) M^{p/2} \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

For various related inequalities, see [6]-[10] and [14]-[15].

We have the following inequality that provides a refinement and a reverse for the celebrated *Young's scalar inequality*

$$\begin{aligned} \frac{1}{2} \nu (1-\nu) \frac{(b-a)^2}{\max\{a, b\}} &\le (1-\nu) a + \nu b - a^{1-\nu} b^\nu \\ &\le \frac{1}{2} \nu (1-\nu) \frac{(b-a)^2}{\min\{a, b\}} \end{aligned} \tag{1.5}$$

for any a, $b > 0$ and $\nu \in [0, 1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

For new recent reverses and refinements of Young's inequality see [2]-[3], [11]-[12], [13] and [19].

By the use of (1.5). we have obtained new refinements and reverses of Hölder-McCarthy operator inequality in the case when $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, has also been performed.

2. Some Refinements and Reverse Results

We have:

Theorem 2.1. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$\begin{aligned}(2.1)\qquad \frac{p(1-p)}{2}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \tfrac{p(1-p)}{2M}\langle Ax,x\rangle\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le 1-\tfrac{\langle A^px,x\rangle}{\langle Ax,x\rangle^p}\\ &\le \tfrac{p(1-p)}{2m}\langle Ax,x\rangle\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le \tfrac{p(1-p)}{2}\tfrac{M}{m}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

In particular,

$$\begin{aligned}(2.2)\qquad \frac{1}{8}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \tfrac{\langle Ax,x\rangle}{8M}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ \le 1-\tfrac{\langle A^{1/2}x,x\rangle}{\langle Ax,x\rangle^{1/2}} &\le \tfrac{\langle Ax,x\rangle}{8m}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le \tfrac{1}{8}\tfrac{M}{m}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right),\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. If $a,\ b \in [m, M]$, then by Cartwright-Field inequality (1.5) we have

$$\frac{1}{2M}p(1-p)(b-a)^2 \le (1-p)a + pb - a^{1-p}b^p \le \frac{1}{2m}p(1-p)(b-a)^2$$

or, equivalently

$$\begin{aligned}(2.3)\qquad \frac{1}{2M}p\left(1-p\right)\left(b^{2}-2ab+a^{2}\right) &\leq \left(1-p\right)a+pb-a^{1-p}b^{p}\\ &\leq \frac{1}{2m}p\left(1-p\right)\left(b^{2}-2ab+a^{2}\right),\end{aligned}$$

for any $p\in(0,1)$.

Fix $a\in[m,M]$ and by using the operator functional calculus for A with $mI\leq A\leq MI$ we have

$$\begin{aligned}(2.4)\qquad \frac{1}{2M}p\left(1-p\right)\left(A^{2}-2aA+a^{2}I\right) &\leq \left(1-p\right)aI+pA-a^{1-p}A^{p}\\ &\leq \frac{1}{2m}p\left(1-p\right)\left(A^{2}-2aA+a^{2}I\right).\end{aligned}$$

Then for any $x\in H$ with $\|x\|=1$ we have from (2.4) that

$$\begin{aligned}(2.5)\qquad &\tfrac{1}{2M}p\left(1-p\right)\left(\left\langle A^{2}x,x\right\rangle-2a\left\langle Ax,x\right\rangle+a^{2}\right)\\ &\leq\left(1-p\right)a+p\left\langle Ax,x\right\rangle-a^{1-p}\left\langle A^{p}x,x\right\rangle\\ &\leq\tfrac{1}{2m}p\left(1-p\right)\left(\left\langle A^{2}x,x\right\rangle-2a\left\langle Ax,x\right\rangle+a^{2}\right),\end{aligned}$$

for any $a\in[m,M]$.

If we choose in (2.5) $a=\langle Ax,x\rangle\in[m,M]$, then we get for any $x\in H$ with $\|x\|=1$ that

$$\begin{aligned}\frac{1}{2M}p\left(1-p\right)\left(\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\right) &\leq\left\langle Ax,x\right\rangle-\left\langle Ax,x\right\rangle^{1-p}\left\langle A^{p}x,x\right\rangle\\ &\leq\tfrac{1}{2m}p\left(1-p\right)\left(\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\right),\end{aligned}$$

and by division with $\langle Ax,x\rangle>0$ we obtain the second and third inequalities in (2.1).

The rest is obvious. ∎

Remark 2.1. It is well known that, if $mI\leq A\leq MI$ with $M>0$, then, see for instance [17, p. 27], we have

$$(1\leq)\,\frac{\left\langle A^{2}x,x\right\rangle}{\left\langle Ax,x\right\rangle^{2}}\leq\frac{(m+M)^{2}}{4mM}$$

for any $x\in H$ with $\|x\|=1$, which implies that

$$(0\leq)\,\frac{\left\langle A^{2}x,x\right\rangle}{\left\langle Ax,x\right\rangle^{2}}-1\leq\frac{(M-m)^{2}}{4mM}.$$

Using (2.1) and by denoting $h=\frac{M}{m}$ we get

$$(2.6)\qquad (0\leq)\,1-\frac{\left\langle A^{p}x,x\right\rangle}{\left\langle Ax,x\right\rangle^{p}}\leq\frac{p\left(1-p\right)}{8}\left(h-1\right)^{2}$$

and, in particular,

$$(0\leq)\,1-\frac{\left\langle A^{1/2}x,x\right\rangle}{\langle Ax,x\rangle^{1/2}}\leq\frac{1}{32}\left(h-1\right)^{2}, \tag{2.7}$$

for any $x\in H$ with $\|x\|=1$.

We consider the *Kantorovich's constant* defined by

$$K\left(h\right):=\frac{\left(h+1\right)^{2}}{4h},\ \ h>0. \tag{2.8}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K\left(h\right)\geq 1$ for any $h>0$ and $K\left(h\right)=K\left(\frac{1}{h}\right)$ for any $h>0$.

Observe that for any $h>0$

$$K\left(h\right)-1=\frac{\left(h-1\right)^{2}}{4h}=K\left(\frac{1}{h}\right)-1.$$

From (2.6) we then have

$$(0\leq)\,1-\frac{\langle A^{p}x,x\rangle}{\langle Ax,x\rangle^{p}}\leq\frac{p\left(1-p\right)}{2}h\left[K\left(h\right)-1\right] \tag{2.9}$$

and, in particular,

$$(0\leq)\,1-\frac{\left\langle A^{1/2}x,x\right\rangle}{\langle Ax,x\rangle^{1/2}}\leq\frac{1}{8}h\left[K\left(h\right)-1\right], \tag{2.10}$$

for any $x\in H$ with $\|x\|=1$.

Also, if $a,\,b>0$ then

$$K\left(\frac{b}{a}\right)-1=\frac{\left(b-a\right)^{2}}{4ab}.$$

Since $\min\left\{a,b\right\}\max\left\{a,b\right\}=ab$ if $a,b>0$, then

$$\frac{\left(b-a\right)^{2}}{\max\left\{a,b\right\}}=\frac{\min\left\{a,b\right\}\left(b-a\right)^{2}}{ab}=4\min\left\{a,b\right\}\left[K\left(\frac{b}{a}\right)-1\right]$$

and

$$\frac{\left(b-a\right)^{2}}{\min\left\{a,b\right\}}=\frac{\max\left\{a,b\right\}\left(b-a\right)^{2}}{ab}=4\max\left\{a,b\right\}\left[K\left(\frac{b}{a}\right)-1\right]$$

and the inequality (1.5) can be written as

$$\begin{aligned}2\nu\left(1-\nu\right)\min\left\{a,b\right\}\left[K\left(\frac{b}{a}\right)-1\right]&\leq\left(1-\nu\right)a+\nu b-a^{1-\nu}b^{\nu}\\&\leq 2\nu\left(1-\nu\right)\max\left\{a,b\right\}\left[K\left(\tfrac{b}{a}\right)-1\right]\end{aligned}$$

for any $a,\,b>0$ and $\nu\in[0,1]$.

Theorem 2.2. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$
\begin{aligned}
(2.11) \qquad & (0 \leq)\, 1 - \tfrac{\langle A^p x,x\rangle}{\langle Ax,x\rangle^p} \\
& \leq p\,(1-p)\,[K\,(h)-1] \left(2 + \tfrac{\langle |A - \langle Ax,x\rangle I| x,x\rangle}{\langle Ax,x\rangle}\right) \\
& \leq p\,(1-p)\,[K\,(h)-1] \left[2 + \left(\tfrac{\langle A^2 x,x\rangle}{\langle Ax,x\rangle^2} - 1\right)^{1/2}\right] \\
& \leq p\,(1-p)\,[K\,(h)-1] \left[2 + (K\,(h)-1)^{1/2}\right]
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

In particular, we have

$$
\begin{aligned}
(2.12) \qquad & (0 \leq)\, 1 - \tfrac{\langle A^{1/2} x,x\rangle}{\langle Ax,x\rangle^{1/2}} \\
& \leq \tfrac{1}{4}\,[K\,(h)-1] \left(2 + \tfrac{\langle |A - \langle Ax,x\rangle I| x,x\rangle}{\langle Ax,x\rangle}\right) \\
& \leq \tfrac{1}{4}\,[K\,(h)-1] \left[2 + \left(\tfrac{\langle A^2 x,x\rangle}{\langle Ax,x\rangle^2} - 1\right)^{1/2}\right] \\
& \leq \tfrac{1}{4}\,[K\,(h)-1] \left[2 + (K\,(h)-1)^{1/2}\right]
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

Proof. From (2.11) we have for any a, $b > 0$ and $p \in [0,1]$ that

$$
(2.13) \qquad (1-p)\,a + pb - a^{1-p} b^p \leq p\,(1-p)\,(a + b + |b-a|) \left[K\left(\frac{b}{a}\right) - 1\right]
$$

since

$$
\max\{a,b\} = \frac{1}{2}\,(a + b + |b-a|)\,.
$$

If a, $b \in [m,M]$, then $\frac{b}{a} \in \left[\frac{m}{M}, \frac{M}{m}\right]$ and by the properties of Kantorovich's constant K, we have

$$
1 \leq K\left(\frac{b}{a}\right) \leq K\left(\frac{M}{m}\right) = K\,(h) \text{ for any } a, b \in [m,M]\,.
$$

Therefore, by (2.13) we have

$$
(1-p)\,a + pb - a^{1-p} b^p \leq p\,(1-p)\,(a + b + |b-a|)\,[K\,(h) - 1]
$$

for any a, $b \in [m,M]$ and $p \in [0,1]$.

Fix $a \in [m,M]$ and by using the operator functional calculus for A with $mI \leq A \leq MI$, we have

$$
(2.14) \qquad (1-p)\,aI + pA - a^{1-p} A^p \leq p\,(1-p)\,[K\,(h) - 1]\,(aI + A + |A - aI|)\,.
$$

Then for any $x \in H$ with $\|x\| = 1$ we get from (2.14) that

$$\begin{aligned} &(1-p)\,a + p\,\langle Ax, x\rangle - a^{1-p}\,\langle A^p x, x\rangle \\ &\leq p\,(1-p)\,[K\,(h) - 1]\,(a + \langle Ax, x\rangle + \langle |A - aI|\,x, x\rangle)\,, \end{aligned} \tag{2.15}$$

for any $a \in [m, M]$ and $p \in [0, 1]$.

Now, if we take $a = \langle Ax, x\rangle \in [m, M]$, where $x \in H$ with $\|x\| = 1$ in (2.15), then we obtain

$$\begin{aligned} &\langle Ax, x\rangle - \langle Ax, x\rangle^{1-p}\,\langle A^p x, x\rangle \\ &\leq p\,(1-p)\,[K\,(h) - 1]\,(2\,\langle Ax, x\rangle + \langle |A - \langle Ax, x\rangle\, I|\,x, x\rangle)\,, \end{aligned}$$

which, by division with $\langle Ax, x\rangle > 0$ provides the first inequality in (2.11).

By Schwarz inequality, we have for $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned} \langle |A - \langle Ax, x\rangle\, I|\,x, x\rangle &\leq \left\langle (A - \langle Ax, x\rangle\, I)^2\, x, x\right\rangle^{1/2} \\ &= \left\langle \left(A^2 - 2\,\langle Ax, x\rangle\, A + \langle Ax, x\rangle^2\, I\right) x, x\right\rangle^{1/2} \\ &= \left(\langle A^2 x, x\rangle - \langle Ax, x\rangle^2\right)^{1/2}, \end{aligned}$$

which proves the second part of (2.11).

Since

$$\frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle^2} - 1 \leq \frac{(M-m)^2}{4mM} = K\,(h) - 1$$

for $x \in H$ with $\|x\| = 1$, then the last part of (2.11) is thus proved. ∎

3. A Comparison for Upper Bounds

We observe that the inequality (2.9) provides for the quantity

$$(0 \leq)\, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p},\ \ x \in H \text{ with } \|x\| = 1,$$

the following upper bound

$$B_1\,(p, h) := \frac{p\,(1-p)}{2} h\,[K\,(h) - 1]\,, \tag{3.1}$$

while the inequality (2.11) gives the upper bound

$$B_2\,(p, h) := p\,(1-p)\,[K\,(h) - 1]\left[2 + (K\,(h) - 1)^{1/2}\right], \tag{3.2}$$

where $p \in (0, 1)$ and $h > 1$.

Now, if we depict the 3D plot for the difference of the bounds B_1 and B_2, namely

$$D(x,y) := B_1(y,x) - B_2(y,x)$$

on the box $[1,8] \times [0,1]$, then we observe that it takes both positive and negative values, showing that the bounds $B_1(p,h)$ and $B_2(p,h)$ can not be compared in general, namely neither of them is better for any $p \in (0,1)$ and $h > 1$.

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NORMALIZATION OF HEALTH RECORDS IN THE SERBIAN LANGUAGE WITH THE AIM OF SMART HEALTH SERVICES REALIZATION *

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Abstract. The development of information technology increases its use in various spheres of human activity, including healthcare. Bundles of data and reports are generated and stored in textual form, such as symptoms, medical history, and doctor's observations of patients' health. Electronic recording of patient data not only facilitates day-to-day work in hospitals, enables more efficient data management and reduces material costs, but can also be used for further processing and to gain knowledge to improve public health. Publicly available health data would contribute to the development of telemedicine, e-health, epidemic control, and smart healthcare within smart cities. This paper describes the importance of textual data normalization for smart healthcare services. An algorithm for normalizing medical data in Serbian is proposed in order to prepare them for further processing (F1-score=0,816), in this case within the smart health framework. By applying this algorithm, in addition to the normalized medical records, corpora of keywords and stop words, which are specific to the medical domain, are also obtained and can be used to improve the results in the normalization of medical textual data.

Keywords: telemedicine; e-health; epidemic control; smart healthcare; medical data mining.

1. Introduction

In medical information systems, a large amount of data is created and stored every day. They allow storing of common information (number of patients examined by a doctor, consumption of materials, prescriptions, etc.), but they also include medical reports that contain patient information such as anamnesis, diagnosis, symptoms, etc. These data collected daily should be used for analyzes and predictions to improve medical information systems. Therefore, it is necessary to

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prepare and appropriately process this data. Well prepared data can be processed for different purposes, for example in smart health services [1]. Smart health services are important for improving the quality of services provided, the efficiency of health services, etc. As such, they are considered the basis for the implementation of smart medical IS as an indispensable part of the concept of smart cities [2], which is becoming more and more relevant today. The fact is that the problem of organizing life and optimization, especially in big cities, is one of the current problems that is being intensively solved. The goal is to provide a range of services that will make life easier and cheaper. Healthcare occupies a significant place in this concept.

A smart city is a place where information and telecommunications technologies are used to enhance traditional services. It is a city that connects physical infrastructure, information technology infrastructure, social infrastructure, and business infrastructure to enhance the collective intelligence of the city. Within the smart city, there are branches such as smart transportation, smart healthcare, energy efficiency, smart technology and infrastructure, smart education, smart management, and smart people. Smart healthcare is e-Health aimed at promoting public health services in smart cities [1]. Smart healthcare uses technological innovations in the health care system. As we mentioned before, part of the medical records could be processed and used for the purposes of smart health and its services, such as epidemic control, the visualization of vaccination data, disease prevention, a self-diagnosis, etc.

The motivation for our paper is creating conditions for building smart health services using text mining techniques. For the purposes of this paper, a corpus that is containing 5,261 medical reports were created. In this paper, the aim was to create the algorithms for deleting the non-relevant data from the input set of medical data and preparing the relevant data for further processing. The relevant data is purified from excessive words, punctuation marks and other data that did not carry any informative value. Obtained redundant and keywords are remembered, in order of using them in the normalization of new data.

The paper is organized in the following way. The second section describes related researches. An overview of smart health services is given in the third section. The fourth section describes the proposed smart health framework for epidemic control. Next, the description of the data set and medical data are given to understand the need for their normalization. The following section presents our approach to normalization of medical data written in Serbian. Then the results of the application of the proposed method on the above-mentioned medical corpus are given. Finally, conclusions and directions for further research are given in the last section.

1.1. Related Work

Papers from our field of research can be divided into two groups. The first group includes papers describing the normalization and processing of medical textual data which is not written in the Serbian language. These papers present universal characteristics of medical data and their normalization, which are independent of the language.

The differences between clinical and ordinary texts and problems related to obtaining information from medical texts are described in [3]. In [4, 5], the methods used in the normalization of electronic medical data are given. In the paper [6], one way of classifying medical data and the application of neural networks in solving this problem is described. Here are processed texts from the Internet that contain a description of the health status of patients, the symptoms and the like. Paper [7] proposes a method for normalizing symptoms written in the Chinese language. A complete system that normalizes and extracts information from medical records and its architecture is described in [8]. In [14] the characteristics of clinical reports are presented, from the corpus of medical reports in Sweden taken from 2014-2015.

On the other hand, the methods described in the previous papers are not sufficient to fully apply to the medical text in the Serbian language, and of course, do not include lexical resources specific for the medical domain. The second group includes papers dealing with the normalization of textual documents and their analysis in the Serbian language, but even here the normalization was not carried out in clinical texts. The papers [10, 11] give the description of the process of normalization of informal documents in the Serbian language with the aim of faster searching. In [9] is presented the normalization of text in Serbian language using n-gram analysis. The specific language resources are needed for different data processing [12, 13]. Some steps from these methods can be used in the normalization of medical data, but most of them need to be adapted. The structure of medical data (reports) and their contents are much different from other types of text documents, so they need to be adapted according to their specificities.

1.2. Smart Health Services and the Smart City

The smart city infrastructure includes physical infrastructure, information and communication technology (ICT) infrastructure and services. Physical infrastructure is the part of a smart city including roads, railways, a water supply system etc. ICT infrastructure consists of computer and information systems, networks, sensors etc. so it is a link between the other two infrastructures of the smart city. It is based on the Internet of Things (IoT) and Big Data [2, 15]. Service infrastructure is based on physical infrastructure and can have some ICT components [1].

The application of mobile and ubiquitous computing enabled the collection of data from the user's environment. These data are contextual, and applications using them are called context-aware applications [1]. The use of smartphones to improve the health status of the user led to the formation of a new sub-feature in electronic healthcare, which is mobile healthcare (m-Health). The synergy of mobile and electronic healthcare with the concept of smart cities has come up with a new term – smart healthcare (s-Health) [1].

M-Health and s-Health are subsets of e-Health, but they also have their intersection. S-Health and m-Health differ in the source information they use, and the flow of this information. For m-Health, the source information comes from the users/patients, while in s-Health, besides this information, it uses the collective data obtained from the infrastructure of the smart city. Regarding the difference in the data, it refers to the fact that after processing, the m-Health returns the response to the user, and in the s-Health, the data are returned to the user, but also affect the collective data of the smart city.

To better understand the flow of data from patients to the ICT infrastructure of the smart city and back, some specific examples of services and their classification are given. Most smart healthcare services can be classified into one of these categories [16]:

- **Services based on the collective intelligence of the city.** They use data collected from sensors located on the territory of a smart city and analyze the obtained data for predictions for different purposes. Such service is suggesting a route with less air pollution for users with a respiratory problem.

- **Context-oriented services** analyze the close user environment, based on the image, the sound of motion detection, and so on. Examples are image processing and analysis for detecting abnormal phenomena in the relevant diseases. Motion detection is used for determining emotions, stress levels, breathing etc.

- **Services based on IoT devices** - They also analyze the close user environment, over the wearable IoT devices. The examples are the measurement of blood sugar, electrocardiogram of the heart, blood pressure and temperature using IoT bracelets.

- **Smart houses for patients** are smart environment (home) care for the elderly and the chronically ill, using technology infrastructure (sensors, cameras, wearable devices, and web services) to respect the wishes of patients to be at home.

- **Services based on crowdsourcing and medical data mining** - Similar to the first group of services, just with a large number of the original information from a wider area, which may be from the outside of the territory of the city. The examples are services for concluding in cases of measuring and obtaining abnormal results in some diseases and for detecting depression levels through crowdsourcing by comparing data from other users and using medical data mining.

The smart health framework which we are developing is based on medical data mining and it is described in the next section.

