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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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NEW INEQUALITIES OF OSTROWSKI TYPE FOR CO-ORDINATED CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS *

Muhammad Aamir Ali, Hüseyin Budak and Zhiyue Zhang

Abstract. In this paper, we have established new inequalities of Ostrowski type for co-ordinated convex function by using generalized fractional integral. We have also discussed some special cases of our established results.

Keywords: inequalities of Ostrowski type; convex function; generalized fractional integral.

1. Introduction

In 1938, A. Ostowski established the following fascinating integral inequality [11].

Theorem 1.1. [11] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) whose derivative is bounded on (a,b), i.e., $\left\|f^{'}(t)\right\|_{\infty}:=\sup\left|f^{'}(t)\right|<\infty$, for all $t\in(a,b)$. Then we have the following integral inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})}{(b-a)^2}\right](b-a)\left\|f^{'}\right\|_{\infty},\tag{1.1}$$

for all $x\in[a,b]$. The $\frac{1}{4}$is the best possible.

The inequality (1.1) can be rewritten in equivalent form as:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left\|f^{'}\right\|_{\infty}.\tag{1.2}$$

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Since 1938 when A. Ostrowski proved his famous inequality, (see, [11]), many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules have been considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [1]-[4], [6]-[15] and the references therein.

Let us consider now a bidimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}^2$ is said to be convex on Δ if the following inequality holds:

$$f(tx+(1-t)z, ty+(1-t)w) \leq tf(x, y)+(1-t)f(z, w), \ \forall\, (x, y),\ (z, w) \in \Delta \text{ and } t \in [0, 1]. \tag{1.3}$$

The mapping f is said to be concave on co-ordinates Δ if (1.3) holds in reversed direction.

A formal definition of co-ordinated convex (concave) functions may be expressed as:

Definition 1.1. [17]A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$ and $t, s \in [0, 1]$, if it satisfies the following inequality:

$$\begin{aligned} & f(tx + (1-t)\ y, su + (1-s)\ v) \\ \leq\ & ts\ f(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v). \end{aligned} \tag{1.4}$$

The mapping f is a co-ordinated concave on Δ if the inequality (1.4) holds in reversed direction for all $t, s \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

In [5], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the* *following inequalities:*

$$(1.5)\qquad f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ dydx$$
$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right.$$
$$\left.+\quad \frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right]$$
$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp. The inequalities in (1.5) holds in reversed direction if the mapping f is a co-ordinated concave.

In [10], Latif et al. established following Ostrowski type inequalities for co-ordinated convex functions:

Theorem 1.3. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be twice partial differentiable mapping on Δ° with $a<b,\ c<d,\ a,c\geq 0$ such that $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on co-ordinates on Δ and $\left|\frac{\partial^2 f}{\partial s\partial t}\right|\leq M,\ (x,y)\in\Delta$, then the following inequality holds:*

$$(1.6)\qquad \left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dvdu-A_1\right|$$
$$\leq\ M\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left[\frac{(y-c)^2+(d-y)^2}{2(d-c)}\right],$$

where
$$A_1=\frac{1}{d-c}\int_c^d f(x,v)dv+\frac{1}{b-a}\int_c^d f(u,y)dy.$$

Theorem 1.4. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be twice partial differentiable mapping on Δ° with $a<b,\ c<d,\ a,c\geq 0$ such that $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex*

on co-ordinates on Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\left|\frac{\partial^2 f}{\partial s\partial t}(x,y)\right|\leq M$, $(x,y)\in\Delta$, *then the following inequality holds:*

$$\begin{aligned}(1.7)\qquad &\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dvdu-A_1\right|\\ \leq\ & \frac{M}{(1+p)^{\frac{2}{p}}}\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left[\frac{(y-c)^2+(d-y)^2}{2(d-c)}\right],\end{aligned}$$

where A_1 *is defined in Theorem 1.3.*

Theorem 1.5. *Let* $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ *be twice partial differentiable mapping on* Δ° *with* $a<b$, $c<d$, $a,c\geq 0$ *such that* $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on co-ordinates on* Δ, $q\geq 1$ *and* $\left|\frac{\partial^2 f}{\partial s\partial t}(x,y)\right|\leq M$, $(x,y)\in\Delta$, *then the following inequality holds:*

$$\begin{aligned}(1.8)\qquad &\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dvdu-A_1\right|\\ \leq\ & \frac{M}{4}\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left[\frac{(y-c)^2+(d-y)^2}{2(d-c)}\right],\end{aligned}$$

where A_1 *is defined in Theorem 1.3.*

Theorem 1.6. *Let* $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ *be twice partial differentiable mapping on* Δ° *with* $a<b$, $c<d$, $a,c\geq 0$ *such that* $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is concave on co-ordinates on* Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\begin{aligned}(1.9)\qquad &\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)dvdu-A_1\right|\\ \leq\ & \frac{1}{(1+p)^{\frac{2}{p}}(b-a)(d-c)}\left[(x-a)^2\left\{(y-c)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|\right.\right.\\ &\left.+(d-y)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right|\right\}\\ &+(b-x)^2\left\{(y-c)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right|\right.\\ &\left.\left.+(d-y)^2\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right|\right\}\right],\end{aligned}$$

where A_1 *is defined in Theorem 1.3.*

In [9], Latif and Hussain established following Ostrowski type inequalities for co-ordinated convex function by using fractional integral:

Theorem 1.7. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b,\ c < d,\ a, c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is convex on co-ordinates on Δ and $\left|\frac{\partial^2 f}{\partial s \partial t}\right| \leq M,\ (x,y) \in \Delta$, then the following inequality holds for fractional integrals, with $\alpha,\ \beta > 0$:*

$$\left|\frac{\left[(x-a)^\alpha + (b-x)^\alpha\right]\left[(y-c)^\beta + (d-y)^\beta\right]}{(b-a)(d-c)} f(x,y) + A_2\right| \tag{1.10}$$
$$\leq \frac{(\alpha\beta + 2\alpha + 2\beta + 4)\left[(x-a)^\alpha + (b-x)^\alpha\right]\left[(y-c)^\beta + (d-y)^\beta\right]}{(b-a)(d-c)(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)} M,$$

where

$$\begin{aligned} A_2 \;=\; & \frac{\Gamma(\alpha+1)\Gamma(\beta+a)}{(b-a)(d-c)}\left[J^{\alpha,\beta}_{x-,y-} f(a,c) + J^{\alpha,\beta}_{x-,y+} f(a,d) + J^{\alpha,\beta}_{x+,y-} f(b,c)\right. \\ & \left. + J^{\alpha,\beta}_{x+,y+} f(b,d)\right] - \frac{\left[(x-a)^\alpha + (b-x)^\alpha\right]\Gamma(\beta+1)}{(b-a)(d-c)}\left[J^{\beta}_{y-} f(x,c) + J^{\beta}_{y+} f(x,d)\right] \\ & - \frac{\left[(y-c)^\beta + (d-y)^\beta\right]\Gamma(\alpha+1)}{(b-a)(d-c)}\left[J^{\alpha}_{x-} f(a,y) + J^{\alpha}_{x+} f(b,y)\right]. \end{aligned}$$

Theorem 1.8. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b,\ c < d,\ a, c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is convex on co-ordinates on Δ, $p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ and $\left|\frac{\partial^2 f}{\partial s \partial t}(x,y)\right| \leq M,\ (x,y) \in \Delta$, then the following inequality holds for fractional integrals, with $\alpha,\ \beta > 0$:*

$$\left|\frac{\left[(x-a)^\alpha + (b-x)^\alpha\right]\left[(y-c)^\beta + (d-y)^\beta\right]}{(b-a)(d-c)} f(x,y) + A_2\right| \tag{1.11}$$
$$\leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \frac{\left[(x-a)^\alpha + (b-x)^\alpha\right]\left[(y-c)^\beta + (d-y)^\beta\right]}{(b-a)(d-c)} M,$$

where A_2 is defined in Theorem 1.7.

Theorem 1.9. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b,\ c < d,\ a, c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is convex on co-ordinates on Δ, $q \geq 1$ and $\left|\frac{\partial^2 f}{\partial s \partial t}(x,y)\right| \leq M,\ (x,y) \in \Delta$, then the following inequality holds for fractional integrals, with $\alpha,\ \beta > 0$:*

$$\left|\frac{\left[(x-a)^\alpha + (b-x)^\alpha\right]\left[(y-c)^\beta + (d-y)^\beta\right]}{(b-a)(d-c)} f(x,y) + A_2\right| \tag{1.12}$$
$$\leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}} \frac{\left[(x-a)^\alpha + (b-x)^\alpha\right]\left[(y-c)^\beta + (d-y)^\beta\right]}{(b-a)(d-c)} M,$$

where A_2 is defined in Theorem 1.7.

Theorem 1.10. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b,\ c < d,\ a,c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is concave on co-ordinates on Δ, $p,q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds for fractional integrals with $\alpha,\ \beta > 0$:*

$$\left|\frac{\left[(x-a)^\alpha+(b-x)^\alpha\right]\left[(y-c)^\beta+(d-y)^\beta\right]}{(b-a)(d-c)}f(x,y)+A_2\right| \tag{1.13}$$
$$\leq \frac{1}{(1+\alpha p)^{\frac{1}{p}}(1+\beta p)^{\frac{1}{p}}(b-a)(d-c)}$$
$$\times\left[(x-a)^{\alpha+1}\left\{(y-c)^{\beta+1}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|\right.\right.$$
$$\left.+(d-y)^{\beta+1}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right|\right\}$$
$$+(b-x)^{\alpha+1}\left\{(y-c)^{\beta+1}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right|\right.$$
$$\left.\left.+(d-y)^{\beta+1}\left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right|\right\}\right],$$

where A_2 is defined in Theorem 1.7.

In [16], Sarikaya and Ertugral defined a new left-sided and right-sided generalized fractional integrals as follows:

$$_{a+}I_\varphi f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt, \quad x > a \tag{1.14}$$

$$_{b-}I_\varphi f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt, \quad x < b \tag{1.15}$$

respectively, where $\varphi : [0,\infty) \to [0,\infty)$ a function which satisfies $\int_0^1 \frac{\varphi(t)}{t}dt < \infty$.

In [17], Yildirim et al. defined generalized fractional integrals for two variable functions as follows:

Definition 1.2. [17] Let $f \in L_1([a,b]\times[c,d])$. The generalized fractional integrals $_{a+,c+}I_{\varphi,\psi}$, $_{a+,d-}I_{\varphi,\psi}$, $_{b-,c+}I_{\varphi,\psi}$ and $_{b-,d-}I_{\varphi,\psi}$ are defined by

$$_{a+,c+}I_{\varphi,\psi}f(x,y) = \int_a^x\int_c^y \frac{\varphi(x-t)}{x-t}\frac{\psi(y-s)}{y-s}f(t,s)dsdt, \quad x > a,\ y > c, \tag{1.16}$$

$$(1.17)\qquad {}_{a+,d-}I_{\varphi,\psi}f(x,y)=\int_a^x\int_y^d \frac{\varphi(x-t)}{x-t}\frac{\psi(s-y)}{s-y}f(t,s)dsdt,\quad x>a,\ y<d,$$

$$(1.18)\qquad {}_{b-,c+}I_{\varphi,\psi}f(x,y)=\int_x^b\int_c^y \frac{\varphi(t-x)}{t-x}\frac{\psi(y-s)}{y-s}f(t,s)dsdt,\quad x<b,\ y>c,$$

and

$$(1.19)\qquad {}_{b-,d-}I_{\varphi,\psi}f(x,y)=\int_x^b\int_y^d \frac{\varphi(t-x)}{t-x}\frac{\psi(s-y)}{s-y}f(t,s)dsdt,\quad x<b,\ y<d.$$

Similar the above definitions, we can give the following integrals:

$$(1.20)\qquad {}_{a+}I_{\varphi}f(x,c)=\int_a^x \frac{\varphi(x-t)}{x-t}f(t,c)dt,\quad x>a$$

$$(1.21)\qquad {}_{a+}I_{\varphi}f(x,d)=\int_a^x \frac{\varphi(x-t)}{x-t}f(t,d)dt,\quad x>a,$$

$$(1.22)\qquad {}_{c+}I_{\psi}f(a,y)=\int_c^y \frac{\psi(y-s)}{y-s}f(a,s)ds,\quad y>c,$$

and

$$(1.23)\qquad {}_{d-}I_{\psi}f(b,y)=\int_y^d \frac{\psi(s-y)}{s-y}f(b,s)ds,\quad y<d.$$

The main objective of this paper is to establish new Ostrowski type inequalities for co-ordinated convex functions similar to [9, 10] by using generalized fractional integrals.

2. Main Results

Throughout this section, for clarity, we have defined

$$\begin{aligned}\Lambda_1(g) &= \int_0^g \frac{\varphi((x-a)t)}{t}dt,\quad \Lambda_2(g)=\int_0^g \frac{\varphi((b-x)t)}{t}dt\\ \Psi_1(h) &= \int_0^h \frac{\psi((y-c)s)}{s}ds,\quad \Psi_2(h)=\int_0^h \frac{\psi((d-y)s)}{s}ds.\end{aligned}$$

Lemma 2.1. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a twice differentiable mapping on Δ° with $a < b,\ c < d$. If $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, $a, c \geq 0$, then following identity holds for all $(x,y) \in \Delta$:*

$$
\begin{aligned}
(2.1)\quad & \frac{[\Lambda_2(1)+\Lambda_1(1)]\,[\Psi_2(1)+\Psi_1(1)]}{(b-a)(d-c)} f(x,y) + A \\
= \ & \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)a, sy+(1-s)c)\,dsdt \\
& - \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)a, sy+(1-s)d)\,dsdt \\
& - \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)b, sy+(1-s)c)\,dsdt \\
& + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)b, sy+(1-s)d)\,dsdt,
\end{aligned}
$$

where

$$
\begin{aligned}
A \ = \ & \left[{}_{x-,y-}I_{\varphi,\psi} f(a,c) + {}_{x-,y+}I_{\varphi,\psi} f(a,d) + {}_{x+,y-}I_{\varphi,\psi} f(b,c) + {}_{x+,y+}I_{\varphi,\psi} f(b,d)\right] \\
& - \Psi_1(1)\left[{}_{x-}I_{\varphi} f(a,y) + {}_{x+}I_{\varphi} f(b,y)\right] - \Psi_2(1)\left[{}_{x-}I_{\varphi} f(a,y) + {}_{x+}I_{\varphi} f(b,y)\right] \\
& - \Lambda_1(1)\left[{}_{y-}I_{\psi} f(x,c) + {}_{y+}I_{\psi} f(x,d)\right] - \Lambda_2(1)\left[{}_{y-}I_{\psi} f(x,c) + {}_{y+}I_{\psi} f(x,d)\right].
\end{aligned}
$$

Proof. Applying integration by parts and change of variables $u = tx + (1-t)a$ and $v = sy + (1-s)c$, we get

$$
\begin{aligned}
(2.2)\quad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)a, sy+(1-s)c)\,dsdt \\
= \ & \int_0^1 \Lambda_1(t) \left\{ \int_0^1 \Psi_1(s) \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)a, sy+(1-s)c)\,ds \right\} dt \\
= \ & \int_0^1 \Lambda_1(t) \left\{ \frac{\Psi_1(1)}{y-c} \frac{\partial}{\partial t} f(tx+(1-t)a, y) \right. \\
& \left. - \frac{1}{y-c} \int_0^1 \frac{\psi((y-c)s)}{s} \frac{\partial}{\partial t} f(tx+(1-t)a, sy+(1-s)c)\,dsdt \right\} \\
= \ & \frac{\Psi_1(1)}{y-c} \int_0^1 \Lambda_1(t) \frac{\partial}{\partial t} f(tx+(1-t)a, y)\,dt \\
& - \frac{1}{y-c} \int_0^1 \frac{\psi((y-c)s)}{s} \left\{ \int_0^1 \Lambda_1(t) \frac{\partial}{\partial t} f(tx+(1-t)a, sy+(1-s)c)\,dt \right\} ds \\
= \ & \frac{\Psi_1(1)}{y-c} \left\{ \frac{1}{x-a}\Lambda_1(1) f(x,y) - \frac{1}{x-a} \int_0^1 \frac{\varphi((x-a)t)}{t} f(tx+(1-t)a, y)\,dt \right\} \\
& - \frac{1}{y-c} \int_0^1 \frac{\psi((y-c)s)}{s} \left\{ \frac{1}{x-a}\Lambda_1(1) f(x, sy+(1-s)c) \right.
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{1}{x-a}\int_0^1 \frac{\varphi((x-a)t)}{t} f(tx+(1-t)a, sy+(1-s)c)dt\Big\} ds \\
= \; & \frac{\Psi_1(1)\Lambda_1(1)}{(y-c)(x-a)} f(x,y) - \frac{\Psi_1(1)}{(x-a)(y-c)}\int_0^1 \frac{\varphi((x-a)t)}{t} f(tx+(1-t)a, y)dt \\
& - \frac{\Lambda_1(1)}{(x-a)(y-c)}\int_0^1 \frac{\psi((y-c)s)}{s} f(x, sy+(1-s)c) \\
& + \frac{1}{(x-a)(y-c)}\int_0^1\int_0^1 \frac{\varphi((x-a)t)}{t}\frac{\psi((y-c)s)}{s} f(tx+(1-t)a, sy+(1-s)c)dsdt \\
= \; & \frac{\Psi_1(1)\Lambda_1(1)}{(y-c)(x-a)} f(x,y) - \frac{\Psi_1(1)}{(x-a)(y-c)}\; {}_{x-}I_{\varphi} f(a,y) \\
& - \frac{\Lambda_1(1)}{(x-a)(y-c)}\; {}_{y-}I_{\psi} f(x,c) + \frac{1}{(x-a)(y-c)}\; {}_{x-,y-}I_{\varphi,\psi} f(a,c).
\end{aligned}
$$

Similarly, applying the integration by parts, we also get

$$
\begin{aligned}
(2.3) \qquad & \int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)a, sy+(1-s)d)dsdt \\
= \; & -\frac{\Lambda_1(1)\Psi_2(1)}{(x-a)(d-y)} f(x,y) + \frac{\Psi_2(1)}{(x-a)(d-y)}\; {}_{x-}I_{\varphi} f(a,y) \\
& + \frac{\Lambda_1(1)}{(x-a)(d-y)}\; {}_{y+}I_{\psi} f(x,d) - \frac{1}{(x-a)(d-y)}\; {}_{x-,y+}I_{\varphi,\psi} f(a,d),
\end{aligned}
$$

$$
\begin{aligned}
(2.4) \qquad & \int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)b, sy+(1-s)c)dsdt \\
= \; & -\frac{\Lambda_2(t)\Psi_1(s)}{(b-x)(y-c)} f(x,y) + \frac{\Psi_1(s)}{(b-x)(y-c)}\; {}_{x+}I_{\varphi} f(b,y) \\
& + \frac{\Lambda_2(t)}{(b-x)(y-c)}\; {}_{y-}I_{\psi} f(x,c) - \frac{1}{(b-x)(y-c)}\; {}_{x+,y-}I_{\varphi,\psi} f(b,c),
\end{aligned}
$$

$$
\begin{aligned}
(2.5) \qquad & \int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)b, sy+(1-s)d)dsdt \\
= \; & \frac{\Lambda_2(t)\Psi_2(s)}{(b-x)(d-y)} f(x,y) - \frac{\Psi_2(s)}{(b-x)(d-y)}\; {}_{x+}I_{\varphi} f(b,y) \\
& - \frac{\Lambda_2(t)}{(b-x)(d-y)}\; {}_{y+}I_{\psi} f(x,d) - \frac{1}{(b-x)(y-c)}\; {}_{x+,y+}I_{\varphi,\psi} f(b,d).
\end{aligned}
$$

From (2.2)-(2.5) and dividing the resultant one by $(b-a)(d-c)$, we get our desired equality (2.1). $\square$

Theorem 2.1. *Let $f:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be twice partial differentiable mapping on Δ° with $a<b,\ c<d,\ a,c\geq 0$ such that $\frac{\partial^2 f}{\partial s\partial t}\in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on co-ordinates on Δ and $\left|\frac{\partial^2 f}{\partial s\partial t}\right|\leq M,\ (x,y)\in\Delta$, then the following inequality holds for generalized fractional integrals:*

$$\begin{aligned}
(2.6)\qquad &\left|\frac{[\Lambda_2(1)+\Lambda_1(1)]\,[\Psi_2(1)+\Psi_1(1)]}{(b-a)(d-c)}f(x,y)+A\right| \\
\leq\ & \frac{M}{(b-a)(d-c)}\left[(x-a)(y-c)I_1+(x-a)(d-y)I_2\right. \\
& \left.+(b-x)(y-c)I_3+(b-x)(d-y)I_4\right],
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)dsdt, \quad I_2=\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)dsdt, \\
I_3 &= \int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)dsdt, \quad I_4=\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)dsdt,
\end{aligned}$$

and A is defined in Lemma 2.1.

Proof. From Lemma 2.1, we get the following inequality that holds for all $(x,y)\in\Delta$:

$$\begin{aligned}
(2.7)\quad &\left|\frac{[\Lambda_2(1)+\Lambda_1(1)]\,[\Psi_2(1)+\Psi_1(1)]}{(b-a)(d-c)}f(x,y)+A\right| \\
\leq\ & \frac{(x-a)(y-c)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\
&+\frac{(x-a)(d-y)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|dsdt \\
&+\frac{(b-x)(y-c)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|dsdt \\
&+\frac{(b-x)(d-y)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|dsdt.
\end{aligned}$$

By the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ on co-ordinates on Δ and $\left|\frac{\partial^2 f}{\partial s\partial t}\right|\leq M,\ (x,y)\in\Delta$, we have following inequalities:

$$\begin{aligned}
(2.8) \qquad & \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)a, sy+(1-s)c) \right| dsdt \\
\leq \;& M \int_0^1 \int_0^1 ts\Lambda_1(t)\Psi_1(s)dsdt + M \int_0^1 \int_0^1 t(1-s)\Lambda_1(s)\Psi_1(s)dsdt \\
& + M \int_0^1 \int_0^1 (1-t)s\Lambda_1(t)\Psi_1(s)dt + M \int_0^1 \int_0^1 (1-t)(1-s)\Lambda_1(t)\Psi_1(s)dsdt \\
= \;& M \int_0^1 \int_0^1 \Lambda_1(t)\Psi_1(s)dsdt.
\end{aligned}$$

Similarly, we have following inequalities

$$\begin{aligned}
(2.9) \qquad & \textstyle\int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)a, sy+(1-s)d) \right| dsdt \\
& \qquad \leq M \textstyle\int_0^1 \int_0^1 \Lambda_1(t)\Psi_2(s)dsdt,
\end{aligned}$$

$$\begin{aligned}
(2.10) \qquad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)b, sy+(1-s)c) \right| dsdt \\
\leq \;& M \int_0^1 \int_0^1 \Lambda_2(t)\Psi_1(s)dsdt,
\end{aligned}$$

$$\begin{aligned}
(2.11) \qquad & \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s) \left| \frac{\partial^2}{\partial s \partial t} f(tx+(1-t)b, sy+(1-s)d) \right| \\
\leq \;& M \int_0^1 \int_0^1 \Lambda_2(t)\Psi_2(s)dsdt.
\end{aligned}$$

Now using (2.8)-(2.11) in (2.7), then we have our required inequality (2.6). □

Remark 2.1. In Theorem 2.1, if we suppose $\varphi(t) = t$ and $\psi(s) = s$, then the inequality (2.6) becomes inequality (1.6).

Remark 2.2. In Theorem 2.1, if we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$, then the inequality (2.6) is reduced to the inequality (1.10).

Theorem 2.2. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b,\ c < d,\ a, c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is convex on co-ordinates on Δ, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\left|\frac{\partial^2 f}{\partial s \partial t}(x,y)\right| \leq M$, $(x,y) \in \Delta$, then the following inequality holds for generalized fractional integrals:*

$$\begin{aligned}
(2.12) \qquad & \left| \frac{[\Lambda_2(1) + \Lambda_1(1)]\,[\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x,y) + A \right| \\
\leq \;& \frac{M}{(b-a)(d-c)} \left[(x-a)(y-c)J_1 + (x-a)(d-y)J_2 \right. \\
& \left. + (b-x)(y-c)J_3 + (b-x)(d-y)J_4 \right],
\end{aligned}$$

where

$$\begin{aligned} J_1 &= \left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}}, \quad J_2 = \left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}, \\ J_3 &= \left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}}, \quad J_4 = \left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}, \end{aligned}$$

and A is defined as in Lemma 2.1.

Proof. From Lemma 2.1 and the Hölder inequality, we have the following inequality that holds for all $(x,y)\in\Delta$:

$$\begin{aligned} &\left|\frac{[\Lambda_2(1)+\Lambda_1(1)]\,[\Psi_2(1)+\Psi_1(1)]}{(b-a)(d-c)}f(x,y)+A\right| \\ \leq\ & \frac{(x-a)(y-c)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|dsdt \\ &+\frac{(x-a)(d-y)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|dsdt \\ &+\frac{(b-x)(y-c)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|dsdt \\ &+\frac{(b-x)(d-y)}{(b-a)(d-c)}\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|dsdt \\ \leq\ & \frac{(x-a)(y-c)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\frac{(x-a)(d-y)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\frac{(b-x)(y-c)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\frac{(b-x)(d-y)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.13}$$

As we know that $\left|\frac{\partial^2 f}{\partial s\partial t}(x,y)\right|^q$ is co-ordinated convex and $\left|\frac{\partial^2 f}{\partial s\partial t}(x,y)\right|^q \leq M,$ for all $(x,y)\in\Delta$, then we have the following inequality:

$$\begin{aligned}(2.14)\qquad &\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}\\ \leq\ & M\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}}.\end{aligned}$$

Analogously, we also have following inequalities

$$\begin{aligned}(2.15)\qquad &\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a, sy+(1-s)cd)\right|^q dsdt\right)^{\frac{1}{q}}\\ \leq\ & M\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}},\end{aligned}$$

$$\begin{aligned}(2.16)\qquad &\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}\\ \leq\ & M\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}},\end{aligned}$$

$$\begin{aligned}(2.17)\qquad &\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b, sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\\ \leq\ & M\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}.\end{aligned}$$

By using (2.14)-(2.17) in (2.13), then we have our desired inequality (2.12). □

Remark 2.3. In Theorem 2.2, if we suppose $\varphi(t)=t$ and $\psi(s)=s$, then the inequality (2.12) becomes inequality (1.7).

Remark 2.4. In Theorem 2.2, if we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$, then the inequality (2.12) is reduced to the inequality (1.11).

Theorem 2.3. *Let $f : \Delta := [a, b] \times [c, d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b$, $c < d$, $a, c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is convex on co-ordinates on Δ, $q \geq 1$ and $\left|\frac{\partial^2 f}{\partial s \partial t}(x, y)\right| \leq M$, $(x, y) \in \Delta$, then the following inequality holds for generalized fractional integrals:*

$$\begin{aligned}
(2.18)\qquad & \left|\frac{[\Lambda_2(1) + \Lambda_1(1)]\,[\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A\right| \\
\leq\; & \frac{M}{(b-a)(d-c)}\left[(x-a)(y-c)I_1 + (x-a)(d-y)I_2\right. \\
& \left.+(b-x)(y-c)I_3 + (b-x)(d-y)I_4\right],
\end{aligned}$$

where I_1, I_2, I_3 and I_4 are same as defined in Theorem 2.1 and A is defined as in Lemma 2.1.

Proof. From Lemma 2.1 and the power mean inequality, we get the following inequality that holds for all $(x, y) \in \Delta$:

$$\begin{aligned}
(2.19)\; & \left|\frac{[\Lambda_2(1) + \Lambda_1(1)]\,[\Psi_2(1) + \Psi_1(1)]}{(b-a)(d-c)} f(x, y) + A\right| \\
\leq\; & \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx + (1-t)a, sy + (1-s)c)\right| dsdt \\
& + \frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx + (1-t)a, sy + (1-s)d)\right| dsdt \\
& + \frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx + (1-t)b, sy + (1-s)c)\right| dsdt \\
& + \frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx + (1-t)b, sy + (1-s)d)\right| dsdt. \\
\leq\; & \frac{(x-a)(y-c)}{(b-a)(d-c)} \left(\int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s) dsdt\right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx + (1-t)a, sy + (1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
& + \frac{(x-a)(d-y)}{(b-a)(d-c)} \left(\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s) dsdt\right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx + (1-t)a, sy + (1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-x)(y-c)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)(d-y)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}.$$

As we know that $\left|\frac{\partial^2 f}{\partial s\partial t}(x,y)\right|^q$ is co-ordinated convex and $\left|\frac{\partial^2 f}{\partial s\partial t}(x,y)\right|\leq M,$ for all $(x,y)\in\Delta$, then we have the following inequality:

$$\int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt \tag{2.20}$$
$$\leq M^q\int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s).$$

Similarly, we have following inequalities:

$$\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt \tag{2.21}$$
$$\leq M^q\int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)$$

$$\int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt \tag{2.22}$$
$$\leq M^q\int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s),$$

$$\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt \tag{2.23}$$
$$\leq M^q\int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s).$$

By using (2.20)-(2.23) in (2.19), we have our desired inequality (2.18). □

Remark 2.5. In Theorem 2.3, if we suppose $\varphi(t)=t$ and $\psi(s)=s$, then the inequality (2.18) becomes inequality (1.8).

Remark 2.6. In Theorem 2.3, if we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$, then the inequality (2.18) reduces to the inequality (1.12).

Theorem 2.4. *Let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be twice partial differentiable mapping on Δ° with $a < b,\ c < d,\ a, c \geq 0$ such that $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is concave on co-ordinates on Δ, $p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, then the following inequality hold for generalized fractional integrals:*

$$\begin{aligned}
(2.24)\qquad & \left|\frac{[\Lambda_2(1)+\Lambda_1(1)]\,[\Psi_2(1)+\Psi_1(1)]}{(b-a)(d-c)} f(x,y) + A\right| \\
\leq\ & \frac{1}{(b-a)(d-c)}\left[(x-a)(y-c)\left|\frac{\partial^2}{\partial s\partial t} f\left(\frac{x+a}{2}, \frac{y+c}{2}\right)\right| J_1\right. \\
& +(x-a)(d-y)\left|\frac{\partial^2}{\partial s\partial t} f\left(\frac{x+a}{2}, \frac{y+d}{2}\right)\right| J_2 \\
& +(b-x)(y-c)\left|\frac{\partial^2}{\partial s\partial t} f\left(\frac{x+b}{2}, \frac{y+c}{2}\right)\right| J_3 \\
& \left.+(b-x)(d-y)\left|\frac{\partial^2}{\partial s\partial t} f\left(\frac{x+b}{2}, \frac{y+d}{2}\right)\right| J_4\right],
\end{aligned}$$

where $J_1,\ J_2,\ J_3$ and J_4 are same as defined in Theorem 2.2.

Proof. From Lemma 2.1 and the Hölder inequality, we have the following inequality that holds for all $(x,y) \in \Delta$:

$$\begin{aligned}
(2.25) & \left|\frac{[\Lambda_2(1)+\Lambda_1(1)]\,[\Psi_2(1)+\Psi_1(1)]}{(b-a)(d-c)} f(x,y) + A\right| \\
\leq\ & \frac{(x-a)(y-c)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_1(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)a, sy+(1-s)c)\right| dsdt \\
& +\frac{(x-a)(d-y)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_1(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)a, sy+(1-s)d)\right| dsdt \\
& +\frac{(b-x)(y-c)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_2(t)\Psi_1(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)b, sy+(1-s)c)\right| dsdt \\
& +\frac{(b-x)(d-y)}{(b-a)(d-c)} \int_0^1\int_0^1 \Lambda_2(t)\Psi_2(s)\left|\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)b, sy+(1-s)d)\right| dsdt \\
\leq\ & \frac{(x-a)(y-c)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_1(s))^p\, dsdt\right)^{\frac{1}{p}} \\
& \times\left(\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial s\partial t} f(tx+(1-t)a, sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(x-a)(d-y)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_1(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)(y-c)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_1(s))^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)(d-y)}{(b-a)(d-c)}\left(\int_0^1\int_0^1 (\Lambda_2(t)\Psi_2(s))^p\,dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is concave on co-ordinates on Δ, so an application of (1.5) with inequalities in reversed direction, we have following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt \le \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|^q, \tag{2.26}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt \le \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right|^q, \tag{2.27}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt \le \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right|^q, \tag{2.28}$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt \le \left|\frac{\partial^2}{\partial s\partial t}f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right|^q. \tag{2.29}$$

By using (2.26)-(2.29) in (2.25), then we have our desired inequality (2.24). □

Remark 2.7. In Theorem 2.4, if we suppose $\varphi(t) = t$ and $\psi(s) = s$, then the inequality (2.24) becomes inequality (1.9).

Remark 2.8. In Theorem 2.4, if we take $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$, then the inequality (2.24) is reduced to the inequality (1.13).

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MULTIDIMENSIONAL FIXED POINT RESULTS FOR CONTRACTION MAPPING PRINCIPLE WITH APPLICATION

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Abstract. The main aim of this article is to study the existence and uniqueness of fixed point for isotone mappings of any number of arguments under contraction mapping principle on a complete metric space endowed with a partial order. As an application of our result, we have studied the existence and uniqueness of the solution to an integral equation. The results we have obtaied will generalize, extend and unify several classical and very recent related results in the literature in metric spaces.
Keywords: fixed point; contraction mapping principle; partially ordered metric space; non-decreasing mapping; integral equation.

1. Introduction

The Banach contraction principle is one of the most popular tools in solving the existence in many problems of mathematical analysis. Due to its simplicity and usefulness, there are a lot of generalizations of this principle in the literature. Ran and Reurings [14] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodriguez-Lopez [12] extended the result of Ran and Reurings [14] and applied their main results to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions.

The concept of multidimensional fixed point was introduced by Roldan et al. in [16], which is an extension of Berzig and Samet's notion given in [2], which extended and generalized the mentioned fixed point results to higher dimensions. However, they used permutations of variables and distinguished between the first and the last variables. For more details one can consult ([3] [4], [5], [6], [7], [8], [9], [10], [11],[13], [16], [17], [18], [19], [20], [21], [22]).

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In this article, we have studied the existence and uniqueness of fixed point for isotone mappings of any number of arguments under contraction mapping principle on a complete metric space endowed with a partial order. As an application of our result we study the existence and uniqueness of the solution to an integral equation. We improve and generalize the results of Alsulami [1], Razani and Parvaneh [15], Su [20] and many other famous results in the literature.

2. Preliminaries

In order to establish our main results, we will use the following notions. If X is a non-empty set, then we denote $X \times X \times ... \times X$ (n times) by X^n, where $n \in \mathbb{N}$ with $n \geq 2$. If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (that is $x \preceq y$ or $y \preceq x$ holds), then we will write $x \asymp y$. Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$, that is, A and B are non-empty subsets of Λ_n such that $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$. We will denote

$$\begin{aligned} \Omega_{A,\ B} &= \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A,\ \sigma(B) \subseteq B\}, \\ \text{and } \Omega'_{A,\ B} &= \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B,\ \sigma(B) \subseteq A\}. \end{aligned}$$

Henceforth, let $\sigma_1, \sigma_2, ..., \sigma_n$ be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, $g(x)$ will be denoted by gx.

A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X, \preceq)$ be a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notations:

$$x \preceq_i y \Rightarrow \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases} \tag{2.1}$$

Consider on the product space X^n the following partial order: for $Y = (y_1, y_2, ..., y_i, ..., y_n)$, $V = (v_1, v_2, ..., v_i, ..., v_n) \in X^n$,

$$Y \sqsubseteq V \Leftrightarrow y_i \preceq_i v_i. \tag{2.2}$$

We say that two points Y and V are comparable, if $Y \sqsubseteq V$ or $V \sqsubseteq Y$. Obviously, $(X^n, \sqsubseteq)$ is a partially ordered set.

Definition 2.1. ([10], [16], [18]). A point $(x_1, x_2, ..., x_n) \in X^n$ is called a $\Upsilon-$fixed point of the mapping $F : X^n \to X$ if

$$F(x_{\sigma_i(1)},\ x_{\sigma_i(2)},\ ...,\ x_{\sigma_i(n)}) = x_i,\ \text{for all } i \in \Lambda_n.$$

This definition extends the notions of coupled, tripled, and quadruple fixed points. In fact, if we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, that is, $\sigma = (\sigma(1), \sigma\ (2), ..., \sigma\ (n))$, then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = (1, 2)$ and $\sigma_2 = (2, 1)$,

(ii) Berinde and Borcut's tripled fixed points are associated with $n = 3$, $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_3 = (3, 2, 1)$,

(iii) Karapinar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.

These cases consider A as the odd numbers in $\{1, 2, ..., n\}$ and B as its even numbers. However, Berzig and Samet [2] use $A = \{1, 2, ..., m\}$, $B = \{m+1, ..., n\}$ and arbitrary mappings."

Definition 2.2. [16]. Let $(X, \preceq)$ be a partially ordered space. We say that F has the mixed monotone property if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B, that is, for all $x_1, x_2, ..., x_n, y, z \in X$ and all i

$$y \preceq z \Rightarrow F(x_1,\ ...,\ x_{i-1},\ y,\ x_{i+1},\ ...,\ x_n) \preceq_i F(x_1,\ ...,\ x_{i-1},\ z,\ x_{i+1},\ ...,\ x_n).$$

Definition 2.3. ([18], [21]). Let (X, d) be a metric space and define $\Delta_n, \rho_n : X^n \times X^n \to [0, +\infty)$, for $Y = (y_1, y_2, ..., y_n)$, $V = (v_1, v_2, ..., v_n) \in X^n$, by

$$\Delta_n(Y,\ V) = \frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i) \text{ and } \rho_n(Y,\ V) = \max_{1\leq i\leq n} d(y_i,\ v_i).$$

Then Δ_n and ρ_n are metric on X^n and (X, d) is complete if and only if (X^n, Δ_n) and (X^n, ρ_n) are complete. It is easy to see that

$$\begin{aligned} \Delta_n(Y^k,\ Y) &\to 0 \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty) \\ \text{and } \rho_n(Y^k,\ Y) &\to 0 \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty),\ i \in \Lambda_n, \end{aligned}$$

where $Y^k = (y_1^k, y_2^k, ..., y_n^k)$ and $Y = (y_1, y_2, ..., y_n) \in X^n$.

Lemma 2.1. *([18], [21], [22]). Let $(X, d, \preceq)$ be an ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Define F_Υ, $G : X^n \to X^n$, for all $y_1, y_2, ..., y_n \in X$, by*

$$F_\Upsilon(y_1,\ y_2,\ ...,\ y_n) = \begin{pmatrix} F(y_{\sigma_1(1)},\ y_{\sigma_1(2)},\ ...,\ y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)},\ y_{\sigma_2(2)},\ ...,\ y_{\sigma_2(n)}), \\ ...,\ F(y_{\sigma_n(1)},\ y_{\sigma_n(2)},\ ...,\ y_{\sigma_n(n)}) \end{pmatrix}, \tag{2.3}$$

and

$$G(y_1,\ y_2,\ ...,\ y_n) = (gy_1,\ gy_2,\ ...,\ gy_n). \tag{2.4}$$

(1) *If F has the mixed $(g, \preceq)-$monotone property, then F_Υ is monotone $(G, \sqsubseteq)-$non-decreasing.*

(2) *If F is $d-$continuous, then F_Υ is also Δ_n-continuous and ρ_n-continuous.*

(3) *If g is $d-$continuous, then G is Δ_n-continuous and ρ_n-continuous.*

(4) *A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(y_1, y_2, ..., y_n)$ is a fixed point of F_Υ.*

(5) *A point $(y_1, y_2, ..., y_n) \in X^n$ is a $\Upsilon-$coincidence point of F and g if and only if $(y_1, y_2, ..., y_n)$ is a coincidence point of F_Υ and G.*

(6) *If $(X, d, \preceq)$ is regular, then $(X^n, \Delta_n, \sqsubseteq)$ and $(X^n, \rho_n, \sqsubseteq)$ are also regular.*

Lemma 2.2. *[8]. Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ be a mapping. Then*

(a) If there exists $y_0^1, y_0^2, ..., y_0^n \in X$ verifying $y_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, ..., y_0^{\sigma_i(n)})$, for $i \in \Lambda_n$, then there exists $Y_0 \in X^n$ such that $Y_0 \sqsubseteq F_\Upsilon(Y_0)$.

(b) If F is a mixed monotone mapping, then F_Υ is an isotone mapping.

(c) If for each $i \in \Lambda_n$ and $y_i, v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i, then there exists $Z \in X^n$ which is $\sqsubseteq -$comparable to Y and V.

Definition 2.4. [20]. A generalized altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfied the following conditions:

(i_ψ) ψ is non-decreasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

3. Main results

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping for which there exist a generalized altering distance function ψ and a right upper semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\psi(d(Tx,\ Ty)) \le \varphi(d(x,\ y)), \tag{3.1}$$

for all $x, y \in X$ with $x \preceq y$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. Suppose either

(a) T is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exists $x_0 \in X$ such that $x_0 \asymp Tx_0$, then T has a fixed point. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is $\preceq -$comparable to x and y, then the fixed point is unique.

We omit the proof of the previous result since its proof is similar to the main theorem in [20].

Put $\psi(t) = t$ and $\varphi(t) = kt$ with $k < 1$ in Theorem 3.1, we get the following result:

Corollary 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx,\ Ty) \leq kd(x,\ y),$$

for all $x, y \in X$ with $x \preceq y$, where $k < 1$. Suppose either

(a) T is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exists $x_0 \in X$ such that $x_0 \asymp Tx_0$, then T has a fixed point. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is $\preceq -$comparable to x and y, then the fixed point is unique.

Next we give an $n-$dimensional fixed point theorem for mixed monotone mappings. For brevity, $(y_1, y_2, ..., y_n)$, $(v_1, v_2, ..., v_n)$ and $(y_0^1, y_0^2, ..., y_0^n)$ will be denoted by Y, V and Y_0 respectively.

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ be a mixed monotone mapping for which there exist a generalized altering distance function ψ and a right upper semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying*

$$\psi(d(F(y_1,\ y_2,\ ...,\ y_n),\ F(v_1,\ v_2,\ ...,\ v_n))) \leq \varphi\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right), \tag{3.2}$$

for which $y_i, v_i \in X$ such that $y_i \preceq_i v_i$ for all $i \in \Lambda_n$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. Also, suppose that either F is continuous or $(X, d, \preceq)$ is regular. If there exists $y_0^1, y_0^2, ..., y_0^n \in X$ such that

$$y_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)}),\ \textit{for } i \in \Lambda_n.$$

Then F has a $\Upsilon-$fixed point. Moreover, if for each $i \in \Lambda_n$ and $y_i, v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i. Then F has a unique $\Upsilon-$fixed point.

Proof. For fixed $i \in A$, we have $y_{\sigma_i(t)} \preceq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. By using (3.2), we have

$$\begin{aligned} &\psi(d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))) \\ (3.3) \qquad \leq\ &\varphi\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right), \end{aligned}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $y_{\sigma_i(t)} \succeq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (3.2) that

$$\begin{aligned} & \psi(d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))) \\ \leq\ & \psi(d(F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}),\ F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}))) \\ \leq\ & \varphi\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right), \end{aligned} \tag{3.4}$$

for all $i \in B$. Now by using (2.2), (2.3), (3.3), (3.4) and by the monotonicity of ψ, we have

$$\psi(\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V))) \leq \varphi(\rho_n(Y,\ V)),$$

for all $Y,\ V \in X^n$ with $Y \sqsubseteq V$. It is only required to apply Theorem 3.1 with the help of Lemma 2.1 and Lemma 2.2 for the mapping $T = F_\Upsilon$ in the ordered metric space $(X^n,\ \rho_n,\ \sqsubseteq)$. ∎

Theorem 3.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ be a mixed monotone mapping for which there exist a generalized altering distance function ψ and a right upper semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} & \psi\left(\frac{1}{n}\sum_{i=1}^{n} d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)}))\right) \\ \leq\ & \varphi\left(\frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i)\right), \end{aligned} \tag{3.5}$$

for all $y_1,\ y_2,\ ...,\ y_n,\ v_1,\ v_2,\ ...,\ v_n \in X$ with $y_i \preceq_i v_i$, for $i \in \Lambda_n$, where $\psi(t) > \varphi(t)$ for all $t > 0$ and $\varphi(0) = 0$. Also, suppose that either F is continuous or $(X,\ d,\ \preceq)$ is regular. If there exists $y_0^1,\ y_0^2,\ ...,\ y_0^n \in X$ verifying $y_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, for $i \in \Lambda_n$, then F has a $\Upsilon-$fixed point. Moreover, if for each $i \in \Lambda_n$ and $y_i,\ v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i. Then F has a unique $\Upsilon-$fixed point.

Proof. Note that the contractive condition (3.5) means that

$$\psi(\Delta_n(F_\Upsilon(Y),\ F_\Upsilon(V))) \leq \varphi(\Delta_n(Y,\ V)),$$

for all $Y,\ V \in X^n$ with $Y \sqsubseteq V$. Therefore, it is only necessary to use Theorem 3.1 with the help of Lemma 2.1 and Lemma 2.2 for the mapping $T = F_\Upsilon$ in the ordered metric space $(X^n,\ \Delta_n,\ \sqsubseteq)$. ∎

In a similar way, we may state the results analogue to Corollary 3.1, for Theorem 3.2 and Theorem 3.3.

4. Applications

In this section we give an application to our results. Consider the integral equation

$$u(t) = \int_0^T K(t,\ s,\ u(s))ds + g(t),\ t \in [0,\ T], \tag{4.1}$$

where $T > 0$. Consider the space:

$$C[0,\ T] = \{u : [0,\ T] \to \mathbb{R} : u \text{ is continuous on } [0,\ T]\},$$

equipped with the metric

$$d(x,\ y) = \sup_{t \in [0,\ T]} |x(t) - y(t)|\,, \text{ for each } x,\ y \in C[0,\ T].$$

It is obvious that $(C[0, T], d)$ is a complete metric space. Furthermore, $C[0, T]$ can be equipped with the following partial order $\preceq$

$$x \preceq y \Longleftrightarrow x(t) \leq y(t),\ \text{for each } x,\ y \in C[0,\ T] \text{ and } t \in [0,\ T].$$

It is clear that $(C[0, T], d, \preceq)$ is regular.

Theorem 4.1. *Suppose that the following hypotheses hold:*

(i) $K : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous.*

(ii) For all $s, t, x, y \in C[0, T]$ *with* $y \preceq x$*, we have*

$$K(t,\ s,\ y(s)) \leq K(t,\ s,\ x(s)).$$

(iii) There exists a continuous function $G : [0, T] \times [0, T] \to [0, +\infty)$ *such that*

$$|K(t,\ s,\ x) - K(t,\ s,\ y)| \leq G(t,\ s) \cdot \frac{|x - y|}{2},$$

for all $s, t \in C[0, T]$ *and* $x, y \in \mathbb{R}$ *with* $y \leq x$*,*

(iv) $\sup_{t\in[0,\ T]} \int_0^T G(t,\ s)^2 ds \leq \frac{1}{T}$.

Then the integral (4.1) has a solution $x^* \in C[0, T]$.

Proof. We, first, define $F : C[0, T] \to C[0, T]$ by

$$Fx(t) = \int_0^T K(t,\ s,\ x(s))ds + g(t),\ \text{for all } t \in [0,\ T] \text{ and } x \in C[0,\ T].$$

Suppose $y \preceq x$, then from (ii), for all $s,\ t \in [0,\ T]$, we have $K(t,\ s,\ y(s)) \le K(t,\ s,\ x(s))$. Thus, we get

$$Fy(t) = \int_0^T K(t,\ s,\ y(s))ds + g(t) \le \int_0^T K(t,\ s,\ x(s))ds + g(t) = Tx(t).$$

Now, for all $u,\ v \in C[0,\ T]$ with $y \preceq x$, due to (iii) and by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} & |Fx(t) - Fy(t)| \\ \le\ & \int_0^T |K(t,\ s,\ x(s)) - K(t,\ s,\ y(s))|\, ds \\ \le\ & \int_0^T G(t,\ s) \cdot \frac{|x(s) - y(s)|}{2} ds \\ \le\ & \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|x(s) - y(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$|Fx(t) - Fy(t)| \le \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|x(s) - y(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}}. \tag{4.2}$$

Taking (iv) into account, we estimate the first integral in (4.2) as follows:

$$\left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \le \frac{1}{\sqrt{T}}. \tag{4.3}$$

For the second integral in (4.2) we proceed in the following way:

$$\left(\int_0^T \left(\frac{|x(s) - y(s)|}{3} \right)^2 ds \right)^{\frac{1}{2}} \le \sqrt{T} \cdot \frac{d(x,\ y)}{2}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we conclude that

$$|Fx(t) - Fy(t)| \le \frac{1}{2} d(x,\ y).$$

Taking supremum for each $t \in [0,\ T]$, we get

$$d(Fx,\ Fy) \le \frac{1}{2} d(x,\ y),$$

for all x, $y \in C[0, T]$ with $y \preceq x$. Thus, the contractive condition of Corollary 3.1 is satisfied with $k = 1/2 < 1$. Hence, all the hypotheses of Corollary 3.1 are satisfied. Thus, F has a fixed point $x^* \in C[0, T]$ which is a solution of (4.1). ∎

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CHARACTERIZATION OF SOME BIDERIVATIONS ON TRIANGULAR BANACH ALGEBRAS

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Abstract. Let A and B be unital Banach algebras, X be a unital A-B-module and T be the triangular Banach algebra associated to A, B and X. The structure of some derivations on triangular Banach algebras was studied by some authors. Note that despite the apparent similarity between derivations and biderivations and also inner derivations and inner biderivations, there are fundamental differences between them. Although there are some studies of biderivations on triangular Banach algebras, any of them do not completely determine the structure of biderivations on triangular Banach algebras. In this paper, we completely characterize biderivations and inner biderivations from $T \times T$ to T^* and we show that the first bicohomology group $BH^1(T, T^*)$ is equal to $BH^1(A, A^*) \oplus BH^1(B, B^*)$.
Keywords: unital Banach algebras; triangular Banach algebra; bicohomology group; biderivations.

1. Introduction

A derivation from a Banach algebra A to a Banach A-module X is a bounded linear mapping $d : A \to X$ such that for each $a, b \in A$, $d(ab) = d(a)b + ad(b)$. For each $x \in X$ the mapping $\delta_x : a \to ax - xa$, $(a \in A)$ is a bounded derivation, called an inner derivation.

Let A be a Banach algebra and X be an A-module. A bounded bilinear mapping $D : A \times A \to X$ is called a biderivation if D is a derivation with respect to both arguments. That is, the mappings ${}_aD : A \to X$ and $D_b : A \to X$ where

$${}_aD(b) = D(a, b) = D_b(a) \quad (a, b \in A),$$

are derivations. We denote the space of such biderivations by $BZ^1(A, X)$.

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Let $x \in Z(A, X) = \{x \in X; ax = xa \ \ \forall a \in A\}$. The map $D_x : A \times A \to X$ that

$$D_x(a, b) = x[a, b] = xab - xba \quad (a, b \in A),$$

is a basic example of a biderivation which is called an inner biderivation. We denote the space of such inner biderivations by $BN^1(A, X)$. Also we define the first bicohomology group $BH^1(A, X)$ as follows,

$$BH^1(A, X) = \frac{BZ^1(A, X)}{BN^1(A, X)}.$$

For more applications and details about biderivations see [6, Section 3]. Also see [5, 8], in which the structures of some biderivations on triangular algebras and generalized matrix algebras and when these biderivations on these algebras are inner, were studied.

Let A and B be Banach algebras and X be an A-B-module. Then the algebra

$$T = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; a \in A, x \in X, b \in B \right\}$$

equipped with the usual addition and multiplication of matrix and with the norm

$$\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|x\| + \|b\|$$

is a Banach algebra which is called triangular Banach algebra associated to X. Then the dual of triangular Banach algebra T is

$$T^* = \left\{ \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}; f \in A^*, h \in X^*, g \in B^* \right\};$$

where $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right) = f(a) + h(x) + g(b)$.

Recall that for every Banach A-module X the dual space X^* is a Banach A-module with module structures $a \cdot f$ and $f \cdot a$ that $a \cdot f(x) = f(xa)$ and $f \cdot a(x) = f(ax)$. So T^* is a T-module with the module actions

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} = \begin{pmatrix} a \cdot f + x \cdot h & b \cdot h \\ 0 & b \cdot g \end{pmatrix}$$

and

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} f \cdot a & h \cdot a \\ 0 & h \cdot x + g \cdot b \end{pmatrix},$$

for every $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T$ and $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \in T^*$.

A Banach algebra A is called weakly amenable if every derivation from A to A^* is an inner derivation. The concept of weak amenability of Banach algebras was introduced by Bade, Curtis and Dales [1] for commutative Banach algebras and then by Johnson [10] for a general Banach algebra.

In this paper, we consider A and B as unital Banach algebras and X as a unital A-B-module, that is, $1_A x = x 1_B = x$, for every $x \in X$. We characterize the biderivations from $T \times T$ to T^*. In particular, we show that $BH^1(T, T^*) = BH^1(A, A^*) \oplus BH^1(B, B^*)$.

2. Biderivations and biamenability of triangular Banach algebras

Similar to the definitions of amenability or weak amenability of Banach algebras we may define the notions of biamenability [2] or weak biamenability of Banach algebras [3].

Definition 2.1. We say that a Banach algebra A is weakly biamenable if

$$BH^1(A, A^*) = \{0\}.$$

Example 2.1. (i) Let $\mathfrak{A}$ be a Banach space and $\theta \in \mathfrak{A}^*$. Then $\mathfrak{A}$ with the product

$$ab = \theta(a)b \quad (a, b \in \mathfrak{A}),$$

is a Banach algebra and θ becomes a homomorphism. Also for each $h \in \mathfrak{A}^*$ and $a, b, c \in \mathfrak{A}$ we have $h \cdot a = \theta(a)h$ and $a \cdot h = h(a)\theta$ and since $\theta(ab) = \theta(ba)$, we have $[a, b]c = [b, a]c$. Now consider $f \in \mathfrak{A}^*$ such that for some $a_0, b_0 \in \mathfrak{A}$, $\theta(a_0)f(b_0) \neq f(a_0)\theta(b_0)$. Define the biderivation $D : \mathfrak{A} \times \mathfrak{A} \to A^*$ by $D(a, b) = \delta_{\delta_{f(a)}}(b)$, for each $a, b \in \mathfrak{A}$. Then since D is non zero and the only inner biderivation from $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{A}^*$ is zero, we conclude that $\mathfrak{A}$ with this product is not weakly biamenable.

(ii) Let $B(H)$ be the Banach algebra of operators on Hilbert space H and $D : B(H) \times B(H) \to B(H)^*$ be a biderivation. Then similar to Lemma 1 of [5] $D(T, S)[U, V] = [T, S]D(U, V)$ for each $T, S, U, V \in B(H)$. Also by Lemma 5.8 of [12] $B(H) = span\{UV - VU; V, U \in B(H)\}$. Therefore there exist $\{U_i\}, \{V_i\}$ in $B(H)$ such that $I = \sum_i [U_i, Vi]$. Now we have

$$\begin{aligned} D(T, S) &= D(T, S)I \\ &= D(T, S) \textstyle\sum_i [U_i, Vi] \\ &= \textstyle\sum_i [T, S] D(U_i, V_i) \\ &= [T, S] \textstyle\sum_i D(U_i, V_i) \end{aligned}$$

and similarly $D(T, S) = \sum_i D(U_i, V_i)[T, S]$. So if we put $x = \sum_i D(U_i, V_i)$, then $x \in Z(B(H), B(H)^*)$ and $D(T, S) = x[T, S]$. That is D is an inner biderivation and so $B(H)$ is weakly biamenable.

Note that despite the apparent similarity between derivations and biderivations and also inner derivations and inner biderivations, there are fundamental differences between them. Especially when a biderivation wants to be an inner bidetivation these differences become more apparent. A part of these differences comes from the nature of biderivations which depend on two components. Another essential part of these differences goes back to the definition of inner biderivations which depends on the implemented elements that should be in $Z(A, X)$. According to this, the concept of amenability and also weak amenability have a different nature from biamenability and weak biamenability, respectively. Indeed, there are examples of Banach algebras that are biamenable while they are not amenable and there are Banach algebras that are amenable while they are not biamenable [2]. Also, if we consider the definition of a biamenable group G such that $G \times G$ is amenable, then we see that the Johnson's theorem [11] is not valid for biamenability. Indeed, each abelian group G is biamenable while the commutative group algebra $L^1(G)$ is not biamenable [2]. Of course, the situation of weak biamenability is better than biamenability, and many similar results of [4] and [7] are valid for weak biamenability of Banach algebras. Also, for each locally compact abelian group G, $L^1(G)$ is weakly biamenable. For more detales, see [3].

The next theorem characterizes all biderivations from $T \times T$ to T^*.

Theorem 2.1. *A bilinear mapping $D : T \times T \to T^*$ is a biderivation if and only if there exist biderivations $d_A : A \times A \to A^*$ and $d_B : B \times B \to B^*$ such that*

$$D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} d_A(a, a') & 0 \\ 0 & d_B(b, b') \end{pmatrix}.$$

Proof. It is easy to verify that if

$$D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} d_A(a, a') & 0 \\ 0 & d_B(b, b') \end{pmatrix},$$

for some biderivations $d_A : A \times A \to A^*$ and $d_B : B \times B \to B^*$, then D is a biderivation.

Conversely, let $D : T \times T \to T^*$ be a biderivation. Since for every $a, a' \in A, b, b' \in B$ and $x, x' \in X$ we have

$$_{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}}D\left(\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) =$$

$$D_{\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right)$$

and $_{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}}D$ and $D_{\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}}$ are derivations, according to [9] there exist derivations $d_{(a',x',b')}, d'_{(a,x,b)} : A \to A^*$ and $\delta_{(a',x',b')}, \delta'_{(a,x,b)} : B \to B^*$ and also

$k_{(a',x',b')}, k'_{(a,x,b)} \in X^*$ such that

$$
\begin{aligned}
&\begin{pmatrix} d_{(a',x',b')}(a) - xk_{(a',x',b')} & k_{(a',x',b')}a - bk_{(a',x',b')} \\ 0 & k_{(a',x',b')}x + \delta_{(a',x',b')}(b) \end{pmatrix} \\
= \ & D_{\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right) \\
= \ & D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= \ & {}_{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}}D\left(\begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= \ & \begin{pmatrix} d'_{(a,x,b)}(a') - x'k'_{(a,x,b)} & k'_{(a,x,b)}a' - b'k'_{(a,x,b)} \\ 0 & k'_{(a,x,b)}x' + \delta'_{(a,x,b)}(b') \end{pmatrix}.
\end{aligned}
$$

In particular

$$
\begin{aligned}
\begin{pmatrix} d_{(a',0,0)}(a) & k_{(a',0,0)}a \\ 0 & 0 \end{pmatrix} &= D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} d'_{(a,0,0)}(a') & k'_{(a,0,0)}a' \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Define $d_A : A \times A \to A^*$ by $d_A(a, a') = d_{(a',0,0)}(a) = d'_{(a,0,0)}(a')$. Then obviously d_A is a bounded biderivation.

Similarly we can define the biderivation $d_B : B \times B \to B^*$ such that $d_B(b, b') = \delta_{(0,0,b')}(b) = \delta'_{(0,0,b)}(b')$. Also we have

$$
\begin{aligned}
\begin{pmatrix} d_{(0,0,b')}(a) & k_{(0,0,b')}a \\ 0 & 0 \end{pmatrix} &= D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & -b'k'_{(a,0,0)} \\ 0 & \delta'_{(a,0,0)}(b') \end{pmatrix}.
\end{aligned}
$$

So

$$
d_{(0,0,b')}(a) = 0, \quad \delta'_{(a,0,0)}(b') = 0 \quad \text{and} \quad k_{(0,0,b')}a = -b'k'_{(a,0,0)}. \tag{2.1}
$$

On the other hand

$$
\begin{aligned}
\begin{pmatrix} d_{(0,x',0)}(a) & k_{(0,x',0)}a \\ 0 & 0 \end{pmatrix} &= D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} -x'k'_{(a,0,0)} & 0 \\ 0 & k'_{(a,0,0)}x' \end{pmatrix}.
\end{aligned}
$$

Therefore $d_{(0,x',0)}(a) = -x'k'_{(a,0,0)}$, $k_{(a',0,0)}a = 0$ and $k'_{(a,0,0)}x' = 0$. In particular for each $x' \in X$, $-x'k'_{(1_A,0,0)} = d_{(0,x',0)}(1_A) = 0$ and since X is a unital A-module, $k'_{(1_A,0,0)} = 0$. On the other hand by (2.1) we have $k_{(0,0,1_B)} = k_{(0,0,1_B)}1_A =$

$-1_B k'_{(1_A,0,0)} = 0$ and hence

$$d_{(0,x',0)}(a) = -x'k'_{(a,0,0)} = -x'1_B k'_{(a,0,0)} = x'k_{(0,0,1_B)}a = 0.$$

So

$$\begin{aligned} D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) &= D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &+ D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x' \\ 0 & 0 \end{pmatrix}\right) \\ &+ D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix}\right) \\ &= \begin{pmatrix} d_A(a,a') & k_{(a',0,0)}a + k_{(0,x',0)}a + k_{(0,0,b')}a \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Now since $k_{(a+a',0,0)} = k_{(a,0,0)} + k_{(a',0,0)}$, $k_{(0,x+x',0)} = k_{(0,x,0)} + k_{(0,x',0)}$ and $k_{(0,0,b+b')} = k_{(0,0,b)} + k_{(0,0,b')}$, we can define the linear mapping

$$\begin{aligned} h : A \oplus X \oplus B &\to X^* \\ (a,x,b) &\mapsto k_{(a,0,0)} + k_{(0,x,0)} + k_{(0,0,b)} \end{aligned}$$

and we have $D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} d_A(a,a') & h(a',x',b')a \\ 0 & 0 \end{pmatrix}$.

Similarly we have

$$D\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} 0 & -bh(a',x',b') \\ 0 & d_B(b,b') \end{pmatrix}$$

and

$$D\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} -xh(a',x',b') & 0 \\ 0 & h(a',x',b')x \end{pmatrix}.$$

So we have

$$\begin{aligned} & D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\ = & \begin{pmatrix} d_A(a,a') - xh(a',x',b') & h(a',x',b')a - bh(a',x',b') \\ 0 & h(a',x',b')x + d_B(b,b') \end{pmatrix}. \end{aligned}$$

Also we can show similarly there is a bounded linear mappings $h' : A \oplus X \oplus B \to X^*$ such that

$$\begin{aligned} & D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\ = & \begin{pmatrix} d_A(a,a') - x'h'(a,x,b) & h'(a,x,b)a' - b'h'(a,x,b) \\ 0 & h'(a,x,b)x' + d_B(b,b') \end{pmatrix}. \end{aligned}$$

Therefore $h(a',x',b')a - bh(a',b',x') = h'(a,x,b)a' - b'h'(a,x,b)$. So

$$\begin{aligned} h(a,x,b) &= h(a,x,b)1_A - 0_B h(a,x,b) \\ &= h'(1_A.0,0)a - bh'(1_A,0,0) \\ &= k'(1_A,0,0)a - bk'(1_A,0,0) \\ &= 0. \end{aligned}$$

That is,

$$D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} d_A(a,a') & 0 \\ 0 & d_B(b,b') \end{pmatrix}.$$

$\square$

Proposition 2.1. *The biderivation $D : T \times T \to T^*$ which is defined for each $a, a' \in A, b, b' \in B$ and $x, x' \in X$, by*

$$D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} d_A(a,a') & 0 \\ 0 & d_B(b,b') \end{pmatrix}$$

is an inner biderivation if and only if d_A and d_B are inner biderivations.

Proof. If d_A and d_B are inner biderivations, then there are $f \in Z(A, A^*)$ and $g \in Z(B, B^*)$ such that for each $a, a' \in A$, $d_A(a,a') = f[a,a'] = faa' - fa'a$ and for each $b, b' \in B$, $d_B(b,b') = g[b,b'] = gbb' - gb'b$. Now we have

$$\begin{aligned} D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) &= \begin{pmatrix} d_A(a,a') & 0 \\ 0 & d_B(b,b') \end{pmatrix} \\ &= \begin{pmatrix} f[a,a'] & 0 \\ 0 & g[b,b'] \end{pmatrix} \\ &= \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right]. \end{aligned}$$

Also it is easy to see that $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in Z(T, T^*)$ if and only if $f \in Z(A, A^*)$ and $g \in Z(B, B^*)$. Hence D is an inner biderivation.

Conversely, if D is an inner biderivation, then there exists $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \in Z(T, T^*)$ such that

$$D\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right].$$

In particular

$$\begin{aligned} \begin{pmatrix} d_A(a,a') & 0 \\ 0 & 0 \end{pmatrix} &= D\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} f[a,a'] & h[a,a'] \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $d_A(a,a') = f[a,a']$ and $h[a,a'] = 0$. On the other hand for each $a \in A$ we have

$$\begin{aligned}\begin{pmatrix} fa & ha \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \\ &= \begin{pmatrix} af & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

That is $f \in Z(A, A^*)$ and $ha = 0$, that is $h = 0$. So d_A is an inner biderivation. Similarly we can show that d_B is an inner biderivation. □

Note that in the latter proposition it is also proved that

$$Z(T, T^*) = \left\{ \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}; f \in Z(A, A^*), g \in Z(B, B^*) \right\}.$$

Theorem 2.2. $BH^1(T \times T, T^*) = BH^1(A \times A, A^*) \oplus BH^1(B \times B, B^*)$

Proof. Define

$$\begin{aligned}\varphi : BZ^1(A \times A, A^*) \oplus BZ^1(B \times B, B^*) &\to BH^1(T \times T, T^*), \\ (d_A, d_B) &\mapsto \left[\begin{pmatrix} d_A & 0 \\ 0 & d_B \end{pmatrix}\right]\end{aligned}$$

where $\left[\begin{pmatrix} d_A & 0 \\ 0 & d_B \end{pmatrix}\right]$ is the equivalent class of $\begin{pmatrix} d_A & 0 \\ 0 & d_B \end{pmatrix}$ in $BH^1(T \times T, T^*)$. Then by Theorem 2.1, φ is onto and by Proposition 2.1 we have

$$\begin{aligned} ker\varphi &= \left\{ (d_A, d_B); \begin{pmatrix} d_A & 0 \\ 0 & d_B \end{pmatrix} \text{is inner} \right\} \\ &= \{(d_A, d_B); d_A \text{ and } d_B \text{ are inner}\} \\ &= BN^1(A \times A, A^*) \oplus BN^1(B \times B, B^*). \end{aligned}$$

Therefore

$$\begin{aligned} BH^1(T \times T, T^*) &= \frac{BZ^1(A \times A, A^*) \oplus BZ^1(B \times B, B^*)}{BN^1(A \times A, A^*) \oplus BN^1(B \times B, B^*)} \\ &= BH^1(A \times A, A^*) \oplus BH^1(B \times B, B^*). \end{aligned}$$

□

Corollary 2.1. *T is weakly biamenable if and only if A and B are weakly biamenable.*

For example if A is a commutative Banach algebra and there is a non zero biderivation from $A \times A$ into A^*, then since the only inner biderivation from $A \times A$ into A^* is zero, A and therefore T are not weakly biamenable.

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SOME FIXED POINT RESULTS FOR CONVEX CONTRACTION MAPPINGS ON $\mathcal{F}-$METRIC SPACES

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Abstract. In this paper, we have established some fixed point theorems for convex contraction mappings in $\mathcal{F}$-metric spaces. Also, we have ntroduced the concept of $(\alpha, \beta)-$convex contraction mapping in $\mathcal{F}$-metric spaces and give some fixed point results for such contractions. Moreover, some examples are given to support our theoretical results.

Keywords: $\mathcal{F}$-Complete; Convex contraction; Fixed point; $\mathcal{F}$-Metric space; Orbital continuity.

1. Introduction

Fixed point theory plays a pivotal role in functional and nonlinear analysis. The Banach contraction principle is an important result of the fixed point theory. In recent years, various extensions of metric spaces have been introduced (see e.g. [1, 4, 6, 9, 10, 12, 16, 18] and references therein). The notion of a $\mathcal{F}$-metric space was firstly introduced and studied by Jleli and Samet in [17] (see e.g. [13, 20] and references therein). We recall some of the basic definitions and results in the sequel.
Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbb{R}$ such that
$\mathcal{F}_1)$ f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.
$\mathcal{F}_2)$ For every sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n\to+\infty} t_n = 0 \textit{ if and only if } \lim_{n\to+\infty} f(t_n) = -\infty.$$

Definition 1.1. [17] Let X be a (nonempty) set. A function $D : X \times X \to [0, \infty)$ is called a $\mathcal{F}$-metric on X if there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that for all $x, y \in X$ the following conditions hold:
(D_1) $D(x, y) = 0$ *if and only if* $x = y$.

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(D_2) $D(x,y) = D(y,x)$.
(D_3) For every $N \in \mathbb{N}, N \geq 2$ and for every $\{u_i\}_i^N \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x,y) > 0 \textit{ implies } f(D(x,y)) \leq f(\sum_{i=1}^{N-1} D(u_i, u_{i+1})) + \alpha.$$

In this case, the pair (X, D) is called a $\mathcal{F}$-metric space.

Example 1.1. [17] Let $X = \mathbb{R}$ and $D : X \times X \to [0, \infty)$ be defined as follows:

$$D(x,y) = \begin{cases} (x-y)^2, & (x,y) \in [0,3] \times [0,3] \\ \\ |x-y|, & \text{otherwise}, \end{cases}$$

and let $f(t) = ln(t)$ for all $t > 0$ and $\alpha = ln(3)$. Then, D is a $\mathcal{F}$-metric on X. Since $D(1,3) = 4 \geq D(1,2) + D(2,3) = 2$, Then D is not a metric on X.

Example 1.2. [17] Let $X = \mathbb{R}$ and $D : X \times X \to [0, \infty)$ be defined as follows:

$$D(x,y) = \begin{cases} e^{|x-y|}, & x \neq y \\ \\ 0, & x = y \end{cases}$$

Then, D is a $\mathcal{F}$-metric on X. Since $D(1,3) = e^2 \geq D(1,2) + D(2,3) = 2e$, Then D is not a metric on X.

Definition 1.2. [17] Let (X, D) be an $\mathcal{F}$-metric space and $\{x_n\}$ be a sequence in X.
1) A sequence $\{x_n\}$ is called $\mathcal{F}$-convergent to $x \in X$, if $\lim_{n\to+\infty} D(x_n, x) = 0$.
2) A sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, if and only if $\lim_{n,m\to+\infty} D(x_n, x_m) = 0$.
3) A $\mathcal{F}$-metric space (X, D) is said to be $\mathcal{F}$-complete, if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to some element in X.

Istratescu [14] introduced the notion of convex contraction and proved that if (X, d) is a complete metric space, then every convex contraction mapping on X has a unique fixed point.

Definition 1.3. [14] Let (X, d) be a metric space. The continuous selfmap T on X is called a convex contraction of order 2 whenever there exist $a_i \in (0,1)$, $i = 1, 2$, with $a_1 + a_2 < 1$ such that for all $x, y \in X$,

$$d(T^2x, T^2y) \leq a_1 d(Tx, Ty) + a_2 d(dx, y). \tag{1.1}$$

Theorem 1.1. *[14] Let (X, d) be a complete metric space. Then any convex contraction mapping of order 2 has a fixed point which is unique.*

Definition 1.4. [14] Let (X, d) be a metric space. The continuous selfmap T on X is called a two-sided convex contraction mapping if there exist $a_i, b_i \in (0,1)$, $i = 1, 2$, with $a_1 + a_2 + b_1 + b_2 < 1$ such that for all $x, y \in X$,

$$d(T^2x, T^2y) \leq a_1 d(x, Tx) + a_2 d(Tx, T^2x) + b_1 d(y, Ty) + b_2 d(Ty, T^2y). \tag{1.2}$$

Theorem 1.2. *[14] Let (X, d) be a complete metric space. Then any two-sided convex contraction mapping has a unique fixed point.*

Remark 1.1. [5] The assumption of continuity condition of Theorem 1.1 and Theorem 1.2 can be replaced by a relatively weaker condition of orbital continuity.

Definition 1.5. [5] Let (X, d) is a metric space. A self maping T on X is called orbitally continuous at a point $x^* \in X$, if for any $\{x_n\} \subseteq O(x, T)$ we have

$$x_n \to x^* \textit{ implies } Tx_n \to Tx^* \textit{ as } n \to +\infty,$$

where $O(x, T) = \{T^n x \mid n = 0, 1, 2, ...\}$.

Recently, a number of fixed point theorems for convex contraction mapping have been obtained by various authors (see e.g. [2, 3, 5, 7, 8, 11, 15, 19, 21] and references therein).

2. Convex Contraction Mappings on $\mathcal{F}-$Metric Spaces

In this section, we prove several fixed point theorems for convex contractions mappings defined on a $\mathcal{F}$-metric space.

Theorem 2.1. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let T be a convex contraction of order 2 on X. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. We can define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. In case $x_m = x_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$, then it is clear that x_m is a fixed point of T. So assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Set $v = \max\{D(x_0, Tx_0), D(Tx_0, T^2x_0)\}$. Using (1.1), we have the following relations:

$$D(T^3x_0, T^2x_0) \leq a_1 D(T^2x_0, Tx_0) + a_2 D(Tx_0, x_0) \leq v(a_1 + a_2),$$

similarly,

$$\begin{aligned} D(T^4x_0, T^3x_0) &\leq a_1 D(T^3x_0, T^2x_0) + a_2 D(T^2x_0, Tx_0) \\ &\leq a_1 v(a_1 + a_2) + a_2 v \\ &\leq v(a_1 + a_2), \end{aligned}$$

as well as

$$\begin{aligned} D(T^5x_0, T^4x_0) &\leq a_1 D(T^4x_0, T^3x_0) + a_2 D(T^3x_0, T^2x_0) \\ &\leq a_1 v(a_1 + a_2) + a_2 v(a_1 + a_2) \\ &= v(a_1 + a_2)^2. \end{aligned}$$

An induction argument shows that

$$D(T^{2m+1}x_0, T^{2m}x_0) \leq v(a_1+a_2)^m, \tag{2.1}$$

and

$$D(T^{2m-1}x_0, T^{2m}x_0) \leq v(a_1+a_2)^{m-1}, \tag{2.2}$$

for all $m \in \mathbb{N}$. Now, we show that $\{T^n x_0\}$ is a $\mathcal{F}$-Cauchy sequence. Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that D_3 is satisfied. Let $\varepsilon > 0$ be fixed. From $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \textit{ implies } f(t) < f(\varepsilon) - \alpha. \tag{2.3}$$

Let $m, n \in \mathbb{N}$ and $n > m$. If $m = 2k$ or $m = 2k+1$, from (2.1) and (2.2), we have

$$\sum_{i=m}^{n-1} D(T^i x_0, T^{i+1}x_0) \leq 2v(a_1+a_2)^k \left(\frac{1}{1-(a_1+a_2)}\right).$$

Since $a_1 + a_2 < 1$, we have

$$\lim_{k\to+\infty} 2v(a_1+a_2)^k \left(\frac{1}{1-(a_1+a_2)}\right) = 0.$$

Then there exists some $N \in \mathbb{N}$ such that

$$0 < 2v(a_1+a_2)^k \left(\frac{1}{1-(a_1+a_2)}\right) < \delta$$

for all $k \geq N$. Using (2.3) and $(\mathcal{F}_1)$, we get

$$\begin{aligned} f\left(\sum_{i=m}^{n-1} D(T^i x_0, T^{i+1}x_0)\right) &\leq f\left(2v(a_1+a_2)^k \left(\frac{1}{1-(a_1+a_2)}\right)\right) \\ &< f(\varepsilon) - \alpha. \end{aligned} \tag{2.4}$$

From (D_3) and (2.4), for $n > m \geq N$, we have

$$f(D(T^m x_0, T^n x_0)) \leq \left(\sum_{i=m}^{n-1} D(T^i x_0, T^{i+1}x_0)\right) + \alpha < f(\varepsilon).$$

Using $(\mathcal{F}_1)$, we obtain $D(T^m x_0, T^n x_0) < \varepsilon$, $n > m \geq N$. So $\{x_n\}$ is $\mathcal{F}$-Cauchy in the $\mathcal{F}$-complete $\mathcal{F}$-metric space X, so there exists $x^* \in X$ such that, $\lim_{n\to\infty} D(x_n, x^*) = 0$. Since T is $\mathcal{F}$-continuous, then, we have

$$Tx^* = T(\lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} Tx_n = x^*,$$

so x^* is the fixed point of T. Finally, we shall show that the fixed point is unique. To this end, we assume that there exists another fixed point z^* and $D(x^*, z^*) > 0$. From (1.1), we have

$$\begin{aligned} D(x^*, z^*) &= D(T^2x^*, T^2z^*) \leq a_1 D(Tx^*, Tz^*) + a_2 D(x^*, z^*) \\ &= (a_1+a_2) D(x^*, z^*). \end{aligned}$$

Since $a_1 + a_2 < 1$, we get $x^* = z^*$. □

Example 2.1. Let $X = [0, \infty)$ be endowed with the $\mathcal{F}$-metric given in Example 1.1. Define $T : X \to X$ by $Tx = \frac{x}{2} + 1$. Hence, for $a_1 = 0$ and $a_2 = \frac{1}{2}$, all the conditions of Theorem 2.1 are satisfied and T has a unique fixed point in X.

Theorem 2.2. *Let (X, D) be $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a two-sided convex contraction mapping on X. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. We define the Picard iteration sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Set $v = \max\{D(x_0, Tx_0), D(Tx_0, T^2x_0)\}$. From (1.2), we have

$$\begin{aligned} D(T^3x_0, T^2x_0) \quad &\leq \quad a_1 D(Tx_0, T^2x_0) + a_2 D(T^2x_0, T^3x_0) \\ &+ \quad b_1 D(x_0, Tx_0) + b_2 D(Tx_0, T^2x_0), \end{aligned}$$

then, we have

$$D(T^3x_0, T^2x_0) \leq (\frac{\lambda}{\gamma})v,$$

where $\lambda = a_1 + b_1 + b_2$ and $\gamma = 1 - a_2$. Similarly we obtain the following relation,

$$D(T^4x_0, T^3x_0) \leq (\frac{\lambda}{\gamma})v,$$

and

$$D(T^5x_0, T^4x_0) \leq (\frac{\lambda}{\gamma})^2 v.$$

Continuing this process, we obtain

$$D(T^{2m+1}x_0, T^{2m}x_0) \leq (\frac{\lambda}{\gamma})^m v, \tag{2.5}$$

and

$$D(T^{2m-1}x_0, T^{2m}x_0) \leq (\frac{\lambda}{\gamma})^{m-1} v. \tag{2.6}$$

Now, we show that $\{T^n x_0\}$ is a $\mathcal{F}$-Cauchy sequence. Let $m, n \in \mathbb{N}$ and $n > m$. If $m = 2k$ or $m = 2k + 1$, from (2.5) and (2.6), we have

$$\sum_{i=m}^{n-1} D(T^i x_0, T^{i+1}x_0) \leq 2v(\frac{\lambda}{\gamma})^k (\frac{1}{1 - \frac{\lambda}{\gamma}}).$$

Since $\frac{\lambda}{\gamma} < 1$, we have

$$\lim_{k \to +\infty} 2v(\frac{\lambda}{\gamma})^k (\frac{1}{1 - \frac{\lambda}{\gamma}}) = 0.$$

Using a similar technique to that in the proof of Theorem 2.1, it is easy to see that $\{x_n\}$ is a $\mathcal{F}$-Cauchy sequence in $\mathcal{F}$-complete $\mathcal{F}$-metric. Then, there exists x^* such that, $\lim_{n\to\infty} D(x_n, x^*) = 0$. Since T is $\mathcal{F}$-continuous, we have

$$Tx^* = T(\lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} Tx_n = x^*,$$

so x^* is the fixed point T. For the uniqueness of the fixed point x^*, assume z^* is another fixed point of T and $D(x^*, z^*) > 0$. From (1.2), we have

$$\begin{aligned} D(x^*, z^*) = D(T^2x^*, T^2z^*) &\leq a_1D(x^*, Tx^*) + a_2D(Tx^*, T^2x^*) \\ &+ b_1D(z^*, Tz^*) + b_2D(Tz^*, T^2z^*) \\ &\leq (a_1 + a_2 + b_1 + b_2)D(x^*, z^*). \end{aligned}$$

Since $a_1 + a_2 + b_1 + b_2 < 1$, we obtain $D(x^*, z^*) = 0$ that is $x^* = z^*$. □

Example 2.2. Let $X = \{0, 1, 2\}$ be endowed with the $\mathcal{F}$-metric given in Example 1.2. Define $T : X \to X$ by $T0 = T2 = 0$ and $T1 = 2$. If $x = 0$ and $y = 1$, then for all $\lambda \in (0, 1)$, we have

$$D(T0, T1) = D(0, 2) = e^2 > \lambda e = \lambda D(0, 1).$$

Hence, T does not satisfy the condition of Banach contraction principle [17]. Since $T^2x = 0$ for all $x \in X$, then, all assumption of Theorem 2.2 are satisfied. Hence T has a unique fixed point.

Now, we give the definitions of $(\alpha, \beta)-$admissible convex contraction of order 2 and two-sided convex contraction mappings in the setting of $\mathcal{F}$-metric space and prove several fixed point theorems for such mappings on $\mathcal{F}$-metric spaces.

Definition 2.1. Let (X, D) be an $\mathcal{F}$-metric space and $T : X \to X$ be a cyclic $(\alpha, \beta)-$admissible mapping. We say that T is a $(\alpha, \beta)-$admissible convex contraction of order 2 if T is orbitally continuous and there exist $a_i \in (0, 1)$, $i = 1, 2$, such that

(2.7) $\quad \alpha(x)\beta(y) \geq 1$ *implies* $D(T^2x, T^2y) \leq a_1D(Tx, Ty) + a_2D(x, y)$,

where $a_1 + a_2 < 1$ for all $x, y \in X$.

Theorem 2.3. *Let (X, D) be an $\mathcal{F}-$complete $\mathcal{F}-$metric space and $T : X \to X$ be a $(\alpha, \beta)-$admissible convex contraction mapping of order 2. Assume, there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary in X. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{m+1} = x_m$ for some $m \in X$, then x_m is a fixed point of T. So, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\beta(x_0) \geq 1$ and $T : X \to X$ is a $(\alpha, \beta)-$admissible mapping then $\alpha(Tx_0) = \alpha(x_1) \geq 1$ which implies $\beta(T^2x_0) = \beta(x_2) \geq 1$. By continuing this process, we have $\beta(T^{2n}x_0) \geq 1$ and

$\alpha(T^{2n-1}x_0) \geq 1$ for all $n \in \mathbb{N}$. Again, since T is a cyclic $(\alpha,\beta)-$admissible mapping and $\alpha(x_0) \geq 1$ by similarly, it can be shown that, $\beta(T^{2n-1}x_0) \geq 1$ and $\alpha(T^{2n}x_0) \geq 1$ for all $n \in \mathbb{N}$. Then, we obtain $\alpha(T^n x_0) = \alpha(x_n) \geq 1$ and $\beta(T^n x_0) = \beta(x_n) \geq 1$ for all $n \in \mathbb{N}$. Let $v = \max\{D(x_0, Tx_0), D(T^2x_0, Tx_0)\}$. Since $\alpha(Tx_0)\beta(x_0) \geq 1$, from the inequlity (2.7), we have

$$\begin{aligned} D(T^3x_0, T^2x_0) &= D(T^2(Tx_0), T^2x_0) \\ &\leq a_1 D(T^2x_0, Tx_0) + a_2 D(Tx_0, x_0) \\ &\leq (a_1 + a_2)v. \end{aligned}$$

Again, $\alpha(T^2x_0)\beta(Tx_0) \geq 1$, then we have

$$\begin{aligned} D(T^4x_0, T^3x_0) &= D(T^2(T^2x_0), T^2(Tx_0)) \\ &\leq a_1 D(T^3x_0, T^2x_0) + a_2 D(T^2x_0, Tx_0) \\ &\leq (a_1 + a_2)v. \end{aligned}$$

By continuing this process and using a similar technique to that in the proof of Theorem 2.1, it is easy to see that

$$D(T^{2m+1}x_0, T^{2m}x_0) \leq v(a_1 + a_2)^m,$$

and

$$D(T^{2m-1}x_0, T^{2m}x_0) \leq v(a_1 + a_2)^{m-1},$$

for all $m \in \mathbb{N}$ and $\{x_n\}$ is $\mathcal{F}$-Cauchy in the $\mathcal{F}$-complete $\mathcal{F}$-metric space X. Then, there exists $x^* \in X$ such that $\lim_{n\to+\infty} D(x_n, x^*) = 0$. Since T is orbitally $\mathcal{F}$-continuous, we have

$$Tx^* = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} x_{n+1} = x^*.$$

We claim that the fixed point of T is unique. Assume that, on contrary, there exists another fixed point $z^* \in X$ of T such that $D(x^*, z^*) > 0$. Since $\alpha(x^*)\beta(z^*) \geq 1$, it follows from (2.7) that

$$D(x^*, z^*) = D(T^2x^*, T^2z^*) \leq a_1 D(Tx^*, Tz^*) + a_2 D(x^*, z^*) = (a_1 + a_2) D(x^*, z^*).$$

Since $a_1 + a_2 < 1$, it follows that $x^* = z^*$. Consequently, T has a unique fixed point. This completes proof of the Theorem 2.3. □

Definition 2.2. Let (X, D) be an $\mathcal{F}$-metric space and $T : X \to X$ be a cyclic $(\alpha,\beta)-$admissible mapping. We say that T is a $(\alpha,\beta)-$admissible two-sided convex contraction if T is orbitally continuous and there exist $a_i, b_i \in (0,1)$, $i = 1, 2$, such that

$$\text{(2.8)} \quad \alpha(x)\beta(y) \geq 1 \textit{ implies } \begin{aligned}[t] D(T^2x, T^2y) &\leq a_1 D(x, Tx) + a_2 D(Tx, T^2x) \\ &+ b_1 D(y, Ty) + b_2 D(Ty, T^2y), \end{aligned}$$

where $a_1 + a_2 + b_1 + b_2 < 1$ for all $x, y \in X$.

Theorem 2.4. *Let (X, D) be an $\mathcal{F}-$complete $\mathcal{F}-$metric space and $T : X \to X$ be a $(\alpha, \beta)-$admissible two-sided convex contraction. Assume, there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.*

Proof. The proof is similar to Theorem 2.2 and Theorem 2.3, therefore we omit it. $\square$

Example 2.3. Consider the $\mathcal{F}$-metric space given in Example 1.1. Let

$$Tx = \begin{cases} -\frac{x}{3}, & x \in [-3, 3] \\ x^3, & \text{otherwise} \end{cases}$$

and $\alpha, \beta : X \to [0, +\infty)$ be given by

$$\alpha(x) = \begin{cases} 1, & x \in [-3, 0] \\ 0, & \text{otherwise} \end{cases} \qquad \beta(x) = \begin{cases} 1, & x \in [0, 3] \\ 0, & \text{otherwise} \end{cases}$$

First, we show that T is an (α, β)-admissible mapping. Let $x \in X$, if $\alpha(x) \geq 1$, then $x \in [-3, 0]$ and so $Tx \in [0, 3]$, that is $\beta(Tx) \geq 1$. Also, if $\beta(x) \geq 1$, then $\alpha(Tx) \geq 1$. Thus T is a cyclic (α, β)-admissible mapping. Let $x, y \in X$ and $\alpha(x)\beta(y) \geq 1$. Then $x \in [-3, 0]$ and $y \in [0, 3]$. Then, we get

$$D(T^2x, T^2y) = D(\frac{x}{9}, \frac{y}{9}) = |\frac{x}{9} - \frac{y}{9}| = \frac{1}{9}D(x, y) \leq \frac{1}{2}D(x, y).$$

Then, all assumption of Theorem 2.3 for $a_1 = 0$ and $a_2 = \frac{1}{2}$ are satisfied. Hence T has a fixed point.

