UNIVERSITY OF NIŠ

ISSN 0352-9665 (Print)
ISSN 2406-047X (Online)
COBISS.SR-ID 5881090
UDC 51+004

FACTA UNIVERSITATIS

Series
MATHEMATICS AND INFORMATICS
Vol. 36, No 1 (2021)

Scientific Journal **FACTA UNIVERSITATIS**
UNIVERSITY OF NIŠ
Univerzitetski trg 2, 18000 Niš, Republic of Serbia
Phone: +381 18 257 095 Telefax: +381 18 257 950
e-mail: facta@ni.ac.rs http://casopisi.junis.ni.ac.rs/

Scientific Journal FACTA UNIVERSITATIS publishes original high scientific level works in the fields classified accordingly into the following periodical and independent series:

Architecture and Civil Engineering
Automatic Control and Robotics
Economics and Organization
Electronics and Energetics
Law and Politics
Linguistics and Literature
Mathematics and Informatics
Mechanical Engineering
Medicine and Biology
Philosophy, Sociology, Psychology and History
Physical Education and Sport
Physics, Chemistry and Technology
Teaching, Learning and Teacher Education
Visual Arts and Music
Working and Living Environmental Protection

SERIES **MATHEMATICS AND INFORMATICS**

Founded in 1986 by **Gradimir V. Milovanović**, Serbian Academy of Sciences and Arts, and Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia
The cover image taken from http://www.pptbackgrounds.net/binary-code-and-computer-monitors-backgrounds.html.
Publication frequency – one volume, five issues per year.
Published by the University of Niš, Republic of Serbia

This publication was in part supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia
Printed by "UNIGRAF-X-COPY" – Niš, Republic of Serbia
Circulation 150

ISSN 0352 – 9665 (Print)
ISSN 2406 – 047X (Online)
COBISS.SR-ID 5881090
UDC 51+004

FACTA UNIVERSITATIS

SERIES **MATHEMATICS AND INFORMATICS**
Vol. 36, No 1 (2021)

UNIVERSITY OF NIŠ

INSTRUCTION FOR AUTHORS

The journal Facta Universitatis: Series Mathematics and Informatics publishes original papers of high scientific value in all areas of mathematics and computer science, with a special emphasis on articles in the field of applied mathematics and computer science. Survey articles dealing with interactions between different fields are welcome.

Papers submitted for publication should be concise and written in English. They should be prepared in LaTeX with the style factaMi.cls in accordance with instructions given in the file instructions.tex (see http://casopisi.junis.ni.ac.rs/index.php/FUMathInf/manager/files/styles/facta-style.zip). Under the title, name(s) of the author(s) should be given, at the end of the paper the full name (with official title, institute or company affiliation, etc.) and exact address should appear. Each paper should be accompanied by a brief summary (50–150 words) and by 2010 Mathematics Subject Classification numbers (http://www.ams.org/msc) or 1998 ACM Computing Classification System codes (http://www.acm.org/class). Figures should be prepared in eps format, and footnotes in the text should be avoided if at all possible.

References should be listed alphabetically at the end of the manuscript, in the same way as the following examples (for a book, a paper in a journal, paper in a contributed volume and for an unpublished paper):

[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

References should be quoted in the text by giving the corresponding number in square brackets.

Electronic submission. Manuscripts prepared in the above form should be submitted via Electronic editorial system, available at http://casopisi.junis.ni.ac.rs/index.php/FUMathInf/index. Authors are encouraged to check the home page of the journal and to submit manuscripts through the editorial system.

Galley proofs will be sent to the author.

FACTA UNIVERSITATIS

Series

Mathematics and Informatics

Vol. 36, No 1 (2021)

Contents

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 1-14
https://doi.org/10.22190/FUMI191108001B
Original Scientific Paper

HERIMITIAN SOLUTIONS TO THE EQUATION $AXA^* + BYB^* = C$, FOR HILBERT SPACE OPERATORS

Amina Boussaid and Farida Lombarkia

Faculty of Mathematics and informatics, Department of Mathematics, University of Batna 2, 05078, Batna, Algeria

Abstract. In this paper, by using generalized inverses we have given some necessary and sufficient conditions for the existence of solutions and Hermitian solutions to some operator equations, and derived a new representation of the general solutions to these operator equations. As a consequence, we have obtained a well-known result of Dajić and Koliha.
Keywords: Hilbert space, operator equations, inner inverse, Hermitian solution.

1. Introduction and basic definitions

Let H and K be infinite complex Hilbert spaces, and $\mathbb{B}(H,K)$ the set of all bounded linear operators from H to K. Throughout this paper, the range and the null space of $A \in \mathbb{B}(H,K)$ are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ respectively. An operator $B \in \mathbb{B}(K,H)$ is said to be the inner inverse of $A \in \mathbb{B}(H,K)$ if it satisfies the equation $ABA = A$, we denote the inner inverse by A^-. An operator A is called regular if A^- exists. It is well known that $A \in \mathbb{B}(H,K)$ is regular if and only if A has closed range. There are many papers in which the basic aim is to find necessary and sufficient conditions for the existence of a solution or Hermitian solution to some matrix or operator equations using generalized inverses. In [15, 16, 18], Mitra and Navarra et al. established necessary and sufficient conditions for the existence of a common solution and gave a representation of the general common solution to the pair of matrix equations

$$A_1XB_1 = C_1 \ \ and \ \ A_2XB_2 = C_2. \tag{1.1}$$

Received November 11, 2019; accepted January 7, 2021.
Corresponding Author: Amina Boussaid, Faculty of Mathematics and informatics, Department of Mathematics,University of Batna 2, 05078, Batna, Algeria | E-mail: boussaid1990@gmail.com
2010 *Mathematics Subject Classification.* 47A05; 47A62; 15A09

In [23], Wang considered the same problem for matrices over regular rings with identity. Furthermore, in [13, 16] Khatri and Mitra determined the conditions for the existence of a Hermitian solution and gave the expression of the general Hermitian solution to the matrix equation

$$AXB = C, \tag{1.2}$$

In [8] J. Groß gave the general Hermitian solution to matrix equation (1.2), where $B = A^*$.

Quaternion matrix equations and its general Hermitian solutions have attracted more attention in recent years. The reason for this is a large number of applications in control theory and many other fields, see [9, 10, 11, 12, 14, 24] and the references therein. Among them, the matrix equation

$$AXA^* + BYB^* = C, \tag{1.3}$$

has been studied by Chang and Wang in [1]. They used the generalized singular value decomposition to find necessary and sufficient conditions for the existence of real symmetric solutions. Also in [27, Corollary 3.1], Xu et al found necessary and sufficient conditions for the equation (1.3) to have a Hermitian solution.

Recently several operator equations have been extended from matrices to infinite dimensional Hilbert space, Banach space and Hilbert $\mathcal{C}^*$-modules, see [3, 4, 21], [6, 17, 22, 25, 26] and the references therein.

In this paper, our main objective is to give necessary and sufficient conditions for the existence of a Hermitian solution to the operator equation $AXA^* + BYB^* = C$. After section one where several basic definitions are assembled, in section 2, we give necessary and sufficient conditions for the existence of a common solution to the operator equations

$$A_1XB_1 = C_1 \ \textit{and} \ A_2XB_2 = C_2.$$

In section 3, we apply the result of section 2 to determine new necessary and sufficient conditions for the existence of a Hermitian solution and give a representation of the general Hermitian solution to the operator equation $AXB = C$. Finally, we give some necessary and sufficient condition for the existence of a Hermitian solution to the operator equation $AXA^* + BYB^* = C$.

2. Common solutions to the operator equations $A_1XB_1 = C$ and $A_2XB_2 = C_2$

In this section, we give necessary and sufficient conditions for the existence of a common solution to the pair of equations

$$A_1XB_1 = C_1, \quad A_2XB_2 = C_2,$$

with A_1, A_2, B_1, B_2, C_1 and C_2 are linear bounded operators defined on Hilbert spaces H, K, E, L, N and G. Before enouncing our main results, we recall the following lemmas

Lemma 2.1. *[2] Let $A, B \in \mathbb{B}(H,K)$ are regular operators and $C \in \mathbb{B}(H,K)$. Then the operator equation*

$$AXB = C$$

has a solution if and only if $AA^-CB^-B = C$, or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A) \text{ and } \mathcal{R}(C^*) \subset \mathcal{R}(B^*).$$

A representation of the general solution is

$$X = A^-CB^- + U - A^-AUBB^-,$$

where $U \in \mathbb{B}(K,H)$ is an arbitrary operator.

Lemma 2.2. *[2] Let $A, B \in \mathbb{B}(H,K)$ are regular operators and $C, D \in \mathbb{B}(H,K)$. Then the pair of operators equations*

$$AX = C \quad \text{and} \quad XB = D$$

has a common solution if and only if

$$AA^-C = C, \quad DB^-B = D \quad \text{and} \quad AD = CB,$$

or equivalently

$$\mathcal{R}(C) \subset \mathcal{R}(A), \quad \mathcal{R}(D^*) \subset \mathcal{R}(B^*) \quad \text{and} \quad AD = CB.$$

A representation of the general solution is

$$X = A^-C + DB^- - A^-ADB + (I_H - A^-A)V(I_H - BB^-),$$

where $V \in \mathbb{B}(H)$ is an arbitrary operator.

The following two lemmas can be deduced from a result of Patrício and Puystjens [20] originally formulated for matrix with entries in an associative ring. A simple modification shows that it applies equally well to Hilbert space operators.

Lemma 2.3. *[20] Let $A \in \mathbb{B}(H,K)$ and $B \in \mathbb{B}(E,K)$ be regular operators. Then $\begin{pmatrix} A & B \end{pmatrix} \in \mathbb{B}(H \times E, K)$ is regular if and only if $S = (I_K - AA^-)B$ is regular. In this case, the inner inverse of $\begin{pmatrix} A & B \end{pmatrix}$ is given by*

$$\begin{pmatrix} A & B \end{pmatrix}^- = \begin{pmatrix} A^- - A^-BS^-(I_K - AA^-) \\ S^-(I_K - AA^-) \end{pmatrix}.$$

Lemma 2.4. *[3] Let $A \in \mathbb{B}(H,K)$ and $B \in \mathbb{B}(H,E)$ be regular operators. Then the regularity of any one of the following operators implies the regularity of the remaining three operators*

$$D = B(I_H - A^- A),\ M = A(I_H - B^- B), \begin{pmatrix} A \\ B \end{pmatrix} \quad and \quad \begin{pmatrix} B \\ A \end{pmatrix}.$$

In this case, the inner inverse of $\begin{pmatrix} A \\ B \end{pmatrix}$ *is given by*

$$\begin{pmatrix} A \\ B \end{pmatrix}^- = \begin{pmatrix} (I_H - B^- B)M^- & B^- - (I_H - B^- B)M^- AB^- \end{pmatrix}.$$

Lemma 2.5. *[2] Suppose that* $A_1 \in \mathbb{B}(H,K)$, $A_2 \in \mathbb{B}(H,E)$, $B_1 \in \mathbb{B}(L,G)$, $B_2 \in \mathbb{B}(N,G)$, $S_1 = A_2(I_H - A_1^- A_1)$ *and* $M_1 = (I_G - B_1 B_1^-)B_2$ *are regular operators. Then*

$$T_1 = (I_E - S_1 S_1^-)A_2 A_1^- \ \ and \ \ D_1 = B_1^- B_2(I_N - M_1^- M_1),$$

are regular with inner inverses $T_1^- = A_1 A_2^-$ *and* $D_1^- = B_2^- B_1$.

In the following theorem, we give necessary and sufficient conditions for the existence of a common solution of the operator equations

$$A_1 X B_1 = C_1, \quad A_2 X B_2 = C_2$$

Theorem 2.1. *Suppose that* $A_1 \in \mathbb{B}(H,K)$, $A_2 \in \mathbb{B}(H,E)$, $B_1 \in \mathbb{B}(L,G)$, $B_2 \in \mathbb{B}(N,G)$, $M_1 = (I_G - B_1 B_1^-)B_2$ *and* $S_1 = A_2(I_H - A_1^- A_1)$ *are regular operators and* $C_1 \in \mathbb{B}(L,K)$, $C_2 \in \mathbb{B}(N,E)$. *Then the following statements are equivalent*

1. *The pair of equations (1.1) have a common solution* $X \in \mathbb{B}(G,H)$.
2. *There exists two operators* $U \in \mathbb{B}(N,K)$ *and* $V \in \mathbb{B}(L,E)$, *such that the operator equation* $AXB = C$ *is solvable, where*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}.$$

3. *For* $i = 1, 2$, $\mathcal{R}(C_i) \subset \mathcal{R}(A_i)$, $\mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*)$ *and*

$$T_1 C_1 D_1 = T_2 C_2 D_2,$$

where $T_1 = (I_E - S_1 S_1^-)A_2 A_1^-$, $T_2 = (I_E - S_1 S_1^-)$, $D_1 = B_1^- B_2(I_N - M_1^- M_1)$ *and* $D_2 = (I_N - M_1^- M_1)$.

Proof.

(1) $\Leftrightarrow$ (2) The equivalence is easily established.

(2) $\Rightarrow$ (3) According to Lemma 2.1, the operator equation $AXB = C$ has a solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \quad and \quad \mathcal{R}(C^*) \subset \mathcal{R}(B^*),$$

then, we deduce that

$$\text{(2.1)} \qquad for \quad i = 1,2, \quad \mathcal{R}(C_i) \subset \mathcal{R}(A_i) \quad and \quad \mathcal{R}(C_i^*) \subset \mathcal{R}(B_i^*).$$

On the other hand, we have

$$T_1C_1D_1 = (I_E - S_1S_1^-)A_2A_1^-C_1B_1^-B_2(I_N - M_1^-M_1)$$

$$\text{(2.2)} \qquad = (I_E - S_1S_1^-)A_2A_1^-A_1X_0B_1B_1^-B_2(I_N - M_1^-M_1),$$

where X_0 is the common solution to the pair of equations (1.1).
Let

$$S_1 = A_2(I_H - A_1^-A_1) \quad and \quad M_1 = (I_G - B_1B_1^-)B_2.$$

This implies that

$$\text{(2.3)} \qquad A_2A_1^-A_1 = A_2 - S_1 \quad and \quad B_1B_1^-B_2 = B_2 - M_1.$$

We insert (2.3) in (2.2) to obtain

$$\text{(2.4)} \qquad T_1C_1D_1 = T_2C_2D_2.$$

From (2.1) and (2.4), we deduce that (2) $\Rightarrow$ (3).