2. The Smart Health Framework

Smart health services that can be created by analyzing patient history data are smart health services based on medical data mining (Figure 2.1). They could be displayed on a public health portal and would include the following reports:

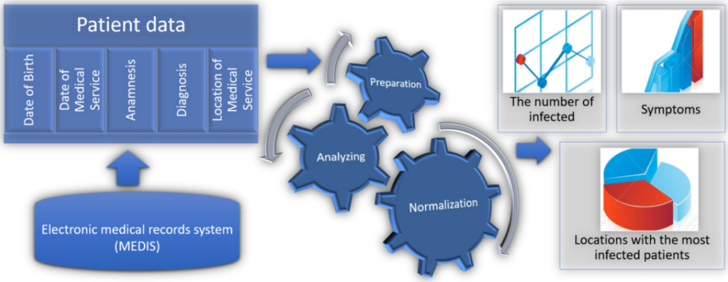

FIG. 2.1: Smart health framework for epidemic control

a) Report on how many people have a specific diagnosis daily in the city, as well as on a weekly, monthly, and annual basis. Based on this service, the user would be aware of the existence of the epidemic disease in his city and its status, whether it took a significant hit depending on whether the number of patients admitted day by day increased or decreased.

b) Reporting how many people have been diagnosed with a disease at a particular health station. Here a citizen could see how many people are ill in his immediate area.

c) Report on the presence of the disease in different age groups, where the citizen could decide if he belongs to the riskier group.

d) Report on the most common symptoms in people diagnosed with the disease. Here, a citizen could see the number of people who showed up with certain symptoms and had a diagnosis of a disease whose epidemic was ongoing. In this way, he could more easily recognize the symptoms and contact the doctor himself if he had any or most of the symptoms.

All these services would help to keep the citizen up to date with the epidemic in his place and take measures to avoid or treat the disease, and in this way the consequences of the epidemic will be reduced.

The first three services require data analysis and visualization, which is not demanding, while the fourth requires specific textual processing for proper symptom extraction. Analyzing the anamnesis we used, we came across abbreviations, misspellings, different word forms, and synonyms for the same symptoms. The anamnesis should be cleansed of words that have no meaning, and the words of significance should be reduced to the same form. Abbreviations should also be processed and preserved. There are also negations of symptoms in anamnesis, so the service would not show the true number of patients who have a symptom if negation were not taken into account. Addressing these issues is a key motive and contribution of this paper.

3. The Data Set

The specificity of the language in which the report is written further complicates the process of normalization. Corpora are required to identify the specifics of medical reports. Medical records are sensitive to research because of the confidentiality of the information they carry, so appropriate medical identification of medical records must be made and all personal data of patients, as well as doctors, are removed. There are several English-language corpora available such as: Informatics for Integrating Biology the Bedside, BioScope Corpus and the Thyme corpus [17]. There is no Serbian corpus in electronic form publicly available. We have used about five thousand medical records written in the Serbian language from 32 outpatients belonging to the Health Center Niš (DZ Niš), collected by the MEDIS.NET information system. The medical reports were written by 169 different doctors. This corpus is made by all ethical standards, with the de-identification of patients and medical staff as well as maintaining links to the affiliation of multiple reports to the same patient.

4. The Description of Medical Data

Medical reports are mostly generated by the hospital's internal needs. Clinical reports are needed for different stakeholders such as: medical staff to keep up with day-to-day activities, patients to document their health status, clinical research (medical researchers, pharmacists, epidemiologists, etc.), hospital management to keep track of finances and inventory planning, budget etc. Medical reports may contain numerical and textual information. Medical data is of mixed type (structured, semi-structured and unstructured) and therefore requires more complex processing that involves the existence of appropriate specialized lexical resources. The structure part contains values of specific variables, so it is the easiest to process (name, surname, year). The semi-structured part gives descriptive values for some parameters, but the structure is still known (temperature, pressure and laboratory analysis). The unstructured part consists of free text that the doctor gives and consists of symptoms, history, observations, conclusions. Unstructured data contains linguistically incomplete, informal and non-standard abbreviations which makes it difficult for computer processing and analysis. For this, it is necessary to pre-process the data before analysis to bring it into a standardized form.

Table 4.1 gives an example of a medical report that we are processing. In it, we can identify the structural part containing the date of service, name of service, diagnosis, diagnosis code, organizational unit in which the service was provided and location of the service. Also, this report contains an unstructured section consisting of an anamnesis. This part is more complex to process because it needs to extract relevant data and transform it into a standardized format, suitable for further processing.

Medical reports are most often written by doctors and nurses. Because of the speed at which they are written, they often contain many errors. Very often, the sentences are incomplete, for example, the auxiliary verbs are omitted as well as

Table 4.1: An example of the used medical record

Date of the service	23-03-18
Name of the service	Re-examination of adults
Anamnesis	Pacijent dobio sinoc osip po koži. Makulopapulozna ospa po kozi iza ušiju, čela i spušta se na trup. Vezikularni disajni šum (en. The patient received a skin rash last night. Maculopapular rash on the skin behind the ears and forehead and going down to the hull. Vesicular breathing noise)
Diagnosis	Morbili – Measles
Diagnosis code	B05
The organizational unit of the service	General medicine
Location of the service	Central building

the subject when it is obvious that the subject is the patient himself. Even the attachments are rarely found in the medical data, only in the description of the symptoms (e.g., fever, sweating, shortness of breath). An example is given in Table 4.2.

Such descriptions are concise and carry the most important information that, in a sense, facilitates the processing of such text. In medical reports, every word has more weight. In addition to deliberately omitting certain words, there are very common mistakes and spelling mistakes such as:

- two words are merged, and instead of space is a letter,

Table 4.2: An example of incomplete sentences in anamnesis

Original anamnesis text	
Serbian	English
“pečenje i svrab po celom telu, difuzna ospa koja svrbi”	”Burning and itching all over the body, diffuse wasp that itches”
Extended meaning	
Pacijent ima pečenje i svrab po celom telu u obliku difuzne ospa koja svrbi	The patient has burning and itching all over the body in the form of diffuse pox that itches

- misspelled word,
- incorrect writing of diacritical symbols, (eg. izvestaj, etc.), and often writes dj instead đ,
- a misspelled or omitted letter often, or a letter of excess e.g. "kašljke" instead of "kašlje",
- there are abbreviations of medical, punctuation marks, brackets, numbers, etc.
- x in anamnesis, (e.g. extremiteti).
- z and y mixed because of the keyboard.

When compared to other types of text according to Ehrentraunt and others [18], twice as many spelling mistakes (10%) are found in medical reports than in manuscripts, newspaper articles, web articles, etc. This is not surprising given the time limit they have for patient screening - average time due to scheduled appointments.

We often find abbreviations in medical reports. Abbreviations are ways of writing longer words without all the letters. It makes the text easier to write and read, but only on the condition that the reader knows the meaning of abbreviations. The problem is non-standardized abbreviations. Sometimes the same abbreviation can have an ambiguous meaning (for example, feb. for February and febrile). It is often the case that different abbreviations are used for the same term: shorten the temperature differently (e.g. T*, t, temp, etc.); In addition to abbreviations, acronyms are often used. Standardized acronyms are used in medical reports. However, there may also be problems because not often do acronyms have more meaning, for example: DIK is an acronym for pediatric infectious clinic (*dečija infektivna klinika*) and disseminated intravascular coagulation(*diseminovana intravaskularna koagulacija*), EEG for the electroencephalography and the electroencephalogram, etc.

The interpretation of acronyms, in this case, is likely to be related to the specialty of the physician who writes them or to the diagnosis. This association can be processed manually but also by machine learning methods with the appropriate corpus of data. Words that have a Latin or Greek root are common in medicine. However, in recent decades, more and more English words have been used for which there is no adequate translation into Serbian, so they are most commonly used in their original form. This is especially striking when it comes to medical techniques, medical devices, and certain surgical procedures. Depending on what the purpose of further processing of medical texts is in normalization, it may be desirable to simplify the litigants used in medical reports (e.g. pulmo – lung, hyperemia – the increase of blood flow to different tissues in the body).

The occurrence of negation in medical texts is very common because it excludes the existence of some symptoms that indicate disease. The presence of negation in the report significantly affects the meaning of the report itself, given the structure of the medical texts (short and concise). An example of the anamnesis with negation

Table 4.3: An example of the anamnesis with negation

Serbian	**English**
"Izbilo ga nešto po telu, pre dva dana, temper. nema, ne kašlje, ne boli ga grlo"	"There was something on the body, two days ago, temper. no, no cough, no pain"

is given in Table 4.3. The importance of the processing of negation in medical texts is demonstrated by the existence of the English corpus BioSope with a radial report in which the negation is manually marked as well as its scope of action [17]. For Serbian, there are rules of negation that can significantly improve data processing [19].

5. The Applied Normalization Method

The method for normalizing medical reports that we propose consists of several steps: tokenization, removal of stop words, processing of negation, removal of punctuation, and cutting off into n-grams. The algorithm we applied to the described data is shown in Figure 5.1.

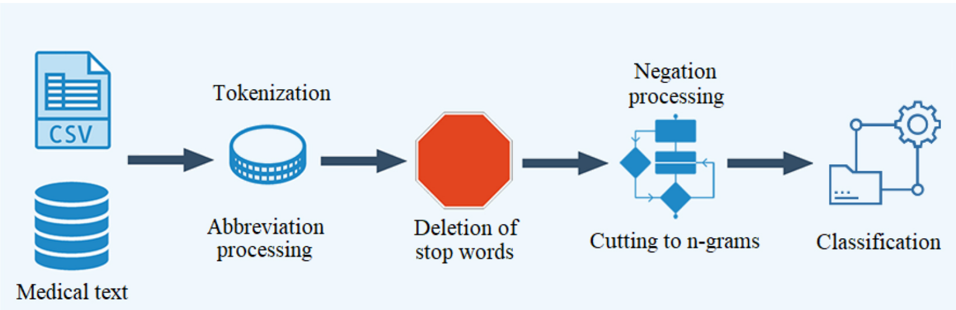

Fig. 5.1: Normalization of medical reports

Tokenization: In the first step of preparing a clinical text for further processing, it is necessary to delete unnecessary elements (multiple blanks, dates, special characters, etc.) from the anamnesis to identify the annotated words. The anamnesis were taken in the Serbian language using a Latin letter containing diacritical symbols ć, č, ž, š, đ, so in this step they are changed by the symbols cx, cw, zx, sx and dx respectively, since x and w are not used in Serbian letter, and proposed combinations are not found in corpus of diseases and related health problems (Serbian and Latin version) [20]. This makes it easier to process diacritical symbols because otherwise they will be transformed into special characters. In this step, the abbreviation processing is also done. Abbreviations belonging to standard medical

Table 5.1: Some of stop words from dictonary

Stop words	**Stop words**
kakva (en. which)	igde (en. where)
iako (en. although)	će (en. will)

abbreviations are generally acronyms and are capitalized. Non-standardized abbreviations specific to the Serbian language are separated and reduced to a single form without a point at the end. Punctuation marks remain in this step, because of their meaning in the processing of negation. When the reports are cleared in this way, then they can be divided into words, that is, tokens, at the level of further processing.

Deletion of stop words: The reports also contain words that are not meaningful, and these words are called stop words. These words should be deleted from medical records to reduce their scope and to retain only relevant information. By reducing the volume of data, the speed of their further processing increases. Stop words are usually adverbs, prepositions, pronouns, conjunctions, words and other words that are not relevant for determining the meaning of the text. The process of removing the stop word begins by creating a dictionary of stop words. The dictionary of stop words in Serbian, which is the result of our previous research [9], contains 3117 stop words. In Table 5.1, some words from this dictionary are given. Some words need to be removed from the set of stop words because they represent medical abbreviations or abbreviations for chemical elements. Their processing will be performed afterwards, considering whether the appropriate elements represent an abbreviation or stop word (whether they are capitalized, do they appear with other keywords, for example, "se" can be the auxiliary verb, and Se is the value of selenium from blood tests). Negation signals are also excluded from the set of stop words in order not to lose information about the negation of the existence of some symptoms.

Negation processing: The specificity of medical reports is that they are written in the third person and are concise, so that not all the negation signals appear. From the negation signals in the processed corpus, next words were found: ne (en. not), nije (en. it is not), nema (en. haven't), bez (en. without) and ni (en. nor). The extent of the action of this negation will be determined by the first non-stop word following the negation signal to the punctuation mark, or if it does not exist then it will be taken first before the negation signal to the punctuation mark. A word in the negation range receives a prefix ne_ and the negation signal is deleted. An example is given in Table 5.2. Punctuation marks can be removed after the negation has been processed.

Cutting off to n-grams: Bearing in mind the richness of the Serbian language, the presence of cases, phonetic changes, and changing verbs by person, gender and number, it happens that words that carry the same meaning can be found in many forms. Word-based grammatical rules would make this problem complex. Also, a morphological dictionary of Serbian, which is not publicly available in electronic

Table 5.2: An anamnesis after processing abbreviations and negations

Before processing	After processing
Izbilo ga nešto po telu, pre dva dana, temper. nema , ne kašlje, ne boli ga grlo	izbilo telu, dana, ne_temperaturu , ne_kasxlje, ne_boli grlo

form, would be necessary. So, we decided on a language-independent variant, which is a cutting to n-grams. An n-gram is a subset consisting of n elements of a given string. For example, the word "učiti" consists of the following n-grams: u-č-i-t-i (length 1), uč-či-it-ti (length 2), uči-čit-iti (length 3), učit-čiti (length 4) and učiti (length 5). N-grams are obtained by moving the frame of length n, whose origin can be in positions 1 to $m - n + 1$, where m is the length of the string [9]. By analyzing the content of the n-gram, the correlation between the appearance of the n-gram and the characteristics of the text can be noticed. N-grams are suitable for use in the analysis of textual documents in natural languages, due to language independence. N-gram analysis is a procedure applied to text, and its result is to obtain a set of n-grams of a given length. In proposed method, we used n-grams which length is 4 (4-grams). When cutting to 4-grams, the negation prefix is ignored (ne_temperaturu is cut to ne_temp). This length was chosen as optimal because it gives the best results for Serbian compared to the analysis of 3-grams and 5-grams (it is compared in next section).

Classifying n-grams is required after normalization, where the n-gram is sorted into keywords and stop words that are specific to the medical domain. Keywords include symptoms that can be remembered for the appropriate type of illness, while stop words are those that occur in all anamnesis, regardless of diagnosis. These are the words that most commonly appear in medical reports ("uput", "doznaka"). Since there are synonyms in keywords, they must be grouped. Classification is a work that requires special attention and will be the subject of extensive research in subsequent papers. Here, the classification is made in a simple way to show the impact of normalization on the results. The classification was made by looking for [20] n-grams among the symptoms and if they were found they are declared as keywords for the appropriate type of illness. Those n-grams which are not found in symptoms were declared as stop words in the medical domain.

6. Results and Discussion

After normalization over the described medical reports, the anamnesis was obtained in the normalized form. Of the 5261 medical reports, 2112 contained text in the non-structural section, while the rest contained only the structured section. After ejecting numbers and stop words, there were 16606 words left to normalize. The normalization with n-grams of lengths 3, 4, and 5 was performed, to determine which n-gram size yields the best results. To compare results precision, recall, and

Table 6.1: The results of different normalization methods

Normalization method	**WN**	**NW**	**CNW**	**Precision**	**Recall**	**F1-score**
3-gram	16606	13011	10213	0.7850	0.6150	0.6897
4-gram	16606	10330	9488	0.9185	0.5714	0.7045
5-gram	16606	8472	7891	0.9314	0.4752	0.6293
Proposed method	16606	13375	12232	0.9145	0.7366	0.8160

F1-score measures were used.

$$Precision = \frac{CNW}{NW} \tag{6.1}$$

$$Recall = \frac{CNW}{WN} \tag{6.2}$$

$$F1 - score = \frac{2 * Precision * Recall}{Precision + Recall} \tag{6.3}$$

Where CNW is the number of corectly normalized words, NW is the number of normalized terms, and WN is the number of words to be normalized.

The normalization results are shown in Table 6.1 and the best F1-score was for 4-grams. From the table it can be seen that increasing the length of n-grams increases the precision and the recall weakens. But for 3-grams the precision is much lower, while for 5-grams the recall is smaller, and therefore their F1-score is smaller compared to 4-grams. The precision for 3-grams of was expected to be lower because if the word had a prefix, the root (as the carrier of its meaning) is removed by normalization. Here, corectly normalized word is word which resulting n-gram was a part of the semantic root, and this was compared to a manualy labeled corpus.

Then we performed the proposed method as it is described in 5th section and the obtained results are also given in the Table 6.1. It can be obtained that the proposed method have best results for F1-score (0.8160). This results are comparable with methods for normalizing medical data in other languages (in [7] F1-score=0.6562). The words were reduced to 4-grams, and then counted in order to find their frequency in the anamnesis. The anamneses are now remembered in purified form, contain only relevant data and their volume has been reduced. Table 6.2 gives an example of anamnesis before and after normalization.

Then the n-grams with the largest number of occurrences in medical histories are divided into two groups, by meaning. The first group consists of keywords, which are those words whose meaning is closely related to the disease and they are shown in Table 6.3. Table 6.3 shows that the number of occurrences of symptoms in the corpus varies significantly depending on the application of the normalization method. So, if n-grams are searched in the corps, and abbreviations, synonyms and

Table 6.2: An example of one normalized anamnesis

Before normalization	**After normalization**
Kontrola: još uvek ima malaksalost, slabost, Savet , kontrola 30. 3. 2018	kont mala slab save kont
Pacijent dobio sinoc osip po koži. Makulopapulozna ospa po koži iza ušiju, čela i spušta se na trup. Vezikularni disajni šum	paci dobi osip kozxi maku ospa kozi usxij cwela spusx trup vezi disa sxum

negation is not considered, we will not get the exact number of symptoms. Table 6.3 shows that the percentage of occurrence for some symptoms is significantly increased (rash, because of synonyms and temperature, as it is often abbreviated). There are also those symptoms whose occurrence is reduced (change, for example) after the application of the proposed method, which is because they were found in negation.

The second group consists of stop words whose meanings do not determine the patient's illness, and they can appear in any medical history or clinical document. As the anamnesis is already in purified form without the stop words that are characteristic of general documents, the stop words thus extracted refer to the medical domain and can be stored to improve the results of normalization of new anamnesis or documents. These are shown in Table 6.4. The most common occurring keywords are related to symptoms of the disease (rash, temperature, cough, etc.) and the most frequent stop words are medical terms that don't indicate the patient's condition (appointment, report, etc.).

Therefore, this normalization can be used in the execution of statistics in the control of epidemics. The simple example is given in Figure 6.1.

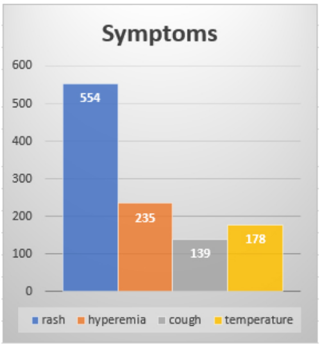

Fig. 6.1: Statistics in the control of epidemics

If we compare the Table 6.3 with the chart (Figure 6.1), it can be concluded

Table 6.3: The percentage of occurrence for the most occured keywords

N-gram	Simple 4-gram	Proposed method	Associated words	Associated n-grams of synonyms
hipe	10.62%	12.07%	hiperemija (en. hyperemia)	
kasxl	5.97%	6.63%	kašalj (en. cough)	kaša
kozxi	5.40%	6.01%	koža (en. skin)	kože, kožn, koži
licu	7.34%	10.89%	lice (en. face)	lica, lice
morb	17.90%	17.85%	morbili (en. morbili)	
ospa	13.92%	25.76%	ospa (en. rash)	osip
prom	5.16%	5.02%	promena (en. change)	
pulm	14.16%	13.92%	pulmo	
telu	10.89%	12.74%	telo (en. body)	tela, telu, telo
temp	8.43%	18.32%	temperatura (en. temperature)	
zxdre	12.36%	12.41%	ždrelo (en. pharynx)	

Table 6.4: The most occured stop words

n-gram	The percentage of occurrence in the anamnesis	Associated words
Kont	21.77%	kontrola (en. appointment)
Izve	13.01%	izveštaj (en. report)
Bolo	11.36%	bolovanje (en. sick leave)
Dana	10.46%	dan (en. day)
Dozn	9.28%	doznaka (en. remittance)
Uput	7.00%	uput (en. refer)
Nala	5.73%	nalaz (en. finding)
Preg	5.30%	pregled (en. examination)

that more information can be extracted to make the symptoms more accurate. For example, skin, face, body are mentioned in the table, which are the rash provisions. This means that the joint occurrence of n-grams in the anamnesis could be determined in order to obtain phrases, which will be the subject of the further research.

7. Conclusion

This paper outlines how medical reports in an electronic form stored daily can be used, as well as problems that can be encountered in their analysis and processing. The importance of text normalization for smart health services was demonstrated and a way to normalize clinical textual information written in Serbian was presented. In the experimental part, the results of the application of the proposed algorithm for normalization over the anamnesis collected using the MEDIS medical information system are presented. The described method extracts relevant data from medical reports so that it can be used for a variety of purposes, including public health and other smart healthcare services. The data stored in a purified form can be used for further processing. Applying the proposed method over the collected data, we obtained the F1-score (0.816). Since there is no adequate method in the Serbian language for comparison, compared to the methods for normalizing medical data in other languages [7], the proposed method produces good results. In this way, a corpus containing the stop words for a medical domain in the Serbian language can be formed, which can be used for processing medical text for various purposes. The subject of our further research will be the classification of medical terms and the labeling of entities in medical records and their application in smart healthcare services.