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SOME NOTES ON KENMOTSU MANIFOLD

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Abstract. In the present paper, we will deal with a Kenmotsu manifold M. Firstly, we will study the notion of torse-forming vector field on such a manifold. Then, we will investigate some curvature conditions such as $Q.\mathcal{M} = 0$ and $C.Q = 0$ on such a manifold and obtain some necessary conditions for such a manifold given as to be Einstein. Also, we will study a Kenmotsu manifold M admitting a Ricci soliton and give an example for this manifold.

Keywords: Kenmotsu manifold; torse-forming vector field; Einstein manifold; Ricci soliton.

1. Introduction

A Riemannian manifold (M, g) is called a Ricci soliton if there exists a constant $\lambda \in \mathbb{R}$ and a vector field $V \in \Gamma(TM)$ such that

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.1}$$

where $\pounds_V g$ denotes the Lie-derivative of the metric tensor g along vector field V, S is the Ricci tensor of M and X, Y are arbitrary vector fields on M. If $\pounds_V g = 0$ and $\pounds_V g = \rho g$, then potential vector field V is said to be Killing and conformal Killing, respectively, where ρ is a function. Also, when V is zero or Killing in (1.1), then the Ricci soliton reduces to Einstein manifold. So, it is considered as a natural generalization of Einstein metric. In addition, a Ricci soliton is called a gradient if the potential vector field V is the gradient of a potential function $-f$ (i.e., $V = -\nabla f$) and is called shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

The notion of Ricci soliton in Riemannian geometry was introduced by Hamilton in 1988 [11]. This notion corresponds to the self-similar solution of Hamilton's Ricci

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flow: $\frac{\partial}{\partial t}g = -2S$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphims and scaling. Also, Ricci solitons model the formation of singularities in the Ricci flow. In the framework of the contact geometry, they have been studied by many mathematicians in some different classes of contact geometry since Sharma applied Ricci solitons to K-contact manifolds [20]. For the recent studies on Ricci solitons, we refer to ([1], [7], [9], [10], [15], [17], [21], [24] and [25]).

On the other hand, torse-forming vector fields were firstly defined and studied by Yano [22]. They appear in many areas of differential geometry and physics. In recent years, they were studied by different authors such as Blaga et al. [2], Crasmareanu [8], Mandal et al. [14], Mihai et al. [16] and many others. According to Yano, a vector field v on a Riemannian manifold (M, g) is called torse-forming if it satisfies the following condition

$$\nabla_X v = fX + \alpha(X)v \tag{1.2}$$

for any $X \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M, α is a $1-$form and f is a smooth function on M. If the $1-$form α vanishes identically in (1.2), the vector field v is called a concircular [6]. If $\alpha = 0$ and $f = 1$ in (1.2), then v is called a concurrent vector field [5]. If $f = -1$ in (1.2), then v is called an irrotational vector field [2]. Also, the vector field v is called a recurrent if it satisfies (1.2) with $f = 0$.

The paper is organized as follows:

Section 1 is concerned with introduction. In section 2, we give some basic notions which are going to be needed. In section 3, we consider a Kenmotsu manifold M endowed with a torse-forming vector field v and find that the vector field v is a pointwise collinear with the structure vector field ξ. In section 4, we study a Kenmotsu manifold M under some curvature conditions and deal with Ricci solitons on such a manifold. Also, we give an example to support our results.

2. Preliminaries

In this section, we shall give a brief review of some fundamental definitions and formulas of almost contact metric manifolds from [3], [13] and [23].

A $(2n+1)-$dimensional smooth manifold M is an almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g) such that φ is a tensor field of type $(1,1)$, ξ is a vector field (called the characteristic vector field) of type $(0,1)$, $1-$ form η is a tensor field of type $(1,0)$ on M and the Riemannian metric g satisfies the following relations:

$$\varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$
$$\eta(\xi) = 1, \tag{2.2}$$
$$\varphi\xi = 0, \tag{2.3}$$
$$\eta \circ \varphi = 0 \tag{2.4}$$

and

$$g(\varphi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.5}$$
$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.6}$$
$$\eta(X) = g(X,\xi) \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$.

Remark that the canonical distribution D is $\varphi-$invariant since $D = Im\varphi$. Also, the characteristic vector field ξ is orthogonal to D and therefore the tangent bundle splits orthogonally:

$$TM = D \oplus \{\xi\}. \tag{2.8}$$

If the following condition is satisfied for an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, then it is called a Kenmotsu manifold

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.9}$$

where ∇ is the Levi-Civita connection on M. For a Kenmotsu manifold, we also have

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.10}$$
$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$
$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.12}$$
$$S(X,\xi) = -2n\eta(X), \tag{2.13}$$
$$S(\xi,\xi) = -2n, \tag{2.14}$$
$$Q\xi = -2n\xi, \tag{2.15}$$

where S and R are the Ricci tensor and Riemann curvature tensor of M, respectively and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

Now, we recall some basic notions from [4], [18], [19], [23] as follows:

The projective curvature tensor $\mathcal{M}$, the extended projective curvature tensor $\mathcal{M}^e$ and the concircular curvature tensor C of a $(2n+1)-$dimensional manifold (M, g) are defined by

$$\begin{aligned} M(X,Y)Z &= R(X,Y)Z - \frac{1}{4n}\{S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY\} \qquad (n \geq 1) \end{aligned} \tag{2.16}$$
$$\begin{aligned} M^e(X,Y)Z &= M(X,Y)Z - \eta(X)M(\xi,Y)Z \\ &\quad -\eta(Y)M(X,\xi)Z - \eta(Z)M(X,Y)\xi \end{aligned} \tag{2.17}$$

and

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{2.18}$$

for any $X, Y, Z \in \Gamma(TM)$, where r stands for the scalar curvature of M. From (2.11), (2.12), (2.13) and (2.15), we also have

$$M(X,Y)\xi = \eta(X)Y - \eta(Y)X - \frac{1}{4n}\{2n\eta(X)Y - 2n\eta(Y)X + \eta(Y)QX - \eta(X)QY\}, \tag{2.19}$$

$$R(QX,Y)\xi = -2n\eta(X)Y - \eta(Y)QX, \tag{2.20}$$

$$R(X,QY)\xi = \eta(X)QY + 2n\eta(Y)X, \tag{2.21}$$

$$R(X,Y)Q\xi = -2n(\eta(X)Y - \eta(Y)X). \tag{2.22}$$

On the other hand, a Riemannian manifold (M, g) is called $\eta-$Einstein if there exists two real constants a and b such that the Ricci tensor field S of M satisfies

$$S = ag + b\eta \otimes \eta.$$

Also, if the constant b is equal to zero, then M is called Einstein.

3. Torse-forming Vector Field on Kenmotsu Manifold

In this section, we deal with a Kenmotsu manifold M endowed with a torse-forming vector field v and give some characterizations for such a vector field.

Now, we begin to this section with the following:

Proposition 3.1. *Let M be a Kenmotsu manifold endowed with a torse-forming vector field v. Then, the vector field v is never on the distribution D of M.*

Proof. Let us assume that the vector field v is on the distribution D. Then, using the fact that $g(v, \xi) = 0$, we have

$$g(\nabla_X v, \xi) + g(v, \nabla_X \xi) = 0 \tag{3.1}$$

for any $X \in \Gamma(TM)$. Since the vector field v is a torse-forming on M, from (1.2), (2.10) and (3.1), one has

$$f\eta(X) + g(X, v) = 0$$

equivalently

$$g(f\xi, X) = -g(X, v).$$

Removing X in the above equation gives

$$v = -f\xi. \tag{3.2}$$

This is a contradiction. Therefore, the vector field v is never on distribution D. □

From (3.2), we can state the following corollary:

Corollary 3.1. *Let M be a Kenmotsu manifold endowed with a torse-forming vector field v. Then, v is a pointwise collinear with the structure vector field ξ.*

Theorem 3.1. *Let M be a Kenmotsu manifold endowed with a torse-forming vector field v such that v is a pointwise collinear with the structure vector field ξ. Then, we have the followings:*

i) The vector field v is a Killing on M.

ii) If M admits a Ricci soliton with potential vector field v , then it is an expanding.

Proof. Let v be a pointwise collinear with the structure vector field ξ. From (3.2), we write $v = -f\xi$. Then, we have

$$\begin{aligned} \nabla_X v &= \nabla_X(-f\xi), \\ &= -X(f)\xi - f\nabla_X\xi, \\ &= -X(f)\xi - f(X - \eta(X)\xi) \end{aligned} \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$. Since the vector field v is a torse-forming on M, from equations (1.2), (3.2) and (3.3), one has

$$-X(f)\xi - f(X - \eta(X)\xi) = fX - f\alpha(X)\xi. \tag{3.4}$$

Also, taking the inner product of (3.4) with ξ and using the equations (2.2), (2.7), we get

$$X(f) = -f\eta(X) + f\alpha(X). \tag{3.5}$$

Again, taking the inner product of (3.4) with the arbitrary vector field Y and using (2.2), (2.5) gives

$$-X(f)\eta(Y) - fg(\varphi X, \varphi Y) = fg(X, Y) - f\alpha(X)\eta(Y). \tag{3.6}$$

By virtue of (2.5), (3.5) and (3.6), we find

$$2fg(\varphi X, \varphi Y) = 0. \tag{3.7}$$

On the other hand, let $\{e_1, e_2, ..., e_{2n}, e_{2n+1} = \xi\}$ be an orthonormal basis of T_pM, $p \in M$. Putting $X = Y = e_i$ in (3.7) and summing over $i = 1, 2, ..., 2n+1$, we obtain

$$f = 0. \tag{3.8}$$

From (3.2) and (3.8), we have $v = 0$. As a result of this, the vector field v is a Killing on M. Therefore, we write

$$(\pounds_v g)(X, Y) = 0$$

for any $X, Y \in \Gamma(TM)$.

Now, let us consider that M admits a Ricci soliton with potential vector field v. Then, the equation (1.1) reduces to

$$S(X,Y) = -\lambda g(X,Y) \tag{3.9}$$

Putting $X = Y = \xi$ in (3.9) and using (2.13), we get $\lambda = 2n$. This shows that the Ricci soliton is expanding. Thus, the proof is completed. □

Using the equation (3.8), we can give the following corollary:

Corollary 3.2. *Let M be a Kenmotsu manifold endowed with a torse-forming vector field v such that v is a pointwise collinear with the structure vector field ξ. Then, the vector field v is never irrotational on M.*

4. Ricci Solitons and Some Curvature Conditions on Kenmotsu Manifold

In this section, we give some important characterizations which classify a Kenmotsu manifold M under some curvature conditions and study Ricci solitons on M.

The first result of this section is the following:

Theorem 4.1. *Let M be a Kenmotsu manifold admiting a Ricci soliton with the potential vector field V. If V is orthogonal to ξ, then the Ricci soliton is expanding.*

Proof. It follows immediately from the definition of Lie-derivative, we have

$$\begin{aligned} (\pounds_V g)(\xi,\xi) &= g(\nabla_\xi V,\xi) + g(\nabla_\xi V,\xi) \\ &= 2g(\nabla_\xi V,\xi). \end{aligned} \tag{4.1}$$

From the fact that $\nabla_\xi \xi = 0$, it is easy to see that

$$\nabla_\xi(g(V,\xi)) = g(\nabla_\xi V,\xi). \tag{4.2}$$

Since M is a Ricci soliton, with the help of (1.1), (4.1) and (4.2), we get

$$\begin{aligned} S(\xi,\xi) &= -\frac{1}{2}(\pounds_V g)(\xi,\xi) - \lambda g(\xi,\xi) \\ &= -g(\nabla_\xi V,\xi) - \lambda \\ &= -\nabla_\xi(g(V,\xi)) - \lambda. \end{aligned} \tag{4.3}$$

Also, making use of (2.14) and (4.3), we find that

$$\nabla_\xi(g(V,\xi)) = 2n - \lambda. \tag{4.4}$$

If the potential vector field V is orthogonal to ξ, then the equation (4.4) becomes

$$\lambda = 2n$$

which shows that the Ricci soliton is expanding. Therefore, this completes the proof of the theorem. $\square$

The next example supports the Theorem 4.1 as follows:

Example 4.1. [12] We consider the three-dimensional Riemannian manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)\}$ and the linearly independent vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z},$$

where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_i, e_i) &= 1 \\ g(e_i, e_j) &= 0 \quad for \quad i \neq j. \end{aligned}$$

and is given by

$$g = \frac{1}{z^2}\Big\{dx \otimes dx + dy \otimes dy + dz \otimes dz\Big\}.$$

Also, let η, φ be the $1-$ form and the $(1, 1)-$tensor field, respectively defined by

$$\eta(Z) = g(Z, e_3), \quad \varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0$$

for any $Z \in \Gamma(TM)$. Hence, $(M, \varphi, \xi, \eta, g)$ becomes an almost contact metric manifold with the characteristic vector field $e_3 = \xi$.

By direct calculations, we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 \quad and \quad [e_2, e_3] = e_2.$$

On the other hand, using Koszul's formula for the Riemannian metric g, we get:

$$\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_2} e_3 = e_2, \quad \nabla_{e_3} e_3 = 0 \tag{4.5}$$

and others

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = 0, \quad \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = -e_3. \tag{4.6}$$

Therefore, the manifold M is a $3-$dimensional Kenmotsu manifold. Using the equations (4.5) and (4.6), we find

$$\begin{aligned} &R(e_1, e_2)e_3 = 0, && R(e_1, e_3)e_2 = 0, && R(e_2, e_3)e_1 = 0, \\ &R(e_1, e_2)e_2 = -e_1, && R(e_1, e_2)e_1 = e_2, && R(e_1, e_3)e_3 = -e_1, \\ &R(e_1, e_3)e_1 = e_3, && R(e_2, e_3)e_3 = -e_2, && R(e_3, e_2)e_2 = -e_3, \end{aligned}$$

which yields

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2 \quad and \quad S(e_i, e_j) = 0 \tag{4.7}$$

for all $i, j = 1, 2, 3$ $(i \neq j)$. In this case, the manifold M is a Ricci soliton with potential vector field e_1 or e_2 which satisifes the equation (1.1) for $\lambda = 2$.

Theorem 4.2. *Let M be a Kenmotsu manifold such that the condition $Q.\mathcal{M}=0$ is satisfied. Then, M is an Einstein manifold.*

Proof. Suppose that M satisfies the condition $(Q.\mathcal{M})(X,Y)Z=0$, namely,

$$Q(\mathcal{M}(X,Y)Z)-\mathcal{M}(QX,Y)Z-\mathcal{M}(X,QY)Z-\mathcal{M}(X,Y)QZ=0 \tag{4.8}$$

for any $X,Y,Z\in\Gamma(TM)$, where Q stands for the Ricci operator defined by $S(X,Y)=g(QX,Y)$. Putting $Z=\xi$ in (4.8) gives

$$Q(\mathcal{M}(X,Y)\xi)-\mathcal{M}(QX,Y)\xi-\mathcal{M}(X,QY)\xi-\mathcal{M}(X,Y)Q\xi=0. \tag{4.9}$$

For the first and second term of (4.9), if we use (2.13), (2.15), (2.19) and (2.20), we get

$$\begin{aligned} Q(M(X,Y)\xi) &= \eta(X)QY-\eta(Y)QX-\frac{1}{4n}\{2n\eta(X)QY-2n\eta(Y)QX \\ &\quad +\eta(Y)Q^2X-\eta(X)Q^2Y\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \mathcal{M}(QX,Y)\xi &= -2n\eta(X)Y-\eta(Y)QX-\frac{1}{4n}\{-4n^2\eta(X)Y-2n\eta(Y)QX \\ &\quad +2n\eta(X)QY+\eta(Y)Q^2X\}. \end{aligned} \tag{4.11}$$

For the third and fourth term of (4.9), making use of (2.13), (2.15), (2.21) and (2.22), we derive

$$\begin{aligned} M(X,QY)\xi &= \eta(X)QY+2n\eta(Y)X-\frac{1}{4n}\{4n^2\eta(Y)X+2n\eta(X)QY \\ &\quad -2n\eta(Y)QX-\eta(X)Q^2Y\}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathcal{M}(X,Y)Q\xi &= -2n(\eta(X)Y-\eta(Y)X)-\frac{1}{4n}\{4n^2\eta(Y)X-4n^2\eta(X)Y \\ &\quad -2n\eta(Y)QX+2n\eta(X)QY\}. \end{aligned} \tag{4.13}$$

If we substitute (4.10)-(4.13) in (4.9), after some calculations we obtain

$$2n\eta(X)Y-2n\eta(Y)X+\eta(X)QY-\eta(Y)QX=0. \tag{4.14}$$

Putting $Y=\xi$ in (4.14) and using the equalities (2.2), (2.15) yields

$$QX=-2nX. \tag{4.15}$$

Taking the inner product of (4.15) with W, we have

$$S(X,W)=-2ng(X,W) \tag{4.16}$$

for any $W\in\Gamma(TM)$. This completes the proof of the theorem. □

Using the equality (4.16), we can give the following corollary.

Corollary 4.1. *Let M be a Kenmotsu manifold such that the condition $Q.\mathcal{M} = 0$ is satisfied. If M admits a Ricci soliton with the potential vector field ξ, then the Ricci soliton is expanding.*

Proof. It follows from the definition of the Lie-derivative and from (2.10), we have

$$(4.17)\qquad \begin{aligned}(\pounds_\xi g)(X,Y) &= g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\\ &= g(X-\eta(X)\xi, Y) + g(Y-\eta(Y)\xi, X)\\ &= 2g(X,Y) - 2\eta(X)\eta(Y)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Since M is a Ricci soliton, from (1.1) we write

$$(4.18)\qquad (\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0.$$

If we use the equalities (4.16) and (4.17) in (4.18), we get

$$(4.19)\qquad (2-4n+2\lambda)g(X,Y) - 2\eta(X)\eta(Y) = 0.$$

Putting $X = Y = \xi$ in (4.19) and using (2.2) gives $\lambda = 2n$ which means that the Ricci soliton is expanding. This result ends the proof of the corollary. □

Theorem 4.3. *Let M be a Kenmotsu manifold such that the condition $C.Q = 0$ is satisfied. Then, M is either of constant scalar curvature or M is an Einstein manifold.*

Proof. Let us suppose that the manifold satisfies the condition $(C(X,Y).Q)Z = 0$, that is,

$$(4.20)\qquad C(X,Y)QZ - Q(C(X,Y)Z) = 0$$

for any $X, Y, Z \in \Gamma(TM)$. Substituting $Y = \xi$ in (4.20), one has

$$(4.21)\qquad C(X,\xi)QZ - Q(C(X,\xi)Z) = 0.$$

Futhermore, with the help of (2.7) and (2.12), we get

$$(4.22)\qquad C(X,\xi)Z = (1 + \frac{r}{2n(2n+1)})(g(X,Z)\xi - \eta(Z)X).$$

Replacing Z by QZ in (4.22) and using (2.7), (2.13) we have

$$(4.23)\qquad C(X,\xi)QZ = (1 + \frac{r}{2n(2n+1)})(S(X,Z)\xi + 2n\eta(Z)X).$$

Applying Q to the both sides of (4.22) and from (2.15), we infer

$$(4.24)\qquad Q(C(X,\xi)Z) = (1 + \frac{r}{2n(2n+1)})(-2ng(X,Z)\xi - \eta(Z)QX).$$

From (4.21), (4.23) and (4.24), we write

$$(1+\frac{r}{2n(2n+1)})(S(X,Z)\xi+2n\eta(Z)X+2ng(X,Z)\xi+\eta(Z)QX)=0.$$

Taking the inner product of the above equation with ξ and making use of (2.7), (2.13), we have

$$(1+\frac{r}{2n(2n+1)})(S(X,Z)+2ng(X,Z))=0$$

which implies that

$$r=-2n(2n+1)$$

or

$$S(X,Z)=-2ng(X,Z).$$

This is the desired result. Thus, the proof is completed. □

Theorem 4.4. *Let M be a Kenmotsu manifold with vanishing extended $\mathcal{M}^e-$projective curvature tensor. Then, the followings are satisfied:*

i) M is an Einstein manifold.

ii) M is locally isometric to the hyperbolic space $H^{(2n+1)}(-1)$ if and only if $\mathcal{M}-$projective curvature tensor vanishes.

iii) If M admits a Ricci soliton with the potential vector field V, then V is a conformal Killing on M.

Proof. Let M be a Kenmotsu manifold with vanishing extended $\mathcal{M}^e-$projective curvature tensor. Then, the equation (2.17) becomes

$$M(X,Y)Z=\eta(X)M(\xi,Y)Z+\eta(Y)M(X,\xi)Z+\eta(Z)M(X,Y)\xi \tag{4.25}$$

for any $X,Y,Z\in\Gamma(TM)$. If we take $X=\xi$ in (4.25), we have

$$\eta(Y)M(\xi,\xi)Z+\eta(Z)M(\xi,Y)\xi=0. \tag{4.26}$$

From the equalities (2.11)-(2.16), we get

$$M(\xi,\xi)Z=0 \tag{4.27}$$

and

$$M(\xi,Y)\xi=Y-\eta(Y)\xi-\frac{1}{4n}(-4n\eta(Y)\xi+2nY-QY). \tag{4.28}$$

Using (4.27) and (4.28) in (4.26) and after simple calculations, one has

$$\eta(Z)Y - \frac{1}{2}\eta(Z)Y + -\frac{1}{4n}\eta(Z)QY) = 0. \tag{4.29}$$

Substituting $Z = \xi$ in (4.29), then the equation (4.29) is reduced to

$$QY = -2nY. \tag{4.30}$$

Also, taking the inner product of (4.30) with W, we have

$$S(Y, W) = -2ng(Y, W). \tag{4.31}$$

for any $W \in \Gamma(TM)$. Therefore, M is an Einstein manifold. Making use of (4.30) and (4.31) in (2.16) gives

$$M(X, Y)Z = R(X, Y)Z - \frac{1}{4n}\{-4ng(Y, Z)X + 4ng(X, Z)Y\},$$

that is,

$$M(X, Y)Z = R(X, Y)Z + \{g(Y, Z)X - g(X, Z)Y\}.$$

This proves *ii*).

On the other hand, let us consider that M is a Ricci soliton with the potential vector field V. Then, from (1.1) and (4.31) we conclude that

$$\begin{aligned} (\pounds_V g)(X, Y) &= -2S(X, Y) - 2\lambda g(X, Y) \\ &= (4n - 2\lambda)g(X, Y) \end{aligned}$$

which implies that the potential vector field V is a conformal Killing on M. Consequently, we get the requested results. $\square$

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ON CONFORMALLY BERWALD M-TH ROOT (α, β)-METRICS

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Abstract. In this paper, we study the class of mth root (α, β)-metrics which is a significant class mixed of two classes of metrics: m-th root metrics and (α, β)-metrics. First, we find the necessary and sufficient condition under which the quartic (α, β)-metrics are conformally Berwald. Then, we find the necessary and sufficient condition under which the cubic (α, β)-metrics are conformally Berwald. Finally, we construct some conformal Finslerian invariants.

Keywords: (α, β)-metrics; Finslerian invariants; conformally Berwald metrics; Riemannian metrics.

1. Introduction

The conformal transformations of the class of Riemannian metrics have been well investigated and developed. The class of Finsler metrics are a natural generalization of the class of Riemannian metrics. The conformal transformation of Finsler metrics was initiated by Knebelman in [10] and studied by Hashiguchi in [4]. Let F and $\bar{F}$ be two Finsler metrics on a manifold M. In [4], Hashiguchi proved that F is conformal to $\bar{F}$ if and only if there exists a scalar function $\kappa = \kappa(x)$ such that $\bar{F} = e^{\kappa} F$. The scalar function κ is called the conformal factor. A Finsler metric is called a conformally flat metric if it is locally conformal to a locally Minkowski metric [26]. There are many efforts to find a conformally invariant curvature tensor similar to the Weyl conformal curvature of a Riemannian metric and to establish the condition for a Finsler metric to be conformally flat. In [20], Szilasi-Vincze gave an intrinsic proof of the Weyl theorem, which states that the projective and conformal properties of a Finsler metric determine its metric properties uniquely. Therefore the conformal properties of Finsler metrics deserve extra attention.

A Berwald metric is much closer to a Riemannian metric than the other class of Finsler metrics because any geodesic of a Berwald metric must be that of a Riemannian metric [17]. A Finsler metric F on a manifold M is said to be a Berwald metric

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if there exists a torsion-free affine connection ∇ on M whose parallel transport preserves F, namely, if $c = c(t)$ is a smooth path in M with the endpoints x_1 and x_2, and $P_c : T_{x_1}M \to T_{x_2}M$ is the ∇-parallel transport along c, then for all $y \in T_xM$, $F_{x_2}\big(P_c(y)\big) = F_{x_1}(y)$ holds. Thus a Riemannian metric viewed as a special Berwald metric, with the associated connection ∇ the Levi-Civita connection.

A Finsler metric conformally related to a Berwald metric is called conformally Berwald metric. In [6], Hashiguchi-Ichijyō proved that a Finsler metric $F = F(x, y)$ on a manifold M is conformal to a Berwald metric if and only if it is a Wagner metric (see also [28]). The Wagner metrics form an important class of the so-called generalized Berwald metrics admitting Finsler connections whose horizontal part depends only on the position - more precisely there exists a linear connection on M such that the indicatrix hypersurfaces are invariant under the parallel transport. Also, Berwald metrics in the classical sense are characterized by a similar property of the canonical Berwald connection. If a Berwald metric has vanishing Riemannian curvature, then it is called a locally Minkowski metric. In [8], Hashiguchi-Ichijyō determined all conformally flat Randers surfaces. Then, Hashiguchi proved that a conformally flat Randers metric is conformally Berwald metric and the associated Riemannian metric is also conformally flat [5]. He also studied the converse problem. In [1], Aikou obtained the conditions for a Finsler metric to be locally or globally conformal to a Berwald metric. In [7], Hōjō-Matsumoto-Okubo found the necessary and sufficient conditions under which a Randers metric and Kropina metric be a conformally Berwald metric. In [27], Vincze discussed the problem whether how we can check the conformality of a Finsler metric to a Berwald metric. His method is based on a differential 1-form constructing on the underlying manifold by the help of integral formulas such that its exterior derivative is conformally invariant. If the Finsler metric is conformal to a Berwald metric, then the exterior derivative vanishes [27]. In [15], Matveev-Nikolayevsky obtained some results regarding locally conformally Berwald closed metrics that are not globally conformally Berwald. In [30], Xia-Zhong found some explicit examples of complex Berwald metrics which are neither Hermitian metrics nor conformal changes of complex Minkowski metrics.

In order to find explicit examples of conformally Berwald metrics, one can investigate the class of m-th root Finsler metrics. Let M be an n-dimensional manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F : TM \to \mathbb{R}$ be a scalar function defined by $F = \sqrt[m]{A}$, where $A := \mathfrak{a}_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$ and $\mathfrak{a}_{i_1\dots i_m}$ is symmetric in all its indices. Then F is called an m-th root Finsler metric on M [19]. For more progress, see [21], [24] and [25]. The fourth root metric is called a quartic metric [22][23]. The significant quartic metric $F = \sqrt[4]{y^iy^jy^ky^l}$ is called Berwald-Moór metric which has important role in the theory of space-time structure and gravitation as well as in unified gauge field theories [2][3][16].

We show that every 4-th root metric $F = \sqrt[4]{\mathfrak{a}_{ijkl}(x)y^iy^jy^ky^l}$ on a manifold M of dimension $n \geq 3$ can be written in the following form

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4},$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian and $\beta = b_i(x)y^i$ is a 1-form on M. For $n = 2$, F can be written as $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2}$. Then, we characterize conformally Berwald 4-th root (α, β)-metric as follows.

Theorem 1.1. *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a non-Riemanian quartic (α, β)-metric on an n-dimensional manifold M, where c_i are nonzero constants. Then F is a conformally Berwald metric if and only if β satisfies following*

$$r_{ij} = \frac{r^s_s}{n-1}\Big(a_{ij} - \frac{1}{b^2}b_ib_j\Big) - \frac{1}{b^2}\Big(b_is_j + b_js_i\Big), \tag{1.1}$$

$$s_{ij} = \frac{1}{b^2}\Big(b_is_j - b_js_i\Big) \tag{1.2}$$

and the conformal factor $\kappa = \kappa(x)$ satisfies

$$\kappa_i = -\frac{1}{b^2}\Big(2s_i + \frac{1}{n-1}r^s_sb_i\Big), \tag{1.3}$$

where $\kappa_i := \partial\kappa/\partial x^i$ and $b := ||\beta||_\alpha = \sqrt{a^{ij}b_ib_j}$.

Suppose that the quartic (α, β)-metric $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ is a Berwald metric. Then by Lemma 2.3, β is parallel with respect to α. Therefore $r_{ij} = s_{ij} = 0$ and F satisfies (1.1) and (1.2). In this case, (1.3) implies that $\kappa = constant$. Thus, we conclude the following.

Corollary 1.1. *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a non-Riemannian Berwald quartic (α, β)-metric. Then F is a conformally Berwald metric if and only if the conformal transformation is homothetic.*

It is remarkable that, the Corollary 1.1 confirms the Vincze's theorem in [27] that say a conformal transformation between two non-Riemannian Berwald metrics must be a homothety.

By the same argument used in proof of Theorem 1.1, one can get the following result.

Corollary 1.2. *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2}$ be a non-Riemanian quartic (α, β)-metric on an n-dimensional manifold M, where c_i are nonzero constants. Then F is a conformally Berwald metric if and only if β satisfies (1.1) and (1.2) and the conformal factor $\kappa = \kappa(x)$ satisfies (1.3).*

The third root metric $F = \sqrt[3]{\mathfrak{a}_{ijk}(x)y^iy^jy^k}$ is called the cubic metric. In [29], Wegener studied cubic Finsler metrics of dimensions two and three. Wegener's paper is only an abstract of his PhD thesis without all details and calculations. In [12], Matsumoto wrote an improved version of Wegener's results. In [13], Matsumoto-Numata proved that every cubic (α, β)-metric on a manifold M of dimension $n \geq 3$ can be written in the following form

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}.$$

For $n = 2$, they showed that F is given by $F = \sqrt[3]{\alpha^2\beta}$. In this paper, we prove the following.

Theorem 1.2. *Let (M, F) be an n-dimensional Finsler manifold. Then the following hold:*

(i) The cubic (α, β)-metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ is a conformally Berwald metric if and only if β satisfies

$$r_{ij} = \frac{1}{b^2}(b_j r_i + b_i r_j) - b^r \bar{f}_r(c_1 a_{ij} + 3c_2 b_i b_j) - a_{ij} b^r k_r, \tag{1.4}$$

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) \tag{1.5}$$

and the conformal factor $\kappa = \kappa(x)$ satisfies

$$\kappa_j = \frac{2}{b^2}(r_j - u b_j) - 2(2c_1 + 3c_2 b^2)\bar{f}_j, \tag{1.6}$$

where c_1 and c_2 are nonzero constants, $\kappa_r := \partial\kappa/\partial x^r$, $f := b_i\kappa_j a^{ij}$, $f_i := \partial f/\partial x^i$, and

$$u := \frac{1}{2}(2c_1\bar{f}_r - \kappa_r)b^r, \quad \bar{f}_j := \frac{1}{3b^2(c_1 + c_2b^2)}(s_j + r_j).$$

(ii) The cubic (α, β)-metric $F = \sqrt[3]{\alpha^2\beta}$ is a conformally Berwald metric if and only if β satisfies

$$r_{ij} = \frac{1}{b^2}(b_j r_i + b_i r_j) - b^r(\kappa_r + \frac{1}{3}\bar{f}_r)a_{ij} - \frac{2h}{b^2}b_i b_j, \tag{1.7}$$

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i) \tag{1.8}$$

and the conformal factor $\kappa = \kappa(x)$ satisfies

$$\kappa_j = \frac{2}{b^2}(r_j - h b_j) - \frac{4}{3}\bar{f}_j, \tag{1.9}$$

where

$$h := \frac{1}{6}(2\bar{f}_r - 3\kappa_r)b^r, \qquad \bar{f}_j = \frac{1}{b^2}(s_j + r_j).$$

2. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark some notions about an (α,β)-metric. An (α,β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a scalar function on an open interval $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. The metric α is called the associated Riemannian metric of the (α,β)-metric F. Throughout this paper, we assume that the associated Riemannian metric of an (α,β)-metric is positive-definite.

For an (α,β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, one can define $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\{\theta_i^j := \gamma^j_{ik}(x)dx^k\}$ denote the Levi-Civita connection forms of the Riemannian metric α. Let us put

$$r_{ij} := \tfrac{1}{2}\big(b_{i|j} + b_{j|i}\big), \quad s_{ij} := \tfrac{1}{2}\big(b_{i|j} - b_{j|i}\big),$$
$$r_j := b^ir_{ij}, \quad r := b^ib^jr_{ij}, \quad s_j := b^is_{ij}, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j,$$
$$r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad s_{i0} := s_{ij}y^j, \quad s^i{}_j := a^{im}s_{mj}, \quad r^i{}_j := a^{im}r_{mj}.$$

Then β is parallel with respect to α if and only if $b_{i|j} = 0$ or equivalently $r_{ij} = s_{ij} = 0$.

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian and $\beta = b_i(x)y^i$ is a 1-form on M. Assume that F is conformally related to a Finsler metric $\bar{F}$ on M, that is, there is a scalar function $\kappa = \kappa(x)$ on M such that $\bar{F} = e^{\kappa(x)}F$. It is easy to see that $\bar{F} = \bar{\alpha}\phi(\bar{\beta}/\bar{\alpha})$ is also an (α,β)-metric, where $\bar{\alpha} = e^{\kappa(x)}\alpha$ and $\bar{\beta} = e^{\kappa(x)}\beta$. Put $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x)y^iy^j}$ and $\bar{\beta} = \bar{b}_i(x)y^i$. Let us define

$$b := \|\beta_x\|_\alpha = \sqrt{a^{ij}b_ib_j}, \quad \bar{b} := \|\bar{\beta}_x\|_{\bar{\alpha}} = \sqrt{\bar{a}^{ij}\bar{b}_i\bar{b}_j}.$$

Thus

$$b = \bar{b}. \tag{2.1}$$

Let (M,F) be a Finsler manifold. A global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$, where $G^i = G^i(x,y)$ are given by

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

The vector field $\mathbf{G}$ is called the associated spray to (M,F). F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ is quadratic in $y \in T_xM$ for any $x \in M$. Then (M,F) is called a Berwald manifold. The important described characteristic of a Berwald manifold is that all its tangent spaces are linearly isometric to a common Minkowski space [18].

In order to prove Theorem 1.1, we need the following.

Lemma 2.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian and $\beta = b_i(x)y^i$ is a 1-form on M. Suppose that F is conformally related to a Finsler metric $\bar{F}$ on M, i.e., $\bar{F} = e^{\kappa(x)}F$, where $\kappa = \kappa(x)$ is scalar function on M. Then the following hold*

$$\bar{r}_{ij} = \frac{e^{\kappa}}{2}\left(2r_{ij} + 2fa_{ij} - b_j\kappa_i - b_i\kappa_j\right), \tag{2.2}$$

$$\bar{s}_{ij} = \frac{e^{\kappa}}{2}\left(2s_{ij} - b_j\kappa_i + b_i\kappa_j\right), \tag{2.3}$$

where $\kappa_i := \partial\kappa/\partial x^i$ and $f := \kappa_t b^t$.

Proof. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric which is conformally related to a Finsler metric $\bar{F}$ on M, that is, there is a scalar function $\kappa = \kappa(x)$ on M such that $\bar{F} = e^{\kappa(x)}F$. If we write $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x)y^iy^j}$ and $\bar{\beta} = \bar{b}_i(x)y^i$, then the following hold

$$\bar{a}_{ij} = e^{2\kappa}a_{ij}, \qquad \bar{b}_i = e^{\kappa}b_i. \tag{2.4}$$

Therefore, we get

$$\bar{a}^{ij} = e^{-2\kappa}a^{ij}, \quad \bar{b}^i = e^{-\kappa}b^i.$$

Let G^i and $\bar{G}^i$ be the spray coefficients of F and $\bar{F}$, respectively. By using the Rapcsák's identity, the following relationship between G^i and $\bar{G}^i$ holds

$$\bar{G}^i = G^i + \frac{\bar{F}_{;m}y^m}{2\bar{F}}y^i + \frac{\bar{F}}{2}\bar{g}^{il}\Big\{\bar{F}_{;k,l}y^k - \bar{F}_{;l}\Big\}, \tag{2.5}$$

where "$;$" and "$,$" denote the horizontal and vertical derivation with respect to the Berwald connection of F. Since $F_{;m} = 0$, then the following hold

$$\bar{F}_{;m} = \kappa_m e^{\kappa}F, \quad \bar{F}_{;m,l} = \kappa_m e^{\kappa}F_{,l}, \quad \bar{g}_{ij} = e^{2\kappa}g_{ij}, \quad \bar{g}^{ij} = e^{-2\kappa}g^{ij}. \tag{2.6}$$

By putting (2.6) in (2.5), we get

$$\bar{G}^i = G^i + \kappa_0 y^i - \frac{1}{2}F^2\kappa^i, \tag{2.7}$$

where $\kappa_0 := \kappa_i y^i$ and $\kappa^i := g^{im}\kappa_m$. Let us put

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}.$$

Then taking twice vertical derivation of (2.7) yields

$$\bar{G}^i{}_{jk} = G^i{}_{jk} + \kappa_j\delta^i_k + \kappa_k\delta^i_j - g_{jk}\kappa^i. \tag{2.8}$$

By (2.4) and (2.8), we get the following

$$\bar{b}_{i||j} = e^{\kappa}(b_{i|j} - b_j\kappa_i + fa_{ij}), \tag{2.9}$$

where "$|$" and "$||$" denote the covariant derivatives with respect to α and $\bar{\alpha}$, respectively. By (2.9), we get (2.2) and (2.3). $\square$

In order to prove Theorem 1.1, we need to the following.

Lemma 2.2. *Let $F = \sqrt[4]{\mathfrak{a}_{ijkl}(x)y^iy^jy^ky^l}$ be a quartic metric on an n-dimensional manifold M. Then the following hold:*

(1) *If $n = 2$, then by choosing suitable quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one form $\beta = b_i(x)y^i$, F is always written in the form*

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2},$$

where c_1 and c_2 are real constants and α^2 may be degenerate.

(2) *If $n \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a one-form $\beta = \beta_i(x)y^i$ which is homogeneous in α and β of degree one, then it is written in the following form*

$$F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4},$$

where c_1, c_2 and c_3 are real constants.

Proof. The proof is very tedious, computational and straightforward. By the same argument used by Matsumoto-Numata for the cubic Finsler metrics in [13], one can get the proof. Here, we omit the process of proof. □

In [9], Kim-Park claimed that using the homogeneousness of a Finsler metric, one can consider the general form of m-th root metric ($m \geq 3$) admitting (α, β)-metric and obtain the following

$$\begin{aligned}
F &= \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3},\\
F &= \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4},\\
&\vdots\\
F &= \sqrt[m]{\Sigma_0^s c_{m-2r}\alpha^{2r}\beta^{m-2r}}, \quad s \leq \frac{m}{2},
\end{aligned}$$

where c_i are constants. They studied quartic metric $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ and proved the following.

Lemma 2.3. ([9]) *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a non-Riemannian quartic metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a non-zero 1-form on M and c_i ($1 \leq i \leq 3$) are non-zero constants. Then F is a Berwald metric if and only if β is parallel with respect to α.*

Proof of Theorem 1.1: By Lemma 2.1, we have

$$\bar{b}_{i||j} = e^{\kappa}(b_{i|j} - \kappa_i b_j + a_{ij}\kappa_m b^m), \tag{2.10}$$

where "|" and "||" denote the covariant derivatives with respect to α and $\bar{\alpha}$, respectively. By assumption, $\bar{F}$ is a Berwald metric. Then by Lemma 2.3, (2.10) reduces to following

$$b_{i|j} - \kappa_i b_j + b^r \kappa_r a_{ij} = 0. \tag{2.11}$$

Multiplying (2.11) with b^i and a^{ij} yield, respectively

$$b^j b_{i|j} = b^2 \kappa_i - b^r \kappa_r b_i, \tag{2.12}$$

$$b^r \kappa_r = -\frac{1}{n-1} a^{ij} b_{i|j}. \tag{2.13}$$

Putting (2.13) in (2.12) yields

$$\kappa_i = \frac{1}{b^2}\Big[b^r b_{i|r} - \frac{1}{n-1} a^{rs} b_{r|s} b_i\Big]. \tag{2.14}$$

It is remarkable that since κ_i is a gradient vector, then

$$\kappa_{i|j} - \kappa_{j|i} = 0.$$

(2.11) can be written as

$$r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) - b^r \kappa_r a_{ij}, \tag{2.15}$$

$$s_{ij} = \frac{1}{2}(\kappa_i b_j - \kappa_j b_i). \tag{2.16}$$

(2.15) and (2.16) give respectively

$$b^r \kappa_r = -\frac{1}{n-1} a^{rs} r_{rs}, \tag{2.17}$$

$$s_j = \frac{1}{2}\Big(\kappa_r b^r b_j - b^2 \kappa_j\Big). \tag{2.18}$$

Putting (2.17) and (2.18) in (2.15) and (2.16) yield, respectively

$$r_{ij} = \frac{r^s_s}{n-1}\Big(a_{ij} - \frac{1}{b^2} b_i b_j\Big) - \frac{1}{b^2}\Big(b_i s_j + b_j s_i\Big), \tag{2.19}$$

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big). \tag{2.20}$$

Now (2.14) can be written as

$$\kappa_i = \frac{1}{b^2}\Big(b^r r_{ir} - s_i - \frac{1}{n-1} a^{rs} r_{rs} b_i\Big). \tag{2.21}$$

and (2.19) gives

$$b^r r_{ir} = -s_i. \tag{2.22}$$

By putting (2.22) in (2.21), we get

$$\kappa_i = -\frac{1}{b^2}\Big(2s_i + \frac{1}{n-1} r^s_s b_i\Big). \tag{2.23}$$

This completes the proof. □

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where open $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Then β is called Killing with respect to α if and only if $r_{ij} = 0$.

Corollary 2.1. *Let $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a non-Riemanian quartic (α,β)-metric on an n-dimensional manifold M, where c_i are nonzero constants and β is a Killing 1-form. Then F is a conformally Berwald metric if and only if it is a Berwald metric.*

Proof. By Theorem 1.1, β satisfies (1.1) and (1.2). Contracting (1.1) with b^i implies that

$$r_i + s_i = 0. \tag{2.24}$$

Let β be a Killing 1-form with respect to α, i.e., $r_{ij} = 0$. Then (2.24) yields $s_i = 0$. Putting it in (1.2) implies that $s_{ij} = 0$. Thus β is parallel with respect to α. By Lemma 2.3, F reduces to a Berwald metric. In this case, by (1.3) one can verify that the conformal change reduces to a homothetic change. □

3. Proof of Theorem 1.2

In this section, we are going to find the necessary and sufficient condition under which a cubic (α,β)-metric is conformally Berwald. For this aim, we remark that the (α,β)-metric $F = \alpha^{m+1}\beta^{-m}$ is called m-Kropina metric. In [13], Matsumoto-Numata studied the class of cubic metrics and proved the following.

Lemma 3.1. (Matsumoto-Numata [13]) *Let $F = \sqrt[3]{\mathfrak{a}_{ijk}(x)y^iy^jy^k}$ be a cubic metric on an n-dimensional manifold M. Then the following hold:*

(i) *If $n = 2$, then by choosing suitable quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one form $\beta = b_i(x)y^i$, F is a $(-\frac{1}{3})$-Kropina metric*

$$F = \sqrt[3]{\alpha^2\beta}, \tag{3.1}$$

where α^2 may be degenerate.

(ii) *If $n \geq 3$ and F is a function of a non-degenerate quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a one-form $\beta = b_i(x)y^i$ and it is homogeneous in α and β of degree one, then it is written in the following form*

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}, \tag{3.2}$$

where c_1 and c_2 are constants.

Also, in [9], Kim-Park studied cubic (α,β)-metrics and proved the following.

Lemma 3.2. (Kim-Park [9]) *Let $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ be a cubic (α,β)-metric on a manifold M, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form on M. Then F is a Berwald metric if and only if there exists functions $f_i=f_i(x)$ on M satisfy following*

$$b_{i|j}=3(c_1+c_2b^2)b_if_j+(c_1+3c_2b^2)b_jf_i-b_kf^k(c_1a_{ij}+3c_2b_ib_j), \tag{3.3}$$

where c_1, c_2 and c_3 are constants and $b^2=b_ib^i$. In this case, f_i are given by following

$$f_j=\frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(b^2)}{c_1+c_2b^2}\right]. \tag{3.4}$$

Now, we can consider the case (i) in Theorem 1.2 and prove the following.

Lemma 3.3. *Let (M,F) be an n-dimensional Finsler manifold. Then the cubic (α,β)-metric $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ is conformally Berwald if and only if β satisfies following*

$$s_{ij}=\frac{1}{b^2}\left(b_is_j-b_js_i\right), \tag{3.5}$$

$$r_{ij}=\frac{1}{b^2}\left(b_jr_i+b_ir_j\right)-(c_1a_{ij}+3c_2b_ib_j)\bar{f}_rb^r-a_{ij}k_rb^r, \tag{3.6}$$

and the conformal factor $\kappa=\kappa(x)$ satisfies

$$\kappa_j=\frac{2}{b^2}(r_j-ub_j)-2(2c_1+3c_2b^2)\bar{f}_j, \tag{3.7}$$

where

$$\bar{f}_j=\frac{1}{3b^2(c_1+c_2b^2)}(s_j+r_j), \qquad u:=\frac{1}{2}(2c_1\bar{f}_r-\kappa_r)b^r.$$

Proof. Let $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ be a cubic metric on a manifold M which is conformally related to the Berwald metric $\bar{F}$, namely, $\bar{F}=e^{\kappa}F$, where $\kappa=\kappa(x)$ is a scalar function on M. Thus $\bar{F}=\sqrt[3]{c_1\bar{\alpha}^2\bar{\beta}+c_2\bar{\beta}^3}$ is also a cubic (α,β)-metric, where $\bar{\alpha}=e^{\kappa(x)}\alpha$ and $\bar{\beta}=e^{\kappa(x)}\beta$. Put $\bar{\alpha}=\sqrt{\bar{a}_{ij}(x)y^iy^j}$ and $\bar{\beta}=\bar{b}_i(x)y^i$. Then by Lemma 3.2, there exist functions $\bar{f}_i=\bar{f}_i(x)$ on M such that $\bar{\beta}$ satisfies following

$$\bar{b}_{i||j}=3(c_1+c_2\bar{b}^2)\bar{b}_i\bar{f}_j+(c_1+3c_2\bar{b}^2)\bar{b}_j\bar{f}_i-\bar{b}_m\bar{f}^m(c_1\bar{a}_{ij}+3c_2\bar{b}_i\bar{b}_j), \tag{3.8}$$

where "$||$" denotes the covariant derivatives with respect to $\bar{\alpha}$ and $\bar{f}_i$ are given by following

$$\bar{f}_i=\frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(\bar{b}^2)}{c_1+c_2\bar{b}^2}\right]=\frac{1}{6c_1}\frac{\partial}{\partial x^i}\left[\frac{\log(b^2)}{c_1+c_2b^2}\right].$$

Here, $\bar{f}^m := \bar{a}^{mk}\bar{f}_k$. On the other hand, by Lemma 2.1 the following holds

$$\bar{b}_{i||j} = e^{\kappa}(b_{i|j} - \kappa_i b_j + b^m \kappa_m a_{ij}), \tag{3.9}$$

where "|" denotes the covariant derivatives with respect to α. By (2.1), (2.4), (3.8) and (3.9), we get

$$\begin{aligned} b_{i|j} - \kappa_i b_j + b^m \kappa_m a_{ij} = 3(c_1 + c_2 b^2) b_i \bar{f}_j + (c_1 + 3c_2 b^2) b_j \bar{f}_i \\ -b^m \bar{f}_m (c_1 a_{ij} + 3c_2 b_i b_j). \end{aligned} \tag{3.10}$$

(3.10) implies that

$$\begin{aligned} r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) + (2c_1 + 3c_2 b^2)(b_i \bar{f}_j + b_j \bar{f}_i) - b^m \bar{f}_m (c_1 a_{ij} + 3c_2 b_i b_j) \\ -b^m \kappa_m a_{ij} \end{aligned} \tag{3.11}$$

and

$$s_{ij} = \frac{1}{2}\left(\kappa_i b_j - \kappa_j b_i\right) + c_1(b_i \bar{f}_j - b_j \bar{f}_i). \tag{3.12}$$

Multiplying (3.12) with b^i yields

$$s_j = \left(c_1 \bar{f}_j - \frac{\kappa_j}{2}\right) b^2 - b_j \left(c_1 \bar{f}_i - \frac{\kappa_i}{2}\right) b^i. \tag{3.13}$$

By (3.12) and (3.13), we get

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big). \tag{3.14}$$

Let us put

$$u := \frac{b^r}{2}\Big(2c_1 \bar{f}_r - \kappa_r\Big).$$

Then contracting (3.11) with b^i gives

$$r_j = u b_j + \Big((2c_1 + 3c_2 b^2)\bar{f}_j + \frac{\kappa_j}{2}\Big) b^2. \tag{3.15}$$

By (3.15), we obtain

$$\kappa_j = 2\left[\frac{r_j - u b_j}{b^2} - (2c_1 + 3c_2 b^2)\bar{f}_j\right]. \tag{3.16}$$

Considering (3.15), the relation (3.11) can be written as follows

$$r_{ij} = \frac{1}{b^2}\Big(b_j r_i + b_i r_j\Big) - b^r \bar{f}_r (c_1 a_{ij} + 3c_2 b_i b_j) - a_{ij} b^r k_r. \tag{3.17}$$

Comparing (3.13) and (3.15) yield

$$\bar{f}_j = \frac{1}{3b^2(c_1 + c_2 b^2)}\Big(s_j + r_j\Big). \tag{3.18}$$

Conversely, we make the conformally changed $\bar{F}$ from F by the conformal change $\bar{F} = e^{\kappa(x)}F$. Suppose that the metric F satisfies (3.5) and (3.6), and the conformal factor κ satisfies (3.7). Then (3.5), (3.6) and (3.7) lead to

$$\begin{aligned} b_{i|j} - \kappa_i b_j + b^m \kappa_m a_{ij} &= r_{ij} + s_{ij} - \kappa_i b_j + \kappa_m b^m a_{ij} \\ &= 3d b_i \bar{f}_j + (c_1 + 3c_2 b^2) b_j \bar{f}_i - b^m \bar{f}_m (c_1 a_{ij} + 3c_2 b_i b_j), \end{aligned} \tag{3.19}$$

where $d := c_1 + c_2 b^2$. By (3.10) and (3.19), $\bar{F}$ is a Berwald metric. It follows that F is a conformally Berwald metric. $\Box$

In [11], Matsumoto studied Kropina metrics and characterized m-Kropina metrics of Berwald-type as follows.

Lemma 3.4. (Matsumoto [11]) *Let $F = \alpha^{m+1}\beta^{-m}$ be the m-Kropina metric on a manifold M. Then F is a Berwald metric if and only if there exists a covariant vector field $f_i = f_i(x)$ such that the following holds*

$$b_{i|j} = m(a_{ij} b_k f^k - b_j f_i) + b_i f_j,$$

where $f^k = a^{lk} f_l$.

Using Lemma 3.4, we prove the following.

Lemma 3.5. *Let (M, F) be an n-dimensional Finsler manifold M. Then the cubic (α,β)-metric $F = \sqrt[3]{\alpha^2\beta}$ is conformally Berwald if and only if β satisfies following*

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big), \tag{3.20}$$

$$r_{ij} = \frac{1}{b^2}\Big(b_j r_i + b_i r_j\Big) - \Big(b^r \kappa_r + \frac{1}{3} b^r \bar{f}_r\Big) a_{ij} - \frac{2h}{b^2} b_i b_j, \tag{3.21}$$

and the conformal factor κ satisfies

$$\kappa_j = \frac{2}{b^2}(r_j - h b_j) - \frac{4}{3}\bar{f}_j, \tag{3.22}$$

where

$$h := \frac{1}{6}(2\bar{f}_r - 3\kappa_r) b^r, \qquad \bar{f}_j = \frac{1}{b^2}(s_j + r_j).$$

Proof. Let $F = \sqrt[3]{\alpha^2\beta}$ be a cubic metric on a manifold M which is conformally related to the Berwald metric $\bar{F}$, namely, $\bar{F} = e^{\kappa}F$, where $\kappa = \kappa(x)$ is a scalar function on M. Thus $\bar{F} = \sqrt[3]{\bar{\alpha}^2\bar{\beta}}$ is also a cubic (α,β)-metric, where $\bar{\alpha} = e^{\kappa(x)}\alpha$ and $\bar{\beta} = e^{\kappa(x)}\beta$. Put $\bar{\alpha} = \sqrt{\bar{a}_{ij}(x) y^i y^j}$ and $\bar{\beta} = \bar{b}_i(x) y^i$. By Lemma 3.4, $F = \sqrt[3]{\alpha^2\beta}$ is a Berwald metric if and only if there exists f_i satisfying

$$\bar{b}_{i||j} = -\frac{1}{3}\bar{a}_{ij}\bar{b}_r \bar{f}^r + \frac{1}{3}\bar{b}_j \bar{f}_i + \bar{b}_i \bar{f}_j, \tag{3.23}$$

where "$||$" denotes the covariant derivatives with respect to $\bar{\alpha}$ and $\bar{f}^k := \bar{a}^{lk}\bar{f}_l$. By Lemma 2.1, the following hold

$$\bar{b}_{i||j} = e^{\kappa}(b_{i|j} - \kappa_i b_j + b^r\kappa_r a_{ij}), \quad \bar{a}_{ij} = e^{2\kappa}a_{ij}, \quad \bar{b}_i = e^{\kappa}b_i. \tag{3.24}$$

where "$|$" denotes the covariant derivatives with respect to α. By (3.23) and (3.24), we get

$$b_{i|j} - \kappa_i b_j + b^r\kappa_r a_{ij} = -\frac{1}{3}a_{ij}b^r\bar{f}_r + \frac{1}{3}b_j\bar{f}_i + b_i\bar{f}_j \tag{3.25}$$

which is equivalent to

$$r_{ij} = \frac{1}{2}\Big(\kappa_i b_j + \kappa_j b_i\Big) - \frac{1}{3}\Big(a_{ij}\bar{f}_r b^r - 2(b_j\bar{f}_i + b_i\bar{f}_j)\Big) - a_{ij}\kappa_r b^r, \tag{3.26}$$

$$s_{ij} = \frac{1}{2}\Big(\kappa_i b_j - \kappa_j b_i\Big) + \frac{1}{3}\Big(b_i\bar{f}_j - b_j\bar{f}_i\Big). \tag{3.27}$$

Multiplying (3.27) with b^i yields

$$s_j = b^2\Big(\frac{\bar{f}_j}{3} - \frac{\kappa_j}{2}\Big) - \Big(\frac{\bar{f}_i}{3} - \frac{\kappa_i}{2}\Big)b^i b_j. \tag{3.28}$$

Consequently, eliminating f_i from (3.27) we obtain

$$s_{ij} = \frac{1}{b^2}\Big(b_i s_j - b_j s_i\Big). \tag{3.29}$$

Let us put

$$h := \frac{1}{6}\Big(2\bar{f}_r - 3\kappa_r\Big)b^r.$$

Then multiplying (3.26) with b^i yields

$$r_j = hb_j + \frac{b^2}{6}\Big(4\bar{f}_j + 3\kappa_j\Big). \tag{3.30}$$

(3.30) implies that

$$\kappa_j = \frac{2}{b^2}(r_j - hb_j) - \frac{4}{3}\bar{f}_j. \tag{3.31}$$

By (3.30) and (3.28), we get

$$\bar{f}_j = \frac{1}{b^2}\Big(s_j + r_j\Big). \tag{3.32}$$

Multiply (3.30) with b_i and construct $(b_j r_i + b_i r_j)/b^2$. By considering (3.26), we get the following

$$r_{ij} = \frac{1}{b^2}\Big(b_j r_i + b_i r_j\Big) - \Big(b^r\kappa_r + \frac{1}{3}b^r\bar{f}_r\Big)a_{ij} - \frac{2h}{b^2}b_i b_j. \tag{3.33}$$

Conversely, we make the conformally changed $\bar{F}$ from F by the conformal change $\bar{F} = e^{\kappa(x)}F$. Suppose that the metric F satisfies (3.20) and (3.21), and the conformal factor κ satisfies (3.22). Then (3.20), (3.21) and (3.22) lead to

$$\begin{aligned} b_{i|j} - \kappa_i b_j \quad &+ b^m \kappa_m a_{ij} = r_{ij} + s_{ij} - \kappa_i b_j + b^m \kappa_m a_{ij} \\ &= b_i\Big(\frac{s_j}{b^2} + \frac{r_j}{b^2}\Big) - b_j\Big(\frac{s_i}{b^2} + \frac{r_i}{b^2}\Big) - 2\frac{r_i}{b^2}b_j - \frac{2h}{b^2}b_ib_j - \frac{1}{3}b^r\bar{f}_r a_{ij} - \kappa_i b_j \\ &= -\frac{1}{3}a_{ij}b^r\bar{f}_r + \frac{1}{3}b_j\bar{f}_i + b_i\bar{f}_j. \end{aligned} \tag{3.34}$$

By (3.25) and (3.34), $\bar{F}$ is a Berwald metric and then F is a conformally Berwald metric. □

Proof of Theorem 1.2: By Lemmas 3.3 and 3.5, we get the proof. □

4. Some Conformal Invariants

In the theory of conformal changes of Riemannian metrics, the Weyl invariant tensor plays important roles. Let $(M, \mathbf{g})$ be a Riemannian manifold of dimension $n \geq 4$. In local coordinate system, the Weyl tensor is written as follows

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}\Big\{g_{il}R_{jk} + g_{jk}R_{il} - g_{ik}R_{jl} - g_{jl}R_{ik}\Big\} - \frac{\mathbf{S}}{(n-1)(n-2)}\Big\{g_{ik}g_{jl} - g_{il}g_{jk}\Big\}.$$

where R_{ijkl} is the Riemann tensor of Riemannian metric $\mathbf{g}$, $R_{ij} = R^k_{\ ikj}$ is the Ricci tensor and $\mathbf{S} = g^{ij}R_{ij} = R^j_j$ is the scalar curvature of $\mathbf{g}$. In dimensions 2 and 3, the Weyl curvature tensor vanishes identically. If the Weyl tensor vanishes in dimension 4, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordström's theory of gravitation, which was a precursor of general relativity. The Weyl tensor is invariant under conformal changes: if $\tilde{\mathbf{g}} = e^{f(x)}\mathbf{g}$ for some positive scalar function $f = f(x)$ then $\tilde{W} = W$. For this reason, the Weyl tensor is also called the *conformal tensor*. It follows that a necessary condition for a Riemannian manifold to be conformally flat is that the Weyl tensor vanish. The existence of this conformal invariant is quite remarkable since there is no known generalization of the Weyl conformal curvature tensor to Finsler geometry [7]. Then the following natural question arises:

Is there any conformal invariant in Finsler Geometry?

Let M be an n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s=t=0}, \quad u, v \in T_xM.$$

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form $dV_{BH} = \sigma_F(x)\omega^1 \wedge \cdots \wedge \omega^n$ is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det\big(g_{ij}(x, y)\big)}}{\sigma_F(x)}.$$

Now, let $\bar{F} = e^{\kappa}F$ be two conformal Finsler metrics on an n-dimensional manifold M, where $\kappa = \kappa(x)$ is a scalar function on M. It is easy to verify that

$$\bar{g}_{ij}(x, y) = e^{2\kappa} g_{ij}(x, y), \quad \det(\bar{g}_{ij}) = e^{2n\kappa}\det(g_{ij}), \quad \sigma_{\bar{F}} = e^{n\kappa}\sigma_F.$$

Thus, we conclude the following.

Lemma 4.1. *Let $\bar{F} = e^{\kappa}F$ be two conformal Finsler metrics on a manifold M. Then $\bar{\tau} = \tau$.*

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0},$$

where $u, v, w \in T_xM$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. Thus $\mathbf{C} = 0$ if and only if F is Riemannian. Using the notion of Cartan torsion, one can define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) = \sum_{i=1}^n g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j)$, where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

At any point $x \in M$, Shen defined the norms of $\mathbf{C}$ and $\mathbf{I}$ in [18] as follows

$$||\mathbf{C}|| = \sup_{y,u \in T_xM_0} \frac{F(y)|\mathbf{C}_y(u, u, u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}} = \sup_{y,u \in I_xM} \frac{|\mathbf{C}_y(u, u, u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}}, \tag{4.1}$$

$$||\mathbf{I}|| = \sup_{y,u \in T_xM_0} \frac{F(y)|\mathbf{I}_y(u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}} = \sup_{y,u \in I_xM} \frac{|\mathbf{I}_y(u)|}{[\mathbf{g}_y(u, u)]^{\frac{3}{2}}}, \tag{4.2}$$

where I_xM is the indicatrix of F at $x \in M$.

For a vector $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) := \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\}.$$

Then F is said to be C-reducible if $\mathbf{M}_y = 0$.

Lemma 4.2. (Matsumoto-Hōjō Lemma) *A Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric if and only if its Matsumoto torsion vanish.*

For a non-zero vector $y \in T_xM_0$, define the torsion $\mathbf{A}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{A}_y(u,v,w) := \mathbf{C}_y(u,v,w) - \frac{P}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w) + \mathbf{I}_y(v)\mathbf{h}_y(u,w) + \mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\} - \frac{Q}{||\mathbf{I}||^2}\mathbf{I}_y(u)\mathbf{I}_y(v)\mathbf{I}_y(w), \tag{4.3}$$

where $P = P(x,y)$ and $Q = Q(x,y)$ are scalar functions on TM and $||\mathbf{I}||^2 = I^iI_i$. A Finsler metric F on an n-dimensional manifold M is called semi-C-reducible if $\mathbf{A}_y = 0$. In [14], Matsumoto-Shibata proved that every (α,β)-metric is semi-C-reducible.

Theorem 4.1. ([14]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

Let us define

$$||\mathbf{M}|| = \sup_{y,u\in T_xM_0} \frac{F(y)|\mathbf{M}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}} = \sup_{y,u\in I_xM} \frac{|\mathbf{M}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}}, \tag{4.4}$$

$$||\mathbf{A}|| = \sup_{y,u\in T_xM_0} \frac{F(y)|\mathbf{A}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}} = \sup_{y,u\in I_xM} \frac{|\mathbf{A}_y(u,u,u)|}{[\mathbf{g}_y(u,u)]^{\frac{3}{2}}}. \tag{4.5}$$

Then, we get the following.

Theorem 4.2. *Let (M,F) be an n-dimensional Finsler manifold. Then the following are conformally invariant:*

(i) $\mathcal{C} := F^2||\mathbf{C}||^2$;

(ii) $\mathcal{M} := F^2||\mathbf{M}||^2$;

(iii) $\mathcal{A} := F^2||\mathbf{A}||^2$.

Proof. We have $\bar{C}_{ijk} = e^{2\kappa}C_{ijk}$. Then $\bar{C}^{ijk} = e^{-4\kappa}C^{ijk}$ which yields

$$||\bar{\mathbf{C}}||^2 = e^{2\kappa}||\mathbf{C}||^2. \tag{4.6}$$

Then $\mathcal{C} = \mathcal{C}(x,y)$ is a conformally invariant.

In local coordinates, the Matsumoto torsion is given by following

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\},$$

where $h_{ij} := FF_{y^iy^j}$ is the angular metric. Since

$$h_{ij} = e^{2\kappa}\bar{h}_{ij}, \quad I_i = \bar{I}_i,$$

then

$$\bar{M}_{ijk} = e^{2\kappa} M_{ijk}$$

which implies that

$$\bar{M}^{ijk} = e^{-4\kappa} M^{ijk}.$$

Then

$$||\bar{\mathbf{M}}||^2 = e^{2\kappa} ||\mathbf{M}||^2.$$

Thus $\mathcal{M} = \mathcal{M}(x,y)$ is a conformally invariant.

Finally, in local coordinates $\mathbf{A}_y$ is written as follows

$$A_{ijk} := C_{ijk} - \frac{P}{1+n}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} - \frac{Q}{||\mathbf{I}||^2} I_i I_j I_k.$$

We get $\bar{A}_{ijk} := e^{2\kappa} A_{ijk}$. Then $||\bar{\mathbf{A}}||^2 = e^{2\kappa}||\mathbf{A}||^2$. Then, $\mathcal{A} = \mathcal{A}(x,y)$ is a conformally invariant. □

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DIRAC OPERATORS ON LIE ALGEBROIDS

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Abstract. We compare the Dirac operator on transitive Riemannian Lie algebroid equipped by spin or complex spin structure with the one defined on to its base manifold. Consequently we derive upper eigenvalue bounds of Dirac operator on base manifold of spin Lie algebroid twisted with the spinor bundle of kernel bundle.

Keywords: Riemannian Lie algebroid; Dirac operator; eigenvalue bounds.

1. Introduction

Let D be a first-order differential operator acting on a vector bundle S over a Riemannian manifold M. If $D^2 = \Delta$, where Δ is the Laplacian of S, then D is called a Dirac operator on S. In high-energy physics, this requirement is often relaxed: only the second-order part of D^2 must equal the Laplacian [4].

A Lie algebroid is a triple $(E, [\cdot,\cdot], \rho)$ consisting of a vector bundle E over a manifold M, together with a Lie bracket $[\cdot,\cdot]$ on its space of sections $\Gamma(E)$ and a morphism of vector bundles $\rho : E \to TM$ called the anchor map, where TM is the tangent bundle of M. The anchor map and the bracket satisfy the Leibniz rule $[X, fY] = \rho(X)f \cdot Y + f[X,Y]$, where $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$ and $\rho(X)f$ is the derivative of f along the vector field $\rho(X)$. It follows that $\rho([X,Y]) = [\rho(X), \rho(Y)]$ for all $X, Y \in \Gamma(E)$ (for more details, see [6]).

In [1], Bär gives upper eigenvalue bounds for the Dirac operator of a closed Riemannian spin manifold M isometrically immersed in a Riemannian spin manifold Q admitting Killing spinors. He provides a "submanifold theory" of Dirac operators and describes the relations between the Dirac operator of the ambient space and the Dirac operator of the submanifold twisted by the spinor bundle of the normal bundle. When the ambient space Q admits a Killing spinor Ψ with real Killing constant α (that is, a spinor field Ψ satisfying the equation $\nabla_X \Psi = \alpha X \cdot \Psi$ for all vector fields X), he shows that there exists at least k eigenvalues of $D_M^{\Sigma N}$, where

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k is the dimension of the space $\Sigma_\alpha Q$ of Killing spinors with constant α unless $dim(M)$ and $codim(M)$ are both odd, and $k = [\frac{1}{2}dim(\Sigma_\alpha Q)]$ otherwise, satisfying the equation

$$\lambda^2 \leq n^2\alpha^2 + \frac{n^2}{4vol(M)} \int |H|^2,$$

where $n := dim(M)$, and H is the mean curvature vector field [1]. Moreover, almost the same result is obtained when α is purely imaginary.

Recently, Balcerzak-Pierzchalski study the Dirac operators on Lie algebroids [2]. They considered the Lie algebroids equipped with a structure of a Clifford module and obtained the Witzenböck formulas for the square of Dirac operators. In this paper, we have considered transitive Lie algebroids on closed spin manifolds. Transitivity property causes that Lie algebroids to be decomposed as $L \oplus E$ of vector bundles, where $L = \ker \rho$, $E = \lambda(TM)$ and λ is a bundle diffeomorphism between TM and E[3]. Further, we suppose the Lie algebroids admit a spin structure. First, we compare the spinor connection of the spin Lie algebroid with the one defined on the base manifold. Then, we obtain the relation between Dirac operators on a Lie algebroid and its base manifold similar to the ideas and methods employed in [1] (see the relation (5.1)). Finally, we derive upper eigenvalue bounds of Dirac operators on Lie algebroids based on calculation of Rilegh-Ritz quotient (see Theorem 6.1 and 6.2).

2. Preliminaries

Let M be a smooth manifold. A Lie algebroid on M is a vector bundle (A, π, M) together with a Lie bracket product on ΓA and a vector bundle map $\rho : A \longrightarrow TM$ called the anchor map of A, such that the following conditions satisfy[6];

1. The induced map $\rho : \Gamma A \longrightarrow TM$ is a homeomorphism of vector bundles.
2. For all $X, Y \in \Gamma A$ and $f \in C^\infty(M)$,

$$[X, fY] = f[X, Y] + (\rho(X)(f))Y.$$

A Lie algebroid $\rho : A \longrightarrow TM$ is called transitive if ρ is surjective. For a transitive Lie algebroid, $L = \ker \rho$ is a bundle of Lie algebroid. In fact, the Lie algebroid On ΓA can be restricted to ΓL and its restriction on L is tensorial, consequently, we have a Lie algebra structure on each fibre of L. So, on a transitive Lie algebroid $\rho : A \longrightarrow TM$ we find the short exact sequence of the following vector bundles

$$0 \longrightarrow L \longrightarrow A \longrightarrow TM \longrightarrow 0.$$

Suppose $\rho : A \longrightarrow TM$ is transitive Lie algebroid, then a vector bundle map $\lambda : TM \longrightarrow A$ such that $\rho \circ \lambda = 1_{TM}$, is a splitting of $\rho : A \longrightarrow TM$, i.e., we can decompose to $L \oplus E$ of vector bundles, where $E = \lambda(TM)$ (and vice versa). It is easy to check that λ is a bundle diffeomorphism between TM and E. Fix a splitting

$\lambda : TM \longrightarrow A$ of ρ. The map λ defines a linear connection on L, and is called an adjoint connection(see [3]).

For each splitting λ the 2-differential form $\Omega^\lambda \in A^2(M, L)$ is defined by

$$\Omega^\lambda(U, V) = [\lambda(U), \lambda(V)] - [\lambda([U, V]).$$

The 2-form Ω^λ is related to the curvature tensor of ∇^λ is given by

$$R^\lambda(U, V)(s) = [2\Omega^\lambda(U, V), s].$$

We can define a Lie bracket on the transitive Lie algebroid sections

$$[\lambda(U) + S_1), \lambda(V) + S_2] = [\lambda(U), \lambda(V)] + \nabla^\lambda_U S_2 - \nabla^\lambda_V S_1 + [S_1, S_2] + \Omega(U, V).$$

For all $U \in \mathcal{X}(M)$, let us put $\lambda(U) = \overline{U}$.

By splitting $A = L \oplus \lambda(TM)$, the Riemannian metric g on transitive Lie algebroid induces a metric on M as follows

$$\forall U, V \in M \quad \langle U, V\rangle_M = \langle \overline{U}, \overline{V}\rangle_A.$$

Now, we define $\Omega^a : \mathcal{X}(M) \times \Gamma L \longrightarrow \mathcal{X}(M)$ by

$$\forall U, V \in \mathcal{X}(M), s \in \Gamma L, \quad \langle \Omega^a(U, s), V\rangle_M = \langle \Omega(U, V), s\rangle_A.$$

3. Spinor Modules

This section is devoted to spinor modules which inspired from [1]. We want to compare the Dirac operators on a Riemannian spin Lie algebroid and its spin base manifold. For this end, we have to compare spinor bundles on Lie algebroid with the spinor bundle of the base manifold. The starting point is decomposing transitive Lie algebroid A to $A = L \oplus \lambda(TM)$, where $L = \ker \rho$ and $\lambda : TM \longrightarrow A$ is splitting. Hence we need to recognize spinor modules on clifford algebra of an Euclidean space with the two factor.

If $\dim E = n$ and $\dim F$=m are even integers, then $\mathbb{C}l(E)$ has precisely one irreducible module that is spinor module ΣE. Denote the clifford multiplication by $\gamma_E : \mathbb{C}l(E) \longrightarrow End(\Sigma E)$. When restricted to the even subalgebra $\mathbb{C}l^0(E)$ the spinor module decomposes in to even and odd half-spinors $\Sigma E = \Sigma^+ E \oplus \Sigma^- E$. The complex volume element $\omega_{\mathbb{C}} = i^{\frac{n}{2}} \gamma_{\mathbb{C}}(e_1 \cdots e_n)$ acts as $+1$ on $\Sigma^+ E$ and as -1 on $\Sigma^- E$.

If n is odd, then there are exactly two irreducible modules, $\Sigma^0 E$ and $\Sigma^1 E$. In this case the dimension of these modules are $2^{\frac{n-1}{2}}$. Clifford multiplication will now be denoted by $\gamma_{E,j} : \mathbb{C}l(E) \longrightarrow End(\Sigma^j E)$.

Similarly to the half spinor spaces in even dimensions, the two modules $\Sigma^0 E$ and $\Sigma^1 E$ can be distinguished by the action of the complex volume element $\omega_{\mathbb{C}} = i^{\frac{n+1}{2}} \gamma_{\mathbb{C}}(e_1 \cdots e_n)$, on $\Sigma^j E$ acts as $(-1)^j, j = 0, 1$. One can pass from $\Sigma^0 E$ to $\Sigma^1 E$ by taking the same underlying vector space $\Sigma^0 E = \Sigma^1 E$ and there exists a vector space isomorphism $\Phi : \Sigma^0 E \longrightarrow \Sigma^1 E$ such that $\Phi \circ \gamma_{E,0}(x) = -\gamma_{E,1}(x) \circ \Phi$ for all $x \in E$. Now let E and F be two oriented Euclidean vector spaces. Assume that $\dim E = n$ and $\dim F = k$.

Now we construct the spinor module of $E \oplus F$ from those of E and F.

Case 1. n and k are even. Let us put $\Sigma := \Sigma E \otimes \Sigma F$, $\gamma : E \oplus F \longrightarrow End(\Sigma)$, $\gamma(x)(\sigma \otimes \tau) = (\gamma_E(x)\sigma) \otimes \tau$ and

$$\gamma(y)(\sigma \otimes \tau) = (-1)^{\deg \sigma} \sigma \otimes (\gamma_F(y)\tau), \tag{3.1}$$

where $x \in E, y \in F, \sigma \in \Sigma E, \tau \in \Sigma F$. Thus

$$deg\, \sigma = \begin{cases} 0 & if \;\; n \;\; or \;\; k \;\; iseven. \\ 1 & o.w. \end{cases}$$

and we have $\gamma(X+Y) \cdot \gamma(X+Y)(\sigma \otimes \tau) = -(X+Y)^2 \cdot (\sigma \otimes \tau)$;

As γ is a Clifford map, it extends to a homomorphism $\mathbb{C}l(E \oplus F) \longrightarrow End(\Sigma)$. Therefore (Σ, γ) is a module on $\mathbb{C}l(E \oplus F)$ of dimension $2^{\frac{n}{2}} \cdot 2^{\frac{k}{2}} = 2^{\frac{n+k}{2}}$. Then Σ is isomorphic to $\Sigma(E \oplus F)$. Hence,

$$\Sigma^+(E \oplus F) = (\Sigma^+ E \otimes \Sigma^+ F) \oplus (\Sigma^- E \otimes \Sigma^- F),$$

$$\Sigma^-(E \oplus F) = (\Sigma^+ E \otimes \Sigma^- F) \oplus (\Sigma^- E \otimes \Sigma^+ F).$$

Case 2. n and k are even and odd, respectively. In this case, dimension $E \oplus F$ is odd and

$$\Sigma^j = \Sigma E \otimes \Sigma^j F, \qquad \gamma_j : E \oplus F \longrightarrow End(\Sigma^j), \quad j = 0, 1.$$

As in the case 1, we make Σ^0 and Σ^1 in to $\mathbb{C}L(E \oplus F)$-modules. Easily one can check that the complex volume element of $\mathbb{C}L(E \oplus F)$ acts on Σ^j as $(-1)^j$. Hence (Σ^j, γ_j) is isomorphic to $(\Sigma^j(E \oplus F), \gamma_{E \oplus F, j})$.

Case 3. n odd k are even. This case is symmetric to the second case. Let us put $\Sigma := \Sigma E \otimes \Sigma F$, $\gamma : E \oplus F \longrightarrow End(\Sigma)$, $\gamma(x)(\sigma \otimes \tau) = (-1)^{\deg \sigma}(\gamma_E(x)\sigma) \otimes \tau$, $\gamma(y)(\sigma \otimes \tau) = \sigma \otimes (\gamma_F(y)\tau)$. Then $x \in E, y \in F, \sigma \in \Sigma E, \tau \in \Sigma F$. Hence (Σ^j, γ_j) is isomorphic to $(\Sigma^j(E \oplus F), \gamma_{E \oplus F, j})$.

Case 4. n and k are odd. In this case, let us put $\Sigma^+ := \Sigma^0 E \otimes \Sigma^0 F$, $\Sigma^- := \Sigma^1 E \otimes \Sigma^1 F$ and $\Sigma := \Sigma^+ \oplus \Sigma^-$. There there exits a vector space isomorphism $\Phi : \Sigma^0 F \longrightarrow \Sigma^1 F$ such that $\phi \circ \gamma_{F,0}(Y) = -\gamma_{F,1}(Y) \circ \phi$ for all $Y \in F$. With respect to splitting $\Sigma = \Sigma^+ \oplus \Sigma^-$, let us define

$$\gamma(x) := \begin{pmatrix} 0, & \gamma_{E,0}(x) \otimes \Phi^{-1} \\ -\gamma_{E,0}(x) \otimes \Phi, & 0 \end{pmatrix}$$

$$\gamma(y) := \begin{pmatrix} 0, & Id \otimes \Phi^{-1} \circ \gamma_{F,1}(y) \\ -Id \otimes \Phi \circ \gamma_{F,0}(y), & 0 \end{pmatrix}. \tag{3.2}$$

Thus $\gamma(X+Y)\circ\gamma(X+Y) = -(X+Y)^2\cdot Id$, and hence γ extends to a representation of $\mathbb{C}L(E\oplus F)$ on Σ. Therefore there is an isomorphism from $(\Sigma(E\oplus F), \gamma_{E\oplus F})$ to (Σ, γ).

4. Spinor Connections

Let $\hat{\nabla}$ be the Levi-Civita connection of the Riemannian transitive Lie algebroid (A, g) and let $\lambda : TM \longrightarrow A$ be a splitting for each $a \in A, s \in \Gamma L, U \in \mathcal{X}(M)$, which we denote by

$$\nabla^A_U a := \hat{\nabla}_{\overline{U}} a$$
$$\nabla^L_U s := (\hat{\nabla}_{\overline{U}} s)^L.$$

The superscript L is the projection to L. Denote ∇^L, ∇^A the Levi-Civita connection which is defined as follows

$$\nabla^A_U \overline{V} = \overline{\nabla}^M_U V + \Omega(U, V)$$
$$\nabla^A_U s = -\overline{\Omega^a(U, V)} + \nabla^L_U s,$$

where ∇^M is the Levi-Civita connection on M. In this case if the Riemannian metric is compatible with A we have $\nabla^L = \nabla^\lambda$.

Let $E \longrightarrow M$ be an oriented Riemannian vector bundle and let $P_{so}(E)$ be bundle of oriented orthonormal frames. Every Riemannian covariant derivative ∇ corresponds to a 1-form connection ω on $P_{so}(E)$(see[5]). Let $e = (e_1, \cdots, e_n)$ be a local section on open set $O \subseteq M$. The local connection form $\omega^e = e^*(\omega)$: $TO \longrightarrow so(n)$ is given by the formula $\omega^e = \sum_{i<j} \omega_{ij} E_{ij}$ where $\omega_{ij} = \langle \nabla e_i, e_j\rangle$ and $E_{ij} \in so(n)$ are the standard basis matrices of Lie algebra $so(n)$. Let $(U_1, \cdots, U_n)$ be a local positively oriented orthonormal tangent frame of M and let $(s_1, \cdots, s_k)$ be a local positively oriented orthonormal frame of L. Then $h := (\overline{U_1}, \cdots, \overline{U_n}, s_1, \cdots, s_k)$ is a local section of $P_{so}(A)$. Now we can write the following matrix forms

$$\Omega(U, \cdot) = ((\langle\Omega(U, U_i), s_j\rangle)_{ij},$$

$$\nabla^A_U - (\overline{\nabla^M_U} \oplus \nabla^L_U) = \begin{pmatrix} 0, & -(\langle\Omega(U, U_i, s_j\rangle)_{ji} \\ (\langle\Omega(U, U_i, s_j\rangle)_{ij}, & 0 \end{pmatrix}. \tag{4.1}$$

Let A be a spin Lie algebroid and M a spin manifold so the bundle L has a spin structure see [5]. If $\Theta : Spin(n+k) \longrightarrow SO(n+k)$ is the spin representation and ω^A, ω^M and ω^L are the induced connection 1-forms on the corresponding spin bundles. By (4.1), we have

$$\Theta_*(\omega^A(dh\cdot U) - (\omega^M \oplus \omega^L)(dh\cdot U)) = \begin{pmatrix} 0, & -(\langle\Omega(U, U_i, s_j\rangle)_{ji} \\ (\langle\Omega(U, U_i, s_j\rangle)_{ij}, & 0 \end{pmatrix}.$$

Using a standard formula for Θ_* and the above equation, we get

$$\omega^A(dh\cdot U) - (\omega^M \oplus \omega^L)(dh\cdot U) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{k}\langle\Omega(U, U_i), s_j\rangle \cdot e_i \cdot f_j, \tag{4.2}$$

where $e_1, \cdots, e_n$ and $f_1, \cdots, f_k$ are the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. If ΣA, ΣM, and ΣL are the spinor bundles of A, M and L, then from the consideration in previous we know that:

$$\Sigma A = \begin{cases} \Sigma M \otimes \Sigma L, & if \quad n \quad or \quad k \quad iseven. \\ \Sigma M \otimes \Sigma L \oplus \Sigma M \otimes \Sigma L, & o.w. \end{cases}$$

Let $\nabla^{\Sigma A}$, $\nabla^{\Sigma M}$, and $\nabla^{\Sigma L}$ be the induced connections on spinor bundles ΣA, ΣM, and ΣL, respectively. Define the product connection $\nabla^{\Sigma M \otimes \Sigma L}$ on ΣA by

$$\nabla^{\Sigma M \otimes \Sigma L} = \begin{cases} \nabla^{\Sigma M} \otimes Id \oplus Id \otimes \nabla^{\Sigma L}, & if \quad n \quad or \quad k \quad iseven. \\ \nabla^{\Sigma M} \otimes Id \oplus Id \otimes \nabla^{\Sigma L} \oplus \nabla^{\Sigma M} \otimes Id \oplus Id \otimes \nabla^{\Sigma L}, & o.w. \end{cases}$$

Equation (3.1) yields

$$\begin{aligned} \nabla_U^{\Sigma A} - \nabla^{\Sigma M \otimes \Sigma L} \quad &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{k} \langle \Omega(U, U_i), s_j \rangle \gamma_A(\overline{U_i} \cdot s_j) \\ &= \frac{1}{2} \sum_{i=1}^{n} \gamma_A(\overline{U_i} \cdot \Omega(U, U_i)). \end{aligned} \tag{4.3}$$

Consider $\omega_k = i^{\frac{k+1}{2}} \gamma_A(s_1 \cdots s_k)$ and put $\omega_\perp = \omega_k$ when k is even and $\omega_\perp = -i\omega_k$ when k is odd.

5. Dirac Operators

Define the Dirac operator $D_M^{\Sigma L} : \Sigma M \otimes \Sigma L \longrightarrow \Sigma M \otimes \Sigma L$ on M twisted with the spinor bundle ΣL by

$$D_M^{\Sigma L} \psi := \sum \overline{U_i} \cdot_M (\nabla_U^{\Sigma M} \otimes Id \oplus Id \otimes \nabla_U^{\Sigma L}) \psi$$

where $\overline{U} \cdot_M \psi = \overline{U} \cdot \omega_\perp \cdot \psi$ and

$$\tilde{D}_M^{\Sigma L} := \begin{cases} D_M^{\Sigma L} & if \quad n \quad or \quad k \quad iseven, \\ D_M^{\Sigma L} \oplus -D_M^{\Sigma L}, & o.w. \end{cases}$$

Also define

$$\begin{aligned} \tilde{D} \quad &:= \sum_{i=1}^{n} \gamma_A(\overline{U}_i) \nabla_{U_i}^{\Sigma M \otimes \Sigma L} \\ \hat{D} \quad &:= \sum_{i=1}^{n} \gamma_A(\overline{U}_i) \nabla_{U_i}^{\Sigma A}. \end{aligned}$$

The three last operators act on sections of ΣA.

Using equation (4.3), we get

$$\begin{aligned}\hat{D}-\tilde{D} &= \frac{1}{2}\sum_{i,j=1}^{n}\gamma_A(\bar{U}_j\cdot\bar{U}_i\cdot\Omega(\bar{U}_j\cdot\bar{U}_i)) \\ &= \sum_{1\le i<j\le n}\gamma_A(\bar{U}_j\cdot\bar{U}_i\cdot\Omega(\bar{U}_j\cdot\bar{U}_i)),\end{aligned} \tag{5.1}$$

because of $\Omega(U,U)=0$ and for $i<j$ we have $U_i\cdot U_j=-U_j\cdot U_i$.

In order to find the relation between $\tilde{D}$ and $\tilde{D}_M^{\Sigma L}$, for different dimensions, we have to consider various cases. In case 1 and case 2 we have

$$\begin{aligned}\tilde{D} &= \sum_{i=1}^{n}\gamma_A(\bar{U}_i)\nabla_{U_i}^{\Sigma M\otimes\Sigma L} \\ &= \sum_{i=1}^{n}(\gamma_M(\bar{U}_i)\otimes Id)\nabla_{U_i}^{\Sigma M\otimes\Sigma L}=D_M^{\Sigma L}=\tilde{D}_M^{\Sigma L}.\end{aligned}$$

In case 3 we get from equation (2) on $\Sigma M\otimes\Sigma^+L$

$$\tilde{D}=D_M^{\Sigma L}=\tilde{D}_M^{\Sigma}L$$

and on $\Sigma M\otimes\Sigma^-L$ we obtain

$$\tilde{D}=-D_M^{\Sigma L}=-\tilde{D}_M^{\Sigma}L.$$

Finally in case 4 have we get from equation (3.2)

$$\tilde{D}=i\begin{pmatrix}0, & D_M^{\Sigma L}\\ -D_M^{\Sigma L}, & 0\end{pmatrix}$$

In all cases we see that $\tilde{D}$ is formally self-adjoint because $D_M^{\Sigma}L$ is and

$$\tilde{D}^2=(D_M^{\Sigma}L)^2.$$

6. Upper Bound for Eigenvalues

Let (A,g) be a spin Lie algebroid and (M,g_M) a spin manifold. The spinor ψ is called a Killing spinor with Killing constant α if it satisfies $\nabla_a^{\Sigma A}\psi=\alpha\cdot\gamma_A(a)\psi$ for all $a\in\Gamma A$. Obviously the set of Killing spinors with Killing constant forms a vector space of dimension $\nu(A,\alpha)$. Let $\mu(A,n,\alpha)$ be the smallest integer greater than or equal to $\nu(A,\alpha)/2$. If dimension n and k are both odd we then put $\mu(A,n,\alpha):=\nu(A,\alpha)$, in this case.
Define $|\Omega|^2:=\sum_{i,j=1}^{n}|\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))|^2$.