Conversely, since

$$T_1C_1D_1 = T_2C_2D_2.$$

Then

$$\mathcal{R}(T_2C_2) \subset \mathcal{R}(T_1) \quad and \quad \mathcal{R}(D_1^*C_1^*) \subset \mathcal{R}(D_2^*).$$

By applying Lemma 2.2, there exist $U \in \mathbb{B}(N,K)$ which is the common solution to the pair of equations

$$\text{(2.5)} \qquad \begin{cases} T_1U = T_2C_2 \\ UD_2 = C_1D_1, \end{cases}$$

given by

$$\text{(2.6)} \qquad U = C_1D_1 + T_1^-(I_E - S_1S_1^-)C_2M_1^-M_1 + (A_1A_1^- - T_1^-T_1)ZM_1^-M_1,$$

where $Z \in \mathbb{B}(N,K)$ is an arbitrary operator.
On other hand, since

$$T_1C_1D_1 = T_2C_2D_2.$$

Then

$$\mathcal{R}(T_1C_1) \subset \mathcal{R}(T_2) \quad and \quad \mathcal{R}(D_2^*C_2^*) \subset \mathcal{R}(D_1^*).$$

It follows from Lemma 2.2 that there exist $V \in \mathbb{B}(L,E)$ which is the common solution to the pair of equations

$$\text{(2.7)} \qquad \begin{cases} T_2V = T_1C_1 \\ VD_1 = C_2D_2, \end{cases}$$

given by

$$V = T_1C_1 + S_1S_1^-C_2(I_N - M_1^-M_1)D_1^- + S_1S_1^-Z'(B_1^-B_1 - D_1D_1^-), \tag{2.8}$$

where $Z' \in \mathbb{B}(L,E)$ is an arbitrary operator.
Thus, there exists $U \in \mathbb{B}(N,K)$ and $V \in \mathbb{B}(L,E)$ solutions to the pair of equations (2.5), (2.7) and as for $i = 1,2$, we have $A_iA_i^-C_i = C_i$ and $C_iB_i^-B_i = C_i$, we obtain

$$\begin{aligned} &AA^-CB^-B = \\ &= \begin{pmatrix} A_1A_1^-C_1B_1^-B_1 & A_1A_1^-(C_1D_1 + UM_1^-M_1) \\ (T_1C_1 + S_1S_1^-V)B_1^-B_1 & T_1(C_1D_1 + UM_1^-M_1) + S_1S_1^-(VD_1 + C_2M_1M_1^- \end{pmatrix} \\ &= C. \end{aligned}$$

So that, the operator equation $AXB = C$ is solvable and (3) $\Rightarrow$ (2). $\square$

Theorem 2.2. *Suppose that $A_1 \in \mathbb{B}(H,K)$, $A_2 \in \mathbb{B}(H,E)$, $B_1 \in \mathbb{B}(L,G)$, $B_2 \in \mathbb{B}(N,G)$, $M_1 = (I_G - B_1B_1^-)B_2$ and $S_1 = A_2(I_H - A_1^-A_1)$ are regular operators and $C_1 \in \mathbb{B}(L,K)$, $C_2 \in \mathbb{B}(N,E)$, when any one of the conditions (2), (3) of Theorem 2.1 holds, a general common solution to the pair of equations (1.1) is given by*

$$\begin{aligned} X &= (A_1^-C_1 + (I_H - A_1^-A_1)S_1^-(V - A_2A_1^-C_1))B_1^-(I_G - B_2M_1^-(I_G - B_1B_1^-)) \\ &+ (A_1^-U + (I_H - A_1^-A_1)S_1^-(C_2 - A_2A_1^-U))M_1^-(I_G - B_1B_1^-) + F \\ &- (A_1^-A_1 + (I_H - A_1^-A_1)S_1^-S_1)F(B_1B_1^- + M_1M_1^-(I_G - B_1B_1^-)), \end{aligned} \tag{2.9}$$

where $F \in \mathbb{B}(G,H)$ is an arbitrary operator and U, V are given by

$$\begin{cases} U = C_1B_1^-B_2(I_N - M_1^-M_1) + A_1A_2^-(I_E - S_1S_1^-)C_2M_1^-M_1 + A_1A_1^-ZM_1^-M_1 \\ \quad -A_1A_2^-(I_E - S_1S_1^-)A_2A_1^-ZM_1^-M_1, \\ \text{and} \\ V = (I_E - S_1S_1^-)A_2A_1^-C_1 + S_1S_1^-C_2(I_N - M_1^-M_1)B_2^-B_1 + S_1S_1^-Z'B_1^-B_1 \\ \quad -S_1S_1^-Z'B_1^-B_2(I_N - M_1^-M_1)B_2^-B_1, \end{cases}$$

where $Z \in \mathbb{B}(N,K)$ and $Z' \in \mathbb{B}(L,E)$ are arbitrary operators.

Proof. From Theorem 2.1, we get that the pair of equations (1.1) has a common solution equivalently the two conditions (2) and (3) holds.
On the other hand, since the pair of equations (1.1) is equivalent to

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix}. \tag{2.10}$$

According to Lemma 2.3 and Lemma 2.4, we have

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in \mathbb{B}(H, K \times E) \quad \text{and} \quad \begin{pmatrix} B_1 & B_2 \end{pmatrix} \in \mathbb{B}(L \times N, G)$$

are regular with inner inverses

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- = \begin{pmatrix} (I_E - A_2^- A_2)S_1^- & A_2^- - (I_E - A_2^- A_2)S_1^- A_1 A_2^- \end{pmatrix}, \tag{2.11}$$

and

$$\begin{pmatrix} B_1 & B_2 \end{pmatrix}^- = \begin{pmatrix} B_1^- - B_1^- B_2 M_1^- (I_G - B_1 B_1^-) \\ M_1^- (I_G - B_1 B_1^-) \end{pmatrix}, \tag{2.12}$$

respectively.
Using Lemma 2.1, we deduce that the general solution of (2.10) is given by

$$\begin{aligned} X &= \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- \begin{pmatrix} C_1 & U \\ V & C_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^- + \\ &\quad + F - \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^- \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} F \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^- . \end{aligned} \tag{2.13}$$

By substituting (2.11) and (2.12) in (2.13), we get the solution X as defined in (2.9) such that U, V are given in (2.6) and (2.8) respectively and $F \in \mathbb{B}(G, H)$ is an arbitrary operator. $\square$

3. Hermitian solutions to the operator equations $AXB = C$ and $AXA^* + BYB^* = C$

Based on Theorem 2.1 and Theorem 2.2, in this section we give necessary and sufficient conditions for the existence of Hermitian solutions to the operator equations

$$AXB = C \quad and \quad AXA^* + BYB^* = C$$

and obtain the general Hermitian solution to those operator equations respectively. Before enouncing our main results we have the following lemma

Lemma 3.1. *Let $A \in \mathbb{B}(H, K)$ and $B \in \mathbb{B}(K, H)$, such that A, B, $S_1 = B^*(I_H - A^- A)$ and $M_1 = (I_H - BB^-)A^*$ are regular. Then the operator equation*

$$AXB = C,$$

has a Hermitian solution if and only if the pair of operator equations

$$AXB = C \quad and \quad B^* X A^* = C^* \tag{3.1}$$

has a common solution, a representation of the general Hermitian solution to $AXB = C$ is of the form

$$X_H = \frac{X + X^*}{2},$$

where X is the representation of the general common solution to the equations (3.1).

Proof. From Theorem 2.1 the pair of operator equations (3.1) has a common solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A) \text{ and } \mathcal{R}(C^*) \subset \mathcal{R}(B^*),$$

and

$$(I_K - S_1 S_1^-) B^* A^- C B^- A^* (I_K - M_1^- M_1) = (I_K - S_1 S_1^-) C^* (I_K - M_1^- M_1).$$

A representation of the general common solution to equations (3.1) is given by (2.9) in Theorem 2.2, where $A_1 = A$, $B_1 = B$, $C_1 = C$, $A_2 = B^*$, $B_2 = A^*$ and $C_2 = C^*$. Clearly X_H is a Hermitian solution to (1.2). □

From the above proof and Theorem 2.2, we obtain the following corollary.

Corollary 3.1. *Let $A \in \mathbb{B}(H,K)$, $B \in \mathbb{B}(K,H)$, $M_1 = (I_H - BB^-)A^*$ and $S_1 = B^*(I_H - A^-A)$ are regular operators and $C \in \mathbb{B}(K)$. Then the operator equation*

$$AXB = C,$$

has a Hermitian solution if and only if

1. $\mathcal{R}(C) \subset \mathcal{R}(A)$ *and* $\mathcal{R}(C^*) \subset \mathcal{R}(B^*)$
2. $(I_K - S_1 S_1^-) B^* A^- C B^- A^* (I_K - M_1^- M_1) = (I_K - S_1 S_1^-) C^* (I_K - M_1^- M_1)$.

In this case, a representation of the general Hermitian solution is of the form

$$X_H = \frac{X + X^*}{2},$$

where

$$\begin{aligned} X &= (A^-C + (I_H - A^-A)S_1^-(V - B^*A^-C))B^-(I_H - A^*M_1^-(I_H - BB^-)) \\ &\quad +(A^-U + (I_H - A^-A)S_1^-(C^* - B^*A^-U))M_1^-(I_H - BB^-) + F \\ &\quad - (A^-A + (I_H - A^-A)S_1^-S_1)F(BB^- + M_1M_1^-(I_H - BB^-), \end{aligned} \tag{3.2}$$

where $F \in \mathbb{B}(H)$ is an arbitrary operator and U, V are given by

$$\begin{cases} U = CB^-A^*(I_K - M_1^-M_1) + A(B^*)^-(I_K - S_1S_1^-)C^*M_1^-M_1 + AA^-ZM_1^-M_1 \\ \quad -A(B^*)^-(I_K - S_1S_1^-)B^*A^-ZM_1^-M_1 \\ \text{and} \\ V = (I_K - S_1S_1^-)B^*A^-C + S_1S_1^-C^*(I_K - M_1^-M_1)(A^*)^-B + S_1S_1^-Z'B^-B \\ \quad -S_1S_1^-Z'B^-A^*(I_K - M_1^-M_1)(A^*)^-B, \end{cases}$$

where $Z, Z' \in \mathbb{B}(K)$ are arbitrary operators.

Corollary 3.2. *Let $A \in \mathbb{B}(H,K)$, $C \in \mathbb{B}(K)$ such that A is regular and $C^* = C$. Then the operator equation*

$$AXA^* = C$$

has a Hermitian solution $X \in \mathbb{B}(H)$ if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A)$$

In this case, a representation of the general Hermitian solution is

$$X_H = A^- C(A^-)^* + F - A^- AF(A^- A)^*, \tag{3.3}$$

where $F \in \mathbb{B}(H)$ is an arbitrary Hermitian operator.

Proof. We put $B = A^*$ in Corollary 3.1 we get the result. □

As a consequence of Corollary 3.1 we obtain the well-known Theorem of Alegra Dajić and J.J. Koliha [3, Theorem 3.1].

Corollary 3.3. *[3, Theorem 3.1] Let $A, C \in \mathbb{B}(H,K)$ such that A is a regular operator. Then the operator equation*

$$AX = C$$

has a Hermitian solution $X \in \mathbb{B}(H)$ if and only if

$$AA^- C = C \quad \textit{and} \quad AC^* \textit{ is Hermitian.}$$

The general Hermitian solution is of the form

$$X_H = A^- C + (I_H - A^- A)(A^- C)^* + (I_H - A^- A)Z'(I_H - A^- A)^*,$$

where $Z' \in \mathbb{B}(H)$ is an arbitrary Hermitian operator.

Proof. By applying Corollary 3.1, the operator equation $AX = C$ has a Hermitian solution if and only if

$$\mathcal{R}(C) \subset \mathcal{R}(A),$$

which is equivalent to

$$AA^- C = C,$$

and

$$(I_H - I_H + A^- A)A^- CA^* = (I_H - I_H + A^- A)C^*,$$

this implies that

$$CA^* = AC^*.$$

Hence, AC^* is Hermitian. In this case,

$$\begin{aligned} X &= [A^- C + (I_H - A^- A)(A^- C + (I_H - A^- A)C^*(A^*)^- + \\ &+ (I_H - A^- A)Z'(I_H - A^- A)^* - A^- C)], \\ &= A^- C + (I_H - A^- A)(A^- C)^* + (I_H - A^- A)Z'(I_H - A^- A)^*. \end{aligned}$$

It follows that,

$$\begin{aligned} X_H &= \frac{X+X^*}{2}, \\ &= A^-C + (I_H - A^-A)(A^-C)^* + (I_H - A^-A)Z'(I_H - A^-A)^*. \end{aligned}$$

□

Theorem 3.1. *Let $A, B \in \mathbb{B}(H,K)$ and $A_1 = (I_K - AA^-)B$, $C_1 = (I_K - AA^-)C$ and $S_2 = B(I_H - A_1^- A_1)$ be all regular and $C \in \mathbb{B}(K)$ is Hermitian. Then the operator equation*

$$AXA^* + BYB^* = C,$$

has a Hermitian solution if and only if

1. $A_1 A_1^- (I_K - AA^-)C(B^*)^- B^* = (I_K - AA^-)C$
2. $(I_K - S_2 S_2^-)[C - BA_1^-(I_K - AA^-)C(B^*)^- B^*](I_K - (A^-)^* A^*) = 0.$

In this case, a representation of the general Hermitian solution is of the form

$$(X_H, Y_H) = \left(\frac{X+X^*}{2}, \frac{Y+Y^*}{2}\right),$$

where X and Y are given by

$$\begin{cases} X = A^-(C - BYB^*)(A^*)^- + F - A^-AF(A^-A)^* \\ \text{and} \\ Y = A_1^-(I_K - AA^-)C(B^*)^- + \\ \quad +(I_H - A_1^-A_1)S_2^-[V - BA_1^-(I_K - AA^-)C](B^*)^- + U \\ \quad -[A_1^-A_1 + (I_H - A_1^-A_1)S_2^-S_2]UB^*(B^*)^-, \end{cases}$$

and

$$\begin{aligned} V &= (I_K - S_2S_2^-)BA_1^-(I_K - AA^-)C + S_2S_2^-C(I_K - (A^-)^*A^*)(A_1^*)^-B^* \\ &+ S_2S_2^-Z(B^*)^-(I_H - A_1^*(A_1^-)^*)B^*, \end{aligned}$$

with $F \in \mathbb{B}(H)$, $U \in \mathbb{B}(H)$ and $Z \in \mathbb{B}(K)$ are arbitrary Hermitian operators.

Proof. The operator equation (1.3) is equivalent to

$$AXA^* = C - BYB^*. \tag{3.4}$$

Applying Corollary 3.2, the operator equation (3.4) has a Hermitian solution if and only if

$$\begin{aligned} \mathcal{R}(C - BYB^*) \subset \mathcal{R}(A) &\Leftrightarrow AA^-(C - BYB^*) = (C - BYB^*), \\ &\Leftrightarrow (I - AA^-)(C - BYB^*) = 0. \end{aligned} \tag{3.5}$$

Then, (3.5) is equivalent to the operator equation

$$A_1 Y B^* = C_1, \tag{3.6}$$

with $A_1 = (I_K - AA^-)B$, and $C_1 = (I_K - AA^-)C$.
From Corollary 3.1, the operator equation (3.6) has a Hermitian solution if and only if

$$\begin{aligned} \mathcal{R}(C_1) \subset \mathcal{R}(A_1) &\Leftrightarrow A_1 A_1^- C_1 = C_1, \\ &\Leftrightarrow A_1 A_1^- (I_K - AA^-)C = (I_K - AA^-)C, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \mathcal{R}(C_1^*) \subset \mathcal{R}(B) &\Leftrightarrow C_1 (B^*)^- B^* = C_1, \\ &\Leftrightarrow (I_K - AA^-)C(B^*)^- B^* = (I_K - AA^-)C. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get

$$A_1 A_1^- (I_K - AA^-)C(B^*)^- B^* = (I_K - AA^-)C.$$

On the other hand, we have

$$(I_K - S_2 S_2^-)BA_1^-(I_K - AA^-)C(B^*)^- A_1^* = (I_K - S_2 S_{2-})C(I_K - (A^-)^* A^*).$$

This implies that

$$(I_K - S_2 S_2^-)[C - BA_1^-(I_K - AA^-)C(B^*)^- B^*](I_K - (A^-)^* A^*) = 0.$$

A representation of the general Hermitian solution to the operator equation (3.6) is of the form

$$Y_H = \frac{Y + Y^*}{2},$$

where Y is given by (3.2) in Corollary 3.1 such that $A = A_1$, $B = B^*$ and $C = C_1$

$$\begin{aligned} Y &= A_1^-(I_K - AA^-)C(B^*)^- + (I_H - A_1^- A_1)S_2^-[V - BA_1^-(I_K - AA^-)C](B^*)^- + \\ &+ U - [A_1^- A_1 + (I_H - A_1^- A_1)S_2^- S_2]UB^*(B^*)^-, \end{aligned}$$

and

$$\begin{aligned} V &= (I_K - S_2 S_2^-)BA_1^-(I_K - AA^-)C + S_2 S_2^- C(I_K - (A^-)^* A^*)(A_1^*)^- B^* + \\ &+ S_2 S_2^- Z(B^*)^-(I_H - A_1^*(A_1^-)^*)B^*, \end{aligned}$$

with $U \in \mathbb{B}(H)$ and $Z \in \mathbb{B}(K)$ are arbitrary Hermitian operators.
We return to the operator equation

$$AXA^* = C - BYB^*,$$

in order to find the Hermitian solution X.

