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[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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BANACH FIXED POINT THEOREM ON ORTHOGONAL CONE METRIC SPACES

Zeinab Eivazi Damirchi Darsi Olia, Madjid Eshaghi Gordji and Davood Ebrahimi Bagha

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Abstract. In this paper, we introduce new concept of orthogonal cone metric spaces. We establish new versions of fixed point theorems in incomplete orthogonal cone metric spaces. As an application, we show the existence and uniqueness of solution of the periodic boundry value problem.

Keywords: Orthogonal set; Fixed point; Orthogonal cone metric space; Differential equation; Solution.

1. Introduction

The concept of standard metric spaces is a fundamental tool in topology, functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces.

In recent years, several generalizations of standard metric spaces have appeared. Huang and Zhang [8] have introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and proved many fixed point theorems of contractive type mappings in cone metric space. In 2010, W.S.Du [2] has shown that many results in fixed point theory on cone metric spaces are equivalent to ordinary metric spaces. Subsequently, many authors in [2, 7, 9] have generalized the results of Huang and Zhang [8].

Huang and Zhang [8] considered the concept of cone metric spaces as follows:

Definition 1.1. *[8] Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:*

1. *P is closed, nonempty, and $P \neq \{0\}$;*

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2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ *then* $ax + by \in P$*;*

3. $x \in P$ *and* $-x \in P$ *then* $x = 0$.

Given a cone $P \subset E$*, we define a partial ordering* $\leqslant$ *with respect to* P *by* $x \leqslant y$ *if and only if* $y - x \in P$*. We shall write* $x < y$ *to indicate that* $x \leqslant y$ *but* $x \neq y$*, while* $x \ll y$ *will stand for* $y - x \in intP$ *,* $intP$ *denotes the interior of* P *. The cone* P *is called normal if there is a number* $K > 0$ *such that for all* $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \parallel x \parallel \leq K \parallel y \parallel .$$

The least positive number satisfying above is called the normal constant of P.

The cone P *is called regular if every increasing sequence which is bounded from above is convergent. That is, if* $\{x_n\}$ *is sequence such that*

$$x_1 \leq x_2 \leq \quad \cdots \quad \leq x_n \leq \quad \cdots \quad \leq y,$$

for some $y \in E$*, then there exists* $x \in E$ *such that* $\parallel x_n - x \parallel \to 0$ *as* $n \longrightarrow \infty$*. Equivalently the cone* P *is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.*

Definition 1.2. *[8] Let* X *be a nonempty set. Suppose the mapping* $d : X \times X \to E$ *satisfies*

- *(d1)* $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*
- *(d2)* $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
- *(d3)* $d(x, y) \leq d(x, z) + d(y, z)$ *for all* $x, y, z \in X$.

Then d *is called a cone metric on* X*, and* (X, d) *is called a cone metric space.*

It is obvious that cone metric spaces generalize metric spaces.

Example 1.1. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metic space.

Definition 1.3. *[8] Let* (X, d) *be a cone metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$*. If for every* $c \in E$ *with* $0 \ll c$ *there is* N *such that for all* $n > N$, $d(x_n, x) \ll c$*, then* $\{x_n\}$ *is said to be convergent and* $\{x_n\}$ *converges to* x *and* x *is the limit of* $\{x_n\}$*. We denote this by*

$$\lim_{n \to \infty} x_n = x \quad or x_n \to x \quad (n \to \infty).$$

Definition 1.4. *[8] Let* (X, d) *be a cone metric space,* $\{x_n\}$ *be a sequence in* X*. If for any* $c \in E$ *with* $0 \ll c$*, there is* N *such that for all* $n, m > N$, $d(x_n, x_m) \ll c$*, then* $\{x_n\}$ *is called a Cauchy sequence in* X.

Definition 1.5. *[8] Let (X, d) be a cone metric space if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.*

Huang and Zhang [8] also proved the following fixed point theorem in cone metric spaces.

Theorem 1.1. *[8] Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$ where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X and for any $x \in X$, an iterative sequence $\{T^n x\}$ converges to the fixed point.

Eshaghi and et.al. [3] introduced the notion of orthogonal sets as follows (also see [11, 1, 4, 5, 6, 10]):

Definition 1.6. *[3] Let $X \neq \phi$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition*

$$\exists x_0;\ ((\forall y; y \perp x_0) \quad or \quad (\forall y; x_0 \perp y)),$$

it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

Definition 1.7. *Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called an orthogonal sequence (briefly O-sequence) if*

$$\Big((\forall n; x_n \perp x_{n+1}) \quad or \quad (\forall n; x_{n+1} \perp x_n)\Big).$$

for more information refer to [3].

Definition 1.8. *[3] Let $(X, d, \perp)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). The space X is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.*

It is easy to see that every complete metric space is O-complete and the converse is not true (see [3]).

Definition 1.9. *[3] Let $(X, d, \perp)$ be an orthogonal metric space and $0 < k < 1$.*

1. *A mapping $f : X \to X$ is said to be orthogonal contractive ($\perp$−contractive) mapping with Lipschitz constant k if*

$$d(fx, fy) \leq kd(x, y) \qquad if \qquad x \perp y.$$

2. *A mapping $f : X \to X$ is called an orthogonal preserving ($\perp$−preserving) mapping if $x \perp y$ then $f(x) \perp f(y)$.*

3. *A mapping $f : X \to X$ is an orthogonal continuous ($\perp-$continuous) mapping in $a \in X$ if for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X if $a_n \to a$ then $f(a_n) \to f(a)$. Also f is $\perp-$continuous on X if f is $\perp-$continuous in each $a \in X$.*

They also, proved the following theorem which can be considered as a real extension of Banach fixed point theorem [11, 1, 3, 4, 5, 6, 10].

Theorem 1.2. *[3] Let $(X, d, \perp)$ be an O-complete metric space (not necessarily complete metric space). Let $f : X \to X$ be $\perp-$continuous, $\perp-$contraction (with Lipschitz constant k) and $\perp-$preserving, then f has a unique fixed point x^* in X. Also, f is a Picard operator, that is,* $\lim f^n(x) = x^*$ *for all $x \in X$.*

Let us consider the following periodic boundry value problem

$$\begin{cases} u^{'}(t) = f(t, u(t)), \\ u(t_0) = u(T), \end{cases} \tag{1.1}$$

where $t \in I = [0, T]$, $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose that there exists $\beta > 0$, $\mu > 0$ with $\mu < \beta$ such that for $x, y \in \mathbb{R}$ we have

$$0 \le \Big|[f(t, y) + \beta y] - [f(t, x) + \beta x]\Big| \le \mu|y - x|. \tag{1.2}$$

Inspired and motivated by the above results, we introduce new concept of orthogonal cone metric space. In such space, we establish new versions of fixed point theorems. As an application, we show the existence and uniqueness of solution of the periodic boundry value problem 1.1.

2. Main Results

In this section, we shall introduce a new definitions to prove the main results. We begin with the following definition. In the following part, we shall suppose E is a Banach space, P is a cone in E with $intP \neq \phi$ and $\le$ is partial ordering with respect to P.

Definition 2.1. *Let $(X, \perp)$ be a nonempty orthogonal set. Suppose the mapping $d : X \times X \to E$ satisfies*

- *(d1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*
- *(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;*
- *(d3) $d(x, y) \le d(x, z) + d(y, z)$ for all $x, y, z \in X$.*

Then d is called a cone metric on $(X, \perp)$ and $(X, d, \perp)$ is called an orthogonal cone metric space.

We have the concept of orthogonal complete cone metric space as follows:

Definition 2.2. *Let $(X, d, \perp)$ be an orthogonal cone metric space, if every Cauchy O-sequence is convergent in X, then X is called an orthogonal complete cone metric space.*

It is easy to see that every complete cone metric space is O-complete and the converse is not true. In the next example, X is O-complete cone metric space and it is not complete.

Example 2.1. Let $E = \mathbb{R}$, $P = [0, \infty)$ and $X = [0, 1)$. Suppose $x \perp y$ if $x \leq y$. $(X, \perp)$ is an O-set. Clearly, X with metric $d : X \times X \to E$ such that $d(x, y) = |x - y|$ is not complete cone metric space but it is O-complete cone metric space. Because if $\{x_k\}$ is an arbitrary Cauchy O-sequence in X, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $\{x_{k_n}\} \leq \frac{1}{2}$ for all n. It follows that $\{x_{k_n}\}$ converges to a $x \in [0, \frac{1}{2}] \subset X$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent.

In the following example, we shall prove a theorem that can be considered as the main result of our paper.

Theorem 2.1. *Let $(X, d, \perp)$ be an orthogonal complete cone metric space (not necessarily complete cone metric space), P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ is $\perp$-preserving, $\perp$-continuous and $\perp$-contraction Lipschitz constant $k \in [0, 1)$. Then T has a unique fixed point in X. In addition T is a picard operator.*

Proof. By definition of orthogonality, there exists $x_0 \in X$ such that

$$(\forall x \in X, x \perp x_0) \qquad \textit{or} \quad (\forall x \in X, x_0 \perp x).$$

It follows that $x_0 \perp T(x_0)$ or $T(x_0) \perp x_0$. Let

$$x_1 := T(x_0), x_2 := T(x_1) = T^2(x_0), \cdots, x_{n+1} := Tx_n = T^n(x_0).$$

We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq kd(x_n, x_{n-1}) \\ &\leq k^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq k^n d(x_1, x_0). \end{aligned}$$

So for $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m-1}, x_m), \\ &\leq (k^{n-1} + k^{n-2} + \cdots + k^m) d(x_1, x_0), \\ &\leq \frac{k^m}{1-k} d(x_1, x_0). \end{aligned}$$

We get $\|d(x_n, x_m)\| \leq \frac{k^m}{1-k} K \|d(x_1, x_0)\|$. This implies that $d(x_n, x_m) \to 0$ $(n, m \to \infty)$. Hence the O-sequence $\{x_n\}$ is Cauchy. By completeness of X, there exists x^* in X such that $x_n \to x^*$ $(n \to \infty)$. On the other hand, T is $\perp$-continuous and hence $Tx_n \to Tx^*$ as n tends to infinity and $T(x^*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$. Therefore x^* is a fixed point of T.

To prove the uniqueness of the fixed point, let $y^* \in X$ be a fixed point of T. Then we have $T^n(y^*) = y^*$ for all $n \in \mathbb{N}$. By our choice of x_0 in the first part of the proof, we have

$$x_0 \perp y^* \quad \text{or} \quad y^* \perp x_0.$$

Since T is $\perp$-preserving, we have

$$T^n(x_0) \perp T^n(y^*) \quad \text{or} \quad T^n(y^*) \perp T^n(x_0),$$

for all $n \in \mathbb{N}$. On the other hand, T is $\perp$-contraction, then we have for all $n \in \mathbb{N}$,

$$\begin{aligned} d(x^*, y^*) &= d(T^n(x^*), T^n(y^*)) \\ &\leq d(T^n(x^*), T^n(x_0)) + d(T^n(x_0), T^n(y^*)) \\ &\leq k^n [d(x^*, x_0) + d(x_0, y^*)]. \end{aligned}$$

Also we have

$$\|d(x^*, y^*)\| \leq K(k^n [\|d(x^*, x_0)\| + \|d(x_0, y^*)\|]).$$

As n goes to infinity, we get $x^* = y^*$.

Finally, we show that T is a Picard operator. Let $x \in X$ be arbitrary. Similarly, then

$$[x_0 \perp x^* \ \text{and} \ x_0 \perp x] \ \text{or} \ [x^* \perp x_0 \ \text{and} \ x \perp x_0].$$

Now, since T is $\perp$-preserving, then

$$[T^n(x_0) \perp T^n(x^*) \ \text{and} \ T^n(x_0) \perp T(x)] \ \text{or} \ [T^n(x^*) \perp T^n(x_0) \ \text{and} \ T(x) \perp T^n(x_0)],$$

for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$, we get

$$d(T^n(x), T^n(x_0)) \leq k d(T^{n-1}(x), T^{n-1}(x_0)) \leq \cdots \leq k^n d(x, x_0).$$

Letting $n \to \infty$ we have $\lim_{n\to\infty} T^n(x) = x^*$. This completes the proof. □

Here, we obtain another fixed point theorem by replacing $\perp$-contractive condition by another slightly modified condition.

Theorem 2.2. *Let* $(X, d, \perp)$ *be an orthogonal complete cone metric space,* P *be a normal cone with normal constant* K*. Let* $T : X \to X$ *be* $\perp$*-preserving,* $\perp$*-continuous mapping satisfying the following* $\perp$*-contractive condition*

$$d(Tx, Ty) \leq a(d(x, y)) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)],$$

for $x, y \in X$ *with* $x \perp y$ *and the constants* $a, b, c \in [0, 1)$ *and* $a + b + c < 1$*. Then* T *has a unique fixed point in* X*.*

Proof. By definition of orthogonality, there exists $x_0 \in X$ such that

$$(\forall x \in X, x \perp x_0) \quad or \quad (\forall x \in X, x_0 \perp x).$$

It follows that $x_0 \perp T(x_0)$ or $T(x_0) \perp x_0$. Let

$$x_1 := T(x_0), x_2 := T(x_1) = T^2(x_0), \cdots, x_{n+1} := Tx_n = T^n(x_0).$$

We have

$$\begin{aligned} & d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \\ & \leq a(d(x_n, x_{n-1})) + b[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] \\ & \leq a(d(x_n, x_{n-1})) + b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + c[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\ & \leq a(d(x_n, x_{n-1})) + b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Therefore,

$$d(x_{n+1}, x_n)(1 - b - c) = d(x_n, x_{n-1})(a + b + c),$$

and we get

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \frac{a + b + c}{1 - b - c}.$$

Substituting $k = \frac{a+b+c}{1-b-c}$ and as $0 \leq k < 1$ we have

$$d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0).$$

For any $m \geq 1$, $p \geq 1$, it follows that

$$\begin{aligned} d(x_{m+p}, x_m) &\leq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m) \\ &\leq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m) \\ &\leq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_{m+p-3}) \\ &+ \cdots + d(x_{m+2}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\leq k^{m+p-1} d(x_1, x_0) + k^{m+p-2} d(x_1, x_0) + k^{m+p-3} d(x_1, x_0) \\ &+ \cdots + k^m d(x_1, x_0) \\ &\leq (k^{m+p-1} + k^{m+p-2} + k^{m+p-3} + \cdots + k^m) d(x_1, x_0) \\ &\leq \frac{k^m}{1 - k} d(x_1, x_0). \end{aligned}$$

So we have

$$\|d(x_{m+p}, x_m)\| \leq K\frac{k^m}{1-k}\|d(x_1, x_0)\|.$$

Letting $m \longrightarrow \infty$ we conclude that $\{x_n\}$ is a Cauchy O-sequence. Since (X, d) is a complete orthogonal cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \longrightarrow \infty$.

Next, we claim that x^* is a fixed point of T.

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, Tx_n) + d(Tx_n, x^*) \\ &\leq d(Tx^*, Tx_n) + d(x_{n+1}, x^*), \end{aligned}$$

and we have

$$\begin{aligned} d(Tx^*, x^*) &\leq a(d(x^*, x_n)) + b[d(x^*, Tx^*) + d(x_n, Tx_n)] \\ &+ c[d(x^*, Tx_n) + d(x_n, Tx^*)] + d(x_{n+1}, x^*) \\ &\leq a((d(x^*, x_n)) + b[d(x^*, Tx^*) + d(x_n, x_{n+1})] \\ &+ c[d(x^*, x_{n+1}) + d(x_n, Tx^*)] + d(x_{n+1}, x^*) \\ &\leq a((d(x^*, x_n)) + b[d(x^*, Tx^*) + d(x_n, x^*) + d(x^*, x_{n+1})] \\ &+ c[d(x^*, x_{n+1}) + d(x_n, x^*) + d(x^*, Tx^*)] + d(x_{n+1}, x^*). \end{aligned}$$

So

$$d(Tx^*, x^*)(1-b-c) \leq d(x^*, x_n)(a+b+c) + d(x^*, x_{n+1})(1+b+c),$$

and

$$d(Tx^*, x^*) \leq \frac{d(x^*, x_n)(a+b+c) + d(x^*, x_{n+1})(1+b+c)}{(1-b-c)}.$$

Therefore

$$\|d(Tx^*, x^*)\| \leq K(\frac{(a+b+c)}{(1-b-c)}\|d(x^*, x_n)\| + \frac{(1+b+c)}{(1-b-c)}\|d(x^*, x_{n+1})\|).$$

Letting $n \longrightarrow \infty$, we have $Tx^* = x^*$.

Finally, we need to prove that the fixed point is unique.

If there is another fixed point y^*, then

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq a(d(x^*, y^*)) + b[d(x^*, Tx^*) + d(y^*, Ty^*)] + c[d(x^*, Ty^*) + d(y^*, Tx^*)] \\ &\leq a(d(x^*, y^*)) + b[d(x^*, Tx^*) + d(y^*, Ty^*)] \\ &+ c[d(x^*, Tx^*) + d(Tx^*, Ty^*) + d(y^*, Ty^*) + d(Ty^*, Tx^*)] \\ &= (a+2c)d(x^*, y^*). \end{aligned}$$

$$(1-a-2c)d(x^*, y^*) \leq 0,$$

this implies that

$$\|d(x^*, y^*)\| = 0.$$

Hence $x^* = y^*$. Therefore the proof is completed. □

3. Application in differential equations

In this section, we apply results in the previous section to show the existence and uniqueness of solution of the following periodic boundary value problem 1.1 where $t \in I = [0, T]$, $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $X = \{u \in C(I, \mathbb{R}); u(t) > 1 \quad \{for \ \ almost \ \ every\} \quad t \in I\}$. Consider the Banach space $E = \mathbb{R}$ and $P = [0, \infty)$. Define partial ordering $\leq$ with respect to P by $a \leq b$ if and only if $b - a \in P$.

Suppose the mapping $d : X \times X \to E$ by

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|,$$

for $x, y \in X$.

Suppose that there exists $\beta > 0$, $\mu > 0$ with $\mu < \beta$ such that for $x, y \in \mathbb{R}$ we have 1.2.

Theorem 3.1. *Under above conditions, for all $T > 0$ the differential equation 1.1 has a unique solution.*

Proof. The problem can be written in integral equation as

$$x(t) = \int_0^T G(t, s)[f(s, x(s)) + \beta x(s)]ds,$$

where

$$G(t, s) = \begin{cases} \frac{e^{\beta(T+s-t)}}{e^{\beta T}-1}, & 0 \leq s \leq t \leq T \\ \frac{e^{\beta(s-t)}}{e^{\beta T}-1}, & 0 \leq t \leq s \leq T \end{cases} \tag{3.1}$$

Define the following orthogonality relation $\perp$ in X:

$$x \perp y \quad \text{if} \quad x(t)y(t) \geq y(t),$$

for almost every $t \in I$. It's easy to see that $(X, d, \perp)$ is a cone metric space. Since every x is a continuous function over a closed and bounded subset of the Euclidean space, this supremum is actually attained in $(X, d, \perp)$. Hence $(X, d, \perp)$ is complete.

Now, we define $A : (X, d, \perp) \to (X, d, \perp)$ as follows:

$$(Ax)(t) = \int_0^T G(t, s)[f(s, x(s)) + \beta x(s)]ds,$$

for all $t \in I$.

Note that the fixed points of A are the solutions of 1.1.

First, we claim that for every $x \in X$, $Ax \in X$. To see this, for every $t \in I$ and $x \in X$, we have

$$\begin{aligned} Ax(t) &= \int_0^T G(t,s)[f(s,x(s)) + \beta x(s)]ds \\ &= \int_0^T G(t,s)f(s,x(s))ds + \int_0^T G(t,s)\beta x(s)ds \\ &> \int_0^T G(t,s)f(s,x(s))ds + \beta \int_0^T G(t,s)ds \\ &= \int_0^T G(t,s)f(s,x(s))ds + \beta \frac{1}{e^{\beta T}-1}(\frac{1}{\beta}e^{\beta(T+s-t)}]_0^t + \frac{1}{\beta}e^{\beta(s-t)}]_t^T) \\ &= \int_0^T G(t,s)f(s,x(s))ds + \beta\frac{1}{\beta} \\ &= \int_0^T G(t,s)f(s,x(s))ds + 1. \end{aligned}$$

one can conclude that $Ax(t) > 1$ and we have $Ax \in X$.

Now, we check that the hypotheses in Theorem 2.1 is satisfied. To this end, we prove the following statements:

1. A is $\perp$-preserving,
2. A is $\perp$-contraction,
3. A is $\perp$-continuous

1. We recall that A is $\perp$-preserving if for every $x, y \in X$, $x \perp y$ we have $Ax \perp Ay$. We have shown above that $Ax(t) > 1$ for all $t \in I$, which implies that $Ax(t)Ay(t) \geq Ay(t)$ for all $t \in I$. So $Ax \perp Ay$ if $x \perp y$.

2. Let $x, y \in X$ and $x \perp y$, we have

$$\begin{aligned} |Ax(t) - Ay(t)| &= \left|\int_0^T G(t,s)[f(s,x(s)) + \beta x(s) - f(s,y(s)) - \beta y(s)]ds\right| \\ &\leq \int_0^T |G(t,s)||\mu(x(t) - y(t))|ds \\ &\leq \mu|x(t) - y(t)| \int_0^T G(t,s)ds \\ &= \frac{\mu}{\beta}|x(t) - y(t)|. \end{aligned}$$

So,

$$d(Ax, Ay) = \sup_{t \in I} |Ax(t) - Ay(t)| \leq \frac{\mu}{\beta} \sup_{t \in I} |x(t) - y(t)| = \frac{\mu}{\beta} d(x, y). \tag{3.2}$$

The inequality 3.2 shows that A is $\perp$-contraction with Lipschitz constant $\lambda = \frac{\mu}{\beta} < 1$.

3. Let $\{x_n\}$ be an O-sequence in X such that $\{x_n\}$ converges to some $x \in X$. Since A is $\perp$-preserving, $\{Ax_n\}$ is an O-sequence. For each $n \in \mathbb{N}$, by (2), we have

$$|Ax_n - Ax| \leq \lambda |x_n - x|.$$

As n tends to infinity, it follows that A is $\perp$-continuous.

The mapping A defined above satisfies the hypotheses of the Theorem 2.1. Thus, the existence and uniqueness of its fixed point $x^* \in X$ has been guaranteed by Theorem 2.1. As we noted above, x^* is a unique solution to differential equation 1.1. □

REFERENCES

1. H. Baghani, M. Eshaghi Gordji and M. Ramezani, *Orthogonal sets: their relation to the axiom of choice and a generalized fixed point theorem*, J. Fixed Point Theory Appl. (2016) 18: 465.
2. W.S. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. 72 (2010) 2259–2261.
3. M. Eshaghi Gordji, M. Ramezani, M. De La Sen and Y. J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory. 18 (2017) no. 2, 569–578.
4. M. Eshaghi and H. Habibi, *Fixed point theory in generalized orthogonal metric space*, J. Linear Topol. Algebr. 06 (2017) no. 03, 251–260.
5. M. Eshaghi, H. Habibi and M. B. Sahabi, *Orthogonal sets; orthogonal contractions*, Asian-European J. Math. Accepted.
6. M. Eshaghi and H. Habibi, *Existence and uniqueness of solutions to a first-order differential equation via fixed point theorem in orthogonal metric space*, FACTA UNIVERSITATIS (NIS) Ser. Math. Inform. Accepted.
7. S. Jankovic, Z. Kadelburg and S. Radenvic, *On cone metric spaces: A survey*, Nonlinear Analysis. 74 (2011) 2591–2601.
8. L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. 332 (2007) 1468 –1476.
9. J. R. Morales and E. Rojas, *Cone metric spaces and fixed point theorems of T-contractive mappings*, Revista Notas de Matematica. vol.4(2), no. 269 (2008) 66–78.
10. M. Ramezani and H. Baghani, *The Mier–Keeler fixed point theorem in incomplete modular spaces with application*, J. Fixed Point Theory Appl. (2017), Volume 19, Issue 4, 2369–2382.
11. A. Bahraini, G. Askari, M. Eshaghi Gordji and R. Gholami, *Stability and hyperstability of orthogonally $*$-m-homomorphisms in orthogonally Lie C^*-algebras: a fixed point approach*, J. Fixed Point Theory Appl. (2018), 20:89.

Zeinab Eivazi Damirchi Darsi Olia
Department of Mathematics
Islamic Azad University
South Tehran Branch, Tehran, Iran
St_z_eivazi@azad.ac.ir

Madjid Eshaghi Gordji
Department of Mathematics
Semnan University
P.O. Box 35195-363
Semnan, Iran
meshaghi@semnan.ac.ir

Davood Ebrahimi Bagha
Department of Mathematics
Islamic Azad University
Central Tehran Branch, Tehran, Iran
E_bagha@yahoo.com

FACTA UNIVERSITATIS (NIŠ)
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ARENS REGULARITY OF PROJECTIVE TENSOR PRODUCT

Mostfa Shams Kojanaghi and Kazem Haghnejad Azar

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Abstract. In this paper, we study some of approximate identity properties, and its application in the Arens regularity of tensor products of Banach algebras with some results in group algebras. W e consider under which sufficient and necessary conditions the Banach algebra $A\widehat{\otimes}B$ is Arens regular.

Keywords: Arens regularity; tensor product; Banach algebras; group algebras.

1. Introduction

Suppose that A and B are Banach algebras. Since 1988 the Arens regularity of $A\widehat{\otimes}B$ has received a great deal of attention by many researchers. Among them, Ülger in [19, 21] showed that $A\widehat{\otimes}B$ is not Arens regular, in general, even when A and B are Arens regular. He introduced a new concept of biregular mapping and showed that a bounded bilinear mapping $m : A \times B \to \mathbb{C}$ is biregular if and only if $A\widehat{\otimes}B$ is Arens regular, where $\mathbb{C}$ is the space of complex numbers. Let X, Y and Z be normed spaces and let $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} that he called m is Arens regular whenever $m^{***} = m^{t***t}$, for more information see [9, 10, 14]. Let A be a Banach algebra, regarding A as a Banach A-bimodule, the operation $\pi : A \times A \longrightarrow A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. The first (left) and second (right) Arens products of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and $a''ob''$, respectively. Let B be a Banach A-bimodule, and let

$$\pi_\ell : \ A \times B \longrightarrow B \quad \text{and} \quad \pi_r : \ B \times A \longrightarrow B,$$

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be the right and left module actions of A on B. By above notation, the transpose of π_r denoted by $\pi_r^t : A \times B \to B$. Then

$$\pi_\ell^* : B^* \times \ A \longrightarrow B^* \quad \text{and} \quad \pi_r^{t*t} : A \times B^* \longrightarrow B^*.$$

Thus B^* is a left Banach A-module and a right Banach A-module with respect to the module actions π_r^{t*t} and π_ℓ^*, respectively. The the second dual B^{**} is a Banach A^{**}-bimodule with the following module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{***} : \ B^{**} \times A^{**} \longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the first Arens product. Similarly, B^{**} is a Banach A^{**}-bimodule with the module actions

$$\pi_\ell^{t****t} : \ A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{t****t} : \ B^{**} \times A^{**} \longrightarrow B^{**},$$

where A^{**} is considered as a Banach algebra with respect to the second Arens product.

Let B be a left Banach A-module and e be a left unit element of A. Then e is a left unit (resp. weakly left unit) for B, if $\pi_\ell(e,b) = b$ (resp. $\langle b', \pi_\ell(e,b)\rangle = \langle b', b\rangle$ for all $b' \in B^*$) where $b \in B$. The definition of right unit (resp. weakly right unit) is similar. A Banach A-bimodule B is called unital, if B has the same left and right unit. In this way, B is called a unitary Banach A-bimodule.

Suppose that A is a Banach algebra and B is a Banach A-bimodule. Since B^{**} is a Banach A^{**}-bimodule, where A^{**} is equipped with the first Arens product, we define the topological center of the right module action of A^{**} on B^{**} as follows:

$$\begin{aligned} Z^\ell_{A^{**}}(B^{**}) = Z(\pi_r) \quad &= \quad \{b'' \in B^{**} : \text{ the map } \ a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**} \\ &\qquad \text{is} \quad \text{weak}^*\text{-weak}^* \text{ continuous}\}. \end{aligned}$$

In this way, we write $Z^\ell_{B^{**}}(A^{**}) = Z(\pi_\ell)$, $Z^r_{A^{**}}(B^{**}) = Z(\pi_\ell^t)$ and $Z^r_{B^{**}}(A^{**}) = Z(\pi_r^t)$, where $\pi_\ell : \ A \times B \to B \quad and \quad \pi_r : \ B \times A \to B$ are the left and right module actions of A on B, for more information related to the Arens regularity of module actions on Banach algebras, see [2, 4, 9, 10]. If we set $B = A$, we write $Z^\ell_{A^{**}}(A^{**}) = Z_1(A^{**}) = Z_1^\ell(A^{**})$ and $Z^r_{A^{**}}(A^{**}) = Z_2(A^{**}) = Z_2^r(A^{**})$, for more information see [12]. Let A be a Banach algebra, A^* and A^{**} be the first and second dual of A, respectively. For $a \in A$ and $a' \in A^*$, by $a'a$ and aa', we mean the functionals in A^* defined by $\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$, respectively. A Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle$ - $\langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$.

2. Main Results

Consider the tensor product, $X \otimes Y$, of the vector space X and Y which can be constructed as a space of linear functional on $B(X \times Y)$. By $X\widehat{\otimes}Y$ we shall denote the projective tensor products of X and Y, where $X\widehat{\otimes}Y$ is the completion of $X \otimes Y$ for the norm

$$\|u\| = \inf \sum_{i=1}^{n} \|x_i\|\|y_i\|,$$

where the infimum is taken over all the representations of u as a finite sum of the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ [5].

The natural multiplication of $A\widehat{\otimes}B$ is the linear extension of the following multiplication on decomposable tensors $(a \otimes b)(\tilde{a} \otimes \tilde{b}) = a\tilde{a} \otimes b\tilde{b}$. For more details, see [16].

A functional a' in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \to a'a$ from A into A^* is weakly compact. Pym in [15] showed that this definition is equivalent with the following condition:

$$\lim_i \lim_j \langle a', a_i b_j\rangle = \lim_j \lim_i \langle a', a_i b_j\rangle,$$

whenever both iterated limits exist, for any two net $(a_i)_i$ and $(b_j)_j$ in $\{a \in A : \|a\| \leq 1\}$. The collection of all weakly almost periodic functionals on A is denoted by $wap(A)$. Also, $a' \in wap(A)$ if and only if $\langle a''b'', a'\rangle = \langle a''ob'', a'\rangle$ for every $a'', b'' \in A^{**}$. Thus, it is clear that A is Arens regular if and only if $wap(A) = A^*$ [9, Theorem 2.6.17]. In the sequel, to show that the projective tensor products $A\widehat{\otimes}B$ is Arens regular, it is sufficient that we show that $wap(A\widehat{\otimes}B) = (A\widehat{\otimes}B)^*$. In all of this section, we regard $A^*\widehat{\otimes}B^*$ as a subset of $(A\widehat{\otimes}B)^*$ and by A_1 and B_1 we mean all elements of $a \in A$ and $b \in B$ such that $\|a\| \leq 1$ and $\|b\| \leq 1$.

Theorem 2.1. *Suppose that A and B are Banach algebras and for every sequence $(x_i)_i$, $(y_j)_j \subseteq A_1$, $(z_i)_i$, $(w_j)_j \subseteq B_1$ and $f \in B(A \times B)$, we have*

$$\lim_j \lim_i f(x_i z_i, y_j w_j) = \lim_i \lim_j f(x_i z_i, y_j w_j).$$

Then $A\widehat{\otimes}B$ is Arens regular.

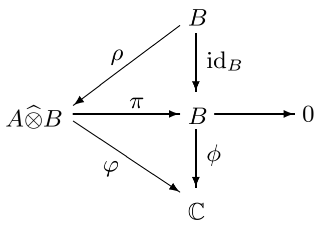
Proof. Assume that $f \in B(A \times B)$. Since $B(A \times B) = (A\widehat{\otimes}B)^*$, it is enough to show that $f \in wap(A\widehat{\otimes}B)$. Let $(x_i)_i$, $(y_j)_j \subseteq A_1$ and $(z_i)_i$, $(w_j)_j \subseteq B_1$, then we have the following equality

$$\begin{aligned}
\lim_j \lim_i \langle f, (x_i \otimes y_j)(z_i \otimes w_j)\rangle &= \lim_j \lim_i \langle f, x_i z_i \otimes y_j w_j\rangle \\
&= \lim_j \lim_i f(x_i z_i, y_j w_j) \\
&= \lim_i \lim_j f(x_i z_i, y_j w_j) \\
&= \lim_i \lim_j \langle f, (x_i \otimes y_j)(z_i \otimes w_j)\rangle,
\end{aligned}$$

for every $f \in (A\widehat{\otimes}B)^*$. This means that $f \in wap(A\widehat{\otimes}B)$, and proof is complete. $\square$

Definition 2.1. Let A be a Banach algebra and let B be a Banach A-bimodule and let $\pi : A\widehat{\otimes}B \longrightarrow B$ such that $\pi(a \otimes b) = ab$ for every $a \in A$, $b \in B$. We say that B is non-trivial on A, if π is surjective and has a bounded right inverse.

Remark 2.1. In the above definition, if A is unital, then π will be surjective. Now, suppose π has a continuous right inverse ρ, e_A and e_B are units of A and B, respectively. Let $\varphi \in (A\widehat{\otimes}B)^*$, then $\varphi \circ \rho$ belongs to B^*. Hence, there is a $\phi \in B^*$ such that $\varphi \circ \rho = \phi$. In other word, in the following diagram, we have $\varphi \circ \rho = \phi \circ \mathrm{id}_B$.



As well as, $\phi \circ \pi$ is in $(A\widehat{\otimes}B)^*$. Thus, there is a $\psi \in (A\widehat{\otimes}B)^*$ such that $\phi \circ \pi = \psi$. Then $\phi = \psi \circ \rho$. For given $a \otimes b \in A\widehat{\otimes}B$ we have

$$\begin{aligned} \phi \circ \pi(a \otimes b) &= \phi(ab) = \varphi \circ \rho(ab) = \varphi \circ \rho(e_A a e_B b) \\ &= \varphi \circ \rho((e_A a)(e_B b)) = \psi \circ \rho((e_A a)(e_B b)) \\ &= \psi(a \otimes b). \end{aligned} \tag{2.1}$$

Then, by (2.1), for every $\varphi \in (A\widehat{\otimes}B)^*$ there is a $\psi \in (A\widehat{\otimes}B)^*$ such that $\varphi \circ \rho = \psi \circ \rho$ and $\psi(a \otimes b) = \psi \circ \rho(ab)$, for every $a \in A$ and $b \in B$. Since A is unital, every element c of B can be written as $c = ab$ where $a \in A$ and $b \in B$. We can define $\rho : B \longrightarrow A\widehat{\otimes}B$ by $\rho(b) = e_A \otimes b$ and $\rho(ab) = \rho((e_A a)b) = a \otimes b$, for every $a \in A$ and $b \in B$. By this definition ρ is injective and it is a unique way to define of ρ. By this definition the above diagram commutes and we have $\phi \circ \pi(a \otimes b) = \varphi(a \otimes b)$, for every $a \in A$ and $b \in B$.

A wide class of Banach algebras which satisfy in the Definition 2.1, are projective and biprojective Banach algebras. A Banach algebra A-bimodule B is called projective if $\pi : A^\sharp\widehat{\otimes}B \longrightarrow B$ has bounded right inverse in ${}_AB(B, A^\sharp\widehat{\otimes}B)$ and the Banach algebra A is called biprojective if $\pi : A\widehat{\otimes}A \longrightarrow A$ has bounded right inverse in ${}_AB(A, A\widehat{\otimes}A)$ (for more details see [18]).

Theorem 2.2. *Let A and B be Banach algebras and B is unital. Suppose B is a Banach A-bimodule. Then*

1. *if $A\widehat{\otimes}B$ is Arens regular, then A is Arens regular.*
2. *if B is non-trivial on A and B be a unitary Banach A-bimodule. Then A and B are Arens regular if and only if $A\widehat{\otimes}B$ is Arens regular.*

Proof. 1. Assume that $A\widehat{\otimes}B$ is Arens regular and $u \in B$ is the unit element of B. We show that $wap(A) = A^*$. Get $(a_i)_i \subseteq A$, $(c_j)_j \subseteq A$ and $a' \in A^*$. Define $\phi = a' \otimes b'$ where $b' \in B^*$ and $b'(u) = 1$. Since $A^* \otimes B^* \subseteq (A\widehat{\otimes}B)^*$ and $A\widehat{\otimes}B$ is Arens regular, we have $a' \otimes b' \in wap(A\widehat{\otimes}B)$. Hence it follows that

$$\begin{aligned}\lim_i \lim_j \langle a', a_i c_j\rangle &= \lim_i \lim_j \langle a' \otimes b', a_i c_j \otimes u\rangle \\ &= \lim_i \lim_j \langle a' \otimes b', (a_i \otimes u)(c_j \otimes u)\rangle \\ &= \lim_j \lim_i \langle a' \otimes b', (a_i \otimes u)(c_j \otimes u)\rangle \\ &= \lim_j \lim_i \langle a', a_i c_j\rangle.\end{aligned}$$

This means that $a' \in wap(A)$, and so A is Arens regular.

2. Let u be a unit element of B and let B be Arens regular. Then $wap(B) = B^*$. Suppose that $(a_i)_i \subseteq A_1$ and $(b_j)_j \subseteq B_1$ whenever both iterated limits exist. Then $(a_i u)_i \subseteq B_1$, and so for every $b' \in B^*$, we have the following equality

$$\lim_i \lim_j \langle b', (a_i u)b_j\rangle = \lim_j \lim_i \langle b', (a_i u)b_j\rangle.$$

Now; let $\varphi \in (A\widehat{\otimes}B)^*$. Then by Remark 2.1, $\pi : A\widehat{\otimes}B \longrightarrow B$ has a continuous right inverse ρ such that $\varphi \circ \rho$ belongs to B^* and there is a $\phi \in B^*$ such that $\varphi \circ \rho = \phi$, and $\phi \circ \pi(a \otimes b) = \varphi(a \otimes b)$, for every $a \otimes b \in A\widehat{\otimes}B$. Now we have

$$\begin{aligned}\lim_i \lim_j \langle \varphi, a_i \otimes b_j\rangle &= \lim_i \lim_j \langle \phi \circ \pi, a_i \otimes b_j\rangle = \lim_i \lim_j \langle \phi, \pi(a_i \otimes b_j)\rangle \\ &= \lim_i \lim_j \langle \phi, a_i b_j\rangle = \lim_i \lim_j \langle \phi, a_i(u b_j)\rangle \\ &= \lim_j \lim_i \langle \phi, (a_i u) b_j\rangle = \lim_j \lim_i \langle \phi, \pi(a_i \otimes b_j)\rangle \\ &= \lim_j \lim_i \langle \varphi, a_i \otimes b_j\rangle.\end{aligned}$$

It follows that $\varphi \in wap(A\widehat{\otimes}B)$, and so $A\widehat{\otimes}B$ is Arens regular. The converse by using the part (1) holds.

□

Corollary 2.1. *Suppose that A and B are unital Banach algebras and B is a unitary Banach A-bimodule. Assume that B is non-trivial on A. If A and B are Arens regular, then every bilinear form $m : A \times B \to \mathbb{C}$ is weakly compact.*

Proof. Apply Theorem 2.2 and Theorem 3.4 of [19]. □

Let A and B be Banach algebras. A bilinear form $m : A \times B \to \mathbb{C}$ is said to be biregular, if for any two pairs of sequence $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j)$$

provided that these limits exist. There are some examples of biregular non regular bilinear form for more information see [19].

Corollary 2.2. *Suppose that A and B are Banach algebras. Then we have the following assertions:*

1. *By the conditions of Theorem 2.1, every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.*
2. *By the conditions of Theorem 2.2 (2), every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.*

In the following, we give a simple proof of Theorem 3.4 of [19].

Theorem 2.3. *[19, Theorem 3.4] Let A and B be Banach algebras and $u : A \to B^*$ be a continuous linear operator. Then the bilinear form $m : A \times B \to \mathbb{C}$ defined by $m(a, b) = \langle u(a), b\rangle$ is biregular.*

Proof. Let $(a_i)_i$, $(\tilde{a}_j)_j$ in A_1 and $(b_i)_i$, $(\tilde{b}_j)_j$ in B_1 such that the following iterated limits exist:

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) \quad \text{and} \quad \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

There are $(a_\alpha)_\alpha$, $(\tilde{a}_\beta)_\beta$ in A and $(b_\alpha)_\alpha$, $(\tilde{b}_\beta)_\beta$ in B such that $a_\alpha \overset{w^*}{\to} a''$ and $\tilde{a}_\beta \overset{w^*}{\to} \tilde{a}''$ in A^{**} and we have $b_\alpha \overset{w^*}{\to} b''$ and $\tilde{b}_\beta \overset{w^*}{\to} \tilde{b}''$ in B^{**}. Since A and B are Arens regular, we have

$$\lim_\alpha \lim_\beta a_\alpha\tilde{a}_\beta = \lim_\beta \lim_\alpha a_\alpha\tilde{a}_\beta = a''\tilde{a}'',$$

and

$$\lim_\alpha \lim_\beta b_\alpha\tilde{b}_\beta = \lim_\beta \lim_\alpha b_\alpha\tilde{b}_\beta = b''\tilde{b}''.$$

Then, since u is continuous, we have

$$\begin{aligned} \lim_\alpha \lim_\beta m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) &= \lim_\alpha \lim_\beta \langle u(a_\alpha\tilde{a}_\beta), b_\alpha\tilde{b}_\beta\rangle \\ &= \langle u''(a''\tilde{a}''), b''\tilde{b}''\rangle. \end{aligned}$$

Similarly, we have

$$\lim_\beta \lim_\alpha m(a_\alpha\tilde{a}_\beta, b_\alpha\tilde{b}_\beta) = \langle u''(a''\tilde{a}''), b''\tilde{b}''\rangle.$$

Consequently, we have

$$\lim_i \lim_j m(a_i\tilde{a}_j, b_i\tilde{b}_j) = \lim_j \lim_i m(a_i\tilde{a}_j, b_i\tilde{b}_j).$$

It follows that m is biregular. □

Example 2.1. [19] Let A be a Banach algebra and $1 < p < \infty$. Then

1. $\ell^p \widehat{\otimes} A$ is Arens regular if and only if A is Arens regular.
2. Let G be a locally compact group. Then, $L^p(G)\widehat{\otimes} A$ is Arens regular if and only if A is Arens regular.

Proof. For prove, we apply Theorem 3.4 of [19] and Theorem 2.7. □

We finish this section with the following problems:

Problem 2.1. Let G be a locally compact group. What can say for the following sets?

$Z^{\ell}_{L^1(G)^{**}}((L^1(G)\widehat{\otimes} L^1(G))^{**}) =?$ $\qquad$ $Z^{\ell}_{L^1(G)^{**}}(L^1(G)^{**}\widehat{\otimes} L^1(G)^{**}) =?$

REFERENCES

1. R. E. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc., **2**(1951), 839-848.
2. A. Bagheri Vakilabad, K. Haghnejad Azar and A. Jabbari, *Arens regularity of module actions and weak amenability of Banach algebras*, Periodica Math. Hung., **71**(2)(2015), 224-235.
3. J. Baker, A. T. Lau and J. S. Pym, *Module homomorphism and topological centers associated with weakly sequentially compact Banach algebras*, J. Func. Anal., **158**(1998), 186-208.
4. A. Bodaghi, K. Haghnejad Azar and A. Jabbari, *Some notes on the topological centers of module actions*, Iran. J. Sci. Tech. (Sciences), **39**(3.1)(2015), 417-431.
5. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin 1973.
6. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math., **11**(1961), 820-847.
7. H. G. Dales, *Banach algebra and automatic continuity*, Oxford 2000.
8. H. G. Dales and A. T-M. Lau, *The second dual of Beurling algebras*, Mem. Amer. Math. Soc., 177 (2005).
9. M. Eshaghi Gordji and M. Filali, *Arens regularity of module actions*, Studia Math. **181**(3)(2007), 237-254.
10. K. Haghnejad Azar, *Arens regularity of bilinear forms and unital Banach module space*, Bull. Iran. Math. Soc., **40**(2)(2014), 505-520.
11. B. E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc., **23**(1991), 281-284.
12. A. T. Lau and A. Ülger, *Topological center of certain dual algebras*, Trans. Amer. Math. Soc., **348**(1996), 1191-1212.
13. T. Miao, *Unital Banach algebras and their subalgebras*, Math. Proc. Camb. Phil. Soc., **143**(2007), 343-347.

14. S. Mohamadzadeh and H. R. E. Vishki, *Arens regularity of module actions and the second adjoint of a derivation*, Bull. Aust. Math. Soc., **77**(2008), 465-476.

15. J. S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math. Soc., **15**(1965), 84-104.

16. R. A. Rayan, *Introduction to tensor product of Banach plgebras*, Springer-Verlag, London 2002.

17. Reiter, H., Stegeman, J. D., *Classical harmonic analysis and locally compact groups*, London Math. Soc. Mono. Ser. Volume 22, Oxford, 2000.

18. V. Runde, *Lectures on Amenability*, Springer, New York 2002.

19. A. Ülger, *Arens regularity of the algebra* $A\widehat{\otimes}B$, Trans. Amer. Math. Soc., **305**(2)(1988), 623-639.

20. A. Ülger, *Arens regularity of weakly sequentialy compact Banach algebras*, Proc. Amer. Math. Soc., **127**(11)(1999), 3839-3839.

21. A. Ülger, *Erratum to Arens regularity of the algebra* $A\widehat{\otimes}B$, Trans. Amer. Math. Soc., **355**(9)(2003), 623-639.

Mostfa Shams Kojanaghi
Department of Mathematics
Islamic Azad University Tehran
Tehran, Iran
`mstafa.shams99@yahoo.com`

Kazem Haghnejad Azar
Faculty of Science
Department of Mathematics
University of Mohaghegh Ardabili, Ardabil, Iran
Ardabil, Iran
`haghnejad@uma.ac.ir`

FACTA UNIVERSITATIS (NIŠ)
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STRONG CONVERGENCE THEOREM FOR UNIFORMLY L−LIPSCHITZIAN MAPPING OF GREGUS TYPE IN BANACH SPACES

Olilima O. Joshua, Mogbademu A. Adesanmi and Adeniran T. Adefemi

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Abstract. In this paper, we introduced a new mapping called uniformly L-Lipschitzian mapping of Gregus type, and used the Mann iterative scheme to approximate the fixed point. A Strong convergence result for the sequence generated by the scheme is shown in real Banach space. Our result generalized and unify many recent results in this area of research. In addition, using Java (jdk 1.8.0_101), we give a numerical example to support our claim.

Key words: Mann iterative scheme; uniformly L-Lipschitzian mapping; normalized duality mapping.

1. Introduction

Let E and E^* be a real Banach space and its dual space respectively. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2, \|x\| = \|f\|\}$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing.

Definition 1.1. [Ofoedu E.U [13]] Let K be a nonempty closed convex subset of a real Banach space E. The mapping $T : K \to E$ is said to be

i) nonexpansive if for all $x, y \in K$

$$\|Tx - Ty\| \leqslant \|x - y\|. \tag{1.1}$$

ii) uniformly L−Lipschitzian if there exists $L > 0$ such that, for any $x, y \in K$

$$\|T^n x - T^n y\| \leqslant L\|x - y\|, \quad \forall n \geqslant 1. \tag{1.2}$$

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iii) asymptotically nonexpansive if there exists $k_n \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for any given $x, y \in K$,

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\|, \qquad \forall n > 1. \tag{1.3}$$

iv) asymptotically pseudocontractive if there exists a sequence $k_n \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$. and there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leqslant k_n \|x - y\|^2, \qquad n \geqslant 1. \tag{1.4}$$

We can easily see from equations (1.2), (1.3), (1.4) that the class of asymptotically non-expansive mappings is a generalization of the class of uniformly L$-$Lipschitzian mapping. And that every asymptotically nonexpansive mappings are asymptotically pseudocontractive, the reason is shown below,

$$\langle T^n x - T^n y, j(x-y)\rangle \leqslant \|T^n x - T^n y\|\|x - y\| \leqslant k_n \|x - y\|^2, \quad \forall n \geqslant 1. \tag{1.5}$$

But the converse is not always true. The example to show that the converse is not true was constructed by Rhoades [15]. The asymptotically nonexpansive mappings and the asymptotically pseudocontractive mappings were introduced by Goebel and Kirk [4] and Schu [16] respectively.

In 1980, Gregus [5] introduced what is now known as the Gregus fixed point theorem. He proved the following theorem.

Theorem 1.1. Gregus [5] *Let K be a closed convex subset of a Banach space E and $T : K \to K$ a mapping that satisfies $\|Tx - Ty\| \leqslant a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$ for all $x, y \in K$ where $0 < a < 1, b, c \geqslant 0$ and $a + b + c = 1$. Then T has a unique fixed point.*

The class of mapping introduced by Gregus [5] is a generalization of non-expansive mapping which is a very important mapping in fixed point theorem and applications, because if $a = 1, b = c = 0$ then we have the mapping in (1.1), and if $a = 0, b = c = \frac{1}{2}$ we have the Kannan mappings introduced by Kannan in [6]. This class of mappings have been extended by many authors in various ways and under different conditions on T. For results on these, see [8, 11, 12, 14] and the references therein.

The trend for uniformly L$-$Lipschitzian mapping and asymptotically pseudo-contractive mapping is given below for better understanding the concept we intend to introduce.

In 1991 Schu [16], proved the following result using the modified Mann iterative scheme

Theorem 1.2. Schu [16] *Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudo-contractive mapping with a sequence $\{k_n\}$ satisfying the following conditions:*

(i) $k_n \to 1$ as $n \to 1$;

(ii) $\sum_{n=1}^{\infty} q_n^2 - 1 < \infty$, where $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $[0,1]$ *such that* $\epsilon < \alpha_n < b$, $\forall n \geqslant 1$ *where* $\epsilon > 0$ *and* $b \in (0, L^{-2}[(1+L^2)^{1/2}-1])$ *are some positive numbers. For any* $x_1 \in K$, *let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geqslant 1. \tag{1.6}$$

Then $\{x_n\}$ *converges strongly to a fixed point of* T *in* K.

In 2000, Chang [1] extended theorem 1.2 from Hilbert space to uniformly smooth Banach space, by proving the following theorem:

Theorem 1.3. Chang [1] *Let* E *be a real uniformly smooth Banach space,* K *be a nonempty bounded closed convex subset of* E, $T : K \to K$ *be an asymptotically pseudocontractive mapping with sequence* $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$, *and* $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. *Let* $\alpha_n \subset [0,1]$ *satisfying the following conditions:*

(i) $\alpha_n \to 0$ as $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, *let* $\{x_n\}$ *be the iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geqslant 0,$$

If there exists a strict increasing function $\Phi : [0,\infty) \to [0,\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\langle T^n x - p, j(x-p)\rangle \leqslant k_n \|x-p\|^2 - \Phi(\|x-p\|),$$

for all $x \in K$ *and* $n \geqslant 0$, *where* $p \in F(T)$, *then* $x_n \to p$ *as* $n \to \infty$.

Recently Ofoedu [13], extended theorem 1.3 from uniformly smooth Banach space to real Banach space and he also dispensed with the boundedness condition imposed by earlier researchers, by stating and proving the following theorem:

Theorem 1.4. Ofoedu [13] *Let* E *be a real Banach space,* K *be a nonempty closed and convex subset of* E, $T : K \to K$ *be a uniformly* $L-$*Lipschitzian asymptotically pseudocontractive mapping with sequence* $\{k_n\}_{n\geqslant 0} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$, *and let* $p \in F(T) = \{x \in K : Tx = x\} \neq \emptyset$. *Let* $\alpha_n \subset [0,1]$ *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geqslant 0.$$

If there exists a strict increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - p, j(x-p)\rangle \leqslant k_n\|x-p\|^2 - \Phi(\|x-p\|)$$

for all $x \in K$ and $n \geqslant 0$, where $p \in F(T)$, then $x_n \to p$ as $n \to \infty$.

Inspired by the above results, we introduce the following concept which generalizes the class of uniformly L−Lipschitzian mappings. The mapping is defined as follows:

Definition 1.2. Let K be a nonempty closed subset of a real Banach space E. The mapping $T : K \to E$ is said to be uniformly L−Lipschitzian mapping of Gregus type if there exists $L > 0$, and the sequences $a_n, b_n \in [0,\infty)$, with $a_n, b_n \to 0$ as $n \to \infty$, such that for any $x, y \in K$,

$$\|T^n x - T^n y\| \leqslant L\|x-y\| + a_n\|x - T^n x\| + b_n\|y - T^n y\|, \quad \forall n \geqslant 1. \tag{1.7}$$

If we set $a_n = b_n = 0 \;\; \forall n \in \mathbb{N}$, equation (1.7) is reduced to (1.2). Clearly, every uniformly L−Lipschitzian mapping is uniformly L−Lipschitaian mapping of Gregus type, but the converse is not generally true. It suffices to construct an example of a map that is uniformly L−Lipschitzian of Gregus type but not uniformly L−Lipschitzian.

Example 1.1. Let $E = \mathbb{R}$ be the set of real numbers with the usual norm, and let $K = [0,\infty)$. Consider the mapping $T : K \to K$ defined by

$$Tx = \frac{x^3}{4(1+x)}, \qquad \forall\, x \in K$$

It is easy to see that T is a monotone increasing function satisfying (1.7), but T does not satisfy inequality (1.2). In fact,

$$\begin{aligned}
|T^n x - T^n y| &\leqslant |Tx - Ty| = \left|\frac{x^3}{4+4x} - \frac{y^3}{4+4y}\right| = \frac{1}{4}\left|\frac{x^3}{1+x} - \frac{y^3}{1+y}\right| \\
&= \frac{1}{4}\left|\frac{x - y + x^3(1+y) - x + y - y^3(1+x)}{(1+x)(1+y)}\right| \\
&\leqslant \frac{1}{4}\left|\frac{x-y}{(1+x)(1+y)}\right| + \left|\frac{x^3(1+y) - x}{4(1+x)(1+y)}\right| + \left|\frac{y - y^3(1+x)}{4(1+x)(1+y)}\right| \\
&\leqslant \frac{1}{4}|x-y| + \left|x - \frac{x^3}{4(1+x)}\right| + \left|y - \frac{y^3}{4(1+y)}\right| \\
&= \frac{1}{4}|x-y| + |x - Tx| + |y - Ty|\,.
\end{aligned}$$

Therefore,

$$|T^n x - T^n y| \leqslant \frac{1}{4}|x-y| + |x-Tx| + |y-Ty|\,. \tag{1.8}$$

Hence, T is uniformly L−Lipschitzian mapping of Gregus type where the sequences $a_n = b_n = 1$, $\forall n \in \mathbb{N}$ and $L = \frac{1}{4}$. But observe that,

$$\frac{x^3}{4(1+x)} > x \qquad \forall\, x > \frac{4+\sqrt{32}}{2},$$

hence we have that,

$$|T^n x - T^n y| \leqslant |Tx - Ty| = \left|\frac{x^3}{4+4x} - \frac{y^3}{4+4y}\right| > |x-y|.$$

Thus, T is not a uniformly L−Lipschitzian mapping. We can now say that the class of uniformly L− Lipschitzian mappings of Gregus type properly includes the class of uniformly L−Lipschitzian mappings. Hence, it is more important to study this class of mappings in fixed point theory and applications.

In particular, If we let x to be any point in K and $y \in F(T)$ then, from (1.8) we have,

$$|T^n x - T^n y| \leqslant \frac{1}{2}|x-0| + \frac{1}{4}|x-Tx| + \frac{1}{4}|0-T0|.$$

but

$$|T^n x - T^n y| = |T^n x - 0| \leqslant |Tx - 0| = \left|\frac{x^3}{4+4x} - 0\right| > |x-0|,$$

for $x > \frac{4+\sqrt{32}}{2}$, $y = 0 \in F(T)$.

It is our aim in this paper to consider the iterative scheme in (1.6) and prove a strong convergence theorem for the newly introduced uniformly L−Lipschitzian mappings of Gregus type to a unique fixed point in real Banach spaces.

2. Preliminaries

We shall need the following Proposition and lemmas in the main theorem.

Proposition 2.1. *Let K be a nonempty closed convex subset of a Banach space and $T : K \to K$ be a Uniformly L−Lipschitzian mapping of Gregus Type with $a_n \in (0, \frac{1}{2})$ such that $a_n \to 0$ as $n \to \infty$. Then for any $x_0 \in K$, let $\{x_n\}$ be an iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$$

for all $n \geqslant 0$, there exists an $M > 0$ such that for $p \in F(T)$, the following hold:

i.) $\|T^n x_n - x_n\| \leqslant M\|x_n - p\|$

ii.) $\|T^n x_{n+1} - x_{n+1}\| \leqslant M\|x_{n+1} - p\|$

Proof. Since T is uniformly L−Lipschitzian mapping of Gregus Type, we have,

$$\begin{aligned}\|x_n - T^n x_n\| &\leqslant \|x_n - p\| + \|T^n x_n - p\| = \|x_n - p\| + \|T^n x_n - T^n p\| \\ &\leqslant \|x_n - p\| + L\|x_n - p\| + a_n\|x_n - T^n x_n\| + b_n\|p - T^n p\| \\ &= (1+L)\|x_n - p\| + a_n\|x_n - T^n x_n\| \\ &\leqslant \frac{(1+L)}{1-a_n}\|x_n - p\|. \end{aligned} \tag{2.1}$$

Since, $a_n \in [0, 1/2)$ we have that, $-a_n > -\frac{1}{2}$. Hence, $1 - a_n > 1 - \frac{1}{2}$, this implies that

$$\frac{1}{1-a_n} < 2.$$

Therefore,

$$\frac{(1+L)}{1-a_n} < 2(1+L).$$

Let $M = 2(1+L) > 1$, (2.1) becomes

$$\|T^n x_n - x_n\| \leqslant M\|x_n - p\|. \tag{2.2}$$

Using similar procedure we can easily get that

$$\|T^n x_{n+1} - x_{n+1}\| \leqslant M\|x_{n+1} - p\|. \tag{2.3}$$

This completes the proof. □

Lemma 2.1. Mogbademu [9] *Let E be a normed linear space then for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$, the following inequality holds:*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 2.2. C. Moore and B.V Nnoli [10] *Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing function with $\Phi(0) = 0$ and let $\{\theta_n\}, \{\lambda_n\}, \{\sigma_n\}$ be any nonnegative real sequences such that $\sigma_n = o(\lambda_n)$, $\sum_{i=0}^{\infty} \lambda_n = \infty$, and $\lim_{n\to\infty} \lambda_n = 0$. Suppose that*

$$\theta_{n+1}^2 \leqslant \theta_n^2 - \lambda_n \Phi(\theta_{n+1}) + \sigma_n, \quad n \geqslant 1,$$

then $\lim_{n\to\infty} \theta_n = 0$.

3. The Main Result

Theorem 3.1. *Let E be a real Banch space, K be a nonempty closed convex subset of E, $T : K \to K$ be a Uniformly L−Lipschitzian mapping of Gregus Type with $F(T) \neq \emptyset$ where $F(T) = \{x \in K : Tx = x\}$ and $p \in F(T)$. Let $\{k_n\} \in [0, \infty)$ be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$, and let $\alpha_n \in [0, 1]$ satisfying the following:*

i) $\alpha_n \to 0$ as $n \to \infty$

ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

For any $x_0 \in K$, *define a sequence* $\{x_n\}$ *by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

for all $n \geqslant 0$. *If there exists a strictly increasing function* $\Phi : [0, +\infty) \to [0, +\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\langle T^n x - p, j(x - p)\rangle \leqslant k_n \|x - p\|^2 - \Phi(\|x - p\|) \tag{3.1}$$

for all $x \in K$. *Then*

i) $\{x_n\}_{n \geqslant 0}$ is bounded;

ii) $x_n \to p$ as $n \to \infty$ where p is a unique fixed point of T.

Proof. This proof shall be divided into two steps. In step 1, we will show boundedness, while in step 2 we will show that the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T say p.

Step 1: Let $k = \sup\{k_n \ : n \geqslant 1\}$, since T is Uniformly L−Lipschitzian of Gregus Type and $\Phi : [0, \infty) \to [0, \infty)$ is a strictly increasing real valued function with $\Phi(0) = 0$, for $x \in K$, $p \in F(T)$ we obtain

$$\Phi(\|x - p\|) \leqslant (k + L + \alpha_n M)\|x - p\|^2.$$

Taking limit as $n \to \infty$ we have

$$\Phi(\|x - p\|) \leqslant (k + L)\|x - p\|^2.$$

Assume that $x_1 \neq Tx_1$ for some $x_1 \in K$ such that

$$(k + L)\|x_1 - p\|^2 \in R(\Phi),$$

we denote that $a_0 = (k + L)\|x_1 - p\|^2$, where $R(\Phi)$ is the range of Φ. Indeed, if $\Phi(a) \to \infty$ as $a \to \infty$, then $a_0 \in R(\Phi)$; if $\sup\{\Phi(a) : a \in [0, \infty)\} = a_1 < +\infty$ with $a_1 < a_0$, then for $p \in K$, there exists a sequence $\{u_n\}$ in K such that $u_n \to p$ as $n \to \infty$ with $u_n \neq p$, thus there exists an $n_0 \in \mathbb{N}$ such that

$$(k + L)\|u_n - p\|^2 < \frac{a_1}{2}$$

for $n \geqslant n_0$. We redefine $x_1 = u_{n_0}$ and $(k + L)\|x_1 - p\|^2 \in R(\Phi)$.

Set $R = \Phi^{-1}(a_0)$. Then we obtain $\|x_1 - p\| \leqslant R$.

Denote

$$B_1 = \{x \in K : \|x - p\| \leqslant R\}, B_2 = \{x \in K : \|x - p\| \leqslant 2R\}$$

Now, we show that $x_n \in B_1$ using mathematical induction for any $n \geqslant 1$. If $n = 1$, then $x_1 \in B_1$. Suppose that thr result is true for some n, that is $x_n \in B_1$. Now we show that $x_{n+1} \in B_1$. Suppose that, $x_{n+1} \notin B_1$, that is, $x_{n+1} > R$. Denote

$$\tau_0 = \min\left\{1, \frac{1}{2M}, \frac{1}{2LM}, \frac{\Phi(R)}{16R(M(l+3)R}, \frac{\Phi(R)}{16R^2}\right\}. \tag{3.2}$$

Since a_n, b_n, α_n and $k_n - 1 \to 0$ as $n \to \infty$. We can let $0 \leqslant a_n, b_n, \alpha_n, k_n - 1 \leqslant \tau_0$ for any $n \geqslant 1$. We obtain the following:

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant \|x_n - p\| + \alpha_n(T^n x_n - x_n) \\ &\leqslant R + \alpha_n MR \\ &\leqslant 2R \end{aligned} \tag{3.3}$$

Using Proposition 2.1 we have the following,

$$\begin{aligned} \|T^n x_n - T^n x_{n+1}\| &\leqslant L\alpha_n\|x_n - x_{n+1}\| + a_n\|x_n - T^n x_n\| + b_n\|x_{n+1} - T^n x_{n+1}\| \\ &\leqslant (L\alpha_n + a_n)M\|x_n - p\| + b_n M\|x_{n+1} - p\| \\ &\leqslant (L\alpha_n + a_n)MR + 2b_n MR \\ &\leqslant \tau_0 MR(L+3) \\ &\leqslant \frac{\Phi(R)}{16R}. \end{aligned} \tag{3.4}$$

Let us consider the following estimate, using Lemma 2.1

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n T^n x_n - p\|^2 \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2. \\ &\leqslant (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle T^n x_n - p, j(x_{n+1} - p)\rangle \\ &= (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p)\rangle \\ &\quad +2\alpha_n\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \\ &\leqslant (1-\alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\|T^n x_n - T^n x_{n+1}\|\|x_{n+1} - p\| \\ &\quad +2\alpha_n\left[k_n\|x_{n+1} - p\|^2 - \Phi(\|x_{n+1} - p\|)\right] \qquad (3.5) \\ &\leqslant (1-\alpha_n)^2 R^2 + 2\alpha_n k_n\|x_{n+1} - p\|^2 + \\ &\quad 2\alpha_n\frac{\Phi(R)}{16R}2R - 2\alpha_n\Phi(R). \qquad (3.6) \end{aligned}$$

Since $\alpha_n, k_n \to 1$ as $n \to \infty$. Thus, without loss of generality, we let $1 - 2\alpha_n k_n > 0$ for any $n \geqslant 1$, since

$$\frac{1}{1 - 2\alpha_n k_n} \leqslant 1 + \frac{2\alpha_n k_n}{1 - 2\alpha_n k_n}, \tag{3.7}$$

from (3.6) we have that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leqslant R^2+\frac{2\alpha_n}{1-2\alpha_n k_n}\left[(k_n-1)+\frac{\alpha_n}{2}\right]R^2+\\
&\quad \frac{\alpha_n\Phi(R)}{4(1-2\alpha_n k_n)}-\frac{2\alpha_n}{1-2\alpha_n k_n}\Phi(R)\\
&\leqslant R^2+\frac{\alpha_n\Phi(R)}{4(1-2\alpha_n k_n)}+\frac{\alpha_n\Phi(R)}{4(1-2\alpha_n k_n)}-\frac{2\alpha_n}{1-2\alpha_n k_n}\Phi(R)\\
&= R^2-\frac{3\alpha_n}{2(1-2\alpha_n k_n)}\Phi(R)\\
&\leqslant R^2, \qquad (3.8)
\end{aligned}$$

which is a contradiction. Hence, the sequence $\{x_n\}$ is bounded.

Step 2: Here, we intend to show that x_n converges uniquely to $p \in F(T)$. Firstly, let us show that p is unique.

Now, we show that p is unique. Suppose for contradiction there exists $p, q \in F(T)$, where $p \neq q$, such that the sequence $\{x_n\}$ converges to p, q hence, we have that,

$$\begin{aligned}
\|p-q\| &= \|T^n p - T^n q\|\\
&\leqslant L\|p-q\|+a_n\|p-T^n p\|+b_n\|q-T^n q\|\\
&= L\|p-q\|
\end{aligned}$$

Therefore,

$$0 \leqslant \|p-q\| \leqslant 0.$$

Hence, $p = q$ is a contradiction.

Next, we show that the sequence $\{x_n\}$ converge to a unique fixed point p of T.

Since $\|x_n - p\|$ is bounded, there exists $M_* > 0$ such that $\|x_n - p\|^2 < M_*$. Hence, from equation (3.5) we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leqslant (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\|T^n x_n - T^n x_{n+1}\|\|x_{n+1}-p\|\\
&\quad +2\alpha_n\left[k_n\|x_{n+1}-p\|^2-\Phi(\|x_{n+1}-p\|)\right]\\
&\leqslant \|x_n-p\|^2-\alpha_n\Phi(\|x_{n+1}-p)\|\\
&\quad +2\alpha_n(k_n-1)M_*^2+\alpha_n^2 M_*^2+2\alpha_n M_* M(L\alpha_n+a_n+b_n)M_* \qquad (3.9)
\end{aligned}$$

Comparing Lemma 2.2 with (3.9) we can let $\theta_{n+1}^2 = \|x_{n+1}-p\|^2, \theta_n^2 = \|x_n - p\|^2, \lambda_n = \alpha_n$ and $\sigma_n = 2\alpha_n(k_n-1)M_*^2 + \alpha_n^2 M_*^2 + 2\alpha_n M_* M(L\alpha_n + a_n + b_n)M_*$. From condition (i), we have that $\sum_{n=1}^{\infty}\lambda_n = \sum_{n=1}^{\infty}\alpha_n = \infty$, therefore, $\|x_n - p\| \to 0$ as $n \to \infty$. □

Example 3.1. Consider $T = \frac{x^3}{4+4x}$, we have shown in Example 1.1, that T is uniformly L−Lipschitzian of Gregus type, and we can easily check that the fixed point of T is $p = 0$.

Now, take $\alpha_n = 1/2$, and the initial guess value $x_0 = 0.5, 1.0, 1.5$ and 2.0. In table 3.1 and figure 3.1, we give a numerical example using Java 2.7, to support our claim that the sequence x_n converges uniquely to its fixed point $p = 0$ for the uniformly L−Lipschitzian mapping of Gregus type T.

Table 3.1: Numerical Example for the uniformly L−Lipschitzian mapping of Gregus type using the iteration in (1.6), $Tx = \frac{x^3}{4(1+x)}$.

S/N		**Modified**	**Mann Iteration**	
0	0.5	1	1.5	2
1	0.260416667	0.5625	0.91875	1.333333333
2	0.131959801	0.295488281	0.509897366	0.793650794
3	0.066233649	0.150233556	0.265923844	0.43166398
4	0.033150889	0.075485267	0.134818761	0.222854726
5	0.016579852	0.037792625	0.0676793	0.112558723
⋮	⋮	⋮	⋮	⋮
25	1.58E-08	3.61E-08	6.46E-08	1.08E-07
26	7.91E-09	1.80E-08	3.23E-08	5.39E-08
27	3.95E-09	9.01E-09	1.62E-08	2.69E-08
28	1.98E-09	4.51E-09	8.08E-09	1.35E-08
⋮	⋮	⋮	⋮	⋮
55	1.47E-17	3.36E-17	6.02E-17	1.00E-16
56	7.36E-18	1.68E-17	3.01E-17	5.02E-17
57	N/A	8.40E-18	1.50E-17	2.51E-17
58	N/A	N/A	7.52E-18	1.25E-17
59	N/A	N/A	N/A	6.27E-18

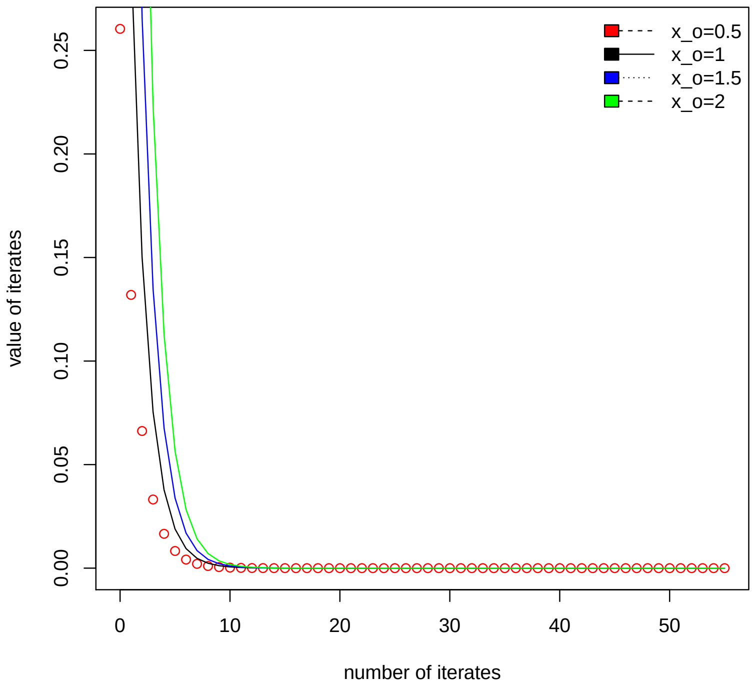

FIG. 3.1: Convergence behaviour of Modified Mann iteration process to the fixed point $p = 0$ with initial guess values taken at $x_0 = 0.5, 1.0, 1.5$ and 2.0.

Competing Interest

The authors declares that they have no competing interest.

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REFERENCES

1. S. S. CHANG. *Some results for asymptotically pseudocontractipings* Proc. Amer. Math. Soc. **129** (2000), 845–853.
2. S. S. CHANG, Y. J. CHO and J. K KIM. *Some results for Uniformly Lipschitzian mappings in Banach spaces.* App. Math. Letters. **22** (2009), 121–125.

3. S. Deimling. *Non linear Functional Analysis*. Springer, Berlin, (1980)
4. K. Goebel and W. A. Kirk. *A fixed point theorem for asymptotically nonexpansive mappings*. Proc. Amer. Math. Soc. **35** (1972), 171–174.
5. M. Gregus. *A fixed point theorem in Banach space*. Bollettindella Unione Matematica Italiana A Vol. **5** (1980), pp. 193–198.
6. R, Kannan. *Some results on fixed points*. Bull. Calcutta Math. Soc. **60** (1968), 71–76.
7. W. R. Mann. *Mean value methods in iteration*. Proc. Amer. Math. Soc. **4** (1953), 506–510.
8. A. K. Marwan, A. Amini-Harandi and N Hussain. *A generalization of a Gregus fixed point in metric spaces*. Journal of Applied Math. Vol. **2014**, Article ID 580297, 5 pages, 2014, https://doi.org/10.1155/2014/580297
9. A. A. Mogbademu. *Fixed points of nearly weak uniformly L-Lipschitzian mappings in real Banach spaces*. CREAT. MATH. INFORM. **27** (2018), No. 1, 63 – 70
10. C. Moore and B. V. Nnoli. *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators*. Comput. Math. Appl. **42** (2001), 131–140.
11. S. Moradi and A. Farajzadeh. *On Olaleru's open problem on Gregus fixed point theorem*. Journal of Global Optimization. Vol. **56**, no. 4 (2013), pp. 1689-1697.
12. R,N Mukherjee and V. Verma. *A note on a fixed point theorem of Gregus*, Mathematica Japonica. Vol. **33** (1988), No. 5, pp. 745–749.
13. E. U. Ofoedu. *Strong Convergence theorem for Uniformly $L-$Lipschitzian asymtotically nonexpansive mapping in Banach space*. J. Math. Anal. Appl. **321** (2006), 722–728.
14. J. O. Olaleru and H. Akewe. *An extenson of Gregus fixed point theorem*. Fixed Point Theory and Applications. Volume **2007**, Article ID 78628, 8 pages
15. B. E. Rhoades. *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.
16. J. Schu. *Iterative construction of fixed point of asymptotically nonexpansive mappings*. J. Math. Anal. Appl. **158** (1991), 407–413.
17. Z. Xu and G. Lv: *Strong convergence theorems for uniformly $L-$Lipschitzian asymptotically pseudocontractive mappings in Banach spaces*. J. of Inequalities and Applications. **2013**, 2013:79.

Olilima O. Joshua
Faculty of Science
Department of Mathematical Sciences
Augustine University Ilara-Epe
Lagos, Nigeria
P. M. Box 1010
joshua.olilima@augustineuniversity.edu.ng

Mogbademu A. Adesanmi
Faculty of Science
Department of Mathematical Sciences

Augustine University Ilara-Epe
Lagos, Nigeria
P. M. Box 1010
amogbademu@unilag.edu.ng

Adeniran T. Adefemi
Faculty of Science
Department of Statistics
University of Ibadan
Oyo State, Nigeria
at.adeniran@mail.ui.edu.ng

FACTA UNIVERSITATIS (NIŠ)
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ON THE FIXED-CIRCLE PROBLEM

Ufuk Çelik and Nihal Özgür

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Abstract. In this paper, we focus on the geometric properties of fixed-points of a self-mapping and obtain new solutions to a recent problem called "fixed-circle problem" in the setting of an S-metric space. For this purpose, we develop various techniques by defining new contractive conditions and using some auxiliary functions. Furthermore, we present new examples to support our theoretical results.
Keywords: fixed-points; S-metric space; self-mapping.

1. Introduction

It is known that the fixed-point theory has been generalized by various approaches. One of these approaches is to generalize the used contractive condition (for example see [2], [5]). The other is to generalize the used metric space (see [1, 8, 21, 23] and the references therein). For example, in [21], Sedghi, Shobe and Aliouche presented the notion of an S-metric space as the generalization of a metric space. Then, some fixed-point theorems have been extensively studied on S-metric spaces (see [6, 7, 9, 13, 15, 18, 19, 21, 22, 24, 25, 27] for more details).

On the other hand, fixed-point theorems have been widely studied with different aspects such as the uniqueness of a fixed-point, common fixed point, etc. If a fixed point is not unique then the investigation of the geometric properties of fixed points of a self-mapping is an interesting problem. As a recent approach, the concept of a fixed circle and the fixed-circle problem have been presented on a metric (resp. an S-metric) space as a new direction of the generalization of known fixed-point results (see [17] and [16]). Then, new fixed circle theorems have been given by various techniques on metric (resp. S-metric) spaces (see [11, 12, 20, 26] for the metric case; [10, 14, 24, 25] for the S-metric case).

Our aim in this paper is to obtain new fixed-circle theorems for self-mappings on an S-metric space. In Section 2., we recall some basic facts about S-metric spaces.

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In Section 3., we give new fixed-circle theorems by introducing new types of the notion of an F_c^S-contraction introduced and used in [10]. In Section 4., we investigate new existence and uniqueness theorems for fixed circles of self-mappings using some auxiliary functions and contractive conditions. We support our theoretical results by illustrative examples.

2. Preliminaries

In this section, we recall some necessary notions and results on S-metric spaces with new examples.

Definition 2.1. [21] Let X be a nonempty set and $\mathcal{S} : X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

1. $\mathcal{S}(x,y,z) = 0$ if and only if $x = y = z$,
2. $\mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$.

Then $\mathcal{S}$ is called an S-metric on X and the pair $(X, \mathcal{S})$ is called an S-metric space.

Example 2.1. [21] Let $X = \mathbb{R}$ (or $\mathbb{C}$) and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z) = |x-z| + |y-z|\,,$$

for all $x, y, z \in \mathbb{R}$ (or $\mathbb{C}$). Then the function $\mathcal{S} : X^3 \to [0,\infty)$ is an S-metric and it is called the usual S-metric on $\mathbb{R}$ (or $\mathbb{C}$).

Lemma 2.1. *[21] Let $(X, \mathcal{S})$ be an S-metric space and $x, y \in X$. Then we have*

$$\mathcal{S}(x,x,y) = \mathcal{S}(y,y,x).$$

It was given the relationships between a metric and an S-metric in the following lemma [7].

Lemma 2.2. *[7] Let (X, d) be a metric space. Then the following properties are satisfied:*

1. *$\mathcal{S}_d(x,y,z) = d(x,z) + d(y,z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in $(X, \mathcal{S}_d)$.*
3. *$\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in $(X, \mathcal{S}_d)$.*
4. *(X, d) is complete if and only if $(X, \mathcal{S}_d)$ is complete.*

The metric $\mathcal{S}_d$ was called as the S-metric generated by d in [13].