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EXISTENCE AND UNIQUENESS RESULTS FOR A COUPLED SYSTEM OF HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS

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Abstract. In this paper, we have studied existence and uniqueness of solutions for a coupled system of multi-point boundary value problems for Hadamard fractional differential equations. By applying principle contraction and Shaefer's fixed point theorem new existence results have been obtained.

Keywords: multi-point boundary value problems; Hadamard fractional differential equations; Shaefer's fixed point theorem.

1. Introduction

Differential equations of fractional order have proved to be very useful in the study of models of many phenomenons in various fields of science and engineering, such as: electrochemistry, physics, chemistry, viscoelasticity, control, image and signal processing. For more details, we refer the reader to [3, 5, 6, 7, 11, 12, 13, 14, 16, 18]. There has been a sign cant progress in the investigation of these equations in recent years, see [3, 8, 17, 18, 19]. More recently, a basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 3, 14, 16, 20, 22]. On the other hand, existence and uniqueness of solutions to boundary value problems for fractional differential equations has attracted the attention of many authors, see for example, [16, 17, 19] and the references therein. Moreover, the study of coupled systems of fractional order is also important in various problems of applied nature [2, 9, 10, 15, 24, 25]. Recently, many people have established the existence and uniqueness for solutions of some fractional systems, see [9, 10, 21, 23, 25] and the reference therein. In the last few decades, much attention has been focused on the study of the existence and uniqueness of solutions for boundary value problems of Riemann-Liouville type or Caputo type fractional

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differential equations, see [21, 23, 24, 25]. There are few papers devoted to the research of the Hadamard fractional differential equations; see [2].

In this paper, we study the existence of solutions for a Hadamard coupled system of nonlinear fractional integro-differential equations given by:

$$(1.1)\qquad \begin{cases} D^{\alpha}x\left(t\right)=f_{1}\left(t,y\left(t\right),D^{\delta}y\left(t\right)\right),1<\alpha\leq 2,t\in\left[1,T\right],\\ D^{\beta}y\left(t\right)=f_{2}\left(t,x\left(t\right),D^{\sigma}x\left(t\right)\right),1<\beta\leq 2,t\in\left[1,T\right],\\ x\left(1\right)=0,\ x\left(T\right)-\sum_{i=1}^{m}\lambda_{i}I^{p}x\left(\eta_{i}\right)=0,\\ y\left(1\right)=0,\ y\left(T\right)-\sum_{i=1}^{m}\mu_{i}I^{q}x\left(\xi_{i}\right)=0,\end{cases}$$

where $\sigma\leq\alpha-1,\delta\leq\beta-1$; $p,q>0$; $1<\eta_i,\xi_i<T$ and $D^{\alpha},D^{\beta},D^{\delta}$ and D^{σ} are the Hadamard fractional derivatives, I^p and I^q are the Hadamard fractional integrals and f_1,f_2 are continuous functions on $[1,T]\times\mathbb{R}^2$.

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1.1). In section 4 an examples are treated illustrating our results.

2. Preliminaries

This section is devoted to the basic concepts of Hadamard type fractional calculus will be used throughout this paper [13].

Definition 2.1. The fractional derivative of $f:[1,\infty[\ \rightarrow\mathbb{R}$ in the sense of Hadamard is defined as:

$$(2.1)\qquad D^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{n-\alpha-1}\frac{f\left(s\right)}{s}ds,n-1<\alpha<n,$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of the real number α and $log(t)=log_e(t)$.

Definition 2.2. The Hadamard fractional integral operator of order $\alpha>0$, for a continuous function f on $[1,\infty[$ is defined as:

$$(2.2)\qquad I^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f\left(s\right)}{s}ds,\alpha>0,$$

where $\Gamma\left(\alpha\right):=\int_{0}^{\infty}e^{-u}u^{\alpha-1}du$.

Lemma 2.1. *Let $\alpha > 0$. Then*

$$I^{\alpha} D^{\alpha} x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{\alpha - i}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n, n = [\alpha] + 1$.

We give also an auxiliary lemma to define the solutions for the problem (1.1).

Lemma 2.2. *Let $g \in C([1, T], \mathbb{R})$, the solution of the boundary value problem*

$$\begin{cases} D^{\alpha} x(t) = g(t), 1 < \alpha \leq 2, t \in [1, T], \\ x(1) = 0, x(T) = \sum_{i=1}^{m} \lambda_i I^p x(\eta_i), \end{cases} \tag{2.4}$$

is given by:

$$\begin{array}{c} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \\ + \frac{(\log t)^{\alpha-1}}{\Pi} \left[\frac{\sum_{i=1}^{m} \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{g(s)}{s} ds \right. \\ \left. - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \right], \end{array} \tag{2.5}$$

where

$$\Pi = \frac{1}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}}. \tag{2.6}$$

Proof. As argued in [13], for $c_i \in \mathbb{R}, i = 1, 2$, and by lemma 3, the general solution of equation of problem (2.4) is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \tag{2.7}$$

Using the boundary conditions for (2.4), we find that $c_2 = 0$.

For c_1, we have

$$\begin{array}{c} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds + c_1 (\log T)^{\alpha-1} \\ = \frac{\sum_{i=1}^{m} \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{g(s)}{s} ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}. \end{array} \tag{2.8}$$

which gives

$$\begin{array}{c} c_1 = \frac{\frac{\sum_{i=1}^{m} \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{g(s)}{s} ds}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}} \\ - \frac{\frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)} \sum_{i=1}^{m} \lambda_i (\log \eta_i)^{p+\alpha-1}} \end{array} \tag{2.9}$$

Substituting the value of c_1 and c_2 in (2.7), we get (2.5). $\square$

3. Main Results

Let us introduce the spaces $X = \{x : x \in C^1([1,T]), D^\sigma x \in C([1,T])\}$ and

$$Y = \{y : y \in C^1([1,T]), D^\delta y \in C([1,T])\}$$

endowed with the norm $\|x\|_X = \|x\| + \|D^\sigma x\|$; with

$$\|x\| = \sup_{t\in[1,T]} |x(t)|, \ \|D^\sigma x\| = \sup_{t\in[1,T]} |D^\sigma x(t)|,$$

and $\|y\|_Y = \|y\| + \left\|D^\delta y\right\|$; with

$$\|y\| = \sup_{t\in[1,T]} |y(t)|, \ \left\|D^\delta y\right\| = \sup_{t\in[1,T]} \left|D^\delta y(t)\right|.$$

Obviously, $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are a Banach spaces. The product space $\left(X \times Y, \|(x,y)\|_{X\times Y}\right)$ is also Banach space with norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$. Let us now introduce the quantities:

$$\begin{aligned}
N_1 &= \tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\tfrac{\sum_{i=1}^m \lambda_i(\log\eta_i)^{p+\alpha}}{\Gamma(p+\alpha+1)} + \tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right),\\
N_2 &= \tfrac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)} + \tfrac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\tfrac{\sum_{i=1}^m \lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)} + \tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right),\\
N_3 &= \tfrac{(\log T)^\beta}{\Gamma(\beta+1)} + \tfrac{(\log T)^{\beta-1}}{|\Delta|}\left(\tfrac{\sum_{i=1}^m \mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)} + \tfrac{(\log T)^\beta}{\Gamma(\beta+1)}\right),\\
N_4 &= \tfrac{(\log T)^{\beta-\delta}}{\Gamma(\beta-\delta+1)} + \tfrac{\Gamma(\beta)(\log t)^{\beta-\delta-1}}{\Gamma(\beta-\delta)|\Delta|}\left(\tfrac{\sum_{i=1}^m \mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)} + \tfrac{(\log T)^\beta}{\Gamma(\beta+1)}\right),
\end{aligned}$$

which

$$\Lambda = \frac{1}{(\log T)^{\alpha-1} - \frac{\Gamma(\alpha)}{\Gamma(p+\alpha)}\sum_{i=1}^m \lambda_i(\log\eta_i)^{p+\alpha-1}},$$

and

$$\Delta = \frac{1}{(\log T)^{\beta-1} - \frac{\Gamma(\beta)}{\Gamma(q+\beta)}\sum_{i=1}^m \mu_i(\log\xi_i)^{q+\beta-1}}.$$

We list also the following hypotheses:

(H1) The functions $f_1, f_2 : [1,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H2) There exists a nonnegative continuous functions $a_i, b_i \in C([1,T]), i = 1,2$ such that for all $t \in [1,T]$ and $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| &\le a_1(t)|x_1-x_2| + b_1(t)|y_1-y_2|,\\
|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)| &\le a_2(t)|x_1-x_2| + b_2(t)|y_1-y_2|,
\end{aligned}$$

with

$$\begin{aligned}
\omega_1 = \sup\nolimits_{t\in[1,T]} a_1(t), \omega_2 = \sup\nolimits_{t\in[1,T]} b_1(t),\\
\varpi_1 = \sup\nolimits_{t\in[1,T]} a_2(t), \ \varpi_2 = \sup\nolimits_{t\in[1,T]} b_2(t).
\end{aligned}$$

(H3) There exists a nonnegative functions $l_1(t)$ and $l_2(t)$ such that

$$|f_1(t,x,y)| \leq l_1(t), |f_2(t,x,y)| \leq l_2(t) \text{ for each } t \in [1,T] \text{ and all } x,y \in \mathbb{R},$$

with

$$L_1 = \sup_{t\in[1,T]} l_1(t), L_2 = \sup_{t\in[1,T]} l_2(t).$$

Our first result is based on Banach contraction principle:

Theorem 3.1. *Suppose that the hypothesis (H2) holds.*

If

$$(N_1+N_2)(\omega_1+\omega_2)+(N_3+N_4)(\varpi_1+\varpi_2)<1, \tag{3.1}$$

then the boundary value problem (1.1) has a unique solution on $[1,T]$.

Proof. Consider the operator $\phi : X \times Y \to X \times Y$ defined by:

$$\phi(x,y)(t) := (\phi_1 y(t), \phi_2 x(t)), t \in [1,T], \tag{3.2}$$

where

$$\begin{aligned} \phi_1 y(t) := \quad & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_1\left(s,y(s),D^\delta y(s)\right)}{s}ds \\ & +\frac{(\log t)^{\alpha-1}}{\Lambda}\left[\frac{\sum_{i=1}^m \lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{f_1\left(s,y(s),D^\delta y(s)\right)}{s}ds\right. \\ & \left.-\frac{1}{\Gamma(\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\beta-1}\frac{f_2(s,x(s),D^\sigma x(s))}{s}ds\right]. \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \phi_2 x(t) := & \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1}\frac{f_2(s,x(s),D^\sigma x(s))}{s}ds \\ & +\frac{(\log t)^{\beta-1}}{\Delta}\left[\frac{\sum_{i=1}^m \mu_i}{\Gamma(q+\beta)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\beta+q-1}\frac{f_2(s,x(s),D^\sigma x(s))}{s}ds\right. \\ +\frac{(\log t)^{\beta-1}}{\Delta} & \left[\frac{\sum_{i=1}^m \mu_i}{\Gamma(q+\beta)}\int_1^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\beta+q-1}\frac{f_2(s,x(s),D^\sigma x(s))}{s}ds\right. \end{aligned} \tag{3.4}$$

We shall prove that ϕ is contraction mapping.

Let $(x,y),(x_1,y_1) \in X \times Y$. Then, for each $t \in [1,T]$, we have:

$$\begin{aligned} |\phi_1 y(t) - \phi_1 y_1(t)| \quad & \leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)-f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s}ds \\ & \quad +\frac{(\log T)^{\alpha-1}}{|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(p+\alpha)}\times\right. \\ & \int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)-f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s}ds \\ & +\left.\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)-f_1\left(s,y_1(s),D^\delta y_1(s)\right)\right|}{s}ds\right]. \end{aligned} \tag{3.5}$$

Thanks to (H2),we obtain

$$
\begin{aligned}
&|\phi_1 y(t) - \phi_1 y_1(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\omega_1 \|y-y_1\| + \omega_2 \left\|D^\delta y - D^\delta y_1\right\|}{s} ds \\
&+ \left|\frac{(\log T)^{\alpha-1}}{\Lambda}\right| \frac{\sum_{i=1}^m \lambda_i}{\Gamma(p+\alpha)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{\omega_1 \|y-y_1\| + \omega_2 \left\|D^\delta y - D^\delta y_1\right\|}{s} ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{\omega_1 \|y-y_1\| + \omega_2 \left\|D^\delta y - D^\delta y_1\right\|}{s} ds.
\end{aligned} \tag{3.6}
$$

Consequently,

$$
\begin{aligned}
&|\phi_1 y(t) - \phi_1 y_1(t)| \\
&\leq \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha-1}}{|\Lambda|} \left(\frac{\sum_{i=1}^m \lambda_i (\log \eta_i)^{p+\alpha}}{\Gamma(\alpha+p+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right] (\omega_1 + \omega_2) \\
&\times \left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right),
\end{aligned}
$$

which implies that

$$
\|\phi_1(y) - \phi_1(y_1)\| \leq N_1 (\omega_1 + \omega_2) \left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right),
$$

$$
\begin{aligned}
&|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \\
&\leq \frac{1}{\Gamma(\alpha-\sigma)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\sigma-1} \frac{\left|f\left(s, y(s), D^\delta y(s)\right) - f\left(s, y_1(s), D^\delta y_1(s)\right)\right|}{s} ds \\
&+ \frac{\Gamma(\alpha)(\log t)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|} \times \\
&\left[\frac{\sum_{i=1}^m \lambda_i}{\Gamma(\alpha+p)} \int_1^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+p-1} \frac{\left|f_1\left(s, y(s), D^\delta y(s)\right) - f_1\left(s, y_1(s), D^\delta y_1(s)\right)\right|}{s} ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{\left|f_1\left(s, y(s), D^\delta y(s)\right) - f_1\left(s, y_1(s), D^\delta y_1(s)\right)\right|}{s} ds\right].
\end{aligned} \tag{3.7}
$$

By (H2), we have

$$
\begin{aligned}
&|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \\
&\leq \frac{(\log T)^{\alpha-\sigma} (\omega_1 + \omega_2)\left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)} \\
&+ \frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|} \left(\frac{\sum_{i=1}^m \lambda_i (\log \eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right) (\omega_1 + \omega_2) \\
&\times \left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).
\end{aligned} \tag{3.8}
$$

Hence,

$$
\begin{aligned}
&|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \\
&\leq \left[\frac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)} + \frac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|} \left(\frac{\sum_{i=1}^m \lambda_i (\log \eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right] \times \\
&(\omega_1 + \omega_2) \left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).
\end{aligned} \tag{3.9}
$$

Therefore,

$$
|D^\sigma \phi_1 y(t) - D^\sigma \phi_1 y_1(t)| \leq N_2 (\omega_1 + \omega_2) \left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \tag{3.10}
$$

Consequently,

$$
\|D^\sigma \phi_1(y) - D^\sigma \phi_1(y_1)\| \leq N_2 (\omega_1 + \omega_2) \left(\|y - y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \tag{3.11}
$$

By (**??**) and (3.11), we can write

$$\|\phi_1(y)-\phi_1(y_1)\|_X \leq (N_1+N_2)(\omega_1+\omega_2)\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right). \tag{3.12}$$

With the same arguments as before, we have

$$\|\phi_2(x)-\phi_2(x_1)\|_Y \leq (N_3+N_4)(\varpi_1+\varpi_2)\left(\|x-x_1\|+\|D^\sigma x-D^\sigma x_1\|\right). \tag{3.13}$$

And by,(3.12) and (3.13), we obtain

$$\begin{aligned}&\|\phi(x,y)-\phi(x_1,y_1)\|_{X\times Y}\\ &\leq \left[(N_1+N_2)(\omega_1+\omega_2)+(N_3+N_4)(\varpi_1+\varpi_2)\right]\|(x-x_1,y-y_1)\|_{X\times Y}.\end{aligned} \tag{3.14}$$

Thanks to (3.1), we conclude that ϕ is contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a fixed point which is a solution of the coupled system (1.1). □

The second main result is the following theorem:

Theorem 3.2. *Assume that the hypotheses (H1) and (H3) are satisfied.*
Then, the coupled system (1.1) has at least a solution on $[1,T]$.

Proof. We shall use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on $X\times Y$. It is to note that ϕ is continuous on $X\times Y$ in view of the continuity of f_1 and f_2 (hypothesis (H1)).

Now, We shall prove that ϕ maps bounded sets into bounded sets in $X\times Y$: Taking $r>0$, and $(x,y)\in B_r$, $B_r:=\{(x,y)\in X\times Y; \|(x,y)\|_{X\times Y}\leq r\}$, then for each $t\in[1,T]$, we have:

$$\begin{aligned}|\phi_1 y(t)| &\leq \tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\\ &+\tfrac{(\log t)^{\alpha-1}}{|\Lambda|}\left[\tfrac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right.\\ &\left.+\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.15}$$

Thanks to (H3), we can write
(3.16)

$$\begin{aligned}|\phi_1 y(t)| &\leq \tfrac{1}{\Gamma(\alpha)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-1}\frac{\sup_{t\in J} l_1(t)}{s}ds\\ &+\tfrac{(\log t)^{\alpha-1}}{|\Lambda|}\left[\tfrac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\sup_{t\in J} l_1(t)}{s}ds\right.\\ &\left.+\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\frac{\sup_{t\in J} l_1(t)}{s}ds\right].\\ &\leq \sup_{t\in J} l_1(t)\left[\tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}+\tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\tfrac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}(3.17)\qquad &|\phi_1 y(t)| \\ &\le L_1\left[\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\tfrac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right], t\in[1,T].\end{aligned}$$

Hence, we have

$$\begin{aligned}(3.18)\qquad &\|\phi_1(y)\| \\ &\le L_1\left[\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\tfrac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\tfrac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]=L_1N_1.\end{aligned}$$

On the other hand,

$$\begin{aligned}|D^{\sigma}\phi_1 y(t)| \le\; & \tfrac{1}{\Gamma(\alpha-\sigma)}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-\sigma-1}\tfrac{\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|}{s}ds \\ &+\tfrac{\Gamma(\alpha)(\log t)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left[\tfrac{\sum_{i=1}^{m}\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\tfrac{\left|f_1\left(s,y(s),D^{\delta}y(s)\right)\right|}{s}ds\right. \\ &\left.+\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\tfrac{\left|f_1\left(s,y(s),D^{\delta}y(s)\right)\right|}{s}ds\right].\end{aligned}$$

By (H3), we have,

$$\begin{aligned}(3.19)\qquad &|D^{\sigma}\phi_1 y(t)| \\ &\le L_1\left[\tfrac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}+\tfrac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\tfrac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right].\end{aligned}$$

Consequently we obtain,

$$(3.20)\qquad |D^{\sigma}\phi_1 y(t)|\le L_1N_2, t\in[1,T].$$

Therefore,

$$(3.21)\qquad \|D^{\sigma}\phi_1(y)\|\le L_1N_2.$$

Combining (3.18) and (3.21), yields

$$(3.22)\qquad \|\phi_1(y)\|_X\le L_1(N_1+N_2).$$

Similarly, it can be shown that,

$$(3.23)\qquad \|\phi_2(x)\|_Y\le L_2(N_3+N_4).$$

It follows from (3.22) and (3.23) that

$$(3.24)\qquad \|\phi(x,y)\|_{X\times Y}\le L_1(N_1+N_2)+L_2(N_3+N_4).$$

Consequently

$$(3.25)\qquad \|\phi(x,y)\|_{X\times Y}<\infty.$$

Next, we will prove that ϕ is equicontinuous on $[1,T]$: For $(x,y)\in B_r$, and $t_1,t_2\in[1,T]$, such that $t_1<t_2$. Thanks hypothesis (H3), we have:

$$\begin{aligned}|\phi_1 y(t_2)-\phi_1 y(t_1)| \leq\ & \tfrac{L_1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\alpha-1}-\left(\log\tfrac{t_2}{s}\right)^{\alpha-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\alpha-1}\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1}{\Gamma(\alpha)}\tfrac{(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}}{\Lambda}\tfrac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\tfrac{\eta_i}{s}\right)^{\alpha+p-1}\tfrac{1}{s}ds \\ & -\tfrac{1}{\Gamma(\alpha)}\int_1^T\left(\log\tfrac{T}{s}\right)^{\alpha-1}\tfrac{1}{s}ds.\end{aligned}$$

Thus,

(3.26)

$$\begin{aligned}|\phi_1 y(t_2)-\phi_1 y(t_1)| \leq\ & \tfrac{L_1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\alpha-1}-\left(\log\tfrac{t_2}{s}\right)^{\alpha-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\alpha-1}\tfrac{1}{s}ds\right| \\ & +L_1\left|\tfrac{(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}}{\Lambda}\right|\left(\tfrac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right).\end{aligned}$$

and using (H3),we obtain:

$$\begin{aligned}& |D^\sigma\phi_1 y(t_2)-D^\sigma\phi_1 y(t_1)| \\ & \leq\tfrac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\alpha-\sigma-1}-\left(\log\tfrac{t_2}{s}\right)^{\alpha-\sigma-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\alpha-\sigma-1}\tfrac{1}{s}ds\right|\end{aligned}\tag{3.27}$$