By Corollary 3.2, the operator equation (3.4) has a Hermitian solution if and only if

$$\mathcal{R}(C - BYB^*) \subset \mathcal{R}(A).$$

So the operator equation (3.4) has a Hermitian solution X_H given by

$$X_H = A^-(C - BYB^*)(A^*)^- + F - A^-AF(A^-A)^*,$$

with $F \in \mathbb{B}(H)$ is an arbitrary Hermitian operator. $\square$

4. Conclusions

This paper gives necessary and sufficient conditions for the existence of a common solution to the pair of equations

$$A_1XB_1 = C_1 \ \ and \ \ A_2XB_2 = C_2;$$

We have applied this result to determine new necessary and sufficient conditions for the existence of Hermitian solution and given a representation of the general Hermitian solution to the operator equation

$$AXB = C.$$

Then, we have given necessary and sufficient conditions for the existence of Hermitian solution and a representation of the general Hermitian solution to the operator equation

$$AXA^* + BYB^* = C.$$

Acknowledgments

The authors are grateful to the referee for several helpful remarks and suggestions concerning this paper.

REFERENCES

1. X. W. Chang and J. Wang: *The symmetric solutions of the matrix equations $AX + YA = C$, $AXA^T + BYB^T = C$ and $(A^TXA, B^TXB) = (C, D)$*. Linear Algebra and Appl. **179** (1993), 171–189.
2. A. Dajić: *Common solution of linear equations in ring, with application.* Electonic Journal of Linear Algebra, **30** (2015), 66–79.
3. A. Dajić and J. J. Koliha: *Positive solution to the equation $AX = C$ and $XB = D$ for hilbert space operators.* J. Math. Anal. Appl. **333** (2007), 567–576.
4. A. Dajić and J. J. Koliha: *Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators.* Linear Algebra and its Appl. **429** (2008), 1779–1809.

5. Y. Deng and X. Hu: *On solutions of matrix equation $AXA^T + BYB^T = C$.* Journal of Computational Math. **23** (2005), 17–26.
6. F. O. Farid, M. S. Moslehian, et al.: *On the Hermitian solutions to a system of adjointable operator equations.* Linear Algebra and its Appl. **437** (2012), 1854–1891.
7. P. A. Fillmore and J. P. Williams: *On operator ranges.* Advances in Math. **7** (1971), 244–281.
8. J. Groß: *A note on the general Hermitian solution to $AXA^* = B$.* Bull. Malaysian Math. Soc. (Second Series), **21** (1998), 57–62.
9. Z. H. He: *Some new results on a system of sylvester-type quaternion matrix equations.* Linear and Multilinear Algebra, DOI: 10.1080/03081087.2019.1704213.
10. Z. H. He, M. Wang and X. Liu; *On the general solutions to some systems of quternion matrix equations.* RACSAM (2020), 114:95.
11. Z. H. He: *Some quaternion matrix equations involving ϕ-Hermicity.* Filomat, **33** (2019), 5097–5112.
12. Z. H. He: *A system of coupled quaternion matrix equations with seven unknowns and its applications.* Adv Appl Clifford Algebras, (2019), 29:38.
13. C. G. Khatri and S. K. Mitra: *Hermitian and nonnegative definite solution of linear matrix equations.* SIAM J. Appl. Math. **31(4)** (1976), 579–585.
14. N. Li, J. Jiang and W. Wang: *Hermitian solution to a quaternion matrix equation.* Applied Mechanics and materials, **50-51** (2011), 391–395.
15. S. K. Mitra: *Common solution to a pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$.* Mathematical Proceedings of the Cambridge Philosophical Society, **74** (1973), 213–216.
16. S. K. Mitra: *A pair of simultaneous linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ and a matrix programming problem.* Linear Algebra and its Appl. **131** (1990), 107–123.
17. Z. Mousavi, R. Eskandari, et al.: *Operator equations $AX+YB = C$ and $AXA^* + BYB^* = C$ in Hilbert $\mathcal{C}^*$-Modules.* Linear Algebra and Its Appl. **517** (2017), 85–98.
18. A. Navarra, P. L. Odell and D. M. Young: *A representation of the general common solution to the matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ with applications.* Computers and Mathematics with Applications, **41** (2001), 929–935.
19. A. B. Ozguler and N. A. Akar: *A common solution to a pair of linear matrix equations over a principal domain.* Linear Algebra and its Appl. **144** (1991), 85–99.
20. P. Patrício and R. Puystjens: *About the von Neumann regularity of triangular block matrices.* Linear Algebra and its Appl. **332-334** (2001), 485-–502.
21. S. V. Phadke and N. K. Thakare: *Generalized inverses and operator equations.* Linear Algebra and its Appl. **23** (1979), 191–199.
22. M. Vosough and M. S. Moslehian: *Solvability of the matrix inequality.* Linear and Multilinear Algebra, **66(9)** (2017), 1799–1818.
23. Q. W. Wang: *A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity.* Linear Algebra and its Appl. **384** (2004), 43–54.
24. Q. W. Wang and Z. H. He: *Some matrix equations with applications.* Linear Multilinear Algebra, **60:11-12** (2012), 1327–1353.
25. Q. Xu: *Common hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$.* Linear Algebra and its Appl. **429** (2008), 1–11.

26. Q. Xu, L. Sheng and Y. Gu: *The solutions to some operator equations.* Linear Algebra and its Appl. **429** (2008), 1997–2024.
27. G. Xu, M. Wei and D. Zheng: *On solutions of matrix equation* $AXB + CYD = F$. Linear Algebra and its Appl. **279** (1998), 93–109.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 15-29
https://doi.org/10.22190/FUMI191220002A

Original Scientific Paper

SOME REMARKS ON THE CLASSICAL PRIME SPECTRUM OF MODULES

Alireza Abbasi and Mohammad Hasan Naderi

Faculty of Science,Department of Mathematics, University of Qom, Qom, Iran, P.O. Box 37161-46611

Abstract. Let R be a commutative ring with identity and let M be an R-module. A proper submodule P of M is called a classical prime submodule if $abm \in P$, for $a, b \in R$, and $m \in M$, implies that $am \in P$ or $bm \in P$. The classical prime spectrum of M, Cl.Spec(M), is defined to be the set of all classical prime submodules of M. We say M is classical primefule if $M = 0$, or the map ψ from Cl.Spec(M) to Spec$(R/\mathrm{Ann}(M))$, defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for all $P \in$ Cl.Spec(M), is surjective. In this paper, we study classical primeful modules as a generalization of primeful modules. Also, we investigate some properties of a topology that is defined on Cl.Spec(M), named the Zariski topology.
Keywords: Classical prime, Classical primeful, Classical top module

1. Introduction

Throughout the paper all rings are commutative with identity and all modules are unital. Let M be an R-module. If N is a submodule of M, then we write $N \leq M$. For any two submodules N and K of an R-module M, the residual of N by K is denoted by $(N : K) = \{r \in R : rK \subseteq N\}$. A proper submodule P of M is called a prime submodule if $am \in P$, for $a \in R$ and $m \in M$, implies that $m \in P$ or $a \in (P : M)$. Also, a proper submodule P of M is called a classical prime submodule if $abm \in P$, for $a, b \in R$ and $m \in M$, implies that $am \in P$ or $bm \in P$(see for example [5]). The set of prime(resp. classical prime) submodules of M is denoted by Spec(M)(resp. Cl.Spec(M)). The class of prime submodules of modules was introduced and studied in 1992 as a generalization of

Received December 20, 2019; accepted June 26, 2020.
Corresponding Author: Mohammad hasan Naderi, Faculty of Science,Department of Mathematics, University of Qom, Qom, Iran, P.O. Box 37161-46611 | E-mail: mh.naderi@qom.ac.ir
2010 *Mathematics Subject Classification.* Primary 13C13; Secondary 13C99, 13A15, 13A99

the class of prime ideals of rings. Then, many generalizations of prime submodules were studied such as primary, classical prime, classical primary and classical quasi primary submodules, see [1, 8, 16, 4] and [7].

For a proper submodule N of an R-module M, the prime radical of N is $\sqrt[p]{N} = \cap\{P|P \in \mathcal{V}^*(N)\}$, where $\mathcal{V}^*(N) = \{P \in \mathrm{Spec}(M) \mid N \subseteq P\}$. Also the classical prime radical of N is $\sqrt[Cl]{N} = \cap\{P|P \in \mathcal{V}(N)\}$, where $\mathcal{V}(N) = \{P \in \mathrm{Cl.Spec}(M) \mid N \subseteq P\}$. If there are no such prime (resp. classical prime) submodules, $\sqrt[p]{N}$ (resp. $\sqrt[Cl]{N}$) is M. We say N is a radical (resp. classical radical) submodule, if $\sqrt[p]{N} = N$(resp. $\sqrt[Cl]{N} = N$).

The set of all maximal submodules of M is denoted by $\mathrm{Max}(M)$. A Noetherian module M is called a semi-local (resp. a local) module if $\mathrm{Max}(M)$ is a non-empty finite (resp. a singleton) set. A non-Noetherian commutative ring R is called a quasisemilocal (resp. a quasilocal) ring if R has only a finite number (resp. a singleton) of maximal ideals. An R-module M is called a multiplication (resp. weak multiplication) module if for every submodule (resp. prime submodule) of M, there exists an ideal I of R such that $N = IM$(see [14] and [2]). If N is a prime submodule of a multiplication R-module M, then $N_1 \cap N_2 \subseteq N$, where $N_1, N_2 \leq M$, implies that $N_1 \subseteq N$ or $N_2 \subseteq N$ (see for more detail [11] and [19]). An R-module M is called compatible if its classical prime submodules and its prime submodules coincide. All commutative rings and multiplicative modules are examples of compatible modules, (see for more detail [8]). A submodule N of M is said to be strongly irreducible if for submodules N_1 and N_2 of M, the inclusion $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. Strongly irreducible submodules have been characterized in [13].

Let M be an R-module. For any subset E of M, we consider classical varieties denoted by $\mathcal{V}(E)$. We define $\mathcal{V}(E) = \{P \in \mathrm{Cl.Spec}(M) : E \subseteq P\}$. Then

(a) If N is a submodule generated by E, then $\mathcal{V}(E) = \mathcal{V}(N)$.

(b) $\mathcal{V}(0_M) = \mathrm{Cl.Spec}(M)$ and $\mathcal{V}(M) = \emptyset$.

(c) $\bigcap_{i\in I} \mathcal{V}(N_i) = \mathcal{V}(\sum_{i\in I} N_i)$, where $N_i \leq M$

(d) $\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L)$, where $N, L \leq M$.

Now, we assume that $\mathcal{C}(M)$ denotes the collection of all subsets $\mathcal{V}(N)$ of $\mathrm{Cl.Spec}(M)$. Then, $\mathcal{C}(M)$ contains the empty set and $\mathrm{Cl.Spec}(M)$, and also $\mathcal{C}(M)$ are closed under arbitrary intersections. However, in general, $\mathcal{C}(M)$ is not closed under finite union. An R-module M is called a classical top module if $\mathcal{C}(M)$ is closed under finite unions, i.e., for every submodules N and L of M, there exists a submodule K of M such that $\mathcal{V}(N) \cup \mathcal{V}(L) = \mathcal{V}(K)$, for in this case, $\mathcal{C}(M)$ satisfies the axioms for the closed subsets of a tological space, then in this case, $\mathcal{C}(M)$ induce a topology on $\mathrm{Cl.Spec}(M)$. We call the induced topology the classical quasi-Zariski topology(see [9]).

In this paper, we introduce the notion of classical primeful modules and also we investigate some properties of classical quasi-Zariski topology of $\mathrm{Cl.Spec}(M)$. In Section 2, we introduce the notion of classical primeful modules as a generalization of primefule modules. In particular, in Proposition 2.3, it is proved that if M is a classical primeful R-module, then $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$. Then we get some properties of classical top modules. In Section 3, we get some properties of classical quasi-Zariski topology of $\mathrm{Cl.Spec}(M)$ and also we get some properties of classical top modules.

2. Classical primeful module

The notion of primeful modules was introduced by Chin P. Lu in [18] as follows:

Definition 2.1. An R-module M is primeful if either $M = (0)$, or $M \neq (0)$ and the map $\phi : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$, defined by $\phi(P) = (P : M)/\mathrm{Ann}(M)$ for all $P \in \mathrm{Spec}(M)$, is surjective.

Now, we extend the notion of primeful modules to classical primeful modules.

Definition 2.2. Suppose $\mathrm{Cl.Spec}(M) \neq \emptyset$, then the map ψ from $\mathrm{Cl.Spec}(M)$ to $\mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for all $P \in \mathrm{Cl.Spec}(M)$, will be called the natural map of $\mathrm{Cl.Spec}(M)$.

An R-module M is classical primeful if either

(i) $M = (0)$, or

(ii) $M \neq (0)$ and the map $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ from above is surjective.

Lemma 2.1. *Let M be a classical top R-module. Then the natural map $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ is injective.*

Proof. Let $P, Q \in \mathrm{Cl.Spec}(M)$. If $\psi(P) = \psi(Q)$, then

$$(P : M)/\mathrm{Ann}(M) = (Q : M)/\mathrm{Ann}(M).$$

So $(P : M) = (Q : M)$ and then $P = Q$. □

Theorem 2.1. *Let M be a classical top R-module. Then, If R satisfies ACC on prime ideals, then M satisfies ACC on classical prime submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of classical prime submodules of M. This induces the following chain of prime ideals, $\psi(N_1) \subseteq \psi(N_2) \subseteq \ldots$, where ψ is the natural map

$$\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M)).$$

Since R satisfies ACC on prime ideals, there exists a positive integer k such that for each $i \in \mathbb{N}$, $\psi(N_k) = \psi(N_{k+i})$. Now by Lemma 2.1, we have $N_k = N_{k+i}$ as required. □

Remark 2.1. *([8, Proposition 5.3])) Let S be a multiplicatively closed subset of R, p a prime ideal of R such that $p \cap S = \emptyset$ and let M be an R-module. If P is a classical p-prime submodule of M with $P_s \neq M_s$, then P_s is also a classical p_s-prime submodule of M_s. Moreover if Q is a prime R_s-submodule of M_s, then*

$$Q^c = \{m \in M : f(m) \in Q\}$$

is a classical prime submodule of M.

Let p be a prime ideal of a ring R, M an R-module and $N \leqslant M$. By the saturation of N with respect to p, we mean the contraction of N_p in M and designate it by $S_p(N)$. It is also known that

$$S_p(N) = \{e \in M | es \in N for\ some\ s \in R \setminus p\}.$$

Saturations of submodules were investigated in detail in [17].