Now we give a new example of an S-metric generated by a metric.

Example 2.2. Let $X \neq \varnothing$, $d : X^2 \to [0,\infty)$ be any metric on X and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z) = \min\{1, d(x,z)\} + \min\{1, d(y,z)\}.$$

Then the function $\mathcal{S} : X^3 \to [0,\infty)$ is an S-metric on X and the pair $(X,\mathcal{S})$ is an S-metric space. Clearly, this S-metric $\mathcal{S}$ is generated by the metric m defined as $m(x,y) = \min\{1, d(x,y)\}$.

There are some examples of an S-metric which is not generated by any metric (see [7], [10], [14] and [13]). We give a new example.

Example 2.3. Let $X = \mathbb{R}$, $d : X^2 \to [0,\infty)$ be any metric on X and the function $\mathcal{S} : X^3 \to [0,\infty)$ be defined by

$$\mathcal{S}(x,y,z) = \min\{1, d(x,z)\} + |y - z|.$$

Then the function $\mathcal{S} : X^3 \to [0,\infty)$ is an S-metric on X which is not generated by any metric and the pair $(X,\mathcal{S})$ is an S-metric space. Conversely, assume that there exists a metric d_1 such that

$$\mathcal{S}(x,y,z) = d_1(x,z) + d_1(y,z),$$

for all $x,y,z \in X$. Then we obtain

$$\mathcal{S}(x,x,z) = 2d_1(x,z) \Rightarrow d_1(x,z) = \frac{1}{2}\min\{1, d(x,z)\} + \frac{1}{2}|x - z|$$

and

$$\mathcal{S}(y,y,z) = 2d_1(y,z) \Rightarrow d_1(y,z) = \frac{1}{2}\min\{1, d(y,z)\} + \frac{1}{2}|y - z|,$$

for all $x,y,z \in X$. So we get

$$\begin{aligned} \min\{1, d(x,z)\} + |y - z| &\neq \tfrac{1}{2}\min\{1, d(x,z)\} + \tfrac{1}{2}|x - z| \\ &+ \tfrac{1}{2}\min\{1, d(y,z)\} + \tfrac{1}{2}|y - z|, \end{aligned}$$

which is a contradiction. Hence $\mathcal{S}$ is not generated by any metric.

Definition 2.2. [16] Let $(X,\mathcal{S})$ be an S-metric space. Then a circle and a disc are defined on an S-metric space as follows, respectively:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x,x,x_0) = r\}$$

and

$$D^S_{x_0,r} = \{x \in X : \mathcal{S}(x,x,x_0) \leq r\}.$$

Example 2.4. Let X be a nonempty set, the function $d : X^2 \to [0,\infty)$ be any metric on X and the S-metric space $(X,\mathcal{S})$ be defined as in Example 2.2. Let us consider the circle $C^S_{x_0,r}$ according to the S-metric $\mathcal{S}$:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x,x,x_0) = 2\min\{1, d(x,x_0)\} = r\}.$$

Then we have the following cases:

Case 1 : If $r = 2$ then $C^S_{x_0,r} = \{x \in X : d(x, x_0) \geq 1\}$.

Case 2 : If $r > 2$ then $C^S_{x_0,r} = \emptyset$.

Case 3 : If $r < 2$ then $C^S_{x_0,r} = C_{x_0,\frac{r}{2}}$, where $C_{x_0,\frac{r}{2}} = \left\{x \in X : d(x, x_0) = \frac{r}{2}\right\}$.

Example 2.5. Let X be a nonempty set, the function $d : X^2 \to [0, \infty)$ be any metric on X and the S-metric space be defined as in Example 2.3. Let us consider the circle $C^S_{x_0,r}$ according to the S-metric:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x, x, x_0) = \min\{1, d(x, x_0)\} + |x - x_0| = r\}.$$

Then we have the following cases:

Case 1 : If $x \in (X \backslash D_{x_0,1}) \cup C_{x_0,1}$ then $C^S_{x_0,r} = \{x \in (X \backslash D_{x_0,1}) \cup C_{x_0,1} : |x - x_0| = r - 1\}$.

Case 2 : If $x \in D_{x_0,1} \backslash C_{x_0,1}$ then $C^S_{x_0,r} = \{x \in D_{x_0,1} \backslash C_{x_0,1} : d(x, x_0) + |x - x_0| = r\}$.

In the following example, the S-metric is not generated by any metric but any circle on this S-metric space is the same as the circle on the usual metric space $\mathbb{R}$(or $\mathbb{C}$).

Example 2.6. Let $X = \mathbb{R}$ (or $\mathbb{C}$) and the function $\mathcal{S} : X^3 \to [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\},$$

for all $x, y, z \in X$. Then the function $\mathcal{S} : X^3 \to [0, \infty)$ is an S-metric on X which is not generated by any metric. For any circle $C^S_{x_0,r}$ on this S-metric space we have $C^S_{x_0,r} = \{x_0 - r, x_0 + r\}$ which is correspond to the circle $C_{x_0,r}$ with the equation $|y - x_0| = r$ on the usual metric space $\mathbb{R}$.

3. Fixed-Circle Theorems via New Types of F^S_c-contractions

In this section, we give new fixed-circle theorems using new types of the notion of an F^S_c-contraction introduced in [10]. At first, we recall the definition of a fixed-circle and the following family of functions which was introduced by Wardowski in [28].

Definition 3.1. [16] Let $(X, \mathcal{S})$ be an S-metric space, $C^S_{x_0,r}$ be a circle on X and $T : X \to X$ be a self-mapping. If $Tx = x$ for every $x \in C^S_{x_0,r}$ then the circle $C^S_{x_0,r}$ is called as the fixed circle of T.

Definition 3.2. [28] Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ such that

$(F1)$ F is strictly increasing,

$(F2)$ For each sequence $\{\alpha_n\}$ in $(0, \infty)$ the following holds $\lim \alpha_n = 0$ if and only if $\lim F(\alpha_n) = -\infty$,

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Some functions that satisfy the conditions $(F1)$, $(F2)$ and $(F3)$ of Definition 3.2 are given in the following example (see [28] for more details).

Example 3.1. [28] The following functions defined by

$$F_1 : (0, \infty) \to \mathbb{R}, F_1(x) = \ln(x),$$

$$F_2 : (0, \infty) \to \mathbb{R}, F_2(x) = \ln(x) + x,$$

$$F_3 : (0, \infty) \to \mathbb{R}, F_3(x) = -\frac{1}{\sqrt{x}}$$

and

$$F_4 : (0, \infty) \to \mathbb{R}, F_4(x) = \ln(x^2 + x)$$

are the examples of Definition 3.2.

Using this family of functions, in [4], some new fixed-point theorems was obtained on S-metric spaces. In [10], it was introduced the following new contraction type to obtain some fixed-circle results on an S-metric space.

Definition 3.3. [10] Let $(X, \mathcal{S})$ be an S-metric space. A self-mapping T on X is said to be an F_c^S-contraction if there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$\mathcal{S}(Tx, Tx, x) > 0 \Longrightarrow t + F(\mathcal{S}(Tx, Tx, x)) \leq F(\mathcal{S}(x, x, x_0)).$$

In [24], Suzuki-Berinde type F_c^S-contractions were introduced for the same purpose. Now we define new types of F_c^S-contractions to get new fixed-circle results. To do this, we use some classical contraction conditions such as Ćirić-type, modified Hardy-Rogers type and Khan-type contractive conditions.

Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. We will use the number r defined by

$$r = \inf \{\mathcal{S}(Tx, Tx, x) : x \in X, x \neq Tx\}, \tag{3.1}$$

in all of our results.

3.1. Ćirić type fixed-circle results on S-metric spaces

At first, we introduce the following Ćirić type F_c^S-contraction.

Definition 3.4. Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. If there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$\mathcal{S}(Tx, Tx, x) > 0 \Longrightarrow t + F(\mathcal{S}(Tx, Tx, x)) \leq F(m(x, x, x_0)),$$

where

$$m(x,x,y) = \max\left\{ \begin{array}{c} \mathcal{S}(x,x,y), \mathcal{S}(x,x,Tx), \mathcal{S}(y,y,Ty), \\ \frac{1}{2}[\mathcal{S}(x,x,Ty) + \mathcal{S}(y,y,Tx)] \end{array} \right\},$$

then the self-mapping T is called a Ćirić type F_c^S-contraction on X.

An immediate consequence of this definition is the following proposition.

Proposition 3.1. *Let $(X,\mathcal{S})$ be an S-metric space. If a self-mapping T on X is a Ćirić-type F_c^S-contraction with $x_0 \in X$ then we have $Tx_0 = x_0$.*

Proof. Assume that $Tx_0 \neq x_0$. From the definition of a Ćirić-type F_c^S-contraction and Lemma 2.1, we get

$$\begin{aligned} \mathcal{S}(Tx_0,Tx_0,x_0) &> 0 \Longrightarrow t + F[\mathcal{S}(Tx_0,Tx_0,x_0)] \leq F(m(x_0,x_0,x_0)) \\ &= F\left(\max\left\{ \begin{array}{c} \mathcal{S}(x_0,x_0,x_0), \mathcal{S}(x_0,x_0,Tx_0), \mathcal{S}(x_0,x_0,Tx_0), \\ \frac{1}{2}[\mathcal{S}(x_0,x_0,Tx_0) + \mathcal{S}(x_0,x_0,Tx_0)] \end{array} \right\}\right) \\ &= F(\mathcal{S}(x_0,x_0,Tx_0)). \end{aligned}$$

This is a contradiction by the fact that $t > 0$. Then we have $Tx_0 = x_0$. □

Using Ćirić type F_c^S-contractions, we give the following fixed-circle theorem.

Theorem 3.1. *Let $(X,\mathcal{S})$ be an S-metric space, T be a Ćirić type F_c^S-contractive self-mapping with $x_0 \in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0) = r$ for all $x \in C_{x_0,r}^S$ then the circle $C_{x_0,r}^S$ is a fixed circle of T. In particular, T fixes every circle $C_{x_0,\rho}^S$ where $\rho < r$ if $\mathcal{S}(Tx,Tx,x_0) = \rho$ for all $x \in C_{x_0,\rho}^S$.*

Proof. Since $\mathcal{S}(Tx,Tx,x_0) = r$, the self-mapping T maps $C_{x_0,r}^S$ into (or onto) itself. Let $x \in C_{x_0,r}^S$ be an arbitrary point. If $Tx \neq x$, by the definition of r we have $\mathcal{S}(Tx,Tx,x) \geq r$. Hence, using the Ćirić-type F_c^S-contractive property, Lemma 2.1, Proposition 3.1 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(\mathcal{S}(Tx,Tx,x)) \leq F(m(x,x,x_0)) - t < F(m(x,x,x_0)) \\ &= F\left(\max\left\{ \begin{array}{c} \mathcal{S}(x,x,x_0), \mathcal{S}(x,x,Tx), \mathcal{S}(x_0,x_0,Tx_0), \\ \frac{1}{2}[\mathcal{S}(x,x,Tx_0) + \mathcal{S}(x_0,x_0,Tx)] \end{array} \right\}\right) \\ &= F(\max\{r, \mathcal{S}(x,x,Tx), 0, r\}) = F(\mathcal{S}(Tx,Tx,x)), \end{aligned}$$

which is a contradiction. Therefore, $\mathcal{S}(Tx,Tx,x) = 0$ and so $Tx = x$. Consequently, $C_{x_0,r}^S$ is a fixed circle of T.

Using the similar arguments, it is easy to see that T also fixes any circle $C_{x_0,\rho}^S$ where $\rho < r$. □

Remark 3.1. 1) Notice that, in Theorem 3.1, Ćirić type F_c^S-contractive self-mapping T fixes the disc $D_{x_0,r}^S$ if $\mathcal{S}(Tx,Tx,x_0)=\rho$ for all $x\in C_{x_0,\rho}^S$ and each $\rho\leq r$.

2) In Theorem 3.1, if $r=0$, then we have $C_{x_0,r}^S=\{x_0\}$ and this is a fixed circle of the self-mapping T by Proposition 3.1.

In the following example, we see that the converse statement of Theorem 3.1 is not always true.

Example 3.2. Let $X=\mathbb{C}$ be the S-metric space with the usual S-metric defined in Example 2.1, $z_0\in\mathbb{C}$ be any point and the self-mapping $T:X\to X$ be defined as

$$Tz=\begin{cases} z & , \quad |z-z_0|\leq\frac{\mu}{2} \\ z_0 & , \quad |z-z_0|>\frac{\mu}{2} \end{cases},$$

for all $z\in\mathbb{C}$ with $\mu>0$. We show that T is not a Ćirić-type F_c^S-contractive self-mapping. Indeed, if $|z-z_0|>\frac{\mu}{2}$ for $z\in\mathbb{C}$, then using Lemma 2.1 and the Ćirić-type F_c^S-contractive property, we get

$$\mathcal{S}(Tz,Tz,z)=\mathcal{S}(z_0,z_0,z)>0\Longrightarrow t+F(\mathcal{S}(z_0,z_0,z))\leq F(m(z,z,z_0)),$$

$$t+F(\mathcal{S}(z_0,z_0,z))\leq F(\mathcal{S}(z,z,z_0))$$

and so

$$t+F(r)\leq F(r)\Longrightarrow t\leq 0.$$

This is a contradiction since $t>0$. Hence T is not a Ćirić-type F_c^S-contractive self-mapping for any $z_0\in\mathbb{C}$. But T fixes every circle $C_{x_0,\rho}^S$ where $\rho\leq\mu$.

Now we give some illustrative examples of Theorem 3.1.

Example 3.3. Let $X=\{z\in\mathbb{C}:|z|=2\}$. Let us consider the S-metric $\mathcal{S}$ defined in Example 2.6 on X and define the self-mapping $T:X\to X$ by

$$Tz=\begin{cases} -2 & , \quad \frac{\pi}{3}\leq\arg(z)\leq\frac{\pi}{2} \\ z & , \quad otherwise \end{cases}.$$

Then the self-mapping T is a Ćirić-type F_c^S-contractive self-mapping with $F=\ln x$, $t=\ln\left(\frac{\sqrt{8+4\sqrt{3}}}{2\sqrt{3}}\right)$ and $z_0=-2i$. Indeed, we obtain

$$\begin{aligned} r &= \inf\{\mathcal{S}(z,z,Tz):z\in X, z\neq Tz\} \\ &= 2\sqrt{2}. \end{aligned}$$

In the case $\mathcal{S}(z,z,Tz)>0$, we find

$$\begin{aligned} m(z,z,-2i) &= \max\left\{\begin{array}{c} \mathcal{S}(z,z,-2i),\mathcal{S}(z,z,-2),\mathcal{S}(-2i,-2i,-2i), \\ \frac{1}{2}[\mathcal{S}(z,z,-2i)+\mathcal{S}(-2i,-2i,-2)] \end{array}\right\} \\ &= \max\left\{|z+2i|,|z+2|,0,\frac{1}{2}\left[|z+2i|+|2i-2|\right]\right\} \\ &= \sqrt{8+4\sqrt{3}} \end{aligned}$$

and hence

$$t+\ln(|z+2|)\leq \ln\left(\sqrt{8+4\sqrt{3}}\right).$$

Clearly, T fixes the circle $C^{S}_{-2i,2\sqrt{2}}=\{-2,2\}$ and the disc $D^{S}_{-2i,2\sqrt{2}}=\left\{z\in X:\mathcal{S}\left(z,z,-2i\right)\leq 2\sqrt{2}\right\}$.

3.2. Modified Hardy–Rogers type fixed-circle results on S-metric spaces

Now we introduce the following modified Hardy-Rogers type F_c^S-contraction.

Definition 3.5. Let $(X,\mathcal{S})$ be an S-metric space and T be a self-mapping on X. If there exist $F\in\mathbb{F}$, $t>0$ and $x_0\in X$ such that for all $x\in X$ the following holds

$$\begin{aligned}\mathcal{S}(Tx,Tx,x) &> 0\Longrightarrow t+F(\mathcal{S}(Tx,Tx,x))\leq \\ & F\left[\begin{array}{c}\alpha\mathcal{S}(x,x,x_0)+\beta\mathcal{S}(Tx_0,Tx_0,x)+\gamma\mathcal{S}(Tx,Tx,x_0)\\ +\eta\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx,Tx,x)]}{[1+\mathcal{S}(Tx_0,Tx_0,x)]}+\lambda\frac{\mathcal{S}(Tx_0,Tx_0,x_0)+\mathcal{S}(Tx,Tx,x_0)}{1+\mathcal{S}(Tx_0,Tx_0,x_0).\mathcal{S}(x_0,x_0,x)}\\ +\mu\frac{\mathcal{S}(Tx,Tx,x)[1+\mathcal{S}(Tx,Tx,x_0)]}{1+\mathcal{S}(x,x,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)}\end{array}\right],\end{aligned}$$

where $\alpha+\beta+\gamma+\eta+\lambda+\mu<\frac{1}{2}$, $\alpha,\beta,\gamma,\eta,\lambda,\mu\geq 0$ and $a\neq 0$, then the self-mapping T is called a modified Hardy-Rogers type F_c^S-contraction on X.

Proposition 3.2. *Let $(X,\mathcal{S})$ be an S-metric space. If a self-mapping T on X is a modified Hardy-Rogers type F_c^S-contraction with $x_0\in X$ then we have $Tx_0=x_0$.*

Proof. Assume that $Tx_0\neq x_0$. By the hypothesis, we obtain

$$\begin{aligned}\mathcal{S}(Tx_0,Tx_0,x_0) &> 0\Longrightarrow t+F(\mathcal{S}(Tx_0,Tx_0,x_0))\leq \\ & F\left[\begin{array}{c}\alpha\mathcal{S}(x_0,x_0,x_0)+\beta\mathcal{S}(Tx_0,Tx_0,x_0)+\gamma\mathcal{S}(Tx_0,Tx_0,x_0)\\ +\eta\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx_0,Tx_0,x_0)]}{[1+\mathcal{S}(Tx_0,Tx_0,x_0)]}+\lambda\frac{\mathcal{S}(Tx_0,Tx_0,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)}{1+\mathcal{S}(Tx_0,Tx_0,x_0).\mathcal{S}(x_0,x_0,x_0)}\\ +\mu\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx_0,Tx_0,x_0)]}{1+\mathcal{S}(x_0,x_0,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)}\end{array}\right]\\ &= F\left[(\beta+\gamma+\eta+2\lambda+\mu)\mathcal{S}(Tx_0,Tx_0,x_0)\right]\\ &< F\left[\mathcal{S}(Tx_0,Tx_0,x_0)\right].\end{aligned}$$

This is a contradiction since $t>0$. Hence we get $Tx_0=x_0$. □

Now using the notion of a modified Hardy-Rogers type F_c^S-contraction condition, we prove the following fixed-circle theorem.

Theorem 3.2. *Let $(X,\mathcal{S})$ be an S-metric space, T be a modified Hardy-Rogers type F_c^S-contractive self-mapping with $x_0\in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0)=r$ for all $x\in C^S_{x_0,r}$ then $C^S_{x_0,r}$ is a fixed circle of T. In particular, T fixes every circle $C^S_{x_0,\rho}$ where $\rho<r$ if $\mathcal{S}(Tx,Tx,x_0)=\rho$ for all $x\in C^S_{x_0,\rho}$.*

Proof. Let $x \in C^S_{x_0,r}$ and $Tx \neq x$. If $r = 0$, then we have $C^S_{x_0,r} = \{x_0\}$ and this is a fixed circle of the self-mapping T by Proposition 3.2. Assume that $r > 0$. Using the modified Hardy-Rogers type F^S_c-contraction property, Proposition 3.2, Lemma 2.1 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(\mathcal{S}(Tx,Tx,x)) \\ &\leq F\left[\begin{array}{c} \alpha\mathcal{S}(x,x,x_0)+\beta\mathcal{S}(Tx_0,Tx_0,x)+\gamma\mathcal{S}(Tx,Tx,x_0) \\ +\eta\frac{\mathcal{S}(Tx_0,Tx_0,x_0)[1+\mathcal{S}(Tx,Tx,x)]}{[1+\mathcal{S}(Tx_0,Tx_0,x)]}+\lambda\frac{\mathcal{S}(Tx_0,Tx_0,x_0)+\mathcal{S}(Tx,Tx,x_0)}{1+\mathcal{S}(Tx_0,Tx_0,x_0).\mathcal{S}(x,x,x_0)} \\ +\mu\frac{\mathcal{S}(Tx,Tx,x)[1+\mathcal{S}(Tx,Tx,x_0)]}{1+\mathcal{S}(x,x,x_0)+\mathcal{S}(Tx_0,Tx_0,x_0)} \end{array}\right]-t \\ &< F[\alpha r+\beta r+\gamma r+\lambda r+\mu\mathcal{S}(Tx,Tx,x)] \\ &\leq F[(\alpha+\beta+\gamma+\lambda+\mu)\mathcal{S}(Tx,Tx,x)] \\ &\leq F[\mathcal{S}(Tx,Tx,x)], \end{aligned}$$

which is a contradiction. Therefore, $\mathcal{S}(Tx,Tx,x) = 0$ and so $Tx = x$. Consequently, $C^S_{x_0,r}$ is a fixed circle of T. Using the similar arguments, it is easy to see that T also fixes any circle $C^S_{x_0,\rho}$ where $\rho < r$. □

Remark 3.2. 1) Let (X, S) be an S-metric space, T be a modified Hardy-Rogers type F^S_c-contractive self-mapping with $x_0 \in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0) = \rho$ for all $x \in C^S_{x_0,\rho}$ and each $\rho \leq r$, then T fixes the disc $D^S_{x_0,r}$.

2) Let us consider the self-mapping T given in Example 3.2. Then it can be easily seen that T is not a modified Hardy-Rogers type F^S_c-contractive self-mapping. But, T fixes every circle $C^S_{x_0,\rho}$ where $\rho \leq r$. Hence the converse statement of Theorem 3.2 is not always true.

Example 3.4. Let $X = \mathbb{R}^+$ and the S-metric $\mathcal{S}$ be the usual S-metric. Let us define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} 2x + \frac{4}{x} &, \quad x \in [1,4) \\ x &, \quad otherwise \end{cases},$$

for all $x \in X$. Then the self-mapping T is a modified Hardy-Rogers type F^S_c-contractive self-mapping with $\alpha = \frac{1}{4}$, $\beta = \frac{1}{25}$, $\gamma = \frac{1}{25}$, $\lambda = \frac{1}{25}$, $\mu = \frac{1}{25}$, $F = \ln x$, $t = \ln \frac{9}{8}$ and $x_0 = 35$. Indeed, in the cases $\mathcal{S}(Tx,Tx,x) > 0$ we find

$$8 \leq \mathcal{S}(Tx,Tx,x) \leq 10$$

and

$$62 \leq \mathcal{S}(x,x,x_0) \leq 68$$

and hence

$$\begin{aligned} t+F\left(2\left|x+\frac{4}{x}\right|\right) &\leq F\left[2\alpha\left|x-35\right|\right] \\ &\leq F\left[\begin{array}{c} 2\alpha\left|x-35\right|+2\beta\left|x-35\right|+2\gamma\left|Tx-35\right| \\ +\eta.0+2\lambda\left|Tx-35\right| \\ +\mu\frac{2\left|x+\frac{4}{x}\right|[1+\left|Tx-35\right|]}{1+2\left|x-35\right|} \end{array}\right]. \end{aligned}$$

Also we have

$$r = \inf\{\mathcal{S}(Tx,Tx,x) : x \neq Tx\} = 8.$$

Therefore, the self-mapping T fixes the circle $C^S_{35,8} = \{31, 39\}$ and the disc $D^S_{35,8} = \{x \in \mathbb{R}^+ : 31 \leq x \leq 39\}$.

3.3. Khan-type fixed-circle results on S-metric spaces

Now we introduce the following Khan-type F^S_c-contraction.

Definition 3.6. Let $(X, \mathcal{S})$ be an S-metric space and T be a self-mapping on X. If there exist $F \in \mathbb{F}$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$\begin{aligned} \mathcal{S}(Tx,Tx,x) &> 0 \Longrightarrow t + F(\mathcal{S}(Tx,Tx,x)) \\ &\leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x) + \mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x) + \mathcal{S}(Tx,Tx,x_0)}\right], \end{aligned}$$

where

$$h \in [0,1), \mathcal{S}(Tx_0,Tx_0,x) + \mathcal{S}(Tx,Tx,x_0) \neq 0.$$

Then the self-mapping T is called Khan-type F^S_c-contraction on X.

Proposition 3.3. *Let $(X, \mathcal{S})$ be an S-metric space. If a self-mapping T on X is a Khan-type F^S_C-contraction with $x_0 \in X$. Then we have $Tx_0 = x_0$.*

Proof. Assume that $Tx_0 \neq x_0$. By the hypothesis, we have

$$\begin{aligned} \mathcal{S}(Tx_0,Tx_0,x_0) &> 0 \Longrightarrow t + F(\mathcal{S}(Tx_0,Tx_0,x_0)) \\ &\leq F\left[h\frac{\mathcal{S}(Tx_0,Tx_0,x_0)\mathcal{S}(Tx_0,Tx_0,x_0) + \mathcal{S}(Tx_0,Tx_0,x_0)\mathcal{S}(Tx_0,Tx_0,x_0)}{\mathcal{S}(Tx_0,Tx_0,x_0) + \mathcal{S}(Tx_0,Tx_0,x_0)}\right] \\ &= F\left[h\frac{\mathcal{S}^2(Tx_0,Tx_0,x_0) + \mathcal{S}^2(Tx_0,Tx_0,x_0)}{2\mathcal{S}(Tx_0,Tx_0,x_0)}\right] \\ &= F\left[h\frac{2\mathcal{S}^2(Tx_0,Tx_0,x_0)}{2\mathcal{S}(Tx_0,Tx_0,x_0)}\right] \\ &< F\left[\mathcal{S}(Tx_0,Tx_0,x_0)\right], \end{aligned}$$

which is contradiction since $t > 0$. Then we have $Tx_0 = x_0$. □

Now using the notion of a Khan-type F^S_c-contraction condition, we prove the following fixed-circle theorem.

Theorem 3.3. *Let $(X, \mathcal{S})$ be an S-metric space, T be a Khan-type F^S_c-contraction with $x_0 \in X$ and r be defined as in (3.1). If $\mathcal{S}(Tx,Tx,x_0) = r$ for all $x \in C^S_{x_0,r}$ then $C^S_{x_0,r}$ is a fixed circle of T. In particular, T fixes every circle $C^S_{x_0,\rho}$ with $\rho < r$ if $\mathcal{S}(Tx,Tx,x_0) = \rho$ for all $x \in C^S_{x_0,\rho}$.*

Proof. Let $x \in C^S_{x_0,r}$ and $Tx \neq x$. If $r = 0$, then we have $C^S_{x_0,r} = \{x_0\}$ and this is a fixed circle of the self-mapping T by Proposition 3.3.

Assume that $r > 0$. Using the Khan-type F^S_C-contractive property, Proposition 3.3, Lemma 2.1 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(\mathcal{S}(Tx,Tx,x)) \\ &\leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}\right]-t \\ &< F\left[h\frac{\mathcal{S}(Tx,Tx,x)r+r^2}{2r}\right] = F\left[h\frac{\mathcal{S}(Tx,Tx,x)+r}{2}\right] \\ &\leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)+\mathcal{S}(Tx,Tx,x)}{2}\right] = F\left[h\mathcal{S}(Tx,Tx,x)\right] \\ &< F[\mathcal{S}(Tx,Tx,x)], \end{aligned}$$

which is a contradiction. Therefore we have $\mathcal{S}(Tx,Tx,x) = 0$ and so $Tx = x$. Consequently, $C^S_{x_0,r}$ is a fixed circle of T.

By the similar arguments, it is easy to verify that T also fixes any circle $C^S_{x_0,\rho}$ where $\rho < r$. □

Remark 3.3. Notice that, in Theorem 3.3, Khan-type F^S_c-contractive self-mapping T fixes the disc $D^S_{x_0,r}$ if $\mathcal{S}(Tx,Tx,x_0) = \rho$ for all $x \in C^S_{x_0,\rho}$ and each $\rho \leq r$. Therefore, the center of any fixed circle is also fixed by T.

Now we give the following illustrative example.

Example 3.5. Let $X = \left\{e^k : k \in \mathbb{N}\right\}$ and the S-metric be defined as in [14] such that

$$\mathcal{S}(x,y,z) = \left|\ln\frac{x}{y}\right| + \left|\ln\frac{xy}{z^2}\right|$$

for all $x,y,z \in X$ (see Example 2.6 on page 12 in [14]). Let us define the self-mapping $T : X \to X$ as

$$Tx = \begin{cases} ex^2 & , \quad x \in \left\{e^1,e^2,e^3,e^4,e^5,e^6,e^7\right\} \\ x & , \quad \text{otherwise} \end{cases},$$

for all $x \in X$. Then the self-mapping T is a Khan-type F^S_c-contractive self-mapping with $F = -\frac{1}{\sqrt{x}}$, $t = \frac{1}{8} - \frac{1}{4\sqrt{5}}$ and $x_0 = e^{23}$. Indeed, in the case $\mathcal{S}(Tx,Tx,x) > 0$, we find

$$\mathcal{S}(Tx,Tx,x) \in \{4,6,8,10,12,14,16\}$$

and

$$20 < h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)},$$

where $h = \frac{20}{21}$. Then we have

$$t + F\left(\mathcal{S}(Tx,Tx,x)\right) \leq F\left[h\frac{\mathcal{S}(Tx,Tx,x)\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx_0,Tx_0,x)\mathcal{S}(Tx,Tx,x_0)}{\mathcal{S}(Tx_0,Tx_0,x)+\mathcal{S}(Tx,Tx,x_0)}\right].$$

We obtain

$$r = \inf\{\mathcal{S}(Tx, Tx, x) : x \neq Tx\} = 4$$

and therefore, the self-mapping T fixes the circle $C^S_{e^{23},4} = \{e^{21}, e^{25}\}$ and the disc $D^S_{e^{23},4} = \{e^{21}, e^{22}, e^{23}, e^{24}, e^{25}\}$.

4. Fixed-Circle Theorems via Auxiliary Functions

In this section, we investigate the existence and uniqueness theorems for fixed circles of self-mappings using some auxiliary functions. Let $r > 0$ be any real number. We consider the function $\varphi_r : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ defined as

$$\varphi_r(u) = \begin{cases} u - r & , \quad u > 0 \\ 0 & , \quad u = 0 \end{cases}, \tag{4.1}$$

for all $u \in \mathbb{R}^+ \cup \{0\}$ [12]. Using the function φ_r we give the following theorem.

Theorem 4.1. *Let $(X, \mathcal{S})$ be an S-metric space and $C^S_{x_0,r}$ be any circle on X. Consider the function φ_r defined in (4.1). If there exists a self-mapping $T : X \to X$ satisfying the conditions*

1. *$\mathcal{S}(Tx, Tx, x_0) = r$ for each $x \in C^S_{x_0,r}$,*
2. *$\mathcal{S}(Tx, Tx, Ty) > r$ for each $x, y \in C^S_{x_0,r}$ and $x \neq y$,*
3. *$\mathcal{S}(Tx, Tx, Ty) \leq \mathcal{S}(x, x, y) - \varphi_r(\mathcal{S}(x, x, Tx))$ for each $x, y \in C^S_{x_0,r}$,*

then the circle $C^S_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C^S_{x_0,r}$ be an arbitrary point. By the condition (1), we have $Tx \in C^S_{x_0,r}$ for all $x \in C^S_{x_0,r}$. Now we prove that x is a fixed point of T. On the contrary, let us assume that $Tx \neq x$. Taking $y = Tx$ and using the condition (2), we find

$$\mathcal{S}(Tx, Tx, T^2x) > r. \tag{4.2}$$

Using the condition (3), we have

$$\begin{aligned} \mathcal{S}(Tx, Tx, T^2x) &\leq \mathcal{S}(x, x, Tx) - \varphi_r(\mathcal{S}(x, x, Tx)) \\ &= \mathcal{S}(x, x, Tx) - \mathcal{S}(x, x, Tx) + r = r. \end{aligned} \tag{4.3}$$

Combining the inequalities (4.2) and (4.3), we get a contradiction. Hence it should be $Tx = x$. Consequently, the circle $C^S_{x_0,r}$ is a fixed circle of T. □

Remark 4.1. Notice that the condition (1) in Theorem 4.1 guarantees that Tx is on the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$, the condition (2) shows that the distance of the images of any two elements on the circle $C^S_{x_0,r}$ can not be less than (or equal to) r.

Now we give an example of a self-mapping which has a fixed-circle on an S-metric space.

Example 4.1. Let $X = \mathbb{R}$ and the metric function $d : X^2 \to [0, \infty)$ be defined by

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ |x| + |y| & , \quad x \neq y \end{cases},$$

for all $x, y \in X$. Let us consider the S-metric defined in Example 2.2. The circle $C^S_{\frac{1}{2},1} = \left\{x \in X : \mathcal{S}(x, x, \frac{1}{2}) = 1\right\} = \{0\}$. If we consider the self-mapping $T_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$T_1 x = \begin{cases} 4 & , \quad x = \frac{1}{2} \\ 0 & , \quad x \neq \frac{1}{2} \end{cases},$$

for all $x \in \mathbb{R}$ then the self-mapping T_4 satisfies the conditions of Theorem 4.1 and T_4 fixes the circle $C^S_{\frac{1}{2},1}$.

In the following example, we see that the converse statement of Theorem 4.1 is not always true.

Example 4.2. Let $X = \mathbb{C}$ and consider the S-metric defined in Example 2.6. Let us consider the circle $C^S_{0,\frac{1}{3}}$ and define the self-mapping $T_2 : \mathbb{C} \to \mathbb{C}$

$$T_2 z = \begin{cases} \frac{1}{9\overline{z}} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$, where $\overline{z}$ denotes the complex conjugate of the complex number z. Clearly, we have $T_2(C^S_{0,\frac{1}{3}}) = (C^S_{0,\frac{1}{3}})$. It can be easily checked that the self mapping T_2 does not satisfy the condition (2) of Theorem 4.1. But, an easy computation shows that T_2 fixes the circle $C^S_{0,\frac{1}{3}}$.

In the following example we see that the circle need not to be fixed even if $T(C^S_{x_0,r}) = C^S_{x_0,r}$.

Example 4.3. Let $(\mathbb{C}, \mathcal{S})$ be the usual S-metric space. Let us consider the circle $C^S_{0,\frac{1}{8}}$ and define the self-mapping $T_3 : \mathbb{C} \to \mathbb{C}$ as

$$T_3 z = \begin{cases} \frac{1}{16z} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then we have $T_3(C^S_{0,\frac{1}{8}}) = C^S_{0,\frac{1}{8}}$. But the self-mapping T_3 does not satisfy the conditions (2) and (3) of Theorem 4.1. Clearly, the circle $C^S_{0,\frac{1}{8}}$ is not a fixed circle of T_3 since $T_3(\frac{i}{4}) = -\frac{i}{4}$ and $T_3(-\frac{i}{4}) = \frac{i}{4}$. More precisely, T_3 fixes only the points $\frac{1}{4}$ and $-\frac{1}{4}$ on the circle $C^S_{0,\frac{1}{8}}$.

In the following example we see that a self mapping can be fix more than one circle.

Example 4.4. Let $X=\mathbb{R}$ and $(X,\mathcal{S})$ be the S-metric space defined in Example 2.6. Let us consider the circles $C_{0,4}^{S}$ and $C_{6,2}^{S}$ and the self-mapping $T_4:\mathbb{R}\to\mathbb{R}$ as

$$T_4x=\begin{cases}\frac{2x+4}{x+5} & , \quad x\in(-\infty,4)\\ \frac{17x+56}{24} & , \quad x\in(4,\infty)\\ 4 & , \quad x=4\end{cases},$$

for all $x\in\mathbb{R}$. It can be easily checked that the self-mapping T_4 satisfies the conditions of Theorem 4.1 and that both of the circles $C_{0,4}^{S}$ and $C_{6,2}^{S}$ are the fixed circles of T_4.

Now we give another existence theorem for fixed circles.

Theorem 4.2. *Let $(X,\mathcal{S})$ be an S-metric space and $C_{x_0,r}^{S}$ be any circle on X. Let us define the mapping*

$$\varphi:X\to[0,\infty)\,,\varphi(x)=\mathcal{S}\left(x,x,x_0\right),$$

for all $x\in X$. If there exists a self-mapping $T:X\to X$ satisfying

1. *$\mathcal{S}(x,x,Tx)\leq\max\{\varphi(x),\varphi(Tx)\}-r$,*
2. *$\mathcal{S}(Tx,Tx,x_0)-h\mathcal{S}(x,x,Tx)\leq r$,*

for all $x\in C_{x_0,r}^{S}$ and $h\in[0,1)$, then $C_{x_0,r}^{S}$ is a fixed circle of T.

Proof. Let $x\in C_{x_0,r}^{S}$. On the contrary, assume that $Tx\neq x$. Then we have the following cases:

Case 1. If $\max\{\varphi(x),\varphi(Tx)\}=\varphi(x)$ then using the condition (1) we have

$$\mathcal{S}(x,x,Tx)\leq\max\{\varphi(x),\varphi(Tx)\}-r=\varphi(x)-r=r-r=0$$

and so $\mathcal{S}(x,x,Tx)=0$, a contradiction. Hence we get $Tx=x$.

Case 2. If $\max\{\varphi(x),\varphi(Tx)\}=\varphi(Tx)$ then we obtain

$$\mathcal{S}(x,x,Tx)\leq\max\{\varphi(x),\varphi(Tx)\}-r=\varphi(Tx)-r,$$

and using the condition (2) we find

$$\mathcal{S}(x,x,Tx)\leq\varphi(Tx)-r\leq h\mathcal{S}(x,x,Tx)+r-r=h\mathcal{S}(x,x,Tx),$$

a contradiction since $h\in[0,1)$. Hence we get $Tx=x$.

Consequently, $C_{x_0,r}^{S}$ is a fixed circle of T. $\square$

Remark 4.2. (1) Notice that the condition (1) in Theorem 4.2 guarantees that Tx is not in the interior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Similarly the condition (2) guarantees that Tx is not exterior of the circle $C^S_{x_0,r}$ for $x \in C^S_{x_0,r}$. Hence $Tx \in C^S_{x_0,r}$ for each $x \in C^S_{x_0,r}$.

(2) Notice that the conditions of Theorem 4.2 are satisfied by the self-mapping T_2.

Now we give the following example.

Example 4.5. Let $X = \mathbb{R}$ be the S-metric space with the usual S-metric defined in Example 2.1. Let us consider the circle $C^S_{0,8}$ and define the self-mapping $T_5 : \mathbb{R} \to \mathbb{R}$ as

$$T_5 x = \begin{cases} 2 & , \quad x \in \left\{-\frac{8}{\sqrt{3}}, 2\right\} \\ \frac{8x+16\sqrt{3}}{\sqrt{3}x+8} & , \quad x \in \mathbb{R} \backslash \left\{-\frac{8}{\sqrt{3}}, 2\right\} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_5 satisfies the conditions (1) and (2) in Theorem 4.2. Hence $C^S_{0,8}$ is a fixed circle of T_5. Notice that $C^S_{3,2}$ is another fixed circle of T_5 and so the number of the fixed circles need not to be unique for a giving self-mapping.

Now, in the following example, we give an example of a self-mapping which satisfies the condition (1) and does not satisfy the condition (2) of Theorem 4.2.

Example 4.6. Let $X = \mathbb{R}$ and the S-metric be defined as in Example 2.6. Let us consider the circle $C^S_{0,6}$ and define the self-mapping $T_6 : \mathbb{R} \to \mathbb{R}$ as

$$T_6 x = \begin{cases} \frac{4x+48\sqrt{3}}{\sqrt{3}x+3} & , \quad x \in (-7, 7) \\ 20 & , \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T_6 satisfies the conditions (1) but does not satisfy the conditions (2) in Theorem 4.2. Consequently $C^S_{0,6}$ is not a fixed circle of T_6.

In the following, we give an example of a self-mapping which satisfies the condition (2) and does not satisfy the condition (1) in Theorem 4.2.

Example 4.7. Let $X = \mathbb{C}$ be the S-metric space with the usual S-metric defined in Example 2.1. Let us consider the circle $C^S_{0,12}$ and define the self-mapping $T_7 : \mathbb{C} \to \mathbb{C}$ as

$$T_7 z = \begin{cases} \frac{Re(z)}{2} & if \quad Re(z) \geq 0 \\ -\frac{Re(z)}{2} & if \quad Re(z) < 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then the self-mapping T_7 satisfies the condition (2) and does not satisfy the condition (1) in Theorem 4.2.

Now we use the following corollaries to obtain a uniqueness theorem for fixed circles of self-mappings.

Corollary 4.1. *[22] Let $(X, \mathcal{S})$ be a complete S-metric space and T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y), \tag{4.4}$$

for some $a, b, c \geq 0, a + b + c < 1$, and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$ then T is continuous at the fixed point.

Corollary 4.2. *[22] Let $(X,\mathcal{S})$ be a complete S-metric space and T be a self-mapping of X, and*

$$S(Tx,Tx,Ty) \leq h\max\{S(Tx,Tx,y),S(Ty,Ty,x)\}, \tag{4.5}$$

for some $h \in \left[0,\frac{1}{3}\right)$ and all $x,y \in X$. Then T has a unique fixed point in X. Moreover, T is continuous at the fixed point.

We give the following theorem.

Theorem 4.3. *Let $(X,\mathcal{S})$ be an S-metric space and $T : X \to X$ be a self-mapping with the fixed circle $C^S_{x_0,r}$. If one of the contractive conditions (4.4) or (4.5) is satisfied for all $x \in C^S_{x_0,r}$, $y \in X \setminus C^S_{x_0,r}$ by T then $C^S_{x_0,r}$ is the unique fixed circle of T.*

Proof. Assume that there exists two fixed circles $C^S_{x_0,r}$ and $C^S_{x_0,\rho}$ of the self-mapping T. Let $x \in C^S_{x_0,r}$ and $y \in C^S_{x_0,\rho}$ be arbitrary points with $x \neq y$. If the contractive condition (4.4) is satisfied by T , then we obtain

$$\begin{aligned} S(x,x,y) &= S(Tx,Tx,Ty) \leq aS(x,x,y) + bS(Tx,Tx,x) + cS(Ty,Ty,y) \\ &= aS(x,x,y), \end{aligned}$$

which is a contradiction since $a+b+c < 1$. Hence it should be $x = y$. Consequently $C^S_{x_0,r}$ is the unique fixed circle of T. Similarly, if the contractive condition (4.5) is satisfied by T then we get

$$S(x,x,y) = S(Tx,Tx,Ty) \leq h\max\{S(Tx,Tx,y),S(Ty,Ty,x)\} = hS(x,x,y),$$

which is a contradiction since $h \in \left[0,\frac{1}{3}\right)$. Hence it should be $x = y$. Consequently $C^S_{x_0,r}$ is the unique fixed circle of T. □

Now we consider the identity map $I_X : X \to X$ defined as $I_X(x) = x$ for all $x \in X$. We note that the identity map satisfies the conditions of Theorem 4.2 but can not satisfy the condition (2) of Theorem 4.1 everywhen. Therefore, we investigate a condition which excludes the identity map in Theorem 4.2 (resp. Theorem 4.1). For this purpose, we obtain the following theorem.

Theorem 4.4. *Let $(X,\mathcal{S})$ be an S-metric space, $T : X \to X$ be a self mapping having a fixed circle $C^S_{x_0,r}$ and the mapping φ_r be defined as in (4.1). The self-mapping $T : X \to X$ satisfies the condition*

$$\mathcal{S}(x,x,Tx) < \varphi_r(\mathcal{S}(x,x,Tx)) + r, \tag{4.6}$$

for all $x \in X$ if and only if $T = I_X$.

Proof. Let $x \in X$ be any point and assume that $Tx \neq x$. Then using the inequality (4.6), we get

$$\mathcal{S}(x,x,Tx) < \varphi_r\left(\mathcal{S}(x,x,Tx)\right) + r = \mathcal{S}(x,x,Tx) - r + r,$$

which is a contradiction. Hence we have $Tx = x$ and $T = I_X$.

Conversely, it is clear that the identity map I_X satisfies the condition (4.6). □

REFERENCES

1. H. Alolaiyan, B. Ali and M. Abbas: *Characterization of a b-metric space completeness via the existence of a fixed point of Ciric-Suzuki type quasi-contractive multivalued operators and applications.* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **27(1)** (2019), 5–33.
2. F. Bojor: *Fixed points of Kannan mappings in metric spaces endowed with a graph.* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **20(1)** (2012), 31–40.
3. Lj. B. Ciric: *A generalization of Banach's contraction principle.* Proc. Amer. Math. Soc. **45** (1974), 267–273.
4. S. Chaipornjareansri: *Fixed point theorems for Fw-contractions in complete s-metric spaces.* Thai J. Math. **14** (2016), Special issue, 98–109.
5. A. Fulga and E. Karapinar: *Revisiting of some outstanding metric fixed point theorems via E-contraction.* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **26(3)** (2018), 73–98.
6. A. Gupta: *Cyclic contraction on S-metric space.* Int. J. Anal. Appl. **3(2)** (2013), 119–130.
7. N. T. Hieu, N. T. Thanh Ly and N. V. Dung: *A generalization of Ćirić quasi-contractions for maps on S-metric spaces.* Thai J. Math. **13(2)** (2015), 369–380.
8. E. Karapinar, A. F. Roldán-López-de-Hierro and S. Bessem: *Matkowski theorems in the context of quasi-metric spaces and consequences on G-metric spaces.* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **24(1)** (2016), 309–333.
9. N. Mlaiki: *α-ψ-contractive mapping on S-metric space.* Math. Sci. Lett. **4(1)** (2015), 9–12.
10. N. Mlaiki, U. Çelik, N. Taş, N. Y. Özgür and A. Mukheimer: *Wardowski type contractions and the fixed-circle problem on S-metric spaces.* J. Math. (2018), Art. ID 9127486, 9 pp.
11. N. Mlaiki, N. Taş and N. Y. Özgür: *On the fixed-circle problem and Khan type contractions.* Axioms **7(4)** (2018), 80.
12. N. Y. Özgür and N. Taş: *Some fixed-circle theorems and discontinuity at fixed circle.* In: AIP Conference Proceedings **1926(1)**, AIP Publishing LLC, 2018, pp. 020048.
13. N. Y. Özgür and N. Taş: *Some new contractive mappings on S-metric spaces and their relationships with the mapping* ($S25$). Math. Sci. (Springer) **11(1)** (2017), 7–16.

14. N. Y. Özgür N. Taş and U. Çelik: *New fixed-circle results on S-metric spaces.* Bull. Math. Anal. Appl. **9(2)** (2017), 10–23.
15. N. Y. Özgür and N. Taş: *Some generalizations of fixed point theorems on S-metric spaces.* Essays in Mathematics and Its Aplications in Honor of Vladimir Arnold, New York, Springer, 2016.
16. N. Y. Özgür and N. Taş: *Fixed-circle problem on S-metric spaces with a geometric viewpoint.* Facta Univ. Ser. Math. Inform. **34(3)** (2019), 459–472.
17. N. Y. Özgür and N. Taş: *Some fixed-circle theorems on metric spaces.* Bull. Malays. Math. Sci. Soc. **42(4)** (2019), 1433–1449.
18. N. Y. Özgür and N. Taş: *Some fixed point theorems on S-metric spaces.* Mat. Vesnik **69(1)** (2017), 39–52.
19. N. Y. Özgür and N. Taş: *The Picard theorem on S-metric spaces.* Acta Math. Sci. Ser. B (Engl. Ed.) **38(4)** (2018), 1245–1258.
20. R. P. Pant, N. Y. Özgür and N. Taş: *On Discontinuity Problem at Fixed Point.* Bull. Malays. Math. Sci. Soc. **43(1)** (2020), 499–517.
21. S. Sedghi, N. Shobe and A. Aliouche: *A generalization of fixed point theorems in S-metric spaces.* Mat. Vesnik **64(3)** (2012), 258–266.
22. S. Sedghi and N. V. Dung: *Fixed point theorems on S-metric spaces.* Mat. Vesnik **66(1)** (2014), 113–124.
23. N. Souayah: *A fixed point in partial S_b-metric spaces.* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **24(3)** (2016), 351–362.
24. N. Taş: *Suzuki-Berinde type fixed-point and fixed-circle results on S-metric spaces.* J. Linear Topol. Algebra **7(3)** (2018), 233–244.
25. N. Taş: *Various types of fixed-point theorems on S-metric spaces.* J. BAUN Inst. Sci. Technol. **20(2)** (2018), 211–223.
26. N. Taş, N. Y. Özgür and N. Mlaiki: *New types of F_C-contractions and the fixed-circle problem.* Mathematics, **6(10)** (2018), 188.
27. N. Taş and N. Y. Özgür: *Common fixed points of continuous mappings on S-metric spaces.* Math. Notes. **104(3-4)** (2018), 587–600.
28. D. Wardowski: *Fixed points of a new type of contractive mappings in complete metric spaces.* Fixed Point Theory Appl. **2012**, (2012):94, 6 pp.

Ufuk Çelik
Faculty of Arts and Sciences
Department of Mathematics
10145 Balıkesir, Turkey
ufuk.celik@baun.edu.tr

Nihal Özgür
Faculty of Arts and Sciences
Department of Mathematics
10145 Balıkesir, Turkey
nihal@balikesir.edu.tr

FACTA UNIVERSITATIS (NIŠ)
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ON PSEUDO-HERMITIAN MAGNETIC CURVES IN SASAKIAN MANIFOLDS

Şaban Güvenç and Cihan Özgür

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Abstract. We define pseudo-Hermitian magnetic curves in Sasakian manifolds endowed with the Tanaka-Webster connection. After we have given a complete classification theorem, we shall construct parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$.

Keywords: magnetic curve; slant curve; Sasakian manifold; the Tanaka-Webster connection.

1. Introduction

The study of the motion of a charged particle in a constant and time-independent static magnetic field on a Riemannian surface is known as the Landau–Hall problem [16]. The main problem is to study the movement of a charged particle moving in the Euclidean plane $\mathbb{E}^2$. The solution of the Lorentz equation (called also the Newton equation) corresponds to the motion of the particle. The trajectory of a charged particle moving on a Riemannian manifold under the action of the magnetic field is a very interesting problem from a geometric point of view [16].

Let (N, g) be a Riemannian manifold, and F a closed 2-form, Φ the Lorentz force, which is a $(1,1)$-type tensor field on N. F is called a *magnetic field* if it is associated to Φ by the relation

$$F(X,Y) = g(\Phi X, Y), \tag{1.1}$$

where X and Y are vector fields on N (see [1], [3] and [8]). Let ∇ be the Riemannian connection on N and consider a differentiable curve $\alpha : I \to N$, where I denotes an open interval of $\mathbb{R}$. α is said to be a *magnetic curve* for the magnetic field F, if it is a solution of the Lorentz equation given by

$$\nabla_{\alpha'(t)}\alpha'(t) = \Phi(\alpha'(t)). \tag{1.2}$$

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From the definition of magnetic curves, it is straightforward to see that their speed is constant. Specifically, unit-speed magnetic curves are called *normal magnetic curves* [9].

In [9], Druţă-Romaniuc, Inoguchi, Munteanu and Nistor studied magnetic curves in a Sasakian manifold. Magnetic curves in cosymplectic manifolds were studied in [10] by the same authors. In [13], 3-dimensional Berger spheres and their magnetic curves were considered by Inoguchi and Munteanu. Magnetic trajectories of an almost contact metric manifold were studied in [14], by Jleli, Munteanu and Nistor. The classification of all uniform magnetic trajectories of a charged particle moving on a surface under the action of a uniform magnetic field was obtained in [19], by Munteanu. Furthermore, normal magnetic curves in para-Kaehler manifolds were researched in [15], by Jleli and Munteanu. In [17], Munteanu and Nistor obtained the complete classification of unit-speed Killing magnetic curves in $\mathbb{S}^2\times\mathbb{R}$. Moreover, in [18], they studied magnetic curves on $\mathbb{S}^{2n+1}$. 3-dimensional normal para-contact metric manifolds and their magnetic curves of a Killing vector field were investigated in [5], by Calvaruso, Munteanu and Perrone. In [20], the present authors studied slant curves in contact Riemannian 3-manifolds with pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian harmonic mean curvature vector field for the Tanaka-Webster connection in the tangent and normal bundles, respectively. The second author gave the parametric equations of all normal magnetic curves in the 3-dimensional Heisenberg group in [21]. Recently, the present authors have also considered slant magnetic curves in S-manifolds in [11].

These studies motivate us to investigate pseudo-Hermitian magnetic curves in $(2n+1)$-dimensional Sasakian manifolds endowed with the Tanaka-Webster connection. In Section 2, we summarize the fundamental definitions and properties of Sasakian manifolds and the unique connection, namely the Tanaka-Webster connection. We give the main classification theorems for pseudo-Hermitian magnetic curves in Section 3. We show that a pseudo-Hermitian magnetic curve cannot have osculating order greater than 3. In the last section, after a brief information on $\mathbb{R}^{2n+1}(-3)$, we obtain the parametric equations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection.

2. Preliminaries

Let N be a $(2n+1)$-dimensional Riemannian manifold satisfying the following equations

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta\circ\phi = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \quad g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y on N, where ϕ is a $(1,1)$-type tensor field, η is a 1-form, ξ is a vector field and g is a Riemannian metric on N. In this case, (N,ϕ,ξ,η,g) is said to be an *almost contact metric manifold* [2]. Moreover, if $d\eta(X,Y) = \Phi(X,Y)$, where $\Phi(X,Y) = g(X,\phi Y)$ is the *fundamental 2-form* of the manifold, then N is said to be a *contact metric manifold* [2].

Furthermore, if we denote the Nijenhuis torsion of ϕ by $[\phi,\phi]$, for all $X, Y \in \chi(N)$, the condition given by

$$[\phi,\phi](X,Y) = -2d\eta(X,Y)\xi$$

is called the *normality condition* of the almost contact metric structure. An almost contact metric manifold turns into a *Sasakian manifold* if the normality condition is satisfied [2].

From Lie differentiation operator in the characteristic direction ξ, the operator h is defined by

$$h = \frac{1}{2}L_\xi\phi.$$

It is directly found that the structural operator h is symmetric. It also validates the equations below, where we denote the Levi-Civita connection by ∇:

$$h\xi = 0, \quad h\phi = -\phi h, \quad \nabla_X\xi = -\phi X - \phi hX, \tag{2.3}$$

(see [2]).

If we denote the Tanaka-Webster connection on N by $\widehat{\nabla}$ ([22], [24]), then we have

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + (\widehat{\nabla}_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi$$

for all vector fields X, Y on N. By the use of equations (2.3), the Tanaka-Webster connection can be calculated as

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi. \tag{2.4}$$

The torsion of the Tanaka-Webster connection is

$$\widehat{T}(X,Y) = 2g(X,\phi Y)\xi + \eta(Y)\phi hX - \eta(X)\phi hY. \tag{2.5}$$

In a Sasakian manifold, from the fact that $h = 0$ (see [2]), the equations (2.4) and (2.5) can be rewritten as:

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{2.6}$$

$$\widehat{T}(X,Y) = 2g(X,\phi Y)\xi.$$

The following proposition states why the Tanaka-Webster connection is unique:

Proposition 2.1. [23] *The Tanaka-Webster connection on a contact Riemannian manifold $N = (N,\phi,\xi,\eta,g)$ is the unique linear connection satisfying the following four conditions:*

(a) $\widehat{\nabla}\eta = 0$, $\widehat{\nabla}\xi = 0$;

(b) $\widehat{\nabla}g = 0$, $\widehat{\nabla}\phi = 0$;

(c) $\widehat{T}(X,Y) = -\eta([X,Y])\xi, \quad \forall X, Y \in D$;

(d) $\widehat{T}(\xi,\phi Y) = -\phi\widehat{T}(\xi,Y), \ \forall Y \in D$.

3. Magnetic Curves with respect to the Tanaka-Webster Connection

Let (N, ϕ, ξ, η, g) be an n-dimensional Riemannian manifold and $\alpha : I \to N$ a curve parametrized by arc-length. If there exists g-orthonormal vector fields $E_1, E_2, ..., E_r$ along α such that

$$\begin{aligned} E_1 &= \alpha', \\ \widehat{\nabla}_{E_1} E_1 &= \widehat{k}_1 E_2, \\ \widehat{\nabla}_{E_1} E_2 &= -\widehat{k}_1 E_1 + \widehat{k}_2 E_3, \\ &... \\ \widehat{\nabla}_{E_1} E_r &= -\widehat{k}_{r-1} E_{r-1}, \end{aligned} \tag{3.1}$$

then α is called a *Frenet curve for* $\widehat{\nabla}$ *of osculating order* r , $(1 \leq r \leq n)$. Here $\widehat{k}_1, ..., \widehat{k}_{r-1}$ are called *pseudo-Hermitian curvature functions of* α and these functions are positive valued on I. A *geodesic for* $\widehat{\nabla}$ (or *pseudo-Hermitian geodesic*) is a Frenet curve of osculating order 1 for $\widehat{\nabla}$. If $r = 2$ and $\widehat{k}_1$ is a constant, then α is called a *pseudo-Hermitian circle.* A *pseudo-Hermitian helix of order* r $(r \geq 3)$ is a Frenet curve for $\widehat{\nabla}$ of osculating order r with non-zero positive constant pseudo-Hermitian curvatures $\widehat{k}_1, ..., \widehat{k}_{r-1}$. If we shortly state *pseudo-Hermitian helix,* we mean its osculating order is 3 [7].

Let $N = \left(N^{2n+1}, \phi, \xi, \eta, g\right)$ be a Sasakian manifold endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Let us denote the fundamental 2-form of N by Ω. Then, we have

$$\Omega(X, Y) = g(X, \phi Y), \tag{3.2}$$

(see [2]). From the fact that N is a Sasakian manifold, we have $\Omega = d\eta$. Hence, $d\Omega = 0$, i.e., it is closed. Thus, we can define a magnetic field F_q on N by

$$F_q(X, Y) = q\Omega(X, Y),$$

namely the *contact magnetic field with strength* q, where $X, Y \in \chi(N)$ and $q \in \mathbb{R}$ [14]. We will assume that $q \neq 0$ to avoid the absence of the strength of magnetic field (see [4] and [9]).

From (1.1) and (3.2), the Lorentz force Φ associated to the contact magnetic field F_q can be written as

$$\Phi = -q\phi.$$

So the Lorentz equation (1.2) is

$$\nabla_{E_1} E_1 = -q\phi E_1, \tag{3.3}$$

where $\alpha : I \to N$ is a curve with arc-length parameter, $E_1 = \alpha'$ is the tangent vector field and ∇ is the Levi-Civita connection (see [9] and [14]). By the use of equations (2.6) and (3.3), we have

$$\widehat{\nabla}_{E_1} E_1 = \left[-q + 2\eta(E_1)\right] \phi E_1. \tag{3.4}$$

Definition 3.1. Let $\alpha : I \to N$ be a unit-speed curve in a Sasakian manifold $N = (N^{2n+1}, \phi, \xi, \eta, g)$ endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Then it is called a *normal magnetic curve with respect to the Tanaka-Webster connection* $\widehat{\nabla}$ (or shortly a *pseudo-Hermitian magnetic curve*) if it satisfies equation (3.4).

If $\eta(E_1) = \cos\theta$ is a constant, then α is called a *slant curve* [6]. From the definition of pseudo-Hermitian magnetic curves, we have the following direct result as in the Levi-Civita case:

Proposition 3.1. *If α is a pseudo-Hermitian magnetic curve in a Sasakian manifold, then it is a slant curve.*

Proof. Let $\alpha : I \to N$ be a pseudo-Hermitian magnetic curve. Then, we find

$$\begin{aligned} \frac{d}{dt} g(E_1, \xi) &= g(\widehat{\nabla}_{E_1} E_1, \xi) + g(E_1, \widehat{\nabla}_{E_1} \xi) \\ &= g([-q + 2\eta(E_1)]\, \phi E_1, \xi) \\ &= 0. \end{aligned}$$

So we obtain

$$\eta(E_1) = \cos\theta = constant,$$

which completes the proof. □

As a result, we can rewrite equation (3.4) as

$$\widehat{\nabla}_{E_1} E_1 = (-q + 2\cos\theta)\, \phi E_1, \tag{3.5}$$

where θ is the contact angle of α. Now, we can state the following theorem:

Theorem 3.1. *Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Then $\alpha : I \to N$ is a pseudo-Hermitian magnetic curve if and only if it belongs to the following list:*

(a) *pseudo-Hermitian non-Legendre slant geodesics (including pseudo-Hermitian geodesics as integral curves of ξ);*

(b) *pseudo-Hermitian Legendre circles with $\widehat{k}_1 = |q|$ and having the Frenet frame field (for $\widehat{\nabla}$)*

$$\{E_1, -sgn(q)\phi E_1\};$$

(c) *pseudo-Hermitian slant helices with*

$$\widehat{k}_1 = |-q + 2\cos\theta| \sin\theta,\ \widehat{k}_2 = |-q + 2\cos\theta|\, \varepsilon \cos\theta$$

and having the Frenet frame field (for $\widehat{\nabla}$)

$$\left\{E_1, \frac{\delta}{\sin\theta}\phi E_1, \frac{\varepsilon}{\sin\theta}(\xi - \cos\theta E_1)\right\},$$

where $\delta = sgn(-q + 2\cos\theta)$, $\varepsilon = sgn(\cos\theta)$ and $\cos\theta \neq \frac{q}{2}$.

Proof. Let us assume that $\alpha : I \to N$ is a normal magnetic curve with respect to $\widehat{\nabla}$. Consequently, equation (3.5) must be validated. Let us assume $\widehat{k}_1 = 0$. Hence, we have $\cos\theta = \frac{q}{2}$ or $\phi E_1 = 0$. If $\cos\theta = \frac{q}{2}$, then α is a pseudo-Hermitian non-Legendre slant geodesic. Otherwise, $\phi E_1 = 0$ gives us $E_1 = \pm\xi$. Thus, α is a pseudo-Hermitian geodesic as an integral curve of $\pm\xi$. So we have just proved that α belongs to (a) from the list, if the osculating order $r = 1$. Now, let $\widehat{k}_1 \neq 0$. From equation (3.5) and the Frenet equations for $\widehat{\nabla}$, we find

$$\widehat{\nabla}_{E_1} E_1 = \widehat{k}_1 E_2 = (-q + 2\cos\theta)\,\phi E_1. \tag{3.6}$$

Since E_1 is unit, the equation (2.2) gives us

$$g(\phi E_1, \phi E_1) = \sin^2\theta. \tag{3.7}$$

By the use of (3.6) and (3.7), we obtain

$$\widehat{k}_1 = |-q + 2\cos\theta|\sin\theta, \tag{3.8}$$

which is a constant. Let us denote $\delta = sgn(-q + 2\cos\theta)$. From (3.8), we can write

$$\phi E_1 = \delta \sin\theta E_2. \tag{3.9}$$

Let us assume $\widehat{k}_2 = 0$, that is, $r = 2$. From the fact that $\widehat{k}_1$ is a constant, α is a pseudo-Hermitian circle. (3.9) gives us

$$\eta(\phi E_1) = 0 = \delta \sin\theta \eta(E_2),$$

which is equivalent to

$$\eta(E_2) = 0.$$

Differentiating this last equation with respect to $\widehat{\nabla}$, we obtain

$$\widehat{\nabla}_{E_1}\eta(E_2) = 0 = g\left(\widehat{\nabla}_{E_1} E_2, \xi\right) + g\left(E_2, \widehat{\nabla}_{E_1}\xi\right).$$

Since $\widehat{\nabla}\xi = 0$ and $r = 2$, we have

$$g(-\widehat{k}_1 E_1, \xi) = 0,$$

that is, $\eta(E_1) = 0$. Hence, α is Legendre and $\cos\theta = 0$. From equation (3.8), we get $\widehat{k}_1 = |q|$. In this case, we also obtain $\delta = -sgn(q)$ and $E_2 = -sgn(q)\phi E_1$. We have proved that α belongs to (b) from the list, if the osculating order $r = 2$. Now, let us assume $\widehat{k}_2 \neq 0$. If we use $\widehat{\nabla}\phi = 0$, we calculate

$$\widehat{\nabla}_{E_1}\phi E_1 = \widehat{k}_1 \phi E_2. \tag{3.10}$$

From (2.1) and (3.9), we find

$$\phi^2 E_1 = -E_1 + \cos\theta\xi = \delta \sin\theta\phi E_2, \tag{3.11}$$

which gives us

$$\phi E_2 = \frac{\delta}{\sin\theta}\left(-E_1 + \cos\theta\xi\right).$$

So equation (3.10) becomes

$$\widehat{\nabla}_{E_1}\phi E_1 = \widehat{k}_1 \frac{\delta}{\sin\theta}\left(-E_1 + \cos\theta\xi\right). \tag{3.12}$$

If we differentiate the equation (3.9) with respect to $\widehat{\nabla}$, we also have

$$\begin{aligned}\widehat{\nabla}_{E_1}\phi E_1 &= \delta\sin\theta\widehat{\nabla}_{E_1}E_2 \\ &= \delta\sin\theta\left(-\widehat{k}_1E_1 + \widehat{k}_2E_3\right).\end{aligned} \tag{3.13}$$

By the use of (3.12) and (3.13), we obtain

$$\widehat{k}_1\cot\theta\left(\xi - \cos\theta E_1\right) = \widehat{k}_2\sin\theta E_3. \tag{3.14}$$

One can easily see that

$$g(\xi - \cos\theta E_1, \xi - \cos\theta E_1) = \sin^2\theta.$$

From (3.14), we calculate

$$\widehat{k}_2 = \left|-q + 2\cos\theta\right|\varepsilon\cos\theta,$$

where we denote $\varepsilon = sgn(\cos\theta)$. As a result, we get

$$E_3 = \frac{\varepsilon}{\sin\theta}\left(\xi - \cos\theta E_1\right), \tag{3.15}$$

$$E_2 = \frac{\delta}{\sin\theta}\phi E_1.$$

If we differentiate (3.15) with respect to $\widehat{\nabla}$, since $\phi E_1 \parallel E_2$, we find $\widehat{k}_3 = 0$. So we have just completed the proof of (c). Considering the fact that $\widehat{k}_3 = 0$, the Gram-Schmidt process ends. Thus, the list is complete.

Conversely, let $\alpha : I \to N$ belong to the given list. It is easy to show that equation (3.5) is satisfied. Hence, α is a pseudo-Hermitian magnetic curve. □

A pseudo-Hermitian geodesic is said to be a pseudo-Hermitian ϕ-curve if the set $sp\{E_1, \phi E_1, \xi\}$ is ϕ-invariant. A Frenet curve of osculating order $r = 2$ is said to be a pseudo-Hermitian ϕ-curve if $sp\{E_1, E_2, \xi\}$ is ϕ-invariant. A Frenet curve of osculating order $r \geq 3$ is said to be a pseudo-Hermitian ϕ-curve if $sp\{E_1, E_2, ..., E_r\}$ is ϕ-invariant.

Theorem 3.2. *Let $\alpha : I \to N$ be a pseudo-Hermitian ϕ-helix of order $r \leq 3$, where $N = \left(N^{2n+1}, \phi, \xi, \eta, g\right)$ is a Sasakian manifold endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Then:*

(a) *If* $\cos\theta = \pm 1$, *then it is an integral curve of* ξ, *i.e. a pseudo-Hermitian geodesic and it is a pseudo-Hermitian magnetic curve for* F_q *for arbitrary* q*;*

(b) *If* $\cos\theta \notin \{-1, 0, 1\}$ *and* $\widehat{k}_1 = 0$, *then it is a pseudo-Hermitian non-Legendre slant geodesic and it is a pseudo-Hermitian magnetic curve for* $F_{2\cos\theta}$*;*

(c) *If* $\cos\theta = 0$ *and* $\widehat{k}_1 \neq 0$, *i.e.* α *is a Legendre* ϕ*-curve, then it is a pseudo-Hermitian magnetic circle generated by* $F_{-\delta\widehat{k}_1}$, *where* $\delta = sgn(g(\phi E_1, E_2))$*;*

(d) *If* $\cos\theta = \frac{\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}$ *and* $\widehat{k}_2 \neq 0$, *then it is a pseudo-Hermitian magnetic curve for* $F_{-\delta\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}+\frac{2\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}}$, *where* $\delta = sgn(g(\phi E_1, E_2))$ *and* $\varepsilon = sgn(\cos\theta)$.

(e) *Except above cases,* α *cannot be a pseudo-Hermitian magnetic curve for any* F_q.

Proof. Firstly, let us assume $\cos\theta = \pm 1$, that is, $E_1 = \pm\xi$. As a result, we have

$$\widehat{\nabla}_{E_1} E_1 = 0, \ \phi E_1 = 0.$$

Hence, equation (3.5) is satisfied for arbitrary q. This proves (a). Now, let us take $\cos\theta \notin \{-1, 0, 1\}$ and $\widehat{k}_1 = 0$. In this case, we obtain

$$\widehat{\nabla}_{E_1} E_1 = 0, \ \phi E_1 \neq 0.$$

So equation (3.5) is valid for $q = 2\cos\theta$. The proof of (b) is over. Next, let us assume $\cos\theta = 0$ and $\widehat{k}_1 \neq 0$. One can easily see that α has the Frenet frame field (for $\widehat{\nabla}$)

$$\{E_1, \delta\phi E_1\}$$

where δ corresponds to the sign of $g(\phi E_1, E_2)$. Consequently, we get

$$\widehat{\nabla}_{E_1} E_1 = \delta\widehat{k}_1 \phi E_1,$$

that is, α is a pseudo-Hermitian magnetic curve for $q = -\delta\widehat{k}_1$. We have just proven (c). Finally, let $\cos\theta = \frac{\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}$ and $\widehat{k}_2 \neq 0$. So α has the Frenet frame field (for $\widehat{\nabla}$)

$$\left\{ E_1, \frac{\delta}{\sin\theta}\phi E_1, \frac{\varepsilon}{\sin\theta}(\xi - \cos\theta E_1) \right\},$$

where $\delta = sgn(g(\phi E_1, E_2))$ and $\varepsilon = sgn(\cos\theta)$. After calculations, it is easy to show that equation (3.5) is satisfied for $q = -\delta\sqrt{\widehat{k}_1^2 + \widehat{k}_2^2} + \frac{2\varepsilon\widehat{k}_2}{\sqrt{\widehat{k}_1^2+\widehat{k}_2^2}}$. Hence, the proof of (d) is completed. Except above cases, from Theorem 3.1, α cannot be a pseudo-Hermitian magnetic curve for any F_q. □

4. Parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$

In this section, our aim is to obtain parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$. To do this, we need to recall some notions from [2]. Let $N=\mathbb{R}^{2n+1}$. Let us denote the coordinate functions of N with $(x_1,...,x_n,y_1,...,y_n,z)$. One may define a structure on N by $\eta=\frac{1}{2}(dz-\sum_{i=1}^{n}y_i dx_i)$, which is a contact structure, since $\eta\wedge(d\eta)^n\neq 0$. This contact structure has the characteristic vector field $\xi=2\frac{\partial}{\partial z}$. Let us also consider a $(1,1)$-type tensor field ϕ given by the matrix form as

$$\phi=\begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$

Finally, let us take the Riemannian metric on N given by $g=\eta\otimes\eta+\frac{1}{4}\sum_{i=1}^{n}((dx_i)^2+(dy_i)^2)$. It is known that (N,ϕ,ξ,η,g) is a Sasakian space form and its ϕ-sectional curvature is $c=-3$. This special Sasakian space form is denoted by $\mathbb{R}^{2n+1}(-3)$ [2]. One can easily show that the vector fields

$$X_i=2\frac{\partial}{\partial y_i},\ X_{n+i}=\phi X_i=2(\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial z}),\ i=\overline{1,n},\ \xi=2\frac{\partial}{\partial z} \tag{4.1}$$

are g-unit and g-orthogonal. Hence, they form a g-orthonormal basis [2]. Using this basis, the Levi-Civita connection of $\mathbb{R}^{2n+1}(-3)$ can be obtained as

$$\nabla_{X_i}X_j=\nabla_{X_{m+i}}X_{m+j}=0,\ \nabla_{X_i}X_{m+j}=\delta_{ij}\xi,\ \nabla_{X_{m+i}}X_j=-\delta_{ij}\xi,$$

$$\nabla_{X_i}\xi=\nabla_\xi X_i=-X_{m+i},\ \nabla_{X_{m+i}}\xi=\nabla_\xi X_{m+i}=X_i,$$

(see [2]). As a result, the Tanaka-Webster connection of $\mathbb{R}^{2n+1}(-3)$ is

$$\begin{aligned}\widehat{\nabla}_{X_i}X_j &= \widehat{\nabla}_{X_{m+i}}X_{m+j}=\widehat{\nabla}_{X_i}X_{m+j}=\widehat{\nabla}_{X_{m+i}}X_j=\\ \widehat{\nabla}_{X_i}\xi &= \widehat{\nabla}_\xi X_i=\widehat{\nabla}_{X_{m+i}}\xi=\widehat{\nabla}_\xi X_{m+i}=0,\end{aligned}$$

which was calculated in [12]. Now, we can investigate the parametric equations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection.

Let $N=\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection $\widehat{\nabla}$. Let $\alpha:I\subseteq\mathbb{R}\to N$, $\alpha=(\alpha_1,\alpha_2,...,\alpha_n,\alpha_{n+1},...,\alpha_{2n},\alpha_{2n+1})$ be a pseudo-Hermitian magnetic curve. Then, the tangential vector field of α can be written as

$$E_1=\sum_{i=1}^{n}\alpha_i'\frac{\partial}{\partial x_i}+\sum_{i=1}^{n}\alpha_{n+i}'\frac{\partial}{\partial y_i}+\alpha_{2n+1}'\frac{\partial}{\partial z}.$$

In terms of the g-orthonormal basis, E_1 is rewritten as

$$E_1=\frac{1}{2}\left[\sum_{i=1}^{n}\alpha_{n+i}'X_i+\sum_{i=1}^{n}\alpha_i'X_{n+i}+\left(\alpha_{2n+1}'-\sum_{i=1}^{n}\alpha_i'\alpha_{n+i}\right)\xi\right].$$

From Proposition 3.1, α is a slant curve. Hence, we have

$$\eta(E_1) = \cos\theta = constant,$$

which is equivalent to

$$\alpha'_{2n+1} = 2\cos\theta + \sum_{i=1}^{n} \alpha'_i \alpha_{n+i}. \tag{4.2}$$

From the fact that α is parametrized by arc-length, we also have

$$g(E_1, E_1) = 1,$$

that is,

$$\sum_{i=1}^{2n} (\alpha'_i)^2 = 4\sin^2\theta. \tag{4.3}$$

Differentiating E_1 with respect to $\widehat{\nabla}$, we obtain

$$\widehat{\nabla}_{E_1} E_1 = \frac{1}{2}\left(\sum_{i=1}^{n} \alpha''_{n+i} X_i + \sum_{i=1}^{n} \alpha''_i X_{n+i} \right).$$

We also easily find

$$\phi E_1 = \frac{1}{2}\left(-\sum_{i=1}^{n} \alpha'_i X_i + \sum_{i=1}^{n} \alpha'_{n+i} X_{n+i} \right).$$

Since α is a pseudo-Hermitian magnetic curve, it must satisfy

$$\widehat{\nabla}_{E_1} E_1 = (-q + 2\cos\theta)\, \phi E_1.$$

Then, we can write

$$\frac{\alpha''_{n+1}}{-\alpha'_1} = ... = \frac{\alpha''_{2n}}{-\alpha'_n} = \frac{\alpha''_1}{\alpha'_{n+1}} = ... = \frac{\alpha''_n}{\alpha'_{2n}} = -\lambda,$$

where $\lambda = q - 2\cos\theta$. From the last equations, we can select the pairs

$$\frac{\alpha''_{n+1}}{-\alpha'_1} = \frac{\alpha''_1}{\alpha'_{n+1}}, \; ... \;, \; \frac{\alpha''_{2n}}{-\alpha'_n} = \frac{\alpha''_n}{\alpha'_{2n}}. \tag{4.4}$$

Firstly, let $\lambda \neq 0$. Solving the ODEs, we have

$$(\alpha'_i)^2 + (\alpha'_{n+i})^2 = c_i^2, i = 1, ..., n$$

for some arbitrary constants c_i $(i = 1, ..., n)$ such that

$$\sum_{i=1}^{n} c_i^2 = 4\sin^2\theta.$$

So we have

$$\alpha_i' = c_i \cos f_i, \ \alpha_{n+i}' = c_i \sin f_i$$

for some differentiable functions $f_i : I \to \mathbb{R}$ $(i = 1, ..., n)$. From (4.4), we get

$$\frac{\alpha_{n+i}''}{-\alpha_i'} = -f_i' = -\lambda,$$

which gives us

$$f_i = \lambda t + d_i$$

for some arbitrary constants d_i $(i = 1, ..., n)$. Here, t denotes the arc-length parameter. Then, we find

$$\alpha_i' = c_i \cos(\lambda t + d_i), \ \alpha_{n+i}' = c_i \sin(\lambda t + d_i).$$

Finally, we obtain

$$\alpha_i = \frac{c_i}{\lambda} \sin(\lambda t + d_i) + h_i,$$

$$\alpha_{n+i} = \frac{-c_i}{\lambda} \cos(\lambda t + d_i) + h_{n+i},$$

$$\begin{aligned} \alpha_{2n+1} &= 2t\cos\theta + \sum_{i=1}^{n} \left\{ \frac{-c_i^2}{4\lambda^2} \left[2(\lambda t + d_i) + \sin(2(\lambda t + d_i)) \right] \right. \\ &\qquad \left. + \frac{c_i h_{n+i}}{\lambda} \sin(\lambda t + d_i) \right\} + h_{2n+1} \end{aligned}$$

for some arbitrary constants h_i $(i = 1, ..., 2n + 1)$.