Hence, by (3.26) and (3.27),we can write

$$\begin{aligned}\|\phi_1 y(t_2)-\phi_1 y(t_1)\|_X \leq\ & \tfrac{L_1}{\Gamma(\alpha)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\alpha-1}-\left(\log\tfrac{t_2}{s}\right)^{\alpha-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\alpha-1}\tfrac{1}{s}ds\right| \\ & +L_1\left|\tfrac{(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}}{\Lambda}\right|\left(\tfrac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ & +\tfrac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\alpha-\sigma-1}-\left(\log\tfrac{t_2}{s}\right)^{\alpha-\sigma-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1}{\Gamma(\alpha-\sigma)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\alpha-\sigma-1}\tfrac{1}{s}ds\right| \\ & +\tfrac{L_1\Gamma(\alpha)}{\Gamma(\alpha-\sigma)}\left|\tfrac{(\log t_1)^{\alpha-\sigma-1}-(\log t_2)^{\alpha-\sigma-1}}{\Lambda}\right|\left(\tfrac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right).\end{aligned}\tag{3.28}$$

With the same arguments as before, we get

(3.29)

$$\begin{aligned}& \|\phi_1 x(t_2)-\phi_1 x(t_1)\|_Y \\ & \leq\tfrac{L_2}{\Gamma(\beta)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\beta-1}-\left(\log\tfrac{t_2}{s}\right)^{\beta-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_2}{\Gamma(\beta)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\beta-1}\tfrac{1}{s}ds\right| \\ & +L_2\left|\tfrac{(\log t_1)^{\beta-1}-(\log t_2)^{\beta-1}}{\Delta}\right|\left(\tfrac{\sum_{i=1}^m\mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)}+\tfrac{(\log T)^{\beta}}{\Gamma(\beta+1)}\right) \\ & +\tfrac{L_2}{\Gamma(\beta-\delta)}\left|\int_1^{t_1}\left(\left(\log\tfrac{t_1}{s}\right)^{\beta-\delta-1}-\left(\log\tfrac{t_2}{s}\right)^{\beta-\delta-1}\right)\tfrac{1}{s}ds\right| \\ & +\tfrac{L_2}{\Gamma(\beta-\delta)}\left|\int_{t_1}^{t_2}\left(\log\tfrac{t_2}{s}\right)^{\beta-\delta-1}\tfrac{1}{s}ds\right| \\ & +\tfrac{L_2\Gamma(\beta)}{\Gamma(\beta-\delta)}\left|\tfrac{(\log t_1)^{\beta-\delta-1}-(\log t_2)^{\beta-\delta-1}}{\Delta}\right|\left(\tfrac{\sum_{i=1}^m\mu_i(\log\xi_i)^{\beta+q}}{\Gamma(\beta+q+1)}+\tfrac{(\log T)^{\beta}}{\Gamma(\beta+1)}\right).\end{aligned}$$

Thanks to (3.28) and (3.29), we can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|_{X\times Y}\to 0$ as $t_2\to t_1$ and by Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

Finally, we shall show that the set Ω defined by

$$\Omega=\{(x,y)\in X\times Y,(x,y)=\rho\phi(x,y),0<\rho<1\}, \tag{3.30}$$

is bounded:

Let $(x,y)\in\Omega$, then $(x,y)=\rho\phi(x,y)$, for some $0<\rho<1$. Thus, for each $t\in[1,T]$, we have:

$$x(t)=\rho\phi_1 y(t),y(t)=\rho\phi_2 x(t). \tag{3.31}$$

Then

$$\begin{aligned}\frac{1}{\rho}|x(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\\ &+\frac{\left|(\log T)^{\alpha-1}\right|}{|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.32}$$

Thanks to (H3), we can write

$$\begin{aligned}\frac{1}{\rho}|x(t)| &\leq \frac{L_1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\\ &+\frac{L_1(\log T)^{\alpha-1}}{|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{1}{s}ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{1}{s}ds\right].\end{aligned} \tag{3.33}$$

Therefore,

$$|x(t)|\leq\rho L_1\left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}+\frac{(\log T)^{\alpha-1}}{|\Lambda|}\left(\frac{\sum_{i=1}^m\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right)\right]. \tag{3.34}$$

Hence,

$$|x(t)|\leq\rho L_1N_1. \tag{3.35}$$

On the other hand,

$$\begin{aligned}&\frac{1}{\rho}|D^\sigma x(t)|\\ &\leq\frac{1}{\Gamma(\alpha-\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\sigma-1}\frac{\left|f\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\\ &+\frac{\Gamma(\alpha)(\log t)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left[\frac{\sum_{i=1}^m\lambda_i}{\Gamma(\alpha+p)}\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+p-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right.\\ &\left.+\frac{1}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{\left|f_1\left(s,y(s),D^\delta y(s)\right)\right|}{s}ds\right].\end{aligned} \tag{3.36}$$

By (H3), we have

$$\begin{aligned} &\tfrac{1}{\rho}\left|D^{\sigma}x\left(t\right)\right| \\ &\leq L_1\left[\tfrac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}+\tfrac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\tfrac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]. \end{aligned} \tag{3.37}$$

Therefore,

$$\begin{aligned} &\left|D^{\sigma}x\left(t\right)\right| \\ &\leq \rho L_1\left[\tfrac{(\log T)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}+\tfrac{\Gamma(\alpha)(\log T)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)|\Lambda|}\left(\tfrac{\sum_{i=1}^{m}\lambda_i(\log\eta_i)^{\alpha+p}}{\Gamma(\alpha+p+1)}+\tfrac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\right]. \end{aligned} \tag{3.38}$$

Thus,

$$\left\|D^{\sigma}\left(x\right)\right\| \leq \rho L_1 N_2. \tag{3.39}$$

From (3.35) and (3.39) we get

$$\|x\|_X \leq \rho L_1\left(N_1+N_2\right). \tag{3.40}$$

Analogously, we can obtain

$$\|y\|_Y \leq \rho L_2\left(N_3+N_4\right). \tag{3.41}$$

It follows from (3.40) and (3.41) that

$$\|(x,y)\|_{X\times Y} \leq \rho\left[L_1\left(N_1+N_2\right)+L_2\left(N_3+N_4\right)\right]. \tag{3.42}$$

Hence,

$$\|\phi\left(x,y\right)\|_{X\times Y} < \infty. \tag{3.43}$$

This shows that the set Ω is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that ϕ has at least a fixed point, which is a solution of coupled system (1.1). $\square$

4. Examples

Example 4.1. Let us consider the Hadamard coupled system:

$$\begin{cases} D^{\frac{5}{3}}x\left(t\right)=\frac{1}{8(t+2)^2}\left(\frac{|y(t)|}{1+|y(t)|}+\frac{t\left|D^{\frac{1}{2}}y(t)\right|}{2\pi\left(1+\left|D^{\frac{1}{2}}y(t)\right|\right)}\right)+\cos\left(1+t+t^2\right), t\in[1,e], \\ D^{\frac{5}{3}}y\left(t\right)=\frac{1}{20\pi+t^2}\left(\sin\left|x\left(t\right)\right|+\frac{t+t^2}{\pi}\sin\left|D^{\frac{1}{3}}x\left(t\right)\right|\right)+\ln\left(2+t^2\right), t\in[1,e], \\ x\left(1\right)=0, x\left(e\right)=2I^{\frac{3}{2}}x(\frac{5}{4})+I^{\frac{3}{2}}x(\frac{4}{3})+\frac{7}{4}I^{\frac{3}{2}}x(\frac{7}{5}), \\ y\left(1\right)=0, y\left(e\right)=\frac{3}{2}I^{\frac{4}{3}}x(\frac{7}{6})+\frac{6}{5}I^{\frac{4}{3}}x(\frac{5}{4})+\frac{7}{6}I^{\frac{4}{3}}x(\frac{3}{2}). \end{cases} \tag{4.1}$$

For this example, we have for $t\in[1,e]$

$$\begin{aligned} f_1(t,x,y) &= \tfrac{1}{8(t+2)^2}\left(\tfrac{|x|}{1+|x|}+\tfrac{t|y|}{2\pi(1+|y|)}\right)+\cos\left(1+t+t^2\right),\ x,y\in\mathbb{R}, \\ f_2(t,x,y) &= \tfrac{1}{20\pi+t^2}\left(\sin|x|+\tfrac{t+t^2}{\pi}\sin|y|\right)+\ln\left(2+t^2\right),\ x,y\in\mathbb{R}. \end{aligned}$$

Taking $x, y, x_1, y_1 \in \mathbb{R}, t \in [1, e]$, then:

$$\begin{array}{c} |f_1(t,x,y) - f_1(t,x_1,y_1)| \leq \frac{1}{8(t+2)^2}|x-x_1| + \frac{t}{16\pi(t+2)^2}|y-y_1|, \\ |f_2(t,x,y) - f_2(t,x_1,y_1)| \leq \frac{1}{20\pi+t^2}|x-x_1| + \frac{t+t^2}{\pi\left(20\pi+t^2\right)}|y-y_1|. \end{array}$$

So, we can take

$$a_1(t) = \frac{1}{8(t+2)^2}, b_1(t) = \frac{t}{16\pi(t+2)^2},$$

and

$$a_2(t) = \frac{1}{20\pi+t^2}, b_2(t) = \frac{t+t^2}{\pi(20\pi+t^2)}.$$

It follows then that

$$\begin{array}{c} \omega_1 = \sup_{t\in[1,e]} a_1(t) = \frac{1}{72}, \omega_2 = \sup_{t\in[1,e]} b_1(t) = \frac{1}{144\pi}, \\ \varpi_1 = \sup_{t\in[1,e]} a_2(t) = \frac{1}{20\pi+1}, \varpi_2 = \sup_{t\in[1,e]} b_2(t) = \frac{e+e^2}{\pi(20\pi+e)}, \\ N_1 = 1,3234, N_2 = 1,5028, N_3 = 1,3153, N_4 = 1,4974. \end{array}$$

and $\Delta = 1.0246, \Lambda = 1.020$,

$$(N_1 + N_2)(\omega_1 + \omega_2) + (N_3 + N_4)(\varpi_1 + \varpi_2) = 0,305\,4 < 1.$$

Hence by Theorem 5, then the system (4.1) has a unique solution on $[1, e]$.

Example 4.2. Consider the following coupled system:

$$(4.2) \qquad \begin{cases} D^{\frac{7}{4}}x(t) = \frac{\sin\left(|y(t)|+\left|D^{\frac{3}{4}}y(t)\right|\right)}{t^2+5t+2}, t \in [1,e], \\ D^{\frac{6}{5}}y(t) = \frac{\cos\left(|x(t)|+\left|D^{\frac{1}{5}}x(t)\right|\right)}{t^2+t+20\pi}, t \in [1,e], \\ x(1) = 0, x(e) = \frac{6}{5}I^{\frac{5}{3}}\left(\frac{5}{4}\right) + \frac{7}{4}I^{\frac{5}{3}}\left(\frac{4}{3}\right) + \frac{7}{4}I^{\frac{5}{3}}\left(\frac{7}{5}\right), \\ y(1) = 0, y(e) = \frac{3}{2}I^{\frac{7}{5}}\left(\frac{7}{6}\right) + \frac{6}{5}I^{\frac{7}{5}}\left(\frac{5}{4}\right) + \frac{7}{6}I^{\frac{7}{5}}\left(\frac{3}{2}\right). \end{cases}$$

Then, we have

$$\begin{array}{c} f_1(t,x,y) = \frac{\sin\left(|y(t)|+\left|D^{\frac{3}{4}}y(t)\right|\right)}{t^2+5t+2}, x, y \in \mathbb{R}, \\ f_2(t,x,y) = \frac{\cos\left(|x(t)|+\left|D^{\frac{1}{5}}x(t)\right|\right)}{t^2+t+20\pi}, x, y \in \mathbb{R}. \end{array}$$

Let $x, y \in \mathbb{R}$ and $t \in [1, e]$. Then

$$|f_1(t,x,y)| \leq \frac{1}{t^2+5t+2}, |f_2(t,x,y)| \leq \frac{1}{t^2+t+20\pi}.$$

So we take

$$l_1(t) = \frac{1}{t^2+5t+2}, l_2(t) = \frac{1}{t^2+t+20\pi}.$$

Then

$$L_1 = 0,1250, L_2 = 0,0154.$$

Thanks to Theorem 6, the system (4.2) has at least one solution on $[1, e]$.

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FIXED POINT THEOREMS USING IMPLICIT RELATION IN PARTIAL METRIC SPACES

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Abstract. This paper aims to establish some C using implicit relation in the framework of complete partial metric spaces, and also, to obtain other well-known results as corollaries to the result. The results presented in this paper extend and generalize several results from the existing literature to the setting of more general metric spaces and contraction conditions.

Keywords: contraction conditions; contraction conditions; complete partial metric spaces.

1. Introduction and Preliminaries

Let (X, d) be a metric space and let $S\colon X \to X$ be a self-mapping.

(i) A point $x \in X$ is called a fixed point of S if $x = Sx$.

(ii) S is called contraction if there exists a fixed constant $0 \le r < 1$ such that

$$d(S(x), S(y)) \le r\, d(x, y) \tag{1.1}$$

for all $x, y \in X$. If X is complete, then every contraction has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X (the Banach contraction principle). Obviously, every contraction is a continuous function. The Banach contraction mappings principle is the opening and vital result in the direction of fixed point theory. Subsequently, several authors have devoted their concentration to expanding and improving this theory (see, e.g., [6, 7, 15, 16, 22, 23, 24]).

Matthews ([12, 13]) launched the notion of partial metric space and proved equivalent result of Banach's theorem in such spaces. Afterwards, a multitude of results was obtained in these spaces (see, e.g., [2, 3, 9, 10, 15, 18, 20, 21]). Also, the concept of PMS provides to study denotational semantics of dataflow networks [12, 13, 17, 19].

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Matthews [12] introduced the notion of partial metric spaces as follows:

Definition 1.1. ([12]) Let X be a nonempty set and let $p\colon X \times X \to \mathbb{R}^+$ be a function satisfy

($p1$) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

($p2$) $p(x,x) \leq p(x,y)$,

($p3$) $p(x,y) = p(y,x)$,

($p4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X,p) is called partial metric space.

It is clear that if $p(x,y) = 0$, then from ($p1$) and ($p2$) we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero. Various applications of this space has been extensively investigated by many authors (see [11], [18] for details).

Example 1.1. ([4]) Let $X = \mathbb{R}^+$ and $p\colon X \times X \to \mathbb{R}^+$ given by $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 1.2. ([4]) Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\Big([a,b],[c,d]\Big) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

Remark 1.1. ([8]) Let (X,p) be a partial metric space.

($a1$) The function $d_M\colon X \times X \to \mathbb{R}^+$ defined as $d_M(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X and (X, d_M) is a (usual) metric space.

($a2$) The function $d_S\colon X \times X \to \mathbb{R}^+$ defined as $d_S(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}$ is a (usual) metric on X and (X, d_S) is a (usual) metric space.

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [12].

Definition 1.2. ([12]) Let (X,p) be a partial metric space. Then

($b1$) a sequence $\{x_n\}$ in (X,p) is said to be convergent to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n, x)$,

($b2$) a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite,

$(b3)$ (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

$(b4)$ A mapping $F\colon X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F\Big(B_p(x_0, \delta)\Big) \subset B_p\Big(F(x_0), \varepsilon\Big)$.

Definition 1.3. ([14]) Let (X, p) be a partial metric space. Then

$(c1)$ a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n) = 0$,

$(c2)$ (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that $p(x, x) = 0$.

Lemma 1.1. *([12, 13]) Let (X, p) be a partial metric space. Then*

(d1) a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_M),

(d2) (X, p) is complete if and only if the metric space (X, d_M) is complete,

(d3) a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 1.2. *([1]) Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Now, an implicit relation has been introduced to investigate some fixed point and common fixed point theorems in partial metric spaces.

Definition 1.4. (**Implicit Relation**) Let Ψ be the family of all real valued continuous functions $\psi\colon \mathbb{R}_+^4 \to \mathbb{R}_+$, for four variables. For some $h \in [0, 1)$, we consider the following conditions.

$(r1)$ For $x, y \in \mathbb{R}_+$, if $y \le \psi(x, x, y, \frac{x+y}{2})$, then $y \le hx$.

$(r2)$ For $x \in \mathbb{R}_+$, if $y \le \psi(0, 0, y, \frac{y}{2})$, then $y = 0$.

$(r3)$ For $x \in \mathbb{R}_+$, if $y \le \psi(y, 0, 0, y)$, then $y = 0$, since $h \in [0, 1)$.

The purpose of this paper is to establish some fixed point and common fixed point theorems in the setting of partial metric spaces using implicit relation. The results of findings extend and generalize several results from the existing literature.

1.1. Main Results

In this section, some fixed point and common fixed point theorems shall be proved using implicit relation in the framework of partial metric spaces.

Theorem 1.1. *Let (X,p) be a complete partial metric space and let $T\colon X \to X$ be a mapping satisfying the inequality*

$$p(Tx,Ty) \leq \psi\Big\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]\Big\}, \tag{1.2}$$

for all $x,y \in X$ and some $\psi \in \Psi$. Then we have

(a) If ψ satisfies the conditions $(r1)$ and $(r2)$, then T has a fixed point. Moreover, for any $x_0 \in X$ and the fixed point z, we have

$$p(Tx_n, z) \leq \Big(\frac{h^n}{1-h}\Big)p(x_0, Tx_0).$$

(b) If ψ satisfies the condition $(r3)$, then T admits a unique fixed point.

Proof. (a) For each $x_0 \in X$ and $n \in \mathbb{N}$, put $x_{n+1} = Tx_n$. It follows from (1.2) and $(p4)$ that

$$\begin{aligned}
p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\
&\leq \psi\Big\{p(x_{n-1},x_n), p(x_{n-1},Tx_{n-1}), p(x_n,Tx_n), \\
&\qquad \frac{1}{2}[p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})]\Big\} \\
&\leq \psi\Big\{p(x_{n-1},x_n), p(x_{n-1},x_n), p(x_n,x_{n+1}), \\
&\qquad \frac{1}{2}[p(x_{n-1},x_{n+1})+p(x_n,x_n)]\Big\} \\
&\leq \psi\Big\{p(x_{n-1},x_n), p(x_{n-1},x_n), p(x_n,x_{n+1}), \\
&\qquad \frac{1}{2}[p(x_{n-1},x_n)+p(x_n,x_{n+1})-p(x_n,x_n)]\Big\} \\
&\leq \psi\Big\{p(x_{n-1},x_n), p(x_{n-1},x_n), p(x_n,x_{n+1}), \\
&\qquad \frac{1}{2}[p(x_{n-1},x_n)+p(x_n,x_{n+1})]\Big\}
\end{aligned} \tag{1.3}$$

Since ψ satisfies the condition $(r1)$, there exists $h \in [0,1)$ such that

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n) \leq h^n p(x_0, x_1). \tag{1.4}$$

Set $A_n = p(x_n, x_{n+1})$ and $A_{n-1} = p(x_{n-1}, x_n)$, then from (1.4), we obtain

$$A_n \leq hA_{n-1} \leq h^2 A_{n-2} \leq \cdots \leq h^n A_0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, then by using $(p4)$ and equation (1.4), we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\ &\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\ &\leq h^n p(x_0, x_1) + h^{n+1} p(x_0, x_1) + \cdots + h^{n+m-1} p(x_0, x_1) \\ &= h^n [p(x_0, x_1) + h p(x_0, x_1) + \cdots + h^{m-1} p(x_0, x_1)] \\ &= h^n [1 + h + \cdots + h^{m-1}] A_0 \\ &\leq h^n \Big(\frac{1 - h^{m-1}}{1 - h}\Big) A_0. \end{aligned}$$

Taking $n, m \to \infty$ in the above inequality, we get $p(x_n, x_m) \to 0$ since $0 < h < 1$, hence $\{x_n\}$ is a Cauchy sequence in X. Thus by Lemma 1.1 this sequence will also Cauchy in (X, d_M). In addition, since (X, p) is complete, (X, d_M) is also complete. Thus there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Moreover by Lemma 1.1,

$$p(z, z) = \lim_{n \to \infty} p(z, x_n) = \lim_{n,m \to \infty} p(x_n, x_m) = 0, \tag{1.5}$$

implies

$$\lim_{n \to \infty} d_M(z, x_n) = 0. \tag{1.6}$$

Moreover, taking the limit as $m \to \infty$ we get

$$p(x_n, z) \leq \Big(\frac{h^n}{1 - h}\Big) p(x_1, x_0).$$

It implies that

$$p(Tx_n, z) \leq \Big(\frac{h^n}{1 - h}\Big) p(x_0, Tx_0).$$

Now, we show that z is a fixed point of T. Notice that due to (1.5), we have $p(z, z) = 0$. By using inequality (1.2), we get

$$\begin{aligned} p(x_{n+1}, Tz) &= p(Tx_n, Tz) \\ &\leq \psi\Big\{p(x_n, z), p(x_n, Tx_n), p(z, Tz), \\ &\qquad \frac{1}{2}[p(x_n, Tz) + p(z, Tx_n)]\Big\} \\ &= \psi\Big\{p(x_n, z), p(x_n, x_{n+1}), p(z, Tz), \\ &\qquad \frac{1}{2}[p(x_n, Tz) + p(z, x_{n+1})]\Big\}. \end{aligned}$$

Note that $\psi \in \Psi$, then taking the limit as $n \to \infty$ and using (1.5) and Lemma 1.2, we get

$$p(z, Tz) \leq \psi\Big\{0, 0, p(z, Tz), \frac{1}{2} p(z, Tz)\Big\}.$$

Since ψ satisfies the condition $(r2)$, then $p(z, Tz) \leq h.0 = 0$. This shows that $z = Tz$. Thus z is a fixed point of T.