Proposition 2.1. *For any nonzero R-module M, the following are equivalent:*

(1) The natural map $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ is surjective;

(2) For every $p \in \mathrm{V}(\mathrm{Ann}(M))$, there exists $P \in \mathrm{Cl.Spec}(M)$ such that $(P : M) = p$;

(3) $pM_p \neq M_p$, for every $p \in \mathrm{V}(\mathrm{Ann}(M))$;

(4) $S_p(pM)$, the contraction of pM_p in M, is a classical p-prime submodule of M for every $p \in \mathrm{V}(\mathrm{Ann}(M))$;

(5) $\mathrm{Cl.Spec_p(M)} \neq \emptyset$; for every $p \in \mathrm{V}(\mathrm{Ann}(M))$.

Proof. (1)$\Longleftrightarrow$(2): It is clear by Definition 2.2.

(2)$\Longrightarrow$(3): Let $p \in \mathrm{V}(\mathrm{Ann}(M))$ and let N be a classical p-prime submodule of M. Then N_p is a classical pR_p-prime submodule of M_p by Remark 2.1. Now, since $pM_p \subseteq N_p \subsetneq M_p$, we conclude that $pM_p \neq M_p$.

(3)$\Longrightarrow$(4): Since pR_p is the maximal ideal of R_p and $pM_p \neq M_p$, $pM_p = (pR_p)M_p$ is a pR_p-prime, and therefore classical pR_p-prime, submodule of M_p. Then $S_p(pM) = (pM_p)^c$, the contraction of pM_p in M, is a classical p-prime submodule of M by Remark 2.1.

(4)$\Longrightarrow$(5) and (5)$\Longrightarrow$(2) are easy. $\square$

Proposition 2.2. *Every finitely generated R-module M is classical primeful.*

Proof. If $M = 0$, evidently the results is true. Now, let M be a nonzero finitely generated R-module. Then $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$, so for every $p \in \mathrm{V}(\mathrm{Ann}(M))$, M_p is a nonzero finitely generated module over the local ring R_p. Then by virtue of Nakayama's Lemma, $pM_p \neq M_p$, for every $p \in \mathrm{V}(\mathrm{Ann}(M))$. Therefore by Proposition 2.1, M is classical primeful. □

For every finitely generated module M, $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$. The next proposition proves that the equality holds even if M is only a classical primeful module.

Proposition 2.3. *(see [18, Proposition 3.4])) If M is a classical primeful R-module, then* $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$.

Proof. If $M = (0)$, then $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M)) = \emptyset$. Now let M be a nonzero classical primeful R-module, so $\mathrm{V}(\mathrm{Ann}(M)) \neq \emptyset$. By Proposition 2.1, if $p \in \mathrm{V}(\mathrm{Ann}(M))$, then $S_p(pM)$ is a classical p-prime submodule of M, so $\mathrm{S}_p(pM) \neq M$. Since $\mathrm{S}_p(0) \subseteq \mathrm{S}_p(pM)$, then $M \neq \mathrm{S}_p(0)$, from which we can see that $M_p \neq (0)$. Thus $\mathrm{V}(\mathrm{Ann}(M)) \subseteq \mathrm{Supp}(M)$. The other inclusion is always true.

For every prime, ideal p of R, R_p is always a quasilocal ring. However, for an arbitrary R-module M, M_p is not necessarily a local R_p-module. But by the next proposition, if M is a nonzero classical top classical primeful R-module, then $R/\mathrm{Ann}(M)$ is a quasilocal ring.

Proposition 2.4. *Let M be a nonzero classical top classical primeful R-module. If M is a semi-local (resp. local) module, then $R/\mathrm{Ann}(M)$ is a quasisemilocal (resp. a quasilocal) ring.*

Proof. Let M be a local module with unique maximal submodule P. Then $p := (P : M) \in \mathrm{Max}(R)$. Now let $\mathrm{Ann}(M) \subseteq q \in \mathrm{Max}(R)$. It is enough to prove $q = p$. To prove this, we note that $S_q(qM)$ is a classical q-prime submodule of M by Proposition 2.1. Now we show that $S_q(qM) \in \mathrm{Max}(M)$. Let $S_q(qM) \subseteq K$ for some submodule K of M. Then we have $q = (S_q(qM) : M) = (K : M)$. Hence $S_q(qM) = K$ by Lemma 2.1. This implies that $S_q(qM) = P$ and therefore $q = p$. For the semi-local case we argue similarly. □

In the rest of this section, we get some properties of classical top modules. First note that every classical top module is a top module([9, Proposition 2.4]). In the next theorem, we introduce some modules that they are classical top modules.

Theorem 2.2. *Let M be an R-module. Then M is a classical top module in each of the following cases:*

(1) M is a multiplication R-module.

(2) M be a module that every classical prime submodule of M is strongly irreducible.

(3) M is an R-module with the property that for any two submodules N and L of M, $(N : M)$ and $(L : M)$ are comaximal.

Proof. (1). Let $P \in \mathcal{V}(N_1 \cap N_2)$ and so $N_1 \cap N_2 \subseteq P$. Since M is compatible, then $(N_1 \cap N_2 : M) \subseteq (P : M)$, so $N_1 \subseteq P$ or $N_2 \subseteq P$. Therefore $P \in \mathcal{V}(N_1)$ or $P \in \mathcal{V}(N_2)$. This implies that M is a classical top module.

(2). Let $P \in \mathcal{V}(N \cap L)$. Since $\mathcal{V}(N) \cup \mathcal{V}(L) \subseteq \mathcal{V}(N \cap L)$, for each submodules N and L of M, then $N \cap L \subseteq P$. Now, since P is strongly irreducible, then $N \subseteq P$ or $L \subseteq P$. Therefore $P \in \mathcal{V}(N) \cup \mathcal{V}(L)$. Thus $\mathcal{C}(M)$ is closed under finite unions. Hence M is a classical top module.

(3). Let P be a classical prime submodule of M with $N \cap L \subseteq P$. Then $(N : M) \cap (L : M) \subseteq (P : M) \in \mathrm{Spec}(R)$. We may assume that $(N : M) \subseteq (P : M)$. Then clearly $(L : M) \nsubseteq (P : M)$ by assumption. Hence $N \subseteq P$. Therefore P is strongly irreducible. This implies that M is a classical top module by (2). □

If Y is a nonempty subset of Cl.Spec(M), then the intersection of the members of Y is denoted by $\mathfrak{T}(Y)$. Thus, if Y_1 and Y_2 are subsets of Cl.Spec(M), then $\mathfrak{T}(Y_1 \cup Y_2) = \mathfrak{T}(Y_1) \cap \mathfrak{T}(Y_2)$. An R-module M is said to be distributive if $(A+B) \cap C = (A \cap C) + (B \cap C)$, for all submodules A, B and C of M(see for example [12]).

Theorem 2.3. *Let M is a classical top module and $\sqrt[cl]{E} = E$ for each submodule E of M. Then M is a distributive module.*

Proof. Let A, B and C be any submodules of M. Then,

$$\begin{aligned}
(A+B) \cap C &= \sqrt[cl]{(A+B) \cap C} \\
&= \cap\{P \in \mathrm{Cl.Spec}(M) | (A+B) \cap C \subseteq P\} \\
&= \cap\{P | P \in \mathcal{V}((A+B) \cap C)\} \\
&= \mathfrak{T}(\mathcal{V}((A+B) \cap C)) \\
&= \mathfrak{T}(\mathcal{V}(A+B) \cup \mathcal{V}(C)) \\
&= \mathfrak{T}((\mathcal{V}(A) \cap \mathcal{V}(B)) \cup \mathcal{V}(C)) \\
&= \mathfrak{T}((\mathcal{V}(A) \cup \mathcal{V}(C)) \cap (\mathcal{V}(B) \cup \mathcal{V}(C))) \\
&= \mathfrak{T}((\mathcal{V}(A \cap C)) \cap (\mathcal{V}(B \cap C))) \\
&= \mathfrak{T}((\mathcal{V}(A \cap C) + (B \cap C))) \\
&= \sqrt[cl]{(A \cap C) + (B \cap C)} \\
&= (A \cap C) + (B \cap C)
\end{aligned}$$

Hence M is a distributive module. □

Proposition 2.5. *Let M be a classical top module. Then for every two submodules A and B of M the equality $\sqrt[cl]{A \cap B} = \sqrt[cl]{A} \cap \sqrt[cl]{B}$ holds.*

Proof. By definition, $\sqrt[cl]{A \cap B} = \mathfrak{T}(\mathcal{V}(A \cap B)) = \mathfrak{T}(\mathcal{V}(A) \cup \mathcal{V}(B))$ $= \mathfrak{T}(\mathcal{V}(A)) \cap \mathfrak{T}(\mathcal{V}(B)) = \sqrt[cl]{A} \cap \sqrt[cl]{B}$. □

3. Some properties of topological space Cl.Spec(M)

In this section, we study some properties of topological space Cl.Spec(M). The closure of Y in Cl.Spec(M) with respect to the classical quasi-Zariski topology denoted by $\overline{Y}$.

Lemma 3.1. *Let M be a classical top module and let Y be a nonempty subset of* Cl.Spec(M). *Then $\overline{Y} = \mathcal{V}(\mathfrak{T}(Y))$. Hence, for every $N \leq M$, $\mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \mathcal{V}(N)$.*

Proof. Suppose $\mathcal{V}(E)$ is a closed set of Cl.Spec(M) containing Y. Then for every classical prime submodule P in Y, $E \subseteq P$. Therefore $E \subseteq \mathfrak{T}(Y)$ and so $\mathcal{V}(\mathfrak{T}(Y)) \subseteq \mathcal{V}(E)$. Since $Y \subseteq \mathcal{V}(\mathfrak{T}(Y))$, then $\mathcal{V}(\mathfrak{T}(Y))$ is the smallest closed subset of Cl.Spec(M) containing Y. Thus $\overline{Y} = \mathcal{V}(\mathfrak{T}(Y))$.

Finally, since $\mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \overline{\mathcal{V}(N)}$, and since $\mathcal{V}(N)$ is a closed subset of Cl.Spec(M), then $\overline{\mathcal{V}(N)} = \mathcal{V}(N)$. Consequntly $\mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \mathcal{V}(N)$. $\square$

Let X be a topological space and let x and y be two points of X. We say that x and y can be separated if each lies in an open set which does not contain the other point. X is a T_1- space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if the singleton set $\{x\}$ is a closed set, for any x in X.

Theorem 3.1. *Let M be an R-module. Then* Cl.Spec(M) *is T_1-space if and only if each classical prime submodule is maximal in the family of all classical prime submodules of M. i.e,* Max(M) = Cl.Spec(M).

Proof. Let P be maximal in Cl.Spec(M) with respect inclution. Then $\overline{\{P\}} = \mathcal{V}(\mathfrak{T}(\{P\})) = \mathcal{V}(P)$, but P is maximal in Cl.Spec(M), so $\overline{\{P\}} = \{P\}$. Then $\{P\}$ is a closed set in Cl.Spec(M). Thus Cl.Spec(M) is a T_1 - space, and vice versa. $\square$

Definition 3.1. Let X be a topological space and Y $\subseteq$ X. Then:

(1) X is irreducible if $X \neq \emptyset$ and for every decomposition $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X$, $i = 1, 2$, we have $A_1 = X$ or $A_2 = X$.

(2) Y is irreducible if Y is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets F, G which are closed in X and satisfy $Y \subseteq F \cup G$, then $Y \subseteq F$ or $Y \subseteq G$[10, Ch. II, p. 119].

Lemma 3.2. *Let M be an R-module. Then for every $P \in$* Cl.Spec(M), *$\mathcal{V}(P)$ is irreducible.*

Proof. Let $\mathcal{V}(P) \subseteq Y_1 \cup Y_2$, for some closed sets Y_1 and Y_2. Since $P \in \mathcal{V}(P)$, either $P \in Y_1$ or $P \in Y_2$. Suppose that $P \in Y_1$. Then $Y_1 = \cap_{i\in I}(\cup_{j=1}^{n_i}\mathcal{V}(N_{ij}))$, for some I, $n_i(i \in I)$ and $N_{ij} \leq M$. Then for all $i \in I$, $P \in \cup_{j=1}^{n_i}\mathcal{V}(N_{ij})$. Thus for all $i \in I$, $\mathcal{V}(P) \subseteq \cup_{j=1}^{n_i}\mathcal{V}(N_{ij}))$, so $\mathcal{V}(P) \subseteq Y_1$. Thus $\mathcal{V}(P)$ is irreducible. $\square$

M. Behboodi and M. R. Haddadi show that if $Y \subseteq \mathrm{Spec}(M)$ and $\mathfrak{T}(Y)$ is a prime submodule of M and $\mathfrak{T}(Y) \in \overline{Y}$, then Y is irreducible([6, Theorem 3.4]). In the next proposition, we extend this fact to classical prime submodules.

Proposition 3.1. *Let M be a classical top module and $Y \subseteq \mathrm{Cl.Spec}(M)$. Then $\mathfrak{T}(Y)$ is a classical prime submodule of M if and only if Y is an irreducible space.*

Proof. Let $P = \mathfrak{T}(Y)$ be a classical prime submodule of M and $P \in Y$, so $\overline{Y} = \mathcal{V}(P)$. If $Y \subseteq Y_1 \cup Y_2$, for closed sets Y_1 and Y_2, then $\overline{Y} \subseteq Y_1 \cup Y_2$. Since $\mathcal{V}(P) \subseteq Y_1 \cup Y_2$ and by Lemma 3.2, $\mathcal{V}(P)$ is irreducible, then $\mathcal{V}(P) \subseteq Y_1$ or $\mathcal{V}(P) \subseteq Y_2$. Now, since $Y \subseteq \mathcal{V}(P)$, then either $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Thus Y is irreducible. For the converse, we can apply [6, Theorem 3.4]. □

Corollary 3.1. *Let M be a classical top module. Then for every classical prime submodule P, $\mathcal{V}(P)$ is an irreducible subspace of* $\mathrm{Cl.Spec}(M)$. *Consequently, $\mathcal{V}(N)$ is irreducible if and only if $\sqrt[Cl]{N}$ is a classical prime submodule.*

Proof. First note that $\mathfrak{T}(\mathcal{V}(P)) = \bigcap\{P | P \in \mathcal{V}(P)\} = \sqrt[cl]{P} = P$. Then $\mathcal{V}(P)$ is an irreducible subspace of $\mathrm{Cl.Spec}(M)$, by Proposition 3.1. Finnaly, it is enough to note that $\sqrt[Cl]{N} = \mathfrak{T}(\mathcal{V}(N))$. □

Proposition 3.2. *Let M be a classical top R-module, $\overline{R} = R/\mathrm{Ann}(M)$ and let $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ be the natural map of* $\mathrm{Cl.Spec}(M)$. *Then ψ is continuous in the classical quasi-Zariski topology.*

Proof. It suffices to prove that $\psi^{-1}(\mathrm{V}(\overline{I})) = \mathcal{V}(IM)$, for every $I \in \mathrm{V}(\underline{\mathrm{Ann}(M)})$. Let $P \in \mathrm{Cl.Spec}(M)$, then $P \in \psi^{-1}(\mathrm{V}(\overline{I}))$, so $\psi(P) \in \mathrm{V}(\overline{I})$, therefore $\overline{(P : M)} \in \mathrm{V}(\overline{I})$. Then $\overline{(P : M)} \in \mathrm{Spec}(\overline{R})$ and $\overline{I} \subseteq \overline{(P : M)}$, so $(P : M) \in \mathrm{Spec}(R)$ and $I/\mathrm{Ann}(M) \subseteq (P : M)/\mathrm{Ann}(M)$. Hence $(P : M) \in \mathrm{Spec}(R)$ and $\mathrm{Ann}(M) \subseteq I \subseteq (P : M)$. Now, since $IM \subseteq (P : M)M \subseteq P$, then $P \in \mathcal{V}(IM)$, which it shows that $\psi^{-1}(\mathrm{V}(\overline{I})) \subseteq \mathcal{V}(IM)$. In similar way, we can show $\mathcal{V}(IM) \subseteq \psi^{-1}(\mathrm{V}(\overline{I}))$ and hence

$$\psi^{-1}(\mathrm{V}(\overline{I})) = \mathcal{V}(IM).\square$$

Lemma 3.3. *Let M be a classical top R-module, $\overline{R} = R/\mathrm{Ann}(M)$ and let ψ be the natural map of* $\mathrm{Cl.Spec}(M)$. *If M is classical primeful, then ψ is both closed and open; more precisely, for every submodule N of M, $\psi(\mathcal{V}(N)) = \mathrm{V}(\overline{(N : M)})$ and*

$$\psi(\mathrm{Cl.Spec}(M) \setminus \mathcal{V}(N)) = \mathrm{Cl.Spec}(R/\mathrm{Ann}(M)) \setminus (\mathrm{V}(\overline{(N : M)}).$$