Secondly, let $\lambda = 0$. In this case, $q = 2\cos\theta$ and $\widehat{k}_1 = 0$. Hence, we have

$$\widehat{\nabla}_{E_1} E_1 = \frac{1}{2} \left(\sum_{i=1}^{n} \alpha_{n+i}'' X_i + \sum_{i=1}^{n} \alpha_i'' X_{n+i} \right) = 0,$$

which gives us

$$\alpha_i = c_i t + d_i, \ i = 1, ..., 2n,$$

$$\alpha_{2n+1} = 2t\cos\theta + \sum_{i=1}^{n} c_i \left(\frac{c_{n+i}}{2} t^2 + d_{n+i} t \right) + c_{2n+1},$$

where c_i $(i = 1, 2, ..., 2n + 1)$ and d_i $(i = 1, 2, ..., 2n)$ are arbitrary constants such that

$$\sum_{i=1}^{2n} c_i^2 = 4\sin^2\theta.$$

To conclude, we can state the following theorem:

Theorem 4.1. *The pseudo-Hermitian magnetic curves on* $\mathbb{R}^{2n+1}(-3)$ *endowed with the Tanaka-Webster connection have the parametric equations*

$$\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^{2n+1}(-3), \alpha = (\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}, ..., \alpha_{2n}, \alpha_{2n+1}),$$

where α_i $(i = 1, ..., 2n+1)$ *satisfies either*

(a)

$$\alpha_i = \frac{c_i}{\lambda} \sin(\lambda t + d_i) + h_i,$$

$$\alpha_{n+i} = \frac{-c_i}{\lambda} \cos(\lambda t + d_i) + h_{n+i},$$

$$\begin{aligned} \alpha_{2n+1} &= 2\cos\theta t + \sum_{i=1}^{n} \left\{ \frac{-c_i^2}{4\lambda^2} \left[2(\lambda t + d_i) + \sin(2(\lambda t + d_i))\right] \right. \\ &\qquad \left. + \frac{c_i h_{n+i}}{\lambda} \sin(\lambda t + d_i) \right\} + h_{2n+1}, \end{aligned}$$

where $\lambda = q - 2\cos\theta \neq 0$, c_i, d_i $(i = 1, ..., n)$ *and* h_i $(i = 1, ..., 2n+1)$ *are arbitrary constants such that*

$$\sum_{i=1}^{n} c_i^2 = 4\sin^2\theta;$$

or

(b)

$$\alpha_i = c_i t + d_i,$$

$$\alpha_{2n+1} = 2t\cos\theta + \sum_{i=1}^{n} c_i \left(\frac{c_{n+i}}{2} t^2 + d_{n+i} t \right) + c_{2n+1},$$

where $q = 2\cos\theta$ *and* c_i $(i = 1, 2, ..., 2n+1)$, d_i $(i = 1, 2, ..., 2n)$ *are arbitrary constants such that*

$$q^2 + \sum_{i=1}^{2n} c_i^2 = 4.$$

REFERENCES

1. T. Adachi: *Curvature bound and trajectories for magnetic fields on a Hadamard surface.* Tsukuba J. Math. **20** (1996), 225–230.
2. D. E. Blair: *Riemannian geometry of contact and symplectic manifolds, Second edition.* Birkhauser Boston, Inc., Boston, MA, 2010.
3. M. Barros, A. Romero, J. L. Cabrerizo and M. Fernández: *The Gauss-Landau-Hall problem on Riemannian surfaces.* J. Math. Phys. **46** (2005), 112905 15 pp.

4. J. L. Cabrerizo, M. Fernandez and J. S. Gomez: *On the existence of almost contact structure and the contact magnetic field.* Acta Math. Hungar. **125** (2009), 191–199.

5. G. Calvaruso, M. I. Munteanu and A. Perrone: *Killing magnetic curves in three-dimensional almost paracontact manifolds.* J. Math. Anal. Appl. **426** (2015), 423–439.

6. J. T. Cho, J. Inoguchi and J. E. Lee: *On slant curves in Sasakian 3-manifolds.* Bull. Austral. Math. Soc. **74** (2006), 359–367.

7. J. T. Cho and J. E. Lee: *Slant curves in contact pseudo-Hermitian 3-manifolds.* Bull. Austral. Math. Soc. **78** (2008), 383–396.

8. A. Comtet: *On the Landau levels on the hyperbolic plane.* Ann. Physics **173** (1987), 185–209.

9. S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor: *Magnetic curves in Sasakian manifolds.* J. Nonlinear Math. Phys. **22** (2015), 428–447.

10. S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu and A. I. Nistor: *Magnetic curves in cosymplectic manifolds.* Rep. Math. Phys. **78** (2016), 33–48.

11. Ş. Güvenç and C. Özgür: *On slant magnetic curves in S-manifolds.* J. Nonlinear Math. Phys. **26** (2019), 536–554.

12. Ş. Güvenç: *On pseudo-Hermitian biharmonic slant curves in Sasakian space forms endowed with the Tanaka–Webster connection.* Bull. Iran. Math. Soc. **46** (2020), 207–223.

13. J. Inoguchi and M. I. Munteanu: *Periodic magnetic curves in Berger spheres.* Tohoku Math. J. **69** (2017), 113–128.

14. M. Jleli, M. I. Munteanu and A. I. Nistor: *Magnetic trajectories in an almost contact metric manifold* $\mathbb{R}^{2N+1}$. Results Math. **67** (2015), 125–134.

15. M. Jleli and M. I. Munteanu: *Magnetic curves on flat para-Kahler manifolds.* Turkish J. Math. **39** (2015), 963–969.

16. L. D. Landau and E. M. Lifshitz: *Course of theoretical physics.* Vol. 1. Mechanics. Third edition. Pergamon Press, Oxford-New York-Toronto, Ont., 1976.

17. M. I. Munteanu and A. I. Nistor: *The classification of Killing magnetic curves in* $S^2 \times R$. J. Geom. Phys. **62** (2012), 170–182.

18. M. I. Munteanu and A. I. Nistor: *A note on magnetic curves on* S^{2n+1}. C. R. Math. Acad. Sci. Paris **352** (2014), 447–449.

19. M. I. Munteanu: *The Landau-Hall problem on canal surfaces.* J. Math. Anal. Appl. **414** (2014), 725–733.

20. C. Özgür and Ş. Güvenç: *On some types of slant curves in contact pseudo-Hermitian 3-manifolds.* Ann. Polon. Math. **104** (2012), 217–228.

21. C. Özgür: *On magnetic curves in the 3-dimensional Heisenberg group.* Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **43** (2017), 278–286.

22. N. Tanaka: *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections.* Japan. J. Math. (N.S.) **2** (1976), 131–190.

23. S. Tanno: *Variational problems on contact Riemannian manifolds.* Trans. Amer. Math. Soc. **314** (1989), 349–379.

24. S. M. Webster: *Pseudo-Hermitian structures on a real hypersurface.* J. Differential Geom. **13** (1978), 25–41.

Şaban Güvenç
Faculty of Arts and Sciences
Department of Mathematics
Campus of Çağış
10145 Balikesir, Turkey
sguvenc@balikesir.edu.tr

Cihan Özgür
Faculty of Arts and Sciences
Department of Mathematics
Campus of Çağış
10145 Balikesir, Turkey
cozgur@balikesir.edu.tr

FACTA UNIVERSITATIS (NIŠ)
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SOME RESULTS ON $*-$RICCI FLOW

Dipankar Debnath and Nirabhra Basu

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Abstract. In this paper we have introduced the notion of $*$-Ricci flow and shown that $*$-Ricci soliton which was introduced by Kaimakamis and Panagiotidou in 2014 is a self similar soliton of the $*$-Ricci flow. We have also found the deformation of geometric curvature tensors under $*$-Ricci flow. In the last two section of the paper, we have found the $\mathfrak{F}$-functional and ω-functional for $*$-Ricci flow respectively.
Keywords: $*-$ Ricci flow, Conformal Ricci flow, $\mathfrak{F}$ functionals, ω functionals.

1. Introduction

A *Ricci soliton* is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\pounds$ denotes the Lie derivative operator, λ is a constant and S is the Ricci tensor of the metric g. Tachibana [3] first introduced $*$-Ricci tensor on almost Hermitian manifolds and Hamada [1] apply this to almost contact manifolds by defining

$$S^*(X, Y) = \frac{1}{2} trace(Z \rightarrow R(X, \phi Y)\phi Z),$$

for any $X, Y \in TM$. In 2014, Kaimakamis and Panagiotidou [2] introduced the concept of $*$-Ricci solitons within the background of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor S in (1.1) with the $*$-Ricci tensor S^*. More precisely, a $*$-Ricci soliton on (M, g) is defined by

$$\pounds_V g + 2S^* + 2\lambda g = 0. \tag{1.2}$$

Inspired by the work of Kaimakamis and Panagiotidou [2], we introduced and studied $*$-Ricci flow on Riemannian manifold and further studied $*$-Ricci solitons. We

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have obtained deformation of geometric curvature tensor under $*$-Ricci flow. We have also provided the rate of change of F-functionals and ω-entropy functional with respect to time under this flow.
We have defined $*$-Ricci flow as follows

$$\frac{\partial g}{\partial t} = -2S^*(X,Y). \tag{1.3}$$

In this paper we have shown that just like Ricci soliton; $*$-Ricci soliton is a self-similar soliton of the $*$-Ricci flow. We have also found the deformation of geometric curvature tensors under $*$-Ricci flow.

Proposition 1.1. Defining $\bar{g(t)} = \sigma(t)\phi_t^*(g) + \sigma(t)\phi_t^*(\frac{\partial g}{\partial t}) + \sigma(t)\varphi_t^*(\pounds_X g)$, we have

$$\frac{\partial \hat{g}}{\partial t} = \acute{\sigma}(t)\psi_t^*(g) + \sigma(t) + \psi_t^*(\frac{\partial g}{\partial t}) + \sigma(t)\psi_t^*(\pounds_X g). \tag{1.4}$$

Proof: This follows from the definition of Lie derivative. If we have a metric g, a vector field Y and $\lambda \in R$ such that

$$-2Ric^*(g_0) = \pounds_Y g_0 - 2\lambda g_0$$

after setting $g(t) = g_0$ and $\sigma(t) = 1-2\lambda t$ and then integrating the t-dependent vector filed $X(t) = \frac{1}{\sigma(t)}Y$. To give a family of deffeomorphism ψ_t with ψ_0 the identity then $\bar{g}$ defined previously is a Ricci flow with

$$\bar{g} = g_0 \frac{\partial \bar{g}}{\partial t} = \sigma'(t)\phi_t^*(g_0) + \sigma(t)\phi_t^*(\pounds_X g_0)$$

$$= \phi_t^*(-2\lambda g_0 + \pounds_Y g_0) = \phi_t^*(-2Ric^*(g_0)) = -2Ric^*(\bar{g}).$$

Proposition 1.2. Under $*-$Ricci flow

$$g(\frac{\partial}{\partial g}\nabla_X Y, Z) = -2(\nabla_X S^*)(Y,Z) + 2S^*(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z).$$

Proof. Let us consider

$$\frac{\partial}{\partial t}\nabla_X Y = \pi(X,Y).$$

Now we can write

$$g(\frac{\partial}{\partial t}\nabla_X Y, Z) = g(\pi(X,Y),Z). \tag{1.5}$$

Again

$$g(\frac{\partial}{\partial t}\nabla_X Y, Z) = \frac{\partial}{\partial t}g(\nabla_X Y, Z) - \frac{\partial g}{\partial t}(\nabla_X Y, Z).$$

$$g(\pi(X,Y),Z) = \frac{\partial}{\partial t}g(\nabla_X, Z) + 2S^*(\nabla_X Y, Z). \tag{1.6}$$

We have

(1.7) $$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

From (1.5) we have

$$g(\pi(X,Y),Z) = \frac{\partial}{\partial t}[Xg(Y,Z) - g(Y,\nabla_X Z)] + 2S^*(\nabla_X Y, Z)$$

$$g(\pi(X,Y),Z) = X\frac{\partial g}{\partial t}(Y,Z) - (\frac{\partial g}{\partial t})(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z)$$

or

$$g(\pi(X,Y),Z = -2(\nabla_X S^*)(Y,Z) + 2S^*(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z)$$

i.e.

(1.8) $$g(\frac{\partial}{\partial g}\nabla_X Y, Z) = -2(\nabla_X S^*)(Y,Z) + 2S^*(Y,\nabla_X Z) + 2S^*(\nabla_X Y, Z).$$

2. The $\mathfrak{F}$-functional for the $*$-Ricci flow

Let M be a fixed closed manifold, g a Riemannian metric and f a function defined on M to the set of real numbers $\mathbb{R}$.

Then the $\mathfrak{F}$-functional on pair (g, f) is defined as

(2.1) $$\mathfrak{F}(g,f) = \int(-1 + |\nabla f|^2)e^{-f}dV.$$

Now, we will establish how the $\mathfrak{F}$-functional changes according to time under $*$-Ricci flow.

Theorem 2.1. *In $*$-Ricci flow the rate of change of $\mathfrak{F}$-functional with respect of time is given by*

$$\frac{d}{dt}\mathfrak{F}(g,f) = \int[-2Ric^*(\nabla f,\nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)$$

$$+ (-1 + |\nabla f|^2)(-\frac{\partial f}{\partial t} + \frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV$$

where

$$\mathfrak{F}(g,f) = \int(-1 + |\nabla f|^2)e^{-f}dV.$$

Proof. We may calculate

(2.2) $$\frac{\partial}{\partial t}|\nabla f|^2 = \frac{\partial}{\partial t}g(\nabla f,\nabla f) = \frac{\partial g}{\partial t}(\nabla f,\nabla f) + 2g(\nabla\frac{\partial f}{\partial t},\nabla f).$$

So using proposition 2.3.12 of [13] we can write

$$\frac{d}{dt}\mathfrak{F}(g,f) = \int [\frac{\partial g}{\partial t}(\nabla f, \nabla f) + 2g(\nabla \frac{\partial f}{\partial t}, \nabla f)]e^{-f}dV$$
$$+ \int (-1 + |\nabla f|^2)[-\frac{\partial f}{\partial t} + \frac{1}{2}tr\frac{\partial g}{\partial t}]e^{-f}dV. \tag{2.3}$$

Using integration by parts of equation(2.2), we get

$$\int 2g(\nabla \frac{\partial f}{\partial t}, \nabla f)e^{-f}dV = -2\int \frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)e^{-f}dV. \tag{2.4}$$

Now putting (2.4) in (2.3), we get

$$\frac{d}{dt}\mathfrak{F}(g,f) = \int [\frac{\partial g}{\partial t}(\nabla f, \nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)$$
$$+ (-1 + |\nabla f|^2)(-\frac{\partial f}{\partial t} + \frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV. \tag{2.5}$$

Using (1.3) in (2.5), we get the following result for conformal Ricci flow, as

$$\frac{d}{dt}\mathfrak{F}(g,f) = \int [-2Ric^*(\nabla f, \nabla f) - 2\frac{\partial f}{\partial t}(\Delta f - |\nabla f|^2)$$
$$+ (-1 + |\nabla f|^2)(-\frac{\partial f}{\partial t} + \frac{1}{2}tr\frac{\partial g}{\partial t})]e^{-f}dV. \tag{2.6}$$

Hence the proof.

3. ω-entropy functional for the $*$- Ricci flow

Let M be a closed manifold, g a Riemannian metric on M and f a smooth function defined from M to the set of real numbers $\mathbb{R}$. We define ω-entropy functional as

$$\omega(g,f,\tau) = \int [\tau(R^* + |\nabla f|^2) + f - n]udV \tag{3.1}$$

where $\tau > 0$ is a scale parameter and u is defined as $u(t) = e^{-f(t)}$;$\int_M udV = 1$.

We would also like to define heat operator acting on the function $f : M \times [0,\tau] \longrightarrow \mathbb{R}$ by $\diamondsuit := \frac{\partial}{\partial t} - \Delta$ and also, $\diamondsuit^* := -\frac{\partial}{\partial t} - \Delta + R^*$, conjugate to $\diamondsuit$.

We choose u, such that $\diamondsuit^* u = 0$.

Now we prove the following theorem.

Theorem 3.1: *If g, f, τ evolve according to*

$$\frac{\partial g}{\partial t} = -2Ric^* \tag{3.2}$$

$$\frac{\partial \tau}{\partial t} = -1 \tag{3.3}$$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R^* + \frac{n}{2\tau} \tag{3.4}$$

and the function v is defined as $v = [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n]u$, the rate of change of ω-entropy functional for conformal Ricci flow is $\frac{d\omega}{dt} = -\int_M \diamondsuit^* v$, where

$$\diamondsuit^* v = 2u(\Delta f - |\nabla f|^2 + R^*) - \frac{un}{2\tau} - v - u\tau[4 < Ric^*, Hessf >$$

$$-2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) + 2|Hessf|^2].$$

Proof: We find that

$$\diamondsuit^* v = \diamondsuit^*(\frac{v}{u}u) = \frac{v}{u}\diamondsuit^* u + u\diamondsuit^*(\frac{v}{u}).$$

We have defined previously that $\diamondsuit^* u = 0$,

so

$$\diamondsuit^* v = u\diamondsuit^*(\frac{v}{u})$$

$$\diamondsuit^* v = u\diamondsuit^*[\tau(2\nabla f - |\nabla f|^2 + R^*) + f - n].$$

We shall use the conjugate of heat operator, as defined earlier as $\diamondsuit^* = -(\frac{\partial}{\partial t} + \Delta - R^*)$.

Therefore

$$\diamondsuit^* v = -u(\frac{\partial}{\partial t} + \Delta - R^*)[\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n]$$

$$\Rightarrow u^{-1}\diamondsuit^* v = -(\frac{\partial}{\partial t} + \Delta)[\tau(2\Delta f - |\nabla f|^2 + R^*)]$$

$$-(\frac{\partial}{\partial t} + \Delta)f - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n].$$

Using equation (3.3), we have

$$u^{-1}\diamondsuit^* v = (2\Delta f - |\nabla f|^2 + R^*) - \tau(\frac{\partial}{\partial t} + \Delta)(2\Delta f - |\nabla f|^2 + R^*)$$

$$-\frac{\partial f}{\partial t} - \Delta f - \frac{v}{u}. \tag{3.5}$$

Now

$$\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\frac{\partial}{\partial t}(\Delta f) - \frac{\partial}{\partial t}|\nabla f|^2.$$

Using proposition (2.5.6) of [13], we have

$$\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta\frac{\partial f}{\partial t} + 4 < Ric^*, Hessf >$$
$$- \frac{\partial g}{\partial t}(\nabla f, \nabla f) - 2g(\frac{\partial}{\partial t}\nabla f, \nabla f).$$

Now using the $*-$Ricci flow equation (1.3), we have

$$\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta\frac{\partial f}{\partial t} + \; 4 < Ric^*, Hessf >$$
$$+ 2Ric^*(\nabla f, \nabla f) - 2g(\frac{\partial}{\partial t}\nabla f, \nabla f). \tag{3.6}$$

Using (3.4) in (3.6), we get

$$\frac{\partial}{\partial t}(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta(-\Delta f + |\nabla f|^2 - R^* + \frac{n}{2\tau}) + 4 < Ric^*, Hessf >$$
$$+ 2Ric^*(\nabla f, \nabla f) - 2g(\frac{\partial}{\partial t}\nabla f, \nabla f). \tag{3.7}$$

Now let us compute

$$\Delta(2\Delta f - |\nabla f|^2 + R^*) = 2\Delta^2 f - \Delta|\nabla f|^2. \tag{3.8}$$

Using (3.7) and (3.8) in (3.5) we obtain after a brief calculation

$$u^{-1}\Diamond^* v = (2\Delta f - |\nabla f|^2 + R^*) - \tau[-2\Delta^2 f + 2\Delta|\nabla f|^2 + 4 < Ric^*, Hessf >$$
$$+2Ric^*(\nabla f, \nabla f) - 2g(\tfrac{\partial}{\partial t}\nabla f, \nabla f) + 2\Delta^2 f - \Delta|\nabla f|^2] - \tfrac{\partial f}{\partial t} - \Delta f - \tfrac{v}{u}$$
$$= \Delta f - |\nabla f|^2 + R^* - \tau[\Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f)$$
$$- 2g(\tfrac{\partial}{\partial t}\nabla f, \nabla f)] - \tfrac{\partial f}{\partial t} - \tfrac{v}{u}$$
$$= \Delta f - |\nabla f|^2 + R^* - \tau[\Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f)$$
$$- 2g(\tfrac{\partial}{\partial t}\nabla f, \nabla f)] + \Delta f - |\nabla f|^2 + R^* - \tfrac{n}{2\tau} - \tfrac{v}{u}$$
$$= 2(\Delta f - |\nabla f|^2 + R^*) - \tfrac{n}{2\tau} - \tfrac{v}{u} - \tau[\Delta|\nabla f|^2 + 4 < Ric^*, Hessf >$$
$$+ 2Ric^*(\nabla f, \nabla f) - 2g(\tfrac{\partial}{\partial t}\nabla f, \nabla f)]$$

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \tfrac{n}{2\tau} - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n] - \tau[\Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f) - 2g(\tfrac{\partial}{\partial t}\nabla f, \nabla f)].$$

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + \Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f) - 2g(\nabla \frac{\partial f}{\partial t}, \nabla f)]. \tag{3.9}$$

Using (3.4), we get

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + \Delta|\nabla f|^2 + 4 < Ric^*, Hessf > +2Ric^*(\nabla f, \nabla f) - 2g(\nabla(-\Delta f + |\nabla f|^2 + \frac{n}{2\tau} - R^*), \nabla f)]. \tag{3.10}$$

We can rewrite (3.10) in the following way

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + 4 < Ric^*, Hessf > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] + \tau[-\Delta|\nabla f|^2 - 2Ric^*(\nabla f, \nabla f) + 2g(\nabla(\Delta f), \nabla f)] \tag{3.11}$$

and using Bochner formula in (3.11) and simplifying it, we get

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - f + n - \tau[2\Delta f - |\nabla f|^2 + R^* + 4 < Ric^*, Hessf > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2.$$

$$\Rightarrow u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - [\tau(2\Delta f - |\nabla f|^2 + R^*) + f - n] - \tau[4 < Ric^*, Hessf > -2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2.$$

i.e.

$$u^{-1}\diamondsuit^* v = 2(\Delta f - |\nabla f|^2 + R^*) - \frac{n}{2\tau} - \frac{v}{u} - \tau[4 < Ric^*, Hessf > - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f)] - 2\tau|Hessf|^2. \tag{3.12}$$

So finally we have

$$\begin{aligned}\diamondsuit^* v = 2u(\Delta f - |\nabla f|^2 + R^*) - \frac{un}{2\tau} - v - u\tau[4 < Ric^*, Hess f > \\ - 2g(\nabla|\nabla f|^2, \nabla f) + 4g(\nabla(\Delta f), \nabla f) + 2|Hess f|^2]. \qquad (3.13)\end{aligned}$$

Now using remark (8.2.7) of [13], we get

$$\frac{d\omega}{dt} = -\int_M \diamondsuit^* v.$$

So the evolution of ω with respect to time can be found by this integration.

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REFERENCES

1. T. Hamada: *Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor*, Tokyo J. Math., 2002, 25, 473-483.
2. G. Kaimakamis and K. Panagiotidou: *$*$-Ricci solitons of real hypersurfaces in non-flat complex space forms*, J. Geom. Phys., 2014, 86, 408-413.
3. S. Tachibana: *On almost-analytic vectors in almost Kählerian manifolds*, Tohoku Math. J., 1959, 11, 247-265.
4. X. Dai, Y. Zhao and U. C. De: *$*$-Ricci soliton on $(\kappa;\mu)'$-almost Kenmotsu manifolds*, Open Math. 17 (2019), 74-882.
5. V. Venkatesha, H. A. Kumara and D. M. Naik: *Almost $*$-Ricci soliton on para Kenmotsu manifolds*, Arab. J. Math. (2019). https://doi.org/10.1007/s 40065- 019-00269-7.
6. A. K. Huchchappa, D. M. Naik and V. Venkatesha: *Certain results on contact metric generalized $(\kappa;\mu)$-space forms*, Commun. Korean Math. Soc. 34 (4) (2019), 1315-1328.
7. G. Perelman: *The entropy formula for the Ricci flow and its geometric applications*, arXiv.org/abs/math/0211159, (2002) 1-39.
8. N. Basu and D. Debnath: *Characteristic of conformal Ricci-soliton and conformal gradient Ricci soliton in LP-Sasakian manifold* , accepted in PJM(Palestine Journal of Mathematics).
9. Venkatesha, D. M. Naik and H. A. Kumara: *$*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds*, arXiv.org .math.arXiv:1901.05222v [Math. DG]16 Jan 2019.
10. G. Perelman: *Ricci flow with surgery on three manifolds*, arXiv.org/abs/math/ 0303109, (2002), 1-22.
11. R. S. Hamilton: *Three Manifold with positive Ricci curvature*, J.Differential Geom.17(2), (1982), 255-306.

12. B. Chow, P. Lu, L. Ni: *Hamilton's Ricci Flow*, American Mathematical Society Science Press, 2006.

13. P. Topping: *Lecture on The Ricci Flow*, Cambridge University Press; 2006.

14. A. E. Fischer: *An introduction to conformal Ricci flow*, Class. Quantum Grav.21(2004), S171 - S218.

15. K. Mandal and S. Makhal: *$*$-Ricci solitons on three dimensional normal almost contact metric manifolds*, Lobachevskii Journal of Mathematics, 40, 189-194,2019.

16. A. Ghosh and D. S. Patra: *$*$-Ricci Soliton within the frame-work of Sasakian and (κ,μ)-contact manifold* , International Journal of Geometric Methods in Modern Physics, Vol. 15, No. 07, 1850120 (2018).

17. K. De and C. Dey: *$*$-Ricci solitons on (ε) - para Sasakian manifolds*, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics . 2019, Vol. 12 Issue 2, p265-274.

18. P. Majhi, U. C. De and Y. J. Suh: *$*$-Ricci solitons on Sasakian 3-manifolds*, Publicationes mathematicae, 2018,93(1-2):241-252.

19. D. Dey and P. Majhi: *$N(k)$-contact metric as $*$-conformal Ricci soliton*, arXiv.org.math. arXiv:2005.02194.

20. D. G. Prakasha and P. Veeresha: *Para-Sasakian manifolds and $*$-Ricci solitons*, Afrika Matematika volume 30, pages. 989–998(2019).

Dipankar Debnath
Department of Mathematics
Bamanpukur High School(H.S)
Bamanpukur, Sree Mayapur
Nabadwip, Nadia
West Bengal, Pin-741313
India
dipankardebnath123@gmail.com

Nirabhra Basu
Department of Mathematics
Bhawanipur Education Society College
Kolkata-700020
West Bengal
India
nirabhra.basu@thebges.edu.in

FACTA UNIVERSITATIS (NIŠ)
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SURFACE FAMILY WITH COMMON LINE OF CURVATURE IN 3-DIMENSIONAL GALILEAN SPACE

Mustafa Altin and İnan Ünal

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Abstract. In this paper we tried to find parametric presentation of a surface family with common line of curvature in 3-dimensional Galilean space. We have obtained necessary and sufficient conditions for the curve to be a common line of curvature on this surface. We have stated examples to visualize our results and also, we have examined a torsion free curve.
Keywords: surface family; curvature; 3-dimensional Galilean space.

1. Introduction

The surface family (or pencil surface) is a notion in differential geometry applied in engineering science such as computer, manufacturing, mechanical engineering [25]. In 2004 Wang et al. [25] gave the definition of a surface family. Their paper is a reverse engineering problem to find a spatial curve to characterize the surface and also the paper contains conditions for a curve to be a geodesic on this surface. Besides, their work could be seen as an example of industrial mathematics. Kasap et al. [10] generalized this study by assumption of more general marching-scale functions. In [13] Li et al studied the approximation minimal surface with geodesics by using Dirichlet function and they minimized the area of surface family by using Dirichlet approach. This method can be used for obtaining minimal cost of material while building surfaces. The surface family notion has been studied by many researchers [1, 2, 9, 10].

There are many special curves on a surface such as geodesics. One of them is the line of curvature. A line of curvature is a curve on a surface whose tangent line at every point is aligned along a principal curvature direction. In [4] Che at al. analysed and computed these curves which are defined on implicit surface and worked on differential geometry of them. Same authors derived a necessary and

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sufficient condition for a given curve to be the line of curvature on the surface. Surface family with common line of curvature has been studied in [7, 8, 12].

Galileo geometry is a type of non-Euclidean geometry based on Galileo principle of relativity [20] and it has many important applications in physics [14]. In the last decades, these kind of spaces have become interesting by geometers because of their significant properties as a non-Euclidean geometry. Curves and surfaces in Galilean geometry has been studied by many authors [3, 5, 6, 15–17, 21]. Surfaces family, especially, in Galilean space have been studied in [22–24].

In this study, we examined a surface family with common line of curvature in 3− dimensional Galilean space. We obtain necessary and sufficient conditions for the curve to be a line of curvature on the surface. We get some results for a torsion free curve. Finally, we present examples and plot their graphs.

2. Preliminaries

A. Cayley and F. Klein discovered that both Euclidean and non-Euclidean geometries can be considered as mathematical structures living inside projective-metric spaces. Their contribution to geometry is called Cayley-Klein geometry and non- Euclidean geometries could be classified by this geometry. In fact, the 3-dimensional Galilean geometry is also a Cayley-Klein space [20].

2.1. Basic Facts in 3D Galilean Space

In this subsection, we recall some fundamental facts from Galilean geometry. For details see [18, 20].

A vector $\omega = (\omega_1, \omega_2, \omega_3)$ in 3-dimensional Galilean space $\mathbb{G}_3$ is called non-isotropic if $\omega_1 \neq 0$, otherwise it is called isotropic.

Let $\omega = (\omega_1, \omega_2, \omega_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$ be two vectors in Galilean space $\mathbb{G}_3$. The inner product and the vector product of ω and η in $\mathbb{G}_3$ are defined by

$$\langle \omega, \eta \rangle = \begin{cases} \omega_1 \eta_1, \ if \ \ \omega_1 \neq 0 \ or \ \eta_1 \neq 0 \\ \omega_2 \eta_2 + \omega_3 \eta_3 \ if \ \omega_1 = 0 \ and \ \eta_1 = 0 \end{cases}$$

and

$$\omega \times \eta = \begin{cases} \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & \omega_2 & \omega_3 \\ 0 & \eta_2 & \eta_3 \end{vmatrix} & \text{if } \omega_1 = \ \eta_1 = 0, \\ \\ \begin{vmatrix} 0 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} & \text{if } \omega_1 \neq 0 \text{ or } \eta_1 \neq 0. \end{cases}$$

respectively.

Let $\gamma : I \to \mathbb{G}_3, \quad I \subset \mathbb{R}$ be a curve in $\mathbb{G}_3$ given by $\gamma(\phi) = (\delta(\phi), \zeta(\phi), \psi(\phi))$. Then the curvature κ_1 and torsion κ_2 of $\gamma(\phi)$ is obtained as

$$(2.1)\ \kappa_1(\phi) = \|\gamma''(\phi)\|, \quad \kappa_2(\phi) = \frac{1}{\kappa_1^2(\phi)} det(\gamma'(\phi), \gamma''(\phi), \gamma'''(\phi)), \kappa_1(\phi) \neq 0$$

where $\|,\|$ is the Galilean norm. Thus, we have Frenet formulas of $\gamma(\phi)$ by

$$(2.2) \qquad \begin{cases} V_1{}' = \kappa_1 V_2, \\ V_2{}' = \kappa_2 V_3, \\ V_3{}' = -\kappa_2 V_2, \end{cases}$$

where V_1, V_2 and V_3 are tangent, normal and binormal vector fields of $\gamma(\phi)$, respectively.

If $\delta'(\phi) = 0$, then $\gamma(\phi)$ is called a non-admissible curve, otherwise it is called an admissible curve. Let $\gamma(\phi)$ be an admissible curve in $\mathbb{G}_3$, given by

$$(2.3) \qquad \gamma(\phi) = (\phi, \zeta(\phi), \psi(\phi)).$$

Then κ_1 and κ_2 can be obtained as

$$\kappa_1(\phi) = \sqrt{\zeta''(\phi)^2 + \psi''(\phi)^2}, \ \kappa_2(\phi) = \frac{1}{(\kappa_1(\phi))^2} \det(\gamma'(\phi), \gamma''(\phi), \gamma'''(\phi))$$

and the Frenet vectors are given by

$$\begin{cases} V_1(\phi) = \gamma'(\phi) = (1, \zeta'(\phi), \psi'(\phi)), \\ V_2(\phi) = \frac{\gamma''(\phi)}{\kappa_1(\phi)} = \frac{1}{\kappa_1(\phi)}(0, \zeta''(\phi), \psi''(\phi)), \\ V_3(\phi) = \frac{1}{\kappa_1(\phi)}(0, -\psi''(\phi), \zeta''(\phi)). \end{cases}$$

2.2. Some facts on Surface Theory in 3D Galilean Space

A surface in $\mathbb{G}_3$ is a parametric mapping from a region R in $\mathbb{R}^2$ to $\mathbb{G}_3$ such as

$$(2.4) \qquad S : R \subset \mathbb{R}^2 \to \mathbb{G}_3, \quad S(\phi, \varphi) = (S_1(\phi, \varphi), S_2(\phi, \varphi), S_3(\phi, \varphi))$$

where S_1, S_2 and S_3 are functions in $C^1(\mathbb{G}_3, \mathbb{R})$. The normal vector field of S is given by

$$(2.5) \qquad \mathcal{N}(\phi, \varphi) = S_\phi \times S_\varphi.$$

where $S_\phi = \frac{\partial S}{\partial \phi}$ and $S_\varphi = \frac{\partial S}{\partial \varphi}$ are partial derivatives of S.

Every surface has its own intrinsic geometry which has been known since Gauss. So, curves on a surface have geometric properties independent from the ambient space. We have a classification for curves on a surface by following definition.

Definition 2.1. Let $\gamma(\phi)$ be a curve on a surface S in 3-dimensional Galilean space $\mathbb{G}_3$. Then $\gamma(\phi)$ is

1. a *line of curvature*, if the tangent vector at any point is in the direction of the principal curvature.
2. a *geodesic* if the normal vector field $V_2(\phi)$ of the curve $\gamma(\phi)$ and the normal $\mathcal{N}(\phi, \varphi_0)$ are parallel.
3. an *asymptotic* if the the binormal $V_3(\phi)$ of $\gamma(\phi)$ and the normal $\mathcal{N}(\phi, \varphi_0)$ of the surface at any point on $\gamma(\phi)$, are parallel to each other.

On the other hand, if $\gamma(\phi)$ is both an asymptotic and a parametric (isoparametric) curve, then it is called *isoasymptotic*; if it is both an geodesic and a parametric (isoparametric) curve , then it is called *isogeodesic.*

The well-known theorem below gives the conditions for any curve on a surface S to be the line of curvature. For proof and details, we refer to reader [19].

Theorem 2.1. (**Monge's Theorem**) *A necessary and sufficient condition for a curve on a surface to be a line of curvature is that the surface normals along the curve form a developable surface [19].*

Let $S(\phi, \varphi)$ be a parametric surface in $\mathbb{G}_3$ is defined as follow;

$$S(\phi, \varphi) = \gamma(\phi) + [\lambda_1(\phi, \varphi)V_1(\phi) + \lambda_2(\phi, \varphi)V_2(\phi) + \lambda_3(\phi, \varphi)V_3(\phi)] \tag{2.6}$$

for $(\phi, \varphi) \in R = [I_1, I_2] \times [I_3, I_4]$, where $\lambda_1(\phi, \varphi)$, $\lambda_2(\phi, \varphi)$ and $\lambda_3(\phi, \varphi)$ are the values of the marching-scale functions in $C^1(S, \mathbb{R})$ and $\{V_1(\phi), V_2(\phi), V_3(\phi)\}$ is the Frenet frame of $\gamma(\phi)$. The surface (2.6) is called surface family with a common curve $\gamma(\phi)$.

A ruled surface formed by the surface normals can be given by

$$\Psi(\phi, \varphi) = \gamma(\phi) + \varphi\mathfrak{n},$$

where φ is the distance of a point on $\Psi(\phi, \varphi)$ to point $\gamma(\phi)$ and $\mathfrak{n} = \cos\theta V_2(\phi) + \sin\theta V_3(\phi)$, the vector functions $V_2(\phi)$, $V_3(\phi)$ are the principal normal and the binormal of $\gamma(\phi)$, respectively. The surface $\Psi(\phi, \varphi)$ is called a normal surface [11].

Thus, by Monge's Theorem, $\gamma(\phi)$ is the line of curvature if and only if $\Psi(\phi, \varphi)$ is developable and $\mathfrak{n}$ is parallel to the normal vector field $\mathcal{N}$ of the surface (2.6). Also by classical differential geometry, it is well known that a surface is developable if and only if $\det(\gamma'(\phi), \mathfrak{n}, \mathfrak{n}') = 0$ (see [19]).

Hence from (2.2), we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\theta'\sin\theta - \kappa_2\sin\theta & \kappa_2\cos\theta + \theta'\cos\theta \end{vmatrix} = 0 \tag{2.7}$$

and so

$$\theta' + \kappa_2 = 0.$$

This means that

$$\theta = -\int_{\phi_0}^{\phi} \kappa_2 d\phi + \theta_0 \tag{2.8}$$

where ϕ_0 is the starting value of arc length and $\theta_0 = \theta_0(\phi_0)$. In this paper, we assume $\phi_0 = 0$. Then by substituting θ in $\mathfrak{n}$ and with parallelity of $\mathfrak{n}$ to $\mathcal{N}$, we obtain the result that $\gamma(\phi)$ is the line of curvature.

3. Surfaces with common line of curvature in 3D Galilean space $\mathbb{G}_3$

In this section, we work on surfaces family in Galilean 3-space $\mathbb{G}_3$. We give if and only if conditions for a unit speed non-isotropic curve, being a line of curvature on a surface family. Furthermore, we give some examples and we present their graphics.

Theorem 3.1. *The curve $\gamma(\phi) = (\phi, \zeta(\phi), \psi(\phi))$ is a line of curvature on the surface defined in (2.6) if and only if*

$$\begin{cases} \lambda_1(\phi, \varphi_0) = \lambda_2(\phi, \varphi_0) = \lambda_3(\phi, \varphi_0) = 0, \\ -\frac{\partial \lambda_3(\phi,\varphi_0)}{\partial \varphi} = \mu(\phi)\cos\theta, \ \frac{\partial \lambda_2(\phi,\varphi_0)}{\partial \varphi} = \mu(\phi)\sin\theta \end{cases} \tag{3.1}$$

where $(\phi, \varphi) \in R = [I_1, I_2] \times [I_3, I_4]$, $\mu(\phi) \neq 0$. The functions $\theta(\phi)$ and $\mu(\phi)$ are called controlling functions.

Proof. Let $S(\phi, \varphi)$ be a surface in $\mathbb{G}_3$ given by (2.6). For a curve $\gamma(\phi)$ on $S(\phi, \varphi)$ which is isoparametric, we have a parameter $\varphi_0 \in [I_3, I_4]$ such that $\gamma(\phi) = S(\phi, \varphi_0)$ with conditions

$$\lambda_1(\phi, \varphi_0) = \lambda_2(\phi, \varphi_0) = \lambda_3(\phi, \varphi_0) = 0, (\phi, \varphi_0) \in R.$$

By direct computations, we have

$$\begin{aligned} \frac{\partial S(\phi, \varphi)}{\partial \phi} &= \left[1 + \frac{\partial \lambda_1(\phi, \varphi)}{\partial \phi}\right] V_1(\phi) \\ &+ \left[\kappa_1 \lambda_1(\phi, \varphi) + \frac{\partial \lambda_2(\phi, \varphi)}{\partial \phi} - \kappa_2 \lambda_3(\phi, \varphi)\right] V_2(\phi) \\ &+ \left[\kappa_2 \lambda_2(\phi, \varphi) + \frac{\partial \lambda_3(\phi, \varphi)}{\partial \phi}\right] V_3(\phi) \end{aligned}$$

and

$$\frac{\partial S(\phi, \varphi)}{\partial \varphi} = \frac{\partial \lambda_1(\phi, \varphi)}{\partial \varphi} V_1(\phi) + \frac{\partial \lambda_2(\phi, \varphi)}{\partial \varphi} V_2(\phi) + \frac{\partial \lambda_3(\phi, \varphi)}{\partial \varphi} V_3(\phi).$$

Thus, we get the normal vector of surface by

$$\mathcal{N}(\phi,\varphi)=\frac{\partial S(\phi,\varphi)}{\partial\phi}\times\frac{\partial S(\phi,\varphi)}{\partial\varphi}. \tag{3.2}$$

So, for $\varphi_0\in[I_3,I_4]$, we have

$$\mathcal{N}(\phi,\varphi_0)=\mathcal{N}_1(\phi,\varphi_0)V_1(\phi)+\mathcal{N}_2(\phi,\varphi_0)V_2(\phi)+\mathcal{N}_3(\phi,\varphi_0)V_3(\phi)$$

where

$$\begin{aligned}
\mathcal{N}_1(\phi,\varphi_0) &= 0\\
\mathcal{N}_2(\phi,\varphi_0) &= \frac{\partial\lambda_1(\phi,\varphi_0)}{\partial\varphi}(\kappa_2\lambda_2(\phi,\varphi_0)+\frac{\partial\lambda_3(\phi,\varphi_0)}{\partial\phi})-(1+\frac{\partial\lambda_1(\phi,\varphi_0)}{\partial\phi})\frac{\partial\lambda_3(\phi,\varphi_0)}{\partial\varphi}\\
\mathcal{N}_3(\phi,\varphi_0) &= (1+\frac{\partial\lambda_1(\phi,\varphi_0)}{\partial\phi})\frac{\partial\lambda_2(\phi,\varphi_0)}{\partial\varphi}-(\kappa_1\lambda_1(\phi,\varphi_0)\\
&\quad+\frac{\partial\lambda_2(\phi,\varphi_0)}{\partial\phi}-\kappa_2\lambda_3(\phi,\varphi_0))\frac{\partial\lambda_1(\phi,\varphi_0)}{\partial\varphi}.
\end{aligned}$$

Suppose that $\gamma(\phi)$ is a line of curvature on $S(\phi,\varphi)$. Thus, for a function $\mu(\phi)\neq 0$ on $S(\phi,\varphi)$, necessary and sufficient condition to provide $\mathfrak{n}(\phi)\parallel\mathcal{N}(\phi,\varphi_0)$ is

$$\begin{aligned}
\mathcal{N}_2(\phi,\varphi_0) &= \mu(\phi)\cos\theta\\
\mathcal{N}_3(\phi,\varphi_0) &= \mu(\phi)\sin\theta.
\end{aligned}$$

So the proof is completed. □

Example 3.1. Let $\gamma(\phi)=(\phi,2\sin(\phi),2\cos(\phi))$ be an admissible curve in $\mathbb{G}_3$. Then, we get the first and the second curvatures of $\gamma(\phi)$ by $\kappa_1=2$ and $\kappa_2=-1$. Thus, the Frenet frame is obtained by

$$V_1(\phi)=(1,2\cos(\phi),-2\sin(\phi)),V_2(\phi)=(0,-\sin(\phi),-\cos(\phi)),V_3(\phi)=(0,\cos(\phi),-\sin(\phi)).$$

If we choose

$$\lambda_1(\phi,\varphi)=\phi^2\varphi,\ \lambda_2(\phi,\varphi)=\phi\sin(\theta)\sin(\phi\varphi),\ \lambda_3(\phi,\varphi)=\cos(\theta)(\phi-\phi e^{\phi\varphi}),$$

then we get surface family $S(\phi,\varphi)$ given by (2.6) with common curve $\gamma(\phi)$. Then, by taking $\mu(\phi)=\phi^2$ and $\varphi_0=0$, the conditions given in (3.1) are satisfied.

Suppose that the normal surface $\Psi(\phi,\varphi)$ of S is developable in $\mathbb{G}_3$. From (2.8), we have $\theta=\phi$. Thus, we get

$$\begin{aligned}
S_1(\phi,\varphi) &= \phi+\phi^2\varphi\\
S_2(\phi,\varphi) &= 2\sin(\phi)+2\phi^2\varphi\cos(\phi)+sin^2(\phi)\phi\sin(\phi\varphi)+cos^2(\phi)(\phi-\phi e^{\phi\varphi})\\
S_3(\phi,\varphi) &= 2\cos(\phi)-2\phi^2\varphi\sin(\phi)+\phi sin(\phi)\cos(\phi)sin(\phi\varphi)-\sin(\phi)cos(\phi)(\phi-\phi e^{\phi\varphi}).
\end{aligned}$$

As seen, all $S_i(\phi,\varphi),\ i=1,2,3$ are in $C^1(S,\mathbb{R})$. Consequently, $\gamma(\phi)$ is a line of curvature on $S(\phi,\varphi)$ with positive curvature and negative torsion .

By taking $R=[0,2\pi]\times[-3,1]$, we visualize the curve $\gamma(\phi)=S(\phi,0)$ in Fig.3.1, the surface $S(\phi,\varphi)$ in Fig. 3.2 and curve on surface in Fig. 3.3.

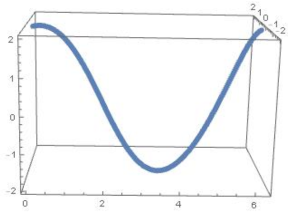

FIG. 3.1: Image of $\gamma(\phi)$

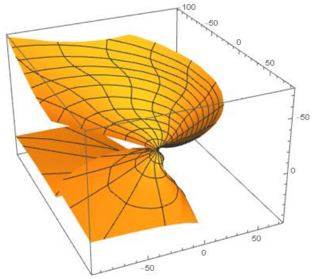

FIG. 3.2: Image of $S(\phi,\varphi)$

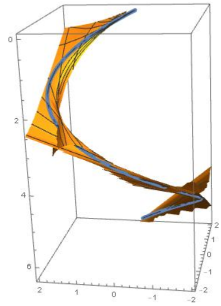

FIG. 3.3: $\gamma(\phi)$ on the $S(\phi,\varphi)$

Let take into consideration the case of the marching-scale functions

$$\begin{aligned}\lambda_1(\phi,\varphi) &= \rho_1(\phi)\Lambda_1(\varphi),\\ \lambda_2(\phi,\varphi) &= \rho_2(\phi)\Lambda_2(\varphi),\\ \lambda_3(\phi,\varphi) &= \rho_3(\phi)\Lambda_3(\varphi)\end{aligned}$$

with under conditions $\lambda_1(\phi,\varphi_0) = \lambda_2(\phi,\varphi_0) = \lambda_3(\phi,\varphi_0) = 0$ and $(\phi,\varphi_0) \in R = [I_1, I_2] \times [I_3, I_4]$, where $\rho_1(\phi), \Lambda_1(\varphi), \rho_2(\phi), \Lambda_2(\varphi), \rho_3(\phi)$ and $\Lambda_3(\varphi)$ are functions in $C^1(S,\mathbb{R})$. Then from Theorem 3.1, we have following corollary:

Corollary 3.1. *The curve $\gamma(\phi) = (\phi, \zeta(\phi), \psi(\phi))$ is a line of curvature on the surface defined in (2.6) if and only if*

$$(3.3) \qquad \begin{cases} \Lambda_1(\varphi_0) = \Lambda_2(\varphi_0) = \Lambda_3(\varphi_0) = 0, \\ -\rho_3(\phi)\frac{d\Lambda_3}{d\varphi}(\varphi_0) = \mu(\phi)\cos\theta,\ \ \rho_2(\phi)\frac{d\Lambda_2}{d\varphi}(\varphi_0) = \mu(\phi)\sin\theta \end{cases}$$

where $(\phi,\varphi_0) \in R = [I_1, I_2] \times [I_3, I_4]$and $\mu(\phi) \neq 0$.

Example 3.2. Let $\gamma(\phi) = (\phi, \cos(\phi), \sin(\phi))$ be an admissible curve in $\mathbb{G}_3$. Then, we get first two curvatures as $\kappa_1 = 1$ and $\kappa_2 = 1$. Also the Frenet frame is given by

$$V_1(\phi) = (1, -\sin(\phi), \cos(\phi)),\quad V_2(\phi) = (0, -\cos(\phi), -\sin(\phi)),\quad V_3(\phi) = (0, \sin(\phi), \cos(\phi)).$$

Thus, we get surface family $S(\phi,\varphi)$ given by (2.6) with common curve $\gamma(\phi)$. Suppose that the normal surface $\Psi(\phi,\varphi)$ of S is developable in $\mathbb{G}_3$. Thus, from (2.8), we have $\theta = -\phi$.

If we choose $\rho_1(\phi) = \phi, \Lambda_1(\phi,\varphi) = (\varphi^2 - 1),\ \rho_2(\phi) = \rho_3(\phi) = 1,\ \Lambda_2(\phi,\varphi) = \sin(\theta)(\varphi - 1),\ \Lambda_3(\phi,\varphi) = \cos(\theta)(1-\varphi)$, and take $\mu(\phi) = 1$, $\varphi_0 = 1$ so that equation (3.1) is satisfied, then a member of surface family in $\mathbb{G}_3$ is obtained by

$$\begin{aligned} S(\phi,\varphi) = \; & (\phi + \phi(\varphi^2 - 1), \cos(\phi) - \phi(\varphi^2 - 1)\sin(\phi), \\ & \sin(\phi) + \phi(\varphi^2 - 1)\cos(\phi) + \sin^2(\phi)(\varphi - 1) + \cos^2(\phi)(1 - \varphi)). \end{aligned}$$

By taking $R = [0, 2\pi] \times [0, 3]$, we visualize the curve $\gamma(\phi) = S(\phi, 0)$ in Fig.3.4, the surface $S(\phi,\varphi)$ in Fig. 3.5 and curve on surface in Fig. 3.6.

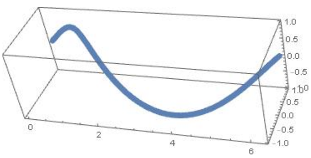

FIG. 3.4: Image of $\gamma(\phi)$

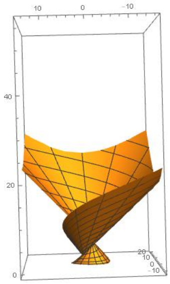

FIG. 3.5: Image of $S(\phi, \varphi)$

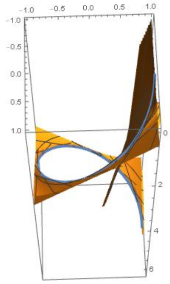

FIG. 3.6: $\gamma(\phi)$ on $S(\phi, \varphi)$

Suppose that the second curvature of $\gamma(\phi)$ vanish, i.e $\kappa_2 = 0$. Then, from (2.8), we have $\theta = \theta_r(constant)$. Thus, from (3.3), we obtain

$$\frac{\mu(\phi)}{\rho_3(\phi)} = -c_1, \quad \frac{\mu(\phi)}{\rho_2(\phi)} = c_2.$$

Considering conditions in (3.3), we get

$$\frac{d\Lambda_3}{d\varphi}(\varphi_0) = c_1 cos\theta_r \text{ and } \frac{d\Lambda_2}{d\varphi}(\varphi_0) = c_2 cos\theta_r.$$

On the other hand, since $\mathfrak{n} \parallel \mathcal{N}$, if $\theta_r = (2m+1)\frac{\pi}{2}$ for any integer m then $V_3 \parallel \mathcal{N}$. Thus, $\gamma(\phi)$ is an isoasymptotic curve on the surface. Also, if $\theta_r = m\pi$ then $V_2 \parallel \mathcal{N}$ meaning $\gamma(\phi)$ is an isogeodesic curve on the surface. Consequently, we obtain the following result.

Corollary 3.2. *Let the curve $\gamma(\phi) = (\phi, \zeta(\phi), \psi(\phi))$ be a line of curvature with torsion free on the surface is defined in (2.6). Then, we have*

if $\theta_r = (2m+1)\frac{\pi}{2}$ for any integer m, then $\gamma(\phi)$ is also isoasympotic,

if $\theta_r = m\pi$ for any integer m, then $\gamma(\phi)$ is also isogeodesic.

Example 3.3. Let $\gamma(\phi) = (\phi, 1 + sin\phi, sin\phi)$ be an admissible curve in $\mathbb{G}_3$. Then, we get first two curvatures as $\kappa_1 = \sqrt{2}sin\phi$ and $\kappa_2 = 0$. Also, the Frenet frame is given by

$$V_1(\phi) = (1, cos(\phi), \cos(\phi)), \quad V_2(\phi) = (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \quad V_3(\phi) = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

If we choose

$$\rho_1(\phi) = \phi^2, \ \Lambda_1(\varphi) = \varphi, \quad \rho_2(\phi) = \phi, \Lambda_2(\varphi) = \varphi sin(\theta_r), \quad \rho_3(\phi) = \phi, \Lambda_3(\phi) = -\varphi cos\theta_r$$

and take $\varphi_0 = 0$, $c_1 = c_2 = 1$, then a member of surface family in $\mathbb{G}_3$ is obtained by

$$\begin{aligned} S(\phi, \varphi) \quad &= \quad (\phi + \phi^2 \varphi, 1 + sin\phi - \frac{1}{\sqrt{2}} \phi\varphi sin(\theta_r) - \frac{1}{\sqrt{2}} \phi\varphi cos(\theta_r), \\ & sin\phi - \frac{1}{\sqrt{2}} \phi\varphi sin\theta_r + \frac{1}{\sqrt{2}} \phi\varphi cos\theta_r). \end{aligned}$$

By taking $R = [0, 2\pi] \times [0, 0.2]$, we visualize the curve $\gamma(\phi) = S(\phi, 0)$ in Fig.3.7 and

1. the surface $S(\phi, \varphi)$ in Fig. 3.8 for $\theta = \frac{\pi}{6}$;
2. curve on surface in Fig. 3.9 for $\theta = \frac{\pi}{6}$, in Fig. 3.10 for $\theta = \frac{\pi}{2}$; in Fig. 3.11 for $\theta = 0$.

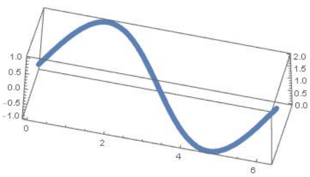

FIG. 3.7: Image of $\gamma(\phi)$

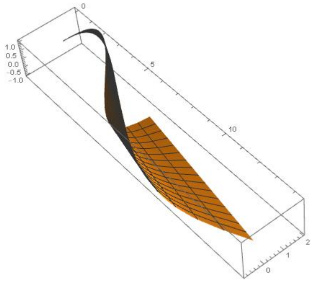

FIG. 3.8: Image of $S(\phi, \varphi)$

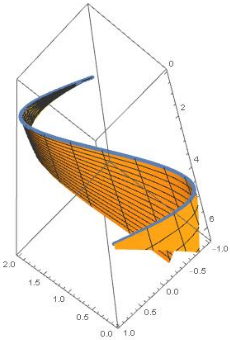

FIG. 3.9: $\gamma(\phi)$ on $S(\phi, \varphi)$

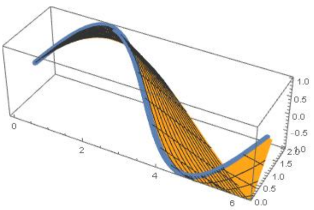

FIG. 3.10: $\gamma(\phi)$ on $S(\phi, \varphi)$

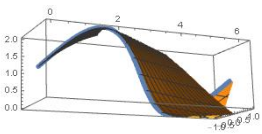

FIG. 3.11: $\gamma(\phi)$ on $S(\phi, \varphi)$

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REFERENCES

1. G.Ş. Atalay and E. Kasap: *Family of surface with a common null geodesic.* International Journal of Physical Sciences 4(8) (2009), 428-433.
2. G.Ş. Atalay and E. Kasap: *Surfaces family with common null asymptotic* Applied Mathematics and Computation 260 (2015), 135-139.
3. M.E. Aydin, M.A. Külahçi and A.O. Öğrenmiş: *Constant Curvature Translation Surfaces in Galilean 3-Space* International Electronic Journal of Geometry 12(1) (2019), 9-19.
4. W. Che, J-C. Paul and X. Zhang : *Lines of curvature and umbilical points for implicit surfaces* Computer Aided Geometric Design **24(7)** (2007), 395-409.
5. M. Dede :*Tubular surfaces in Galilean space* Mathematical Communications 18(1) (2013), 209-217.
6. B. Divjak, Z. Milin-Sipus: *Special curves on ruled surfaces in Galilean and pseudo-Galilean space* Acta Math. Hungar. 98 (2003), 203-215.
7. E. Ergün, E. Bayram, and E. Kasap : *Surface pencil with a common line of curvature in Minkowski 3-space* Acta Mathematica Sinica, English Series 30(12) (2014), 2103-2118.
8. E. Ergün, E. Bayram, and E. Kasap : *Surface family with a common natural line of curvature lift* Journal of Science and Arts 15(4) (2015), 321.
9. E. Kasap and F.T. Akyildiz: *Surfaces with common geodesic in Minkowski 3-space* Applied mathematics and computation 177(1) (2006), 260-270.
10. E. Kasap, F.T. Akyildiz and K. Orbay: *A generalization of surfaces family with common spatial geodesic* Applied Mathematics and Computation 201(1-2) (2008), 781-789.
11. C-Y. Li, R-H. Wang and C-G Zhu: *Parametric representation of a surface pencil with a common line of curvature* Computer-Aided Design 43(9) (2011), 1110-1117.
12. C-Y. Li, R-H. Wang, and C-G Zhu: *A generalization of surface family with common line of curvature* Applied Mathematics and Computation 219(17) (2013), 9500-9507.
13. C-Y. Li, R-H. Wang, and C-G Zhu: *Designing approximation minimal parametric surfaces with geodesics* Applied mathematical modelling 37(9) (2013), 6415-6424.
14. Z.E. Musielak and J. L. Fry : *Physical theories in Galilean space-time and the origin of Schrödinger-like equations* Annals of Physics, 324-2 (2009), 296-308.
15. A.O. Ögrenmis, M. Ergut and M. Bektas :*On the helices in the Galilean space* $\mathbb{G}_3$ Iran.J. Sci. Technol. Trans. A Sci. 31(2) (2007), 177-181.
16. A.O. Öğrenmiş, H. Öztekin and M. Ergüt: *Bertrand curves in Galilean space and their characterizations* Kragujevac Journal of Mathematics 32(32) (2009), 139-147.
17. H. Öztekin : *Special Bertrand curves in 4D Galilean space* Mathematical Problems in Engineering Hindawi Publishing Corporation 2014 (2014), 1-7.
18. O. Röschel, :*Die geometrie des Galileischen raumes Forschungszentrum Graz* Mathematisch-Statistische Sektion, Graz. 1(2) (1985).

19. D.J. Struik: *Lectures on classical differential geometry* Courier Corporation, 1961.
20. I.M. Yaglom: *A simple non-Euclidean geometry and its physical basis: An elementary account of Galilean geometry and the Galilean principle of relativit* Springer Science Business Media, 2012.
21. D.W. Yoon, J.W. Lee and C.W. Lee : *Osculating curves in the Galilean 4-Space* International Journal of Pure and Applied Mathematics 100(4) (2015) 497-506.
22. D.W. Yoon and Z.K. Yüzbasi : *An approach for hypersurface family with common geodesic curve in the 4D Galilean space* $\mathbb{G}_4$, The Pure and Applied Mathematics 25(4) (2018) 229-241.
23. Z.K. Yüzbaşi and M. Bektaş : *On the construction of a surface family with common geodesic in Galilean space* $\mathbb{G}_3$ Open Physics 14(1) (2016) 360-363.
24. Z.K. Yüzbasi : *On a family of surfaces with common asymptotic curve in the Galilean space* $\mathbb{G}_3$, J. Nonlinear Sci. Appl. 9 (2016) 518-523.
25. G-J. Wang, K. Tang and C-L. Tai : *Parametric representation of a surface pencil with a common spatial geodesic* Computer-Aided Design 36(5) (2004) 447-459.

Mustafa Altin
Technical Sciences Vocational School
Bingol University
12000 Bingöl, Turkey
maltin@bingol.edu.tr

İnan Ünal
Department of Computer Engineering
Munzur University
62000 Tunceli, Turkey
inanunal@munzur.edu.tr

FACTA UNIVERSITATIS (NIŠ)
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A CLASSIFICATION OF SOME ALMOST α-PARA-KENMOTSU MANIFOLDS *

Quanxiang Pan and Ximin Liu

Abstract. In this paper, we mainly study local structures and curvatures of the almost α-para-Kenmotsu manifolds. In particular, locally symmetric almost α-para-Kenmotsu manifolds satisfying certain nullity conditions are classified.
Key words: curvatures; α-para-Kenmotsu manifolds; nullity conditions.

1. Introduction

One of the recent topics in the theory of almost contact metric manifolds is the study of the so-called nullity distributions. In [5], E. Boeckx studied the full classification of contact (κ, μ)-spaces, later in [11] and [12], P. Dacko and Z. Olszak gave a systematic study of almost cosymplectic (κ, μ, ν)-spaces and almost cosymplectic $(-1, \mu, 0)$-spaces. G. Dileo and A. M. Pastore in [8] studied nullity distributions on almost Kenmotsu manifolds. In recent years, many authors have turned to the study of almost paracontact geometry due to an unexpected relationship between contact (κ, μ)-spaces and paracontact geometry that was found in [3].

The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [14] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in [16] by Zamkovoy. In fact, such manifolds were studied earlier in [17],[18],[6],[15] and in these papers the authors called such structures almost para-cohermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained in [13],[10],[16].

In [2], a complete study of paracontact metric manifolds satisfying a certain nullity condition has been carried out, later, in [9], the authors gave a complete study of almost α-cosymplectic manifolds, where α is a function, basic properties of such manifolds are obtained and general curvature identities are proved. It is

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also showed that almost α-para-Kenmotsu (κ, μ, ν)-spaces have para-Kähler leaves. Motivated by [7], [8] and [9], the aim of this paper is devoted to investigate local symmetry and nullity distributions on almost α-para-Kenmotsu manifolds.

This paper is organized in the following way. In section 2, some preliminaries and properties about almost α-para-Kenmotsu manifolds are given. In section 3, we characterize almost paracontact metric manifolds which are $\mathcal{CR}$-integrable almost α-para-Kenmotsu through the existence of a suitable linear connection, and in section 4, we investigate almost α-para-Kenmotsu manifolds which are locally symmetric and give some properties. In section 5, we study almost α-para-Kenmotsu manifolds satisfying some nullity distributions and give some properties and classification theorems of them.

2. Almost α-para-Kenmotsu manifolds

Now, we recall some basic notions of almost paracontact manifold (see [9]). A $2n+1$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\varphi^2 = \mathrm{Id} - \eta \otimes \xi, \quad \eta(\xi) = 1,$

(ii) the tensor field φ induces to an almost paracomplex structure on each fibre of $\mathcal{D}$ =Ker(η), i.e. the $\pm1-$eigendistributions $\mathcal{D}^{\pm} := \mathcal{D}_{\varphi}(\pm 1)$ of φ have equal dimension n.

From the definition, it follows that $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and rank$(\varphi) = 2n$. When the tensor field $\mathcal{N}_{\varphi} := [\varphi, \varphi] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.1}$$

for any vector fields $X, Y \in \Gamma(TM)$, then we say that $(M^{2n+1}, \varphi, \xi, \eta, g)$ is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n, n+1)$. For an almost paracontact metric manifold, there always exists an orthogonal basis$\{\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ such that $g(X_i, X_j) = \delta_{ij}, g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \varphi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\varphi-$basis. Moreover,we can define a skew-symmetric tensor field 2-form Φ by $\Phi(X, Y) := g(X, \varphi Y)$,which is usually called the fundamental form.

Lemma 2.1. ([*16*]) *For an almost paracontact structure (φ, ξ, η, g), the covariant derivative $\nabla\varphi$ of φ with respect to the Levi-Civita connection ∇ is given by*

$$\begin{aligned}2g((\nabla_X\varphi)Y, Z) = &- 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) - g(\mathcal{N}^{(1)}(Y, Z), \varphi X)\\ &+ \mathcal{N}^{(2)}(Y, Z)\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y).\end{aligned}$$

Definition 2.1. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold, if it satisfies

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

where α=const.$\neq 0$, then M^{2n+1} is called an almost α-para-Kenmotsu manifold.

Let M be an almost α-para-Kenmotsu manifold with structure (φ, ξ, η, g). Since the 1-form η is closed, then the distribution $\mathcal{D} = \ker(\eta)$ is integrable, we have $L_\xi \eta = 0$, and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. Then, using Lemma 2.1, the Levi-Civita connection is given by

$$2g((\nabla_X \varphi)Y, Z) = -2\alpha g(\eta(Y)\varphi X + g(X, \varphi Y)\xi, Z) - g(\mathcal{N}(Y, Z), \varphi X) \tag{2.2}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. If we replace X by ξ, it follows $\nabla_\xi \varphi = 0$, which implies that $\nabla_\xi \xi = 0$ and $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$.

The tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ and $h^{'} = h \cdot \varphi$ are symmetric operators anticommuting with φ and $h\xi = 0 = h^{'}\xi$, and we note that $\nabla_\xi h^{'} = 0$ if and only if $\nabla_\xi h = 0$. Let $Y = \xi$ in (2.2) we obtain

$$\nabla_X \xi = \alpha \varphi^2 X + \varphi h X \tag{2.3}$$

Proposition 2.1. *An almost α-para-Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ has para-Kähler leaves if and only if*

$$(\nabla_X \varphi)Y = g(\alpha\varphi X + hX, Y) - \eta(Y)(\alpha\varphi X + hX).$$

Theorem 2.1. *([9]) Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold with para-Kähler leaves. Then M^{2n+1} is para-Kenmotsu $(\alpha = 1)$ if and only if $\nabla_X \xi = \varphi^2 X$.*

Proposition 2.2. *([9]) Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y, Z \in \Gamma(TM^{2n+1})$,*

$$R(\xi, X)\xi = \alpha^2 \varphi^2 X + 2\alpha\varphi h X - h^2 X + \varphi(\nabla_\xi h)X, \tag{2.4}$$

$$\frac{1}{2}(R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) = \alpha^2 \varphi^2 X - h^2 X, \tag{2.5}$$

$$\begin{aligned} R(X, Y)\xi =& \alpha\eta(X)(\alpha Y + \varphi h Y) \\ & - \alpha\eta(Y)(\alpha X + \varphi h X) + (\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X, \end{aligned} \tag{2.6}$$

$$\begin{aligned} & g(R(\xi, X)Y, Z) + g(R(\xi, X)\varphi Y, \varphi Z) - g(R(\xi, \varphi X)\varphi Y, Z) \\ & - g(R(\xi, \varphi X)Y, \varphi Z) = 2(\nabla_{hX}\Phi)(Y, Z) + 2\alpha^2\eta(Y)g(X, Z) \\ & - 2\alpha^2\eta(Z)g(X, Y) - 2\alpha\eta(Z)g(\varphi h X, Y) + 2\alpha\eta(Y)g(\varphi h X, Z). \end{aligned} \tag{2.7}$$

Proposition 2.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Then, for any $X, Y \in \Gamma(TM^{2n+1})$,*

$$g(\mathcal{N}(\varphi X, Y), \xi) = 0. \tag{2.8}$$

Proof. By direct computations one has

$$g(\mathcal{N}(\varphi X, Y), \xi) = g([\varphi X, \varphi Y], \xi) = g((\nabla_X \varphi)Y - (\nabla_{\varphi Y} \varphi)\varphi X, \xi),$$

which implies (2.8) by using (2.2) and $[\xi, X] = -2\varphi h X$.

Theorem 2.2. *([9]) Let M^{2n+1} be an almost α-para-Kenmotsu manifold with $h = 0$. Then, M^{2n+1} is locally a warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kähler manifold, M_1 ia an open interval with coordinate t and $f^2 = we^{2\alpha t}$ for some positive constant.*

3. $\mathcal{CR}$-integrability

For an almost α-para-Kenmotsu manifold we have $[X, Y] - [\varphi X, \varphi Y] \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$, since $d\eta = 0$ and thus $\mathcal{D}$ is integrable. Hence, the structure (φ, ξ, η, g) is $\mathcal{CR}$-integrable if and only if $\mathcal{N}(X, Y) = [X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] = 0$ on $\mathcal{D}$, that is to the request that the integral manifolds of $\mathcal{D}$ are para-Kähler.

Theorem 3.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold. Then, M^{2n+1} is a $\mathcal{CR}$-integrable almost α-para-Kenmotsu manifold if and only if there exists a linear connection $\tilde{\nabla}$ such that*

1) $\tilde{\nabla}\varphi = 0$, $\tilde{\nabla} g = 0$, $\tilde{\nabla}\eta = 0$.

2) the torsion $\tilde{T}$ satisfies

a) $\tilde{T}(X, Y) = 0$ for any $X, Y \in D$,

b) $\tilde{T}(\xi, X) = X + h' X$ for any $X \in D$,

c) $\tilde{T}_\xi$ is selfadjoint.

Moreover, such a connection is uniquely determined by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(\alpha X - h' X, Y)\xi - \eta(Y)(\alpha X - h' X), \tag{3.1}$$

∇ being the Levi-Civita connection.

Proof. Let M^{2n+1} is a $\mathcal{CR}$-integrable almost α-para-Kenmotsu manifold. We put $\tilde{\nabla} = \nabla + H$, where the tensor field H of type (1,2) is defined by

$$H(X, Y) = g(\alpha X - h' X, Y)\xi - \eta(Y)(\alpha X - h' X).$$

Since $H(X, \varphi Y) - \varphi(H(X, Y)) = -(g(\alpha\varphi X + hX, Y) - \eta(Y)(\alpha\varphi X + hX)) = -(\nabla_X \varphi)Y$, owing to Proposition 2.1. By direct calculations, we get $g(H(X, Y), Z) + g(H(X, Z), Y) = 0$ and $(\nabla_X \eta)Y - \eta(H(X, Y)) = 0$, moreover, we get $\tilde{\nabla}\varphi = 0$, $\tilde{\nabla} g = 0$, $\tilde{\nabla}\eta = 0$. Since $\tilde{T}(X, Y) = \eta(X)(\alpha Y - h' Y) - \eta(Y)(\alpha X - h' X) = 0$ for any $X, Y \in \mathcal{D}$, and $\tilde{T}(\xi, X) = \alpha X - h' X$ for any $X \in \mathcal{D}$, hence $\tilde{T}_\xi$ is selfadjoint. As for the uniqueness and the vice versa part, the proof is similar with Theorem 3.1 in [8].
□

Corollary 3.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $\mathcal{CR}$-integrable almost α-para-Kenmotsu manifold. Then M^{2n+1} is a α-para-Kenmotsu manifold if and only if the linear connection $\tilde{\nabla}$ verifies $\tilde{T}_\xi \circ \varphi = \varphi \circ \tilde{T}_\xi$.*

Proof. Since $\tilde{T}_\xi \varphi X - \varphi \tilde{T}_\xi X = \tilde{T}(\xi, \varphi X) - \varphi \tilde{T}(\xi, X) = -2hX$ for any $X \in \mathcal{D}$, hence, Corollary 3.1 is easily followed by Theorem 3.1. □

4. Almost α-para-Kenmotsu manifolds and local symmetrys

In this section, we investigate almost α-para-Kenmotsu manifolds which are locally symmetric, that is, almost α-para-Kenmotsu manifolds satisfying the condition $\nabla R = 0$, which is a natural generalization of almost α-para-Kenmotsu manifold of constant curvature.

By similar proof as that of proposition 6 in [7], we get the following lemma

Lemma 4.1. *Let M^{2n+1} be a locally symmetric almost α-para-Kenmotsu manifold. Then, $\nabla_\xi h = 0$.*

Theorem 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost α-para-Kenmotsu manifold. Then, M^{2n+1} is an α-para-Kenmotsu manifold if and only if $h = 0$. Moreover, if any of the above equivalent conditions holds, M^{2n+1} has constant sectional curvature $c = -\alpha^2$.*

Proof. First, assuming that M^{2n+1} is an α-para-Kenmotsu manifold, by Theorem 2.1. we have $\nabla_X \xi = \alpha \varphi^2 X$, comparing with (2.3) it follows that $h = 0$ and by (2.6), we easily obtain $R(X, Y)\xi = -\alpha^2(\eta(Y)X - \eta(X)Y)$, let ∇_Z acting on the above equation and by the local symmetry, we have $R(X, Y)Z = -\alpha^2(g(Y, Z)X - g(X, Z)Y)$, it follows then M is of constant sectional curvature $c = -\alpha^2$. Now, supposing M' is the integral manifold of $\mathcal{D}$ and ∇' is the corresponding connection on M'. Then $\nabla_X Y = \nabla'_X Y + h(X, Y)$, then $h(X, Y) = g(\nabla_X Y, \xi)\xi = -\alpha g(X, Y)\xi$, this implies $H = -\alpha\xi$ thus $h(X, Y) = g(X, Y)H$, and M' is a totally umbilical submanifold of M^{2n+1}. What is more, it is not difficult to see that $R'(X, Y) = R(X, Y) + \alpha^2(g(Y, Z)X - g(X, Z)Y) = 0$, we know that M' is flat and the sectional curvature of M' vanishes. This means that M' is a flat para-Kähler manifold. For another part of the proof, noticing that $\nabla_Z \xi = \alpha\varphi^2 Z = \alpha Z$ if and only if $h = 0$, by Theorem 2.1 we prove that M^{2n+1} is an α-para-Kenmotsu manifold. At last, it is obvious from the proof of the equivalence that if any of the above equivalent conditions holds, M^{2n+1} has constant sectional curvature $c = -\alpha^2$. Thus, we complete the proof. □

Theorem 4.2. *An almost α-para-Kenmotsu manifold of constant curvature c is an α-para-Kenmotsu manifold and $c = -\alpha^2$.*

Proof. Supposing M^{2n+1} is an almost α-para-Kenmotsu manifold of constant sectional curvature c, it is obvious that

$$R(X, Y)Z = c(\eta(Y)X - \eta(X)Y). \tag{4.1}$$

∇_W acting on (4.1) we get $\nabla_W R = 0$, thus, M^{2n+1} is locally symmetric, by Lemma 4.1, we get $\nabla_\xi h = 0$. Comparing (2.6) with (4.1), we obtain

$$(c+\alpha^2)(\eta(Y)X-\eta(X)Y)+\alpha(\eta(Y)\varphi hX-\eta(X)\varphi hY)-(\nabla_X\varphi h)Y+(\nabla_Y\varphi h)X=0.$$

Choosing $X=\xi$ and $Y\in\mathcal{D}$ and by Lemma 4.1, we get

$$-(c+\alpha^2)Y-2\alpha\varphi hY+h^2X=0. \tag{4.2}$$

Now, if Y is an eigenvector of h with eigenvalue λ, then (4.2) becomes $-(c+\alpha^2)Y-2\alpha\lambda\varphi Y+\lambda^2X=0$. We get $\lambda=0$ and $c=-\alpha^2$ since Y and φY are linearly independent. Hence $h=0$ and $c=-\alpha^2$, by Theorem 4.1, we know M^{2n+1} is an α-para-Kenmotsu manifold of constant curvature $c=-\alpha^2$. Thus, we complete the proof. □

5. Almost α-para-Kenmotsu manifolds and nullity distributions

In this section, we study almost α-para-Kenmotsu manifolds under the assumption that ξ belongs to the (κ,μ)-nullity distribution and $(\kappa,\mu)'$-nullity distribution.

First, we consider the (κ,μ)-nullity distribution. if ξ belongs to the (κ,μ)-nullity distribution, $(\kappa,\mu)\in R^2$, denoted by $\mathcal{N}(\kappa,\mu)$,which is given by putting for each $p\in M^{2n+1}$,

$$\begin{aligned}\mathcal{N}_p(\kappa,\mu)=\{&Z\in\Gamma(T_pM^{2n+1})|R(X,Y)Z\\ &=\kappa(g(Y,Z)X-g(X,Z)Y)+\mu(g(Y,Z)hX-g(X,Z)hY)\}.\end{aligned}$$

So, if $\xi\in\mathcal{N}(\kappa,\mu)$, that is, for any X,Y$\in\Gamma(TM^{2n+1})$

$$R(X,Y)\xi=\kappa(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY).$$

Proposition 5.1. *([9]) Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu (κ,μ)-space. Then the following identities hold:*

$$h^2X=(\kappa+\alpha^2)\varphi^2X, \tag{5.1}$$

$$R(\xi,X)Y=\kappa(g(X,Y)\xi-\eta(Y)X)+\mu(g(X,hY)\xi-\eta(Y)hX), \tag{5.2}$$

$$Q\xi=-2nk\xi, \tag{5.3}$$

$$(\nabla_X\varphi)Y=g(\alpha\varphi X+hX,Y)-\eta(Y)(\alpha\varphi X+hX). \tag{5.4}$$

Theorem 5.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold. Let us suppose that $\xi\in\mathcal{N}(\kappa,\mu)$. Then, $\kappa=-1,h=0$ and M^{2n+1} is locally a warped product of an almost paraKähler manifold and an open interval. Moreover, assuming the local symmetry, M^{2n+1} is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$ of constant curvature $-\alpha^2$.*

Proof. $\xi \in \mathcal{N}(\kappa, \mu)$ means that $R(X,\xi)\xi = \kappa X + \mu hX$, for any unit vector field X orthogonal to ξ. Combining with (2.5), it follows that $h^2X = (\alpha^2 + \kappa)X$. Now, if X is a unit eigenvector of h with eigenvalue λ, we get $\lambda^2 = \alpha^2 + \kappa \geq 0$. It follows that $\kappa \geq -\alpha^2$ and $Spec(h) = \{0, \lambda, -\lambda\}$. Computing $R(X,\xi)\xi$ by means of (2.6), we easily obtain

$$R(X,\xi)\xi = -\alpha^2 X - 2\alpha\lambda\varphi X + \lambda^2 X - \lambda\varphi\nabla_\xi X + \varphi h\nabla_\xi X,$$

thus we have

$$(\kappa + \lambda\mu + \alpha^2 - \lambda^2)X + 2\alpha\lambda\varphi X + \lambda\varphi\nabla_\xi X - \varphi h\nabla_\xi X = 0,$$

and taking the scalar product with φX, we obtain $\alpha\lambda = 0$. Since $\alpha = const. \neq 0$, it follows that $\lambda = 0, h = 0, \kappa = -\alpha^2$ and thus $K(X,\xi) = -\alpha^2$.