Theorem 6.1. *Let A be a Riemannian spin Lie algebroid on M and M be a closed Riemannian spin manifold. Suppose that the bundle L carry the induced spin structure and $\alpha \in \mathbb{R}$. Then there are at least $\mu = \mu(A, n, \alpha)$ eigenvalues $\lambda_1, \cdots, \lambda_\mu$ of the Dirac operator on $D_M^{\Sigma L}$ such that*

$$|\lambda_k| \leq n|\alpha| + \frac{1}{2}\|\Omega\|_{L^\infty(M)}$$

Proof. Now, let ψ be a Killing spinor on A with Killing constant $\alpha \in \mathbb{R}$. Such Killing spinors have constant length and we may assume that $|\psi| = 1$. We compute the Rayleigh quotient of $\tilde{D}_M^{\Sigma L}$ using the previous notation. Then, we get the following

$$\begin{aligned}
\frac{\Big((\tilde{D}_M^{\Sigma L})^2\psi, \psi\Big)_{L^2(M)}}{(\psi, \psi)_{L^2(M)}} &= \frac{\Big(\tilde{D}^2\psi, \psi\Big)_{L^2(M)}}{vol(M)} \\
&= \frac{\Big(\tilde{D}\psi, \tilde{D}\psi\Big)_{L^2(M)}}{vol(M)} \\
&= \frac{\Big\|\hat{D}\psi - \frac{1}{2}\sum_{i,j=1}^n \gamma_A(\overline{U_j} \cdot \overline{U_i} \cdot \Omega(U_j, U_i))\psi\Big\|^2_{L^2(M)}}{vol(M)} \\
&= \frac{1}{vol(M)}\{\|\hat{D}\psi\|^2_{L(M)} \\
&-\frac{1}{2}(\hat{D}\psi, \sum_{i,j=1}^n \gamma_A(\overline{U_j} \cdot \overline{U_i} \cdot \Omega(U_j, U_i))\psi)_{L^2(M)} \\
&-\frac{1}{2}(\sum_{i,j=1}^n \gamma_A(\overline{U_j} \cdot \overline{U_i} \cdot \Omega(U_j, U_i))\psi, \hat{D}\psi)_{L^2(M)} \\
&+\frac{1}{4}\|\sum_{i,j=1}^n \gamma_A(\overline{U_j} \cdot \overline{U_i} \cdot \Omega(U_j, U_i))\psi\|^2_{L^2(M)}\}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\hat{D}\psi &= \sum_{i=1}^n \gamma_A(\overline{U_i})\nabla^{\Sigma L}_{U_i}\psi \\
&= \sum_{i=1}^n \gamma_A(\overline{U_i})\alpha\gamma_A(\overline{U_i})\psi \\
&= -n\alpha\psi.
\end{aligned}$$

Note also that

$$(a \cdot \psi, \varphi) + (\psi, a \cdot \varphi) = 0, \; for\; each\, a \in A.$$

Thus, we get

$$
\begin{aligned}
frac\Big((\tilde{D}_M^{\Sigma L})^2\psi,\psi\Big)_{L^2(M)}(\psi,\psi)_{L^2(M)} \quad &= \frac{1}{vol(M)}\{n^2\alpha^2 vol(M) \\
&+\frac{n\alpha}{2}(\psi,\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi)_{L^2(M)} \\
&+\frac{n\alpha}{2}(\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi,\psi)_{L^2(M)} \\
&+\frac{1}{4}\|\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi\|^2_{L^2(M)}\} \\
&= n^2\alpha^2 + n\alpha(\psi,\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi)_{L^2(M)} \\
&+\frac{1}{4vol(M)}\|\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi\|^2_{L^2(M)}
\end{aligned}
$$

By considering the following inequality,

$$
\begin{aligned}
|\alpha(\psi,\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi)_{L^2(M)}| \qquad &\le |\alpha|\cdot\int_M|\psi|^2|\Omega| \\
&\le |\alpha|\cdot\|\psi\|^2_{L^2(M)}\cdot\|\Omega\|_{L^\infty(M)}.
\end{aligned}
$$

the min-max principle implies the assertion. □

Theorem 6.2. *Let A be a Riemannian spin Lie algebroid on M and M be a closed Riemannian spin manifold. Suppose that the bundle L carry the induced spin structure and $\alpha\in i\mathbb{R}$. Then there are at least $\mu=\mu(A,n,\alpha)$ eigenvalues $\lambda_1,\cdots,\lambda_\mu$ of the Dirac operator on $D_M^{\Sigma L}$ such that*

$$
\lambda_k^2 \le n^2|\alpha|^2+\frac{1}{4vol(M)}\int_M|\Omega|^2,
$$

Proof. Now, let ψ be a Killing spinor on A with Killing constant $\alpha\in i\mathbb{R}$. Such Killing spinors have constant length and we may assume that $|\psi|=1$. We compute the Rayleigh quotient of $\tilde{D}_M^{\Sigma L}$ using the previous notation. The same computations as in the proof of the previous Theorem, we get the following

$$
\begin{aligned}
\frac{\left((\tilde{D\psi}_M^{\Sigma L})^2\psi,\psi\right)_{L^2(M)}}{(\psi,\psi)_{L^2(M)}} &= \frac{1}{vol(M)}\{\|\hat{D}\psi\|^2_{L(M)} \\
&-\frac{1}{2}(\hat{D}\psi, \sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi)_{L^2(M)} \\
&-\frac{1}{2}(\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi, \hat{D}\psi)_{L^2(M)} \\
&+\frac{1}{4}\|\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi\|^2_{L^2(M)}\} \\
&= \frac{1}{vol(M)}\{n^2|\alpha|^2 vol(M) \\
&+\frac{n\alpha}{2}(\psi, \sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi)_{L^2(M)} \\
&+\frac{n\bar{\alpha}}{2}(\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi, \psi)_{L^2(M)} \\
&+\frac{1}{4}\|\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi\|^2_{L^2(M)}\} \\
&= n^2|\alpha|^2 + \frac{1}{4vol(M)}\|\sum_{i,j=1}^{n}\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi\|^2_{L^2(M)} \\
&\le n^2|\alpha|^2 + \frac{1}{4vol(M)}\sum_{i,j=1}^{n}\|\gamma_A(\overline{U_j}\cdot\overline{U_i}\cdot\Omega(U_j,U_i))\psi\|^2_{L^2(M)} \\
&= n^2|\alpha|^2 + \frac{1}{4vol(M)}\int_M|\Omega|^2|\psi|^2 \\
&= n^2|\alpha|^2 + \frac{1}{4vol(M)}\int_M|\Omega|^2.
\end{aligned}
$$

The min-max principle implies the assertion. □

Example 6.1. *Let rank $L = 1$ (i.e., L is the trivial line bundle since it is orientable). In this case, the Dirac operator on M twisted by L is ordinary Dirac operator and the above theorem become as follows: If (A, g_A) is a Lie algebroid with spin structure that $A = TM \oplus \mathbb{R}$ and spin manifold is close, there exist at least $\mu = \mu(A, n, \alpha)$ eigenvalues $\lambda_1 \cdots \lambda_\mu$ of D_M that $|\lambda_j| \le n|\alpha|$. This is because $\Omega = 0$.*

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W_2-CURVATURE TENSOR ON K-CONTACT MANIFOLDS

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Abstract. The object of the present paper is to obtain sufficient conditions for a K-contact manifold to be a Sasakian manifold.

Keywords: Sasakian manifold; K-contact manifold; W_2 curvature tensor.

1. Introduction

The inclination of existent mathematics is abstractions, generalizations and applications. In the offering exposition, we are entering an era of new concepts and some generalizations which play a functional role in contemporary mathematics. Contact geometry has been matured from the mathematical formalism of classical mechanics. A complete regular contact metric manifold M^{2n+1} carries a K-contact structure (ϕ, ξ, η, g), defined in terms of the almost Kähler structure (J, G) of the base manifold M^{2n}. Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2n}, J, G) is Kählerian. If (M^{2n}, J, G) is only almost Kähler, then (ϕ, ξ, η, g) is only K-contact [3]. It is to be noted that a K-contact manifold is intermediate between a contact metric manifold and a Sasakian manifold. K-contact and Sasakian manifolds have been studied by several authors such as ([2], [7], [8], [10],[18], [20],) and many others. It is well known that every Sasakian manifold is K-contact, but the converse is not true, in general. However, a three-dimensional K-contact manifold is Sasakian [9].

On the other hand, Pokhariyal and Mishra [14] have introduced new tensor fields, called W_2 and E-tensor fields, in a Riemannian manifold, and studied their relativistic properties. Then, Pokhariyal [13] and De [6] have studied some properties of this tensor fields in a Sasakian manifold and Trans-Sasakian manifold respectively.

The curvature tensor W_2 is defined by

$$W_2(X,Y,U,V) \quad = \quad R(X,Y,U,V)$$

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$$+\frac{1}{n-1}[g(X,U)S(Y,V)-g(Y,U)S(X,V)], \tag{1.1}$$

where S is a Ricci tensor of type (0,2).

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([16],[11]) if $R(X,Y).R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X,Y).W_2 = 0$, then the manifold is said to be W_2 semi-symmetric manifold.

The object of the present paper is to enquire under what conditions a K contact manifold will be a Sasakian manifold.

The present paper is organized as follows:

After a brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. Section 3 is devoted to the study of K-contact manifolds satisfying $\tilde{Z}(X,Y).W_2 = 0$ and prove that the manifold is Sasakian. In section 4, we consider K-contact manifolds satisfying $R(\xi,X).W_2 = 0$ and $W_2(\xi,X).R = 0$.

2. Priliminaries

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ,ξ,η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([3], [4])

$$\phi^2 = -I + \eta\otimes\xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta\circ\phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ,ξ,η), structure, that is,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{2.4}$$

and

$$g(X,\xi) = \eta(X),$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g).

An almost contact metric structure becomes a contact metric structure if

$$g(X,\phi Y) = d\eta(X,Y), \tag{2.5}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.

If ξ is a Killing vector field, then M^{2n+1} is said to be a K-contact manifold ([3], [15]). A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.6}$$

Besides the above relations in K-contact manifold the following relations hold ([1], [3], [15]):

$$\nabla_X \xi = -\phi X. \tag{2.7}$$

$$\tilde{R}(\xi, X, Y, \xi) = \eta(R(\xi, X)Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.8}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi. \tag{2.9}$$

$$S(X,\xi) = 2n\eta(X). \tag{2.10}$$

$$(\nabla_X \phi)Y = R(\xi, X)Y, \tag{2.11}$$

for any vector fields X, Y.
Again a K-contact manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant.

A transformation of a n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([12], [21]). A concircular transformation is always a conformal transformation [12]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{Z}$. It is defined by ([19], [22])

$$\tilde{Z}(X,Y)U = R(X,Y)U - \frac{r}{n(n-1)}(g(Y,U)X - g(X,U)Y). \tag{2.12}$$

where $X, Y, W \in T(M)$. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In a K-contact manifold, using (2.6) equation (2.12) reduce to

$$\tilde{Z}(\xi, X)Y = (1 - \frac{r}{n(n-1)})\{g(X,Y)\xi - \eta(Y)X\}. \tag{2.13}$$

A K-contact manifold is said to be W_2 flat if W_2 curvature vanishes at each point of the manifold. From the definition of the W_2 curvature tensor, it can be easily proved that a W_2 flat manifold implies the manifold is an Einstein manifold. It is known that [5] a compact K-contact Einstein manifold is Sasakian. Thus we have the following:

Proposition 2.1. *A W_2 flat compact K-contact manifold is Sasakian.*

3. K-contact manifolds satisfying $\tilde{Z}(X,Y).W_2 = 0$

In this section we consider a K-contact manifolds satisfying the curvature condition

$$\tilde{Z}(X,Y).W_2 = 0. \tag{3.1}$$

This equation implies

$$\begin{aligned}&\tilde{Z}(X,Y)W_2(Z,U)V - W_2(\tilde{Z}(X,Y)Z,U)V\\ &-W_2(Z,\tilde{Z}(X,Y)U)V - W_2(Z,U)\tilde{Z}(X,Y)V = 0.\end{aligned} \tag{3.2}$$

Putting $X = \xi$ in (3.2) we obtain

$$\begin{aligned}&\tilde{Z}(\xi,Y)W_2(Z,U)V - W_2(\tilde{Z}(\xi,Y)Z,U)V\\ &-W_2(Z,\tilde{Z}(\xi,Y)U)V - W_2(Z,U)\tilde{Z}(\xi,Y)V = 0.\end{aligned} \tag{3.3}$$

Using (2.13) in (3.3), we obtain

$$\begin{aligned}&(1-\frac{r}{n(n-1)})\{g(Y,W_2(Z,U)V)\xi - g(W_2(Z,U)V,\xi)Y\\ &-g(Y,Z)W_2(\xi,U)V + \eta(Z)W_2(Y,U)V - g(Y,U)W_2(Z,\xi)V\\ &\eta(U)W_2(Z,U)V - g(Y,V)W_2(Z,U)\xi + \eta(V)W_2(Z,U)Y\} = 0.\end{aligned} \tag{3.4}$$

Taking the inner product with ξ and using (2.13) in (3.4), we have

$$(1-\frac{r}{n(n-1)})g(Y,W_2(Z,U)V) = 0. \tag{3.5}$$

Again from (2.13) we have $(1-\frac{r}{n(n-1)}) \neq 0$. Hence we have

$$W_2(Z,U,V,Y) = 0. \tag{3.6}$$

From the Proposition 2.1 we have

Theorem 3.1. *A K-contact manifold satisfying the curvature condition*

$$\tilde{Z}(X,Y).W_2 = 0,$$

is Sasakian.

4. K-contact manifolds satisfying $R(\xi,X).W_2 = 0$ and $W_2(\xi,X).R = 0$

In this section we first proof a proposition

Proposition 4.1. *In an n-dimensional K-contact manifold, $\eta(W_2(X,Y)Z) = 0$.*

Proof. From equation (1.1), we have

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)QY - g(Y,Z)QX]. \tag{4.1}$$

Taking the inner product of above equation with ξ and using equations (2.8) and (2.10), we get

$$\eta(W_2(X,Y)Z) = 0. \tag{4.2}$$

□

Theorem 4.1. *In an n-dimensional K-contact manifold , $R(\xi, X)W_2 = 0$ if and only if $W_2 = 0$.*

Proof. Let in an n-dimensional K-contact manifold the curvature condition

$$R(\xi, X).W_2 = 0 \tag{4.3}$$

holds. This equation implies

$$\begin{aligned} &R(\xi, X)W_2(Y,Z)U - W_2(R(\xi, X)Y, Z)U \\ &-W_2(Y, R(\xi, X)Z)U - W_2(Y,Z)R(\xi, X)U = 0. \end{aligned} \tag{4.4}$$

Using equation (2.8) and taking the inner product of above equation with ξ, we get

$$\begin{aligned} &W_2(Y,Z,U,X) - \eta(W_2(Y,Z)U)\eta(X) - g(X,Y)\eta(W_2(\xi,Z)U) \\ &+\eta(Y)\eta(W_2(X,Z)U) + \eta(Z)\eta(W_2(Y,X)U) - g(X,Z)\eta(W_2(Y,\xi)U) \\ &+\eta(U)\eta(W_2(Y,Z)X) - g(X,U)\eta(W_2(Y,Z)\xi) = 0, \end{aligned} \tag{4.5}$$

which on using equation (4.2) gives
$W_2(Y,Z,U,X) = 0$,
that is $W_2 = 0$.
Conversely, suppose $W_2 = 0$, then from equation (4.4), we have $R(\xi, X)W_2 = 0$.
This completes the proof. □

Theorem 4.2. *An n-dimensional K-contact manifold satisfying $W_2(\xi, X).R = 0$, is an Einstein manifold.*

Proof. Let the curvature condition $W_2(\xi, X).R = 0$ holds, then we have

$$\begin{aligned} &W_2(\xi, X)R(Y,Z)U - R(W_2(\xi, X)Y, Z)U - R(Y, W_2(\xi, X)Z)U \\ &-R(Y,Z)W_2(\xi, X)U = 0. \end{aligned} \tag{4.6}$$

Now putting $X = \xi$ in equation (4.1) and using equations (2.8) and (2.10), we obtain

$$W_2(\xi, Y)Z = \eta(Z)[\frac{QY}{n-1} - Y]. \tag{4.7}$$

Now from equations (4.6) and (4.7), we have

$$\eta(R(Y,Z)U)[\frac{QX}{n-1} - X] - \eta(Y)[\frac{1}{n-1}R(QX,Z)U - R(X,Z)U]$$
$$-\eta(Z)[\frac{1}{n-1}R(Y,QX)U - R(Y,X,)U] -$$
$$\eta(U)[\frac{1}{n-1}R(Y,Z)QX - R(Y,Z)X] = 0, \tag{4.8}$$

which on taking the inner product with ξ and using equations (2.10) gives

$$\eta(Y)\eta(X)g(Z,U) - \eta(Z)\eta(X)g(Y,U) + \eta(Y)\eta(U)g(X,Z)$$
$$-\eta(U)\eta(Z)g(X,Y) - \frac{1}{n-1}[S(X,Y)g(Z,U) - S(X,Z)g(Y,U) +$$
$$\eta(U)\eta(Y)S(Z,X) - \eta(U)\eta(Z)S(X,Y)] = 0. \tag{4.9}$$

Putting $U = Z = \xi$ in above equation , we get

$$S(X,Y) = (n-1)g(X,Y), \tag{4.10}$$

which shows that the manifold is an Einstein Manifold.

□

It is known that [5] a compact K-contact Einstein manifold is Sasakian. Thus we get the following:

Corollary 4.1. *A compact K-contact manifold satisfying the curvature condition $W_2(\xi,X).R = 0$, is Sasakian.*

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ON $\mathcal{T}$-HYPERSURFACES OF A PARASASAKIAN MANIFOLD

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Abstract. The main purpose of this paper is to study transversal hypersurface (briefly, $\mathcal{T}$-hypersurface) P of a paraSasakian manifold M. We derive results allied with totally geodesic and totally umbilical $\mathcal{T}$-hypersurface of M. The necessary and sufficient condition for normality of $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure is established. Examples of $\mathcal{T}$-hypersurface are also illustrated.

Keywords: ParaSasakian manifold;Pseudo-metric; Hypersurface; $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure; Geodesic.

1. Introduction

The study of hypersurface in pseudo-Riemannian manifold is one of the potent aspects of the theory of pseudo-Riemannian geometry. It has ample significance in general theory of relativity, black holes and quantum mechanics ([1–3]). Therefore, several researchers showed interest in studying the geometry of hypersurface in different ambient spaces (c.f., [4–7]).

On the other hand, transversal hypersurface (briefly, $\mathcal{T}$-hypersurface) of contact Riemannian manifold is a hypersurface such that ξ, the *characteristic vector field* (or *Reeb vector field*) of manifold never tangent to the hyperplane. The concept of $\mathcal{T}$-hypersurface is introduced by K.Yano in 1972 [8]. After that transversal hypersurfaces were investigated by several authors in different ambient manifolds (c.f., [9–11]).

A systematic study of transversal hypersurfaces of paraSasakian manifold has not been undertaken yet, however paraSasakian manifolds have many analogies and differences with the Sasakian manifolds due to the fact that the geometry of hypersurfaces of pseudo-Riemannian manifold behave differently (for more details see, [12]). In the present paper, we consider an almost paracontact pseudo-metric manifold

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M. We obtain that every $\mathcal{T}$-hypersurface of M admits an almost paraHermitian structure as well as a $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure, and derive results allied with totally geodesic and totally umbilical transversal hypersurface. Finally, the condition of normality of $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure is obtained in a paraSasakian manifold. Examples of $\mathcal{T}$-hypersurface with $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure are also illustrated.

2. Preliminaries

Let a manifold M of dimension $(2n+1)$ be C^∞ and paracompact, and $\Gamma(TM)$ denotes the section of tangent bundle TM of manifold. Then M is said to be an *almost paracontact manifold* if it admits a tensor field φ of $(1,1)$-type, a 1-form η and a characteristic vector field ξ such that

$$\varphi^2 + \eta \otimes \xi = \mathcal{I} \text{ and } \eta(\xi) = 1, \tag{2.1}$$

where φ induces an *almost paracomplex structure* on the distribution $\mathcal{D} = ker(\eta)$, that is, the eigenspaces corresponding to eigenvalues ± 1 have equal dimension and $\mathcal{I}$ being the identity operator on tangent bundle of M. Equation (2.2) yields

$$\varphi\xi = 0, \ \operatorname{rank}(\varphi) = 2n \text{ and } \eta \circ \varphi = 0. \tag{2.2}$$

A *pseudo-metric* $\tilde{\mathfrak{g}}$ is known as *compatible* with structure (φ, ξ, η) if for any vector fields Y and Z, we have

$$\tilde{\mathfrak{g}}(Y, Z) = \eta(Y)\eta(Z) - \tilde{\mathfrak{g}}(\varphi Y, \varphi Z) \tag{2.3}$$

where signature of $\tilde{\mathfrak{g}}$ is necessarily $(n+1, n)$ and $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ is known as an *almost paracontact pseudo-metric* $(2n+1)$*-manifold*. Here, $\tilde{\mathfrak{g}}(Y, \xi) = \eta(Y)$. In view of equations (2.1) and (2.2), we have

$$\tilde{\mathfrak{g}}(Y, \varphi Z) = -\tilde{\mathfrak{g}}(\varphi Y, Z). \tag{2.4}$$

Let us consider $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ be an almost paracontact pseudo-metric $(2n+1)$-manifold. Let $\left(Z, \nu\frac{d}{dx}\right)$ be any tangent vector on $M \times \mathbb{R}$, where $Z \in \Gamma(TM)$, x denotes standard coordinate on $\mathbb{R}$ and ν is a smooth function. Then the *almost paracomplex structure* J on product manifold $M \times \mathbb{R}$ is given by $J\left(Z, \nu\frac{d}{dx}\right) = \left(\varphi Z + \nu\xi, \eta(Z)\frac{d}{dx}\right)$ and M is called *normal* if and only if J is *integrable* i.e., M is *normal* if and only if

$$d\eta(Y, Z)\xi = \frac{1}{2}N_\varphi(Y, Z), \tag{2.5}$$

where N_φ being the *Nijenhuis torsion* of endomorphism φ which is given as follows:

$$N_\varphi(Y, Z) = (\nabla_{\varphi Y}\varphi)Z - (\nabla_{\varphi Z}\varphi)Y + \varphi((\nabla_Z\varphi)Y - (\nabla_Y\varphi)Z), \tag{2.6}$$

for any tangent vectors Y, Z on M. Let Φ denotes the fundamental 2-form on M then it is defined by $\Phi(Y, Z) = \tilde{\mathfrak{g}}(Y, \varphi Z)$. If $\Phi(Y, Z) = d\eta(Y, Z)$then $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ is said to be a *paracontact pseudo-metric manifold* (c.f., [13–18]).

Definition 2.1. Let $(M;\varphi,\xi,\eta,\tilde{\mathfrak{g}})$ be a $(2n+1)$-dimensional almost paracontact pseudo-metric manifold, then it is called:

- *paracosympletic* if Φ and η are parallel, that is $\nabla\Phi = 0$ and $\nabla\eta = 0$.
- *paraSasakian* if and only if

$$(\nabla_Z\varphi)Y = \eta(Y)Z - \tilde{\mathfrak{g}}(Z,Y)\xi. \tag{2.7}$$

From equation (2.7), we can deduce that

$$\nabla_Z\xi = -\varphi Z, \tag{2.8}$$

$$\Phi(Z,Y) = (\nabla_Z\eta)Y. \tag{2.9}$$

Let $\mathcal{L}$ denotes *Lie-derivative* then for every paraSasakian manifold we have $\mathcal{L}_\xi\tilde{\mathfrak{g}} = \mathcal{L}_\xi\varphi = 0$ (see also, [15, 18–20]).

3. $\mathcal{T}$-hypersurfaces

Let $(M;\varphi,\xi,\eta,\tilde{\mathfrak{g}})$ be an almost paracontact pseudo-metric manifold, P be a smooth connected $2n$-manifold and $\iota : P \to M$ be an *immersion*. Then $i(P)$ is known as an *immersed hypersurface* of M. Let ι induces a symmetric tensor field $\mathfrak{g}$ on the immersed hypersurface $\iota(P)$ which satisfies $\mathfrak{g}(Y,Z)|_p = \tilde{\mathfrak{g}}(\iota_*Y,\iota_*Z)|_{\iota(p)}$, $\forall Y,Z \in T_pP$, where ι_* is the *pushforward map* (or *differential map*) of ι defined by $\iota_* : T_pP \to T_{\iota(p)}M$ and $(\iota_*Z)(\beta) = Z(\beta\circ\iota)$ for any smooth function β in a vicinity of $\iota(p)$ of $\iota(P)$. Hereafter, we put p and P in place of $\iota(p)$ and $\iota(P)$. In view of *causal character* of vector fields of manifold, we have three types of hypersurface P, specifically, *pseudo-Riemannian*, *Riemannian* and *null* (or *lightlike*) and metric $\mathfrak{g}$ is a *non-degenerate* or a *degenerate* according as P is pseudo-Riemannian (Riemannian) hypersurface and lightlike hypersurface respectively [12, p. 42].

Let us suppose that $(P,\mathfrak{g})$ be a pseudo-Riemannian hypersurface of M. Then normal bundle of P is given by $TP^\perp = \{Y \in \Gamma(TM)|\mathfrak{g}(Y,Z) = 0,\ \forall Z \in \Gamma(TM)\}$. Here $\dim(\mathrm{T_pP^\perp}) = 1$, due to the fact that P is a hypersurface. The orthogonal complementary decomposition is given by $TM = TP^\perp \perp TP$, $TP^\perp \cap TP = \{0\}$.

The hypersurface P is said to be a $\mathcal{T}$-hypersurface of M if the characteristic vector field ξ is never tangent to the hyperplane. Here, ξ can be considered as affine normal to P. Now, ξ and $Y \in \Gamma(TP)$ are linearly independent, therefore $\varphi(Y)$ can be written as

$$\varphi Y = JY + \alpha(Y)\xi, \tag{3.1}$$

where J is a tensor field of type $(1,1)$ and α is a 1-form on P. Operating φ on (3.1) and using equation (2.2), we have $\varphi^2 Y = \varphi JY$. Employing equations (2.1) and (3.1), this expression yields

$$Y - \eta(Y)\xi = J^2Y + (\alpha\circ J)(Y)\xi.$$

Considering normal and tangential parts from above relation, we obtain

$$J^2 = \mathcal{I},\ \ \alpha \circ J = -\eta. \tag{3.2}$$

From above equation, we can deduce that

$$\eta \circ J = -\alpha. \tag{3.3}$$

Therefore, we have a paracomplex structure J on $\mathcal{T}$-hypersurface P. From equation (3.1), $\forall\ Y, Z \in \Gamma(TP)$ we have

$$\mathfrak{g}(\varphi Y, \varphi Z) = \mathfrak{g}(JY, JZ) + \alpha(Y)\mathfrak{g}(\xi, JZ) + \alpha(Z)\mathfrak{g}(JY, \xi) + \alpha(Y)\alpha(Z)\mathfrak{g}(\xi, \xi).$$

Employing equations (2.1)-(2.3) and (3.3) in the above expression, we attain that

$$\mathfrak{g}(JY, JZ) + \mathfrak{g}(Y, Z) = \eta(Y)\eta(Z) + \alpha(Y)\alpha(Z). \tag{3.4}$$

Let us define

$$H(Y, Z) = \mathfrak{g}(\varphi Y, \varphi Z). \tag{3.5}$$

We claim that H is paraHermitian metric. From equation (3.5), we find

$$H(JY, JZ) = \mathfrak{g}(\varphi JY, \varphi JZ).$$

In light of (2.3), above expression can be written as

$$H(JY, JZ) + \mathfrak{g}(JY, JZ) = \eta(JY)\eta(JZ)$$

using equations (3.3) and (3.4) in the above relation, we have

$$H(JY, JZ) = \mathfrak{g}(Y, Z) - \eta(Y)\eta(Z) = -H(Y, Z).$$

This shows that H is a paraHermitian metric. Thus, we are in position to give the following result:

Proposition 3.1. *Let P be a $\mathcal{T}$-hypersurface of an almost paracontact pseudo-metric manifold. Then P admits an almost paraHermitian structure.*

Let P be a orientable $\mathcal{T}$-hypersurface of M, D denotes the induced Levi-Civita connection on P and N be a unit normal vector field to the hypersurface P. Then the formulas of *Gauss* and *Weingarten* formulas are given respectively by

$$\nabla_Y N = -A_N Y, \tag{3.6}$$

$$\nabla_Y Z = D_Y Z + h(Y, Z)N, \tag{3.7}$$

where

$$h(Y, Z) = \mathfrak{g}(A_N Y, Z) \tag{3.8}$$

is a *second fundamental form* and A_N is the *shape operator* allied with the normal section N. The hypersurface P is *totally geodesic* in M if second fundamental form vanishes identically. A point p of P is called *umbilical* if $h(Y,Z)|_p = \rho\mathfrak{g}(Y,Z)|_p$, $\forall\ Y, Z \in T_pM$, where $\rho \in \mathbb{R}$ and depends on p. The hypersurface P is said to be *totally umbilical* if every point of P is umbilical, that is, $h = \zeta\mathfrak{g}$, where ζ is a smooth function (see, [1,15,21]).

Given $Y \in \Gamma(TP)$, the vector field φY does not belong to $\Gamma(TP)$. Therefore, φY can be decomposed as follows

$$\varphi Y = \mathfrak{f}Y + \mu(Y)N, \tag{3.9}$$

where $\mathfrak{f}$ is a $(1,1)$-type tensor field and μ is a non-zero 1-form.
Next, we define

$$\varphi N = -U,\ \xi = V + \delta N,\ \eta(Y) = \upsilon(Y),\ \eta(N) = \delta, \tag{3.10}$$

where $U, V \in \Gamma(TP)$, υ is a 1-form and δ is a smooth function on P. Clearly $\delta \neq 0$ because if $\delta = 0$ then $\mathfrak{g}(\xi, N) = 0$, this implies that ξ is perpendicular to N so we have $\xi \in \Gamma(TP)$, which contradicts the fact that P is a $\mathcal{T}$-hypersurface. Substituting U in place of Y in (3.9), we get

$$\varphi U = \mathfrak{f}U + \mu(U)N,$$

in the light of (3.10), we obtain

$$-\varphi^2 N = \mathfrak{f}U + \mu(U)N.$$

Now employing (2.2) in above expression, we have

$$-N + \eta(N)\xi = \mathfrak{f}U + \mu(U)N,$$

applying (3.10) in above relation, we arrive at

$$-N + \delta V + \delta^2 N = \mathfrak{f}U + \mu(U)N,$$

considering normal and tangential parts of above expression, we obtain

$$\mathfrak{f}U = \delta V,\ \mu(U) = \delta^2 - 1. \tag{3.11}$$

On the other hand, substituting $X = V$ in (3.9), we get

$$\varphi V = \mathfrak{f}V + \mu(V)N.$$

Using (3.10), above equation takes the form

$$\varphi(\xi - \delta N) = \mathfrak{f}V + \mu(V)N,$$

comparing normal and tangential parts from the above equality, we find

$$\mathfrak{f}V = \delta U,\ \mu(V) = 0. \tag{3.12}$$

By the consequences of equations (3.9) and (3.10), we get $\mu(Y) = \mathfrak{g}(U, Y)$ and

$$\mu(\mathfrak{f}Y) = \mathfrak{g}(\mathfrak{f}Y, U) = \mathfrak{g}(\varphi(Y) - \mu(Y)N, -\varphi(N)),$$

employing (2.3) in above relation, we achieve that

$$\mu \circ \mathfrak{f} = -\delta \upsilon. \tag{3.13}$$

Similarly, we can find

$$\upsilon \circ f = -\delta\mu, \tag{3.14}$$
$$\upsilon(U) = 0, \ \upsilon(V) = 1 - \delta^2. \tag{3.15}$$

Replacing Y by fY in (3.9), we have

$$\varphi(\mathfrak{f}Y) = \mathfrak{f}(\mathfrak{f}Y) + \mu(\mathfrak{f}Y)N,$$

again using (3.9) in above equation, we obtain

$$\varphi^2(Y) - \mu(Y)\varphi N = \mathfrak{f}^2(Y) - \delta\upsilon(Y)N.$$

Employing (2.2) and (3.10) in the above relation, we conclude that

$$Y - \eta(Y)\xi + \mu(Y)U = \mathfrak{f}^2(Y) - \delta\upsilon(Y)N,$$

reusing (3.10) in above expression, we have

$$\mathfrak{f}^2 = I - \upsilon \otimes V + \mu \otimes U. \tag{3.16}$$

With the help of (2.3) and (3.9), we find that g satisfying

$$\mathfrak{g}(\mathfrak{f}Y, \mathfrak{f}Z) + \mathfrak{g}(Y, Z) = \upsilon(Y)\upsilon(Z) - \mu(Y)\mu(Z), \tag{3.17}$$

and

$$\mathfrak{g}(Y, \mathfrak{f}Z) + \mathfrak{g}(\mathfrak{f}Y, Z) = 0, \tag{3.18}$$

$\forall Y, Z \in \Gamma(TP)$. The above computations lead to the following result:

Proposition 3.2. *Let P be a $\mathcal{T}$-hypersurface of an almost paracontact pseudo-metric manifold M. Then P admits a $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$-structure.*

Example 3.1. *Let $M = (\mathbb{R} - \{0, 1\}) \times \mathbb{R}^4_2 \subset \mathbb{R}^5_2$ with standard Cartesian coordinates $(x_1, x_2, x_3, x_4, x_5)$. Define φ, ξ, η and $\tilde{\mathfrak{g}}$ on M by*

$$\varphi\partial_{x_1} = \partial_{x_2},\ \varphi\partial_{x_2} = \partial_{x_1},\ \varphi\partial_{x_3} = \partial_{x_4},\ \varphi\partial_{x_4} = \partial_{x_3},\ \varphi\partial_{x_5} = 0,$$
$$\xi = \partial_{x_5},\ \eta = dx_5 \text{ and } \tilde{\mathfrak{g}} = x_1^2(dx_2^2 - dx_1^2) + x_1(dx_4^2 - dx_3^2) + \eta \otimes \eta,$$

where $\partial_{x_j} = \frac{\partial}{\partial x_j}(j \in \{1,2,3,4,5\})$. *Then from simple computations, we find that* $(M;\varphi,\xi,\eta,\tilde{\mathfrak{g}})$ *is an almost paracontact pseudo-metric 5-manifold. Consider* $(P,\mathfrak{g})$ *be a pseudo-Riemannian hypersurface of* M *which is given by*

$$\mathfrak{F}(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4,x_1).$$

Then the local basis of tangent hyperplane of P *is given by*

$$X_1 = \partial_{x_1} + \partial_{x_5},\ X_2 = \partial_{x_2},\ X_3 = \partial_{x_3},\ X_4 = \partial_{x_4}$$

and normal vector field N *of the hypersurface is given by* $N = \partial_{x_1} + x_1^2\partial_{x_5}$. *Here, it is clear that* ξ_p, $p \in P$ *is not tangent to the hypersurface. Therefore,* P *is a* $\mathcal{T}$*-hypersurface of* M. *Here, we find*

$$\eta(N) = x_1^2 = \delta,\ V = -x_1^2\partial_{x_1} + (1-x_1^4)\partial_{x_5}\ \text{and}\ U = -\partial_{x_2}.$$

Further, any tangent vector field of the hypersurface P *can be expressed as* $X = \sum_{i=1}^4 a_iX_i$, *where* a_1,a_2,a_3 *and* a_4 *are smooth functions. Operating* φ *on both the sides, we have*

$$\begin{aligned}\varphi X &= a_2(1+x_1^2)\partial_{x_1} + a_1\partial_{x_2} + a_4\partial_{x_3} + a_3\partial_{x_4} + a_2x_1^4\partial_{x_5} - x_1^2a_2N\\ &= \mathfrak{f}X + \mu(X)N,\end{aligned}$$

where $\mu(X) = -x_1^2a_2$ *and* $\mathfrak{f}$ *is given by*

$$\mathfrak{f} = \begin{pmatrix} 0 & 1+x_1^2 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & x_1^4 & 0 & 0 & 0\end{pmatrix}.$$

Hence, P *is a* $\mathcal{T}$*-hypersurface of* M *which admits a* $(\mathfrak{f},\mathfrak{g},\mu,\upsilon,\delta)$*-structure.*

Lemma 3.1. *If* P *be a* $\mathcal{T}$*-hypersurface of an almost paracontact pseudo-metric manifold* M. *Then, we have*

$$\delta\alpha = \mu, \tag{3.19}$$

$$J = \mathfrak{f} - \tfrac{1}{\delta}\mu \otimes V, \tag{3.20}$$

$$H(\cdot, J\cdot) = -\mathfrak{g}(\cdot, \mathfrak{f}\cdot), \tag{3.21}$$

$$JU = \tfrac{1}{\delta}V, \tag{3.22}$$

$$\mu \circ J = \mu \circ \mathfrak{f} = -\delta\upsilon, \tag{3.23}$$

$$JV = \mathfrak{f}V = \delta U. \tag{3.24}$$

Proof. Using (3.10) in equation (3.1), we obtain $\varphi Y = JY + \alpha(Y)V + \delta\alpha(Y)N$. Now with the help of (3.9), we achieve that $fY + \mu(Y)N = JY + \alpha(Y)V + \delta\alpha(Y)N$. Comparing tangential and normal parts from above relation, we find (3.19) and

$$\mathfrak{f} = J + \alpha \otimes V.$$

In view of (3.19), the above expression yields (3.20). By the virtue of equations (3.3) and (3.5), we have

$$H(Y, JZ) + \mathfrak{g}(Y, JZ) + \alpha(Z)\eta(Y) = 0. \tag{3.25}$$

Using equations (3.19) and (3.20) in (3.25), we get (3.21). Now from (3.20), we conclude

$$JU = \mathfrak{f}U - \frac{1}{\delta}\mu(U)V.$$

Employing (3.11) in above equality, we achieve (3.22). Now, we have $\mu(JY) = \mu(\mathfrak{f}Y) - \alpha(Y)\mu(V)$, by the consequences of equations (3.12) and (3.13), we derive (3.23). Further, (3.24) follows from equations (3.12) and (3.20). These completes the proof. $\square$

Lemma 3.2. *Let P be a $\mathcal{T}$-hypersurface of an almost paracontact pseudo-metric manifold. Then, we have*

$$(\nabla_Y\varphi)Z = (D_Y\mathfrak{f})Z - \mu(Z)A_NY + h(Y, Z)U + \{(D_Y\mu)Z + h(Y, \mathfrak{f}Z)\}N, \tag{3.26}$$

$$\nabla_Y\xi = D_YV - \delta A_NY + \{h(Y, V) + Y.\delta\}N, \tag{3.27}$$

$$(\nabla_Y\varphi)N = -D_YU + \mathfrak{f}A_NY + (u(A_NY) - h(U, Y))N, \tag{3.28}$$

$$(\nabla_Y\eta)Z = (D_Yv)Z - \delta h(Y, Z), \tag{3.29}$$

for any $Y, Z \in \Gamma(TP)$.

Proof. We have $(\nabla_Y\varphi)Z = \nabla_Y\varphi Z - \varphi\nabla_YZ$, by the consequence of (3.7) this expression reduces to

$$(\nabla_Y\varphi)Z = D_Y\varphi Z + h(Y, \varphi Z)N - \varphi(D_YZ + h(Y, Z)N).$$

Employing equations (3.9) and (3.10) in the above relation, we find

$$\begin{aligned}(\nabla_Y\varphi)Z = \quad & (D_Y\mathfrak{f})Z + \mu(Z)D_YN + (Y.\mu(Z))N \\ & + h(Y, \mathfrak{f}Z)N - \mu(D_YZ)N + h(Y, Z)U.\end{aligned}$$

In view of (3.6), the above equation leads to (3.26). From (3.10), we get

$$\nabla_Y\xi = \nabla_Y(V + \delta N) = \nabla_YV + Y.\delta N + \delta\nabla_YN.$$

Now employing (3.6) and (3.7), we find (3.27). We have $(\nabla_Y\varphi)N = \nabla_Y\varphi N - \varphi(\nabla_YN)$, by the virtue of (3.7) and (3.9), this expression yields (3.28). Since $(\nabla_Y\eta)Z = \mathfrak{g}(\nabla_Y\xi, Z)$, therefore using equations (3.6), (3.7) and (3.10) we obtain (3.29). This completes the proof of lemma. $\square$

As a direct consequence of above lemma, we obtain the following result:

Proposition 3.3. *Let P be a $\mathcal{T}$-hypersurface of a paracosympletic manifold, then we have*

$$(D_Y\mathfrak{f})Z = \mu(Z)A_NY - h(Y,Z)U, \tag{3.30}$$

$$(D_Y\mu)Z = -h(Y,\mathfrak{f}Z), \tag{3.31}$$

$$D_YV = \delta A_NY, \tag{3.32}$$

$$(D_Y\upsilon)Z = \delta h(Y,Z), \tag{3.33}$$

$$Y.\delta = h(Y,V), \tag{3.34}$$

$$D_YU = \mathfrak{f}A_NY. \tag{3.35}$$

Remark 3.1. *Let the vector field U be parallel on $\mathcal{T}$-hypersurface P of a paracosympletic manifold M, then from (3.35) we receive that $\mathfrak{f}A_NY = 0$, which shows that 0 is an eigen value of $\mathfrak{f}$.*

Remark 3.2. *(a) If $\mathfrak{f}$ is parallel that is, $(D_Y\mathfrak{f})Z = 0$, then by equation (3.30) we obtain that $h(Y,Z)U = \mu(A_NY)\mu(Z)$.*

(b) If υ is parallel then from equation (3.33), we have $h(X,Y) = 0$ that is, P is a totally geodesic, since $\delta \neq 0$.

4. $\mathcal{T}$-hypersurface of a paraSasakian manifold

Here, we consider a $\mathcal{T}$-hypersurface P of a paraSasakian manifold M.

Theorem 4.1. *Let P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold, then we have*

$$(D_Y\mathfrak{f})Z = \mu(Z)A_NY + \upsilon(Z)Y - h(Z,Y)U - \mathfrak{g}(Z,Y)V, \tag{4.1}$$

$$(D_Y\mu)Z = -\delta\mathfrak{g}(Y,Z) - h(Y,\mathfrak{f}Z), \tag{4.2}$$

$$D_YV - \delta A_NY + \mathfrak{f}Y = 0, \tag{4.3}$$

$$h(Y,V) + \mu(Y) + Y.\delta = 0, \tag{4.4}$$

$$D_YU + \delta Y - \mathfrak{f}A_NY = 0, \tag{4.5}$$

$$(D_Y\eta)Z - \delta h(Z,Y) - g(\mathfrak{f}Z,Y) = 0, \tag{4.6}$$

for any $Z,Y \in \Gamma(TP)$.

Proof. Using equation (2.7) in (3.26), we get

$$\begin{aligned} -\mathfrak{g}(Z,Y)\xi + \eta(Z)Y = \quad & (D_Y\mathfrak{f})Z - \mu(Z)A_NY + h(Z,Y)U \\ & +\{(D_Y\mu)Z + h(Y,\mathfrak{f}Z)\}N. \end{aligned}$$

In view of (3.10) above equation reduces to the following form

$$\begin{aligned} -\mathfrak{g}(Z,Y)V - \delta\mathfrak{g}(Z,Y)N + \upsilon(Z)Y = \quad & (D_Y\mathfrak{f})Z - \mu(Z)A_NY + h(Z,Y)U \\ & +\{(D_Y\mu)Z + h(\mathfrak{f}Z,Y)\}N. \end{aligned}$$

Considering normal and tangential parts from above expression, we receive (4.1) and (4.2). By the virtue of equations (2.8), (3.9) and (3.27), we obtain (4.3) and (4.4). In view of equations (2.7) and (3.28), we have (4.5). equation (4.6) follows from (2.9) and (3.29). Hence this completes the proof of the theorem. □

Using $h(Z,Y) = \zeta\mathfrak{g}(Z,Y)$ in equation (4.4), we obtain following result:

Corollary 4.1. *If P be a totally umbilical $\mathcal{T}$-hypersurface of a paraSasakian manifold, then necessary and sufficient condition for P to be a totally geodesic is that*

$$\mu(Z) + Z.\delta = 0. \tag{4.7}$$

equation (4.6) leads to the following remark:

Remark 4.1. *Let P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold M. Then P is a totally geodesic $\iff$ $(D_Y\eta)Z = \mathfrak{g}(\mathfrak{f}Z,Y)$, $\forall\ Y,Z \in \Gamma(TP)$.*

Let us consider the fundamental 2-form $\mathfrak{F}$ on P, given by $\mathfrak{F}(Y,Z) = H(Y,JZ)$. Using the equation (3.21), this reduces to $\mathfrak{F}(Y,Z) = \mathfrak{g}(Y,\mathfrak{f}Z)$. From equation (4.1), we have

$$(D_Y\mathfrak{F})(Z,W) = \mu(W)h(Z,Y) + \upsilon(W)\mathfrak{g}(Y,Z) - \mu(Z)h(Y,W) - \upsilon(Z)\mathfrak{g}(Y,W).$$

In view of the above equation, we find

$$(D_W\mathfrak{F})(Y,Z) + (D_Y\mathfrak{F})(Z,W) + (D_Z\mathfrak{F})(W,Y) = 0.$$

This implies that $\mathfrak{F}$ is closed. Now differentiating (3.20) covariantly along X and using equations (4.1)-(4.4), we get

$$(D_YJ)Z = \upsilon(Z)Y - h(Z,Y)U + \frac{1}{\delta}(h(JZ,Y) + \mu(Z)JY). \tag{4.8}$$

In view of the above equation, we find that the Nijenhuis tensor N_J formed with J satisfies $N_J(Y,Z) = 0$. These lead to the following proposition:

Proposition 4.1. *Every $\mathcal{T}$-hypersurface of a paraSasakian manifold admits paraKäehlerian structure.*

Let the tensor field $\mathfrak{f}$ be parallel then from (4.1), we have

$$h(Z,Y)U = \mu(Y)A_NZ + \upsilon(Y)Z - \mathfrak{g}(Z,Y)V. \tag{4.9}$$

Operating μ on (4.9) and using (3.11), we find

$$(\delta^2 - 1)h(Z,Y) = \mu(A_NZ)\mu(Y) + \upsilon(Y)\mu(Z). \tag{4.10}$$

Replacing Z by V and employing (3.11), the above equation reduces to

$$h(Y,V) + \mu(Y) = 0. \tag{4.11}$$

In view of equations (4.4) and (4.11), we obtain that $Y.\delta = 0$. This leads to the following proposition:

Proposition 4.2. *Let P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold M and the tensor field $\mathfrak{f}$ be parallel. Then δ is a non-zero constant.*

Let $S_{\mathfrak{f}}$ denotes the torsion tensor of $\mathfrak{f}$ defined by

$$S_{\mathfrak{f}}(Z,Y) = N_{\mathfrak{f}}(Z,Y) + d\mu(Z,Y)U + d\upsilon(Z,Y)V, \tag{4.12}$$

where $N_{\mathfrak{f}}$ is the Nijenhuis torsion of $\mathfrak{f}$, and

$$d\mu(Z,Y) = (D_Z\mu)Y - (D_Y\mu)Z,$$
$$d\upsilon(Z,Y) = (D_Z\upsilon)Y - (D_Y\upsilon)Z.$$

If $S_{\mathfrak{f}}$ vanishes identically, then the structure $(\mathfrak{f},\mathfrak{g},\mu,\upsilon,\delta)$ is said to be *normal*. Let P be a $\mathcal{T}$-hypersurface of paraSasakian manifold and the structure $(\mathfrak{f},\mathfrak{g},\mu,\upsilon,\delta)$ be normal. Then, we find

$$\eta\left(N_{\mathfrak{f}}(Z,Y)\right) + \left(1-\delta^2\right) d\eta(Z,Y) = 0,\ \forall\ Z,Y \in \Gamma(TP). \tag{4.13}$$

Theorem 4.2. *If P be a $\mathcal{T}$-hypersurface of a paraSasakian manifold. Then the structure $(\mathfrak{f},\mathfrak{g},\mu,\upsilon,\delta)$ is normal if and only if the shape operator A_N of P satisfies $A_N\mathfrak{f} = \mathfrak{f}A_N$.*

Proof. Employing equations (3.18) and (4.1), we have

$$\begin{aligned} N_{\mathfrak{f}}(Z,Y) = \quad & \mu(Y)(A_N\mathfrak{f}Z - \mathfrak{f}A_NZ) - \mu(Z)(A_N\mathfrak{f}Y - \mathfrak{f}A_NY) \\ & +(h(Z,\mathfrak{f}Y) - h(\mathfrak{f}Z,Y))U - 2\mathfrak{g}(Z,\mathfrak{f}Y)V. \end{aligned} \tag{4.14}$$

In light of equations (4.2) and (4.3), we get

$$d\mu(Z,Y) = \quad h(\mathfrak{f}Z,Y) - h(Z,\mathfrak{f}Y), \tag{4.15}$$

$$d\upsilon(Z,Y) = \quad 2\mathfrak{g}(Z,\mathfrak{f}Y). \tag{4.16}$$

Using equations (4.14)-(4.16) in (4.12), we obtain

$$S_{\mathfrak{f}}(Z,Y) = \mu(Y)(A_N\mathfrak{f}Z - \mathfrak{f}A_NZ) - \mu(Z)(A_N\mathfrak{f}Y - \mathfrak{f}A_NY). \tag{4.17}$$

This completes the proof. □

Example 4.1. *Let $M = \mathbb{R}^3_1$ with coordinates (x,y,z). Define φ,ξ and η on M by*

$$\varphi\partial_x = \partial_y - 2x\partial_z,\ \varphi\partial_y = \partial_x,\ \varphi\partial_z = 0, \xi = \partial_z, \text{ and } \eta = 2xdy + dz,$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$. Then (φ,ξ,η) is an almost paracontact structure on M. By simple computations, it can be seen that the structure is normal. Now, we consider $\tilde{\mathfrak{g}} = -dx^2 + dy^2 + \eta\otimes\eta$. Using φ and the metric $\tilde{\mathfrak{g}}$, we find

$\tilde{\mathfrak{g}}(\varphi Y, \varphi Z) + \tilde{\mathfrak{g}}(Y, Z) = \eta(Y)\eta(Z)$ *and* $\eta(Y) = \tilde{\mathfrak{g}}(Y, \xi)$*, and thus* $(M; \varphi, \xi, \eta, \tilde{\mathfrak{g}})$ *is a normal almost paracontact pseudo-metric 3-manifold. With respect to* $\tilde{\mathfrak{g}}$*, we have*

$$\begin{gathered}\nabla_{\partial_x}\partial_x = 0, \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 2x\partial_y + (1-4x^2)\partial_z,\\ \nabla_{\partial_x}\partial_z = \nabla_{\partial_z}\partial_x = \partial_y - 2x\partial_z, \nabla_{\partial_y}\partial_y = 4x\partial_x,\\ \nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = \partial_x, \nabla_{\partial_z}\partial_z = 0.\end{gathered}$$

Using equation (2.7) and the above expressions, we find that M *is a paraSasakian manifold. Let* $(P, \mathfrak{g})$ *be a pseudo-Riemannian hypersurface of* M *which is defined by*

$$\mathfrak{F}(r, \vartheta) = (r, \sinh\vartheta, \cosh\vartheta),$$

where $r, \vartheta \in \mathbb{R}$*. Then the local basis of tangent bundle of* P *is given by the vector fields*

$$Z_1 = \partial_x, \text{ and } Z_2 = \cosh\vartheta\,\partial_y + \sinh\vartheta\,\partial_z.$$

The normal vector field N *of the hypersurface is expressed as*

$$N = \partial_y - \frac{(4r^2+1) + 2r\tanh\vartheta}{2r + \tanh\vartheta}\partial_z.$$

Here, it is clear that ξ *is never tangent to the hypersurface. Therefore,* P *is a* $\mathcal{T}$*-hypersurface of* M*. Now, we obtain that*

$$\begin{gathered}\eta(N) = -\tfrac{1}{2r+\tanh\vartheta} = \delta,\\ V = \tfrac{1}{2r+\tanh\vartheta}\partial_y + \left(\tfrac{2r\tanh\vartheta - \operatorname{sech}^2\vartheta}{(2r+\tanh\vartheta)^2}\right)\partial_z \text{ and } U = -\partial_x.\end{gathered}$$

Further, any tangent vector field of the hypersurface P *can be expressed as* $Z = b_1 Z_1 + b_2 Z_2$*, where* b_1 *and* b_2 *are smooth functions. Operating* φ *on both the sides, we have*

$$\varphi Z = \mathfrak{f}Z + \mu(Z)N,$$

where $\mu(Z) = b_1$ *and* $\mathfrak{f}$ *is given by*

$$\mathfrak{f} = \begin{pmatrix} 0 & \cosh\vartheta & 0\\ 0 & 0 & 0\\ \frac{1}{2r+\tanh\vartheta} & 0 & 0\end{pmatrix}.$$

Hence, P *is a* $\mathcal{T}$*-hypersurface of a paraSasakian manifold* M *and admits* $(\mathfrak{f}, \mathfrak{g}, \mu, \upsilon, \delta)$*-structure.*

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ON f-KENMOTSU MANIFOLDS AND THEIR SUBMANIFOLDS WITH QUARTER SYMMETRIC METRIC CONNECTIONS *

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Abstract. The object of the present paper is to study invariant submanifolds of f-Kenmotsu manifolds with respect to quarter symmetric metric connections. Some necessary and sufficient conditions for such submanifolds to be totally geodesic have been deduced. Also we have constructed an example of a submanifold of a five-dimensional f-Kenmotsu manifold to justify our results.

Keywords: f-Kenmotsu manifold; quarter symmetric metric connection.

1. Introduction

In 1924, Friedman and Schouten [10] introduced the notion of semi-symmetric metric connections on a manifold and the notion of quarter symmetric metric connections was defined and studied by Golab [11]. The quarter-symmetric metric connections are generalizations of the semi-symmetric metric connections. A linear connection $\bar{\nabla}$ in a Riemannian manifold is said to be a quarter symmetric metric connection [11] if the torsion tensor T defined by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.1}$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

for any vector field X, Y on the manifold. Here η is a 1-form and ϕ is a (1,1) tensor field. If $\phi X = X$, then the quarter symmetric connection is reduced to a semi-symmetric connection. If the quarter symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0,$$

for any vector field X, Y, Z on the manifold, then the connection $\bar{\nabla}$ is said to be quarter symmetric metric connection; otherwise, it is non-metric connection.

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Quarter symmetric connections have been characterized by several authors ([3], [16], [17], [18], [26], [28]). Recently, P-Sasakian manifolds admitting a quarter symmetric metric connections have been studied by De et all [7].

The notion of invariant submanifolds is an important topic of study in differential geometry. If in a submanifold of an almost contact manifold the structure tensor maps tangent vector fields to tangent vector fields, then the submanifold is called invariant [5]. Invariant submanifolds of Sasakian manifolds were studied by M. Kon [14]. Invariant submanifolds of contact and para contact manifolds have been studied by several authors ([8], [20], [21], [24], [25], [30]).

In 1982, Olszak and Rosca [22] introduced f-Kenmotsu manifolds and gave their geometric interpretations, they also proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold. Several authors ([4], [6], [29]) studied f-Kenmotsu manifolds. In the present paper we would like to study invariant submanifolds of f-Kenmotsu manifolds with respect to quarter symmetric metric connections. In fact, we have obtained the conditions for such submanifolds to be totally geodesic. The present paper is organized as follows:

Section 1, is introductory. After preliminaries in Section 2, we obtain the relations between the curvature tensor, Ricci tensor and scalar curvature of the manifold with respect to Levi-Civita connection and quarter symmetric metric connection in Section 3. Next we study invariant submanifolds of an f-Kenmotsu manifold and construct an example of a submanifold of a five-dimensional f-Kenmotsu manifold to justify our results. Finally, we obtain the conditions for such submanifolds to be totally geodesic.

2. Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$-dimensional differentiable manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on the manifold, satisfying the relations

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1,$$

$$\eta(X) = g(X, \xi),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

$$\phi\xi = 0, \quad \eta o\phi = 0, \quad g(X, \phi Y) = -g(\phi X, Y)$$

for any vector fields X, Y on the manifold $\widetilde{M}$.

The manifold $\widetilde{M}$ is called an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies the relation [29]

$$(\widetilde{\nabla}_X \phi)Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.2}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the f-Kenmotsu manifolds and f is a C^{∞}-function on the manifold. If $f = \beta = \text{constant} \neq 0$, then the manifold is β-Kenmotsu manifold [13] and if $f = 0$, then the manifold reduces to cosymplectic manifold [13]. Moreover f-Kenmotsu manifold is called regular if $f^2 + f' \neq 0$, where $f' = \xi f$.

Form (2.2), we get

$$\widetilde{\nabla}_X \xi = f(X - \eta(X)\xi). \tag{2.3}$$

Let M^{2m+1} $(m < n)$ be a submanifold of a contact metric manifold $\widetilde{M}^{2n+1}$. Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of M and $\widetilde{M}$, respectively. Then for any vector fields $X, Y \in \chi(M)$, the second fundamental form h is defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

and for any vector field V of normal bundle $T^{\perp}M$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V. \tag{2.5}$$

The second fundamental form h and the shape operator A_V are related by [27]

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.6}$$

A submanifold M of a f-Kenmotsu manifold is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H \tag{2.7}$$

for any vector field X, Y on M; H is the mean curvature of M given by

$$H = \frac{1}{2m+1} \sum_{i=1}^{2m+1} h(e_i, e_i). \tag{2.8}$$

Moreover, if $h(X, Y) = 0$ for all $X, Y \in \chi(M)$, then the submanifold is called totally geodesic. If $H = 0$, then the submanifold M is minimal in $\widetilde{M}$.

Covariant derivative of order $p, p \geq 1$ of a $(0, k)$ tensor field is denoted by $\nabla^p T$. According to [23] the tensor T is said to be recurrent and 2-recurrent if

$$(\nabla T)(X_1, X_2, ..., X_k; X)T(Y_1, Y_2, ...Y_k) = (\nabla T)(Y_1, Y_2, ...Y_k; X)T(X_1, X_2, ..., X_k), \tag{2.9}$$

and

$$\begin{aligned} &(\nabla^2 T)(X_1, X_2, ..., X_k; X, Y)T(Y_1, Y_2, ...Y_k) = \\ &\quad (\nabla^2 T)(Y_1, Y_2, ...Y_k; X, Y)T(X_1, X_2, ..., X_k), \end{aligned} \tag{2.10}$$

where $X, Y, X_1, X_2, ..., X_k, Y_1, Y_2, ...Y_k \in \chi(\widetilde{M})$. If T is non-zero then there exists a unique 1-form π and a $(0, 2)$ tensor ψ, such that

$$\nabla T = T \otimes \pi, \qquad \pi = d(\log ||T||), \tag{2.11}$$

and

$$\nabla^2 T = T \otimes \psi, \tag{2.12}$$

where $||T|| = g(T,T)$.

In a $(2n+1)$ dimensional f-Kenmotsu manifold, we have [22]

$$\begin{aligned} R(X,Y)\xi &= f^2(\eta(X)Y - \eta(Y)X) \\ &+ (Yf)\phi^2 X - (Xf)\phi^2 Y \end{aligned} \tag{2.13}$$

$$S(X,\xi) = -(2nf^2 - \xi f)\eta(X) - (2n-1)Xf, \tag{2.14}$$

$$S(\xi,\xi) = -2n(f^2 - \xi f), \tag{2.15}$$

$$Q\xi = -(2nf^2 - \xi f)\xi - (2n-1)\mathrm{grad}f, \tag{2.16}$$

where R, S and Q denote the Riemann curvature tensor, Ricci tensor and Ricci operator respectively.

In a 3-dimensional f- Kenmotsu manifold we also have [22]

$$R(X,Y)Z = (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z \tag{2.17}$$

$$- (\frac{r}{3} + 3f^2 + 2f')(\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z), \tag{2.18}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{3} + 3f^2 + 2f')\eta(X)\eta(Y), \tag{2.19}$$

where r is the scalar curvature and $f' = \xi f$.

On a manifold $\widetilde{M}$, for a $(0,k)$-type tensor field $T(k \geq 1)$ and a $(0,2)$-type tensor field E, we denote by $Q(E,T)$ a $(0,k+2)$-type tensor field [12] defined as follows

$$\begin{aligned} Q(E,T)(X_1, X_2, ..., X_k; X, Y) = &- T((X \wedge_E Y)X_1, X_2, ..., X_n) \\ &- T(X_1, (X \wedge_E Y)X_2, ..., X_k) - ... \\ &- T(X_1, ..., (X \wedge_E Y)X_k), \end{aligned} \tag{2.20}$$

where $(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y$.
The submanifold M of $\widetilde{M}$ is pseudo parallel ([1], [2], [9]) if

$$\widetilde{R}(X,Y).h = (\widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]})h = L_1 Q(g,h) \tag{2.21}$$

for any vector field X, Y tangent to M and L_1 is a function on the subset U on M, where $U = \{x \in M : Q(g,h) \neq 0 \mathrm{at} x\}$. Again if $L_1 = 0$, then the manifold is said to be semiparallel [15]. The submanifold is Ricci generalized pseudoparallel [19] if its second fundamental form h satisfies

$$\widetilde{R}(X,Y).h = L_2 Q(S,h), \tag{2.22}$$

where L_2 is a function on the subset V of M, where $V = \{x \in M : Q(S,h) \neq 0 \mathrm{at} x\}$.

3. f-Kenmotsu manifolds with respect to quarter symmetric metric connection

Let $\tilde{\nabla}$ and $\overline{\tilde{\nabla}}$ be the Levi-Civita and quarter symmetric metric connections of an f-Kenmotsu manifold $\widetilde{M}$ of dimension $(2n+1)$. Then we have [11]

$$\overline{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + U(X,Y), \tag{3.1}$$

where $U(X,Y)$ is (1,1) tensor field and $X,Y \in \chi(\widetilde{M})$. The tensor U is defined by

$$U(X,Y) = \frac{1}{2}(T(X,Y) + T'(X,Y) + T'(Y,X)), \tag{3.2}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y) \tag{3.3}$$

for $X,Y,Z \in \chi(\widetilde{M})$.
Now from (1.2) and (3.3) we infer that

$$T'(X,Y) = g(X,\phi Y)\xi - \eta(X)\phi(Y). \tag{3.4}$$

Using (1.2) and (3.4) in (3.2), we obtain

$$U(X,Y) = -\eta(X)\phi(Y). \tag{3.5}$$

Therefore, the relation between quarter symmetric metric connection $\overline{\tilde{\nabla}}$ and the Levi-Civita connection $\tilde{\nabla}$ in an f-Kenmotsu manifold is given by

$$\overline{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y - \eta(X)\phi(Y). \tag{3.6}$$

Let $\overline{\tilde{R}}$ be the curvature tensor of an f-Kenmotsu manifold $\widetilde{M}$ with respect to quarter symmetric metric connection $\overline{\tilde{\nabla}}$. Then $\overline{\tilde{R}}$ is defined by

$$\overline{\tilde{R}}(X,Y)Z = \overline{\tilde{\nabla}}_X \overline{\tilde{\nabla}}_Y Z - \overline{\tilde{\nabla}}_Y \overline{\tilde{\nabla}}_X Z - \overline{\tilde{\nabla}}_{[X,Y]} Z. \tag{3.7}$$

With the help of (2.2), (2.3) and (3.6) we obtain

$$\begin{aligned} \overline{\tilde{\nabla}}_X \overline{\tilde{\nabla}}_Y Z &= \tilde{\nabla}_X \tilde{\nabla}_Y Z - \eta(X)\phi(\tilde{\nabla}_Y Z) - (g(\tilde{\nabla}_X Y,\xi) + fg(X,Y) \\ &- f\eta(X)\eta(Y))\phi(Z) - \eta(Y)(\tilde{\nabla}_X \phi(Z) - \eta(X)Z - \eta(X)\eta(Z)\xi) \end{aligned}$$

and

$$\begin{aligned} \overline{\tilde{\nabla}}_Y \overline{\tilde{\nabla}}_X Z &= \tilde{\nabla}_Y \tilde{\nabla}_X Z - \eta(Y)\phi(\tilde{\nabla}_X Z) - (g(\tilde{\nabla}_Y Z,\xi) + fg(X,Y) \\ &- f\eta(X)\eta(Y))\phi(Z) - \eta(X)(\tilde{\nabla}_Y \phi Z - \eta(Y)Z - \eta(Y)\eta(Z)\xi) \end{aligned}$$

and

$$\overline{\tilde{\nabla}}_{[X,Y]} Z = \tilde{\nabla}_{[X,Y]} Z - \eta(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X)\phi(Z).$$

Using these results in (3.7) we have

$$\begin{aligned} \overline{\tilde{R}}(X,Y)Z &= \tilde{R}(X,Y)Z + f(\eta(Y)\phi(X) - \eta(X)\phi(Y))\eta(Z) \\ &+ f(\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z))\xi, \end{aligned} \tag{3.8}$$

where $\tilde{R}$ is curvature tensor with respect to Levi-Civita connection.

Let $\overline{\tilde{S}}$ and $\tilde{S}$ be Ricci curvature tensors of $\widetilde{M}$ with respect to quarter symmetric and Levi-Civita connections. Then $\overline{\tilde{S}}$ is defined by

$$\overline{\tilde{S}}(X,Y) = \sum_{i=1}^{2n+1} g(\overline{\tilde{R}}(e_i,X)Y,e_i), \tag{3.9}$$

where $\{e_1,e_2,....,e_{2n+1}\}$ is a local orthonormal basis on $\widetilde{M}$. Using the relations in (2.1) and (3.8) we have

$$\overline{\tilde{S}}(X,Y) = \tilde{S}(X,Y) + fg(\phi X,Y). \tag{3.10}$$

Let $\overline{\tilde{Q}}$ and $\tilde{Q}$ be the Ricci operators on $\widetilde{M}$ with respect to the connections $\overline{\tilde{\nabla}}$ and $\tilde{\nabla}$ respectively. Then using (3.10) we have

$$\overline{\tilde{Q}}X = \tilde{Q}X + f\phi X. \tag{3.11}$$

Let $\overline{\tilde{r}}$ and $\tilde{r}$ be the scalar curvature in $\widetilde{M}$ with respect to the connections $\overline{\tilde{\nabla}}$ and $\tilde{\nabla}$ respectively. Then

$$\overline{\tilde{r}} = \tilde{r}. \tag{3.12}$$

Now for $X,Y \in \chi(\widetilde{M})$ we obtain from the previous results

$$\overline{\tilde{R}}(X,Y)\xi = \tilde{R}(X,Y)\xi + f(\eta(Y)\phi(X) - \eta(X)\phi(Y)), \tag{3.13}$$

$$\overline{\tilde{S}}(X,\xi) = \tilde{S}(X,\xi) \tag{3.14}$$

and

$$\overline{\tilde{Q}}X = \tilde{Q}X. \tag{3.15}$$

Now we prove the following:

Theorem 3.1. *In an f-Kenmotsu manifold $\widetilde{M}$ with respect to quarter symmetric metric connection $\overline{\tilde{\nabla}}$ we have*

$$\overline{\tilde{R}}(X,Y)Z + \overline{\tilde{R}}(Y,Z)X + \overline{\tilde{R}}(Z,X)Y = 0.$$

Proof. Using (3.8) we obtain the theorem. □

4. Invariant submanifolds of f-Kenmotsu manifolds with respect to quarter symmetric metric connection

Let M be a $(2m+1)$-dimensional invariant submanifold of a f-Kenmotsu manifold $\widetilde{M}$ of dimension $(2n+1)$ (where $n > m$). Generally the submanifold M is said to be invariant submanifold of $\widetilde{M}$ if $\phi(TM) \subset TM$. Let $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ be the Levi-Civita and quarter symmetric metric connections of $\widetilde{M}$. Let ∇ and $\bar{\nabla}$ be the induced connections on M form the connections $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$.

Let h and $\bar{h}$ be the second fundamental forms of the submanifold with respect to Levi-Civita connections and quarter symmetric metric connections respectively.

Then we have

$$\overline{\widetilde{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X,Y). \tag{4.1}$$

Using the equation (3.6) in (4.1) we have

$$\bar{\nabla}_X Y + \bar{h}(X,Y) = \nabla_X Y + h(X,Y) - \eta(X)\phi(Y). \tag{4.2}$$

Since the submanifold is invariant, therefore comparing tangential and normal components, we have

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi(Y), \tag{4.3}$$

$$\bar{h}(X,Y) = h(X,Y). \tag{4.4}$$

Thus the second fundamental forms $\bar{h}$ and h of the submanifold with respect to the quarter symmetric metric connection and the Levi-Civita connection are the same. From (3.6) and (4.3) we can say that an invariant submanifold admits quarter symmetric connections. Hence we have the following:

Lemma 4.1. *Let M be an invariant submanifold of a f-Kenmotsu manifold $\widetilde{M}$, and $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ are the Levi-Civita and quarter symmetric metric connections of $\widetilde{M}$. If ∇ and $\bar{\nabla}$ are the induced connections on M form the connections $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ of $\widetilde{M}$ respectively, then M admits a quarter symmetric metric connection and the second fundamental forms with respect to $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ are same.*

Theorem 4.1. *Any invariant submanifold of an f-Kenmotsu manifold is totally geodesic with respect to the Levi-Civita connections if and only if it is so with respect to quarter symmetric metric connections.*

Proof. The above theorem follows from the Lemma 4.1. □

Using (2.8) and (4.4), we can say that the mean curvature vector with respect to the Levi-Civita connection and quarter symmetric metric connection are same. Thus we have the following:

Theorem 4.2. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\widetilde{M}$. Then the mean curvature vector with respect to the Levi-Civita connection and quarter symmetric metric connection are same.*

We may state the following:

Corollary 4.1. *An invariant submanifold of a f-Kenmotsu manifold is totally umbilical with respect to the Levi-Civita connection if and only if it is totally umbilical* *with respect to the quarter symmetric metric connection.*

Corollary 4.2. *An invariant submanifold of a f-Kenmotsu manifold is minimal with respect to the Levi-Civita connection if and only if it is minimal with respect to the quarter symmetric metric connection.*

Example 4.1. We consider a five-dimensional manifold $\widetilde{M} = \{(x_1, x_2, x_3, x_4, t) \in \mathbb{R}^5 : t \neq 0\}$, where (x_1, x_2, x_3, x_4, t) are the standard coordinates in $\mathbb{R}^5$. Let us choose the vector fields

$$e_1 = t^2\frac{\partial}{\partial x_1}, \quad e_2 = t^2\frac{\partial}{\partial x_2}, \quad e_3 = t^2\frac{\partial}{\partial x_3}, \quad e_4 = t^2\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial t},$$

which are linearly independent at each point of $\widetilde{M}$. We define the metric g such that $\{e_1, e_2, e_3, e_4, e_5\}$ is an orthonormal basis of $\widetilde{M}$ i.e.,

$$\begin{aligned} g(e_i, e_j) &= 1 \quad \text{if} \quad i = j \\ &= 0 \quad \text{if} \quad i \neq j, \quad \text{where} \quad 1 \leqslant i, j \leqslant 5. \end{aligned}$$

We consider a 1-form η defined by

$$\eta(X) = g(X, e_5), \quad X \in \chi(\widetilde{M}).$$

That is, we choose $e_5 = \xi$. We define the tensor field ϕ by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = -e_4, \quad \phi(e_4) = e_3, \quad \phi(e_5) = 0.$$

The linearity property of g and ϕ shows that

$$\eta(e_5) = 1, \qquad \phi^2(X) = -X + \eta(X)e_5,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on $\widetilde{M}$. Then $\widetilde{M}(\phi, \xi, \eta, g)$ forms an almost contact manifold with $e_5 = \xi$. Let $\tilde{\nabla}$ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_5, e_i] = \frac{2}{t}e_i, \quad i = 1, 2, 3, 4, \quad \text{and} \quad [e_i, e_j] = 0, \quad \text{otherwise.}$$

Now by Koszul's formula, we can obtain the following

$$\begin{gathered} \tilde{\nabla}_{e_1}e_1 = \frac{2}{t}e_5, \quad \tilde{\nabla}_{e_1}e_5 = -\frac{2}{t}e_1, \quad \tilde{\nabla}_{e_2}e_5 = -\frac{2}{t}e_2, \quad \tilde{\nabla}_{e_2}e_2 = \frac{2}{t}e_5, \\ \tilde{\nabla}_{e_3}e_3 = \frac{2}{t}e_5, \quad \tilde{\nabla}_{e_3}e_5 = -\frac{2}{t}e_3, \quad \tilde{\nabla}_{e_4}e_4 = \frac{2}{t}e_5, \quad \tilde{\nabla}_{e_4}e_5 = -\frac{2}{t}e_4, \\ \tilde{\nabla}_{e_i}e_j = 0, \quad \text{otherwise.} \end{gathered}$$

The above relations imply that the manifold satisfies

$$\tilde{\nabla}_X\xi = f\{X - \eta(X)\xi\}$$

for $\xi = e_5$, and $f = -\frac{2}{t}$. Hence we can say that $\widetilde{M}$ is an f-Kenmotsu manifold. Again since $f^2 + f' \neq 0$, so the manifold is regular f-Kenmotsu manifold.
Let M be a subset of $\widetilde{M}$ and consider the immersion $h : M \to \widetilde{M}$ defined by

$$h(x_1, x_2, t) = (x_1, x_2, 0, 0, t).$$

It is easy to prove that $M = \{(x_1, x_2, t) \in \mathbb{R}^3 : t \neq 0\}$ is a submanifold of $\widetilde{M}$, where (x_1, x_2, t) are the standard coordinates of $\mathbb{R}^3$. We choose the vector fields

$$e_1 = t^2 \frac{\partial}{\partial x_1}, \quad e_2 = t^2 \frac{\partial}{\partial x_2}, \quad e_5 = \frac{\partial}{\partial t}.$$

We define g_1 such that $\{e_1, e_2, e_5\}$ is an orthonormal basis of M. That is,

$$\begin{aligned} g_1(e_i, e_j) &= 1 \qquad \text{if} \quad i = j \\ &= 0 \qquad \text{if} \quad i \neq j, \quad \text{where} \quad i, j = 1, 2, 5. \end{aligned}$$

We define a 1-form η_1 and a (1,1) tensor ϕ_1 respectively by

$$\eta_1 = g_1(X, e_5), \quad \text{and} \quad \phi_1(e_1) = -e_2, \quad \phi_1(e_2) = e_1, \quad \phi_1(e_5) = 0.$$

The linearity property of g_1 and ϕ_1 shows that

$$\eta_1(e_5) = 1, \qquad \phi_1^2(X) = -X + \eta_1(X)e_5,$$

$$g_1(\phi_1 X, \phi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y)$$

for any vector fields X, Y on $M(\phi_1, \xi, \eta_1, g_1)$. It is seen that M is an invariant submanifold of $\widetilde{M}$ with $e_5 = \xi$. Moreover, let ∇ be the Levi-Civita connection with respect to the metric g_1. Then we have

$$[e_5, e_i] = \frac{2}{t} e_i, \quad i = 1, 2, \quad \text{and} \quad [e_i, e_j] = 0, \quad \text{otherwise}.$$

Now by Koszul's formula, we can obtain the following

$$\nabla_{e_1} e_1 = \frac{2}{t} e_5, \quad \nabla_{e_1} e_5 = -\frac{2}{t} e_1, \quad \nabla_{e_2} e_5 = -\frac{2}{t} e_2, \quad \nabla_{e_2} e_2 = \frac{2}{t} e_5,$$
$$\nabla_{e_i} e_j = 0, \quad \text{otherwise}.$$

Let us consider $\bar{\tilde{\nabla}}$ and $\bar{\nabla}$ be the quarter symmetric metric connections on $\widetilde{M}$ and M respectively. Using (3.6) we can find $\bar{\tilde{\nabla}}_{e_i} e_j$ and $\bar{\nabla}_{e_i} e_j$.
Let h and $\bar{h}$ be the second fundamental forms with respect to Levi-Civita connection and quarter symmetric metric connections. By using (2.4) we have

$$h(X, Y) = 0, \qquad \text{and} \qquad \bar{h}(X, Y) = 0$$

for any vector field on the manifold. Thus the submanifold is totally geodesic with respect to Levi-Civita connection and quarter symmetric metric connection. Hence the Theorem 4.1 is verified.

5. Invariant submanifolds of f-Kenmotsu manifolds with certain curvature conditions on the second fundamental form

Now from (2.3) and (2.4) we have

$$\nabla_X \xi + h(X, \xi) = f(X - \eta(X)\xi). \tag{5.1}$$

Comparing normal and tangential components, we have

$$h(X, \xi) = 0, \tag{5.2}$$

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{5.3}$$

Using (4.4) and (5.2) we can say that

$$\bar{h}(X, \xi) = 0. \tag{5.4}$$

From (2.2) and (2.4), we obtain

$$(\nabla_X \phi)Y - h(X, \phi Y) + \phi(h(X, Y)) = f(g(\phi X, Y)\xi - \eta(Y)\phi X). \tag{5.5}$$

Comparing tangential components, we get

$$h(X, \phi Y) = \phi(h(X, Y)). \tag{5.6}$$

Theorem 5.1. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\widetilde{M}$. Then h is recurrent with respect to the quarter symmetric metric connection if and only if it is totally geodesic.* *Proof.* If h is recurrent with respect to quarter symmetric metric connection, then from (2.11) we have

$$(\overline{\tilde{\nabla}}_X h)(Y, Z) = \pi(X)h(Y, Z).$$

Putting $Z = \xi$ and using (5.2) we have

$$h(Y, \bar{\nabla}_X \xi) = 0. \tag{5.7}$$

From (2.3), (5.2) and the above equation we obtain $fh(X, Y) = 0$. Consequently $h(X, Y) = 0$, for any $X, Y \in \chi(M)$. The converse is trivial.
This proves the theorem. $\square$

Theorem 5.2. *Let M be an invariant submanifold of a f-Kenmotsu manifold $\widetilde{M}$. Then M has parallel third fundamental form with respect to the quarter symmetric metric connection if and only if it is totally geodesic.*

Proof. Let M has parallel third fundamental form with respect to quarter symmetric metric connection. Then we have

$$(\overline{\tilde{\nabla}}_X \overline{\tilde{\nabla}}_Y h)(Z, W) = 0. \tag{5.8}$$

Substituting $W = Z = \xi$ and using the equations (2.1), (5.2) we have from above

$$2h(\bar{\nabla}_X\xi, \bar{\nabla}_Y\xi) = 0. \tag{5.9}$$

Now we use the result in (2.3) and we get $f^2h(X, Y) = 0$, thus we have $h(X, Y) = 0$, for any $X, Y \in \chi(M)$. Therefore, M is totally geodesic. The converse statement is trivially true.
This completes the proof. □

Theorem 5.3. *Let M be an invariant submanifold of an f-Kenmotsu manifold $\widetilde{M}$. Then h is 2-recurrent with respect to the quarter symmetric metric connection if and only if it is totally geodesic.*

Proof. Let h be 2-recurrent with respect to quarter symmetric metric connection. Then from (2.12) we have

$$(\overline{\tilde{\bar{\nabla}}}_X\overline{\tilde{\bar{\nabla}}}_Y h)(Z, W) = \psi(X, Y)h(Z, W). \tag{5.10}$$

Putting $Z = \xi$ and using the equation (5.2) we have

$$(\overline{\tilde{\bar{\nabla}}}_X\overline{\tilde{\bar{\nabla}}}_Y h)(\xi, W) = 0. \tag{5.11}$$

Then by previous theorem we can say M is totally geodesic. The converse is trivially true.
This finishes the proof. □

Theorem 5.4. *An invariant submanifold of an f-Kenmotsu manifold is totally geodesic if and only if $Q(S, \overline{\tilde{\bar{\nabla}}}_X h) = 0$, provided $f^2 \neq \xi f$.*

Proof. Let M be an invariant submanifold of an f-Kenmotsu manifold $\widetilde{M}$ satisfying $Q(S, \overline{\tilde{\bar{\nabla}}}_X h) = 0$. Then

$$Q(S, \overline{\tilde{\bar{\nabla}}}_X h)(W, K; U, V) = 0$$

for the vector fields $X, W, K, U, V \in \chi(M)$. By the above equation and (2.20), we have

$$\begin{aligned} 0 = \; & - (\overline{\tilde{\bar{\nabla}}}_X h)(S(V, W)U, K) + (\overline{\tilde{\bar{\nabla}}}_X h)(S(U, W)V, K) \\ & - (\overline{\tilde{\bar{\nabla}}}_X h)(W, S(V, K)U) + (\overline{\tilde{\bar{\nabla}}}_X h)(W, S(U, K)V). \end{aligned}$$

Hence,

$$\begin{aligned} 0 = \; & - \bar{\nabla}^{\perp}_X h(S(V, W)U, K) + h(\bar{\nabla}_X S(V, W)U, K) + h(S(V, W)U, \nabla_X K) \\ & + \bar{\nabla}^{\perp}_X h(S(U, W)V, K) - h(\bar{\nabla}_X S(U, W)V, K) - h(S(U, W)V, \bar{\nabla}_X K) \\ & - \bar{\nabla}^{\perp}_X h(W, S(V, K)U) + h(\bar{\nabla}_X W, S(V, K)U) + h(W, \bar{\nabla}_X S(V, K)U) \\ & + \bar{\nabla}^{\perp}_X h(W, S(U, K)V) - h(\bar{\nabla}_X W, S(U, K)V) - h(W, \bar{\nabla}_X S(U, K)V). \end{aligned}$$

Substituting $K = V = W = \xi$ in the above equation and using equation (5.2) we can obtain

$$S(\xi, \xi)h(U, \bar{\nabla}_X\xi) = 0. \tag{5.12}$$

Using the equations (2.3), (2.15) in the above equation, we have

$$(2n)(f^2 - \xi f) f h(U, \phi X) = 0. \tag{5.13}$$

With the help of (5.6) we obtain $h(U, X) = 0$, provided $f^2 \neq \xi f$, for any $U, X \in \chi(M)$. Hence the submanifold is totally geodesic. Converse is trivially true. This proves the theorem. □

Theorem 5.5. *Let M an invariant submanifold of an f-Kenmotsu manifold $\widetilde{M}$. Then M is totally geodesic if and only if the submanifold is semiparallel with respect to quarter symmetric connection, provided $f^2 \neq \xi f$.*

Proof. If the submanifold M is semiparallel then

$$\bar{\tilde{R}}(X, Y) h(U, V) = 0. \tag{5.14}$$

The above equation gives

$$R^N(X, Y) h(U, V) - h(\bar{R}(X, Y)U, V) - h(U, \bar{R}(X, Y)V) = 0. \tag{5.15}$$

Putting $U = X = \xi$ in the forgoing equation and using (5.2) we have

$$h(\bar{R}(\xi, Y)\xi, V) = 0. \tag{5.16}$$

Then using (3.13) we get

$$\{f^2 - \xi f\} h(Y, V) = 0. \tag{5.17}$$

With the help of (5.6) we obtain $h(Y, V) = 0$, provided $f^2 \neq \xi f$, for any $Y, V \in \chi(M)$. Hence the submanifold is totally geodesic. Converse is trivially true. This completes the proof. □

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ANTI-INVARIANT RIEMANNIAN SUBMERSIONS FROM LOCALLY CONFORMAL KAEHLER MANIFOLDS *

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Abstract. Recently, Sahin [10] studied the anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. In present work, these notions of anti-invariant and Lagrangian Riemannian submersions have been extended to locally conformal Kaehler manifolds. Certain decomposition results and the geometry of foliation have also been investigated.
Keywords: anti-invariant Riemannian submersions; almost Hermitian manifolds; Riemannian manifolds; Kaehler manifolds.

1. Introduction

Locally conformal Kaehler manifolds (shortly, l.c.K. manifolds) have been rich source of attraction for many years. Many geometers considered these manifolds and their submanifolds in different settings (for details see, [3] and [13]). On the other side, for any Riemannian manifold $\mathcal{M}$ and Riemannian manifold $\mathcal{B}$, the Riemannian submersion π from $\mathcal{M}$ onto $\mathcal{B}$ was studied for very first time by B. O'Neil [6]. Gray [4], Ianus [5], Park ([7], [8]), Sahin ([11], [12]), Choudhary [2] etc. have also taken into consideration the geometry of Riemannian submersions for different structures on differentiable manifolds. Recently, anti-invariant Riemannian submersions have been taken into study from almost Hermitian manifolds onto Riemannian manifolds by B. Sahin [10].

In present work, these notions of anti-invariant and Lagrangian Riemannian submersions have been extended to locally conformal Kaehler manifolds. Certain decomposition results and the geometry of foliation have also been investigated.

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2. Preliminaries

This section is preliminary in nature wherein we collect definitions and formulas that are to be used. We start with l.c.K. manifold.

Definition 2.1. [3] For Hermitian manifold $(\tilde{\mathcal{M}}, g)$ of dimension-$2m$ and Kaehler 2-form Ω holding for the relation

$$\Omega(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, J\mathcal{Y}),$$

for all $\mathcal{X}, \mathcal{Y} \in \chi(\tilde{\mathcal{M}})$ and closed 1-form ω defined globally on manifold $\tilde{\mathcal{M}}$ such that

$$d\Omega = \omega \wedge \Omega,$$

the manifold $\tilde{\mathcal{M}}$ is known as locally conformal Kaehler manifold.

Here, ω is sign of the Lee form of $\tilde{\mathcal{M}}$. We have the following cases for ω:

- when ω is exact, $\tilde{\mathcal{M}}$ is globally conformal Kahler (g.c.K.) manifold,
- when $\omega = 0$, $\tilde{\mathcal{M}}$ is Kaehler manifold.

One can observe that any l.c.K. manifold becomes g.c.K. manifold provided it is simply connected. Let us use $\sharp$ to represent the rising of the indices in association with the metric g, then for any l.c.K. manifold $\tilde{\mathcal{M}}$, $B_1 = \omega^\sharp$ indicates the Lee vector field and satisfies

$$g(\mathcal{X}, B_1) = \omega(\mathcal{X}); \forall \mathcal{X} \in \chi(\tilde{\mathcal{M}}).$$

[3] When we use $\theta = \omega o J$ for anti-Lee form and $A = -JB_1$ for anti-Lee vector field, respectively. Then

$$(\tilde{\nabla}_{\mathcal{X}} J)\mathcal{Y} = \frac{1}{2}\{\theta(\mathcal{Y})\mathcal{X} - \omega(\mathcal{Y})J\mathcal{X} - g(\mathcal{X}, \mathcal{Y})A - \Omega(\mathcal{X}, \mathcal{Y})B_1\}, \tag{2.1}$$

$\forall \mathcal{X}, \mathcal{Y} \in \chi(\tilde{\mathcal{M}})$, where, $\tilde{\nabla}$ is used for the Levi Civita connection of $(\tilde{\mathcal{M}}, g)$.

Any map π of m-dimensional Riemannian manifold $(\mathcal{M}^m, g)$ onto a b'-dimensional Riemannian manifold $(\mathcal{B}^{b'}, g_{\mathcal{B}})$ with $m > b'$ stands for a Riemannian submersion if π has maximal rank and the lengths of horizontal vectors are preserved by differential π_*.

It is known that $\pi^{-1}(q'), q' \in \mathcal{B}$ is an $(m - b')$ dimensional submanifold of Riemannian manifold $\mathcal{M}$ and named as fibers. A vector field on $\mathcal{M}$ is said to be

- vertical provided it is always tangent to $\pi^{-1}(q')$;
- horizontal provided it is always orthogonal to $\pi^{-1}(q')$.

Next, we have

Definition 2.2. [10] Let $\mathcal{X}$ represents a vector field on a Riemannian manifold $\mathcal{M}$, then $\mathcal{X}$ is known as basic if

- it is horizontal
- it is π-related to a vector field $\mathcal{X}_*$ on $\mathcal{B}$, that is, $\pi_*\mathcal{X}_{p_1} = \mathcal{X}_{*\pi(p_1)}, \forall p_1 \in \mathcal{M}$.

Let us use $\mathcal{V}$ and $\mathcal{H}$ to denote the projection morphisms on $ker\pi_*$ and $(ker\pi_*)^\perp$, respectively. Then

Lemma 2.1. *[6] When $\pi : \mathcal{M} \to \mathcal{B}$ represents a Riemannian submersion from a Riemannian manifold $\mathcal{M}$ onto a Riemannian manifold $\mathcal{B}$. Then*

(a) $g(\mathcal{X}, \mathcal{Y}) = g_{\mathcal{B}}(\mathcal{X}_*, \mathcal{Y}_*) o\pi$,

(b) $\mathcal{H}[\mathcal{X}, \mathcal{Y}]$ *of* $[\mathcal{X}, \mathcal{Y}]$ *is basic vector field corresponding to* $[\mathcal{X}_*, \mathcal{Y}_*]$*, i.e.,* $([\mathcal{X}, \mathcal{Y}]^{\mathcal{H}}) = (\mathcal{X}_*, \mathcal{Y}_*)$,

(c) *when V is vertical vector, $[V, \mathcal{X}]$ is also vertical,*

(d) *when ∇^* be the Levi-Civita connection on $\mathcal{B}$, $\mathcal{H}(\nabla_{\mathcal{X}}\mathcal{Y})$ will be the basic vector field that corresponds to $\nabla^*_{\mathcal{X}_*}\mathcal{Y}_*$.*

Here, $\mathcal{X}, \mathcal{Y}$ are considered as basic vector fields on $\mathcal{M}$.