Proof. First we show that $\psi(\mathcal{V}(N)) = \mathrm{V}(\overline{(N : M)})$, for every $N \leq M$, whenever M is classical primeful. Since ψ is continuous, as we have seen in Proposition 3.2,

$$\psi^{-1}(\mathrm{V}(\overline{(N : M)})) = \mathcal{V}((N : M)M) = \mathcal{V}(N).$$

Hence, $\psi(\mathcal{V}(N)) = \psi \circ \psi^{-1}(\mathrm{V}(\overline{(N : M)}) = \mathrm{V}(\overline{(N : M)}$, since ψ is surjective and M is classical primeful. Consequently:

$$\psi(\mathrm{Cl.Spec}(M) \setminus \mathcal{V}(N)) = \mathrm{Spec}(R/\mathrm{Ann}(M)) \setminus (\mathrm{V}(\overline{(N : M)}).\square$$

Corollary 3.2. *Let M be a classical top R-module, $\overline{R} = R/\mathrm{Ann}(M)$ and let ψ be the natural map of* Cl.Spec(M)*. Then ψ is bijective if and only if it is a homeomorphism.*

Proof. This follows from Proposition 3.2 and Lemma 3.3. □

Proposition 3.3. *Let M be a classical top R-module and let Y be a subset of* Cl.Spec(M)*. If Y is irreducible, then $T = \{(P : M) | P \in Y\}$ is an irreducible subset of* Spec(R)*, with respect to Zariski topology.*

Proof. Let $\overline{R} = R/\mathrm{Ann}(M)$, ψ the natural map of Cl.Spec(M) and let Y be a subset of Cl.Spec(M). Since ψ is continuous by proposition 3.2, Then $\psi(Y) = \overline{Y}$ is an irreducible subset of Spec$(R/\mathrm{Ann}(M))$. Therefore

$$\mathfrak{T}(\overline{Y}) = (\mathfrak{T}(Y) : M)/\mathrm{Ann}(M) \in \mathrm{Spec}(R/\mathrm{Ann}(M)).$$

Therefore $\mathfrak{T}(T) = (\mathfrak{T}(Y) : M)$ is a prime ideal of R, then by Proposition 3.1, T is an irreducible subset of Spec(R). □

Clearly the next lemma is true(see for example [8], page 10).

Lemma 3.4. *If $\{P_i\}_{i \in I}$ is a chain of classical prime submodules of an R-module M, then $\bigcap_{i \in I} P_i$ is a classical prime submodule of M.*

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = Cl(\{y\})$, where $Cl(\{y\})$ is the closure of $\{y\}$ in Y. Note that a generic point of a closed subset Y of a topological space is unique if the topological space is a T_0-space.

Theorem 3.2. *Let M be a classical primeful R-module. If M is a classical top module, then a subset Y of* Cl.Spec(M) *is an irreducible closed subset if and only if $Y = \mathcal{V}(P)$, for some $P \in$* Cl.Spec(M)*. Thus every irreducible closed subset of* Cl.Spec(M) *has a generic point.*

Proof. By Corollary 3.1, for every $P \in$ Cl.Spec(M), $Y = \mathcal{V}(P)$ is an irreducible closed subset of Cl.Spec(M). Conversely, if Y is an irreducible closed subset of Cl.Spec(M), then $Y = \mathcal{V}(N)$, for some $N \leq M$. Now, since $Y = \mathcal{V}(N) = \mathcal{V}(\sqrt[Cl]{N})$, then $\mathfrak{T}(Y) = \mathfrak{T}(\mathcal{V}(N)) = \sqrt[Cl]{N}$ is a classical prime submodule of M by Lemma 3.4. Then $\mathcal{V}(\mathfrak{T}(Y)) = \mathcal{V}(\mathfrak{T}(\mathcal{V}(N))) = \mathcal{V}(\sqrt[Cl]{N})$, so by Theorem 3.1, $Y = \mathcal{V}(N) = \mathcal{V}(\sqrt[Cl]{N})$, with $\sqrt[Cl]{N} \in$ Cl.Spec(M). □

A maximal irreducible subset Y of X is called an irreducible component of X and it is always closed. In the next theorem, we show that there exists a bijection map from the set of irreducible components of Cl.Spec(M) to the set of minimal classical prime submodules of M.

Theorem 3.3. *Let M be a classical top R-module. Then the map $\mathcal{V}(P) \longmapsto P$ is a bijection from the set of irreducible components of* Cl.Spec(M) *to the set of minimal classical prime submodules of M.*

Proof. Let Y be an irreducible component of Cl.Spec(M). By Theorem 3.2, each irreducible component of Cl.Spec(M) is a maximal element of the set $\{\mathcal{V}(Q)|Q \in$ Cl.Spec$(M)\}$, so for some $P \in$ Cl.Spec(M), $Y = \mathcal{V}(P)$. Obviously, P is a minimal classical prime submodule of M. Suppose T is a classical prime submodule of M that $T \subseteq P$, then $\mathcal{V}(P) \subseteq \mathcal{V}(T)$, so $P = T$. Now, let P be a minimal classical prime submodule of M, so for every $Q \in$ Cl.Spec(M), $P \subseteq Q$. Then for all $Q \in$ Cl.Spec(M), $\mathcal{V}(Q) \subseteq \mathcal{V}(P)$. Thus $\mathcal{V}(P)$ is a maximal irreducible subset of Cl.Spec(M). □

Theorem 3.4. *Consider the following statements for a nonzero classical top primeful R-module M:*

1. Cl.Spec(M) *is an irreducible space.*
2. Supp(M) *is an irreducible space.*
3. $\sqrt{\mathrm{Ann}(M)}$ *is a prime ideal of R.*
4. Cl.Spec$(M) = \mathcal{V}(pM)$, *for some* $p \in$ Supp(M).

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). In addition, if M is a multiplication module, then all of the four statements are equivalent.

Proof. (1) $\Longrightarrow$ (2): By Proposition 3.2, the natural map ψ is continuous and by assumption ψ is surjective. Therefore $Im(\psi) = \mathrm{Spec}(R/\mathrm{Ann}(M))$ is also irreducible. Now by Proposition 2.3, $\mathrm{Supp}(M) = \mathrm{V}(\mathrm{Ann}(M))$ is homeomorphic to $\mathrm{Spec}(R/\mathrm{Ann}(M))$. Therefore Supp$(M)$ is an irreducible space.

(2) $\Longrightarrow$ (3): By Proposition 3.1, $\mathfrak{T}(\mathrm{Supp}(M))$ is a prime ideal of R. Then $\mathfrak{T}(\mathrm{Supp}(M)) = \mathfrak{T}(\mathrm{V}(\mathrm{Ann}(M))) = \sqrt{\mathrm{Ann}(M)}$ is a prime ideal of R.

(3) $\Longrightarrow$ (4) Let $a \in \sqrt{\mathrm{Ann}(M)}$. So for some integer $n \in N$, $a^n M = 0$. Therefore for every classical prime submodule P of M, $a \in (P : M)$. Then for each $P \in$ Cl.Spec(M), $\mathrm{Ann}(M) \subseteq \sqrt{\mathrm{Ann}(M)} \subseteq (P : M)$. Since M is classical primeful, there exists a classical prime submodule Q of M such that $(Q : M) = \sqrt{\mathrm{Ann}(M)}$. Then,

$$\begin{aligned} \mathrm{Cl.Spec}(M) &= \{P \in \mathrm{Cl.Spec}(M)|(Q : M) \subseteq (P : M)\} \\ &= \mathcal{V}((Q : M)M) \\ &= \mathcal{V}(\sqrt{\mathrm{Ann}(M)}M). \end{aligned}$$

It is clear that $p := \sqrt{\mathrm{Ann}(M)} \in \mathrm{Supp}(M)$. Therefore $\mathrm{Cl.Spec}(M) = \mathcal{V}(pM)$.

Now, let M be a multiplication module and let $\mathrm{Cl.Spec}(M) = \mathcal{V}(pM)$, for some $p \in \mathrm{Supp}(M)$. Since M is classical primeful, there exists $P \in \mathrm{Cl.Spec}(M)$, such that $(P : M) = p$. Since M is multiplication, we have $\mathrm{Cl.Spec}(M) = \mathcal{V}(pM) = \mathcal{V}((P : M)M) = \mathcal{V}(P)$. This implies that Cl.Spec(M) is an irreducible space by Corollary 3.1. □

Let M be an R-module. For each subset N of M, we denote $\mathrm{Cl.Spec}(M) - \mathcal{V}(N)$ by $\mathcal{U}(N)$. Further for each element $m \in M$, $\mathcal{U}(\{m\})$ is denoted by $\mathcal{U}(m)$. Hence

$$\mathcal{U}(m) = \mathrm{Cl.Spec}(M) - \mathcal{V}(\{m\}).$$

Moreover, for any family $\{N_i\}_{i\in I}$ of submodules of M, we have

$$\mathcal{U}(\sum_{i\in I} N_i) = \mathcal{U}(\bigcup_{i\in I} N_i).$$

Theorem 3.5. *Let M be a classical top module. Then for every $m \in M$, the sets $\mathcal{U}(m)$ form a base for* $\mathrm{Cl.Spec}(M)$.

Proof. Let $\mathcal{U}(N)$ be an open set in $\mathrm{Cl.Spec}(M)$, where N is a submodule of M. Then:

$$\begin{aligned}
\mathcal{U}(N) = \mathcal{U}(\bigcup_{n\in N}\{n\}) &= \mathrm{Cl.Spec}(M) - \mathcal{V}(\bigcup_{n\in N}\{n\}) \\
&= \mathrm{Cl.Spec}(M) - \bigcap_{n\in N}\mathcal{V}(\{n\}) \\
&= \bigcup_{n\in N}(\mathrm{Cl.Spec}(M) - \mathcal{V}(\{n\})) \\
&= \bigcup_{n\in N}\mathcal{U}(n)
\end{aligned}$$

Then for every $m \in M$, the sets $\mathcal{U}(m)$ form a base of $\mathrm{Cl.Spec}(M)$. $\square$

For a submodule N of an R-module M, we put:

$$\mathcal{FG}(N) := \{L | L \subseteq N \ and\ L\ is\ finitely\ generated\}$$

Lemma 3.5. *Let M be an R-module and N be a submodule of M. Then $\mathcal{V}(N) = \bigcap_{L\in\mathcal{FG}(N)}\mathcal{V}(L)$ and $\mathcal{U}(N) = \bigcup_{L\in\mathcal{FG}(N)}\mathcal{U}(L)$.*

Proof. Suppose that $P \in \mathcal{V}(N)$. If $L \in \mathcal{FG}(N)$, then $L \subseteq N \subseteq P$. Then $P \in \mathcal{V}(L)$, and $\mathcal{V}(N) \subseteq \bigcap_{L\in\mathcal{FG}(N)}\mathcal{V}(L)$. Now, let for every $L \in \mathcal{FG}(N)$, $P \in \mathcal{V}(L)$ and $P \notin \mathcal{V}(N)$. Since $N \nsubseteq P$, then there exists $x \in N \setminus P$. Then $Rx \subseteq N$ and Rx is finitely generated, hence $Rx \in \mathcal{FG}(N)$. Therefore $x \in Rx \subseteq P$, a contradiction. Thus $\bigcap_{L\in\mathcal{FG}(N)}\mathcal{V}(L) \subseteq \mathcal{V}(N)$. $\square$

Theorem 3.6. *Let M be a classical top R-module. Then every quasi-compact open subset of* $\mathrm{Cl.Spec}(M)$ *is of the form $\mathcal{U}(N)$, for some finitely generated submodule N of M.*

Proof. Suppose $\mathcal{U}(B) = \mathrm{Cl.Spec}(M) \setminus \mathcal{V}(B)$ is a quasi-compact open subset of Cl.Spec(M). Then by Lemma 3.5, $\mathcal{U}(B) = \bigcup_{L\in\mathcal{FG}(B)}\mathcal{U}(L)$. Now, since $\mathcal{U}(B)$ is quasi-compact, then every open covering of $\mathcal{U}(B)$ has a finite subcovering, therefore $\mathcal{U}(B) = \mathcal{U}(L_1) \cup ... \cup \mathcal{U}(L_n) = \mathcal{U}(\sum_{i=1}^{n} L_i)$. □

Proposition 3.4. *Let M be a classical top R-module. If* Spec(R) *is a T_1-space, then* Cl.Spec(M) *is also a T_1-space.*

Proof. Suppose Q is a classical prime submodule of M. Then $Cl(\{Q\}) = \mathcal{V}(Q)$. If $P \in \mathcal{V}(Q)$, then by Theorem 3.1, every prime ideal of R is a maximal ideal, so $(Q : M) = (P : M)$, then by Lemma 2.1, $Q = P$. Therefore $Cl(\{Q\}) = \{Q\}$ and this implies that Cl.Spec(M) is a T_1-space. □

Definition 3.2. A topological space X is Noetherian provided that the open (respectively, closed) subsets of X satisfy the ascending (respectively, descending) chain condition (see for example [3], page 79 or [10], §4.2).

Theorem 3.7. *An R-module M has Noetherian calssical spectrum if and only if the ACC for classical radical submodules of M holds.*

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ be an ascending chain of classical radical submodules of M. Since for all $i \in \mathbb{N}$, $\sqrt[Cl]{N_i} = N_i$, then equivalently

$$\sqrt[Cl]{N_1} \subseteq \sqrt[Cl]{N_2} \subseteq \sqrt[Cl]{N_3} \subseteq \dots$$

is an ascending chain of classical radical submodules of M. Then equivalently

$$\mathfrak{T}(\mathcal{V}(N_1)) \subseteq \mathfrak{T}(\mathcal{V}(N_2)) \subseteq \mathfrak{T}(\mathcal{V}(N_3)) \subseteq \dots$$

is an ascending chain of classical radical submodules of M. Therefore

$$\mathcal{V}(N_1) \supseteq \mathcal{V}(N_2) \supseteq \mathcal{V}(N_3) \supseteq \dots$$

is a descending chain of closed sets $\mathcal{V}(N_i)$ of Cl.Spec(M). Now, R-module M has Noetherian spectrum if and only if Cl.Spec(M) is a Noetherian topological space if and only if there exists a positive integer k such that $\mathcal{V}(N_k) = \mathcal{V}(N_{k+n})$ for each $n = 1, 2, ...$ if and only if $\sqrt[Cl]{N_k} = \sqrt[Cl]{N_{k+n}}$ if and only if $N_k = N_{k+n}$ if and only if the ACC for classical radical submodules of M holds.□

Theorem 3.8. *Let M be a classical top R-module such that* Cl.Spec(M) *is a Noetherian space. Then the following statements are true.*

1. *Every ascending chain of classical prime submodules of M is stationary.*
2. *The set of minimal classical prime submodules of M is finite. In particular,* Cl.Spec$(M) = \bigcup_{i=1}^{n} \mathcal{V}(P_i)$, *where P_i are all minimal classical prime submodules of M.*

Proof. (1). Suppose $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$ is an ascending chain of classical prime submodules of M. Therefore $\mathcal{V}(N_1) \supseteq \mathcal{V}(N_2) \supseteq \dots$ is a descending chain of closed subsets of Cl.Spec(M), which is stationary by assumption. There exists an integer $k \in \mathbb{N}$ such that $\mathcal{V}(N_k) = \mathcal{V}(N_{k+i})$, for each $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $N_k = N_{k+i}$.