Being $h = 0$, Theorem 2.2 ensures that M^{2n+1} is locally a warped product of an almost para-Kähler manifold and an open interval. Furthermore, if M^{2n+1} is locally symmetric, by Theorem 4.1, it is an α-para-Kenmotsu manifold locally isometric to $H^{2n+1}(-\alpha^2)$. Thus, we complete the proof. □

From Theorem 5.1 we know for almost α-para-Kenmotsu manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$, if $\xi \in \mathcal{N}(\kappa, \mu)$, then $\kappa = -1, h = 0$ and M^{2n+1} is locally a warped product of an almost para-Kähler manifold and an open interval. Therefore, we consider the $(\kappa, \mu)'$-nullity distribution, $(\kappa, \mu)' \in R^2$, as the distribution $\mathcal{N}(\kappa, \mu)'$ is given by putting for each $p \in M^{2n+1}$,

$$\begin{aligned} \mathcal{N}_p(\kappa, \mu)' = \{&Z \in \Gamma(T_pM^{2n+1}) | R(X,Y)Z \\ &= \kappa(g(Y,Z)X - g(X,Z)Y) + \mu(g(Y,Z)h'X - g(X,Z)h'Y)\}. \end{aligned} \tag{5.5}$$

So, if $\xi \in \mathcal{N}(\kappa, \mu)'$, that is, for any X,Y$\in \Gamma(TM^{2n+1})$

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y). \tag{5.6}$$

Theorem 5.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, \mu)'$ and $h' \neq 0$. Then, $\kappa < -\alpha^2, \mu = 2\alpha$ and $Spec(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigenvalue and $\lambda = \sqrt{-(\alpha^2 + \kappa)}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves.*

Proof. Let X be a unit vector field orthogonal to ξ, we have $R(X,\xi)\xi = kX + \mu h'X$ and if we suppose $X \in [\lambda]'$, since $h'^2 = -h^2$, combing with (2.5), we get $\lambda^2 = -(\kappa + \alpha^2) \geq 0$, then $\kappa \leq -\alpha^2$. $Spec(h') = \{0, \lambda, -\lambda\}$. Using (2.6) to compute $R(X,\xi)\xi$, we have

$$(\kappa + \lambda\mu + \alpha^2 - 2\alpha\lambda + \lambda^2)X - \lambda\nabla_\xi X + h'\nabla_\xi X = 0. \tag{5.7}$$

let (5.7) take the scalar product with X and φX respectively, we get $\lambda(\mu - 2\alpha) = 0$ and $\lambda g(\nabla_\xi X, \varphi X) = 0$. If $\lambda = 0$, then $h' = 0$ or equivalently $h = 0$, $N(\kappa, \mu) =$

$N(\kappa,\mu)'$ and Theorem 5.1 applies. Therefore, assuming $\lambda \neq 0$, it follows that $\kappa < -\alpha^2$ and $\mu = 2\alpha$, $g(\nabla_\xi X, \varphi X) = 0$ for any unit $X \in [\lambda]'$. Let (5.7) take the scalar product with any $Y \in [-\lambda]'$, we get $g(\nabla_\xi X, Y) = 0$ and thus $\nabla_\xi X \in [\lambda]'$. Analogously $\nabla_\xi Y \in [-\lambda]'$ and we obtain $\nabla_\xi h' = 0$. Comparing (5.6) with (2.6) for any $X, Y \in \mathcal{D}$, we have

$$(\nabla_X h')Y - (\nabla_Y h')X = 0. \tag{5.8}$$

If $X \in [\lambda]'$, by (2.3) we have $\nabla_X \xi = \alpha X - h'X = (\alpha - \lambda)X \in [\lambda]'$, and since $\nabla_\xi h' = 0$, we easily get $\nabla_\xi X \in [\lambda]'$. By (5.8) we have

$$0 = (\nabla_X h')Z - (\nabla_Z h')X = -\lambda \nabla_X Z - h'\nabla_X Z - \lambda \nabla_Z X + h'\nabla_Z X. \tag{5.9}$$

let (5.9) take the scalar product with $Y \in [-\lambda]'$, we get $g(\nabla_Z X, Y) = 0$, therefore $\nabla_Z X \in [\lambda]'$ since $g(\nabla_X Z, \xi) = 0$. For any $X, W \in [\lambda]'$, $Y, Z \in [-\lambda]'$ it follows that $\nabla_X W \in [\xi] \oplus [\lambda]'$ since $g(\nabla_X W, \xi) = (\lambda - \alpha) g(X, W)$. Hence, we get $g([X, W], \xi) = g(\nabla_X W - \nabla_W X, \xi) = 0$ and $g([X, W], Y) = g(\nabla_X W - \nabla_W X, Y) = 0$, thus $[X, W] \in [\lambda]'$. Similarly, it holds $[Y, Z] \in [-\lambda]'$. Therefore, the distributions $[\xi] \oplus [\lambda]'$,$[\xi] \oplus [-\lambda]'$, $[\lambda]'$ and $[-\lambda]'$ are integrable. It is easy to see that the distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are totally geodesic leaves. Now we prove the distribution $[\lambda]'$ is totally umbilical, we choose a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, with $e_i \in [\lambda]'$. The second fundamental form $h(e_i, e_j) = g(\nabla_{e_i} e_j, \xi)\xi = (\lambda - \alpha)\delta_{ij}\xi$, so the mean curvature vector field is $H = (\lambda - \alpha)\xi$, hence $h(X, W) = g(X, W)H$ and thus $[\lambda]'$ is totally umbilical. Similarly, we can get $[-\lambda]'$ is also totally umbilical with the mean curvature vector field is $H' = (\lambda + \alpha)\xi$ and $h'(Y, Z) = g(Y, Z)H'$. Thus, we complete the proof. □

Theorem 5.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, \mu)'$ and $h' \neq 0$. Then, the integral manifolds of $\mathcal{D}$ are para-Kähler manifolds.*

Proof. For any $X, Y, Z \in \mathcal{D}$, if $\xi \in \mathcal{N}(\kappa, \mu)'$, then $R(X, Y)\xi = 0$, (2.7) in Proposition 2.2 gives that $(\nabla_{hX}\Phi)(Y, Z) = 0$. Replacing X by hX, we get $(\nabla_{h^2 X}\Phi)(Y, Z) = 0$ or equivalently, $-\lambda^2(\nabla_X \Phi)(Y, Z) = 0$ since $h^2 X = -h'^2 X = -\lambda^2 X$ if X is a unit eigenvector of h' with eigenvalue λ. Being $\lambda \neq 0$, we get $(\nabla_X \Phi)(Y, Z) = 0$. Using (2.2) we obtain $g(N(Y, Z), \varphi X) = 0$, which together with (2.8) in Proposition 2.3 gives $\mathcal{N}(Y, Z) = 0$ for any $Y, Z \in \mathcal{D}$, therefore the integral manifolds of $\mathcal{D}$ are para-Kähler. Thus, we complete the proof. □

Corollary 5.1. *Any almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, \mu)'$, $\kappa < -\alpha^2$, is a $\mathcal{CR}$-manifold.*

Theorem 5.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a locally symmetric almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, \mu)'$ and $h' \neq 0$. Then, M^{2n+1} is locally isometric to $H^{n+1}(-(\lambda - \alpha)^2) \times R^n$.*

Proof. As proved in Theorem 5.2, the distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves and the distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. It follows that M^{2n+1} is locally isometric to the product of an integral manifold M_1^{n+1} of $[\xi]\oplus[\lambda]'$ and an integral manifold M_2^n of $[-\lambda]'$. Therefore, we can choose coordinates $(u^0,\dots,u^{2n})$ such that $\frac{\partial}{\partial u^0}\in[\xi]$, $\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^n}\in[\lambda]'$ and $\frac{\partial}{\partial u^{n+1}},\dots,\frac{\partial}{\partial u^{2n}}\in[-\lambda]'$. Now, we set $X_i=\frac{\partial}{\partial u^i}$ for any $i\in\{1,\dots,n\}$, so that the distribution $[-\lambda]'$ is spanned by the vector fields $\varphi X_1,\dots,\varphi X_n$. it is easy to see that $X_i\in[\lambda]'$ is projectable and $\varphi X_i\in[-\lambda]'$ is vertical, then $[X_i,\varphi X_j]$ is vertical by [1], hence $[X_i,\varphi X_j]\in[-\lambda]'$. Taking the scalar product with any $Z\in[\lambda]'$, since $\nabla_{X_i}\varphi X_j\in[-\lambda]'$, we get $g(\nabla_{\varphi X_j}X_i,Z)=0$ and then $\nabla_{\varphi X_j}X_i=0$. Applying $(\nabla_{\varphi X}\varphi)\varphi Y-(\nabla_X\varphi)Y=\alpha(\eta(Y)\varphi X+2g(X,\varphi Y)\xi)+\eta(Y)hX$ (appeared in [9]), we have $(\nabla_{X_i}\varphi)X_j+\varphi(\nabla_{\varphi X_i}\varphi X_j)=0$, which implies $(\nabla_{X_i}\varphi)X_j=0$, $\nabla_{\varphi X_i}\varphi X_j=0$, since the two part belong to $[-\lambda]'$ and $[\lambda]'$ respectively. $\nabla_{\varphi X_i}\varphi X_j=0$ means that M_2^n of $[-\lambda]'$ is flat. Now we compute the curvature of M_1^{n+1}. Applying φ to $(\nabla_{X_i}\varphi)X_j=0$ gives

$$\nabla_{X_i}X_j-\varphi\nabla_{X_i}\varphi X_j=(\lambda-\alpha)g(X_i,X_j)\xi.$$

Derivating with respect to X_k yields:

$$\begin{aligned}&\nabla_{X_k}\nabla_{X_i}X_j-(\nabla_{X_k}\varphi)(\nabla_{X_i}\varphi X_j)-\varphi\nabla_{X_k}\nabla_{X_i}\varphi X_j\\ =\;&(\lambda-\alpha)X_k(g(X_i,X_j))\xi-(\lambda-\alpha)^2g(X_i,X_j)X_k.\end{aligned}$$

taking the scalar product with X_l on both sides of the above equality and taking into account $g((\nabla_{X_k}\varphi)(\nabla_{X_i}\varphi X_j),X_l)=-g(\nabla_{X_i}\varphi X_j,(\nabla_{X_k}\varphi)X_l)=0$, we obtain

$$g(\nabla_{X_k}\nabla_{X_i}X_j,X_l)+g(\nabla_{X_k}\nabla_{X_i}\varphi X_j,\varphi X_l)=-(\lambda-\alpha)^2g(X_i,X_j)g(X_k,X_l).$$

Interchanging i and k, subtracting and being $[X_i,X_k]=0$ we have

$$\begin{aligned}&g(R(X_k,X_i)X_j,X_l)+g(R(X_k,X_i)\varphi X_j,\varphi X_l)\\ =\;&-(\lambda-\alpha)^2g(X_i,X_j)g(X_k,X_l)+(\lambda-\alpha)^2g(X_k,X_j)g(X_i,X_l).\end{aligned}$$

Since $\nabla_{\varphi X_i}\varphi X_j=0$ and $[\varphi X_i,\varphi X_j]=0$, by a straightforward calculation we obtain

$$g(R(X_k,X_i)\varphi X_j,\varphi X_l)=g(R(\varphi X_j,\varphi X_l)X_k,X_i)=0,$$

and thus

$$g(R(X_k,X_i)X_j,X_l)=-(\lambda-\alpha)^2[g(X_i,X_j)g(X_k,X_l)-g(X_k,X_j)g(X_i,X_l)].$$

Moreover, since $R(X,Y)\xi=0$ for any $X,Y\in\mathcal{D}$, we get $g(R(X_i,X_j)\xi,X_k)=0$. By (2.4) in Proposition 2.2, and $\nabla_\xi h=0$ because of the symmetry, we get $g(R(X_i,\xi)\xi,X_j)=-(\lambda-\alpha)^2g(X_i,X_j)$. Therefore, we conclude that the integral manifold M_1^{n+1} of $[\xi]\oplus[\lambda]'$ is a space of constant curvature $-(\lambda-\alpha)^2$. Thus, we complete the proof. □

Lemma 5.1. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold such that $\xi\in N(\kappa,2\alpha)'$ and $h'\neq 0$. Then, for any $X,Y\in\Gamma(TM^{2n+1})$,*

$$(\nabla_X h')Y = g(h'^2X-\alpha h'X,Y)\xi+\eta(Y)(h'^2X-\alpha h'X). \tag{5.10}$$

Proof. We choose a local orthonormal frame $\{\xi,e_i,\varphi e_i\}$ with $e_i\in[\lambda]'$.

1) If $X,Y\in[\lambda]'$, we know that $\nabla_XY\in[\xi]\oplus[\lambda]'$ from Theorem 5.2. It is easy to get

$$\nabla_XY=g(\nabla_XY,e_i)e_i+g(\nabla_XY,\xi)\xi=(\lambda-\alpha)g(X,Y)\xi+g(\nabla_XY,e_i)e_i,$$

and thus

$$(\nabla_Xh')Y=\nabla_Xh'Y-h'\nabla_XY=\lambda\nabla_XY-\lambda g(\nabla_XY,e_i)e_i=\lambda(\lambda-\alpha)g(X,Y)\xi.$$

2) If $X,Y\in[-\lambda]'$, we know that $\nabla_XY\in[\xi]\oplus[-\lambda]'$ from Theorem 5.2. Similarly we have

$$\nabla_XY=g(\nabla_XY,\varphi e_i)\varphi e_i+g(\nabla_XY,\xi)\xi=-(\lambda+\alpha)g(X,Y)\xi+g(\nabla_XY,\varphi e_i)\varphi e_i,$$

and

$$(\nabla_Xh')Y=\lambda(\lambda+\alpha)g(X,Y)\xi.$$

3) If $X\in[\lambda]'$, $Y\in[-\lambda]'$, since $g(\nabla_XY,\xi)=(\lambda-\alpha)g(X,Y)=0$, and for any $Z\in[\lambda]'$, $g(\nabla_XY,Z)=Xg(Y,Z)-g(Y,\nabla_XZ)=0$, thus we get $\nabla_XY\in[-\lambda]'$ and $(\nabla_Xh')Y=\nabla_Xh'Y-h'\nabla_XY=0$, therefore we have $(\nabla_Yh')X=0$ since $(\nabla_Xh')Y-(\nabla_Yh')X=0$.

Therefore, for any $X\in\Gamma(TM^{2n+1})$, we write $X=X_\lambda+X_{-\lambda}+\eta(X)\xi$, with $X_\lambda\in[\lambda]'$ and $X_{-\lambda}\in[-\lambda]'$, since $\nabla_\xi h'=0$, we get

$$\begin{aligned}(\nabla_Xh')Y &= (\nabla_{X_\lambda}h')Y_\lambda+\eta(Y)(\nabla_{X_\lambda}h')\xi+(\nabla_{X_{-\lambda}}h')Y_{-\lambda}+\eta(Y)(\nabla_{X_{-\lambda}}h')\xi\\ &= \lambda(\lambda-\alpha)g(X_\lambda,Y_\lambda)\xi+\lambda(\lambda-\alpha)\eta(Y)X_\lambda+\lambda(\lambda+\alpha)g(X_{-\lambda},Y_{-\lambda})\xi\\ &\quad+\lambda(\lambda+\alpha)\eta(Y)X_{-\lambda}\\ &= -\alpha\lambda\{g(X_\lambda,Y_\lambda)-g(X_{-\lambda},Y_{-\lambda})\}\xi+\lambda^2\{g(X_\lambda,Y_\lambda)+g(X_{-\lambda},Y_{-\lambda})\}\xi\\ &\quad+\eta(Y)(-\alpha\lambda X_\lambda+\alpha\lambda X_{-\lambda}+\lambda^2X_\lambda-\lambda^2X_{-\lambda})\\ &= g(h'^2X-\alpha h'X,Y)\xi+\eta(Y)(h'^2X-\alpha h'X).\end{aligned}$$

Lemma 5.2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold such that $\xi\in\mathcal{N}(\kappa,2\alpha)'$ and $h'\neq 0$. Then, for any $X,Y\in\mathcal{D}$,*

$$R(X,Y)h'Z-h'R(X,Y)Z=(\kappa+2\alpha^2)[g(Y,Z)h'X-g(X,Z)h'Y-g(h'Y,Z)X+g(h'X,Z)Y].$$

Proof. We know from Lemma 5.1 that $(\nabla_X h')Y = g(h'^2X - \alpha h'X, Y)\xi$ for any $X, Y \in \mathcal{D}$, by direct calculation we obtain

$$\begin{aligned}
&R(X,Y)h'Z - h'R(X,Y)Z \\
&= \nabla_X\nabla_Y h'Z - \nabla_Y\nabla_X h'Z - \nabla_{[X,Y]}h'Z - h'R(X,Y)Z \\
&= g((\nabla_X h'^2)Y - (\nabla_Y h'^2)X - \alpha((\nabla_X h')Y - (\nabla_Y h')X), Z) + g(h'^2Y - \alpha h'Y, Z)\nabla_X\xi \\
&- g(h'^2X - \alpha h'X, Z)\nabla_Y\xi - g(\nabla_Y\xi, Z)(h'^2X - \alpha h'X) + g(\nabla_X\xi, Z)(h'^2Y - \alpha h'Y).
\end{aligned}$$

It follows that for any $X, Y \in \mathcal{D}$, we know from $h'^2X = -h^2X = -(\kappa+\alpha^2)X$, and $(\nabla_X h'^2)Y = -(\kappa+\alpha^2)\eta(\nabla_X Y)\xi$, hence, $(\nabla_X h'^2)Y - (\nabla_Y h'^2)X = 0$ since $\mathcal{D}$ is integrable, and from Lemma 5.1, we get $(\nabla_X h')Y - (\nabla_Y h')X = 0$. Lemma 5.2 is followed by direct computation. Thus, we complete the proof. □

Lemma 5.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, 2\alpha)'$ and $h' \neq 0$. Then, for any $X, Y, Z \in \mathcal{D}$, we have*

$$\begin{aligned}
& R(X,Y)\varphi Z - \varphi R(X,Y)Z \\
= \;& g(\alpha X - h'X, \varphi Z)(\alpha Y - h'Y) - g(\alpha X - h'X, Z)(\alpha\varphi Y - \varphi h'Y) \\
& + g(\alpha Y - h'Y, Z)(\alpha\varphi X - \varphi h'X) - g(\alpha Y - h'Y, \varphi Z)(\alpha X - h'X).
\end{aligned}$$

Proof. Since the Weingarten operator for an integral manifold M' of $\mathcal{D}$ is given by

$$AX = -\nabla_X\xi = -(\alpha X - h'X),$$

by Theorem 2.3 in [4] we get the Guass equation

$$R(X,Y)Z = R'(X,Y)Z + g(\alpha X - h'X, Z)(\alpha Y - h'Y) - g(\alpha Y - h'Y, Z)(\alpha X - h'X).$$

By Theorem 5.3, the integral manifolds of $\mathcal{D}$ are para-Kähler manifolds, and from Lemma 10.1 of [4], we know $R'(X,Y)\varphi Z - \varphi R'(X,Y)Z = 0$. Combining with the above two equations, we get the required formula for R and φ. Thus, we complete the proof. □

Proposition 5.2. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, 2\alpha)'$ and $h' \neq 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the curvature tensor R satisfies:*

$$\begin{gathered}
R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \\
R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0, \\
R(X_\lambda, Y_{-\lambda})Z_\lambda = (\kappa + 2\alpha^2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(\kappa + 2\alpha^2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
R(X_\lambda, Y_\lambda)Z_\lambda = (\kappa + 2\alpha\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (\kappa - 2\alpha\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{gathered}$$

Proof. For any $X \in [\lambda]'$, and $Y, Z \in [-\lambda]'$, by Lemma 5.2 we have

$$-\lambda R(X,Y)Z - h'R(X,Y)Z = 2\lambda(\kappa + 2\alpha^2)g(Y,Z)X.$$

Taking the scalar product with $W \in [\lambda]'$, we obtain

$$g(R(X,Y)Z,W) = -(\kappa + 2\alpha^2)g(Y,Z)g(X,W). \tag{5.11}$$

Lemma 5.2 implies that $R(X,Y)Z \in [\lambda]'$ for any $X, Y, Z \in [\lambda]'$ and $R(X,Y)Z \in [-\lambda]'$ for any $X, Y, Z \in [-\lambda]'$. Now, in order to compute $R(X_\lambda, Y_\lambda)Z_{-\lambda}$, we consider a local orthonormal frame $\{\xi, e_i, \varphi e_i\}$, with $e_i \in [\lambda]'$. Condition $\xi \in N(\kappa, 2\alpha)'$ means that $g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \xi) = g(R(X_\lambda, Y_\lambda)\xi, Z_{-\lambda}) = 0$, and since $R(X_\lambda, Y_\lambda)e_i \in [\lambda]'$, thus $g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, e_i) = 0$. Using the first Bianchi identity and (5.11), we have

$$\begin{aligned} g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \varphi e_i) &= g(R(Y_\lambda, Z_{-\lambda})\varphi e_i, X_\lambda) - g(R(X_\lambda, Z-\lambda)\varphi e_i, Y_\lambda) \\ &= -(\kappa + 2\alpha^2)[g(Z_{-\lambda}, \varphi e_i)g(X_\lambda, Y_\lambda) - g(Z_{-\lambda}, \varphi e_i)g(X_\lambda, Y_\lambda)] \\ &= 0, \end{aligned}$$

so that $R(X_\lambda, Y_\lambda)Z_{-\lambda}=0$. The terms $R(X_{-\lambda}, Y_{-\lambda})Z_\lambda$, $R(X_\lambda, Y_{-\lambda})Z_\lambda$ and $R(X_\lambda, Y_{-\lambda})Z_{-\lambda}$ are computed in a similar manner. By Lemma 5.3, using $R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0$, we get

$$R(X_\lambda, Y_\lambda)\varphi Z_\lambda = -(\alpha - \lambda)^2[g(Y_\lambda, \varphi Z_{-\lambda})X_\lambda - g(X_\lambda, \varphi Z_{-\lambda})Y_\lambda]$$

Replacing $Z_{-\lambda}$ by $\varphi Z_\lambda \in [\lambda]'$, and since $-(\alpha - \lambda)^2 = \kappa + 2\alpha\lambda$, we have

$$R(X_\lambda, Y_\lambda)Z_\lambda = R(X_\lambda, Y_\lambda)\varphi(\varphi Z_\lambda) = (\kappa + 2\alpha\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda].$$

In the same manner, we obtain $R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (\kappa - 2\alpha\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]$. Thus, we complete the proof. □

Proposition 5.3. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa, 2\alpha)'$ and $h' \neq 0$. Then,we have*

1) $K(X, \xi) = \kappa + 2\alpha\lambda$, if $X \in [\lambda]'$;
$K(X, \xi) = \kappa - 2\alpha\lambda$, if $X \in [-\lambda]'$;
2) $K(X, Y) = \kappa + 2\alpha\lambda$, if $X, Y \in [\lambda]'$;
$K(X, Y) = \kappa - 2\alpha\lambda$, if $X, Y \in [-\lambda]'$;
$K(X, Y) = -(\kappa + 2\alpha^2)$, if$X \in [\lambda]', Y \in [-\lambda]'$.
3) $r = 8\alpha\lambda n - 4\alpha^2 n^2 - 2kn$.

Proof. The proof for the sectional curvature is easily followed by Proposition 5.2. In order to compute the scalar curvature, we choose a orthonormal frame $\{\xi, e_i, \varphi e_i\}$ with $e_i \in [\lambda]'$, by direct calculations we have

$$Ric(\xi, \xi) = \sum_{i=1}^{n} R(\xi, e_i, e_i, \xi) - \sum_{i=1}^{n} R(\xi, \varphi e_i, \varphi e_i, \xi) = 4\alpha\lambda n,$$

$$\begin{aligned} Ric(e_i,e_i) &= \sum_{i=1}^{n} R(e_i,\xi,\xi,e_i) + \sum_{j\neq i=1}^{n} R(e_i,e_j,e_j,e_i) - \sum_{j=1}^{n} R(e_i,\varphi e_j,\varphi e_j,e_i) \\ &= n(\kappa+2\alpha\lambda) + n(\kappa+2\alpha^2), \end{aligned}$$

$$Ric(\varphi e_i,\varphi e_i) = (\kappa-2\alpha\lambda)(2-n) + n(\kappa+2\alpha^2),$$

and it is easy to get the scalar curvature $r = 8\alpha\lambda n - 4\alpha^2 n^2 - 2kn$. □

Proposition 5.4. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost α-para-Kenmotsu manifold such that $\xi \in \mathcal{N}(\kappa,2\alpha)'$ and $h' \neq 0$. Then, M^{2n+1} is locally isometric to the warped products*

$$S^{n+1}(\kappa+2\alpha\lambda)\times_f R^n, \quad \text{or} \quad B^{n+1}(\kappa-2\alpha\lambda)\times_{f'} R^n,$$

where $S^{n+1}(\kappa+2\alpha\lambda)$ is a space of constant positive curvature $\kappa+2\alpha\lambda$, $B^{n+1}(\kappa-2\alpha\lambda)$ is a space of constant negative curvature $\kappa-2\alpha\lambda$, $f = ce^{-(\lambda+\alpha)t}$, $f' = c'e^{(\alpha-\lambda)t}$, with c, c' positive constants.

Proof. By Theorem 5.2, we get that the distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves, the distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. First, we consider that M^{2n+1} is locally a warped product $S\times_f F$ such that $TS = [\xi]\oplus[\lambda]'$ and $TF = [-\lambda]'$. Now, we compute the function f. We have denoted by $\breve{g}$ and $\hat{g}$ the pseudo-Riemannian metrics on S and F, respectively, such that the warped metric is given by $\breve{g} + f^2\hat{g}$. Then, the projection $\pi : S\times_f F \to S$ is a submersion with horizontal distribution $[\xi]\oplus[\lambda]'$ and vertical distribution $[-\lambda]'$. From Theorem 5.2 we know that the mean curvature vector field for the immersed submanifold $(F,\hat{g})$ is $H' = (\lambda+\alpha)\xi$. By Proposition 4.1 in [4], we get for any $Y,Z \in [-\lambda]'$, $nor(\nabla_Y Z) = h(Y,Z) = -\frac{g(Y,Z)}{f}\, grad_{\breve{g}} f$. And since $h(Y,Z) = g(Y,Z)H'$, we get $-(\lambda+\alpha)f\xi = grad_{\breve{g}} f$. We choose local coordinates $\{t,x^1,\ldots,x^n\}$ on B such that $\xi = \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i} \in [\lambda]'$ for any $i = 1,\ldots,n$. After direct computation we get $f = ce^{-(\lambda+\alpha)t}, c > 0$. Since $\xi \in \mathcal{N}(\kappa,2\alpha)'$, we have $R(X,Y)\xi = 0$, and $R(X,\xi)\xi = (\kappa+2\alpha\lambda)X$, also by $\xi \in \mathcal{N}(\kappa,2\alpha)'$, we get $R(\xi,X)Y = \kappa(g(X,Y)\xi - \eta(Y)X) + 2\alpha(g(h'X,Y)\xi - \eta(Y)'hX)$, thus, we get $R(\xi,X)Y = (\kappa+2\alpha\lambda)g(X,Y)\xi$. Applying Proposition 5.2, we get $R(X,Y)Z = (\kappa+2\alpha\lambda)[g(Y,Z)X - g(X,Z)Y]$, hence, we conclude that S is a space of constant curvature $\kappa+2\alpha\lambda > 0$. Next, we compute the curvature R^F of $(F,\hat{g})$, by Proposition 4.2 in [4], for any $U,V,W \in [-\lambda]'$, it holds

$$R^F(V,W)U = R(V,W)U - \frac{g(\text{grad}f,\text{grad}f)}{f^2}\{g(V,U)W - g(W,U)V\}.$$

Since $\text{grad}f = -(\lambda+\alpha)f\xi$, we get that $g(\text{grad}f,\text{grad}f) = (\lambda+\alpha)^2 f^2 = (2\alpha\lambda-\kappa)f^2$, and by Proposition 5.2, we get $R(V,W)U = (2\alpha\lambda-\kappa)\{g(V,U)W - g(W,U)V\}$. Then, $R^F(V,W)U = 0$, and thus the fibers of the warped product are flat spaces.

Similar discussions for horizontal distribution $[\xi]\oplus[-\lambda]'$ and vertical distribution $[\lambda]'$. In this case, the mean curvature vector field for the immersed submanifold

$(F, \hat{g})$ is $H' = (\lambda - \alpha)\xi$ and computing the warping function, we obtain $f' = c'e^{(\alpha-\lambda)t}, c' > 0$. Moreover, we can also prove that the base manifold of the warped product is a space of constant curvature $\kappa - 2\alpha\lambda < 0$ and the fibers are flat spaces. Thus, we complete the proof. □

REFERENCES

1. A. L. Besse: *E*instein manifolds. Springer-verlag, 1987.
2. B. C. Montano, I. K. Erken, C. Murathan: *N*ullity conditions in paracontact geometry. Diff. Geom. Appl. **30** (2012), 665–693.
3. B. C. Montano, L. D. Terlizzi: *G*eometric structures associated to a contact metric $(\kappa, \mu)-$space. Pacific J. Math. **246**(2) (2010), 257–292.
4. B. Y. Chen: *Pseudo-Riemannian geometry, $\delta-$ invariants and applications.* World Scientific, London, 2011
5. E. Boeckx: A full classification of contact $(\kappa, \mu)-$spaces, Illinois J. Math. **44**(2000) 212-219 *Pseudo-Riemannian geometry, $\delta-$ invariants and applications.* World Scientific, London, 2011
6. G. Calvaruso: *Harmonicity properties of invariant vector fields on three dimensional Lorentzian Lie groups.* J. Geom. Phys. **61** (2011), 498-515
7. G. Dileo, A. M. Pastore: *Almost kenmotsu manifolds and local symmetry* . Bull. Belg. math. Soc. Simon Stevin. **14**(2007), 343-354
8. G. Dileo, A. M. Pastore: *Almost kenmotsu manifolds and nullity distributions* . J.Geom. **93**(2009), 46-61
9. I. K. Erken , P. Dacko and C. Murathan: *Almost a-paracosymplectic manifolds.* J. Geom. Phys. **88**(2015), 30-51
10. J. Welyczko: *On basic curvature identities for almost (para)contact metric manifolds.* Available in Arxiv: 1209.4731[math. DG]
11. P. Dacko, Z. Olszak: *On almost cosymplectic $(\kappa, \mu, \nu)-$spaces.* Pdes Submanifolds and Affine Differential Geometry. 2005: 211-220
12. P. Dacko, Z. Olszak: *On almost cosymplectic $(-1, \mu, 0)-$spaces.* Centr. Eur. J. Math. **3** (2) (2005) 318-330
13. P. Dacko: *On almost para-cosymplectic manifolds.* Tsukuba J. Math. **28** (2004), 193-213
14. S. Kaneyuki, F. L. Williams: *Almost paracontact and parahodge structures on manifolds.* Nagoya Math. J. **99**(1985), 173-187
15. R. Rossca, L. Vanhecke: *Súr une variété presque paracokählérienne munie d'une connecxion self-orthogonale involutive.* Ann. Sti. Univ. "Al. I. Cuza" Ia si **22**(1976), 49-58
16. S. Zamkovoy: *Canonical connections on paracontact manifolds.* Ann.Glob. Anal. Geom. **36**(2009), 37-60 *A class of contact Riemannian manifold.* Tohoku Math. J. **24**(1972), 93-103
17. K. Buchner, R. Rosca: *Variétes para-coKählerian á champ concirculaire horizontale.* C. R. Acad. Sci. Paris 285(1997), ser. A,723-726

18. K. Buchner, R. Rosca: *Co-isotropic submanifolds of a para-coKählerian manifold with concicular vector field*. J. Geometry **25**(1985), 164-177

D. Allen: *Relations between the local and global structure of fnite semigroups*. Ph. D. Thesis, University of California, Berkeley, 1968.

19. P. Erdős: *On the distribution of the roots of orthogonal polynomials*. In: Proceedings of a Conference on Constructive Theory of Functions (G. Alexits, S. B. Steckhin, eds.), Akademiai Kiado, Budapest, 1972, pp. 145–150.
20. A. Ostrowski: *Solution of Equations and Systems of Equations*. Academic Press, New York, 1966.
21. E. B. Saff , R. S. Varga: *On incomplete polynomials II*. Pacific J. Math. **92** (1981), 161–172.

Quanxiang Pan
Henan Institute of Techonology
School of Science
453000 Xinxiang, China
panquanxiang@dlut.edu.cn

Ximin Liu
Dalian University of Technology
School of Mathematical Sciences
116024,Dalian, China
ximinliu@dlut.edu.cn

FACTA UNIVERSITATIS (NIŠ)
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APPROXIMATION BY JAIN-SCHURER OPERATORS

Nursel Çetin and Gülen Başcanbaz-Tunca

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Abstract. In this paper we deal with Jain-Schurer operators. We give an estimate, related to the degree of approximation, via moduli of smoothness of the first and the second order. Also, we present a Voronovskaja-type result. Moreover, we show that the Jain-Schurer operators preserve the properties of a modulus of continuity. Finally, we study monotonicity of the sequence of the Jain-Schurer operators when the attached function is convex and non-decreasing.

Keywords: Jain-Schurer operators; monotonicity; moduli of smoothness; Voronovskaja-type result.

1. Introduction

In [19], Schurer constructed the following linear positive operators

$$S_{n,p}(f;x) = e^{-(n+p)x} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(n+p)^k x^k}{k!}, \tag{1.1}$$

where $x \in [0,b],\ b < \infty,\ n \in \mathbb{N},\ p \geq 0$, and f is real valued and bounded function on $[0,\infty)$. The case $p = 0$ gives the the well known Szász-Mirakjan operators. There are a number of generalizations of Szász-Mirakjan operators, here we cite only a few ([4], [6], [10], [11]) with references therein. Some works concerning Schurer's setting can be found in [3], [14], [20], [16] and [17]. Motivated by these statements, we extend the well known Jain operators in the Schurer's design. Recall that in [12], Jain constructed the following linear positive operators

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) w_\beta(k;nx), \quad x \in (0,\infty), \tag{1.2}$$

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and $P_n^{[\beta]}(f;0) = f(0)$, where $n \in \mathbb{N}$, $\beta \in [0,1)$, $f \in C[0,\infty)$, and for $0 < \alpha < \infty$, $w_\beta(k;\alpha)$ is given by

$$w_\beta(k;\alpha) := \frac{\alpha(\alpha+k\beta)^{k-1}}{k!} e^{-(\alpha+k\beta)}, \quad k \in \mathbb{N} \cup \{0\} \tag{1.3}$$

and it satisfies $\sum_{k=0}^{\infty} w_\beta(k;\alpha) = 1$. In the paper, the author studied convergence properties and the order of approximation by the sequence of these operators on any finite closed interval of $[0,\infty)$ by taking β as a sequence β_n such that $\beta_n \to 0$ as $n \to \infty$. For some interesting papers concerning Jain operators, we refer to [1], [2], [7], [9], [18], [23] and references therein. Obviously, the case $\beta = 0$ gives the well known Szász-Mirakjan operators [22].

In this work, for a fixed $p \in \mathbb{N} \cup \{0\}$, we consider the linear positive operators denoted by $S_{n,p}^{\beta}$, $n \in \mathbb{N}$, and defined as

$$S_{n,p}^{\beta}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) w_\beta(k;(n+p)x), \quad x \in (0,\infty) \tag{1.4}$$

and $S_{n,p}^{\beta}(f;0) = f(0)$, for $f \in C_B[0,\infty) := \{f \in C[0,\infty) : f \text{ is bounded}\}$, $\beta \in [0,1)$, and $w_\beta(k;(n+p)x)$ given by (1.3). We call $S_{n,p}^{\beta}$ as Jain-Schurer operators. Note that, each $S_{n,p}^{\beta}$ maps $C_B[0,\infty)$ into itself, and the case $p = 0$ covers the Jain operators: $S_{n,0}^{\beta} = P_n^{[\beta]}$, $n \in \mathbb{N}$. On the other hand, in the case $\beta = 0$, $S_{n,p}^{\beta}$ reduces to the Schurer extension of the Szász-Mirakjan operators given by (1.1). We obtain an estimate, which will be used next for the rate of convergence, with the help of the modulus of smoothness of a bounded and continuous function, and prove a Voronovskaja-type result. Moreover, we show that each Jain-Schurer operator preserves the properties of a general modulus of continuity. Finally, we investigate the monotonicity of the sequence of the Jain-Schurer operators $S_{n,p}^{\beta}(f)$, with respect to n, when the function f is convex and non-decreasing.

Now, denoting $e_j(t) = t^j$, $j \in \mathbb{N} \cup \{0\}$ and $\varphi_x^j(t) := (t-x)^j$, $j \in \mathbb{N}$, for the Jain operators $P_n^{[\beta]}$ we have (see, e.g., [11, Lemma 1])

Lemma 1.1. *For the operators $P_n^{[\beta]}$ given by (1.2), one has*

$$\begin{aligned}
P_n^{[\beta]}(e_0;x) &= 1, \\
P_n^{[\beta]}(e_1;x) &= \frac{x}{1-\beta}, \\
P_n^{[\beta]}(e_2;x) &= \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}, \\
P_n^{[\beta]}(e_3;x) &= \frac{x^3}{(1-\beta)^3} + \frac{3x^2}{n(1-\beta)^4} + \frac{(1+2\beta)x}{n^2(1-\beta)^5}, \\
P_n^{[\beta]}(e_4;x) &= \frac{x^4}{(1-\beta)^4} + \frac{6x^3}{n(1-\beta)^5} + \frac{(8\beta+7)x^2}{n^2(1-\beta)^6} + \frac{\left(6\beta^2+8\beta+1\right)x}{n^3(1-\beta)^7}.
\end{aligned}$$

Making use of Lemma 1.1, straightforward computation shows that moments and central moments of the Jain-Schurer operators are obtained as in the following lemmas, respectively:

Lemma 1.2. *For the operators $S_{n,p}^{\beta}$ given by (1.4), one has*

$$S_{n,p}^{\beta}\left(e_j;x\right) = P_n^{[\beta]}\left(e_j;\left(\frac{n+p}{n}\right)x\right), \quad j=0,1,\ldots.$$

Lemma 1.3. *For the operators $S_{n,p}^{\beta}$ given by (1.4), one has*

$$\begin{aligned}
S_{n,p}^{\beta}\left(\varphi_x^1;x\right) &= \left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta},\\
S_{n,p}^{\beta}\left(\varphi_x^2;x\right) &= \left(\beta+\frac{p}{n}\right)^2\frac{x^2}{\left(1-\beta\right)^2}+\left(1+\frac{p}{n}\right)\frac{x}{n\left(1-\beta\right)^3},\\
S_{n,p}^{\beta}\left(\varphi_x^4;x\right) &= \left(\beta+\frac{p}{n}\right)^4\frac{x^4}{\left(1-\beta\right)^4}+6\left(\beta+\frac{p}{n}\right)^2\left(1+\frac{p}{n}\right)\frac{x^3}{n\left(1-\beta\right)^5}\\
&\quad+\left(1+\frac{p}{n}\right)\frac{\left(4n\beta+3n+8p\beta+7p+8n\beta^2\right)}{n^3\left(1-\beta\right)^6}x^2+\left(1+\frac{p}{n}\right)\frac{\left(6\beta^2+8\beta+1\right)}{n^3\left(1-\beta\right)^7}x\,.
\end{aligned}$$

2. Modulus of smoothness K-Functional

In this part of the paper, we extend the result proved by Agratini for the Jain operators [2, Theorem 2] to the Jain-Schurer operators. To this aim, we recall the terminology that will be used in the results. As usual, let $C_B\left[0,\infty\right)$ denote the space of real valued, bounded and continuous functions defined on $[0,\infty)$ equipped with the norm given by

$$\|f\| = \sup_{x\in[0,\infty)}\left|f\left(x\right)\right|$$

for $f\in C_B\left[0,\infty\right)$. Also, let $UC_B\left[0,\infty\right)$ denote the space of all real valued bounded and uniformly continuous functions on $[0,\infty)$.

For a bounded, real valued function f on $[0,\infty)$ and $\delta>0$, the first modulus of smoothness, modulus of continuity, of f is defined by

$$\omega_1\left(f,\delta\right) = \sup_{|h|\le\delta}\ \sup_{x,x+h\in[0,\infty)}\left|f\left(x+h\right)-f\left(x\right)\right|$$

and second modulus of smoothness of f is defined by

$$\omega_2\left(f,\delta\right) = \sup_{|h|\le\delta}\ \sup_{x+2h\in[0,\infty)}\left|f\left(x+2h\right)-2f\left(x+h\right)+f\left(x\right)\right|.$$

We have the following well known property of the modulus of smoothness (see, e.g., [3, p. 266, Lemma 5.1.1]).

Remark 2.1. *If* $f \in UC_B[0,\infty)$, *then* $\lim_{\delta\to 0^+}\omega_k(f,\delta)=0$ for $k=1,2$.

For convenience, we need the following Peetre's K-functional defined by

$$K(f,\delta)=\inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\|+\delta\|g''\|\right\},$$

where $\delta>0$ and

$$C_B^2[0,\infty)=\left\{g\in C_B[0,\infty): g', g''\in C_B[0,\infty)\right\}.$$

Note that the modulus of smoothness and the K-functional of an $f\in C_B[0,\infty)$ are related to each other as in the following sense: *There exist positive constants* C_1 *and* C_2 *such that*

$$C_1\omega_2(f,\delta)\le K\left(f,\delta^2\right)\le C_2\omega_2(f,\delta) \tag{2.1}$$

(see, e.g., [8, p. 177, Theorem 2.4]).

Below, we present a quantitative estimate to reach to the subsequent result concerning the rate of the approximation by $\left\{S_{n,p}^{\beta_n}(f;x)\right\}_{n\ge 1}$.

Theorem 2.1. *Let* $p\in\mathbb{N}_0$ *be fixed,* $0\le\beta<1$ *and* $f\in C_B[0,\infty)$. *Then, for each* $x\in(0,\infty)$, *one has*

$$\left|S_{n,p}^{\beta}(f;x)-f(x)\right|\le\omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)+C\omega_2\left(f,\delta_{n,p}^{\beta}(x)\right), \tag{2.2}$$

where $C>0$ *is a positive constant and*

$$\delta_{n,p}^{\beta}(x):=\frac{1}{2}\sqrt{\left(\beta+\frac{p}{n}\right)^2\frac{x^2}{(1-\beta)^2}+\left(1+\frac{p}{n}\right)\frac{x}{2n(1-\beta)^3}}. \tag{2.3}$$

Proof. Consider an auxiliary operator

$$\overline{S}_{n,p}^{\beta}(f;x):=S_{n,p}^{\beta}(f;x)+f(x)-f\left(\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}\right) \tag{2.4}$$

for $f\in C_B[0,\infty)$, $n\in\mathbb{N}$. In this case, $\overline{S}_{n,p}^{\beta}$ are linear and positive and each operator preserves the linear functions. Now, let $g\in C_B^2[0,\infty)$. From Taylor's formula about an arbitrary fixed point x, one has

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-y)g''(y)\,dy \tag{2.5}$$

for $t\in[0,\infty)$. Application of the operators $\overline{S}_{n,p}^{\beta}$ on both sides of (2.5) gives that

$$\overline{S}_{n,p}^{\beta}(g;x)-g(x)=g'(x)\overline{S}_{n,p}^{\beta}(t-x;x)+\overline{S}_{n,p}^{\beta}\left(\int_x^t(t-y)g''(y)\,dy;x\right). \tag{2.6}$$

Taking (2.4) into account for $f(t) = \int_x^t (t-y)\, g''(y)\, dy$, expression (2.6) reduces to

$$\overline{S}_{n,p}^{\beta}(g;x) - g(x) = S_{n,p}^{\beta}\left(\int_x^t (t-y)\, g''(y)\, dy; x\right) - \int_x^{\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}} \left[\left(1+\frac{p}{n}\right)\frac{x}{1-\beta} - y\right] g''(y)\, dy.$$

Using the fact

$$\left|\int_x^{\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}} \left[\left(1+\frac{p}{n}\right)\frac{x}{1-\beta} - y\right] g''(y)\, dy\right| \leq \frac{1}{2}\left(S_{n,p}^{\beta}\left(\varphi_x^1; x\right)\right)^2 \|g''\|,$$

by Lemma 1.3, we obtain

$$\begin{aligned}
&\left|\overline{S}_{n,p}^{\beta}(g;x) - g(x)\right| \\
\leq\ & S_{n,p}^{\beta}\left(\left|\int_x^t (t-y)\, g''(y)\, dy\right|; x\right) + \left|\int_x^{\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}} \left[\left(1+\frac{p}{n}\right)\frac{x}{1-\beta} - y\right] |g''(y)|\, dy\right| \\
\leq\ & \frac{\|g''\|}{2}\left[S_{n,p}^{\beta}\left(\varphi_x^2; x\right) + \left(S_{n,p}^{\beta}\left(\varphi_x^1; x\right)\right)^2\right] \\
\text{(2.7)}\quad & \frac{\|g''\|}{2}\left[2\left(\beta + \frac{p}{n}\right)^2 \frac{x^2}{(1-\beta)^2} + \left(1+\frac{p}{n}\right)\frac{x}{n(1-\beta)^3}\right].
\end{aligned}$$

On the other hand, from (2.4) and Lemma 1.2, it can be easily obtained that

$$\left|\overline{S}_{n,p}^{\beta}(f;x)\right| \leq \left|S_{n,p}^{\beta}(f;x)\right| + 2\|f\| \leq 3\|f\| \tag{2.8}$$

for $f \in C_B[0,\infty)$. Thus, taking (2.4), (2.7) and (2.8) into account, for $f, g \in C_B[0,\infty)$ one has

$$\begin{aligned}
&\left|S_{n,p}^{\beta}(f;x) - f(x)\right| \\
\leq\ & \left|\overline{S}_{n,p}^{\beta}(f-g;x) - (f-g)(x)\right| + \left|\overline{S}_{n,p}^{\beta}(g;x) - g(x)\right| \\
& + \left|f(x) - f\left(\left(1+\frac{p}{n}\right)\frac{x}{1-\beta}\right)\right| \\
\leq\ & \omega_1\left(f, \left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right) \\
& + 4\left\{\|f-g\| + \frac{1}{4}\left[\left(\beta+\frac{p}{n}\right)^2 \frac{x^2}{(1-\beta)^2} + \left(1+\frac{p}{n}\right)\frac{x}{2n(1-\beta)^3}\right]\|g''\|\right\}.
\end{aligned}$$

Finally, taking infimum over all $g \in C_B^2[0,\infty)$ on the right hand-side of the last inequality and applying (2.1), we get

$$\begin{aligned} \left|S_{n,p}^{\beta}(f;x)-f(x)\right| &\leq \omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)+K\left(f,\left(\delta_{n,p}^{\beta}(x)\right)^2\right) \\ &\leq \omega_1\left(f,\left(\beta+\frac{p}{n}\right)\frac{x}{1-\beta}\right)+C\omega_2\left(f,\delta_{n,p}^{\beta}(x)\right), \end{aligned}$$

where $\delta_{n,p}^{\beta}(x)$ is given by (2.3). □

Note that the case $p=0$ in the above theorem reduces to Theorem 2 in [2].

Taking Remark 2.1 and (2.2) into account, we reach to the following conclusion:

Corollary 2.1. *i) If β is taken as a sequence β_n such that $0 \leq \beta_n < 1$ and $lim_{n\to\infty}\beta_n = 0$ and $f \in UC_B[0,\infty)$, then one gets $\lim_{n\to\infty} S_{n,p}^{\beta_n}(f;x) = f(x)$ on $[0,\infty)$ and the order of the approximation does not exceed to that of $\omega_1\left(f,\left(\beta_n+\frac{p}{n}\right)\frac{x}{1-\beta_n}\right)+C\omega_2\left(f,\delta_{n,p}^{\beta_n}(x)\right)$.*

ii) If β is taken as a sequence β_n such that $0 \leq \beta_n < 1$ and $lim_{n\to\infty}\beta_n = 0$ and $f \in C_B[0,\infty)$, then $\left\{S_{n,p}^{\beta_n}(f)\right\}_{n\geq 1}$ converges uniformly to f on $[a,b]$, $0 \leq a < b < \infty$, by the well known Korovkin theorem.

3. A Voronovskaja-type result

In [9], Farcaş obtained the following Voronovskaja-type result for the Jain operator $P_n^{[\beta]}$ given by (1.2):

$$\lim_{n\to\infty} n\left\{P_n^{[\beta_n]}(f;x)-f(x)\right\} = \frac{x}{2}f''(x), \qquad x>0,$$

for $f \in C_2[0,\infty)$, the space of all continuous functions having continuous second order derivative, where $0 \leq \beta_n < 1$ is a sequence such that $\lim_{n\to\infty}\beta_n = 0$.

Note that a Voronovskaja-type result for a generalization of the Jain operators was obtained by Olgun et al. [18]. On the other hand, a Voronovskaja-type theorem as well as its a generalized form for Schurer setting of the Szász-Mirakjan operators were obtained by Sikkema in [20, p. 333].

In this part, we investigate a Voronovskaja-type result for the Jain-Schurer operators $S_{n,p}^{\beta}$, $n \in \mathbb{N}$.

Theorem 3.1. *Let $p \in \mathbb{N}_0$ be fixed and $0 \leq \beta_n < 1$ be a sequence such that $lim_{n\to\infty} n\beta_n = 0$. If f is bounded and continuous on $[0,\infty)$ and has the second order derivative at some $x \in (0,\infty)$, then one has*

$$\lim_{n\to\infty} n\left\{S_{n,p}^{\beta_n}(f;x)-f(x)\right\} = pxf'(x)+\frac{x}{2}f''(x).$$

Proof. From Taylor's formula, one has

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + h(t-x)(t-x)^2, \tag{3.1}$$

at the fixed point $x \in [0,\infty)$, where $h(t-x)$ is bounded for all $t \in [0,\infty)$ and $\lim_{t\to x} h(t-x) = 0$. Application of the operators $S_{n,p}^{\beta}$ to (3.1) implies

$$\begin{aligned} n\left[S_{n,p}^{\beta_n}(f;x) - f(x)\right] &= f'(x)\, nS_{n,p}^{\beta_n}(t-x;x) + \frac{1}{2} f''(x)\, nS_{n,p}^{\beta_n}\left((t-x)^2;x\right) \\ &\quad + nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right). \end{aligned}$$

Using the facts $\lim_{n\to\infty} n\beta_n = 0$ and Lemma 1.3, it readily follows that

$$\lim_{n\to\infty} nS_{n,p}^{\beta_n}(t-x;x) = px$$

and

$$\lim_{n\to\infty} nS_{n,p}^{\beta_n}\left((t-x)^2;x\right) = x.$$

Hence, we have

$$\lim_{n\to\infty} n\left(S_{n,p}^{\beta_n}(f;x) - f(x)\right) = pxf'(x) + \frac{x}{2} f''(x) + \lim_{n\to\infty} nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right).$$

It suffices to prove that $\lim_{n\to\infty} nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right) = 0$. Indeed, defining $h(0) = 0$ and taking the fact $\lim_{t\to x} h(t-x) = 0$ into account, we get that h is continuous at x. Hence, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|h(t-x)| < \varepsilon$ for all t satisfying $|t-x| < \delta$. On the other hand, since $h(t-x)$ is bounded on $[0,\infty)$, there is an $M > 0$ such that $|h(t-x)| \le M$ for all t. Therefore, we may write $|h(t-x)| \le M\frac{(t-x)^2}{\delta^2}$ when $|t-x| \ge \delta$. So, these arguments enable one to write $|h(t-x)| \le \varepsilon + M\frac{(t-x)^2}{\delta^2}$ for all t. The monotonicity and linearity of $S_{n,p}^{\beta_n}$ give that

$$\begin{aligned} S_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right) &\le \varepsilon S_{n,p}^{\beta_n}\left((t-x)^2;x\right) + \frac{M}{\delta^2} S_{n,p}^{\beta_n}\left((t-x)^4;x\right) \\ &= \varepsilon S_{n,p}^{\beta_n}\left(\varphi_x^2;x\right) + \frac{M}{\delta^2} S_{n,p}^{\beta_n}\left(\varphi_x^4;x\right). \end{aligned}$$

Making use of Lemma 1.3, with $\beta = \beta_n$,

$$\lim_{n\to\infty} nS_{n,p}^{\beta_n}\left(h(t-x)(t-x)^2;x\right) = 0,$$

by the hypothesis on β_n, which completes the proof. □

4. A Retaining Property

Recall that *A continuous and non-negative function* ω *defined on* $[0,\infty)$ *is called a modulus of continuity, if each of the following conditions is satisfied:*

i) $\omega(u+v) \le \omega(u) + \omega(v)$ *for* $u, v, u+v \in [0,\infty)$, *i.e.,* ω *is semi-additive,*

ii) $\omega(u) \ge \omega(v)$ *for* $u \ge v > 0$, *i.e.,* ω *is non-decreasing,*

iii) $\lim_{u \to 0^+} \omega(u) = \omega(0) = 0$, ([15, p. 106]).

In [13], Li proved that each Bernstein polynomial preserves the properties of modulus of continuity on $[0,1]$. Motivated by this result, in this section we will show that each Jain-Schurer operator has this preservation property as well. In the proof, we need the following Jensen formula

$$(4.1)\quad (u+v)(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u (u+k\beta)^{k-1} v [v+(m-k)\beta]^{m-k-1},$$

where u, v, and $\beta \in \mathbb{R}$ (see, e.g., [3, p. 326]).

Theorem 4.1. *Let* $p \in \mathbb{N}_0$ *be fixed and* $0 \le \beta < 1$. *If* ω *is a bounded modulus of continuity on* $[0,\infty)$, *then for each* $n \in \mathbb{N}$, $S_{n,p}^{\beta}(\omega; x)$ *is also a modulus of continuity.*

Proof. Let $x, y \in [0,\infty)$ *and* $x \le y$. *From the definition of* S_n^{β}, *we have*

$$(4.2)\qquad S_{n,p}^{\beta}(\omega; y) = \sum_{j=0}^{\infty} \omega\left(\frac{j}{n}\right) \frac{(n+p)\,y[(n+p)\,y + j\beta]^{j-1}}{j!} e^{-[(n+p)y+j\beta]}.$$

Taking $u = (n+p)\,x$, $v = (n+p)\,y - (n+p)\,x$ *and* $m = j$ *in (4.1), we obtain*

$$\begin{aligned} &(n+p)\,y\,((n+p)\,y + j\beta)^{j-1} \\ = &\sum_{k=0}^{j} \binom{j}{k} (n+p)\,x\,[(n+p)\,x + k\beta]^{k-1} \\ &\times (n+p)(y-x)\,[(n+p)(y-x) + (j-k)\,\beta]^{j-k-1}. \end{aligned}$$

Substituting this expression into (4.2) we get

$$\begin{aligned} S_{n,p}^{\beta}(\omega;y) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\omega\left(\frac{j}{n}\right)\binom{j}{k}\frac{1}{j!}(n+p)\,x\left[(n+p)\,x+k\beta\right]^{k-1} \\ &\quad\times(n+p)(y-x)\left[(n+p)(y-x)+(j-k)\beta\right]^{j-k-1}e^{-[(n+p)y+j\beta]} \\ &= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\omega\left(\frac{j}{n}\right)\frac{1}{k!\,(j-k)!}(n+p)\,x\left[(n+p)\,x+k\beta\right]^{k-1} \\ &\quad\times(n+p)(y-x)\left[(n+p)(y-x)+(j-k)\beta\right]^{j-k-1}e^{-[(n+p)y+j\beta]} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{k+l}{n}\right)\frac{1}{k!l!}(n+p)\,x\left[(n+p)\,x+k\beta\right]^{k-1} \\ &\quad\times(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}e^{-[(n+p)y+k\beta+l\beta]}. \end{aligned} \tag{4.3}$$

On the other hand, from (1.3), we have

$$e^{(n+p)(y-x)}=\sum_{l=0}^{\infty}\frac{(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}}{l!}e^{-l\beta}.$$

Therefore, $S_{n,p}^{\beta}(\omega;x)$ may be written as

$$\begin{aligned} S_{n,p}^{\beta}(\omega;x) &= \sum_{k=0}^{\infty}\omega\left(\frac{k}{n}\right)\frac{(n+p)\,x[(n+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+p)x+k\beta]} \\ &= \sum_{k=0}^{\infty}\omega\left(\frac{k}{n}\right)\frac{(n+p)\,x[(n+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+p)y+k\beta]}e^{(n+p)(y-x)} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{k}{n}\right)\frac{1}{k!l!}(n+p)\,x\left[(n+p)\,x+k\beta\right]^{k-1} \\ &\quad\times(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}e^{-[(n+p)y+k\beta+l\beta]}. \end{aligned} \tag{4.4}$$

Subtracting (4.4) from (4.3), we obtain

$$\begin{aligned} &S_{n,p}^{\beta}(\omega;y)-S_{n,p}^{\beta}(\omega;x) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\omega\left(\frac{k+l}{n}\right)-\omega\left(\frac{k}{n}\right)\right\}\frac{1}{k!l!}(n+p)\,x\left[(n+p)\,x+k\beta\right]^{k-1} \\ &\quad\times(n+p)(y-x)\left[(n+p)(y-x)+l\beta\right]^{l-1}e^{-[(n+p)y+(k+l)\beta]}. \end{aligned} \tag{4.5}$$

Using the semi-additivity property of ω, *we get*

$$
\begin{aligned}
& S_{n,p}^{\beta}(\omega; y) - S_{n,p}^{\beta}(\omega; x) \\
\leq\ & \sum_{k=0}^{\infty} \frac{(n+p)\, x\, [(n+p)\, x + k\beta]^{k-1}}{k!} e^{-k\beta} \\
& \times \sum_{l=0}^{\infty} \omega\left(\frac{l}{n}\right) \frac{(n+p)(y-x)[(n+p)(y-x)+l\beta]^{l-1}}{l!} e^{-[(n+p)y+l\beta]} \\
=\ & e^{(n+p)x} \sum_{l=0}^{\infty} \omega\left(\frac{l}{n}\right) \frac{(n+p)(y-x)[(n+p)(y-x)+l\beta]^{l-1}}{l!} e^{-[(n+p)y+l\beta]} \\
=\ & \sum_{l=0}^{\infty} \omega\left(\frac{l}{n}\right) \frac{(n+p)(y-x)[(n+p)(y-x)+l\beta]^{l-1}}{l!} e^{-[(n+p)(y-x)+l\beta]} \\
=\ & S_{n,p}^{\beta}(\omega; y-x),
\end{aligned}
$$

which shows the semi-additivity of $S_{n,p}^{\beta}$. *From (4.5) it readily follows that* $S_{n,p}^{\beta}(\omega; y) \geq S_{n,p}^{\beta}(\omega; x)$ *for* $y \geq x$, *i.e.,* $S_{n,p}^{\beta}$ *is non-decreasing. Moreover, since the series is uniformly convergent, it follows that* $\lim_{x\to 0^+} S_{n,p}^{\beta}(\omega; x) = S_{n,p}^{\beta}(\omega; 0) = \omega(0) = 0$. *This comples the proof.* □

5. Monotonicity of the sequence of the Jain-Schurer operators

In [5], Cheney and Sharma proved that the sequence of Szász-Mirakjan operators $P_n^{[0]}(f)$ is non-increasing in n, when f is convex. The purpose of this section is to observe the monotonicity of the sequence of the Jain-Schurer operators when the attached function is convex and non-decreasing and $p \neq 0$. In the case $p = 0$, we obtain monotonicity of the sequence of Jain operators in n when f is convex. For the proof, we further need the following Abel-Jensen formula

$$(u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} (u+k\beta)^k\, v\, [v+(m-k)\beta]^{m-k-1} \tag{5.1}$$

for non-negative real number β, where $u, v \in \mathbb{R}$ and $m \geq 1$ (see, e.g., [21]). Reasoning as in [5], we present the following result:

Theorem 5.1. *Let* f *be a non-decreasing and convex function on* $[0,\infty)$. *Then, for all* n, $S_{n,p}^{\beta}(f)$ *is non-increasing in* n *when* $p \neq 0$. *For the case* $p = 0$, *the same result holds when* f *is only convex on* $[0,\infty)$.

Proof. From (1.3), with $\alpha = x$, it is obvious that

$$e^x = \sum_{k=0}^{\infty} \frac{x\,(x+k\beta)^{k-1}}{k!} e^{-k\beta}. \tag{5.2}$$

Since $S_{n,p}^{\beta}(f;0)=f(0)$, we study only for $x>0$. Taking the definition of $S_{n,p}^{\beta}$ and (5.2) into consideration, one has

$$\begin{aligned}
&S_{n,p}^{\beta}(f;x)-S_{n+1,p}^{\beta}(f;x)\\
=\;& e^{x}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(n+p)\,x[(n+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\\
&-\sum_{k=0}^{\infty}f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\\
=\;& \sum_{l=0}^{\infty}\frac{x\,(x+l\beta)^{l-1}}{l!}e^{-l\beta}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(n+p)\,x[(n+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\\
&-\sum_{k=0}^{\infty}f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}.
\end{aligned}$$

By simple calculations, one can write

$$\begin{aligned}
&S_{n,p}^{\beta}(f;x)-S_{n+1,p}^{\beta}(f;x) \qquad (5.3)\\
=\;& \sum_{l=0}^{\infty}\frac{x\,(x+l\beta)^{l-1}}{l!}e^{-l\beta}\sum_{k=l}^{\infty}f\left(\frac{k-l}{n}\right)\frac{(n+p)\,x[(n+p)\,x+(k-l)\,\beta]^{k-l-1}}{(k-l)!}e^{-[(n+1+p)x+(k-l)\beta]}\\
&-\sum_{k=0}^{\infty}f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\\
=\;& \sum_{k=0}^{\infty}e^{-[(n+1+p)x+k\beta]}\left\{\sum_{l=0}^{k}f\left(\frac{k-l}{n}\right)\frac{(n+p)\,x[(n+p)\,x+(k-l)\,\beta]^{k-l-1}}{(k-l)!}\frac{x\,(x+l\beta)^{l-1}}{l!}\right.\\
&\left.-f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}\right\}\\
=\;& \sum_{k=0}^{\infty}e^{-[(n+1+p)x+k\beta]}\left\{\sum_{l=0}^{k}f\left(\frac{l}{n}\right)\frac{(n+p)\,x[(n+p)\,x+l\beta]^{l-1}}{l!}\frac{x\,[x+(k-l)\,\beta]^{k-l-1}}{(k-l)!}\right.\\
&\left.-f\left(\frac{k}{n+1}\right)\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}\right\}\\
=\;& \sum_{k=0}^{\infty}\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\times\\
&\left\{\sum_{l=0}^{k}\binom{k}{l}\frac{(n+p)\,x[(n+p)\,x+l\beta]^{l-1}x\,[x+(k-l)\,\beta]^{k-l-1}}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}f\left(\frac{l}{n}\right)-f\left(\frac{k}{n+1}\right)\right\}.
\end{aligned}$$

Now, it only remains to show that the curly bracket in the last formula must be non-negative. For this, we denote

$$\alpha_l:=\binom{k}{l}\frac{(n+p)\,x[(n+p)\,x+l\beta]^{l-1}x\,[x+(k-l)\,\beta]^{k-l-1}}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}>0,$$

and

$$x_l=\frac{l}{n}$$

for $l=0,1,...,k$. Now, replacing u with $(n+p)\,x,\ v$ with $x,\ m$ with k and k with l in (4.1) we evidently get

(5.4)

$$\sum_{l=0}^{k}\alpha_l=\frac{1}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}\,(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}=1.$$

On the other hand, it follows that

$$\begin{aligned}\sum_{l=0}^{k}\alpha_l x_l &= \frac{1}{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}\times\\ &\quad\sum_{l=0}^{k}\binom{k}{l}\frac{l}{n}\,(n+p)\,x[(n+p)\,x+l\beta]^{l-1}x\,[x+(k-l)\,\beta]^{k-l-1}\\ &= \frac{k\,(n+p)\,x}{n\,(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}\times\\ &\quad\sum_{l=0}^{k-1}\binom{k-1}{l}[(n+p)\,x+\beta+l\beta]^{l}x\,[x+(k-l-1)\,\beta]^{k-l-2}.\end{aligned}\tag{5.5}$$

Making use of the Abel-Jensen formula given by (5.1) for $u=(n+p)\,x+\beta,\ v=x,\ k=l,\ m=k-1$, (5.5) reduces to

$$\begin{aligned}\sum_{l=0}^{k}\alpha_l x_l &= \frac{k\,(n+p)\,x}{n\,(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}[(n+1+p)\,x+k\beta]^{k-1}\\ &= \frac{k\,(n+p)}{n\,(n+1+p)}.\end{aligned}\tag{5.6}$$

Taking into account (5.4), (5.6) and the convexity of f, (5.3) reduces to

$$\begin{aligned}&(5.7)\,S_{n,p}^{\beta}(f;x)-S_{n+1,p}^{\beta}(f;x)\\ &=\sum_{k=0}^{\infty}\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\left\{\sum_{l=0}^{k}\alpha_l f\left(\frac{l}{n}\right)-f\left(\frac{k}{n+1}\right)\right\}\\ &\geq\sum_{k=0}^{\infty}\frac{(n+1+p)\,x[(n+1+p)\,x+k\beta]^{k-1}}{k!}e^{-[(n+1+p)x+k\beta]}\left\{f\left(\frac{k\,(n+p)}{n\,(n+1+p)}\right)-f\left(\frac{k}{n+1}\right)\right\}.\end{aligned}$$

It is obvious that when $p=0$, (5.7) gives the non-negativity of $S_{n,0}^{\beta}(f;x)-S_{n+1,0}^{\beta}(f;x)$ under the convexity of f, which means that the sequence of Jain operators is non-increasing in n under the convexity of the function. On the other hand, for $p\in\mathbb{N}$ it follows that

$$\frac{k\,(n+p)}{n\,(n+1+p)}=\frac{k}{n+1}\,\frac{n+1}{n}\,\frac{n+p}{n+1+p}=\frac{k}{n+1}\,\frac{1+\frac{p}{n}}{\left(1+\frac{p}{n+1}\right)}.$$

Hence, one has

$$\frac{k\,(n+p)}{n\,(n+1+p)} \geq \frac{k}{n+1}$$

by the fact that $\frac{1+\frac{p}{n}}{1+\frac{p}{n+1}} \geq 1$. Then, the result follows directly from the non-decreasingness of f. $\square$

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REFERENCES

1. U. ABEL and O. AGRATINI: *Asymptotic behaviour of Jain operators.* Numer. Algor. **71** (2016), 553–565.
2. O. AGRATINI: *Approximation properties of a class of linear operators.* Math. Meth. Appl. Sci. **36**, (2013), no.17, 2353–2358.
3. F. ALTOMARE and M. CAMPITI: *Korovkin-type approximaton theory and its applications.* Walter de Gruyter, Berlin-New York, 1994.
4. M. BODUR, Ö. G. YILMAZ and A. ARAL: *Approximation by Baskakov-Szász-Stancu Operators Preserving Exponential Functions.* Constr. Math. Anal **1** (2018), no. 1, 1-8.
5. E. W. CHENEY and A. CHARMA: *Bernstein power series.* Can. J. Math. **16** (1964), 241-252.
6. N. ÇETİN and N. İSPİR: *Approximation by complex modified Szász-Mirakjan operators.* Studia Sci. Math. Hungar. **50** (2013), no. 3, 355–372.
7. E. DENİZ: *Quantitative estimates for Jain-Kantorovich operators.* Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **65** (2016), no. 2, 121–132.
8. R. A. DEVORE and G. G. LORENTZ: *Constructive Approximation.* Springer, Berlin, 1993.
9. A. FARCAŞ: *An asymptotic formula for Jain's operators.* Stud. Univ. Babeş-Bolyai Math. **57** (2012), no. 4, 511–517.
10. V. GUPTA and T. M. RASSIAS: *Direct estimates for certain Szász type operators.* Appl. Math. Comput. **251** (2015), 469–474.
11. V. GUPTA and G. C. GREUBEL: *Moment estimations of new Szász–Mirakyan–Durrmeyer operators.* Appl. Math. Comput. **271** (2015), 540-547.
12. G.C. JAIN: *Approximation of functions by a new class of linear operators.* J. Austral. Math. Soc. **13** (1972), no.3, 271–276.
13. Z. LI: *Bernstein polynomials and modulus of continuity.* J. Approx. Theory **102** (2000), no.1, 171-174.
14. A. J. LÓPEZ-MORENO, J. JÓDAR and F. J. MUÑOZ-DELGADO: *Exponential type moments and localization results for multivariate Baskakov-Schurer operators.* Int. J. Differ. Equ. Appl. **6** (2002), no.1, 53–67.

15. H. N. MHASKAR and D. V. PAI: *Fundamentals of approximation theory.* CRC Press, Boca Raton, FL, Narosa Publishing House, New Delhi, 2000.
16. D. MICLĂUŞ and O. T. POP: *The generalization of certain results for Szász-Mirakjan-Schurer operators.* Creat. Math. Inform. **21** (2012), no. 1, 79–85.
17. V. N. MISHRA and P. SHARMA: *On approximation properties of Baskakov-Schurer-Szász operators.* Appl. Math. Comput. **281** (2016), 381–393.
18. A. OLGUN, F. TAŞDELEN and A. ERENÇİN: *A generalization of Jain's operators.* Appl. Math. Comput. **266** (2015), 6–11.
19. F. SCHURER: *On Linear positive operators in approximation theory.* Thesis Delft, 1965.
20. P. C. SIKKEMA: *On some linear positive operators.* Indag. Math. **32** (1970), 327–337.
21. D. D. STANCU and E. I. STOICA: *On the use Abel-Jensen type combinatorial formulas for construction and investigation of some algebraic polynomial operators of approximation*, Stud. Univ. Babeş-Bolyai Math. **54** (2009), no.4, 167-182.
22. O. SZÁSZ: *Generalization of S. Bernstein's polynomials to infinite interval.* J. Research Nat. Bur. Standards **45** (1950), 239–245.
23. S. TARABIE: *On Jain-Beta linear operators.* Appl. Math. Inf. Sci. **6** (2012), no.2, 213–216.

Nursel Çetin
Turkish State Meteorological Service
Research Department
06560, Ankara, Turkey
nurselcetin07@gmail.com

Gülen Başcanbaz-Tunca
Ankara University
Faculty of Science
Department of Mathematics
06100, Tandogan Ankara, Turkey
tunca@science.ankara.edu.tr

FACTA UNIVERSITATIS (NIŠ)
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AUTHOMATED METHOD FOR DESIGNING FUZZY SYSTEMS *

Ivana Micić, Nada Damljanović and Zorana Jančić

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Abstract. The paper presents a method for building fuzzy systems using the input-output data that can be obtained from the examples. Using this method, a rule-based system is created, where fuzzy logic depends on the opinions and preferences of decision-makers involved in the process. Some advantages of the proposed method are highlighted. We have provided a practical example to illustrate the application of the process.
Keywords: fuzzy systems; rule-based system; fuzzy rule; membership function.

1. Introduction

Zadeh's fuzzy rule based systems deal with fuzzy rules instead of classical logic rules. Nowadays, they have been successfully used for modeling and control in different fields and industries [1, 2, 5, 15, 16].

Fuzzy rule based systems with fuzzifier and defuzzifier introduced by Mamdani [9, 10] are commonly known as fuzzy logic controllers. Mamdani fuzzy rule based systems deal with real-valued inputs and outputs, and therefore, they can be used in a wide range of real-world applications. The behavior of the system is guided by linguistic rules with the "IF-THEN" form whose premises and consequents are composed of fuzzy logic statements [3, 12, 14]. More on linguistic Mamdani-type fuzzy rule-based systems can be found in [17].

One of drawbacks of Mamdani fuzzy rule based systems can be viewed in a fact that good performance on input-output training data do not nonsensically led to good performance on novel inputs [4, 6, 11]. Therefore, a construction of fuzzy functions and corresponding base of rules based on inclusion of expert knowledge into the process is proposed.

The model presented in this paper is shown to be very good, because of its flexibility, therefore it can be very easy for implementing and application in various

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fields. In this work, trough one illustrative example, it will be shown that greater number of functions for presenting the input data will give much better results, and therefore, the flexibility od the model is limited by the lower bound in the number of functions for presenting input data.

2. Automated method for designing fuzzy systems based on learning from example

The input of the observed fuzzy system is a set of N input-output pairs, of the form

$$\{(X_0^p, y_0^p)\}, \quad p \in 1, \ldots, N, \tag{2.1}$$

where $X_0^p \in U = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subset R^n$ and $y_0^p \in V = [\alpha_y, \beta_y] \subset R$. Clearly, the input of a fuzzy system (2.1) is the collection of data given by Table 2.1.

Table 2.1: Input-output data collection

Input	C_1	C_2	...	C_i	...	C_n	Output
X_0^1	x_{01}^1	x_{02}^1	...	x_{0i}^1	...	x_{0n}^1	y_0^1
X_0^2	x_{01}^2	x_{02}^2	...	x_{0i}^2	...	x_{0n}^2	y_0^2
			...		...		
X_0^p	x_{01}^p	x_{02}^p	...	x_{0i}^p	...	x_{0n}^p	y_0^p
			...		...		
X_0^N	x_{01}^N	x_{02}^N	...	x_{0i}^N	...	x_{0n}^N	y_0^N

Designing the fuzzy system based on these input-output data collection can be described in the the following five steps.

Step 1. *Experts opinion*

For each $i \in \{1, 2, \ldots, n\}$ and corresponding attribute C_i, values represented in the ith column of Table 2.1 can have different importance to a decision expert. Some values are extremely important, while others are totally unacceptable. On the other hand, different decision experts can have different intuition and preferences on what's important. Therefore, N_i decision experts are involved to express their preference on attribute C_i.

For each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, N_i\}$, the jth expert on attribute C_i choose four elements $a_i^j, b_i^j, c_i^j, d_i^j \in [\alpha_i, \beta_i]$.

Table 2.2: Expert's preference on atributes

a_i^j	b_i^j	c_i^j	d_i^j

For example, let attribute C_i represent price of some article (or service) which can generally range between α_i and β_i. Given values a_i^j, b_i^j, c_i^j and d_i^j have the following meaning: If the price is lower than a_i or it is higher than d_i, then we are not interested in buying that article (too cheap or too expensive items are not interesting to us). If the price is between b_i^j and c_i^j, then we are absolutely interested in buying the article (shopping surely). As price goes from a_i to b_i, we are increasingly interested for buying it, and if price goes from c_i to d_i our interest in the purchase of item drops.

In this way, for each attribute C_i $(i = 1, 2, \ldots, n)$, we have determined N_i fuzzy sets

$$A_i^j : [\alpha_i, \beta_i] \to [0,1], \qquad j = 1, 2, \ldots, N_i, \tag{2.2}$$

as follows:

$$A_i^j(x) = \begin{cases} 0, & \alpha_i \leqslant x \leq a_i^j \text{ or } d_i^j \leqslant x \leq \beta_i; \\ \frac{x - a_i^j}{b_i^j - a_i^j}, & a_i^j \leqslant x \leq b_i^j; \\ 1, & b_i^j \leqslant x \leq c_i^j; \\ \frac{x - d_i^j}{c_i^j - d_i^j}, & c_i^j \leqslant x \leq d_i^j. \end{cases}$$

It is assumed that, for each $i = 1, 2, \ldots, n$, the set of fuzzy functions (2.2) is complete in $[\alpha_i, \beta_i]$, i.e., for every $x_i \in [\alpha_i, \beta_i]$, there exists A_i^j such that $\mu_{A_i^j}(x_i) \neq 0$.

With similar arguments, N_y decision experts are involved to express their preference on output column $(y_0^1, y_0^2, \ldots, y_0^N)^T$.

Table 2.3: Expert's preference on output

a_i	b_i	c_i	d_i

Consequently, N_y fuzzy sets

$$B^j : [\alpha_y, \beta_y] \to [0,1], \qquad j = 1, 2, \ldots, N_y, \tag{2.3}$$

are defined in the following way:

$$B^j(x) = \begin{cases} 0, & \alpha_y \leqslant x \leq a^j \text{ or } d^j \leqslant x \leqslant \beta_y; \\ \frac{x - a^j}{b^j - a^j}, & a^j \leqslant x \leq b^j; \\ 1, & b^j \leqslant x \leq c^j; \\ \frac{x - d^j}{c^j - d^j}, & c^j \leqslant x \leq d^j, \end{cases}$$

Again, the assumption is that they are complete in $[\alpha_y, \beta_y]$.

One can notice that in the case of incompleteness of obtained fuzzy sets (2.2) or (2.3), the number of experts being examined must increase. Also, let us notice that obtained fuzzy sets are trapezoidal, and in a naturally way they can be transformed to triangular fuzzy sets or singletons.

Step 2. *Rules generated by input-output data*

In this step, for every input-output pair

$$(X_0^p, y_0^p), \qquad p = 1, 2, \ldots, N,$$

and corresponding inputs and output

$$x_{0i}^p, \quad i = 1, 2, \ldots, n \quad \text{and} \quad y_0^p,$$

we will determine the membership values

$$A_i^j(x_{0i}^p), \qquad j = 1, 2, \ldots, N_i,$$

and the membership values

$$B^l(y_0^p), \qquad l = 1, 2, \ldots, N_y.$$

Then for every input variable x_{0i}^p, $i = 1, 2, \ldots, n$, we will determine the fuzzy set in which x_{0i}^p has the largest membership value, that is, we will determine A_i^{j*} such that

$$A_i^{j*}(x_{0i}^p) \geqslant A_i^j(x_{0i}^p), \quad j = 1, 2, \ldots, N_i.$$

Similarly, we will determine B^{l*} such that

$$B^{l*}(y_0^p) \geqslant B^l(y_0^p), \quad 1 = 1, 2, \ldots, N_y.$$

Finally, we obtain a fuzzy IF-THEN rule as

$$\text{IF} \quad x_1 = A_1^{j*} \quad \text{and} \quad \cdots \quad \text{and} \quad x_n = A_n^{j*} \quad \text{THEN} \quad y = B^{l*}. \tag{2.4}$$

Step 3. *Degrees of fuzzy rules*

Since the number of input-output pairs is usually large, and for every pair one rule is generated, it is highly likely that there are conflicting rules, i.e., there are rules with the same IF part and different THEN part. In order to overcome this conflict, the degree to each rule generated in Step 2 is assigned and only one rule from a conflicting group that has the maximum degree is chosen. That procedure resolves the conflict problem, but also reduced the number of rules.

The degree of the rule, denoted by D, is defined as follows: Let the rule (2.4) be generated by a pair (X_0^p, y_0^p), then its degree is defined by:

$$D(rule) = \prod_{i=1}^{n} A_1^{j*}(x_{0i}^p) \cdot B^{l*}(y_0^p) \tag{2.5}$$

If the input-output pairs have different reliability and we can determine a number to asses it, we may incorporate this information into the degrees of the rules.