[6] Let us denote by the symbols $\mathcal{T}$ and $\mathcal{A}$, O'Neills tensors for vector fields E, F on $\mathcal{M}$ and by ∇ the Levi-Civita connection of g such that the following hold

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{2.2}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F. \tag{2.3}$$

The necessary and sufficient condition for Riemannian submersion $\pi : \mathcal{M} \to \mathcal{B}$ to be totally geodesic fibres is that $\mathcal{T}$ vanishes identically. Now, let us suppose that $\Gamma(T\mathcal{M})$ denotes the set of all sections on the tangent bundle $T\mathcal{M}$, then for any $E \in \Gamma(T\mathcal{M}), \mathcal{T}_E$ and $\mathcal{A}_E$ represent skew-symmetric operators on $(\Gamma(T\mathcal{M}), g)$ reversing the horizontal and vertical distributions. One can observe that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$ and hold for the following ([6], [10])

$$\mathcal{T}_{\mathcal{U}}\mathcal{W} = \mathcal{T}_{\mathcal{W}}\mathcal{U}, \forall \mathcal{U}, \mathcal{W} \in \Gamma(ker\pi_*) \tag{2.4}$$

$$\mathcal{A}_{\mathcal{X}}\mathcal{Y} = -\mathcal{A}_{\mathcal{Y}}\mathcal{X} = \frac{1}{2}\mathcal{V}[\mathcal{X}, \mathcal{Y}], \forall \mathcal{X}, \mathcal{Y} \in (\Gamma(ker\pi_*)^\perp). \tag{2.5}$$

Now we state the following lemma [10]

Lemma 2.2. *When* $\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp)$ *and* $\mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$*, we have the following relations:*

(a) $\nabla_{\mathcal{W}}\mathcal{W}' = \mathcal{T}_{\mathcal{W}}\mathcal{W}' + \hat{\nabla}_{\mathcal{W}}\mathcal{W}'$

(b) $\nabla_{\mathcal{W}}\mathcal{X} = \mathcal{H}\nabla_{\mathcal{W}}\mathcal{X} + \mathcal{T}_{\mathcal{W}}\mathcal{X}$

(c) $\nabla_{\mathcal{X}}\mathcal{W} = \mathcal{A}_{\mathcal{X}}\mathcal{W} + \mathcal{V}\nabla_{\mathcal{X}}\mathcal{W}$

(d) $\nabla_{\mathcal{X}}\mathcal{Y} = \mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} + \mathcal{A}_{\mathcal{X}}\mathcal{Y}$

where $\hat{\nabla}_{\mathcal{W}}\mathcal{W}' = \mathcal{V}\nabla_{\mathcal{W}}\mathcal{W}'$. *Moreover,* $\mathcal{H}\nabla_{\mathcal{W}}\mathcal{X} = \mathcal{A}_{\mathcal{X}}\mathcal{W}$*, when* $\mathcal{X}$ *is basic.*

3. Anti-invariant and Lagrangian Riemannian submersions

This section deals with the anti-invariant and Lagrangian Riemannian submersion. Certain conditions to show these submersions to be totally geodesic maps are also discussed. A diffeomorphism f of Riemannian manifold $(\mathcal{M}, g)$ onto another Riemannian manifold $(\mathcal{B}, g')$ is said be geodesic map if image of any geodesic arc in $\mathcal{M}$ under f is a geodesic arc in $\mathcal{B}$ and image of any geodesic arc in $\mathcal{B}$ under f^{-1} is a geodesic arc in $\mathcal{M}$. A map is said to be totally geodesic if its hessian vanishes.

Now, recall anti-invariant Riemannian submersion by the following way.

Definition 3.1. [10] Let $(\mathcal{M}, g_{\mathcal{M}}, J)$ represents a complex almost Hermitian manifold of dimension m and $(\mathcal{B}, g_{\mathcal{B}})$ be a Riemannian manifold. Then, any Riemannian submersion $\pi : \mathcal{M} \to \mathcal{B}$ is said to be anti-invariant Riemannian submersion if $J(ker\pi_*) \subseteq (ker\pi_*)^\perp$.

For an anti-invariant Riemannian submersion π from an almost Hermitian manifold $(\mathcal{M}, g_{\mathcal{M}}, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, above definition implies $J(ker\pi_*)^\perp \cap (ker\pi_*) \neq 0$, and that produces

$$(ker\pi_*)^\perp = J(ker\pi_*) \oplus \mu, \tag{3.1}$$

here μ is used for the orthogonal complementary distribution to $J(ker\pi_*)$ in $(ker\pi_*)^\perp$. So,

$$J\mathcal{X} = B\mathcal{X} + C\mathcal{X}, \quad \mathcal{X} \in \Gamma((ker\pi_*)^\perp), B\mathcal{X} \in \Gamma(ker\pi_*), C\mathcal{X} \in \Gamma(\mu). \tag{3.2}$$

For Riemannian submersion π, (3.2) and $\pi_*((ker\pi_*)^\perp) = T\mathcal{B}$ indicate

$$g_{\mathcal{B}}(\pi_* JV, \pi_* C\mathcal{X}) = 0, \ \ \forall \mathcal{X} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*),$$

implying

$$T\mathcal{B} = \pi_*(J(ker\pi_*)) \oplus \pi_*(\mu). \tag{3.3}$$

[1] Let $\phi^{'} : \mathcal{M} \to \mathcal{B}$ be smooth map from Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}})$ onto $(\mathcal{B}, g_{\mathcal{B}})$. Then, any section of the bundle $\mathrm{Hom}(T\mathcal{M}, \phi^{'-1}(T\mathcal{B})) \to \mathcal{M}$ can be thought by differential $\phi^{'}_{*}$, $\phi^{'-1}(T\mathcal{B})$ being the pullback bundle having fibres $(\phi^{'-1}(TB))_p = T_{\phi^{'}(p)}B, p \in \mathcal{M}$. Thanks to pullback connection and the Levi-Civita connection $\nabla^{\mathcal{M}}$, one can induce a connection ∇ for $Hom(T\mathcal{M}, \phi^{'-1}(T\mathcal{B}))$. Hence, define the second fundamental form of $\phi^{'}$ by

$$(\nabla\phi^{'}_{*})(\mathcal{X}, \mathcal{Y}) = \nabla^{\phi^{'}}_{\mathcal{X}}\phi^{'}_{*}(\mathcal{Y}) - \phi^{'}_{*}(\nabla^{\mathcal{M}}_{\mathcal{X}}\mathcal{Y}), \forall \mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M}), \tag{3.4}$$

here, $\Gamma(T\mathcal{M})$ represents set of all sections on the tangent bundle $T\mathcal{M}$ and $\nabla^{\phi^{'}}$ is the pullback connection.

Next, we give the following result.

Lemma 3.1. *When $\pi : \mathcal{M} \to \mathcal{B}$ represents anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ to a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, and ω be closed 1-form defined globally on $\mathcal{M}$, then for all $\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*)$, we have*

(i) $g(C\mathcal{Y}, J\mathcal{W}) = 0$

(ii) $g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})$

(iii) $g(\nabla_{\mathcal{W}}B\mathcal{Y}, C\mathcal{X}) = g(C\mathcal{X}, \mathcal{T}_{\mathcal{W}}B\mathcal{Y}) = -g(B\mathcal{Y}, \mathcal{T}_{\mathcal{W}}C\mathcal{X})$.

Proof (i) Let $\mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$ and $\mathcal{W} \in \Gamma(ker\pi_*)$, then in the light of (3.2), we get

$$\begin{aligned} g(C\mathcal{Y}, J\mathcal{W}) &= g(J\mathcal{Y} - B\mathcal{Y}, J\mathcal{W}) \\ &= g(J\mathcal{Y}, J\mathcal{W}) \end{aligned}$$

where the fact $B\mathcal{Y} \in \Gamma(ker\pi_*)$ and $J\mathcal{W} \in \Gamma((ker\pi_*)^{\perp})$ was used. Moreover, $g(J\mathcal{Y}, J\mathcal{W}) = g(\mathcal{Y}, \mathcal{W}) = 0$ and this completes the proof.

(ii) Let us assume $B_1 \in \Gamma(ker\pi_*)$, then taking view of (2.1) and part (i), we get

$$\begin{aligned} g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) &= -g(C\mathcal{Y}, \nabla_{\mathcal{X}}J\mathcal{W}) \\ &= -g(C\mathcal{Y}, J\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, J\mathcal{X}) \end{aligned}$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*)$. Thanks to (3.2), we arrive

$$\begin{aligned} g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) &= -g(C\mathcal{Y}, J\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, B\mathcal{X} + C\mathcal{X}) \\ &= -g(C\mathcal{Y}, J\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) \end{aligned}$$

because $C\mathcal{Y} \in \Gamma(\mu)$ and $B\mathcal{X} \in \Gamma(ker\pi_*)$. Taking use of Lemma 2.2 produces

$$g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) - (C\mathcal{Y}, J\mathcal{V}\nabla_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})$$

that simplifies to

$$g(\nabla_{\mathcal{X}} C\mathcal{Y}, J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}),$$

here, we used $J\mathcal{V}\nabla_{\mathcal{X}}\mathcal{W} \in \Gamma(Jker\pi_*)$. □

From here, we assume that $B_1 \in (ker\pi_*)$. We also assume horizontal vector fields to be basic whenever needed in the proofs. Now, let us move to study the integrability results of the horizontal distribution $(ker\pi_*)^{\perp}$. Also, note that $ker\pi_*$ is integrable.

Theorem 3.1. *When $\pi : \mathcal{M} \to \mathcal{B}$ represents anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) $(ker\pi_*)^{\perp}$ *is integrable*

(b) $$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{Y}, B\mathcal{X}), \pi_*J\mathcal{W}) = g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, B\mathcal{Y}), \pi_*J\mathcal{W}) + g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) - g(C\mathcal{X}, J\mathcal{A}_{\mathcal{Y}}\mathcal{W}) - \tfrac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \tfrac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W})$$

(c) $$g(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + g(C\mathcal{X}, J\mathcal{A}_{\mathcal{Y}}\mathcal{W}) + \tfrac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \tfrac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W})$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp})$,$\mathcal{W} \in \Gamma(ker\pi_*)$.

Proof. Taking account of definition 3.1, we see $J\mathcal{Y} \in \Gamma(ker\pi_* \oplus \mu)$ and $J\mathcal{W} \in \Gamma((ker\pi_*)^{\perp})$ and hence with the help of (2.1) for $\mathcal{X} \in \Gamma((ker\pi_*)^{\perp})$, we reach at

$$\begin{aligned} g([\mathcal{X}, \mathcal{Y}], \mathcal{W}) &= g(J[\mathcal{X}, \mathcal{Y}], J\mathcal{W}) \\ &= g(J\nabla_{\mathcal{X}}\mathcal{Y}, J\mathcal{W}) - g(J\nabla_{\mathcal{Y}}\mathcal{X}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) \\ &\quad -g(\nabla_{\mathcal{Y}}J\mathcal{X}, J\mathcal{W}) + \frac{1}{2}\theta(\mathcal{X})g(\mathcal{Y}, J\mathcal{W}), \end{aligned}$$

$\forall \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*)$. Here $\theta = \omega o J$, $\Omega(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, J\mathcal{Y})$ and $g(\mathcal{X}, B_1) = \omega(\mathcal{X})$, then $B_1 \in \Gamma(ker\pi_*)$ and (3.2) produce

$$\begin{aligned} g([\mathcal{X}, \mathcal{Y}], \mathcal{W}) &= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - g(\nabla_{\mathcal{Y}}J\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) \\ &\quad +\frac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) + g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) - g(\nabla_{\mathcal{Y}}B\mathcal{X}, J\mathcal{W}) \\ &\quad -g(\nabla_{\mathcal{Y}}C\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W}). \end{aligned}$$

Because π represents a Riemannian submersion, we conclude

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g(\pi_*\nabla_{\mathcal{X}}B\mathcal{Y},\pi_*J\mathcal{W}) + g(\nabla_{\mathcal{X}}C\mathcal{Y},J\mathcal{W}) - g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{Y}}B\mathcal{X},\pi_*J\mathcal{W}) \\ &\quad -g(\nabla_{\mathcal{Y}}C\mathcal{X},J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}). \end{aligned}$$

Taking into account Lemma 3.1, we arrive at

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g_{\mathcal{B}}(-(\nabla\pi_*)(\mathcal{X},B\mathcal{Y}) + (\nabla\pi_*)(\mathcal{Y},B\mathcal{X}),\pi_*J\mathcal{W}) \\ &\quad -g(C\mathcal{Y},J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + g(C\mathcal{X},J\mathcal{A}_{\mathcal{Y}}\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y},B_1)g(\mathcal{X},J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X},B_1)g(\mathcal{Y},J\mathcal{W}) \end{aligned}$$

proving (a)$\Leftrightarrow$(b).

Next, taking into consideration Lemma 2.2, we derive

$$\begin{aligned} (\nabla\pi_*)(\mathcal{X},B\mathcal{Y}) &- (\nabla\pi_*)(\mathcal{Y},B\mathcal{X}) \\ &= -\pi_*(\nabla_{\mathcal{X}}B\mathcal{Y}) + \pi_*(\nabla_{\mathcal{Y}}B\mathcal{X}) \\ &= -\pi_*(\nabla_{\mathcal{X}}B\mathcal{Y} - \nabla_{\mathcal{Y}}B\mathcal{X}) \\ &= \pi_*(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y}), \end{aligned}$$

$\forall\mathcal{X},\mathcal{Y}\in\Gamma((ker\pi_*)^{\perp}),\mathcal{W}\in\Gamma(ker\pi_*)$. Simplification reduces to

$$\begin{aligned} g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X},B\mathcal{Y}) &- (\nabla\pi_*)(\mathcal{Y},B\mathcal{X}),\pi_*J\mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y}),\pi_*J\mathcal{W}) \\ &= g(\mathcal{A}_{\mathcal{Y}}B\mathcal{X} - \mathcal{A}_{\mathcal{X}}B\mathcal{Y},J\mathcal{W}), \end{aligned}$$

moreover, $\mathcal{A}_{\mathcal{X}}B\mathcal{Y} - \mathcal{A}_{\mathcal{Y}}B\mathcal{X}\in\Gamma((ker\pi_*)^{\perp})$, it establishes (b)$\Leftrightarrow$(c). □

Definition 3.2. [10] Let π represents an anti-invariant Riemannian submersion such that $J(ker\pi_*) = (ker\pi_*)^{\perp}$. Then, π is known as Lagrangian Riemannian submersion. Moreover, when $\mu\neq\{0\}$, π is called as proper anti-invariant Riemannian submersion.

Thanks to Theorem 3.1, we write the following.

Corollary 3.1. *When $\pi:\mathcal{M}\to\mathcal{B}$ represents a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M},g,J)$ onto a Riemannian manifold $(\mathcal{B},g_{\mathcal{B}})$, then the following are equivalent:*

(a) $(ker\pi_*)^{\perp}$ *is integrable*

(b) $(\nabla\pi_*)(\mathcal{X},J\mathcal{Y}) = (\nabla\pi_*)(\mathcal{Y},J\mathcal{X}) - \frac{1}{2}g(B\mathcal{Y},B_1)\mathcal{X} + \frac{1}{2}g(B\mathcal{X},B_1)\mathcal{Y}$

(c) $\pi_*(\mathcal{A}_{\mathcal{X}}J\mathcal{Y} - \mathcal{A}_{\mathcal{Y}}J\mathcal{X}) = \frac{1}{2}g(B\mathcal{Y},B_1)\mathcal{X} - \frac{1}{2}g(B\mathcal{X},B_1)\mathcal{Y}$, $\forall\mathcal{X},\mathcal{Y}\in\Gamma((ker\pi_*)^{\perp})$.

Proof. Let us assume that $\mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp)$ and $\mathcal{W} \in \Gamma(ker\pi_*)$. Then, $J\mathcal{X} \in \Gamma(ker\pi_*)$ and $J\mathcal{W} \in \Gamma((ker\pi_*)^\perp)$. Hence, taking into light (2.1), we derive

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g(J[\mathcal{X},\mathcal{Y}], J\mathcal{W}) \\ &= g(J\nabla_{\mathcal{X}}\mathcal{Y}, J\mathcal{W}) - g(J\nabla_{\mathcal{Y}}\mathcal{X}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - g(\nabla_{\mathcal{Y}}J\mathcal{X}, J\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W}). \end{aligned}$$

Use of (3.4) produces

$$\begin{aligned} g([\mathcal{X},\mathcal{Y}],\mathcal{W}) &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) - g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{Y}}J\mathcal{X}, \pi_*J\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W}) \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) + g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{Y}, J\mathcal{X}), \pi_*J\mathcal{W}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W}) \end{aligned}$$

thus, $(ker\pi_*)^\perp$ is integrable iff

$$\begin{aligned} g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) &= g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{Y}, J\mathcal{X}), \pi_*J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) \\ &\quad +\frac{1}{2}g(B\mathcal{X}, B_1)g(\mathcal{Y}, J\mathcal{W}) \end{aligned}$$

establishing (a)⇔(b).

Next, with the help of (3.4) we get

$$\begin{aligned} (\nabla\pi_*)(\mathcal{Y}, J\mathcal{X}) \quad - \quad & (\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}) \\ &= -\pi_*(\nabla_{\mathcal{Y}}J\mathcal{X}) + \pi_*(\nabla_{\mathcal{X}}J\mathcal{Y}) \\ &= \pi_*(\mathcal{H}(\nabla_{\mathcal{X}}J\mathcal{Y}) - \mathcal{H}(\nabla_{\mathcal{Y}}J\mathcal{X})) \\ &= \pi_*(\mathcal{A}_{\mathcal{X}}J\mathcal{Y} - \mathcal{A}_{\mathcal{Y}}J\mathcal{X}), \end{aligned}$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp)$. This concludes (b)⇔(c). □

4. Geometry of leaves

The geometry of leaves of $(ker\pi_*)$ and $(ker\pi_*)^\perp$ of anti-invariant and Lagrangian Riemannian submersions are studies here. We have

Theorem 4.1. *When $\pi : \mathcal{M} \to \mathcal{B}$ represents an anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation on $\mathcal{M}$ is defined by $(ker\pi_*)^\perp$*

(b) $g(\mathcal{A}_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) = g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) - \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) + \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

(c) $g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) = -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$.

Proof. Taking into account (2.1), (3.2), Lemma 2.2 and Lemma 3.1, we write the following

$$\begin{aligned}
g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{W}) &= g(J\nabla_{\mathcal{X}}\mathcal{Y}, J\mathcal{W})\\
&= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})\\
&= g(\nabla_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) + g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W})\\
&\quad -\frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})\\
&= g(\mathcal{A}_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) - g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})\\
&\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})
\end{aligned}$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$. In this way, a totally geodesic foliation on $\mathcal{M}$ is defined by $(ker\pi_*)^\perp$ iff

$$\begin{aligned}
g(\mathcal{A}_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) &= g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) - \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X})\\
&\quad +\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})
\end{aligned}$$

concluding (a)⇔(b). Next, with the help of (3.4), we derive

$$\begin{aligned}
g(\mathcal{A}_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W}) &= g(\nabla_{\mathcal{X}}B\mathcal{Y}, J\mathcal{W})\\
&= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W})\\
&= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) - g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W})\\
&= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) + g_{\mathcal{B}}(\nabla^{\pi}_{\mathcal{X}}\pi_*(J\mathcal{Y}), \pi_*J\mathcal{W})\\
&\quad -g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W})\\
&= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) + g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W}) - g(\nabla_{\mathcal{X}}C\mathcal{Y}, J\mathcal{W})\\
&= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W})
\end{aligned}$$

proving (b)⇔(c). □

For Lagrangian Riemannian submersion, we have the following corollary.

Corollary 4.1. *When π denotes a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation is defined by* $(ker\pi_*)^\perp$ *on manifold* $\mathcal{M}$

(b) $g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) = \frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

(c) $g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) = -\frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$.

Proof. Thanks to (2.1), we write

$$\begin{aligned} g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{W}) &= g(J\nabla_{\mathcal{X}}\mathcal{Y}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}), \pi_*J\mathcal{W}) - \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}), \end{aligned}$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W} \in \Gamma(ker\pi_*)$. This way, a totally geodesic foliation is defined by $(ker\pi_*)^\perp$ on the manifold $\mathcal{M}$ iff

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) = \frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}).$$

Therefore, (a)⇔(b). Next, taking help of (3.4) it follows

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) &= g_{\mathcal{B}}(\nabla_{\mathcal{X}}J\mathcal{Y}, J\mathcal{W}) \\ &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{X}}J\mathcal{Y}, \pi_*J\mathcal{W}) \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) \end{aligned}$$

establishing

$$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_*J\mathcal{W}) = -\frac{1}{2}\theta(\mathcal{Y})g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$$

and that proves (b)⇔(c). □

Now, taking into consideration (3.4) to get

$$(\nabla\pi_*)(\mathcal{W}, \mathcal{X}) = \nabla^\pi_{\mathcal{W}}\pi_*\mathcal{X} - \pi_*\nabla_{\mathcal{W}}\mathcal{X}, \quad \mathcal{X} \in \Gamma(\mu), \mathcal{W} \in \Gamma(ker\pi_*).$$

Also,

$$(\nabla\pi_*)(\mathcal{X}, \mathcal{W}) = \nabla^\pi_{\mathcal{X}}\pi_*\mathcal{W} - \pi_*\nabla_{\mathcal{X}}\mathcal{W}.$$

We use above two equations and symmetric property of second fundamental form to get

$$\nabla^\pi_{\mathcal{W}}\pi_*\mathcal{X} = 0. \tag{4.1}$$

Next, we state the following Theorem.

Theorem 4.2. *When π denotes an anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation on $\mathcal{M}$ is defined by $(ker\pi_*)$*

(b) $\mathcal{T}_{\mathcal{W}} B\mathcal{X} + \mathcal{A}_C \mathcal{X}\mathcal{W} = 0$ *or* $\mathcal{T}_{\mathcal{W}} B\mathcal{X} + \mathcal{A}_C \mathcal{X}\mathcal{W} \in \Gamma(\mu)$

(c) $g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, J\mathcal{X}), \pi_* J\mathcal{W}') = 0$, $\forall \mathcal{X} \in \Gamma((ker\pi_*)^\perp), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Proof. Taking into use (2.1), We obtain

$$\begin{aligned} g(\nabla_{\mathcal{W}}\mathcal{W}', \mathcal{X}) &= g(J\nabla_{\mathcal{W}}\mathcal{W}', J\mathcal{X}) \\ &= g(\nabla_{\mathcal{W}} J\mathcal{W}', J\mathcal{X}) \\ &= -g(J\mathcal{W}', \nabla_{\mathcal{W}} J\mathcal{X}), \quad \mathcal{X} \in \Gamma((ker\pi_*)^\perp), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*), \end{aligned}$$

where orthogonality between $(ker\pi_*)$ and $(ker\pi_*)^\perp$ has been used. Taking help of (3.2) and Lemma 2.2, above equation reduces to

$$\begin{aligned} g(\nabla_{\mathcal{W}}\mathcal{W}', \mathcal{X}) &= -g(J\mathcal{W}', \nabla_{\mathcal{W}} B\mathcal{X}) - g(J\mathcal{W}', \nabla_{\mathcal{W}} C\mathcal{X}) \\ &= -g(J\mathcal{W}', \mathcal{T}_{\mathcal{W}} B\mathcal{X}) - g(J\mathcal{W}', \mathcal{A}_{C\mathcal{X}}\mathcal{W}) \\ &= -g(J\mathcal{W}', \mathcal{T}_{\mathcal{W}} B\mathcal{X} + \mathcal{A}_{C\mathcal{X}}\mathcal{W}) \end{aligned}$$

implying (a)$\Leftrightarrow$(b). Furthermore, (3.4) produces

$$\begin{aligned} g(\mathcal{T}_{\mathcal{W}} B\mathcal{X}, J\mathcal{W}') &+ g(\mathcal{A}_{C\mathcal{X}}\mathcal{W}, J\mathcal{W}') \\ &= g(\mathcal{H}(\nabla_{\mathcal{W}} B\mathcal{X}), J\mathcal{W}') + g(\mathcal{H}(\nabla_{\mathcal{W}} C\mathcal{X}), J\mathcal{W}') \\ &= g(\nabla_{\mathcal{W}} B\mathcal{X}, J\mathcal{W}') + g(\nabla_{\mathcal{W}} C\mathcal{X}, J\mathcal{W}') \\ &= g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{W}} B\mathcal{X}, \pi_* J\mathcal{W}') + g_{\mathcal{B}}(\pi_*\nabla_{\mathcal{W}} C\mathcal{X}, \pi_* J\mathcal{W}') \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, B\mathcal{X}), \pi_* J\mathcal{W}') - g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, C\mathcal{X}), \pi_* J\mathcal{W}') \\ &+ g_{\mathcal{B}}(\nabla^\pi_{\mathcal{W}}\pi_* C\mathcal{X}, \pi_* J\mathcal{W}'). \end{aligned}$$

Taking into consideration (4.1), we get

$$\begin{aligned} g(\mathcal{T}_{\mathcal{W}} B\mathcal{X}, J\mathcal{W}') &+ g(\mathcal{A}_{C\mathcal{X}}\mathcal{W}, J\mathcal{W}') \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, B\mathcal{X}), \pi_* J\mathcal{W}') - g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, C\mathcal{X}), \pi_* J\mathcal{W}') \\ &= -g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, J\mathcal{X}), \pi_* J\mathcal{W}') \end{aligned}$$

concluding (b)$\Leftrightarrow$(c). $\square$

Now, for a Lagrangian Riemannian submersion π, (3.3) interprets $T\mathcal{B} = \pi_*(J(ker\pi_*))$.

Corollary 4.2. *When π represents a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$, then the following are equivalent:*

(a) *totally geodesic foliation on $\mathcal{M}$ is defined by* $(ker\pi_*)$

(b) $\mathcal{T}_{\mathcal{W}} J\mathcal{W}' = 0$

(c) $(\nabla\pi_*)(\mathcal{W}, J\mathcal{X}) = 0$

for $\mathcal{X} \in \Gamma((ker\pi_*)^{\perp})$ *and* $\mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Proof. In the light of Theorem 4.2, $(a) \Leftrightarrow (b)$ is obvious. For the proof of $(b) \Leftrightarrow (c)$, consider that $(ker\pi_*)$ and $(ker\pi_*)^{\perp}$ are orthogonal, then we write

$$\begin{aligned} g(\nabla_V J\mathcal{W}, J\mathcal{X}) &= -g(J\mathcal{W}, \nabla_V J\mathcal{X}) \\ &= -g_{\mathcal{B}}(\pi_* J\mathcal{W}, \pi_* \nabla_V J\mathcal{X}) \\ &= g_{\mathcal{B}}(\pi_* J\mathcal{W}, (\nabla\pi_*)(V, J\mathcal{X})) \\ g(\mathcal{T}_V J\mathcal{W}, J\mathcal{X}) &= g_{\mathcal{B}}(\pi_* J\mathcal{W}, (\nabla\pi_*)(V, J\mathcal{X})), \end{aligned}$$

here, we have taken help of (3.4) and Lemma 2.2. Further, $\mathcal{T}_V J\mathcal{W} \in \Gamma(ker\pi_*)$ that provides the required result $(b) \Leftrightarrow (c)$. □

Definition 4.1. [1] For a differential map π from a Riemannian manifold $\mathcal{M}$ onto a Riemannian manifold $\mathcal{B}$, if $\nabla\pi_* = 0$ holds, then π is said to be is called totally geodesic.

Next, we have

Theorem 4.3. *When π is used to denote a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then π represents a totally geodesic map iff*

$$\mathcal{T}_{\mathcal{W}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{W} + \frac{1}{2}g(\mathcal{W}, \mathcal{W}')A = 0$$

and

$$\mathcal{A}_{\mathcal{X}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{X} + \frac{1}{2}\Omega(\mathcal{X}, \mathcal{W}')B_1 = 0,$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Proof. The following holds for a Riemannian submersion π

$$(\nabla\pi_*)(\mathcal{X}, \mathcal{Y}) = 0 \qquad \forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}) \tag{4.2}$$

In the light of (2.1), (3.4) and (4.1), we derive

$$\begin{aligned} (\nabla\pi_*)(\mathcal{W}, \mathcal{W}') &= \nabla^{\pi}_{\mathcal{W}} \pi_*(\mathcal{W}') - \pi_*(\nabla_{\mathcal{W}} \mathcal{W}') \\ &= -\pi_*(\nabla_{\mathcal{W}} \mathcal{W}') \\ &= \pi_*(J(J\nabla_{\mathcal{W}} \mathcal{W}')) \\ &= \pi_*(J(\nabla_{\mathcal{W}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{W} + \frac{1}{2}g(\mathcal{W}, \mathcal{W}')A)) \\ &= \pi_*(J(\mathcal{T}_{\mathcal{W}} J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{W} + \frac{1}{2}g(\mathcal{W}, \mathcal{W}')A)), \end{aligned} \tag{4.3}$$

$\forall \mathcal{W}, \mathcal{W}' \in (ker\pi_*)$.

Further, use of (3.4) produces

$$\begin{aligned}(\nabla\pi_*)(\mathcal{X},\mathcal{W}') &= \nabla^{\pi}_{\mathcal{X}}\pi_*(\mathcal{W}') - \pi_*(\nabla_{\mathcal{X}}\mathcal{W}') \\ &= -\pi_*(\nabla_{\mathcal{X}}\mathcal{W}') \\ &= \pi_*(J(J\nabla_{\mathcal{X}}\mathcal{W}')) \\ &= \pi_*(J(\nabla_{\mathcal{X}}J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{X} + \frac{1}{2}\Omega(\mathcal{X},\mathcal{W}')B_1)) \\ &= \pi_*(J(\mathcal{A}_{\mathcal{X}}J\mathcal{W}' + \frac{1}{2}\omega(\mathcal{W}')J\mathcal{X} + \frac{1}{2}\Omega(\mathcal{X},\mathcal{W}')B_1)) \qquad (4.4)\end{aligned}$$

$\forall \mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}' \in (ker\pi_*)$. Hence, the result holds in view of (4.2),(4.3) and (4.4) and singularity of J. □

5. Decomposition theorems

[14] Let us use $\mathcal{M}$ to represent a manifold whose dimension is m and by (χ^t) a system of coordinate neighborhoods used to cover $\mathcal{M}$ in such a way that if (χ^t) and (χ^{t_1}) be any two coordinate neighborhoods, then in their intersection we obtain

$$\chi^{a_1} = \chi^{a_1}(\chi^a), \chi^{x_1} = \chi^{x_1}(\chi^x),$$

with

$$|\delta_a\chi^{a_1}| \neq 0, |\delta_x\chi^{x_1}| \neq 0,$$

here all the indices $a, b, ...$ run over $1, 2, ..., p$ and $x, y, z, ...$ over $p+1, ..., p+q = m$. This type of system of coordinate neighborhoods is known as separating coordinate system and if such a system of coordinate neighborhoods exists then it defines a locally product structure on the manifold $\mathcal{M}$. A manifold $\mathcal{M}$ equipped with a locally product structure is known as locally product manifold.

Next, we define

Definition 5.1. [9] When $N = \mathcal{M} \times \mathcal{B}$ is a manifold with Riemannian metric tensor g and $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ be the canonical foliations intersecting perpendicularly everywhere. Then

(i) the necessary and sufficient condition for g to represent the metric tensor of a warped product $\mathcal{M} \times_{f'} \mathcal{B}$ is that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ denote the totally geodesic and spherical foliations, respectively.

(ii) the necessary and sufficient condition for g to be metric tensor of a twisted product $\mathcal{M} \times_{f'} \mathcal{B}$ is that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ represent the totally geodesic and totally umbilical foliations, respectively

(iii) the necessary and sufficient condition for g to be metric tensor of a usual product of Riemannian manifolds is that $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{B}}$ are totally geodesic foliations.

Thanks to Theorems 4.1 and 4.2, we have

Theorem 5.1. *When π is used to denote an anti-invariant Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the necessary and sufficient condition for $\mathcal{M}$ to be locally product manifold is that the following hold*

$$\begin{aligned} g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) &= -g(C\mathcal{Y}, J\mathcal{A}_{\mathcal{X}}\mathcal{W}) + \frac{1}{2}\omega(\mathcal{W})g(C\mathcal{Y}, C\mathcal{X}) \\ &\quad -\frac{1}{2}g(B\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W}) \end{aligned}$$

and

$$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{W}, J\mathcal{X}), \pi_* J\mathcal{W}') = 0$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

Thanks to Corollaries 4.1 and 4.2, we have

Theorem 5.2. *When π is used to denote a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the necessary and sufficient condition for $\mathcal{M}$ to be locally product manifold is that the following hold*

$$g_{\mathcal{B}}((\nabla\pi_*)(\mathcal{X}, J\mathcal{Y}), \pi_* J\mathcal{W}) = -\frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) - \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$$

and

$$\mathcal{T}_{\mathcal{W}} J\mathcal{W}' = 0$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$.

For twisted product manifold, we get

Theorem 5.3. *When π represents a Lagrangian Riemannian submersion from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$. Then the necessary and sufficient condition for $\mathcal{M}$ to be locally twisted product manifold of the form $\mathcal{M}_{(ker\pi_*)^\perp} \times_{f'} \mathcal{M}_{(ker\pi_*)}$ is that the following relations hold*

$$\mathcal{T}_{\mathcal{W}} J\mathcal{X} = -g(\mathcal{X}, \mathcal{T}_{\mathcal{W}}\mathcal{W})\|\mathcal{W}\|^{-2} J\mathcal{W}$$

and

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{X}} J\mathcal{Y}, J\mathcal{W}) = \frac{1}{2}g(J\mathcal{Y}, B_1)g(\mathcal{X}, J\mathcal{W}) + \frac{1}{2}g(\mathcal{X}, \mathcal{Y})g(B_1, \mathcal{W})$$

$\forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^\perp), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*)$. *Here, $\mathcal{M}_{(ker\pi_*)^\perp} \times_{f'} \mathcal{M}_{(ker\pi_*)}$ denote the integral manifold of the distributions $(ker\pi_*)^\perp$ and $(ker\pi_*)$.*

Proof. With the help of (2.1) and Lemma 2.2, we write

$$\begin{aligned} g(\nabla_{\mathcal{W}}\mathcal{W}', \mathcal{X}) &= -g(\nabla_{\mathcal{W}}\mathcal{X}, \mathcal{W}') \\ &= -g(J\nabla_{\mathcal{W}}\mathcal{X}, J\mathcal{W}') \\ &= -g(\nabla_{\mathcal{W}}J\mathcal{X}, J\mathcal{W}') \\ &= -g(\mathcal{T}_{\mathcal{W}}J\mathcal{X}, J\mathcal{W}'), \quad \forall \mathcal{X} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W}, \mathcal{W}' \in \Gamma(ker\pi_*), \end{aligned}$$

where orthogonality of $(ker\pi_*)^{\perp}$ and $(ker\pi_*)$ has been used. Hence, we conclude that for any function λ on $\mathcal{M}$, the condition of totally umbilicity holds for $(ker\pi_*)$ iff

$$\mathcal{T}_{\mathcal{W}}J\mathcal{X} = -\mathcal{X}(\lambda)J\mathcal{W}. \tag{5.1}$$

Therefore, taking in use (2.1), we obtain

$$\begin{aligned} g(-\mathcal{X}(\lambda)J\mathcal{W}, J\mathcal{W}) &= g(\mathcal{T}_{\mathcal{W}}J\mathcal{X}, J\mathcal{W}) \\ -\mathcal{X}(\lambda)\|\mathcal{W}\|^2 &= g(\mathcal{T}_{\mathcal{W}}J\mathcal{X}, J\mathcal{W}) \\ &= g(\nabla_{\mathcal{W}}J\mathcal{X}, J\mathcal{W}) \\ &= g(J\nabla_{\mathcal{W}}\mathcal{X}, J\mathcal{W}) \\ &= -g(\mathcal{X}, \mathcal{T}_{\mathcal{W}}\mathcal{W}) \\ \mathcal{X}(\lambda) &= g(\mathcal{X}, \mathcal{T}_{\mathcal{W}}\mathcal{W})\|\mathcal{W}\|^{-2}. \end{aligned} \tag{5.2}$$

In this way, (5.1) and (5.2) produce

$$\mathcal{T}_{\mathcal{W}}J\mathcal{X} = -g(\mathcal{X}, \mathcal{T}_{\mathcal{W}}\mathcal{W})\|\mathcal{W}\|^{-2}J\mathcal{W}$$

and that proves the result with the help of Corollary 4.1. □

Next, we give a non existence result of a twisted product manifold $\mathcal{M}_{(ker\pi_*)^{\perp}} \times_{f'} \mathcal{M}_{(ker\pi_*)}$.

Theorem 5.4. *There does not exist Lagrangian Riemannian submersion π from l.c.K. manifold $(\mathcal{M}, g, J)$ onto a Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$ such that $\mathcal{M}$ is a locally proper twisted product manifold $\mathcal{M}_{(ker\pi_*)^{\perp}} \times_{f'} \mathcal{M}_{(ker\pi_*)}$.*

Proof. Let π denotes a Lagrangian Riemannian submersion from l.c.K. manifold $\mathcal{M}$ onto a Riemannian manifold $\mathcal{B}$ and $\mathcal{M}$ be representing a locally twisted product $\mathcal{M}_{(ker\pi_*)^{\perp}} \times_{f'} \mathcal{M}_{(ker\pi_*)}$. Then, due to definition 5.1, $\mathcal{M}_{(ker\pi_*)}$ and $\mathcal{M}_{(ker\pi_*)^{\perp}}$ will be representing totally geodesic and totally umbilical foliations, respectively. When h denotes the second fundamental form of $\mathcal{M}_{(ker\pi_*)^{\perp}}$, we write

$$g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{W}) = g(h(\mathcal{X}, \mathcal{Y}), \mathcal{W}), \quad \forall \mathcal{X}, \mathcal{Y} \in \Gamma((ker\pi_*)^{\perp}), \mathcal{W} \in \Gamma(ker\pi_*).$$

When H is used for the mean curvature vector field of $\mathcal{M}_{(ker\pi_*)^{\perp}}$, then we deduce

$$g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{W}) = g(H, \mathcal{W})g(\mathcal{X}, \mathcal{Y}). \tag{5.3}$$

Taking (2.1) and lemma 2.2 into consideration, we present

$$\begin{aligned} g(\nabla_{\mathcal{X}}\mathcal{Y},\mathcal{W}) &= -g(\mathcal{Y},\nabla_{\mathcal{X}}\mathcal{W}) \\ &= -g(J\mathcal{Y},J\nabla_{\mathcal{X}}\mathcal{W}) \\ &= -g(J\mathcal{Y},\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X}), \end{aligned} \tag{5.4}$$

here we used the orthogonal property between $(ker\pi_*)^\perp$ and $(ker\pi_*)$. Therefore, (5.3) and (5.4) generate the following

$$\begin{aligned} g(H,\mathcal{W})g(\mathcal{X},\mathcal{Y}) &= -g(J\mathcal{Y},\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X}) \\ g(H,\mathcal{W})g(J\mathcal{Y},J\mathcal{X}) &= -g(J\mathcal{Y},\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X}) \\ -g(H,\mathcal{W})\|\mathcal{X}\|^2 &= g(\mathcal{A}_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X},J\mathcal{X}) \\ &= g(\nabla_{\mathcal{X}}J\mathcal{W}+\frac{1}{2}\omega(\mathcal{W})J\mathcal{X},J\mathcal{X}) \\ &= g(J\nabla_{\mathcal{X}}\mathcal{W},J\mathcal{X}) \\ &= -g(\mathcal{W},\nabla_{\mathcal{X}}\mathcal{X}) \end{aligned}$$

Finally, we reach to

$$g(H,\mathcal{W})\|\mathcal{X}\|^2 = g(\mathcal{W},\mathcal{A}_{\mathcal{X}}\mathcal{X}).$$

So, use of (2.5) shows $\mathcal{A}_{\mathcal{X}}\mathcal{X}=0$, that is $g(H,\mathcal{W})\|\mathcal{X}\|^2=0$. But, $H\in\Gamma(ker\pi_*)$ with Riemannian metric g supply $H=0$ and that that means $(ker\pi_*)^\perp$ is totally geodesic. That proves $\mathcal{M}$ to be usual product of Riemannian manifolds. □

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SOME CHARACTERIZATIONS OF THREE-DIMENSIONAL f-KENMOTSU RICCI SOLITONS

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Abstract. The aim of the present paper is to give some characterizations of f-Kenmotsu Ricci soliton with a supporting example.

Keywords: f-Kenmotsu manifold; Ricci almost soliton; gradient Ricci soliton.

1. Introduction

The revolutionary concept of Ricci flow was introduced by Hamilton [5] in order to solve Poincare conjecture. The conjecture was fully solved by Perelman [11] using Hamilton's Ricci flow technique. After the work of Perelman, the study of Ricci flow has become an important topic in differential geometry. A Ricci flow is a weak parabolic heat type partial differential equation of the following form

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij}, \tag{1.1}$$

$$g(0) = g_0. \tag{1.2}$$

Here g_{ij} denotes the components of Riemannian metric g and S_{ij} denotes the components of Ricci tensor S. A Ricci soliton is a solution of the above equation which is constant up to diffeomorphism and scaling. A Ricci soliton on a Riemannian manifold is characterized by the equation

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0.$$

Here λ is a constant, called soliton constant and the vector field V is called soliton vector field. A Ricci soliton is called expanding, shrinking or steady while λ is positive, negative or zero. A Ricci soliton is called Ricci almost soliton if λ is

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considered as a function instead of a constant [12]. A Ricci soliton is called gradient Ricci soliton if the soliton vector field is gradient of a potential function [13]. The study of Ricci solitons on almost contact manifolds was first initiated by Ramesh Sharma [16]. The Ricci solitons on almost contact manifolds have been studied by several authors ([4], [13], [15]). Ricci soliton on (κ, μ) contact metric manifold has been studied by the present authors in [14]

The notion of Kenmotsu manifold was introduced by K. Kenmotsu and was subsequently generalized to f-Kenmotsu manifolds. For details we refer to [8] and [9]. Ricci solitons on Kenmotsu manifold have been studied in [6]. The notion of ϕ-Ricci symmetric manifolds was introduced by U. C. De and A. Sarkar [2]. The notion of ϕ-symmetric manifolds was introduced by T. Takahashi [17]. Later several authors studied ϕ-symmetric manifolds. Three dimensional quasi-Sasakian manifolds with cyclic parallel and η-parallel Ricci tensor have been studied by U. C. De and A. Sarkar [3].

The objective of the present paper is to give some characterizations of f-Kenmotsu manifolds with Ricci solitons and hence establish the relations between such manifolds with locally ϕ-symmetric manifolds and manifolds with cyclic parallel and η-parallel Ricci tensors.

The present paper is organised as follows:
After the introduction, we give will required preliminaries in Section 2. In Section 3, we will study three dimensional f-Kenmotsu manifolds admitting Ricci soliton. Section 4 contains a supporting example.

2. Preliminaries

An odd dimensional smooth manifold M is said to be an almost contact metric manifold, if there exists a (1,1) tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g on M such that [1]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi(X)) = 0. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y \in \chi(M)$. Such a manifold of dimension (2n+1) is denoted by M^{2n+1} (ϕ, ξ, η, g) . Also M^{2n+1} (ϕ, ξ, η, g) is called an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies

$$(\nabla_X \phi)Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.3}$$

where $f \in C^\infty$(M) is such that $df \wedge \eta = 0$ ([8], [9]). If $f = \beta$ is nonzero constant, then the manifold is a β-Kenmotsu manifold [7]. If $f = 0$, then the manifold is cosymplectic [7]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$. For an f-Kenmotsu manifold, it follows from (2.3)

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{2.4}$$

The condition $df \wedge \eta = 0$ holds only for dim $M \geq 5$ [10]. In a three dimensional f-Kenmotsu manifold, we have

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z \\ &- (\frac{r}{2} + 3f^2 + 3f')\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\}, \end{aligned} \tag{2.5}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.6}$$

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi, \tag{2.7}$$

where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, also R, S and r are Riemannian curvature tensor, Ricci curvature tensor and scalar curvature on M respectively [9]. From (2.5) and (2.6) we get

$$R(X,Y)\xi = -(f^2 + f')(\eta(Y)X - \eta(X)Y), \tag{2.8}$$

$$S(X,\xi) = -2(f^2 + f')\eta(X), \tag{2.9}$$

$$S(\xi,\xi) = -2(f^2 + f'), \tag{2.10}$$

$$Q\xi = -2(f^2 + f')\xi. \tag{2.11}$$

As a consequence of (2.4), we also have

$$(\nabla_X \eta)(Y) = fg(\phi X, \phi Y). \tag{2.12}$$

Also from (2.9) it follows that

$$S(\phi X, \phi Y) = S(X,Y) + 2(f^2 + f')\eta(X)\eta(Y) \tag{2.13}$$

for all vector fields $X, Y \in \chi(M)$.

An f-Kenmotsu manifold $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be ϕ-symmetric if its curvature tensor R bears the condition

$$\phi^2(\nabla_X R)(Y,Z)W = 0, \tag{2.14}$$

for all vector fields $X, Y, Z, W \in \chi(M)$ [17]. In particular, if X, Y, Z, W are orthogonal to ξ, then $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be locally ϕ-symmetric. An f-Kenmotsu manifold $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be ϕ-Ricci symmetric if its Ricci operator Q bears the condition

$$\phi^2(\nabla_X Q)Y = 0 \tag{2.15}$$

for all vector fields $X, Y \in \chi(M)$. If X and Y are orthogonal to ξ, then $M^{(2n+1)}$ (ϕ, ξ, η, g) is said to be locally ϕ-Ricci symmetric. It may be noted that ϕ-symmetric implies ϕ-Ricci symmetric, but the converse is not valid in general.

Ricci tensor S of a Riemannian manifold (M, g) is called η-parallel if

$$g((\nabla_X S)Y, Z) = 0$$

for all vector fields X, Y, Z tangent to M and orthogonal to ξ where g and ∇ denote Riemannian metric and Riemannian connection respectively.

Ricci tensor S of a Riemannian manifold (M,g) is called cyclic-parallel if

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{2.16}$$

for all vector fields X, Y, Z tangent to M. Here ∇ denotes Riemannian connection.

3. Three-dimensional f-Kenmotsu manifolds with Ricci soliton

In this section we prove the following:

Theorem 3.1. *In a three-dimensional f Kenmotsu Ricci soliton, if f is constant and the soliton vector field is Killing, then the soliton is expanding.*

Proof. For a three-dimensional f-Kenmotsu manifold, from (2.7), we get

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi. \tag{3.1}$$

Differentiating covariantly along Y and using (2.4) and (2.12) we obtain

$$\begin{aligned}(\nabla_Y Q)X &= (\frac{dr(Y)}{2} + 2fdf(Y) + df'(Y))X + (\frac{r}{2} + f^2 + f')\nabla_Y X \\ &- (\frac{dr(Y)}{2} + 6fdf(Y) + 3df'(Y))\eta(X)\xi \\ &- (\frac{r}{2} + 3f^2 + 3f')fg(\phi X, \phi Y)\xi - (\frac{r}{2} + 3f^2 + 3f') \\ & \quad \eta(X)f(Y - \eta(Y)\xi).\end{aligned} \tag{3.2}$$

Taking inner product of (3.2) with Y we have

$$\begin{aligned}g((\nabla_Y Q)X, Y) &= (\frac{dr(Y)}{2} + 2fdf(Y) + df'(Y))g(X,Y) \\ &+ (\frac{r}{2} + f^2 + f')g(\nabla_Y X, Y) \\ &- (\frac{dr(Y)}{2} + 6fdf(Y) + 3df'(Y))\eta(X)\eta(Y) \\ &- (\frac{r}{2} + 3f^2 + 3f')fg(\phi X, \phi Y)\eta(Y) \\ &- (\frac{r}{2} + 3f^2 + 3f')\eta(X)g(Y,Y)f \\ &+ (\frac{r}{2} + 3f^2 + 3f')\eta(X)(\eta(Y))^2 f.\end{aligned} \tag{3.3}$$

Let $\{e_1, e_2, \xi\}$ be an orthonormal ϕ-basis at any point of a tangent space. It is known that

$$div(Q)X = g((\nabla_{e_1} Q)X, e_1) + g((\nabla_{e_2} Q)X, e_2) + g((\nabla_{e_3} Q)X, e_3). \tag{3.4}$$

Using (3.3) in (3.4) we get

$$\begin{aligned} div(Q)X &= (\frac{dr(e_1)}{2} + 2fdf(e_1) + df'(e_1))g(X, e_1) \\ &+ (\frac{r}{2} + f^2 + f')g(\nabla_{e_1}X, e_1) \\ &- (\frac{dr(e_2)}{2} + 6fdf(e_2) + 3df'(e_2))g(X, e_2) \\ &+ (\frac{r}{2} + 3f^2 + 3f')g(\nabla_{e_2}X, e_2) \\ &+ (\frac{dr(\xi)}{2} + 2fdf(\xi) + df')\eta(X) \\ &+ (\frac{r}{2} + f^2 + f')g(\nabla_{\xi}X, \xi) \\ &- (\frac{dr(\xi)}{2} + 2fdf(\xi) + df')\eta(X). \end{aligned} \tag{3.5}$$

We know that $div(Q)X = \frac{1}{2}dr(X)$. Putting $X = \xi$ in (3.5) we obtain

$$\frac{1}{2}dr\xi = 2(\frac{r}{2} + f^2 + f')f - 4fdf(\xi) - 2df'(\xi). \tag{3.6}$$

If f-Kenmotsu manifold admits Ricci soliton then

$$S(X, Y) = -\frac{1}{2}((\mathcal{L}_V g)(X, Y) - \lambda g(X, Y)). \tag{3.7}$$

If V is a Killing vector field, from (3.7) we get $r = -3\lambda =$ constant. Therefore, from (3.6)

$$(\frac{r}{2} + f^2 + f')f = 2fdf(\xi) - df'(\xi). \tag{3.8}$$

If f is a non-zero constant then

$$r = -2f^2. \tag{3.9}$$

Consequently, $\lambda = \frac{2}{3}f^2$. This completes the proof. □

We know from [6] that a three-dimensional non cosymplectic f-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with f being constant, is locally ϕ-Ricci symmetric if and only if the scalar curvature is constant. So we get the following corollary

Corollary 3.1. *If a three-dimensional f-Kenmotsu manifold with constant f admits a Ricci soliton with Killing soliton vector field, then it is ϕ-Ricci symmetric, and hence ϕ-symmetric.*

Again we know from [6] that in a three-dimensional non cosymplectic f-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with f being constant, the Ricci tensor is η-parallel if and only if the scalar curvature is constant. Hence we get

Corollary 3.2. *If a three-dimensional f-Kenmotsu manifold with constant f admits Ricci soliton with Killing soliton vector field, then its Ricci tensor is η-parallel.*

From [6] we know that a three-dimensional non cosymplectic f-Kenmotsu manifold $M^3(\phi, \xi, \eta, g)$ with f being constant, satisfies cyclic parallel Ricci tensor if and only if the scalar curvature is constant. So, we can state the following:

Corollary 3.3. *If a three-dimensional f-Kenmotsu manifold with constant f admits Ricci soliton with Killing soliton vector field, then its Ricci tensor is cyclic parallel.*

4. Example

Example 4.1. Let $M = \{(u, v, w) \in R^3 : u, v, w(\neq 0) \in R\}$ be a Riemannian manifold, where (u, v, w) denotes the standard coordinates of a point in R^3. Let us suppose that

$$e_1 = 3w\frac{\partial}{\partial u}, \quad e_2 = 3w\frac{\partial}{\partial v}, \quad e_3 = -3w\frac{\partial}{\partial w} \tag{4.1}$$

are three linearly independent vector fields at each point of M and therefore it forms a basis for the tangent space $\chi(M)$. We also define the Riemannian metric g of the manifold M given by

$$g = \frac{1}{w^2}[du \odot du + dv \odot dv + dw \odot dw]. \tag{4.2}$$

Let η be the one form satisfying

$$\eta(U) = g(U, e_3) \tag{4.3}$$

for any $U \in \chi(M)$ and let ϕ be the (1, 1) tensor field defined by $\phi e_1 = -e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$. By the linear properties of ϕ and g, we can easily verify the following relations

$$\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3 \tag{4.4}$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V) \tag{4.5}$$

for arbitrary vector fields $U, V \in \chi(M)$. This shows that $\xi = e_3$ the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M. If ∇ is the Livi-Civita connection with respect to the Riemannian metric g, then with the help of above, we can easily calculate that

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 3e_1, \quad [e_2, e_3] = 3e_2. \tag{4.6}$$

Now we recall Koszul's formula as

$$\begin{aligned} 2g(\nabla_U V, W) &= U(g(V, W)) + V(g(W, X)) - W(g(U, V)) \\ &- g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]) \end{aligned}$$

for arbitrary vector fields $U, V, W \in \chi(M)$. Making use of Koszul's formula, we get the following:

$$\nabla_{e_2} e_3 = 3e_2 \quad \nabla_{e_2} e_2 = 3e_3 \quad \nabla_{e_2} e_1 = 0 \tag{4.7}$$

$$\nabla_{e_3} e_3 = 0 \quad \nabla_{e_3} e_2 = 0 \quad \nabla_{e_3} e_1 = 0 \tag{4.8}$$

$$\nabla_{e_1} e_3 = 3e_1 \quad \nabla_{e_1} e_2 = 0 \quad \nabla_{e_1} e_1 = 3e_3. \tag{4.9}$$

From the above calculation, it is clear that M satisies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = 3$ is a non-zero constant. Thus we conclude that M leads to an f-Kenmotsu manifold. Also $f^2 + f'$ is non-zero. This implies that M is a three-dimensional regular f-Kenmotsu manifold. We find the components of curvature tensor and Ricci tensor as follows:

$$R(e_2, e_3)e_3 = -3e_2, \qquad R(e_3, e_2)e_2 = -3e_3, \tag{4.10}$$

$$R(e_1, e_3)e_3 = -3e_1, \qquad R(e_3, e_1)e_1 = -3e_3, \tag{4.11}$$

$$R(e_1, e_2)e_2 = -3e_1, \qquad R(e_1, e_2)e_3 = 0, \tag{4.12}$$

$$R(e_2, e_1)e_1 = -3e_2, \qquad R(e_3, e_1)e_2 = 0, \tag{4.13}$$

$$S(e_1, e_1) = -6, \quad S(e_2, e_2) = -6, \quad S(e_3, e_3) = -6, \tag{4.14}$$

$$S(\phi e_1, \phi e_1) = -6, \quad S(\phi e_2, \phi e_2) = -6, \quad S(\phi e_3, \phi e_3) = -0, \tag{4.15}$$

$S(\phi e_i, \phi e_j) = 0$ for all $i, j = 1, 2, 3 (i \neq j)$. From the above consequence, it is clear that $\phi^2\{(\nabla_U Q)(V)\} = 0$ for all vector fields $U, V \in \chi(M)$. Hence M is locally ϕ-Ricci symmetric. From above we get $r = -18$, this implies the scalar curvature is constant. Moreover, $(\nabla_X S)(\phi e_i, \phi e_j) = 0$ for $X \in \chi(M) i, j = 1, 2, 3$. So M is η-parallel, cyclic parallel. This example is also satisfying the Ricci soliton equation if $\lambda = 6$. Hence $\lambda = \frac{2}{3} f^2$ is verified. So the soliton is expanding. Thus, Theorem 3.1 and the associated corollaries are verified by this example.

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EIGHTY ONE RICCI-TYPE IDENTITIES *

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Abstract. In this manuscript, the identities of Ricci Type with respect to a non-symmetric affine connection space are obtained and simplified. The components of commutation formulae are discussed.

Key words: covariant derivative; identities of Ricci Type; commutation formula.

1. Introduction

An N-dimensional manifold $\mathcal{M}_N$ equipped with an affine connection with torsion ∇ is the non-symmetric affine connection space $\mathbb{GA}_N$ (see L. P. Eisenhart [1], S. M. Minčić [4–6, 6–8]), M. S. Stanković [13], Lj. S. Velimirović [10, 11], M. Lj. Zlatanović [13, 14], M. Z. Petrović [9–11]. The non-symmetric affine connection spaces are subjects of research for many other authors but our aim is to examine some basic facts about these spaces in this paper.

The affine connection coefficients for the affine connection ∇ are L^i_{jk}. These coefficients are non-symmetric by indices j and k. Hence, their symmetric and anti-symmetric parts are defined as

$$L^i_{\underline{jk}} = \frac{1}{2}\big(L^i_{jk} + L^i_{kj}\big) \quad \text{and} \quad L^i_{\underset{\vee}{jk}} = \frac{1}{2}\big(L^i_{jk} - L^i_{kj}\big). \tag{1.1}$$

Four kinds of covariant derivatives with respect to the non-symmetric affine connection ∇ are defined. Coordinately, these four types (for a tensor a^i_j of the type $(1,1)$) are [4–11, 13, 14]

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$$
\begin{aligned}
&a^i_{j\underset{1}{|}k}=a^i_{j,k}+L^i_{\alpha k}a^\alpha_j-L^\alpha_{jk}a^i_\alpha, \quad a^i_{j\underset{2}{|}k}=a^i_{j,k}+L^i_{k\alpha}a^\alpha_j-L^\alpha_{kj}a^i_\alpha,\\
&a^i_{j\underset{3}{|}k}=a^i_{j,k}+L^i_{\alpha k}a^\alpha_j-L^\alpha_{kj}a^i_\alpha, \quad a^i_{j\underset{4}{|}k}=a^i_{j,k}+L^i_{k\alpha}a^\alpha_j-L^\alpha_{jk}a^i_\alpha.
\end{aligned}
\tag{1.2}
$$

In the case of $L^i_{\underset{\vee}{jk}}=0$, the four kinds of covariant derivatives (1.2) reduce to one kind [2,12]

$$
a^i_{j|k}=a^i_{j\underset{0}{|}k}=a^i_{j,k}+L^i_{\underline{\alpha k}}a^\alpha_j-L^\alpha_{\underline{jk}}a^i_\alpha,
\tag{1.3}
$$

Proposition 1.1. *The fourth kind of the covariant derivative expressed in* (1.2) *and the covariant derivative with respect to the symmetric affine connection given by* (1.3) *satisfy the equalities*

$$
\begin{aligned}
a^i_{j\underset{4}{|}k}&=a^i_{j\underset{1}{|}k}+a^i_{j\underset{2}{|}k}-a^i_{j\underset{3}{|}k},\\
a^i_{j|k}&=\frac{1}{2}a^i_{j\underset{1}{|}k}+\frac{1}{2}a^i_{j\underset{2}{|}k}.
\end{aligned}
\tag{1.4}
$$

If $L^i_{\underset{\vee}{jk}}\not\equiv 0$, the geometrical objects $a^i_{j\underset{1}{|}k}$, $a^i_{j\underset{2}{|}k}$, $a^i_{j\underset{3}{|}k}$ are linearly independent.

Proof. With respect to the equalities $L^i_{jk}=L^i_{\underline{jk}}+L^i_{\underset{\vee}{jk}}$, $L^i_{\underset{\vee}{jk}}=-L^i_{\underset{\vee}{kj}}$ and the equation (1.3), one gets

$$
\begin{aligned}
&a^i_{j\underset{1}{|}k}=a^i_{j|k}+L^i_{\underset{\vee}{\alpha k}}a^\alpha_j-L^\alpha_{\underset{\vee}{jk}}a^i_\alpha, \quad a^i_{j\underset{2}{|}k}=a^i_{j|k}-L^i_{\underset{\vee}{\alpha k}}a^\alpha_j+L^\alpha_{\underset{\vee}{jk}}a^i_\alpha,\\
&a^i_{j\underset{3}{|}k}=a^i_{j|k}+L^i_{\underset{\vee}{\alpha k}}a^\alpha_j+L^\alpha_{\underset{\vee}{jk}}a^i_\alpha, \quad a^i_{j\underset{4}{|}k}=a^i_{j|k}-L^i_{\underset{\vee}{\alpha k}}a^\alpha_j-L^\alpha_{\underset{\vee}{jk}}a^i_\alpha,
\end{aligned}
\tag{1.5}
$$

From the expressions (1.5), one obtains [9,10]

$$
a^i_{j\underset{4}{|}k}=a^i_{j\underset{1}{|}k}+a^i_{j\underset{2}{|}k}-a^i_{j\underset{3}{|}k} \quad \text{and} \quad a^i_{j|k}=\frac{1}{2}a^i_{j\underset{1}{|}k}+\frac{1}{2}a^i_{j\underset{2}{|}k},
$$

which proves the first part of this proposition.

Furthermore, the geometrical objects $a^i_{j\underset{1}{|}k}$, $a^i_{j\underset{2}{|}k}$, $a^i_{j\underset{3}{|}k}$ expressed as in the equation (1.5) may be considered as the vectors $v_1=(1,1,-1)$, $v_2=(1,-1,1)$, $v_3=(1,1,1)$. These vectors are linearly independent, which completes the proof for this proposition. □

Curvatures of the space $\mathbb{GA}_N$ are $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, for $v_1, v_2, w_1, w_2 \in \{0,1,2,3,4\}$. We will study the curvatures of the space $\mathbb{GA}_N$ obtained with respect to the first three kinds of covariant derivatives (1.2) in this paper.

Our purpose is to coordinately express the curvatures of the space $\mathbb{GA}_N$ with respect to first three kinds of covariant derivatives (1.2) in this paper. We will obtain the coordinates of the differences $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, for $v_1, v_2, w_1, w_2 \in \{1,2,3\}$. The pseudocurvature tensors as possible components of these differences will be discussed. The number of linearly independent geometrical objects $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{1,2,3\}$, will be obtained. At the end of the paper, we will list all of the commutation formulae with respect to $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{1,2,3\}$.

2. Identities of Ricci type

With respect to the equations (1.3, 1.5), one gets

$$a^i_{j\underset{v}{|}k} = a^i_{j|k} + c_v L^i_{\underset{\vee}{\alpha k}} a^\alpha_j + d_v L^\alpha_{\underset{\vee}{jk}} a^i_\alpha, \tag{2.1}$$

for $v = 0, \dots, 4$ and $c_0 = 0$, $c_1 = 1$, $c_2 = -1$, $c_3 = 1$, $c_4 = -1$, $d_0 = 0$, $d_1 = -1$, $d_2 = 1$, $d_3 = 1$, $d_4 = -1$.

Moreover, it holds the equation

$$\begin{aligned}
a^i_{j\underset{v}{|}m\underset{w}{|}n} &= a^i_{j|m|n} + c_v L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + c_w L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} + d_v L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + d_w L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + d_w L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} \\
&\quad + a^\alpha_j \big(c_v L^i_{\underset{\vee}{\alpha m}|n} + c_v c_w L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + c_v (c_w + d_w) L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - c_v d_w L^\beta_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\beta\alpha}} \big) \\
&\quad - a^i_\alpha \big(- d_v L^\alpha_{\underset{\vee}{jm}|n} - d_v (c_w + d_w) L^\beta_{\underset{\vee}{jm}} L^\alpha_{\underset{\vee}{\beta n}} - d_v d_w L^\beta_{\underset{\vee}{jn}} L^\alpha_{\underset{\vee}{\beta m}} + d_v d_w L^\beta_{\underset{\vee}{mn}} L^\alpha_{\underset{\vee}{\beta j}} \big) \\
&\quad + a^\alpha_\beta \big(c_w d_v L^\beta_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}} + c_v d_w L^\beta_{\underset{\vee}{jn}} L^i_{\underset{\vee}{\alpha m}} \big),
\end{aligned} \tag{2.2}$$

for $v, w \in \{0,1,2,3,4\}$.

The next theorem holds.

Theorem 2.1. First Ricci-Type Identities Theorem *The family of identities of the Ricci Type with respect to a non-symmetric affine connection* ∇ *is*

$$\begin{aligned}
a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m} = {} & (c_{v_1}-c_{w_2})L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n} + (c_{w_1}-c_{v_2})L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m} + (d_{v_1}-d_{w_2})L^\alpha_{\underset{\vee}{jm}}a^i_{\alpha|n}\\
& + (d_{w_1}-d_{v_2})L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} + (d_{w_1}+d_{w_2})L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha}\\
& + a^\alpha_j\Big\{R^i_{\alpha mn} + c_{v_1}L^i_{\underset{\vee}{\alpha m}|n} - c_{v_2}L^i_{\underset{\vee}{\alpha n}|m}\\
& \qquad + \big[c_{v_1}c_{w_1} - c_{v_2}(c_{w_2}+d_{w_2})\big]L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}}\\
& \qquad + \big[c_{v_1}(c_{w_1}+d_{w_1}) - c_{v_2}c_{w_2}\big]L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\\
& \qquad - (c_{v_1}d_{w_1}+c_{v_2}d_{w_2})L^\beta_{\underset{\vee}{mn}}L^i_{\underset{\vee}{\beta\alpha}}\Big\}\\
& - a^i_\alpha\Big\{R^\alpha_{jmn} - d_{v_1}L^\alpha_{\underset{\vee}{jm}|n} + d_{v_2}L^\alpha_{\underset{\vee}{jn}|m}\\
& \qquad - \big[d_{v_1}(c_{w_1}+d_{w_1}) - d_{v_2}d_{w_2}\big]L^\beta_{\underset{\vee}{jm}}L^\alpha_{\underset{\vee}{\beta n}}\\
& \qquad - \big[d_{v_1}d_{w_1} - d_{v_2}(c_{w_2}+d_{w_2})\big]L^\beta_{\underset{\vee}{jn}}L^\alpha_{\underset{\vee}{\beta m}}\\
& \qquad + (d_{v_1}d_{w_1}+d_{v_2}d_{w_2})L^\beta_{\underset{\vee}{mn}}L^\alpha_{\underset{\vee}{\beta j}}\Big\}\\
& + a^\alpha_\beta\{(c_{w_1}d_{v_1}-c_{v_2}d_{w_2})L^\beta_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + (c_{v_1}d_{w_1}-c_{w_2}d_{v_2})L^\beta_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}\},
\end{aligned}\tag{2.3}$$

for $v_1,v_2,w_1,w_2\in\{0,1,2,3,4\}$. □

From this theorem, we obtain that just tensors are components of the curvatures for the space $\mathbb{GA}_N$.

The rank of the matrix of the type 81×19 whose rows are composed of the elements

$$\begin{aligned}
& c_{v_1}-c_{w_2},\quad c_{w_1}-c_{v_2},\quad d_{v_1}-d_{w_2},\quad d_{w_1}-d_{v_2},\quad d_{w_1}+d_{w_2},\\
& 1,\quad c_{v_1},\quad -c_{v_2},\quad c_{v_1}c_{w_1}-c_{v_2}(c_{w_2}+d_{w_2}),\quad c_{v_1}(c_{w_1}+d_{w_1})-c_{v_2}c_{w_2},\quad -(c_{v_1}d_{w_1}+c_{v_2}d_{w_2}),\\
& -1,\quad d_{v_1},\quad -d_{v_2},\quad d_{v_1}(c_{w_1}+d_{w_1})-d_{v_2}d_{w_2},\quad d_{v_1}d_{w_1}-d_{v_2}(c_{w_2}+d_{w_2}),\quad -(d_{v_1}d_{w_1}+d_{v_2}d_{w_2}),\\
& c_{w_1}d_{v_1}-c_{v_2}d_{w_2},\quad c_{v_1}d_{w_1}-c_{w_2}d_{v_2},
\end{aligned}$$

for $v_1,v_2,w_1,w_2\in\{1,2,3\}$, is 15.

In this way, we proved the next theorem.

Theorem 2.2. $1-2-3$-Commutation Formulae Theorem *Fifteen of the geometrical objects* $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, *for* $v_1,v_2,w_1,w_2\in\{1,2,3\}$, *are linearly independent.* □

One may check that the geometrical objects

$$\begin{aligned}\mathcal{B}^{i}_{(1).jmn} &= a^{i}_{j\underset{1}{|}m\underset{1}{|}n} - a^{i}_{j\underset{1}{|}n\underset{1}{|}m}\\ &= -2L^{\alpha}_{\underset{\vee}{mn}}a^{i}_{j|\alpha} + a^{\alpha}_{j}\big(R^{i}_{\alpha mn} + L^{i}_{\underset{\vee}{\alpha m}|n} - L^{i}_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^{i}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^{i}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{i}_{\underset{\vee}{\alpha j}}\big)\\ &\quad - a^{i}_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\alpha n}} - L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\alpha m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{\alpha}_{\underset{\vee}{\beta j}}\big),\end{aligned}\tag{2.4}$$

$$\begin{aligned}\mathcal{B}^{i}_{(2).jmn} &= a^{i}_{j\underset{1}{|}m\underset{1}{|}n} - a^{i}_{j\underset{1}{|}n\underset{2}{|}m}\\ &= 2L^{i}_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n} - 2L^{\alpha}_{\underset{\vee}{jm}}a^{i}_{\alpha|n} - 2a^{\alpha}_{\beta}\big(L^{\beta}_{\underset{\vee}{jm}}L^{i}_{\underset{\vee}{\alpha n}} + L^{\beta}_{\underset{\vee}{jn}}L^{i}_{\underset{\vee}{\alpha m}}\big)\\ &\quad + a^{\alpha}_{j}\big(R^{i}_{\alpha mn} + L^{i}_{\underset{\vee}{\alpha m}|n} - L^{i}_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^{i}_{\underset{\vee}{\beta n}} + L^{i}_{\underset{\vee}{\alpha n}}L^{i}_{\underset{\vee}{\beta m}}\big)\\ &\quad - a^{i}_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big)\end{aligned}\tag{2.5}$$

$$\begin{aligned}\mathcal{B}^{i}_{(3).jmn} &= a^{i}_{j\underset{1}{|}m\underset{1}{|}n} - a^{i}_{j\underset{1}{|}n\underset{3}{|}m}\\ &= -2L^{\alpha}_{\underset{\vee}{jm}}a^{i}_{\alpha|n} - 2a^{\alpha}_{\beta}L^{\beta}_{\underset{\vee}{jm}}L^{i}_{\underset{\vee}{\alpha n}}\\ &\quad + a^{\alpha}_{j}\big(R^{i}_{\alpha mn} + L^{i}_{\underset{\vee}{\alpha m}|n} - L^{i}_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^{i}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^{i}_{\underset{\vee}{\beta m}}\big)\\ &\quad - a^{i}_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} - 3L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big),\end{aligned}\tag{2.6}$$

$$\begin{aligned}\mathcal{B}^{i}_{(4).jmn} &= a^{i}_{j\underset{1}{|}m\underset{1}{|}n} - a^{i}_{j\underset{2}{|}n\underset{1}{|}m}\\ &= 2L^{i}_{\underset{\vee}{\alpha n}}a^{\alpha}_{j|m} - 2L^{\alpha}_{\underset{\vee}{jn}}a^{i}_{\alpha|m} - 2L^{\alpha}_{\underset{\vee}{mn}}a^{i}_{j|\alpha} - 2a^{\alpha}_{\beta}\big(L^{\beta}_{\underset{\vee}{jm}}L^{i}_{\underset{\vee}{\alpha n}} + L^{\beta}_{\underset{\vee}{jn}}L^{i}_{\underset{\vee}{\alpha m}}\big)\\ &\quad + a^{\alpha}_{j}\big(R^{i}_{\alpha mn} + L^{i}_{\underset{\vee}{\alpha m}|n} + L^{i}_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^{i}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^{i}_{\underset{\vee}{\beta m}}\big)\\ &\quad - a^{i}_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big),\end{aligned}\tag{2.7}$$

$$\begin{aligned}\mathcal{B}^{i}_{(5).jmn} &= a^{i}_{j\underset{1}{|}m\underset{1}{|}n} - a^{i}_{j\underset{2}{|}n\underset{2}{|}m}\\ &= 2L^{i}_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n} + 2L^{i}_{\underset{\vee}{\alpha n}}a^{\alpha}_{j|m} - 2L^{\alpha}_{\underset{\vee}{jm}}a^{i}_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{jn}}a^{i}_{\alpha|m}\\ &\quad + a^{\alpha}_{j}\big(R^{i}_{\alpha mn} + L^{i}_{\underset{\vee}{\alpha m}|n} + L^{i}_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^{i}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^{i}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{i}_{\underset{\vee}{\beta\alpha}}\big)\\ &\quad - a^{i}_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{\alpha}_{\underset{\vee}{\beta j}}\big),\end{aligned}\tag{2.8}$$

$$\begin{aligned}\mathcal{B}^{i}_{(6).jmn} &= a^{i}_{j\underset{1}{|}m\underset{1}{|}n} - a^{i}_{j\underset{2}{|}n\underset{3}{|}m}\\ &= 2L^{i}_{\underset{\vee}{\alpha n}}a^{\alpha}_{j|m} - 2L^{\alpha}_{\underset{\vee}{jm}}a^{i}_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{jn}}a^{i}_{\alpha|m} - 2a^{\alpha}_{\beta}L^{\beta}_{\underset{\vee}{jn}}L^{i}_{\underset{\vee}{\alpha m}}\\ &\quad + a^{\alpha}_{j}\big(R^{i}_{\alpha mn} + L^{i}_{\underset{\vee}{\alpha m}|n} + L^{i}_{\underset{\vee}{\alpha n}|m} + 3L^{\beta}_{\underset{\vee}{\alpha m}}L^{i}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^{i}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{i}_{\underset{\vee}{\beta\alpha}}\big)\\ &\quad - a^{i}_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{\alpha}_{\underset{\vee}{\beta j}}\big),\end{aligned}\tag{2.9}$$

$$\begin{aligned}
\mathcal{B}^i_{(7).jmn} &= a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m}\\
&= -2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} - 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha} - 2a^{\alpha}_{\beta}L^{\beta}_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^i_{\underset{\vee}{\beta\alpha}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big),
\end{aligned}\tag{2.10}$$

$$\begin{aligned}
\mathcal{B}^i_{(8).jmn} &= a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m}\\
&= 2L^i_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n} - 2L^{\alpha}_{\underset{\vee}{jm}}a^i_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} - 2a^{\alpha}_{\beta}L^{\beta}_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{\alpha}_{\underset{\vee}{\beta j}}\big),
\end{aligned}\tag{2.11}$$

$$\begin{aligned}
\mathcal{B}^i_{(9).jmn} &= a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m}\\
&= -2L^{\alpha}_{\underset{\vee}{jm}}a^i_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} - 2a^{\alpha}_{\beta}\big(L^{\beta}_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + L^{\beta}_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}\big)\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}} + 2L^{\beta}_{\underset{\vee}{mn}}L^{\alpha}_{\underset{\vee}{\beta j}}\big),
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
\mathcal{B}^i_{(10).jmn} &= a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= -2L^i_{\underset{\vee}{\alpha n}}a^{\alpha}_{j|m} + 2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2a^{\alpha}_{\beta}\big(L^{\beta}_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + L^{\beta}_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}\big)\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big),
\end{aligned}\tag{2.13}$$

$$\begin{aligned}
\mathcal{B}^i_{(11).jmn} &= a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= 2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2a^{\alpha}_{\beta}L^{\beta}_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + 3L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big),
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
\mathcal{B}^i_{(12).jmn} &= a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= -2L^i_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n} + 2L^{\alpha}_{\underset{\vee}{jm}}a^i_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2a^{\alpha}_{\beta}\big(L^{\beta}_{\underset{\vee}{jm}}L^i_{\underset{\vee}{\alpha n}} + L^{\beta}_{\underset{\vee}{jn}}L^i_{\underset{\vee}{\alpha m}}\big)\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\beta}_{\underset{\vee}{jm}}L^{\alpha}_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{jn}}L^{\alpha}_{\underset{\vee}{\beta m}}\big),
\end{aligned}\tag{2.15}$$

$$
\begin{aligned}
\mathcal{B}^i_{(13).jmn} &= a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= -2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} + 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m}\\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^\beta_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\beta\alpha}}\big)\\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\beta_{\underset{\vee}{jm}} L^\alpha_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{jn}} L^\alpha_{\underset{\vee}{\beta m}} + 2L^\beta_{\underset{\vee}{mn}} L^\alpha_{\underset{\vee}{\beta j}}\big),
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
\mathcal{B}^i_{(14).jmn} &= a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= -2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2a^\alpha_\beta L^\beta_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}}\\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - 3L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^\beta_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\beta\alpha}}\big)\\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\beta_{\underset{\vee}{jm}} L^\alpha_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{jn}} L^\alpha_{\underset{\vee}{\beta m}} + 2L^\beta_{\underset{\vee}{mn}} L^\alpha_{\underset{\vee}{\beta j}}\big),
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
\mathcal{B}^i_{(15).jmn} &= a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m}\\
&= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2a^\alpha_\beta L^\beta_{\underset{\vee}{jm}} L^i_{\underset{\vee}{\alpha n}}\\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^\beta_{\underset{\vee}{mn}} L^i_{\underset{\vee}{\beta\alpha}}\big)\\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\beta_{\underset{\vee}{jm}} L^\alpha_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{jn}} L^\alpha_{\underset{\vee}{\beta m}}\big),
\end{aligned}
\tag{2.18}
$$

are a base of the vector spaces generated by the differences $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{1, 2, 3\}$.

With respect to the equation (2.3), we obtain that many curvature tensors but no one curvature pseudotensor may be obtained with respect to the identities of Ricci Type presented in the First Ricci-Type Identities Theorem.

Vice versa, any linear combination of the geometrical objects $b^i_{(k)jmn}$, $k = 1, \dots, 16$, corresponds to infinitely many linear combinations of the differences $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\}$.

To obtain curvature pseudotensors for the space $\mathbb{GA}_N$, we need to consider the base $(c^i_{(k)jmn}) = (b^i_{(k)jmn} + \mathcal{L}^i_{(k)jmn})$, $k = 1, \dots, 16$, where the geometrical objects $\mathcal{L}^i_{(k)jmn}$ are linear combinations of the products $L^i_{\underline{\alpha n}} L^\alpha_{\underset{\vee}{jm}}$, $L^i_{\underline{\alpha m}} L^\alpha_{\underset{\vee}{jn}}$, $L^i_{\underline{\alpha j}} L^\alpha_{\underset{\vee}{mn}}$, $L^i_{\underset{\vee}{\alpha n}} L^\alpha_{\underline{jm}}$, $L^i_{\underset{\vee}{\alpha m}} L^\alpha_{\underline{jn}}$, $L^i_{\underset{\vee}{\alpha j}} L^\alpha_{\underline{mn}}$.

Any linear combination of the geometrical objects $c^i_{(k)jmn}$ does not correspond to a linear combination of the differences $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\}$.

For this reason, the geometrical objects $b^i_{(k)jmn}$ are components of a base for the space of differences $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\}$ unlike the geometrical objects $c^i_{(k)jmn}$.

Remark 2.1. *Any identity of Ricci Type where the curvature pseudotensors of the space $\mathbb{GA}_N$ are obtained may be simplified and reduced to the form* (2.3)*.*

2.1. Eighty one Ricci-Type identities

With respect to the First Ricci-Type Identities Theorem, and for $v_1, v_2, w_1, w_2 \in \{1, 2, 3\}$, we obtain the next identities of Ricci Type.

$$
\begin{aligned}
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= -2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big),\\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad + 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} + L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big),\\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}},\\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad + 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}} + L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big),
\end{aligned}
$$

$$
\begin{aligned}
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^i_{\underset{\vee}{\alpha m}} a^{\alpha}_{j|n} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad + 2a^{\alpha}_{\beta} L^i_{\underset{\vee}{\alpha n}} L^{\beta}_{\underset{\vee}{jm}}, \\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{mn}} a^i_{j|\alpha} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^{\alpha}_{\beta} L^i_{\underset{\vee}{\alpha n}} L^{\beta}_{\underset{\vee}{jm}}, \\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^i_{\underset{\vee}{\alpha n}} a^{\alpha}_{j|m} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad + 2a^{\alpha}_{\beta} L^i_{\underset{\vee}{\alpha m}} L^{\beta}_{\underset{\vee}{jn}}, \\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{1}{|}m} &= 2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{jn}} a^i_{\alpha|m} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad + 2a^{\alpha}_{\beta} \big(L^i_{\underset{\vee}{\alpha n}} L^{\beta}_{\underset{\vee}{jm}} + L^i_{\underset{\vee}{\alpha m}} L^{\beta}_{\underset{\vee}{jn}}\big), \\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^i_{\underset{\vee}{\alpha m}} a^{\alpha}_{j|n} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}}\big) \\
&\quad - 2a^{\alpha}_{\beta} \big(L^i_{\underset{\vee}{\alpha m}} L^{\beta}_{\underset{\vee}{jn}} + L^i_{\underset{\vee}{\alpha n}} L^{\beta}_{\underset{\vee}{jm}}\big), \\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^{\alpha}_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^{\alpha}_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^{\alpha}_{j|m} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}} + 2L^{\alpha}_{\underset{\vee}{j\beta}} L^{\beta}_{\underset{\vee}{mn}}\big), \\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^{\alpha}_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^{\alpha}_{j|n} \\
&\quad + a^{\alpha}_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_{\alpha} \big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} - L^{\alpha}_{\underset{\vee}{jn}|m} - L^{\alpha}_{\underset{\vee}{\beta n}} L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}} L^{\beta}_{\underset{\vee}{jn}} + 2L^{\alpha}_{\underset{\vee}{j\beta}} L^{\beta}_{\underset{\vee}{mn}}\big) \\
&\quad - 2a^{\alpha}_{\beta} L^i_{\underset{\vee}{\alpha n}} L^{\beta}_{\underset{\vee}{jm}},
\end{aligned}
$$

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\begin{aligned}
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}}L^\beta_{\underset{\vee}{mn}}\big),\\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad - 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha m}}L^\beta_{\underset{\vee}{jn}} + L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}}\big),\\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}}L^\beta_{\underset{\vee}{jn}},\\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= 2L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}}L^\beta_{\underset{\vee}{jn}},\\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}},\\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}}a^i_{\alpha|n}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}},
\end{aligned}
$$

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\begin{aligned}
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad + 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big), \\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= -2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big), \\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big), \\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big),
\end{aligned}
$$

$$
\begin{aligned}
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{1}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} - L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad - 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}} + L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}\big), \\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big), \\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big), \\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad - 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} + L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big), \\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}},
\end{aligned}
$$

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\begin{aligned}
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&- 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&- 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big), \\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big), \\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&+ 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} + L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big), \\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&+ 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&- a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&+ 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} + L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big),
\end{aligned}
$$

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\begin{aligned}
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big),\\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}},\\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}},\\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}},\\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad + 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}} + L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}},\\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}},
\end{aligned}
$$

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\begin{aligned}
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}}a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} - 2L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}},\\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}}L^\beta_{\underset{\vee}{jn}},\\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}}a^\alpha_{j|n} + 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha m}}L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}}\big),\\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad - 2a^\alpha_\beta\big(L^i_{\underset{\vee}{\alpha m}}L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}}\big),\\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}}L^\beta_{\underset{\vee}{mn}}\big),\\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{2}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha n}}a^\alpha_{j|m}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} + L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}}L^\beta_{\underset{\vee}{jm}},\\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}}a^i_{\alpha|m} - 2L^\alpha_{\underset{\vee}{mn}}a^i_{j|\alpha}\\
&\quad + a^\alpha_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}}L^\beta_{\underset{\vee}{mn}}\big)\\
&\quad - a^i_\alpha\big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}}L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}}L^\beta_{\underset{\vee}{jn}}\big)\\
&\quad - 2a^\alpha L^i \; L^\beta
\end{aligned}
$$

$$
\begin{aligned}
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= -2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{jn|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&+ 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big), \\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&+ 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} \big) \\
&- 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} \big) \\
&- 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}} \big), \\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} - 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} - 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} + 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}} \big), \\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} \big),
\end{aligned}
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\begin{aligned}
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{1}{|}m} &= 2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} - L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} - 2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jm}} a^i_{\alpha|n} + 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} + L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}\big), \\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}}\big) \\
&\quad + 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&\quad + a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} + L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}}\big) \\
&\quad - a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}}\big) \\
&\quad - 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}},
\end{aligned}
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\begin{aligned}
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}}L^{\beta}_{\underset{\vee}{mn}}\big),\\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} + 2L^i_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n} - 2L^i_{\underset{\vee}{\alpha n}}a^{\alpha}_{j|m}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&+ 2a^{\alpha}_{\beta}\big(L^i_{\underset{\vee}{\alpha m}}L^{\beta}_{\underset{\vee}{jn}} - L^i_{\underset{\vee}{\alpha n}}L^{\beta}_{\underset{\vee}{jm}}\big),\\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{2}{|}m} &= 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha} + 2L^i_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} + L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&+ 2a^{\alpha}_{\beta}L^i_{\underset{\vee}{\alpha m}}L^{\beta}_{\underset{\vee}{jn}},\\
a^i_{j\underset{1}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jm}}a^i_{\alpha|n} - 2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} - L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}} - 2L^{\alpha}_{\underset{\vee}{j\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&- 2a^{\alpha}_{\beta}\big(L^i_{\underset{\vee}{\alpha n}}L^{\beta}_{\underset{\vee}{jm}} + L^i_{\underset{\vee}{\alpha m}}L^{\beta}_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jm}}a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha n}}a^{\alpha}_{j|m}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}}\big),\\
a^i_{j\underset{1}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jm}}a^i_{\alpha|n} + 2L^{\alpha}_{\underset{\vee}{mn}}a^i_{j|\alpha}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} + L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}}\big)\\
&- 2a^{\alpha}_{\beta}L^i_{\underset{\vee}{\alpha n}}L^{\beta}_{\underset{\vee}{jm}},\\
a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= -2L^{\alpha}_{\underset{\vee}{jn}}a^i_{\alpha|m} - 2L^i_{\underset{\vee}{\alpha m}}a^{\alpha}_{j|n}\\
&+ a^{\alpha}_j\big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^{\beta}_{\underset{\vee}{\alpha m}}L^i_{\underset{\vee}{\beta n}} - L^{\beta}_{\underset{\vee}{\alpha n}}L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}}L^{\beta}_{\underset{\vee}{mn}}\big)\\
&- a^i_{\alpha}\big(R^{\alpha}_{jmn} - L^{\alpha}_{\underset{\vee}{jm}|n} + L^{\alpha}_{\underset{\vee}{jn}|m} + L^{\alpha}_{\underset{\vee}{\beta n}}L^{\beta}_{\underset{\vee}{jm}} + L^{\alpha}_{\underset{\vee}{\beta m}}L^{\beta}_{\underset{\vee}{jn}}\big),
\end{aligned}
$$

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\begin{aligned}
a^i_{j\underset{2}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- 2a^\alpha_\beta \big(L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}} + L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}} \big), \\
a^i_{j\underset{2}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha m}} a^\alpha_{j|n} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} - L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{3}{|}m\underset{1}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= -2L^\alpha_{\underset{\vee}{jn}} a^i_{\alpha|m} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} + L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} \big) \\
&- 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha m}} L^\beta_{\underset{\vee}{jn}}, \\
a^i_{j\underset{3}{|}m\underset{2}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} - 2L^i_{\underset{\vee}{\alpha n}} a^\alpha_{j|m} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} - L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- 2a^\alpha_\beta L^i_{\underset{\vee}{\alpha n}} L^\beta_{\underset{\vee}{jm}}, \\
a^i_{j\underset{3}{|}m\underset{3}{|}n} - a^i_{j\underset{3}{|}n\underset{3}{|}m} &= 2L^\alpha_{\underset{\vee}{mn}} a^i_{j|\alpha} \\
&+ a^\alpha_j \big(R^i_{\alpha mn} + L^i_{\underset{\vee}{\alpha m}|n} - L^i_{\underset{\vee}{\alpha n}|m} + L^\beta_{\underset{\vee}{\alpha m}} L^i_{\underset{\vee}{\beta n}} - L^\beta_{\underset{\vee}{\alpha n}} L^i_{\underset{\vee}{\beta m}} + 2L^i_{\underset{\vee}{\alpha\beta}} L^\beta_{\underset{\vee}{mn}} \big) \\
&- a^i_\alpha \big(R^\alpha_{jmn} - L^\alpha_{\underset{\vee}{jm}|n} + L^\alpha_{\underset{\vee}{jn}|m} + L^\alpha_{\underset{\vee}{\beta n}} L^\beta_{\underset{\vee}{jm}} - L^\alpha_{\underset{\vee}{\beta m}} L^\beta_{\underset{\vee}{jn}} - 2L^\alpha_{\underset{\vee}{j\beta}} L^\beta_{\underset{\vee}{mn}} \big).
\end{aligned}
$$

3. Conclusion

This manuscript conducted the research of the components of curvatures for the non-symmetric affine connection space $\mathbb{GA}_N$ with respect to three of four plus one kinds of covariant derivatives (1.2, 1.3).