(b) Let z_1, z_2 be fixed points of T with $z_1 \neq z_2$. We shall prove that $z_1 = z_2$. It follows from equation (1.2) and (1.5) that

$$\begin{aligned} p(z_1, z_2) &= p(Tz_1, Tz_2) \\ &\leq \psi\Big\{p(z_1, z_2), p(z_1, Tz_1), p(z_2, Tz_2), \\ &\qquad \frac{1}{2}[p(z_1, Tz_2) + p(z_2, Tz_1)]\Big\} \\ &= \psi\Big\{p(z_1, z_2), p(z_1, z_1), p(z_2, z_2), \\ &\qquad \frac{1}{2}[p(z_1, z_2) + p(z_2, z_1)]\Big\} \\ &= \psi\Big\{p(z_1, z_2), 0, 0, p(z_1, z_2)\Big\}. \end{aligned}$$

Since ψ satisfies the condition $(r3)$, then we get

$$\begin{aligned} p(z_1, z_2) &\leq h\, p(z_1, z_2) \\ &\Rightarrow p(z_1, z_2) = 0, \ \text{ since } 0 < h < 1. \end{aligned}$$

This shows that $z_1 = z_2$. Thus, the fixed point of T is unique. This completes the proof. □

Theorem 1.2. *Let T_1 and T_2 be two self-maps on a complete partial metric space (X, p) and*

$$\begin{aligned} p(T_1x, T_2y) &\leq \psi\Big\{p(x, y), p(x, T_1x), p(y, T_2y), \\ &\qquad \frac{p(x, T_2y) + p(y, T_1x)}{2}\Big\} \end{aligned} \tag{1.7}$$

for all $x, y \in X$ and some $\psi \in \Psi$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. For each $x_0 \in X$. Put $x_{2n+1} = T_1 x_{2n}$ and $x_{2n+2} = T_2 x_{2n+1}$ for $n =$

$0, 1, 2, \ldots$. It follows from (1.7), $(p4)$ and Lemma 1.1 that

$$\begin{aligned} p(x_{2n+1}, x_{2n}) &= p(T_1x_{2n}, T_2x_{2n-1}) \\ &\leq \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, T_1x_{2n}), p(x_{2n-1}, T_2x_{2n-1}), \\ &\qquad \frac{p(x_{2n}, T_2x_{2n-1}) + p(x_{2n-1}, T_1x_{2n})}{2}\Big\} \\ &= \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\ &\qquad \frac{p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})}{2}\Big\} \\ &\leq \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\ &\qquad \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n})}{2}\Big\} \\ &\leq \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\ &\qquad \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})}{2}\Big\}. \end{aligned} \tag{1.8}$$

Since ψ satisfies the condition $(r1)$, there exists $h \in [0, 1)$ such that

$$p(x_{2n+1}, x_{2n}) \leq hp(x_{2n}, x_{2n-1}) \leq h^{2n}p(x_1, x_0). \tag{1.9}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, then by using $(p4)$ and equation (1.9), we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\ &\quad -p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\ &\leq h^n p(x_0, x_1) + h^{n+1}p(x_0, x_1) + \cdots + h^{n+m-1}p(x_0, x_1) \\ &= h^n[p(x_0, x_1) + hp(x_0, x_1) + \cdots + h^{m-1}p(x_0, x_1)] \\ &= h^n[1 + h + \cdots + h^{m-1}]p(x_0, x_1) \\ &\leq h^n\Big(\frac{1 - h^{m-1}}{1 - h}\Big)p(x_0, x_1). \end{aligned}$$

Taking $n, m \to \infty$ in the above inequality, we get $p(x_n, x_m) \to 0$ since $0 < h < 1$, hence $\{x_n\}$ is a Cauchy sequence in X. Thus, by Lemma 1.1 this sequence will also Cauchy in (X, d_M). In addition, since (X, p) is complete, (X, d_M) is also complete. Thus there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Moreover by Lemma 1.1,

$$p(u, u) = \lim_{n\to\infty} p(u, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \tag{1.10}$$

implies

$$\lim_{n\to\infty} d_M(u, x_n) = 0. \tag{1.11}$$

Now, we have to prove that u is a common fixed point of T_1 and T_2. For this, consider

$$\begin{aligned} p(x_{2n+1}, T_1u) &= p(T_1x_{2n}, T_1u) \\ &\leq \psi\Big\{p(x_{2n}, u), p(x_{2n}, T_1x_{2n}), p(u, T_1u), \\ &\qquad \frac{p(x_{2n}, T_1u) + p(u, T_1x_{2n})}{2}\Big\} \\ &= \psi\Big\{p(x_{2n}, x), p(x_{2n}, x_{2n+1}), p(u, T_1u), \\ &\qquad \frac{p(x_{2n}, T_1u) + p(u, x_{2n+1})}{2}\Big\}. \end{aligned}$$

Note that $\psi \in \Psi$, then using (1.10), Lemma 1.2 and taking the limit as $n \to \infty$, we get

$$p(u, T_1u) \leq \psi\Big(0, 0, p(u, T_1u), \frac{p(u, T_1u)}{2}\Big).$$

Since ψ satisfies the condition $(r2)$, then $p(u, T_1u) \leq h.0 = 0$. This shows that $u = T_1u$ for all $u \in X$. Similarly, we can show that $u = T_2u$. Thus, u is a common fixed point of T_1 and T_2.

Now, to show that the common fixed point of T_1 and T_2 is unique. For this, let u' be another common fixed point of T_1 and T_2, that is, $T_1u' = T_2u' = u'$ with $u' \neq u$. Then we have to show that $u = u'$. It follows from equation (1.7) and (1.10) that

$$\begin{aligned} p(u, u') &= p(T_1u, T_2u') \\ &\leq \psi\Big\{p(u, u'), p(u, T_1u), p(u', T_2u'), \\ &\qquad \frac{p(u, T_2u') + p(u', T_1u)}{2}\Big\} \\ &= \psi\Big\{p(u, u'), p(u, u), p(u', u'), \\ &\qquad \frac{p(u, u') + p(u', u)}{2}\Big\} \\ &= \psi\Big\{p(u, u'), 0, 0, p(u, u')\Big\}. \end{aligned}$$

Since ψ satisfies the condition $(r3)$, then we get

$$\begin{aligned} p(u, u') &\leq h\, p(u, u') \\ &\Rightarrow p(u, u') = 0, \quad \text{since } 0 < h < 1. \end{aligned}$$

Thus, we get $u = u'$. This shows that u is the unique common fixed point of T_1 and T_2. This completes the proof. $\square$

Theorem 1.3. *Let T_1 and T_2 be two continuous self-maps on a complete partial metric space (X,p) and*

$$\begin{aligned} p(T_1^m x, T_2^n y) \quad &\leq \quad \psi\Big\{p(x,y), p(x,T_1^m x), p(y,T_2^n y), \\ &\qquad \frac{p(x,T_2^n y)+p(y,T_1^m x)}{2}\Big\} \end{aligned} \tag{1.12}$$

for all $x,y \in X$, where m and n are some integers and some $\psi \in \Psi$. Then T_1 and T_2 have a unique common fixed point in X.

Proof. Since T_1^m and T_2^n satisfy the conditions of Theorem 1.2. So T_1^m and T_2^n have a unique common fixed point. Let z be the common fixed point. Then, we have

$$\begin{aligned} T_1^m z = z &\Rightarrow T_1(T_1^m z) = T_1 z \\ &\Rightarrow T_1^m(T_1 z) = T_1 z. \end{aligned}$$

If $T_1 z = z_0$, then $T_1^m z_0 = z_0$. So, $T_1 z$ is a fixed point of T_1^m. Similarly, $T_2(T_2^n z) = T_2 z$. Now, using equation (1.12) and Lemma 1.1, we obtain

$$\begin{aligned} p(z,T_1 z) \quad &= \quad p(T_1^m z, T_1^m(T_1 z)) \\ &\leq \quad \psi\Big\{p(z,T_1 z), p(z,T_1^m z), p(T_1 z, T_1^m(T_1 z)), \\ &\qquad \frac{p(z,T_1^m(T_1 z))+p(T_1 z, T_1^m z)}{2}\Big\} \\ &= \quad \psi\Big\{p(z,T_1 z), p(z,z), p(T_1 z, T_1 z), \\ &\qquad \frac{p(z,T_1 z)+p(T_1 z, z)}{2}\Big\} \\ &= \quad \psi\Big\{p(z,T_1 z), 0, 0, \frac{p(z,T_1 z)+p(z,T_1 z)}{2}\Big\} \\ &= \quad \psi\Big\{p(z,T_1 z), 0, 0, p(z,T_1 z)\Big\}. \end{aligned}$$

Since ψ satisfies the condition $(r3)$, then we get

$$\begin{aligned} p(z,T_1 z) \quad &\leq \quad h\, p(z,T_1 z) \\ &\Rightarrow \quad p(z,T_1 z) = 0, \quad \text{since } 0 < h < 1. \end{aligned}$$

Thus, we have $z = T_1 z$ for all $z \in X$. Similarly, we can show that $z = T_2 z$. This shows that z is a common fixed point of T_1 and T_2. For the uniqueness of z, let $z' \neq z$ be another common fixed point of T_1 and T_2. Then clearly z' is also a common fixed point of T_1^m and T_2^n which implies $z' = z$. Hence T_1 and T_2 have a unique common fixed point. This completes the proof. □

Theorem 1.4. *Let $\{F_\alpha\}$ be a family of continuous self mappings on a complete partial metric space (X,p) satisfying*

$$\begin{aligned} p(F_\alpha x, F_\beta y) \quad &\leq \quad \psi\Big\{p(x,y), p(x,F_\alpha x), p(y,F_\beta y), \\ &\qquad \frac{p(x,F_\beta y)+p(y,F_\alpha x)}{2}\Big) \end{aligned} \tag{1.13}$$

for $\alpha, \beta \in \Psi$ with $\alpha \neq \beta$ and $x, y \in X$. Then there exists a unique $u \in X$ satisfying $F_\alpha u = u$ for all $\alpha \in \Psi$.

Proof. For $x_0 \in X$, we define a sequence as follows:

$$x_{2n+1} = F_\alpha x_{2n},\ x_{2n+2} = F_\beta x_{2n+1},\ n = 0, 1, 2, \ldots.$$

It follows from (1.13), ($p4$) and Lemma 1.1 that

$$\begin{aligned}
p(x_{2n+1}, x_{2n}) &= p(F_\alpha x_{2n}, F_\beta x_{2n-1}) \\
&\leq \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, F_\alpha x_{2n}), p(x_{2n-1}, F_\beta x_{2n-1}), \\
&\qquad \frac{p(x_{2n}, F_\beta x_{2n-1}) + p(x_{2n-1}, F_\alpha x_{2n})}{2}\Big\} \\
&= \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\
&\qquad \frac{p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})}{2}\Big\} \\
&\leq \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\
&\qquad \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n})}{2}\Big\} \\
&\leq \psi\Big\{p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), \\
&\qquad \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})}{2}\Big\}. \qquad (1.14)
\end{aligned}$$

Since ψ satisfies the condition ($r1$), there exists $h \in (0, 1)$ such that

$$p(x_{2n+1}, x_{2n}) \leq h p(x_{2n}, x_{2n-1}) \leq h^{2n} p(x_1, x_0). \qquad (1.15)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, then by using ($p4$) and equation (1.15), we have

$$\begin{aligned}
p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_m) \\
&\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1}) \\
&\leq h^n p(x_0, x_1) + h^{n+1} p(x_0, x_1) + \cdots + h^{n+m-1} p(x_0, x_1) \\
&= h^n [p(x_0, x_1) + h p(x_0, x_1) + \cdots + h^{m-1} p(x_0, x_1)] \\
&= h^n [1 + h + \cdots + h^{m-1}] p(x_0, x_1) \\
&\leq h^n \Big(\frac{1 - h^{m-1}}{1 - h}\Big) p(x_0, x_1).
\end{aligned}$$

Taking $n, m \to \infty$ in the above inequality, we get $p(x_n, x_m) \to 0$ since $0 < h < 1$, hence $\{x_n\}$ is a Cauchy sequence in X. Thus, by Lemma 1.1 this sequence will also Cauchy in (X, d_M). In addition, since (X, p) is complete, (X, d_M) is also complete. Thus there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. Moreover by Lemma 1.1,

$$p(v, v) = \lim_{n\to\infty} p(v, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \qquad (1.16)$$

implies

$$\lim_{n\to\infty} d_M(v, x_n) = 0. \tag{1.17}$$

By the continuity of F_α and F_β, it is clear that $F_\alpha v = F_\beta v = v$. Therefore v is a common fixed point of F_α for all $\alpha \in \Psi$.

In order to prove the uniqueness, let us take another common fixed point v' of F_α and F_β where $v \neq v'$. Then from equation (1.13) and (1.16), we obtain

$$\begin{aligned} p(v, v') &= p(F_\alpha v, F_\beta v') \\ &\leq \psi\Big\{p(v, v'), p(v, F_\alpha v), p(v', F_\beta v'), \\ &\qquad\qquad \frac{p(v, F_\beta v') + p(v', F_\alpha v)}{2}\Big\} \\ &= \psi\Big\{p(v, v'), p(v, v), p(v', v'), \\ &\qquad\qquad \frac{p(v, v') + p(v', v)}{2}\Big\} \\ &= \psi\Big\{p(v, v'), 0, 0, p(v, v')\Big\}. \end{aligned}$$

Since ψ satisfies the condition $(r3)$, then we get

$$\begin{aligned} p(v, v') &\leq h\, p(v, v') \\ &\Rightarrow p(v, v') = 0, \text{ since } 0 < h < 1. \end{aligned}$$

Thus, we get $v = v'$ for all $v \in X$. This shows that v is a unique common fixed point of F_α for all $\alpha \in \Psi$. This completes the proof. □

Next, we give analogues of fixed point theorems in metric spaces for partial metric spaces by combining Theorem 1.1 with $\psi \in \Psi$ and ψ satisfies conditions $(r1)$, $(r2)$ and $(r3)$. The following corollary is an analogue of Banach's contraction principle.

Corollary 1.1. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$p(Tx, Ty) \leq a\, p(x, y)$$

for all $x, y \in X$, where $a \in [0, 1)$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1.1 with $\psi(u_1, u_2, u_3, u_4) = au_1$ for some $a \in [0, 1)$ and all $u_1, u_2, u_3, u_4 \in \mathbb{R}_+$. □

The following corollary is an analogue of R. Kannan's result [7].

Corollary 1.2. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$p(Tx, Ty) \le b\,[p(x, Tx) + p(y, Ty)]$$

for all $x, y \in X$, where $b \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1.1 with $\psi(u_1, u_2, u_3, u_4) = b(u_2 + u_3)$ for some $b \in [0, \frac{1}{2})$ and all $u_1, u_2, u_3, u_4 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, x, y, \frac{x+y}{2}) = b(x + y)$. So, if $y \le \psi(x, x, y, \frac{x+y}{2})$, then $y \le \Big(\frac{b}{1-b}\Big)x$ with $\Big(\frac{b}{1-b}\Big) < 1$. Thus, T satisfies the condition $(r1)$.

Next, if $y \le \psi(0, 0, y, \frac{y}{2}) = b(0 + y) = by$, then $y = 0$, since $b < \frac{1}{2} < 1$. Thus, T satisfies the condition $(r2)$.

Finally, if $y \le \psi(y, 0, 0, y) = b.0 = 0$, then $y = 0$. Thus, T satisfies the condition $(r3)$. $\square$

The following corollary is an analogue of S. K. Chatterjae's result [6].

Corollary 1.3. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$p(Tx, Ty) \le c\,[p(x, Ty) + p(y, Tx)]$$

for all $x, y \in X$, where $c \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1.1 with $\psi(u_1, u_2, u_3, u_4) = cu_4$ for some $c \in [0, 1)$ and all $u_1, u_2, u_3, u_4 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, x, y, \frac{x+y}{2}) = c\Big(\frac{x+y}{2}\Big)$. So, if $y \le \psi(x, x, y, \frac{x+y}{2})$, then $y \le \Big(\frac{c}{2-c}\Big)x$ with $\Big(\frac{c}{2-c}\Big) < 1$. Thus, T satisfies the condition $(r1)$.

Next, if $y \le \psi(0, 0, y, \frac{y}{2})$, then $y = 0$ since $c < 1$. Thus, T satisfies the condition $(r2)$.

Finally, if $y \le \psi(y, 0, 0, y) = cy$, then $y = 0$ since $c < 1$. Thus, T satisfies the condition $(r3)$. $\square$

The following corollary is an analogue of S. Reich's result [16].

Corollary 1.4. *Let (X, p) be a complete partial metric space. Suppose that the mapping $T\colon X \to X$ satisfies the following condition:*

$$p(Tx, Ty) \le L_1\,p(x, y) + L_2\,p(x, Tx) + L_3\,p(y, Ty)$$

for all $x, y \in X$, where $L_1, L_2, L_3 \ge 0$ are constants with $L_1 + L_2 + L_3 < 1$. Then T has a unique fixed point in X. Moreover, if $L_3 < \frac{1}{2}$, then T is continuous at the fixed point.

Proof. The assertion follows using Theorem 1.1 with $\psi(u_1, u_2, u_3, u_4) = L_1u_1 + L_2u_2 + L_3u_3$ for some $L_1, L_2, L_3 \geq 0$ are constants with $L_1 + L_2 + L_3 < 1$ and all $u_1, u_2, u_3, u_4 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, x, y, \frac{x+y}{2}) = L_1x + L_2x + L_3y$. So, if $y \leq \psi(x, x, y, \frac{x+y}{2})$, then $y \leq \left(\frac{L_1+L_2}{1-L_3}\right)x$ with $\left(\frac{L_1+L_2}{1-L_3}\right) < 1$. Thus, T satisfies the condition $(r1)$.

Next, if $y \leq \psi(0, 0, y, \frac{y}{2}) = L_1.0 + L_2.0 + L_3.y = L_3y$, then $y = 0$ since $L_3 < 1$. Thus, T satisfies the condition $(r2)$.

Finally, if $y \leq \psi(y, 0, 0, y) = L_1.y + L_2.0 + L_3.0 = L_1y$, then $y = 0$ since $L_1 < 1$. Thus, T satisfies the condition $(r3)$. □

Example 1.3. Let $X = [0, 1]$. Define $p\colon X \times X \to \mathbb{R}^+$ as $p(x, y) = \max\{x, y\}$ with $T\colon X \to X$ by $T(x) = \frac{x}{3}$. Clearly (X, p) is a partial metric space. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(Tx, Ty) = \frac{y}{3}$, $p(x, y) = y$, $p(x, Tx) = x$, $p(y, Ty) = y$, $p(x, Ty) = x$, $p(y, Tx) = y$.

(i) Now, we consider

$$p(Tx, Ty) = \frac{y}{3} \leq ay,$$

or $a \geq \frac{1}{3}$. If we take $0 \leq a < 1$, then T satisfies all the conditions of Corollary 1.1. Hence, applying Corollary 1.1, T has a unique fixed point. Here it is seen that $0 \in X$ is the unique fixed point of T.

(ii) Now, we consider

$$p(Tx, Ty) = \frac{y}{3} \leq b(x + y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{3} \leq \frac{3}{2}b,$$

or $b \geq \frac{2}{9}$. If we take $0 \leq b < \frac{1}{2}$, then T satisfies all the conditions of Corollary 1.2. Hence, applying Corollary 1.2, T has a unique fixed point and the unique fixed point T is $0 \in X$.

(iii) Now, we consider

$$p(Tx, Ty) = \frac{y}{3} \leq c(x + y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{3} \leq \frac{3}{2}c,$$

or $c \geq \frac{2}{9}$. If we take $0 \leq c < \frac{1}{2}$, then T satisfies all the conditions of Corollary 1.3. Hence, applying Corollary 1.3, T has a unique fixed point and it is $0 \in X$.

(iv) Now, we consider

$$p(Tx, Ty) = \frac{y}{3} \leq L_1 y + L_2 x + L_3 y,$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{3} \leq L_1 + \frac{1}{2} L_2 + L_3,$$

If we take (1) $L_1 = \frac{1}{3}$, $L_2 = \frac{1}{2}$ and $L_3 = 0$ (2) $L_1 = \frac{1}{2}$, $L_2 = 0$ and $L_3 = \frac{1}{3}$ and (3) $L_1 = 0$, $L_2 = \frac{1}{4}$ and $L_3 = \frac{1}{5}$, then T satisfies all the conditions of Corollary 1.4. Hence, applying Corollary 1.4, T has a unique fixed point and it is $0 \in X$.