Specifically, suppose the input-output pair (X_0^p, y_0^p) has the degree $\mu^p \in [0,1]$, then the degree of the rule generated by a pair (X_0^p, y_0^p) is defined by:

$$D(rule) = \prod_{i=1}^{n} A_1^{j*}(x_{0i}^p) \cdot B^{l*}(y_0^p) \cdot \mu^p. \tag{2.6}$$

In practice, an expert may check the data (if the number of input-output pairs is small) and estimate the degree μ^p. If we cannot tell the difference among the input-output pairs, we simply choose all μ^p value 1, in that way (2.6) is reduced to (2.5).

Step 4. *Fuzzy rule base*

The fuzzy rule base consists of the following set of rules:

1. The rules generated in Step 2 that do not conflict with any other rules;
2. The rule from a conflicting group that has the maximum degree, where a group of conflicting rules consists of rules with the same IF parts;
3. Linguistic rules from human experts (due to conscious knowledge).

Step 5. *Fuzzy system*

In this step of algorithm, the fuzzy system is constructed based on the fuzzy rule base obtained in Step 4 (see [13, 17]).

In the sequel, we present an simple example, with a small amount of input-output data, which will illustrate working of the previous procedure and problems that may occur when using this method.

3. Example

In order to rate the quality of service offered by the hotel, one hotel booking site measures two components - cleanliness and comfort. Cleanliness takes values from interval $[0,6]$, while comfort takes values from interval $[0,11]$. According to these components, as a result the rating of hotel, which takes values from 1 to 5, is obtained. The following table presents the rate of the quality that customers specified based on the ratings they gave for cleanliness and comfort:

For this two input - one output space system, we will present how using of automated method for designing fuzzy systems based on learning from example works. Moreover, we will compare the results for certain value, which is obtained when for the same system, we change only the number of membership functions used for presenting the input data.

Table 3.1: Input-output data of example

Id	Clean	Comfort	Rate
1	2.8	2	2
2	3.9	8.2	4
3	1.2	5	2
4	2	8.4	3
5	5	10.3	5
6	5	9.2	4
7	4	4	3
8	3.7	1	1
9	4	9.8	5
10	4	8.7	4
11	2	9	3
12	1.3	5.7	2
13	0.8	4.1	1
14	3	9.4	3
15	3.1	9.9	4

For presenting the cleanliness and the comfort the trapezoid functions will be used. The input space is $U_x = [0,6] \times [0,11]$. For presenting the rate, singleton functions will be used, and the output space is $U_y = \{1,..,5\}$.

In the first case, we will have 4 membership functions for the cleanliness and 6 for comfort. Trapezoid functions $A_1, A_2, A_3, A_4 : [0,6] \to [0,1]$ for cleanliness:

$$A_1(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1; \\ \frac{2-x}{1}, & 1 \leqslant x \leqslant 2; \\ 0, & \text{otherwise.} \end{cases} \qquad A_2(x) = \begin{cases} \frac{x-1}{0.5}, & 1 \leqslant x \leqslant 1.5; \\ 1, & 1.5 \leqslant x \leqslant 2.5; \\ \frac{3-x}{0.5}, & 2.5 \leqslant x \leqslant 3; \\ 0, & \text{otherwise.} \end{cases}$$

$$A_3(x) = \begin{cases} \frac{x-2}{1}, & 2 \leqslant x \leqslant 3; \\ 1, & 3 \leqslant x \leqslant 4; \\ \frac{5-x}{1}, & 4 \leqslant x \leqslant 5; \\ 0, & \text{otherwise.} \end{cases} \qquad A_4(x) = \begin{cases} \frac{x-3}{2}, & 3 \leqslant x \leqslant 5; \\ 1, & 5 \leqslant x \leqslant 6; \\ 0, & \text{otherwise.} \end{cases}$$

Trapezoid functions $B_1, .., B_6 : [0,11] \to [0,1]$ for comfort:

$$B_1(x_2) = \begin{cases} 1, & 0 \leqslant x_2 \leqslant 1; \\ \frac{2.5-x}{1.5}, & 1 \leqslant x_2 \leqslant 2.5; \\ 0, & \text{otherwise.} \end{cases} \qquad B_2(x_2) = \begin{cases} \frac{x-0.5}{1.5}, & 0.5 \leqslant x_2 \leqslant 2; \\ 1, & 2 \leqslant x_2 \leqslant 3; \\ \frac{4.5-x}{1.5}, & 3 \leqslant x_2 \leqslant 4.5; \\ 0, & \text{otherwise.} \end{cases}$$

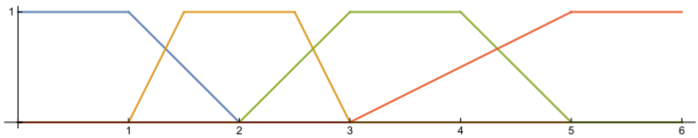

FIG. 3.1: Membership functions A_1 (blue), A_2 (yellow), A_3 (green) and A_4 (red).

$$B_3(x_2) = \begin{cases} \frac{x_2-2.5}{1.5}, & 2.5 \leqslant x_2 \leqslant 4; \\ 1, & 4 \leqslant x_2 \leqslant 5; \\ \frac{6.5-x_2}{1.5}, & 5 \leqslant x_2 \leqslant 6.5; \\ 0, & \text{otherwise}. \end{cases} \quad B_4(x_2) = \begin{cases} \frac{x_2-4.5}{1.5}, & 4.5 \leqslant x_2 \leqslant 6; \\ 1, & 6 \leqslant x_2 \leqslant 7; \\ \frac{6.5-x_2}{1.5}, & 6.5 \leqslant x \leqslant 8; \\ 0, & \text{otherwise}. \end{cases}$$

$$B_5(x_2) = \begin{cases} \frac{x_2-6.5}{1.5}, & 6.5 \leqslant x_2 \leqslant 8; \\ 1, & 8 \leqslant x_2 \leqslant 9; \\ \frac{10.5-x_2}{1.5}, & 9 \leqslant x_2 \leqslant 10.5; \\ 0, & \text{otherwise}. \end{cases} \quad B_6(x_2) = \begin{cases} \frac{x_2-8.5}{1.5}, & 8.5 \leqslant x_2 \leqslant 10; \\ 1, & 10 \leqslant x_2 \leqslant 11; \\ 0, & \text{otherwise}. \end{cases}$$

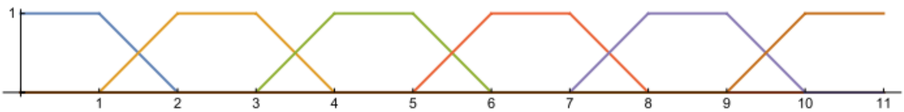

FIG. 3.2: Membership functions B_1 (blue), B_2 (yellow), B_3 (green), B_4 (red), B_5 (purple) and B_6 (brown).

In the second case, we will have 2 membership functions for the cleanliness and 3 for comfort. Trapezoid functions $A'_1, A'_2 : [0, 6] \to [0, 1]$ for cleanliness:

$$A'_1(x_1) = \begin{cases} 1, & 0 \leqslant x_1 \leqslant 2; \\ \frac{3.5-x_1}{1.5}, & 2 \leqslant x_1 \leqslant 3.5; \\ 0, & \text{otherwise}. \end{cases} \quad A'_2(x_1) = \begin{cases} \frac{x_1-1.5}{1.5}, & 1.5 \leqslant x_1 \leqslant 3; \\ 1, & 3 \leqslant x_1 \leqslant 5; \\ 0, & \text{otherwise}. \end{cases}$$

Trapezoid functions $B'_1, .., B'_3 : [0, 11] \to [0, 1]$ for comfort:

$$B'_1(x_2) = \begin{cases} 1, & 0 \leqslant x_2 \leqslant 3; \\ \frac{6-x_2}{3}, & 3 \leqslant x_2 \leqslant 6; \\ 0, & \text{otherwise}. \end{cases} \quad B'_2(x_2) = \begin{cases} \frac{x_2-2}{3}, & 2 \leqslant x_2 \leqslant 5; \\ 1, & 5 \leqslant x_2 \leqslant 7; \\ \frac{10-x_2}{3}, & 7 \leqslant x_2 \leqslant 10; \\ 0, & \text{otherwise}. \end{cases}$$

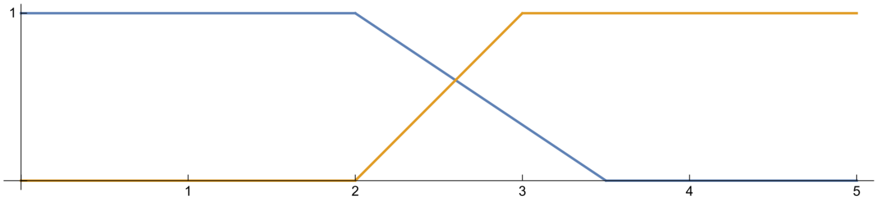

FIG. 3.3: Membership functions A'_1 (blue) and A'_2 (yellow).

$$B'_3(x_2) = \begin{cases} \frac{x_2-6}{3}, & 6 \leqslant x_2 \leqslant 9; \\ 1, & 9 \leqslant x_2 \leqslant 11; \\ 0, & \text{otherwise.} \end{cases}$$

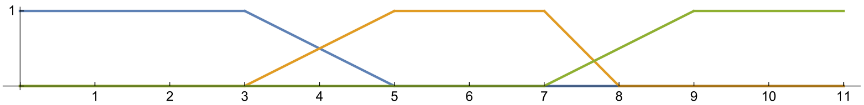

FIG. 3.4: Membership functions B'_1 (blue), B'_2 (yellow) and B'_3 (green).

For presenting the rate of the hotel, in the both cases, 5 singleton functions $C_i : U_y \to \{0, 1\}$, $i \in \{1, .., 5\}$ will be used:

$$C_i(y) = \begin{cases} 1, & y = i; \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy rule base, constructed from input-output data, in the first case is given by Table 3.2, and in the second case, fuzzy rule base constructed from input-output data is given by Table 3.3.

As we can see from Table 3.3, there is a lot of conflict rules. When we solve all the conflicts, and get rid of double rules, we obtain reduced Model 2., presented in Table 3.4

Fuzzy inference engine used here is Minimum Inference Engine, that is: individual-rule based inference with union combination, Mamdani's minimum implication, and min for all the t-norm operators and max for all the s-norm operators [7, 8]:

$$O(y) = max_{l=1}^{M}[sup_{(x_1,x_2)\in U_x} \min(I(x_1, x_2), A^l(x_1), B^l(x_2), C^l(y))].$$

where M is the number of rules. Fuzzifier $I(x_1, x_2)$, used here, is the the singleton fuzzifier, i.e. for the given input (x_1^0, x_2^0):

$$I(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) = (x_1^0, x_2^0); \\ 0, & \text{otherwise.} \end{cases}$$

Table 3.2: Model 1.

Rule 1	IF x_1 is A_2 and x_2 is B_2 then y is C_2
Rule 2	IF x_1 is A_3 and x_2 is B_5 then y is C_4
Rule 3	IF x_1 is A_1 and x_2 is B_3 then y is C_2
Rule 4	IF x_1 is A_2 and x_2 is B_5 then y is C_3
Rule 5	IF x_1 is A_4 and x_2 is B_6 then y is C_5
Rule 6	IF x_1 is A_4 and x_2 is B_5 then y is C_4
Rule 7	IF x_1 is A_3 and x_2 is B_3 then y is C_3
Rule 8	IF x_1 is A_3 and x_2 is B_1 then y is C_1
Rule 9	IF x_1 is A_3 and x_2 is B_6 then y is C_5
Rule 10	IF x_1 is A_3 and x_2 is B_5 then y is C_4
Rule 11	IF x_1 is A_2 and x_2 is B_5 then y is C_3
Rule 12	IF x_1 is A_1 and x_2 is B_4 then y is C_2
Rule 13	IF x_1 is A_1 and x_2 is B_3 then y is C_1
Rule 14	IF x_1 is A_2 and x_2 is B_5 then y is C_3
Rule 15	IF x_1 is A_2 and x_2 is B_6 then y is C_4

The outputs obtained by both methods are given in Table 3.5. As we can see from Table 3.5 and Table 3.1, better approximation is obtained by Model 1. For example, in the first row of input output Table 3.1 cleanliness is valued by 2.8, comfort by 2 and the overall impression rate is 2, and in the second row of Table 3.2 cleanliness is valued by 2, comfort by 3 and the overall impression rates are 2 and 1, by Model 1 and Model 2, respectively. Similarly, fifth row of Table 3.1 corresponds to fourth row of Table 3.4, and again we can see that better approximation is achieved by Model 1. The reason for this lies in the fact that Model 1 consider higher number of fuzzy functions for cleanliness and comfort (in Model 1 there are four fuzzy functions for cleanliness and six fuzzy functions for comfort, while in Model 2 we have only two fuzzy functions for cleanliness and tree for comfort). Therefore Model 1 provides sophisticated and finer fuzzy partition of the universe of the discourse. On the other hand, the Model 2, due to insufficient number of input functions, will never rate the quality of a hotel with a rating of 2 or 5, which is a serious disadvantage of this model. So, the suggestion is that there must be a lower bound on the number of functions that represent the input data-set.

Table 3.3: Model 2.

Rule 1	IF x_1 is A'_2 and x_2 is B'_2 then y is C_2
Rule 2	IF x_1 is A'_2 and x_2 is B'_3 then y is C_4
Rule 3	IF x_1 is A'_1 and x_2 is B'_2 then y is C_2
Rule 4	IF x_1 is A'_2 and x_2 is B'_2 then y is C_3
Rule 5	IF x_1 is A'_2 and x_2 is B'_3 then y is C_5
Rule 6	IF x_1 is A'_2 and x_2 is B'_3 then y is C_4
Rule 7	IF x_1 is A'_2 and x_2 is B'_2 then y is C_3
Rule 8	IF x_1 is A'_2 and x_2 is B'_1 then y is C_1
Rule 9	IF x_1 is A'_2 and x_2 is B'_3 then y is C_5
Rule 10	IF x_1 is A'_2 and x_2 is B'_3 then y is C_4
Rule 11	IF x_1 is A'_1 and x_2 is B'_3 then y is C_3
Rule 12	IF x_1 is A'_1 and x_2 is B'_2 then y is C_1
Rule 13	IF x_1 is A'_1 and x_2 is B'_2 then y is C_1
Rule 14	IF x_1 is A'_2 and x_2 is B'_3 then y is C_3
Rule 15	IF x_1 is A'_2 and x_2 is B'_3 then y is C_4

Table 3.4: Reduced model 2.

Rule 1	IF x_1 is A'_2 and x_2 is B'_2 then y is C_3
Rule 2	x_1 is A'_1 and x_2 is B'_2 then y is C_1
Rule 3	x_1 is A'_2 and x_2 is B'_3 then y is C_4
Rule 4	x_1 is A'_2 and x_2 is B'_1 then y is C_1
Rule 5	x_1 is A'_1 and x_2 is B'_3 then y is C_3

Table 3.5: Output of the algorithm.

Id	Input	Model 1	Model 2
1	(2, 3)	2	1
2	(2.5, 1.9)	2	1
3	(3, 8)	3	4
4	(5, 10.6)	5	4
5	(4, 6)	3	3

4. Conclusion

The paper presents an algorithm for designing fuzzy systems based on learning from examples. The concept uses Mamdani's fuzzy rule systems with a fuzzifier and defuzzifier. Particular attention has been given to the preferences of decision makers involved in the process as experts with extensive practical experience. Based on their opinion, corresponding fuzzy functions that express the importance of attributes in the model are defined. An example to illustrate the process has also been provided. Moreover, through this example, the importance of determining the lower number of functions which represent the input data set is highlighted. In other words, it is shown that the number of these functions significantly influences the quality of the solution.

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REFERENCES

1. A. Bardossy and L. Duckstein: *Fuzzy rule-based modeling with application to geophysical, biological and engineering systems.* CRC Press, Boca Raton, 1995.
2. Z. Chi, H. Yan and T. Pham: *Fuzzy algorithms: with applications to image processing and pattern recognition.* World Scientific, 1996.
3. D. Dubois and H. Prade: *What are fuzzy rules and how to use them*, Fuzzy Sets Syst. **84** (1996), 169–185.
4. S. Guillaume: *Designing Fuzzy Inference Systems from Data: An Interpretability-Oriented Review.* IEEE Trans. on Fuzzy Syst. **9(3)** (2001), 426–443.
5. K. Hirota: *Industrial applications of fuzzy technology.* Springer-Verlag, 1993.
6. S. S. Izquierd and L. R. Izquierdo: *Mamdani Fuzzy Systems for Modelling and Simulation: A Critical Assessment.* Journal of Artificial Societies and Social Simulation **21(3)**, 2018.
7. E.P. Klement, R. Mesiar and E. Pap: *Triangular Norms.* Dordrecht: Kluwer, 2000.
8. W. Van Leekwijck and E.E. Kerre: *Defuzzification: criteria and classification.* Fuzzy Sets Syst. **108** (1999), 159–178.
9. E.H. Mamdani: *Applications of fuzzy algorithm for control a simple dynamic plant.* In: Proc. of the IEEE **121** (1974), 585–1588.
10. E.H. Mamdani and S. Assilian: *An experiment in linguistic synthesis with a fuzzy logic controller.* Int J Man-Mach Stud **7(1)** (1975), 1–13.
11. C. Moraga: *An Essay on the Interpretability of Mamdani Systems*, in: E. Trillas, P.P. Bonissone, L. Magdalena, J. Kacprzyk (Eds.), Comb. Exp. Theory SE - 5, Springer Berlin Heidelberg, 2012, 61–72.

12. V. Novak: *Reasoning about mathematical fuzzy logic and its future* Fuzzy Sets Syst. **192** (2012), 25–44.
13. V. Novak, I. Perfilieva and A.Dvorak: *Insite into fuzzy modeling.* John Wiley& Sons, Inc., 2015.
14. S.K. Pal and D.P. Mandal: *Fuzzy Logic and Approximate Reasoning: An Overview.* IETE J. Res. **37** (2015), 548–560.
15. W. Pedrycz: *Fuzzy Modelling: Paradigms and Practice.* Kluwer Academic Press, 1996.
16. T. Takagi and M. Sugeno: *Fuzzy identification of systems and its applications to modeling and control.* IEEE Trans Syst Man Cybern **15(1)**(1985), 116–132.
17. L.-X. Wang: *A Course In Fuzzy Systems and Control.* Prentice Hall PTR, 1997.

Ivana Micić
University of Niš
Faculty of Sciences and Mathematics
Višegradska 33, P. O. Box 224
18000 Niš, Serbia
ivanajancic84@gmail.com

Nada Damljanović
University of Kragujevac
Faculty of Technical Sciences in Čačak
Svetog Save 65, P. O. Box 227
32000 Čačak, Serbia
nada.damljanovic@ftn.kg.ac.rs

Zorana Jančić
University of Niš
Faculty of Sciences and Mathematics
Višegradska 33, P. O. Box 224
18000 Niš, Serbia
ivanajancic84@gmail.com

FACTA UNIVERSITATIS (NIŠ)
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ON DISCRETE WEIGHTED STATISTICAL CONVERGENCE

Sinan Ercan, Yavuz Altın and Rifat Çolak

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Abstract. In the present paper, the notion of discrete weighted mean method of summability have been extended over the concept of statistical convergence. We have also given the notion of statistical $(M, P_\lambda)-$ summability and $[M, P_\lambda]_q$-summability. We have introduced some properties of these modes of convergence.
Keywords: statistical convergence; weighted statistical convergence; statistical $(\bar{N}, p_n)$-summability.

1. Introduction

Zygmund [16] introduced the idea of statistical convergence in 1935. Fast and Steinhaus introduced statistical convergence to assign limit to sequences which are not convergent in the usual sense in the same year (see [4],[14]). They used the asymptotic density of a set $A \subset \mathbb{N}$ which is defined as follows:

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right|,$$

whenever the limit exists. $|\{.\}|$ indicates the cardinality of the enclosed set. A sequence $x = (x_k)$ of numbers is called statistically convergent to a number ℓ provided that for $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : |x_k - \ell| \geq \varepsilon\}\right| = 0.$$

In this case, we write $S - \lim_{k\to\infty} x_k = \ell$. S indicates the set of all statistically convergent sequences. This notion is used an effective tool to resolve many problems in ergodic theory, fuzzy set theory, trigonometric series and Banach spaces. It was studied in summability theory by Kolk *et al.* (see [8]). Also,. many researchers

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studied related topics with summability theory (see[2], [3], [5], [12], [13]). Furthermore, another type of Cesaro summability was studied by Armitage and Maddox [1].

Moricz and Orhan [11] defined the notion of statistical $(\bar{N}, p_n)$ – summability as: Let $p = (p_k)$ be a sequence of nonnegative real numbers such that $p_0 > 0$, $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$ and $t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k$, $n = 0, 1, 2, \ldots$. The sequence $x = (x_k)$ is statistically summable to ℓ by the weighted mean method determined by the sequence (p_k) or briefly statistically $(\bar{N}, p_n)$ – summable if

$$st - \lim_{n\to\infty} t_n = \ell.$$

$(\bar{N}, st)$ indicates the set of statistically $(\bar{N}, p_n)$ – summable sequences.

Weighted statistical convergence is introduced by Karakaya and Chisti in [7]. Also Küçükaslan studied this notion in [9]. Then the modified definition is given by Mursaleen *et al.* in [10] as follows:

A sequence $x = (x_k)$ is weighted statistically convergent (or $S_{\bar{N}}$-convergent) to ℓ if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : p_k |x_k - \ell| \geq \varepsilon\}$ has weighted density zero, i.e.

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \leq P_n : p_k |x_k - \ell| \geq \varepsilon\}| = 0,$$

where $\mathbb{N} = \{1, 2, ...\}$. This limit is indicated by $S_{\bar{N}} - \lim_{k\to\infty} x_k = \ell$. $S(\bar{N})$ denotes the set of these kind of sequences.

The concept of weighted statistical convergence of order α is studied by Ghosal in [6]. Watson introduce the notion of discrete weighted mean method of summability in [15] as follows:

A sequence (x_k) is limitable to ℓ by the discrete weighted mean method, if

$$\tau_n = t_{[\lambda_n]} = \frac{1}{P_{[\lambda_n]}} \sum_{k=0}^{[\lambda_n]} p_k x_k \to \ell$$

as $n \to \infty$ where (λ_n) is a real sequence satisfies $1 \leq \lambda_0 < \lambda_1 < ... \to \infty$ and $P_{[\lambda_n]} = \sum_{k=0}^{[\lambda_n]} p_k \to \infty$ as $n \to \infty$ for $p_0 > 0$ and $[\lambda_n]$ denotes the integer part of the number λ_n. The set of these kind sequences denoted by (M_{P_λ}).

2. Main Results

In this part, first we give the concepts of discrete weighted statistical convergence, $[M, P_\lambda]_q$-summability and statistical (M, P_λ) −summability. Then we establish the relationship between these concepts. The discrete weighted density of a set $K \subseteq \mathbb{N}$ is defined by

$$\delta_M(K) = \lim_{n\to\infty} \frac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : k \in K\right\}\right|.$$

In particular, if we choose $\lambda_n = n$ and $p_n = 1$, then the discrete weighted density reduces to the natural density.

Throughout this paper (p_k), is a sequence of nonnegative real numbers with $p_1 > 0$ and $P_{[\lambda_n]} = \sum_{k=1}^{[\lambda_n]} p_k \to \infty$ as $n \to \infty$ and $\lambda = (\lambda_n)$ is a real sequence satisfies $1 \leq \lambda_1 < \lambda_2 < ... \to \infty$ as $n \to \infty$ and we use the notations Λ, $\widehat{P}$, $E(\lambda_n)$, $E(\lambda)$ such as

$$\Lambda = \{\lambda = (\lambda_n) : 1 \leq \lambda_1 < \lambda_2 < ... \to \infty \text{ as } n \to \infty\},$$

$$\widehat{P} = \left\{p = (p_k) : p_1 > 0,\ p_k \geq 0,\ k = 2, 3, ... \text{ and } P_{[\lambda_n]} = \sum_{k=1}^{[\lambda_n]} p_k \to \infty \text{ as } n \to \infty\right\},$$

$$E(\lambda_n) = \{k \leq [\lambda_n] : k \in \mathbb{N}\}$$

and

$$E(\lambda) = \{[\lambda_n] : n \in \mathbb{N}\} = \{[\lambda_1], [\lambda_2], [\lambda_3], ...\}$$

for a sequence $\lambda = (\lambda_n) \in \Lambda$. Also $A\Delta B = (A\backslash B) \cup (B\backslash A)$.

Definition 2.1 Let $\lambda = (\lambda_n) \in \Lambda$ and $p = (p_k) \in \widehat{P}$ be given. A sequence $x = (x_k)$ is said to be discrete weighted statistically convergent (briefly $S(M_{P_\lambda})$-convergent) to ℓ if the set $\{k \in \mathbb{N} : p_k |x_k - \ell| \geq \varepsilon\}$ has discrete weighted density zero for every $\varepsilon > 0$, i.e.

$$\lim_{n\to\infty} \frac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : p_k |x_k - \ell| \geq \varepsilon\right\}\right| = 0.$$

It is indicated by $S(M_{P_\lambda}) - \lim_{k\to\infty} x_k = \ell$. $S(M_{P_\lambda})$ indicates the set of these kind of sequences.

Definition 2.2 Let $\lambda = (\lambda_n) \in \Lambda$ and $p = (p_k) \in \widehat{P}$ be given. A sequence $x = (x_k)$ is called $[M, P_\lambda]_q$-summable $(0 < q < \infty)$ to the limit ℓ if

$$\lim_{n\to\infty} \frac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k |x_k - \ell|^q = 0.$$

ℓ is said to be $[M,P_\lambda]_q$-limit of x. $[M,P_\lambda]_q$ indicates the set of these kind of sequences.

Definition 2.3 Let $\lambda=(\lambda_n)\in\Lambda$ and $p=(p_k)\in\widehat{P}$ be given. A sequence $x=(x_k)$ is said to be statistically summable to ℓ by the discrete weighted mean method or briefly statistically (M,P_λ) −summable if

$$st-\lim_{n\to\infty}\tau_n=\ell.$$

It is indicated by $(M,P_\lambda)-\lim_{k\to\infty}x_k=\ell$ and we denote by (M,P_λ) the set of such sequences.

Note that for any $(\lambda_n)=(n+r)\in\Lambda$, where $0\le r<1$ is a fixed number.

(i) $[M,P_\lambda]_q$-summable reduces to $\left[\bar{N},p_n\right]_q$-summable which is given in [10],

(ii) Discrete weighted statistical convergence is reduced to weighted statistical convergence which is given in [10],

(iii) Statistical (M,P_λ) −summability is reduced to statistical $\left(\bar{N},p_n\right)$ − summability which is given in [11].

As a result of (i), (ii) and (iii) we have that $[M,P_\lambda]_q$ includes $\left[\bar{N},p_n\right]_q$, $S\left(M_{P_\lambda}\right)$ includes $S\left(\bar{N}\right)$ and (M,P_λ) includes $\bar{N}\left(st\right)$, respectively.

We first begin with the following property. In the proof of the following Theorem we use the technique used by Watson in [15].

Theorem 2.4 Let $\lambda=(\lambda_n),\mu=(\mu_n)\in\Lambda$ be given. Then

(i) $[M,P_\lambda]_q\subseteq[M,P_\mu]_q$ if $E(\mu)\backslash E(\lambda)$ is finite,

(ii) If $p_k>0$ for each k and if $[M,P_\lambda]_q\subseteq[M,P_\mu]_q$ holds, then $E(\mu)\backslash E(\lambda)$ is finite.

Proof. (i) Assume that $E(\mu)\backslash E(\lambda)$ is finite. Then we have an integer n_0 such that $\{[\mu_n]:n\ge n_0\}\subseteq E(\lambda)$. That is, there is an increasing sequence (j_n) of positive integers such that $j_n\to\infty$ and $[\mu_n]=[\lambda_{j_n}]$ for $n\ge n_0$. If a sequence $x=(x_n)$ is statistically $[M,P_\lambda]_q$ − summable to ℓ, then we have

$$\frac{1}{P_{[\mu_n]}}\sum_{k=1}^{[\mu_n]}p_k\left|x_k-\ell\right|^q=\frac{1}{P_{[\lambda_{j_n}]}}\sum_{k=1}^{[\lambda_{j_n}]}p_k\left|x_k-\ell\right|^q$$

for $n\ge n_0$, which gives that $x=(x_n)$ is statistically $[M,P_\mu]_q$ −summable to ℓ $(0<q<\infty)$.

(ii) Suppose that $[M,P_\lambda]_q\subseteq[M,P_\mu]_q$ but $E(\mu)\backslash E(\lambda)$ is infinite. Then there is a strictly increasing sequence $\left(\left[\mu_{n_j}\right]\right)$ such that $\left[\mu_{n_j}\right]\notin E(\lambda)$, for $j=1,2,3,\ldots$.

Consider that $\tau_n = t_{[\lambda_n]} = \frac{1}{P_{[\lambda_n]}}\sum_{k=1}^{[\lambda_n]} p_k x_k$. Then define a sequence (τ_n) as follows.

$$\tau_n = \begin{cases} 0 & \text{if} \quad [\lambda_n] \neq \left[\mu_{n_j}\right], \\ (-1)^j & \text{if} \quad [\lambda_n] = \left[\mu_{n_j}\right]. \end{cases}$$

Using the equality $P_{[\lambda_n]}t_{[\lambda_n]} - P_{[\lambda_n]-1}t_{[\lambda_n]-1} = P_{[\lambda_n]}x_{[\lambda_n]}$, we have $x = (x_n) \in [M, P_\lambda]_q$ since $t_{[\lambda_n]} = 0$ for all n. But the sequence is not in $[M, P_\mu]_q$.

We have the below results from Theorem 2.4.

Corollary 2.5 Let $\lambda = (\lambda_n), \mu = (\mu_n) \in \Lambda$ be given and assume that $p_k > 0$ for every k. Then

(i) $[M, P_\lambda]_q \subseteq [M, P_\mu]_q$ if and only if $E(\mu) \backslash E(\lambda)$ is a finite set.

(ii) $[M, P_\lambda]_q = [M, P_\mu]_q$ if and only if $E(\lambda) \Delta E(\mu)$ is finite set.

Corollary 2.6 Let $\lambda = (\lambda_n), \mu = (\mu_n) \in \Lambda$ be given and assume that $p_k > 0$ for every k. Then

(i) $(M, P_\lambda) \subseteq (M, P_\mu)$ if and only if $E(\mu) \backslash E(\lambda)$ is a finite set.

(ii) $(M, P_\lambda) = (M, P_\mu)$ if and only if $E(\lambda) \Delta E(\mu)$ is a finite set.

The following property has been given formerl, but it is also seen clearly from Theorem 2.4 by taking $\lambda = (\lambda_n) = (n)$.

Corollary 2.7 For any $(\mu_n) \in \Lambda$ the inclusion $\left[\bar{N}, p_n\right]_q \subseteq [M, P_\mu]_q$ is satisfied, where $0 < q < \infty$.

Corollary 2.8 For any $(\lambda_n) \in \Lambda$ $[M, P_\lambda]_q \subseteq \left[\bar{N}, p_n\right]_q$ is satisfied if $\{1, 2, 3, ...\} \backslash E(\lambda)$ is finite.

The following results can be obtained from Corollary 2.5 and Corollary 2.6.

Corollary 2.9 *(i)* $\left(\bar{N}, p_n\right) \subseteq (M, P_\mu)$ for any $\mu = (\mu_n) \in \Lambda$.

(ii) $(M, P_\lambda) \subseteq \left(\bar{N}, p_n\right)$ if $\{1, 2, 3, ...\} \backslash E(\lambda)$ is a finite set.

Theorem 2.10 If a sequence $x = (x_k)$ is (M_{P_λ})-summable to ℓ, then it is $S(M_{P_\lambda})$-convergent to ℓ. The inverse implication need not be true.

Proof. Let $x \in (M_{P_\lambda})$ and define the set $K_{P_\lambda}(\varepsilon) = \left\{k \leq P_{[\lambda_n]} : p_k |x_k - \ell| \geq \varepsilon\right\}$ for $\varepsilon > 0$. Hence the inequality which we have

$$\frac{1}{P_{[\lambda_n]}}\sum_{k=1}^{[\lambda_n]} p_k |x_k - \ell| = \frac{1}{P_{[\lambda_n]}}\left(\sum_{k \in K_{P_\lambda}(\varepsilon)} + \sum_{k \notin K_{P_\lambda}(\varepsilon)}\right) p_k |x_k - \ell|$$

$$\geq \frac{1}{P_{[\lambda_n]}} \sum_{k \in K_{P_\lambda}(\varepsilon)} p_k |x_k - \ell|$$
$$\geq \frac{1}{P_{[\lambda_n]}} \sum_{k \in K_{P_\lambda}(\varepsilon)} \varepsilon$$
$$= \varepsilon . \frac{1}{P_{[\lambda_n]}} \left| \left\{ k \leq P_{[\lambda_n]} : p_k |x_k - \ell| \geq \varepsilon \right\} \right| .$$

This implies that $x \in S(M_{P_\lambda})$. To see the converse implication is not true, consider $\lambda_n = n$, $p_k = 1$ for all k and define $x = (x_k)$ by

$$x_k = \begin{cases} m^3, & k = m^2 \\ 0, & k \neq m^2 \end{cases} \quad m = 1, 2, \dots \ .$$

Now we have

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : 1. |x_k - 0| \geq \varepsilon\}| = 0 \text{ and } \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1 |x_k - 0| = \infty.$$

This means $x \in S(M_{P_\lambda})$ but $x \notin (M_{P_\lambda})$.

Theorem 2.11 Assume that $p_n \geq 1$ for all n and

$$1 \leq \lim_{n \to \infty} \frac{P_{[\lambda_n]}}{n} < \infty \tag{2.1}$$

holds . Then $x \in S$ if $x \in S(M_{P_\lambda})$. The opposite case is not true.

Proof. Suppose that $p_n \geq 1$ and (2.1) holds for all $n \in \mathbb{N}$. Let $x \in S(M_{P_\lambda})$, then we have

$$\begin{aligned} \tfrac{1}{n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| & \leq \tfrac{1}{n} |\{k \leq n : p_k |x_k - \ell| \geq \varepsilon\}| \\ & \leq \tfrac{P_{[\lambda_n]}}{n} \tfrac{1}{P_{[\lambda_n]}} \left| \left\{ k \leq P_{[\lambda_n]} : p_k |x_k - \ell| \geq \varepsilon \right\} \right| \end{aligned}$$

for $\varepsilon > 0$. As $n \to \infty$ we obtain $x \in S$.

To see the opposite case is not true, consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1 & \text{if } k = m^2 \\ \frac{1}{\sqrt{k}} & \text{if } k \neq m^2 \end{cases} \quad m \in \mathbb{N} \ .$$

The sequence $x = (x_k)$ is a statistically convergent to 0, but it is not discrete weighted statistical convergent to 0, while $p_k = k$ for $k \in \mathbb{N}$.

The sufficient condition to be true the converse implication is given the following theorem.

Theorem 2.12 Assume that $p_n < 1$ and

$$1 \leq \lim_{n\to\infty} \frac{n}{P_{[\lambda_n]}} < \infty \tag{2.2}$$

for all $n \in \mathbb{N}$. If $x \in S$, then $x \in S\left(M_{P_\lambda}\right)$.

Proof. Suppose that $p_n < 1$ and (2.2) holds for all $n \in \mathbb{N}$. Let $x \in S$ then we have

$$\begin{aligned} \tfrac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : p_k \left|x_k - \ell\right| \geq \varepsilon\right\}\right| &\leq \tfrac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : \left|x_k - \ell\right| \geq \varepsilon\right\}\right| \\ &\leq \tfrac{n}{P_{[\lambda_n]}} \tfrac{1}{n} \left|\left\{k \leq n : \left|x_k - \ell\right| \geq \varepsilon\right\}\right| \end{aligned}$$

for any $\varepsilon > 0$. Hence we obtain the desired result as $n \to \infty$.

Theorem 2.13 $S\left(M_{P_\lambda}\right)$-lim of an $S\left(M_{P_\lambda}\right)$-convergent sequence is unique.

Proof. Suppose the sequence $x = (x_k)$ is $S\left(M_{P_\lambda}\right)$-convergent both to ℓ_1 and ℓ_2. If possible let $\ell_1 \neq \ell_2$ and choose $\varepsilon = \frac{1}{2}\left|\ell_1 - \ell_2\right| > 0$ and $p_k > c > 0$ for all k. Then

$$\begin{aligned} 1 &\leq \tfrac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : p_k \left|\ell_1 - \ell_2\right| \geq \varepsilon c\right\}\right| \\ &\leq \tfrac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : p_k \left|x_k - \ell_1\right| \geq \tfrac{\varepsilon c}{2}\right\}\right| + \tfrac{1}{P_{[\lambda_n]}} \left|\left\{k \leq P_{[\lambda_n]} : p_k \left|x_k - \ell_2\right| \geq \tfrac{\varepsilon c}{2}\right\}\right|. \end{aligned}$$

This is impossible because right hand side tends to 0 as $n \to \infty$. Hence we have desired result $\ell_1 = \ell_2$.

Theorem 2.14 Let the sequence $\left(p_k\left|(x_k - \ell)\right|\right)$ be bounded. Then $x = (x_k)$ is statistically $\left(M_{P_\lambda}\right)-$ summable to ℓ if it is $S\left(M_{P_\lambda}\right)$-convergent to ℓ, but the opposite case is not true.

Proof. Suppose $p_k\left|(x_k - \ell)\right| \leq T$ for every k, for some constant T and assume that the sequence $x = (x_k)$ is $S\left(M_{P_\lambda}\right)$-convergent to ℓ. We have

$$\begin{aligned} \left|\tau_n - \ell\right| &= \left|\tfrac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k x_k - \ell\right| \\ &= \left|\tfrac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k \left(x_k - \ell\right)\right| \end{aligned}$$

Theorem 2.15 Let the sequence $\left(p_k\left|(x_k - \ell)\right|\right)$ be bounded. Then $x = (x_k)$ is statistically $\left(M_{P_\lambda}\right)-$ summable to ℓ if it is $S\left(M_{P_\lambda}\right)$-convergent to ℓ, but the opposite case is not true.

Proof. Suppose $p_k |(x_k - \ell)| \leq T$ for every k, for some constant T and assume that the sequence $x = (x_k)$ is $S(M_{P_\lambda})$-convergent to ℓ. We have

$$\begin{aligned}|\tau_n - \ell| &= \left| \frac{1}{P_{[\lambda n]}} \sum_{k=1}^{[\lambda n]} p_k x_k - \ell \right| \\ &= \left| \frac{1}{P_{[\lambda n]}} \sum_{k=1}^{[\lambda n]} p_k (x_k - \ell) \right|\end{aligned}$$

$$\begin{aligned}&= \left| \frac{1}{P_{[\lambda n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda n]} p_k (x_k - \ell) + \frac{1}{P_{[\lambda n]}} \sum_{\substack{k=1 \\ k \in K^c_{P_\lambda}(\varepsilon)}}^{[\lambda n]} p_k (x_k - \ell) \right| \\ &\leq \left| \frac{1}{P_{[\lambda n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda n]} p_k (x_k - \ell) \right| + \left| \frac{1}{P_{[\lambda n]}} \sum_{\substack{k=1 \\ k \in K^c_{P_\lambda}(\varepsilon)}}^{[\lambda n]} p_k (x_k - \ell) \right| \\ &\leq \frac{1}{P_{[\lambda n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda n]} p_k |(x_k - \ell)| + \frac{1}{P_{[\lambda n]}} \sum_{\substack{k=1 \\ k \in K^c_{P_\lambda}(\varepsilon)}}^{[\lambda n]} p_k |(x_k - \ell)| \\ &\leq \frac{1}{P_{[\lambda n]}} .T. |K_{P_\lambda}(\varepsilon)| + \frac{1}{P_{[\lambda n]}} \sum_{k \in K^c_{P_\lambda}} \varepsilon \\ &= \frac{1}{P_{[\lambda n]}} .T. |K_{P_\lambda}(\varepsilon)| + \varepsilon \frac{\left|K^c_{P_\lambda}(\varepsilon)\right|}{P_{[\lambda n]}} \to 0\end{aligned}$$

as $n \to \infty$, where $K_{P_\lambda}(\varepsilon) = \left\{ k \leq P_{[\lambda_n]} : p_k |x_k - \ell| \geq \varepsilon \right\}$. This means that $\tau_n \to \ell$ as $n \to \infty$. That is, x is (M_{P_λ})-summable to ℓ and hence it is statistically (M_{P_λ}) −summable to ℓ.

To see that the opposite case is not true, let $p_k = 1$ for every $k \in \mathbb{N}$. Consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1 & \text{if } k = m^2 - m, m^2 - m + 1, ..., m^2 - 1; \\ -m & \text{if } k = m^2; \\ 0 & \text{otherwise.} \end{cases}$$

where $m = 2, 3, 4, ...$. Then we have, for $s = 0, 1, 2, ..., m - 1; m = 2, 3, ...$

$$\tau_n = t_{[\lambda_n]} = \frac{1}{[\lambda_n] + 1} \sum_{k=0}^{[\lambda_n]} x_k = \begin{cases} \frac{s+1}{[\lambda_n]+1} & \text{if } [\lambda_n] = m^2 - m + s \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\lim_{n\to\infty} \tau_n = 0$ and hence $st - \lim_{n\to\infty} \tau_n = 0$, i.e. $x = (x_k)$ is statistically (M_{P_λ}) −summable to 0. On the other hand $st - \lim_{k\to\infty} \inf x_k = 0$ and $st - \lim_{k\to\infty} \sup x_k = 1$, because

$$\delta\left(\left\{k : k = m^2,\ m = 1, 2, 3, ...\right\}\right) = 0,$$

$$\delta\left(\left\{k : k \neq m^2 - m, m^2 - m + 1, ..., m^2 - 1, m;\ m = 2, 3, ...\right\}\right) \neq 0$$

and

$$\delta\left(\left\{k : k = m^2 - m, m^2 - m + 1, ..., m^2 - 1;\ m = 2, 3, ...\right\}\right) \neq 0.$$

Hence $x = (x_k)$ is not $S\left(M_{P_\lambda}\right)$-convergent.

Theorem 2.16 Let a sequence $x = (x_k)$ be $\left[M_{P_\lambda}\right]_q$-summable to ℓ. If $0 < q < 1$ and $0 \leq |x_k - \ell| < 1$ or $1 \leq q < \infty$ and $1 \leq |x_k - \ell| < \infty$, then $x = (x_k)$ is $S\left(M_{P_\lambda}\right)$-statistically convergent to ℓ.

Proof. We have $p_k |x_k - \ell|^q \geq p_k |x_k - \ell|$ for both cases. Then

$$\begin{aligned} \frac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k |x_k - \ell|^q &\geq \frac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k |x_k - \ell| \\ &\geq \frac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell| \\ &\geq \frac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} \varepsilon \\ &= \varepsilon \frac{\left|K_{P_\lambda}(\varepsilon)\right|}{P_{[\lambda_n]}}. \end{aligned}$$

Since $\frac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k |x_k - \ell|^q \to 0$ as $n \to \infty$ we have $\lim_{n\to\infty} \frac{1}{P_{[\lambda_n]}} \left|K_{P_\lambda}(\varepsilon)\right| = 0$. This means that $x = (x_k)$ is $S\left(M_{P_\lambda}\right)$-convergent to ℓ.

Theorem 2.17 Let the sequence $(p_k |(x_k - \ell)|)$ be bounded and let a sequence $x = (x_k)$ be $S\left(M_{P_\lambda}\right)$-convergent to ℓ. If $0 < q < 1$ and $0 \leq T < \infty$ or $1 \leq q < \infty$ and $0 \leq T < 1$, then $x = (x_k)$ is $\left[M_{P_\lambda}\right]_q$ $-$summable to ℓ.

Proof. Assume that $x = (x_k)$ is $S\left(M_{P_\lambda}\right)$-convergent to ℓ. Since $p_k |x_k - \ell| \leq T$ $(k = 1, 2, ...)$ for some $T \geq 0$, we have

$$\begin{aligned} \frac{1}{P_{[\lambda_n]}} \sum_{k=1}^{[\lambda_n]} p_k |x_k - \ell|^q &= \frac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \notin K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell|^q + \frac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell|^q \\ &= s_1([\lambda_n]) + s_2([\lambda_n]) \end{aligned}$$

where

$$s_1([\lambda_n]) = \frac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \notin K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell|^q$$

and

$$s_2([\lambda_n]) = \frac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell|^q .$$

Now we have

$$\begin{aligned} s_1([\lambda_n]) &= \tfrac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \notin K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell|^q \\ &\leq \tfrac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \notin K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell| = \varepsilon \tfrac{1}{P_{[\lambda_n]}} \left| K^c_{P_\lambda}(\varepsilon) \right| \to 0 \end{aligned}$$

and

$$\begin{aligned} s_2([\lambda_n]) &= \tfrac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell|^q \leq \tfrac{1}{P_{[\lambda_n]}} \sum_{\substack{k=1 \\ k \in K_{P_\lambda}(\varepsilon)}}^{[\lambda_n]} p_k |x_k - \ell| \\ &\leq (\sup_k p_k |x_k - \ell|) \left(|K_{P_\lambda}(\varepsilon)| / P_{[\lambda_n]} \right) \leq T |K_{P_\lambda}(\varepsilon)| / P_{[\lambda_n]} \to 0 \end{aligned}$$

as $n \to \infty$. Hence (x_k) is $[M_{P_\lambda}]_q$-summable to ℓ.

REFERENCES

1. D. H. Armitage, I. J.Maddox: *A new type of Cesàro mean.* Analysis, 9(1-2) (1989), 195-206.
2. B.T. Bilalov, S. R., Sadigova: *On μ-statistical convergence.* Proc. Amer. Math. Soc. 143(9) (2015), 3869-3878.
3. J. Connor: *The statistical and strong $p-$Cesáro convergence of sequences.* Analysis, 8 (1988), 47-63.
4. H. Fast: *Sur la convergence statistique.* Colloquium Math. 2 (1951), 241-244.
5. J.A. Fridy: *On statistical convergence.* Analysis, 5 (1985), 301-313.
6. S. Ghosal: *Weighted statistical convergence of order α and its applications.* J. Egyptian Math. Soc. 24(1) (2016), 60-67.
7. V. Karakaya, T. A. Chishti: *Weighted statistical convergence.* Iran. J. Sci. Technol. Trans. A Sci. 33 (2009), 219-223.
8. E. Kolk: *The statistical convergence in Banach spaces.* Acta et Comment. Univ. Tartu. 928 (1991), 41-52.
9. M. Küçükaslan: *Weighted statistical convergence.* International Journal of Science and Technology, 2 (2012), 2-10.
10. M. Mursaleen, V. Karakaya, M. Ertürk, F. Gürsoy: *Weighted statistical convergence and its application to Korovkin type aprroximation theorem.* Appl. Math. Comput. 218 (2012), 9132-9137.

11. F. Moricz, C. Orhan: *Tauberian conditions under which statistical convergence follows from statistical summability by weighted means.* Studia Sci. Math. Hungar. 41 (2004), 391-403.

12. D. Rath, B. C. Tripathy: *On statistically convergent and statistically convergent and statistically Cauchy sequences.* Indian J. Pure. Appl. Math. 25(4) (1994), 381-386.

13. T. Šalăt: *On statistically convergent sequences of real numbers.* Math. Slovaca, 30 (1980), 139-150.

14. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique.* Colloquium Math. 2 (1951), 73-74.

15. B. Watson: *Discrete weighted mean methods.* Indian J. Pure Appl. Math. 30(12) (1999), 1223-1227.

16. A. Zygmund: *Trigonometrical Series.* Monogr. Mat., Vol. 5. Warszawa-Lwow 1935.

Sinan Ercan
Department of Mathematics,
Firat University Elazig, Türkiye
sinanercan45@gmail.com

Yavuz Altın
Department of Mathematics,
Firat University, Elazig, Türkiye
yaltin23@yahoo.com

Rifat Çolak
Department of Mathematics
Firat University, Elazig, Türkiye
rftcolak@gmail.com

FACTA UNIVERSITATIS (NIŠ)
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APPLICATIONS OF MATRIX TRANSFORMATIONS TO ABSOLUTE SUMMABILITY

Mehmet Ali Sarıgöl

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Abstract. Rhoades and Savaş [6], [11] established necessary conditions for inclusions of the absolute matrix summabilities under additional conditions. In this paper, we determine necessary or sufficient conditions for some classes of infinite matrices, and using this, we get necessary or sufficient conditions for more general absolute summabilities applied to all matrices.
Keywords: matrix summability; infinite matrices; Cesàro matrices; triangular matrix.

1. Introduction

Let X and Y be two sequence spaces of the space ω, the set of all sequences with real or complex terms. Let $A = (a_{nv})$ be an infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series are convergent for $v, n \geq 0$. If $A(x) \in Y$ for all $x \in X$, then A is called a matrix transformation from X into Y, and denoted by (X, Y).

In many cases, since an infinite matrix can be considered as a linear operator between two sequence spaces, the theory of matrix transformations in sequence spaces has aroused interest for many years, of which purpose is to provide the necessary and sufficient conditions for a matrix to map a sequence space into another.

X is called a BK-space, if it is a Banach space on which all coordinate functionals defined by $p_n(x) = x_n$ are continuous.

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Let Σa_v be a given infinite series with n-th partial sum s_n and let (γ_n) be a sequence of nonnegative numbers. By $(A_n(s))$, we denote the A-transform of the sequence $s=(s_n)$. The series Σx_v is said to be summable $|A,\gamma_n|_k, k\geq 1$, if (see [7])

$$\sum_{n=1}^{\infty}\gamma_n^{k-1}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty. \tag{1.1}$$

Note that, for $\gamma_n=n$, $|A,\gamma_n|_k=|A|_k$ [12] , Also, if A is chosen as the matrices of the weighted mean (R,p_n) $(resp.\gamma_n=P_n/p_n)$ and Cesàro mean (C,α) together with $\gamma_n=n$, then, it reduces to the summabilities $|R,p_n|_k$ [8] $\left(resp.|\overline{N},p_n|_k\ \text{[1]}\right)$ and $|C,\alpha|_k$ [2], respectively. By the weighted and Cesàro matrices we mention

$$a_{nv}=\begin{cases}\frac{p_v}{P_n}, & 0\leq v\leq n\\ 0, & v>n,\end{cases}$$

and

$$a_{nv}=\begin{cases}\frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}, & 0\leq v\leq n\\ 0, & v>n.\end{cases}$$

respectively, where (p_n) is a sequence of positive numbers with $P_n=p_0+p_1+...+p_n\to\infty$, and

$$A_n^{\alpha}=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!},n\geq 1,\ \ A_0^{\alpha}=1$$

$$|A_n^{\alpha}|\leq A(\alpha)n^{\alpha}\text{ for all }\alpha$$

$$A_n^{\alpha}\geq A(\alpha)n^{\alpha}\text{ and }A_n^{\alpha}>0\text{ for }\alpha>-1.$$

Let $A=(a_{nv})$ be a lower triangular matrix, we derive the matrices $\overline{A}=(\overline{a}_{nv})$ and $\widehat{A}=(\widehat{a}_{nv})$ from the matrix A as follows:

$$\begin{aligned}\overline{a}_{00} &= \widehat{a}_{00}=a_{00}\\ \overline{a}_{nv} &= \sum_{r=v}^{n}a_{nr};\ \ n,v=0,1,...\\ \widehat{a}_{nv} &= \overline{a}_{nv}-\overline{a}_{n-1,v},\ \ \overline{a}_{n-1,n}=0.\end{aligned}$$

Then, $\widehat{A}$ is a triangular matrix and has unique inverse which is also triangular (see [13]). We will denote its inverse $\widehat{A}'$. Hence, it can be written that

$$A_n(x)=\sum_{v=0}^{n}a_{nv}s_v=\sum_{r=0}^{n}\left(\sum_{v=r}^{n}a_{nv}\right)x_r=\sum_{v=0}^{n}\overline{a}_{nv}x_v$$

and

$$\widehat{A}_n(x) = A_n(x)-A_{n-1}(x)=\sum_{v=0}^{n}\left(\overline{a}_{nv}-\overline{a}_{n-1,v}\right)x_v=\sum_{v=0}^{n}\widehat{a}_{nv}x_v. \tag{1.2}$$

which means that the summability $|A,\gamma_n|_k$ is equivalent to

$$\sum_{n=0}^{\infty}\gamma_n^{k-1}\left|\widehat{A}_n(x)\right|^k<\infty. \tag{1.3}$$

By $|\gamma A|_k$, we define the set of all series summable by $|A,\gamma_n|_k$. Then, a series Σx_v is summable $|A,\gamma_n|_k$ iff $x=(x_v)\in|\gamma A|_k$, i.e.,

$$|\gamma A|_k=\left\{x=(x_v):\widetilde{A}(x)=\left(\widetilde{A}_n(x)\right)\in\ell_k\right\} \tag{1.4}$$

where $\widetilde{A}_n(x)=\gamma_n^{1-1/k}\widehat{A}_n(x)$ for all $n\geq 0$ and ℓ_k is the set of all k-absolutely convergent series.

We note that, since $\widetilde{A}=(\widetilde{a}_{nv})$ is a triangle matrix, it is routine to show that $|\gamma A|_k$ is a BK- space if normed by

$$\|x\|_{|\gamma A|_k}=\left\|\widetilde{A}(x)\right\|_{\ell_k},\ \ 1\leq k<\infty. \tag{1.5}$$

Dealing with the absolute weighted mean summability of infinite series, Bor and Thorpe [1] established sufficient conditions in order that all $\left|\overline{N},p_n\right|_k$ summable series is also summable $\left|\overline{N},q_n\right|_k$, and conversely. The author [10] showed that Bor and Thorpe's conditions are not only sufficient but also necessary for the conclusion. Also, these results of the author [10] were extended by Rhoades and Savaş [6] using a triangle matrix instead of weighted mean matrix as follows.

Theorem 1.1. Let $1<k\leq s<\infty$, (p_n) be a sequence satisfying

$$\sum_{n=v+1}^{\infty}n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k=O\left(\frac{1}{P_n^k}\right). \tag{1.6}$$

Let B be a lower triangular matrix. Then, necessary conditions for Σx_v summable $\left|\overline{N},p_n\right|_k$ to imply Σx_v is summable $|B|_s$ are

$$\frac{P_v\,|b_{vv}|}{p_v}=O\left(v^{1/s-1/k}\right),$$

$$\sum_{n=v+1}^{\infty}n^{s-1}\left|\Delta_v\widehat{b}_{nv}\right|^s=O\left(v^{s-s/k}\frac{p_v}{P_v}\right),$$

$$\sum_{n=v+1}^{\infty}n^{s-1}\left|\widehat{b}_{n,v+1}\right|^s=O\left(1\right).$$

This result has also been extended by Savaş [11] to the matrix methods as follows

Theorem 1.2. Let $1 < k \leq s < \infty$, A and B be two lower triangular matrices. A satisfying

$$\sum_{n=v+1}^{\infty} n^{k-1} \left|\Delta_v \widehat{a}_{nv}\right|^k = O\left(|a_{vv}|^k\right). \tag{1.7}$$

Then necessary conditions for Σx_v summable $|A|_k$ to imply Σx_v is summable $|B|_s$ are

$$|b_{vv}| = O\left(v^{1/s-1/k} |a_{vv}|\right),$$

$$\sum_{n=v+1}^{\infty} n^{s-1} \left|\Delta_v \widehat{b}_{nv}\right|^s = O\left(v^{s-s/k} |a_{vv}|^s\right)$$

and

$$\sum_{n=v+1}^{\infty} n^{s-1} \left|\widehat{b}_{n,v+1}\right|^s = O\left(\sum_{n=v+1}^{\infty} n^{k-1} \left|\widehat{a}_{n,v+1}\right|^k\right)^{s/k}.$$

2. Main results

We note that Theorem 1.1 and Theorem 1.2 give necessary conditions for the triangle matrices under the conditions (1.6) and (1.7). In the present paper, we determine necessary or sufficient conditions for a matrix $T \in \left(|\gamma A|_k, |\phi B|_s\right)$, $1 \leq k \leq s < \infty$. Also, in the special case, we get some more general results that do not include the conditions (1.6) and (1.7). More precisely, we give the following theorems.

Theorem 2.1. Let A, B be infinite triangle matrix and T be any infinite matrix of complex numbers. Further, let (γ_n) and (ϕ_n) be two sequences of positive numbers. Then, the necessary conditions for $T \in \left(|\gamma A|_k, |\phi B|_s\right)$, $1 < k \leq s < \infty$, are

$$\bar{l}_{nr} = \gamma_r^{-1/k^*} \sum_{i=r}^{\infty} t_{ni} \widehat{a}'_{ir} \text{ converges for } n, r \geq 0 \tag{2.1}$$

$$\sup_m \sum_{v=0}^{m} \frac{1}{\gamma_r} \left| \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr} \right|^{k^*} < \infty \text{ for } n, r \geq 0 \tag{2.2}$$

$$\sum_{n=m}^{\infty} \phi_n^{s-1} \left| \sum_{v=0}^{n} \sum_{i=m}^{\infty} \widehat{b}_{nv} t_{vi} \widehat{a}'_{im} \right|^s = O(\gamma_m^{s/k^*}), \tag{2.3}$$

where k^* is the conjugate of k, i.e., $k^* = k/(k-1)$.

Theorem 2.2. Let A, B be infinite triangle matrix and T be any infinite matrix of complex numbers. Further, let (ϕ_n) be a sequences of positive numbers. Then, the necessary and sufficient conditions for $T \in (|A|, |\phi B|_s), 1 = k \leq s < \infty$, are

$$\bar{l}_{nr} = \sum_{i=r}^{\infty} t_{ni} \widehat{a}'_{ir} \text{ converges for all } n, r \geq 0 \tag{2.4}$$

$$\sup_{m,r} \left| \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr} \right| < \infty \tag{2.5}$$

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{n} \widetilde{b}_{nv} \bar{l}_{vr} \right|^s = O(1). \tag{2.6}$$

Note that for $1 < k \leq s < \infty$, the characterization of the class of all matrices (ℓ_k, ℓ_s) are not known. Hence one can not expect to get a set of necessary and sufficient conditions for Theorem 2.1.

We require the following lemmas for the proof of our theorems.

Lemma A. Let X and Y be BK spaces, and A be an infinite matrix of complex numbers. If A is a matrix transformation from X into Y, *i.e.*, $A \in (X, Y)$, then it is a bounded linear operator [13].

Lemma B. Let $1 < k < \infty$ and A be an infinite matrix of complex numbers. Then

a-) $A \in (\ell, c)$ iff

$$i\text{-) } \lim_n a_{nv} \text{ exists for all } v \geq 0, \text{ and } \quad ii\text{-) } \sup_{n,v} |a_{nv}| < \infty,$$

b-) $A \in (\ell_k, c)$ iff

$$i\text{-) } (i) \text{ is satisfied, and } \quad ii\text{-) } \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty,$$

where c is the set of all convergent sequences, and $1/k + 1/k^* = 1$ [13].

Lemma C. Let $1 \leq s < \infty$ and A be an infinite matrix. Then $A \in (\ell_1, \ell_s)$ iff

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^s < \infty$$

where ℓ_s is the set of all s- absolutely convergent sequences [3].

Proof of the Theorem 2.1. Let $1 < k \leq s < \infty$. Suppose, $T \in (|\gamma A|_k, |\phi B|_s)$. Then, $T(x)$ exists and $T(x) \in |\phi B|_s$ for all $x \in |\gamma A|_k$. Now, $x \in |\gamma A|_k$ iff $y = \widetilde{A}(x) \in$

ℓ_k,where $y_n = \widetilde{A}_n(x) = \gamma_n^{1/k^*} \widehat{A}_n(x)$, and $\widehat{A}_n(x)$ is defined by (1.2). By the inverse of (1.2), we have

$$x_n = \sum_{r=0}^{n} \widehat{a}'_{nr} \widehat{A}_r(x) = \sum_{r=0}^{n} \widehat{a}'_{nr} \gamma_r^{-1/k^*} y_r,$$

and so

$$\begin{aligned}\sum_{v=0}^{m} t_{nv} x_v &= \sum_{v=0}^{m} t_{nv} \sum_{r=0}^{v} \widehat{a}'_{vr} \gamma_r^{-1/k^*} y_r \\ &= \sum_{r=0}^{m} \left(\gamma_r^{-1/k^*} \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr} \right) y_r = \sum_{r=0}^{\infty} l_{mr}^{(n)} y_r \\ &= L_m^{(n)}(y)\end{aligned}$$

where

$$l_{mr}^{(n)} = \begin{cases} \gamma_r^{-1/k^*} \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr}, & 0 \le r \le m \\ 0, & r > m. \end{cases}$$

This implies that $T(x)$ exists for all $x \in |\gamma A|_k$ iff $L^{(n)}(y)$ exists for $y \in \ell_k$, or equivalently, $L^{(n)} = \left(l_{mr}^{(n)} \right) \in (\ell_k, c)$. So, it follows from Lemma B that $T(x)$ exists iff (2.1) and (2.2) are satisfied. Further,

$$\begin{aligned}T_n(x) &= \sum_{v=0}^{\infty} t_{nv} x_v = \sum_{r=0}^{\infty} \lim_{m\to\infty} l_{mr}^{(n)} y_r \\ &= \sum_{r=0}^{\infty} \overline{l}_{nr} y_r = \overline{L}_n(y),\end{aligned}$$

which means $T(x) = \overline{L}(y)$. On the other hand, since $x \in |\phi B|_s$ iff $\widetilde{B}_n(x) \in \ell_s$, $T(x) \in |\phi B|_s$ iff $\widetilde{B}_n(T(x)) \in \ell_s$, *i.e.*, $C(y) \in \ell_s$, where

$$c_{nr} = \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \text{ for } n, r \ge 0,$$

because, for each $n \ge 0$,

$$\begin{aligned}C_n(y) &= \sum_{v=0}^{\infty} c_{nr} y_r = \sum_{r=0}^{\infty} \left(\sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \right) y_r \\ &= \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{L}_v(y) = \sum_{v=0}^{n} \widetilde{b}_{nv} T_v(x) \\ &= \widetilde{B}_n(T(x)).\end{aligned}$$

Also, it can be seen that $C = \widetilde{B}.\overline{L}$. So, by combining the above calculations we get $C \in (\ell_k, \ell_s)$. On the other hand, since ℓ_k is BK space for $k \ge 1$, then, by

Lemma A , the matrix C defines a bounded linear operator $L_C : \ell_k \to \ell_s$ such that $L_C(x) = (C_n(x))$ for all $x \in \ell_k$, and so there exists a constant M such that

$$\|L_C(x)\|_{\ell_s} \leq M \|x\|_{\ell_k} \text{ for all } x \in \ell_k. \tag{2.7}$$

Now in particular we put $x_m = 1$ and $x_n = 0$ for $n \neq m$. Then, we obtain

$$C_n(x) = \begin{cases} 0, & n < m \\ c_{nm}, & n \geq m \end{cases}$$

and

$$\|L_C(x)\|_{\ell_s} = \left(\sum_{n=m}^{\infty} \left| \phi_n^{1/s^*} \gamma_m^{-1/k^*} \sum_{v=0}^{n} \sum_{i=m}^{\infty} \widehat{b}_{nv} t_{vi} \widehat{a}'_{im} \right|^s \right)^{1/s}.$$

So, it follows from (2.7) that (2.3) holds. This completes the proof.

Proof of the Theorem 2.2. Let $1 = k \leq s < \infty$. Then, $T \in (|A|, |\phi B|_s)$ iff $T(x)$ exists and $T(x) \in |\phi B|_s$ for all $x \in |A|$. Now, $x \in |A|$ iff $y \in \ell$, where $y_n = \widehat{A}_n(x)$ and $\widehat{A}_n(x)$ is defined by (1.2). Then, by the inverse of (1.2), we have

$$x_n = \sum_{r=0}^{n} \widehat{a}'_{nr} \widehat{A}_r(x) = \sum_{r=0}^{n} \widehat{a}'_{nr} y_r,$$

and so

$$\begin{aligned} \sum_{v=0}^{m} t_{nv} x_v &= \sum_{v=0}^{m} t_{nv} \sum_{r=0}^{v} \widehat{a}'_{vr} \gamma_r^{-1/k^*} y_r \\ &= \sum_{r=0}^{m} \left(\sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr} \right) y_r = \sum_{r=0}^{\infty} l_{mr}^{(n)} y_r \\ &= L_m^{(n)}(y) \end{aligned}$$

where

$$l_{mr}^{(n)} = \begin{cases} \sum_{v=r}^{m} t_{nv} \widehat{a}'_{vr}, & 0 \leq r \leq m \\ 0, & r > m. \end{cases}$$

This implies that $T(x)$ exists for all $x \in |A|$ iff $L^{(n)}(y) \in (\ell, c)$, or equivalently, by Lemma B, (2.4) and (2.5) are satisfied. Further, we have

$$\begin{aligned} T_n(x) &= \sum_{v=0}^{\infty} t_{nv} x_v = \sum_{r=0}^{\infty} \lim_{m \to \infty} l_{mr}^{(n)} y_r \\ &= \sum_{r=0}^{\infty} \overline{l}_{nr} y_r = \overline{L}_n(y), \end{aligned}$$

which also means $T(x) = \overline{L}(y)$. On the other hand, since $T(x) = \overline{L}(y)$, then, $T(x) \in |\phi B|_s$ iff $C(y) \in \ell_s$, where

$$c_{nr} = \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{l}_{vr} \text{ for } n, r \geq 0,$$

because,

$$\begin{aligned} C_n(y) &= \sum_{r=0}^{\infty} c_{nr} y_r = \sum_{r=0}^{\infty} \left(\sum_{v=0}^{n} \widetilde{b}_{nv} \bar{l}_{vr} \right) y_r \\ &= \sum_{v=0}^{n} \widetilde{b}_{nv} \overline{L}_v(y) = \sum_{v=0}^{n} \widetilde{b}_{nv} T_v(x) \\ &= \widetilde{B}_n(T(x)). \end{aligned}$$

Thus it follows from Lemma C that

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{n} \widetilde{b}_{nv} \bar{l}_{vr} \right|^s = O(1),$$

which completes the proof.

We note that in the special case $T = I$, identity matrix, then $I \in (|\gamma A|_k, |\phi B|_s)$ means that if a series is summable $|A, \gamma_n|_k$, then it is also summable $|B, \phi_n|_s$, and also, conditions $(2.1), (2.2)$ hold and (2.3) reduces to

$$\phi_m^{s-1} \left| \frac{b_{mm}}{a_{mm}} \right|^s + \sum_{n=m+1}^{\infty} \phi_n^{s-1} \left| \sum_{i=m}^{n} \widehat{b}_{ni} \widehat{a}'_{im} \right|^s = O(\gamma_m^{s/k^*}).$$

So, as consequences of Theorem 2.1-2.2, we have many results. Now we list some of them.

Corollary 2.3. Let A and B be infinite triangle matrix of complex numbers. Further, let (γ_n) and (ϕ_n) be two sequences of positive numbers.

a-) If $1 < k \leq s < \infty$, then, the necessary conditions in order that a series by summable $|A, \gamma_n|_k$ is also summable $|B, \phi_n|_s$ are

$$\phi_m^{1/s^*} \left| \frac{b_{mm}}{a_{mm}} \right| = O(\gamma_m^{1/k^*}) \tag{2.8}$$

$$\sum_{n=m+1}^{\infty} \phi_n^{s-1} \left| \sum_{i=m}^{n} \widehat{b}_{ni} \widehat{a}'_{im} \right|^s = O(\gamma_m^{s/k^*}). \tag{2.9}$$

b-) If $1 = k \leq s < \infty$, then, the necessary and sufficient conditions in order that a series by summable $|A|$ is also summable $|B, \phi_n|_s$ are that (2.8) and (2.9) with $k = 1$ are satisfied.

Let us take $\phi_n = \gamma_n = n$ for all n. Since $|A, \gamma_n|_k = |A|_k$ and $|B, \phi_n|_s = |B|_s$, then, Corollary 2.3 reduces to the following result which do not include the additional condition (1.7) of Theorem 1.2.

Corollary 2.4. Let $1 < k \leq s < \infty$, A and B be triangle matrix of complex numbers. Then necessary conditions in order that a series by summable $|A|_k$ is also summable $|B|_s$ are

$$m^{1/k-1/s}\left|\frac{b_{mm}}{a_{mm}}\right| = O(1)$$

and

$$\sum_{n=m+1}^{\infty} n^{s-1}\left|\sum_{i=m}^{n}\widehat{b}_{ni}\widehat{a}'_{im}\right|^{s} = O\left(m^{s/k^*}\right).$$

If $1 = k \leq s < \infty$, by Theorem 2.2, these conditions with $k = 1$ are also necessary and sufficient for the conclusion to satisfy.

Also, if we put $A = I$ and $\gamma_v = v$ for all $v \geq 1$, then the summability $|A, \gamma_n|_k$ is equivalent to the condition

$$\sum_{n=1}^{\infty} n^{k-1}|x_n|^k < \infty.$$

Hence the following result is deduced by theorem 2.1, which is due to Sarıgöl [9].

Corollary 2.5. Let $1 \leq s < \infty$ and B be triangle matrix of complex numbers. Then, the necessary and sufficient conditions in order that an absolutely convergent series is also summable $|B|_s$ are

$$\sum_{n=v}^{\infty} n^{s-1}\left|\widehat{b}_{nv}\right|^{s} = O(1).$$

Further, if A and B are the matrix of weighted means (R, p_n) and (R, q_n) then, it is easily seen that $\widehat{a}_{nv} = p_n P_{v-1}/P_n P_{n-1}, 1 \leq v \leq n$, and zero otherwise, $\widehat{a}'_{vv} = P_v/p_v$, $\widehat{a}'_{v,v-1} = -P_{v-2}/p_{v-1}$ and $\widehat{a}'_{n,v} = 0$ for $n \neq v, v+1$, and also, $\widehat{b}_{nv} = q_n Q_{v-1}/Q_n Q_{n-1}, 1 \leq v \leq n$, and zero otherwise. So the following result follows immediately from Theorem 2.2, of which sufficiency for the case $\phi_v = \gamma_v = v$ and $k = s$ is due to Orhan and Sarıgöl [5].

Corollary 2.6. Let $1 = k \leq s < \infty$ and B be triangle matrix of complex numbers. Then, necessary and sufficient conditions in order that a series by summable $|R, p_n|$ is also summable $|R, q_n|_s$ are

$$v^{1-1/s}\left|\frac{P_v q_v}{p_v Q_v}\right| = O(1)$$

and

$$\left|Q_{v-1}\frac{P_v}{p_v} - Q_v\frac{P_{v-1}}{p_v}\right|^{s}\sum_{n=v+1}^{\infty} n^{s-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^{s} = O(1).$$

Let A and B be Cesàro matrices (C,α) and (C,β), respectively. In this case, it is well known that $\widehat{a}_{nv} = vA_{n-v}^{\alpha-1}/nA_n^{\alpha}$, $\widehat{b}_{nv} = vA_{n-v}^{\beta-1}/nA_n^{\beta}$, and $\widehat{a}'_{nv} = vA_{n-v}^{-\alpha-1}A_v^{\alpha}/n$. So, (2.1) is equivalent to

$$v^{\alpha-\beta+1/k-1/s} = O\left(1\right),$$

or $\beta \geq \alpha + 1/k - 1/s$. Also, since (see, Lemma 5, Mehdi [4])

$$\sum_{n=v}^{\infty} \frac{1}{n} \left| \frac{A_{n-r}^{\beta-\alpha-1}}{A_n^{\beta}} \right|^s = \begin{cases} O(v^{-s\beta-1}), & s\left(\beta-\alpha-1\right) < -1 \\ O(v^{-s\beta-1}\log v), & s\left(\beta-\alpha-1\right) = -1 \\ O(v^{-s(\alpha+1)}), & s\left(\beta-\alpha-1\right) > -1 \end{cases}$$

we have

$$\begin{aligned} E_v &= \sum_{n=v}^{\infty} n^{s-1} \left| \sum_{r=v}^{n} \widehat{b}_{nr}\widehat{a}'_{rv} \right|^s = \left(vA_v^{\alpha}\right)^s \sum_{n=v}^{\infty} n^{s-1} \left| \frac{1}{nA_n^{\beta}} \sum_{r=v}^{n} A_{n-r}^{\beta-1} A_{r-v}^{-\alpha-1} \right|^s \\ &= \left(vA_v^{\alpha}\right)^s \sum_{n=v}^{\infty} \frac{1}{n} \left| \frac{A_{n-v}^{\beta-\alpha-1}}{A_n^{\beta}} \right|^s = O\left(v^{s-s/k}\right). \end{aligned}$$

In fact, since $\beta \geq \alpha + 1/k - 1/s$, it is clear that $s\left(\beta-\alpha-1\right) + s + 1 - s/k \geq 0$. So, it is easy to see from Mehdi's lemma that (2.8) is satisfied, because, E_v is equal to $O(1)v^{-s(\beta-\alpha-1)-1-s+s/k}$, $O(1)v^{-s(\beta-\alpha-1)-1-s+s/k}\log v$ and $O(1)v^{-s+s/k}$ for $s\left(\beta-\alpha-1\right) < -1$, $s\left(\beta-\alpha-1\right) = -1$ and $s\left(\beta-\alpha-1\right) > -1$, respectively. So Theorem 2.1 reduces to the following result of which sufficiency was proved by Flett [2].

Corollary 2.7. Let $1 < k \leq s < \infty$, and $\alpha > -1$. Then, necessary conditions in order that a series by summable $\left|C,\alpha\right|_k$ is also summable $\left|C,\beta\right|_s$ are $\beta \geq \alpha + 1/k - 1/s$.

REFERENCES

1. H. Bor and B. Thorpe: *On some absolute summability methods.* Analysis. **7** (1987), 145-152.
2. T. M. Flett: *On an extension of absolute summability and some theorems of Littlewood and Paley.* Proc. London Math. Soc. **7** (1957), 113-141.
3. I. J. Maddox: *Elements of functional analysis.* Cambridge University Press, London,New York, 1970.
4. M. R. Mehdi: *Summability factors for generalized absolute summability I.* Proc. London Math. Soc. **10** (1960), 180-199.
5. C. Orhan and M. A. Sarıgöl: *On absolute weighted mean summability.* Rocky Moun. J. Math. **23 (3)** (1993), 1091-1097.
6. B. E. Rhoades and E. Savas: *On inclusion relations for absolute summability.* Int. J. Math. Math. Sci. **32 (3)** (2002), 129–138.

7. M. A. Sarigöl: *On local properties of factored Fourier series.* App. Math. Comput. **216** (2010), 3386–3390.

8. M. A. Sarigöl: *On two absolute Riesz summability factors of infinite series.* Proc. Amer. Math. Soc. **118 (2)** (1993), 485-488.

9. M. A. Sarigöl: *On* $|T|_k$ *summability and absolute Norlund summability.* Math. Slovaca, **42 (3)** (1992), 325-329.

10. M. A. Sarigöl: *On absolute weighted mean summability methods.* Proc. Amer. Math. Soc. **115(1)** (1992), 157–160

11. E. Savas: *Necessary conditions for inclusion relations for absolute summability. Appl. Math. Comput.* **151** (2004), 523–531.

12. N. Tanovič-Miller: *On strong summability.* Glas. Mat. **34** (1979), 87–97.

13. A. Wilansky: *Summability through Functional Analysis.* North–Holland Mathematical Studies 85, Elsevier Science Publishers, Amsterdam, New York, Oxford, 1984.

Mehmet Ali Sarıgöl

Pamukkale University

Department of Mathematics

Denizli, Turkey

msarigol@pau.edu.tr

FACTA UNIVERSITATIS (NIŠ)
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CESÀRO AND STATISTICAL DERIVATIVE

Fatih Nuray

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Abstract. In this study, we introduce the notions of Cesàro, strongly Cesàro and statistical derivatives for real valued functions. These notions are based on the concepts of Cesàro and statistical convergence of a sequence. Then we establish some relationships between strongly Cesàro derivative and statistical derivative.
Keywords: Cesàro derivative; statistical derivative; Cesàro continuity; real valued functions; convergence of a sequence.

1. Introduction

In mathematical analysis, the concepts of limit, continuity and derivative for a function are given respectively. In the literature, the concept of Cesàro limit has been known for many years. Later, Cesàro continuity, statistical limit and statistical continuity concepts were given (see [5]). In [3] strongly sequentially continuous functions were defined and studied. Cesàro derivative and statistical derivative definitions do not appear in the literature. We will introduce the concepts of Cesàro derivative and statistical derivative in this study to fill the gap in the literature.

A sequence $x = (x_k)$ is said to be Cesàro summable to the number u if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = u,$$

in this case we write $(C,1) - \lim x_n = u$, strongly Cesàro summable to the number u if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - u| = 0,$$

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in this case we write $[C,1]-\lim x_n = u$, and statistically convergent to the number u if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \quad |x_k - u| \geq \epsilon\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set, in this case we write $st - \lim x_n = u$.

Let (a_n) and (b_n) be two sequences of real numbers such that $(C,1)-\lim a_n = a$ and $(C,1) - \lim b_n = b$. It is known that

$$(C,1) - \lim a_n.b_n = a.b \quad and \quad (C,1) - \lim(a_n + b_n) = a + b.$$

The idea of statistical convergence was introduced by Steinhaus in [13] and Fast in [6] independently and since then has been studied by other authors including [4, 7, 11] and [14]. Recently, the articles [1], [2], [8], [9] and [10] have been published on statistical convergence and its applications.

2. Cesàro Derivative

Very basic finite difference formulas approximates the derivative $f'(x)$ using a sequence $x_n > 0$ such that $\lim_{n\to\infty} x_n = 0$. Two basic formulas for derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x_0 are

$$\lim_{n\to\infty} \frac{f(x_0+x_n)-f(x_0)}{x_n} = f'(x_0) \text{ and } \lim_{n\to\infty} \frac{f(x_0+x_n)-f(x_0-x_n)}{2x_n} = f'(x_0).$$

The first formula is Newton's difference quotient and determines the slope of a secant line of the graph of f. The second formula is the symmetric difference quotient and determines the slope of a cord of the graph of f. For more detail (see [12]).
With the similar approach we will now define the Cesàro derivative.