(2). This follows from Theorem 3.3 and the fact that if X is a Noetherian space, then the set of irreducible components of X is finite(see for example [10, Proposition 10]). □

Recall that if M is a Noetherian module, then each open subset of Spec(M) is quasi-compact(see for example [15, Theorem 3.3]). The next theorem shows that the same result is true for Cl.Spec(M) in Noetherian classical top modules.

Theorem 3.9. *Let M be a Noetherian classical top module. Then each open subset of* Cl.Spec(M) *is quasi-compact.*

Proof. Let for every submodule N of $M, \mathcal{U}(N)$ be an open subset of Cl.Spec(M). Also, let $\{\mathcal{U}(n_i)\}_{n_i \in N}$ be a basic open cover for $\mathcal{U}(N)$. We show that there exist a finite subfamily of $\{\mathcal{U}(n_i)\}_{n_i \in N}$ which covers Cl.Spec(M). Since $\mathcal{U}(N) \subseteq \bigcup_{n_i \in N} \mathcal{U}(n_i) = \mathcal{U}(\bigcup_{n_i \in N} n_i)$, then for every submodule K of M that is generated by the set $A = \{n_i\}_{i \in I}$, $\mathcal{U}(N) \subseteq \mathcal{U}(K)$. Since M is a Noetherian module, then $K =< k_1, k_2, ..., k_n >$, for some $k_i \in K$, therefore $b_i = \sum_{j=1}^{n} r_{ij} n_{ij}$, where $i = 1, ..., n$ and $n_{ij} \in A$. Thus there exists the subset $\{n_{i1}, ..., n_{in}\} \subseteq A$ such that $K =< n_{i1}, ..., n_{in} >$. So $\mathcal{U}(N) \subseteq \mathcal{U}(K) = \mathcal{U}(< n_{i1}, ..., n_{in} >)$. Then

$$\mathcal{U}(N) \subseteq \mathcal{U}(\bigcup_{i=1}^{n} n_i) = \bigcup_{i=1}^{n} \mathcal{U}(n_i).$$

consequently, $\mathcal{U}(N)$ is quasi-compact. □

Recall that a function Φ between two topological spaces X and Y is called an open map if, for any open set U in X, the image $\Phi(U)$ is open in Y. Also, Φ is called a homeomorphism if it has the following properties

(i) Φ is a bijection;

(ii) Φ is continuous;

(iii) Φ is an open map

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. By Hochster's characterization [15], a topology τ on a set X is spectral if and only if the following axioms hold:

(i) X is a T_0-space.

(ii) X is quasi-compact and has a basis of quasi-compact open subsets.

(iii) The family of quasi-compact open subsets of X is closed under finite intersections.

(iv) X is a sober space; i.e., every irreducible closed subset of X has a generic point.

For any ring R, it is is well-known that Spec(R) satisfies these conditions(cf. [10], Chap. II, 4.1 - 4.3]). We show that Cl.Spec(M) is necessarily a spectral space in the classical quasi-Zariski topology for every module M.

We remark that any closed subset of a spectral space is spectral for the induced topology.

Theorem 3.10. *Let M be a classical top primful R-module, $\overline{R} = R/\mathrm{Ann}(M)$ and let ψ be the natural map of* Cl.Spec(M). *Then ψ is a homeomorphism.*

Proof. It is clear by Lemma 2.1, Proposition 3.2, Lemma 3.3 and Corollary 3.2. □

Corollary 3.3. *Let M be a classical top primful R-module. Then* Cl.Spec(M) *with classical quasi-Zariski topology is a spectral space.*

Lemma 3.6. *Let M be a classical top R-module. Then the following statements are equivalent:*

(a) the natural map $\psi : \mathrm{Cl.Spec}(M) \longrightarrow \mathrm{Spec}(R/\mathrm{Ann}(M))$ *is injective.*

(b) Cl.Spec(M) *is a T_0-space.*

Proof. We recall that a topological space is T_0 if and only if the closures of distinct points are distinct. Now, the result follows from

$$P = Q \Longleftrightarrow \mathcal{V}(P) = \mathcal{V}(Q), \quad \forall P, Q \in \mathrm{Cl.Spec}(M).\square$$

Corollary 3.4. *Let M be a Noetherian classical primeful top module. Then the following statements are holed:*

(i) Cl.Spec(M) *is a T_0-space.*

(ii) Cl.Spec(M) *is quasi-compact and has a basis of quasi-compact open subsets.*

(iii) The family of quasi-compact open subsets of Cl.Spec(M) *is closed under finite intersections.*

(iv) Cl.Spec(M) *is a sober space; i.e., every irreducible closed subset of* Cl.Spec(M) *has a generic point.*

Proof. It is clear by Lemma 3.6, Theorem 3.5, Theorem 3.9, Theorem 3.2. □

Acknowledgment

The authors would like to thank the referee for his/her helpful comments.

REFERENCES

1. M. Ali: *Multiplication modules and homogeneous idealization II*. Beitr Algebra Geom. **48** (2007), 321–343.
2. A. Azizi: *Weak multiplication modules*. Czechoslovak Math. J. **53** (2003), no. 128, 529–534.
3. M. F. Atiyah and I. Macdonald: *Introduction to commutative algebra*. Longman Higher Education, New York, 1969.
4. M. Baziar and M. Behboodi: *Classical primary submodules and decomposition of modules*. J Algebra Appl. **8** (3) (2009), 351–362.
5. M. Behboodi: *A generalization of Bear's lower nilradical for modules*. J. Algebra and its Appl. **6**(2) (2007), 337-353.
6. M. Behboodi and M. R. haddadi: *Classical Zariski topology of modules and spectral spaces*. I. International Electronic Journal of Algebra. **4** (2008), 104–130.
7. M. Behboodi, R. Jahani-Nezhad, and M.H. Naderi, Classical quasi-primary submodules, *Bull. Iranian Math. Soc.* **37** (4) (2011) 51-71.
8. M. Behboodi and H. Koohy: *Weakly prime modules*. Vietnam J. Math. **32**(2) (2004), 185–195.
9. M. Behboodi and M. J. Noori: *Zariski-like topology on the classical prime spectrum*. Bulletin of the Iranian Mathematical Society. **35** (1)(2009), 253–269.
10. N. Bourbaki: *Algebra commutative*. Chap, 1. 2, Hermann, Paris, 1961.
11. F. Callialp and U. Teki: *On unions of prime submodules*. Southeast Asian Bull. Math. **28**(2004), no.2, 213–218.
12. V. Camillo : *Distributive modules*. Journal of Algebra **36**(1975), 16-25.
13. S. Ebrahimi Atani: *Strongly irreducible submodules*. Bull. Korean Math. Soc. **42** (2005), no. 1, 121–131.
14. Z. A. EL-Bast and P. F. Smith: *Multiplication modules*. Comm. Algebra. **16** (1988), 755–779.
15. M. Hochster: *Prime ideal structure in commutative rings*. Trans. Amer. math. Soc. **137** (1969), 43–60.
16. C. P. Lu: *Prime submodules of modules*. Comment Math Univ St Paul. **33** (1) (1984), 61–69.
17. C. P. Lu: *Saturations of submodules*. Comm. Algebra. **31** (6) (2003), 2655-2673.
18. C. P. Lu: *A module whose prime spectrum has the surjecive natural map*. Houston J. Math. **31** (1) (2007), 125-143.
19. A. A. Tuganbaev: *Flat multiplication modules*, (Russian) Uspekhi Mat. Nauk. **60** (361)(2005), no.1, 173 – 174, translation in Russian Math. Surveys. **60** (2005) no.1, 171–172.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 31-42
https://doi.org/10.22190/FUMI191227003K
Original Scientific Paper

ON BINOMIAL SUMS WITH THE TERMS OF SEQUENCES$\{g_{kn}\}$ AND $\{h_{kn}\}$

Sibel Koparal, Neşe Ömür and Cemile D. Çolak

Kocaeli University, Department of Mathematics, 41380 İzmit Kocaeli, Turkey

Abstract. In this paper, we derive sums and alternating sums of products of terms of the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ with binomial coefficients. For example,

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}g_{k(n-ti)}h_{kti}$$
$$= 2^{n-m}n^{\underline{m}}g_{kn} - n^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{n(1-t)}h_{kt}^{n-m}g_{k(tm+tn-n)},$$

where c is a nonzero real number, t is any integer and m is a nonnegative integer.
Keywords: Binomial sums, alternating sums, generalized Fibonacci numbers, recurrence relation.

1. Introduction

Define the second order linear recursive sequences $\{u_n\}$ and $\{v_n\}$ for $n \geq 1$ and nonzero integers p, q, by

$$u_{n+1} = pu_n + qu_{n-1} \text{ and } v_{n+1} = pv_n + qv_{n-1}$$

with initials $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = p$, respectively.

When $q = 1$, $u_n = U_n$ (the nth generalized Fibonacci number) and $v_n = V_n$ (the nth generalized Lucas number). Also, when $p = q = 1$, $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number). If α and β are the roots of the

Received Decembar 27, 2019; accepted September 3, 2020.
Corresponding Author: Sibel Koparal, Kocaeli University, Department of Mathematics, 41380 İzmit Kocaeli, Turkey | E-mail: sibelkoparal1@gmail.com
2010 *Mathematics Subject Classification.* Primary 11B39; Secondary 05A10, 05A15, 05A19

equation $x^2 - px - q = 0$, the Binet formulae of the sequences $\{u_n\}$ and $\{v_n\}$ have the forms

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

respectively, where $\alpha, \beta = \left(p \pm \sqrt{\Delta}\right)/2$ and $\Delta = p^2 + 4q$. From [3], Kılıç and Stanica derived the following recurrence relations for the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ for $k \geq 0$, $n > 0$. It is clear that

$$u_{k(n+1)} = v_k u_{kn} + (-1)^{k+1} q^k u_{k(n-1)} \text{ and } v_{k(n+1)} = v_k v_{kn} + (-1)^{k+1} q^k v_{k(n-1)},$$

where the initial conditions are 0, u_k, and 2, v_k, respectively. The Binet formulae of the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ are given by

$$u_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \text{ and } v_{kn} = \alpha^{kn} + \beta^{kn},$$

respectively. It is clearly seen that $u_{-kn} = (-1)^{kn+1} u_{kn}$ and $u_{2kn} = u_{kn} v_{kn}$.

In [9], Komatsu obtained the two binomial sums of the generalized Fibonacci numbers as follows:

$$\sum_{i=0}^{n} \binom{n}{i} c^i u_i = g_n \quad (n \geq 0)$$

which satisfies the recurrence relation

$$g_{n+1} = (pc + 2)\, g_n + \left(qc^2 - pc - 1\right) g_{n-1} \quad (n \geq 1)$$

with $g_0 = 0$, $g_1 = c$ and

$$\sum_{i=0}^{n} \binom{n}{i} c^{n-i} d^i u_i = h_n \quad (n \geq 0)$$

which satisfies the recurrence relation

$$h_{n+1} = (pd + 2c)\, h_n + \left(qd^2 - pcd - c^2\right) h_{n-1} \quad (n \geq 1)$$

with $h_0 = 0$ and $h_1 = d$, where c, d are nonzero real numbers. Also he gave several Fibonacci identities including binomial coefficients by using the method of ordinary power series generating functions.

In [1], Cook et al. obtained some binomial summation identities including the terms of the sequence $\{g_n\}$ in [9]. For example,

$$\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i \left(qc^2 - pc - 1\right)^{2n-i} g_{2i+1} = (pc + 2)^{2n} g_{2n+1},$$

and

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i\left(qc^2-pc-1\right)^{2(2n-i)}g_{4i} = c^{2n}(pc+2)^{2n}\left(p^2+4q\right)^n g_{4n}.$$

In [10], Ömür et al. defined the subsequences $\{g_{kn}\}$ and $\{h_{kn}\}$ with binomial sums $g_{kn}=\sum\limits_{i=0}^{n}\binom{n}{i}c^{ki}u_{ki}$ and $h_{kn}=\sum\limits_{i=0}^{n}\binom{n}{i}c^{ki}v_{ki}$. These subsequences satisfy the following relations

$$g_{k(n+1)}=\left(c^kv_k+2\right)g_{kn}-\left(c^{2k}(-q)^k+c^kv_k+1\right)g_{k(n-1)}$$

and

$$h_{k(n+1)}=\left(c^kv_k+2\right)h_{kn}-\left(c^{2k}(-q)^k+c^kv_k+1\right)h_{k(n-1)}$$

in which $g_0=0$, $g_k=c^ku_k$ and $h_0=2$, $h_k=2+c^kv_k$, respectively. The Binet formulae of the sequences $\{g_{kn}\}$ and $\{h_{kn}\}$ are

$$g_{kn}=\frac{\left(c^k\alpha^k+1\right)^n-\left(c^k\beta^k+1\right)^n}{\alpha-\beta} \text{ and } h_{kn}=\left(c^k\alpha^k+1\right)^n+\left(c^k\beta^k+1\right)^n,$$

respectively. The authors obtained some binomial summation identities of sequence $\{g_{kn}\}$. For example, for $n>0$,

$$\sum_{i=0}^{2n}\binom{2n}{i}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{k(2i+1)}=\left(c^kv_k+2\right)^{2n}g_{k(2n+1)},$$

and

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{2n-i}g_{k(2i+1)}$$
$$=c^{2kn}\left(v_k^2+4q^k(-1)^{k+1}\right)^n g_{k(2n+1)}.$$

In [7], Kılıç et al. introduced sums and alternating sums of products of terms of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ as follows: for odd number n,

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(e+fi)} &= D^{(n-1)/2}\Big(U_{k(b+f)/2}^nU_{k(n(b+f)/2+a+e)}\\ &\quad+(-1)^{e+(b-f)/2}U_{k(b-f)/2}^nU_{k(n(b-f)/2+a-e)}\Big),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^iU_{k(a+bi)}U_{k(e+fi)} &= \frac{1}{D}\Big((-1)^nV_{k(b+f)/2}^nV_{k(n(b+f)/2+a+e)}\\ &\quad-(-1)^{e+n(b-f)/2}V_{k(b-f)/2}^nV_{k(n(b-f)/2+a-e)}\Big),\end{aligned}$$

where a, b, e, f are any integers, $b+f \equiv 2 (\bmod 4)$ and $D = p^2 + 4$.

In [5, 6], Kılıç considered and computed the alternating binomial sums of the forms

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i f(n,i,k,t) \text{ and } \sum_{i=0}^{n} \binom{n}{i} g(n,i,k,t),$$

where $f(n,i,k,t)$ is $U_{kti}V_{k(n-ti)}$ and $U_{kti}V_{(k+1)tn-(k+2)ti}$, and, $g(n,i,k,t)$ is $U_{ki}U_{k(tn+i)}$, $U_{ki}V_{k(tn+i)}$, $V_{ki}V_{k(tn+i)}$ and $V_{ki}U_{k(tn+i)}$ for positive integers t and n. For example, for odd k,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti}V_{k(n-ti)} = U_{kt}^n \begin{cases} (-1)^t V_{kn(t-1)}D^{(n-1)/2} & \text{if } n \text{ is odd,} \\ U_{kn(t-1)}D^{n/2} & \text{if } n \text{ is even,} \end{cases}$$

where D is defined as before.