Here, it was elaborated that curvature pseudotensors are not components of the differences $a^i_{j\underset{v_1}{|}m\underset{w_1}{|}n} - a^i_{j\underset{v_2}{|}n\underset{w_2}{|}m}$, $v_1, v_2, w_1, w_2 \in \{0, 1, 2, 3, 4\}$.

In future work, we will study the commutation formulae obtained with respect to all triples of linearly independent geometrical objects $a^i_{j\underset{p}{|}k}$, $p = 0, \ldots, 4$.

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ON THE NUMERICAL RANGE OF EP MATRICES

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Abstract. In this work, we will study the numerical range $W(T)$ of EP matrices or operators having a canonical form $T = U(A \oplus 0)U^*$ in the case when $0 \notin W(A)$. As a result, we will define a kind of distance $d(W(A,T))$ between the sets $W(A)$ and $W(T)$ and investigate their connenctions, giving also upper and lower bounds for the distance $d(W(A^{-1}, T^{\dagger}))$. Finally we will present the form of their angular numerical range $F(T)$ and its connection with $F(T^{\dagger})$.
Keywords: Numerical Range, Angular numerical range, EP matrices, Moore-Penrose inverse.

1. Introduction- Preliminaries and notation

For the sake of simplicity, we will use the following notation for the unit ball of $\mathbf{C}^n$: $N_1 = \{x \in \mathbf{C}^n,\ \|x\| = 1\}$. All the definitions presented below can be found in [5, 7].
The numerical range of a square matrix $T \in \mathbf{C}^{n\times n}$ is the subset of the complex plane $\mathbf{C}$ defined as:

$$W(T) = \{\langle Tx, x\rangle : \quad x \in N_1 \subset \mathbf{C}^n\}.$$

The numerical radius $r(T)$ is defined as:

$$r(T) = \sup\{|\lambda|, \quad \lambda \in W(T)\}.$$

Another tool used in this work is, in the case of EP matrices $T = U(A \oplus 0)U^*$, the distance between the origin and the set $W(A)$, called the inner numerical radius, $\hat{r}(T)$ defined as:

$$\hat{r}(T) = \inf\{|\lambda|, \quad \lambda \in \partial W(T)\}$$

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Finally, the field angle $\Theta(W(T))$ is the angle formed by the two support lines of $W(T)$ coming from the origin. If $0 \in W(T)$ then $\Theta(W(T)) = 2\pi$. If 0 is on the boundary of $W(T)$ and there is a unique tangent to the boundary at 0 then $\Theta(W(T)) = \pi$. For more on the field angle see [7] .
When T is symmetric, it holds that $r(T) = \rho(T) = \|T\|$, where $\rho(T)$ is the spectral radius of T. For more details, see e.g. [10].
In addition, by taking into account that when T is not invertible then $N(T)$ the null space of T is non zero and for any vector $u \in N(T)$ we have $\langle Tu, u\rangle = 0$. Therefore, we conclude that $0 \in W(T)$ in the case of singular matrices.
The following result is well known and can be used for calculations and/or algorithmic purposes:

Proposition 1.1. *For any given $x \in \mathbf{C}^n$, the numerical range $W(T)$ is equal to:*

$$W(T) = \{\lambda = \langle Tx, x\rangle : \quad x \in N_1 \subset \mathbf{C}^n\} = \left\{\lambda = \frac{1}{\| x \|^2}\langle Tx, x\rangle : \quad x \in \mathbf{C}^n\right\}$$

Since for any $x \in \mathbf{C}^n$ we have

$$\left\{\langle Tx, x\rangle = \| x \|^2 \langle T(\frac{x}{\| x \|}), \frac{x}{\| x \|}\rangle = \| x \|^2 \lambda, \quad \lambda \in W(T) \quad x \in \mathbf{C}^n\right\}$$

The numerical range of a matrix is known to be a compact and convex subset of $\mathbf{C}$. Many interesting properties arise from this set and various properties of T can be deduced from W(T).
When the corresponding matrix is real, $W(T)$ is symmetric with respect to the real axis. On the other hand, $W(T) \subset \mathbf{R}$ if and only if T is Hermitian, i.e., $T^* = T$; in this case, the endpoints of $W(T)$ coincide with the minimum and the maximum eigenvalues of T. Furthermore, $W(T)$ is a line segment in the complex plane if and only if the matrix T is normal and has collinear eigenvalues. We will present an example of the real case and one of of the complex case in Figure 1.1. These two examples will be used again in the sequel. Note that in both cases, the origin does not belong to $W(T)$.

Another tool used in this work is the generalized inverse (Moore-Penrose) of a singular square matrix. (Since the Numerical Range is defined only for square matrices.) In the case when T is a complex $m \times n$ matrix of rank r, Penrose showed that there is a unique matrix satisfying the four Penrose equations, called the pseudo-inverse of T, denoted by $T^\dagger$ such that

$$(1.1) \qquad TT^\dagger = (TT^\dagger)^*, \quad T^\dagger T = (T^\dagger T)^*, \quad TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger,$$

where T^* denotes the conjugate transpose of T.

It is easy to see that $TT^\dagger$ is the orthogonal projection of $\mathbf{C}^n$ onto the range $\mathcal{R}(T)$, of T, denoted by P_T, and that $T^\dagger T$ is the orthogonal projection of $\mathbf{C}^m$ onto $\mathcal{R}(T^*)$,

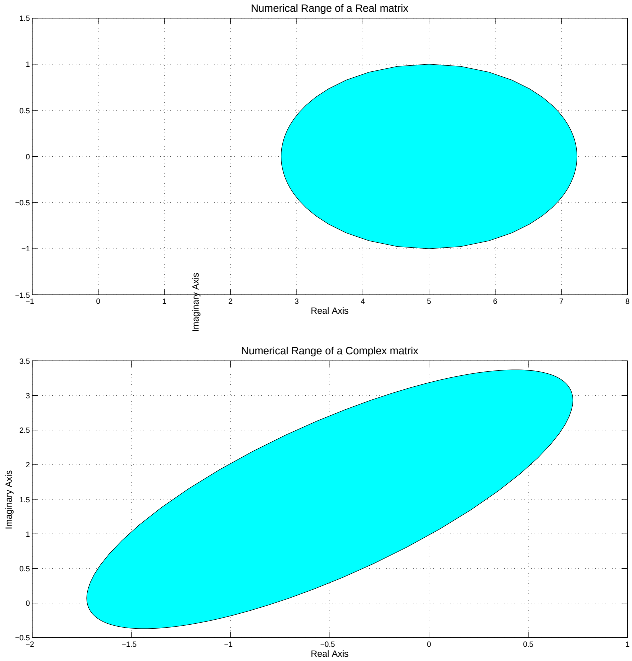

FIG. 1.1: The numerical Range of (a) a real and (b) a complex matrix.

denoted by P_{T^*}. It is also well known that it holds $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$. Standard reference books on generalized inverses are [1, 2, 4].

The matrix T is called EP matrix if $TT^\dagger = T^\dagger T$. The set of EP matrices of rank r are usually denoted by EP_r. We take advantage of the fact that EP matrices have a simple canonical form according to the decomposition $\mathbf{C}^m = \mathcal{R}(T) \oplus \mathcal{N}(T)$. Indeed, an EP_r matrix T has the simple block matrix form

$$T = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* = U(A \oplus 0)U^*, \tag{1.2}$$

where the matrix $A : R(T) \to R(T)$ is invertible with $rank(A) = r$ and U is unitary. So, T can aslo be seen as a dilation of the matrix A.
The generalized inverse $T^\dagger$ of the matrix T defined in (1.2) has the form

$$T^\dagger = U \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U(A^{-1} \oplus 0)U^*. \tag{1.3}$$

This decompositon is called the URU^* decomposition of a matrix, and is a special case of the URV decomposition.
According to another characterization, a square complex matrix T is said to be EP if T and its conjugate transpose T^* have the same range. EP operators matrices

constitute a wide class, which includes the self adjoint, the normal and the invertible matrices. For more details about on EP matrices we refer to [2, 3, 11]. Various characterizations of EP matrices were also collected in [12].
All the previous results are also valid for bounded EP operators on Hilbert space with the appropriate modifications.
Throughout this paper, $\mathcal{H}$ will denote a separable Hilbert space of infinite dimension and the set of all bounded operators acting on $\mathcal{H}$ is denoted by $B(\mathcal{H})$. When the space is finite dimensional, $\mathcal{H}$ will be replaced by $\mathbf{C}^n$.

2. The Numerical Range of EP matrices

An important role, studying numerical ranges, plays whether or not zero belongs to the numerical range. Necessary and sufficient conditions such that the origin belongs to the numerical range of a complex matrix may be found e.g. in [9]. In this work, we will examine the special case of EP matrices, $T = U(A \oplus 0)U^*$ such that $0 \notin W(A)$.
For any matrix $A \in \mathbf{C}^{n\times n}$, and any principal submatrix A_1 of A, $W(A_1) \subseteq W(A)$. In the case of an EP operator, we have the following:

Theorem 2.1. *Let T be an EP operator with the simple canonical form according to the decomposition $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$, having the form : $T = U(A \oplus 0)U^*$. Then $W(T) = co(W(A) \cup (0))$.*

Proof. It is obvious that $W(T) \supseteq co(W(A) \cup (0))$. For the contrary, since the space can be decomposed as $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$, then for every $x \in \mathcal{H}, x = x_1 + x_2$.
If $\lambda \in W(T), \lambda \neq 0$ then $\lambda = \langle Tx, x\rangle = \langle Ax_1, x_1\rangle$ for some x, $\|x\| = 1$, therefore $\lambda = \langle Ax_1, x_1\rangle = \lambda'\|x_1\|^2$ with $\|x_1\| \le 1$ and $\lambda' \in W(A)$. This is equal to

$$\lambda = \lambda'\|x_1\|^2 + (1 - \|x_1\|^2)0,$$

so $\lambda \in co(W(A) \cup (0))$. Obviously, when $0 \in W(A) \Rightarrow W(A) = W(T)$.
□

The above result shows clearly the fact that when a number λ is a corner of $W(A)$ then λ is an eigenvalue of A (see [8]).

Example 2.1. We give two particular examples of the above proposition in the next two figures.
1. Let

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}, \quad T = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A is the real matrix whose Numerical range was shown in Figure 1.1(a)
We have seen that $0 \notin W(A)$. The numerical range of A is presented in blue in Figure 2.1. The numerical range of $T = U(A \oplus 0)U^*$ is presented in green. $W(A)$ is an ellipse, with foci the eigenvalues of A, $\lambda_1 = 3, \lambda_2 = 7$ shown in red. We can see that $W(T)$ is the convex hull of $W(A) \cup (0)$ and that the origin is a corner point of $W(T)$.

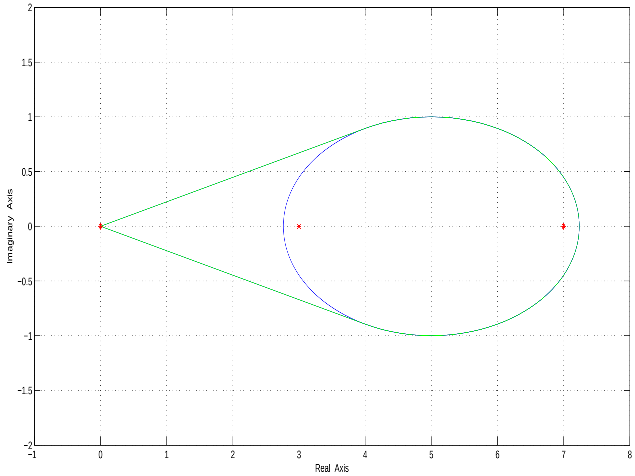

FIG. 2.1: Numerical Range of $W(T) = co(W(A) \cup (0))$ in green, $W(A)$ in blue.

2. Let

$$B = \begin{bmatrix} -1 & i \\ 2 & 3i \end{bmatrix}, \quad S = \begin{bmatrix} -1 & i & 0 \\ 2 & 3i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

B is the complex matrix whose Numerical range was shown in Figure 1.1(b). As we have seen in Figure 1.1(b), $0 \notin W(B)$ and $W(B)$ is an ellipse, with foci the eigenvalues of B, $\lambda_1 = 1.508 - 0.236i$, $\lambda_2 = 0.508 + 3.24i$. The numerical range $W(S)$ of $S = U(B \oplus 0)U^*$ is presented in green in Figure 2.2 and the eigenvalues are shown in red. We can see again that the origin is a corner point of $W(S)$ and that $W(S)$ is the convex hull of $W(B) \cup 0$.

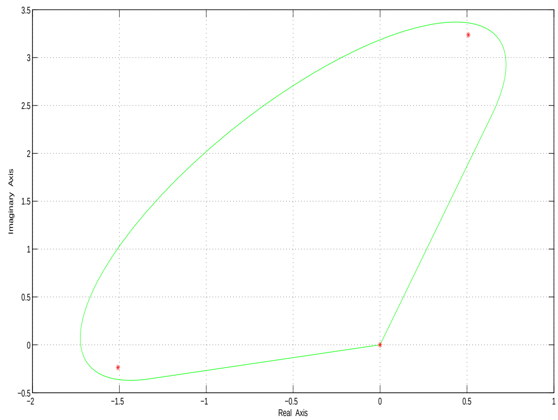

FIG. 2.2: Numerical Range of $W(S) = co(W(B) \cup (0))$ in green.

3. The distance between $W(T)$, $W(A)$

As we can see from the previous results for EP operators the numerical range $W(T)$ is an extension of the numerical range $W(A)$. To what extend? To have a measure of this we define a kind of distance between the sets $W(T)$ and $W(A)$. In fact, we use the distance $d(W(A,T))$ of the origin to $W(A)$. Whenever $0 \in W(A)$ then the two sets coincide, so the distance is equal to zero. When $0 \notin W(A)$ then $d(W(A,T)) = \hat{r}(A)$, the inner numerical radius of A, which is defined as $\hat{r}(A) = min|z|, z \in W(A)$. We give the following definition:

Definition 3.1. Let $T \in B(\mathcal{H})$ be an EP operator having a canonical form $T = U(A \oplus 0)U^*$. Then the distance between the numerical ranges $W(A), W(T)$ is defined as:

$$d(W(A,T)) = \begin{cases} 0, & 0 \in W(A) \\ \hat{r}(A), & 0 \notin W(A) \end{cases}$$

As we can see, this type of distance may be seen as a special case of the Hausdorff distance between subsets of a metric space.
Using the matrices presented in Example 2.1, we have that

$$d(W(A,T)) = 2.7639, \quad d(W(B,S)) = 0.5711$$

The calculation of the above values was made using a modified Matlab code presented in [10].

A natural question then would be: What about the distance $d(W(A^{-1}, T^\dagger))$, and its relation with $d(W(A,T))$? Equivalently, what is the relation between $\hat{r}(A), \hat{r}(A^{-1})$ when $0 \notin W(A)$?
We obviously have that $T^\dagger = U \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$.
It also holds that $W(T^\dagger) = co(W(A^{-1}) \cup (0))$. Since in general there is no connection between $W(A)$ and $W(A^{-1})$, neither between the numerical radius of these matrices, the only relation we can have is an inequality connecting $d(W(A^{-1}, T^\dagger))$, and $d(W(A,T))$.
An answer can be given using a result from [6]:

Proposition 3.1. *Let a nonsingular square matrix A, $\hat{r}(A)$ denote its inner numerical radius and $r(A)$ its numerical radius. Then it holds that:*

$$\frac{\hat{r}(A)}{\|A\|^2} \leq \hat{r}(A^{-1}) \leq min\{\frac{\hat{r}(A)}{\sigma^2_{min}(A)}, \frac{r(A)}{\|A\|^2}\}$$

Based on the above result, we conclude the following Proposition:

Proposition 3.2. *Let an EP matrix T with the canonical form $T = U(A \oplus 0)U^*$. If $0 \notin W(A)$ then we have*

$$\frac{d(W(A,T))}{\|A\|^2} \leq d(W(A^{-1}, T^\dagger)) \leq min\{\frac{d(W(A,T))}{\sigma^2_{min}(A)}, \frac{r(A)}{\|A\|^2}\}$$

Example 3.1. Using the matrices from Example 2.1:

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}, \quad T = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T^{\dagger} = \begin{bmatrix} 0.2857 & -0.0476 & 0 \\ -0.1429 & 0.1905 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have that :
$r(A) = 7.2361$ and $\hat{r}(A) = 2.7639$, while $\|A\| = 7.3351$, $\sigma_{min}(A) = 3$.
So, replacing in the above inequality we can see that

$$\frac{2.7639}{7.3351^2} \leq d(W(A^{-1}, T^{\dagger})) \leq min\{\frac{2.7639}{9}, \frac{7.2361}{7.3351^2}\} \Leftrightarrow$$

$$0.0514 \leq d(W(A^{-1}, T^{\dagger})) \leq 0.1345$$

Using Matlab we get that $d(W(A^{-1}, T^{\dagger})) = \hat{r}(A^{-1}) = 0.1316$ which satisfies the above inequality.

Another question on the relation between $W(A), W(T)$ is how much the boundary $\partial(A)$ has been changed to give $\partial(T)$. The distance $d(W(A,T))$ can give us a measure of course, but we can also perform a statistical analysis of the boundaries of numerical ranges as another index of the change performed. We expect the variance of $\partial(T)$ to increase more with respect to $\partial(A)$, as the distance $d(W(A,T))$ increases.

Example 3.2. In this example we will use the matrices B, S from Example 2.1 and the following matrices C, R:

$$C = \begin{bmatrix} 4 & 0 & 0 & -1 \\ -1 & 4 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix} \quad R = \begin{bmatrix} 4 & 0 & 0 & -1 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have that $d(W(C,R)) = \hat{r}(C) = 3$ and we can calculate the variance of the boundaries of C and R:

$$Var(\partial(C)) = 0.520, \quad Var(\partial(R)) = 4.946.$$

On the other hand, using the matrices B, S from Example 2.1 with $S = U(B \oplus 0)U^*$, having a much lower distance of the Numerical Ranges, $d(W(B,S)) = 0.5711$ we can see that:

$$Var(\partial(B)) = 0.949, \quad Var(\partial(S)) = 0.90958.$$

As another possible index of change, we can also calculate the field angles $\Theta(W(R)) = 0.49$ radians and $\Theta(W(S)) = 2.074$ radians, using the matlab code drawing the Numerical range found in [10].

From the above results we cannot have a clear picture of the relation between $d(W(A,T)$ and the variance of the boundaries or the field angle. So, we will return to this question in the last section (discussion) of this paper.

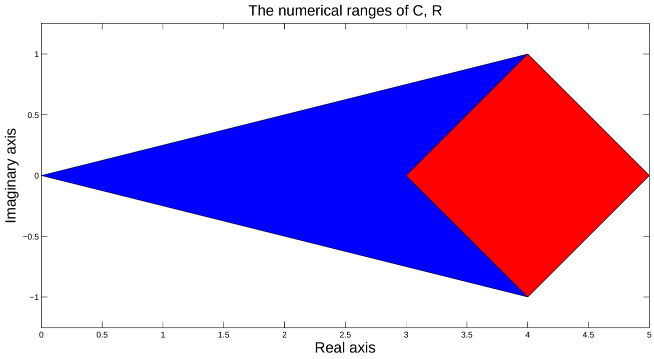

FIG. 3.1: Numerical Range of $W(R) = co(W(C) \cup (0))$ in blue and red, $W(C)$ in red.

4. The angular numerical range of EP operators and matrices

As we have seen, the origin is a sharp point of $W(T)$ in the above examples. This property leads us to the notion of the Angular Numerical Range, a cone of the complex plane and we will discuss it in this section. The angular opening of the smallest angular sector including $W(T)$ is the field angle $\Theta(W(T))$ and is also the angular opening of the Angular Field of values $F(T)$:

Definition 4.1. The Angular Field of Values (or Angular Numerical range) of an operator T, denoted by $F(T)$ is the subset of the complex plane $\mathbf{C}$ defined as:

$$F(T) = \{\langle Tx, x\rangle : \quad x \in \mathbf{C}^n, x \neq 0\}.$$

It is clear that $F(T)$ is a cone or a sector in $\mathbf{C}$ having its top at the origin. It is known that $F(T)$ is spanned by $W(T)$ and that $\Theta(W(T)) = \Theta(F(T))$.
In the case of nonsingular operators or matrices, we know that there is no connection between the numerical range $W(T)$ and of $W(T^{-1})$. But when it comes to the angular numerical range, $F(T)$, then $F(T) = \overline{F(T^{-1})}$, where $\overline{F(T^{-1})}$ denotes the conjugate set. (See [7] page 66). It is clear from this result that for real matrices we have $F(T^{-1}) = F(T)$.
But, what happens if we replace T^{-1} with $T^{\dagger}$ in the case of a singular operator or matrix?
In this case we can notice the following: When T is singular, then 0 always belongs to $W(T)$ (take a vector x in the kernel of T). In the general case when 0 is in the interior of the numerical range, then $F(T)$ is the entire complex plane, and then $F(T) = F(T^{\dagger}) = \mathbf{C}$.
We will examine the non trivial case, the class of EP operators, because then, 0 is an angular point of the numerical range. In this case we have the following theorem:

Theorem 4.1. *When T is a singular EP operator, $T = U(A \oplus 0)U^*$, $\quad 0 \notin W(A)$. Then $F(T^{\dagger}) = \overline{F(T)}$. In the case of real matrices, it holds that $F(T^{\dagger}) = F(T)$.*

Proof. By taking the canonical form of T, there exists a unitary operator U and an invertible operator (or an $r \times r$ matrix) A, such that

$$T = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^*$$

Using this factorization, and using the fact that $F(T)$ is preserving congruence (See [7], page 13) we can see that $F(T) = F(A)$. Indeed, from the definition of $F(T)$, we have that (since $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$):

$$F(T) = F(A \oplus 0) = \begin{bmatrix} x^* & y^* \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^*Ax = F(A)$$

Now, using the fact that

$$T^\dagger = U \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$$

and that $F(T) = \overline{F(T^{-1})}$, we have that $F(T^\dagger) = F(A^{-1}) = \overline{F(A)} = \overline{F(T)}$. $\square$

In Figure 4.1 that follows we can see that the Field Angles and the Angular Numerical Ranges (the blue and green cones anchored in the origin) coincide for the matrices $R, R^\dagger$ used in Example 3.2, and also presented in Figure 3.1. As we can see, the angular opening of the sectors $F(R), F(R^\dagger)$ is the same:

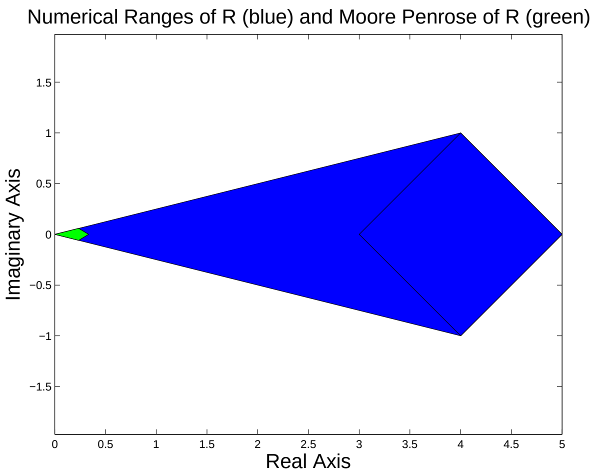

FIG. 4.1: Numerical Ranges and Angular Numerical Ranges of the real matrices $R, R^\dagger$ in blue and green respectively. We can see that $F(R) = F(R^\dagger)$.

Table 5.1: Different values of $d(W(A,T)), \Theta(W(A)), Var(A)$ where $T = U(A \oplus 0)U^*$

Matrix	$d(W(A,T))$	$\Theta(W(T))$	$Var(\partial(T))$
P	15.52	1.6562	3.4704
R	3	0.49	4.95
T	2.764	0.44	10.81
S	0.571	2.075	0.909

5. Discussion- Conclusion

In this work, we presented a detailed analysis of the numerical range of EP operators and matrices and its connection with their canonical form. We defined the distance $d(W(T,A))$ when T can be decomposed as $T = U(A \oplus 0)U^*$.
From all the above discussion, we can say that for EP operators and matrices, the Numerical range $W(T)$ can be seen as an pertrubation of $W(A)$ when $0 \notin W(A)$. But, to what extend is this pertrubation? One can think by geometric intuition that in general, when $d(W(A,T))$ gets larger then the field angle $\Theta(W(T))$ gets smaller, while the variance of the numerical boundary gets larger also.
Investigating this assumption, we used some numerical examples of matrices with different sizes and various values of the distance $d(W(A,T))$ defined in this section. In the following table we can see the corresponding values for the matrices presented in the examples of this paper. The matrix P is a random 7×7 EP matrix of the form $P = U(P_1 \oplus 0)U^*, 0 \notin W(P_1)$.

From the results presented in the above table we can say that this question needs a deeper investigation, since we can see that the general trend that we expected is satisfied but not always. More factors have to be taken into account in order to have a more clear picture of the connection between $d(W(A,T)), \Theta(W(T))$. This gives us a motivation for the extention of this work in the future.
The other topic that was presented in this work was the Angular Numerical range $F(T)$. The connection of $F(T)$ and $F(T^{-1})$ has already been investigated (see [7]), and in this work we extended this result for $F(T^\dagger)$ for the class of EP operators and matrices.

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COMPARATIVE STUDY OF MUTATION OPERATORS IN THE GENETIC ALGORITHMS FOR THE K-MEANS PROBLEM *

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Abstract. The k-means problem and the algorithm of the same name are the most commonly used clustering model and algorithm. Being a local search optimization method, the k-means algorithm falls to a local minimum of the objective function (sum of squared errors) and depends on the initial solution which is given or selected randomly. This disadvantage of the algorithm can be avoided by combining this algorithm with more sophisticated methods such as the Variable Neighborhood Search, agglomerative or dissociative heuristic approaches, the genetic algorithms, etc. Aiming at the shortcomings of the k-means algorithm and combining the advantages of the k-means algorithm and revolutionary approach, a genetic clustering algorithm with the cross-mutation operator was designed. The efficiency of the genetic algorithms with the tournament selection, one-point crossover and various mutation operators (without any mutation operator, with the uniform mutation, DBM mutation and new cross-mutation) are compared on the data sets up to 2 millions of data vectors. We used data from the UCI repository and special data set collected during the testing of the highly reliable semiconductor components. In this paper, we do not discuss the comparative efficiency of the genetic algorithms for the k-means problem in comparison with the other (non-genetic) algorithms as well as the comparative adequacy of the k-means clustering model. Here, we focus on the influence of various mutation operators on the efficiency of the genetic algorithms only..

Keywords: k-means problem; Variable Neighborhood Search; genetic algorithms; cross-mutation operator.

1. Introduction

With the increasing popularity of 5G commercialization and the development of the IT hardware, data has grown exponentially in recent years. According to

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statistics, the global data usage has reached 40 Zb [1], and researchers in various fields are increasingly interested in big data research. One of the most important directions of intelligent data processing is cluster analysis. Clustering algorithm is a technique for statistical data analysis and is widely used in many fields, including machine learning, data mining, image analysis, and biological information processing. In the commercial field, it can be used in a recommendation system to improve efficiency of the company, and it can also solve problems such as reducing the size of the initial data set and pattern recognition [2, 3]. Cluster analysis, also known as group analysis or automated grouping, is a statistical analysis method performed on a set of several patterns (data set), usually a pattern which is a metric vector, or a point in a multidimensional space. In this paper, we call them data vectors. The criterion to estimate the result of most clustering algorithms is the distance between the elements in the same cluster and the distance between different clusters [4]. According to the principle of clustering objects, similar objects are grouped into a subset so that objects in the same subset are as close as possible, and the distance between different subsets is as far as possible.

The k-means algorithm is one of the most popular clustering algorithms due to its simplicity and remarkable effect [5]. However, the k-means algorithm is a local search method which depends on the initial solution that is given or generated randomly. Genetic algorithms have global optimization capabilities. The Genetic k-Means algorithm is designed by combining the advantages of both. This article summarizes the current research status of genetic clustering algorithms based on the k-means optimization model.

Although the idea of the k-means clustering goes back to Steinhaus in 1957 [6], the term "k-means" was first used by James MacQueen in 1967 [7]. The k-means algorithm is one of the most widely used clustering algorithms due to its simple and convenient implementation principle and good experimental results. This algorithm accepts the parameter k, randomly selectsk cluster centers (called centroids) among the data vectors (if they are not given), and calculates the distance between the N data in the data set and the closest center points. This experiment uses Euclidean distance in a d-dimensional space [8]:

$$d(X,Y) = \sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}.$$

Here, $X = (x_1, ..., x_d)$ and $Y = (y_1, ..., y_d)$ are two given points (vectors).

The minimized objective function (sum of squared distances also called sum of squared errors, SSE) of the k-means optimization model and algorithm is as follows:

$$f(C_1, ..., C_k) = \sum_{i=1}^{N} min_{j=\overline{1,k}} d^2(A_i, C_j). \tag{1.1}$$

Here, $A_1, ..., A_N$ are the clustered data vectors, and $C_1, ..., C_k$ are the searched cluster centers (centroids) which must be found.

According to the obtained results, each data sample is assigned to its nearest cluster, and the cluster center with the newest average value of the data samples in each cluster is calculated. The cluster center is updated repeatedly until the convergence condition is reached. The running process of the k-means algorithm is as follows:

Step 1: Select k objects from the data object as the initial cluster center (this step is optional, only is the initial solution is not given);

Step 2: Calculate the distance of each object to each cluster center separately, and assign the object to the nearest cluster;

Step 3: Recalculate the center of k clusters after all object assignments are completed;

Step 4: Compare with the k cluster centers obtained in the previous calculation. If the cluster center changes, then turn to Step 2, otherwise output the clustering results.

The advantages and disadvantages of the k-means algorithm are obvious. The advantage is that the algorithm is simple and the convergence speed is fast. The results (local optima) obtained with different initial cluster centroid positions may vary significantly. It is easy to get a local optimal solution instead of a global optimal solution.

In order to solve the shortcomings of the k-means algorithm, the k-means++ algorithm proposed by Arthur in 2007 improved the initialization step of the k-means algorithm [9]. This improvement can be intuitively understood so that the k initial cluster centers should be separated from each other as much as possible. However, the k-means++ algorithm and similar "smart initialization" algorithms [10, 11] are still random search methods which fall into a local minimum.

The Genetic Algorithm (GA) was first proposed by Holland of the United States in the 1970s [12]. The algorithm was designed and proposed according to the evolutionary laws of organisms in nature. In 1967, Bagley, a student of Professor Holland [13], first proposed the term "Genetic Algorithm" in his doctoral dissertation and discussed the application of the GAs in games, but early research lacked guiding theory and the development of computing tools. In 1975, Holland et al. [14] proposed a model theory that is extremely important for the study of genetic algorithm theory.

The genetic algorithm has several basic frameworks of coding, fitness function, and initial group selection [15, 16, 17]. Many genetic algorithms for the k-means problem [18, 19, 3] use the direct coding: the chromosome (a solution in a population of solutions) is the set of the coordinates of the cluster centers (centroids).

The fitness function is used to express the adaptability of an individual to the environment. In this research, we use directly (1.1) as the fitness function.

The basic operation process of genetic algorithm is as follows [20]:

Step 1 (Initialization): set the evolution algebra counter $t = 0$, set the maximum evolution algebra T, and randomly generate n individuals as the initial group $P(0)$.

Step 2 (Individual evaluation): Calculate the fitness of each individual in the group $P(t)$.

Step 3 (Selection operation): Apply the selection operator to the group. The purpose of selection is to directly inherit the optimized individuals to the next generation or to generate new individuals through pairing and crossover to the next generation. The selection operation is based on the assessment of the fitness of the individuals in the group.

Step 4 (Crossover operation): Apply crossover operator to the group. The crossover operator plays a central role in genetic algorithms.

Step 5 (Mutation operation): Apply mutation operators to groups. That is, the gene values at certain loci of individual strings in the group are changed. After selection, crossover and mutation operations, the population $P(t)$ obtains the next generation population $P(t + 1)$.

Step 7 (Termination condition judgement): if $t = T$, the individual with the maximum fitness obtained in the evolution process is used as the optimal solution output to terminate the calculation. Instead of the limitaion of generations, the time limitation can be used.

Since the k-means is an NP-hard optimization problem [21, 22, 23], the results are easily stuck by the local optimal solution. The genetic algorithms are popular instruments for global optimization. Krishna and Morty proposed a new clustering method called Genetic K-means Algorithm (GKA) [17] combining the global search capabilities of genetic algorithms with traditional k-means algorithms.

The flow of GA-k-means algorithm [17, 24, 25, 26, 27] is as follows.

Step 1: K samples are randomly selected from data set as the cluster center, and the k cluster centers are considered as a chromosome. This operation is repeated n times to obtain a population of size n.

Step 2: Use ordinary k-means algorithm to cluster data set with each chromosome as the cluster center. Get the new clustering center and the fitness function value corresponding to each chromosome.

Step 3: Get the next generation through selection, crossover, and mutation operations, and retain the best chromosomes from the previous generation. Repeat Steps 2 and 3 until the termination condition is met.

Each chromosome is a sequence of real numbers representing k cluster centers. For a d-dimensional data set, the length of the chromosome is kd. The sum of the squared distances within the clusters in the data set (1.1) is used as the fitness function.

Such algorithms can use two main selection operations. The first one is the most commonly used method of proportional (roulette wheel) selection. The main idea is that the probability that an individual is selected depends directly on the corresponding fitness of the individual. Another one is tournament selection strategy.

For the k-means problem, after several iterations, the fitness function values of all individuals become very close to each other. Thus, the roulette wheel selection is inefficient, and we use the tournament approach. The algorithm randomly selects 3 chromosomes from the population, and then selects an individual with the highest fitness value from these 3 chromosomes [28]. Since we need to select two "parent" chromosomes for the crossover operator, the second one can be chosen using the same approach or selected randomly from the population with equal probabilities.

The crossover is a random process. The first type is a single-point crossover [24]. A point is randomly selected as the crossover point in the range of 1 to chromosome length, and the two chromosomes are exchanged to the right of the cross point. Two different points are randomly selected as the intersection point in the range of 1 to chromosome length, and the middle part of the two points is exchanged to obtain two new offspring. One of them is randomly selected to survive and enter next generation. The third type is uniform crossover. For each node on the chromosome, there is a certain chance that the crossover operation will occur. After the entire process is completed, two new chromosomes will be obtained, and one will be randomly selected to survive. However, in this paper we use the one-point crossover only and focus on the efficiency of various mutation operators.

In the genetic algorithm, the mutation operation is to imitate the mutation link of biological evolution in nature to change the individual. Although the chance of mutation is relatively small, it is an indispensable link to generate new species. Constantly fine-tune the new individuals generated by the crossover operator to increase species diversity and search area. Traditionally, such genetics algorithms algorithms with real-coded chromosomes for the k-means problem do not use any mutation operators [18, 3] mutation operations commonly used in other GAs. However, this reduces the population diversity and may make the final results premature and converge prematurely [25].

We compare several mutation operators and propose a simple idea of using the one-point crossover operator as the mutation operator. The efficiency of such approach is proved experimentally.

Our comparison is performed only with respect to squared distance between points and centroids (1.1). There are lots of other clustering quality measures and lots of clustering models (minimized or maximized objective functions), and there is a perpetual question, which clustering model is more adequate. In this paper, we do not compare the adequacy of the clustering models. We do not use any internal or external criteria [29, 30, 31, 32, 33] which allow us to compare the adequacy (preciseness) of various clustering models. The only aim of this research was to improve the solution of the k-means optimization problem (not the accuracy of the clustering result), i.e., to build algorithms which allow us to obtain better values of the sum of squared distances.

2. Mutation in the Genetic k-Means Algorithm

In comparison with the crossover operator, the mutation operator in standard genetic algorithms is usually considered as a secondary operator with low probability [34]. Nevertheless, some evolutionary algorithms without any crossover operator are able to work better than standard genetic algorithms due to mutation and selection [35, 36, 37, 38].

The majority of the bibliographical sources describe the evolutionary algorithms for the k-means problems which use integer or binary chromosome encoding [39, 40, 30, 41, 42, 43], and thus these approaches cannot be implemented in our study because the considered greedy heuristic algorithms use the real encoding only. The other part of the sources propose the algorithms which actually solve a problem other than k-means (other than sum of squared distances minimization) while we focused on the improvement of the k-means problem solution only without any change in the clustering model. The third part of the sources including authors' papers do not use any mutation operators at all or use special operators called mutation which actually run local search algorithms [18, 3, 44, 19, 45].

The mutation operator [46] changes each allele (a part of the chromosome) a_n ($n = 1, ..., k$) to a new value a'_n (a'_n might be equal to a_n) with probability M_P independently, where $0 < M_P < 1$ is a parameter called the mutation probability that is specified by the user. weak mutation, average mutation, and strong mutation. Usually, the probabilities are $1/5n$ (weak mutation) , $1/n$ (average mutation), and $5/n$ (strong mutation), and n is the length of the individual (the number of alleles in the chromosome). There are two purposes for introducing mutations into genetic algorithms: one is to make the genetic algorithm have local random search ability. When the genetic algorithm is close to the optimal solution neighborhood through the crossover operator, the local random search ability using the mutation operator can accelerate the convergence to the optimal solution. Obviously, the probability of mutation in this case should be a small value, otherwise the building blocks close to the optimal solution will be destroyed by the mutation. The second is to enable genetic algorithms to maintain group diversity to prevent immature convergence. The termination condition of the genetic algorithm is that the individual's fitness reaches a preset threshold, or the fitness does not rise any more, or the number of iterations reaches the preset algebra.

In [26], Sheng proposed a simple inversion approach. Generate randomly a number within 0-1. If this number is less than the probability of mutation, then perform an inversion operation on a value in the chromosome. This method has certain limitations and can only be used for populations whose chromosomes are binary-coded, but not for populations whose chromosomes are real-coded. For the real-valued chromosomes, only few approaches were proposed.

Maulik proposed the Uniform random mutation in [27]. Randomly generate a number from 0 to 1, if the number is less than mutation probability, the point on the chromosome will mutate. The mutation strategy is as follow. Randomly generate the number b from 0 to 1, if the value at a gene (cenreoid coordinate) position is v, after mutation it becomes:

$$v \leftarrow \begin{cases} v \pm 2bv, & v \neq 0, \\ \pm 2b, & v = 0 \end{cases} .$$

Positive and negative signs have the same probability. This mutation operation is simple, however, the disadvantage is that when the value of a certain data is very large or very small, the impact of this mutation will also be very large or very small, which does not conform to the principle of mutation. If the range of the initial data is very large, the gap between the mutated data and the initial data will be very large.

In 1999, Krishna and Murty proposed a mutation strategy called distance-based mutation (DBM) [28, 46]. Authors believed that the mutation must change the allele value according to the distance of the cluster centroid from the corresponding data point. Each allele (part of the chromosome) corresponds to a data point, and its value represents the cluster to which the data point belongs. Define operators so that if the corresponding cluster center is closer to the data point, the allele value is more likely to be changed to the cluster number. After determining that an allele is about to mutate, replace the allele with a randomly selected value from the following distribution:

$$P_j = \frac{c_m d_{max} - d_j}{\sum_{(i=1)}^{K} (c_m d_m ax - d_i)}.$$

Here, $d_j = d(A_i, C_j)$ is the Euclidean distance between point x_i and centroid c_j, and c_m is a constant.

3. Idea of the Cross-Mutation Operator

The idea of our new mutation operator is very simple: to implement the crossover operator to the individual being mutated and to a randomly generated individual.

We call this new mutation operator the cross-mutation. A randomly generated result improved by the standard k-means algorithm is used as an input of the mutation operator. The solution being mutated is the second input. For this two input solutions (chromosomes), we implement the single-point crossover and then run the k-means algorithm again to improve the result. Similar ideas are used in known variable neighborhood search algorithms [47]. Observe the performance of the genetic clustering algorithm using cross-mutation-like operators by comparing the cross-mutation-like operators with the other three mutation operators.

The cross-mutation operator can be described as follows:

Required: Chromosome to be mutated S.

Step 1: Randomly generate a chromosome (set of centroids) S';

Step 2: $S' \leftarrow kmeans(S')$;

Step 3: $S \leftarrow crossover(S, S')$;

Step 4: $S = kmeans(S)$.

Our computational experiments show that the genetic algorithms based on this idea are able to outperform both the algorithms without any mutation and the algorithms with the uniform random mutation and DBM algorithms.

4. Computational Experiments

In our experiments, we used data sets from the UCI repository [48] and data collected during the process of testing the highly reliable electronic components (semiconductor devices 140UD25) [49] in a specialized testing center [50]. The aim of clustering the highly reliable semicinductor devices is to detect the homogeneous production batches in a mixed lot of the shipped devices.

The semiconductor device data set contains 1125 objects of dimensionality 18 (18 tests), and each dimension represents a certain attribute of the tested device.

Five clustering algorithms are used: k-means algorithm in the multi-start mode, Genetic k-Means algorithm clustering algorithm without any mutation operator, Genetic k-means algorithm with the uniform random mutation operator, Genetic k-means algorithm with cross-mutation operator, and the DBM genetic clustering algorithm.

For distance measure, Euclidean distance. For all data sets, we used the 0-1 normalization. All algorithms ran 30 times limited by 150 generations. Population size is equal to 20.

All the experiments were performed with the average mutation probability $1/n$ where n is the length of the chromosome ($n = k$ for the Genetic k-Means algorithm).

As the result of a randomized algorithm may be accidental. In order to make the experimental results statistically significant, run the entire experiment 30 times and record the experimental results. The averaged results for the semiconductor device data set are summarized in Tables 4.1 and 4.2.

Table 4.1: Computational experiments with semiconductor testing data set (1125 data vectors of dimensionality 18), 150 generations, 30 attempts

Mutation strategy	Obj. function (1.1) value Average	Median
Without mutation	92.219	92.255
Uniform random mutation	91.862	91.905
Crossover-like mutation	91.638	91.635
DBM mutation	91.909	91.815

Table 4.2: Statistical significance of the difference in results (Mann–Whitney U test) for the semiconductor testing data set, 30 attempts

Mutation strategies	Significance level	Conclusion
Without mutation vs. uniform	0.012	Significant difference
Without mutation vs. cross-mutation	0.005	Significant difference
Uniform mutation vs. cross-mutation	0.011	Significant difference
DBM vs. uniform	0.110	Difference is insignificant
DBM vs. cross-mutation	0.008	Significant difference

Fig. 4.1 shows that the convergence speed of two algorithms is almost the same. However, the median convergence speed of 30 runs is better for the GA with our new mutation operator, and this difference is statistically significant.

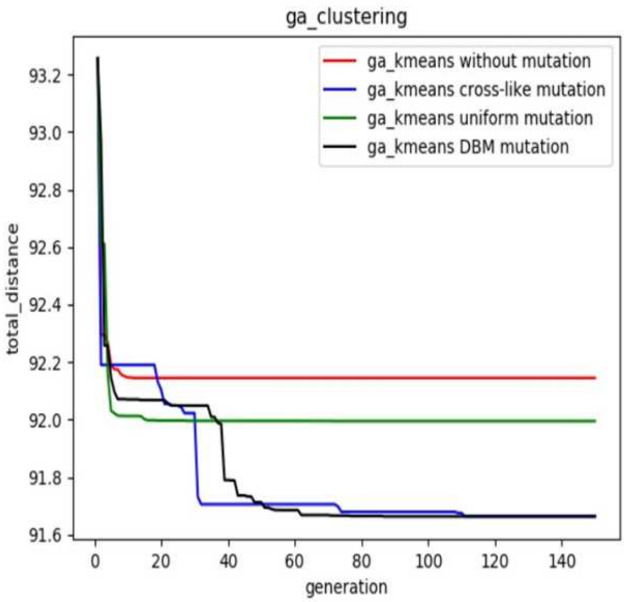

Fig. 4.1: Comparative convergence speed of four genetic algorithms with various mutation operators on the semiconductor testing data set

The advantage of the new mutation operator over the three other variants is statistically significant.

Clustering algorithms can be used in recommendation systems, based on user portraits, to identify products or videos that may be of interest to users. For example, in the e-commerce industry and short video industry that have emerged in recent years, by using real-time recommendation systems using big data tech-

Table 4.3: Computational experiments with household power consumption data set (2075259 data vectors of dimensionality 6), 150 generations, 30 attempts

Mutation strategy	Obj. function (1.1) value Average	Median
Without mutation	13468.54	13470.05
Uniform random mutation	13485.99	13487.90
Crossover-like mutation	13457.07	13457.70
DBM mutation	14674.96	13468.35

nology, by analyzing user behavior, making user portraits and clustering users to recommend more products to users, this method brought a lot of revenue to many companies [51, 52].

The second experiment uses data on the household power consumption. The power consumprion information may be a simplest but important indicator of the behaviour of people. The second data set contains data of electric power consumption in the households with a one-minute sampling rate over a period of almost 4 years [48]. Different electrical quantities and some sub-metering values are available. This archive contains 2075259 measurements gathered in a house located in Sceaux (7km of Paris, France) between December 2006 and November 2010 (47 months). Each data contains 8 attributes, namely data, time, global active power, global reactive power, voltage, submeterings. Date and time attributes were removed, and the other attributes were 0-1 normalized.

The results of running our algorithms are shown in Fig. 4.2 and Tables 4.3, 4.4. As it can be seen from the above figure, the genetic clustering algorithm without mutation operator has the fastest convergence speed, and has converged in about 10 generations, and the result stays at 13471.5. For the DBM mutation, it started to decline particularly fast. From the 5th generation to the 25th generation, the downward trend began to become slow. It converged around the 65th generation, and the result stayed at 13469.3. The uniform mutation operator declined very rapidly before the 15th generation, and the downward trend slowed down from the 15th generation to the 35th generation, it converged at the 75th generation. The cross-like mutation operator also had a process of hormonal decline before the 5th generation, and it has been steadily decreasing after the 5th generation until it converges at the 110th generation, and the result is better than the other three operators. Repeat this procedure for 15 times, and record the final results of various mutation operators each time and record them.

For this comparatively large data set, the overall conclusion is the same: our new mutation operator outperforms the other three versions of the genetic algorithms.

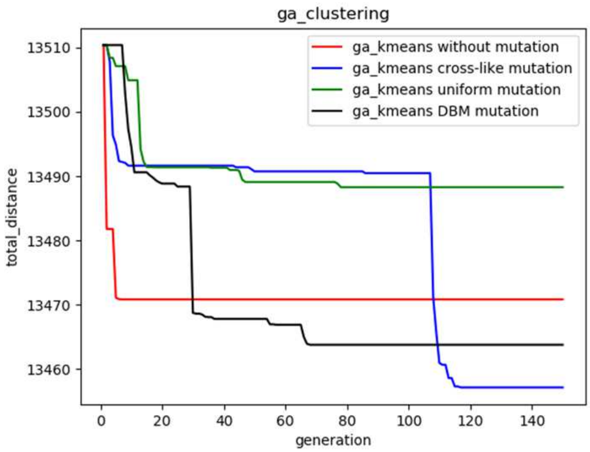

FIG. 4.2: Comparative convergence speed of four genetic algorithms with various mutation operators on household power consumption data set [48]

Table 4.4: Statistical significance of the difference in results (Mann–Whitney U test) for the household power consumption data set, 30 attempts

Mutation strategies	Significance level	Conclusion
Without mutation vs. uniform	0.013	Significant difference
Without mutation vs. cross-mutation	0.005	Significant difference
Uniform mutation vs. cross-mutation	0.011	Significant difference
DBM vs. uniform	0.010	Significant difference
DBM vs. cross-mutation	0.008	Significant difference

5. Conclusions

The modern scientific literature offers only few approaches to building the mutation operator for the genetic algorithms with real coded chromosomes for solving the k-means problem. Traditionally, these algorithms do not use any mutation. However, the simple idea of using the same single-point crossover operator for both crossover and mutation is able to improve the results of the genetic algorithm. In this case, the one-point crossover is applied to the chromosome being mutated and a randomly generated chromosome improved by running the k-means algorithm. This new mutation operator is efficient for both small and large data sets.

However, investigation of the new operator efficiency with various mutation probabilities and various quantity of clusters as well as its applicability for the other crossover operators are subject of our further research.

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STATISTICAL INFERENCE FOR GEOMETRIC PROCESS WITH THE GENERALIZED RAYLEIGH DISTRIBUTION

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Abstract. In the present paper, the statistical inference problem is considered for the geometric process (GP) by assuming the distribution of the first arrival time with generalized Rayleigh distribution with the parameters α and λ. We have used the maximum likelihood method for obtaining the ratio parameter of the GP and distributional parameters of the generalized Rayleigh distribution. By a series of Monte-Carlo simulations evaluated through the different samples of sizes - small, moderate and large, we have also compared the estimation performances of the maximum likelihood estimators with the other estimators available in the literature such as modified moment, modified L-moment, and modified least squares. Furthermore, wehave presented two real-life datasets analyses to show the modeling behavior of GP with generalized Rayleigh distribution.

Keywords: Monotone processes; non-parametric estimation; parametric estimation; stochastic process; data with trend.

1. Introduction

In 1988, Lam [18] introduced the geometric process (GP) as a simple monotonic stochastic process. In order to model a successive inter-arrival times dataset with a monotone trend, the GP is a quite important alternative to the alpha series process and the nonhomogeneous Poisson process with a monotone intensity function. Since it has a simple form which is easily applied to the many real-life problems from different areas such as science, health, engineering etc., see [17], its popularity increases day by day according to its alternatives. Some key features of the GP and its advantages, which the GP provides in the modeling of the arrival times data with a trend, studied by Lam [16], Lam [18], Lam et al.[19] and Braun et al. [9], [10]. The GP is given by the following definition, see [17].

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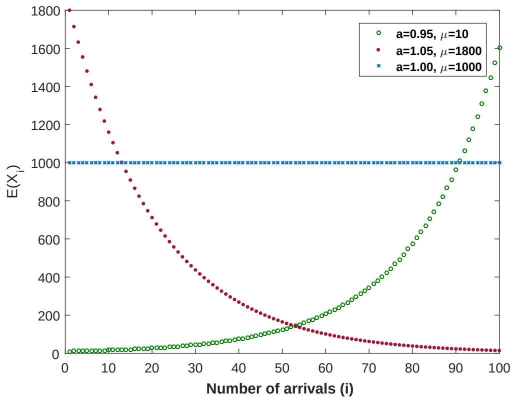

FIG. 1.1: Behavior of the GP

Definition 1.1. Let X_i be the arrival time between the $(i-1)$th and ith events of a counting process $\{N(t), t \geq 0\}$ for $i = 1, 2, ...$ The process $\{X_i, i = 1, ..., n\}$ is said to be a GP with parameter a if there exists a real number $a > 0$ such that $Y_i = a^{i-1}X_i, i = 1, 2, ...,$ are independently and identically distributed (iid) random variables which have any continuous distribution supported on positive real interval. Where a is called the ratio parameter of the GP.

In a general concept, there are three important parameter types in a GP. The first of these parameter types is the ratio parameter a. The second type of them is mean and variance of the first arrival time X_1. In the GP, determining the mean and variance of the first arrival time is quite important because of the fact that the means and variances of the random variables $X_i, i = 1, 2, ...$ are easily represented by the mean and variance of the first arrival time. Assume that $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$ for a GP with the ratio parameter a. By these notations, the mean and variance of the random variable X_i, $(i = 1, 2, \cdots, n)$, are given by following forms:

$$E\left(X_i\right) = \frac{\mu}{a^{i-1}} \ ,i = 1, 2, ... \tag{1.1}$$

$$Var\left(X_i\right) = \frac{\sigma^2}{a^{2(i-1)}}, i = 1, 2, ... \tag{1.2}$$

Hence, by using the relation given by equation 1.1, we can provide Figure 1.1 to illustrate the monotonic behavior of the GP, where the $E\left(X_i\right)$ is plotted against the arrival number $i, (i = 1, 2, \cdots,)$ for a fixed μ.

By the Figure 1.1, the process has a monotone increasing behavior when $a < 1$ and has a monotone decreasing behavior when $a > 1$. If $a = 1$ then the process is a Renewal process (RP) [17].

The last type of the important parameters is the distributional parameters of the first occurrence time X_1. In the literature, one can find many published studies related to the parameter estimation problem for both the ratio parameter a and distributional parameters of GP. Lam [16] obtained some non-parametric estimators for parameter a. Several studies that take into account some specific lifetime distributions for first occurrence time X_1 and focus on estimating the distributional parameters of GP are as follows: Gamma [12], Weibull [3], log-normal [18], inverse Gaussian [13], Lindley [7], power Lindley [4], Rayleigh [8], two-parameter Rayleigh [5] and two-parameter Lindley [6] distribution for the GP.

The main motivation of this study is to estimate the parameters of GP when the distribution of first occurrence time is Generalized Rayleigh (GR) also known as two-parameter Burr Type X distribution. We are motivated to the GR distribution for the distribution of the first occurrence time because it is an important alternative to the other famous distributions used in reliability analysis such as the Gamma, Weibull, exponential. In accordance with the purpose of this study, we employ the maximum likelihood (ML), modified moments (MM), modified L-moments (MLM) and modified least-squares (MLS) methods to obtain estimators of the unknown parameters of GP.

The rest of the paper is organized as follows: In section 2, we shall overview the GR distribution. In section 3, we shall obtain the ML estimators of the unknown parameters of GP with the GR distribution. Furthermore, we will investigate some modified estimators for distributional parameters of GP considering the non-parametric estimate of the ratio parameter a. In section 4, some Monte-Carlo simulation studies which compare the efficiencies of the ML estimators obtained in section 3 with the MM, the MLM, and the MLS estimators are performed. Section 5 covers two real-life examples which illustrate the modeling capability of a GP with GR distribution. Section 6 concludes the study.

2. An overview to GR distribution

The GR distribution, also known as two-parameter Burr Type X distribution, was originally studied by Surles and Padgett [22]. Later on, the distribution was renamed as the GR by Raqab and Kundu [21]. The GR is a commonly used probability model in the modeling of positive and non-symmetric data observed from various areas such as communication, health, engineering, reliability etc. Since the distribution is applicable to the modeling of data measured from a wide variety of areas, the interest in the theory and methods related to GR distribution is progressive.

The probability density function (pdf) of the GR distribution with the parameters α and λ is

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\left(1 - e^{-(\lambda x)^2}\right)^{\alpha-1},\ x > 0, \tag{2.1}$$

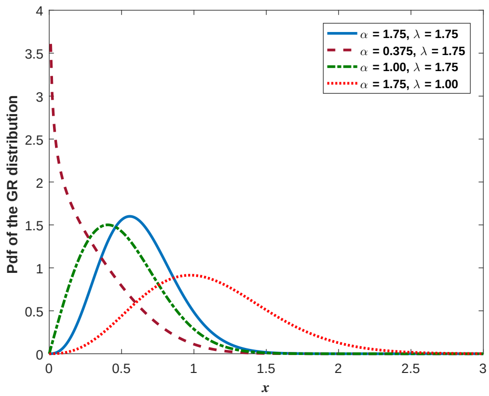

FIG. 2.1: Pdf of the GR distribution for the different values of the parameters

and the corresponding cumulative distribution (cdf) is

$$F\left(x,\alpha,\lambda\right)=\left(1-e^{-(\lambda x)^2}\right)^{\alpha},\ \ x>0, \tag{2.2}$$

where α and λ are the positive and real valued scale and shape parameters of the distribution, respectively [14]. When $\alpha = 1$, the GR distribution is a Rayleigh with parameter λ. If $\lambda = 1$, then the distribution is reduce to the one-parameter Burr Type X distribution with parameter α. The GR distribution is a unimodal and its pdf is skew to the right when $\alpha > \frac{1}{2}$ and is a decreasing function otherwise [21]. Figure 2.1 below lucidly show the behaviors of the pdf of the GR distribution discussed in here.

The expectation and variance of the GR distribution are not available in the explicit forms, however, they can be easily obtained for selected values of the parameters by using a numeric method.

3. Inference for GP

In this section, in addition to obtaining the ML estimators of the GP with GR distribution, we will also investigate some modified estimators when the ratio parameter of the process is estimated by using a non-parametric estimator.

3.1. ML Estimates

Let us $X_1, X_2, ..., X_n$ be a random sample taken from a GP with ratio a and $X_1 \sim GR(\alpha, \lambda)$ with the pdf (2.1). By considering the equation (2.1) and Definition 1.1, the log-likelihood function for the random variables X_i, $(i = 1, 2, ..., n)$ can be written as

$$\begin{aligned} \ln L(a,\alpha,\lambda) &= n\,(n-1)\ln a + n\ln 2 + 2n\ln\lambda + n\ln\alpha - \lambda^2 \textstyle\sum_{i=1}^{n}\left(a^{i-1}x_i\right)^2 \\ &\quad + \textstyle\sum_{i=1}^{n}\ln x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\left(\lambda a^{i-1}x_i\right)^2}\right). \end{aligned} \tag{3.1}$$

If the first derivatives of Equation (3.1) according to a, α and λ are taken, we have

$$\frac{\partial \ln L(a,\alpha,\lambda)}{\partial a} = \tfrac{(n-1)n}{a} + 2(\alpha-1)\textstyle\sum_{i=1}^{n}\frac{(i-1)\lambda^2 a^{2i-3}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} = 0 \tag{3.2}$$

$$\frac{\partial \ln L(a,\alpha,\lambda)}{\partial \lambda} = \tfrac{2}{\lambda} + (\alpha-1)\textstyle\sum_{i=1}^{n}\frac{2\lambda a^{2i-2}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} = 0 \tag{3.3}$$

$$\frac{\partial \ln L(a,\alpha,\lambda)}{\partial \alpha} = \tfrac{n}{\alpha} + \textstyle\sum_{i=1}^{n}\log\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right) \tag{3.4}$$

analytical expressions for the ML estimators of the parameters a, λ and α can not be obtained from equations (3.2)-(3.4). However, equations (3.2)-(3.4) can be simultaneously solved using a numerical method such as well-known Newton's method.

Let $\theta = \begin{bmatrix} a \\ \lambda \\ \alpha \end{bmatrix}$ be the parameter vector and likelihood equations given by (3.2)-(3.3) and (3.4) are represented by a gradient vector $\nabla(\theta)$ as

$$\nabla(\theta) = \begin{bmatrix} \frac{\partial \ln L(a,\alpha,\lambda)}{\partial a} \\ \frac{\partial \ln L(a,\alpha,\lambda)}{\partial \lambda} \\ \frac{\partial \ln L(a,\alpha,\lambda)}{\partial \alpha} \end{bmatrix}. \tag{3.5}$$

Thus, in order to estimate of the parameter vector θ, the iterative method given by 3.6 can be used by starting from an initial estimation such as $\hat{\theta}_0$.

$$\theta_{m+1} = \theta_m - H^{-1}(\theta_m)\nabla(\theta_m) \tag{3.6}$$

where $H^{-1}(\theta)$ is the inverse of the Hessian matrix $H(\theta)$.The elements of the matrix $H(\theta)$ are the second derivatives of the log-likelihood function (3.1) with respect to a, λ and α. Let h_{ij} be the $(i,j)\,th$ $(i,j=1,2,3)$ element of the matrix $H(\theta)$. The h_{ij}'s are obtained as below

$$\begin{aligned} h_{11} = &-\tfrac{(n-1)n}{a^2} + (\alpha-1)\textstyle\sum_{i=1}^{n}\left(\frac{(2i-3)(2i-2)\lambda^2 a^{2i-4}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}\right. \\ &\left. -\frac{(2i-2)^2\lambda^4 a^{4i-6}x_i^4 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} - \frac{(2i-2)^2\lambda^4 a^{4i-6}x_i^4 e^{-2\lambda^2 a^{2i-2}x_i^2}}{\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right)^2}\right) \end{aligned} \tag{3.7}$$

$$h_{12} = (\alpha-1)\sum_{i=1}^{n}\left(\frac{2(2i-2)\lambda a^{2i-3}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}\right.$$

$$\left.-\frac{2(2i-2)\lambda^3 a^{4j-5}x_i^4 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}-\frac{2(2i-2)\lambda^3 a^{4j-5}x_i^4 e^{-2\lambda^2 a^{2i-2}x_i^2}}{\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right)^2}\right) \tag{3.8}$$

$$h_{13} = \sum_{i=1}^{n}\frac{(2i-2)\lambda^2 a^{2i-3}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} \tag{3.9}$$

$$h_{22} = -\frac{2}{\lambda^2}+(\alpha-1)\sum_{i=1}^{n}\left(\frac{2a^{2i-2}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}-\right.$$

$$\left.\frac{4\lambda^2 a^{4i-4}x_i^4 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}-\frac{4\lambda^2 a^{4i-4}x_i^4 e^{-2\lambda^2 a^{2i-2}x_i^2}}{\left(1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}\right)^2}\right) \tag{3.10}$$

$$h_{23} = \sum_{i=1}^{n}\frac{2\lambda a^{2i-2}x_i^2 e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}}{1-e^{\lambda^2\left(-a^{2i-2}\right)x_i^2}} \tag{3.11}$$

$$h_{33} = -\frac{n}{\alpha^2}. \tag{3.12}$$

Note that inverse of the matrix H is calculated as

$$H^{-1} = \frac{1}{Det\,(H)}\begin{bmatrix} h_{22}h_{33}-h_{23}h_{32} & -h_{12}h_{33}-h_{13}h_{32} & h_{12}h_{23}-h_{13}h_{22} \\ -h_{21}h_{33}-h_{31}h_{23} & h_{11}h_{33}-h_{13}h_{31} & -h_{11}h_{23}-h_{21}h_{13} \\ h_{21}h_{32}-h_{22}h_{31} & -h_{11}h_{32}-h_{12}h_{31} & h_{11}h_{22}-h_{12}h_{21} \end{bmatrix},$$

where $Det\,(H) = h_{11}h_{22}h_{33} - h_{11}h_{23}h_{32} - h_{12}h_{21}h_{33} + h_{12}h_{31}h_{23} + h_{21}h_{13}h_{32} - h_{13}h_{22}h_{31}$ is determinant of the matrix H. In the Newton method, iterations continue until $\|\theta_{m+1}-\theta_m\| < \varepsilon$ where ε is a predetermined small constant and $\|.\|$ is the Euclidean norm of a vector. Thus, ML estimators of the parameters of GP with GR distribution, say $\hat{a}_{ML}, \hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$, are obtained from respective elements of θ_{m+1}.

Now we investigate the asymptotic features of the estimators $\hat{a}_{ML}, \hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$. The joint distribution of $\hat{a}_{ML}, \hat{\alpha}_{ML}$ and $\hat{\lambda}_{ML}$ is asymptotic-Normal (AN) with mean vector (a, λ, α) and covariance I^{-1}, where matrix I refers to Fisher information defined as

$$I = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a^2}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\lambda}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\alpha}\right) \\ E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\lambda}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial\lambda^2}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial\lambda\partial\alpha}\right) \\ E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial a\partial\alpha}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial \ln L(a,\lambda,\alpha)}{\partial\alpha^2}\right) \end{bmatrix}. \tag{3.13}$$

The elements of the matrix I are written from elements of the Hessian matrix.

3.2. Modified Methods

Lam [16] introduced a non-parametric estimator to estimate only the ratio parameter of the process without making a specific distribution assumption for the GP. The non-parametric estimator of the ratio parameter a is given by, see [16],

$$\hat{a}_{NP} = \exp\left(\frac{6}{(n-1)\,n\,(n+1)}\sum_{i=1}^{n}(n-2i+1)\ln X_i\right). \tag{3.14}$$

The distributional parameters of the GP are easily estimated using the available estimators in the literature when the ratio parameter a is estimated as $\hat{a}_{NP}$. This approximation is known as modified estimation technique in the literature. Now we examine the estimates of the distributional parameters of GP with the GR distribution by assuming that the parameter a is estimated as $\hat{a}_{NP}$. Let $X_1, X_2, ..., X_n$ be a random sample from a GP with ratio a and $X_1 \sim GR(\alpha, \lambda)$, and the parameter a is known as $\hat{a}_{NP}$, from Definition 1.1, we have

$$\hat{Y}_i = \hat{a}_{NP}^{i-1} X_i \tag{3.15}$$

and $\hat{Y}_i \sim GR(\alpha, \lambda)$. Thus, the MM, MLM, and MLS estimators of the α and λ parameters can be obtained as follows by taking into account the moments, L-moments, and least-squares estimators given in [14] and along with the predicted $\hat{Y}_i$.

MM Estimators: The MM estimate of the parameters α, say $\hat{\alpha}_{MM}$ can be obtained from numerical solution of the equation

$$\frac{\psi'(1) - \psi'(\alpha+1)}{\left(\psi(\alpha+1) - \psi(1)\right)^2} - \frac{V}{U^2} = 0 \tag{3.16}$$

where $U = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i^2$, $V = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i^4 - U^2$ and $\psi(.)$ is the digamma function, (cf. [1]). Also, by considering $\hat{\alpha}_{MM}$, MM estimates of the parameter λ, say $\hat{\lambda}_{MM}$ is obtained as follows

$$\hat{\lambda}_{MM} = \sqrt{\frac{\psi(\hat{\alpha}_{MM}+1) - \psi(1)}{U}} \tag{3.17}$$

MLM Estimators: The MLM estimates of the parameters α and λ, say $\hat{\alpha}_{MLM}$ and $\hat{\lambda}_{MLM}$, respectively, are obtained by numerical solution of non-linear equation

$$\frac{\psi(2\alpha+1) - \psi(\alpha 1)}{\psi(\alpha+1) - \psi(1)} - \frac{l_2}{l_1} = 0,$$

where $l_1 = \frac{1}{n}\sum_{i=1}^{n}\hat{Y}_{(i)}^2$ and $l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)\hat{Y}_{(i)}^4 - l_1$ and notation $\hat{Y}_{(i)}$ indicates the ith observation of ordered sample, where $i = 1, 2, ...n$.

MLS Estimators: The MLS estimates of the parameters α and λ, $\hat{\alpha}_{MLS}$ and $\hat{\lambda}_{MLs}$, respectively, are obtained by minimizing the quadratic function $Q(\alpha, \lambda)$

$$Q\left(\alpha,\lambda\right)=\sum_{i=1}^{n}\left(\left(1-e^{-\left(\lambda\hat{Y}_{(i)}\right)^{2}}\right)^{\alpha}-\frac{i}{n+1}\right)^{2} \tag{3.18}$$

with respect to α and λ.

For details on deriving these estimators, we refer to [14].

4. Monte-Carlo Simulation Study

In this section, we run some Monte-Carlo simulations to show the estimation performance of ML and modified estimators obtained in the previous section. The main goal of these Monte-Carlo studies, besides displaying the estimation performance of the ML estimators, compare its efficiency with the other estimators. Throughout the Monte-Carlo studies, we set the parameter values as $\lambda = 1$, $\alpha = 0.5, 1$ and 2, and $a = 0.90, 0.95, 1.05, 1.10$. By the 1000 times replicated simulations conducted on the different samples of sizes $n = 30, 50, 100$, we compute the means, biases and $n\times$ mean squared errors ($n\times$MSE) for the ML, MM, MLM and MLS estimates for each collection of parameters. The simulated results are presented in Tables 1-3.

According to the simulation results in Tables 4.1-4.3, we can clearly say that the performances of all estimators are quite satisfactory in all cases. Besides, as the sample size n increases, bias and $n\times$MSE values of all estimators decrease. Thus, we can say that all estimators are asymptotically unbiassed and consistent. In addition, ML estimators outperform the other estimators in small, moderate and large sample sizes.

Table 4.1: The simulated Means, Biases and nxMSEs for the ML, MLS, MM and MLM estimators of the parameters a, α and λ, when $\alpha = 0.5$ and $\lambda = 1$.

			$\hat{a}$			$\hat{\alpha}$			$\hat{\lambda}$		
a	n	Method	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE
0.90	30	ML	0.9021	0.0021	0.0419	0.5526	0.0526	5.5639	1.0555	0.0555	20.2610
		MLS	0.9014	0.0014	0.1107	0.5099	0.0099	4.8498	1.0573	0.0573	52.0585
		MM	0.9014	0.0014	0.1107	0.5675	0.0675	16.1491	1.0724	0.0724	42.9262
		MLM	0.9014	0.0014	0.1107	0.4845	0.0155	6.6621	1.0213	0.0213	36.3018
	50	ML	0.8999	0.0001	0.0123	0.5280	0.0280	2.4407	1.0667	0.0667	16.1984
		MLS	0.9000	0.0000	0.0255	0.5073	0.0073	2.9507	1.0523	0.0523	26.8384
		MM	0.9000	0.0000	0.0255	0.5301	0.0301	8.9010	1.0528	0.0528	27.8249
		MLM	0.9000	0.0000	0.0255	0.4878	0.0122	3.8533	1.0313	0.0313	24.6382
	100	ML	0.8998	0.0002	0.0012	0.5172	0.0172	1.1802	1.0423	0.0423	6.5389
		MLS	0.8996	0.0004	0.0034	0.5057	0.0057	1.4203	1.0512	0.0512	15.3660
		MM	0.8996	0.0004	0.0034	0.5238	0.0238	3.9173	1.0601	0.0601	15.8429
		MLM	0.8996	0.0004	0.0034	0.4981	0.0019	1.5968	1.0451	0.0451	13.9685
0.95	30	ML	0.9507	0.0007	0.0571	0.5590	0.0590	5.5451	1.1014	0.1014	29.9822
		MLS	0.9519	0.0019	0.1450	0.5160	0.0160	6.0655	1.0577	0.0577	52.0999
		MM	0.9519	0.0019	0.1450	0.5943	0.0943	16.8724	1.1040	0.1040	51.0806
		MLM	0.9519	0.0019	0.1450	0.4981	0.0019	6.6786	1.0463	0.0463	44.1838
	50	ML	0.9504	0.0004	0.0107	0.5335	0.0335	3.1944	1.0481	0.0481	14.6898
		MLS	0.9501	0.0001	0.0270	0.5089	0.0089	3.3424	1.0425	0.0425	27.6539
		MM	0.9501	0.0001	0.0270	0.5536	0.0536	9.9358	1.0675	0.0675	30.7541
		MLM	0.9501	0.0001	0.0270	0.4943	0.0057	4.3293	1.0319	0.0319	26.2168
	100	ML	0.9500	0.0000	0.0014	0.5207	0.0207	1.1473	1.0396	0.0396	6.5734
		MLS	0.9501	0.0001	0.0037	0.5094	0.0094	1.3092	1.0314	0.0314	13.4648
		MM	0.9501	0.0001	0.0037	0.5307	0.0307	3.9233	1.0442	0.0442	13.9762
		MLM	0.9501	0.0001	0.0037	0.5037	0.0037	1.4159	1.0291	0.0291	12.2366
1.05	30	ML	1.0508	0.0008	0.0596	0.5529	0.0529	5.1782	1.0974	0.0974	24.8192
		MLS	1.0494	0.0006	0.1488	0.5149	0.0149	6.0798	1.0998	0.0998	57.9221
		MM	1.0494	0.0006	0.1488	0.5785	0.0785	16.3929	1.1198	0.1198	49.7872
		MLM	1.0494	0.0006	0.1488	0.4939	0.0061	6.8040	1.0689	0.0689	40.4968
	50	ML	1.0500	0.0000	0.0140	0.5296	0.0296	2.6421	1.0665	0.0665	13.1142
		MLS	1.0509	0.0009	0.0360	0.5079	0.0079	2.8651	1.0349	0.0349	28.8999
		MM	1.0509	0.0009	0.0360	0.5451	0.0451	8.0653	1.0553	0.0553	28.4833
		MLM	1.0509	0.0009	0.0360	0.4952	0.0048	3.2639	1.0263	0.0263	24.7169
	100	ML	1.0501	0.0001	0.0015	0.5150	0.0150	1.1322	1.0254	0.0254	6.3604
		MLS	1.0502	0.0002	0.0041	0.5037	0.0037	1.3424	1.0147	0.0147	11.7915
		MM	1.0502	0.0002	0.0041	0.5226	0.0226	3.5821	1.0238	0.0238	12.4582
		MLM	1.0502	0.0002	0.0041	0.4954	0.0046	1.4584	1.0079	0.0079	11.0757
1.10	30	ML	1.1017	0.0017	0.0721	0.5657	0.0657	5.3227	1.0710	0.0710	21.8885
		MLS	1.1013	0.0013	0.1583	0.5205	0.0205	4.2594	1.0488	0.0488	41.9511
		MM	1.1013	0.0013	0.1583	0.5946	0.0946	16.2766	1.0973	0.0973	51.3694
		MLM	1.1013	0.0013	0.1583	0.5024	0.0024	5.8000	1.0408	0.0408	40.9797
	50	ML	1.1007	0.0007	0.0155	0.5321	0.0321	2.7774	1.0389	0.0389	12.7543
		MLS	1.1005	0.0005	0.0321	0.5114	0.0114	3.4238	1.0277	0.0277	23.1304
		MM	1.1005	0.0005	0.0321	0.5653	0.0653	10.4034	1.0609	0.0609	24.3047
		MLM	1.1005	0.0005	0.0321	0.5028	0.0028	4.2720	1.0227	0.0227	20.2996
	100	ML	1.1000	0.0000	0.0018	0.5140	0.0140	1.0566	1.0329	0.0329	5.8668
		MLS	1.1003	0.0003	0.0049	0.4964	0.0036	1.2135	1.0060	0.0060	12.3035
		MM	1.1003	0.0003	0.0049	0.5206	0.0206	3.7712	1.0252	0.0252	14.6663
		MLM	1.1003	0.0003	0.0049	0.4928	0.0072	1.4634	1.0076	0.0076	12.4369

Table 4.2: The simulated Means, Biases and nxMSEs for the ML, MLS, MM and MLM estimators of the parameters a, α and λ, when $\alpha = 1$ and $\lambda = 1$.

			$\hat{a}$			$\hat{\alpha}$			$\hat{\lambda}$		
a	n	Method	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE
0.90	30	ML	0.8995	0.0005	0.0226	1.1673	0.1673	34.3936	1.0710	0.0710	12.8072
		MLS	0.9002	0.0002	0.0364	1.0807	0.0807	37.1783	1.0255	0.0255	18.3058
		MM	0.9002	0.0002	0.0364	1.2559	0.2559	84.4448	1.0717	0.0717	18.4970
		MLM	0.9002	0.0002	0.0364	1.0727	0.0727	32.7471	1.0313	0.0313	15.3820
	50	ML	0.9002	0.0002	0.0050	1.0705	0.0705	12.2203	1.0226	0.0226	5.9189
		MLS	0.9006	0.0006	0.0081	1.0206	0.0206	15.2559	0.9911	0.0089	8.0817
		MM	0.9006	0.0006	0.0081	1.1228	0.1228	30.5716	1.0241	0.0241	9.9016
		MLM	0.9006	0.0006	0.0081	1.0159	0.0159	13.6941	0.9961	0.0039	7.9823
	100	ML	0.8999	0.0001	0.0006	1.0335	0.0335	5.5753	1.0233	0.0233	2.8179
		MLS	0.9000	0.0000	0.0009	1.0135	0.0135	7.6169	1.0095	0.0095	4.2659
		MM	0.9000	0.0000	0.0009	1.0668	0.0668	15.4622	1.0268	0.0268	4.8785
		MLM	0.9000	0.0000	0.0009	1.0127	0.0127	7.2168	1.0134	0.0134	3.9930
0.95	30	ML	0.9486	0.0014	0.0302	1.1563	0.1563	32.0993	1.0953	0.0953	14.9887
		MLS	0.9482	0.0018	0.0468	1.0824	0.0824	46.1606	1.0610	0.0610	21.3359
		MM	0.9482	0.0018	0.0468	1.2546	0.2546	75.3895	1.1168	0.1168	21.4723
		MLM	0.9482	0.0018	0.0468	1.0828	0.0828	40.3227	1.0746	0.0746	18.0953
	50	ML	0.9500	0.0000	0.0061	1.0781	0.0781	16.7709	1.0411	0.0411	7.9158
		MLS	0.9499	0.0001	0.0101	1.0414	0.0414	25.0638	1.0266	0.0266	12.0983
		MM	0.9499	0.0001	0.0101	1.1250	0.1250	41.2553	1.0479	0.0479	11.8192
		MLM	0.9499	0.0001	0.0101	1.0263	0.0263	19.4910	1.0258	0.0258	10.3355
	100	ML	0.9500	0.0000	0.0007	1.0217	0.0217	4.4096	1.0115	0.0115	3.0601
		MLS	0.9501	0.0001	0.0012	1.0065	0.0065	6.4943	1.0016	0.0016	5.1477
		MM	0.9501	0.0001	0.0012	1.0593	0.0593	12.7412	1.0177	0.0177	4.8222
		MLM	0.9501	0.0001	0.0012	1.0033	0.0033	6.0221	1.0035	0.0035	4.3940
1.05	30	ML	1.0494	0.0006	0.0265	1.1192	0.1192	21.9043	1.0618	0.0618	9.0571
		MLS	1.0495	0.0005	0.0429	1.0300	0.0300	19.9858	1.0261	0.0261	12.3809
		MM	1.0495	0.0005	0.0429	1.1917	0.1917	60.6033	1.0675	0.0675	14.1094
		MLM	1.0495	0.0005	0.0429	1.0257	0.0257	23.3553	1.0277	0.0277	11.3251
	50	ML	1.0500	0.0000	0.0075	1.0763	0.0763	10.4700	1.0345	0.0345	6.8339
		MLS	1.0497	0.0003	0.0105	1.0247	0.0247	13.3111	1.0186	0.0186	9.5556
		MM	1.0497	0.0003	0.0105	1.1251	0.1251	29.2849	1.0503	0.0503	10.9266
		MLM	1.0497	0.0003	0.0105	1.0229	0.0229	12.6927	1.0246	0.0246	9.1466
	100	ML	1.0502	0.0002	0.0007	1.0364	0.0364	4.7974	1.0122	0.0122	3.0109
		MLS	1.0502	0.0002	0.0013	1.0038	0.0038	5.7562	0.9983	0.0017	4.4178
		MM	1.0502	0.0002	0.0013	1.0683	0.0683	14.2285	1.0192	0.0192	5.3255
		MLM	1.0502	0.0002	0.0013	1.0061	0.0061	6.1276	1.0032	0.0032	4.4140
1.10	30	ML	1.1003	0.0003	0.0353	1.1368	0.1368	27.3646	1.0665	0.0665	12.3611
		MLS	1.0998	0.0002	0.0460	1.0416	0.0416	30.0485	1.0269	0.0269	15.1012
		MM	1.0998	0.0002	0.0460	1.2349	0.2349	70.7584	1.0873	0.0873	17.8739
		MLM	1.0998	0.0002	0.0460	1.0502	0.0502	29.7390	1.0422	0.0422	14.3413
	50	ML	1.0999	0.0001	0.0075	1.1114	0.1114	15.9703	1.0530	0.0530	6.8821
		MLS	1.1001	0.0001	0.0113	1.0520	0.0520	17.7979	1.0204	0.0204	9.3748
		MM	1.1001	0.0001	0.0113	1.1819	0.1819	39.7175	1.0613	0.0613	10.0945
		MLM	1.1001	0.0001	0.0113	1.0592	0.0592	17.1944	1.0308	0.0308	8.3747
	100	ML	1.1001	0.0001	0.0010	1.0332	0.0332	4.9204	1.0139	0.0139	3.3641
		MLS	1.1000	0.0000	0.0017	1.0073	0.0073	6.3576	1.0052	0.0052	5.1185
		MM	1.1000	0.0000	0.0017	1.0602	0.0602	15.3053	1.0208	0.0208	5.3203
		MLM	1.1000	0.0000	0.0017	1.0062	0.0062	6.4213	1.0080	0.0080	4.5479

Table 4.3: The simulated Means, Biases and nxMSEs for the ML, MLS, MM and MLM estimators of the parameters a, α and λ, when $\alpha = 2$ and $\lambda = 1$.

			$\hat{a}$			$\hat{\alpha}$			$\hat{\lambda}$		
a	n	Method	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE	Mean	Bias	$n\times$MSE
0.90	30	ML	0.8999	0.0001	0.0134	2.3764	0.3764	185.0170	1.0691	0.0691	8.3586
		MLS	0.9001	0.0001	0.0172	2.2418	0.2418	399.8574	1.0313	0.0313	8.9383
		MM	0.9001	0.0001	0.0172	2.3288	0.3288	116.3478	1.0636	0.0636	9.1816
		MLM	0.9001	0.0001	0.0172	2.2279	0.2279	308.3195	1.0429	0.0429	8.3048
	50	ML	0.8996	0.0004	0.0031	2.1643	0.1643	70.3287	1.0346	0.0346	4.9931
		MLS	0.8997	0.0003	0.0037	2.0955	0.0955	105.0502	1.0178	0.0178	5.8255
		MM	0.8997	0.0003	0.0037	2.1857	0.1857	92.2488	1.0319	0.0319	5.8339
		MLM	0.8997	0.0003	0.0037	2.0743	0.0743	84.8229	1.0191	0.0191	5.3602
	100	ML	0.9000	0.0000	0.0003	2.0767	0.0767	22.3545	1.0111	0.0111	1.6830
		MLS	0.9000	0.0000	0.0004	2.0189	0.0189	29.2812	1.0009	0.0009	2.1476
		MM	0.9000	0.0000	0.0004	2.1333	0.1333	48.6809	1.0162	0.0162	2.3114
		MLM	0.9000	0.0000	0.0004	2.0183	0.0183	26.2329	1.0037	0.0037	1.8638
0.95	30	ML	0.9503	0.0003	0.0124	2.3595	0.3595	168.6435	1.0486	0.0486	7.1311
		MLS	0.9502	0.0002	0.0153	2.1344	0.1344	159.8323	1.0111	0.0111	8.0305
		MM	0.9502	0.0002	0.0153	2.3240	0.3240	120.3577	1.0480	0.0480	7.8498
		MLM	0.9502	0.0002	0.0153	2.1655	0.1655	146.1292	1.0262	0.0262	7.5533
	50	ML	0.9498	0.0002	0.0030	2.1902	0.1902	74.3277	1.0317	0.0317	4.1722
		MLS	0.9498	0.0002	0.0035	2.0753	0.0753	84.4408	1.0100	0.0100	4.7112
		MM	0.9498	0.0002	0.0035	2.2398	0.2398	83.9141	1.0374	0.0374	4.8533
		MLM	0.9498	0.0002	0.0035	2.0776	0.0776	70.6279	1.0171	0.0171	4.3557
	100	ML	0.9501	0.0001	0.0004	2.1205	0.1205	38.9492	1.0084	0.0084	1.9344
		MLS	0.9501	0.0001	0.0005	2.0379	0.0379	40.9310	0.9959	0.0041	2.3678
		MM	0.9501	0.0001	0.0005	2.1333	0.1333	62.1434	1.0075	0.0075	2.6137
		MLM	0.9501	0.0001	0.0005	2.0401	0.0401	38.3668	0.9983	0.0017	2.2154
1.05	30	ML	1.0503	0.0003	0.0185	2.2911	0.2911	164.7333	1.0347	0.0347	7.3818
		MLS	1.0505	0.0005	0.0218	2.0555	0.0555	211.5846	0.9842	0.0158	8.1717
		MM	1.0505	0.0005	0.0218	2.2707	0.2707	129.7693	1.0283	0.0283	8.5876
		MLM	1.0505	0.0005	0.0218	2.0674	0.0674	141.8400	1.0018	0.0018	7.7984
	50	ML	1.0499	0.0001	0.0032	2.1497	0.1497	67.0813	1.0352	0.0352	3.4191
		MLS	1.0500	0.0000	0.0039	2.0691	0.0691	97.2026	1.0112	0.0112	4.3589
		MM	1.0500	0.0000	0.0039	2.2045	0.2045	82.0182	1.0357	0.0357	4.0980
		MLM	1.0500	0.0000	0.0039	2.0698	0.0698	87.2862	1.0183	0.0183	3.7935
	100	ML	1.0500	0.0000	0.0005	2.0791	0.0791	26.5465	1.0111	0.0111	2.1213
		MLS	1.0500	0.0000	0.0007	2.0259	0.0259	39.8430	1.0009	0.0009	2.6829
		MM	1.0500	0.0000	0.0007	2.1326	0.1326	55.8938	1.0161	0.0161	2.8456
		MLM	1.0500	0.0000	0.0007	2.0264	0.0264	33.9811	1.0047	0.0047	2.4850
1.10	30	ML	1.0998	0.0002	0.0174	2.3212	0.3212	194.9628	1.0429	0.0429	6.9826
		MLS	1.0997	0.0003	0.0214	2.1402	0.1402	165.2003	1.0096	0.0096	7.5882
		MM	1.0997	0.0003	0.0214	2.3034	0.3034	121.5048	1.0441	0.0441	8.2323
		MLM	1.0997	0.0003	0.0214	2.1614	0.1614	156.1144	1.0235	0.0235	7.5732
	50	ML	1.1003	0.0003	0.0034	2.1643	0.1643	79.7157	1.0159	0.0159	2.9575
		MLS	1.1001	0.0001	0.0046	2.1007	0.1007	110.5767	1.0066	0.0066	4.3802
		MM	1.1001	0.0001	0.0046	2.1697	0.1697	97.4672	1.0176	0.0176	4.0407
		MLM	1.1001	0.0001	0.0046	2.0696	0.0696	90.7552	1.0067	0.0067	3.7026
	100	ML	1.0997	0.0003	0.0006	2.1008	0.1008	30.4018	1.0329	0.0329	2.0447
		MLS	1.0997	0.0003	0.0007	2.0582	0.0582	40.5492	1.0243	0.0243	2.5109
		MM	1.0997	0.0003	0.0007	2.1426	0.1426	50.4873	1.0366	0.0366	2.8699
		MLM	1.0997	0.0003	0.0007	2.0540	0.0540	36.9373	1.0263	0.0263	2.4028

5. Application

In this section, we analyze two real-life datasets called No.3 data and Software data to illustrate the estimation procedures the ML, the MM, the MLM and the MLS. To compare the RP and GPs with the ML, the MM, the MLM and the MLS estimators, we use the mean-squared error (MSE*) criterion defined as, see [?],

- MSE*$= (1/n) \sum_{k=1}^{n} \left(X_k - \hat{X}_k \right)^2$,

where $\hat{X}_k$ is calculated by

$$\hat{X}_k = \begin{cases} \hat{\mu}_{(ML)} \hat{a}_{ML}^{1-k} & \text{GP with the ML estimators,} \\ \hat{\mu}_{(MLS)} \hat{a}_{NP}^{1-k} & \text{GP with the MLS estimators,} \\ \hat{\mu}_{(MM)} \hat{a}_{NP}^{1-k} & \text{GP with the MM estimators,} \\ \hat{\mu}_{(MLM)} \hat{a}_{NP}^{1-k} & \text{GP with the MLM estimators,} \\ \hat{\mu}_{(ML)} & \text{RP with the ML estimators,} \end{cases} \tag{5.1}$$

and $\hat{\mu}_{(.)}$ is estimate of the expected value of the first occurrence time under the fitted GR distribution with the ML, MM, MLM and MLS estimators and can be numerically calculated from

$$\hat{\mu}_{(.)} = \int_0^\infty x f\left(x, \hat{\alpha}_{(.)}, \hat{\lambda}_{(.)} \right) dx$$

No.3 data:

In the No.3 data set, there are 71 observations, which are regarding the unscheduled maintenance actions for U.S.S. Halfbeak No.3 main propulsion diesel engine [2]. This data set was found to be consistent with a GP in which the ratio parameter is greater than 1, see [16].