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ has a Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(x_0 + x_k) - f(x_0)}{x_k} = w$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.1 as follows:

Definition 2.2. A function $f : \mathbb{R} \to \mathbb{R}$ has a Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{f(x_0 + x_k) - f(x_0 - x_k)}{2x_k} = w$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

A function $f : \mathbb{R} \to \mathbb{R}$ is Cesàro continuous at a point x_0 if

$$(C,1) - \lim f(x_0 + x_n) = f(x_0)$$

holds for each sequence $(x_n) \to 0$.

Theorem 2.1. *Let a function $f : \mathbb{R} \to \mathbb{R}$ has a Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ then f is Cesàro continuous at the point x_0.*

Proof. Let $\lim x_n = 0$. Clearly

$$f(x_0 + x_n) - f(x_0) = \frac{f(x_0 + x_n) - f(x_0)}{x_n} x_n$$

holds for each $n \in \mathbb{N}$. Since $\lim x_n = 0$ implies $(C,1) - \lim x_n = 0$, we can write

$$(C,1) - \lim(f(x_0 + x_n) - f(x_0)) = (C,1) - \lim \frac{f(x_0 + x_n) - f(x_0)}{x_n} (C,1) - \lim x_n.$$

Hence, from the assumption we have

$$(C,1) - \lim f(x_0 + x_n) = f(x_0)$$

so f is Cesàro continuous at the point x_0. □

Definition 2.3. A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{f(x_0 + x_k) - f(x_0)}{x_k} - w \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 2.3 as follows:

Definition 2.4. A function $f : \mathbb{R} \to \mathbb{R}$ has a strongly Cesàro derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{f(x_0 + x_k) - f(x_0 - x_k)}{2x_k} - w \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

It is clear from the definitions of Cesàro and strongly Cesàro derivatives that if a function has a strongly Cesàro derivative at point x_0, it has a Cesàro derivative at that point.

3. Statistical Derivative

In this section, we first give the definition of statistical derivative and then we establish some relationships between the strongly Cesàro derivative and statistical derivative.

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ has a statistical derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{f(x_0 + x_k) - f(x_0)}{x_k} - w \right| \ge \epsilon \right\} \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

An equivalent definition to the Definition 3.1 as follows:

Definition 3.2. A function $f : \mathbb{R} \to \mathbb{R}$ has a statistical derivative $w \in \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{f(x_0 + x_k) - f(x_0 - x_k)}{2x_k} - w \right| \ge \epsilon \right\} \right| = 0$$

holds whenever $x_n > 0$ and $\lim_{n\to\infty} x_n = 0$.

If a function has derivative it has statistical derivative but converse may not be true.

Theorem 3.1. *a) If a function $f : \mathbb{R} \to \mathbb{R}$ has strongly Cesàro derivative at a point $x_0 \in \mathbb{R}$ then it has statistical derivative at the point x_0.*

b) If $\left(\frac{f(x_0+x_k)-f(x_0)}{x_k}\right)$ is bounded for each $k \in \mathbb{N}$ and f has statistical derivative at a point $x_0 \in \mathbb{R}$ then f has strongly Cesàro derivative at the point x_0.

Proof. Let's write y_k instead of $\frac{f(x_0+x_k)-f(x_0)}{x_k}$ for simplicity.

a) Let f has strongly Cesàro derivative at a point $x_0 \in \mathbb{R}$. For an arbitrary $\epsilon > 0$, we get

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} |y_k - w| &= \left(\frac{1}{n} \sum_{k=1 \, |y_k - w| \ge \epsilon}^{n} |y_k - w| + \frac{1}{n} \sum_{k=1 \, |y_k - w| < \epsilon}^{n} |y_k - w| \right) \\
&\ge \frac{1}{n} \sum_{k=1 \, |y_k - w| \ge \epsilon}^{n} |y_k - w| \\
&\ge \frac{1}{n} |\{1 \le k \le n : |y_k - w| \ge \epsilon\}| \epsilon.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\frac{1}{n}|\{1\le k\le n : |y_k-w|\ge\epsilon\}|=0$$

that is, f has a statistical derivative at the point x_0.

b) Now suppose that f has a statistical derivative at the point x_0 and bounded, since $\left(\frac{f(x_0+x_k)-f(x_0)}{x_k}\right)$ is bounded for each $k\in\mathbb{N}$, say $|y_k-w|\le K$ for all k. Given $\epsilon>0$, we get

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}|y_k-w| &= \frac{1}{n}\left(\sum_{k=1_{|y_k-w|\ge\epsilon}}^{n}|y_k-w|+\sum_{k=1_{|y_k-w|<\epsilon}}^{n}|y_k-w|\right)\\ &\le \frac{1}{n}\left(K\sum_{k=1_{|y_k-w|\ge\epsilon}}^{n}1+\sum_{k=1_{|y_k-w|<\epsilon}}^{n}|y_k-w|\right)\\ &\le K\frac{1}{n}|\{1\le k\le n : \quad |y_k-w|\ge\epsilon\}|+\frac{1}{n}\sum_{k=1}^{n}\epsilon\end{aligned}$$

hence we have,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|y_k-w|=0,$$

that is f has strongly Cesàro derivative at the point x_0. □

REFERENCES

1. T. Acar and S. A. Mohiuddine: *Statistical (C, 1)(E, 1) summability and Korovkin's theorem.* Filomat **30**(2) (2016) 387–393.
2. C. Belen, and S. A. Mohiuddine: *Generalized weighted statistical convergence and application.* Appl. Math. Comput. 219 (2013) 9821–9826.
3. A. Borichev, R. Deville and E. Matheron: *Strongly sequentially continuous functions.* Quaestiones Mathematicae, **24** (2001), 535–548.
4. J. S. Connor: *The statistical and strong p-Cesáro of sequences.* Analysis, **8**(1988), 47–63.
5. J. Connor, and K.G. Grosse-Erdmann: *Sequential definitions of continuity for real functions.* Rocky Mountain J. Math., bf 33, 1, (2003), 93–121.
6. H. Fast: *Sur la convergence statistique* Cooloq. Math. **2**(1951), 241–244.
7. J. A. Fridy: *On statistical convergence* Analysis, **5**(1985), 301–313.
8. U. Kadak and S. A. Mohiuddine: *Generalized statistically almost convergence based on the difference operator which includes the (p, q)-gamma function and related approximation theorems.* Results Math. **73**(1) (2018), Article 9.

9. S. A. Mohiuddine: *Statistical weighted A-summability with application to Korovkin's type approximation theorem*. J. Inequal. Appl. (2016) 2016:101.

10. S. A. Mohiuddine and B. A. S. Alamri: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM **113**(3) (2019) 1955–1973.

11. T. Šalát: *On statistically convergent sequences of real numbers*. Math. Slovaca, **30**(1980), 139–150.

12. S. Pedersen and J. P. Sjoberg: *Sequential Derivatives*, arXiv: 1801.04039v1 [math. CA] 12 Jan 2018.

13. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math., **2**(1951), 73–74.

14. I. J. Schoenberg: *The integrability of certain functions and related summability methods*. Amer. Math. Monthly, **66** (1959), 361–375.

Fatih Nuray
Faculty of Science
Department of Mathematics
Afyon Kocatepe University
Afyonkarahisar, Turkey
fnuray@aku.edu.tr

FACTA UNIVERSITATIS (NIŠ)
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A HYBRID ALGORITHM FOR THE UNCERTAIN INVERSE p-MEDIAN LOCATION PROBLEM

Akram Soltanpour, Fahimeh Baroughi and Behrooz Alizadeh

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Abstract. In this paper, we investigate the inverse p-median location problem with variable edge lengths and variable vertex weights on networks in which the vertex weights and modification costs are the independent uncertain variables. We propose a model for the uncertain inverse p-median location problem with tail value at risk objective. Then, we show that it is NP-hard. Therefore, a hybrid particle swarm optimization algorithm has been presented to obtain the approximate optimal solution of the proposed model. The algorithm contains expected value simulation and tail value at risk simulation.
Keywords: p-median location problem; inverse optimization; Hybrid algorithm; nonlinear programming.

1. Introduction

One of the important aspects of location problems which has recently been studied by many researchers is the p-median location problem which can be stated as follows. Let $N = (V, E)$ be an undirected connected network with vertex set V, $|V| = n$, and edge set E, $|E| = m$. The distance between two points on N is equal to the length of the shortest path connecting these two points. Each vertex is associated with a nonnegative weight that is the demand of the client at this vertex. In a p-median problem on a network, the aim is to find p locations for establishing facilities on edges or vertices of the network such that the sum of the weighted distances from the clients to the closest facility becomes minimum. In the context of the p-median location problems, the interested reader is referred to papers [1, 7, 8, 11, 12, 13, 17, 20, 21, 28, 37, 39].

In recent years, inverse location problems have found an increasing interest. In an inverse location problem the goal is to modify parameters of the problem at

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minimum cost such that a prespecified solution becomes optimal. Burkard et al. investigated the inverse 1-median problem with variable vertex weights on a tree network and also on a plane and presented algorithms in $O(n \log n)$ time for them [9]. Also they proposed an algorithm in $O(n^2)$ time for the problem under investigation on cycles [10]. Baroughi et al. [3] proved that the inverse p-median location problem (IpMLP) on general networks is NP-hard. For a survey on the inverse p-median location problems, we refer the interested reader to [16, 18, 19, 23, 30, 36].

In the real life, we are usually faced with various types of uncertainty. For example, in location problems, we are usually not sure of the vertex weights, the travel times between vertices, the establishing costs of facilities and the vertex weights or edge lengths modification costs of a network. The uncertainty theory that proposed by Liu [25] is a suitable tool to deal with these parameters. Some researchers applied the uncertainty theory to deal with the location problems,for example Gao [14] modeled the single facility location problems with uncertain demands. Wen et al. [43] investigated the capacitated facility location-allocation problem with uncertain demands and also Nguyen and Chi [31] studied inverse 1-median problem on a tree with uncertain costs and showed that the inverse distribution function of the minimum cost can be obtained at $O(n^2 \log n)$ time. For a survey on uncertain location problems, we refer the interested reader to [15, 22, 27, 34, 40, 46].

The uncertainty leads to the risk. Liu in [26] introduced the risk concept in the uncertain environment. Measuring the risk is one of the important steps in the decision making process. The risk metrics contain techniques and data sets used to calculate the risk value of the problem under investigation. Tail value at risk (TVaR) metric [32] is one of the measures of the risk that is widely acceptable among industry segments and market participants.

In the risk management related to location problems, Berman et al. [6] studied the effect of a decision maker's risk attitude on the median and center location problems, with uncertain demand in the mean-variance framework. Wang et al. [41] investigated a two-stage fuzzy facility location problem with value at risk. Wagner et al. [42] developed and examined a new algorithm for solving the p-median problem when the demands are probabilistic and correlated. For a survey on the risk management in the location problems with fuzzy variables, see, e.g. [5, 44].

In this paper, we concentrate on IpMLP with variable edge lengths and variable vertex weights on networks. We assume that the vertex weights and modification costs are the independent uncertain variables. We propose a model for the uncertain inverse p-median location problem with tail value at risk objective and expected value constraints and show that the problem is NP-hard. Considering the uncertain and NP-hard nature in uncertain IpMLP (UIpMLP), evolutionary and meta-heuristics algorithms can be used to UIpMLP for successful generation of optimal solutions. Hence, we present a hybrid particle swarm optimization algorithm which contains expected value simulation and tail value at risk simulation to obtain the approximate optimal solution of the proposed model.

Based on our knowledge, there are two papers on the implementation of metaheuristic algorithms to the inverse location problems until now. Alizadeh and

Bakhteh [2] studied the general IpMLPs on networks and presented a modified firefly algorithm for the problem under investigation. Mirzapolis Adeh et al. [29] investigated the general inverse ordered p-median location problem on crisp networks and designed a modified particle swarm optimization (PSO) algorithm for it. There is no scientific paper on implementation of hybrid metaheuristic algorithms on IpMLPs in uncertain networks. However, many papers can be found in the literature for other classical location problems on uncertain networks. Bashiri et al. [4] modeled fuzzy capacitated p-hub center problem and presented a genetic algorithm for the problem. Huang and Hao [22] modeled uncapacitated facility location problem with uncertain customers positions and provided a hybrid intelligent algorithm for solving it. In 2018 Rahmaniani et al. [35] proposed an efficient hybrid solution algorithm for the capacitated facility location-allocation problem under uncertainty. Yang et al. [45] presented an improved hybrid particle swarm optimization algorithm for fuzzy p-hub center problem.

The article is organized as follows: In the next section, we first introduce uncertainty theory and TVaR metric in an uncertain environment. Then, we discuss uncertain optimization model and present a new model with TVaR objective and expected value constraints. In Section 3., we first introduce IpMLP with variable edge lengths and variable vertex weights on networks and then investigate the problem with uncertain vertex weights and uncertain modification costs. A model for the uncertain inverse p-median location problem (UIpMLP) with TVaR objective is presented and it is shown that the problem under investigation is NP-hard. Then, we present a hybrid PSO algorithm to obtain the approximate optimal solution of the proposed model, which it contains expected value simulation and TVaR simulation. Finally, to show the effectiveness of the proposed hybrid PSO algorithm, we give a numerical example. Section 4. gives a brief conclusion to this paper.

2. Preliminaries

In this section, we first present some definitions and theorems of the uncertainty theory and TVaR metric in an uncertain environment. Then, we introduce the uncertain optimization model and present a new model with TVaR objective and expected value constraints.

2.1. Uncertainty theory

Let Γ be a nonempty set and Θ be a σ-algebra over Γ. An uncertain measure is a set function $\mathcal{M} : \Theta \to [0, 1]$ that satisfies in normality, duality and subadditivity axioms. The triple $(\Gamma, \Theta, \mathcal{M})$ is called an uncertainty space.

Definition 2.1. (*Liu*[25]). Let $(\Gamma, \Theta, \mathcal{M})$ be an uncertainty space. A measurable function θ from $(\Gamma, \Theta, \mathcal{M})$ to the set of real numbers is called an uncertain variable.

Definition 2.2. (*Liu*[25]). Let θ be an uncertain variable. For any real number x, the function $\Upsilon(x) = \mathcal{M}\{\theta \leq x\}$ is called an uncertainty distribution of θ.

Definition 2.3. (*Liu*[25]). Let θ_i, $i = 1, \ldots, n$, be the uncertain variables. We call θ_i, $i = 1, \ldots, n$, independent if for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers,

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\theta_i \in B_i\}\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\theta_i \in B_i\}.$$

Definition 2.4. (*Liu*[25]). The expected value of the uncertain variable θ is defined as

$$E[\theta] = \int_0^{+\infty} \mathcal{M}\{\theta \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\theta \leq r\}dr,$$

provided that at least one of the two integral is finite.

A real valued function $f(x_1, x_2, ..., x_n)$ is said to be strictly increasing if $f(x_1, x_2, \ldots, x_n) > f(y_1, y_2, \ldots, y_n)$ when $x_i > y_i$ for $i = 1, 2, \ldots, n$.

Theorem 2.1. (*Liu*[*25*]). *Let θ_i, $i = 1, 2, \ldots, n$, be the independent uncertain variables and Υ_i^{-1}, $i = 1, 2, \ldots, n$, be the inverse uncertainty distributions of θ_i. Also, let $f(x_1, x_2, \ldots, x_n)$ be a strictly increasing function with respect to x_i, $i = 1, 2, \ldots, n$. Then the uncertain variable $\vartheta = f(\theta_1, \theta_2, \ldots, \theta_n)$ has the following inverse uncertainty distribution*

$$\Upsilon^{-1}(\alpha) = f(\Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)),$$

and also it has the following expected value

$$E[\vartheta] = \int_0^1 f(\Upsilon_1^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha))d\alpha.$$

2.2. TVaR metric in an uncertain environment

The risk demonstrates a situation, in which there is a chance of loss or danger. The quantification of the risk is a key step towards the management and mitigation of the risk. In this section, we introduce the definition of the TVaR metric to account the probability of loss and the severity of the loss in an uncertain environment [32].

In order to define the TVaR metric, we first introduce the definition of the loss function.

Definition 2.5. (*Liu*[26]). Consider θ_i, $i = 1, 2, \ldots, n$, as the uncertain factors of a system. A function f is said to be a loss function if some specified loss occurs if and only if

$$f(\theta_1, \theta_2, \ldots, \theta_n) > 0.$$

In the uncertain environment, TVaR of the loss function is defined as follows.

Definition 2.6. (*Peng*[32]). Let θ_i, $i = 1, 2, \ldots, n$, be the uncertain factors and f be the loss function of a system. Then TVaR of f is defined as

$$TVaR_\beta = \frac{1}{\beta} \int_0^\beta sup \{\lambda \mid \mathcal{M} \{f(\theta_1, \theta_2, \ldots, \theta_n) \geq \lambda\} \geq \gamma\} \, d\gamma,$$

for each given risk confidence level $\beta \in (0, 1]$.

Theorem 2.2. (*Peng*[*32*]). *Let θ_i, $i = 1, 2, \ldots, n$, be the uncertain factors of a system and Υ_i^{-1}, $i = 1, 2, \ldots, n$, be the inverse uncertainty distributions of θ_i. Also assume that the loss function $f(x_1, x_2, \ldots, x_n)$ is a strictly increasing function with respect to x_i, $i = 1, 2, \ldots, n$. Then, for each risk confidence level $\beta \in (0, 1]$, we have*

$$TVaR_\beta = \frac{1}{\beta} \int_0^\beta f(\Upsilon_1^{-1}(1-\gamma), \Upsilon_2^{-1}(1-\gamma), \ldots, \Upsilon_n^{-1}(1-\gamma)) d\gamma.$$

2.3. Uncertainty optimization

Let $x = (x_1, x_2, \ldots, x_n)$ be a decision vector, and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ be an uncertain vector. Consider the following optimization model.

$$\begin{array}{lll} \min & f(x,\theta) & \\ \text{s.t.} & g_j(x,\theta) \leq 0 & j = 1, \ldots, p, \\ & z_l(x) \leq 0 & l = 1, \ldots, m, \\ & x \geq 0, & \end{array} \tag{2.1}$$

where f and g_j, $j = 1, \ldots, p$ are uncertain functions and z_l, $l = 1, \ldots, m$ are crisp functions.

Since the objective function of the model (2.1) involves uncertainty, it cannot be directly optimized. Therefore, by considering $f(x, \theta)$ as a loss function, we minimize its TVaR. In addition, since the uncertain constraints do not define a crisp feasible set, we use the expected value of constraints. Thus, the model (2.1) can be reformulated as

$$\begin{array}{lll} \min & TVaR_\alpha(f(x,\theta)) & \\ \text{s.t.} & E(g_j(x,\theta)) \leq 0 & j = 1, \ldots, p, \\ & z_l(x) \leq 0 & l = 1, \ldots, m, \\ & x \geq 0. & \end{array} \tag{2.2}$$

According to Theorems 2.1 and 2.2, we can rewrite the problem (2.2) as follows:

$$\begin{array}{lll} \min & \frac{1}{\beta}\int_0^\beta f(x, \Upsilon_1^{-1}(1-\gamma), \Upsilon_2^{-1}(1-\gamma), \ldots, \Upsilon_n^{-1}(1-\gamma)) d\gamma & \\ \text{s.t.} & \int_0^1 g_j(x, \Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha), \ldots, \Upsilon_n^{-1}(\alpha)) d\alpha \leq 0 & j = 1, \ldots, p, \\ & z_l(x) \leq 0 \quad l = 1, \ldots, m, & \\ & x \geq 0, & \end{array} \tag{2.3}$$

where $g_j(x, \theta_1, \theta_2, \ldots, \theta_n)$ is strictly increasing with respect to $\theta_1, \theta_2, \ldots, \theta_n$.

3. Problem definition

In this section, we first introduce IpMLP with variable edge lengths and variable vertex weights on networks and then investigate the problem with uncertain vertex weights and uncertain modification costs. A model for UIpMLP with TVaR objective and expected value constraints is presented. To solve the proposed model, we present a hybrid PSO algorithm which contains expected value simulation and TVaR simulation.

3.1. UIpMLP on networks

We can express IpMLP with variable edge lengths and variable vertex weights as follows: Let $N = (V, E)$ with $|V| = n$ and $|E| = m$ be a connected network. Also let vertex $v \in V$ have a positive weight $w(v)$ and edge $e \in E$ have a positive length ℓ_e. In an IpMLP on networks, a set of vertices $\{m_1, \ldots, m_p\}$ is given. The goal is to modify $w(v)$, $v \in V$, and ℓ_e, $e \in E$, at minimum total cost such that the given set becomes a p-median of modified location problem. Let us consider nonnegative costs c_e^+ and c_v^+, if ℓ_e and $w(v)$ are increased by one unit, respectively. Also we consider nonnegative costs c_e^- and c_v^-, if ℓ_e and $w(v)$ are decreased by one unit, respectively. Let p_e, q_e, p_v and q_v be the amounts by which the edge length ℓ_e and the vertex weight $w(v)$ are increased and decreased, respectively. We let p_e, q_e, p_v and q_v obey the upper bounds u_e^+, u_e^-, u_v^+, u_v^-. In addition, assume that $\mathcal{S}$ is the set of all subsets $S \subseteq V$ with $|S| = p$. Thus, IpMLP on N can be stated as follows.

Change ℓ_e, $e \in E$, to $\tilde{\ell}_e = \ell_e + p_e - q_e$ and $w(v)$, $v \in V$, to $\tilde{w}(v) = w(v) + p_v - q_v$ such that

(i) The set $\{m_1, \ldots, m_p\}$ becomes a p-median of N with respect to $\tilde{\ell}$ and $\tilde{w}(v)$, i.e.,

$$\sum_{v \in V} \tilde{w}(v) \min_{i=1,\ldots,p} d_{\tilde{\ell}}(v, m_i) \leq \sum_{v \in V} \tilde{w}(v) \min_{k \in S} d_{\tilde{\ell}}(v, v_k) \quad \forall\, S \in \mathcal{S}, \tag{3.1}$$

(ii) The bound constraints are satisfied:

$$0 \leq p_e \leq u_e^+, \quad 0 \leq q_e \leq u_e^- \quad \forall\, e \in E, \tag{3.2}$$

$$0 \leq p_v \leq u_v^+, \quad 0 \leq q_v \leq u_v^- \quad \forall\, v \in V, \tag{3.3}$$

(iii) The objective function

$$\sum_{e \in E} (c_e^+ p_e + c_e^- q_e) + \sum_{v \in V} (c_v^+ p_v + c_v^- q_v)$$

becomes minimum.

This formulation of IpMLP is a nonlinear programming model. In the following, we consider IpMLP with uncertain vertex weights and uncertain modification costs.

Let $N = (V, E)$ be a network with independent uncertain vertex weights $\theta_v, v \in V$. Also let $w(v)$ be a parameter on each vertex $v \in V$, which will be changed to $\tilde{w}(v)$. In addition, suppose that θ_v relates to this parameter, i.e., for each vertex $v \in V$, we have an original weight $\theta(w(v))$ and also a new weight $\theta(\tilde{w}(v))$. Let ϑ_v^+ and ϑ_v^- be the independent uncertain variables with respect to the costs c_v^+ and c_v^-, for all $v \in V$, and ϑ_e^+ and ϑ_e^- be the independent uncertain variables with respect to the costs c_e^+ and c_e^-, for all $e \in E$, respectively.

Let us assume that we are given a set of vertices $\{m_1, \ldots, m_p\}$. In an UIpMLP, the goal is to find $\tilde{\ell}_e = \ell_e + p_e - q_e$ and $\tilde{w}(v) = w(v) + p_v - q_v$ such that $\{m_1, \ldots, m_p\}$ becomes a p-median of the problem with respect to $\theta_v(\tilde{w}(v))$ and $\tilde{\ell}_e$, $v \in V$, $e \in E$, and the total cost

$$\sum_{v \in V} (\vartheta_v^+ p_v + \vartheta_v^- q_v) + \sum_{e \in E} (\vartheta_e^+ p_e + \vartheta_e^- q_e)$$

is minimized.

Therefore, we can model UIpMLP as follows.

$$\begin{array}{ll} \min & \left[\sum_{v \in V} (\vartheta_v^+ p_v + \vartheta_v^- q_v) + \sum_{e \in E} (\vartheta_e^+ p_e + \vartheta_e^- q_e)\right] \\ s.t. & \\ (3.4) & \left[\sum_{v \in V} \theta(\tilde{w}(v)) \left(\min_{i=1,\ldots,p} d_{\tilde{\ell}}(v, m_i) - \min_{k \in S} d_{\tilde{\ell}}(v, v_k)\right)\right] \leq 0 \quad \forall S \in \mathcal{S}, \\ & 0 \leq p_e \leq u_e^+, \quad 0 \leq q_e \leq u_e^- \quad \forall e \in E, \\ & 0 \leq p_v \leq u_v^+, \quad 0 \leq q_v \leq u_v^- \quad \forall v \in V. \end{array}$$

Definition 3.1. Let $p = (p_e)_{e \in E}$ and $q = (q_v)_{v \in V}$ be the vectors that satisfies in (3.2) and (3.3). Then (p, q) is called expected solution of (3.4) if and only if $\forall S \in \mathcal{S}$

$$\sum_{v \in V} E\left[\theta(\tilde{w}(v))\right] \left(\min_{i=1,\ldots,p} d_{\tilde{\ell}}(v, m_i) - \min_{k \in S} d_{\tilde{\ell}}(v, v_k)\right) \leq 0.$$

Now, let (p, q) be a expected solution of (3.4). Define

$$f(p, q) = \sum_{v \in V} (\vartheta_v^+ p_v + \vartheta_v^- q_v) + \sum_{e \in E} (\vartheta_e^+ p_e + \vartheta_e^- q_e).$$

Definition 3.2. For a risk confidence level $\beta \in (0, 1]$, a expected solution (p^*, q^*) is called optimal solution with minimum $TVaR$ if

$$TVaR_\beta(f(p^*, q^*)) \leq TVaR_\beta(f(p, q)),$$

holds for any expected solution (p, q).

Therefore, we can find an optimal expected solution with minimum TVaR as follows:

Let $(\Psi_v^+)^{-1}$, $v \in V$, and $(\Psi_e^+)^{-1}$, $e \in E$ be the inverse uncertainty distributions of ϑ_v^+ and ϑ_e^+, respectively. Also let $(\Psi_v^-)^{-1}$, $v \in V$, and $(\Psi_e^-)^{-1}$, $e \in E$ be the inverse uncertainty distributions of ϑ_v^- and ϑ_e^-, respectively. Assume that Υ_v^{-1}, $v \in V$, is the inverse uncertainty distribution of θ_v. Then, for a risk confidence level $\beta \in (0,1]$, the optimal expected solution with minimum TVaR is the optimal solution of the following model:

$$
(3.5)\quad \begin{aligned}
\min \quad & \sum_{v \in V} \left[\left(\frac{1}{\beta} \int_0^\beta (\Psi_v^+)^{-1}(1-\gamma) d\gamma \right) p_v + \left(\frac{1}{\beta} \int_0^\beta (\Psi_v^-)^{-1}(1-\gamma) d\gamma \right) q_v \right] \\
& + \sum_{e \in E} \left[\left(\frac{1}{\beta} \int_0^\beta (\Psi_e^+)^{-1}(1-\gamma) d\gamma \right) p_e + \left(\frac{1}{\beta} \int_0^\beta (\Psi_e^-)^{-1}(1-\gamma) d\gamma \right) q_e \right] \\
s.t. \quad & \\
& \sum_{v \in V} \left(\int_0^1 \Upsilon_v^{-1}(\tilde{w}(v), \alpha) d\alpha \right) \left(\min_{i=1,\dots,p} d_{\tilde{\ell}}(v, m_i) - \min_{k \in S} d_{\tilde{\ell}}(v, v_k) \right) \le 0 \\
& \hspace{8cm} \forall S \in \mathcal{S}, \\
& 0 \le p_e \le u_e^+, \quad 0 \le q_e \le u_e^- \quad \forall e \in E, \\
& 0 \le p_v \le u_v^+, \quad 0 \le q_v \le u_v^- \quad \forall v \in V.
\end{aligned}
$$

The above model is a deterministic inverse p-median problem formulation with vertex weights

$$
\int_0^1 \Upsilon_v^{-1}(\tilde{w}(v), \alpha) d\alpha, \ \ \forall v \in V,
$$

vertex weight modification costs

$$
\frac{1}{\beta} \int_0^\beta (\Psi_v^+)^{-1}(1-\gamma) d\gamma, \ \frac{1}{\beta} \int_0^\beta (\Psi_v^-)^{-1}(1-\gamma) d\gamma,
$$

and edge length modification costs

$$
\frac{1}{\beta} \int_0^\beta (\Psi_v^+)^{-1}(1-\gamma) d\gamma, \ \frac{1}{\beta} \int_0^\beta (\Psi_v^-)^{-1}(1-\gamma) d\gamma.
$$

Baroughi et al. in [3] showed that IpMLP on general networks is NP-hard. Thus we immediately conclude the following proposition.

Proposition 3.1. *UIpMLP with TVaR criterion on general networks is NP-hard.*

The above proposition implies that it is not possible to present exact polynomial time methods to solve UIpMLP on general networks. Therefore, we propose an efficient hybrid PSO algorithm for approximating the optimal solution of UIpMP on networks.

3.2. Hybrid PSO algorithm

Kennedy and Eberhart in 1995 [24] developed the PSO algorithm as a nature-inspired evolutionary computation algorithm. Consider the following model

$$\begin{array}{ll} \min & f(x) \\ s.t. & x \in X, \end{array}$$

where X is the restricted region. In PSO algorithm, a potential solution is presented as a particle $x_j \in X$ and a direction $v_j \in \mathbb{R}$ in which the particle will move. A swarm of particles is defined as a set $\{x_1, x_2, \ldots, x_N\}$, in which N is number of particles. Each particle x_j retains a record of the position of its previous best performance in a vector called $P_{best,j}$. The particle with best performance r in the population has been maintained in a vector G_{best}. An iteration involves evaluating of each particle x_j, then randomly setting of v_j in the direction of particle $x_j^{'}$s best previous position $P_{best,j}$ and the best previous position G_{best} of any particle in the population.

Since in UIpMLP the aim is to modify the vertex weights and edge lengths with respect to modification bounds. Thus, we consider a particle of the problem as $x_j = (x_{1,j}, x_{2,j}, \ldots, x_{2m+2n,j})$ where

$$\begin{array}{l} (x_{1,j}, x_{2,j}, \ldots, x_{m,j}) = (p_e)_{e\in E}, \\ (x_{m+1,j}, x_{m+2,j}, \ldots, x_{2m,j}) = (q_e)_{e\in E}, \\ (x_{2m+1,j}, x_{2m+2,j}, \ldots, x_{2m+n,j}) = (p_v)_{v\in V}, \\ (x_{2m+n+1,j}, x_{2m+n+2,j}, \ldots, x_{2m+2n,j}) = (q_v)_{v\in V}. \end{array} \tag{3.6}$$

Therefore, x_j represents the decision vector of UIpMLP that used in PSO. In addition, according to the orthogonality condition

- if $q_e > p_e$, then $q_e = q_e - p_e$, $p_e = 0$,
- if $q_e < p_e$, then $p_e = p_e - q_e$, $q_e = 0$,
- if $q_v > p_v$, then $q_v = q_v - p_v$, $p_v = 0$,
- if $q_v < p_v$, then $p_v = p_v - q_v$, $q_v = 0$.

For checking the feasibility of particle x_j, we calculate the expected value of constraints by using the following uncertain simulation algorithm [33]. Let $S \in \mathcal{S}$.

Algorithm 1 (Expected value simulation)

1. Set $E = 0$.
2. For $k = 1, \ldots, 99$ do

   compute

$$E_k = 0.01 \sum_{v\in V} \left(\Upsilon_v^{-1}(\tilde{w}(v), 0.0k)\right) \left(\min_{i=1,\ldots,p} d_{\tilde{\ell}}(v, m_i) - \min_{k\in S} d_{\tilde{\ell}}(v, v_k)\right),$$

and $E := E + E_k$.

3. Report E.

Therefore, if the particle $x_j = (x_{1,j}, x_{2,j}, \ldots, x_{2m+2n,j})$ is defined as (3.6) and for each $S \in \mathcal{S}$, $E \leq 0$, then x_j is feasibile.

Based on Theorem 2.2, we present the following uncertain simulation procedure for computing TVaR of objective function for each feasible particle x_j and given $\beta \in (0, 1]$.

Algorithm 2 (TVaR simulation)

1. Set $T_\beta = 0$.
2. For $j = 1, \ldots, M$ do

   compute

$$T_\beta^j = \sum_{v \in V} \left[\left((\Psi_v^+)^{-1}(1 - \frac{j}{M}\beta) \right) p_v + \left((\Psi_v^-)^{-1}(1 - \frac{j}{M}\beta) \right) q_v \right]$$

$$+ \sum_{e \in E} \left[((\Psi_e^+)^{-1}(1 - \frac{j}{M}\beta))p_e + \left((\Psi_e^-)^{-1}(1 - \frac{j}{M}\beta) \right) q_e \right],$$

   and $T_\beta = T_\beta + \frac{j}{M}\beta T_\beta^j$.

3. Compute $TVaR_\beta = \frac{1}{\beta} T_\beta$.
4. Report $TVaR_\beta$.

To solve the model (3.5) with hybrid PSO algorithm, we first randomly generate the particle x_j by checking the feasibility of it using expected value simulation. Repeat this process N times. We get N initial feasible particles $x_1, x_2, \ldots, x_N$. Then, we assume that the fitness of each x_j is the minus of TVaR, i.e.,

$$Fit(x_j) = -TVaR_\beta(x_j).$$

Thus, the particle with higher fitness has smaller objective value. The fitness of each particle is obtained by using TVaR simulation.

In the process of updating $(i + 1)$th iteration, we first denote $P_{best,j}(i)$ for each particle x_j and $G_{best}(i)$, then we obtain the new directs and the positions of the particles by using the following two equations:

$$v_j(i + 1) = v_j(i) + C_1 r_1[P_{best,j}(i) - x_j(i)] + C_2 r_2[G_{best}(i) - x_j(i)], \tag{3.7}$$

$$x_j(i+1) = x_j(i) + v_j(i+1), \tag{3.8}$$

where, $P_{best,j}(i) = x_j(i)$ if

$$Fit(x_j(i)) \geq Fit(x_j(i-1))$$

and $P_{best,j}(i) = P_{best,j}(i-1)$ otherwise, and $G_{best}(i) = P_{best,k}(i)$, with

$$k = argmin\{P_{best,j}(i) : j = 1, \ldots, N\}.$$

In addition r_1 and r_2 are uniformly distributed random numbers in the interval $[0, 1]$ and C_1 and C_2 are learning rates, to well adjust the convergence of the particles. The values of C_1 and C_2 are usually assumed to be 2.

If the updated x_j is feasible, then we consider it as a new particle of the next generation. Otherwise, as long as a feasible new particle is found, we re-update (3.7) and (3.8).

We obtain a new generation of particles by repeating the above process N times.

If MaxIt indicate the number of generations of the PSO algorithm, then based on all the explanations above, we summarize the hybrid PSO algorithm for solving the model (3.5) as follows.

Algorithm 3 (Hybrid PSO algorithm)

1. Initialize the feasibile particles $x_1, \ldots, x_N$ (use expected value simulation).
2. Compute the fitness for all particles by using TVaR simulation, and evaluate each particle according to it.
3. Update all the particles by using equations (3.7) and (3.8).
4. As long as a new feasible population is found, re-update (3.7) and (3.8).
5. Repeat Steps 2 to 4 for MaxIt times.
6. Return G_{best} as the optimal solution of the model (3.5), and

$$TVaR_\beta(G_{best}) = -Fit(G_{best})$$

as the corresponding optimal value.

3.3. An illustrative example

In this subsection, we give a numerical example to illustrate the hybrid PSO algorithm. The result of the numerical experiment is obtained on a PC with processor Intel(R) Core(TM) i3 CPU 2.27GHZ and 4GB of RAM under windows 7.

We apply the hybrid PSO algorithm for solving UIpMLP with TVaR criteria at a risk confidence level of $\beta = 0.8$ on the given network N in Figure 3.1. Let the cost coefficients be linear uncertain variables (see Table 3.3). Also let the vertex weights θ be the linear uncertain variables with respect to, $\tilde{w}(v)$, i.e.,

$$\theta = \theta(\tilde{w}(v)) = \mathcal{L}(\tilde{w}(v) - 10, \tilde{w}(v) + 10).$$

The input data of the network are given in Tables 3.1 and 3.3.

Note that if $\theta = \mathcal{L}(a, b)$ is the linear uncertain variable, then for a risk confidence level $\beta \in (0, 1]$

$$TVaR_\beta(\theta) = \frac{\beta}{2}(a - b) + b,$$

and

$$E[\theta] = \frac{(a + b)}{2}.$$

In the following, we show the computational results of the hybrid PSO algorithm's performance on an example of UI2MLP on the given network.

Note that the goal is to change $w(v)$ and ℓ_e with respect to modification bounds so that $\{v_2, v_3\}$ becomes a 2-median at minimum total cost under the new vertex weights and edge lengths.

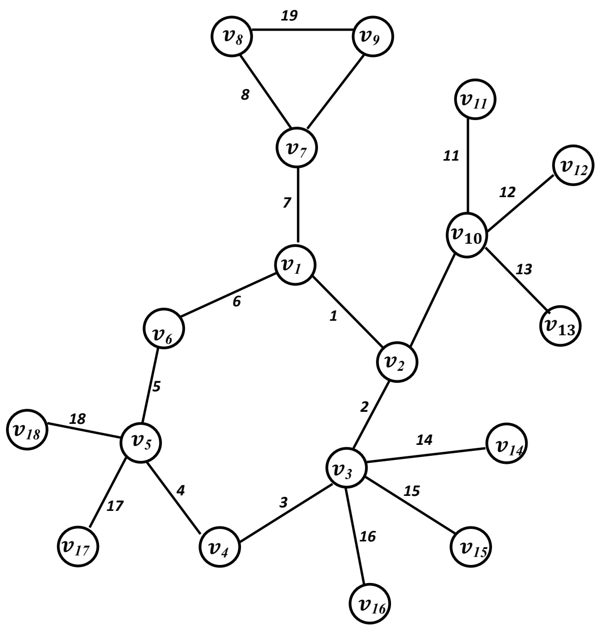

FIG. 3.1: Network N

Table 3.1: The input data for UI2MLP

ℓ_e	(14, 34, 25, 7, 22, 10, 8, 20, 12, 7, 10, 26, 12, 6, 10, 23, 31, 21, 22)
u_e^+	(5, 4, 5, 4, 7, 2, 5, 6, 3, 9, 2, 13, 1, 3, 5, 1, 7, 4, 1)
u_e^-	(10, 30, 15, 3, 17, 8, 4, 10, 8, 4, 5, 20, 6, 1, 8, 13, 2, 3, 13)
$w(v)$	(34, 18, 14, 13, 21, 11, 13, 20, 40, 22, 9, 17, 13, 6, 24, 14, 15, 12)
u_v^+	(20, 11, 4, 1, 2, 4, 7, 8, 15, 32, 13, 5, 15, 1, 2, 4, 1, 1)
u_v^-	(3, 2, 11, 9, 10, 8, 1, 1, 6, 7, 2, 2, 2, 4, 15, 10, 13, 11)

The hybrid PSO algorithm is run for the problem with 100, 200, 300 and 400 generations, respectively. Table 3.2 shows the best solutions of the problem.

Table 3.4, shows the best solutions of UI2MLP using hybrid PSO algorithm. Furthermore, the convergence of the objective values with population sizes 10,15,20,25 and $MaxIt = 100$ is shown in Figure 3.2. The convergence of the objective values with $N = 10$ and $MaxIt = 100, 200, 300, 400$ is given in Figure 3.3.

Table 3.2: The results of the performance of hybrid PSO algorithm

N, MaxIt	Objective value	N, MaxIt	Objective value
10, 100	-6880	10, 200	-6635
15, 100	- 8240	10, 300	-7609
20, 100	-8520	10, 400	-9150
25, 100	-10496		

Table 3.3: Uncertain cost coefficients

ϑ_e^+	$(\mathcal{L}(8,10), \mathcal{L}(18,21), \mathcal{L}(19,21), \mathcal{L}(4,6), \mathcal{L}(3,4), \mathcal{L}(14,16), \mathcal{L}(28,30), \mathcal{L}(10,12), \mathcal{L}(17,18), \mathcal{L}(6,8), \mathcal{L}(22,24), \mathcal{L}(7,9), \mathcal{L}(15,17), \mathcal{L}(18,21), \mathcal{L}(26,28), \mathcal{L}(28,30), \mathcal{L}(16,18), \mathcal{L}(4,6), \mathcal{L}(4,6))$
ϑ_e^-	$(\mathcal{L}(18,20), \mathcal{L}(14,15), \mathcal{L}(10,12), \mathcal{L}(24,26), \mathcal{L}(17,18), \mathcal{L}(15,17), \mathcal{L}(27,29), \mathcal{L}(8,10), \mathcal{L}(22,24), \mathcal{L}(22,24), \mathcal{L}(11,13), \mathcal{L}(2,4), \mathcal{L}(1,3), \mathcal{L}(15,17), \mathcal{L}(24,25), \mathcal{L}(28,30), \mathcal{L}(3,5), \mathcal{L}(17,19), \mathcal{L}(17,19))$
ϑ_v^+	$(\mathcal{L}(24,26), \mathcal{L}(27,28), \mathcal{L}(3,5), \mathcal{L}(27,28), \mathcal{L}(19,20), \mathcal{L}(1,4), \mathcal{L}(8,10), \mathcal{L}(16,18), \mathcal{L}(29,30), \mathcal{L}(29,30), \mathcal{L}(4,6), \mathcal{L}(30,31), \mathcal{L}(29,30), \mathcal{L}(14,16), \mathcal{L}(24,26), \mathcal{L}(4,6), \mathcal{L}(12,13), \mathcal{L}(27,28))$
ϑ_v^-	$(\mathcal{L}(19,21), \mathcal{L}(22,24), \mathcal{L}(12,13), \mathcal{L}(19,21), \mathcal{L}(5,6), \mathcal{L}(20,22), \mathcal{L}(1,2), \mathcal{L}(7,10), \mathcal{L}(2,3), \mathcal{L}(3,4), \mathcal{L}(24,26), \mathcal{L}(20,21), \mathcal{L}(19,20), \mathcal{L}(29,30), \mathcal{L}(2,3), \mathcal{L}(13,15), \mathcal{L}(12,13), \mathcal{L}(22,24))$

Table 3.4: The obtained G_{best} for UI2MLP by using hybrid PSO algorithm

N, MaxIt	G_{best}
10, 100	(3.92, 0.90, 4.88, 0, 4.84, 0, 0, 5.82, 1.25, 0, 0, 0.33, 0, 0, 0.99, 0.94, 0, 0.82, 0.69, 0, 0, 0, 2.75, 0, 2.50, 3.22, 0, 0, 2.80, 1.08, 0, 4.60, 0.23, 0, 0, 1.46, 0, 0, 0, 9.51, 3.40, 0.32, 1.38, 0, 0, 0, 0, 0, 0, 4.94, 0, 0, 1.54, 3.26, 0.61, 0, 0.22, 0, 0, 0, 0, 6.31, 0.34, 0.72, 4.28, 5.64, 0.01, 0, 0.34, 2.56, 0, 0, 0, 10.73)
15, 100	(0.24, 2.85, 4.07, 0, 4.07, 1.29, 1.95, 4.12, 0, 8.92, 0, 8.38, 0.97, 0.30, 3.2, 0, 3.42, 3.49, 0.08, 0, 0, 0, 2.12, 0, 0, 0, 0, 6.16, 0, 2.68, 0, 0, 0, 0, 7.84, 0, 0, 0, 0.50, 0, 1.95, 0.30, 0.94, 3.70, 6.81, 0, 0, 7.51, 5.21, 1.95, 0, 0, 0, 0, 0, 0, 0, 0.94, 0, 0, 0, 0, 0, 0.31, 0.48, 0, 0, 0, 0.8, 1.08, 4.82, 8.40, 5.80, 10.03)
20, 100	(3.43, 3.91, 1.20, 0, 4.02, 0, 0, 4, 2.57, 0, 1.48, 11.32, 0, 0, 3.03, 0, 0, 0, 0, 0, 0, 0, 1.67, 0, 3.10, 1.55, 0, 0, 3.70, 0, 0, 0.86, 0.96, 0, 7.83, 1.09, 1.22, 2.76, 0.52, 0, 0.78, 0, 0.44, 1.79, 6.47, 0, 11.03, 0, 0, 0, 4.89, 0.02, 0, 0, 0.29, 0, 0, 0.11, 0, 4.76, 0, 0, 0, 0.10, 0, 5.28, 1.77, 0.68, 0, 0, 4.57, 5.31, 0, 10)
25, 100	(2.32, 2.36, 2.88, 0, 4.76, 0, 0.28, 0, 0.80, 0, 1.78, 0, 0.16, 0, 2.61, 0, 1.70, 0, 0.91, 0, 0, 0, 2.04, 0, 1.45, 0, 9.33, 0, 0.50, 0, 8.00, 0, 0.39, 0, 7.15, 0, 1.06, 0, 0, 8.81, 0.69, 0, 0, 0.93, 1.37, 4.04, 3.59, 26.46, 0, 0.82, 0, 0, 0, 0, 0.45, 0, 0.62, 0,0, 4.63, 4.86, 0, 0, 0, 0, 0, 1.35, 0, 0.12, 0.72, 12.62, 3.096, 0, 10.65)
10, 200	(3.25, 2.75, 3.77, 0, 5.40, 1.71, 2.35, 0, 2.47, 4.18, 0, 8.17, 0, 0, 4.96, 0, 0, 0, 0.56, 0, 0, 0, 1.39, 0, 0, 0, 9.88, 0, 0, 4.93, 0, 4.93, 0.29, 0, 3.01, 1.57, 1.16, 0, 0, 1.42, 0, 0.85, 0, 3.90, 2.67, 5.46, 0, 0, 0, 0.95, 0.58, 0.72, 0, 0, 0.53, 0, 2.24, 0, 4.77, 0, 4.97, 0, 0, 0, 5.12, 6.30, 1.55, 0, 0, 0, 9.99, 1.85, 0, 10.17)
10, 300	4.42, 0.91, 2.20, 0, 2.82, 0, 0, 0, 2.54, 0, 0, 1.42, 0, 0, 0, 0, 4.70, 0, 0.77, 0, 0, 0, 1.51, 0, 1.05, 1.30, 5.94, 0, 0.89, 2.83, 0, 2.91, 0.20, 1.74, 2.71, 0, 2.60, 0, 0, 3.66, 0, 0, 0, 0, 6.34, 6.11, 9.43, 0, 0, 1.10, 0, 0.20, 1.64, 0.36, 0, 0, 2.31, 0, 1.27, 2.86, 6.34, 5.40, 0, 0, 0, 4.33, 0.79, 0, 0.50, 0, 0, 0, 0.81, 10.66)
10, 400	(1.80, 2.94, 1.64, 0, 3.60, 0, 2.13, 4.58, 0.53, 0, 0.36, 11.20, 0.42, 1.08, 0, 0, 0, 2.98, 0, 0, 0, 0, 1.94, 0, 2.28, 0, 0, 0, 3.89, 0, 0, 0, 0, 7.56, 5.33, 1.80, 0, 9.03, 1.69, 8.02, 1.17, 0, 0, 3.97, 1.14, 7.81, 0, 1.61, 4.24, 1.71, 0, 0.71, 0.71, 0, 0.66, 0, 0, 0, 0, 8.40, 3.35, 0, 0, 0, 4.31, 0, 0, 0, 1.43, 0, 0, 4.72, 0, 10.97)

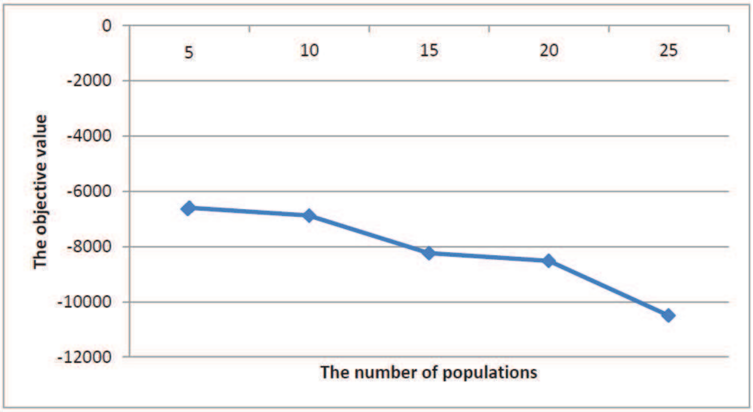

FIG. 3.2: The convergence of TVaR, MaxIt=100

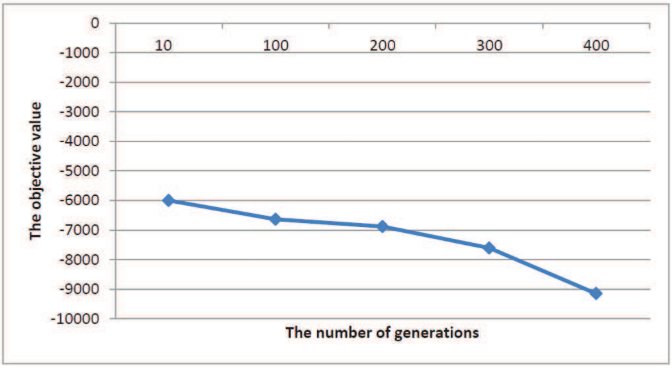

FIG. 3.3: The convergence of TVaR, N=10

4. Conclusion

In this paper, we investigated IpMLP with variable edge lengths and variable vertex weights on a network in which the vertex weights and modification costs are the independent uncertain variables. We proposed a model for UIpMLP with TVaR objective and expected value constraints and showed that it is NP-hard. Thus, we presented a hybrid PSO algorithm for approximating the optimal solutions, which it contains expected value simulation and TVaR simulation. Finally, by computational experiments, the efficiency of the algorithm is illustrated.

REFERENCES

1. O. Alp, E. Erkut and Z. Drezner: *An efficient genetic algorithm for the p-median problem.* Ann. Oper. Res. **10** (2003), 1387-1395.
2. B. Alizadeh and S. Bakhteh: *A modified firefly algorithm for general inverse p-median location problems under different distance norms.* Opsearch. **54** (2017), 618-636.
3. F. Baroughi, R. E. Burkard and E. Gassner: *Inverse p-median problems with variable edge lengths.* Math. Meth. Oper. Res. **73** (2011), 263-280.
4. M. Bashiri, M. Mirzaei and M. Randall: *Modeling fuzzy capacitated p-hub center problem and a genetic algorithm solution.* Appl. Math. Model. **37(5)** (2013), 3513-3525.
5. X. Bai and Y. Liu: *Minimum risk facility location-allocation problem with type-2 fuzzy variables.* sci. world j. **2014** (2014), 1-9.
6. O. Berman, N. Sanajian and J. Wang: *Location choice and risk attitude of a decision maker.* Omega **66** (2017), 170-181.
7. R. Benkoczi and B. Bhattacharya: *A new template for solving p-median problems for trees in sub-quadratic time (extended abstract).* LNCS. **3669** (2005), 271-282.
8. R. E. Burkard and J. Krarup: *A linear algorithm for the pos/neg-weighted 1-median problem on a cactus.* Computing. **60** (1998), 193-215.
9. R. E. Burkard, C. Pleschiutschnig and J. Zhan: *Inverse median problems. Discrete Optim.* **1** (2004), 23-39.
10. R. E. Burkard, C. Pleschiutschnig and J. Zhan: *The inverse 1-median problem on a cycle.* Discrete Optim. **5** (2008), 242-253.
11. Z. Drezner: *The planar two-center and two-median problems.* Transportation Sci. **18** (1984), 351-361.
12. S. Eilon, C. D. T. Watson-Gandy and N. Christofides: *Distribution management: mathematical modeling and practical analysis.* New York: Hafner (1971).
13. H. A. Eiselt and V. Marianov: *Foundations of location analysis.* OR/MS, NewYork: Springer (2011).
14. Y. Gao : *Uncertain models for single facility location problems on networks.* Appl. Math. Model. **36** (2012), 2592-2599.
15. Y. Gao and Z. Qin: *A chance constrained programming approach for uncertain p-hub center location problem.* Comput. Ind. Eng. **102** (2016), 10-20.
16. M. Galavii: *The inverse 1-median problem on a tree and on a path.* Electron. Notes Discret. Math. **36** (2010), 1241-1248.
17. A. J. Goldman: *Optimal center location in simple networks.* Transportation Sci. **5** (1971), 212-221.
18. X. Guan and B. Zhang: *Inverse 1-median problem on trees under weighted Hamming distance.* J Global Optim. **54** (2012), 75-82.
19. X. Guan and B. Zhang: *Inverse 1-median problem on trees under weighted l_∞ norm.* LNCS. **6124** (2010), 150-160.
20. S. L. Hakimi: *Optimum locations of switching centers and the absolute centers and medians of a graph.* Oper. Res. **12** (1964), 450-459.

21. S. L. Hakimi: *Optimum distribution of switching centers in a communication network and some related graph theoretic problems.* Oper. Res. **13** (1965), 462-475.

22. X. Huang and D. Hao: *Modelling uncapacitated facility location problem with uncertain customers' positions.* J. Intell. Fuzzy Syst. **28** (2015), 2569-2577.

23. J. Hatzl: *2-balanced flows and the inverse 1-median problem in the Chebyshev space.* Discrete Optim. **9** (2012), 137-148.

24. J. Kennedy and R. C. Eberhart: *Particle swarm optimization.* ICNN'97. **4** (1995), 1942-1948.

25. B. Liu: *Uncertainty theory.* 2nd ed., Springer-Verlag, Berlin (2007).

26. B. Liu: *Uncertain risk analysis and uncertain reliability analysis.* J. Uncertain Syst. **4 (3)** (2010), 163-170.

27. X. Liu: *Uncertain programming model for location problem of multi-product logistics distribution centers.* Appl. Math. Sci. **9** (2015), 6543-6558.

28. R. F. Love: *Facilities location: models and methods.* Oper. Res. Ser. (1988).

29. I. Mirzapolis Adeh, F. Baroughi and B. Alizadeh: *A modified particle swarm optimization algorithm for general inverse ordered p-median location problem on network.* Facta Universitatis, Ser. Math. Inform. **32** (2017), 447-468.

30. K. T. Nguyen: *Inverse 1-median problem on block graphs with variable vertex weights.* j. optimiz. theory appl. **168** (2016), 944-957.

31. K. T. Nguyen and N. T. L. Chi: *A model for the inverse 1-median problem on trees under uncertain costs.* Opusc. Math. **36** (2016), 513-523.

32. J. Peng: *Risk metrics of loss function for uncertain system.* Fuzzy Optim. Decis. Ma. **12** (2013), 53-64.

33. J. Peng, B. Zhang and S. Li: *Towards uncertain network optimization.* J. Uncertain. Anal. Appl. **3** (2015), Article 4.

34. Z. Qin and Y. Gao: *Uncapacitated p-hub location problem with fixed costs and uncertain flows.* J. Intell. Manuf. **28** (2017), 705-716.

35. R. Rahmaniani, M. Saidi-Mehrabad and A. Ghaderi: *An efficient hybrid solution algorithm for the capacitated facility location-allocation problem under uncertainty.* JOIE. DOI: 10.22094/joie.2018.538339 (2018).

36. A. R. Sepasian and F. Rahbarnia: *An $O(n \log n)$ algorithm for the inverse 1-median problem on trees with variable vertex weights and edge reductions.* Optimization. **64** (2015), 595-602.

37. A. Schobel and D. Scholz: *The big cube small cube solution method for multidimensional facility location problems.* Comput. Oper. Res. **37** (2010), 115-122.

38. H. D. Sherali and C. H. Tuncbilek: *A squared Euclidean distance location-allocation problem.* NRL. **39** (1992), 447-469.

39. A. Tamir: *An $O(pn^2)$ algorithm for the p-median and related problems on tree graphs.* Oper. Res. Lett. **19** (1996), 59-64.

40. K. E. Wang and Q. Yang: *Hierarchical facility location for the reverse logistics network design under uncertainty.* J. Uncertain Syst. **8** (2014), 255-270.

41. Sh. Wang, J. Watada and W. Pedrycz: *Value at risk based two-stage fuzzy facility location problems.* TII. **5** (2009), 465-482.

42. M. R. Wagner, J. Bhadury and S. Peng: *Risk management in uncapacitated facility location models with random demands*. Comput. Oper. Res. **36** (2009), 1002-1011.
43. M. Wen, Z. Qin, R. Kang and Y. Yang: *The capacitated facility location-allocation problem under uncertain environment*. J. Intell. Fuzzy Syst. **29** (2015), 2217-2226.
44. K. Yang, Y. Liu and G. Yang: *Optimizing fuzzy p-hub center problem with generalizedvalue at risk criterion*. Appl. Math. Model. **38** (2014), 3987-4005.
45. K. Yang, Y. Liu and G. Yang: *An improved hybrid particle swarm optimization algorithm for fuzzy p-hub center problem*. Comput. Ind. Eng. **64** (2013a), 133-142.
46. B. Zhang, J. Peng and S. Li: *Covering location problem of emergency service facilities in an uncertain environment*. Appl. Math. Model. **51** (2017), 429-447.

Akram Soltanpour
Faculty of Basic Sciences
Department of Applied Mathematics
Sahand University of Technology, Tabriz, Iran

Fahimeh Baroughi
Faculty of Basic Sciences
Department of Applied Mathematics
Sahand University of Technology, Tabriz, Iran
baroughi@sut.ac.ir
(Corresponding author)

Behrooz Alizadeh
Faculty of Basic Sciences
Department of Applied Mathematics
Sahand University of Technology, Tabriz, Iran

FACTA UNIVERSITATIS (NIŠ)
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MULTIPLE USE OF BACKTRACKING LINE SEARCH IN UNCONSTRAINED OPTIMIZATION

Branislav Ivanov, Bilall I. Shaini and Predrag S. Stanimirović

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Abstract. The class of gradient methods is a very efficient iterative technique for solving unconstrained optimization problems. Motivated by recent modifications of some variants of the SM method, this study proposed two methods that are globally convergent as well as computationally efficient. Each of the methods is globally convergent under the influence of a backtracking line search. Results obtained from the numerical implementation of these methods and performance profiling show that the methods are very competitive with respect to well-known traditional methods.
Keywords: unconstrained optimization; gradient methods; line search.

1. Introduction

The following unconstrained optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}), \tag{1.1}$$

is ubiquitous in all areas of science and practical engineering applications. In (1.1), the function $f:\mathbb{R}^n \to \mathbb{R}$ is uniformly convex (UC) and twice continuously differentiable (TCD).

The most frequent iterations for solving (1.1) is the gradient descent (GD) iterative scheme

$$\mathbf{x}_{k+1}^{GD} = \mathbf{x}_k^{GD} - t_k \mathbf{g}_k, \tag{1.2}$$

where $t_k > 0$ is the stepsize and $\mathbf{g}_k := \nabla f(\mathbf{x}_k)$ corresponds to the gradient of f. The step length t_k is mainly calculated using the backtracking line search (BLS).

The Newton iterations stabilized by the line search are defined as

$$\mathbf{x}_{k+1} = \mathbf{x}_k - t_k G_k^{-1} \mathbf{g}_k, \tag{1.3}$$

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wherein G_k^{-1} means the inverse of the Hessian matrix $G_k := \nabla^2 f(\mathbf{x}_k)$. In order to avoid time consuming computation of the Hessian and its inverse, practical numerical methods for solving unconstrained optimization problem are derived from the usage of appropriate approximations H_k of G_k^{-1}. The general scheme of quasi-Newton type with line search [16] is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k - t_k H_k \,\mathbf{g}_k. \tag{1.4}$$

In order to define efficient class of quasi-Newton methods, we use the simplest scalar approximation of the Hessian with respect to known classifications from [5, 8]:

$$B_k := \gamma_k I,\ \ \gamma_k > 0, \tag{1.5}$$

where I is an identity matrix of appropriate order and $\gamma_k > 0$ is a real parameter. The choice (1.5) leads to the iterative prototype

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \gamma_k^{-1} t_k \mathbf{g}_k, \tag{1.6}$$

where t_k denotes the basic step size and γ_k^{-1} is an additional step size which should be defined appropriately. Clearly, the value $\gamma_k^{-1} t_k$ can be considered as a composite step size, so that iterations (1.6) are GD methods. The iterations (1.6) are known as *improved gradient descent* (IGD) methods.

Andrei in [1, 3] originated so called *Accelerated Gradient Descent* (AGD) iterations in the form

$$\mathbf{x}_{k+1}^{AGD} = \mathbf{x}_k^{AGD} - \theta_k^{AGD} t_k \mathbf{g}_k. \tag{1.7}$$

The AGD process (1.7) was improved into the *Modified AGD* (MAGD) method [7] as

$$\mathbf{x}_{k+1}^{MAGD} = \mathbf{x}_k^{MAGD} - \theta_k (t_k + t_k^2 - t_k^3) \mathbf{g}_k. \tag{1.8}$$

A few variants of the IGD class (1.6) were proposed in [7, 10, 11, 14, 15]. The *SM* method belongs to the class IGD methods. It was originated in [14] by the iterative process

$$\mathbf{x}_{k+1}^{SM} = \mathbf{x}_k^{SM} - t_k \left(\gamma_k^{SM}\right)^{-1} \mathbf{g}_k, \tag{1.9}$$

where $t_k > 0$ is the basic step size and $\gamma_k^{SM} > 0$ is the gain parameter determined as in

$$\gamma_{k+1}^{SM} = 2\gamma_k^{SM} \frac{\gamma_k^{SM}\left[f(\mathbf{x}_{k+1}^{SM}) - f(\mathbf{x}_k^{SM})\right] + t_k \|\mathbf{g}_k\|^2}{t_k^2 \|\mathbf{g}_k\|^2}.$$

The ADSS model from [10] is defined as

$$\mathbf{x}_{k+1}^{ADSS} = \mathbf{x}_k^{ADSS} - \left(t_k \left(\gamma_k^{ADSS}\right)^{-1} + l_k\right) \mathbf{g}_k, \tag{1.10}$$

where t_k and l_k are determined by BLSs. The TADSS method [15] is defined by the iterative rule

$$\mathbf{x}_{k+1}^{TADSS} = \mathbf{x}_k^{TADSS} - \left(t_k\left((\gamma_k^{TADSS})^{-1} - 1\right) + 1\right)\mathbf{g}_k.$$

The next scheme was proposed as the modified SM (MSM) method in [7]:

$$\mathbf{x}_{k+1}^{MSM} = \mathbf{x}_k^{MSM} - (t_k + t_k^2 - t_k^3)(\gamma_k^{MSM})^{-1}\mathbf{g}_k. \tag{1.11}$$

The acceleration parameters in ADD, ADSS, TADSS and MSM methods are summarized in Table 1.1.

Table 1.1: Acceleration parameters γ_{k+1} in variants SM method.

Method	**Acceleration parameter** γ_{k+1}	**Reference**
ADD	$\gamma_{k+1}^{ADD} = 2\frac{f(\mathbf{x}_{k+1}^{ADD}) - f(\mathbf{x}_k^{ADD}) - t_k(\mathbf{g}_k^{ADD})^{\mathrm{T}}\left(t_k\mathbf{d}_k^{ADD} - (\gamma_k)^{-1}\mathbf{g}_k\right)}{\left(t_k\mathbf{d}_k^{ADD} - \gamma_k^{-1}\mathbf{g}_k\right)^{\mathrm{T}}\left(t_k\mathbf{d}_k^{ADD} - (\gamma_k^{ADD})^{-1}\mathbf{g}_k\right)}$	(2014) [11]
ADSS	$\gamma_{k+1}^{ADSS} = 2\frac{f(\mathbf{x}_{k+1}^{ADSS}) - f(\mathbf{x}_k^{ADSS}) + \left(t_k(\gamma_k)^{-1} + l_k\right)\Vert\mathbf{g}_k\Vert^2}{\left(t_k(\gamma_k^{ADSS})^{-1} + l_k\right)^2\Vert\mathbf{g}_k\Vert^2}$	(2015) [10]
TADSS	$\gamma_{k+1}^{TADSS} = 2\frac{f(\mathbf{x}_{k+1}^{TADSS}) - f(\mathbf{x}_k^{TADSS}) + \psi_k\Vert\mathbf{g}_k\Vert^2}{\psi_k^2\Vert\mathbf{g}_k\Vert^2}$, $\psi_k = t_k((\gamma_k^{TADSS})^{-1} - 1) + 1$	(2015) [15]
MSM	$\gamma_{k+1}^{MSM} = 2\gamma_k\frac{\gamma_k\left[f(\mathbf{x}_{k+1}^{MSM}) - f(\mathbf{x}_k^{MSM})\right] + (t_k + t_k^2 - t_k^3)\Vert\mathbf{g}_k\Vert^2}{(t_k + t_k^2 - t_k^3)^2\Vert\mathbf{g}_k\Vert^2}$	(2019) [7]

The main goal of this research is to study the impact of multiple usage of backtracking line search in modified SM method [7] and practical computational performance of two new methods. Our intention is to propose and investigate improvements of the MSM method. Globally, we investigate possibility to multiple use backtracking line search in the modified MSM method.

Main results of the present study can be highlighted as follows:

(1) A novel iterative scheme is proposed using the idea of computing the step parameters t_k, t_k^2 and t_k^3 in the MSM method by means of multiple BLS procedures. The resulting iterations will be denoted as TMSM and DMSM.

(2) Convergence behavior of the proposed iterations are investigated on appropriate quadratic functions.

(3) Numerical experiments compare introduced methods with existing iterations and analyze three main performances: number of iterative steps and function evaluations and CPU time.

The remainder of the paper is developed according to the following hierarchy of sections. Two modifications of the MSM methods, termed as TMSM and DMSM methods, are introduced in Section 2. Section 3. investigates the convergence of the presented TMSM and DMSM methods. In Section 4., we perform a number of numerical experiments and compare main performances of the novel methods with similar available methods. Final remarks are presented in Section 5.

2. Multiple use of backtracking line search in modified SM method

The MSM method is based on the iteration

$$\mathbf{x}_{k+1}^{MSM} = \mathbf{x}_k^{MSM} - t_k^{MSM}(\gamma_k^{MSM})^{-1}\mathbf{g}_k, \tag{2.1}$$

where $t_k^{MSM} = t_k + t_k^2 - t_k^3$. The leading idea in defining t_k^{MSM} arises from the observation $t_k + t_k^2 > t_k^{MSM} > t_k$, which means that the MSM method proposes a slightly greater step size with respect to the SM iterations. Since t_k arises from the BLS procedure, which ensures $t_k \in (0,1)$, it implies

$$t_k \le t_k^{MSM} \le t_k + t_k^2.$$

Our intention in current research is to improve behaviour of iterations (2.1) using two or three appropriately defined step-parameters. Following this idea, a method based on triple usage of the BLS in the MSM method is obtained when t_k^2 is substituted with l_k^2 and t_k^3 is substituted with j_k^3 in (2.1), where t_k, l_k and j_k are defined by independent LS procedures: the first BLS (Algorithm 1) calculates t_k, another BLS (Algorithm 2) calculates l_k, while the third BLS (Algorithm 3) determines j_k.

Replacing the above changes gives the expression of the TMSM iteration:

$$\mathbf{x}_{k+1}^{TMSM} = \mathbf{x}_k^{TMSM} - t_k^{TMSM}\left(\gamma_k^{TMSM}\right)^{-1}\mathbf{g}_k, \tag{2.2}$$

where

$$t_k^{TMSM} = \begin{cases} t_k + l_k^2 - j_k^3, & t_k + l_k^2 - j_k^3 > t_k \\ t_k, & t_k + l_k^2 - j_k^3 \le t_k. \end{cases} \tag{2.3}$$

The second order Taylor development of $f(\mathbf{x}_{k+1}^{TMSM})$ gives

$$\begin{aligned} f(\mathbf{x}_{k+1}^{TMSM}) \approx f(\mathbf{x}_k^{TMSM}) &- t_k^{TMSM}\left(\gamma_k^{TMSM}\right)^{-1}\mathbf{g}_k^{\mathrm{T}}\mathbf{g}_k \\ &+ \frac{1}{2}\left(t_k^{TMSM}\right)^2\left((\gamma_k^{TMSM})^{-1}\mathbf{g}_k\right)^{\mathrm{T}}\nabla^2 f(\xi)\left(\gamma_k^{TMSM}\right)^{-1}\mathbf{g}_k. \end{aligned} \tag{2.4}$$

The parameter ξ in (2.4) fulfills the condition $\xi \in [\mathbf{x}_k^{TMSM}, \mathbf{x}_{k+1}^{TMSM}]$. One possible choice is

$$\begin{aligned} \xi &= \mathbf{x}_k^{TMSM} + \delta(\mathbf{x}_{k+1}^{TMSM} - \mathbf{x}_k^{TMSM}) \\ &= \mathbf{x}_k^{TMSM} - \varphi\, t_k^{TMSM}\left(\gamma_k^{TMSM}\right)^{-1}\mathbf{g}_k, \quad 0 \le \varphi \le 1. \end{aligned} \tag{2.5}$$

According to [14], $\nabla^2 f(\xi)$ is approximated as $\gamma_{k+1}^{TMSM} I$. So, (2.4) reduces to

$$\begin{aligned} f(\mathbf{x}_{k+1}^{TMSM}) \approx & f(\mathbf{x}_k^{TMSM}) - t_k^{TMSM}\left(\gamma_k^{TMSM}\right)^{-1}\|\mathbf{g}_k\|^2 \\ &+ \frac{1}{2}\left(t_k^{TMSM}\right)^2\gamma_{k+1}^{TMSM}(\gamma_k^{TMSM})^{-2}\|\mathbf{g}_k\|^2. \end{aligned} \tag{2.6}$$

Then γ_{k+1}^{TMSM} can be obtained from (2.6) as

$$\gamma_{k+1}^{TMSM} = 2\gamma_k^{TMSM} \frac{\gamma_k^{TMSM}\left[f(\mathbf{x}_{k+1}^{TMSM}) - f(\mathbf{x}_k^{TMSM})\right] + t_k^{TMSM}\|\mathbf{g}_k\|^2}{\left(t_k^{TMSM}\right)^2\|\mathbf{g}_k\|^2}. \tag{2.7}$$

The improper situation $\gamma_{k+1}^{TMSM} < 0$ can be resolved by taking $\gamma_{k+1}^{TMSM} = 1$.

The BLS method is implemented in the Algorithm 1 from [14]. Algorithm 1 defines t_k starting from $t = 1$ and subsequently decreases values of t so that it reduces the value of the objective f enough.

Algorithm 1 The backtracking line search calculates t_k.

Require: A real function $f(\mathbf{x})$, appropriate search direction $\mathbf{d}_k$ at the point $\mathbf{x}_k$ and the positive real numbers $0 < \sigma < 0.5$ and $\beta \in (0, 1)$.
1: $t = 1$.
2: While $f(\mathbf{x}_k + t\mathbf{d}_k) > f(\mathbf{x}_k) + \sigma t \mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k$, do $t := t\beta$.
3: Output $t_k := t$.

Algorithm 2 The second backtracking line search calculates l_k.

Require: Objective function $f(\mathbf{x})$, the search direction $\mathbf{d}_k$ at the point $\mathbf{x}_k$ and positive real numbers $0 < \sigma_l < 0.5$ and $\beta_l \in (0, 1)$.
1: $l = 1$.
2: While $f(\mathbf{x}_k + l\mathbf{d}_k) > f(\mathbf{x}_k) + \sigma_l l \mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k$, take $l := l\beta_l$.
3: Return $l_k = l$.

Algorithm 3 The third backtracking line search calculates j_k.

Require: Objective function $f(\mathbf{x})$, the search direction $\mathbf{d}_k$ at the point $\mathbf{x}_k$ and positive real numbers $0 < \sigma_j < 0.5$ and $\beta_j \in (0, 1)$.
1: $j = 1$.
2: While $f(\mathbf{x}_k + j\mathbf{d}_k) > f(\mathbf{x}_k) + \sigma_j j \mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k$, take $j := j\beta_j$.
3: Return $j_k = j$.

Finally, the TMSM method is described in Algorithm 4.

It is expectable that the total number of iterations (NofI) required by the TMSM method will be smaller than the number of iterations of the MSM method, but an increase in the number of function evaluations (NofFE) and the CPU time (CPUT)is expectable. Based on these indicators, we came up with the idea to omit one line search in the TMSM method. This would drastically reduce the CPUT and the NofFE. Following this idea, a method of double use backtracking line search in modified SM method is obtained. In this way, we get a new expression of the DMSM iteration:

$$\mathbf{x}_{k+1}^{DMSM} = \mathbf{x}_k^{DMSM} - t_k^{DMSM}\left(\gamma_k^{DMSM}\right)^{-1}\mathbf{g}_k, \tag{2.8}$$

Algorithm 4 Triple use of backtracking line search in the MSM method (the TMSM method)

Require: Objective function $f(\mathbf{x})$, initial point $\mathbf{x}_0^{TMSM} \in \mathrm{dom}(f)$ and parameters $0 < \lambda < 1$, $0 < \nu < 1$.

1: Put $k = 0$, evaluate $f(\mathbf{x}_0^{TMSM})$, $\mathbf{g}_0 = \nabla f(\mathbf{x}_0^{TMSM})$, and put $\gamma_0^{TMSM} = 1$.

2: If

$$\|\mathbf{g}_k\| \leq \lambda \quad \text{and} \quad \frac{|f(\mathbf{x}_{k+1}^{TMSM}) - f(\mathbf{x}_k^{TMSM})|}{1 + |f(\mathbf{x}_k^{TMSM})|} \leq \nu,$$

STOP; else go to Step 3.

3: (The first backtracking) Compute $t_k \in (0, 1]$ using Algorithm 1.

4: (The second backtracking) Compute $l_k \in (0, 1]$ using Algorithm 2.

5: (The third backtracking) Compute $j_k \in (0, 1]$ using Algorithm 3.

6: Determine t_k^{TMSM} using (2.3).

7: Compute $\mathbf{x}_{k+1}^{TMSM} = \mathbf{x}_k^{TMSM} - (\gamma_k^{TMSM})^{-1} t_k^{TMSM} \mathbf{g}_k$.

8: Compute $f(\mathbf{x}_{k+1}^{TMSM})$ and $\mathbf{g}_{k+1} = \nabla f(\mathbf{x}_{k+1}^{TMSM})$.

9: Determine γ_{k+1}^{TMSM} using (2.7).

10: If $\gamma_{k+1}^{TMSM} < 0$, then take $\gamma_{k+1}^{TMSM} = 1$.

11: Set $k := k + 1$, go to the step 2.

12: Return $\mathbf{x}_{k+1}^{TMSM}$ and $f(\mathbf{x}_{k+1}^{TMSM})$.

where

$$t_k^{DMSM} = \begin{cases} t_k + t_k^2 - j_k^3, & t_k + t_k^2 - j_k^3 > t_k \\ t_k, & t_k + t_k^2 - j_k^3 \leq t_k. \end{cases} \tag{2.9}$$

In exactly the same way as for the TMSM method, we arrive at

$$\gamma_{k+1}^{DMSM} = 2\gamma_k^{DMSM} \frac{\gamma_k^{DMSM}\left[f(\mathbf{x}_{k+1}^{DMSM}) - f(\mathbf{x}_k^{DMSM})\right] + t_k^{DMSM}\|\mathbf{g}_k\|^2}{\left(t_k^{TMSM}\right)^2 \|\mathbf{g}_k\|^2}. \tag{2.10}$$

The difficulty $\gamma_{k+1}^{DMSM} < 0$ can be resolved using $\gamma_{k+1}^{DMSM} = 1$.

The DMSM method is presented in Algorithm 5:

Algorithm 5 Double use backtracking line search in the MSM method (the DMSM method)

Require: Function $f(\mathbf{x})$, chosen initial point $\mathbf{x}_0^{DMSM} \in \mathrm{dom}(f)$ and parameters $0 < \lambda < 1$, $0 < \nu < 1$.

1: Put $k = 0$, evaluate $f(\mathbf{x}_0^{DMSM})$, $\mathbf{g}_0 = \nabla f(\mathbf{x}_0^{DMSM})$ and take $\gamma_0^{DMSM} = 1$.
2: If
$$\|\mathbf{g}_k\| \le \lambda \quad \text{and} \quad \frac{|f(\mathbf{x}_{k+1}^{DMSM}) - f(\mathbf{x}_k^{DMSM})|}{1 + |f(\mathbf{x}_k^{DMSM})|} \le \nu,$$
STOP; else go to Step 3.
3: (The first backtracking) Compute $t_k \in (0, 1]$ using Algorithm 1.
4: (The second backtracking) Compute $j_k \in (0, 1]$ using Algorithm 3.
5: Determine t_k^{DMSM} using (2.9).
6: Compute $\mathbf{x}_{k+1}^{DMSM} = \mathbf{x}_k^{DMSM} - (\gamma_k^{DMSM})^{-1} t_k^{DMSM} \mathbf{g}_k$.
7: Compute $f(\mathbf{x}_{k+1}^{DMSM})$ and $\mathbf{g}_{k+1} = \nabla f(\mathbf{x}_{k+1}^{DMSM})$.
8: Determine the scalar approximation $\gamma_{k+1}^{DMSM} I$ of the Hessian of f at the point $\mathbf{x}_{k+1}^{DMSM}$ using (2.10).
9: If $\gamma_{k+1}^{DMSM} < 0$, then take $\gamma_{k+1}^{DMSM} = 1$.
10: Put $k := k + 1$, go to the step 2.
11: Return $\mathbf{x}_{k+1}^{DMSM}$ and $f(\mathbf{x}_{k+1}^{DMSM})$.