Open Question: Can we extend the results for graphic contraction as defined in Younis et al. [22, 23, 24]?

2. Conclusion

In this paper, we have established some fixed point and common fixed point theorems using implicit relation in the framework of complete partial metric spaces, and also obtained the well-known Banach contraction principle, Kannan contraction, Chatterjae contraction and Reich contraction as corollaries to the result. The results extend, unify and generalize several results from the existing literature to the setting of a more general class of metric spaces and contraction conditions.

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ON THE CAYLEY GRAPHS OF BOOLEAN FUNCTIONS

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Abstract. A Boolean function is a function $f : \mathbb{Z}_n^2 \to \{0, 1\}$ and we denote the set of all n-variable Boolean functions by BF_n. For $f \in BF_n$ the vector $[\mathrm{W}_f(a_0), \ldots, \mathrm{W}_f(a_{2n-1})]$ is called the Walsh spectrum of f, where $\mathrm{W}_f(a) = \sum_{x\in V}(-1)^{f(x)\oplus ax}$, where V_n is the vector space of dimension n over the two-element field F_2. In this paper, we shall consider the Cayley graph Γ_f associated with a Boolean function f. We shall also find a complete characterization of the bent Boolean functions of order 16 and determine the spectrum of related Cayley graphs. In addition, we shall enumerate all orbits of the action of automorphism group on the set BF_n.
Keywords: Boolean function; Walsh spectrum; Cayley graph; automorphism group.

1. Introduction

Suppose V_n is the vector space of dimension n over the two-element field F_2, namely the set of all n-tuples of elements in the field F_2 and $\oplus$ denotes the addition operator over both F_2 and the vector space V_n, where V_n is the vector space of dimension n over the two-element field F_2. A Sylvester-Hadamard matrix of order 2^n denoted by H_n is defined recursively as

$$H_0 = 1,\ H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ldots, H_n = H_1 \otimes H_{n-1},\ n = 1, 2, ... \tag{1.1}$$

where $\otimes$ denotes to Kronecker product or Tensor product.

For two vectors $a, b \in \mathbb{Z}_2^n$, where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we define their scalar product as $a \cdot b = a_1b_1 \oplus \ldots \oplus a_nb_n$. A Boolean function f on n-variables is a map from V_n to V_1. Suppose the vectors $v_0 = (0, 0, ..., 0)$, $v_1 = (0, 0, ..., 1)$, …, $v_{2^n-1} = (1, 1, ..., 1)$ are ordered by lexicographical order. The $(0, 1)$ sequence $(f(v_0), f(v_1), ..., f(v_{2^n-1}))$ is called the truth table of f and BF_n denotes the set of all n-variable Boolean functions.

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2. Walsh spectrum of Boolean functions

For the Boolean function *f*, the support of f denoted by Ω_f is $\Omega_f = \{x \in \mathbb{Z}_2^n,\ f(x) = 1\}$. The Walsh transform of an n-variable Boolean function f is an integer valued function $\mathrm{W}_f : V_n \to [-2^n, 2^n]$ defined by

$$\mathrm{W}_f(u) = \sum_{x \in F_2^n} f(x)(-1)^{u.x}.$$

For the Boolean function *f*, the vector $[\mathrm{W}_f(0), ..., \mathrm{W}_f(2^n - 1)]$ is called the Walsh spectrum of f, see [4,7,15,16]

Consider the Cayley graph $\Gamma_f = Cay(\mathbb{Z}_2^n, \Omega_f)$, the vertex set of the Cayley graph Γ_f is V_n and two vertices $u, v \in V_n$ are adjacent if and only if $f(u \oplus v) = 1$. This means that $E_f = \{(u, v)|u, v \in V_n,\ f(u \oplus v) = 1\}$. Since for every $a \in V_n$, $a \oplus a = 0$, one can verify that for $\Omega_f \subseteq V_n$, we have $x = -x \in \Omega_f$. We denote this class of Cayley graphs constructed by a Boolean function as B-Cayley graphs. In Appendix *I*, all Boolean functions of order 16 (where $|\Omega_f| = 2$) and the spectra of B- Cayley graphs are given. In Appendix II, the characterization of Boolean functions in terms of spectrum of B-Cayley graph Γ_f associated with f is given. In [5], it is proved that for given Boolean function *f*, the Walsh spectrum of B-Cayley graph $Cay(\mathbb{Z}_2^n, \Omega_f)$ is equal with $H_n.f$. For example, let f = [1,0,0,1] be a Boolean function. Then the Walsh spectrum of B-Cayley graph $Cay(\mathbb{Z}_2^n, \Omega_f)$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = [0^2, 2^2].$$

For g = [1,1,1,1] we have $H_n.g = [0^3, 4]$ and for h = [0,0,1,1] we have $H_n.h = [-2, 0^2, 2]$.

Theorem 2.1. *Let f be a Boolean function whose related B-Cayley graph Γ_f is a bipartite regular graph with exactly three distinct eigenvalues and -2 is the smallest eigenvalue. Then $f \in F_2$ and $f = (0, 1, 1, 0)$.*

Proof. Suppose f satisfies in above conditions. We can suppose the spectrum of Γ_f is $Spec(\Gamma_f) = \{[-2]^{m_1}, [\lambda_1]^{m_2}, [\lambda_2]^{m_3}\}$. Since, Γ_f is bipartite, $\lambda_1 = 0$ and $\lambda_1 = 2$, see [6]. On the other hand, Γ_f is regular and so $m_3 = m_1 = 1$. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of a graph, it is a well-known fact that $\sum_{i=1}^{n} \lambda_i^2 = 2m$. This implies that $2m = 8$ and so m=4. Since Γ_f is 2-regular, it is isomorphic with the cycle graph C_4. In addition, suppose $V_n = \{00, 01, 10, 11\}$ is the set of vertices of a square as depicted in Figure 2.1. Then we have $00 + 01 = 01 \in \Omega_f$ and $11 + 01 = 10 \in \Omega_f$. Hence, $f(00) = f(11) = 0$ and $f(01) = f(10) = 1$ which yields that f=(0,1,1,0). □

Example 1. Suppose $V_3 = \{000, 001, 010, 100, 011, 101, 110, 111\}$. If $a = 110$ and f = (0,0,0,0,1,1,1,10), then the related Walsh spectrum is reported in Table 1.

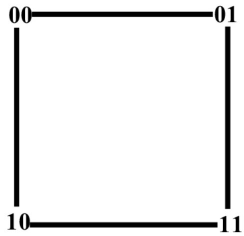

FIG. 2.1: The labeling of vertices of a square.

Table 2.1: The Walsh spectrum of f = (0,0,0,0,1,1,1,10).

x	$a.x$	f	$f(x)\oplus a.x$	$(-1)^{f(x)\oplus a.x}$	W$(f)(a)$
000	0	0	0	1	
001	0	0	0	1	
010	1	0	1	-1	
100	1	0	1	-1	0
011	1	1	0	1	
101	1	1	0	1	
110	0	1	1	-1	
111	0	1	1	-1	

3. Coloring the B-Cayley graphs

Let G be a group and X a nonempty set. An action of G on X is denoted by $(G|\mathrm{X})$ and X is called a G-set. It induces a group homomorphism φ from G into the symmetric group S_X on X, where $\varphi(g)x = gx$ for all $x \in \mathrm{X}$. The orbit of x will be indicated as x^G and defines as the set of all $\varphi(g)x$, $g \in G$. Suppose g is a permutation of n symbols with exactly λ_1 orbits of size 1, λ_2 orbits of size 2, $\ldots$, and λ_n orbits of size n. Then the cycle type of g is defined as $1^{\lambda 1}\ 2^{\lambda 2}\ldots\ n^{\lambda n}$. Let $G = \mathbb{Z}_2^n$ and $f \in BF_n$ is a Boolean function. In [17] it is showed that the automorphism group Aut(G) acts on the set BF_n as follows:

$$\forall x_i \in \mathbb{Z}_2^n,\ \alpha \in \mathrm{Aut}(G):\ f^{\alpha}(x_i) = f(\alpha(x_i)).$$

Hence, the conjugacy class of f under this action can be computed directly from the definition and it is $[f] = f^{\mathrm{Aut}(G)} = \{f^{\alpha} : \alpha \in \mathrm{Aut}(G)\}$.

Let x_1, x_2, ..., x_n be distinct colors. Denote by $C_{m,n}$ the set of all functions f: $\{1, 2, \ldots, m\} \rightarrow \{x_1, x_2, ..., x_n\}$. The action of $p \in S_m$ induced on $C_{m,n}$ is defined by $\hat{p}(f) = fop^{-1}$, $f \in C_{m,n}$. Treating the colors x_1, x_2, $\ldots$, x_n that comprise the range of $f \in C_{m,n}$ as independent variables the weight of f is W(f)=$\Pi_i f(i)$. Evidently, W(f) is a monomial of (total) degree m. Suppose G is a permutation group of degree m, $\hat{G}$=$\{\hat{p}$:$p \in G\}$, $\hat{p}$ is as defined above. Let p_1, p_2,$\ldots$,p_t be the distinct orbits of $\hat{G}$.The weight of p_i is the common value of W(f), $f \in p_i$. The sum of the weights of the orbits is the pattern inventory

$$\mathrm{W_G}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{t} w(p_i).$$

Theorem 2 [14]. (Pólya's Theorem) If G is a subgroup of S_m, then the pattern inventory for the orbits of $C_{m,n}$ modula $\hat{G}$ is

$$\mathrm{W}_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{p \in G} M_1^{C_1(p)} M_2^{C_2(p)} \ldots M_m^{C_m(p)},$$

where $M_k = x_1^k + x_2^k + \ldots + x_n^k$, the k^{th} power sum of the x's, and $(C_1(p), \ldots, C_m(p))$ is the cycle type of the permutation p. We now introduce the notion of cycle index. Let G be a permutation group. The cycle index of G acting on X is the polynomial $\mathrm{Z}(G,X)$ over Q in terms of in determinates x_1, x_2, $\ldots$, x_t, $t = |\mathrm{X}|$, defined by

$$\mathrm{Z}(G,\ \mathrm{X}) = \frac{1}{|G|} \sum_{C \in Conj(G)} |C| \prod_{i=1}^{t} {x_i}^{C_i(g_c)}, \tag{3.1}$$

where $\mathrm{Conj}(G)$ is the set of all conjugacy classes of G with representatives $g_C \in C$.

Let us consider the number of ways of assigning one of the colors green or blue to each corner of a square. Since there are two colors and four corners there are basically $2^4 = 16$ possibilities. However, when we take account of the symmetry of the square we see that some of the possibilities are essentially the same. For example, the first coloring, as in Figure 2 is the same as the second one after rotation through 180^0.

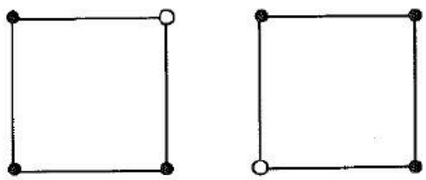

FIG. 3.1: Two indistinguishable colorings.

From above, we regard two colorings as being indistinguishable if one is transformed into the other by symmetry of the square. It is easy to find the distinguishable colorings (in this example) by trial and error: there are just six of them, as shown in the Figure 3.

Now consider a n bead necklace. Let us each corner of it to be colored green or blue. How many different colorings are there? One could argue for 2^n. For example,

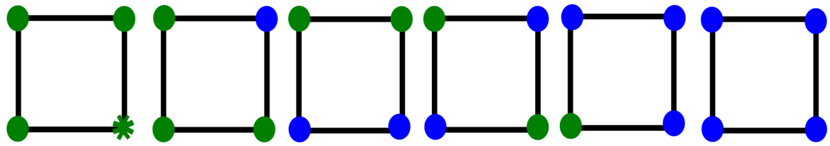

FIG. 3.2: The six distinguishable colorings.

if $n = 4$ and the corners are numbered 0,1,2,3 in clockwise order around the necklace, then there are only 6 ways of coloring the necklace *RRRR*, *BBBB*, *RRRB*, *RBBB*, *RRBB* and *RBRB*, see Figure 4. On the other hand, all colorings *RBBB*, *BRBB*, *BBRB*, *BBBR* are in the same class. We say that they are equivalent. In other words, the number of all non-equivalent colorings is six. This relation introduces an equivalence relation. All equivalences are

$$\begin{aligned}
&\{BBBB\}\,,\\
&\{BBBR\}\,,\{BBRB\}\,,\{BRBB\}\,,\{RBBB\}\,,\\
&\{BBRR\}\,,\{BRRB\}\,,\{RBBR\}\,,\{RRBB\}\,,\\
&\{BRBR\}\,,\{RBRB\}\,,\\
&\{BRRR\}\,,\{RBRR\}\,,\{RRBR\}\,,\{RRRB\}\,,\\
&\{RRRR\}\,.
\end{aligned} \tag{3.2}$$

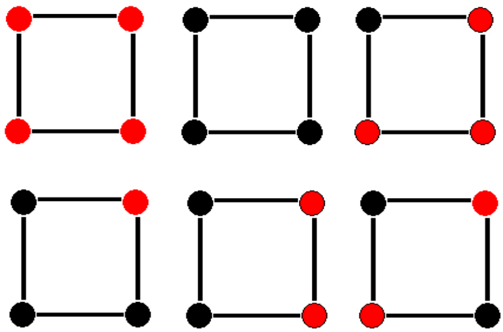

FIG. 3.3: Distinguish colorings of 4 bead necklace.

Hence, any Boolean function can be considered as a coloring of a hyper cube by two colors 0 and 1. The different colorings yields that there are 2^{2^n} Boolean functions on n variables.

Definition 3.1. Consider the Boolean function f and B-Cayley $\Gamma_f = Cay(\mathbb{Z}_2^n, \Omega_f)$. Then f is permutational symmetric (PS) if and only if for any $(x_1, \ldots, x_n) \in \mathrm{V}_n$, we have $f(\alpha(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n)$, for any $\alpha \in \mathrm{Aut}(\Gamma_f)$.

Note that there are $2n$ different input values corresponding to a function. From the above definition, it is clear that for PS functions, the function f possesses the same value corresponding to each of the subsets generated from the automorphism group. As example, for $n = 4$, one gets the following partitions:

$$\begin{aligned}
&\{(0,0,0,0)\},\\
&\{(0,0,0,1)\},\{(0,0,1,0)\},\{(0,1,0,0)\},\{(1,0,0,0)\},\\
&\{(0,0,1,1)\},\{(0,1,1,0)\},\{(1,0,0,1)\},\{(1,1,0,0)\},\\
&\{(0,1,0,1)\},\{(1,0,1,0)\},\\
&\{(0,1,1,1)\},\{(1,0,1,1)\},\{(1,1,0,1)\},\{(1,1,1,0)\},\\
&\{(1,1,1,1)\},
\end{aligned} \tag{3.3}$$

Therefore, there are six different subsets which partition the 16 input patterns and any 4-variable PS Boolean function can have a specific value corresponding to each subset. If we replace in Eq.(3.3) 0 by R and 1 by B, then all above partitions are corresponded to the different colorings of the 2-cube or cycle C_4 as given in Eq.(3.2). Hence, there is a 1-1 correspondence between non-equivalent colorings of a n-cube and 4-variable PS Boolean functions. Let us denote

$$\Lambda_n(x_1,\ldots,x_n) = \{f(\alpha(x_1,\ldots,x_n)) \;=\; f(x_1,\ldots,x_n):\; \alpha \in \mathrm{Aut}(\Gamma_f)\}$$

that is, the orbit of $(x_1, \ldots, x_n)$ under the action of $\mathrm{Aut}(\Gamma_f)$ on V_n. It is clear that $\Lambda_n(x_1,\ldots,x_n)$ generates a partition in the set V_n. Let $\lambda_n = |\Lambda_n(x_1,\ldots,x_n)|$. It is clear that there are 2^{λ_n} number of n-variable PS Boolean functions. Let Γ_f is B-Cayley graph constructed by given Boolean function f. From Polya's Theorem, we get that

$$\lambda_n = \frac{1}{|\mathrm{Aut}(\Gamma_f)|}\sum_{C\in Conj(G)} |C|\prod_{i=1}^{t} 2^{C_i(g_c)}$$

in which every variable in Eq. (3.1) is replaced by 2. Hence, we proved the following theorem.

Theorem 3.1. *For given Boolean function f, the number of PS Boolean functions is*

$$\lambda_n = \frac{1}{|\mathrm{Aut}(\Gamma_f)|}\sum_{C\in Conj(G)} |C|\prod_{i=1}^{t} 2^{C_i(g_c)}.$$

4. Application in chemistry: Enumeration of hetero-fullerenes

Enumeration of chemical compounds has been accomplished by various methods. The Polya-Redfield theorem [14] has been a standard method for combinatorial enumerations of graphs, polyhedra, chemical compounds, and so forth. Combinatorial enumerations have found a wide-ranging application in chemistry, since chemical structural formulas can be regarded as graphs or three-dimensional objects, see [9]. Ghorbani et al. in a series of papers in [1-3,10-12] enumerated the number of hetero-fullerenes with different orders.

The fullerene era was started in 1985 with the discovery of a stable C_{60} cluster and its interpretation as a cage structure with the familiar shape of a soccer ball, by Kroto and his co-authors, see [8,13]. The well-known fullerene, the C_{60} molecule, is a closed-cage carbon molecule with three-coordinate carbon atoms tiling the spherical or nearly spherical surface with a truncated icosahedral structure formed by 20 hexagonal and 12 pentagonal rings. Such molecules made up entirely of n carbon atoms and having 12 pentagonal and $(n/2-10)$ hexagonal faces, where $n \neq 22$ is a natural number equal or greater than 20, see [22-30]. As an application of Polya-Theorem in fullerenes, in Appendix III, the number of hetero-fullerenes of molecule C_{60} is given.

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Appendix I

Boolean functions of order 16 where $\Omega_f = 2$ and spectra of their B-Cayley graphs.