In the first stage of data analysis, we investigate whether the data set follows a GR distribution. Linear regression model

$$\ln X_i = \tau - (i-1) \ln a + \varepsilon_i \tag{5.2}$$

can be employed to this aim, see [13] for further information on derivation of this regression model. Where $\tau = E(\ln Y_i)$, $Y_i = a^{i-1} X_i$ and $\exp(\varepsilon_i) \sim GR(\theta, \beta)$. The error term ε_i given in equation (5.2) can be easily estimated by

$$\hat{\varepsilon}_i = \ln X_i - \hat{\tau} - (i-1) \ln \hat{a}_{NP} \tag{5.3}$$

where $\hat{\tau} = \frac{n(n-1)}{2} \ln \hat{a}_{NP} + \sum_{i=1}^{n} \ln X_i$. Thus, we can say that the data set is consistent with a GR distribution if the exponential errors follow a GR distribution. The parameters estimations of the exponential errors are $\hat{\theta}_{ML} = 0.2410$ and

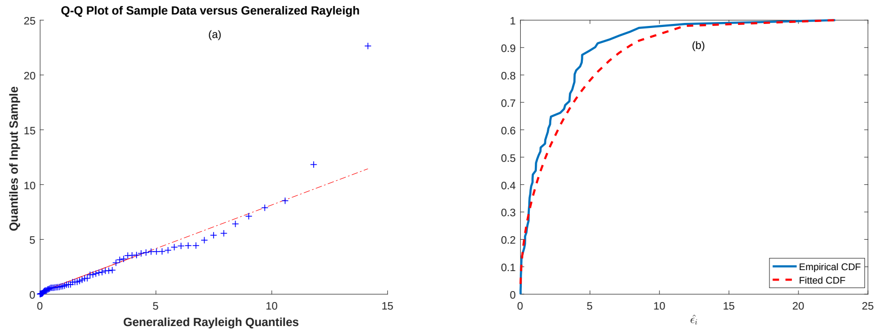

FIG. 5.1: QQ plot for the exponential errors (a), empirical and fitted cdf for the exponential errors (b)

Table 5.1: Estimation of parameters for the No 3 data set

Process	Method	$\hat{a}$	$\hat{\alpha}$	$\hat{\lambda}$	MSE/10^5
GP	ML	1.04272	0.12795	0.0002	1.93257
	MLS		0.2596	0.0004	2.0208
	MM	1.0416	0.0700	0.0002	2.2717
	MLM		0.1277	0.0002	2.0210
RP	ML	1.0000	0.1910	0.0007	3.3945

$\hat{\beta}_{ML} = 0.1330$ and also the value of Kolmogorov-Smirnov (K-S) test is 0.1286 and corresponding p-value is 0.1751. Hence, result of the K-S test, we can say that the No. 3 dataset consistent with a GR distribution. To confirm this result, we present Figure 5.1(a) and Figure 5.1 (b). Figure 5.1(a) displays the Q-Q plot of quantiles of the data versus $GR(\theta, \beta)$. Figure 5.1 (b) display both the empirical and fitted cdf. As it can be clearly seen from Figure 5.1 (a), the quantiles of the data fall approximately on the straight line. In Figure 5.1 (b), the fitted cdf closely follows to empirical cdf.

If the GP with the GR is applied to this data, the parameter estimates obtained by using the employed estimators in the paper and the corresponding MSE values are presented in Table 5.1

From Table 5.1, it is seen that the GP outperform the RP for this data set. Besides, the GP with ML estimators have the lowest MSE value relative to other models. We present the Figure 5.2 to show the relative performances of the four GPs with the ML, the MM, the MLM and the MLS estimators and the RP. Figure 5.2 display the plots of S_k, $S_k = X_1 + X_2 + ... + X_k$, $k = 1, 2, ..., n$ and its estimates $\hat{S}_k$, $\hat{S}_k = \sum_{j=1}^{k} \hat{X}_k$, against the k, $k = 1, 2, ..., n$, where $\hat{X}_k$ can obtained by using (5.1).

According to Figure 5.2, it can be concluded that GPs follow true values more accurately than RP.

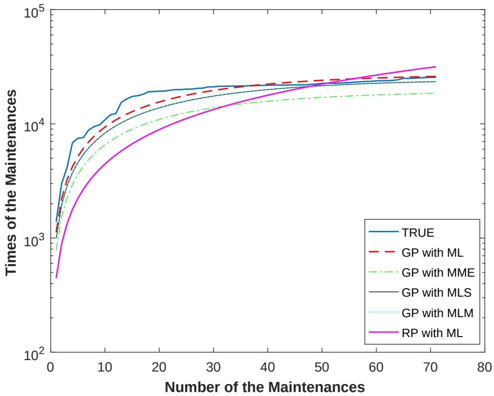

Fig. 5.2: The plots of the observed and estimated maintenance times for the No. 3 data set

Table 5.2: Estimates and evaluated MSE* values of the different GP models for the No. 3 data

	Model									
	G. Rayleigh		Gamma		Log-Normal		Weibull		Inv. Gaussian	
MSE*/10^5	1.93257		2.15623		2.46508		2.11300		1.93442	
Parameter Est.	$\hat{a}$	1.04272	$\hat{a}$	1.03547	$\hat{a}$	1.04165	$\hat{a}$	1.03659	$\hat{a}$	1.04274
	$\hat{\alpha}$	0.12795	$\hat{k}_G$	0.66991	$\hat{\mu}_{LN}$	6.06255	$\hat{\theta}_W$	777.7413	$\hat{\mu}_{IG}$	1118.4
	$\hat{\lambda}$	0.0002	$\hat{\theta}_G$	1290.572	$\hat{\sigma}_{LN}$	1.68506	$\hat{\lambda}_W$	0.7730	$\hat{\sigma}_{IG}$	1781.1

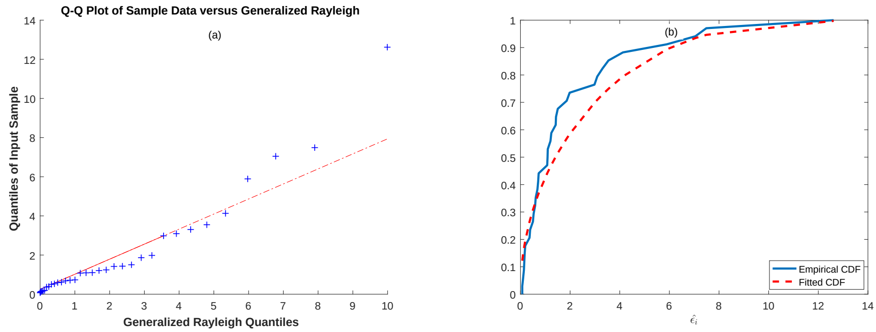

Fig. 5.3: QQ plot for the exponential errors (a), emprical and fitted cdf for the exponential errors (b)

Table 5.3: Estimation of parameters for the software data

Process	Method	$\hat{a}$	$\hat{\alpha}$	$\hat{\lambda}$	MSE/10^3
GP	ML	0.9094	0.3108	0.1319	1.8027
	MLS		0.3032	0.1023	2.1646
	MM	0.9370	0.1352	0.0493	2.0867
	MLM		0.1293	0.0483	2.0965
RP	ML	1.0000	0.1845	0.0087	2.6559

Software data:

Software data set includes 34 observations. These data represent the time between successive failures of a piece of software developed as part of a large data system [11]. Braun et al. [9] showed that this data set consistent by a GP with the ratio parameter $a < 1$. Thus we can apply a GP with the GR distribution to this data. First, we investigate whether the underlying distribution of the data is consistent with a GR distribution, as in the No. 3 data. When the regression given by (5.2) is applied to this data, estimates of the parameters for the exponential errors are $\hat{\theta}_{ML} = 0.2381$ and $\hat{\beta}_{ML} = 0.1677$. For this data, K-S test is 0.1615 and corresponding p-value is 0.3040. Thus, we can say that the software data set consistent with a GR distribution. In addition, we present the Q-Q plot and the fitted and empirical cdf of the exponential errors by the Figure 5.3 to support the result of K-S test.

When a GP with the GR distribution is applied to software data set, estimates of the parameters a, α and λ and the corresponding MSE values are given in Table 5.3

Acording to Table 5.3, GP outperform the RP since it has lower MSE. Furthermore, GP with the ML estimates has the best performance among all GPs. Furthermore, relative performances of the GPs with the all estimators and RP can be seen from Figure 5.4. Figure 5.4 include the plots of the S_k and $\hat{S}_k$'s against the

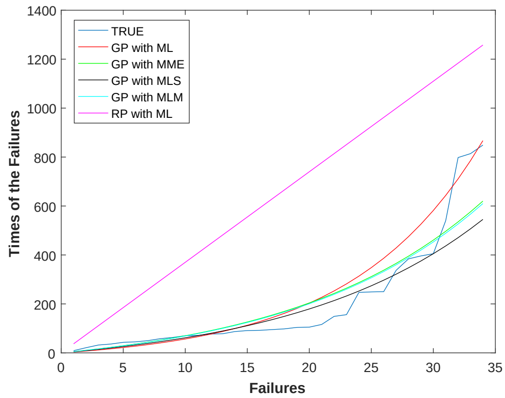

FIG. 5.4: The plots of the observed and estimated failure times for the Software data

Table 5.4: Estimates and evaluated MSE* values of the different GP models for the Software data

	Model									
	G. Rayleigh		Gamma		Log-Normal		Weibull		Inv. Gaussian	
MSE*/10^3	1.8027		1.8763		1.9887		1.8890		2.1314	
Parameter Est.	$\hat{a}$	0.9094	$\hat{a}$	0.9172	$\hat{a}$	0.9370	$\hat{a}$	0.9186	$\hat{a}$	0.9504
	$\hat{\alpha}$	0.3108	$\hat{k}_G$	0.8533	$\hat{\mu}_{LN}$	1.0017	$\hat{\theta}_W$	3.6726	$\hat{\mu}_{IG}$	7.5144
	$\hat{\lambda}$	0.1319	$\hat{\theta}_G$	4.4649	$\hat{\sigma}_{LN}$	1.2742	$\hat{\lambda}_W$	0.8856	$\hat{\sigma}_{IG}$	14.4192

k, $k = 1, 2, ..., n$, where S_k and $\hat{S}_k$ are defined as in the previous example.

As in the previous example, we can easily seen from Figure 5.4 that four GPs follow true values more accurately than RP.

6. Conclusion

The GP with the GR distribution considered by this article has many potential uses for modeling of successive arrival times observed from many fields. The process is very suitable for modeling applications of arrival times with the monotonic ascending or descending behavior as highlighted in the paper. The monotonic behavior of the GP is controlled by a positive-valued ratio parameter a, which is an essential feature of this process. In the paper, for the different values of the parameter a, the behavior of the process has been clearly illustrated in Figure 1.1. In addition to the ratio parameter a, the parameters of the distribution of the first arrival time are other key parameters that regulate the behavior of the process. In order to achieve an optimal modeling performance from the GP, the solution of the estimation problem of these parameters is crucial. The estimation problem for a, α and λ parameters of GP with the GR distribution is solved by employing the

ML methodology in the paper. The results of numerical studies which compare the efficiency of the ML estimators and modified estimators considered in this paper are presented in the tables. Tabulated results display that the ML estimators produce more efficient estimations in all cases with respect to bias and MSE criterion.

In order to demonstrate the phases of data modeling by a GP with the GR distribution and comparing its modeling performance against the RP, in the paper, two examples are carried out on real-world datasets called the No.3 and Software. In both examples, the GP with the GR distribution outperforms the RP with smaller MSE values. Furthermore, by the analysis of the results in the paper, it can be concluded that fitting by a GP with the GR distribution to both data sets is better than fitting by a GP with the possible alternatives of the GR distribution such as Gamma, Log-Normal, inverse Gaussian and Weibull.

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ON (p, q)-STANCU-SZÁSZ-BETA OPERATORS AND THEIR APPROXIMATION PROPERTIES

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Abstract. In the present paper, we have introduced the generalized form of (p, q)-analogue of the Szász-Beta operators with Stancu type parameters. We have studied the local approximation properties of these operators and obtained the convergence rate and weighted approximation.

Keywords: Szász-Beta operators; Stancu type parameters; weighted approximation.

1. Introduction and preliminaries

In the last two decades, the applications of q-calculus emerged as a new area in the field of approximation theory. The development of q-calculus has led to the discovery of various modifications of Bernstein polynomials involving q-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupaş [11] introduced the first q-analogue of the classical Bernstein operators and investigated its approximating and shape preserving properties. Another q-generalization of the classical Bernstein polynomial is due to Phillips [20]. Several generalization of well known positive linear operators based on q-integers were introduced and their approximation properties have been studied by several researchers.

Recently, Mursaleen *et al* introduced the use of (p, q)-calculus in approximation theory and constructed the (p, q)-analogue of Bernstein operators [13] and (p, q)-analogue of Bernstein-Stancu operators [15]. Most recently, the (p, q)-analogue of

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some more operators have been studied in [1]- [3], [5], [12], [14], [16], [17], [18] and [19].

The (p,q)-integer was introduced to generalize or unify several forms of q-oscillator algebras well known in the Physics literature related to the representation theory of single parameter quantum algebras. The (p,q)-integer is defined by

$$(1.1)\quad [n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q} & (p \neq q \neq 1) \\ \frac{1-q^n}{1-q} & (p=1) \\ n & (p=q=1) \end{cases}$$

The (p,q)-binomial expansion is

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1-x)^n_{p,q} := (1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x).$$

The (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

The definite integral of a function f is defined by

$$\int_0^a f(t) d_{p,q}t = (q-p)a \sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}}a\right) \frac{p^k}{q^{k+1}}, \qquad if \mid \frac{p}{q} \mid < 1,$$

$$\int_0^a f(t) d_{p,q}t = (p-q)a \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right) \frac{q^k}{p^{k+1}}, \qquad if \mid \frac{q}{p} \mid < 1.$$

There are two (p,q)-analogues of the classical exponential function defined as follows

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)}{2}} x^n}{[n]_{p,q}!},$$

and

$$E_{p,q}(x)=\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}x^n}{[n]_{p,q}!},$$

which satisfy the equality $e_{p,q}(x)E_{p,q}(-x)=1$. For $p=1$, $e_{p,q}(x)$ and $E_{p,q}(x)$ reduce to q-exponential functions.

For $m,n\in\mathbb{N}$, the (p,q)-Beta and the (p,q)-Gamma functions are defined by

$$B_{p,q}(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}d_{p,q}x,$$

and

$$\Gamma_{p,q}(n)=\int_0^{\infty}p^{\frac{n(n-1)}{2}}E_{p,q}(-qx)d_{p,q}x,\qquad \Gamma_{p,q}(n+1)=[n]_{p,q}!$$

respectively. The two functions are connected through

$$B_{p,q}(m,n)=q^{\frac{2-m(m-1)}{2}}p^{\frac{-m(m+1)}{2}}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

For $p=1$, all the notions of the (p,q)-calculus reduce to those of q-calculus.

Based on (p,q)-calculus, very recently Acar [1] defined the (p,q) analogue of Szász operators as

$$S_{n,p,q}(f;x)=\sum_{k=0}^{n}s_{n,k}^{p,q}(x)f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right) \tag{1.2}$$

for $x\in[0,\infty), 0<q<p\leq 1$, where

$$s_{n,k}^{p,q}(x)=\frac{q^{\frac{k(k-1)}{2}}}{E_{p,q}([n]_{p,q}x)}\frac{([n]_{p,q}x)^k}{[k]_{p,q}!}.$$

Gupta and Noor [9] proposed Szász-Beta operators and obtained some direct results in simultaneous approximation. Gupta and Aral [8] extended the studies and they proposed the q-analogue of Szász-Beta operators. Later on Aral and Gupta [4] introduced the (p,q)-analogue of the Szász-Beta operators as follows

$$D_n^{(p,q)}(f;x)=\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}f(p^{k+1}qt)d_{p,q}t \tag{1.3}$$

where $s_{n,k}^{p,q}(x)$ is defined in (1.2). In this paper, we have generalized this operator (1.3) with Stancu type parameters. Assuming that $0 \leq \alpha \leq \beta$, for $x \in [0,\infty), 0 < q < p \leq 1$, we define

$$D_{n,p,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right) d_{p,q}t. \tag{1.4}$$

2. Auxiliary results

Lemma 2.1. *For $x \in [0,\infty), 0 < q < p \leq 1$, we have*

$$\begin{aligned}
(i) D_n^{p,q}(1;x) &= 1,\\
(ii) D_n^{p,q}(t;x) &= x,\\
(iii) D_n^{p,q}(t^2;x) &= \frac{[2]_{p,q}qx}{p[n-1]_{p,q}} + \frac{p[n]_{p,q}x^2}{[n-1]_{p,q}},\\
(iv) D_n^{p,q}(t^3;x) &= \frac{p^3[n]_{p,q}^2}{q^6[n-1]_{p,q}[n-2]_{p,q}}x^3\\
&+\left(\frac{(p[2]_{p,q}+p^2)[n]_{p,q}}{p^2q^6([n-1]_{p,q}[n-2]_{p,q}} + \frac{(p^2q+2pq^2)[n]_{p,q}}{q^6([n-1]_{p,q}[n-2]_{p,q}}\right)x^2\\
&+\left(\frac{[2]_{p,q}}{p^3q^5([n-1]_{p,q}[n-2]_{p,q}} + \frac{(p[2]_{p,q}+p^2)}{p^3q^5([n-1]_{p,q}[n-2]_{p,q}}\right)x,\\
(v) D_n^{p,q}(t^4;x) &= \frac{p^6[n]_{p,q}^3}{q^{12}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4\\
&+\frac{[n]_{p,q}^2(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)}{q^{11}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^3\\
&+\frac{[n]_{p,q}}{p^5q^9[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big(p^8+3p^7q+5p^6q^2\\
&+5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3+2p^2q^4+pq^5\Big)x^2\\
&+\frac{(p^6+2p^5q+p^4q^2+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5+pq^3+q^6)}{p^6q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x.
\end{aligned}$$

Lemma 2.2. *Let $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$. For $x \in [0,\infty)$, $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$, we have*

$$\begin{aligned}
(i) D_{n,p,q}^{(\alpha,\beta)}(e_0;x) &= 1,\\
(ii) D_{n,p,q}^{(\alpha,\beta)}(e_1;x) &= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta},
\end{aligned}$$

$$
\begin{aligned}
(iii) D_{n,p,q}^{(\alpha,\beta)}(e_2;x) &= \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q}+\beta)^2}x^2 + \frac{[n]_{p,q}(q(p+q)[n]_{p,q}+2\alpha p[n-1]_{p,q})}{p([n]_{p,q}+\beta)^2[n-1]_{p,q}}x \\
&+\frac{\alpha^2}{([n]_{p,q}+\beta)^2}, \\
(iv) D_{n,p,q}^{(\alpha,\beta)}(e_3;x) &= \frac{p^3[n]_{p,q}^5}{q^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^3 \\
&+\frac{[n]_{p,q}^3([n]_{p,q}(p^3q+2p^2q^2+2p+q)+3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^2 \\
&+\frac{[n]_{p,q}}{([n]_{p,q}+\beta)^3}\left(\frac{[n]_{p,q}^2([2]_{p,q}+p[2]_{p,q}+p^2)}{q^5p^3[n-1]_{p,q}[n-2]_{p,q}} + \frac{3q\alpha[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}} + 3\alpha^3\right)x \\
&+\frac{\alpha^3}{([n]_{p,q}+\beta)^3}, \\
(v) D_{n,p,q}^{(\alpha,\beta)}(e_4;x) &= \frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4 \\
&+\frac{[n]_{p,q}^5([n]_{p,q}(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)+4\alpha p^3q^5[n-3]_{p,q})}{q^{11}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^3 \\
&+\frac{[n]_{p,q}^3}{p^5q^9([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Bigg([n]_{p,q}^2(p^8+3p^7q+5p^6q^2
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & 5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3+2p^2q^4+pq^5) \\
&+(4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4+2p^6q^5+2p^5q^3+p^4q^4)) \\
&+(6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q})\Bigg)x^2 \\
&+\frac{[n]_{p,q}}{p^6q^6([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Bigg([n]_{p,q}^3(p^6+2p^5q+p^4q^2 \\
&+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5+pq^3+q^6)+4\alpha[n]_{p,q}^2[n-3]_{p,q} \\
&(2p^5q+p^4q^2+p^4q+p^3q^2)+6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7+p^5q^8) \\
&+\alpha^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}p^6q^6\Bigg)x+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}.
\end{aligned}
$$

Proof. Using Lemma 2.1, we can easily say, $(i)D_{n,p,q}^{(\alpha,\beta)}(e_0;x)=1$. Moreover

$$
\begin{aligned}
(ii) D_{n,p,q}^{(\alpha,\beta)}(e_1;x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)d_{p,q}t \\
&= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{p^{k+1}q}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t \\
&\quad +\frac{\alpha}{[n]_{p,q}+\beta}\sum_{k=0}^{\infty} s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t
\end{aligned}
$$

$$\begin{aligned} &= \frac{[n]_{p,q}}{[n]_{p,q}+\beta} D_n^{p,q}(e_1;x) + \frac{\alpha}{[n]_{p,q}+\beta} D_n^{p,q}(e_0;x) \\ &= \frac{[n]_{p,q}}{[n]_{p,q}+\beta} x + \frac{\alpha}{[n]_{p,q}+\beta}. \end{aligned}$$

$$\begin{aligned} (iii) D_{n,p,q}^{(\alpha,\beta)}(e_2;x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q}+\beta} \right)^2 d_{p,q}t \\ &= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2}q^2}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &\quad + \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1}q}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &\quad + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &+ \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} D_n^{p,q}(e_2;x) + \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} D_n^{p,q}(e_1;x) \\ &\quad + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} D_n^{p,q}(e_0;x) \end{aligned}$$

$$\begin{aligned} &= \frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q}+\beta)^2} x^2 + \frac{[n]_{p,q}(q(p+q)[n]_{p,q} + 2\alpha p[n-1]_{p,q})}{p([n]_{p,q}+\beta)^2[n-1]_{p,q}} x \\ &\quad + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}. \end{aligned}$$

$$\begin{aligned} (iv) D_{n,p,q}^{(\alpha,\beta)}(e_3;x) &= \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} \left(\frac{[n]_{p,q} p^{k+1} qt + \alpha}{[n]_{p,q}+\beta} \right)^3 d_{p,q}t \\ &= \frac{[n]_{p,q}^3}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{3k+3}q^3}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k+2}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &\quad + \frac{3\alpha[n]_{p,q}^2}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2}q^2}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &\quad + \frac{3\alpha^2[n]_{p,q}}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1}q}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &\quad + \frac{\alpha^3}{([n]_{p,q}+\beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}} d_{p,q}t \\ &+ \frac{[n]_{p,q}^3}{([n]_{p,q}+\beta)^3} D_n^{p,q}(e_3;x) + \frac{3\alpha[n]_{p,q}^2}{([n]_{p,q}+\beta)^3} D_n^{p,q}(e_2;x) \end{aligned}$$

$$
\begin{aligned}
&+\frac{3\alpha^2[n]_{p,q}}{([n]_{p,q}+\beta)^3}D_n^{p,q}(e_1;x)+\frac{\alpha^3}{([n]_{p,q}+\beta)^3}D_n^{p,q}(e_0;x)\\
=\ &\frac{p^3[n]_{p,q}^5}{q^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^3\\
&+\frac{[n]_{p,q}^3([n]_{p,q}(p^3q+2p^2q^2+2p+q)+3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}x^2\\
&+\frac{[n]_{p,q}}{([n]_{p,q}+\beta)^3}\left(\frac{[n]_{p,q}^2([2]_{p,q}+p[2]_{p,q}+p^2)}{q^5p^3[n-1]_{p,q}[n-2]_{p,q}}+\frac{3q\alpha[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}}+3\alpha^3\right)x\\
&+\frac{\alpha^3}{([n]_{p,q}+\beta)^3}.
\end{aligned}
$$

$$
\begin{aligned}
(v)D_{n,p,q}^{(\alpha,\beta)}(e_4;x)\ =\ &\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^\infty\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)^4d_{p,q}t\\
=\ &\frac{[n]_{p,q}^4}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{4k+4}q^4}{B_{p,q}(k,n+1)}\int_0^\infty\frac{t^{k+3}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t\\
=\ &\frac{4\alpha[n]_{p,q}^3}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{3k+3}q^3}{B_{p,q}(k,n+1)}\int_0^\infty\frac{t^{k+2}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t\\
&+\frac{6\alpha^2[n]_{p,q}^2}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{2k+2}q^2}{B_{p,q}(k,n+1)}\int_0^\infty\frac{t^{k+1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t\\
+\ &\frac{4\alpha^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{p^{k+1}q}{B_{p,q}(k,n+1)}\int_0^\infty\frac{t^{k}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^\infty\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t\\
+\ &\frac{[n]_{p,q}^4}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_4;x)+\frac{4\alpha[n]_{p,q}^3}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_3;x)+\frac{6\alpha^2[n]_{p,q}^2}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_2;x)\\
&+\frac{4\alpha^3[n]_{p,q}}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_1;x)+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}D_n^{p,q}(e_0;x)\\
=\ &\frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^4\\
&+\frac{[n]_{p,q}^5([n]_{p,q}(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)+4\alpha p^3q^5[n-3]_{p,q})}{q^{11}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}x^3\\
&+\frac{[n]_{p,q}^3}{p^5q^9([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big([n]_{p,q}^2(p^8+3p^7q+5p^6q^2
\end{aligned}
$$

$$
+\quad 5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3+2p^2q^4+pq^5)
$$

$$+(4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4+2p^6q^5+2p^5q^3+p^4q^4))$$
$$+(6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q})\Big)x^2$$
$$+\frac{[n]_{p,q}}{p^6q^6([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\Big([n]_{p,q}^3(p^6+2p^5q+p^4q^2$$
$$+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5+pq^3+q^6)+4\alpha[n]_{p,q}^2[n-3]_{p,q}$$
$$(2p^5q+p^4q^2+p^4q+p^3q^2)+6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7+p^5q^8)$$
$$+\alpha^3[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}p^6q^6\Big)x+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}.$$

□

We readily obtain the following lemma.

Lemma 2.3. *For $x\in[0,\infty)$, $0<q<p\leq 1$, $0\leq\alpha\leq\beta$, we have*

$$(i)D_{n,p,q}^{\alpha,\beta}((t-x);x) = \left(\frac{[n]_{p,q}}{([n]_{p,q}+\beta)}-1\right)x+\frac{\alpha}{([n]_{p,q}+\beta)},$$

$$(ii)D_{n,p,q}^{\alpha,\beta}((t-x)^2;x) = \left(\frac{p[n]_{p,q}^3}{[n-1]_{p,q}([n]_{p,q}+\beta)^2}-\frac{2[n]_{p,q}}{([n]_{p,q}+\beta)}+1\right)x^2$$
$$+\left(\frac{[n]_{p,q}}{([n]_{p,q}+\beta)^2}\left(\frac{2[2]_{p,q}[n]_{p,q}}{p[n-1]_{p,q}}+2\alpha\right)-\frac{2\alpha}{([n]_{p,q}+\beta)}\right)x$$
$$+\frac{\alpha^2}{([n]_{p,q}+\beta)^2},$$

$$(iii)D_{n,p,q}^{\alpha,\beta}((t-x)^4;x) = \left(\frac{p^6[n]_{p,q}^7}{q^{12}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\right.$$

$$-\frac{4p^3[n]_{p,q}^5}{q^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}$$
$$+\frac{6p[n]_{p,q}^3}{([n]_{p,q}+\beta)^2[n-1]_{p,q}}-\frac{4[n]_{p,q}}{([n]_{p,q}+\beta)}+1\Bigg)x^4$$
$$+\Bigg(\frac{[n]_{p,q}^5}{q^{11}([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}$$
$$\Big[[n]_{p,q}(p^5+3p^3q^2+2p^3q+2p^2q^3+pq^4+q^3)+4\alpha p^3q^5[n-3]_{p,q}\Big]$$
$$-\frac{4[n]_{p,q}^3([n]_{p,q}(p^3q+2p^2q^2+2p+q)+3p^2q^6\alpha[n-2]_{p,q})}{pq^6([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}$$
$$+\frac{6[n]_{p,q}([n]_{p,q}(pq+q^2)+2\alpha p[n-1]_{p,q})}{p([n]_{p,q}+\beta)^2[n-1]_{p,q}}-\frac{4\alpha}{([n]_{p,q}+\beta)}\Bigg)x^3$$

$$
\begin{aligned}
&+\Bigg(\frac{[n]^3_{p,q}}{p^5q^9([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\\
&\Big[[n]^2_{p,q}(p^8+3p^7q+5p^6q^2+5p^5q^3+2p^4q^4+p^4q^2+p^3q^4+2p^3q^3\\
&+2p^2q^4+pq^5)+4\alpha[n]_{p,q}[n-3]_{p,q}(p^7q^4+2p^6q^5+2p^5q^3+p^4q^4)\\
+\quad &6\alpha^2p^6q^9[n-2]_{p,q}[n-3]_{p,q}\Big]-\frac{4[n]_{p,q}}{p^3q^5([n]_{p,q}+\beta)^3[n-1]_{p,q}[n-2]_{p,q}}\\
&\Big[[n]^2_{p,q}(2p^2+pq+p+q)+3\alpha[n-2]_{p,q}(p^3q^6+p^2q^7)\\
&+3\alpha^2p^3q^5[n-1]_{p,q}[n-2]_{p,q}\Big]+\frac{6\alpha^2}{([n]_{p,q}+\beta)^2}\Bigg)x^2\\
&+\Bigg(\frac{[n]_{p,q}}{p^6q^6([n]_{p,q}+\beta)^4[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}\\
&\Big[[n]^3_{p,q}(p^6+2p^5q+p^4q^2+p^3q^3+p^3q^2+p^3q+2p^2q^4+2p^2q^2+2pq^5\\
&+pq^3+q^6)+4\alpha[n]^2_{p,q}[n-3]_{p,q}(2p^5q+p^4q^2+p^4q+p^3q^2)\\
&+6\alpha^2[n]_{p,q}[n-2]_{p,q}[n-3]_{p,q}(p^6q^7+p^5q^8)\\
&+\alpha^3p^6q^6[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}\Big]-\frac{4\alpha^3}{([n]_{p,q}+\beta)^3}\Bigg)x\\
&+\frac{\alpha^4}{([n]_{p,q}+\beta)^4}.
\end{aligned}
$$

3. Local approximation

In this section, we present local approximation theorem for operators $D^{\alpha,\beta}_{n,p,q}$. By $C_B[0,\infty)$, we denote the space of all real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\le x<\infty} \mid f(x) \mid .$$

Further, let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\| f-g \| +\delta \| g^{''} \|\}$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty): g^{'}, g^{''}\in C_B[0,\infty)\}$. By Theorem 2.4 of [6], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}) \tag{3.1}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)} \mid f(x+2h)-2f(x+h)+f(x) \mid$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \mid f(x+h) - f(x) \mid .$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$ and $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$. Then for all $n \in \mathbb{N}$, there exists an absolute constant $C > 0$ such that*

$$\mid D^{\alpha,\beta}_{n,p,q}(f;x) - f(x) \mid \leq C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

where

$$\delta_n(x) = \sqrt{D^{\alpha,\beta}_{n,p,q}((t-x)^2;x) + (\alpha_n(x))^2}, \quad \alpha_n(x) = \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta} - x.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\bar{D}^*_n$ defined by

$$\bar{D}^*_n(f;x) = D^{\alpha,\beta}_{n,p,q}(f;x) + f(x) - f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta}\right).$$

From Lemma 2.2 (i), (ii) and Lemma 2.3 (i), we observe that the operators $\bar{D}^*_n(f;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned}
\bar{D}^*_n(1;x) &= D^{\alpha,\beta}_{n,p,q}(1;x) + 1 - 1 = 1, \\
\bar{D}^*_n(t;x) &= D^{\alpha,\beta}_{n,p,q}(t;x) + x - \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta}\right) = x, \\
\bar{D}^*_n((t-x);x) &= \bar{D}^*_n(t;x) - x\bar{D}^*_n(1;x) = 0.
\end{aligned}$$

Let $x \in [0,\infty)$ and $g \in W^2$. Using the Taylor's formula

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying $\bar{D}^*_n$ to both sides of the above equation, we have

$$\begin{aligned}
\bar{D}^*_n(g;x) - g(x) &= g'(x)\bar{D}^*_n((t-x);x) + \bar{D}^*_n\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\
&= D^{\alpha,\beta}_{n,p,q}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\
&\quad - \int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}} \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x + \frac{\alpha}{[n]_{p,q}+\beta} - u\right) g''(u)\mathrm{d}u.
\end{aligned}$$

On the other hand, since

$$\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| \leq \int_x^t \mid t-u \mid\mid g''(u) \mid \mathrm{d}u \leq \parallel g'' \parallel \int_x^t \mid t-u \mid \mathrm{d}u \leq (t-x)^2 \parallel g'' \parallel$$

and

$$\left|\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}}\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-u\right)g''(u)\mathrm{d}u\right|$$

$$\leq \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-x\right)^2 \parallel g'' \parallel .$$

We conclude that

$$\begin{aligned}
\left|\bar{D}_n^*(g;x)-g(x)\right| &\leq \left|D_{n,p,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right)\right. \\
&\quad \left.-\int_x^{\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}}\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-u\right)g''(u)\mathrm{d}u\right| \\
&\leq \parallel g'' \parallel D_{n,p,q}^{\alpha,\beta}((t-x)^2;x)+\parallel g'' \parallel \left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}-x\right)^2 \\
&= \parallel g'' \parallel \delta_n^2(x).
\end{aligned}$$

Now, taking into account boundedness of $\bar{D}_n^*$, we have

$$\mid \bar{D}_n^*(f;x) \mid \quad \leq \quad \mid D_{n,p,q}^{\alpha,\beta}(f;x) \mid +2 \parallel f \parallel \leq 3 \parallel f \parallel .$$

Therefore

$$\begin{aligned}
\mid D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) \mid &\leq \mid \bar{D}_n^*(f-g;x)-(f-g)(x) \mid +\left|f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right| \\
&\quad + \mid \bar{D}_n^*(g;x)-g(x) \mid \\
&\leq \mid \bar{D}_n^*(f-g;x) \mid + \mid (f-g)(x) \mid +\left|f\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta}x+\frac{\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right| \\
&\quad + \mid \bar{D}_n^*(g;x)-g(x) \mid \\
&\leq 4 \parallel f-g \parallel +\omega(f,\alpha_n(x))+\delta_n^2(x) \parallel g'' \parallel .
\end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g \in W^2$, we have the following result

$$\mid D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) \mid \quad \leq \quad 4K_2(f,\delta_n^2(x))+\omega(f,\alpha_n(x)).$$

In view of the property of K-functional, we get

$$\mid D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) \mid \quad \leq \quad C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)).$$

This completes the proof of the theorem. □

4. Approximation properties in weighted spaces

Let $B_\rho[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $\mid f(x) \mid \leq M_f\rho(x)$, where M_f is a constant depending only on f and $\rho(x)$ is a weight function.

Let $C_\rho[0,\infty)$ be the space of all continuous functions in $B_\rho[0,\infty)$ with the norm
$\|f\|_\rho = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ and

$$C_\rho^0 = \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{\mid f(x) \mid}{\rho(x)} < \infty \right\}.$$

In what follows, we assume the weight function as $\rho(x) = 1 + x^2$.

Theorem 4.1. *Let $0 < q = q_n < p = p_n \leq 1$ such that $q_n \to 1$, $p_n \to 1$, as $n \to \infty$. For each $f \in C_\rho^0$, we have*

$$\lim_{n\to\infty} \|D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x)\|_\rho = 0.$$

Proof. With elementary calculations, it can be easily followed that $\lim\limits_{n\to\infty} \|D_{n,p_n,q_n}^{\alpha,\beta}(e_i;\cdot) - e_i\|_\rho = 0$, where $e_i(x) = x^i, i = 0,1,2$. By weighted Korovkin theorem given in [7], we get the required result. □

Next we give the following theorem to approximate all functions in C_ρ^0. This type of result is discussed in [10] for locally integrable functions.

Theorem 4.2. *Let $0 < q = q_n < p = p_n \leq 1$ such that $q_n \to 1$, $p_n \to 1$, $q_n^n \to 1$, $p_n^n \to 1$ as $n \to \infty$. For each $f \in C_\rho^0$ and $a > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} &\leq \sup_{x\leq x_0} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} + \sup_{x\geq x_0} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x) \mid}{(1+x^2)^{1+a}} \\
&\leq \|D_{n,p_n,q_n}^{\alpha,\beta}(f;x) - f(x)\|_{C[0,x_0]} \\
&\quad + \|f\|_\rho \sup_{x\geq x_0} \frac{\mid D_{n,p_n,q_n}^{\alpha,\beta}(1+t^2;x) \mid}{(1+x^2)^{1+a}} \\
&\quad + \sup_{x\geq x_0} \frac{\mid f(x) \mid}{(1+x_0^2)^{1+a}}
\end{aligned}$$

$$= I_1 + I_2 + I_3. \tag{4.1}$$

Since $| f(x) | \leq \|f\|_\rho (1 + x^2)$, we have

$$I_3 = \sup_{x \geq x_0} \frac{| f(x) |}{(1+x^2)^{1+a}} \leq \sup_{x \geq x_0} \frac{\|f\|_\rho}{(1+x^2)^a} \leq \frac{\|f\|_\rho}{(1+x_0^2)^a}$$

Let $\epsilon > 0$ be arbitrary. There exists $n_1 \in \mathbb{N}$ such that

$$\|f\|_\rho \frac{| D^{\alpha,\beta}_{n,p_n,q_n}(1+t^2;x) |}{(1+x^2)^{1+a}} < \frac{1}{(1+x^2)^{1+a}} \|f\|_\rho \left((1+x^2) + \frac{\epsilon}{3\|f\|_\rho} \right), \qquad \forall n \geq n_1$$

$$< \frac{\|f\|_\rho}{(1+x^2)^a} + \frac{\epsilon}{3} \qquad \forall n \geq n_1. \tag{4.2}$$

Hence

$$\|f\|_\rho \sup_{x \geq x_0} \frac{| D^{\alpha,\beta}_{n,p_n,q_n}(1+t^2;x) |}{(1+x^2)^{1+a}} < \frac{\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Thus

$$I_2 + I_3 < \frac{2\|f\|_\rho}{(1+x_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Now, let us choose x_0 to be so large that $\frac{\|f\|_\rho}{(1+x^2)^a} < \frac{\epsilon}{6}$.

Then,

$$I_2 + I_3 < \frac{2\epsilon}{3}, \qquad \forall n \geq n_1. \tag{4.3}$$

$$I_1 = \|D^{\alpha,\beta}_{n,p_n,q_n}(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3}, \qquad \forall n \geq n_2. \tag{4.4}$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (4.1)-(4.4), we get

$$\sup_{x \in [0,\infty)} \frac{| D^{\alpha,\beta}_{n,p_n,q_n}(f;x) - f(x) |}{(1+x^2)^{1+a}} < \epsilon, \qquad \forall n \geq n_0.$$

This completes the proof. □

Now we present ordinary approximation in terms of Lipschitz constant defined by

$$lip_M(\gamma) = \left\{ f \in C_B[0,\infty) : | f(t) - f(x) | \leq M \frac{| t-x |^\gamma}{(t+x)^{\frac{\gamma}{2}}} \right\}, \tag{4.5}$$

where M is a positive constant and $0 < \gamma \leq 1$.

Theorem 4.3. *Let be $f \in C_B[0,\infty)$, $0<q<p\leq 1$, $0\leq \alpha\leq \beta$, then for any $x\in(0,\infty)$, the following inequality holds:*

$$|D^{\alpha,\beta}_{n,p,q}(f;x)-f(x)| \leq M\left(\frac{\varphi^{(\alpha,\beta)}_{n,p,q}(x)}{x}\right)^{\frac{\gamma}{2}},$$

where $\varphi^{(\alpha,\beta)}_{n,p,q}(x)=D^{\alpha,\beta}_{n,p,q}((e_1-x)^2;x)$.

Proof. First, we prove that the result is true for $\gamma=1$. Then, for $f\in lip_M(\gamma)$, we obtain

$$\begin{aligned}|D^{\alpha,\beta}_{n,p,q}(f;x)-f(x)| &\leq \sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)^{n+k+1}_{p,q}} \\ &\quad\times\left|f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|d_{p,q}t \\ &\leq M\sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)^{n+k+1}_{p,q}} \\ &\quad\times\frac{\left|\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}-x\right|}{\sqrt{\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}+x}}d_{p,q}t.\end{aligned}$$

Using $\sqrt{x}<\sqrt{\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}+x}$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|D^{\alpha,\beta}_{n,p,q}(f;x)-f(x)| &\leq \frac{M}{\sqrt{x}}\sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)^{n+k+1}_{p,q}} \\ &\quad\times\left|\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}-x\right|d_{p,q}t \\ &= \frac{M}{\sqrt{x}}D^{\alpha,\beta}_{n,p,q}((e_1-x)^2;x)\leq M\sqrt{\frac{\varphi^{(\alpha,\beta)}_{n,p,q}(x)}{x}}.\end{aligned}$$

Therefore, the result is true for $\gamma=1$. We prove that the result is true for $0<\gamma\leq 1$, applying Hölder's inequality with $p=\frac{2}{\gamma}, q=\frac{1}{2-\gamma}$,

$$\begin{aligned}|D^{\alpha,\beta}_{n,p,q}(f;x)-f(x)| &\leq \sum_{k=0}^{\infty} s^{p,q}_{n,k}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)^{n+k+1}_{p,q}} \\ &\quad\times\left|f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|d_{p,q}t \\ &\leq \sum_{k=0}^{\infty}\left\{s^{p,q}_{n,k}(x)\left(\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)^{n+k+1}_{p,q}}\right.\right.\end{aligned}$$

$$\times\left|f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|d_{p,q}t\Big)^{\frac{2}{\gamma}}\Bigg\}^{\frac{\gamma}{2}}$$
$$\times\left\{\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}d_{p,q}t\right\}^{\frac{2-\gamma}{2}}$$

$$\leq \left\{\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\right.$$
$$\left.\times\left|f\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}\right)-f(x)\right|^{\frac{2}{\gamma}}d_{p,q}t\right\}^{\frac{\gamma}{2}}.$$

Since $f\in lip_M(\gamma)$, we have

$$\begin{aligned}|D_{n,p,q}^{\alpha,\beta}(f;x)-f(x) &\leq \frac{M}{x^{\frac{\gamma}{2}}}\left\{\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)\frac{1}{B_{p,q}(k,n+1)}\int_0^{\infty}\frac{t^{k-1}}{(1+pt)_{p,q}^{n+k+1}}\right.\\ &\quad\left.\times\left(\frac{[n]_{p,q}p^{k+1}qt+\alpha}{[n]_{p,q}+\beta}-x\right)^2d_{p,q}t\right\}^{\frac{\gamma}{2}}\\ &= \frac{M}{x^{\frac{\gamma}{2}}}\left(D_{n,p,q}^{\alpha,\beta}((e_1-x))^2;x)\right)^{\frac{\gamma}{2}}\leq M\left(\sqrt{\frac{\varphi_{n,p,q}^{(\alpha,\beta)}(x)}{x}}\right)^{\gamma}.\end{aligned}$$

Therefore, the proof is completed. □

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GENUINE MODIFIED BASKAKOV-DURRMEYER OPERATORS

Gulsum Ulusoy Ada

Abstract. The present paper deals with genuine Baskakov Durrmeyer operators which have preserved certain functions. We have obtained quantitative Voronovskaya and quantitative Grüss type Voronovskaya theorems using the weighted modulus of continuity. These results include the preservation properties of the classical genuine Baskakov Durrmeyer operators.
Keywords: Genuine Baskakov Durrmeyer operators; weighted modulus of continuity; Grüss Voronovskaya theorem.

1. Introduction

In a recent paper [22], Patel et al. considered a new construction of Baskakov operators on the unbounded interval $[0, \infty)$,

$$V_n^{\vartheta}(g;x) = \sum\nolimits_{l=0}^{\infty} (g \circ \vartheta^{-1}) \left(\frac{k}{n}\right) P_{n,k}^{\vartheta}(x), \tag{1.1}$$

where $P_{n,k}^{\vartheta}(x) = \binom{n+k-1}{k} \frac{(\vartheta(x))^k}{(1+\vartheta(x))^{n+k}}, n \in \mathbb{N}, x \in [0, \infty)$, ϑ is a continuous infinite times differentiable function satisfying the condition $\vartheta(1) = 0, \vartheta(0) = 0$ and $\vartheta'(x) > 0$ for $x \in [0, \infty)$. They investigated some direct theorems, asymptotic formula and A -statistical convergence. This function ϑ not only characterizes the operators but also characterizes the Korovkin set $\{1, \vartheta, \vartheta^2\}$ in a weighted function space. Inspired by this idea, many researchers studied in this direction, we can refer the readers to [[2],[3],[4],[5],[9],].

Very recently, Ada [8] have introduced Durrmeyer modifications of the operators (1.1):

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$$G_n^{\vartheta}(g;x) = (n-1)\sum_{l=0}^{\infty} P_{n,k}^{\vartheta}(x)\int_0^{\infty}(g\circ\vartheta^{-1})(u)p_{n,k}(u)du, \tag{1.2}$$

where $p_{n,k}(u)=\binom{n+k-1}{k}\frac{u^k}{(1+u)^{n+k}}$.

The operators defined in (1.2) are linear and positive. In case of $\vartheta(x)=x$, the operators in (1.2) reduce to well known Baskakov Durrmeyer operators.

Other useful modifications of positive linear operators are genuine types in approximation theory. These modifications for Bernstein durrmeyer operators were first considered by Chen [11]. Since then, many researchers have conducted studies in this field. Among the others, we refer the readers to [[10],[16],[19],[20],[21]].

In [7], the authors introduced a genuine type modification of the operators in (1.2) defined as

$$\begin{aligned} D_n^{\vartheta}(g;x) &= \sum_{k=1}^{\infty} P_{n,k}^{\vartheta}(x)\frac{1}{\beta(k,n+1)}\int_0^{\infty}\left(g\circ\vartheta^{-1}\right)(t)\,\frac{t^{k-1}}{(1+t)^{n+k+1}}dt \\ &\quad + P_{n,0}^{\vartheta}(x)\left(g\circ\vartheta^{-1}\right)(0). \end{aligned} \tag{1.3}$$

In this paper, we will continue to study further approximation properties of the operators (1.3). To describe the pointwise convergence of the operators, we prove a quantitative Voronovskaya type theorem. This quantitative Voronovskaya theorem tells us the rate of pointwise convergence and an upper bound for the error of the approximation. For some other quantitative versions of Voronovskaya's theorem, we can refer the readers to [1],[13],[14].

To prove the main results, we need following moments and central moments of our new operators.

2. Auxiliary results

Lemma 2.1. *We have*

$$D_n^{\vartheta}(1;x)=1,\ D_n^{\vartheta}(\vartheta;x)=\vartheta(x), \tag{2.1}$$

$$D_n^{\vartheta}(\vartheta^2;x)=\frac{\vartheta^2(x)(n+1)+2\vartheta(x)}{n-1}, \tag{2.2}$$

$$D_n^{\vartheta}(\vartheta^3;x)=\frac{\vartheta^3(x)(n+1)(n+2)+6\vartheta^2(x)(n+1)+6\vartheta(x)}{(n-1)(n-2)} \tag{2.3}$$

Lemma 2.2. *If we describe the central moment operator by*

$$M_{n,m}^{\vartheta}(x) = D_n^{\vartheta}\left((\vartheta(t) - \vartheta(x))^m; x\right)$$

then we get

$$M_{n,0}^{\vartheta}(x) = 1, \quad M_{n,1}^{\vartheta}(x) = 0 \tag{2.4}$$

$$M_{n,2}^{\vartheta}(x) = \frac{2\vartheta(x)(\vartheta(x)+1)}{n-1}. \tag{2.5}$$

$$M_{n,3}^{\vartheta}(x) = \frac{12\vartheta^3(x) + 18\vartheta^2(x) + 6\vartheta(x)}{(n-1)(n-2)}$$

$$M_{n,4}^{\vartheta}(x) = \frac{12\left[\vartheta^4(x)(n+7) + 2\vartheta^3(x)(n+7) + \vartheta^2(x)(n+9) + 2\vartheta(x)\right]}{(n-1)(n-2)(n-3)}$$

$$\begin{aligned} M_{n,6}^{\vartheta}(x) \quad = \quad & \tfrac{120}{(n-1)(n-2)(n-3)(n-4)(n-5)}\left[\vartheta^6(x)n^2 + 33n + 62\right) \\ & +3\vartheta^5(x)n^2 + 33n + 62) \\ & +3\vartheta^4(x)(n^2 + 36n + 75) \\ & +\vartheta^3(x)(n^2 + 51n + 140) \\ & +9\vartheta^2(x)(n+5) \\ & +6\vartheta(x)] \end{aligned}$$

for all $n, m \in \mathbb{N}$.

We suppose that:

(p_1) ϑ is a continuously differentiable function on $[0, \infty)$

(p_2) $\vartheta(0) = 0$, $\inf_{x\in[0,\infty)} \vartheta'(x) \geq 1$.

Let $\psi(x) = 1 + \vartheta^2(x)$ and $B_{\psi}(\mathbb{R}^+) = \{f : \ |f(x)| \leq M_f \psi(x)\}$, where M_f is constant which may depend only on f. $C_{\psi}(\mathbb{R}^+)$ denote the subspace of all continuous functions in $B_{\psi}(\mathbb{R}^+)$. By $C_{\psi}^*(\mathbb{R}^+)$, we denote the subspace off all functions $f \in C_{\psi}(\mathbb{R}^+)$ for which $\lim_{x\to\infty} f(x)/\psi(x)$ is finite. Also let $U_{\psi}(\mathbb{R}^+)$ be the space of functions $f \in C_{\psi}(\mathbb{R}^+)$ such that f/ψ is uniformly continuous. $B_{\psi}(\mathbb{R}^+)$ is the linear normed space with the norm $\|f\|_{\psi} = \sup_{x\in\mathbb{R}^+} |f(x)|/\psi(x)$.

The weighted modulus of continuity defined in [17] is as follows

$$\omega_{\vartheta}(f;\delta) = \sup_{\substack{x,t\in\mathbb{R}^+ \\ |\vartheta(t)-\vartheta(x)|\leq\delta}} \frac{|f(t) - f(x)|}{\psi(t) + \psi(x)}$$

for each $f \in C_{\psi}(\mathbb{R}^+)$ and for every $\delta > 0$. We observe that $\omega_{\vartheta}(f;0) = 0$ for every $f \in C_{\psi}(\mathbb{R}^+)$ and the function $\omega_{\vartheta}(f;\delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_{\psi}(\mathbb{R}^+)$ and also $\lim_{\delta\to 0} \omega_{\vartheta}(f;\delta) = 0$ for every $f \in U_{\psi}(\mathbb{R}^+)$.

Lemma 2.3. *([17])For every $f \in U_{\psi}(\mathbb{R}^{+})$, $\lim_{\delta\to 0}\omega_{\vartheta}(f;\delta)=0$ and*

$$|f(y)-f(x)| \leq (\psi(y)+\psi(x))\left(2+\frac{|\vartheta(y)-\vartheta(x)|}{\delta}\right)\omega_{\vartheta}(f,\delta). \tag{2.6}$$

Remark 2.1. If $\vartheta(x)=x$, then ω_{ϑ} is equivalent with Ω_2 given in [18]

$$\Omega_2(f,\delta)=\sup_{\substack{x,y\geq 0\\ |h|\leq\delta}}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

3. Main Results

Theorem 3.1. *If the function ϑ satisfies the conditions (p_1), (p_2) and $g''/\left(\vartheta'\right)^2, g'.\vartheta''/\left(\vartheta'\right)^3 \in C_{\psi}(\mathbb{R}^{+})$, then we get for any $x\in\mathbb{R}^{+}$ that*

$$\begin{aligned}
& n\left[D_n^{\vartheta}(g;x)-g(x)\right]-\left(\vartheta^2(x)+\vartheta(x)\right)D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x)) \\
\leq\ & 12\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^2 \\
& \times\left\{\omega_{\vartheta}\left(\frac{g''}{(\vartheta')^2},\delta_n^{\vartheta}(x)\right)+\omega_{\vartheta}\left(\frac{g'\vartheta''}{(\vartheta')^3},\delta_n^{\vartheta}(x)\right)\right\},
\end{aligned}$$

where $\delta_n^{\vartheta}(x)=\left(\frac{120(1+\vartheta(x))^6(n+16)^2}{(n-5)^4}\right)^{\frac{1}{4}}$.

Proof. By the Taylor expansion of $g\circ\vartheta^{-1}$ we get

$$\begin{aligned}
\left(g\circ\vartheta^{-1}\right)(\vartheta(t)) &= \left(g\circ\vartheta^{-1}\right)(\vartheta(x))+D\left(g\circ\vartheta^{-1}\right)(\vartheta(x))(\vartheta(t)-\vartheta(x)) \\
&\quad+\frac{D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x))(\vartheta(t)-\vartheta(x))^2}{2} \\
&\quad+h(t,x)(\vartheta(t)-\vartheta(x))^2,
\end{aligned} \tag{3.1}$$

where

$$h(t,x)=\frac{D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(\epsilon))-D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x))}{2}$$

and ϵ is a number between $\vartheta(x)$ and $\vartheta(t)$. We can get

$$\begin{aligned}
& \left|D_n^{\vartheta}(g;x)-g(x)-\frac{D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x))}{2}M_{n,2}^{\vartheta}(x)\right| \\
\leq\ & D_n^{\vartheta}\left(|h(t,x)|(\vartheta(t)-\vartheta(x))^2;x\right).
\end{aligned}$$

and using Lemma 2.2 we write

$$\begin{aligned} & \left| D_n^{\vartheta}(g;x) - g(x) - \frac{2\vartheta(x)(\vartheta(x)+1)}{n-1} \frac{D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x))}{2} \right| \\ \leq \; & D_n^{\vartheta}\left(|h(t,x)| \left(\vartheta(t)-\vartheta(x)\right)^2 ; x \right). \end{aligned}$$

In order to complete the proof, we estimate the $D_n^{\vartheta}\left(|h(t,x)| \left(\vartheta(t)-\vartheta(x)\right)^2 ; x \right)$. Since

$$\left(g\circ\vartheta^{-1}\right)''(\vartheta(t)) = \frac{g''(t)}{\left(\vartheta'(t)\right)^2} - g'(t)\frac{\vartheta''(t)}{\left(\vartheta'(t)\right)^3}$$

and we have

$$\begin{aligned} & \frac{\left(g\circ\vartheta^{-1}\right)''(\vartheta(\epsilon)) - \left(g\circ\vartheta^{-1}\right)''(\vartheta(x))}{2} \\ = \; & \frac{1}{2}\left\{ \frac{g''(\epsilon)}{\left(\vartheta'(\epsilon)\right)^2} - g'(\epsilon)\frac{\vartheta''(\epsilon)}{\left(\vartheta'(\epsilon)\right)^3} - \frac{g''(x)}{\left(\vartheta'(x)\right)^2} + g'(x)\frac{\vartheta''(x)}{\left(\vartheta'(x)\right)^3} \right\} \\ = \; & \frac{1}{2}\left\{ \frac{g''(\epsilon)}{\left(\vartheta'(\epsilon)\right)^2} - \frac{g''(x)}{\left(\vartheta'(x)\right)^2} + g'(x)\frac{\vartheta''(x)}{\left(\vartheta'(x)\right)^3} - g'(\epsilon)\frac{\vartheta''(\epsilon)}{\left(\vartheta'(\epsilon)\right)^3} \right\} \\ & \leq \left(\psi(t)+\psi(x)\right)\left(2+\frac{|\vartheta(t)-\vartheta(x)|}{\delta}\right) \\ & \times \left\{ \omega_{\vartheta}\left(\frac{g''}{\left(\vartheta'\right)^2},\delta\right) + \omega_{\vartheta}\left(\frac{g'\vartheta''}{\left(\vartheta'\right)^3},\delta\right) \right\}. \end{aligned}$$

In addition, since $\psi(t)+\psi(x) \leq \delta^2+2\vartheta^2(x)+2\vartheta(x)\delta+2$ whenever $|\vartheta(t)-\vartheta(x)| \leq \delta$, we have

$$\begin{aligned} |h(t,x)| \leq \; & 3\left(\delta^2+2\vartheta^2(x)+2\vartheta(x)\delta+2\right) \\ & \times \left\{ \omega_{\vartheta}\left(\frac{g''}{\left(\vartheta'\right)^2},\delta\right) + \omega_{\vartheta}\left(\frac{g'\vartheta''}{\left(\vartheta'\right)^3},\delta\right) \right\} \end{aligned}$$

and since $\psi(t)+\psi(x) \leq \left(\frac{\vartheta(t)-\vartheta(x)}{\delta}\right)^2\left(\delta^2+2\vartheta^2(x)+2\vartheta(x)\delta+2\right)$ whenever $|\vartheta(t)-\vartheta(x)| > \delta$, we have

$$\begin{aligned} |h(t,x)| \leq \; & 3\left(\delta^2+2\vartheta^2(x)+2\vartheta(x)\delta+2\right)\frac{|\vartheta(t)-\vartheta(x)|^4}{\delta^4} \\ & \times \left\{ \omega_{\vartheta}\left(\frac{g''}{\left(\vartheta'\right)^2},\delta\right) + \omega_{\vartheta}\left(\frac{g'\vartheta''}{\left(\vartheta'\right)^3},\delta\right) \right\}. \end{aligned}$$

Choosing $\delta < 1$ we deduce

$$\begin{aligned} |h(t,x)| &\leq 6\left(\vartheta^2(x)+\vartheta(x)+2\right)\left(\frac{(\vartheta(t)-\vartheta(x))^4}{\delta^4}+1\right) \\ &\times\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2},\delta\right)+\omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3},\delta\right)\right\}. \end{aligned}$$

Using Lemma 2.2 we have

$$\begin{aligned} & n\left[D_n^\vartheta(g;x)-g(x)\right]-\left(\vartheta^2(x)+\vartheta(x)\right)D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x)) \\ \leq\ & 6n\left(2+\vartheta(x)+\vartheta^2(x)\right)\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2},\delta\right)+\omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3},\delta\right)\right\} \\ & \times M_{n,2}^\vartheta(x)\left(1+\frac{1}{\delta^4}M_{n,6}^\vartheta(x)\right) \\ \leq\ & 6\left(2+\vartheta(x)+\vartheta^2(x)\right)\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2},\delta\right)+\omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3},\delta\right)\right\} \\ & \times\left\{2\vartheta(x)(\vartheta(x)+1)+\frac{1}{\delta^4}\left(\frac{120(1+\vartheta(x))^6(n+16)^2}{(n-5)^4}\right)\right\} \end{aligned}$$

and if we choose $\delta_n^\vartheta = \left(\frac{120(1+\vartheta(x))^6(n+16)^2}{(n-5)^4}\right)^{\frac{1}{4}}$ we get

$$\begin{aligned} & n\left[D_n^\vartheta(g;x)-g(x)\right]-\left(\vartheta^2(x)+\vartheta(x)\right)D^2\left(g\circ\vartheta^{-1}\right)(\vartheta(x)) \\ \leq\ & 6\left(2+\vartheta(x)+\vartheta^2(x)\right)\left(2\vartheta^2(x)+\vartheta(x)+1\right) \\ & \times\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2},\delta_n^\vartheta(x)\right)+\omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3},\delta_n^\vartheta(x)\right)\right\} \\ \leq\ & 12\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^2 \\ & \times\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2},\delta_n^\vartheta(x)\right)+\omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3},\delta_n^\vartheta(x)\right)\right\} \end{aligned}$$

which completes the proof. □

Corollary 3.1. *One has the following:*

1. *Let $g'' \in C_\psi(\mathbb{R}^+)$. The choice of $\vartheta(x)=x$ in Theorem 1 gives a quantitative Voronovskaya type theorem for T_n which defined in [12]*

$$\left|n\left[T_n(g;x)-g(x)\right]-\left(x^2+x\right)g''(x)\right| \leq 12(1+x)^4\,\Omega_2(g'';\delta_n(x))$$

where $\delta_n(x) = \left(\frac{120(1+x)^6(n+16)^2}{(n-5)^4}\right)^{\frac{1}{4}}$.

2. *Let $g''/(\vartheta')^2, g'\vartheta''/(\vartheta')^3 \in U_\psi(\mathbb{R}^+)$. If we take limit with $n \to \infty$ in Theorem 3.1 , we get the Voronovskaya theorem for D_n^ϑ*

$$\lim_{n\to\infty} n\left[D_n^\vartheta(g;x) - g(x)\right] = \left(\vartheta^2(x) + \vartheta(x)\right) D^2\left(g\circ\vartheta^{-1}\right)\vartheta(x).$$

3. *Let $g''/(\vartheta')^2, g'\vartheta''/(\vartheta')^3 \in U_\psi(\mathbb{R}^+)$. If $n \to \infty$ with $\vartheta(x) = x$ in Theorem 1, we obtain the Voronovskaya theorem for $T_n$which defined in [12]*

$$\lim_{n\to\infty} n\left[T_n(g;x) - g(x)\right] = \left(x^2 + x\right) g''(x).$$

The following results is a quantitative Grüss Voronovskaya type theorems. For some applications of Grüss inequalities in approximation theory, one can refer to [6],[15].

Theorem 3.2. *If $g, h, \frac{g'\vartheta''}{(\vartheta')^3}, \frac{h'\vartheta''}{(\vartheta')^3}, \frac{g''}{(\vartheta')^2}, \frac{h''}{(\vartheta')^2} \in C_\psi(\mathbb{R}^+)$ such that $\frac{(gh)'\vartheta''}{(\vartheta')^3} \frac{(gh)''\vartheta''}{(\vartheta')^2} \in C_\psi(\mathbb{R}^+)$, then we get at any point $x \in \mathbb{R}^+$ that*

$$\begin{aligned}
& n\left|D_n^\vartheta(gh;x) - D_n^\vartheta(g;x)D_n^\vartheta(h;x) - \frac{\mu_{n,2}^\vartheta(x)}{(\vartheta'(x))^2}\left\{g'(x)h'(x) - \frac{\vartheta''(x)(gh)'(x)}{\vartheta'(x)}\right\}\right| \\
\le\ & 12\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^2 \\
& \times\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2}, \delta_n^\vartheta(x)\right) + \omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3}, \delta_n^\vartheta(x)\right)\right\} \\
\le\ & 12\|g\|_\psi\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^3 \\
& \times\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2}, \delta_n^\vartheta(x)\right) + \omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3}, \delta_n^\vartheta(x)\right)\right\} \\
& +12\|h\|_\psi\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^3 \\
& \times\left\{\omega_\vartheta\left(\frac{g''}{(\vartheta')^2}, \delta_n^\vartheta(x)\right) + \omega_\vartheta\left(\frac{g'\vartheta''}{(\vartheta')^3}, \delta_n^\vartheta(x)\right)\right\} \\
& +nI_n(g)I_n(h),
\end{aligned}$$

where $I_n(g) = \frac{\psi(x)\left\|\left(g\circ\vartheta^{-1}\right)''\right\|_\psi}{2}\left(2\mu_{n,2}^\vartheta(x) + \frac{2\vartheta(x)}{\psi(x)}\mu_{n,3}^\vartheta(x) + \frac{1}{\psi(x)}\mu_{n,4}^\vartheta(x)\right)$ and $I_n(h)$ is the analogues one.

Proof. For $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
& D_n^\vartheta(gh;x) - D_n^\vartheta(g;x)D_n^\vartheta(h;x) - \mu_{n,2}^\vartheta(x)\frac{g'(x)h'(x)}{(\vartheta'(x))^2} \\
& -\mu_{n,2}^\vartheta(x)\frac{h(x)g'(x)\vartheta''(x)}{(\vartheta'(x))^3} - \mu_{n,2}^\vartheta(x)\frac{h'(x)g(x)\vartheta''(x)}{(\vartheta'(x))^3}
\end{aligned}$$

$$
\begin{aligned}
&= D_n^{\vartheta}(gh;x) - g(x)h(x) - \frac{\mu_{n,2}^{\vartheta}(x)}{2}\left(gh\circ\vartheta^{-1}\right)''(\vartheta(x)) \\
&\quad -g(x)\left[D_n^{\vartheta}(h;x) - h(x) - \frac{\mu_{n,2}^{\vartheta}(x)}{2}\left(h\circ\vartheta^{-1}\right)''(\vartheta(x))\right] \\
&\quad -h(x)\left[D_n^{\vartheta}(g;x) - g(x) - \frac{\mu_{n,2}^{\vartheta}(x)}{2}\left(g\circ\vartheta^{-1}\right)''(\vartheta(x))\right] \\
&\quad +\left(h(x) - D_n^{\vartheta}(h;x)\right)\left(D_n^{\vartheta}(g;x) - g(x)\right)
\end{aligned}
$$

so using (2.5) we can write

$$
\begin{aligned}
&\left| D_n^{\vartheta}(gh;x) - D_n^{\vartheta}(g;x)\, D_n^{\vartheta}(h;x) - \frac{\mu_{n,2}^{\vartheta}(x)}{(\vartheta'(x))^2}\left\{h'(x)g'(x) - \frac{\vartheta''(x)(gh)(x)}{(\vartheta'(x))}\right\}\right| \\
\leq\ & |A_1| + |A_2| + |A_3| + |A_4|.
\end{aligned}
$$

By Theorem 1, we have the estimates

$$
\begin{aligned}
|A_1| &\leq 12\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^2 \\
&\quad \times\left\{\omega_{\vartheta}\left(\frac{g''}{(\vartheta')^2}, \delta_n^{\vartheta}(x)\right) + \omega_{\vartheta}\left(\frac{g'\vartheta''}{(\vartheta')^3}, \delta_n^{\vartheta}(x)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
|A_2| &\leq 12\left\|g\right\|_{\psi}\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^3 \\
&\quad \times\left\{\omega_{\vartheta}\left(\frac{g''}{(\vartheta')^2}, \delta_n^{\vartheta}(x)\right) + \omega_{\vartheta}\left(\frac{g'\vartheta''}{(\vartheta')^3}, \delta_n^{\vartheta}(x)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
|A_3| &\leq 12\left\|h\right\|_{\psi}\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^3 \\
&\quad \times\left\{\omega_{\vartheta}\left(\frac{g''}{(\vartheta')^2}, \delta_n^{\vartheta}(x)\right) + \omega_{\vartheta}\left(\frac{g'\vartheta''}{(\vartheta')^3}, \delta_n^{\vartheta}(x)\right)\right\}.
\end{aligned}
$$

In addition we can write

$$
D_n^{\vartheta}(g;x) - g(x) = \left(g\circ\vartheta^{-1}\right)(\vartheta(x))\,\mu_{n,1}^{\vartheta}(x) + \frac{1}{2}D_n^{\vartheta}\left(\left(g\circ\vartheta^{-1}\right)''(\vartheta(\epsilon))\left(\vartheta(t)-\vartheta(x)\right)^2; x\right)
$$

hence we have

$$
\begin{aligned}
&\left|D_n^{\vartheta}(g;x) - g(x)\right| \\
\leq\ & \frac{1}{2}D_n^{\vartheta}\left(\left|\left(g\circ\vartheta^{-1}\right)''(\epsilon)\right|\left(\vartheta(t)-\vartheta(x)\right)^2; x\right) \\
\leq\ & \left\|\left(g\circ\vartheta^{-1}\right)''\right\|_{\psi}\frac{1}{2}D_n^{\vartheta}\left(\left(1+\vartheta^2(\epsilon)\right)\left(\vartheta(t)-\vartheta(x)\right)^2; x\right),
\end{aligned}
$$

where ϵ is an number between t and x. If $t<\epsilon<x$, then $1+\vartheta^2(\epsilon)\leq 1+\vartheta^2(x)$. In this case we get

$$\left|D_n^{\vartheta}(g;x)-g(x)\right| \leq \frac{\left\|\left(g\circ\vartheta^{-1}\right)''\right\|_{\psi}\psi(x)}{2}\mu_{n,2}^{\vartheta}(x)$$

or if $x<\epsilon<t$, then $1+\vartheta^2(\epsilon)\leq 1+\vartheta^2(t)$. In this case we get

$$\begin{aligned}\left|D_n^{\vartheta}(g;x)-g(x)\right| &\leq \frac{\left\|\left(g\circ\vartheta^{-1}\right)''\right\|_{\psi}}{2}D_n^{\vartheta}\left(\left(1+\vartheta^2(t)\right)\left(\vartheta(t)-\vartheta(x)\right)^2;x\right)\\ &= \frac{\left\|\left(g\circ\vartheta^{-1}\right)''\right\|_{\psi}}{2}\left(\left(1+\vartheta^2(x)\right)\mu_{n,2}^{\vartheta}(x)+2\vartheta(x)\mu_{n,3}^{\vartheta}(x)+\mu_{n,4}^{\vartheta}(x)\right).\end{aligned}$$

Therefore, for two cases of $\vartheta(\epsilon)$ we obtain

$$\begin{aligned}\left|D_n^{\vartheta}(g;x)-g(x)\right| &\leq \frac{\left\|\left(g\circ\vartheta^{-1}\right)''\right\|_{\psi}\psi(x)}{2}\left\{2\mu_{n,2}^{\vartheta}(x)+\frac{2\vartheta(x)}{\psi(x)}\mu_{n,3}^{\vartheta}(x)+\frac{1}{\psi(x)}\mu_{n,4}^{\vartheta}(x)\right\}.\\ &: = I_n(g)\end{aligned}$$

□

Corollary 3.2. *The following hold:*

1. *If $g,h,\ g'',h''\in C_{\psi}\left(\mathbb{R}^+\right)$ such that $(gh)''\in C_{\psi}\left(\mathbb{R}^+\right)$. The choice of $\vartheta(x)=x$ in Theorem 2 gives a quantitative Grüss Voronovskaya type theorem for T_n which defined in [12]*

$$\begin{aligned}&n\left|T_n(gh;x)-T_n(g;x)T_n(h;x)-(x^2+x)g'(x)h'(x)\right|\\ \leq\ & 12(2+x+x^2)(1+x)^2\Omega_2\left((gh)'';\delta_n(x)\right)\\ &+12\left\|g\right\|_{\psi}\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^3\Omega_2\left(g'';\delta_n(x)\right)\\ &+12\left\|h\right\|_{\psi}\left(2+\vartheta(x)+\vartheta^2(x)\right)(1+\vartheta(x))^3\Omega_2\left(h'';\delta_n(x)\right)\\ &+nI_n(g)I_n(h)\end{aligned}$$

$\delta_n(x)=\left(\frac{120(1+\vartheta(x))^6(n+16)^2}{(n-5)^4}\right)^{\frac{1}{4}}$.

2. *Let $g,h,\ g'',h''\in U_{\psi}\left(\mathbb{R}^+\right)$ such that $(gh)''\in U_{\psi}\left(\mathbb{R}^+\right)$. If $n\to\infty$ in Theorem 2, we obtain the Grüss Voronovskaya type theorem for D_n^{ϑ}:*

$$n\left|D_n^{\vartheta}(gh;x)-D_n^{\vartheta}(g;x)D_n^{\vartheta}(h;x)=\frac{\left(\vartheta(x)+\vartheta^2(x)\right)}{\left(\vartheta'(x)\right)^2}\left\{g'(x)h'(x)-\frac{\vartheta''(x)(gh)'(x)}{\vartheta'(x)}\right\}\right|$$

3. *Let $g,h,\ g'',h''U_{\psi}\left(\mathbb{R}^+\right)$ such that $(gh)''\in U_{\psi}\left(\mathbb{R}^+\right)$. If $n\to\infty$ with we select $\vartheta(x)=x$ in Theorem 2, we get the Grüss Voronovskaya type theorem for the operators T_n which defined in [12]:*

$$\lim_{n\to\infty}n\left|T_n(gh;x)-T_n(g;x)T_n(h;x)=(x^2+x)g'(x)h'(x)\right|.$$

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CHARACTERIZATION OF ORDERED SEMIGROUPS BASED ON ($\mathbb{k}$,q_k)-QUASI-COINCIDENT WITH RELATION

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Abstract. Based on generalized quasi-coincident with relation, new types of fuzzy bi-ideals of an ordered semigroup S are introduced. Level subset and characteristic functions are used to linked ordinary bi-ideals and $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideals of an ordered semigroup S. Further, upper/lower parts of $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideals of S are determined. Finally, some well-known classes of ordered semigroups like regular, left (resp. right) regular and completely regular ordered semigroups are characterized by the properties of $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideals..

Keywords: fuzzy bi-ideals; ordered semigroup; level subset; characteristic functions.

1. Introduction

Over the last few decades, the use of fuzzy set theory [29] has been accomplishing landmark achievements in contemporary mathematics. Several mathematical problems involving uncertainties in various fields like decision making, automata theory, coding theory, computer sciences, control engineering and economics cannot be dealt with through classical set theory (ordinary mathematical tools) due to crisp in nature. Crisp means dichotomous i.e., yes or no type rather than more or less type. In set theory, an element can either belong or not belong to a set. Zadeh's paper [29] on fuzzy sets has opened a new direction for researchers to tackle problems of uncertainties with a more appropriated mathematical tool. Presently, around the globe, the latest research and new investigations of fuzzy set theory is much productive due to the diverse applications in the aforementioned fields.

In algebraic framework, Rosenfeld [25] was the first to apply Zadeh's idea of fuzzy sets and introduce fuzzy subgroups. The inception of fuzzy subgroups provides a platform for other researchers to use this pioneering idea in other algebraic

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structures along with several diverse applications. Among other algebraic structures, semigroups (especially ordered semigroups) are having a lot of applications in error correcting codes, control engineering, performance of super computer and information sciences. Mordeson *et al.* idea in [23] gave birth to an up to date account of fuzzy subsemigroups and fuzzy ideals of semigroups while Kehayopulu and Tsingelis [10-12] used fuzzy sets in ordered semigroups to develop a fuzzy ideal theory. Shabir and Khan [28] gave a characterization of ordered semigroups by the properties of fuzzy ideals and fuzzy generalized bi-ideals.

In 1996, the idea of a quasi-coincidence of a fuzzy point with a fuzzy set [1, 2] was presente. It played a vital role in generating different types of fuzzy subgroups. Bhakat and Das [1] gave the concept of (α, β)-fuzzy subgroups and introduced $(\in, \in \vee q)$-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. In fact, this is an important and useful generalization of the Rosenfeld's idea of fuzzy subgroup [25]. Since then a verity of research has been carried out using this icebreaking idea. More precisely, (α, β)-fuzzy concept was used by Ma *et al.* [21, 22] in R_0-algebras, Davvaz and Mozafar [3] in Lie algebra, Davvaz and coauthors used the idea of generalized fuzzy sets in rings [4-6], Jun *et al.* [7], Khan and Shabir [13] and Khan *et al.* [14] in ordered semigroups, Shabir *et al.* [26, 27] in semigroups. In 2009, Jun *et al.* [8] initiated a more general form of quasi-coincident with relation (q) and provide (q_k) where $k \in [0, 1)$. The notion has been further strengthened by applying it at various algebraic structures [15-18]. Recently, Jun *et al.* [9] have presented another comprehensive generalization of fuzzy subgroups in light of generalized quasi-coincident with relation. Further, Khan *et al.* [19] elaborated ordered semigroups in terms of fuzzy generalized bi-ideals using this idea [9]. Also, Khan *et al.* [20] determined fuzzy filters of ordered semigroups for the said notion.

In this paper, we apply Jun's idea [9] in ordered semigroups to build a new sort of fuzzy bi-ideals and fuzzy left (resp. right) ideals i.e., $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals and $(\in, \in \vee (\Bbbk, q_k))$-fuzzy left (resp. right) ideals. Further, bridging between ordinary bi-ideals and $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals through level subsets and characteristic functions is a key milestone of the present paper. Moreover, the lower/upper parts of $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals are determined. Finally, several classes of ordered semigroups like regular, left and right regular, and completely regular ordered semigroups are characterized by the properties of $(\in, \in \vee (\Bbbk, q_k))$-fuzzy bi-ideals.

2. Mathematical Formulas

Due to the overarching role of algebraic structures like ordered semigroups in advanced fields such as computer sciences, error correcting codes, automata theory, robotics, control engineering and formal languages, the researchers often develop new structures to tackle the complicated problems faced in the aforementioned fields. The research at hand is a part of new contributions.

In this section, we present some fundamental definitions and results which will be used later on.

A structure $(S, \cdot, \leq)$ is called an **ordered semigroup** if it satisfies the following conditions:

$\cdot \longrightarrow (S, \cdot)$ is a semigroup,

$\cdot \longrightarrow (S, \leq)$ is a poset,

$\cdot \longrightarrow a \leq b \longrightarrow ax \leq bx$ and $a \leq b \longrightarrow xa \leq xb$ for all $a, b, x \in S$.

For subsets A, B of an ordered semigroup S, we denote by $AB = \{ab \in S | a \in A, b \in B\}$. If $A \subseteq S$ we denote $(A] = \{t \in S \mid t \leq h$ for some $h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. If $A, B \subseteq S$, then $A \subseteq (A]$, $(A](B] \subseteq (AB]$, and $((A]] = (A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a **subsemigroup** of S if $A^2 \subseteq A$. A non-empty subset A of S is called **left** (resp. **right**) ideal of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $SA \subseteq A$ (resp. $AS \subseteq A$).

A non-empty subset A of S is called an **ideal** if it is both left and right ideal of S.

A non-empty subset A of an ordered semigroup S is called a **bi-ideal** of S if

(i) $(\forall a \in S)(\forall b \in A)$ $(a \leq b \longrightarrow a \in A)$,

(ii) $A^2 \subseteq A$,

(iii) $ASA \subseteq A$.

An ordered semigroup S is **regular** if for every $a \in S$ there exists, $x \in S$ such that $a \leq axa$, or equivalently, we have (i) $a \in (aSa]$ $\forall a \in S$ and (ii) $A \subseteq (ASA]$ $\forall A \subseteq S$. An ordered semigroup S is called **left** (resp. **right**) **regular** if for every $a \in S$ there exists $x \in S$, such that $a \leq xa^2$(resp. $a \leq a^2x$), or equivalently, (i) $a \in (Sa^2]$(resp. $a \in (a^2S]$) $\forall a \in S$ and (ii) $A \subseteq (SA^2]$(resp. $A \subseteq (A^2S]$) $\forall A \subseteq S$. An ordered semigroup S is called **left** (resp. **right**) **simple** if for every left (resp. right) ideal A of S we have $A = S$ and S is called **simple** if it is both left and right simple. An ordered semigroup S is called **completely regular**, if it is left regular, right regular and regular.

Before 1965, the researchers were using traditional mathematical tools for modeling. Traditional tools are often dichotomous in nature. Dichotomous means yes "1" or no "0", therefore it could not handle problems involving uncertainties. In 1965, Zadeh was the first to introduce fuzzy sets (a new mathematical approach for dealing such problems of uncertainties). A function $\xi : S \longrightarrow [0, 1]$ is called a **fuzzy subset** of S. Since in classical set, the range of the function is $\{0, 1\}$ while in Zadeh's fuzzy set the range is $[0, 1]$, therefore, fuzzy sets are the generalizations of ordinary sets. The study of fuzzification of algebraic structures started in the pioneering paper of Rosenfeld [25] in 1971.

If ξ_1 and ξ_2 are fuzzy subsets of S, then $\xi_1 \preceq \xi_2$ means $\xi_1(x) \leq \xi_2(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets:

$$\xi_1 \wedge \xi_2 \quad : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \wedge \xi_2)(x) = \xi_1(x) \wedge \xi_2(x)$$
$$\xi_1 \vee \xi_2 \quad : S \longrightarrow [0,1] | x \longmapsto (\xi_1 \vee \xi_2)(x) = \xi_1(x) \vee \xi_2(x),$$

for all $x \in S$. A fuzzy subset ξ of S is called a **fuzzy subsemigroup** if $\xi(xy) \geq \xi(x) \wedge \xi(y)$ for all $x, y \in S$. A fuzzy subset ξ of S is called a **fuzzy left (resp. right)-ideal** of S if (i) $x \leq y \longrightarrow \xi(x) \geq \xi(y)$, (ii) $\xi(xy) \geq \xi(y)$ (resp. $\xi(xy) \geq \xi(x)$) for all $x, y \in S$. A fuzzy subset ξ of S is called a **fuzzy ideal** if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subsemigroup ξ is called a **fuzzy bi-ideal** of S if (i) $x \leq y \longrightarrow \xi(x) \geq \xi(y)$, (ii) $\xi(xyz) \geq \xi(x) \wedge \xi(z)$ for all $x, y, z \in S$. Let S be an ordered semigroup and ξ is a fuzzy subset of S. Then, for all $t \in (0,1]$, the set $U(\xi; t) = \{x \in S | \xi(x) \geq t\}$ is called a **level set** of ξ.

Theorem 2.1. *[7] A fuzzy subset ξ of an ordered semigroup S is a fuzzy bi-ideal of S if and only if $U(\xi; t)\,(\neq \emptyset)$ where $t \in (0,1]$ is a bi-ideal of S.*

Proof. If S is an ordered semigroup and A be any subset of S, then the characteristic function $\mathbb{C}_A$ of A is a function i.e., $\mathbb{C}_A : S \longrightarrow [0,1]$ and defined as

$$\begin{array}{cc} \{1 & if x \in A \\ 0 & otherwise\} \end{array}$$

□

Theorem 2.2. [7] A non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if the characteristic function $\mathbb{C}_A$ of A is a fuzzy bi-ideal of S.

Proof. if $a \in S$ and $\mathbb{A}$ is a non empty subset of $\mathbb{S}$. then,

$$A_a = \{(y, z) \in \mathbb{S} \times \mathbb{S} | a \leq yz\}$$

if ξ_1 and ξ_2 are two fuzzy subset of $\mathbb{S}$, then the product $\xi_1 \circ \xi_2$ of ξ_1 and ξ_2 is a function i.e, $\xi_1 \circ \xi_2 : \mathbb{S} \longmapsto [0,1]$ and defined as

$$(\xi_1 \circ \xi_2)(a) = \bigvee_{(y,z) \in A_a} (\xi_1(y) \cap \xi_2(z)) \;\; if A_a \neq \emptyset$$
$$0 \qquad\qquad otherwise$$

Let ξ be a fuzzy subset of $\mathbb{S}$, then the set of the form

$$\begin{array}{ll} \xi(y) = a \in (0,1] & if y = x \\ 0 & if y \neq x \end{array}$$

is called a **fuzzy point** [24] with support x and value a and is denoted by x_a. A fuzzy point x_a is said to **belong to** (resp. **quasi-coincident with**) a fuzzy set ξ, written as $x_a \in \xi$ (resp. $x_a q \xi$) if $\xi(x) \geq a$ (resp. $\xi(x) + a > 1$). If $x_a \in \xi$ or $x_a q \xi$, then we write $x_a \in \vee q \xi$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold. □

3. Fuzzy bi-ideals based on $(\Bbbk, q_k)$-quasi-coincident with relation

Aiming to describe the fuzzy ideals of an ordered semigroups in a more realistic way, the idea of quasi-coincident with relation [1, 2] has been proposed. The said notion played a vital role in generating several type of fuzzy subsystems which have already been used in a variety of productive research [3, 5-9, 13-21, 26, 27]. Jun *et al.* [9] further generalized his idea [8] and initiated an essential generalization of $(\in, \in \vee q)$-fuzzy subgroups. Keeping in view Jun's idea [9], we have introduced a new generalization of quasi-coincident with relation called $(\Bbbk, q_k)$-quasi-coincident with relation. In this section, new classification of an ordered semigorup S based on $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals where $\Bbbk$, $k \in [0,1]$ such that $0 \leq k < \Bbbk \leq 1$ is determined. A fuzzy point x_a is said to belong to (resp. $(\Bbbk, q)$-quasi-coincident with) a fuzzy set ξ, written as $x_a \in \xi$ (resp. $x_a(\Bbbk, q)\xi$) if $\xi(x) \geq a$ (resp. $\xi(x) + a > \Bbbk$). If $x_a \in \xi$ or $x_a(\Bbbk, q)\xi$, then we write $x_a \in \vee(\Bbbk, q)\xi$. The symbol $\overline{\in \vee(\Bbbk, q)}$ means $\in \vee(\Bbbk, q)$ does not hold. In ordered semigroups, generalizing the concept of $x_a(\Bbbk, q)\xi$, we define $x_a(\Bbbk, q_k)\xi$, as $\xi(x) + a + k > \Bbbk$, where $k, \Bbbk \in [0, 1)$ and $0 \leq k < \Bbbk \leq 1$. Note that $x_a q_k \xi$ implies $x_a(\Bbbk, q_k)\xi$, but the converse of the statement is not always true. Particularly, if $\Bbbk = 1$, then every $(\Bbbk, q_k)$-quasi-coincident with relation will lead to quasi-coincident with relation, symbolically $x_a(1, q_k)\xi = x_a q_k \xi$. Also, $x_a \in \bigvee(\Bbbk, q_k)\xi$ (resp. $x_a \in \wedge(\Bbbk, q_k)\xi$) means that $x_a \in \xi$ or $x_a(\Bbbk, q_k)\xi$) (resp. $x_a \in \xi$ and $x_a(\Bbbk, q_k)\xi$). In what follows, let S denote an ordered semigroup unless otherwise stated.

Definition 3.1. A fuzzy subset ξ of S is called an **$(\in, \in \vee(K, q_k))$-fuzzy bi-ideal** of S if it satisfies the conditions: (1) $(\forall x, y \in S)(\forall a \in (0,1])(x \leq y, y_a \in \xi \longrightarrow x_a \in \vee(K, q_k)\xi)$,
(2) $(\forall x, y \in S)(\forall a, b \in (0,1])(x_a \in \xi, y_a \in \xi \longrightarrow (xy)_{a\wedge b} \in \vee(K, q_k)\xi)$,
(3) $(\forall x, y, z \in S)(\forall a, b \in (0,1])(x_a \in \xi, z_b \in \xi \longrightarrow (xyz)_{a\wedge b} \in \vee(K, q_k)\xi)$.

Theorem 3.1. *Let A be a bi-ideal of S and ξ a fuzzy subset in S defined by:*

$$\xi(x) = \begin{cases} \geq \dfrac{\Bbbk - k}{2} & if x \in A, \\ 0 & otherwise \end{cases}$$

Then (1) ξ is a $((\Bbbk, q), \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

(2) ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

Proof. (1) Assume that $x, y \in S$, $x \leq y$ and $a \in (0,1]$ such that $y_a(\Bbbk, q)\xi$. Then $y \in A$, $\xi(y) + a > \Bbbk$. Since A is a bi-ideal of S and $x \leq y \in A$, therefore $x \in A$. Thus $\xi(x) \geq \frac{\Bbbk - k}{2}$. If $a \leq \frac{\Bbbk - k}{2}$, then $\xi(x) \geq a$ and so $x_a \in \xi$. If $a > \frac{\Bbbk - k}{2}$, then $\xi(x) + a + k > \frac{\Bbbk - k}{2} + \frac{\Bbbk - k}{2} + k = \Bbbk$ and so $x_a(\Bbbk, q_k)\xi$. Therefore, $x_a \in \vee(\Bbbk, q_k)\xi$.