3. Convergence analysis

The content of this section is the convergence analysis of the TMSM and DMSM methods. In the following part, we restate and derive some basic statements which will be used in the convergence analysis of Algorithms 4 and 5. The proofs can be found in [1, 9, 12, 13, 14] and have been omitted:

(H_1) the function f is bounded below on $B_0 = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \le f(\mathbf{x}_0)\}$;

(H_2) the gradient $\mathbf{g}$ is Lipschitz continuous in an open convex set $B \supseteq B_0$:

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|, \quad \forall\, \mathbf{x}, \mathbf{y} \in B,\ L > 0. \tag{3.1}$$

Proposition 3.1. [1, 13] *Let $\mathbf{d}_k$ be a descent direction and the gradient $\mathbf{g}_k$ satisfies the Lipschitz condition* (3.1)*. If t_k is determined by the BLS in Algorithm 1, then*

$$t_k \ge \min\left\{1, -\frac{\beta(1-\sigma)}{L}\frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k}{\|\mathbf{d}_k\|^2}\right\}. \tag{3.2}$$

Lemma 3.1. *If the function f is UC and TCD on $\mathbb{R}^n$ then there exist m, M such that*

$$0 < m \le 1 \le M, \tag{3.3}$$

then $f(\mathbf{x})$ *possesses a minimizer* $\mathbf{x}^*$ *and*

$$m\|\mathbf{y}\|^2 \le \mathbf{y}^{\mathrm{T}}\nabla^2 f(\mathbf{x})\mathbf{y} \le M\|\mathbf{y}\|^2, \quad \forall\ \mathbf{x},\mathbf{y}\in\mathbb{R}^n; \tag{3.4}$$

$$\frac{1}{2}m\|\mathbf{x}-\mathbf{x}^*\|^2 \le f(\mathbf{x})-f(\mathbf{x}^*) \le \frac{1}{2}M\|\mathbf{x}-\mathbf{x}^*\|^2, \quad \forall\ \mathbf{x}\in\mathbb{R}^n; \tag{3.5}$$

$$m\|\mathbf{x}-\mathbf{y}\|^2 \le (\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{y}))^{\mathrm{T}}(\mathbf{x}-\mathbf{y}) \le M\|\mathbf{x}-\mathbf{y}\|^2, \quad \forall\ \mathbf{x},\mathbf{y}\in\mathbb{R}^n. \tag{3.6}$$

Lemma 3.2. [14] *The following inequality holds for a TCD and UC function* f *and for the IGD sequence* $\{\mathbf{x}_k\}$ *generated by* (1.6)*:*

$$f(\mathbf{x}_k) - f(\mathbf{x}_{k+1}) \ge \mu\|\mathbf{g}_k\|^2, \tag{3.7}$$

with

$$\mu = \min\left\{\frac{\sigma}{M}, \frac{\sigma(1-\sigma)}{L}\beta\right\}. \tag{3.8}$$

In further, it is assumed in this section that $\mathbf{d}_k$ is a descent direction. Further, the scalar approximation of Hessian is TCD. Moreover, instead of (3.4) and (3.3) it is sufficient to assume:

$$m \le \gamma_k \le M, \quad 0 < m \le 1 \le M, \quad m, M \in \mathbb{R}. \tag{3.9}$$

So, all values $\gamma_k < 0$ will be replaced by $\gamma_k = 1$, while the cases $\gamma_k > M$ will be resolved by $\gamma_k = M$.

Theorem 3.1. *Let* (H_1) *and* (H_2) *and* (3.9) *be true and the mapping* f *is UC. Then the sequence* $\{\mathbf{x}_k^{DMSM}\}$ *fulfils* (3.7)–(3.8).

Proof. From (2.8), it can be concluded

$$\begin{aligned}
\mathbf{x}_{k+1}^{DMSM} &= \mathbf{x}_k^{DMSM} - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k \\
&= \mathbf{x}_k^{DMSM} - t_k\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k \\
&= \mathbf{x}_k^{DMSM} + t_k\mathbf{d}_k,
\end{aligned}$$

where $\mathbf{d}_k = -\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k$.

Based on the stopping condition of the backtracking line search (Algorithm 1), we conclude

$$f(\mathbf{x}_k^{DMSM}) - f(\mathbf{x}_{k+1}^{DMSM}) \ge -\sigma t_k {\mathbf{g}_k}^{\mathrm{T}}\mathbf{d}_k. \quad \forall\ k\in\mathbb{N}. \tag{3.10}$$

In the situation $t_k < 1$, by putting expression for $\mathbf{d}_k$ into (3.10), the following inequalities can be derived:

$$\begin{aligned}
f(\mathbf{x}_k^{DMSM}) - f(\mathbf{x}_{k+1}^{DMSM}) &\ge -\sigma t_k {\mathbf{g}_k}^{\mathrm{T}}\mathbf{d}_k \\
&= -\sigma t_k {\mathbf{g}_k}^{\mathrm{T}}\left(-\frac{t_k^{DMSM}}{t_k}\left(\gamma_k^{DMSM}\right)^{-1}\mathbf{g}_k\right) \\
&= \sigma t_k \frac{t_k^{DMSM}}{t_k}\left(\gamma_k^{DMSM}\right)^{-1}\|\mathbf{g}_k\|^2.
\end{aligned} \tag{3.11}$$

Now, from (3.2), it follows that

$$
\begin{aligned}
t_k &\geq -\frac{\beta(1-\sigma)}{L} \cdot \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k}{\|\mathbf{d}_k\|^2} \\
&= -\frac{\beta(1-\sigma)}{L} \cdot \frac{\mathbf{g}_k^{\mathrm{T}}\left(-\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k\right)}{\left\|-\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k\right\|^2} \\
&= \frac{\beta(1-\sigma)}{L} \cdot \frac{\mathbf{g}_k{}^{\mathrm{T}}\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k}{\left(\frac{t_k^{DMSM}}{t_k}\right)^2(\gamma_k^{DMSM})^{-2}\|\mathbf{g}_k\|^2} \\
&= \frac{\beta(1-\sigma)}{L} \cdot \frac{\|\mathbf{g}_k\|^2}{\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\|\mathbf{g}_k\|^2} \\
&= \frac{(1-\sigma)\beta}{L} \cdot \frac{t_k\gamma_k^{DMSM}}{t_k^{DMSM}}.
\end{aligned} \tag{3.12}
$$

By applying inequality (3.12) to (3.11), we obtain

$$
\begin{aligned}
&f(\mathbf{x}_k^{DMSM}) - f(\mathbf{x}_{k+1}^{DMSM}) \geq \sigma t_k \frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\|\mathbf{g}_k\|^2 \\
&\geq \sigma\frac{(1-\sigma)\beta}{L} \cdot \frac{\gamma_k^{DMSM}}{\frac{t_k^{DMSM}}{t_k}} \frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\|\mathbf{g}_k\|^2 \\
&\geq \sigma\frac{(1-\sigma)\beta}{L}\|\mathbf{g}_k\|^2.
\end{aligned} \tag{3.13}
$$

In the case $t_k = 1$, based on (3.9) and (3.10) the following inequality holds

$$
\begin{aligned}
f(\mathbf{x}_k^{DMSM}) - f(\mathbf{x}_{k+1}^{DMSM}) &\geq -\sigma\mathbf{g}_k^{\mathrm{T}}\mathbf{d}_k \\
&= -\sigma\mathbf{g}_k^{\mathrm{T}}\left(-\frac{t_k^{DMSM}}{t_k}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k\right) \\
&= \frac{\sigma}{\gamma_k^{DMSM}}\frac{t_k^{DMSM}}{t_k}\|\mathbf{g}_k\|^2.
\end{aligned} \tag{3.14}
$$

According to (2.9), it follows that $t_k^{DMSM} \geq t_k$, which implies

$$
\begin{aligned}
f(\mathbf{x}_k^{DMSM}) - f(\mathbf{x}_{k+1}^{DMSM}) &\geq \frac{\sigma}{\gamma_k^{DMSM}}\|\mathbf{g}_k\|^2 \\
&\geq \frac{\sigma}{M}\|\mathbf{g}_k\|^2.
\end{aligned} \tag{3.15}
$$

Finally, from (3.13) and (3.15) we get (3.8). □

Theorem 3.2. *Let* (H_1) *and* (H_2) *are valid in conjunction with* (3.9) *and* f *be a UC function.*
(a) *The sequence* $\{\mathbf{x}_k^{DMSM}\}$ *satisfies* $\lim\limits_{k\to\infty} \|\mathbf{g}_k\|=0$, *and* $\{\mathbf{x}_k^{DMSM}\}$ *converges to* $\mathbf{x}^*$.
(b) *The sequence* $\{\mathbf{x}_k^{TMSM}\}$ *satisfies* $\lim\limits_{k\to\infty} \|\mathbf{g}_k\|=0$, *and* $\{\mathbf{x}_k^{TMSM}\}$ *converges to* $\mathbf{x}^*$.

Proof. Analogously as the proof of [14, Theorem 4.1]. □

Lemma 3.3 confirms the convergence of the DMSM method on the strictly convex quadratic (SCQ) functions

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{T}} A\mathbf{x} - \mathbf{b}^{\mathrm{T}}\mathbf{x}, \tag{3.16}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $\mathbf{b} \in \mathbb{R}^n$. The eigenvalues of A are ordered as $\lambda_1 \le \cdots \le \lambda_n$.

Lemma 3.3. *The DMSM iterations* (2.8) *applied on a SCQ function* f *given by the expression* (3.16) *satisfy the inequality*

$$\lambda_1 \le \frac{\gamma_{k+1}^{DMSM}}{t_{k+1}} \le \frac{2\lambda_n}{\sigma}, k \in \mathbb{N}. \tag{3.17}$$

Proof. Simple verification gives

$$\begin{aligned} f(\mathbf{x}_{k+1}^{DMSM}) - f(\mathbf{x}_k^{DMSM}) =& \frac{1}{2}(\mathbf{x}_{k+1}^{DMSM})^{\mathrm{T}} A\mathbf{x}_{k+1}^{DMSM} - \mathbf{b}^{\mathrm{T}}\mathbf{x}_{k+1}^{DMSM} \\ &- \frac{1}{2}(\mathbf{x}_k^{DMSM})^{\mathrm{T}} A\mathbf{x}_k^{DMSM} + \mathbf{b}^{\mathrm{T}}\mathbf{x}_k^{DMSM}. \end{aligned} \tag{3.18}$$

The substitute of (2.8) in (3.18) gives

$$\begin{aligned} f(\mathbf{x}_{k+1}^{DMSM}) - f(\mathbf{x}_k^{DMSM}) =& \frac{1}{2}\left[\mathbf{x}_k^{DMSM} - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k\right]^{\mathrm{T}} \\ &\times A\left[\mathbf{x}_k^{DMSM} - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k\right] \\ &- \mathbf{b}^{\mathrm{T}}[\mathbf{x}_k^{DMSM} - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k] \\ &- \frac{1}{2}(\mathbf{x}_k^{DMSM})^{\mathrm{T}} A\mathbf{x}_k^{DMSM} + \mathbf{b}^{\mathrm{T}}\mathbf{x}_k^{DMSM} \\ =& -\frac{1}{2}t_k^{DMSM}(\gamma_k^{DMSM})^{-1}(\mathbf{x}_k^{DMSM})^{\mathrm{T}} A\mathbf{g}_k \\ &- \frac{1}{2}t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\,\mathbf{g}_k^{\mathrm{T}} A\mathbf{x}_k^{DMSM} \\ &+ \frac{1}{2}\left(t_k^{DMSM}\right)^2(\gamma_k^{DMSM})^{-2}\mathbf{g}_k^{\mathrm{T}} A\mathbf{g}_k \\ &+ t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{b}^{\mathrm{T}}\mathbf{g}_k. \end{aligned} \tag{3.19}$$

Since the gradient of the function (3.16) corresponding to DMSM is equal to

$$\mathbf{g}_k = A\mathbf{x}_k^{DMSM} - \mathbf{b}, \tag{3.20}$$

one can verify

$$\begin{aligned} f(\mathbf{x}_{k+1}^{DMSM}) - f(\mathbf{x}_k^{DMSM}) &= t_k^{DMSM}(\gamma_k^{DMSM})^{-1}[\mathbf{b}^{\mathrm{T}}\mathbf{g}_k - (\mathbf{x}_k^{DMSM})^{\mathrm{T}}A\mathbf{g}_k] \\ &\quad + \frac{1}{2}\left(t_k^{DMSM}\right)^2(\gamma_k^{DMSM})^{-2}\mathbf{g}_k^{\mathrm{T}}A\mathbf{g}_k \\ &= t_k^{DMSM}(\gamma_k^{DMSM})^{-1}[\mathbf{b}^{\mathrm{T}} - (\mathbf{x}_k^{DMSM})^{\mathrm{T}}A]\mathbf{g}_k \\ &\quad + \frac{1}{2}\left(t_k^{DMSM}\right)^2(\gamma_k^{DMSM})^{-2}\mathbf{g}_k^{\mathrm{T}}A\mathbf{g}_k \\ &= -t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k^{\mathrm{T}}\mathbf{g}_k \\ &\quad + \frac{1}{2}\left(t_k^{DMSM}\right)^2(\gamma_k^{DMSM})^{-2}\mathbf{g}_k^{\mathrm{T}}A\mathbf{g}_k. \end{aligned} \tag{3.21}$$

After substitute (3.21) into (2.10), the parameter γ_{k+1}^{DMSM} becomes

$$\begin{aligned} \gamma_{k+1}^{DMSM} &= 2\gamma_k^{DMSM}\frac{\gamma_k^{DMSM}\left[f(\mathbf{x}_{k+1}^{DMSM}) - f(\mathbf{x}_k^{DMSM})\right] + t_k^{DMSM}\|\mathbf{g}_k\|^2}{\left(t_k^{DMSM}\right)^2\|\mathbf{g}_k\|^2} \\ &= \frac{\mathbf{g}_k^{\mathrm{T}}A\mathbf{g}_k}{\|\mathbf{g}_k\|^2}. \end{aligned} \tag{3.22}$$

Therefore, the following inequalities are valid:

$$\lambda_1 \le \gamma_{k+1}^{DMSM} \le \lambda_n, \ k \in \mathbb{N}. \tag{3.23}$$

The inequality in (3.17) follows from (3.23) in conjunction with $0 < t_{k+1} \le 1$. In order to verify the right hand side inequality in (3.17), it suffices to observe the upper bound caused by the BLS

$$t_k \ge \frac{\beta(1-\sigma)\gamma_k}{L},$$

which implies

$$\frac{\gamma_{k+1}^{DMSM}}{t_{k+1}} < \frac{L}{\beta(1-\sigma)}. \tag{3.24}$$

Using $\mathbf{g}(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$ in common with the fact that A symmetric, it follows that

$$\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})\| = \|A\mathbf{x} - A\mathbf{y}\| = \|A(\mathbf{x} - \mathbf{y})\| \le \|A\|\|\mathbf{x} - \mathbf{y}\| = \lambda_n\|\mathbf{x} - \mathbf{y}\|. \tag{3.25}$$

The Lipschitz constant L in (3.24) can be equal to the largest eigenvalue λ_n. Using $\sigma \in (0, 0.5)$, $\beta \in (\sigma, 1)$ one obtains

$$\frac{\gamma_{k+1}^{DMSM}}{t_{k+1}} < \frac{L}{\beta(1-\sigma)} = \frac{\lambda_n}{\beta(1-\sigma)} < \frac{2\lambda_n}{\sigma}. \tag{3.26}$$

So, the right inequality in (3.17) is verified. □

Theorem 3.3. *Let f be a SCQ function defined in* (3.16)*. In the case $\lambda_n < 2\lambda_1$ the DMSM method* (2.8) *satisfies*

$$(d_i^{k+1})^2 \leq \delta^2 (d_i^k)^2, \tag{3.27}$$

where

$$\delta = \max\left\{1 - \frac{\sigma\lambda_1}{2\lambda_n}, \frac{\lambda_n}{\lambda_1} - 1\right\}. \tag{3.28}$$

In addition,

$$\lim_{k\to\infty} \|\mathbf{g}_k\| = 0. \tag{3.29}$$

Proof. Let $\{v_1, \ldots, v_n\}$ be orthonormal eigenvectors of A. On the basis of (3.20), there exist real quantities $d_1^k, d_2^k, \ldots, d_n^k$ satisfying

$$\mathbf{g}_k = \sum_{i=1}^{n} d_i^k v_i. \tag{3.30}$$

Now, using (2.8) one can simply deduce

$$\begin{aligned}
\mathbf{g}_{k+1} &= A\mathbf{x}_{k+1}^{DMSM} - \mathbf{b} \\
&= A(\mathbf{x}_k^{DMSM} - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\mathbf{g}_k) - \mathbf{b} \\
&= \mathbf{g}_k - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}A\mathbf{g}_k \\
&= \left(I - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}A\right)\mathbf{g}_k.
\end{aligned}$$

Using the simple linear approximation of $\mathbf{g}_{k+1}$ as in (3.30), we get

$$\mathbf{g}_{k+1} = \sum_{i=1}^{n} d_i^{k+1} v_i = \sum_{i=1}^{n} \left(1 - t_k^{DMSM}(\gamma_k^{DMSM})^{-1}\lambda_i\right) d_i^k v_i. \tag{3.31}$$

To prove (3.27), it is enough to show that $\left|1 - \frac{\lambda_i}{\gamma_k^{DMSM}\left(t_k^{DMSM}\right)^{-1}}\right| \leq \delta$. Two cases can be observed. First, if $\lambda_i \leq \frac{\gamma_k^{DMSM}}{t_k^{DMSM}}$ implying (3.17), we can conclude the following:

$$1 > \frac{\lambda_i}{\gamma_k^{DMSM}\left(t_k^{DMSM}\right)^{-1}} \geq \frac{\sigma\lambda_1}{2\lambda_n} \Longrightarrow 1 - \frac{\lambda_i}{\gamma_k^{DMSM}\left(t_k^{DMSM}\right)^{-1}} \leq 1 - \frac{\sigma\lambda_1}{2\lambda_n} \leq \delta. \tag{3.32}$$

Now, let us examine another case $\frac{\gamma_k^{DMSM}}{t_k^{DMSM}} < \lambda_i$. Since

$$1 < \frac{\lambda_i}{\gamma_k^{DMSM}\left(t_k^{DMSM}\right)^{-1}} \leq \frac{\lambda_n}{\lambda_1}, \tag{3.33}$$

it follows that

$$(3.34) \qquad \left|1-\frac{\lambda_i}{\gamma_k^{DMSM}\left(t_k^{DMSM}\right)^{-1}}\right| \le \frac{\lambda_n}{\lambda_1}-1\le\delta.$$

Now, in order to prove $\lim\limits_{k\to\infty}\|\mathbf{g}_k\| = 0$, it suffices to use the orthonormality of $\{v_1,\dots,v_n\}$ in common with (3.30) and conclude

$$(3.35) \qquad \|\mathbf{g}_k\|^2=\sum_{i=1}^{n}(d_i^k)^2.$$

Since (3.27) is valid and $0<\delta<1$ holds, (3.35) initiates that (3.30). □

4. Numerical results

All the considered methods are coded in Matlab R2017a programming language and executed on the notebook with Intel Core i3 2.0 GHz CPU, 8 GB RAM and Windows 10 operating system. The number of iterations (NofI), number of function evaluations (NofFE) and the CPU time (CPUT) are analyzed in numerical experiments.

Numerical testing is based on 24 test functions from [2], where a lot of the problems are taken over from CUTEr collection [4]. For each of tested functions in Tables 4.1, 4.2 and 4.3, 12 numerical testings are performed with 100, 200, 300, 500, 1000, 2000, 3000, 5000, 7000, 8000, 10000 and 15000 unknowns. Tables 4.1, 4.2 and 4.3 arrange summary numerical results for AGD, MAGD, MSM, SM, DMSM and TMSM, tested on 24 functions.

For each of six tested methods (AGD, MAGD, SM, MSM, DMSM and TMSM), the same stopping criteria are used:

$$\|\mathbf{g}_k\|\le 10^{-6} \quad\text{and}\quad \frac{|f(\mathbf{x}_{k+1})-f(\mathbf{x}_k)|}{1+|f(\mathbf{x}_k)|}\le 10^{-16}.$$

The BLS parameters for AGD, MAGD, MSM and SM methods are $\sigma=0.0001$ and $\beta=0.8$. The backtracking procedures in the DMSM method are implemented using $\sigma=0.0001$ and $\beta=0.8$ for Algorithm 1 and $\sigma_j=0.00015$ and $\beta_j=0.85$ for Algorithm 3.

The backtracking procedures in the TMSM method are developed using $\sigma=0.0001$ and $\beta=0.8$ for Algorithm 1, $\sigma_l=0.0002$ and $\beta_l=0.9$ for Algorithm 2 and $\sigma_j=0.00015$ and $\beta_j=0.85$ for Algorithm 3.

Table 4.4 contains average values of NofI, the NofFE and the CPUT for all 288 numerical experiments.

Based on the values for NofI given in Table 4.4, it can be concluded that the DMSM and TMSM methods gives superior results with respect to MAGD, AGD, MSM and SM methods.

Table 4.1: Numerical results of the AGD, MAGD, MSM, SM, DMSM and TMSM methods for the NofI.

Test function	*MAGD*	*TMSM*	*MSM*	*DMSM*	*SM*	*AGD*
Perturbed Quadratic	352325	31269	34828	31386	59908	353897
Raydan 1	58504	30148	26046	17238	14918	22620
Diagonal 3	119719	6767	7030	7077	12827	120416
Generalized Tridiagonal 1	647	332	346	350	325	670
Extended Tridiagonal 1	692219	685	1370	728	4206	3564
Extended TET	455	191	156	156	156	443
Diagonal 4	8084	96	96	96	96	120
Diagonal 5	48	72	72	72	72	48
Extended Himmelblau	302	312	260	264	196	396
Perturbed quadratic diagonal	1060824	36640	37454	31662	44903	2542050
Quadratic QF1	362896	32099	36169	33138	62927	366183
Extended quadratic penalty QP1	229	338	369	298	271	210
Extended quadratic penalty QP2	356357	1735	1674	990	3489	395887
Quadratic QF2	71647	31745	32727	30642	64076	100286
Extended Tridiagonal 2	1665	694	659	583	543	1657
ARWHEAD (CUTE)	12834	328	430	302	270	5667
Almost Perturbed Quadratic	354369	30790	33652	32902	60789	356094
LIARWHD (CUTE)	925138	1257	3029	1726	18691	1054019
ENGVAL1 (CUTE)	822	623	461	434	375	743
QUARTC (CUTE)	177	302	217	220	290	171
Generalized Quartic	229	191	181	186	189	187
Diagonal 7	159	144	147	111	108	72
Diagonal 8	154	120	120	109	118	60
Full Hessian FH3	63	63	63	63	63	45

Performance profiles from [6] are used in comparing the selected methods. As usual, the NofI, NofFE and CPUT profiles are used. All numerical results are represented in Figures 4.1 and 4.2. Figure 4.1 (left) shows the performances of compared methods related to NofI. Figure 4.1 (right) illustrates the performance of these methods relative to NofFE. Graphs in Figure 4.2 illustrate the behavior of considered methods with respect to to CPUT.

From the results displayed in Tables 4.1, 4.2 and 4.3 and according to graphs in Figures 4.1 and Figure 4.2, the following can be observed.

(1) The DMSM and TMSM methods give better results compared to other methods when we compare the number of iterations.

(2) The SM, MSM, DMSM and TMSM exhibit better performances than the AGD and MAGD methods.

From Figure 4.1 (left), it is observable that the graph of the *DMSM* method comes first to the top, which signifies that the DMSM outperforms other considered methods with respect to the NofI.

Table 4.2: Numerical results of the AGD, MAGD, MSM, SM, DMSM and TMSM methods for the NofFE.

Test function	*MAGD*	*TMSM*	*MSM*	*DMSM*	*SM*	*AGD*
Perturbed Quadratic	13855459	645704	200106	370595	337910	13916515
Raydan 1	1282162	1305952	311260	326766	81412	431804
Diagonal 3	4244404	131307	38158	80193	69906	4264718
Generalized Tridiagonal 1	9057	2934	1191	2061	1094	9334
Extended Tridiagonal 1	2077341	14797	10989	9147	35621	14292
Extended TET	4130	1689	528	948	528	3794
Diagonal 4	133440	2316	636	1320	636	1332
Diagonal 5	108	300	156	228	156	108
Extended Himmelblau	5192	3636	976	1908	668	6897
Perturbed quadratic diagonal	38728371	1309740	341299	629088	460028	94921578
Quadratic QF1	13192789	661661	208286	392426	352975	13310016
Extended quadratic penalty QP1	2939	6400	2196	5421	2326	2613
Extended quadratic penalty QP2	8846145	44962	11491	14058	25905	9852040
Quadratic QF2	2810965	642829	183142	364257	353935	3989239
Extended Tridiagonal 2	9613	9779	2866	4951	2728	8166
ARWHEAD (CUTE)	468970	15416	5322	8503	3919	214284
Almost Perturbed Quadratic	13936462	639129	194876	393591	338797	14003318
LIARWHD (CUTE)	41619197	39788	27974	33271	180457	47476667
ENGVAL1 (CUTE)	8332	10120	2285	4319	2702	6882
QUARTC (CUTE)	414	1412	494	780	640	402
Generalized Quartic	1244	1311	493	836	507	849
Diagonal 7	745	930	504	696	335	333
Diagonal 8	740	805	383	546	711	304
Full Hessian FH3	1955	2160	566	1263	631	1352

Figure 4.1 (right) confirms that all six methods are able to solve all test cases. Further, the MSM method is superior in 58.33% of all tests with respect to MAGD (4.17%), TMSM(0%), DMSM(4.17%), SM(29.17%) and AGD(16.67%).

Graphs in Figure 4.2 again confirm that all the methods are able to solve test problems, and the MSM is winer in 54.17% of the tests with respect to MAGD (4.17%), TMSM(0%), DMSM(4.17%), SM(37.50%) and AGD(4.17%).

According to individual data arranged in the tables 4.1-4.3, generated average values as well as the presented graphs, the conclusion is that the DMSM method is winer concerning the NofI.

Compared to the previous numerical results obtained during the testing of AGD, MAGD, MSM, SM, DMSM and TMSM methods, in the next test for parameter values in the second and third backtracking line search we take the values that are less than the values in primary backtracking. The aim of this test is to answer the question: Does the choice of higher or lower parameter values in the second and third backtracking line search in relation to the primary backtracking line search

Table 4.3: Numerical results of the AGD, MAGD, MSM, SM, DMSM and TMSM methods for the CPUT.

Test function	*MAGD*	*TMSM*	*MSM*	*DMSM*	*SM*	*AGD*
Perturbed Quadratic	6049.531	344.172	116.281	198.328	185.641	6756.047
Raydan 1	334.266	388.156	31.906	67.344	36.078	158.359
Diagonal 3	6401.969	199.547	52.609	120.406	102.875	5527.844
Generalized Tridiagonal 1	7.781	4.641	1.469	3.625	1.203	11.344
Extended Tridiagonal 1	8853.172	26.828	29.047	17.297	90.281	55.891
Extended TET	2.766	1.703	0.516	1.203	0.594	3.219
Diagonal 4	16.172	0.719	0.203	0.359	0.141	0.781
Diagonal 5	0.313	0.750	0.344	0.734	0.328	0.391
Extended Himmelblau	1.031	1.094	0.297	0.703	0.188	1.953
Perturbed quadratic diagonal	22820.172	534.750	139.625	273.188	185.266	44978.750
Quadratic QF1	6846.453	258.938	81.531	168.453	138.172	12602.563
Extended quadratic penalty QP1	1.063	2.234	1.000	3.516	0.797	1.266
Extended quadratic penalty QP2	1872.797	12.578	3.516	8.063	6.547	3558.734
Quadratic QF2	768.563	243.938	73.438	153.109	132.703	1582.766
Extended Tridiagonal 2	2.531	4.938	1.047	2.375	1.031	3.719
ARWHEAD (CUTE)	138.000	6.422	1.969	4.609	1.359	95.641
Almost Perturbed Quadratic	7086.563	285.563	73.047	153.891	133.516	13337.125
LIARWHD (CUTE)	15372.625	10.203	9.250	12.641	82.016	27221.516
ENGVAL1 (CUTE)	2.641	4.328	1.047	2.375	1.188	3.906
QUARTC (CUTE)	2.078	4.531	1.844	3.297	2.313	2.469
Generalized Quartic	0.500	0.734	0.281	0.375	0.188	0.797
Diagonal 7	0.688	0.953	0.547	1.469	0.375	0.625
Diagonal 8	0.656	0.781	0.469	1.078	0.797	0.438
Full Hessian FH3	1.188	1.672	0.391	1.234	0.391	1.438

Table 4.4: Average numerical outcomes for 24 test functions tested on 12 numerical experiments.

Average performances	MAGD	TMSM	MSM	DMSM	SM	AGD
Number of iterations	182494.42	8622.54	9064.83	7947.21	14575.25	221896.04
No. of fun.evaluation	5885007.25	228961.54	64424.04	110298.83	93938.63	8434868.21
CPU time (sec)	3190.98	97.51	25.90	49.99	46.00	4829.4

directly affect the numerical results of DMSM and TMSM methods?

The primary BLS uses the same parameters $\sigma = 0.0001$ and $\beta = 0.8$ as in the first test for AGD, MAGD, MSM and SM methods. The BLS procedures in the DMSM method are implemented using $\sigma = 0.0001$ and $\beta = 0.8$ for Algorithm 1 and $\sigma_j = 0.00005$ and $\beta_j = 0.7$ for Algorithm 3. Also, the BLS in the TMSM method are implemented using $\sigma = 0.0001$ and $\beta = 0.8$ for Algorithm 1, $\sigma_l = 0.00001$ and $\beta_l = 0.6$

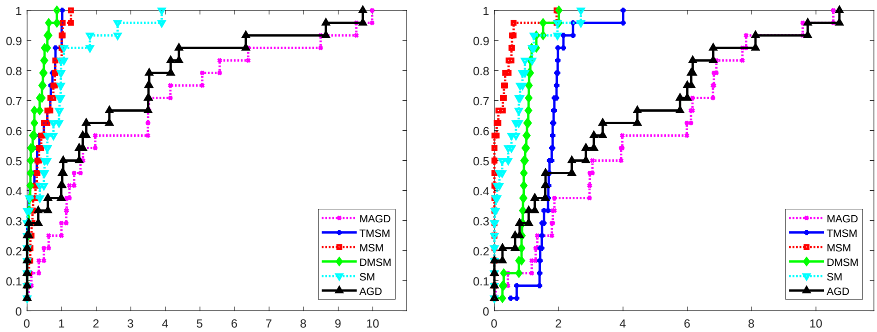

FIG. 4.1: Performance profiles based on the NofI (left) and NofFE (right).

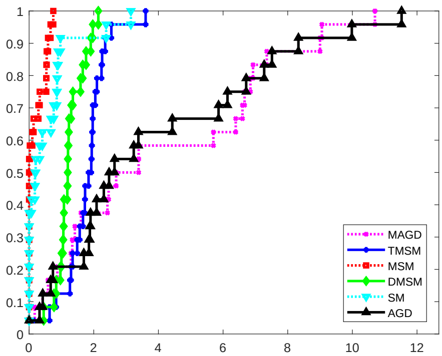

FIG. 4.2: Performance profiles based upon CPUT.

for Algorithm 2 and $\sigma_j = 0.00005$ and $\beta_j = 0.7$ for Algorithm 3.

All other conditions (stop criteria and number of variables) remain the same as in the first numerical experiment.

The obtained numerical results are shown in the Tables 4.5, 4.6 and 4.7.

Table 4.8 includes the average values of NofI, the NofFE and the CPUT in a second numerical experiment.

According to the NofI values given in Table 4.8, it can be notified that the DMSM method gives better results and in the second numerical experiment compared to MAGD, AGD, MSM, SM and TMSM methods.

All numerical results from Tables 4.5, 4.6 and 4.7 are represented in Figures 4.3 and 4.4. Figure 4.3 (left) shows the NofI performances of compared methods. Figure 4.3 (right) demonstrates the NofFE profile of these methods. Figure 4.4

Table 4.5: Numerical results of the AGD, MAGD, MSM, SM, DMSM and TMSM methods for the NofI.

Test function	$MAGD$	MSM	SM	AGD	$TMSM$	$DMSM$
Perturbed Quadratic	352325	34828	59908	353897	35697	28487
Raydan 1	58504	26046	14918	22620	9801	17594
Diagonal 3	119719	7030	12827	120416	8372	6409
Generalized Tridiagonal 1	647	346	325	670	342	348
Extended Tridiagonal 1	692219	1370	4206	3564	907	760
Extended TET	455	156	156	443	156	156
Diagonal 4	8084	96	96	120	96	96
Diagonal 5	48	72	72	48	72	72
Extended Himmelblau	302	260	196	396	288	294
Perturbed quadratic diagonal	1060824	37454	44903	2542050	31031	37331
Quadratic QF1	362896	36169	62927	366183	39619	26585
Extended quadratic penalty QP1	229	369	271	210	303	362
Extended quadratic penalty QP2	356357	1674	3489	395887	2047	1908
Quadratic QF2	71647	32727	64076	100286	39452	28651
Extended quadratic exponential EP1	67	100	73	48	107	107
Extended Tridiagonal 2	1665	659	543	1657	528	615
ARWHEAD (CUTE)	12834	430	270	5667	304	281
Almost Perturbed Quadratic	354369	33652	60789	356094	35755	26274
LIARWHD (CUTE)	925138	3029	18691	1054019	1340	3543
ENGVAL1 (CUTE)	822	461	375	743	418	482
QUARTC (CUTE)	177	217	290	171	289	275
Generalized Quartic	229	181	189	187	197	195
Full Hessian FH3	63	63	63	45	63	63
Diagonal 9	325609	10540	13619	329768	10219	11229

shows the performance CPUT.

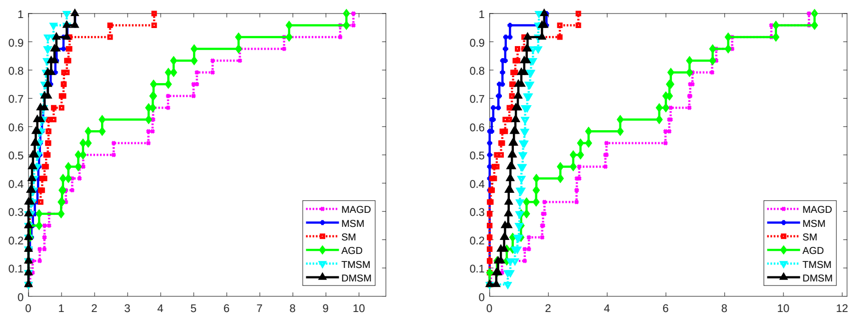

Fig. 4.3: Performance profiles based on the NofI (left) and NofFE (right).

Table 4.6: Numerical results of the AGD, MAGD, MSM, SM, DMSM and TMSM methods for the NofFE.

Test function	*MAGD*	*MSM*	*SM*	*AGD*	*TMSM*	*DMSM*
Perturbed Quadratic	13855459	200106	337910	13916515	423496	260678
Raydan 1	1282162	311260	81412	431804	124905	280011
Diagonal 3	4244404	38158	69906	4264718	95962	54865
Generalized Tridiagonal 1	9057	1191	1094	9334	2408	2153
Extended Tridiagonal 1	2077341	10989	35621	14292	13562	6800
Extended TET	4130	528	528	3794	1080	828
Diagonal 4	133440	636	636	1332	1284	996
Diagonal 5	108	156	156	108	300	228
Extended Himmelblau	5192	976	668	6897	2136	2418
Perturbed quadratic diagonal	38728371	341299	460028	94921578	619938	529154
Quadratic QF1	13192789	208286	352975	13310016	472273	243573
Extended quadratic penalty QP1	2939	2196	2326	2613	5073	3895
Extended quadratic penalty QP2	8846145	11491	25905	9852040	29847	21345
Quadratic QF2	2810965	183142	353935	3989239	444580	257674
Extended quadratic exponential EP1	1513	894	661	990	2083	1617
Extended Tridiagonal 2	9613	2866	2728	8166	4446	4456
ARWHEAD (CUTE)	468970	5322	3919	214284	9038	6761
Almost Perturbed Quadratic	13936462	194876	338797	14003318	424470	237534
LIARWHD (CUTE)	41619197	27974	180457	47476667	22254	53306
ENGVAL1 (CUTE)	8332	2285	2702	6882	6064	4442
QUARTC (CUTE)	414	494	640	402	1264	909
Generalized Quartic	1244	493	507	849	1043	798
Full Hessian FH3	1955	566	631	1352	1152	957
Diagonal 9	12984028	68189	89287	13144711	131327	125119

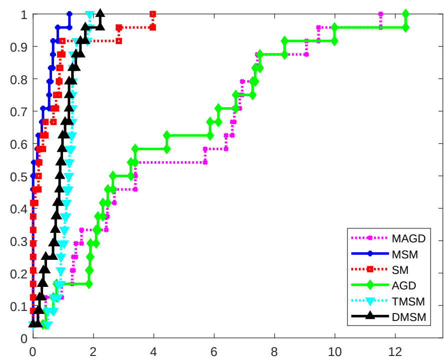

FIG. 4.4: Performance profiles arising from CPUT.

Table 4.7: Numerical results of the AGD, MAGD, MSM, SM, DMSM and TMSM methods for the CPUT.

Test function	*MAGD*	*MSM*	*SM*	*AGD*	*TMSM*	*DMSM*
Perturbed Quadratic	6049.531	116.281	185.641	6756.047	219.328	134.781
Raydan 1	334.266	31.906	36.078	158.359	44.828	66.484
Diagonal 3	6401.969	52.609	102.875	5527.844	129.734	96.688
Generalized Tridiagonal 1	7.781	1.469	1.203	11.344	2.969	2.969
Extended Tridiagonal 1	8853.172	29.047	90.281	55.891	25.672	12.609
Extended TET	2.766	0.516	0.594	3.219	1.234	0.938
Diagonal 4	16.172	0.203	0.141	0.781	0.344	0.172
Diagonal 5	0.313	0.344	0.328	0.391	0.594	0.516
Extended Himmelblau	1.031	0.297	0.188	1.953	0.688	0.875
Perturbed quadratic diagonal	22820.172	139.625	185.266	44978.750	263.953	220.719
Quadratic QF1	6846.453	81.531	138.172	12602.563	173.953	91.047
Extended quadratic penalty QP1	1.063	1.000	0.797	1.266	2.781	1.813
Extended quadratic penalty QP2	1872.797	3.516	6.547	3558.734	8.750	5.906
Quadratic QF2	768.563	73.438	132.703	1582.766	169.266	98.141
Extended quadratic exponential EP1	0.844	0.688	0.438	0.750	1.000	0.859
Extended Tridiagonal 2	2.531	1.047	1.031	3.719	1.828	1.922
ARWHEAD (CUTE)	138.000	1.969	1.359	95.641	2.813	2.625
Almost Perturbed Quadratic	7086.563	73.047	133.516	13337.125	158.156	92.578
LIARWHD (CUTE)	15372.625	9.250	82.016	27221.516	5.250	17.406
ENGVAL1 (CUTE)	2.641	1.047	1.188	3.906	2.578	2.391
QUARTC (CUTE)	2.078	1.844	2.313	2.469	4.625	3.203
Generalized Quartic	0.500	0.281	0.188	0.797	0.422	0.500
Full Hessian FH3	1.188	0.391	0.391	1.438	1.063	0.891
Diagonal 9	6662.984	43.609	38.672	6353.172	61.984	114.703

Table 4.8: Average numerical results in the second numerical experiment.

Average performances	*MAGD*	*MSM*	*SM*	*AGD*	*TMSM*	*DMSM*
Number of iterations	196051.21	9497.04	15136.33	235632.88	9058.46	8004.88
No. of fun.evaluation	6426009.58	67265.54	97642.88	8982579.21	118332.71	87521.54
CPU time (sec)	3468.58	27.71	47.58	5094.18	53.49	40.45

In accordance with obtained numerical data generated in the second numerical experiment, we can give an answer to the question, that independently of the choice of parameter values in the second and third backtracking line search, the DMSM iterations has the best results in relation to NofI. Also, if we compare the average results obtained in Tables 4.4 and 4.8, we can see that there is a slight percentage decrease in the average numerical results of the NofFE and CPUT, the DMSM method compared to the MSM method in the second numerical experiment.

5. Conclusion

Multiple usage of the backtracking line search in the modified SM (MSM) method lead to two improvements of the MSM scheme, denoted as the TMSM and DMSM methods. Proposed iterations are investigated both theoretically and numerically. The linear convergence of the defined model is proved for UC and for a subset of SCQ functions. Numerical experiments confirm that the derived TMSM and DMSM methods outperform the SM, AGD, MAGD and the MSM with respect to the number of iterations. Numerical values arranged in Tables 4.1-4.8 confirm the better performance of presented accelerated gradient descent method. Finally, the obtained TMSM and DMSM methods can be used as a motivation for different possibilities of deriving new efficient schemes for unconstrained optimization.

REFERENCES

1. N. Andrei: *An acceleration of gradient descent algorithm with backtracking for unconstrained optimization.* Numer. Algor. **42** (2006), 63–73.
2. N. Andrei: *An unconstrained optimization test functions collection.* Adv. Model. Optim. **10** (2008), 147—161.
3. N. Andrei: *Relaxed gradient descent and a new gradient descent methods for unconstrained optimization.* https://camo.ici.ro/neculai/newgrad.pdf, Visited October 29, 2020.
4. I. Bongartz, A. R. Conn, N. I. M. Gould and Ph. L. Toint: *CUTEr: constrained and unconstrained testing environments.* ACM Trans. Math. Softw. **21** (1995), 123-–160.
5. C. Brezinski: *A classification of quasi-Newton methods.* Numer. Algor. **33** (2003), 123-–135.
6. E. D. Dolan and J. J. Moré: *Benchmarking optimization software with performance profiles.* Math. Program. **91** (2002), 201—213.
7. B. Ivanov, P. S. Stanimirović, G. V. Milovanović, S. Djordjević and I. Brajević: *Accelerated multiple step-size methods for solving unconstrained optimization problems.* Optimization Methods and Software (2019), https://doi.org/10.1080/10556788.2019.1653868.
8. J. Nocedal and S. J. Wright: *Numerical Optimization.* Springer-Verlag New York, Inc, 1999.
9. J. M. Ortega and W. C. Rheinboldt: *Iterative Solution of Nonlinear Equation in Several Variables.* Academic Press, New York, London, 1970.
10. M. J. Petrović: *An accelerated Double Step Size method in unconstrained optimization.* Applied Math. Comput. **250** (2015), 309–319.
11. M. J. Petrović and P. S. Stanimirović: *Accelerated Double Direction method for solving unconstrained optimization problems.* Mathematical Problems in Engineering **2014** (2014), Article ID 965104, 8 pages.
12. R. T. Rockafellar: *Convex Analysis.* Princeton University Press, Princeton, 1970.
13. Z. -J. Shi: *Convergence of line search methods for unconstrained optimization.* App. Math. Comput. **157** (2004), 393–405.
14. P. S. Stanimirović and M. B. Miladinović: *Accelerated gradient descent methods with line search.* Numer. Algor. **54** (2010), 503–520.

15. P. S. Stanimirović, G. V. Milovanović and M. J. Petrović: *A transformation of accelerated double step size method for unconstrained optimization.* Mathematical Problems in Engineering **2015** (2015), Article ID 283679, 8 pages.

16. W. Sun and Y. -X. Yuan: *Optimization Theory and Methods: Nonlinear Programming.* Springer, Berlin, 2006.

Branislav Ivanov
University of Belgrade, Technical Faculty in Bor
Department of Management
Vojske Jugoslavije 12, 19210 Bor, Serbia
ivanov.branislav@gmail.com

Bilall I. Shaini
State University of Tetova
Rr. e Ilindenit, p.n., Tetovo
R. Macedonia
bilall.shaini@unite.edu.mk

Predrag S. Stanimirović
University of Niš, Faculty of Sciences and Mathematics
Department of Computer Science
Višegradska 33, 18000 Niš, Serbia
pecko@pmf.ni.ac.rs

FACTA UNIVERSITATIS (NIŠ)
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SOLITARY WAVE SOLUTIONS FOR SPACE-TIME FRACTIONAL COUPLED INTEGRABLE DISPERSIONLESS SYSTEM VIA GENERALIZED KUDRYASHOV METHOD

Ahmed A. Gaber and Hijaz Ahmad

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Abstract. In this article, space-time fractional coupled integrable dispersionless system has been considered, and we have used fractional derivative in the sense of modified Riemann-Liouville. The fractional system has been reduced to an ordinary by fractional transformation and the generalized Kudryashov method is applied to obtain exact solutions. We also testify performance as well as the precision of the applied method by means of numerical tests for obtaining solutions. The obtained results have been graphically presented to show the properties of the solutions.
Keywords. integrable dispersionless system; fractional derivative; differential system.

1. Introduction

In recent years, fractional differential equations have gained much attention from researchers due to thier numerous applications in many fields of sciences and engineering. These equations are widely used to describe various phenomena in many fields such as the fluid flow, electro chemistry, scattering theory, transport theory, probability, elasticity, control theory, potential theory, signal processing, image processing, diffusion theory, kinetic theory, systems identification, biology and other areas [1, 2]. The first application of fractional calculus was introduced by Abel [3] in the solution of an integral equation that was arisen in the formulation of the tautochronous problem. This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along with a particle of mass m can fall in a time that is independent of the starting position [4].

Travelling wave methods have an important role to obtain solutions that are described and explained these natural phenomena. Most famous of by these effective methods are (G′/G)-expansion method [5 − 6], variational iteration algorithm-I

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[7 − 8], Exp-function method [9], fractional iteration algorithm [10 − 11], Generalization of He's Exp-Function Method [12], reproducing Kernal method [13], a new extended Auxiliary equation method [14], variational iteration algorithm-II [15 − 16] and Modified Kudryashov method [17 − 19]. In this paper, we use generalized Kudryashov method for finding the exact solutions of space-time fractional coupled integrable dispersionless system.

2. Properties of fractional derivatives

In this paper, we consider the most common definition named in modified Riemann-Liouville derivative which is defined [20 − 26]

$$D_t^{\gamma}u(t) = \begin{cases} \frac{1}{\Gamma(-\gamma)}\int\limits_0^t (t-\tau)^{-\gamma-1}(u(\tau)-u(t))d\tau & ,\gamma<0, \\ \frac{1}{\Gamma(-\gamma)}\frac{d}{dt}\int\limits_0^t (t-\tau)^{-\gamma-1}(u(\tau)-u(t))d\tau, & 0<\gamma\leq 1, \\ (u^{(n-1)}(\tau))^{(\gamma-n-1)}, & n-1<\gamma\leq n,\ n\geq 2 \end{cases} \tag{1}$$

where $u : R \to R$, $t \to u(t)$, denotes a continuous function.

Property 1,

$$D_t^{\gamma}t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\gamma)}t^{r-\gamma}, r>0 \tag{2}$$

Property 2,

$$D_t^{\gamma}(u(t)g(t)) = g(t)D_t^{\gamma}u(t) + u(t)D_t^{\gamma}g(t). \tag{3}$$

Property 3,

$$D_t^{\alpha}u(g(t)) = \frac{du(g(t))}{dg(t)}D_t^{\alpha}g(t) \tag{4}$$

3. Description of the method for FDEs

Consider a given nonlinear wave equation

$$N(u, D_t^{\alpha}u, D_x^{\alpha}u, D_x^{2\alpha}u_{xx}, D_t^{2\alpha}u, D_t^{\alpha}D_x^{\alpha}u, ...) = 0, \tag{5}$$

we seek its wave solutions

$$u = U(\eta), \quad \eta = \frac{h_i x_i^{\alpha}}{\Gamma(1+\alpha)} + \frac{gt^{\alpha}}{\Gamma(1+\alpha)}, \quad i = 1,2,... \tag{6}$$

Consequently, Eq. (5) is reduced to the ordinary differential equation (ODE) by transformation:

$$U(u, gu', hu', g^2u'', h^2u'', ...) = 0. \tag{7}$$

The **generalized Kudryashov method (GKM)** is based on the assumption that the travelling wave solutions can be expressed in the following form

$$u(\eta) = \sum_{i=0}^{m} \frac{a_i}{(1+\phi(\eta))^i}, \tag{8}$$

where m is positive integer which are unknown to be further determined, a_i are unknown constants. In addition, $\phi(\ \eta)$ satisfies Riccati equation

$$\phi'(\eta) = A + B\phi(\eta) + C\phi^2(\eta). \tag{9}$$

We obtained a type of solutions of Eq. (9)

Family 1: A and B are free constants, $C \neq 0$

$$\phi(\eta) = \frac{-B + \sqrt{4AC - B^2}\tan(\frac{1}{2}(\sqrt{4AC - B^2}(\eta + d_0)))}{2C}.$$

Family 2: $A = 0$, $B \neq 0$, and C is a free constant

$$\phi(\eta) = \frac{-B\exp(B\eta + Bd_0)}{C\exp(B\eta + Bd_0) - 1}.$$

Family 3: A is free constant, $B \neq 0$, and $C = 0$

$$\phi(\eta) = \frac{-A}{B} + \frac{1}{B}\exp(B\eta).$$

Family 4: $A = 0$, $B = -1$ and C$= -1$

$$\phi(\eta) = \frac{-d_0}{\exp(\eta) + d_0}.$$

4. Space-time fractional coupled Integrable Dispersionless system

We consider the space-time fractional coupled Integrable Dispersionless (CID) system

$$\begin{aligned} \frac{\partial^{2\alpha} u}{\partial t^\alpha \partial x^\alpha} + \frac{\partial^\alpha}{\partial x^\alpha}(vw) &= 0, \\ \frac{\partial^{2\alpha} v}{\partial t^\alpha \partial x^\alpha} - 2v\frac{\partial^\alpha u}{\partial x^\alpha} &= 0, \end{aligned}$$

$$\frac{\partial^{2\alpha} w}{\partial t^\alpha \partial x^\alpha} - 2w\frac{\partial^\alpha u}{\partial x^\alpha} = 0, \tag{10}$$

where u, v and w are all functions of x and t. Eqs. (10) describes the current-fed string within an external magnetic field [27, 28]. This equations wase presented and solved by the inverse scattering method [29], the exp-function method [30] and residue harmonic balance [31].

We perform the transformation $\eta = \frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}$, Eq. (10) can be reduced into an ODEs

$$\begin{aligned} ghU'' + h(VW' + WV') &= 0, \\ ghV' - 2hVU' &= 0, \end{aligned}$$

$$ghW' - 2hWU' = 0, \tag{11}$$

where $U' = \frac{\partial U}{\partial \eta}$.

We can freely know that the solution does not depend on the balancing the highest order linear and nonlinear terms [32]. For simplicity, we set i=2, we have:

$$\begin{aligned} U(\eta) &= a_0 + \frac{a_1}{1+\phi(\eta)} + \frac{a_2}{(1+\phi(\eta))^2}, \\ V(\eta) &= b_0 + \frac{b_1}{1+\phi(\eta)} + \frac{b_2}{(1+\phi(\eta))^2}, \\ W(\eta) &= r_0 + \frac{r_1}{1+\phi(\eta)} + \frac{r_2}{(1+\phi(\eta))^2}. \end{aligned} \tag{12}$$

Substituting Eq. (12) into Eq. (11), equating to zero the coefficients of all powers of $\phi(\eta)$ yields a set of algebraic equations for a_i, b_i, r_i.

$$h(6BCr_0a_1 + 2Ar_0a_1 + 3gr_1AB + 4Br_0a_2 + gr_1B^2C + 2BCr_1a_1 + 4gr_2B^2) = 0,$$
$$h(2Ab_0a_1 + 6Bb_0a_1 + 4gb_2B^2 + 3gb_1AB + gb_1B^2 + 4BCb_0a_2 + 2Bb_1a_1) = 0,$$
$$h(2Br_0a_1 + gr_1B^2) = 0, h(gb_1B^2 + 2Bb_0a_1) = 0, h(ga_1B^2 - Bb_0r_1 - Bb_1r_0) = 0,$$
$$h(ga_1B^2 - Ab_1r_0 + 3ga_1AB - 2BCb_0r_2 - Abr_1 - 2Bb_2r - 2Bb_1r_1 - 3BCb_0r_1$$
$$+4ga_2B^2 - 3Bb_1r_0) = 0,$$
$$h(2gr_1A^2 + 6gr_2A^2 + 2Ar_0a_1 + 4Ar_0a_2 + 2Ar_1Ca_1 + 4Ar_1a_2 + 2ACr_2a_1$$
$$+4Ar_2a_2 - 2gr_2AB - gr_1AB) = 0,$$
$$h(4Ab_1a_2 + 2Ab_2a_1 + 4Ab_2a_2 + 2Ab_0a_1 + 4Ab_0a_2 + 6gb_2A^2 - gb_1AB$$
$$-2gb_2AB + 2Ab_1a_1 + 2gb_1A^2) = 0,$$
$$h(-2Ab_1r_1 + 2ga_1A^2 + 6ga_2A^2 - ACb_1r_0 - 4Ab_2r_2 - 3ACb_1r_2 - 2Ab_2r_0$$

$$-3Ab_2r_1 - Ab_0r_1 - ga_1AB - 2Ab_0r_2 - 2ga_2AB) = 0,$$

(13)

Solving the system of algebraic equations with the help of Maple, we obtain the solutions organized in the following cases:

Case (1)

$$\begin{aligned} a_1 &= -gA - gC + gB, b_0 = \frac{-1}{4r_0}g^2(4C^2 - 4BC + B^2), \\ b_1 &= \frac{-1}{2r_0}g^2(-2AC - 2C^2 + 3BC + AB - B^2), \\ r_1 &= \frac{2r_0(A + C - B)}{-2C + B}, \ a_0 \text{ is arbitrary, } b_2{=}r_2{=}a_2{=}0. \end{aligned}$$

(14)

Substituting these results into (11) and with the aid of families 1-4, we obtain the following multiple soliton-like and periodic solutions for space-time fractional CID system

$$u(x,t) = a_0 + \frac{-gA - gC + gB}{1 + \frac{-B+\sqrt{4AC-B^2}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\eta+d_0)))}{2C}},$$
$$v(x,t) = \frac{-1}{4r_0}g^2(4C^2 - 4BC + B^2) + \frac{-2Cg^2(-2AC - 2C^2 + 3BC + AB - B^2)}{2r_0(2C - B + \sqrt{4AC - B^2}\tan(\frac{1}{2}(\sqrt{4AC - B^2}(\eta + d_0))))},$$
$$w(x,t) = r_0 + \frac{4Cr_0(A + C - B)}{(-2C + B)(2C + -B + \sqrt{4AC - B^2}\tan(\frac{1}{2}(\sqrt{4AC - B^2}(\eta + d_0))))},$$

(15)

where $\eta = \frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}$.

Case (2)

$$\begin{aligned} a_1 &= gC, b_1 = \frac{g^2C^2}{r_1}, B = 2C, r_1 \text{ is arbitrary} \\ r_0 &= b_0 = a_2 = b_2 = r_2 = A = 0. \end{aligned} \tag{16}$$

Substituting these results into (11) and with the aid of families 1-4, we obtain the following multiple soliton-like and periodic solutions for space-time fractional CID system

$$\begin{aligned} u(x,t) &= a_0 - \frac{gC(C\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}) + Bd_0) - 1)}{C\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}) + Bd_0) + 1}, \\ v(x,t) &= b_0 - \frac{g^2C^2(C\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}) + Bd_0) - 1)}{r_1(C\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}) + Bd_0) + 1)}, \end{aligned}$$

$$w(x,t) = r_0 - \frac{r1(C\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}) + Bd_0) - 1)}{C\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}) + Bd_0) + 1}. \tag{17}$$

Case (3)

$$a_1 = g(B-A), b_0 = \frac{-g^2B^2}{4r_0}, r_1 = \frac{-2r_0(A-B)}{B}, a_0 \text{ is arbitrary}$$
$$b_1 = \frac{-g^2B(A-B)}{2r_0}, a_2 = b_2 = r_2 = 0 \tag{18}$$

Substituting these results into (11) and with the aid of families 1-4, we obtain the following multiple soliton-like and periodic solutions for space-time fractional CID system

$$u(x,t) = a_0 + \frac{g(B-A)}{1 - \frac{A}{B} + \frac{1}{B}\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)}))},$$
$$v(x,t) = \frac{-g^2B^2}{4r_0} - \frac{g^2B(A-B)}{2r_0(1 - \frac{A}{B} + \frac{1}{B}\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)})))},$$
$$w(x,t) = r_0 - \frac{-2r_0(A-B)}{B(1 - \frac{A}{B} + \frac{1}{B}\exp(B(\frac{hx^\alpha}{\Gamma(1+\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)})))}. \tag{19}$$

5. Discussion

In this section, we discuss the physical explanations of the obtained solutions. Note that, the plots of the solutions (15), (17), and (19) are presented in figures 1 to 6 at specific values of the free constants. It appears that the solutions of (15), (17) and (19) depend on the sign of the magnitude $4\mathrm{AC} - \mathrm{B}^2$. In the case of $4\mathrm{AC} - \mathrm{B}^2 > 0$, the solution (15) is expressed in terms of the trigonometric tan function and hence an anti-kink wave is produced as shown by Fig. 7.1(a). Similarly the solution (19) in Fig. 7.3(a). On the other side, $4\mathrm{AC} - \mathrm{B}^2 < 0$, the solution (17) can be expressed in terms of the hyperbolic tan function and accordingly a kink wave is resulted as displayed in Fig. 7.2(b).

On the other hand, the solutions are affected by the fractional derivatives α, in Fig. 7.1(b) the anti-kink wave increases with increasing of α but the reverse effect is observed a little near off the plate ($x > 0.5$). Similarly the solution (19) in Fig. 7.1(b). Finally, Fig. 7.2(b) describes the u-solution in (17) , the kink wave increases with increasing of α at $0 < x \leq 1$ and then it stabilizes with different value of α at $x > 1$. Accordingly, this method is capable of producing a different types of wave solutions for partial differential equations

6. Conclusion

In the present paper, GKM has been successfully employed to obtain the exact solution of space-time fractional coupled integrable dispersionless system. New travelling wave technique is applied to search for the exact solitary solutions. The main advantage of the proposed method over the others is the fact that it can be applied to a wide class of nonlinear evolution equations. The modified Kudryashov [16] is special case of this technique (family 3, take A=0 and B=1). Finally, the obtained results have been graphically presented to show the properties of the obtained solutions.

7. Acknowledgments

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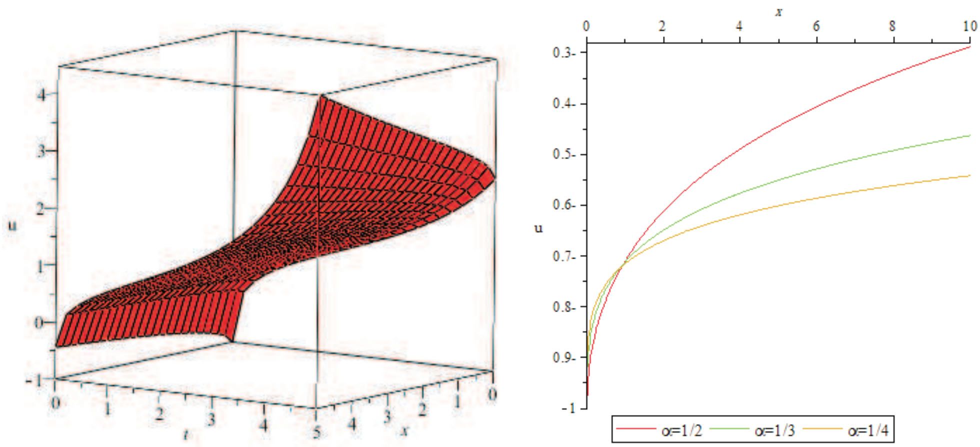
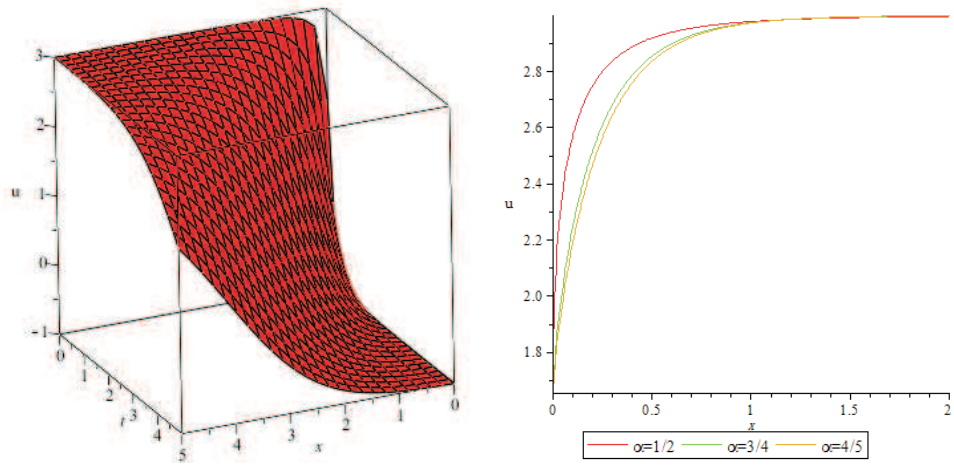


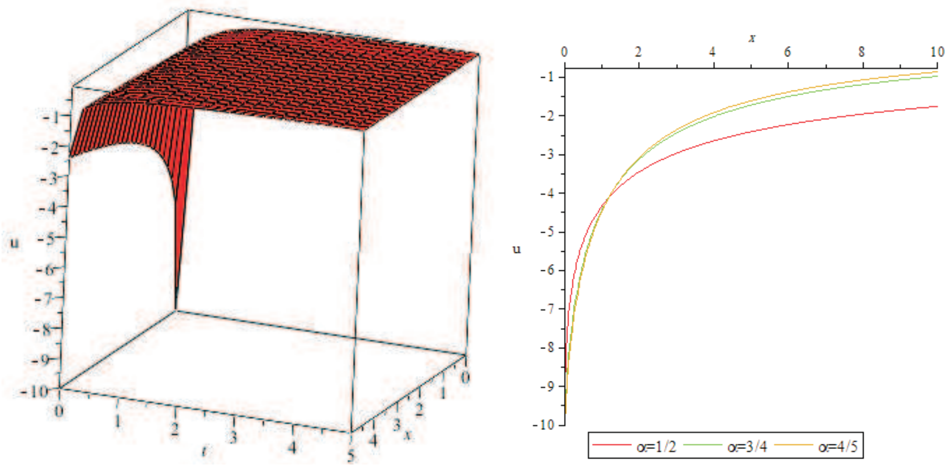
(a) (b)

Fig. 7.1: (a) Anti-Kink wave solution of Eq. (17) where $a_0 = a_1 = g = A = B = C = 1, d_0 = 0$, $h = 0.1$ and $\alpha = \frac{1}{2}$
(b) Anti-Kink wave solution of Eq. (17) where t=0, $a_0 = a_1 = g = A = B = C = 1$, $d_0 = 0$ and $h = 0.1$



(a) (b)

FIG. 7.2: (a) Kink wave solution of Eq. (15) where $a_0 = 1$, $C = 2$, g=-1, $d_0 = 0$, $h = 1$ and $\alpha = \frac{3}{4}$.
(b) Kink wave solution of Eq. (15) where t=0, $a_0 = 1$, $C = 1$, g=-1, $d_0 = 0$ and $h = 1$.



(a) (b)

FIG. 7.3: (a) (5) Anti-Kink wave solution of Eq. (19) where $a_0 = \mathrm{r}_0 = 1$, b_1 =-0.1, $g = B = 1$, $h = 0.1$ and $\alpha = \frac{1}{2}$.
(b) Anti-Kink wave solution of Eq. (19) where $a_0 = \mathrm{r}_0 = 1$, b_1 =-0.1, $g = B = 1$, $h = 0.1$

REFERENCES

1. Zheng Bin, (G′/G)-Expansion Method for Solving Fractional Partial Differential Equations in the Theory of Mathematical Physics, Commun. Theor. Phys. 58 (2012) 623–630.
2. Mohd. Farman Ali, Manoj Sharma, Renu Jain, An Application of Fractional Calculus in Electrical Engineering, Adv. Eng. Tec. Appl. 5 (2016), 41-45.
3. N. H. Abel, Solution De Quelques Problems a L'aide D'integrals Definites. Oeuvres Completes, Grondahl Christiania, Norway,1 (1881),16-18.
4. Mehdi Dalir, Majid Bashour, Applications of Fractional Calculus, App. Math. Sci., (4) (2010), 1021 - 1032.
5. Yokus A, Durur H, Ahmad H. Hyperbolic type solutions for the couple Boiti-Leon-Pempinelli system. Facta Universitatis, Series: Mathematics and Informatics. 2020;35(2):523-31.
6. Yokus A, Durur H, Ahmad H, Yao SW. Construction of Different Types Analytic Solutions for the Zhiber-Shabat Equation. Mathematics. 2020;8(6):908.
7. Ahmad H, Seadawy AR, Khan TA, Thounthong P. Analytic approximate solutions for some nonlinear Parabolic dynamical wave equations. Journal of Taibah University for Science. 2020;14(1):346-58.
8. Ahmad H, Khan TA, Stanimirovic PS, Ahmad I. Modified Variational Iteration Technique for the Numerical Solution of Fifth Order KdV Type Equations. Journal of Applied and Computational Mechanics. 2020;6(SI):1220-1227
9. Kamran Ayub, M. Yaqub Khan, Qazi Mahmood-Ul-Hassan, Solitary and periodic wave solutions of Calogero–Bogoyavlenskii–Schiff equation via exp-function methods, Computers & Math. App., 74 (2017), 3231-3241.
10. Ahmad, H., Khan, T. A., Ahmad, I., Stanimirović, P. S., Chu, Y.-M., A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations, Results in Physics, 2020, 103462. doi: https://doi.org/10.1016/j.rinp.2020.103462.
11. Ahmad, H., Akgül, A., Khan, T. A., Stanimirović, P. S., Chu, Y.-M., New Perspective on the Conventional Solutions of the Nonlinear Time-Fractional Partial Differential Equations, Complexity, 2020,2020, 8829017. doi:10.1155/2020/8829017.
12. A. A. Gaber, Symmetry analysis and Solitary wave solutions of nonlinear ion-acoustic waves equation. Int. J. Ana. App. 18 (2020) 448-460.
13. Akgül A. and Ahmad H., Reproducing Kernel Method for Fangzhuâ€™s Oscillator for Water Collection from Air, Mathematical Methods in the Applied Sciences. 2020, DOI:10.1002/mma.6853
14. Arzu Akbulut, Melike Kaplan, Auxiliary equation method for time-fractional differential equations with conformable derivative, Comput. & Math. with Appl., 75 (2018), 876-882.
15. Ahmad, H., Khan, T. A., Stanimirović, P. S., Chu, Y.-M., Ahmad, I., Modified Variational Iteration Algorithm-II: Convergence and Applications to Diffusion Models, Complexity, 2020, 8841718. doi:10.1155/2020/8841718.
16. Ahmad H, Seadawy AR, Khan TA. Study on numerical solution of dispersive water wave phenomena by using a reliable modification of variational iteration algorithm. Mathematics and Computers in Simulation. 2020;177:13-23

17. A. A. Gaber and et., The generalized version of Kudryashov method for nonlinear space-time fractional partial differential equations of Burgers type, Nonlinear Dynamics, Nonlinear Dyn. 95 (2019), 361-368.
18. K. Hosseini, A. Bekir, R. Ansari, New exact solutions of the conformable time-fractional Cahn–Allen and Cahn–Hilliard equations using the modified Kudryashov method, Optik,132 (2017), 203-209.
19. Dipankar Kumar, Aly R. Seadawy, Atish Kumar Joardar, Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology, Chinese Journal of Physics, 56 (2018), 75-85.
20. Abdelhalim Ebaid, Doaa M. M. ElSayed, Fractional Calculus Model for Damped Mathieu Equation Approximate Analytical Solution, App. Math. Sci., 6 (2012), 4075 - 4080.
21. Amit Prakash, Manoj Kumar, Kapil K. Sharma, Numerical method for solving fractional coupled Burgers equations, App. Math. Comp. 260 (2015) 314–320
22. R. Sahadevan, P. Prakash, On Lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations, Chaos, Solitons &Fractals 104 (2017) 107–120.
23. Qing Huang, Renat Zhdanov, Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann–Liouville derivative, Phy. A 409 (2014) 110–118.
24. Ozkan Guner, Esin Aksoy, Ahmet Bekir, Adem C. Cevikel,Different methods for (3 + 1)-dimensional space–time fractional modified KdV–Zakharov–Kuznetsov equation, Comp. & Math. with Applications, 71(2016), 1259-1269.
25. I. Naeem, M.D. Khan, Symmetry classification of time-fractional diffusion equation, Commun Nonlinear Sci Numer Simulat 42 (2017) 560–570.
26. S. Sahoo, S. Saha Ray, Invariant analysis with conservation laws for the time fractional Drinfeld–Sokolov–Satsuma–Hirota equations,Chaos, Solitons and Fractals 104 (2017) 725–733.
27. H. Kakuhata and K. Konno, Loop Soliton Solutions of String Interacting with External Field, J. Phys. Soc. Jpn. 68 (1999) 757.
28. WANG Pan, TIAN Bo, LIU Wen-Jun , QU Qi-Xing, and JIANG Yan, Conservation Laws and Analytic Soliton Solutions for Coupled Integrable Dispersionless Equations with Symbolic Computation.
29. K. Konno and H. Oono, New Coupled Integrable Dispersionless Equations, J. Phys. Soc. Jpn. 63 (1994) 377.
30. DAI Chao-Qing, YANG Qin, and WANG Yue-Yue, New Exact Solutions of (1 + 1)-Dimensional Coupled Integrable Dispersionless System, Commun. Theor. Phys. 55 (2011) 622–628.
31. A.Y.T. Leung, H.X. Yang, Z.J. Guo, Periodic wave solutions of coupled integrable dispersionless equations by residue harmonic balance, Commun Nonlinear Sci Numer Simulat 17 (2012) 4508–4514.
32. A. Ebaid, An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms, J. Math. Anal. Appl. 392 (2012) 1-5.

Ahmed A. Gaber
College of Science and Humanities at Howtat Sudair
Department of Mathematics, Majmaah University,
Majmaah 11952, Saudi Arabia.
Faculty of Education, Department of Mathematics,
Ain Shams University, Heliopolis, Roxy, Egypt.
a.gaber@mu.edu.sa and aagaber6@gmail.com

Hijaz Ahmad
Section of Mathematics, International Telematic University Uninettuno,
Corso Vittorio Emanuele II, 39, 00186 Roma, Italy.
Department of Basic Sciences, University of Engineering and Technology
Peshawar, 25000, Pakistan
hijaz555@gmail.com

FACTA UNIVERSITATIS (NIŠ)
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AN IDENTITY-BASED ENCRYPTION SCHEME USING ISOGENY OF ELLIPTIC CURVES

Mojtaba Bahramian* and Elham Hajirezaei

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Abstract. Identity-Based Encryption is a public-key cryptosystem that uses the receiver identifier information such as email address, IP address, name, etc, to compute a public and a private key in a cryptosystem and encrypt a message. A message receiver can obtain the secret key corresponding with his privacy information from private key generator and he can decrypt the ciphertext. In this paper, we review Boneh-Franklin's scheme and use a bilinear map and Weil pairing's properties to propose an identity-based cryptography scheme based on isogeny of elliptic curves.
Keywords: Identity-based encryption; elliptic curves; isogeny of elliptic curves.

1. Introduction

Public key encryption (PKE), involves two distinct keys, public key, and private key. The public key can be widely distributed without compromising its corresponding private key. Identity-Based Encryption (IBE) is a public-key encryption scheme in which the public key can be an arbitrary string. Identity-based encryption is a cryptographic scheme, which enables any pair of users to communicate securely without exchanging secret or public keys. Actually by the identity-based scheme, if you know somebody's name or email address you can send him a message which only he can read. This issue has now been particularly attended by cryptographic researchers and so far, many cryptography schemes are based on it has been presented.

The basic identity scheme was first proposed by Shamir [11] in 1984. The scheme is specified by four phases:

1. **Setup:** In this phase, general system parameters and master-key are created.
2. **Extraction:** In this algorithm, the private key associated with an arbitrary public key string $ID \in \{0,1\}^*$ is created by using the master-key.

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3. **Encryption:** A message is encrypted using the public key ID.

4. **Decryption:** An encrypted message is decrypted having the corresponding private key.

When the sender, Alice, sends an e-mail to the receiver, Bob, at bob@email.com, she simply encrypts her message having the public key string "bob@email.com". In this method, we need a trusted third party known as "Private Key Generator" (PKG), which computes a master private key and a public key. The PKG has a privileged position by knowing some secret information that enables it to compute the private keys for all the users in the system. Thus, when Bob receives the encrypted message by his e-mail, he contacts to the PKG, authenticates himself to it in the same way, then he obtains his private key from the PKG, and he can read his e-mail [1,6]. The problem of constructing an IBE was an open problem for many years. Finally, Boneh and Franklin [1] proposed an IBE scheme using bilinear maps in 2001. Soon after Boneh and Franklin's announcement, it was detected that Clifford Cocks, had designed a simple IBE years earlier.

Boneh and Franklin presented a functional IBE scheme in which the performance of their approach is similar to the performance of ElGamal encryption in $\mathbb{F}_q^*$, and the security of their scheme is based on the Computational Diffie-Hellman (CDH) hypothesis on elliptic curves.

In this paper, we propose an identity-based encryption scheme based on the isogenies between elliptic curves. The security of our scheme is based on the hardness of the isogeny problem that is finding an isogeny between two given isogenous elliptic curves. In our proposed scheme we use the endomorphism ring of an ordinary elliptic curve E, (End(E)), and some its properties such as the commutativity of $End(E)$.

Basic Concepts of IBE. As mentioned earlier, in the IBE scheme Alice can use the receiver's identifier information which is presented by any string, such as email address or IP address, even a digital image [10], to encrypt a message. Bob obtains a private key corresponding to his identifier information from the trusted third party, then he can decrypt the ciphertext (Fig. 1.1).

Universally an identity-based encryption scheme is specified by four randomized algorithms:

1. **Setup:** First, the PKG creates a public key pk_{PKG} and a master private key sk_{PKG}, then he publishes pk_{PKG} as a public key.

2. **Extraction:** Bob authenticates himself to the PKG and receives his private key sk_{Bob} corresponding to his identity, ID_{Bob}.

3. **Encryption:** Alice encrypts her message, M to the ciphertext C using ID_{Bob} and pk_{PKG}.

4. **Decryption:** Bob decrypts the ciphertext C, using his private key, sk_{Bob} and reconstruct the message M.

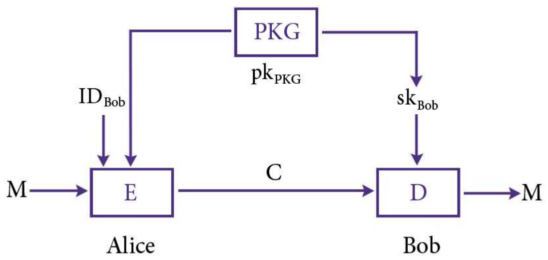

FIG. 1.1: Identity-based encryption scheme

The rest of the paper is organized as follows: Section 2 contains a summary of some preliminaries on elliptic curves, isogenies, and basic properties of the Weil pairing. In section 3, we give a review of Boneh and Franklin's IBE scheme. Our proposed identity-based encryption scheme is given in Section 4. Finally, we dedicate the security analysis of our scheme in Section 5.

2. Preliminaries

In this section, we first briefly introduce elliptic curves, isogenies and Weil pairing (see [12, 15]).

2.1. Elliptic Curves

Elliptic Curve Cryptography (ECC) was introduced by Koblitz [5] and Miller [8] in 1985. They proposed completely different cryptographic use of elliptic curves. The main reason for the attractiveness of ECC is the fact that there is no sub-exponential algorithm known for solving the Discrete Logarithm Problem (DLP) on a properly chosen elliptic curve. We will refer to it later.

Definition 2.1. Let K be a field of characteristic not equal to 2 and 3. An elliptic curve E over K is a curve given by a (short) Weierstrass equation of the form

$$y^2 = x^3 + Ax + B \tag{2.1}$$

where $A, B \in \overline{K}$, and its discriminant, $\Delta = -16(4A^3 + 27B^2)$ is nonzero. The j-invariant of the elliptic curve E is defined by

$$j = j(E) = 1728\frac{4A^3}{4A^3 + 27B^2}$$

furthermore, any elliptic curve E can be determined by its j-invariant. In other words, two elliptic curves with the same j-invariant are isomorphic over K.

We say that the elliptic curve $E : y^2 = x^3 + Ax + B$ is defined over K, where $A, B \in K$. For the elliptic curve E defined over K, the set of K-rational points of E is defined by

$$E(K) = \{(x, y) \in K^2 : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\},$$

where, $\mathcal{O}$ is the point at infinity.

The set $E(K)$ forms an abelian additive group with identity element $\mathcal{O}$. Let $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ be two points on the curve. The sum of P and Q is defined as $R = P + Q = (x_R, y_R)$ where,

1. If $x_P \neq x_Q$, then $x_R = m^2 - x_P - x_Q$ and $y_R = m(x_P - x_R) - y_P$, where $m = (y_Q - y_P)/(x_Q - x_P)$.
2. If $x_P = x_Q$ and $y_P \neq y_Q$, then $R = \mathcal{O}$.
3. If $P = Q$ and $y_P \neq 0$, then $x_R = m^2 - 2x_P$ and $y_R = m(x_P - x_R) - y_P$, where, $m = (3x_P^2 + A)/2y_P$.
4. If $P = Q$ and $y_P = 0$, then $R = \mathcal{O}$.
5. If $Q = \mathcal{O}$, then $R = P$.

For the Weierstrass equation described by (2.1), if $P = (x, y)$, then $-P = (x, -y)$.

Suppose E is an elliptic curve defined over a field K and Let n be a positive integer, the n-torsion subgroup of E defined as follows

$$E[n] = \{P \in E(\overline{K}) \mid nP = \mathcal{O}\}.$$

If the characteristic of K does not divide n, or is zero, then $E[n] \cong \mathbb{Z}_n \times \mathbb{Z}_n$, and if the characteristic of K is $p > 0$, $n = p^r n'$ with $p \nmid n$, then $E[n] \cong \mathbb{Z}_{n'} \times \mathbb{Z}_n$ or $\mathbb{Z}_{n'} \times \mathbb{Z}_{n'}$. For the elliptic curve E defined over the finite field $\mathbb{F}_q$, $q = p^r$ for some prime p, we say that E is supersingular if $E[p] = \{\mathcal{O}\}$, and E is called ordinary if $E[P] \cong \mathbb{Z}_p$.