f:=[1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 0]
f:=[1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 2, 2, 0, 0, 2, 2, 0, 0, 2, 2, 0, 0]
f:=[1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 2, 0, 0, 2, 2, 0, 0, 2, 2, 0, 0, 2]
f:=[1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 0]
f:=[1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 0, 2, 0, 2, 2, 0, 2, 0, 0, 2, 0, 2]
f:=[1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 0, 2, 2]
f:=[1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 0, 2, 2, 0, 2, 0, 0, 2, 0, 2, 2, 0]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 2, 0, 2, 0, 0, 2, 0, 2, 0, 2, 0, 2]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 2, 2, 0, 0, 0, 0, 2, 2, 0, 0, 2, 2]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 2, 0, 0, 2, 0, 2, 2, 0, 0, 2, 2, 0]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, 0, 2, 0, 0, 2, 0, 2, 0, 2, 0, 2, 2, 0, 2, 0]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 2, 0, 0, 0, 0, 2, 2, 0, 0, 2, 2, 2, 2, 0, 0]
f:=[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, 0, 0, 2, 0, 2, 2, 0, 0, 2, 2, 0, 2, 0, 0, 2]
f:=[0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 2, 0, 0, -2, 2, 0, 0, -2, 2, 0, 0, -2]
f:=[0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 2, -2, 0, 0, 2, -2, 0, 0, 2, -2, 0, 0]
f:=[0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 0, -2, 0, -2, 2, 0, 2, 0, 0, -2, 0, -2]
f:=[0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 2, -2, 0, 0, 0, 0, 2, -2, 2, -2, 0, 0, 0, 0]
f:=[0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 0, -2, 2, 0, 2, 0, 0, -2, 0, -2, 2, 0]
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f:=[0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 2, 0, 2, 0, 0, -2, 0, -2, 0, -2, 0, -2]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 2, -2, 2, -2, 2, -2, 0, 0, 0, 0, 0, 0, 0, 0]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 2, 0, 0, -2, 0, -2, 2, 0, 0, -2, 2, 0]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 2, -2, 0, 0, 0, 0, 2, -2, 0, 0, 2, -2]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 0, -2, 0, -2, 0, -2, 0, -2, 2, 0, 2, 0]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, -2, 2, -2, 0, 0, 0, 0, 0, 0, 0, 0, 2, -2, 2, -2]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 0, 0, -2, 0, -2, 2, 0, 0, -2, 2, 0, 2, 0, 0, -2]
f:=[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, -2, 0, 0, 0, 0, 2, -2, 0, 0, 2, -2, 2, -2, 0, 0]
f:=[0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, 0]
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f:=[0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 0, 2, -2, 0, 2, 0, 0, -2, 0, 2, -2, 0]
f:=[0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, -2, -2, 0, 0, 0, 0, 2, 2, -2, -2, 0, 0, 0, 0]
f:=[0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 0, 2, 0, -2, 2, 0, -2, 0, 0, 2, 0, -2]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 2, 2, 0, 0, 0, 0, -2, -2, 0, 0, -2, -2]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 2, 0, 0, -2, 0, 2, -2, 0, 0, 2, -2, 0]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 2, -2, -2, 2, 2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 2, 0, -2, 0, 0, 2, 0, -2, 0, 2, 0, -2]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 0, 0, -2, -2, 0, 0, -2, -2, 2, 2, 0, 0]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, 0, 0, -2, 0, 2, -2, 0, 0, 2, -2, 0, 2, 0, 0, -2]
f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, -2, -2]

f:=[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, 0, -2, 0, 0, 2, 0, -2, 0, 2, 0, -2, 2, 0, -2, 0]

f:=[0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 0, -2, -2, 0, 2, 0, 0, 2, 0, -2, -2, 0]

f:=[0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 0, 0, -2, 2, 2, -2, 0, 0, 0, 0, -2, 2]

f:=[0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 0, -2, 0, 2, 2, 0, -2, 0, 0, -2, 0, 2]

f:=[0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, -2, 2, 0, 0, 0, 0, 2, -2, -2, 2, 0, 0, 0, 0]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 2, 0, 0, 2, 0, -2, -2, 0, 0, -2, -2, 0]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 2, -2, 0, 0, 0, 0, -2, 2, 0, 0, -2, 2]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 2, 0, -2, 0, 0, -2, 0, 2, 0, -2, 0, 2]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, -2, -2, 2, 2, -2, -2, 2, 0, 0, 0, 0, 0, 0, 0, 0]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 0, -2, -2, 0, 0, -2, -2, 0, 2, 0, 0, 2]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, -2, 0, 0, 0, 0, -2, 2, 0, 0, -2, 2, 2, -2, 0, 0]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 0, -2, 0, 0, -2, 0, 2, 0, -2, 0, 2, 2, 0, -2, 0]

f:=[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, -2, -2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 2, -2, -2, 2]

f:=[0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, -2, 0, -2, 0, 2, 0, 2, 0, -2, 0, -2, 0]

f:=[0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, -2, -2, 0, 0, 2, 2, 0, 0, -2, -2, 0, 0]

f:=[0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, -2, 0, 0, -2, 2, 0, 0, 2, -2, 0, 0, -2]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, -2, -2]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 0, -2, 0, -2, 0, 2, 0, 2, -2, 0, -2, 0]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 0, 0, -2, -2, 0, 0, 2, 2, -2, -2, 0, 0]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 0, -2, -2, 0, 0, 2, 2, 0, -2, 0, 0, -2]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 2, 2, 2, -2, -2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, 0, 2, 0, -2, 0, -2, 0, 0, 2, 0, 2, 0, -2, 0, -2]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 2, 0, 0, -2, -2, 0, 0, 0, 0, 2, 2, 0, 0, -2, -2]

f:=[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, 0, 0, 2, -2, 0, 0, -2, 0, 2, 2, 0, 0, -2, -2, 0]

f:=[0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, -2, 0, 0, 2, 2, 0, 0, -2, -2, 0, 0, 2]

f:=[0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, -2, 2, 0, 0, 2, -2, 0, 0, -2, 2, 0, 0]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 0, 2, 0, 2, 0, -2, 0, -2, -2, 0, -2, 0]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 2, -2, 0, 0, 0, 0, 0, 0, 0, 0, -2, 2, -2, 2]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 0, 2, -2, 0, 0, -2, 2, 0, -2, 0, 0, 2]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 0, 0, -2, 2, 0, 0, 2, -2, -2, 2, 0, 0]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 0, 2, 0, -2, 0, -2, 0, 0, -2, 0, -2, 0, 2, 0, 2]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, -2, 2, -2, -2, 2, -2, 2, 0, 0, 0, 0, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 0, 0, -2, -2, 0, 0, 2, 0, -2, 2, 0, 0, 2, -2, 0]

f:=[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, -2, 0, 0, -2, 2, 0, 0, 0, 0, 2, -2, 0, 0, -2, 2]

f:=[0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, -2, 0, 2, 0, 2, 0, -2, 0, -2, 0, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 0, 0, 2, 2, 0, 0, -2, -2, -2, -2, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 0, -2, 2, 0, 0, 2, -2, 0, -2, 0, 0, 2]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 2, -2, -2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, 2, 2]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 0, -2, 0, 2, 0, 2, 0, -2, -2, 0, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 2, 0, 0, -2, -2, 0, 0, 0, 0, -2, -2, 0, 0, 2, 2]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, 0, 0, -2, -2, 0, 0, 2, 0, 2, -2, 0, 0, -2, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 2, -2, -2, -2, -2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, 0, -2, 0, -2, 0, 2, 0, 0, 2, 0, -2, 0, -2, 0, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 0, 2, 2, 0, 0, -2, -2, 0, -2, 0, 0, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 0, 0, 2, -2, 0, 0, -2, 2, -2, 2, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 0, 2, 0, -2, 0, -2, 0, 2, -2, 0, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, -2, -2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, 2, 2, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 0, 0, 2, -2, 0, 0, -2, 0, -2, -2, 0, 0, 2, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, -2, 0, 0, -2, 2, 0, 0, 0, 0, -2, 2, 0, 0, 2, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 0, -2, 0, -2, 0, 2, 0, 0, -2, 0, 2, 0, 2, 0, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, -2, -2, 2, -2, 2, 2, -2, 0, 0, 0, 0, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 2, 0, 2, 0, -2, 0, -2, 0, -2, 0, -2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 2, 2, 0, 0, -2, -2, 0, 0, -2, -2, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 2, 0, 0, 2, -2, 0, 0, -2, -2, 0, 0, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 2, 2, 2, 0, 0, 0, 0, -2, -2, -2, -2, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0] Spec(G):=[2, 0, 2, 0, 0, 2, 0, 2, -2, 0, -2, 0, 0, -2, 0, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 2, 0, 0, 0, 0, 2, 2, -2, -2, 0, 0, 0, 0, -2, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1] Spec(G):=[2, 0, 0, 2, 0, 2, 2, 0, -2, 0, 0, -2, 0, -2, -2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0] Spec(G):=[2, 0, 0, -2, 2, 0, 0, -2, -2, 0, 0, 2, -2, 0, 0, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0] Spec(G):=[2, -2, 0, 0, 2, -2, 0, 0, -2, 2, 0, 0, -2, 2, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0] Spec(G):=[2, 0, 2, 0, 0, -2, 0, -2, -2, 0, -2, 0, 0, 2, 0, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0] Spec(G):=[2, -2, 2, -2, 0, 0, 0, 0, -2, 2, -2, 2, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0] Spec(G):=[2, 0, 0, -2, 0, -2, 2, 0, -2, 0, 0, 2, 0, 2, -2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1] Spec(G):=[2, -2, 0, 0, 0, 0, 2, -2, -2, 2, 0, 0, 0, 0, -2, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0] Spec(G):=[2, 0, -2, 0, 2, 0, -2, 0, -2, 0, 2, 0, -2, 0, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0] Spec(G):=[2, 2, 0, 0, 0, 0, -2, -2, -2, -2, 0, 0, 0, 0, 2, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0] Spec(G):=[2, 0, 0, -2, 0, 2, -2, 0, -2, 0, 0, 2, 0, -2, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0] Spec(G):=[2, 2, -2, -2, 0, 0, 0, 0, -2, -2, 2, 2, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1] Spec(G):=[2, 0, -2, 0, 0, 2, 0, -2, -2, 0, 2, 0, 0, -2, 0, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0] Spec(G):=[2, 0, 0, 2, 0, -2, -2, 0, -2, 0, 0, -2, 0, 2, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0] Spec(G):=[2, -2, 0, 0, 0, 0, -2, 2, -2, 2, 0, 0, 0, 0, 2, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0] Spec(G):=[2, 0, -2, 0, 0, -2, 0, 2, -2, 0, 2, 0, 0, 2, 0, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1] Spec(G):=[2, -2, -2, 2, 0, 0, 0, 0, -2, 2, 2, -2, 0, 0, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0] Spec(G):=[2, 0, 2, 0, -2, 0, -2, 0, -2, 0, -2, 0, 2, 0, 2, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0] Spec(G):=[2, 2, 0, 0, -2, -2, 0, 0, -2, -2, 0, 0, 2, 2, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1] Spec(G):=[2, 0, 0, 2, -2, 0, 0, -2, -2, 0, 0, -2, 2, 0, 0, 2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0] Spec(G):=[2, 0, 0, -2, -2, 0, 0, 2, -2, 0, 0, 2, 2, 0, 0, -2]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1] Spec(G):=[2, -2, 0, 0, -2, 2, 0, 0, -2, 2, 0, 0, 2, -2, 0, 0]

f:=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1] Spec(G):=[2, 0, -2, 0, -2, 0, 2, 0, -2, 0, 2, 0, 2, 0, -2, 0]

Appendix II: All Boolean functions of order 16.

Boolean function *f*	**Eigenvalues**
(0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)	$-2^4,0^8,2^4$
(0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0)	$-1^{12},3^4$
(0,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0)	$-3^2,-1^6,1^6,3^2$
(0,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0)	$-1^{12},3^4$
(0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0)	$-2^6,0^6,2^2,4^2$
(0,1,1,0,0,1,1,0,0,0,0,0,0,0,0,0)	$-4^2,0^{12},4^2$
(0,1,1,0,0,0,1,0,0,0,0,0,0,1,0,0)	$-4,-2^4,0^6,2^4,4$
(0,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0)	$-3^2,-1^8,1^4,5^2$
(0,1,1,1,1,0,0,0,1,0,0,0,0,0,0,0)	$-3^2,-1^8,1^4, 5^2$
(0,1,1,1,1,0,0,0,1,0,0,0,0,0,0,0)	$-3^3.-1^6,1^4,3^2,5$
(0,1,1,0,1,0,0,1,0,1,0,0,0,0,0,0)	$-5,-3,-1^6,1^6,3,5$
(0,1,0,1,1,0,0,0,1,0,0,0,0,0,1,0)	$-3^5,1^{10},5$
(0,1,1,1,1,1,0,0,1,0,0,0,0,0,0,0)	$-4,-2^5,0^6, 2^2,4,6$
(0,1,1,1,1,0,0,0,1,0,0,0,1,0,0,0)	$-2^9,2^6,6$
(0,1,1,1,1,0,0,0,1,0,0,0,0,1,0,0)	$-4^2,-2^3,2^4,0^6,6$
(0,1,0,0,1,0,1,0,1,0,1,0,0,1,0,0)	$-6,-2^3,0^8,2^3,6$
(0,1,0,0,0,0,1,1,0,0,1,1,1,0,0,0)	$-2^6,0^8,6^2$
(0,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0)	$-1^{14},7^2$
(0,1,1,1,1,1,1,0,1,0,0,0,0,0,0,0)	$-3^2,-1^7,1^4,5,7$
(0,1,1,1,1,1,0,0,1,1,0,0,0,0,0,0)	$-5,-1^{11},3^3,7$
(0,1,1,1,1,1,0,0,1,0,1,0,0,0,0,0)	$-3^4,-1^5,1^4,3^2,7$
(0,1,1,1,1,1,0,0,1,0,0,0,0,0,0,1)	$-5,-3^2,-1^5,1^6,3,7$
(0,0,0,0,0,0,0,0,1,0,1,1,1,1,1,1)	$-7,-1^7,1^7,7$
(0,1,1,1,1,1,1,1,0,1,0,0,0,0,0,0)	$-2^7,0^7,6,8$
(0,1,1,1,1,1,1,0,1,1,0,0,0,0,0,0)	$-4,-2^6,0^5,2^2,4,8$
(0,1,1,1,1,1,1,0,1,0,0,0,0,0,0, 1)	$-4^3,0^{11},4,8$
(0,1,1,1,1,1,0,0,1,1,0,0,0,0,1,0)	$-6,-2^4,0^7,2^3,8$
(0,1,0,1,1,0,1,0,1,0,1,0,0,1,0,1)	$-8,0^{14},8$
(0,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0)	$-3^3,-1^8,1^3,5,9$
(0,1,1,1,1,1,1,0,1,0,1,1,0,0,0,0)	$-3^4,-1^6,1^3,3^2,9$
(0,1,1,1,1,1,1,0,1,1,0,0,0,0,1,0)	$-5,-3^2,-1^6,1^5,3,9$
(0,1,1,0,1,0,1,1,1,0,1,1,0,0,0,1)	$-3^6,1^9,9$
(0,1,1,0,0,1,1,1,1,0,0,1,1,0,0,1)	$-7,-1^8,1^6,9$
(0,1,0,1,1,1,1,0,1,1,0,1,0,0,0,1)	$-3^6,1^9,9$
(0,1,1,1,1,1,1,1,1,1,0,1,0,0,0,0)	$-4,-2^6,0^6,2,4,10$
(0,1,1,1,1,1,1,0,1,1,0,1,0,1,0,0)	$-2^{10},2^5,10$
(0,1,1,1,1,1,1,0,1,1,0,0,0,0,1,1)	$-6,-2^4,0^8,2^2,10$
(0,1,1,1,1,1,1,0,1,0,1,0,1,0,0 ,1)	$-4^2,-2^4,0^6,2^3,10$
(0,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0)	$-5,-1^{12},3^2,11$
(0,1,1,1,1,1,1,1,1,1,1,0,1,0,0,0)	$-3^4,-1^6,1^4,3,11$
(0,1,1,1,1,1,1,0,1,1,1,0,0,0,1,1)	$-5,-3^2,-1^6,1^6,11$
(0,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0)	$-4,-2^6,0^6,2,12$
(0,1,1,1,1,1,1,0,1,1,1,0,0,1,1,1)	$-4^3,0^{12},12$
(0,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0)	$-3^3,-1^8,1^4,13$
(0,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1)	$-2^7,0^8,14$
(0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	$-1^{15},15$

Appendix III: The number of $C_{60-k}B_k$ molecules.

$k, 60-k$	**Number of hetero-fullerenes** $C_{60-k}B_k$
0,60	1
1,59	1
2,58	37
3,57	577
4,56	8236
5,55	91030
6,54	835476
7,53	6436782
8,52	42650532
9,51	246386091
10,50	1256602779
11,49	5711668755
12,48	23322797475
13,47	86114390460
14,46	289098819780
15,45	886568158468
16,44	2493474394140
17,43	6453694644705
18,42	15417163018725
19,41	34080036632565
20,40	69864082608210
21,39	133074428781570
22,38	235904682814710
23,37	389755540347810
24,36	600873146368170
25,35	865257299572455
26,34	1164769471671687
27,33	1466746704458899
28,32	1728665795116244
29,31	1907493251046152
30,30	1971076398255692

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WEIGHTED STATISTICAL CONVERGENCE OF REAL VALUED SEQUENCES

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Abstract. Functions defined in the form "$g : \mathbb{N} \to [0, \infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{g(n)} = 0$" are called weight functions. Using the weight function, the concept of weighted density, which is a generalization of natural density, was defined by Balcerzak, Das, Filipczak and Swaczyna in the paper "Generalized kinsd of density and the associated ideals", Acta Mathematica Hungarica 147(1) (2015), 97-115.

In this study, the definitions of g-statistical convergence and g-statistical Cauchy sequence for any weight function g are given and it is proved that these two concepts are equivalent. Also, some inclusions of the sets of all weight g_1-statistical convergent and weight g_2-statistical convergent sequences for g_1, g_2 which have the initial conditions are given.

Keywords: weight functions; natural density; statistical convergent sequences.

1. Introduction

In [5], Fast introduced the concept of statistical convergence. In [15], Schoenberg gave some basic properties of statistically convergence and also studied the concept as a summability method. After this works many Mathematician have used these concept in their studies [8, 9, 10, 11]. In [2, 3], the authors proposed a modified version of density by replacing n by n^α where $0 < \alpha \leq 1$. In [1], the authors defined a more general kind of density by replacing n^α by a function $g : \mathbb{N} \to [0, \infty)$ with $\lim_{n\to\infty} g(n) = \infty$. In this paper, we will study the weighted g-statistically convergence concept.

Let K be a subset of natural numbers. Natural density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K(n)|$$

where $K(n) = \{k \leq n : k \in K\}$ and the vertical bars denotes the number of elements of $K(n)$.

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Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. Let us remember that the definition of density of weight $g(n)$.

Definition 1.1. The density of weight g defined by the formula

$$d_g(A) = \lim_{n\to\infty} \frac{|A(n)|}{g(n)}$$

for $A \subset \mathbb{N}$ [1, 4].

After the study [1], the concept of g-density was applied to various problems related to sequences and interesting results were obtained in [4, 7, 12, 13, 14].

Basically in this study, it will be shown that the results given in [6] can be re-examined by using g-density.

In this paper, we are concerned with the subsets of natural numbers having weight $g(n)$ density zero. To facilitate this, we have introduced the following notation: If x is a sequence such that x_k satisfies property P for all k except a set of weight $g(n)$ density zero, then we say that x_k satisfies P for (weight g almost all k) and it is denoted by $(g - a.a.k)$ for simplicity.

Definition 1.2. Let $x = (x_k)$ be a real valued sequence. x is weight g-statistical convergent to the number L if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - L| < \varepsilon \quad (g - a.a.k)$. In this case we write $g - st - \lim x_k = L$.

C_g^{st} denotes the set of all weight g-statistical convergent sequences.

If we take the function $g(n) = n$ we obtain the usual statistical convergence.

It is clear that every convergent sequence is also weight g-statistical convergent. But the converse is not true in general.

Example 1.1. Let us define the function $g(n) = 2n$ and the sequence as

$$x_k = \begin{cases} 3, & k = m^2, \quad m \in \mathbb{N}, \\ 0, & k \neq 0. \end{cases}$$

Then $|k \leq n : x_k \neq 0| \leq \sqrt{n}$. So, $g - st - lim x_k = 0$.

Theorem 1.1. *If the sequence (x_n) is weight-g-statistical convergent to L then there is a set $K = \{k_1 < k_2 < ...\}$ such that $d_g(K) = d_g(\mathbb{N})$ and $\lim_{n\to\infty} x_{k_n} = L$.*

Proof. Let us assume that $g - st - lim x_k = L$. Take $K_i := \{n \in \mathbb{N} : |x_n - L| < \frac{1}{i}\}$, $(i = 1, 2, ...)$. Then by definition we have $d_g(K_i^c) = 0$ and it is clear that $d_g(K_i) = d_g(\mathbb{N})$, $(i = 1, 2, ...)$. Also it is easy to control that

$$... \subset K_{i+1} \subset K_i \subset ... \subset K_2 \subset K_1 \tag{1.1}$$

Let $\{T_j\}_{j\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers. Let choose an arbitrary number $a_1 \in K_1$. By (1.1) we can choose an element $a_2 \in K_2$, $a_2 > a_1$ such that for each $n \geq a_2$ we have $\frac{K_2(n)}{g(n)} > T_2$. Moreover choose $a_3 > a_2$, $a_3 \in K_3$ such that for each $n \geq a_3$ we have $\frac{K_3(n)}{g(n)} > T_3$. If we proceed in this way we obtain a sequence $a_1 < a_2... < a_i < ...$ of positive integers such that

$$a_i \in K_i,\ (i = 1, 2, ...) \text{ and } \frac{K_i(n)}{g(n)} > T_i \tag{1.2}$$

for each $n \geq a_i$, $i = 1, 2, ...$

Let us establish the set K as follows: each natural number of the interval $[1, a_1]$ belong to K, moreover, any natural number of the interval $[a_i, a_{i+1}]$ belongs to K if and only if it belongs to K_i $(i = 1, 2, ...)$. From (1.1) and (1.2) we have

$$\frac{K(n)}{g(n)} \geq \frac{K_i(n)}{g(n)} > T_i$$

for each n, $a_i \leq n < a_{i+1}$. By last inequality it is clear that $\overline{d}_g(K) = \infty$.

Let $\varepsilon > 0$, and choose i such that $\frac{1}{i} < \varepsilon$. Let $n \geq a_i$, $n \in K$. There exists a number $t \geq i$ such that $a_t \leq n < a_{t+1}$. But from the definition of K, $n \in K_t$. Thus $|x_n - L| < \frac{1}{t} \leq \frac{1}{i} < \varepsilon$. Hence, $\lim_{n\to\infty} x_{k_n} = L$. $\square$

Remark 1.1. The converse of Theorem 1.1 is not true.

Example 1.2. Let us consider the sequence

$$(x_k) := \begin{cases} 1, & k = n^2, \\ 0, & k \neq n^2, \end{cases}$$

and $g(n) = n^{1/4}$. It is clear that the set $K = \{k : k = n^2, n \in \mathbb{N}\} \subset \mathbb{N}$ has the property $\overline{d}_g(K) = \infty$. But $g - st - \lim x_k \neq 1$.

Let us note that every statistical convergent sequence is also weight-g-statistical convergent to the same number. But the converse of this situation is not true.

Example 1.3. Let $a_k = 2^{2^k}$, and

$$g(n) := \begin{cases} a_{2k}, & n \in [a_k, a_{k+1}),\ k = 1, 2, ... \\ 1, & n < 4. \end{cases}$$

Let $A_k := \{n \in \mathbb{N} : a_k \leq n < 2a_k\}$ and $A := \cup_{k\geq 1} A_k$. Let us take account the sequence

$$x_n := \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

It is clear that $\frac{1}{2}a_k \leq |A_k| \leq a_k$. Let us check that $x_n \nrightarrow 0(st)$. If we put $m_k = \max A_k$, we obtain

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n} = \frac{|\{k \leq n : x_k \in A\}|}{n} = \frac{|A|}{m_k} \geq \frac{|A_k|}{m_k} \geq \frac{\frac{1}{2}a_k}{2a_k} = \frac{1}{4}$$

for all $k \geq 1$.