In [4],inspired by the works of [5, 6, 8], Kılıç et al. gave rising factorial of the summation index instead of its powers. Clearly, they considered and computed the generalized alternating weighted binomial sums :

$$\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i f(n,i,k,t),$$

where $f(n,i,k,t)$ as before and m is a nonnegative integer and $x^{\underline{m}}$ stands for the falling factorial defined by $x^{\underline{m}} = x(x-1)...(x-m+1)$. These kinds of binomial sums (except some special cases of k and t) have not been considered according to our best literature acknowledgement. For example, for any integers k and t,

$$\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i U_{kti}V_{k(n-ti)} = (-1)^{kn(t+1)+m} n^{\underline{m}} U_{kt}^{n-m}$$
$$\times \begin{cases} U_{k(tn+tm-n)}D^{(n-m)/2} & \text{if } n \equiv m (\bmod 2), \\ -V_{k(tn+tm-n)}D^{(n-m-1)/2} & \text{if } n \equiv m+1 (\bmod 2), \end{cases}$$

where m is a nonnegative integer.

2. Sums of certain products with the terms of $\{g_{kn}\}$ and $\{h_{kn}\}$

In this section, firstly, we will start with some lemmas for further use.

Lemma 2.1. *For any integers m and n, we have*

$$g_{k(m+n)} + g_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} h_{km}g_{kn} & \text{if } n \text{ is odd,} \\ g_{km}h_{kn} & \text{if } n \text{ is even,} \end{cases}$$

$$g_{k(m+n)} - g_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} g_{km} h_{kn} & \textit{if } n \textit{ is odd,} \\ h_{km} g_{kn} & \textit{if } n \textit{ is even,} \end{cases}$$

$$h_{k(m+n)} - h_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} h_{km} h_{kn} & \textit{if } n \textit{ is odd,} \\ \Delta g_{km} g_{kn} & \textit{if } n \textit{ is even,} \end{cases}$$

$$h_{k(m+n)} + h_{k(m-n)} \left(q^k (-1)^{k+1} c^{2k} - c^k v_k - 1\right)^n = \begin{cases} \Delta g_{km} g_{kn} & \textit{if } n \textit{ is odd,} \\ h_{km} h_{kn} & \textit{if } n \textit{ is even,} \end{cases}$$

where c is a nonzero real number.

Proof. By the Binet formulae of $\{g_{kn}\}$ and $\{h_{kn}\}$, the claimed equalities are obtained. □

We recall some facts for the readers convenience: For any real numbers m and n,

$$(2.1) \qquad (m+n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i \left(m^{t-2i} + n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i \left(m^{t-2i} + n^{t-2i}\right) + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases}$$

and

$$(2.2) \qquad (m-n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i \left(m^{t-2i} - n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i \left(m^{t-2i} + n^{t-2i}\right) + \binom{t}{t/2} (-1)^{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases}$$

where t is a positive integer.

Lemma 2.2. *For any integers r and s, we have*

$$\sum_{i=0}^{n} \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{r(n-i)} h_{k(2ri+s)} = h_{k(rn+s)} h_{kr}^n,$$

$$\sum_{i=0}^{n} \binom{n}{i} \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{r(n-i)} g_{k(2ri+s)} = g_{k(rn+s)} h_{kr}^n,$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{r(n-i)} g_{k(2ri+s)} = \begin{cases} -\Delta^{(n-1)/2} g_{kr}^n h_{k(rn+s)} & \textit{if } n \textit{ is odd,} \\ \Delta^{n/2} g_{kr}^n g_{k(rn+s)} & \textit{if } n \textit{ is even,} \end{cases}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{r(n-i)}h_{k(2ri+s)} = \begin{cases} -\Delta^{(n+1)/2}g_{kr}^{n}g_{k(rn+s)} & \text{if } n \text{ is odd,} \\ \Delta^{n/2}g_{kr}^{n}h_{k(rn+s)} & \text{if } n \text{ is even,} \end{cases} \tag{2.3}$$

where c is a nonzero real number.

Proof. From (2.1), (2.2) and Lemma 2.1, the proof is obtained. □

Lemma 2.3. *[2]Let n and m be integers such that $0 \leq m < n$. For $z \neq -1$,*

$$\sum_{k=0}^{n}\binom{n}{k}k^{\underline{m}}z^{k} = z^{m}n^{\underline{m}}(1+z)^{n-m} \quad .$$

Theorem 2.1. *Let a, b and e be any integers. Then*

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}&\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}\\ &= \frac{1}{\Delta}\Bigg(h_{k(an+b+e)}h_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an}\\ &\qquad - 2^{n}h_{k(b-e)}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{e}\Bigg),\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}&\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}h_{k(ai+b)}h_{k(ai+e)}\\ &= h_{k(an+b+e)}h_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an}\\ &\qquad + 2^{n}h_{k(b-e)}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{e},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}&\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}h_{k(ai+e)}\\ &= g_{k(an+b+e)}h_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an}\\ &\qquad + 2^{n}g_{k(b-e)}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{e},\end{aligned}$$

where c is a nonzero real number.

Proof. Consider that

$$
\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}\\
&=\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}\\
&\times\frac{\left[\left(c^{k}\alpha^{k}+1\right)^{ai+b}-\left(c^{k}\beta^{k}+1\right)^{ai+b}\right]\left[\left(c^{k}\alpha^{k}+1\right)^{ai+e}-\left(c^{k}\beta^{k}+1\right)^{ai+e}\right]}{(\alpha-\beta)^{2}}\\
&=\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}\\
&\times\Big(\left(c^{k}\alpha^{k}+1\right)^{2ai+b+e}-\left(c^{k}\alpha^{k}+1\right)^{ai+b}\left(c^{k}\beta^{k}+1\right)^{ai+e}\\
&\qquad+\left(c^{k}\beta^{k}+1\right)^{2ai+b+e}-\left(c^{k}\alpha^{k}+1\right)^{ai+e}\left(c^{k}\beta^{k}+1\right)^{ai+b}\Big)\\
&=\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}h_{k(2ai+b+e)}\\
&\qquad-\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{e}h_{k(b-e)}.
\end{aligned}
$$

From Lemma 2.2, the desired result is obtained. Similarly, the other cases are given. □

Theorem 2.2. *Let a, b and e be any integers. Then*

$$
\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}\\
&=\begin{cases}-\Delta^{(n-1)/2}g_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is odd},\\ \Delta^{(n-2)/2}h_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is even},\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-ai}h_{k(ai+b)}h_{k(ai+e)}\\
&=\begin{cases}-\Delta^{(n+1)/2}g_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is odd},\\ \Delta^{n/2}h_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-an} & \text{if } n \text{ is even},\end{cases}
\end{aligned}
$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}g_{k(ai+b)}h_{k(ai+d)}$$
$$=\begin{cases}-\Delta^{(n-1)/2}h_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-an} & \text{if } n \text{ is odd,}\\ \Delta^{n/2}g_{k(an+b+e)}g_{ka}^{n}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-an} & \text{if } n \text{ is even,}\end{cases}$$

where c is a nonzero real number.

Proof. Consider that

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}g_{k(ai+b)}g_{k(ai+e)}\\ &=\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}\\ &\times\frac{\left[\left(c^k\alpha^k+1\right)^{ai+b}-\left(c^k\beta^k+1\right)^{ai+b}\right]\left[\left(c^k\alpha^k+1\right)^{ai+e}-\left(c^k\beta^k+1\right)^{ai+e}\right]}{(\alpha-\beta)^2}\\ &=\frac{1}{(\alpha-\beta)^2}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}\\ &\times\left(\left(c^k\alpha^k+1\right)^{2ai+b+e}-\left(c^k\alpha^k+1\right)^{ai+b}\left(c^k\beta^k+1\right)^{ai+e}\right.\\ &\qquad\left.+\left(c^k\beta^k+1\right)^{2ai+b+e}-\left(c^k\alpha^k+1\right)^{ai+e}\left(c^k\beta^k+1\right)^{ai+b}\right)\\ &=\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-ai}h_{k(2ai+b+e)}\\ &\qquad-\frac{1}{\Delta}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^{e}h_{k(b-e)}.\end{aligned}$$

From (2.3), the desired result is obtained. Similarly, using Lemma 2.2, the other cases can be obtained. □

Theorem 2.3. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i g_{k(n-ti)}h_{kti}=(-1)^m n^{\underline{m}}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{n(1-t)}g_{kt}^{n-m}$$
$$\times\begin{cases}-\Delta^{(n-m)/2}g_{k(tn+tm-n)} & \text{if } n\equiv m(\text{mod } 2),\\ \Delta^{(n-m-1)/2}h_{k(tn+tm-n)} & \text{if } n\equiv m+1(\text{mod } 2),\end{cases}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i g_{kti} h_{k(n-ti)} = (-1)^m n^{\underline{m}} \left(c^{2k} (-q)^k + c^k v_k + 1\right)^{n(1-t)} g_{kt}^{n-m}$$

$$\times \begin{cases} \Delta^{(n-m)/2} g_{k(tn+tm-n)} & \text{if } n \equiv m (\text{mod } 2), \\ -\Delta^{(n-m-1)/2} h_{k(tn+tm-n)} & \text{if } n \equiv m+1 (\text{mod } 2), \end{cases}$$

where c is a nonzero real number and m is a nonnegative integer.

Proof. Observe that

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i g_{k(n-ti)} h_{kti} \\
&= \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i \left(\left(c^k\alpha^k+1\right)^{n-ti} - \left(c^k\beta^k+1\right)^{n-ti}\right) \\
&\times \left(\left(c^k\alpha^k+1\right)^{ti} + \left(c^k\beta^k+1\right)^{ti}\right) \\
&= \frac{\left(c^k\alpha^k+1\right)^n}{\alpha-\beta} \left(\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i + \sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i \left(\frac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{ti}\right) \\
&- \frac{\left(c^k\beta^k+1\right)^n}{\alpha-\beta} \left(\sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i + \sum_{i=0}^{n} \binom{n}{i} i^{\underline{m}} (-1)^i \left(\frac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{ti}\right),
\end{aligned}$$

which by Lemma 2.3, equals

$$\begin{aligned}
&\tfrac{\left(c^k\alpha^k+1\right)^n}{\alpha-\beta} \left\{(-1)^m \left(\tfrac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{tm} n^{\underline{m}} \left(1-\left(\tfrac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^t\right)^{n-m}\right\} \\
&-\tfrac{\left(c^k\beta^k+1\right)^n}{\alpha-\beta} \left\{(-1)^m \left(\tfrac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{tm} n^{\underline{m}} \left(1-\left(\tfrac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^t\right)^{n-m}\right\} \\
&= (-1)^m n^{\underline{m}} \tfrac{1}{\alpha-\beta} \left\{\left(c^k\alpha^k+1\right)^n \left(\tfrac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{tm} \left(1-\left(\tfrac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^t\right)^{n-m}\right. \\
&\quad \left. - \left(c^k\beta^k+1\right)^n \left(\tfrac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{tm} \left(1-\left(\tfrac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^t\right)^{n-m}\right\} \\
&= (-1)^m n^{\underline{m}} (\alpha-\beta)^{n-m-1} \\
&\times \left\{\left(c^k\alpha^k+1\right)^n \left(\tfrac{c^k\beta^k+1}{c^k\alpha^k+1}\right)^{tm} \left(\tfrac{\left(c^k\alpha^k+1\right)^t-\left(c^k\beta^k+1\right)^t}{(\alpha-\beta)\left(c^k\alpha^k+1\right)^t}\right)^{n-m}\right. \\
&\left. -(-1)^{n-m} \left(c^k\beta^k+1\right)^n \left(\tfrac{c^k\alpha^k+1}{c^k\beta^k+1}\right)^{tm} \left(\tfrac{\left(c^k\alpha^k+1\right)^t-\left(c^k\beta^k+1\right)^t}{(\alpha-\beta)\left(c^k\beta^k+1\right)^t}\right)^{n-m}\right\},
\end{aligned}$$

which by the Binet formulae, gives us the claimed result. □

Similar to the proof method of Theorem just above, we have the following results without proof.

Theorem 2.4. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}g_{k(n-ti)}h_{kti}$$
$$=2^{n-m}n^{\underline{m}}g_{kn}-n^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{n(1-t)}h_{kt}^{n-m}g_{k(tm+tn-n)},$$

and

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}g_{kti}h_{k(n-ti)}$$
$$=2^{n-m}n^{\underline{m}}g_{kn}+n^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{n(1-t)}h_{kt}^{n-m}g_{k(tm+tn-n)},$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.5. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{i}g_{kti}h_{kn(t+1)-ki(t+2)}$$
$$=n^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{m}h_{k}^{n-m}g_{k(nt-m)}+n^{\underline{m}}h_{k(t+1)}^{n-m}g_{km(t+1)},$$

and

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{i}g_{kn(t+1)-ki(t+2)}h_{kti}$$
$$=n^{\underline{m}}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{m}h_{k}^{n-m}g_{k(nt-m)}-n^{\underline{m}}h_{k(t+1)}^{n-m}g_{km(t+1)},$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.6. *Let k and t be any integers. Then*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(-1\right)^{i}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{i}g_{kti}h_{kn(t+1)-ki(t+2)}$$
$$=\left(-1\right)^{m}n^{\underline{m}}\Delta^{(n-m)/2}$$
$$\times\begin{cases}\begin{array}{c}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{m}\\ \times g_{k}^{n-m}g_{k(tn-m)}+g_{k(t+1)}^{n-m}g_{km(t+1)}\end{array} & \text{if } n\equiv m(\operatorname{mod}\,2),\\[2ex] \Delta^{-1/2}\left\{\begin{array}{c}\left(c^{2k}\left(-q\right)^{k}+c^{k}v_{k}+1\right)^{m}\\ \times g_{k}^{n-m}h_{k(tn-m)}-g_{k(t+1)}^{n-m}h_{km(t+1)}\end{array}\right\} & \text{if } n\equiv m+1(\operatorname{mod}\,2),\end{cases}$$

and

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^i g_{kn(t+1)-ki(t+2)}h_{kti}\\&=(-1)^m n^{\underline{m}}\Delta^{(n-m)/2}\\&\times\begin{cases}\begin{array}{c}\left(c^{2k}(-q)^k+c^kv_k+1\right)^m\\ \times g_k^{n-m}g_{k(tn-m)}-g_{k(t+1)}^{n-m}g_{km(t+1)}\end{array} & \text{if } n\equiv m(\text{mod } 2),\\ \\ \begin{array}{c}\Delta^{-1/2}\Big\{\left(c^{2k}(-q)^k+c^kv_k+1\right)^m\\ \times g_k^{n-m}h_{k(tn-m)}+g_{k(t+1)}^{n-m}h_{km(t+1)}\Big\}\end{array} & \text{if } n\equiv m+1(\text{mod } 2),\end{cases}\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.7. *Let k and t be any integers. Then*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(c^{2k}(-q)^k+c^kv_k+1\right)^i g_{kti}h_{kn-ki(t+2)}\\&=n^{\underline{m}}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-nt}h_{k(t+1)}^{n-m}g_{k(mt+nt+m)}-n^{\underline{m}}h_k^{n-m}g_{km},\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(c^{2k}(-q)^k+c^kv_k+1\right)^i g_{kn-ki(t+2)}h_{kti}\\&=-n^{\underline{m}}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-nt}h_{k(t+1)}^{n-m}g_{k(mt+nt+m)}-n^{\underline{m}}h_k^{n-m}g_{km},\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

Theorem 2.8. *Let k and t be any integers. Then*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i\left(c^{2k}(-q)^k+c^kv_k+1\right)^i g_{kti}h_{kn-ki(t+2)}\\&=(-1)^m n^{\underline{m}}\Delta^{(n-m)/2}\\&\times\begin{cases}\begin{array}{c}\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-nt}\\ \times g_{k(t+1)}^{n-m}g_{k(mt+nt+t)}-g_k^{n-m}g_{km}\end{array} & \text{if } n\equiv m(\text{mod } 2),\\ \\ \begin{array}{c}-\Delta^{-1/2}\Big\{\left(c^{2k}(-q)^k+c^kv_k+1\right)^{-nt}\\ \times g_{k(t+1)}^{n-m}h_{k(mt+nt+t)}-g_k^{n-m}h_{km}\Big\}\end{array} & \text{if } n\equiv m+1(\text{mod } 2),\end{cases}\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{i}g_{kn-ki(t+2)}h_{kti}\\
&=(-1)^{m}n^{\underline{m}}\Delta^{(n-m)/2}\\
&\times\begin{cases}
\begin{array}{l}-\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt}\\ \times g_{k(t+1)}^{n-m}g_{k(mt+nt+t)}-g_{k}^{n-m}g_{km}\end{array} & \text{if } n\equiv m(\text{mod } 2),\\[2ex]
\begin{array}{l}\Delta^{-1/2}\Big\{\left(c^{2k}(-q)^{k}+c^{k}v_{k}+1\right)^{-nt}\\ \times g_{k(t+1)}^{n-m}h_{k(mt+nt+t)}+g_{k}^{n-m}h_{km}\Big\}\end{array} & \text{if } n\equiv m+1(\text{mod } 2),
\end{cases}
\end{aligned}$$

where c is a nonzero real number and m is a nonnegative integer.