Let the elliptic curve E defined over the field $\mathbb{F}_q$. Then $E(\mathbb{F}_q) \cong Z_n$ for some integer $n \geq 1$, or $E(\mathbb{F}_q) \cong Z_{n_1} \times Z_{n_2}$ for some integers $n_1, n_2 \geq 1$ with n_1 dividing n_2. By Hasse's theorem, for elliptic curve E over the finite field $\mathbb{F}_q$, the order of E satisfies $\mid q + 1 - \#E(\mathbb{F}_q) \mid \leq 2\sqrt{q}$. The trace of the elliptic curve E denoted by a_q, is $a_q = q + 1 - \#E(Fq)$. The elliptic curve E is supersingular if and only if $a_q \equiv 0 \pmod{p}$, it means that $\#E(\mathbb{F}_q) \equiv 1 \pmod{p}$.

Discrete Logarithm Problem: Let E be an elliptic curve defined over the finite field $\mathbb{F}_q$, $P \in E$ and $Q \in \langle P \rangle$. The Elliptic Curve Discrete Logarithm Problem (ECDLP) is the problem of finding integer n such that $Q = nP$. It is Well-known that the fastest known algorithm to solve the ECDLP over an arbitrary curve is Pollard's rho method, which has exponential time complexity. [9].

2.2. Isogeny of Elliptic Curves

Definition 2.2. Let K be a field and let E_1 and E_2 be two elliptic curves defined over K. An isogeny is a non-constant morphism $\varphi : E_1(\overline{K}) \to E_2(\overline{K})$ satisfying $\varphi(\mathcal{O}_{E_1}) = \mathcal{O}_{E_2}$. The isogeny φ can be displayed by

$$\varphi : (x, y) \to \left(\frac{p(x)}{q(x)}, \frac{r(x)}{s(x)}y\right)$$

with polynomials $p(x)$, $q(x)$, $r(x)$ and $s(x)$ such that $p(x)$ and $q(x)$ do not have a common factor. The degree of isogeny φ denoted by $\deg(\varphi)$, is the maximum degree of the polynomials $p(x)$ and $q(x)$. Also, we define $\deg(\mathbf{0}) = 0$. The isogeny φ is called separable, if $\deg(\varphi) = \#ker(\varphi)$. We say that two elliptic curves E_1 and E_2 are l-isogenous when there exists a nonzero isogeny of degree l from E_1 to E_2. If $\varphi : E_1 \to E_2$ is an isogeny of degree l, then the dual of φ denoted by $\hat{\varphi}$, is a unique isogeny from E_2 to E_1 of the same degree l, such that $\hat{\varphi} \circ \varphi = [l]_{E_1}$, the multiplication by l map on E_1 and also, $\varphi \circ \hat{\varphi} = [l]_{E_2}$. By Tate's theorem [9], two elliptic curves E_1 and E_2 are isogenous over the finite field $\mathbb{F}_q$, if and only if $\#E_1(\mathbb{F}_q) = \#E_2(\mathbb{F}_q)$. We denote the set of isogenies from E_1 to E_2 by $Hom(E_1, E_2)$. The sum of two isogenies φ and ψ is defined by $(\varphi + \psi)(P) = \varphi(P) + \psi(P)$, for each $P \in E$. It implies that $\varphi + \psi$ is an isogeny, and thus $Hom(E_1, E_2)$ is a group. If $E_1 = E_2$, then we can also compose isogenies. If E is an elliptic curve, we let $End(E) = Hom(E, E)$ be the ring whose addition law is as given above and whose multiplication is composition, $(\varphi\psi)(P) = \varphi(\psi(P))$. The ring $End(E)$ is called the endomorphism ring of E. The Frobenius endomorphism τ_q is defined by $\tau_q(x, y) = (x^q, y^q)$. It is an endomorphism of E (see [15]).

2.3. Bilinear Map

Let G_1 be an additive group of order r and G_2 be a multiplicative group of the same order. A function $e : G1 \times G1 \to G_2$ is said to be a bilinear pairing if the following properties hold

1. **Bilinearity**: for all $P, Q \in G_1$ and $a, b \in \mathbb{Z}_r^*$, $e(aP, bQ) = e(P, Q)^{ab}$.
2. **Non-degeneracy**: there exist $P, Q \in G_1$ such that $e(P, Q) \neq 1$.
3. **Computability**: for all $P, Q \in G_1$, there exists an efficient algorithm to compute $e(P, Q)$.

As we will say in section 2.4, the example of an efficiently computable non-degenerate the bilinear map is the Weil pairing.

2.4. Weil Pairing

As already mentioned if E be an elliptic curve over a field K and let n be an integer not divisible by the characteristic of K, Then $E[n] \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Let

$$\mu_n = \{x \in \overline{K} | x^n = 1\},$$

be the group of n-th roots of unity in K.Since n is not divided by the characteristic of K, the equation $x^n = 1$ has no multiple roots so, it has n distinct roots in $\overline{K}$, Therefore, μ_n is a cyclic group of order n. Any generator γ of μ_n is called a primitive nth root of unity. This is equivalent to saying that $\gamma^k = 1$ if and only if k divided by n.

Definition 2.3. Let E be an elliptic curve over a field K and let n be a positive integer not divisible by the characteristic of K. Then there is a pairing

$$e_n = E[n] \times E[n] \to \mu_n \tag{2.2}$$

called the Weil Pairing. This concept satisfies the following properties:

1. e_n is bilinear in each variable. This means that

   $$e_n(S_1 + S_2, T) = e_n(S_1, T)e_n(S_2, T) \tag{2.3}$$

   and

   $$e_n(S, T_1 + T_2) = e_n(S, T_1)e_n(S, T_2) \tag{2.4}$$

   for all $S, S_1, S_2, T, T_1, T_2 \in E[n]$.

2. e_n is nondegenerate in each variable. This means that if $e_n(S, T) = 1$ for all $T \in E[n]$ then $S = \infty$ and also that if $e_n(S, T) = 1$ for all $S \in E[n]$ then $T = \infty$.

3. $e_n(T, T) = 1$ for all $T \in E[n]$.

4. $e_n(S, T) = e_n(T, S)^{-1}$ for all $S, T \in E[n]$.

5. $e_n(\sigma(S), \sigma(T)) = \sigma(e_n(S, T))$. For all automorphism σ of $\overline{K}$ such that σ is the identity map on the coefficient of E (if E is in Weiratrass form, this means that $\sigma(A) = A$ and $\sigma(B) = B$).

6. $e_n(\alpha(S), \alpha(T) = e_n(S, T)^{\deg(\alpha)}$ for all separable endomorphisms α of E.

If the coefficient of E lie in a finite field $\mathbb{F}_q$, the statement also holds when α is the Frobenius endomorphism τ_Q. (Actually, the statement holds for all endomorphism α, separable or not.)

Now we say that the isogenies φ and $\hat{\varphi}$ are dual (or adjoint) concerning the Weil pairing. Let $\varphi : E_1 \to E_2$ be an isogeny of elliptic curves and let $\hat{\varphi}$ be its dual, and let e_n be a Weil pairing. Then $e_n(\varphi(S), T) = e_n(S, \hat{\varphi}(T))$ for all n-torsion points $S \in E_1[n]$ and $T \in E_2[n]$ (see [7]).

3. Boneh-Franklin Scheme

Boneh and Franklin's Scheme can be built from any bilinear map $\hat{e} : G_1 \times G_1 \to G_2$ between two groups G_1 and G_2 as long as a variant of the computational Diffie-Hellman problem in G_1 is hard. They use the Weil pairing on elliptic curves as an example of such a map. They describe the scheme in four phases:

1. **Setup:** The PKG specifies an elliptic curve E over $\mathbb{F}_p$. It Chooses an arbitrary $P \in E/\mathbb{F}_p$ of order q. The PKG also specifies two hash functions $H_1 : \mathbb{F}_{p^2} \to \{0,1\}^n$ and $H_2 : \{0,1\}^* \to \mathbb{F}_p$. The PKG picks a random $s \in Z_q^*$ as a master key and denoted it by pk_{PKG}. Then it computes a public key $pk_{PKG} = sP$. The PKG publishes $\{E, \mathbb{F}, P, H_1, H_2, pk_{PKG}\}$.
2. **Extraction:** Bob contacts the PKG to get his private key. The PKG first maps, Bob's identity, $ID_{Bob} \in \{0,1\}^*$ to a point $Q_{ID} \in E/\mathbb{F}_p$ of order q, then it computes $sk_{Bob} = sQ_{ID}$ where $Q_{ID} = H_1(ID)$ and s is the master key.
3. **Encryption:** Alice encrypt her message $M \in \{0,1\}^l$ (where l denotes the length of M). under the public key, pk_{PKG} and ID_{Bob} which is mapped to a point $Q_{ID} \in E/\mathbb{F}_p$ of order q. She computes $U = rP$ and $V = H_2(\hat{e}(Q_{ID}, pk_{PKG})^r) \oplus M$, where r is chosen at random from Z_q and $Q_{ID} = H_1(ID)$. The resulting ciphertext $C = (U, V)$ is sent to Bob.
4. **Decryption:** Bob receives the ciphertext C, and checks it. If $U \in E/\mathbb{F}_p$ is not a point of order q rejects the ciphertext. Otherwise, to decrypt C using his private key, sk_{Bob} and computes:

   $$V \oplus H_2(\hat{e}(sk_{Bob}, U)) = M \tag{3.1}$$

   This completes the description as follows:

   $$\begin{aligned} \hat{e}(sk_{Bob}, U) &= \hat{e}(sQ_{ID}, rP) \\ &= \hat{e}(Q_{ID}, P)^{sr} \\ &= \hat{e}(Q_{ID}, pk_{PKG})^r \end{aligned}$$

   Thus, applying decryption after encryption produces the original message M as required.

4. Proposed Scheme

This section details our newly proposed identity-based encryption using isogeny of elliptic curves.

Let $\mathbb{F}_q$ be the field of order q, where q is a power of a prime number p and n be a positive integer coprime to p. Let E be an ordinary elliptic curve over $\mathbb{F}_q$, and let

$e_n : E[n] \times E[n] \to \mu_n$ be the Weil e_n-pairing. In our scheme, we use an algorithm $\mathcal{A}$ to convert a string $ID_{Bob} \in \{0,1\}^*$ to a point $Q_{ID} \in E$ of order n.

The phases in the proposed scheme are Setup phase, Extraction phase, Encryption phase and Decryption phase. The procedure of our scheme is described in detail as follows:

1. **Setup:** The PKG randomly chooses an isogeny $\varphi \in End(E)$ as its master key and maps $ID_{Bob} \in \{0,1\}^*$ to a point $Q_{ID} \in E[n]$ by using algorithm $\mathcal{A}$. The PKG computes a public key as follows:

$$pk_{PKG} = \varphi(Q_{ID}),$$

   and publishes $\{E, q, \varphi(Q_{ID})\}$.

2. **Extraction:** The PKG computes Bob's private key $sk_{Bob} = \hat{\varphi}(Q_{ID})$, and sends it to Bob.

3. **Encryption:** Alice encrypts the message M using Bob's public key, ID_{Bob}, by performing the following steps:

   a) She uses algorithm $\mathcal{A}$ to map ID_{Bob} into the point $Q_{ID} \in E[n]$.

   b) She chooses an isogeny $\psi \in End(E)$.

   c) She sets the ciphertext to be $C = (u, v)$, where

$$u = \psi(Q_{ID}),\ v = e_n(\varphi(Q_{ID}), \hat{\psi}(Q_{ID})) + M,$$

   then she sends $C = (u, v)$ to Bob.

4. **Decryption:** Upon receiving $C = (u, v)$, Bob computes

$$\begin{aligned} e_n(u, sk_{Bob}) &= e_n(\psi(Q_{ID}), \hat{\varphi}(Q_{ID})) \\ &= e_n(Q_{ID}, \hat{\psi}(\hat{\varphi}Q_{ID})) \\ &= e_n(Q_{ID}, \hat{\varphi}(\hat{\psi}(Q_{ID})) \\ &= e_n(\varphi(Q_{ID}), \hat{\psi}(Q_{ID})), \end{aligned}$$

   and extracts the original message $M = v - e_n(\psi(Q_{ID}), \hat{\varphi}(Q_{ID}))$ as required.

5. Security analysis

In this section, we analyze the security of our proposed scheme, which is based on the hardness of some isogeny problems as stated in the following.

Problem 1 (Isogeny Problem): For two given isogenous elliptic curves E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$.

Problem 2 (Isogeny Logarithm Problem): Let E_1 and E_2 be two isogenous elliptic curves, $P \in E_1$ and $Q \in E_2$. Find an isogeny $\varphi : E_1 \to E_2$ such that $Q = \varphi(P)$.

Problem 1 is a hard problem that has been studied by many researchers [2, 3, 4, 6, 13]. The hardness of this problem over ordinary curves is as hard as the discrete logarithm problem, so its security is at the same level. Problem 2 is even harder than problem 1 because it must satisfy the extra term $Q = \varphi(P)$.

Generally, as mentioned earlier, there is no efficient algorithm to find an isogeny between two elliptic curves and it seems hard to determine the structure of $Hom(E_1, E_2)$ and also $End(E)$. Furthermore according to isogeny logarithm problem there is no efficient algorithm to find an isogeny φ by having P and $Q = \varphi(P)$.

Forward secrecy: Recall that in our proposed scheme, the public parameters are $\{E, \mathbb{F}_q, \varphi(Q_{ID})\}$. Suppose Eve (the adversary) knows $pk_{PKG} = \varphi(Q_{ID})$. To extract a message M, he must compute $e(\varphi(Q_{ID}), \hat{\psi}(Q_{ID}))$. But having E, he could get no knowledge of isogeny $\varphi \in End(E)$. Without the knowledge, this is exactly an isogeny problem that Eve is not able to solve, hence he cannot compute $e(\varphi(Q_{ID}), \hat{\psi}(Q_{ID}))$.

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REFERENCES

1. D. Boneh, M. Franklin, Identity-Based Encryption from the Weil Pairing, Proceedings of CRYPTO 2001, LNCS 2139, Springer-Verlag, 2001, 213-229.

2. A. Childs, D. Jao, V. Soukharev, Constructing elliptic curve isogenies in quantum subexponential time, J. Math. Cryptol. 8134 (2014), no. 1, 1-29.

3. S. D. Galbraith, Constructing isogenies between elliptic curves over finite fields, LMS J. Comput. Math. 2 (1999) 118-138.

4. S. D. Galbraith, A. Stolbunov, Improved algorithm for the isogeny problem for ordinary elliptic curves, Appl. Algebra Engrg. Comm. Comput. 24 (2013), no. 2, 107-131.

5. N. Koblitz, Elliptic curve cryptosystems, Mathematics of computation, 48(177), (1987), 203-209.

6. D. Kohel, Endomorphism rings of elliptic curves over finite fields, PhD Thesis, University of California at Berkeley, 1996.

7. A. J. Menezes, S. A. Vanstone, Elliptic curve cryptosystems and their implementation. Journal of Cryptology, 6(4): (1993), 209-224.

8. V. S. Miller, Use of elliptic curves in cryptography, In Advances in Cryptology CRYPTO85 Proceedings, Springer (1986), 417-426.

9. J.M. Pollard, Monte Carlo Methods for index Computation (mod p). Math. Comp. 32(143) (1978) 918–924.

10. A. Sahai, B. Waters, Fuzzy Identity Based Encryption, IACR ePrint Archive, Report 2004/086. (http://eprint.iacr.org/).

11. A. Shamir, Identity-based Cryptosystems and Signature Schemes, Proceedings of CRYPTO '84, LNCS 196, Springer-Verlag (1984), 47-53.

12. J. H. Silverman, The arithmetic of elliptic curves, Vol. 106 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1986.

13. A. Stolbunov, Constructing public-key cryptographic schemes based on class group action on a set of isogenous elliptic curves, Adv. Math. Commun. 4 (2010), no. 2, 215-235.

14. J. Tate, Endomorphisms of abelian varieties over finite fields, Invent. Math 2 (1966) 134-144.

15. L.C. Washington, Elliptic Curves, Number Theory and Cryptography, CRC Press, 2nd edition, 2008.

M. Bahramian
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, I. R. Iran
e-mail: bahramianh@kashanu.ac.ir

E. Hajirezaei
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, I. R. Iran
e-mail: elhamhajrezaei021@gmail.com

FACTA UNIVERSITATIS (NIŠ)
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NEW SUBCLASS OF MEROMORPHIC FUNCTIONS BY THE GENERALIZATION OF THE q-DERIVATIVE OPERATOR

Mohammad Hassn Golmohammadi, Shahram Najafzadeh and Mohammad Reza Foroutan

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Abstract. In this paper, we introduce a new subclass of meromorphic functions, using the exponent q-derivative operator. Afterwards, coefficient estimates, extreme points, convex linear combination, radii of starlikeness and convexity and finally partial sum property have been investigated.

Keywords: Meromorphic functions; q-derivative; coefficient bound; extreme point; convex set; Hadamard product.

1. Introduction

Fractional calculus have started to appear more and more frequently for the modelling of relevant systems in several fields of applied sciences. For more details, one may refer to the books [6, 7, 9] and the recent papers on the subject. The theory of q-analysis has attracted a considerable effort of researches due to its application in many branches of mathematics and physics and q-theory has an important role in various branches of mathematics and physics as for example, in the areas of special functions, ordinary fractional calculus, optimal control problems, q-difference, q-integral equations, q-transform analysis and in quantum physics (see for instance, [1, 2, 3, 4, 5, 8, 10, 16]).

The theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus as the q-integral and q-derivative have been used to construct several subclasses of analytic functions (see, for example, [12, 13, 14, 15, 17]).

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k z^{k-1}, \tag{1.1}$$

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which are analytic in the punctured unit disk

$$\triangle^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\}.$$

Gasper and Rahman [7] defined the q- derivative of a function $f(z)$ of the form equation 1.1 by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}. \tag{1.2}$$

where $z \in \triangle^*$ and $0 < q < 1$.
Therefore, the q- derivative of $f(z) = z^{k-1}$ is given by

$$D_q z^{k-1} = \frac{(zq)^{k-1} - z^{k-1}}{(q-1)z} = [k-1]_q z^{k-2}$$

Our aim in this paper is to introduce a new operator and a new class of functions given by equation 1.1. So we have

$$D_q^n f(z) = \frac{(-1)^n \prod_{k=1}^n (1+q+q^2+\cdots+q^{k-1})}{q^{\binom{n+1}{n-1}} z^{n+1}} + \sum_{k=1}^{\infty} \prod_{i=1}^{n} [k-i]_q a_k z^{k-n-1} \tag{1.3}$$

$$(z \in \triangle^*, n \in \mathbb{N} = \{1, 2, \cdots\})$$

where

$$\prod_{k=1}^{n} (1+q+q^2+\cdots+q^{k-1}) = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{k-1}) \tag{1.4}$$

and

$$\prod_{i=1}^{n} [k-i]_q := \Big(\frac{1-q^{k-1}}{1-q}\Big)\Big(\frac{1-q^{k-2}}{1-q}\Big)\cdots\Big(\frac{1-q^{k-n}}{1-q}\Big). \tag{1.5}$$

also $\prod_{i=1}^n [k-i]_q \to \prod_{i=1}^n (k-i)$ as $q \to \overline{1}$. So we conclude

$$\lim_{q \to \overline{1}} D_q^n f(z) = f^{(n)}(z) \quad , \quad z \in \triangle^*,$$

see also [11].
For $n \in \mathbb{N}$, $0 < q < 1, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1$ and $\beta > 0$, let $\sum_q(n; \lambda, \alpha, \beta)$ be the subclass of $\sum$ consisting of functions f of the form equation 1.1 and satisfying the condition

(1.6)

$$\left| \frac{z^{n+3}\left(D_q^n f(z)\right)'' + z^{n+2}\left(D_q^n f(z)\right)' - \frac{(-1)^n (n+1)^2}{q^{\binom{n+1}{n-1}}} \prod_{k=1}^n (1+q+\cdots+q^{k-1})}{\lambda z^{n+1}\left(D_q^n f(z)\right) + \frac{(-1)^n \prod_{k=1}^n (1+q+\cdots+q^{k-1})}{q^{\binom{n+1}{n-1}}} + \frac{(1+\lambda)\alpha}{q^{\binom{n+1}{n-1}}}} \right|$$

$< \beta$.

We also derive some results given various coefficient inequalities, Radii condition and Hadamard product.

2. Main Results

Unless otherwise mentioned, we suppose throughout this paper that $n \in \mathbb{N}$, $0 < q < 1, 0 \leq \lambda < 1, 0 < \alpha < 1$ and $\beta > 0$. First we state coefficient estimates on the class $\sum_q(n;\lambda,\alpha,\beta)$.

Theorem 2.1. *Let $f(z) \in \sum$, then $f(z) \in \sum_q(n;\lambda,\alpha,\beta)$ is and only if*

(2.1)

$$\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)a_k \leq \frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}.$$

Proof. Let $f(z) \in \sum_q(n;\lambda,\alpha,\beta)$, then equation 1.6 holds true. So by replacing equation 1.3 in equation 1.6 we have

$$\left|\frac{\sum_{k=1}^{+\infty}\left(\prod_{i=1}^{n}[k-i]_q(k-n-1)(k-n-2)+\prod_{i=1}^{n}[k-i]_q(k-n-1)\right)a_kz^k)}{\frac{(1+\lambda)}{q^{\binom{n+1}{n-1}}}(-1)^n\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})+\lambda\sum_{k=1}^{\infty}\prod_{i=1}^{n}[k-i]_qa_kz^k+\frac{(1+\lambda)\alpha}{q^{\binom{n+1}{n-1}}}}\right| < \beta.$$

or

$$\left|\frac{\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q(k-n-1)^2a_kz^k}{\frac{(1+\lambda)}{q^{\binom{n+1}{n-1}}}\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)-\lambda\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_qa_zz^k}\right| < \beta.$$

Since $Re(z) \leq |z|$ for all z, therefore

$$Re\left\{\frac{\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q(k-n-1)^2a_kz^k}{\frac{(1+\lambda)}{q^{\binom{n+1}{n-1}}}\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)-\lambda\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_qa_zz^k}\right\} < \beta.$$

By letting $z \to \overline{1}$ through real values, we have

$$\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)a_k \leq \frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}.$$

Conversely, Let equation 2.1 holds true, by equation 1.6 it is enough to show that

$$X(f) = \left|\frac{z^{n+3}\left(D_q^n f(z)\right)'' + z^{n+2}\left(D_q^n f(z)\right)' - \frac{(-1)^n(n+1)^2}{q^{\binom{n+1}{n-1}}}\prod_{k=1}^n(1+q+\cdots+q^{k-1})}{\lambda z^{n+1}\left(D_q^n f(z)\right) + \frac{(-1)^n\prod_{k=1}^n(1+q+\cdots+q^{k-1})}{q^{\binom{n+1}{n-1}}} + \frac{(1+\lambda)\alpha}{q^{\binom{n+1}{n-1}}}}\right| < \beta,$$

or

$$\begin{aligned} X(f) = & \left|z^{n+3}\left(D_q^n f(z)\right)'' + z^{n+2}\left(D_q^n f(z)\right)' - \frac{(-1)^n(n+1)^2}{q^{\binom{n+1}{n-1}}}\prod_{k=1}^n(1+q+\cdots+q^{k-1})\right| \\ - & \beta\left|\lambda z^{n+1}\left(D_q^n f(z)\right) + \frac{(-1)^n\prod_{k=1}^n(1+q+\cdots+q^{k-1})}{q^{\binom{n+1}{n-1}}} + \frac{(1+\lambda)\alpha}{q^{\binom{n+1}{n-1}}}\right| \\ < & \ 0. \end{aligned}$$

But for $0<|z|=r<1$ we have

$$\begin{aligned} X(f) &= \left|\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q(k-n-1)^2 a_k z^k\right| \\ &- \beta\left|\frac{(1+\lambda)}{q^{\binom{n+1}{n-1}}}\left((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\right)\right. \\ &- \left.\lambda\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q a_k z^k\right| \\ &\le \sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q(k-n-1)^2|a_k|r^k \\ &- \frac{\beta(1+\lambda)\left((-1)^{n-1}\prod_{k=1}^n(1+q+\cdots+q^{k-1})-\alpha\right)}{q^{\binom{n+1}{n-1}}} \\ &+ \lambda\beta\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q|a_k|r^k \le \sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2-\lambda\beta\right)|a_k|r^k \\ &- \frac{\beta(1+\lambda)\left((-1)^{n-1}\prod_{k=1}^n(1+q+\cdots+q^{k-1})-\alpha)\right)}{q^{\binom{n+1}{n-1}}}. \end{aligned}$$

Since the above inequality holds for all r $(0<r<1)$, by letting $r\to\overline{1}$ and using equation 2.1 we obtain $X(f)\le 0$, and this completes the proof. □

Corollary 2.1. *If function $f(z)$ of the form equation 1.1 belongs to $\sum_q(n;\lambda,\alpha,\beta)$ then*

$$a_k \leq \frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\big((k-n-1)^2+\lambda\beta\big)}.$$

This result is sharp for $H(z)$ given by

$$H(z)=\frac{1}{z}+\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\big((k-n-1)^2+\lambda\beta\big)}z^{k-1}.$$

Next we obtain extreme points and convex linear combination property for $f(z)$ belongs to $\sum_q(n;\lambda,\alpha,\beta)$.

Theorem 2.2. *The function $f(z)$ of the form equation 1.1 belongs to $\sum_q(n;\lambda,\alpha,\beta)$ if and only if it can be expressed by $f(z)=\sum_{k=0}^{\infty}\sigma_k f_k(z), \sigma_k\geq 0, \sum_{k=0}^{\infty}\sigma_k=1$ where $f_0(z)=\frac{1}{z}$ and*

$$f_k(z)=\frac{1}{z}+\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\big((k-n-1)^2+\lambda\beta\big)}z^{k-1}, (k=1,2,\ldots).$$

Proof. Let

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}\sigma_k f_k(z) \\ &= \sigma_0 f_0(z) \\ &+ \sum_{k=1}^{\infty}\sigma_k\Big[\frac{1}{z}+\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\big((k-n-1)^2+\lambda\beta\big)}z^{k-1}\Big] \\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\big((k-n-1)^2+\lambda\beta\big)}\sigma_k z^{k-1}. \end{aligned}$$

Now by using Theorem 2.1 we conclude that $f(z)\in\sum_q(n;\lambda,\alpha,\beta)$.
Conversely, if $f(z)$ given by equation 1.1 belongs to $\sum_q(n;\lambda,\alpha,\beta)$, by letting $\sigma_0=1-\sum_{k=1}^{+\infty}\sigma_k$, where

$$\sigma_k=\frac{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)}{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}a_k, \quad (k=1,2,\ldots).$$

we conclude the required result. □

Theorem 2.3. *Let for $n=1,2,\ldots,m$, $f_n(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}a_{k,n}z^{k-1}$ belongs to $\sum_q(n;\lambda,\alpha,\beta)$, then $F(z)=\sum_{n=1}^{m}\sigma_n f_n(z)$ is also in the same class, where $\sum_{n=1}^{m}\sigma_n=1$. (Hence $\sum_q(n;\lambda,\alpha,\beta)$ is a convex set.)*

Proof. According to Theorem 2.1 for every $n = 1, 2, \ldots, m$ we have

$$\sum_{k=1}^{+\infty} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \lambda\beta\right) a_{k,n} \leq \frac{\beta(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1}) - \alpha\right)}{q^{\binom{n+1}{n-1}}}.$$

But

$$\begin{aligned} F(z) &= \sum_{n=1}^{m} \sigma_n f_n(z) \\ &= \sum_{n=1}^{m} \sigma_n \Big(\frac{1}{z} + \sum_{k=1}^{\infty} a_{k,n} z^{k-1}\Big) \\ &= \frac{1}{z} \sum_{n=1}^{m} \sigma_n + \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{m} \sigma_n a_{k,n}\Big) z^{k-1} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{m} \sigma_n a_{k,n}\Big) z^{k-1}. \end{aligned}$$

Since :

$$\begin{aligned} &\sum_{k=1}^{+\infty} \prod_{i=1}^{n} [k-i]_q \big((k-n-1)^2 + \lambda\beta\big) \Big(\sum_{n=1}^{m} \sigma_n a_{k,n}\Big) \\ &= \sum_{n=1}^{m} \sigma_n \Big(\sum_{k=1}^{+\infty} \prod_{i=1}^{n} [k-i]_q \big((k-n-1)^2 + \lambda\beta\big)\Big) a_{k,n} \\ &\leq \sum_{n=1}^{m} \sigma_n \frac{\beta(1+\lambda)\big((-1)^{n-1} \prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1}) - \alpha\big)}{q^{\binom{n+1}{n-1}}} \\ &= \frac{\beta(1+\lambda)\big((-1)^{n-1} \prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1}) - \alpha\big)}{q^{\binom{n+1}{n-1}}} \sum_{n=1}^{m} \sigma_n \\ &= \frac{\beta(1+\lambda)\big((-1)^{n-1} \prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1}) - \alpha\big)}{q^{\binom{n+1}{n-1}}} \end{aligned}$$

then by Theorem 2.1 the proof is complete. □

3. Radii condition and partial sum property

In this section we obtain radii of starlikeness and convexity and investigate about partial sum property.

Theorem 3.1. *if the function $f(z)$ defined by equation 1.1 is in the class $\sum_q(n;\lambda,\alpha,\beta)$, then $f(z)$ is meromorphically univalent starlike of order γ in disk $|z|<R_1$, and it is meromorphically univalent convex of order γ in disk $|z|<R_2$ where*

(3.1)

$$R_1 = \inf_k \left\{ \frac{q^{\binom{n+1}{n-1}} \prod_{i=1}^{n}[k-i]_q \big((k-n-1)^2+\lambda\beta\big)(1-\gamma)}{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)(k+1+\gamma)} \right\}^{\frac{1}{k}}$$

(3.2)

$$R_2 = \inf_k \left\{ \frac{q^{\binom{n+1}{n-1}} \prod_{i=1}^{n}[k-i]_q \big((k-n-1)^2+\lambda\beta\big)(1-\gamma)}{\beta(k-1)(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)(k+1+\gamma)} \right\}^{\frac{1}{k}}$$

Proof. For starlikeness it is enough to show that

$$\left|\frac{zf(z)'+f(z)}{f(z)}\right| < 1-\gamma,$$

but

$$\left|\frac{zf(z)'+f(z)}{f(z)}\right| = \left|\frac{\sum_{k=1}^{+\infty} ka_k z^k}{1+\sum_{k=1}^{+\infty} a_k z^k}\right| \leq \frac{\sum_{k=1}^{+\infty} ka_k|z|^k}{1-\sum_{k=1}^{+\infty} a_k|z|^k} \leq 1-\gamma,$$

or

$$\sum_{k=1}^{+\infty} ka_k|z|^k \leq (1-\gamma)-(1-\gamma)\sum_{k=1}^{+\infty} a_k|z|^k,$$

or

$$\sum_{k=1}^{+\infty} \frac{k+1-\gamma}{1-\gamma} a_k|z|^k \leq 1. \tag{3.3}$$

By using equation 2.1 and equation 3.3 we obtain

$$\frac{k+1-\gamma}{1-\gamma}|z|^k \leq \frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}} \prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)}.$$

So, it is enough to suppose

$$|z|^k \leq \frac{q^{\binom{n+1}{n-1}} \prod_{i=1}^{n}[k-i]_q \left((k-n-1)^2+\lambda\beta\right)(1-\gamma)}{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)(k+1-\gamma)}.$$

Hence we get the required result equation 3.1. For convexity, by using the Alexander,s Theorem(If f be an analytic function in the unit disk and normalized by $f(0)=f'(0)-1=0$, then $f(z)$ is convex if and only if $zf'(z)$ is starlike.) and applying an easy calculation we conclude the required result equation 3.2. So the proof is complete. □

Theorem 3.2. *Let $f(z) \in \sum$, and define*

$$S_1(z) = \frac{1}{z} \quad , \quad S_m(z) = \frac{1}{z} + \sum_{k=1}^{m-1} a_k z^{k-1} \quad , \quad (m = 2, 3, \ldots). \tag{3.4}$$

Also suppose $\sum_{k=1}^{+\infty} x_k a_k \leq 1$, where

$$x_k = \frac{q^{\binom{n+1}{n-1}} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \lambda\beta\right)}{\beta(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n} (1+q+q^2+\cdots+q^{k-1}) - \alpha\right)}, \tag{3.5}$$

then

$$Re\left(\frac{f(z)}{S_m(z)}\right) > 1 - \frac{1}{x_m} \quad , \quad Re\left(\frac{S_m(z)}{f(z)}\right) > \frac{x_m}{1+x_m} \tag{3.6}$$

Proof. Since $\sum_{k=1}^{+\infty} x_k a_k \leq 1$, they by Theorem 2.1, $f(z) \in \sum_q(n; \lambda, \alpha, \beta)$. Also by equation 1.4 and equation 1.5 we have

$$\frac{\prod_{i=1}^{n} [k-i]_q}{(-1)^{n-1} \prod_{k=1}^{n} (1+q+q^2+\cdots+q^{k-1}) - \alpha} \geq 1,$$

so

$$x_k > \frac{q^{\binom{n+1}{n-1}} \left((k-n-1)^2 + \lambda\beta\right)}{\beta(1+\lambda)},$$

and $\{x_k\}$ is an increasing sequence, therefore we obtain

$$\sum_{k=1}^{m-1} a_k + x_m \sum_{k=m}^{+\infty} a_k \leq 1. \tag{3.7}$$

Now by putting

$$X(z) = x_m \left[\frac{f(z)}{S_m(z)} - (1 - \frac{1}{x_m})\right],$$

and making use of equation 3.7 we obtain

$$Re\left(\frac{X(z)-1}{X(z)+1}\right) \leq \left|\frac{X(z)-1}{X(z)+1}\right| = \left|\frac{x_m f(z) - x_m S_m(z)}{x_m f(z) - x_m S_m(z) + 2S_m(z)}\right|$$

By a simple calculation we get $Re\,(X(z)) > 0$, therefore $Re\left(\frac{X(z)}{x_m}\right) > 0$, or equivalently $Re\left[\frac{f(z)}{S_m(z)} - (1 - \frac{1}{x_m})\right] > 0$, and this gives the first inequality in equation 3.6. For the second inequality we consider

$$Y(z) = (1+x_m)[\frac{S_m(z)}{f(z)} - \frac{x_m}{1+x_m}],$$

and by using equation 3.7 we have $\left|\frac{Y(z)-1}{Y(z)+1}\right| \leq 1$, and Hence $Re\,(Y(z)) > 0$, therefore $Re\left(\frac{Y(z)}{1+x_m}\right) > 0$, or equivalently $Re\left[\frac{S_m(z)}{f(z)} - \frac{x_m}{1+x_m}\right] > 0$, and this shows the second inequality in equation 3.6. So the proof is complete. □

4. Some properties of $\sum_q(n;\lambda,\alpha,\beta)$

Theorem 4.1. *Let $f(z), g(z) \in \sum_q(n;\lambda,\alpha,\beta)$ and given by $f(z) = \dfrac{1}{z} + \sum_{k=1}^{+\infty} a_k z^{k-1}$, $g(z) = \dfrac{1}{z} + \sum_{k=1}^{+\infty} b_k z^{k-1}$. Then the function $h(z) = \dfrac{1}{z} + \sum_{k=1}^{+\infty}(a_k^2 + b_k^2) z^{k-1}$ is also in $\sum_q(n;\gamma,\alpha,\beta)$ where $\gamma \le \frac{\lambda}{2} - \frac{(k-n-1)^2}{2\beta}$.*

Proof. Since $f(z), g(z) \in \sum_q(n;\lambda,\alpha,\beta)$ therefore we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\Big[\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)\Big]^2 a_k^2\\
\le\ &\Big[\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)a_k\Big]^2\\
\le\ &\Big[\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}\Big]^2
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{+\infty}\Big[\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)\Big]^2 b_k^2\\
\le\ &\Big[\sum_{k=1}^{+\infty}\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)b_k\Big]^2\\
\le\ &\Big[\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}\Big]^2.
\end{aligned}$$

The above inequalities yield us

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{2}\Big[\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)\Big]^2(a_k^2+b_k^2)\\
\le\ &\Big[\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}\Big]^2.
\end{aligned}$$

Now we must show

$$\begin{aligned}
&\sum_{k=1}^{\infty}\Big[\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\gamma\beta\Big)\Big]^2(a_k^2+b_k^2)\\
\le\ &\Big[\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}\Big]^2.
\end{aligned}$$

But above inequalities holds if

$$\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\gamma\beta\Big) \le \tfrac{1}{2}\Big[\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)\Big]$$

or equivalently

$$2(k-n-1)^2+2\gamma\beta \le (k-n-1)^2+\lambda\beta$$

or

$$\gamma \le \frac{\lambda}{2} - \frac{(k-n-1)^2}{2\beta}.$$

□

Theorem 4.2. *The class* $\sum_q(n;\lambda,\alpha,\beta)$ *is a convex set.*

Proof. Let

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^{k-1},$$

and

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^{k-1},$$

be in the class $\sum_q(n;\lambda,\alpha,\beta)$. For $t \in (0,1)$, it is enough to show that the function $h(z) = (1-t)f(z) + tg(z)$ is in the class $\sum_q(n;\lambda,\alpha,\beta)$. Since

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\Big((1-t)a_k + tb_k\Big)z^{k-1},$$

then

$$\sum_{k=1}^{\infty}\Big[\prod_{i=1}^{n}[k-i]_q\Big((k-n-1)^2+\lambda\beta\Big)\Big]\Big((1-t)a_k+tb_k\Big)$$
$$\le \frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}}$$

so $h(z) \in \sum_q(n;\lambda,\alpha,\beta)$. □

Corollary 4.1. *Let* $f_j(z)$ $(j=1,2,\ldots,n)$*, defined by* $f_j(z) = \dfrac{1}{z} + \sum_{k=1}^{\infty} a_{k,j}z^{k-1}$ *be in the class* $\sum_q(n;\lambda,\alpha,\beta)$*, then the function* $F(z) = \sum_{j=1}^{n} c_j f_j(z)$ *is also in* $\sum_q(n;\lambda,\alpha,\beta)$ *where* $\sum_{j=1}^{n} c_j = 1$.

5. Hadamard product

For the functions $f(z), g(z) \in \Sigma$ is given by equation 1.1, we denote by $(f * g)(z)$ the Hadamard product (or convolution) of the functions $f(z), g(z)$, that is

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k b_k z^{k-1} = (g * f)(z).$$

Theorem 5.1. *If $f(z), g(z)$ defined by equation 1.1 is in the class $\sum_q(n; \lambda, \alpha, \beta)$ then $(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k b_k z^{k-1}$ in the class $\sum_q(n; \gamma, \alpha, \beta)$ where*

$$\gamma \leq \frac{q^{\binom{n+1}{n-1}} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \lambda\beta\right)^2}{\beta^2(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1}) - \alpha\right)} - \frac{(k-n-1)^2}{\beta}.$$

Proof. Since $f(z), g(z) \in \sum_q(n; \lambda, \alpha, \beta)$, so by equation 2.1

$$\sum_{k=1}^{\infty} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \lambda\beta\right) a_k \leq \frac{\beta(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n}(1+q+\cdots+q^{k-1}) - \alpha\right)}{q^{\binom{n+1}{n-1}}} \tag{5.1}$$

and

$$\sum_{k=1}^{\infty} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \lambda\beta\right) b_k \leq \frac{\beta(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n}(1+q+\cdots+q^{k-1}) - \alpha\right)}{q^{\binom{n+1}{n-1}}} \tag{5.2}$$

By using the equation 5.1, equation 5.2 and Cauchy-Schwarts inequality we have

$$\sum_{k=1}^{\infty} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \lambda\beta\right) \sqrt{a_k b_k} \leq \frac{\beta(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n}(1+q+\cdots+q^{k-1}) - \alpha\right)}{q^{\binom{n+1}{n-1}}}. \tag{5.3}$$

we must find the smallest γ such that

$$\sum_{k=1}^{\infty} \prod_{i=1}^{n} [k-i]_q \left((k-n-1)^2 + \gamma\beta\right) a_k b_k \leq \frac{\beta(1+\lambda)\left((-1)^{n-1} \prod_{k=1}^{n}(1+q+\cdots+q^{k-1}) - \alpha\right)}{q^{\binom{n+1}{n-1}}}. \tag{5.4}$$

Now it is enough to show that

$$\begin{aligned}&\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\gamma\beta\right)a_kb_k\\ &\leq \prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)\sqrt{a_kb_k}\end{aligned} \tag{5.5}$$

or equivalently

$$\sqrt{a_kb_k}\leq \frac{(k-n-1)^2+\lambda\beta}{(k-n-1)^2+\gamma\beta}.$$

But from equation 5.3,

$$\sqrt{a_kb_k}\leq \frac{\beta(1+\lambda)\big((-1)^{n-1}(\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)}$$

so it is enough that

$$\begin{aligned}&\frac{\beta(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)}\\ &\leq \frac{(k-n-1)^2+\lambda\beta}{(k-n-1)^2+\gamma\beta}\end{aligned} \tag{5.6}$$

By using the equation 5.6 we have

$$\begin{aligned}\gamma \;\leq\;& \frac{q^{\binom{n+1}{n-1}}\prod_{i=1}^{n}[k-i]_q\left((k-n-1)^2+\lambda\beta\right)^2}{\beta^2(1+\lambda)\big((-1)^{n-1}\prod_{k=1}^{n}(1+q+q^2+\cdots+q^{k-1})-\alpha\big)}\\ -\;& \frac{(k-n-1)^2}{\beta}.\end{aligned} \tag{5.7}$$

□

REFERENCES

1. M. H. Abu-Risha, M. H. Annaby, M. E. H. Ismail and Z. S. Mansour: *Linear q-difference equations*, Z. Anal. Anwend. 26 (4) (2007), 481–494.
2. B. Ahmad and M. Arif: *New subfamily of meromorphic convex functions in circular domain involving q−operators*, Int. J. Anal. Appl. 1 (2018), 75-82.
3. H. Aldweby and M. Darus: *A subclass of harmonic univalent functions associated with q−analogue of Dziok Srivastava operator*, ISRN Math. Anal. 6 (2013), 382-312.
4. D. Albayrak, S. D. Purohit and F. Ucar: *On q-analogues of sumudu transforms*, JAn. Stiint. Univ. Ovidius Constanta Ser. Mat. 21 (1) (2013), 239–260.

5. G. Bangerezako: *Variational calculus on q-nonuniform lattices*, J. Math. anal. Appl. 306 (2005), no. 161-179.

6. T. Ernst: *A comprehensive treatment of q-calculus*, JBirkhauser/Springer Basel AG, Basel, 2012.

7. G. Gasper and M. Rahman: *Basic Hypergeometric series*, Cambridge University Press, Cambridge, 1990.

8. F. H. Jackson: *On q−functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, 46(2) (1909), 253-281.

9. V. Kasand P. Cheung: *Quantum calclus*, Universitext, Springer, New York, 2002.

10. Z. S. I. Mansour: *Linear sequential q-difference equations of fractional order*, Fract. Calc. Appl. Anal. 12(2) (2009), 159–178.

11. A. O. Mostafa, M. K. Aouf, H. M. Zayed and T. Bulboaca: *Convolution conditions for subelasse of mermorphic functions of complex order associated with basic Bessel functions*, J. Egyptian Math. Soc. 25 (2017), 286–290.

12. H. E. Ozkan Ucar: *coefficient inequalties for q-starlike functions*, Appl. Math. Comput. 276 (2016), 122-–126.

13. Y. Polatoglu: *Growth and distortion theorems for generalized q-starlike functions*, Adv. Math. Sci. J. 5 (2016), 7-–12.

14. S. D. Purohit and R. K. Raina: *Certain subclass of analytic functions associated with fractional q-calculus operators*, Math. Scand. 109 (2011), 55-–70.

15. S. D. Purohit: textitA new class of multivalently analytic functions associated with fractional q-calculus operators, Fractional Differ. Calc. 2(2) (2012), 129-138.

16. P. M. Rajković, S. D. Marinković and M. S. Stanković: *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math., 1 (2007), 311–323.

17. T. M. Seoudy and M. K. Aouf: *Convolution properties for certain classes of analytic functions defined by q-derivative operator*, Abster. Appl. Anal., 2014 (2014), Art. ID 846719, 1-7.

Mohammad Hassn Golmohammadi
Faculty of Mathematical Sciences
Department of Pure Mathematics
Payame Noor University, P. O. Bax: 19395 - 3697, Tehran, Iran,
golmohamadi@pnu.ac.ir

Shahram Najafzadeh
Faculty of Mathematical Sciences
Department of Pure Mathematics
Payame Noor University, P. O. Bax: 19395 - 3697, Tehran, Iran,
najafzadeh1234@yahoo.ie

Mohammad Reza Foroutan
Faculty of Mathematical Sciences
Department of Pure Mathematics
Payame Noor University, P. O. Bax: 19395 - 3697, Tehran, Iran,
foroutan$_m$ohammadreza@yahoo.com

FACTA UNIVERSITATIS (NIŠ)
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NONLINEAR NEUTRAL CAPUTO-FRACTIONAL DIFFERENCE EQUATIONS WITH APPLICATIONS TO LOTKA-VOLTERRA NEUTRAL MODEL

Mouataz Billah Mesmouli, Abdelouaheb Ardjouni and Ahcene Djoudi

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Abstract. In this paper, we consider a nonlinear neutral fractional difference equations. By applying Krasnoselskii's fixed point theorem, sufficient conditions for the existence of solutions are established. Also, the uniqueness of a solution is given. As an application of the main theorems, we provide the existence and uniqueness of the discrete fractional Lotka-Volterra model of neutral type. Our main results extend and generalize the results that are obtained in [6].
Key words: Existence and uniqueness; fractional difference equations; Krasnoselskii fixed point theorem; contraction operator; Arzela-Ascoli's theorem; neutral discrete fractional Lotka-Volterra model.

1. Introduction and preliminaries

Fractional difference equations have received a special attention during the last years. Indeed, some mathematicians have recently taken the lead to develop the qualitative properties of fractional difference equations. We recall, for instance, the study made by Atici et. al. [7], [8], [9] and Abdeljawad et. al. [1], [2] (see also [4], [12], [15], [19]-[23], [25] and reference therein) who developed the transform methods, properties of initial value problems and studied applications of these equations.

Let $\mathbb{N}_0 = [0, T_1] \cap \mathbb{Z}$ where $T_1 \in [2, +\infty) \cap \mathbb{Z}$. Alzabut, Abdeljawad and Baleanu [6] discussed the existence of solutions for the difference equation

$$\begin{cases} {}^c\nabla_0^\alpha x(t) = f(t, x(t), x(t-\tau_1)), \ t \in \mathbb{N}_0, \\ x(t) = \psi(t), \ t \in [-\tau_1, 0] \cap \mathbb{Z}, \end{cases} \tag{1.1}$$

where $\tau_1 \in \mathbb{N}$, $\psi : [-\tau_1, 0] \cap \mathbb{Z} \to \mathbb{R}$, $f : \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ${}^c\nabla_0^\alpha$ denotes the Caputo's fractional difference of order $\alpha \in (0, 1)$. By employing the Krasnoselskii fixed point theorem, the authors obtained existence results.

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In this paper, we are interested in the analysis of qualitative theory of the problems of the existence and uniqueness of solutions to nonlinear neutral fractional difference equations

$$\begin{cases} {}^c\nabla_0^\alpha x(t) = f\left(t, x(t), x(t-\tau_1), {}^c\nabla_0^\alpha x(t-\tau_2)\right), \ t \in \mathbb{N}_0, \\ x(t) = \psi(t), \ t \in [-\tau, 0] \cap \mathbb{Z}, \end{cases} \tag{1.2}$$

where $\tau_1, \tau_2 \in \mathbb{N}$, $\tau = \max(\tau_1, \tau_2)$, $\psi : [-\tau, 0] \cap \mathbb{Z} \to \mathbb{R}$, $f : \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ${}^c\nabla_0^\alpha$ denotes the Caputo's fractional difference of order $\alpha \in (0, 1)$. To prove our main results, we employ the Krasnoselskii and Banach fixed point theorems and the Arzelá-Ascoli's theorem. Moreover, we apply the main theorems to the discrete fractional Lotka-Volterra of neutral type

$$\begin{cases} {}^c\nabla_0^\alpha x(t) = x(t)\left[a(t) - b(t) x(t-\tau_1) - c(t) {}^c\nabla_0^\alpha x(t-\tau_2)\right], \ t \in \mathbb{N}_0, \\ x(t) = \psi(t), \ t \in [-\tau, 0] \cap \mathbb{Z}, \end{cases} \tag{1.3}$$

where a, b and c are sequences fulfill some of the conditions described below, which are medically and biologically feasible.

Now, we present some basic definitions, notations and results of fractional difference calculus [16], [17] which are used throughout this paper. For any $\alpha, t \in \mathbb{R}$, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t)}, \ t \in \mathbb{R} - \{..., -2, -1, 0\}, \ 0^{\overline{\alpha}} = 0, \tag{1.4}$$

where Γ is the well known Gamma function satisfying $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Definition 1.1. Let $x : \mathbb{N}_0 \to \mathbb{R}$, $\rho(s) = s - 1$, $\alpha \in \mathbb{R}^+$ and $\mu > -1$. Then

1) The nabla difference of x is defined by

$$\nabla x(t) = x(t) - x(t-1), \ t \in \mathbb{N}_1 = [1, T_1] \cap \mathbb{Z}.$$

2) The Riemann-Liouville's sum operator of x of order $\alpha > 0$ is defined by

$$\nabla_0^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha - 1}}, \ t \in \mathbb{N}_1. \tag{1.5}$$

3) The Riemann-Liouville's difference operator of x of order $0 < \alpha < 1$ is defined by

$${}^c\nabla_0^\alpha x(t) = \nabla_0^{-(1-\alpha)} \nabla x(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{-\alpha}} \nabla x(s), \ t \in \mathbb{N}_1. \tag{1.6}$$

4) The power rule is defined by

$$\nabla_0^{-\alpha} t^{\overline{\mu}} = \frac{\Gamma(\mu + 1 - \alpha)}{\Gamma(\mu + \alpha + 1)} t^{\overline{\mu + \alpha}}, \ t \in \mathbb{N}_1. \tag{1.7}$$

Let $\mathbb{N}_{-\tau} = [-\tau, T_1] \cap \mathbb{Z}$ where $T_1 \in [3, +\infty) \cap \mathbb{Z}$, and $B(\mathbb{N}_{-\tau}, \mathbb{R})$ be the Banach space of all bounded sequences with respect to the maximum norm.

Definition 1.2. A set D of sequences in $B(\mathbb{N}_{-\tau}, \mathbb{R})$ is uniformly Cauchy if for every $\epsilon > 0$, there exists an integer N^* such that $|x(t) - x(s)| < \epsilon$ whenever $t, s > N^*$ for any $x = \{x(n)\}$ in D.

The following discrete version of Arzelá-Ascoli's theorem has a crucial role in the proof of our main theorems.

Theorem 1.1. Arzelá-Ascoli's theorem *A bounded, uniformly Cauchy subset D of $B(\mathbb{N}_{-\tau}, \mathbb{R})$ is relatively compact.*

The proof of the main theorem is achieved by employing the following fixed point theorem.

Theorem 1.2. Krasnoselskii's fixed point theorem [10] *Let D be a nonempty, closed, convex and bounded subset of a Banach space $(X, \|\cdot\|)$. Suppose that $A_1 : D \to X$ and $A_2 : D \to X$ are two operators such that*

(i) A_1 is a contraction,

(ii) A_2 is continuous and $A_2(D)$ resides in a compact subset of X,

(iii) for any $x, y \in D$, $A_1 x + A_2 y \in D$.

Then the operator $A_1 + A_2$ has a fixed point $x \in D$.

2. Existence and uniqueness of solutions

In this section, we give the equivalence of the problem (1.2). So, by an alternative way used in [3], [5] and [14], we turn the problem (1.2) into an equivalent equation, then, the solvability of this equivalent equation implies the existence and uniqueness of solution to the problem (1.2).

Lemma 2.1. *x denotes a solution of the equation (1.2) if and only if it admits the following representation*

$$x(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} z_x(s), \tag{2.1}$$

where $z_x(t) = {}^c\nabla_0^\alpha x(t)$ and $x(t) = \psi(t)$, $t \in [-\tau, 0] \cap \mathbb{Z}$.

Proof. By the same way used in [3], we get for $t \in \mathbb{N}_{-\tau}$, the initial value problem (1.2) is equivalent to the following equation

$$x(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f\left(t, x(s), x(s - \tau_1), {}^c\nabla_0^\alpha x(s - \tau_2)\right). \tag{2.2}$$

By the techniques used in [5] and [14], let

$$z_x(t) = {}^c\nabla_0^\alpha x(t) \text{ and } x(t) = \psi(t) \text{ for } t \in [-\tau, 0] \cap \mathbb{Z}.$$

Then, the equation (2.2) is equivalent to the equation (2.1), with

$$z_x(t) = f(t, x(t), x(t-\tau_1), z_x(t-\tau_2)). \tag{2.3}$$

□

We prove our main results under the following assumptions

(A1) For $t \in \mathbb{N}_{-\tau}$,

$$\begin{aligned} z_x(t) &= f(t, x(t), x(t-\tau_1), z_x(t-\tau_2)) \\ &= f_1(t, x(t)) + f_2(t, x(t), x(t-\tau_1)) + f_3(t, x, z_x(t-\tau_2)), \end{aligned}$$

where f_1, f_2 and f_3 are Lipschitz functions with Lipschitz constants L_{f_i}, $i = 1, 2, 3$, with $L_{f_3} < 1$.

(A2) For $t \in \mathbb{N}_{-\tau}$,

$$\begin{aligned} |f_1(t, u(t))| &\leq M_1 |u(t)|, \\ |f_2(t, u(t), v(t))| &\leq M_2 |u(t)| |v(t)|, \\ |f_3(t, u(t), v(t))| &\leq M_3 |u(t)| |v(t)|, \end{aligned}$$

for any positive numbers M_i, $i = 1, 2, 3$.

Define the set

$$D = \{u \in B(\mathbb{N}_{-\tau}, \mathbb{R}), \ \|u\| \leq r, \ u(t) = \psi(t) \text{ for } t \in [-\tau, 0] \cap \mathbb{Z}\}, \tag{2.4}$$

where r satisfies

$$|\psi(0)| + \frac{M_1 r + M_2 r^2 + M_3 L r^2}{\Gamma(\alpha)} C(\alpha) \leq r, \tag{2.5}$$

and $C(\alpha) = \dfrac{\Gamma(T_1 + \alpha)}{\alpha \Gamma(T_1)}$ is a positive constant depending on the order α and satisfies the inequality

$$L_{f_1} C(\alpha) < \Gamma(\alpha). \tag{2.6}$$

Lemma 2.2. *Suppose that the assumption (A1) holds. Then, for* $t \in \mathbb{N}_{-\tau}$, z_x *satisfies the following inequality*

$$|z_x(t) - z_y(t)| \leq L \|x - y\| \text{ for all } x, y \in B(\mathbb{N}_{-\tau}, \mathbb{R}),$$

where

$$L = \frac{L_{f_1} + 2L_{f_2} + L_{f_3}}{1 - L_{f_3}}.$$

Proof. For all $x, y \in B(\mathbb{N}_{-\tau}, \mathbb{R})$, since (A1) holds, then

$$\begin{aligned} & \left|z_x(t) - z_y(t)\right| \\ \leq\ & L_{f_1}\left|x(t) - y(t)\right| + L_{f_2}\left|x(t) - y(t)\right| + L_{f_2}\left|x(t-\tau_1) - y(t-\tau_1)\right| \\ & + L_{f_3}\left|x(t) - y(t)\right| + L_{f_3}\left|z_x(t-\tau_2) - z_y(t-\tau_2)\right| \\ \leq\ & \left(L_{f_1} + 2L_{f_2} + L_{f_3}\right)\|x - y\| + L_{f_3}\|z_x - z_y\|. \end{aligned}$$

Thus,

$$\left|z_x(t) - z_y(t)\right| \leq \frac{L_{f_1} + 2L_{f_2} + L_{f_3}}{1 - L_{f_3}}\|x - y\|.$$

□

Now, to apply Krasnoselskii's fixed point 1.2, by Lemma 2.1 can define the operators A_1 and A_2 on D by

$$(A_1 x)(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_1(s, x(s)), \tag{2.7}$$

and

$$\begin{aligned} (A_2 x)(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_2(s, x(s), x(s-\tau_1)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f_3(s, x(s), z_x(s-\tau_2)). \end{aligned} \tag{2.8}$$

It is clear that x is a solution of (1.2) if it is a fixed point of the operator $A = A_1 + A_2$.

Theorem 2.1. *Let conditions (A1), (A2), (2.5) and (2.6) hold. Then, the equation (1.2) has a solution in the set D.*

Proof. From the assumptions on the set D, one can easily see that D is a nonempty, closed, convex and bounded set.

Step 1. We prove that the A_1 defined by (2.7) is contraction. We can easily see that for $x, y \in D$

$$\begin{aligned} & \left|(A_1 x)(t) - (A_1 y)(t)\right| \\ \leq\ & \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \left|f_1(s, x(s)) - f_1(s, y(s))\right| \\ \leq\ & L_{f_1} \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \left|x(s) - y(s)\right| \\ \leq\ & \frac{L_{f_1}}{\Gamma(\alpha)} \|x - y\| \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}}. \end{aligned}$$

By virtue of (1.4), (1.5), (1.7) and since $(t-0)^{\overline{0}} = 1$, one can see that

$$\sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (t-0)^{\overline{0}} = \Gamma(\alpha) \nabla_0^{-\alpha} (t-0)^{\overline{0}} = \frac{\Gamma(t+\alpha)}{\alpha\Gamma(t)}.$$

Therefore,

$$|(A_1x)(t) - (A_1y)(t)| \leq \frac{C(\alpha)}{\Gamma(\alpha)} L_{f_1} \|x-y\|, \ t \leq T_1.$$

By the assumption (2.6), we conclude that A_1 is contraction mapping on D.

Furthermore, we obtain for $x \in D$

$$\begin{aligned} & |(A_1x)(t) + (A_2x)(t)| \\ \leq & \left|\psi(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f_1(s, x(s))\right| \\ & + \left|\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f_2(s, x(s), x(s-\tau_1))\right| \\ & + \left|\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f_3(s, x(s), z_x(s-\tau_2))\right| \\ \leq & |\psi(0)| + \frac{M_1\|x\| + M_2\|x\|^2 + M_3L\|x\|^2}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \\ \leq & |\psi(0)| + \frac{M_1r + M_2r^2 + M_3Lr^2}{\Gamma(\alpha)} C(\alpha) \\ \leq & r, \end{aligned} \tag{2.9}$$

which implies that $A_1x + A_2x \in D$. For $x \in D$, we also get

$$\begin{aligned} |(A_2x)(t)| \leq & \left|\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f_2(s, x(s), x(s-\tau_1))\right| \\ & + \left|\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f_3(s, x(s), z_x(s-\tau_2))\right| \\ \leq & \frac{M_2\|x\|^2 + M_3L\|x\|^2}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \\ \leq & \frac{M_2r^2 + M_3Lr^2}{\Gamma(\alpha)} C(\alpha) \\ \leq & r, \end{aligned}$$

which implies that $A_2(D) \subset D$.

Step 2. We prove that A_2 is continuous. Let a sequence x_n converge to x. Taking the norm of $A_2x_n - A_2x$, we have

$$\begin{aligned}
&\left|\left(A_2x_n\right)(t)-\left(A_2x\right)(t)\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_2\left(s,x_n(s),x_n\left(s-\tau_1\right)\right)-f_2\left(s,x(s),x\left(s-\tau_1\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\left|f_3\left(s,x_n(s),z_{x_n}\left(s-\tau_2\right)\right)-f_3\left(s,x(s),z_x\left(s-\tau_2\right)\right)\right|\\
=\ & \frac{C(\alpha)}{\Gamma(\alpha)}\left(2L_{f_2}+L_{f_3}+L_{f_3}L\right)\left\|x_n-x\right\|.
\end{aligned}$$

Then, we conclude that whenever $x_n \to x$, $A_2x_n \to A_2x$. This proves the continuity of A_2. To prove that $A_2(D)$ resides in a compact subset of $B(\mathbb{N}_{-\tau},\mathbb{R})$, i.e., $A_2(D)$ is a relatively compact subset. We let $t_1 \leq t_2 \leq T_1$ to get

$$\begin{aligned}
&\left|\left(A_2x\right)\left(t_2\right)-\left(A_2x\right)\left(t_1\right)\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\left|\sum_{s=1}^{t_2}\left(t_2-\rho(s)\right)^{\overline{\alpha-1}}f_2\left(s,x(s),x\left(s-\tau_1\right)\right)\right.\\
&\left.-\sum_{s=1}^{t_1}\left(t_1-\rho(s)\right)^{\overline{\alpha-1}}f_2\left(s,x(s),x\left(s-\tau_1\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\left|\sum_{s=1}^{t_2}\left(t_2-\rho(s)\right)^{\overline{\alpha-1}}f_3\left(s,x(s),z_x\left(s-\tau_2\right)\right)\right.\\
&\left.-\sum_{s=1}^{t_1}\left(t_1-\rho(s)\right)^{\overline{\alpha-1}}f_3\left(s,x(s),z_x\left(s-\tau_2\right)\right)\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}\left|\left(t_2-\rho(s)\right)^{\overline{\alpha-1}}-\left(t_1-\rho(s)\right)^{\overline{\alpha-1}}\right|\left|f_2\left(s,x(s),x\left(s-\tau_1\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}\left|\left(t_2-\rho(s)\right)^{\overline{\alpha-1}}\right|\left|f_2\left(s,x(s),x\left(s-\tau_1\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}\left|\left(t_2-\rho(s)\right)^{\overline{\alpha-1}}-\left(t_1-\rho(s)\right)^{\overline{\alpha-1}}\right|\left|f_3\left(s,x(s),z_x\left(s-\tau_2\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}\left|\left(t_2-\rho(s)\right)^{\overline{\alpha-1}}\right|\left|f_3\left(s,x(s),z_x\left(s-\tau_2\right)\right)\right|.
\end{aligned}$$

By the assumption (A2) and Lemma 2.2, we obtain

$$\begin{aligned}
&\left|(A_2x)(t_2)-(A_2x)(t_1)\right| \\
\leq\ & \left(M_2r^2+M_3Lr^2\right)\left[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t_1}\left|(t_2-\rho(s))^{\overline{\alpha-1}}-(t_1-\rho(s))^{\overline{\alpha-1}}\right|\right. \\
&\left.+\frac{1}{\Gamma(\alpha)}\sum_{s=t_1+1}^{t_2}\left|(t_2-\rho(s))^{\overline{\alpha-1}}\right|\right].
\end{aligned}$$

By using (1.5), we get

$$\begin{aligned}
&\left|(A_2x)(t_2)-(A_2x)(t_1)\right| \\
\leq\ & \left(M_2r^2+M_3Lr^2\right)\left((t_2)^{\overline{0}}-(t_1)^{\overline{0}}+(t_2-t_1)^{\overline{0}}\right).
\end{aligned}$$

From (1.7), it follows that

$$\begin{aligned}
&\left|(A_2x)(t_2)-(A_2x)(t_1)\right| \\
\leq\ & \frac{\left(M_2r^2+M_3Lr^2\right)}{\Gamma(\alpha+1)}\left(\nabla_0^{-\alpha}(t_2-0)^{\overline{0}}-\nabla_0^{-\alpha}(t_1-0)^{\overline{0}}+\nabla_{t_1}^{-\alpha}(t_2-t_1)^{\overline{0}}\right).
\end{aligned}$$

This implies that $A_2(D)$ is uniformly bounded subset of $B(\mathbb{N}_{-\tau},\mathbb{R})$. Thus, by virtue of the discrete Arzelá-Ascoli's theorem 1.1, we conclude that A_2 is compact.

Step 3. It remains to show that for any $x,y\in D$, we have $A_1x+A_2y\in D$. If $x,y\in D$, then we have

$$\begin{aligned}
&\left|(A_1x)(t)+(A_2y)(t)\right| \\
\leq\ & \left|\psi(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f_1(s,x(s))\right| \\
&+\left|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f_2(s,y(s),y(s-\tau_1))\right| \\
&+\left|\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f_3(s,y(s),z_y(s-\tau_2))\right| \\
\leq\ & |\psi(0)|+\frac{M_1\|x\|+M_2\|y\|^2+M_3L\|y\|^2}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \\
\leq\ & |\psi(0)|+\frac{M_1r+M_2r^2+M_3Lr^2}{\Gamma(\alpha)}C(\alpha) \\
\leq\ & r,
\end{aligned}$$

which implies that $A_1x+A_2y\in D$.

By employing the Krasnoselskii fixed point theorem, we conclude that there exists $x\in D$ such that $x=Ax=A_1x+A_2x$ which is a fixed point of A. Hence, the equation (1.2) has at least one solution in D. □

Remark 2.1. Note that, when $f_3 \equiv 0$ Theorem 2.1 becomes the same Theorem 3 in [6], and this confirms the generality of the results.

It is worth noting that, the authors in [6] stated that they studied the uniqueness of solutions for the equation (1.2), but in reality they did not, because Krasnoselskii's theorem only gives us the existence of solutions, it may be only a written error. So, in this paper, we will study the uniqueness of solutions as well.

Theorem 2.2. *Let conditions (A1), (A2), (2.5) and*

$$\frac{C(\alpha)}{\Gamma(\alpha)}\left(L_{f_1}+2L_{f_2}+L_{f_3}+L_{f_3}L\right)<1, \tag{2.10}$$

hold. Then, the equation (1.2) has a unique solution in D.

Proof. Since the equation (1.2) is equivalent to (2.1), for $x \in D$ define

$$Ax = A_1x + A_2x.$$

Step 1. We must prove that A maps D into itself, then by the condition (2.5) and the same way in (2.9)

$$\begin{aligned} |(Ax)(t)| &= |(A_1x)(t)+(A_2x)(t)| \\ &\le |\psi(0)| + \frac{M_1r+M_2r^2+M_3Lr^2}{\Gamma(\alpha)}C(\alpha) \\ &\le r, \end{aligned}$$

which implies that $Ax \in D$.

Step 2. We prove that A is contraction. We can see that for $x, y \in D$

$$\begin{aligned} &|(Ax)(t)-(Ay)(t)| \\ \le\ & \frac{L_{f_1}}{\Gamma(\alpha)}\|x-y\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} + \frac{2L_{f_2}}{\Gamma(\alpha)}\|x-y\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \\ &+\frac{(L_{f_3}+L_{f_3}L)}{\Gamma(\alpha)}\|x-y\|\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}. \end{aligned}$$

Therefore,

$$|(Ax)(t)-(Ay)(t)| \le \frac{C(\alpha)}{\Gamma(\alpha)}\left(L_{f_1}+2L_{f_2}+L_{f_3}+L_{f_3}L\right)\|x-y\|,\ t \le T_1.$$

By the assumption (2.10), we conclude that A is contraction mapping on D.

By employing the Banach fixed point theorem, we conclude that there exists a unique $x \in D$ such that $x = Ax$ which is a unique fixed point of A. Hence, the equation (1.2) has a unique solution in D. □

Remark 2.2. Note that, when $f_3 \equiv 0$ Theorem 2.2 gives the uniqueness of the solution of the equation (1.1).

Now, we can replace the assumptions (A2) and (2.5) by the following, which provide us the existence and uniqueness too.

($\overline{\mathbf{A2}}$) For $t \in \mathbb{N}_{-\tau}$, we assume that $f_1(t,0) = f_2(t,0,0) = f_3(t,0,0) \equiv 0$ and

$$|\psi(0)| + \frac{(L_{f_1} + 2L_{f_2} + L_{f_3}(1+L))\, r}{\Gamma(\alpha)} C(\alpha) \le r. \tag{2.11}$$

Then we get the following theorems.

Theorem 2.3. *Let conditions (A1), ($\overline{A2}$), (2.11) and (2.6) hold. Then, the equation (1.2) has a solution in the set D.*

Proof. The proof is based on the following estimation, since f_1, f_2 and f_3 satisfy the assumptions (A1) and ($\overline{\text{A2}}$), then

$$\begin{aligned} |f_1(t,x(t))| &= |f_1(t,x(t)) - f_1(t,0)| \\ &\le L_{f_1}\|x\|, \end{aligned}$$

$$\begin{aligned} |f_2(t,x(t),x(t-\tau_1))| &= |f_2(t,x(t),x(t-\tau_1)) - f_2(t,0,0)| \\ &\le 2L_{f_2}\|x\|, \end{aligned}$$

and

$$\begin{aligned} |f_3(t,x(t),z_x(t-\tau_2))| &= |f_3(t,x(t),z_x(t-\tau_2)) - f_3(t,0,0)| \\ &\le L_{f_3}(\|x\| + \|z_x\|) \\ &\le L_{f_3}(\|x\| + L\|x\|) \\ &= L_{f_3}(1+L)\|x\|. \end{aligned}$$

The remaining steps of the proof are the same as in Theorem 2.1. □

Theorem 2.4. *Let conditions (A1), ($\overline{A2}$), (2.11) and (2.10) hold. Then, the equation (1.2) has a unique solution in D.*

Proof. The steps of the proof is given by the same way in Theorem 2.2. □

Remark 2.3. The results of this paper can be carried out for the equation

$$\begin{cases} \nabla_0^\alpha x(t) = f(t,x(t),x(t-\tau_1),{}^c\nabla_0^\alpha x(t-\tau_2)),\ t \in \mathbb{N}_2 = [2,T_1] \cap \mathbb{Z}, \\ x(t) = \psi(t),\ t \in [-\tau,1] \cap \mathbb{Z}, \end{cases} \tag{2.12}$$

where $\tau_1, \tau_2 \in \mathbb{N}$, $\tau = \max(\tau_1,\tau_2)$, $T_1 \in [4,+\infty) \cap \mathbb{Z}$, $\psi : [-\tau,1] \cap \mathbb{Z} \to \mathbb{R}$, $f : \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and ∇_0^α denotes the Riemann-Liouville's fractional difference of order $\alpha \in (0,1)$. The solution of the equation (2.12) has the form

$$x(t) = \frac{t^{\overline{\alpha-1}}}{\Gamma(\alpha)}\psi(1) + \frac{1}{\Gamma(\alpha)}\sum_{s=2}^{t}(t-\rho(s))^{\overline{\alpha-1}} z_x(s).$$

3. Discrete fractional Lotka-Volterra model of neutral type

Because it is very interesting to study the neutral delay population model. So, the Lotka-Volterra model has been extensively investigated by many authors see ([6], [9], [13], [11], [18], [24]) and others, through different approaches. But, all the above works studied the Lotka-Volterra model, or the neutral model with integer order. Then, there is no literature on the type of discrete neutral fractional Lotka-Volterra model.

In this section, we employ Theorems 2.1 and 2.2 to prove the existence and uniqueness results for the solutions of (1.3), that represents an interspecific competition in single species with τ denotes the maturity time period.

For a bounded sequence u on $\mathbb{N}_0$, we define u^+ and u^- as follows

$$u^- = \inf_{t\in\mathbb{N}_0} u(t) \text{ and } u^+ = \sup_{t\in\mathbb{N}_0} u(t),$$

and denote

$$\begin{aligned} f_1(t, x(t)) &= a(t)\, x(t), \\ f_2(t, x(t), x(t-\tau_1)) &= -b(t)\, x(t)\, x(t-\tau_1), \\ f_3(t, x, z_x(t-\tau_2)) &= -c(t)\, x(t)\, z_x(t-\tau_2), \end{aligned}$$

where the coefficients a, b and c satisfy the boundedness relations

$$a^- \leq a(t) \leq a^+,\ b^- \leq b(t) \leq b^+,\ c^- \leq c(t) \leq c^+.$$

From the conditions (A1) and ($\overline{\text{A2}}$), it is easy to see that

$$L_{f_1} = a^+,\ L_{f_2} = rb^+,\ L_{f_3} = rc^+L,$$

and

$$M_1 = L_{f_1},\ M_2 = b^+,\ M_3 = c^+.$$

Theorem 3.1. *Let conditions (2.5), (2.6) and*

$$L_{f_3} = rc^+L < 1,$$

hold. Then, the model (1.3) has a solution in the set D.

Theorem 3.2. *Let condition (2.5), (2.10) and*

$$L_{f_3} = rc^+L < 1,$$

hold. Then, the model (1.3) has a unique solution in the set D.

Remark 3.1. The above theorems can be extended to n species neutral competitive Lotka-Volterra system of the form

(3.1)
$$\begin{cases} {}^{c}\nabla_0^{\alpha} x_i(t) = x_i(t)\left[a_i(t) - \sum\limits_{j=1}^{n} b_{ij}(t)\, x(t-\tau_{ij}) - \sum\limits_{j=1}^{n} c_{ij}(t)^{c}\,\nabla_0^{\alpha} x(t-\tau_{ij})\right], \ t \in \mathbb{N}_0, \\ x_i(t) = \psi_i(t), \ t \in [-\tau, 0] \cap \mathbb{Z}, \end{cases}$$

where $\tau_{ij} \in \mathbb{N}$, $\tau = \max_{1 \le i,j \le n} \tau_{ij}$, $\alpha \in (0,1)$, $\psi_i : [-\tau, 0] \cap \mathbb{Z} \to \mathbb{R}$, $a^- \le a_i(t) \le a^+$, $b^- \le b_{ij}(t) \le b^+$, $c^- \le c_{ij}(t) \le c^+$, $i = 1, 2, ..., n$.

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REFERENCES

1. T. Abdeljawad, F. M. Atici: *On the definitions of nabla fractional operators.* Abstr. Appl. Anal. **2012** (2012), 1–13.
2. T. Abdeljawad, D. Baleanu: *Fractional differences and integration by Parts.* J. Comput. Anal. Appl. **13(5)** (2011), 574–582.
3. T. Abdeljawad: *On delta and nabla Caputo fractional differences and dual identities.* Discrete Dyn. Nat. Soc. **2013** (2013), 1–12.
4. B. Ahmad, D. Garout, S. K. Ntouyas, A. Alsaedi: *Caputo fractional differential inclusions of arbitrary order with nonlocal integro-multipoint boundary conditions.* Miskolc Mathematical Notes **20(2)** (2019), 683–699.
5. A. Ardjouni, A. Djoudi: *Initial-value problems for nonlinear hybrid implicit Caputo fractional differential equations.* Malaya Journal of Matematik **7(2)** (2019), 314–317.
6. J. Alzabut, T. Abdeljawad, D. Baleanu: *Nonlinear delay fractional difference equations with applications on discrete fractional Lotka-Volterra competition model.* J. Computational Analysis and Applications **25(5)** (2018), 889–898.
7. F. M. Atici, P. W. Eloe: *A transform method in discrete fractional calculus.* Int. J. Difference Equ. **2(2)** (2007), 165–176.
8. F. M. Atici, P. W. Eloe: *Discrete fractional calculus with the nabla operator.* Electron. J. Qual. Theory Differ. Equ. **2009(3)** (2009), 1–12.
9. F. M. Atici, S. Şengül: *Modeling with factorial difference equations.* J. Math. Anal. Appl. **369(1)** (2010), 1–9.
10. T. A. Burton: *A fixed point theorem of Krasnoselskii fixed point theorem.* Appl. Math. Lett. **11** (1998), 85–88.
11. F. Capone, R. D. Luca, S. Rionero: *On the stability of non-autonomous perturbed Lotka-Volterra models.* Appl. Math. Comp. **219(12)** (2013), 6868–6881.
12. A. Cuspilici, P. Monforte, M. A. Ragusa: *Study of Saharan dust influence on PM10 measures in Sicily from 2013 to 2015.* Ecological Indicators **76** (2017), 297–303.
13. F. D. Chen, X. X. Chen: *The n-competing Volterra-Lotka almost periodic systems with grazing rates.* J. Biomath. **18(4)** (2003), 411–416.

14. J. Dong, Y. Feng, J. Jiang: *A note on implicit fractional differential equations.* Mathematica Aeterna **7(3)** (2017), 261–267.

15. A. Duro, V. Piccione, M. A. Ragusa, V. Veneziano: *New Enviromentally Sensitive Patch Index - ESPI - for MEDALUS protocol.* AIP Conference Proceedings **1637** (2014), 305–312.

16. L. Erbe, C. S. Goodrich, B. Jia, A. Peterson: *Survey of the qualitative properties of fractional difference operators: monotonicity, convexity and asymptotic behavior of solutions.* Adv. Differ. Equ. **2016(43)** (2016), 1–31.

17. C. Goodrich, A. Peterson: *Discrete Fractional Calculus.* Springer 2015.

18. K. Gopalsamy, P. X. Weng: *Feedback regulation of logistic growth.* Int. J. Math. Sci. **16(1)** (1993), 177–192.

19. E. Guariglia: *Fractional Derivative of the Riemann Zeta Function.* In: Fractional Dynamics, C. Cattani, H.M. Srivastava, X.-Y. Yang (Eds.), De Gruyter, 2015, pp. 357–368.

20. E. Guariglia: *Riemann zeta fractional derivative - functional equation and link with primes.* Advances in Difference Equations, **2019** (2019), 261.

21. E. Guariglia, S. Silvestrov: *A functional equation for the Riemann zeta fractional derivative.* AIP Conference Proceedings **1798** (2017), 020063.

22. S. Hasan, M. Al-Smadi, A. Freihet, S. Momani: *Two computational approaches for solving a fractional obstacle system in Hilbert space.* Adv. Differ. Equ. **2019** (2019), 55.

23. C. Li, X. Dao, P. Guo: *Fractional derivatives in complex planes.* Nonlinear Analysis: Theory, Methods & Applications, **71** (2009), 1857–1869.

24. J. X. Li, J. R. Yan: *Persistence for Lotka-Volterra patch-system with time delay.* Nonlinear Anal. **9(2)** (2008), 490–499.

25. M. D. Ortigueira, F. Coito: *From differences to derivatives.* Fract. Calc. Appl. Anal. **7(4)** (2004), 459–471.

Mouataz Billah Mesmouli
Faculty of Sciences, Department of Mathematics
University of Ha'il
Kingdom of Saudi Arabia
mesmoulimouataz@hotmail.com, m.mesmouli@uoh.edu.sa

Abdelouaheb Ardjouni
Faculty of Sciences and Technology
Department of Mathematics and Informatics
University of Souk-Ahras
P.O. Box 1553
Souk-Ahras, 41000, Algeria
abd_ardjouni@yahoo.fr

Ahcene Djoudi
Faculty of Sciences, Department of Mathematics
University of Annaba
P.O. Box 12
Annaba 23000, Algeria
adjoudi@yahoo.com

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