Moreover, $g - st - \lim x_k = 0$. For sufficiently large n, we have

$$\begin{aligned} \frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{g(n)} &= \frac{|\{k \leq n : x_k \in A\}|}{g(n)} = \frac{|A|}{g(n)} \\ &= \frac{|\{k \leq m_k : x_k \in A\}|}{g(m_k)} \\ &\leq \frac{|A_k|}{a_{2k}} \leq \frac{a_k}{a_{2k}} \to 0. \end{aligned}$$

Definition 1.3. Let $x = (x_k)$ be a real valued sequence. x is weight g-statistical Cauchy sequence if for each $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - x_N| \geq \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - x_N| < \varepsilon \quad (g - a.a.k)$. In this case we write x is weight g-Cauchy sequence.

Lemma 1.1. *The following statements are equivalent:*

(i) x is a weight g-statistically convergent sequence,

(ii) x is a weight g-statistically Cauchy sequence,

(iii) x is a sequence for which there is a convergent sequence y such that $x_k = y_k$ $(g - a.a.k)$.

Proof. $(i) \Rightarrow (ii)$ Let us assume that x is a weight g-statistical convergent sequence. Suppose $\varepsilon > 0$ and $g - st - \lim x = L$. Then $|x_k - L| < \frac{\varepsilon}{2}$ $(g - a.a.k)$ holds.

If we choose a natural number N such that $|x_N - L| < \frac{\varepsilon}{2}$, then we have

$$|x_k - x_N| < |x_k - L| + |x_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ (g - a.a.k).$$

Hence, x is a weight g-statistical Cauchy sequence.

$(ii) \Rightarrow (iii)$ Let us assume that x is a weight g-statistical Cauchy sequence. Choose $N(1)$ such that the interval $I = [x_{N(1)} - 1, x_{N(1)} + 1]$ contains x_k $(g - a.a.k)$. Also apply (ii) to choose M such that $I' = [x_M - \frac{1}{2}, x_M + \frac{1}{2}]$ contains x_k $(g - a.a.k)$. We claim that

$$I_1 = I \cap I' \text{ contains } x_k \ (g - a.a.k),$$

for

$$\{k \le n : x_k \notin I \cap I'\} = \{k \le n : x_k \notin I\} \cup \{k \le n : x_k \notin I'\}.$$

Thus,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{g(n)} |\{k \le n : x_k \notin I \cap I'\}| \le \\ \le \quad &\lim_{n\to\infty} \frac{1}{g(n)} |\{k \le n : x_k \notin I\}| + \lim_{n\to\infty} \frac{1}{g(n)} |\{k \le n : x_k \notin I'\}| = 0. \end{aligned}$$

So, I_1 is closed interval of length less than or equal to 1 and contains x_k $(g - a.a.k)$. Now we continue by choosing $N(2)$ such that $I'' = [x_{N(2)} - \frac{1}{4}, x_{N(2)} + \frac{1}{4}]$ contains x_k $(g - a.a.k)$, by the previously argument $I_2 = I_1 \cap I''$ contains x_k $(g - a.a.k)$, and I_2 has length less than or equal to $\frac{1}{2}$. Proceeding inductively we construct a sequence$\{I_m\}_{m=1}^{\infty}$ of closed intervals such that for each m, $I_{m+1} \subseteq I_m$, and the length of I_m is not greater than 2^{1-m}, and $x_k \in I_m$ $(g - a.a.k)$. From the Nested Interval Theorem there is a number α such that $\alpha = \cap_{m=1}^{\infty} I_m$. If we use $x_k \in I_m$ $(g - a.a.k)$, we can choose an increasing positive sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{g(n)} |\{k \le n : x_k \notin I_m\}| < \frac{1}{g(m)} \ if \ n > \ T_m. \tag{1.3}$$

Next define a subsequence z of x consisting of all terms x_k such that $k > T_1$ and if $T_m < k \le T_{m+1}$ then $x_k \notin I_m$.

Now define the sequence y by

$$y_k = \begin{cases} \alpha, & \text{if } x_k \text{ is a term of } z, \\ x_k, & \text{otherwise.} \end{cases}$$

Then $\lim y_k = \alpha$; for , if $\varepsilon > \frac{1}{g(m)} > 0$ and $k > T_m$ then either x_k is a term of z, which means $y_k = \alpha$ or $y_k = x_k \in I_m$ and $|y_k - \alpha| \le$ length of $I_m < 2^{1-m}$. We also assert that $x_k = y_k$ $(g - a.a.k)$. To confirm this we observe that if $T_m < n < T_{m+1}$ then

$$\{k \le n : y_k \ne x_k\} \subseteq \{k \le n : x_k \notin I_m\}$$

so from (1.3)

$$\frac{1}{g(n)} |\{k \le n : y_k \ne x_k\}| \le \frac{1}{g(n)} |\{k \le n : x_k \notin I_m\}| < \frac{1}{g(m)}$$

is obtained. Thus, the limit as $n \to \infty$ is 0 and $x_k = y_k$ $(g - a.a.k)$.

$(iii) \Rightarrow (i)$ Let us assume that $x_k = y_k$ $(g - a.a.k)$ and $\lim y_k = L$. Suppose $\varepsilon > 0$. Then for each n,

$$\{k \le n : |x_k - L| > \varepsilon\} \subseteq \{k \le n : x_k \ne y_k\} \cup \{k \le n : |y_k - L| > \varepsilon\}$$

from the assumption $\lim y_k = L$, the second set contains a fixed number of integers, say $l = l(\varepsilon)$. So,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n:|x_k-L|>\varepsilon\}| &\le \lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n:x_k\ne y_k\}|+\\ &+ \lim_{n\to\infty}\frac{l}{g(n)}=0\end{aligned}$$

because $x_k = y_k$ $(g - a.a.k)$. Hence, $|x_k - L| \le \varepsilon$ $(g - a.a.k)$. So, the proof is complete. □

Corollary 1.1. *Let x be a real valued sequence. If $g - st - \lim x_k = L$, then x has a subsequence y such that $\lim y_k = L$.*

2. Inclusion Between Two $g - st-$Convergence

Let G denotes the set of all functions $g : \mathbb{N} \to [0, \infty)$ satisfying the condition $g(n) \to \infty$ and $\frac{n}{g(n)} \nrightarrow 0$. In this section, we will introduce some inclusions between various $g \in G$.

Lemma 2.1. *Let $g_1, g_2 \in G$ such that there exist M, $m > 0$ and $k_0 \in \mathbb{N}$ such that $m \le \frac{g_1(n)}{g_2(n)} \le M$ for all $n \ge k_0$. Then $C^{st}_{g_1}(x) = C^{st}_{g_2}(x)$.*

Proof. Suppose the sequence x is weight g_1-statistical convergence to L. This implies that for each $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{|\{k\le n:|x_k-L|\ge\varepsilon\}|}{g_1(n)}=0.$$

Together with the fact that $\frac{g_1(n)}{g_2(n)} \le M$, this implies that

$$\frac{|\{k\le n:|x_k-L|\ge\varepsilon\}|}{Mg_2(n)}\le\frac{|\{k\le n:|x_k-L|\ge\varepsilon\}|}{g_1(n)}.$$

for all $n \ge k_0$. This implies

$$\lim_{n\to\infty}\frac{|\{k\le n:|x_k-L|\ge\varepsilon\}|}{Mg_2(n)}\le\lim_{n\to\infty}\frac{|\{k\le n:|x_k-L|\ge\varepsilon\}|}{g_1(n)}=0.$$

From the hypothesis we obtain

$$\lim_{n\to\infty}\frac{|\{k\le n:|x_k-L|\ge\varepsilon\}|}{g_2(n)}=0.$$

Thus, the sequence x is weight g_2-statistical convergent to L. So, $C^{st}_{g_1}(x) \subset C^{st}_{g_2}(x)$. We can prove the iclusion $C^{st}_{g_2}(x) \subset C^{st}_{g_1}(x)$ by similar way. □

Lemma 2.2. *For each function $f \in G$ there exists a nondecreasing function $g \in G$ such that $C_f^{st}(x) = C_g^{st}(x)$. Moreover,*

$$g(n) \leq f(n) \tag{2.1}$$

for all $n \in \mathbb{N}$.

Proof. If f is nondecreasing, it is nclear. Otherwise, define the related function $g : \mathbb{N} \to [0, \infty)$ as follows. Let $a_1 = \min\{f(n) : n \in \mathbb{N}\}$, $i_1 = \max\{i \in \mathbb{N} : f(i) = a_1\}$ and $g(i) = a_1$ for $0 \leq i \leq i_1$. Next, let $a_2 = \min\{f(n) : n > i_1\}$, $i_2 = \max\{i \in N : f(i) = a_2\}$ and $g(i) = a_2$ for $i_1 < i \leq i_2$. Rest of the function g is established by induction.

Obviously, the function g is nondecreasing and $g(n) \to \infty$. By the construction, $g(n) \leq f(n)$, for all $n \in \mathbb{N}$. Hence $\frac{n}{f(n)} \leq \frac{n}{g(n)}$ for all n which implies that $\frac{n}{g(n)} \nrightarrow 0$. Thus $g \in G$.

Let (x_n) be a weight g-statistical convergent sequence to L. So, for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0$$

holds. From (2.1) we have following inequality

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \leq \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)}.$$

If we take limit when $n \to \infty$ we obtain $f - st - \lim x_k = L$. Thus, the inclusion $C_g^{st} \subset C_f^{st}$.

By construction, for each $n \in \mathbb{N}$ there exist $m \geq n$ such that $g(n) = g(m) = f(m)$. Suppose that $x_n \nrightarrow L$ $(g - st)$. Then there exists a, where $a \in \mathbb{R}^+ \cup \{+\infty\}$ and an inreasing sequence (n_i) of indices such that

$$\lim_{i\to\infty} \frac{|\{k \leq n_i : |x_k - L| \geq \varepsilon\}|}{g(n_i)} = a > 0.$$

For each $i \in \mathbb{N}$ we can find $m_i \geq n_i$ such that $g(n_i) = g(m_i) = f(m_i)$. Hence

$$\frac{|\{k \leq n_i : |x_k - L| \geq \varepsilon\}|}{g(n_i)} \leq \frac{|\{k \leq m_i : |x_k - L| \geq \varepsilon\}|}{f(m_i)}$$

holds. So, $x_n \nrightarrow L$ $(f - st)$. $\square$

Lemma 2.3. *Let $f \in G$ be such that $\frac{n}{f(n)} \to \infty$, L, ε real numbers with $\varepsilon > 0$. Then there exists a sequence (x_n) such that $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded but not convergent to zero.*

Proof. Firstly, let us assume that f is nondecreasing. Take to the smallest non negative integer, k_0, such that for $n \geq k_0$, $f(n) > 2$. Let us define a set $A \subset \mathbb{N}\backslash\{0, 1, 2, ...k_0 - 1\}$ inductively, deciding whether $n \geq k_0$ should belong to A or not. Let $n \notin A$ for all $n < k_0$. Suppose that $n \geq k_0$ and then we have defined $A(n)$. If $\frac{|A(n)|}{f(n+1)} < 1$ then let $n \in A$. Otherwise, let $n \notin A$. So, we construct the set A. From this construction and the condition $f(n) \to \infty$, A is infinite.

We assert that $\mathbb{N}\backslash A$ is also infinite. Let us assume that it is finite and choose $n_0 \in \mathbb{N}$ such that $n \in A$ for all $n \geq n_0$. Then, we have

$$\frac{n - n_0}{f(n+1)} \leq \frac{|A(n)|}{f(n+1)} < 1$$

for all $n \geq n_0$. But this is impossible because of the assumption, $\frac{n-n_0}{f(n+1)} \to \infty$. Now, we will show that $\frac{|A(n)|}{f(n)} < 2$ for each $n \geq k_0$. It is clear that if $n = k_0$ it is true. Suppose that $\frac{|A(n)|}{f(n)} < 2$ for a fixed $n \geq k_0$.

If $\frac{|A(n)|}{f(n+1)} < 1$, we have

$$\begin{aligned} \frac{|A(n+1)|}{f(n+1)} &= \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n+1)} \\ &\leq \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n)} \\ &\leq 1 + \frac{1}{2} < 2. \end{aligned}$$

If $\frac{|A(n)|}{f(n+1)} > 1$, then $n \notin A$ and so,

$$\frac{|A(n+1)|}{f(n+1)} = \frac{|A(n)|}{f(n+1)} \leq \frac{|A(n)|}{f(n)} < 2.$$

Now, let us define a sequence (x_n) as follows:

$$x_n := \begin{cases} n & n \in A \\ L & n \notin A \end{cases}$$

where $L \in \mathbb{R}$ is a fixed number. It is clear that the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded from the first part of this proof.

Now, we will show that the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is not convergent to 0. For this aim consider any $n \geq k_0$. We will find $m \geq n$ such that $\frac{|A(m)|}{f(m)} \geq 1$. If $\frac{|A(n)|}{f(n)} \geq 1$, put $m := n$. Otherwise, choose the smallest $m \geq n$ such that $m \in \mathbb{N}\backslash A$. Then $\frac{|A(m)|}{f(m+1)} \geq 1$ and so, $\frac{|A(m)|}{f(m)} \geq 1$. Thus, the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is not convergent to 0.

Now, let us back to the general case where $f \in G$ need not be nondecreasing. Then we assume the associated function $g \in G$ from Lemma 2.2. Note that $\frac{n}{g(n)} \to \infty$ since $\frac{n}{g(n)} \geq \frac{n}{f(n)}$ for all n and $\frac{n}{f(n)} \to \infty$. By the above reasons we obtain the respective set A for g. Thus, $\frac{|A(n)|}{g(n)} \nrightarrow 0$ and the sequence $(\frac{|A(n)|}{g(n)})$ is bounded. Then $\frac{|A(n)|}{f(n)} \nrightarrow 0$, and the sequence $\left(\frac{|\{k\leq n:|x_k-L|\geq\varepsilon\}|}{f(n)}\right)$ is bounded since $g(n) \leq f(n)$ for all $n \in \mathbb{N}$. $\square$

Theorem 2.1. *If g_1, g_2 belong to G such that $\frac{g_2(n)}{g_1(n)} \to \infty$ then $C_{g_1}^{st}(x) \subsetneq C_{g_2}^{st}(x)$. If $g \in G$ and $\frac{n}{g(n)} \to \infty$ then $C_g^{st}(x) \subsetneq C^{st}(x)$.*

Proof. To prove the first claim note that the inclusion $C_{g_1}^{st}(x) \subset C_{g_2}^{st}(x)$ follows from Lemma 2.1. Set $f := \sqrt{g_1.g_2}$. Then

$$\lim_{n\to\infty} \frac{f(n)}{g_1(n)} = \lim_{n\to\infty} \frac{g_2(n)}{f(n)} = \infty. \tag{2.2}$$

Also we have

$$\frac{n}{g_1(n)} = \frac{n}{g_2(n)}.\frac{g_2(n)}{g_1(n)} \to \infty.$$

So $\frac{n}{f(n)} = \sqrt{\frac{n^2}{g_1(n)g_2(n)}} \to \infty$. Hence f have the assumption of Lemma 2.3. Take the sequence (x_n) obtained in this lemma. Then $x_n \in C_{g_2}^{st}(x)$ but $x_n \notin C_{g_1}^{st}(x)$. Indeed, using (2.2) we have

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_2(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}.\frac{f(n)}{g_2(n)} \to 0$$

because $\left(\frac{|\{k\leq n:|x_k-L|\geq\varepsilon\}|}{f(n)}\right)_{n\in\mathbb{N}}$ is bounded from Lemma 2.3. Thus, $x_n \in C_{g_2}^{st}(x)$. To prove that $x_n \notin C_{g_1}^{st}(x)$ observe that

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\frac{f(n)}{g_1(n)}.$$

So, $x_n \notin C_{g_1}^{st}(x)$ because $\frac{|\{k\leq n:|x_k-L|\geq\varepsilon\}|}{f(n)} \nrightarrow 0$, and $\frac{f(n)}{g_1(n)} \to \infty$ from (2.2).

If we take $g_2(n) = n$, for all $n \in \mathbb{N}$, second assertion proved easily from the same way. $\square$

Corollary 2.1. *Let $0 < \alpha < \beta \leq 1$ and $g_1(n) = n^\alpha$, $g_2 = n^\beta$ for $n \in \mathbb{N}$. Then $C_{g_1}^{st}(x) \subsetneq C_{g_2}^{st}(x)$.*

Example 2.1. Let

$$g_1(n) = \begin{cases} n, & \text{for even } n \in \mathbb{N} \\ \sqrt{n}, & \text{for odd } n \in \mathbb{N} \end{cases}$$

and $g_2(n)=\sqrt{n}$ for $n\in\mathbb{N}$. It is clear that, $\limsup_{n\to\infty}\frac{g_1(n)}{g_2(n)}=\infty$. However, $C^{st}_{g_1}(x)=C^{st}_{g_2}(x)$. Indeed, construct a nondecreasing function $g\in G$ such that $C^{st}_g(x)=C^{st}_{g_1}(x)$, according to the method used in the proof of Lemma 2.1. Then it follows from simple calculations that g is given by

$$g(n)=\begin{cases}\sqrt{n+1} & \text{for even } n\in\mathbb{N}\\ \sqrt{n} & \text{for odd } n\in\mathbb{N}.\end{cases}$$

Obviously, $\frac{1}{2}\le\frac{g(n)}{g_2(n)}\le 2$ for all $n\ge 1$. Therefore, by Lemma 2.1 we have $C^{st}_g(x)=C^{st}_{g_1}(x)$.

Theorem 2.2. *There exists a function $g\in G$ such that C^{st}_g is different from $C^{st}_{n^\alpha}$ with $0<\alpha<1$.*

Proof. Let a_k and $g(n)$ defined as in Example 1.3. Let $A_k:=\{n\in\mathbb{N}: a_{k+1}-(a_{k+1})^{1/4}\le n<a_{k+1}\}$ and $A=\cup_{k\ge 2}A_k$. Let us take account the sequence

$$x_n=\begin{cases} n, & n\in A\\ 0, & n\notin A.\end{cases}$$

It is clear that $\frac{1}{2}(a_{k+1})^{1/4}\le|B_k|\le(a_{k+1})^{1/4}$. Let us check that $g-st-\lim x_k\neq 0$. For $k>0$ we have

$$\frac{|\{k\le a_{k+1}-1:|x_k-0|\ge\varepsilon\}|}{g(a_{k+1}-1)}\ge\frac{\frac{1}{2}|B_k|}{g(a_k)}\ge\frac{\frac{1}{4}(a_{k+1})^{1/4}}{(a_{k+1})^{1/4}}=\frac{1}{4},$$

so, $g-st-\lim x_k\neq 0$. Furthermore,

$$|\{k\le a_{k+1}:|x_k-0|\ge\varepsilon\}|\le(a_k)^{1/4}+(a_{k+1})^{1/4}\le 2(a_{k+1})^{1/4}$$

and so,

$$\frac{|\{k\le a_{k+1}:|x_k-0|\ge\varepsilon\}|}{(a_{k+1})^{1/3}}\le\frac{2(a_{k+1})^{1/4}}{(a_{k+1})^{1/3}}=2(a_{k+1})^{-1/12}\to 0,\ (k\to\infty)$$

holds.

Now, fix any $n\ge 4$ and choose a unique $k\in\mathbb{N}$ such that $n\in[a_k,a_{k+1})$. If $n<a_{k+1}-(a_{k+1})^{1/4}$ then

$$\begin{aligned}\frac{|\{k\le n:|x_k-0|\ge\varepsilon\}|}{n^{1/3}} &= \frac{|\{k\le a_k:|x_k-0|\ge\varepsilon\}|}{n^{1/3}}\\ &\le \frac{|\{k\le n:|x_k-0|\ge\varepsilon\}|}{(a_k)^{1/3}}\le 2(a_k)^{-1/12}.\end{aligned}$$

If $a_{k+1}-(a_{k+1})^{1/4}\le n<a_{k+1}$ then for $b>a>0$, the function

$$f(x):=\frac{a+x}{(b+x)^{1/3}},\ x\ge 0$$

is increasing, thus

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n^{1/3}} \leq \frac{|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon\}|}{(a_{k+1})^{1/3}}.$$

So, $x_n \in C^{st}_{n^{1/3}}(x)$.

Now, let $0 < \alpha < 1$, $\alpha \neq \frac{1}{3}$. If $\alpha < \frac{1}{3}$ then from Corollary 2.1 $C^{st}_{n^\alpha} \subsetneq C^{st}_{n^{1/3}}$ and $C^{st}_g \backslash C^{st}_{n^\alpha} \neq \emptyset$ because $C^{st}_g \backslash C^{st}_{n^{1/3}} \neq \emptyset$. If $\alpha > \frac{1}{3}$ then $C^{st}_{n^\alpha} \backslash C^{st}_g \neq \emptyset$. By the same way we can show that $x_n \in C^{st} \backslash C^{st}_g$. So $C^{st}_g \subsetneq C^{st}$. $\square$

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