Let $x, y \in S$ and $a, b \in (0,1]$ be such that $x_a(\Bbbk, q)\xi$ and $y_b(\Bbbk, q)\xi$. Then $x, y \in A$, so $\xi(x) + a > \Bbbk$ and $\xi(y) + b > \Bbbk$. Since A is a bi-ideal of S, hence $xy \in A$. Thus $\xi(xy) \geq \frac{\Bbbk - k}{2}$. If $a \wedge b > \frac{\Bbbk - k}{2}$, then $\xi(xy) + a \wedge b + k > \frac{\Bbbk - k}{2} + \frac{\Bbbk - k}{2} + k = \Bbbk$

and so $(xy)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq a \wedge b$ and so $(xy)_{a\wedge b} \in \xi$. It implies that, $(xy)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Let $x, y, z \in S$ and $a, b \in (0, 1]$ be such that $x_a(\Bbbk, q)\xi$ and $z_b(\Bbbk, q)\xi$. Then $x, z \in A$, $\xi(x) + a > \Bbbk$ and $\xi(z) + b > \Bbbk$. Since A is a bi-ideal of S, so $xyz \in A$. Hence $\xi(xyz) \geq \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xyz)+a\wedge b+k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $(xyz)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a\wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq a\wedge b$ and so $(xyz)_{a\wedge b} \in \xi$.Therefore, $(xyz)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Implies that ξ is a $((\Bbbk, q), \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. (2) Let $x, y \in S$, $x \leq y$ and $t \in (0, 1]$ be such that $y_a \in \xi$. Then $\xi(y) \geq a$ and $y \in A$. Since A is a bi-ideal of S and $x \leq y \in A$, we have $x \in A$. Thus $\xi(x) \geq \frac{\Bbbk-k}{2}$. If $a \leq \frac{\Bbbk-k}{2}$, then $\xi(x) \geq a$ and so $x_a \in \xi$. If $a > \frac{\Bbbk-k}{2}$, then $\xi(x) + a + k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $x_a(\Bbbk, q_k)\xi$. Therefore, $x_a \in \vee(\Bbbk, q_k)\xi$. Let $x, y \in S$ and $a, b \in (0, 1]$ be such that $x_a \in \xi$ and $y_b \in \xi$. Then $x, y \in A$. Since A is a bi-ideal of S, it leads to $xy \in A$. Thus $\xi(xy) \geq \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xy) + a \wedge b + k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $(xy)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq a \wedge b$ and so $(xy)_{a\wedge b} \in \xi$. Thus, $(xy)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Let $x, y, z \in S$ and $a, b \in (0, 1]$ be such that $x_a \in \xi$ and $z_b \in \xi$. Then $x, z \in A$. Since A is a bi-ideal of S, we have, $xyz \in A$. Hence $\xi(xyz) \geq \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xyz)+a\wedge b+k > \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$ and so $(xyz)_{a\wedge b}(\Bbbk, q_k)\xi$. If $a\wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq a\wedge b$ and so $(xyz)_{a\wedge b} \in \xi$. Hence, $(xyz)_{a\wedge b} \in \vee(\Bbbk, q_k)\xi$. Consequently, ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. □

If we take $\Bbbk = 1$ in Theorem (3.1), then we get the following corollary:

Corollary 3.1. *Let A be a bi-ideal of S and ξ a fuzzy subset in S defined by $\xi(x) \geq \frac{1-k}{2}$ if $x \in A, \xi(x) = 0$ if $x \notin A$, then, ξ is both $(q, \in \vee q_k)$ and $(\in, \in \vee q_k)$ type of fuzzy bi-ideal of S.*

Theorem 3.2. *Let ξ be a fuzzy subset of S. Then the following conditions are equivalent: (1) ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. (2) (i) $(\forall x, y \in S)(x \leq y \longrightarrow \xi(x) \geq \xi(y)\wedge \frac{\Bbbk-k}{2})$, (ii) $(\forall x, y \in S)(\xi(xy) \geq \xi(x)\wedge\xi(y)\wedge \frac{\Bbbk-k}{2})$, (iii) $(\forall x, y, z \in S)(\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2})$.*

Proof. (1) $\Longrightarrow$ (2): Suppose ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. On the contrary assume that, there exists $x, y \in S$, $x \leq y$ such that $\xi(x) < \xi(y) \wedge \frac{\Bbbk-k}{2}$. Choose $a \in (0, 1]$ such that $\xi(x) < a \leq \xi(y) \wedge \frac{\Bbbk-k}{2}$. Then, $y_a \in \xi$, but $\xi(x) < a$ and $\xi(x)+a+k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, so $x_a \overline{\in \vee(\Bbbk, q_k)}\xi$, which is a contradiction. Hence, $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$ for all $x, y \in S$ with $x \leq y$. If there exists $x, y \in S$ such that $\xi(xy) < \xi(x)\wedge\xi(y)\wedge\frac{\Bbbk-k}{2}$. Choose $a \in (0, 1]$ such that $\xi(xy) < a \leq \xi(x)\wedge\xi(y)\wedge \frac{\Bbbk-k}{2}$. Then, $x_a \in \xi, y_a \in \xi$ but $\xi(xy) < a$ and $\xi(xy) + a + k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, so $(xy)_a\overline{(\Bbbk, q_k)}\xi$. Thus, $(xy)_a\overline{\in \vee(\Bbbk, q_k)}\xi$, again a contradiction. Therefore, $\xi(xy) \geq \xi(x) \wedge \xi(y) \wedge \frac{\Bbbk-k}{2}$ for all $x, y \in S$. Now if there exists $x, y, z \in S$ such that $\xi(xyz) < \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$. Then, for $a \in (0, 1]$ such that $\xi(xyz) < a \leq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$, we have, $x_a \in \xi$ and $z_a \in \xi$ but $\xi(xyz) < a$ and $\xi(xyz) + a + k < \frac{\Bbbk-k}{2} + \frac{\Bbbk-k}{2} + k = \Bbbk$, so $(xyz)_a\overline{(\Bbbk, q_k)}\xi$. Thus,$(xyz)_a\overline{\in \vee(\Bbbk, q_k)}\xi$. Therefore, $\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$ for all $x, y, z \in S$.

(2)$\Longrightarrow$ (1): Let $y_a \in \xi$ for some $a \in (0,1]$. Then, $\xi(y) \geq a$. Now, $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2} \geq a \wedge \frac{\Bbbk-k}{2}$. If $a > \frac{\Bbbk-k}{2}$, then $\xi(x) \geq \frac{\Bbbk-k}{2}$ and $\xi(x)+a+k > \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k=\Bbbk$, it follows that $x_a(\Bbbk, q_k)\xi$. If $a \leq \frac{\Bbbk-k}{2}$, then $\xi(x) \geq a$ and so $x_a \in \xi$. Thus, $x_a \in \vee(\Bbbk, q_k)\xi$. Let $x_a \in \xi$ and $y_b \in \xi$ for some $a, b \in (0,1]$, then $\xi(x) \geq a$ and $\xi(y) \geq b$. Thus, $\xi(xy) \geq \xi(x) \wedge \xi(y) \wedge \frac{\Bbbk-k}{2} \geq a \wedge b \wedge \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq \frac{\Bbbk-k}{2}$ and $\xi(xy)+a \wedge b+k > \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k=\Bbbk$ and so $(xy)_{a \wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xy) \geq a \wedge b$ and hence, $(xy)_{a \wedge b} \in \xi$. Thus, $(xy)_{a \wedge b} \in \vee(\Bbbk, q_k)\xi$. Now let $x_a \in \xi$ and $z_b \in \xi$, then $\xi(x) \geq a$ and $\xi(z) \geq b$. Thus, $\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2} \geq a \wedge b \wedge \frac{\Bbbk-k}{2}$. If $a \wedge b > \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq \frac{\Bbbk-k}{2}$ and $\xi(xyz)+a \wedge b+k > \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k=\Bbbk$ and so $(xyz)_{a \wedge b}(\Bbbk, q_k)\xi$. If $a \wedge b \leq \frac{\Bbbk-k}{2}$, then $\xi(xyz) \geq a \wedge b$ and hence, $(xyz)_{a \wedge b} \in \xi$. Thus, $(xyz)_{a \wedge b} \in \vee(\Bbbk, q_k)\xi$. Consequently, ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. $\square$

taking $\Bbbk=1$, Theorem (3.2) leads to the result in [15].

Theorem 3.3. *If S is an ordered semigroup and ξ is fuzzy subset of S, then the following conditions are equivalent:*

(1) A fuzzy subset ξ of S is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

(2) $U(\xi; a)(\neq \emptyset)$ is a bi-ideal of S for all $a \in (0, \frac{\Bbbk-k}{2}]$.

Proof. (1) $\Longrightarrow$ (2): Suppose that ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S and let $x, y, z \in S$ be such that $x, z \in U(\xi; a)$ for some $a \in (0, \frac{\Bbbk-k}{2}]$. Then $\xi(x) \geq a$ and $\xi(z) \geq a$ and by hypothesis

$$\begin{aligned} \xi(xyz) & \geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk-k}{2} \\ & \geq a \wedge a \wedge \tfrac{\Bbbk-k}{2}=a. \end{aligned}$$

Hence, $xyz \in U(\xi; a)$. Also, by similar way, if $x, y \in S$ be such that $x, y \in U(\xi; a)$ for some $a \in (0, \frac{\Bbbk-k}{2}]$, then $xy \in U(\xi; a)$. Now let $x, y \in S$ be such that $y \in U(\xi; a)$ for some $a \in (0, \frac{\Bbbk-k}{2}]$. Then $\xi(y) \geq a$ and by hypothesis

$$\begin{aligned} \xi(x) & \geq \xi(y) \wedge \tfrac{\Bbbk-k}{2} \\ & \geq a \wedge \tfrac{\Bbbk-k}{2}=a. \end{aligned}$$

Hence $x \in U(\xi; a)$.
(2) $\Longrightarrow$ (1): Assume that $U(\xi; a)(\neq \emptyset)$ is a bi-ideal of S for all $a \in (0, \frac{\Bbbk-k}{2}]$. If there exists $x, y, z \in S$ such that $\xi(xyz) < \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$, then choose $a \in (0, \frac{\Bbbk-k}{2}]$ such that $\xi(xyz) < a \leq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$. Thus, $x, z \in U(\xi; t)$ but $xyz \notin U(\xi; a)$, a contradiction. Hence, $\xi(xyz) \geq \xi(x) \wedge \xi(z) \wedge \frac{\Bbbk-k}{2}$ for all $x, y, z \in S$ and $0 \leq k < \Bbbk \leq 1$.
Let $x, y \in S$ be such that $\xi(x) < \xi(y) \wedge \frac{\Bbbk-k}{2}$. Choose $r \in (0, \frac{\Bbbk-k}{2}]$ such that $\xi(x) < r \leq \xi(y) \wedge \frac{\Bbbk-k}{2}$ then $\xi(y) \geq r$ implies that $y_r \in \xi$ but $x_r \overline{\in} \xi$. Now $\xi(x)+r+k < \frac{\Bbbk-k}{2}+\frac{\Bbbk-k}{2}+k=\Bbbk$, which implies that $x_r \overline{(\Bbbk, q_k)} \xi$, a contradiction. Hence, $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$.

Table 3.1: Hasse diagram for $:\leq \{(a_1, a_2)\}$

$\cdot$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_2	a_1
a_4	a_1	a_1	a_2	a_2

By similar way, $\xi(xy) \geq \xi(x) \wedge \xi(y) \wedge \frac{\Bbbk - k}{2}$ for $x, y \in S$. Therefore, ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. $\square$

Example 3.1. Consider the ordered semigroup $S = \{a_1, a_2, a_3, a_4\}$ with the following multiplication and order relation

$$\leq := \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_1, a_2)\}.$$

The covering relation $\preceq := \{(a_1, a_2)\}$ is represented by table (3.1)

Then $\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}$ and $\{a_1, a_2, a_3, a_4\}$ are bi-ideals of S. Define a fuzzy subset ξ of S as follows:

S	a_1	a_2	a_3	a_4
$\xi(x)$	0.70	0.20	0.30	0.60

Then

$a \in (0, 1]$	$U(\xi; a)$
$0 < a \leq 0.20$	S
$0.20 < a \leq 0.30$	$\{a_1, a_3, a_4\}$
$0.30 < a \leq 0.60$	$\{a_1, a_4\}$
$0.60 < a \leq 1$	$\emptyset$

Therefore, using Theorem (3.3), ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S for $a \in (0, \frac{\Bbbk - k}{2}]$ with $\Bbbk = 0.8$ and $k = 0.4$.

If S is an ordered semigroup and ξ is a fuzzy subset of S, then define a set ξ_0 of S as follows:

$$\xi_0 = \{x \in S | \xi(x) > 0\}.$$

Proposition 3.1. *If ξ is a nonzero $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, then the subset ξ_0 of S is a bi-ideal of S.*

Proof. Let ξ be an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. If $x, y \in S$ such that $x \leq y$ and $y \in \xi_0$, then $\xi(y) > 0$. Since ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, therefore

$$\xi(x) \geq \xi(y) \wedge \frac{\Bbbk - k}{2} > 0,$$

Thus $\xi(x) > 0$ and so $x \in \xi_0$. Let $x, y \in \xi_0$. Then, $\xi(x) > 0$ and $\xi(y) > 0$. Now,

$$\begin{aligned} \xi(xy) \quad & \geq \xi(x) \wedge \xi(y) \wedge \tfrac{\Bbbk - k}{2} \\ & > 0, (\xi(x) > 0 and \xi(y) > 0). \end{aligned}$$

Thus $xy \in \xi_0$. For $x, z \in \xi_0$ we have

$$\begin{aligned} \xi(xyz) \quad & \geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk - k}{2} \\ & > 0. \end{aligned}$$

so $xyz \in \xi_0$. Consequently ξ_0 is a bi-ideal of S. $\square$

Lemma 3.1. A non-empty subset A of S is a bi-ideal if and only if the characteristic function $\mathbb{C}_A$ of A is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.

Proof. The proof follows from Theorem (3.3). $\square$

If $\{\xi_i\}_{i \in I}$ is an indexed family of fuzzy subsets of an ordered semigroup S, then the intersection $\bigcap\limits_{i \in I} \xi_i$ of ξ_i is defined as

$$\left(\bigcap_{i \in I} \xi_i \right)(x) = \{\xi_{i_1}(x) \wedge \xi_{i_2}(x) \wedge \xi_{i_3}(x) \wedge ... \mid i_i \in I\} = \bigwedge_{i \in I} (\xi_i(x)).$$

Proposition 3.2. *If $\{\xi_i : i \in I\}$ is a family of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of an ordered semigroup S. Then $\bigcap\limits_{i \in I} \xi_i$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.*

Proof. Let $\{\xi_i\}_{i \in I}$ be a family of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Let $x, y, z \in S$. Then,

$$\begin{aligned} \left(\bigcap_{i \in I} \xi_i \right)((xyz) \quad & = \bigwedge_{i \in I} \xi_i((xyz) \geq \bigwedge_{i \in I} (\xi_i(x) \wedge \xi_i(z) \wedge \tfrac{\Bbbk - k}{2}) \\ & = \left(\bigwedge_{i \in I} \left(\xi_i(x) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \bigwedge_{i \in I} \left(\xi_i(z) \wedge \tfrac{\Bbbk - k}{2}\right) \right) \\ & = \left(\bigcap_{i \in I} \xi_i \right)(x) \wedge \left(\bigcap_{i \in I} \xi_i \right)(z) \wedge \tfrac{\Bbbk - k}{2}. \end{aligned}$$

The remaining conditions for $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S can be proved in a similar way. Thus $\bigcap\limits_{i \in I} \xi_i$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. $\square$

For $\Bbbk = 1$, the Proposition (3.1,3.2), and Lemma (3.1) leads to [Proposition 3.8, Proposition 3.10, Lemma 3.9, [15]] respectively.

4. Upper and lower parts of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals

In this section, we first define the $(\Bbbk, q_k)$-upper/lower parts of an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Then by using the properties of bi-ideals, we characterize regular and intra-regular ordered semigroups in terms of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals.

Definition 4.1. *Let ξ_1 and ξ_2 be a fuzzy subsets of S. Then the fuzzy subsets* $\left(\overline{\xi_1}\right)_k^{\Bbbk}$, $(\xi_1 (\wedge)_k^{\Bbbk} \xi_2)^-$, $(\xi_1 (\vee)_k^{\Bbbk} \xi_2)^-$, $(\xi_1 (\circ)_k^{\Bbbk} \xi_2)^-$, $\left({\xi_1}^+\right)_k^{\Bbbk}$, $(\xi_1 (\wedge)_k^{\Bbbk} \xi_2)^+$, $(\xi_1 (\vee)_k^{\Bbbk} \xi_2)^+$ *and* $(\xi_1 (\circ)_k^{\Bbbk} \xi_2)^+$ *of S are defined as follows:*

$$\begin{aligned}
\left(\overline{\xi_1}\right)_k^{\Bbbk} \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1)_k^{\Bbbk}(x) = \xi_1(x) \wedge \tfrac{\Bbbk - k}{2},\\
(\xi_1 (\wedge)_k^{\Bbbk} \xi_2)^- \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 (\wedge)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \wedge \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2},\\
(\xi_1 (\vee)_k^{\Bbbk} \xi_2)^- \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 (\vee)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \vee \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2},\\
(\xi_1 (\circ)_k^{\Bbbk} \xi_2)^- \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 (\circ)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \circ \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2},
\end{aligned}$$

and

$$\begin{aligned}
\left({\xi_1}^+\right)_k^{\Bbbk} \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1)_k^{\Bbbk}(x) = \xi_1(x) \vee \tfrac{\Bbbk - k}{2},\\
(\xi_1 (\wedge)_k^{\Bbbk} \xi_2)^+ \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 (\wedge)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \wedge \xi_2)(x) \vee \tfrac{\Bbbk - k}{2},\\
(\xi_1 (\vee)_k^{\Bbbk} \xi_2)^+ \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 (\vee)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \vee \xi_2)(x) \vee \tfrac{\Bbbk - k}{2},\\
(\xi_1 (\circ)_k^{\Bbbk} \xi_2)^+ \quad & : S \longrightarrow [0,1] | x \longmapsto (\xi_1 (\circ)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \circ \xi_2)(x) \vee \tfrac{\Bbbk - k}{2},
\end{aligned}$$

for all $x \in S$.

Lemma 4.1. Let ξ_1 and ξ_2 be fuzzy subsets of S. Then the following hold:
(i) $\left(\xi_1 (\wedge)_k^{\Bbbk} \xi_2\right)^- = \left(\left(\overline{\xi_1}\right)_k^{\Bbbk} \wedge \left(\overline{\xi_2}\right)_k^{\Bbbk}\right)$,
(ii) $\left(\xi_1 (\vee)_k^{\Bbbk} \xi_2\right)^- = \left(\left(\overline{\xi_1}\right)_k^{\Bbbk} \vee \left(\overline{\xi_2}\right)_k^{\Bbbk}\right)$,
(iii) $\left(\xi_1 (\circ)_k^{\Bbbk} \xi_2\right)^- = \left(\left(\overline{\xi_1}\right)_k^{\Bbbk} \circ \left(\overline{\xi_2}\right)_k^{\Bbbk}\right)$.

Proof. (i) Let $x \in S$ and ξ_1 and ξ_2 be fuzzy subsets of an ordered semigroup S, then

$$\begin{aligned}
\left(\xi_1 (\wedge)_k^{\Bbbk} \xi_2\right)^- \quad & = (\xi_1 (\wedge)_k^{\Bbbk} \xi_2)(x) = (\xi_1 \wedge \xi_2)(x) \wedge \tfrac{\Bbbk - k}{2}\\
& = \xi_1(x) \wedge \xi_2(x) \wedge \tfrac{\Bbbk - k}{2}\\
& = \xi_1(x) \wedge \xi_2(x) \wedge \tfrac{\Bbbk - k}{2} \wedge \tfrac{\Bbbk - k}{2}\\
& = \left\{\xi_1(x) \wedge \tfrac{\Bbbk - k}{2}\right\} \wedge \left\{\xi_2(x) \wedge \tfrac{\Bbbk - k}{2}\right\}\\
& = (\xi_1)_k^{\Bbbk}(x) \wedge (\xi_2)_k^{\Bbbk}(x) = \left((\xi_1)_k^{\Bbbk} \wedge (\xi_2)_k^{\Bbbk}\right)(x).
\end{aligned}$$

The proof of part (ii) and (iii) is similar to the proof of part (i). □

Lemma 4.2. *Let ξ_1 and ξ_2 be fuzzy subsets of S. Then the following hold:*
(i) $\left(\xi_1 (\wedge)_k^{\Bbbk} \xi_2\right)^+ = \left(\left({\xi_1}^+\right)_k^{\Bbbk} \wedge \left({\xi_2}^+\right)_k^{\Bbbk}\right)$,
(ii) $\left(\xi_1 (\vee)_k^{\Bbbk} \xi_2\right)^+ = \left(\left({\xi_1}^+\right)_k^{\Bbbk} \vee \left({\xi_2}^+\right)_k^{\Bbbk}\right)$,
(iii) $\left(\xi_1 (\circ)_k^{\Bbbk} \xi_2\right)^+ \succeq \left(\left({\xi_1}^+\right)_k^{\Bbbk} \circ \left({\xi_2}^+\right)_k^{\Bbbk}\right)$ *if* $A_x = \emptyset$ *and* $\left(\xi_1 (\circ)_k^{\Bbbk} \xi_2\right)^+ = \left(\left({\xi_1}^+\right)_k^{\Bbbk} \circ \left({\xi_2}^+\right)_k^{\Bbbk}\right)$ *if* $A_x \neq \emptyset$.

Proof. The proof follows from Lemma (4.1). □

Let A be a non-empty subset of S, then the upper and lower parts of the characteristic function $\mathbb{C}_A$ are defined as follows:

$$\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk} : S \longrightarrow [0,1] | x \longmapsto \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \begin{cases} \frac{\Bbbk-k}{2} & if x \in A \\ 0 & otherwise. \end{cases}$$

$$\left({\mathbb{C}_A}^+\right)_k^{\Bbbk} : S \longrightarrow [0,1] | x \longmapsto \left({\mathbb{C}_A}^+\right)_k^{\Bbbk}(x) = \begin{cases} 1 & if x \in A \\ \frac{\Bbbk-k}{2} & otherwise. \end{cases}$$

Lemma 4.3. *Let A and B be non-empty subset of S. Then the following hold:*
(1) $\left(\mathbb{C}_A (\wedge)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}$,
(2) $\left(\mathbb{C}_A (\vee)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{A\cup B}\right)_k^{\Bbbk}$,
(3) $\left(\mathbb{C}_A (\circ)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}$.

Proof. (1) Let $x \in S$, if $x \in A \cap B$, then $\left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$. Also, since $x \in A \cap B$, implies that $x \in A$ and $x \in B$. Therefore, $\mathbb{C}_A(x) = 1$ and $\mathbb{C}_B(x) = 1$. Hence,

$$\begin{aligned} \left(\mathbb{C}_A (\wedge)_k^{\Bbbk} \mathbb{C}_B\right)^- &= \left(\mathbb{C}_A (\wedge)_k^{\Bbbk} \mathbb{C}_B\right)(x) = \mathbb{C}_A(x) \wedge \mathbb{C}_B(x) \wedge \tfrac{\Bbbk-k}{2} \\ &= 1 \wedge 1 \wedge \tfrac{\Bbbk-k}{2} = \tfrac{\Bbbk-k}{2}. \end{aligned}$$

Now if $x \notin A \cap B$, then $\left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}(x) = 0$. Assume that $x \notin A$, then $\left(\mathbb{C}_A (\wedge)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\mathbb{C}_A (\wedge)_k^{\Bbbk} \mathbb{C}_B\right)(x) = \mathbb{C}_A(x) \wedge \mathbb{C}_B(x) \wedge \frac{\Bbbk-k}{2} = 0 \wedge \mathbb{C}_B(x) \wedge \frac{\Bbbk-k}{2} = 0$.
Thus $\left(\mathbb{C}_A (\wedge)_k^{\Bbbk} \mathbb{C}_B\right)^- = \left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}$. The proof of part (2) follows from part (1).
(3) Assume $x \in S$, if $x \in (AB]$, then $\left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$ and $x \leq yz$ for some $y \in A$ and $z \in B$. Hence $(y, z) \in A_x$, so

$$\begin{aligned} \left(\mathbb{C}_A (\circ)_k^{\Bbbk} \mathbb{C}_B\right)(x) &= \left(\mathbb{C}_A \circ \mathbb{C}_B\right)(x) \wedge \tfrac{\Bbbk-k}{2} \\ &= \left\{ \bigvee_{(a,b)\in A_x} \left(\mathbb{C}_A(a) \wedge \mathbb{C}_B(b)\right) \right\} \wedge \tfrac{\Bbbk-k}{2} \\ &\geq \mathbb{C}_A(y) \wedge \mathbb{C}_B(z) \wedge \tfrac{\Bbbk-k}{2} \\ &= 1 \wedge 1 \wedge \tfrac{\Bbbk-k}{2} = \tfrac{\Bbbk-k}{2} = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}(x). \end{aligned}$$

conversely, since $\mathbb{C}_A \circ \mathbb{C}_B)(x) \leq 1$ for all $x \in S$. Therefore, $(\mathbb{C}_A\,(\circ)_k^{\mathbb{k}}\,\mathbb{C}_B)(x) = (\mathbb{C}_A \circ \mathbb{C}_B)(x) \wedge \frac{\mathbb{k}-k}{2} \leq 1 \wedge \frac{\mathbb{k}-k}{2} = \frac{\mathbb{k}-k}{2} = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\mathbb{k}}(x)$. Hence, $(\mathbb{C}_A\,(\circ)_k^{\mathbb{k}}\,\mathbb{C}_B)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\mathbb{k}}$ hold for $x \in (AB]$. By similar way, the require result also hold for $x \notin (AB]$. Consequently, $(\mathbb{C}_A\,(\circ)_k^{\mathbb{k}}\,\mathbb{C}_B)^- = \left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\mathbb{k}}$. □

Lemma 4.4. A non-empty subset A of an ordered semigroup S is a bi-ideal of S if and only if the lower part $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}$ of the characteristic function $\mathbb{C}_A$ of A is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideal of S.

Proof. If A is a bi-ideal of S, then by Theorem 2.2 and Lemma (3.1), $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}$ is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideal of S.
Conversely, suppose that $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}$ is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideal of S. Let $x, y \in S$ such that $x \leq y$. If $y \in A$, then $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(y) = \frac{\mathbb{k}-k}{2}$. Since $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}$ is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideal of S, and $x \leq y$, we have, $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(x) \geq \left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(y) \wedge \frac{\mathbb{k}-k}{2} = \frac{\mathbb{k}-k}{2}$. It follows that $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(x) = \frac{\mathbb{k}-k}{2}$ and so $x \in A$. Let $x, y \in A$. Then, $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(x) = \frac{\mathbb{k}-k}{2}$ and $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(y) = \frac{\mathbb{k}-k}{2}$. Now,

$$\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(xy) \geq \left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(x) \wedge \left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(y) \wedge \frac{\mathbb{k}-k}{2} = \frac{\mathbb{k}-k}{2}.$$

Hence, $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(xy) = \frac{\mathbb{k}-k}{2}$ and so $xy \in A$. Now, let $x, z \in A$ and $y \in S$. Then, $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(x) = \frac{\mathbb{k}-k}{2}$ and $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(z) = \frac{\mathbb{k}-k}{2}$, therefore we have,

$$\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(xyz) \geq \left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(x) \wedge \left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(z) \wedge \frac{\mathbb{k}-k}{2} = \frac{\mathbb{k}-k}{2}.$$

Hence, $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}(xyz) = \frac{\mathbb{k}-k}{2}$ and so $xyz \in A$. Therefore, A is a bi-ideal of S. □

Lemma 4.5. *A non-empty subset A of an ordered semigroup S is a left (resp. right)-ideal of S if and only if the lower part $\left(\overline{\mathbb{C}}_A\right)_k^{\mathbb{k}}$ of the characteristic function $\mathbb{C}_A$ of A is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy left (resp. right)-ideal of S.*

Proof. The proof follows from Lemma (4.4). □

In the following proposition, we show that if ξ is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideal of S, then $\left(\overline{\xi}\right)_k^{\mathbb{k}}$ is a fuzzy bi-ideal of S.

Proposition 4.1. *If ξ is an $(\in, \in \vee(\mathbb{k}, q_k))$-fuzzy bi-ideal of S, then $\left(\overline{\xi}\right)_k^{\mathbb{k}}$ is a fuzzy bi-ideal of S.*

Proof. Let $x, y \in S$, $x \leq y$. Since ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S and $x \leq y$, we have $\xi(x) \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$. It follows that $\xi(x) \wedge \frac{\Bbbk-k}{2} \geq \xi(y) \wedge \frac{\Bbbk-k}{2}$, and hence $\left(\overline{\xi}\right)_k^{\Bbbk}(x) \geq \left(\overline{\xi}\right)_k^{\Bbbk}(y)$. For $x, y \in S$, we have

$$\begin{aligned} \xi(xy) & \geq \xi(x) \wedge \xi(y) \wedge \tfrac{\Bbbk-k}{2} \\ \xi(xy) \wedge \frac{\Bbbk-k}{2} & \geq \xi(x) \wedge \xi(y) \wedge \tfrac{\Bbbk-k}{2} \wedge \tfrac{\Bbbk-k}{2} \\ & = \left(\xi(x) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \left(\xi(y) \wedge \tfrac{\Bbbk-k}{2}\right), \end{aligned}$$

and so $\left(\overline{\xi}\right)_k^{\Bbbk}(xy) \geq \left(\overline{\xi}\right)_k^{\Bbbk}(x) \wedge \left(\overline{\xi}\right)_k^{\Bbbk}(y)$.
Now for $x, y, z \in S$, we have

$$\begin{aligned} \xi(xyz) & \geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk-k}{2} \\ \xi(xy) \wedge \frac{\Bbbk-k}{2} & \geq \xi(x) \wedge \xi(z) \wedge \tfrac{\Bbbk-k}{2} \wedge \tfrac{\Bbbk-k}{2} \\ & = \left(\xi(x) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \left(\xi(z) \wedge \tfrac{\Bbbk-k}{2}\right), \end{aligned}$$

so $\left(\overline{\xi}\right)_k^{\Bbbk}(xyz) \geq \left(\overline{\xi}\right)_k^{\Bbbk}(x) \wedge \left(\overline{\xi}\right)_k^{\Bbbk}(z)$. Consequently, $\left(\overline{\xi}\right)_k^{\Bbbk}$ is a fuzzy bi-ideal of S. □

Numerous classes of ordered semigroups like regular, left (right) simple, completely regular and intra-regular ordered semigroups provide in-depth knowledge of semigroup theory. In the following, we characterize regular, left and right simple and completely regular ordered semigroups in terms of $(\in, \in \vee(\Bbbk, q_k))$-fuzzy left (resp. right) and $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals.

Lemma 4.6. [12] An ordered semigroup S is completely regular if and only if for every $A \subseteq S, A \subseteq (A^2SA^2]$ or for $a \in S$, we have $a \in (a^2Sa^2]$.

Theorem 4.1. *If S is an ordered semigroup, then the following conditions are equivalent:*
(1) S is completely regular.
(2) For every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal ξ of S, $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$ for all $a \in S$.

Proof. (1)$\Longrightarrow$ (2): Suppose S is completely regular and $a \in S$, by Lemma (4.6), $a \in (a^2Sa^2]$. Then there exists $x \in S$; such that $a \leq a^2xa^2$. Since ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, therefore

$$\begin{aligned} \xi(a) & \geq \xi(a^2xa^2) \wedge \tfrac{\Bbbk-k}{2} \\ & \geq \left(\xi(a^2) \wedge \xi(a^2) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} \\ & = \left(\xi(a^2 = a.a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} \\ & \geq \left(\xi(a) \wedge \xi(a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} \\ & = \left(\xi(a) \wedge \tfrac{\Bbbk-k}{2}\right) \end{aligned}$$

it follows that $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \xi(a) \wedge \frac{\Bbbk - k}{2} \geq \left(\xi(a^2) \wedge \frac{\Bbbk - k}{2}\right) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right) \geq \xi(a) \wedge \frac{\Bbbk - k}{2}$. Hence $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$ for all $a \in S$.

(2)$\Longrightarrow$ (1): Let $a \in S$, consider the bi-ideal $B\left(a^2\right) = (a^2 \cup a^4 \cup a^2 S a^2]$ of S generated by a^2, then by Lemma (3.1), $\mathbb{C}_{B(a^2)}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. By (2) $\left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}(a) = \left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}\left(a^2\right)$. Since $a^2 \in B\left(a^2\right)$, so $\left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}\left(a^2\right) = \frac{\Bbbk - k}{2}$. Thus $\left(\overline{\mathbb{C}}_{B(a^2)}\right)_k^{\Bbbk}(a) = \frac{\Bbbk - k}{2}$. Hence $a \in B\left(a^2\right)$ it implies that $a \leq a^2$ or $a \leq a^4$ or $a \leq a^2 x a^2$ for some $x \in S$. If $a \leq a^2$, then $a \leq a^2 = a.a \leq a^2.a^2 = a.a.a^2 \leq a^2 a a^2 \in a^2 S a^2$ and $a \in (a^2 S a^2]$. Similarly, if $a \leq a^4$ or $a \leq a^2 x a^2$ we get $a \in (r^2 S s^2]$ for some $r, s \in S$. Thus S is completely regular. □

An equivalence relation ρ on S is called congruence if $(x, y) \in \rho$ implies $(xz, yz) \in \rho$ and $(zx, zy) \in \rho$ for every $z \in S$. A congruence ρ on S is called semi lattice congruence [12] if $\left(x, x^2\right) \in \rho$ and $(xy, yx) \in \rho$. An ordered semigroup S is called a semi lattice of left and right simple semigroups if there exists a semi lattice congruence ρ on S such that the ρ-class $(x)_\rho$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$, or equivalently, there exists a semilattice Y and a family $\{S_i : i \in Y\}$ of left and right simple subsemigroups of S such that

$$S_i \cap S_j = \emptyset i \neq j, S = \bigcup_{i \in Y} S_i, S_i S_j \subseteq S_{ij} \forall i, j \in Y.$$

A subset P of S is called semiprime [7], if for every $a \in S$ such that $a^2 \in P$, we have $a \in P$, or equivalently, for each subset A of S, such that $A^2 \subseteq P$, implies that have $A \subseteq P$.

Let $\mathbb{N}$ be the equivalence relation on S which is denoted by

$$\mathbb{N} = \{(a, b) \in S \times S \mid N(x) = N(y)\}.$$

Lemma 4.7. [12] Let S be an ordered semigroup, then the following conditions are equivalent:

(i) $(x)_{\mathbb{N}}$ is a left (resp. right) simple subsemigroup of S, for every $x \in S$.

(ii) Every left (resp. right) ideal of S is a right (resp. left) ideal of S and semiprime.

Lemma 4.8. [12] An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for all bi-ideals A and B of S, we have $(A^2] = A$ and $(B^2] = B$.

Theorem 4.2. *An ordered semigroup S is a semilattice of left and right simple semigroups if and only if for every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal ξ of S, then for all $a, b \in S$,*

(i) $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}\left(a^2\right)$.

(ii) $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}(ba)$.

Proof. If ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal, then by hypothesis, there exists a semilattice Y and a family $\{\xi_i : i \in Y\}$of left and right simple subsemigroups of S such that $S_i \cap S_j = \emptyset$, $i \neq j, S = \bigcup_{i \in Y} S_i, S_i S_j \subseteq S_{ij}$ for all $i, j \in Y$. (i): Let $a \in S$, then there exists Y such that $a \in S_i$, as S_i is left and right simple, thus $(S_i a] = S_i$ and $(aS_i] = S_i$. Therefore, $S_i = (aS_i] = (a((S_i a]] = (aS_i a]$. Since $a \in (aS_i a]$ so there exists $x \in S_i$ such that $a \leq axa$. Since $x \in S_i$, $x \leq aya$ for some $y \in S_i$. Therefore, $a \leq axa \leq a(aya)a = a^2ya^2$ implies that $a \in (a^2Sa^2]$. Hence by Lemma (4.6) and Theorem (4.1), $\left(\overline{\xi}\right)_k^{\Bbbk}(a) = \left(\overline{\xi}\right)_k^{\Bbbk}(a^2)$.
(ii) Now to prove $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}(ba)$, let $a, b \in S$, then by (i), $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^2\right) = \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^4\right)$. Also, by Lemma (4.8),

$$\begin{aligned}
(ab)^4 &= (ab)^2\,(ab)^2 = (ab)(ab)(ab)(ab)\\
&= (aba)(babab) \in B(aba)B(babab)\\
&\subseteq (B(aba)B(babab)] = (B(babab)B(aba)]\\
&= (B(babab)\,(B(aba))^2] = (B(babab)B(aba)B(aba)]\\
&\subseteq ((babab)S(aba)(aba)] \subseteq (bababSaba].
\end{aligned}$$

Hence $(ab)^4 \leq (babab)z(aba)$ for some $z \in S$. As ξ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, hence

$$\begin{aligned}
\xi((ab)^4) &\geq \xi\left((babab)z(aba)\right) \wedge \tfrac{\Bbbk-k}{2}\\
&= \xi((ba)(babza)(ba)) \wedge \tfrac{\Bbbk-k}{2}\\
&\geq \left(\xi((ba) \wedge (ba)) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2}\\
&= \xi(ba) \wedge \tfrac{\Bbbk-k}{2}\\
\xi((ab)^4) \wedge \frac{\Bbbk-k}{2} &\geq \left(\xi(ba) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2}
\end{aligned}$$

implies that $\left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^4\right) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ba))$. Thus

$$\begin{aligned}
\left(\overline{\xi}\right)_k^{\Bbbk}(ab) &= \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^2\right)\\
&= \left(\overline{\xi}\right)_k^{\Bbbk}\left((ab)^4\right) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ba))
\end{aligned}$$

leads to $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ba))$. In a similar way $\left(\overline{\xi}\right)_k^{\Bbbk}(ba) \geq \left(\overline{\xi}\right)_k^{\Bbbk}((ab))$ can be shown. Hence $\left(\overline{\xi}\right)_k^{\Bbbk}(ab) = \left(\overline{\xi}\right)_k^{\Bbbk}(ba)$.
Conversely, we know that $\mathbb{N}$ is a semilattice of left and right simple semigroups, so by Lemma (4.7), it is enough to prove that every left (resp. right) ideal of S is an ideal of S. Let L be a left ideal of S and let $a \in L$ and $t \in S$. Since L is a left ideal of S, by Lemma (4.4), $\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy left ideal of S. Hence $\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(at) = \left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(ta)$. As $ta \in SL \subseteq L$ it implies that $\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(ta) = \frac{\Bbbk-k}{2}$. So $at \in L$ that is $LS \subseteq L$. Thus, L is right ideal. if $a^2 \in L$, by hypothesis

$\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}\left(a^2\right)=\frac{\Bbbk-k}{2}=\left(\overline{\mathbb{C}}_L\right)_k^{\Bbbk}(a)$. Thus $a\in L$, so L is semiprime. Similarly, we can prove that right ideal R is left ideal of S and semiprime. □

Proposition 4.2. *If $\{\xi_i : i\in I\}$ is a family of $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideals of an ordered semigroup S. Then, $\bigcap_{i\in I}\bigcap_{i\in I}\left(\overline{\xi_i}\right)_k^{\Bbbk}$ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S.*

Corollary 4.1. *Let S be an ordered semigroup and f_1 and f_2 be fuzzy subsets of S. Then, $\left(f_1\,(\wedge)_k^{\Bbbk}\,f_2\right)^-$ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S.*

Definition 4.2. *An $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal ξ of S is called idempotent if $\xi\,(\wedge)_k^{\Bbbk}\,\xi=\xi$.*

Theorem 4.3. *If ξ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S, then $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-\preceq\left(\overline{\xi}\right)_k^{\Bbbk}$.*

Proof. Suppose that ξ is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S and $a\in S$. If $A_a=\emptyset$, then $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-(a)=(\xi\circ\xi)(a)\wedge\frac{\Bbbk-k}{2}=0\wedge\frac{\Bbbk-k}{2}=0\leq\left(\overline{\xi}\right)_k^{\Bbbk}(a)$. Thus, $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-\preceq\left(\overline{\xi}\right)_k^{\Bbbk}$ hold in this case. Now let $A_a\neq\emptyset$, then

$$\begin{aligned}\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-(a) &= (\xi\circ\xi)(a)\wedge\tfrac{\Bbbk-k}{2}\\ &=\left\{\bigvee_{y,z\in A}\left(\xi(y)\wedge\xi(z)\right)\right\}\wedge\tfrac{\Bbbk-k}{2}\\ &\leq\left\{\bigvee_{y,z\in A}\xi(yz)\right\}\wedge\tfrac{\Bbbk-k}{2}\\ &\leq\left\{\bigvee_{y,z\in A}\xi(a)\right\}\wedge\tfrac{\Bbbk-k}{2}=\xi(a)\wedge\tfrac{\Bbbk-k}{2}=\left(\overline{\xi}\right)_k^{\Bbbk}(a).\end{aligned}$$

Hence, $\left(\xi\,(\circ)_k^{\Bbbk}\,\xi\right)^-\preceq\left(\overline{\xi}\right)_k^{\Bbbk}$. □

Lemma 4.9. *Every $(\in,\in\vee(\Bbbk,q_k))$-fuzzy one-sided ideal of S is an $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideal of S.*

Proof. The proof is straightforward. □

If S is an ordered semigroup, then we define the fuzzy subsets "1" and "0" as follows:

$$\begin{aligned}1 &: S\longrightarrow[0,1]|x\longrightarrow 1(x)=1,\\ 0 &: S\longrightarrow[0,1]|x\longrightarrow 0(x)=0,\end{aligned}$$

for all $x\in S$.

Lemma 4.10. *Let S be an ordered semigroup and f_1 and f_2 be fuzzy subsets of S. Then, $(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \preceq (1 \,(\circ)_k^{\Bbbk}\, f_2)^-$ (resp. $(f_1 \,(\circ)_k^{\Bbbk}\, f_2)^- \preceq (f_1 \,(\circ)_k^{\Bbbk}\, 1)^-$).*

Proof. The proof is straightforward. □

Lemma 4.11. *Let S be an ordered semigroup and f an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Then, $(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^- \preceq \left(\overline{f}\right)_k^{\Bbbk}$.*

Proof. Let $a \in S$. If $A_a = \emptyset$, then

$$(f \,(\circ)_k^{\Bbbk}\, 1 \,(\circ)_k^{\Bbbk}\, f)^-(a) = (f \circ 1 \circ f)(a) \wedge \frac{\Bbbk - k}{2} = 0 \wedge \frac{\Bbbk - k}{2} = 0 \leq \left(\overline{f}\right)_k^{\Bbbk}(a).$$

Let $A_a \neq \emptyset$, then

$$\begin{aligned}
(f \circ^k 1 \circ^k f)^-(a) &= (f \circ 1 \circ f)(a) \wedge \tfrac{\Bbbk - k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} (f(y) \wedge (1 \circ f)(z)\right] \wedge \tfrac{\Bbbk - k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \left(f(y) \wedge \left\{\bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} (1(p) \wedge f(q))\right\}\right)\right] \wedge \tfrac{\Bbbk - k}{2} \\
&= \bigvee\nolimits_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} (f(y) \wedge 1(p) \wedge f(q)) \wedge \tfrac{\Bbbk - k}{2} \\
&= \bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} (f(y) \wedge f(q)) \wedge \tfrac{\Bbbk - k}{2} \\
&= \bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} \left(\left(f(y) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \tfrac{\Bbbk - k}{2}\right)\right) \wedge \tfrac{\Bbbk - k}{2}.
\end{aligned}$$

Since $a \leq yz \leq y(pq) = ypq$ and f is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S, so we have,

$$\begin{aligned}
f(a) &\geq f(ypq) \wedge \tfrac{\Bbbk - k}{2} \geq \left(f(y) \wedge f(q) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \tfrac{\Bbbk - k}{2} \\
&= \left\{\left(f(y) \wedge \tfrac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \tfrac{\Bbbk - k}{2}\right)\right\} \wedge \tfrac{\Bbbk - k}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} \bigvee_{(p,q)\in A_z} \bigvee_{(p,q)\in A_z} \left(\left(f(y) \wedge \frac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \frac{\Bbbk - k}{2}\right)\right) \wedge \frac{\Bbbk - k}{2} \\
&\leq \bigvee_{(y,pq)\in A_a} \bigvee_{(y,pq)\in A_a} \left(\left(f(y) \wedge \frac{\Bbbk - k}{2}\right) \wedge \left(f(q) \wedge \frac{\Bbbk - k}{2}\right)\right) \wedge \frac{\Bbbk - k}{2} \\
&\leq \bigvee_{(y,pq)\in A_a} \bigvee_{(y,pq)\in A_a} f(a) \wedge \frac{\Bbbk - k}{2} = \left(\overline{f}\right)_k^{\Bbbk}(a).
\end{aligned}$$

□

Lemma 4.12. [7] Let S be an ordered semigroup. Then the following are equivalent:
(i) S is regular,
(ii) $B = (BSB]$ for all bi-ideals B of S,
(iii) $B(a) = (B(a)SB(a)]$ for every $a \in S$.

Theorem 4.4. *If S is an ordered semigroup and f is a fuzzy subset of S, then the following conditions are equivalent:*
(1) S *is regular.*
(2) *For every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal f of S,* $(f\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f)^- = \left(\overline{f}\right)_k^{\Bbbk}$.

Proof. (1) $\Longrightarrow$ (2): Let S is regular and f is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Assume $a \in S$. then, there exists $x \in S$ such that $a \leq axa \leq ax(axa) = a(xaxa)$. So $(a, xaxa) \in A_a$, and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
(f\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f)^-(a) &= (f \circ 1 \circ f)(a) \wedge \tfrac{\Bbbk-k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} (f(y) \wedge (1 \circ f)(z))\right] \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \left(f(a) \wedge (1 \circ f)(xaxa) \wedge \tfrac{\Bbbk-k}{2}\right) \\
&= \left[f(a) \wedge \bigvee_{(p,q)\in A_{xaxa}} \bigvee_{(p,q)\in A_{xaxa}} \{1(p) \wedge f(q)\}\right] \wedge \tfrac{\Bbbk-k}{2} \\
&\geq (f(a) \wedge \{1(xax) \wedge f(a)\}) \wedge \tfrac{\Bbbk-k}{2} \\
&= (f(a) \wedge \{1 \wedge f(a)\}) \wedge \tfrac{\Bbbk-k}{2} \\
&= (f(a) \wedge f(a)) \wedge \tfrac{\Bbbk-k}{2} = f(a) \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(\overline{f}\right)_k^{\Bbbk}(a).
\end{aligned}$$

On the other hand, by Lemma (4.11), we have, $(f\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f)^-(a) \leq \left(\overline{f}\right)_k^{\Bbbk}(a)$. Therefore, $(f\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f)^-(a) = \left(\overline{f}\right)_k^{\Bbbk}(a)$.
(2) $\Longrightarrow$ (1): Suppose that $(f\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f)^- = \left(\overline{f}\right)_k^{\Bbbk}$ for every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal f of S. To prove that S is regular, by Lemma (4.12), it is enough to prove that

$$A = (ASA] \forall\, bi-ideals\, A\, of\, S.$$

Let $x \in A$. Since A is a bi-ideal of S, by Lemma (4.4), $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. By hypothesis, $(\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^-(x) = \left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x)$. Since $x \in A$, we have $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$. Thus, $(\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^-(x) = \frac{\Bbbk-k}{2}$. But, by Lemma (4.3), we have $(\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^- = \left(\overline{\mathbb{C}}_{(ASA]}\right)_k^{\Bbbk}$, and $\left(\overline{\mathbb{C}}_{(ASA]}\right)_k^{\Bbbk}(x) = \frac{\Bbbk-k}{2}$, hence we have, $x \in (ASA]$ and so $A \subseteq (ASA]$. On the other hand, since A is a bi-ideal of S, we have $(ASA] \subseteq (A] = A$. Hence, $A = (ASA]$. Therefore, S is regular. $\square$

Lemma 4.13. *Let f_1 and f_2 be $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Then $(f_1\,(\circ)_k^{\Bbbk}\, f_2)^-$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S.*

Proof. The proof is straightforward. $\square$

Lemma 4.14. *Let S be an ordered semigroup. Then the following are equivalent:*
(i) S is both regular and intra-regular,
(ii) $A = (A^2]$ for every bi-ideals A of S,
(iii) $A \cap B = (AB] \cap (BA]$ for all bi-ideals A, B of S.

Theorem 4.5. *Let S be an ordered semigroup. Then the following are equivalent:*
(i) S is both regular and intra-regular,
(ii) $(f\,(\circ)_k^{\Bbbk}\, f)^- = \left(\overline{f}\right)_k^{\Bbbk}$ for every $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals f of S,
(iii) $(f_1\,(\wedge)_k^{\Bbbk}\, f_2)^- = \Big((f_1\,(\circ)_k^{\Bbbk}\, f_2) \wedge_k^{\Bbbk} (f_2\,(\circ)_k^{\Bbbk}\, f_1)^-\Big)^-$ for all $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals f_1 and f_2 of S.

Proof. (i)$\Longrightarrow$(ii). Let F be an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S and $a \in S$. Since S is regular and intra-regular, there exist $x, y, z \in S$ such that $a \leq axa \leq axaxa$ and $a \leq ya^2z$. Then, $a \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa)$ and $(axya, azxa) \in A_a$. Thus,

$$\begin{aligned}
(f\,(\circ)_k^{\Bbbk}\, f)^-(a) &= (f \circ f)(a) \wedge \tfrac{\Bbbk-k}{2} \\
&= \left[\bigvee_{(y,z)\in A_a} \bigvee_{(y,z)\in A_a} (f(y) \wedge f(z))\right] \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \{(f(axya) \wedge f(azxa))\} \wedge \tfrac{\Bbbk-k}{2} \\
&\geq \left\{\left(f(a) \wedge f(a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \left(f(a) \wedge f(a) \wedge \tfrac{\Bbbk-k}{2}\right)\right\} \wedge \tfrac{\Bbbk-k}{2} \\
&= \left(f(a) \wedge \tfrac{\Bbbk-k}{2}\right) \wedge \tfrac{\Bbbk-k}{2} = \left(f(a) \wedge \tfrac{\Bbbk-k}{2}\right) = \left(\overline{f}\right)_k^{\Bbbk}(a).
\end{aligned}$$

On the other hand, by Theorem (4.3), $(f\,(\circ)_k^{\Bbbk}\, f)^-(a) \leq \left(\overline{f}\right)_k^{\Bbbk}(a)$.
(ii)$\Longrightarrow$(iii). Let f_1 and f_2 be $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S. Then, by Corollary (4.1), $(f_1\,(\wedge)_k^{\Bbbk}\, f_2)^-$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. By (ii),

$$(f_1\,(\wedge)_k^{\Bbbk}\, f_2)^- = \Big((f_1\,(\wedge)_k^{\Bbbk}\, f_2)^-\,(\circ)_k^{\Bbbk}\,(f_1\,(\wedge)_k^{\Bbbk}\, f_2)^-\Big)^- \preceq (f_1\,(\circ)_k^{\Bbbk}\, f_2)^-.$$

In a similar way, one can prove that, $(f_1\,(\wedge)_k^{\Bbbk}\, f_2)^- \preceq (f_2\,(\circ)_k^{\Bbbk}\, f_1)^-$.
Thus, $(f_1\,(\wedge)_k^{\Bbbk}\, f_2)^- \preceq ((f_1\,(\circ)_k^{\Bbbk}\, f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\, f_1)^-)^-$. Moreover, $(f_1\,(\circ)_k^{\Bbbk}\, f_2)^-$ and $(f_2\,(\circ)_k^{\Bbbk}\, f_1)^-$ are $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideals of S by Corollary (4.1), and hence,
$(f_1\,(\circ)_k^{\Bbbk}\, f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\, f_1)^-$ is an $(\in, \in \vee(\Bbbk, q_k))$-fuzzy bi-ideal of S. Using (ii),

we have,

$$\begin{aligned}
&((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-)^-\\
&=\left(((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-)\,(\circ)_k^{\Bbbk}\,((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-)\right)^-\\
&\preceq\left((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\circ)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-\right)^-=(f_1\,(\circ)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_2)\,(\circ)_k^{\Bbbk}\,f_1)^-\\
&=(f_1\,(\circ)_k^{\Bbbk}\,f_2\,(\circ)_k^{\Bbbk}\,f_1)^- as (f_2\,(\circ)_k^{\Bbbk}\,f_2)^-=\left(\overline{f_2}\right)_k^{\Bbbk} by (i) above\\
&\preceq(f_1\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f_1)^-=\left(\overline{f}\right)_k^{\Bbbk} as (f_1\,(\circ)_k^{\Bbbk}\,1\,(\circ)_k^{\Bbbk}\,f_1)^-=\left(\overline{f}\right)_k^{\Bbbk} by theorem (4.4)
\end{aligned}$$

In a similar way, one can prove that, $((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-)^-\preceq\left(\overline{f_2}\right)_k^{\Bbbk}$. Consequently, $((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-)^-\preceq\left(\overline{f_1}\right)_k^{\Bbbk}\wedge\left(\overline{f_2}\right)_k^{\Bbbk}=(f_1\,(\wedge)_k^{\Bbbk}\,f_2)^-$. Therefore, we get $(f_1\,(\wedge)_k^{\Bbbk}\,f_2)^-=((f_1\,(\circ)_k^{\Bbbk}\,f_2)^-\,(\wedge)_k^{\Bbbk}\,(f_2\,(\circ)_k^{\Bbbk}\,f_1)^-)^-$.
(iii)$\Longrightarrow$(i). To prove that S is both regular and intra-regular, by Lemma (4.14), it is enough to prove that $A\cap B=(AB]\cap(BA]$ for all bi-ideals A and B of S. Let $x\in A\cap B$. Then, $x\in A$ and $x\in B$. By Lemma (4.4), $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}$ and $\left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}$ are $(\in,\in\vee(\Bbbk,q_k))$-fuzzy bi-ideals of S. Using (iii), we have

$$\begin{aligned}
((\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,\mathbb{C}_B)^-\,(\wedge)_k^{\Bbbk}\,(\mathbb{C}_B\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^-)^-(x)\quad&=(\mathbb{C}_A\,(\wedge)_k^{\Bbbk}\,\mathbb{C}_B)^-(x)\\
&=\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x)\wedge\left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}(x).
\end{aligned}$$

Since $x\in A$ and $x\in B$, we have $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x)=\frac{\Bbbk-k}{2}$ and $\left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}(x)=\frac{\Bbbk-k}{2}$. Thus, $\left(\overline{\mathbb{C}}_A\right)_k^{\Bbbk}(x)\wedge\left(\overline{\mathbb{C}}_B\right)_k^{\Bbbk}(x)=\frac{\Bbbk-k}{2}\wedge\frac{\Bbbk-k}{2}=\frac{\Bbbk-k}{2}$. It follows that

$$((\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,\mathbb{C}_B)^-\,(\wedge)_k^{\Bbbk}\,(\mathbb{C}_B\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^-)^-(x)=\frac{\Bbbk-k}{2}.$$

By Lemma (4.3) , we have $((\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,\mathbb{C}_B)^-\,(\wedge)_k^{\Bbbk}\,(\mathbb{C}_B\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^-)^-=\left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}\wedge\left(\overline{\mathbb{C}}_{(BA]}\right)_k^{\Bbbk}=\left(\overline{\mathbb{C}}_{(AB]\cap(BA]}\right)_k^{\Bbbk}$. Thus, $\left(\overline{\mathbb{C}}_{(AB]\cap(BA]}\right)_k^{\Bbbk}(x)=\frac{\Bbbk-k}{2}$ and $x\in(AB]\cap(BA]$. Moreover, if $x\in(AB]\cap(BA]$, then,

$$\begin{aligned}
\frac{\Bbbk-k}{2}\qquad&=\left(\overline{\mathbb{C}}_{(AB]\cap(BA]}\right)_k^{\Bbbk}(x)\\
&=\left(\left(\overline{\mathbb{C}}_{(AB]}\right)_k^{\Bbbk}\wedge\left(\overline{\mathbb{C}}_{(BA]}\right)_k^{\Bbbk}\right)(x)\\
&=((\mathbb{C}_A\,(\circ)_k^{\Bbbk}\,\mathbb{C}_B)^-\,(\wedge)_k^{\Bbbk}\,(\mathbb{C}_B\,(\circ)_k^{\Bbbk}\,\mathbb{C}_A)^-)^-(x)\\
&=(\mathbb{C}_A\,(\wedge)_k^{\Bbbk}\,\mathbb{C}_B)^-(x)(by(iii))\\
&=\left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}(x).
\end{aligned}$$

Thus, $\left(\overline{\mathbb{C}}_{A\cap B}\right)_k^{\Bbbk}(x)=\frac{\Bbbk-k}{2}$ and $x\in A\cap B$. Therefore, $A\cap B=(AB]\cap(BA]$, consequently, S is both regular and intra-regular. $\square$

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COMPARISON OF VARIOUS FRACTIONAL BASIS FUNCTIONS FOR SOLVING FRACTIONAL-ORDER LOGISTIC POPULATION MODEL

This paper is dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th Birthday

Mohammad Izadi

Abstract. Three types of orthogonal polynomials (Chebyshev, Chelyshkov, and Legendre) are employed as basis functions in a collocation scheme to solve a nonlinear cubic initial value problem arising in population growth models. The method reduces the given problem to a set of algebraic equations consist of polynomial coefficients. Our main goal is to present a comparative study of these polynomials and to asses their performances and accuracies applied to the logistic population equation. Numerical applications are given to demonstrate the validity and applicability of the method. Comparisons are also made between the present method based on different basis functions and other existing approximation algorithms..

Keywords: Liouville-Caputo fractional derivative; Chebyshev and Chelyshkov polynomials; Collocation method; Logistic population model; Legendre polynomial.

1. Introduction

In the present work, we are aiming to find the approximate solutions of the fractional-order growth equation of single species with multiplicative Allee effect. This equation is governed by the following nonlinear ordinary differential equation [1]

$$D_*^{(\mu)} y(t) = r\, y(t) \left(1 - \frac{y(t)}{k}\right) (y(t) - m), \quad 0 < t \leq R < \infty, \tag{1.1}$$

with the initial condition

$$y(0) = \lambda \geq 0. \tag{1.2}$$

Here, r, m, and k are positive constants denoting respectively per capita growth rate, Allee effect threshold and the carrying capacity of the environment. Here, $D_*^{(\mu)}$

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is the standard Liouville-Caputo fractional derivative operator and $0 < \mu \leq 1$. The fractional model (1.1) can be obtained by using the fractional derivative operator on the corresponding inter-order equation. The investigation of the stability of equilibrium points of (1.1) along with the sufficient conditions to ensure the existence and uniqueness of the coresponding solution are considered in [1]. To the best of our knowledge, the following approximative and numerical schemes are developed for the model problem (1.1)-(1.2). These include the Adams-type predictor-corrector method [1], Bessel-collocation method [27], and the spectral tau method based on shifted Jacobi polynomials [10].

The logistic population model is considered as an important type of nonlinear differential equations due to its ability to model several biological and social phenomena. Different variations of the population modelling are considered in the literature [19]. Among others, the following linear and nonlinear models can be mentioned, cf. [20, 10, 13, 26]

$$D_*^{(\mu)} y(t) = r^\mu \, y(t), \tag{1.3}$$

$$D_*^{(\mu)} y(t) = r \, y(t) \left(1 - y(t)\right), \tag{1.4}$$

$$D_*^{(\mu)} y(t) = r^\mu \, y(t) \left(1 - y(t)\right). \tag{1.5}$$

Historically, the origin of fractional differential equations traced back to Newton and Leibniz more than three centuries ago. To model many real world problems, it has turned out the use of fractional-order derivatives are more adequate rather than integer-order ones. That is due to the fact that fractional derivatives and integrals enable the description of the memory properties of various materials and processes [21, 15]. Therefore, one needs to extend the concept of ordinary differentiation as well as integration to an arbitrary non-integer order. The resulting fractional-order equations can be rarely solved exactly or analytically. Consequently, approximate and numerical techniques are playing an important role in identifying the solutions behaviour of such fractional equations. Indeed, the exact analytical solution of the aforementioned population models is not known except for the linear model (1.3) whose solution is written in terms of Mittag-Leffer infinite series, cf. [26].

Recently, considerable attention has been given to the establishment of techniques for the solution of the fractional differential equations using orthogonal functions. The main characteristic of this technique is that it reduces the solution of differential equations to the solution of a system of algebraic equations. Historically this approach originated from the use of Fourier [18], Walsh [7] and block-pulse functions [22] and was later extended to other classical orthogonal polynomials such as Chebyshev, Legendre, Hermite, and Laguerre polynomials [23]. In most of the presented works, the use of numerical techniques in conjunction with operational matrices for differentiation and integration operators of some orthogonal polynomials, for the solution of fractional differential equations on finite and infinite intervals, produced highly accurate solutions for such equations, see [3] for a recent review.

As already mentioned, the model problem (1.1)-(1.2) is known to possess no exact solutions in general. In this manuscript, we will propose approximation methods as extension of the previous works [17], [11, 12], [27], [14], and [25] for solving (1.1)-(1.2). We use the fractional-order polynomials including the Chebyshev, Chelyshkov, and Legendre functions to approximate the solution of (1.1) accurately on the interval $[0, R]$. The main idea of the proposed technique based on using these (orthogonal) functions along with collocation points is that it converts the differential or integral operator involved in (1.1)-(1.2) to an algebraic form, thus greatly reducing the computational effort.

Our manuscript is organized as follows. In the next section, some fundamental definitions of fractional calculus and relevant properties are presented. Then, in subsequent subsections a brief review of the properties of the Chebyshev, Chelyshkov, and Legendre polynomials is outlined. Section 3. is devoted to the presentation of the proposed collocation scheme applied to nonlinear logistic population initial value problem. Hence, the error estimation technique based on the residual function is developed for the present method. In computational Section 4., we apply the proposed method to the some test problems and report our numerical findings. We end the paper with few concluding remarks in Section 5.

2. Basic definitions

In this section, first some properties of the fractional calculus theory are presented. Afterwards, the definitions of fractional Chebyshev, Chelyshkov, and Legendre polynomials are recalled and some properties of them required for our subsequent sections are reviewed.

2.1. Fractional calculus

Definition 2.1. Suppose that $f(t)$ is n-times continuously differentiable. The fractional derivative $D_*^{(\mu)}$ of $f(t)$ of order $\mu > 0$ in the Liouville-Caputo's sense is defined as

$$D_*^{(\mu)} f(t) = \begin{cases} I^{n-\mu} f^{(n)}(t), & \text{if} \quad n-1 < \mu < n, \\ f^{(n)}(t), & \text{if} \quad \mu = n, \ n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where

$$I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} \, ds, \quad t > 0.$$

The properties of the operator $D_*^{(\mu)}$ can be found in [21, 15]. We make use of the followings

(2.2) $D_*^{(\mu)}(C) = 0 \quad (C \text{ is a constant}),$

(2.3) $D_*^{(\mu)} t^{\gamma} =$

$$\begin{cases} \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} t^{\gamma-\mu}, & \text{for} \quad \gamma \in \mathbb{N}_0 \text{ and } \gamma \geq \lceil \mu \rceil, \text{ or } \gamma \notin \mathbb{N}_0 \text{ and } \gamma > \lfloor \mu \rfloor, \\ 0, & \text{for} \quad \gamma \in \mathbb{N}_0 \text{ and } \gamma < \lceil \mu \rceil. \end{cases}$$

We have used the ceiling function $\lceil \mu \rceil$ to denote the smallest integer greater than or equal to μ, and the floor function $\lfloor \mu \rfloor$ to denote the largest integer less than or equal to μ.

2.2. Chebyshev functions

It is known that the classical Chebyshev polynomials are defined on $[-1, 1]$. Starting with $T_0(z) = 1$ and $T_1(z) = z$, these polynomials satisfy the following recurrence relation [2]

$$T_{n+1}(z) = 2z\, T_n(z) - T_{n-1}(z), \quad n = 1, 2, \ldots.$$

By introducing the change of variable $z = 1 - 2(\frac{t}{R})^\alpha$, $\alpha > 0$, one obtains the shifted version of the polynomials defined on $[0, R]$ and will be denoted by $T_n^\alpha(t) = T_n(z)$. The explicit analytical form of $T_n^\alpha(t)$ of degree (αn) is given for $n = 0, 1, \ldots$

$$\text{(2.4)} \; T_n^\alpha(t) = \sum_{k=0}^{n} c_{n,k}\, t^{\alpha k}, \quad c_{n,k} = (-1)^k \frac{n\, 2^{2k}\, (n+k-1)!}{(n-k)!\, R^{\alpha k}\, (2k)!}, \quad k = 0, 1, \ldots, n,$$

with $c_{0,k} = 1$ for all $k = 0, 1, \ldots, n$. It is proved in [17] that the set of fractional polynomial functions $\{T_0^\alpha, T_1^\alpha, \ldots\}$ is orthogonal on $[0, R]$ with respect to the weight function $w(t) = \frac{t^{\alpha/2-1}}{\sqrt{R^\alpha - t^\alpha}}$; i.e.

$$\int_0^R T_n^\alpha(t)\, T_m^\alpha(t) w(t) dt = \frac{\pi}{2\alpha} d_n \delta_{mn}, \quad n, m \geq 0.$$

Here, δ_{mn} is Kronecker delta function, $d_0 = 2$ while $d_n = 1$ for $n \geq 1$. Our aim is to find an approximate solution of model (1.1) expressed in the truncated Chebyshev series form (3.1)

$$y_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, T_n^\alpha(t), \quad 0 \leq t \leq R, \tag{2.5}$$

where the unknown coefficients a_n, $n = 0, 1, \ldots, N$ are sought. To proceed, we write $T_n^\alpha(t)$, $n = 0, 1, \ldots, N$ in the matrix form as follows

$$\mathbf{T}_\alpha(t) = \mathbf{B}_\alpha(t)\, \mathbf{D}_1 \Leftrightarrow \mathbf{T}_\alpha^t(t) = \mathbf{D}_1^t\, \mathbf{B}_\alpha^t(t), \tag{2.6}$$

here, a superscript t denotes the matrix transpose operation and

$$\mathbf{T}_\alpha(t) = \begin{bmatrix} T_0^\alpha(t) & T_1^\alpha(t) & \ldots & T_N^\alpha(t) \end{bmatrix}, \quad \mathbf{B}_\alpha(t) = \begin{bmatrix} 1 & t^\alpha & t^{2\alpha} & \ldots & t^{N\alpha} \end{bmatrix}.$$

The upper triangular $(N+1)\times(N+1)$ matrix $\mathbf{D}_1$ takes the form

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & c_{1,1} & c_{2,1} & c_{3,1} & \dots & c_{N-1,1} & c_{N,1} \\ 0 & 0 & c_{2,2} & c_{3,2} & \dots & c_{N-1,2} & c_{N,2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & c_{N-1,N-1} & c_{N,N-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & c_{N,N} \end{bmatrix}.$$

By means of (2.6) one can write the relation (2.5) in the matrix form

$$y_{N,\alpha}(t) = \mathbf{B}_\alpha(t)\,\mathbf{D}_1\,\mathbf{A}, \tag{2.7}$$

where the vector of unknown is $\mathbf{A} = [a_0 \quad a_1 \quad \dots \quad a_N]^t$.

2.3. Chelyshkov functions

The Chelyshkov polynomials were originally introduced by Chelyshkov [6, 5]. These polynomials are orthogonal over the interval $[0,1]$ with respect to the weight function $w(x)=1$, and are explicitly defined by

$$C_{n,N}(t) = \sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k}\binom{N+n+k+1}{N-n} t^{n+k}, \quad n = 0,1,\dots,N. \tag{2.8}$$

These polynomials satisfy the following orthogonality relation

$$\int_0^1 C_{n,N}(t)\, C_{m,N}(t) dt = \frac{\delta_{nm}}{n+m+1}.$$

Moreover, they can be obtained through the Jacobi polynomials $P_m^{\alpha,\beta}(t)$, where $\alpha,\beta > -1$, and $m \geq 0$ as

$$C_{n,N}(t) = (-1)^{N-n}\, t^n\, P_{N-n}^{0,2n+1}(t).$$

Now, we construct the fractional-order version of (2.8) by replacing $t \to t^\alpha$ as follows [25]

$$C_{n,N}^\alpha(t) = \sum_{k=n}^{N} (-1)^{k-n} \binom{N-n}{k-n}\binom{N+k+1}{N-n} \left(\frac{t^\alpha}{R}\right)^k, \quad n = 0,1,\dots,N. \tag{2.9}$$

It also is not a difficult task to show that the set of fractional polynomial functions $\{C_{0,N}^\alpha, C_{1,N}^\alpha, \dots\}$ is orthogonal on $[0,R]$ with respect to the weight function $w(t) \equiv t^{\alpha-1}$. This implies that

$$\int_0^R C_{n,N}^\alpha(t)\, C_{m,N}^\alpha(t)\, w(t) dt = \frac{R\delta_{nm}}{\alpha(2n+1)}, \quad n,m \geq 0.$$

The Chelyshkov basis polynomials given by equation (2.9) can be written in the matrix form [16, 25]

$$\mathbf{C}_\alpha(t) = \begin{bmatrix} C^\alpha_{0,N}(t) & C^\alpha_{1,N}(t) & \dots & C^\alpha_{N,N}(t) \end{bmatrix} = \mathbf{B}_\alpha(t)\,\mathbf{D}_2, \tag{2.10}$$

where $\mathbf{D}_2$ is an $(N+1)\times(N+1)$ matrix. If N is odd, the matrix $\mathbf{D}_2$ becomes

$$\mathbf{D}_2 = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \dots & 0 & 0 \\ -r\binom{N}{1}\binom{N+2}{N} & r\binom{N-1}{0}\binom{N+2}{N-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{N-1}\binom{N}{N-1}\binom{2N}{N} & -r^{N-1}\binom{N-1}{N-2}\binom{2N}{N-1} & \dots & r^{N-1}\binom{1}{0}\binom{2N}{1} & 0 \\ -r^N\binom{N}{N}\binom{2N+1}{N} & r^N\binom{N-1}{N-1}\binom{2N+1}{N-1} & \dots & r^N\binom{1}{1}\binom{2N+1}{1} & r^N \end{bmatrix},$$

where we have used $r = 1/R$. If N is even we have

$$\mathbf{D}_2 = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \dots & 0 & 0 \\ -r\binom{N}{1}\binom{N+2}{N} & r\binom{N-1}{0}\binom{N+2}{N-1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -r^{N-1}\binom{N}{N-1}\binom{2N}{N} & r^{N-1}\binom{N-1}{N-2}\binom{2N}{N-1} & \dots & r^{N-1}\binom{1}{0}\binom{2N}{1} & 0 \\ r^N\binom{N}{N}\binom{2N+1}{N} & -r^N\binom{N-1}{N-1}\binom{2N+1}{N-1} & \dots & -r^N\binom{1}{1}\binom{2N+1}{1} & r^N \end{bmatrix}.$$

Analogously, we approximate $y(t)$ in terms of the truncated Chelyshkov series form as $y_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, C^\alpha_{n,N}(t)$. Using (2.10) one may rewrite $y_{N,\alpha}(t)$ as follows

$$y_{N,\alpha}(t) = \mathbf{B}_\alpha(t)\,\mathbf{D}_2\,\mathbf{A}. \tag{2.11}$$

2.4. Legendre functions

The orthogonal Legendre polynomials are originally defined on $[-1,1]$. Utilizing the change of variable $x = (\frac{2t}{R} - 1)$ one can obtain the shifted Legendre polynomials defined in $[0,R]$ and satisfies in the following recurrence relation [2]

$$P_{n+1}(t) = \frac{2n+1}{n+1}\Big(\frac{2t}{R} - 1\Big)\,P_n(t) - \frac{n}{n+1}P_{n-1}(t), \quad n = 1,2,\dots,$$

with $P_0(t) = 1$ and $P_1(t) = \frac{2t}{R} - 1$. The analytical form of $P_n(t)$ is explicitly defined for $n = 0,1,\dots$

$$P_n(t) = \sum_{k=0}^{n} l_{n,k}\, t^k, \quad l_{n,k} = (-1)^{n+k}\frac{(n+k)!}{(n-k)!\,R^k\,(k!)^2}, \ k = 0,1,\dots,n. \tag{2.12}$$

Based on the shifted Legendre polynomials (2.12) one generates an orthogonal set of fractional-order Legendre functions by setting $t \to t^{\alpha}$ for $0 < \alpha \leq 1$, see [14]. They take the form

$$P_n^{\alpha}(t) = \sum_{k=0}^{n} l_{n,k}\, t^{k\alpha}, \quad n = 0, 1, \ldots. \tag{2.13}$$

It is proved in [14] that the set of fractional polynomial functions $\{P_0^{\alpha}, P_1^{\alpha}, \ldots\}$ is orthogonal on $[0, R]$ with respect to the weight function $w(t) \equiv t^{\alpha-1}$; i.e.

$$\int_0^R P_n^{\alpha}(t)\, P_m^{\alpha}(t) w(t) dt = \frac{R}{\alpha(2n+1)} \delta_{nm}, \quad n, m \geq 0.$$

The main important properties of the fractional-order Legendre functions can be found in [14] and [24].

Now, let us approximate the solution $y(t)$ of (1.1) in terms of fractional-order Legendre functions. Thus one gets $y_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, P_n^{\alpha}(t)$ or equivalently

$$y_{N,\alpha}(t) = \mathbf{P}_{\alpha}(t)\, \mathbf{A}, \quad \mathbf{P}_{\alpha}(t) = [P_0^{\alpha}(t) \quad P_1^{\alpha}(t) \quad \ldots \quad P_N^{\alpha}(t)]. \tag{2.14}$$

In a similar way as the Chebyshev and Chelyshkov functions, we write $P_n^{\alpha}(t)$ in the matrix form as follows

$$\mathbf{P}_{\alpha}(t) = \mathbf{B}_{\alpha}(t)\, \mathbf{D}_3^t \Leftrightarrow \mathbf{P}_{\alpha}^t(t) = \mathbf{D}_3\, \mathbf{B}_{\alpha}^t(t), \tag{2.15}$$

where the monomial basis vector $\mathbf{B}_{\alpha}(t)$ is previously defined in (2.6). Moreover, the matrix $\mathbf{D}_3$ in this case is a lower triangular matrix whose entries are obtained via (2.12) and has the form

$$\mathbf{D}_3 = \begin{bmatrix} l_{0,0} & l_{1,0} & l_{2,0} & \ldots & l_{N-1,0} & l_{N,0} \\ 0 & l_{1,1} & l_{2,1} & \ldots & l_{N-1,1} & l_{N,1} \\ 0 & 0 & l_{2,2} & \ldots & l_{N-1,2} & l_{N,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & l_{N-1,N-1} & l_{N,N-1} \\ 0 & 0 & 0 & \ldots & 0 & l_{N,N} \end{bmatrix}.$$

Therefore, an equivalent form of (2.14) can be written as

$$y_{N,\alpha}(t) = \mathbf{B}_{\alpha}(t)\, \mathbf{D}_3\, \mathbf{A}. \tag{2.16}$$

Ultimately, to obtain a solution in the form (2.11), (2.11), or (2.16) of the problem (1.1) on the interval $0 < t \leq R$, we will use the collocation points defined by

$$t_i = \frac{R}{N} i, \quad i = 0, 1, \ldots, N. \tag{2.17}$$

3. Description of the method

Now, suppose that we approximate the solution $y(t)$ of the nonlinear logistic population equation (1.1) in terms of $(N+1)$-terms Chebyshev, Chelyshkov or Legendre polynomials series denoted by $y_{N,\alpha}(t)$ on the interval $[0, R]$. As previously stated, in the vector form one may write

$$y(t) \approx y_{N,\alpha}(t) = \mathbf{B}_\alpha(t)\,\mathbf{U}\,\mathbf{A}. \tag{3.1}$$

Depending on which polynomial basis function we use in the approximation, the matrix $\mathbf{U}$ can be either $\mathbf{D}_1$, $\mathbf{D}_2$ or $\mathbf{D}_3$. These matrices are previously defined in (2.6), (2.10), and (2.15) respectively. Putting the collocation points (2.17) into (3.1), we arrive at a system of matrix equations

$$y_{N,\alpha}(t_i) = \mathbf{B}_\alpha(t_i)\,\mathbf{U}\,\mathbf{A}, \quad i = 0, 1, \ldots, N.$$

These equations can be written in a single and compact representation as follows

$$\mathbf{Y} = \mathbf{B}\,\mathbf{U}\,\mathbf{A}, \tag{3.2}$$

where

$$\mathbf{Y} = \begin{bmatrix} y_{N,\alpha}(t_0) \\ y_{N,\alpha}(t_1) \\ \vdots \\ y_{N,\alpha}(t_N) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_\alpha(t_0) \\ \mathbf{B}_\alpha(t_1) \\ \vdots \\ \mathbf{B}_\alpha(t_N) \end{bmatrix}.$$

By taking the fractional derivative of order μ from the both sides of (3.1), we get

$$D_*^{(\mu)} y_{N,\alpha}(t) = D_*^{(\mu)}\,\mathbf{B}_\alpha(t)\,\mathbf{U}\,\mathbf{A}. \tag{3.3}$$

The calculation of $D_*^{(\mu)}\,\mathbf{T}_\alpha(t)$ can be easily obtained via the property (2.2) and (2.3) as follows

$$\mathbf{B}_\alpha^{(\mu)}(t) = D_*^{(\mu)}\,\mathbf{B}_\alpha(t) = [0 \quad D_*^{(\mu)}\,t^\alpha \quad \ldots \quad D_*^{(\mu)}\,t^{\alpha N}].$$

To obtain a system of matrix equations for the fractional derivative, we insert the collocation points (2.17) into (3.3) to get

$$D_*^{(\mu)} y_{N,\alpha}(t_i) = \mathbf{B}_\alpha^{(\mu)}(t_i)\,\mathbf{U}\,\mathbf{A}, \quad i = 0, 1 \ldots, N,$$

which can be written in the matrix form

$$\mathbf{Y}^{(\mu)} = \mathbf{B}^{(\mu)}\,\mathbf{U}\,\mathbf{A}, \tag{3.4}$$

where

$$\mathbf{Y}^{(\mu)} = \begin{bmatrix} D_*^{(\mu)} y_{N,\alpha}(t_0) \\ D_*^{(\mu)} y_{N,\alpha}(t_1) \\ \vdots \\ D_*^{(\mu)} y_{N,\alpha}(t_N) \end{bmatrix}, \quad \mathbf{B}^{(\mu)} = \begin{bmatrix} \mathbf{B}_\alpha^{(\mu)}(t_0) \\ \mathbf{B}_\alpha^{(\mu)}(t_1) \\ \vdots \\ \mathbf{B}_\alpha^{(\mu)}(t_N) \end{bmatrix}.$$

To continue, we approximate the nonlinear term $y^2(t)$. By substituting the collocation points into $y^2_{N,\alpha}(t)$ we arrive at the following matrix representation

$$\mathbf{Y}^2 = \begin{bmatrix} y^2_{N,\alpha}(t_0) \\ y^2_{N,\alpha}(t_1) \\ \vdots \\ y^2_{N,\alpha}(t_N) \end{bmatrix} = \begin{bmatrix} y_{N,\alpha}(t_0) & 0 & \dots & 0 \\ 0 & y_{N,\alpha}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_{N,\alpha}(t_N) \end{bmatrix} \begin{bmatrix} y_{N,\alpha}(t_0) \\ y_{N,\alpha}(t_1) \\ \vdots \\ y_{N,\alpha}(t_N) \end{bmatrix},$$

which is equivalent to

$$\mathbf{Y}^2 = \widehat{\mathbf{Y}}\,\mathbf{Y}. \tag{3.5}$$

Also, the matrix $\widehat{\mathbf{Y}}$ can be written as a product of three block diagonal matrices as

$$\widehat{\mathbf{Y}} = \widehat{\mathbf{B}}\,\widehat{\mathbf{Q}}\,\widehat{\mathbf{A}}, \tag{3.6}$$

where

$$\widehat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_\alpha(t_0) & 0 & \dots & 0 \\ 0 & \mathbf{B}_\alpha(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}_\alpha(t_N) \end{bmatrix}, \quad \text{and}$$

$$\widehat{\mathbf{Q}} = \begin{bmatrix} \mathbf{U} & 0 & \dots & 0 \\ 0 & \mathbf{U} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{U} \end{bmatrix}, \quad \widehat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 & \dots & 0 \\ 0 & \mathbf{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{A} \end{bmatrix}.$$

Similarly, by inserting the collocation points (2.17) into the $y^3(t)$ we arrive at the following matrix representation

$$\mathbf{Y}^3 = \begin{bmatrix} y^3_{N,\alpha}(t_0) \\ y^3_{N,\alpha}(t_1) \\ \vdots \\ y^3_{N,\alpha}(t_N) \end{bmatrix} = \begin{bmatrix} y^2_{N,\alpha}(t_0) & 0 & \dots & 0 \\ 0 & y^2_{N,\alpha}(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y^2_{N,\alpha}(t_N) \end{bmatrix} \begin{bmatrix} y_{N,\alpha}(t_0) \\ y_{N,\alpha}(t_1) \\ \vdots \\ y_{N,\alpha}(t_N) \end{bmatrix},$$

which implies that

$$\mathbf{Y}^3 = (\widehat{\mathbf{Y}})^2\,\mathbf{Y}, \tag{3.7}$$

where $\widehat{\mathbf{Y}}$ is defined in (3.6).

Now, we are able to compute the Chebyshev, Chelyshkov, and Legendre solutions of (1.1). The collocation procedure is based on calculating these polynomial coefficients by means of collocation points defined in (2.17). To proceed, inserting the collocation points into the fractional logistic population differential equation to get the system

$$D^{(\mu)}_*\, y(t_i) = -rm\, y(t_i) + r(1+\frac{m}{k})\, y^2(t_i) - \frac{r}{k}\, y^3(t_i), \quad i = 0, 1, \dots, N.$$

In the matrix form we may write the above equations as

$$\mathbf{Y}^{(\mu)} + \mathbf{M}\,\mathbf{Y} - \mathbf{N}\,\mathbf{Y}^2 + \mathbf{K}\,\mathbf{Y}^3 = \mathbf{Z}, \tag{3.8}$$

where the coefficient matrices $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{K}$ of size $(N+1)\times(N+1)$ and the vector $\mathbf{Z}$ of size $(N+1)\times 1$ have the following forms

$$\mathbf{M} = \begin{bmatrix} rm & 0 & \dots & 0 \\ 0 & rm & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & rm \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} r(1+\frac{m}{k}) & 0 & \dots & 0 \\ 0 & r(1+\frac{m}{k}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r(1+\frac{m}{k}) \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \frac{r}{k} & 0 & \dots & 0 \\ 0 & \frac{r}{k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{r}{k} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By putting the relations (3.2), (3.4), and (3.5), (3.7) into (3.8), the fundamental matrix equation is obtained

$$\mathbf{W}\,\mathbf{A} = \mathbf{Z}, \tag{3.9}$$

where

$$\mathbf{W} := \mathbf{B}^{(\mu)}\,\mathbf{U} + \mathbf{M}\,\mathbf{B}\,\mathbf{U} - \mathbf{N}\,\widehat{\mathbf{B}}\,\widehat{\mathbf{Q}}\,\widehat{\mathbf{A}}\,\mathbf{B}\,\mathbf{U} + \mathbf{K}\,(\widehat{\mathbf{B}}\,\widehat{\mathbf{Q}}\,\widehat{\mathbf{A}})^2\,\mathbf{B}\,\mathbf{U}.$$

Obviously, (3.9) is a nonlinear matrix equation with a_n, $n = 0, 1, \dots, N$, being the unknowns Chebyshev, Chelyshkov, or Legendre coefficients. To take into account the initial condition $y(0) = \lambda$, we tend $t \to 0$ in (3.1) to get the following matrix representation

$$\widetilde{\mathbf{Y}}_0\,\mathbf{A} = \lambda, \qquad \widetilde{\mathbf{Y}}_0 := \mathbf{B}_\alpha(0)\,\mathbf{U} = [y_{00} \quad y_{01} \quad \dots \quad y_{0N}]^t.$$

Consequently, by replacing the first row of the augmented matrix $[\mathbf{W}; \mathbf{Z}]$ by the row matrix $[\widetilde{\mathbf{Y}}_0; \lambda]$, we arrive at the nonlinear algebraic system

$$\widetilde{\mathbf{W}}\,\mathbf{A} = \widetilde{\mathbf{Z}}.$$

Thus, the unknown Chebyshev, Chelyshkov, or Legendre coefficients in (3.1) will be calculated via solving this nonlinear system of equations. This task can be performed using for instance the Newton's iterative method.

3.1. Accuracy of solutions

Since the exact solution of the fractional logistic population differential equation is not known, we need to measure the accuracy of the proposed collocation scheme.

Due to the fact that the truncated Chebyshev, Chelyshkov, and Legendre series (2.5), (2.8), and (2.12) are approximate solutions of (1.1), we expect that the residual obtained by inserting the computed approximated solutions $y_{N,\alpha}(t)$ into the differential equation becomes approximately small. This implies that for $t = t_s \in [0, R]$, $s = 0, 1, \ldots$

$$E_{N,\alpha}(t_s) = D_*^{(\mu)}\, y_{N,\alpha}(t_s) + C_0\, y_{N,\alpha}(t_s) - C_1\, y^2_{N,\alpha}(t_s) + C_2\, y^3_{N,\alpha}(t_s) \cong 0, \tag{3.10}$$

where $C_0 = rm$, $C_1 = r + rm/k$, $C_2 = r/k$, and $E_{N,\alpha}(t_s) \leq 10^{-\ell_s}$ (ℓ_s is positive integer). If $\max 10^{-\ell_s} \leq 10^{-\ell}$ (ℓ positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,\alpha}(t_s)$ at each of the points becomes smaller than the prescribed $10^{-\ell}$, see [4, 27]. Here, we note that the μth-order fractional derivative of the approximate solution (3.10) is computed by using the property (2.3). As the error function is clearly zero at the collocation points (2.17), one expect that $E_{N,\alpha}(t)$ tend to zero as N increased. This says that the smallness of the residual error function means that the approximate solutions are close to the exact solution.

4. Numerical Applications

To illustrate the accuracy and effectiveness of the proposed polynomials collocation methods, two test examples are solved in this section. For comparison, we also implement the collocation spectral method based on the Bessel functions of the first kind in [27].

To start, we take $\mu = 1/3$ in (1.1) and set $\alpha = 10/21$ as the order of basis functions. The parameters are considered as $\lambda = 0.8$, $r = 1/2$, $m = 1$, and $k = 10$. The approximate solutions $y_{N,\alpha}(t)$ of this model problem using Chebyshev, Chelyshkov, and Legendre basis functions for $N = 6$ in the interval $0 \leq t \leq 5$ are obtained as follows, respectively:

$$\begin{aligned} y^{Cheb}_{6,\frac{10}{21}}(t) &= 0.000403175741883\, t^{\frac{20}{7}} - 0.0437836398275\, t^{\frac{10}{7}} - 0.129582980375\, t^{\frac{10}{21}} \\ &\quad + 0.0581143648443\, t^{\frac{20}{21}} + 0.0188356028419\, t^{\frac{40}{21}} - 0.00426036069079\, t^{\frac{50}{21}} + 0.8, \end{aligned}$$

$$\begin{aligned} y^{Chel}_{6,\frac{10}{21}} &= 0.000431604305758\, t^{\frac{20}{7}} - 0.0459657062667\, t^{\frac{10}{7}} - 0.137940445153\, t^{\frac{10}{21}} \\ &\quad + 0.0610146767185\, t^{\frac{20}{21}} + 0.0200210715840\, t^{\frac{40}{21}} - 0.00455595754387\, t^{\frac{50}{21}} + 0.8, \end{aligned}$$

$$\begin{aligned} y^{Leg}_{6,\frac{10}{21}} &= 0.000403170590320\, t^{\frac{20}{7}} - 0.04378450703\, t^{\frac{10}{7}} - 0.129583270558\, t^{\frac{10}{21}} \\ &\quad + 0.0581151779775\, t^{\frac{20}{21}} + 0.0188359825223\, t^{\frac{40}{21}} - 0.00426039646904\, t^{\frac{50}{21}} + 0.8. \end{aligned}$$

The corresponding approximation by means of Bessel function of the first kind takes the form [27]

$$\begin{aligned} y^{Bes}_{6,\frac{10}{21}} &= 0.000431603553833\, t^{\frac{20}{7}} - 0.0459656939040\, t^{\frac{10}{7}} - 0.137940444198\, t^{\frac{10}{21}} \\ &+ 0.0610146708563\, t^{\frac{20}{21}} + 0.020021055753\, t^{\frac{40}{21}} - 0.0045559512912\, t^{\frac{50}{21}} + 0.8. \end{aligned}$$

The above results show clearly a similarity between the solutions obtained by the Chebyshev and Legendre collocation schemes. The same conclusion can be made from the two others polynomials obtained via Chelyshkov and Bessel functions. To further justify this fact, we plot the above approximations in Fig. 4.1. To validate our results, we also employ the predictor-corrector PECE method of Adams-Bashforth-Moulton type described in [8] using $\mu = 1/3$ and step size $h = 1/100$.

Furthermore, we calculate the error function defined in (3.10) for the above approximations. The results are depicted in Fig. 4.2, left plot, in which we used $\mu = 1/3$ and $\alpha = 10/21$. If one uses the same μ as α, a slightly better result is obtained; the right plot in Fig. 4.2 shows the corresponding error functions.

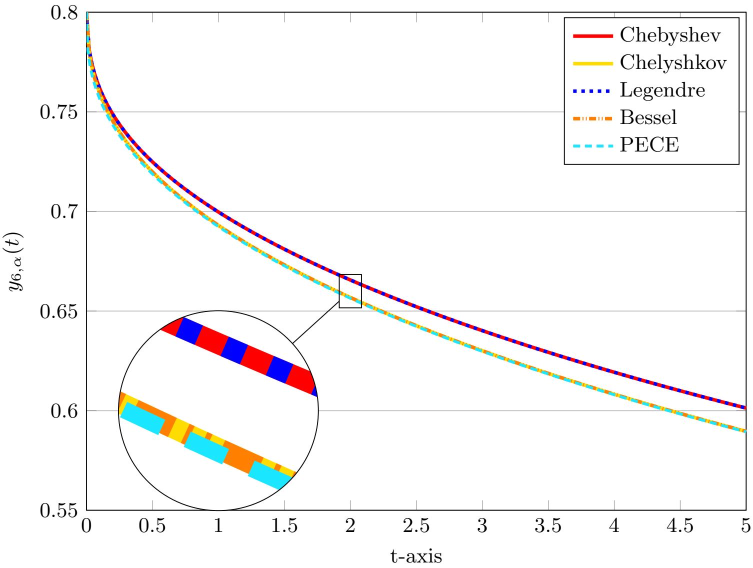


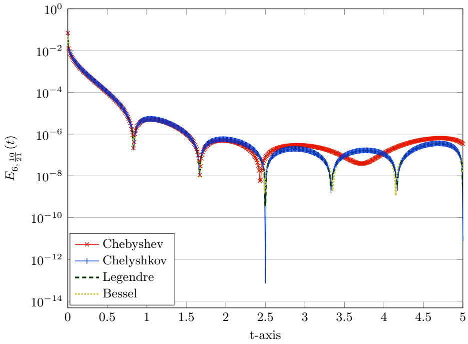
FIG. 4.1: The approximated Chebyshev/Chelyshkov/Legendre/Bessel series solutions $y_{6,\alpha}(t)$ using $\mu = 1/3$, $\alpha = 10/21$ for $r = 1/2$, $m = 1$, and $k = 10$.

Indeed, using μ equals to α give rises to the following approximations

$$
\begin{aligned}
y^{Cheb}_{6,\frac{1}{3}}(t) &= 0.00145592739178\, t - 0.000408618142451\, t^2 - 0.0770406217645\, t^{\frac{1}{3}} \\
&\quad -0.0235193760879\, t^{\frac{2}{3}} - 0.00399643402926\, t^{\frac{4}{3}} + 0.00264600147359\, t^{\frac{5}{3}} + 0.8,
\end{aligned}
$$

$$
\begin{aligned}
y^{Chel}_{6,\frac{1}{3}} &= -0.0008400274797318\, t - 0.000452218849990\, t^2 - 0.08228036902327\, t^{\frac{1}{3}} \\
&\quad -0.02412078745183\, t^{\frac{2}{3}} - 0.002486450422357\, t^{\frac{4}{3}} + 0.002555336195047\, t^{\frac{5}{3}} + 0.8,
\end{aligned}
$$

$$
\begin{aligned}
y^{Leg}_{6,\frac{1}{3}} &= 0.0007238288842966\, t - 0.000383633654034\, t^2 - 0.0772002548625\, t^{\frac{1}{3}} \\
&\quad -0.02298707225931\, t^{\frac{2}{3}} - 0.00348653436761\, t^{\frac{4}{3}} + 0.002467593603573\, t^{\frac{5}{3}} + 0.8,
\end{aligned}
$$



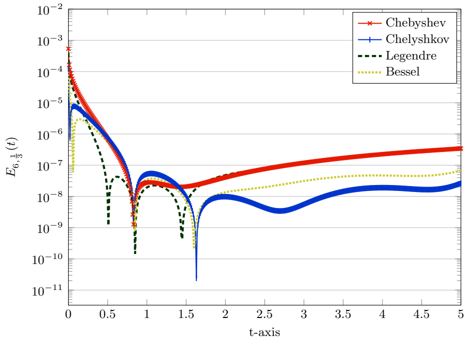


FIG. 4.2: Comparison of the error functions using Chebyshev, Chelyshkov, Legendre, and Bessel functions with $\mu = 1/3$, $\alpha = 10/21$ (left) and μ, $\alpha = 1/3$ (right) for $r = 1/2$, $m = 1$, $k = 10$ and $N = 6$.

$$
\begin{aligned}
y^{Bes}_{6,\frac{1}{3}} &= -0.00055789583397651\,t - 0.0004622063664707\,t^2 - 0.08222082885753\,t^{\frac{1}{3}} \\
&-0.02432274380530\,t^{\frac{2}{3}} - 0.002685716064483\,t^{\frac{4}{3}} + 0.002625907960449\,t^{\frac{5}{3}} + 0.8.
\end{aligned}
$$

In Table 4.1, we report the numerical results correspond to $N = 11$ obtained by the Chebyshev, Chelyshkov, and Legendre-collocation procedures using using $\mu = 1/3$ and $\alpha = 10/21$ at some points $t \in [0, 5]$. A comparison in this table is made with the Bessel polynomials approach from [27].

In the second experiment, we set $\mu = 8/10$, $\alpha = 6/7$ and use the parameters $r = 1/2$, $m = 1$, $k = 10$ as for the first case. In this case, we first consider the approximate solutions $y_{3,\alpha}(t)$ obtained via (3.9) of the model (1.1) for different polynomials in the interval $[0, 5]$. These polynomials of fractional order $\alpha = 6/7$ are obtained as follows

$$
\begin{aligned}
y^{Cheb}_{3,\frac{6}{7}}(t) &= 0.001353089575150 58\,t^{18/7} - 0.01418635085499 24\,t^{12/7} \\
&- 0.0530254117658248\,t^{6/7} + 0.8,
\end{aligned}
$$

$$
\begin{aligned}
y^{Leg}_{3,\frac{6}{7}}(t) &= 0.00135308957514853\,t^{18/7} - 0.0141863508549652\,t^{12/7} \\
&- 0.0530254117658408\,t^{6/7} + 0.8,
\end{aligned}
$$

$$
\begin{aligned}
y^{Chel}_{3,\frac{6}{7}}(t) &= 0.00239567782739856\,t^{18/7} - 0.0181886989840546\,t^{12/7} \\
&- 0.0713242913743184\,t^{6/7} + 0.8,
\end{aligned}
$$

$$
\begin{aligned}
y^{Bes}_{3,\frac{6}{7}}(t) &= 0.00239567440428439\,t^{18/7} - 0.0181886759461094\,t^{12/7} \\
&- 0.0713243387363622\,t^{6/7} + 0.8.
\end{aligned}
$$

Table 4.1: Comparison of numerical approximations in fractional Chebyshev, Chelyshkov, and Legendre-collocation methods for $N = 11$, $\mu = 1/3$, and $\alpha = 10/21$ with $r = 1/2$, $m = 1$, $k = 10$.

t	Chebyshev	Chelyshkov	Legendre	Bessel [27]
0.0	0.800000000000000	0.799999278128293	0.800000002346789	0.8
0.1	0.756938514122078	0.750720355971572	0.756800106479977	0.757299929343
0.5	0.718356585092500	0.717065283660215	0.718315644610171	0.719053533865
0.8	0.701146284741409	0.700526184555178	0.701115558982678	0.701988430676
1.1	0.687275920912118	0.687016466289264	0.687250477599117	0.688230723198
1.5	0.671748232619617	0.671810217070104	0.671726923772954	0.672823131171
1.8	0.661581827887138	0.661820098256081	0.661562638423935	0.662731306189
2.1	0.652336198269439	0.652716988678142	0.652318667387550	0.653550389695
2.5	0.641119373021142	0.641655992776889	0.641103504792762	0.642407766286
2.8	0.633377305676302	0.634012066595954	0.633362385608193	0.634713985075
3.1	0.626111073800506	0.626830874127975	0.626096980389975	0.627490786996
3.5	0.617051206659663	0.617867674288841	0.617038187117911	0.618481357894
3.8	0.610663124253371	0.611542868110921	0.610650880180289	0.612126624513
4.1	0.604580099793391	0.605518095678698	0.604568481573332	0.606073610313
4.5	0.596890007487173	0.597899498980620	0.596878825634452	0.598418968647
4.8	0.591404597688423	0.592459282464779	0.591393696691324	0.592957119829
5.0	0.587869951320244	0.588945877644788	0.587859669371108	0.589436884397

In the next experiments, we fix $N = 3$ and $\mu = 8/10$, $\alpha = 6/7$. We employ the error function (3.10) and compare the results obtained by different polynomial functions. Table 4.2 demonstrates the numerical values of these error functions at some points $t \in [0, 5]$. As the above approximations show, the errors $E_{3,\frac{6}{7}}(t)$ for the Chebyshev and Legendre as well as Chelyshkov and Bessel (our implementation) are approximately similar. Note, in the last column we reports the results from [27]. To see whether the error function $E_{N,\alpha}(t)$ is a decreasing function of N or not, we fix $\mu = 8/10$ and $\alpha = 6/7$ as above but use various $N = 3, 6$ and $N = 10$ in simulation. We select the Chebyshev and Chelyshkov as the basis functions among others. The results are visualized in Fig. 4.3. While the left picture illustrates the Chebyshev error functions, the right one is obtained via Chelyshkov collocation procedure.

Next, to see the effect of using various values of $\alpha \geq \mu$, we fix $N = 7$ and $\mu = 8/10$. Hence, we exploit several values of $\alpha = \mu$, 58/70, 6/7 and compute the numerical solutions at some points in $[0, 5]$. The results are shown in Table. 4.3 while using the Chelyshkov basis functions. To justify our results we compare the computed solutions in this table with Bessel collocation approach [27]. The last two columns are obtained using $\mu = 8/10, \alpha = 6/7$ and $n = 6, 11$ respectively. Looking at Table 4.3 reveals that using Chelyshkov collocation method with $N = 7$ but

Table 4.2: Comparison of error functions in fractional Chebyshev/Legendre and Chelyshkov/Bessel collocation methods for $N = 3$, $\mu = 8/10$, and $\alpha = 6/7$ with $r = 1/2$, $m = 1$, $k = 10$.

t	Chebyshev	Chelyshkov	Legendre	Bessel	Bessel [27]
0.0	5.8666667_{-02}	7.3600000_{-02}	5.8666667_{-02}	7.3600000_{-02}	7.36000000000_{-02}
0.1	1.2372352_{-02}	1.1880958_{-02}	1.2372352_{-02}	1.1880924_{-02}	8.91567095017_{-03}
0.5	5.5046468_{-03}	4.3507888_{-03}	5.5046468_{-03}	4.3507667_{-03}	1.99296107666_{-03}
0.8	3.0589969_{-03}	2.1210125_{-03}	3.0589969_{-03}	2.1209978_{-03}	8.46185079361_{-04}
1.1	1.5039416_{-03}	9.0326860_{-04}	1.5039416_{-03}	9.0325952_{-04}	3.41262782597_{-04}
1.5	2.9615144_{-04}	1.4128415_{-04}	2.9615144_{-04}	1.4128047_{-04}	5.49335472238_{-05}
1.8	1.6982381_{-04}	6.4923105_{-05}	1.6982381_{-04}	6.4924075_{-05}	2.79352687459_{-05}
2.1	3.7838526_{-04}	1.0764412_{-04}	3.7838526_{-04}	1.0764336_{-04}	5.62760899174_{-05}
2.5	3.9162970_{-04}	5.7880301_{-05}	3.9162970_{-04}	5.7878565_{-05}	5.21372970094_{-05}
2.8	2.8292735_{-04}	1.1166955_{-05}	2.8292735_{-04}	1.1165393_{-05}	3.51686446321_{-05}
3.1	1.2750623_{-04}	9.5410734_{-06}	1.2750623_{-04}	9.5417305_{-06}	1.49559607016_{-05}
3.5	8.4963558_{-05}	2.0244083_{-05}	8.4963558_{-05}	2.0245750_{-05}	9.35893830234_{-06}
3.8	2.0838829_{-04}	7.7239067_{-05}	2.0838829_{-04}	7.7243331_{-05}	2.21148978971_{-05}
4.1	2.7510980_{-04}	1.4112847_{-04}	2.7510980_{-04}	1.4113610_{-04}	2.83498750927_{-05}
4.5	2.4829257_{-04}	1.7908045_{-04}	2.4829257_{-04}	1.7909384_{-04}	2.48914327358_{-05}
4.8	1.2896874_{-04}	1.1530618_{-04}	1.2896874_{-04}	1.1532493_{-04}	1.27732345915_{-05}
5.0	5.6388995_{-10}	9.6146640_{-11}	5.6379307_{-10}	2.2949767_{-08}	0

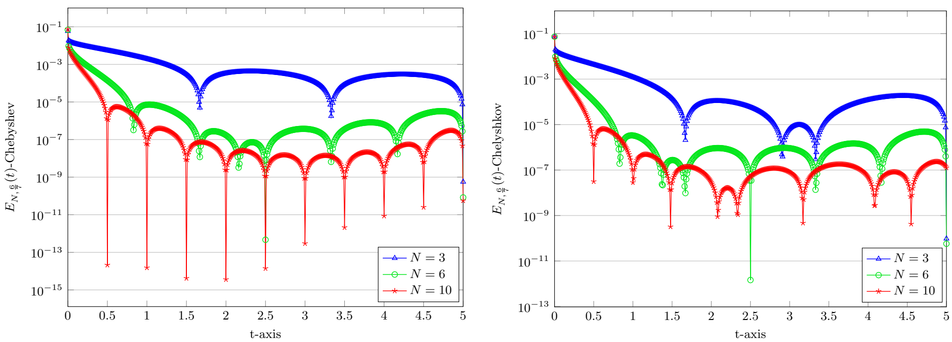


FIG. 4.3: comparison of error functions using Chebyshev (left) and Chelyshkov functions (right) with $\mu = 8/10$, $\alpha = 6/7$, and different $N = 3, 6, 10$.

$\alpha = 58/70$ one gets a comparable result while using Bessel basis functions with $N = 11$.

Table 4.3: Comparison of numerical solutions in Chelyshkov collocation method for $N = 7$, $\mu = 8/10$, and different $\alpha = 8/10$, $58/70$, $6/7$ with $r = 1/2$, $m = 1$, $k = 10$.

	Chelyshkov $(\mu = \frac{8}{10}, N = 7)$			Bessel [27] $(\mu = \frac{8}{10}, \alpha = \frac{6}{7})$	
t	$\alpha = \frac{8}{10}$	$\alpha = \frac{58}{70}$	$\alpha = \frac{6}{7}$	$N = 6$	$N = 11$
0.0	0.80000000000	**0.8000000000**	0.80000000000	0.8	**0.8**
0.1	0.78718882912	**0.787**58463172	0.78802089791	0.788007903475	**0.787**696000559
0.5	0.74982396960	**0.7502**5364375	0.75075459981	0.750739706221	**0.7502**42595254
0.8	0.72356379677	**0.7239**8067700	0.72446611403	0.724456716829	**0.7239**72085247
1.1	0.69746594429	**0.69787**925083	0.69835946985	0.698352878145	**0.69787**5832476
1.5	0.66253413522	**0.66294**840744	0.66342949839	0.663422970008	**0.66294**6544268
1.8	0.63620080581	**0.63661**604251	0.63709835065	0.637090911256	**0.63661**4170560
2.1	0.60981684229	**0.61023**179673	0.61071381759	0.610705745410	**0.61023**0091937
2.5	0.57473447564	**0.57514**638100	0.57562483235	0.575616794783	**0.57514**5107178
2.8	0.54865002906	**0.54905**746525	0.54953072965	0.549523095432	**0.549056**408163
3.1	0.52290039893	**0.52330**146175	0.52376737213	0.523760058308	**0.52330**0477415
3.5	0.48930161944	**0.48969**130186	0.49014406249	0.490136706629	**0.48969**0377445
3.8	0.46480191614	**0.46518**106852	0.46562160232	0.465614022991	**0.46518**0289715
4.1	0.44101520794	**0.44138**236008	0.44180893517	0.441801414133	**0.44138**1760632
4.5	0.41055630328	**0.41090**564596	0.41131160135	0.411305250793	**0.410905**049742
4.8	0.38874292431	**0.38907**789139	0.38946730819	0.389462357675	**0.389077**139783
5.0	0.37472016558	**0.37504**515130	0.37542301726	0.375418410434	**0.37504**4417235

5. Conclusions

In this manuscript, an approximation algorithm based on different polynomials is developed for solving the nonlinear fractional-order logistic population equation modelling the single species multiplicative Allee effect. Exploiting the fractional Chebyshev, Chelyshkov, and Legendre functions along with the collocation points we convert the differential equation into an algebraic system of nonlinear equations. Numerical test problems are given to demonstrate efficiency and accuracy of the proposed method. Moreover, the performance of these three basis functions has assessed and a comparison between them and other existing schemes is made. Furthermore, the reliability of the proposed technique is checked through defining the residual error functions. Referring to graphs and tables we conclude that using the fractional Chelyshkov function produces a more accurate result compared to Chebyshev and Legendre basis functions. The proposed technique can be easily applied to other logistic population models (1.3)-(1.5) and other problems in science and engineering.

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A NEW STUDY ON ABSOLUTE CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper, we have generalized a known theorem dealing with $\varphi-\mid C,\alpha,\mid_k$ summability factors of infinite series to the $\varphi-\mid C,\alpha,\beta\mid_k$ summability method under weaker conditions. Also, some new and known results have been obtained.
Keywords: summability factors; infinite series; Cesàro mean; Hölder's inequality; Minkowsk's inequality; almost increasing sequences.

1. Introduction

A positive sequence (b_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [2]). Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α,β), with $\alpha+\beta>-1$, of the sequence (na_n), that is (see [8])

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v, \tag{1.1}$$

where

$$A_n^{\alpha+\beta}=O(n^{\alpha+\beta}),\quad A_0^{\alpha+\beta}=1 \quad \texttt{and} \quad A_{-n}^{\alpha+\beta}=0 \quad \texttt{for} \quad n>0. \tag{1.2}$$

Let $(\omega_n^{\alpha,\beta})$ be a sequence defined by (see [5])

$$\omega_n^{\alpha,\beta}=\begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha=1,\beta>-1,\\ \max_{1\leq v\leq n}\left|t_v^{\alpha,\beta}\right|, & 0<\alpha<1,\beta>-1. \end{cases} \tag{1.3}$$

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Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n t_n^{\alpha,\beta} \mid^k < \infty. \tag{1.4}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - \mid C, \alpha, \beta \mid_k$ summability is the same as $\mid C, \alpha, \beta \mid_k$ summability (see [9]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - \mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha, \beta; \delta \mid_k$ summability (see [7]). If we take $\beta = 0$, then we have $\varphi - \mid C, \alpha \mid_k$ summability (see [1]). If we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C, \alpha \mid_k$ summability (see [10]). Finally, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$ and $\beta = 0$, then we obtain $\mid C, \alpha; \delta \mid_k$ summability (see [11]).

2. Known Result

The following theorem is known dealing with the $\varphi - \mid C, \alpha \mid_k$ summability factors of infinite series.

Theorem 2.1 ([3]). Let $0 < \alpha \leq 1$. Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{2.4}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non increasing and if the sequence (ω_n^α) defined by (see [13])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha = 1) \\ \max_{1 \leq v \leq n} |t_v^\alpha| & (0 < \alpha < 1) \end{cases} \tag{2.5}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k} (|\varphi_n| \, \omega_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{2.6}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid C, \alpha \mid_k$, $k \geq 1$ and $(\alpha + \epsilon) > 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 for $\varphi - \mid C, \alpha, \beta \mid_k$ summability method under weaker conditions by using an almost increasing sequence instead of a positive non-decreasing sequence. Now we shall prove the following theorem.

Theorem 3.1. Let $0 < \alpha \leq 1$ and let (X_n) be an almost increasing sequence. Let there be sequences (β_n) and (λ_n) such that conditions (2.1)-(2.4) of Theorem 2.1 are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \left|\varphi_n\right|^k)$ is non increasing and if the sequence $(\omega_n^{\alpha,\beta})$ defined by (1.3), satisfies the condition

$$\sum_{n=1}^{m} n^{-k}(|\varphi_n| \, \omega_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty, \tag{3.1}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - \mid C, \alpha, \beta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\beta > -1$, and $(\alpha + \beta)k + \epsilon > 1$.

Remark. It should be noted that, obviously every increasing sequence is almost increasing. However, the converse need not be true (see [12]).

We need the following lemmas for the proof of our theorem.

Lemma 3.1 ([5]). If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{3.2}$$

Lemma 3.2 ([4]). Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (2.3) is satisfied

$$n \beta_n X_n = O(1) \quad as \quad n \to \infty \tag{3.3}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.4}$$

4. Proof of Theorem 3.1. Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean of the sequence $(n a_n \lambda_n)$. Then, by (1.1), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 3.1, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$\begin{aligned} \mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \omega_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \omega_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}. \end{aligned}$$

To complete the proof of the theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad \text{for} \quad r=1,2.$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+2} n^{-k} \mid \varphi_n T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\beta})^{-k} \mid \varphi_n \mid^k \{\sum_{v=1}^{n-1} A_v^{\alpha+\beta} \omega_v^{\alpha,\beta} \mid \Delta\lambda_v \mid\}^k \\
&\leq \sum_{n=2}^{m+1} n^{-k} n^{-(\alpha+\beta)k} \mid \varphi_n \mid^k \sum_{v=1}^{n-1} (A_v^{\alpha+\beta})^k (\omega_v^{\alpha,\beta})^k \beta_v . \{\sum_{v=1}^{n-1} \beta_v\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\omega_v^{\alpha,\beta})^k \beta_v \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} \mid \varphi_n \mid^k}{n^{(\alpha+\beta)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\omega_v^{\alpha,\beta})^k \beta_v v^{\epsilon-k} \mid \varphi_v \mid^k \sum_{n=v+1}^{m+1} \frac{1}{n^{(\alpha+\beta)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} (\omega_v^{\alpha,\beta})^k v^{\epsilon-k} \mid \varphi_v \mid^k \beta_v \int_v^{\infty} \frac{dx}{x^{(\alpha+\beta)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v\beta_v v^{-k} (\omega_v^{\alpha,\beta} \mid \varphi_v \mid)^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{-k} (\omega_r^{\alpha,\beta} \mid \varphi_r \mid)^k \\
&\quad + O(1) m\beta_m \sum_{v=1}^{m} v^{-k} (\omega_v^{\alpha,\beta} \mid \varphi_v \mid)^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by the hypotheses of Theorem 3.1 and Lemma 3.2. Since, $\mid \lambda_n \mid= O(1)$ by (2.4), finally we have that

$$\begin{aligned}
&\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\beta} \mid^k = O(1) \sum_{n=1}^{m} |\lambda_n| \, |\lambda_n|^{k-1} n^{-k} (\omega_n^{\alpha,\beta} \mid \varphi_n \mid)^k \\
= \quad &O(1)) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{-k} (\omega_v^{\alpha,\beta} |\varphi_v|)^k
\end{aligned}$$

$$+O(1)|\lambda_m|\sum_{n=1}^{m} n^{-k}(\omega_n^{\alpha,\beta}|\varphi_n|)^k = O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m$$

$$= \quad O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,$$

by the hypotheses of Theorem 3.1 and Lemma 3.2. This completes the proof of Theorem 3.1. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C, \alpha, \beta \mid_k$ summability factors of infinite series. If we take $\epsilon = 1$, $\beta = 0$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we have a new result dealing with the $\mid C, \alpha; \delta \mid_k$ summability factors of infinite series. Also, if we take (X_n) as a positive non-decreasing sequence and $\beta = 0$, then we obtain Theorem 2.1.

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ENCRYPTION OF 3D PLANE IN GIS USING VORONOI-DELAUNAY TRIANGULATIONS AND CATALAN NUMBERS

Faruk Selimović, Predrag Stanimirović, Muzafer Saračević, and Selver Pepić

Abstract. A method for encryption of the 3D plane in Geographic Information Systems (GIS) is presented. The method is developed using Voronoi-Delaunay triangulation and properties of Catalan numbers. The Voronoi-Delaunay incremental algorithm is presented as one of the most commonly used triangulation techniques for the random point selection. On the basis of the multiple application of Catalan numbers in solving combinatorial problems and their "bit-balanced" characteristic, the process of encrypting and decrypting the coordinates of points using the Lattice Path method (walk on the integer lattice) or LIFO model is given. The triangulation of the plane started using decimal coordinates of a set of given planar points. Afterward, the resulting decimal values of the coordinates are converted to corresponding binary records and the encryption process starts by random selection of the Catalan key according to the LIFO model. These binary coordinates are again converted into their original decimal values, which enables the process of encrypted triangulation. The original triangulation of the plane can be generated by restarting the triangulation algorithm. Due to its exceptional efficiency, *Java* programming language enables efficient implementation of the proposed method.
Keywords: Encryption of 3D plane; Voronoi-Delaunay triangulation; Catalan numbers; Lattice Path method; Java Net- Beans environment.

1. Introduction

Owing to the achieved progress of the GPS navigation systems and robotics, the encryption of a 3D plane takes an important role in the field of data protection in the development of GIS (*Geographic Information Systems*). The increasing role of the GIS in processing and analysis of spatial data as well as in control systems of defense and public security, the *Delaunay triangulation* represents the basic model

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for creating of TIN (*Triangulated Irregular Network*) in the process of obtaining a digital model of terrain (DMT) [1]. In fact, a DMT is an organized set of data on terrain heights recorded in a digital form.

The subject of research in the paper is to investigate possibilities, properties, and applications of the Catalan numbers in generating keys for encryption of the 3D plane triangulation with the Voronoi-Delaunay triangulation. Our intention is to consider and explain the application of the existing knowledge of Catalan numbers in the process of encryption and decryption of the TIN network of the 3D plane.

Catalan numbers (C_n) are most commonly used entities in geometry. They also appear in solutions to a large number of combinatorial problems. Catalan numbers are calculated according to the following formula [2]:

$$C_n = \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}, \ n \geq 0. \tag{1.1}$$

Many combinatorial problems are based on the Catalan sequence, such as: the *ballot* problem, the problem of roads in the network (Lattice Path), the problem of paired parenthesis [7, 8, 9]. An original contribution of our research is the usage of the sequence of the Catalan numbers as a key generator for encryption and decryption of coordinates of the 3D points in a GIS. We note that the integer n is a basis of the generated keys, and C_n is the number of all key combinations on that basis.

For example, the basis $n = 28$ implies the space of $C_{28} = 263747951750360$ keys satisfying the *bit-balance* property. It is known that the key space is growing by increasing the base. In order to verify the validity of the Catalan numbers property, we will exploit their binary records. The fundamental property that one number must satisfy to be labeled as a Catalan number is the *bit-balance bits* property in the binary file corresponding to a specified number C_n. In other words, the binary record of any Catalan number involves identical number of bits "0" and "1" and starts with the bit "1".

If a binary record of a Catalan number is associated with the balanced parenthesis notation, then the bit "1" becomes an open parenthesis, while the bit "0" represents a closed parenthesis. Moreover, each left parenthesis is closed, which implies that each bit "1" assumes its own pair which is just the bit "0". The binary record of an arbitrary Catalan number can also be represented in the form of stack permutations. In this case, the bit "1" represents the *PUSH* command while the "0" is the *POP* command.

For example, the set of $C_n = 14$ values satisfy the Catalan numbers property for $n = 4$: 170, 172, 178, 180, 184, 202, 204, 210, 212, 216, 226, 228, 232, 240. Based on their binary records 10101010, 10101100, 10110010, 10110100, 10111000, 11001010, 11001100, 11010010, 11010100, 11011000, 11100010, 11100100, 11101000, 11110000 we determine the *bit-balance* property corresponding to the Catalan number.

Observing the binary notations of the given numbers, we can notice that each number has the same number of bits "1" and "0"; in other words, there is a balance between them, which is the main property of Catalan numbers. In addition, the number of pairs 1 and 0 is basically n, while the length of the key is always $2n$. In this example, the base is 4, which means that the key length is 8 bits.

As it was already mentioned, the Catalan number can be modeled by many combinatorial problems [2], such as paired parenthesis "(() () (()))" or a ballot record "AABBABAB", graphically in the form of walking through an integer network (*Lattice Path*) or through the stack permutation. Below we present Stack Permutation as a method for encoding the coordinates of $3D$ points.

The remainder sections of the paper are presented in the following order. The encryption of 3D plane coordinates by means of Catalan numbers is described in Section 2. Section 3 is intended to a description of the Voronoi diagram and Delaunay triangulation of the 3D plane. Also, we describe the main reasons for using this kind of triangulation in the proposed method. Section 4 presents Spatial Data Structure in GIS. Section 5 describes the implementation of the 3D plane encryption algorithm in the Java-Net Beans environment. It is also aimed to the analysis of the Java source code and experimental results.

2. Encryption of 3D plane coordinates with Catalan numbers

The stack is an abstract type of data structure that is based on the principle *LIFO* (*last in, first out*) and on two basic operations *push* and *pop*. The stack permutation, as a method for solving combinatorial issues, can be generated using Catalan numbers.

In [3], it was shown that a number of permutations satisfying the given conditions correspond to Catalan numbers. On the basis of this, it is possible to map each binary record (or equivalent Ballot record) of length $2n$ to the corresponding permutation of the length n by applying a stack.

Consider an example of encrypting one of the 3D coordinates (x, y, z) using Stack Permutations. The x coordinate is $x = 1430$, its binary record is $1430_{10} = 10110010110_2$ with $n = 11$ bits. The value of the Catalan number (below the key) is $K = 2816098$. His binary record is $2816098_{10} = 1010101111100001100010_2$, consisting of $2n = 22$ bits. Figure 2.1 describes the details.

The decryption process is analogous to the corresponding encryption. The key and the code in the decryption are read in the reverse order. The *Balanced Parentheses* method is equivalent to a stack permutation [4]. Figure 2.2 represents the encoding process.

3. Voronoi diagram and Delaunay triangulation of 3D plane

The goal of Delaunay triangulation is the decomposition of a certain surface into non-crossing triangular elements. The angular points of the triangles are main points of the surface, and each anchor represents the corner of the least one triangle. Triangulation is a procedure that is used to process points that have a random

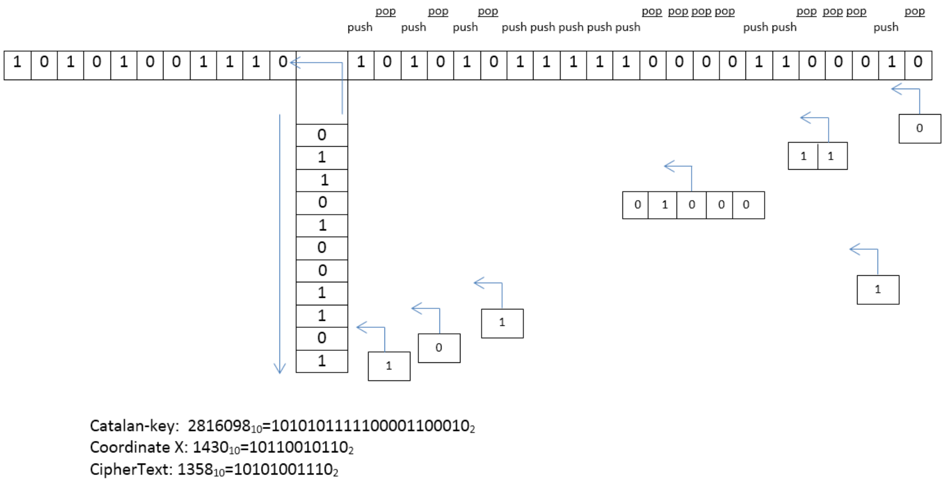


Fig. 2.1: Coordinates encryption example based on Stack Permutation principle

Key	1	0	1	0	1	0	1	1	1	1	1	0	0	0	0	1	1	0	0	0	1	0
Balanced Parentheses	(	)	(	)	(	)	(	(	(	(	(	)	)	)	)	(	(	)	)	)	(	)
Coordinate X	1		0		1		1	0	0	1	0					1	1				0	
Cipher Text		1		0		1						0	1	0	0			1	1	1		0

Fig. 2.2: Coordinates encryption example based on Balanced Parentheses

distribution [10]. *Voronoi polygon* is the geometric place of the closest points of one particular point in the finite set of points. Union of all Voronoi polygons in the set of points in the plane defines the *Voronoi diagram.*

Essentially, the Voronoi diagram as a geometric structure is used for determining the distance between points and the closest points. The Voronoi polygon points separate any point from their nearest neighboring points. The sides of a Voronoi polygon consist the bisectors of the segment line obtained by connecting a point with adjacent points, where each point is combined with adjacent points in order to obtain the Delaunay triangulation. Each cell of the Voronoi diagram presented in Figure 3.1 possesses its own center.

Some of useful properties of an arbitrary Delaunay triangulation are listed bellow:

- Uniqueness and independence from the starting point.
- Formed triangles are in the form of equilateral triangles.

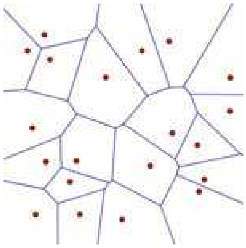


Fig. 3.1: Voronoi diagram - partitions of the plane in the cells

- There is no other point in the circumcircles of the triangles (property of the circumcircle).
- The convex hull is triangulated.
- A line segment that is obtained from the closest pairs of points is in the triangulation.
- A line segment obtained from the point and its nearest point is the side of the triangle in the triangulation

4. Spatial Data Structure in GIS

Spatial data are most important in each GIS. They are geo-referenced by their location on the surface of the earth. Geo-referencing implies a precisely recorded location in a particular coordinate system. Since the GPS system is the backbone of locating and monitoring targets on the surface of the earth, the security or protection of GPS signals sent to earth stations is of great importance in the process of creating business navigation applications. In that sense, 3D level encryption using Catalan numbers and Delaunay triangulation is just one of the geographic data protection models.

4.1. Remote Detection - Global positioning GPS system

The method for collecting and interpreting information about remote objects without physical contact with any of them is termed as *remote sensing*. Common platforms for observations in remote sensing are planes, space probes, and satellites. This method will most often focus on two narrow areas: teledetections and photogrammetry. *Teledetective* is a remote sensing in which information about the earth's surface is collected with the help of the devices located in satellites. *Photogrammetry* means a technique of measurement by which the shape, size, and position of the recorded object are performed on the basis of photographic images.

Basically, GPS satellites send signals to their receivers about their latitude, length, and height, i.e., they send signals for three coordinates (x, y, z). The procedure for obtaining these coordinates is based on the principle of intersection (*trilateration*) of the spheres emitting three satellites. GPS application is multiple [5]. First, it was developed for military purposes, and later in the 1980s, it began to be used for civilian purposes. Navigation of planes, boats, cars without GPS is inconceivable. In the process of signal protection, it is required to have a mechanism (algorithm) for the encryption of coordinates of the points (receiver positions) in the satellite.

So, the a cryptographic signal and an encryption key are sent by a receiver. On the other hand, the receiver should have a decryption mechanism (algorithm) that is capable of returning the received signal (encrypted with the key) to its original value. This algorithm will be explained with more details in the next section.

4.2. Modeling of 3D plane - TIN model

The standard way to represent the terrain surface in digital form is done via *Digital Modeling of Terrain* (DMT). The representation of the surface of the plane is enabled by a mathematical model based on the correct height network (*GRID*) or on the *Triangulated Irregular Network* (TIN). The TIN is formed on the basis of known positions of points and their heights, i.e, coordinates (x, y, z) of given points. The incremental algorithm of Delaunay triangulation is used in the process of network formation.

Based on the TIN model, all the desired calculations can be performed: the value of the inclination at a given point, the height for the given position in the horizontal sense, the direction of the maximum inclination, the curvature of the surfaces at the given point, the visualization of the terrain model, geostatistic analysis and others. Today, TIN models are used in designing traffic, hydraulic engineering, underground facilities, military geographic analysis, etc [6].

Given the wide use of the TIN model, it is necessary to allow encryption of coordinates of points in the moment of electronic transmission as well during storing the model on a certain medium. In general, the algorithm presented in the next section gives the TIN model in conjunction with other (encrypted) coordinate values.

5. Implementation of the 3D plane encryption algorithm in the Java-Net Beans environment

The process of encrypting the coordinate begins by generating a sequence of the total number of randomly selected triangles of the TIN model. After that, the incremental Delaunay triangulation algorithm is applied. In the second step, each decimal value of the vertices coordinate (x, y, z) of the formed triangulation is remembered in the integer string. Then their conversion from decimal to binary form is done because the Catalan key is assigned in binary form. By using the *Stack Permutation* method, the obtained binary coordinate format is converted to another text encoded by the principle *LIFO*, which, after re-conversion to decimal form, is actually *ENCRYPT*, i.e., the result of Algorithm 1.

Algorithm 1 LegalizeEdge $(p_r, \overline{p_i p_j}, \mathcal{T})$

Require: P is set of n points in a plane.
Let p_{-1}, p_{-2} i p_{-3} three points in triangle which consists all other points from set P .
Initial triangulation $\mathcal{T}$ consist the triangle p_{-1}, p_{-2} i p_{-3}
for $r = 1$ to n **do** (Put in p_r u $\mathcal{T}$)
Find the triangle $p_i p_j p_k \in \mathcal{T}$, which consist p_r.
put the p_r integer array K.
return $\mathcal{T}$
for $r = 1$ to n **do** (Access p_r in the array K)
Convert p_r in binary record
Put in the **Stack permutation (LIFO)** method on basis of **Catalan key** from C_n
Convert p_s in decimal record (after permutaton bit p_r it become p_s)
Put the p_s in array K_s
for $s = 1$ to n **do** (Put p_s in $\mathcal{T}_s$)
Find the triangle $p_i p_j p_k \in \mathcal{T}_s$, which consists p_s.
return $\mathcal{T}_s$.
Output : Encrypted n points from set P (encrypted TIN model in plane)

The encoding of the plane points is clearly explained in Algorithm 1. However, this encoding of points implies encryption of their coordinates (x, y, z). In addition to the Delaunay triangulation method, the following methods are used for the implementation of the above steps in the algorithm:

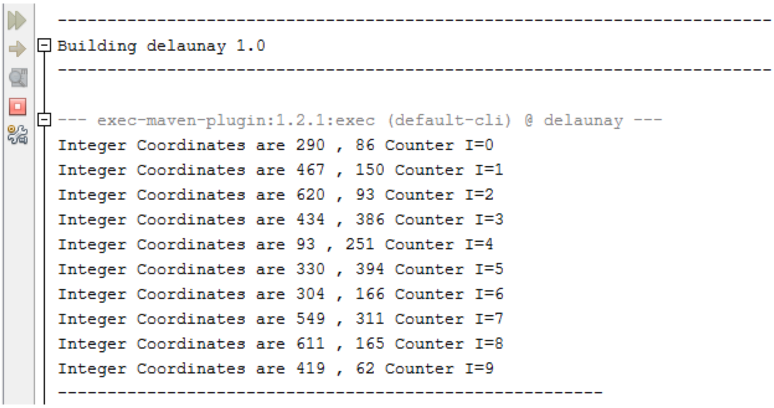
$Convert\_in\_Binary\_Record$, $Binary\_Encoding\_Coordinate$,

$Convert\_Binary\_in\_Integer$ and class *DelaunayAp.ja-va.*

Since the application is done in the *NetBeans* environment, it is possible to present the plane only in 2D form. However, the way of the encryption of the third coordinate z is the same as for *x, y*. Below we present this algorithm in more details.

5.1. Structure of Java source code

Java GUI application starts by the execution of the executable method `main()` class *DelaunayAp.java.* It is necessary to enter the n (number of points in the plane) from the set P. After that, the random coordinates (x, y) are assigned by clicking on the level panel in the large initial triangle $p_i p_j p_k$. The position of the point p_r in the level is determined in this way. The incremental Delaunay triangulation algorithm lies in the background of the constructed method such that all points in the plane are in separated positions. After that, the TIN network of the triangles to be encrypted is created. The decimal values of the coordinates (x, y) are presented in Figure 5.1.


```
------------------------------------------------------------------------
Building delaunay 1.0
------------------------------------------------------------------------

--- exec-maven-plugin:1.2.1:exec (default-cli) @ delaunay ---
Integer Coordinates are 290 , 86 Counter I=0
Integer Coordinates are 467 , 150 Counter I=1
Integer Coordinates are 620 , 93 Counter I=2
Integer Coordinates are 434 , 386 Counter I=3
Integer Coordinates are 93 , 251 Counter I=4
Integer Coordinates are 330 , 394 Counter I=5
Integer Coordinates are 304 , 166 Counter I=6
Integer Coordinates are 549 , 311 Counter I=7
Integer Coordinates are 611 , 165 Counter I=8
Integer Coordinates are 419 , 62 Counter I=9
-------------------------------------------------------
```

FIG. 5.1: (x, y)coordinate values of triangles

A TIN model of the level with introduced points (triangles) of the triangles given in decimal form is presented in Figure 5.2. Previously described events correspond to Algorithm 1 up to step 6, where we get a series of K with the coordinate inputs of x, y points."Encrypt the TIN model" are called the methods $Encoding\_X\_Y\_Coordinates()$. Within this method, the first one is the $Convert\_U\_Binary\_Entry()$, where each dot coordinate in the K sequence is accessed and its conversion from decimal to binary form is executed (step 8 in the algorithm). The method $Binary\_Encoding\_Coordinate()$ is started after the conversion.

Application of this method is explained in more detail in Section 2. The result of the application of this method is the implementation of steps from 9 to 14 in Algorithm 1. It should be noted that the resolution of the monitor in such an environment is a limiting factor. The number of coordinate bits is the exponent of 2 and must always be within the range relative to the resolution of the monitor. For example, in the case of resolution of 1440 x 900 pixels, the number 1440 exceeds the value of 1024 (2^{10}=1024) and due to this in the representation of coordinate values larger than 1024, the exponent must be 11. Also, the Catalan key length always is $2n$, where n is the number of the bits from the coordinates. In our case, the length is of 22 bits. When it comes to 3D modeling and coordinate values which GPS satellites send to Earth stations, this condition is not true. In this case, after the conversion of the decimal value of the coordinates into the binary value, the Catalan key is chosen according to the model $2n$. The result of this method is given in Figure 5.3. Figure 5.4 shows the encrypted TIN level model.

The encoding process is similar to the decoding process, only the encoded and original coordinates change the place. The Catalan key keeps the same value, and

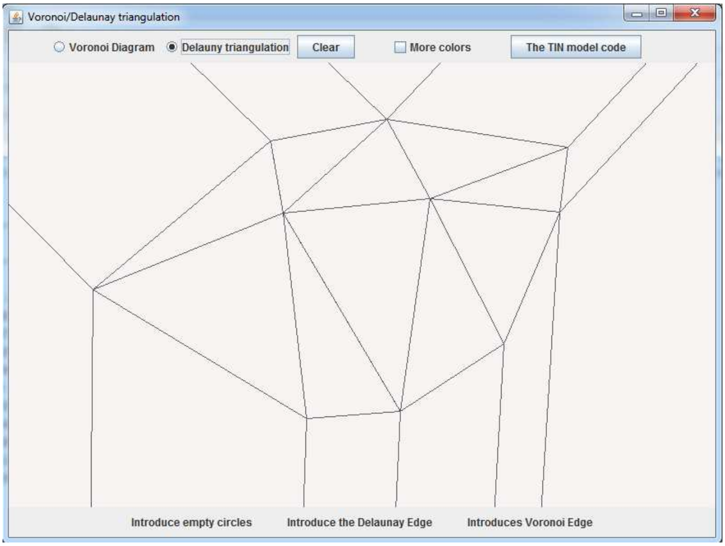


FIG. 5.2: TIN network of irregular triangles

```
--------------------------------------------------------------------------------
Integer Cooordinate: X = 611, Y =165
Catalan Key for N=11: 1010101111100001100010
Binary Coordinate: X = 01001100011 ,Y = 00010100101
Encrypted Binary Coordinates: X = 01000111001 Y = 00000100111
Encrypted Binary Coordinates: X = 569 Y = 39
--------------------------------------------------------------------------------
Integer Cooordinate: X = 419, Y =62
Catalan Key for N=11: 1010101111100001100010
Binary Coordinate: X = 00110100011 ,Y = 00000111110
Encrypted Binary Coordinates: X = 00100101011 Y = 00011101100
Encrypted Binary Coordinates: X = 299 Y = 236
--------------------------------------------------------------------------------
End of Encryption Coordinates----------------------------------------------------
 Encrypted Coordinates X = 296 Y= 92
 Encrypted Coordinates X = 347 Y= 78
 Encrypted Coordinates X = 692 Y= 213
 Encrypted Coordinates X = 362 Y= 266
 Encrypted Coordinates X = 213 Y= 251
 Encrypted Coordinates X = 408 Y= 394
 Encrypted Coordinates X = 352 Y= 46
 Encrypted Coordinates X = 549 Y= 365
 Encrypted Coordinates X = 569 Y= 39
 Encrypted Coordinates X = 299 Y= 236
Execution time: 20 millisecond
```

FIG. 5.3: Values of binary coordinates and their encryptions

reading of the key length and the cipher is done from right to left, ie, in reverse order of encryption. Figure 5.5 shows the descriptive coordinates corresponding to

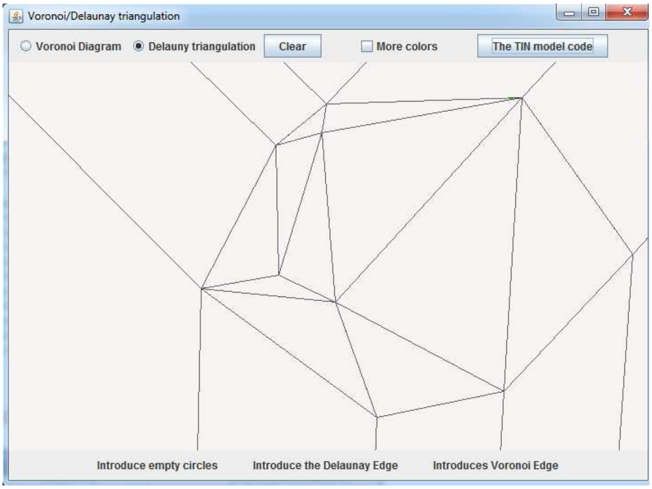


FIG. 5.4: Encrypted TIN model

the original values of the coordinates in Figure 5.3.

```
Decryption Binary Coordinates: X = 01000100101 Y = 00100110111
Described - Original Integer Coordinates: X = 549 Y = 311
--------------------------------------------------------------------------------
Encrypted Coordinates: X = 569, Y =39
Catalan Key for N=11: 1010101111100001100010
Encrypted Binary Coordinates: X = 01000111001 ,Y = 00000100111
Decryption Binary Coordinates: X = 01001100011 Y = 00010100101
Described - Original Integer Coordinates: X = 611 Y = 165
--------------------------------------------------------------------------------
Encrypted Coordinates: X = 299, Y =236
Catalan Key for N=11: 1010101111100001100010
Encrypted Binary Coordinates: X = 00100101011 ,Y = 00011101100
Decryption Binary Coordinates: X = 00110100011 Y = 00000111110
Described - Original Integer Coordinates: X = 419 Y = 62
--------------------------------------------------------------------------------
End of Deciphering Coordinate--------------------------------------------------------------
Described Coordinates X= 290 Y= 86
Described Coordinates X= 467 Y= 150
Described Coordinates X= 620 Y= 93
Described Coordinates X= 434 Y= 386
Described Coordinates X= 93 Y= 251
Described Coordinates X= 330 Y= 394
Described Coordinates X= 304 Y= 166
Described Coordinates X= 549 Y= 311
Described Coordinates X= 611 Y= 165
Described Coordinates X= 419 Y= 62
```

FIG. 5.5: Values of decrypted coordinates

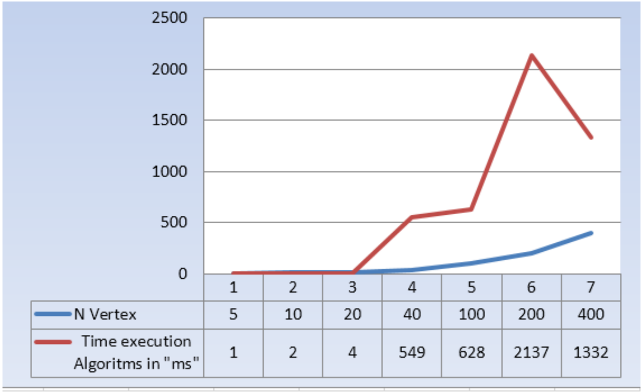


FIG. 5.6: Encryption time of coordinate (x, y, z)

5.2. Experimental results - Encryption time

The encryption time was tested on the vertices $N = \{5,10,20,40,100,200,400\}$. Since *JavaNetBeans* compiles and interprets simultaneously, the capabilities of our algorithm were examined in this environment.

N Vertex	*Time execution of Algoritms in "ms"*
5	1
10	2
20	4
40	549
100	628
200	2137
400	1332

Table1: Encription time

If we also present this data graphically, it can be noticed that the encryption time is not in direct proportion to the number of vertices of the triangles. This fact is a good indicator, because the encryption time does not grow on the basis of increasing the number of vertices. Corresponding results are presented in Figure 5.6.

Considering this low time for encryption, encrypted coordinates can be stored in a database, which will further increase the efficiency of this algorithm. The numerical testing was done on a computer with the next performances: *Intel Core i5-CPU 2.6 GHz, RAM -4 GigaBytes, Operating system: Windows 7 Microsoft -64 bits.*

6. Conclusion and further works

The proposed method is a combination of the computational geometry, geographic information systems and cryptography. A new method for encoding coordinates is based on the Catalan-key. If an integer n is a basis for generated keys, then C_n is total space of different keys, i.e, the number of different binary records. For a 64-bit key, there exist a huge number of total valid values which fulfil the bit-balance property (for base $n = 32$, the space of 64-bit Catalan keys is $C_{32} = 55534064877048198$). In order to provide all Catalan numbers and store them on a disk, it is required a memory space of 44 427 251 901 MB or about 42 369 TB. So, this procedure is very demanding with respect to memory requirements. Further, if it is necessary to find all 64-bit Catalan numbers, and if $1ms$ is necessary to access each element in the set of all C_n, then the CPU time would spend about 176097 years. Average time will be $176097/2 = 88048$ years. Our strategy is usage of some larger bases to generate Catalan-key spaces to prove that the Catalan-key space drastically increases even after a small increase in the base.

In fact, the construction of a large space of Catalan keys assures the security of the presented cryptosystem. The proposed methods of encryption may have wide applications. GIS is the most promising information technology today, due to wide spectrum of possibilities and the scope of its applications. It is almost impossible to efficiently conduct a geographic analysis of the terrain without GIS. Especially, its application in the military analysis of the field is expressed, i.e. in the creation of digital modeling of terrain (DMT). The TIN model is one of the most common methods for presenting DMT, i.e., a network of irregular triangles with vertices in the points with known heights on the terrain [6]. In addition to the application of GIS or DMT for military purposes and in monitoring and navigation devices, it is applied in other areas, such as construction, hydro-engineering, generating maps for flood risks, etc. Lately, there is an increasingly important role in hydraulic modeling. Given this wide application of DMT, the TIN modeling is of great importance. Cryptography is a very dynamic domain and in this paper, only some of its basic mathematical concepts are covered.

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THE LEVINSON-TYPE FORMULA FOR A CLASS OF STURM-LIOUVILLE EQUATION

Sertac Goktas and Khanlar R. Mamedov

Abstract. The boundary value problem

$$-\psi'' + q(x)\psi = \lambda^2\psi, \quad 0 < x < \infty,$$

$$\psi'(0) - (\alpha_0 + \alpha_1\lambda)\psi(0) = 0$$

is considered, where λ is a spectral parameter, $q(x)$ is real-valued function such that

$$\int_0^\infty (1+x)|q(x)|dx < \infty$$

with $\alpha_0, \alpha_1 \geq 0$ ($\alpha_0, \alpha_1 \in \mathbb{R}$).

In this paper, for above-mentioned boundary value problem, the scattering data is considered and the characteristics properties (such as continuity of the scattering function $S(\lambda)$ and giving the Levinson-type formula) of this data are studied.

Keywords: Scattering data; scattering function; Gelfand-Levitan-Marchenko equation; Levinson-type formula.

1. Introduction

Consider the boundary value problem

$$-\psi'' + q(x)\psi = \lambda^2\psi, \quad 0 < x < \infty, \tag{1.1}$$

$$\psi'(0) - (\alpha_0 + \alpha_1\lambda)\psi(0) = 0, \tag{1.2}$$

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where $q(x)$ is real valued function such that

$$\int_0^\infty (1+x)|q(x)|dx < \infty \tag{1.3}$$

and α_0, α_1 are real numbers, also $\alpha_0, \alpha_1 \geq 0$.

Spectral analysis when the spectral parameter appearing linearly on the half line for the boundary value problem (1.1) was studied in [3, 4],(1.2). In the case $q(x) \equiv 0$, this boundary value problem is given by application to the heat transmission problem in [2]. In the wave theory of mathematical physics and geophysics, the applications of the problems can also be found [1, 5, 20, 21, 22, 23].

It is known [15, 16] that the function which can be unique represented in the from

$$e(x,\lambda) = e^{i\lambda x} + \int_0^\infty K(x,t)e^{i\lambda t}dt, \tag{1.4}$$

is a Jost solution of the equation (1.1) for any λ on closed upper half plane, where the kernel $K(x,t)$ satisfies the relation

$$|K(x,t)| \leq \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x+t}{2}\right)\right\}$$

with

$$\sigma(x) \equiv \int_x^\infty |q(t)|dt, \qquad \sigma_1(x) \equiv \int_x^\infty \sigma(t)dt$$

and

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)dt.$$

The function $e(x,-\lambda)$ satisfies the equation (1.1) for each $\lambda \in \mathbb{R} \setminus \{0\}$ and the functions $e(x,\lambda)$ and $e(x,-\lambda)$ form a fundamental set of solutions for the differential equation (1.1). Their Wronskian is as follows:

$$W\{e(x,\lambda), e(x,-\lambda)\} = e'(x,\lambda)e(x,-\lambda) - e(x,\lambda)e'(x,-\lambda) = 2i\lambda.$$

Let $\varpi(x,\lambda)$ denote the a special solution of the equation (1.1) that satisfies the initial conditions

$$\varpi(0,\lambda) = 1, \qquad \varpi'(0,\lambda) = \alpha_0 + \alpha_1\lambda.$$

The following lemma 1.1 and lemma 1.2 which have been proved in [9] should be given in order to achieve the aim of the manuscript:

Lemma 1.1. *The identity*

$$\frac{2i\lambda\varpi(x,\lambda)}{e'(0,\lambda)-(\alpha_0+\alpha_1\lambda)e(0,\lambda)}=e(x,-\lambda)-S(\lambda)e(x,\lambda)$$

holds for all real $\lambda\neq 0$ *where*

$$S(\lambda)=\frac{e'(0,-\lambda)-(\alpha_0+\alpha_1\lambda)e(0,-\lambda)}{e'(0,\lambda)-(\alpha_0+\alpha_1\lambda)e(0,\lambda)} \tag{1.5}$$

and

$$|S(\lambda)|=1.$$

Here, the function $S(\lambda)$ is represented by the formula (1.5). This function is called the scattering function of the boundary value problem (1.1)-(1.3).

The function $S(\lambda)$ is meromorphic function on the upper half plane ($Im\lambda>0$). The zeros of the function $\varphi(\lambda)\equiv e'(0,\lambda)-(\alpha_0+\alpha_1\lambda)e(0,\lambda)$ are the poles of the function $S(\lambda)$.

Lemma 1.2. *The function* $\varphi(\lambda)$ *may have only a finite number of zeros* $\lambda_1,\lambda_2,...,\lambda_n$ *on the half plane* $Im\lambda>0$ *and all these zeros don't lie on the imaginary axis. The zeros* $\varphi(\lambda)$ *and* $\varphi_1(\lambda)\equiv e'(0,-\lambda)-(\alpha_0+\alpha_1\lambda)e(0,-\lambda)$ *are complex conjugate each other and the number of these zeros is equal.*

The number m_k is referred to the multiplicity of the zeros λ_k, $(k=1,2,...,n)$ of the equation $\varphi(\lambda)=0$. These λ_k is called the singular values of the boundary value problem (1.1)-(1.3).

We denote

$$f_j(x)=i\operatorname*{Res}_{\lambda=\lambda_j}\frac{\varphi_2(\lambda)}{\varphi(\lambda)}e^{i\lambda x},$$

where $\varphi_2(\lambda)=\hat{e}\,'(0,\lambda)-(\alpha_0+\alpha_1\lambda)\hat{e}(0,\lambda)$ and $\hat{e}(x,\lambda)$ is a solution of the equation (1.1) (see [18, p.299]). We shall call the polynomial

$$P_k(x)=e^{-i\lambda_k x}f_k(x),\qquad k=1,2,...,n,$$

with degree of m_k-1 the normalization poliynomial for boundary value problem (1.1)-(1.3).

Let

$$F_s(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}[S_0-S(\lambda)]d\lambda,$$

$$F(x)=\sum_{k=1}^{n}f_k(x)+F_s(x), \tag{1.6}$$

where $S_0=\frac{a+i}{a-i}$.

The kernel function $K(x,t)$ of the special solution (1.4) satisfies the integral equation

$$F(x+y)+K(x,y)+\int_x^\infty K(x,t)F(t+y)dt=0, \qquad x<y<\infty \tag{1.7}$$

for each $x \geq 0$.

The equation (1.7) is called the main equation (also called Gelfand-Levitan-Marchenko equation) of the inverse boundary value problem (1.1)-(1.3). This main equation admits a uniquely solution $K(x,t)$ in the space $L_1(x,\infty)$ [9].

The set of values $\{S(\lambda),\lambda_k,P_k(x),(k=\overline{1,n})\}$ is referred to as the scattering data of the boundary value problem (1.1)-(1.3) (see [8]). The inverse scattering problem consists in uniquely recovering the coefficient $q(x)$ from the scattering data. Given the scattering data, we can use formula (1.6) to construct the function $F(x)$ and write out to main equation (1.7) for the unknown function $K(x,y)$. The main equation has a unique solution for every $x \geq 0$. Solving this equation, we find the kernel $K(x,y)$ of the solution (1.7) and hence potential $q(x)=-2\frac{dK(x,x)}{dx}$.

Note that the inverse problem of scattering theory on the half line for the boundary value problem (1.1)-(1.3) in the case $\alpha_1=0$ was completely solved in [6, 7, 15, 16]. Inverse problems in the half line with spectral parameter contained in the boundary conditions was investigated according to spectral function in [19], according to Weyl function in [21]-[23], and acording to scattering data [10]-[13]. In the case of non-selfadjoint, the similar problem was solved in [8]. The uniqueness of solution of inverse scattering problem for boundary value problem (1.1)-(1.3) is given in [9] by using the methods of [8] and [15]. Different from the classical case the zeros of Jost function not lie imaginary axis, lie complex plane and these zeros not simple. The boundary value problem (1.1)-(1.3) is not selfadjoint and for this reason, scattering data is differently defined. Therefore, the properties of the scattering data have to be investigated. The present work is devoted to give the properties of the scattering data of boundary problem (1.1)(1.3). Similar problem in the self-adjoint case was studied in [14, 17].

Let us give a brief description of the structure of our study. In Section 2, we prove the continuity of the scattering function on the whole axis. In Section 3, we derive the Levinson type formula.

2. The continuity of the scattering function

In this section, the continuity of the scattering function $S(\lambda)$ defined by (1.5) will be investigated.

Theorem 2.1. *The scattering function $S(\lambda)$ is continuous for all real points λ.*

Proof. It follows from lemma 1.1 that $\varphi(\lambda) \neq 0$ for all $\lambda \neq 0$. The continuity of the function $S(\lambda)$ can be obtained from hence.

When $\varphi(0) \neq 0$, the function $S(\lambda)$ is continuous for $\lambda = 0$ and $S(0) = 1$.

Let $\varphi(0) = 0$. Namely,

$$\begin{aligned}\varphi(0) &= e'(0,0) - \alpha_0 e(0,0) \\ &= -K(0,0) + \int_0^\infty K_x(0,t)dt - \alpha_0\left[1 + \int_0^\infty K(0,t)dt\right] = 0. \end{aligned} \tag{2.1}$$

To complete proof, we shall investigate the continuity of the function $S(\lambda)$ in this case.

Now, putting $x = 0$ in the main equation (1.7), we obtain

$$K(0,y) + F(y) + \int_0^\infty K(0,t)F(t+y)dt = 0. \tag{2.2}$$

Substituting $x = 0$ after differentiating the main equation (1.7) with respect to x, we get

$$K_x(0,y) + F'(y) - K(0,0)F(y) + \int_0^\infty K_x(0,t)F(t+y)dt = 0. \tag{2.3}$$

After multiplying the equation (2.2) throughout by $-\alpha_0$ and adding to the equality (2.3), we have

(2.4)

$$K_x(0,y) - \alpha_0 K(0,y) - (\alpha_0 + K(0,0))F(y) + F'(y) + \int_0^\infty [K_x(0,t) - \alpha_0 K(0,t)]F(t+y)dt = 0.$$

Then, integrating the equality (2.4) with respect to y from z to ∞, we obtain

$$\begin{aligned}\int_z^\infty [K_x(0,y) - \alpha_0 K(0,y)]dy - (\alpha_0 + K(0,0))\int_z^\infty F(y)dy - F(z) \\ + \int_0^\infty [K_x(0,t) - \alpha_0 K(0,t)]\int_{z+t}^\infty F(\xi)d\xi dt = 0.\end{aligned}$$

Put $K_1(z) = \int\limits_z^\infty [K_x(0,y) - \alpha_0 K(0,y)]dy$. Then, from the last equality, the following relation is obtained:

$$K_1(z) - (\alpha_0 + K(0,0))\int_z^\infty F(y)dy - F(z) - \int_0^\infty \left(\int_{z+t}^\infty F(\xi)d\xi\right) dK_1(t) = 0.$$

Using integration by parts and considering the following process

$$\int_0^\infty K'_x(x,t)|_{x=0} \int_{t+z}^\infty F(\xi)d\xi dt = \int_0^\infty K'_x(x,t)|_{x=0} \int_z^\infty F(y)dydt$$
$$- \int_0^\infty F(t+z) \int_t^\infty K'_x(x,\xi)|_{x=0} d\xi dt,$$

$$\int_z^\infty F'(y)dy - K(0,0)\int_z^\infty F(y)dy + \int_z^\infty K'_x(x,y)|_{x=0}dy$$
$$+ \int_0^\infty K'_x(x,t)|_{x=0} \int_z^\infty F(y)dydt - \int_0^\infty F(t+z)\int_t^\infty K'_x(x,\xi)|_{x=0}d\xi dt = 0,$$

we get

$$K_1(z) - (\alpha_0 + K(0,0) + K_1(0)) \int_z^\infty F(y)dy - F(z) - \int_0^\infty K_1(t)F(t+z)dt = 0.$$

Hence, when $\varphi(0) = 0$ (from (2.1)), $K_1(z)$ is the bounded solution of the equation

$$K_1(z) - \int_0^\infty K_1(t)F(t+z)dt = F(z), \qquad (0 \le z < \infty).$$

It is evident that the bounded solution of this equation is integrable on the half axis. It means that $K_1(z) \in L_1(0,\infty)$ (see [15], p. 211).

Returning to the representation $\varphi(\lambda)$, we have

$$\varphi(\lambda) = i\lambda - K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt - (\alpha_0 + \alpha_1\lambda)\left[1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right]$$
$$= i\lambda K(0,0) + \int_0^\infty K_x(0,t)e^{i\lambda t}dt - \alpha_0\left[1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right]$$
$$- \alpha_1\lambda\left[1 + \int_0^\infty K(0,t)e^{i\lambda t}dt\right],$$

where

$$
\begin{aligned}
&-K(0,0)+\int_0^\infty K_x(0,t)e^{i\lambda t}dt-\alpha_0-\alpha_0\int_0^\infty K(0,t)e^{i\lambda t}dt=\\
&=-K(0,0)-\int_0^\infty e^{i\lambda t}d\left(\int_t^\infty K(0,y)dy\right)-\alpha_0+\alpha_0\int_0^\infty e^{i\lambda t}d\left(\int_t^\infty K(0,y)dy\right)\\
&=-K(0,0)+\int_x^\infty K_x(0,y)dy-\alpha_0\int_0^\infty K(0,y)dy+i\lambda\int_0^\infty e^{i\lambda t}\int_t^\infty K_x(0,y)dydt\\
&-i\alpha_0\lambda\int_0^\infty e^{i\lambda t}\int_0^\infty K(0,y)dydt\\
&=i\lambda\int_0^\infty\int_t^\infty (K_x(0,y)-\alpha_0K(0,y))dye^{i\lambda t}dt\\
&=i\lambda\int_0^\infty K_1(t)e^{i\lambda t}dt.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\varphi(\lambda)&=i\lambda\left[1+\int_0^\infty K_1(t)e^{i\lambda t}dt-i\alpha_1\left(1+\int_0^\infty K(0,t)e^{i\lambda t}dt\right)\right]\\
&=i\lambda\widetilde{K}(\lambda).
\end{aligned}
\tag{2.5}
$$

where

$$\widetilde{K}(\lambda)=1-i\alpha_1+\int_0^\infty K_1(t)e^{i\lambda t}dt-i\alpha_1\int_0^\infty K(0,t)e^{i\lambda t}dt.$$

Similarly, we get

$$\varphi_1(\lambda)=-i\lambda\widetilde{K_1}(\lambda), \tag{2.6}$$

where

$$\widetilde{K_1}(\lambda)=1+i\alpha_1+\int_0^\infty K_1(t)e^{-i\lambda t}dt-i\alpha_1\int_0^\infty K(0,t)e^{-i\lambda t}dt.$$

Consequently, from the equality (1.5)

$$S(\lambda)=-\frac{\widetilde{K_1}(\lambda)}{\widetilde{K}(\lambda)}.$$

Taking into account lemma 1.1 (see [9]) and by using the formulas (2.5) and (2.6), we can write

$$2\varpi(x,\lambda) = \widetilde{K}(\lambda)[e(x,-\lambda) - S(\lambda)e(x,\lambda)].$$

It can be seen that $\widetilde{K}(\lambda) \neq 0$, otherwise it would be $\varphi(x,0) = 0$. But, this can not be true since $\varphi(0,0) = 1$. So, $S(\lambda)$ is continuous at $\lambda = 0$ and by condition (2.1) it holds $S(\lambda) = -\frac{\widetilde{K_1}(0)}{\widetilde{K}(0)}$.

This completes the proof the theorem. □

3. The Levinson-Type formula

We give the formula that expresses the relation between the increment of the argument of the scattering function $S(\lambda)$ and the singular number λ_k of boundary value problem (1.1)-(1.3).

Theorem 3.1. *The following formlua is valid:*

$$-\frac{1-S(0)}{2} - \frac{1}{2\pi}\arg S(\lambda)\Big|_{-\infty}^{\infty} + 1 = 2[m_1 + m_2 + ... + m_n], \tag{3.1}$$

where m_k $(k = 1,2,...,n)$ is the multiplicity of the singular number λ_j $(j = 1,2,...,n)$.

Proof. For sufficiently little $\varepsilon > 0$ and sufficiently large $R > 0$, let

$$\Gamma_{R,\varepsilon} = C_R^+ \cup C_\varepsilon^- \cup [-R,-\varepsilon] \cup [\varepsilon, R],$$

where C_R^+ and C_ε^- are circles with centers in origin and corresponding radius of R and ε, respectively (Fig. 1). Orientation on the C_R^+ is positive and on the C_ε^- negative.

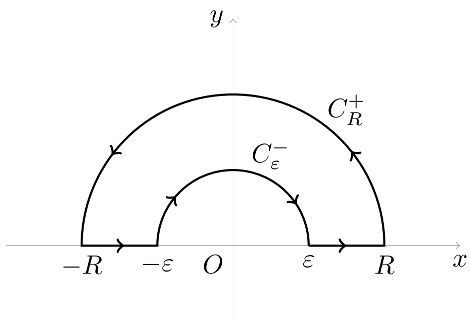

Figure 1: The Graph of $\Gamma_{R,\varepsilon}$.

Let us apply argument principle to $\varphi(\lambda)$ function. This function is regular on the upper half plane and continuous on the closed half plane $Im\lambda \geq 0$. When moving from $-\infty$ to ∞ on the whole real axis and passing origin from top along with half circle with radius ε, the change in the argument of $\varphi(\lambda)$ is equal to number of its pole points times 2π:

$$\arg\varphi(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]} + \arg\varphi(\lambda)|_{\Gamma_\varepsilon} + \arg\varphi(\lambda)|_{\Gamma_R} = 2\pi[m_1 + m_2 + ... + m_n] \tag{3.2}$$

or

$$\frac{1}{2\pi i}\left(\int\limits_{C_R^+} + \int\limits_{C_\varepsilon^-} + \int\limits_{\varepsilon}^{R} + \int\limits_{-R}^{-\varepsilon}\right) d\ln\varphi(\lambda) = m_1 + m_2 + ... + m_n.$$

From (1.5), the scattering function $S(\lambda)$ has the form

$$S(\lambda) = \frac{\varphi_1(\lambda)}{\varphi(\lambda)}$$

for real λ. It is clear from here that $\arg S(\lambda) = -2\arg\varphi(\lambda)$.

Using the last equality, we have

$$\arg\varphi(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]} = -\frac{1}{2}\arg S(\lambda). \tag{3.3}$$

Considering (3.3) in the equality (3.2), we obtain

$$-\frac{1}{2}\arg S(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]} + \arg\varphi(\lambda)|_{C_\varepsilon^-} + \arg\varphi(\lambda)|_{C_R^+} = 2\pi[m_1+m_2+...+m_n]. \tag{3.4}$$

According to theorem 2.1, the function $S(\lambda)$ is continuous on the whole real axis. Hence,

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\left\{-\frac{1}{2}\arg S(\lambda)|_{[-R,-\varepsilon]\cup[\varepsilon,R]}\right\} = -\frac{1}{2}\arg S(\lambda)\Big|_{-\infty}^{\infty}, \tag{3.5}$$

$$\lim_{\varepsilon\to 0}\arg\varphi(\lambda)|_{C_\varepsilon^-} = \begin{cases} 0, & \text{if } \varphi(0)\neq 0, \\ -\pi, & \text{if } \varphi(0) = 0, \end{cases} = -\frac{\pi(1-S(0))}{2} \tag{3.6}$$

and

$$\lim_{R\to\infty}\arg\varphi(\lambda) = \pi \tag{3.7}$$

from lemma 1.1.

Taking into account the equalities (3.5), (3.6) and (3.7) in the equality (3.4), we have

$$-\frac{1}{2}\arg S(\lambda)\Big|_{-\infty}^{\infty}+\pi+\begin{cases}0, & \text{if } \varphi(0)\neq 0,\\ -\pi, & \text{if } \varphi(0)=0,\end{cases}=2\pi[m_1+m_2+...+m_n]$$

From this last equality, the formula (3.1) is obtained, which proves the theorem. □

The note that, this formula is called the Levinson-type formula for the boundary value problem (1.1)-(1.3).

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GRUNDY DOMINATION SEQUENCES IN GENERALIZED CORONA PRODUCTS OF GRAPHS

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Abstract. For a graph $G = (V, E)$, a sequence $S = (v_1, \dots, v_k)$ of distinct vertices of G it is called a *dominating sequence* if $N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \varnothing$ and is called a *total dominating sequence* if $N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ for each $2 \leq i \leq k$. The maximum length of (total) dominating sequence is denoted by (γ_{gr}^t) $\gamma_{gr}(G)$. In this paper we compute (total) dominating sequence numbers for generalized corona products of graphs.
Keywords: dominating sequence; total dominating sequence; generalized corona products.

1. Introduction

In this paper, G is a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For notation and graph theoretical terminology, we generally follow [8]. The order $|V|$ and the size $|E|$ of G is denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ of v is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = d_G(v) = |N_G(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write P_n for the path of order n, C_n for the cycle of order n, and K_n for the complete graph of order n. A subset D of $V(G)$ is called a *dominating set* of G if every vertex of G is either in D or adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A *total dominating set* of G is a set D of vertices of G such that every vertex is adjacent to a vertex in D. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A dominating set of cardinality $\gamma(G)$ $(\gamma_t(G))$ is called a γ-*set*

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(γ_t-*set*). For further information about various domination sets in graphs, we refer reader to [9, 10].

Let G be a graph of order n and let $H_1, H_2, \ldots, H_n$, be n graphs. The *generalized corona product*, is the graph obtained by taking one copy of graphs $G, H_1, H_2, \cdots, H_n$ and joining the ith vertex of G to every vertex of H_i. This product is denoted by $G \circ \wedge_{i=1}^{n} H_i$. If each H_i is isomorphic to a graph H, then generalized corona product is called the *corona product* of G and H and is denoted by $G \circ H$.

Let G be a graph of size m and H be a graph. The *edge corona product*, denoted by $G \diamond H$, is the graph obtained by taking one copy of G and m copies of H, and then joining two end-vertices of the ith edge e_i of G to every vertex of ith copy of H. The *neighborhood corona*, denoted by $G \star H$, is the graph obtained by taking n copies of H and for each $i, 1 \leq i \leq n$, the ith copy of H being adjacent to vertices of $N_G[v_i]$. It is not difficult to see that $G \diamond H$ is the same as $G \circ \wedge_{i=1}^{n} H_i$, where each H_i is a disjoint union of $\deg(v_i)$ copies of H and $G \star H$ is the same as $G \circ \wedge_{i=1}^{n} H_i$, where each H_i is a disjoint union of $\deg(v_i) + 1$ copies of H.

Based on the domination number and the total domination number, various Grundy domination invariants have been introduced in recent years by some authors [1, 5, 6] and then they continued the study of these concepts in [3, 2, 4, 7].

In [5] the first type of Grundy dominating sequence was introduced. Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of a graph G. The corresponding set $\{v_1, \ldots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. A sequence $S = (v_1, \ldots, v_k)$ is called a *closed neighborhood sequence* if, for each i,

$$N_G[v_i] \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

If for a closed neighborhood sequence S, the set $\widehat{S}$ is a dominating set of G, then S is called a *dominating sequence* of G. Clearly, if $S = (v_1, v_2, \ldots, v_k)$ is a dominating sequence for G, then $k \geq \gamma(G)$. We call the maximum length of a dominating sequence in G the *Grundy domination number* of G and denote it by $\gamma_{gr}(G)$. The corresponding sequence is called a Grundy dominating sequence of G or γ_{gr}-sequence of G.

Total dominating sequences were introduced in [6], when G is a graph without isolated vertices. Using the same notation as in the previous paragraph, we say that a sequence $S = (v_1, \ldots, v_k)$ is called an *open neighborhood sequence* if, for each $2 \leq i \leq k$,

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G(v_j) \neq \emptyset.$$

Any open neighborhood sequence S, where $\widehat{S}$ is a total dominating set is called a *total dominating sequence*. The maximum length of a total dominating sequence in G is called the *Grundy total domination number* of G and denoted by $\gamma_{gr}^{t}(G)$.

The corresponding sequence is called a *Grundy total dominating sequence* of G or a γ_{gr}^t-total sequence.

An additional variant of the Grundy (total) domination number was introduced in [1]. Let G be a graph without isolated vertices. A sequence $S = (v_1, \ldots, v_k)$, where $v_i \in V(G)$, is called a $Z-$ *sequence* if for each i,

$$N_G(v_i) \setminus \bigcup_{j=1}^{i-1} N_G[v_j] \neq \emptyset.$$

Then the Z-Grundy domination number $\gamma_{gr}^Z(G)$ of the graph G is the length of a longest Z-sequence.

Let $S_1 = (v_1, \ldots, v_n)$ and $S_2 = (u_1, \ldots, u_m), n, m \geq 1$, be two sequences in G, with $\widehat{S_1} \cap \widehat{S_2} = \emptyset$. The *concatenation* of S_1 and S_2 is defined as the sequence $S_1 \oplus S_2 = (v_1, \ldots, v_n, u_1, \ldots, u_m)$. Clearly $\oplus$ is an associative operation on the set of all sequences, but is not commutative. If $S_2 = \{v\}$, then $S_1 \oplus S_2$ is denoted by $S_1 \oplus v$.

In the next section, we compute Grundy domination numbers for generalized corona products of graphs and based on, we find Grundy domination numbers of edge and neighborhood corona products of graphs.

2. Main Results

In this section we give the exact value of (total) Grundy domination numbers for generalized corona products, and compute them for corona product of some special graphs. First we state two necessary known propositions.

Proposition 2.1. *[6] For $n \geq 4$ even, $\gamma_{gr}^t(P_n) = n$ and $\gamma_{gr}^t(C_n) = n - 2$, while for $n \geq 3$ odd, $\gamma_{gr}^t(P_n) = \gamma_{gr}^t(C_n) = n - 1$.*

Proposition 2.2. *[5, 1] For $n \geq 3$, $\gamma_{gr}(C_n) = \gamma_{gr}^Z(C_n) = n - 2$, while for $n \geq 2$, $\gamma_{gr}(P_n) = \gamma_{gr}^Z(P_n) = n - 1$.*

we are now state and proof the our first main result.

Theorem 2.1. *Let G and $H_1, H_2, \ldots, H_n$ be $n+1$ graphs without isolated vertices. Then*

$$\gamma_{gr}(G \circ \wedge_{i=1}^n H_i) = \sum_{i=1}^n \gamma_{gr}(H_i) + \gamma_{gr}^Z(G).$$

Proof. Set $K = G \circ \wedge_{i=1}^n H_i$. Let $S = (v_1, \ldots, v_k)$ be a Z-Grundy sequence of G and S_i be a γ_{gr}-sequence of H_i for $1 \leq i \leq n$. It is not difficult to see that

$$S_1 \oplus v_1 \oplus S_2 \oplus v_2 \oplus \ldots \oplus S_k \oplus v_k \oplus S_{k+1} \oplus S_{k+2} \oplus \ldots \oplus S_n$$

is a dominating sequence for K. This implies that $\gamma_{gr}(K) \geq \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^{Z}(G)$.

Let T be a γ_{gr}-sequence of K such that $|\widehat{T} \bigcap V(G)|$ is minimum among all γ_{gr}-sequences. Suppose that $\widehat{T} \bigcap V(G) = \{v_1, \ldots, v_t\}$, where $(v_1, \ldots, v_t)$ is a subsequence of T. If $t > \gamma_{gr}^{Z}(G)$, then $(v_1, \ldots, v_t)$ is not a Z-sequence for G and thus, there exists $1 \leq l \leq t$ such that $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \varnothing$. But $N_K[v_l] \setminus \bigcup_{i=1}^{l-1} N_K[v_i] \neq \varnothing$, since $(v_1, \ldots, v_t)$ is a sub-sequence of T. If $\widehat{T} \bigcap V(H_l) \neq \varnothing$, then there exists an element $z \in V(H_l)$ such that one of the $(v_1, \ldots, v_l, z)$ or $(v_1, \ldots, v_{i-1}, z, v_i, \ldots, v_l)$ is a subsequence of T. If $(v_1, \ldots, v_l, z)$ is a subsequence of T, then $N_K[z] \setminus \bigcup_{i=1}^{l} N_K[v_i] = \emptyset$, which is a contradiction. Hence $(v_1, \ldots, v_{i-1}, z, v_i, \ldots, v_l)$ is a subsequence of T. Therefore there exists $x \in N_K[v_l] \setminus \bigcup_{i=1}^{l-1} N_K[v_i] \bigcup N_K[z]$. Since $v_l \in N_K[z]$ and $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \emptyset$, we conclude that $x \neq v_l$. In addition, $x \in V(H_l)$ and x, z are not adjacent vertices, and $x \notin \widehat{T}$. Now, by replacing v_l by x in T, we obtain a γ_{gr}-sequence T', such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. Hence $\widehat{T} \bigcap V(H_l) = \emptyset$. Again consider a vertex $x \in V(H_l)$ and put x instead of v_l in T. Then we obtain a γ_{gr}-sequence T' such that the size of intersection of $\widehat{T'}$ and $V(G)$ is less than the size of intersection of $\widehat{T}$ and $V(G)$. This is a contradiction and so we conclude that $|\hat{T} \bigcap V(G)| \leq \gamma_{gr}^{Z}(G)$. It is not difficult to see $|\widehat{T} \bigcap V(H_i)| \leq \gamma_{gr}(H_i)$ for $1 \leq i \leq n$ and thus $\gamma_{gr}(K) \leq \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^{Z}(G)$. □

The following corollary is an easy consequence of Theorem 2.1 and Proposition 2.2.

Corollary 2.1. *For $n, m \geq 3$*

$$\gamma_{gr}(C_n \circ C_m) = n(m-1) - 2, \gamma_{gr}(P_n \circ P_m) = mn - 1,$$

$$\gamma_{gr}(C_n \circ P_m) = nm - 2, \gamma_{gr}(P_n \circ C_m) = n(m-1) - 1.$$

we are now stat and proof our second main result.

Theorem 2.2. *Let G and $H_1, H_2, \ldots, H_n$ be graphs without isolated vertices. Then*

$$\gamma_{gr}^{t}(G \circ \wedge_{i=1}^{n} H_i) = \sum_{i=1}^{n} \gamma_{gr}^{t}(H_i) + \gamma_{gr}^{Z}(G).$$

Proof. Consider the sequence

$$T = S_1 \oplus v_1 \oplus S_2 \oplus v_2 \oplus \ldots \oplus S_k \oplus v_k \oplus S_{k+1} \oplus S_{k+2} \oplus \ldots \oplus S_n,$$

where $S = (v_1, \ldots, v_k)$ is a Z-Grundy sequence of G and S_i's are γ_{gr}^{t}-sequences of H_i's for $1 \leq i \leq n$. We show that T is a γ_{gr}^{t}-sequence for $K = G \circ \wedge_{i=1}^{n} H_i$. Let $x \in \widehat{T}$. Hence there exists either $1 \leq i \leq n$ such that $x \in \widehat{S}_i$ or $1 \leq j \leq k$ for which $x = v_j$. If $x = v_j$, then there exists $y \in N_G(v_j) \setminus \bigcup_{t=1}^{j-1} N_G[v_t]$. Hence $y \neq v_t$ for

$1 \le t \le j-1$ and therefore $y \in N_K(v_j) \setminus \bigcup_{t=1}^{j-1} N_K[v_t] \bigcup (\bigcup_{t=1}^{j} N_k[S_t])$. This implies that

$$N_K(v_j) \setminus \bigcup_{t=1}^{j-1} N_K[v_t] \bigcup (\bigcup_{t=1}^{j} N_K[S_t]) \neq \emptyset.$$

The same argument can be apply when $x \in \widehat{S}_i$. Since clearly $\widehat{T}$ is a total dominating set, we conclude that T is a total dominating sequence of G. Hence

$$\gamma_{gr}^t(K) \geq \sum_{i=1}^{n} \gamma_{gr}^t(H_i) + \gamma_{gr}^Z(G).$$

Now suppose that T is a γ_{gr}^t-sequence of K such that $|\widehat{T} \bigcap V(G)|$ is minimum among all γ_{gr}^t-sequences of G. Suppose that $\widehat{T} \bigcap V(G) = \{v_1, \ldots, v_t\}$ and $t > \gamma_{gr}^Z(G)$. Hence $(v_1, \ldots, v_t)$ is not a Z-sequence for G. Therefore, there exists $1 \le l \le t$ such that $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \emptyset$. If $\widehat{T} \bigcap V(H_l) = \emptyset$, then by replacing v_l by $x \in V(H_l)$, we can construct a γ_{gr}^t-sequence T' such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. Hence $\widehat{T} \bigcap V(H_l) \neq \emptyset$. If there exists $x \in \widehat{T} \bigcap V(H_l)$ such that x appears after v_l in the sequence T, then (v_l, x) is a subsequence of T and $N_K(x) \setminus N_K(v_l) \neq \emptyset$. Since $N_G(v_l) \setminus \bigcup_{i=1}^{l-1} N_G[v_i] = \emptyset$, we conclude that $N_K(x) \setminus N_K(v_l) = \{v_l\}$ and hence $\widehat{T} \bigcap V(H_l) = \{x\}$. Now choose $y \in N(x)$ and replace v_l by y in T. Again we obtain a γ_{gr}^t-sequence T' such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. Hence all elements of $\widehat{T} \bigcap V(H_l)$ appear before v_l in the sequence T. Hence there exists $y \in V(H_l)$ such that $y \in N_K(v_l) \setminus \bigcup_{x \in \widehat{T} \bigcap V(H_l)} N_K(x)$. Since $\deg_{H_l}(y) \geq 1$, there exists $z \in V(H_l)$ which is adjacent to y. Clearly $z \notin \widehat{T}$ and by changing v_l with z, we get a γ_{gr}^t-sequence T' such that $|\widehat{T'} \bigcap V(G)| < |\widehat{T} \bigcap V(G)|$, which is a contradiction. This argument implies that $|\widehat{T} \bigcap V(G)| \le \gamma_{gr}^Z(G)$. One can easily check that $|\widehat{T} \bigcap V(H_i)| \le \gamma_{gr}^t(H_i)$ for $1 \le i \le n$ and so we conclude that $\gamma_{gr}^t(K) \le \sum_{i=1}^{n} \gamma_{gr}^t(H_i) + \gamma_{gr}^Z(G)$.

□

Corollary 2.2. *Let G be a graph of order n and size m and H be a graph without isolated vertices. Then $\gamma_{gr}(G \diamond H) = 2m\gamma_{gr}(H) + \gamma_{gr}^Z(G)$ and $\gamma_{gr}^t(G \star H) = (2m + n)\gamma_{gr}^t(H) + \gamma_{gr}^Z(G)$.*

Proof. Note that $G \diamond H$ is the same as $G \circ \wedge_{i=1}^{n} H_i$, where H_i is the disjoint union of $\deg(v_i)$ copies of H. Hence by Theorem 2.1,

$$\gamma_{gr}(G \diamond H) = \gamma_{gr}(G \circ \wedge_{i=1}^{n} H_i) = \sum_{i=1}^{n} \gamma_{gr}(H_i) + \gamma_{gr}^Z(G) = 2m\gamma_{gr}(H) + \gamma_{gr}^Z(G).$$

The proof of the second part of the corollary is similar. □

Corollary 2.3. *Let G be a connected graph of order n. Then $\gamma_{gr}^t(G \circ K_1) = 2n$.*

Proof. Suppose that $V(G) = \{v_1, \ldots, v_n\}$ is the vertex set of G. It is not difficult to see that sequence $(u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n)$, where u_i is the vertex of K_1, which is adjacent to v_i, is a Grundy total domination sequence of $G \circ K_1$. $\square$

Corollary 2.4. *Let G be a nontrivial connected graph of order n. Then $\gamma_{gr}^t(G \circ H) = n\gamma_{gr}^t(H) + \gamma_{gr}^Z(G)$, for any nontrivial connected graph H.*

As a similar argument to proof of Theorem 2.1, we can find the Z-Grundy domination number of corona product of graphs.

Theorem 2.3. *Let G and $H_1, H_2, \ldots, H_n$ be $n+1$ graphs without isolated vertices. Then*

$$\gamma_{gr}^Z(G \circ \wedge_{i=1}^n H_i) = \sum_{i=1}^n \gamma_{gr}^Z(H_i) + \gamma_{gr}^Z(G).$$

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