REFERENCES

1. C. K. Cook and T. Komatsu: *Some identities for sequences of binomial sums of generalized Fibonacci numbers*. The Fibonacci Quart. **54(2)** (2016), 105–111.
2. H. W. Gould: *Combinatorial identities*. Morgantown, W. Va., 1972.
3. E. Kiliç and P. Stanica: *Factorizations and representations of second order linear recurrences with indices in arithmetic progressions*. Bulletin of the Mexican Mathematical Society **15(1)** (2009), 23–36.
4. E. Kiliç, N. Ömür and S. Koparal: *On alternating weighted binomial sums with falling factorials*. Bulletin of Analysis and Applications **9(1)** (2017), 58–64.
5. E. Kiliç and N. Ömür: *Some weighted binomial sums by generating function methods*. Integers #A27 (2013).
6. E. Kiliç: *Some classes of alternating weighted binomial sums*. Annals of the Alexandru Ioan Cuza University Mathematics **3(2)** (2016), 835–843.
7. E. Kiliç, N. Ömür and Y. T. Ulutaş: *Binomial sums whose coefficients are products of terms of binary sequences*. Utilitas Math. **84** (2011), 45–52.
8. E. Kiliç, Y. T. Ulutaş and N. Ömür: *Formulas for weighted binomial sums with the powers of terms of binary recurrences*. Miskolc Math. Notes **13(1)** (2012), 53–65.
9. T. Komatsu: *Some generalized Fibonacci identities including powers and binomial coefficients*. The Fibonacci Quart. **52(1)** (2014), 50–60.
10. N. Ömür and C. D. Şener: *On identities for sequences of binomial sums with the terms of sequences $\{u_{kn}\}$ and $\{v_{kn}\}$*. European Journal of Pure and Applied Mathematics **10(3)** (2017), 506–515.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 43-55
https://doi.org/10.22190/FUMI200117004A
Original Scientific Paper

$I-$LACUNARY STATISTICAL CONVERGENCE OF ORDER β OF DIFFERENCE SEQUENCES OF FRACTIONAL ORDER

Nazlım Deniz Aral[1] and Hacer Şengül Kandemir[2]

[1] **Faculty of Arts and Sciences, Department of Mathematics, Bitlis Eren University, Bitlis, 13000, Turkey**
[2] **Faculty of Education, Department of Mathematics, Harran University, Osmanbey Campus 63190, Şanlıurfa, 23119, Turkey**

Abstract. In this paper, we have introduced the concepts of ideal $\Delta^{\alpha}-$lacunary statistical convergence of order β with the fractional order α and ideal $\Delta^{\alpha}-$lacunary strongly convergence of order β with the fractional order α (where $0 < \beta \leq 1$) and given some relations about these concepts.
Keywords: I-convergence, lacunary sequence, difference sequence.

1. Introduction

The idea of statistical convergence was formerly given under the name "almost convergence" by Zygmund [53] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [48] and Fast [24] and later reintroduced by Schoenberg [45]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Çakallı et al. ([7],[8],[9]). Caserta et al. [10], Çınar et al. [12], Connor [11], Et et al. ([20],[23]), Fridy [26], Fridy and Orhan [27], Isik et al. ([29],[30],[31]), Mursaleen [40], Salat [47], Mohiuddine et al. ([5],[6],[33],[38],[39],[41]) and many others.

The idea of statistical convergence depends upon the density of subsets of the

Received January 17, 2020; accepted May 17, 2020.
Corresponding Author: Nazlım Deniz Ara, Faculty of Arts and Sciences, Department of Mathematics, Bitlis Eren University, Bitlis, 13000, Turkey | E-mail: ndaral@beu.edu.tr
2010 *Mathematics Subject Classification.* 40A05, 40C05, 46A45

set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \text{ provided that the limit exists.}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

Recently, Çolak [13] have generalized the statistical convergence by ordering the interval $(0, 1]$ and defined the statistical convergence of order β and strong $p-$Cesàro summability of order β, where $0 < \beta \leq 1$ and p is a positive real number. Şengül and Et ([19],[49]) generalized the concepts such as lacunary statistical convergence of order β and lacunary strong $p-$Cesàro summability of order β for sequences of real numbers.

The notation of I-convergence is a generalization of the statistical convergence. Kostyrko et al. ([36]) introduced the notation of I-convergence. Some further results connected with the notation of I-convergence can be found in ([14],[15],[37],[43],[44],[52]).

Let X be non-empty set. Then a family sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in X, $X \neq \phi$, then the family of sets

$F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with* I. Throughout this study, I will stand for a non-trivial admissible ideal of $\mathbb{N}$ and by a sequence we always mean a sequence of real numbers.

Difference sequence spaces were defined by Kızmaz [35] and the concept was generalized by Et et al. ([16],[17]) as follows:

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \tag{1.1}$$

$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. After then, some properties of difference sequence spaces have been studied in ([1],[2],[21],[22],[34],[44]).

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha : w \to w$ defined by

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}. \tag{1.2}$$

In particular, we have $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6}\cdots$

$\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6}\cdots$

$\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} - \frac{154}{6561}x_{k+6}\cdots$

$\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} - \frac{91}{6561}x_{k+6}\cdots$

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r \notin \{0,-1,-2,-3,...\}$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r) = \int_0^\infty e^{-t}t^{r-1}dt.$$

From the definition, it is observed that:

(i) For any natural number n, $\Gamma(n+1) = n!$,

(ii) For any real number n and $n \notin \{0,-1,-2,-3,...\}, \Gamma(n+1) = n\Gamma(n)$,

(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, ...$

Without loss of generality, we assume throughout that the series defined in (1.2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (1.2) reduces to a finite sum i.e., $\sum_{i=0}^{\alpha}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$. In fact, this operator is generalized the difference operator introduced by Et and Çolak [16].

Recently, using fractional operator Δ^α (fractional order of α) Baliarsingh et al. ([3],[4],[42]) defined the sequence space $\Delta^\alpha(X)$ such as:

$$\Delta^\alpha(X) = \{x = (x_k) : (\Delta^\alpha x_k) \in X\},$$

where X is any sequence space.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([7],[8],[9],[25],[27],[28],[32],[46],[50],[51]).

1.1. Definitions and Main Results

Definition 1 Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction. The sequence $x = (x_k)$ is said to be $(\Delta^\alpha, I)-$lacunary statistically convergent of order β (or $\Delta^\alpha(S_\theta^\beta, I)-$convergent) to the number L, if there is a real number L such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \delta\right\} \in I$$

for each $\varepsilon > 0$ and $\delta > 0$. In this case, we write $x_k \to L(\Delta^\alpha(S_\theta^\beta, I))$. The set of all $(\Delta^\alpha, I)-$lacunary statistically convergent of order β sequences will be denoted by $\Delta^\alpha(S_\theta^\beta, I)$. If $\theta = (2^r)$, then we write $\Delta^\alpha(S^\beta, I)$ instead of $\Delta^\alpha(S_\theta^\beta, I)$. In the special cases $\theta = (2^r)$ and $\beta = 1$, we write $\Delta^\alpha(S, I)$ instead of $\Delta^\alpha(S_\theta^\beta, I)$.

In particular, $\Delta^\alpha(S_\theta^\beta, I)-$convergence includes many special cases; for example, in case of $\alpha = m \in \mathbb{N}$, $(\Delta^\alpha, I)-$lacunary statistical convergence of order β reduces to the $(\Delta^m, I)-$lacunary statistical convergence which was defined and studied by Et and Şengül [18].

Definition 2 Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$, α be a fixed proper fraction and $p \geq 1$ be a real number. A sequence $x = (x_k)$ is said to be $\Delta^\alpha(N_\theta^\beta, I)-$summable to L (or ideal $\Delta^\alpha-$lacunary strongly summable of order β) if

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\beta}\sum_{k \in I_r}|\Delta^\alpha x_k - L|^p \geqslant \varepsilon\right\} \in I.$$

In this case we write $x_k \to L(\Delta^\alpha(N_\theta^\beta, p, I))$. We denote the class of all ideal $\Delta^\alpha-$lacunary strongly summable sequences of order β by $\Delta^\alpha(N_\theta^\beta, p, I)$.

Theorem 1 Let $0 < \beta \leqslant \gamma \leqslant 1$. If $x_k \to L(\Delta^\alpha(S_\theta^\beta, I))$, then $x_k \to L(\Delta^\alpha(S_\theta^\gamma, I))$.

Proof. The inclusion part of the proof is trivial. The following example shows that the inclusion is strict. Let $\alpha \in \mathbb{N}$ and define a sequence $\Delta^\alpha x_k$ by

$$\Delta^\alpha x_k = \begin{cases} k & k = n^3 \\ \frac{1}{3} & \text{otherwise} \end{cases}.$$

Then $x \in (\Delta^\alpha(S_\theta^\gamma, I))$ for $\frac{1}{3} < \gamma \leqslant 1$ but $x \notin (\Delta^\alpha(S_\theta^\beta, I))$ for $0 < \beta \leqslant \frac{1}{3}$ by (1.1).

Theorem 2 If $x_k \to L(\Delta^\alpha(N_\theta^\beta, p, I))$, then $x_k \to L(\Delta^\alpha(N_\theta^\gamma, p, I))$ and the inclusion is proper.

Proof. The inclusion part of the proof is easy. The following example shows that the inclusion is strict. Let $\alpha \in \mathbb{N}$ and define a sequence $\Delta^\alpha x_k$ by

$$\Delta^\alpha x_k = \begin{cases} 1 & k = n^2 \\ 0 & \text{otherwise} \end{cases}.$$

Then $x \in (\Delta^{\alpha}(N_{\theta}^{\gamma}, p, I))$ for $\frac{1}{2} < \gamma \leqslant 1$ but $x \notin (\Delta^{\alpha}(N_{\theta}^{\beta}, p, I))$ for $0 < \beta \leqslant \frac{1}{2}$ by (1.1).

Theorem 3 If $x_k \to L(\Delta^{\alpha}(N_{\theta}^{\beta}, p, I))$, then $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$ and the inclusion is proper.

Proof. Taking $p = 1$ and $L = 0$, we show the strictness of the inclusion. Let $\alpha \in \mathbb{N}$ and define a sequence $\Delta^{\alpha} x_k$ by

$$\Delta^{\alpha} x_k = \begin{cases} \left[\sqrt[3]{h_r}\right] & k = 1, 2, 3, \cdots, \left[\sqrt[3]{h_r}\right] \\ 0 & \text{otherwise} \end{cases}.$$

Then we have for every $\varepsilon > 0$ and $\frac{1}{3} < \beta \leqslant 1$,

$$\frac{1}{h_r^{\beta}} |\{k \in I_r : |\Delta^{\alpha} x_k - 0| \geq \varepsilon\}| \leqslant \frac{\left[\sqrt[3]{h_r}\right]}{h_r^{\beta}},$$

and for any $\delta > 0$ we get

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^{\beta}} |\{k \in I_r : |\Delta^{\alpha} x_k - 0| \geq \varepsilon\}| \geqslant \delta \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{\left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \geqslant \delta \right\}$$

and so $x_k \to 0(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$ for $\frac{1}{3} < \beta \leqslant 1$ by (1.1). On the other hand, for $0 < \beta \leqslant \frac{2}{3}$,

$$\frac{1}{h_r^{\beta}} \sum_{k \in I_r} |\Delta^{\alpha} x_k - 0| = \frac{\left[\sqrt[3]{h_r}\right] \left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \to \infty$$

and for $\alpha = \frac{2}{3}$,

$$\frac{\left[\sqrt[3]{h_r}\right] \left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \to 1.$$

$\left\{ r \in \mathbb{N} : \frac{1}{h_r^{\beta}} \sum_{k \in I_r} |\Delta^{\alpha} x_k - 0| \geqslant 1 \right\} = \left\{ r \in \mathbb{N} : \frac{\left[\sqrt[3]{h_r}\right]\left[\sqrt[3]{h_r}\right]}{h_r^{\beta}} \geqslant 1 \right\} = \{a, a+1, a+2, ...\} \in F(I)$ for some $a \in \mathbb{N}$, since I is admissible. Thus $x_k \nrightarrow 0(\Delta^{\alpha}(N_{\theta}^{\beta}, p, I))$ by (1.1).

The proof of the following theorems is straightforward, so we choose to state these results without proof.

Theorem 4 If $\liminf_r q_r > 1$, then $x_k \to L(\Delta^{\alpha}(S^{\beta}, I))$ implies $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$.

Theorem 5 If $\liminf_r \frac{h_r^{\alpha}}{k_r} > 0$, then $x_k \to L(\Delta^{\alpha}(S, I))$ implies $x_k \to L(\Delta^{\alpha}(S_{\theta}^{\beta}, I))$.

Theorem 6 $\Delta^{\alpha}(S_{\theta}^{\beta}, I) \cap \ell_{\infty}(\Delta^{\alpha})$ is closed subset of $\ell_{\infty}(\Delta^{\alpha})$ for $0 < \beta \leqslant 1$.

Theorem 7 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ (for all $r \in \mathbb{N}$) and $\beta, \gamma \in (0, 1]$ be real numbers such that $\beta \leqslant \gamma$ and α be a proper fraction.

Theorem 8 i) If

$$\lim_{r\to\infty}\inf\frac{h_r^{\beta}}{\ell_r^{\gamma}}>0, \tag{1.3}$$

then $\Delta^{\alpha}(S_{\theta'}^{\gamma},I)\subseteq\Delta^{\alpha}(S_{\theta}^{\beta},I)$

ii) If

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^{\gamma}}=1, \tag{1.4}$$

then $\Delta^{\alpha}(S_{\theta}^{\beta},I)\subseteq\Delta^{\alpha}(S_{\theta'}^{\gamma},I)$.

Proof. i) Omitted.

ii) Let $x=(x_k)\in\Delta^{\alpha}(S_{\theta}^{\beta},I)$ and be (1.4) satisfied. Since $I_r\subset J_r$, for $\varepsilon>0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^{\gamma}}|\{k\in J_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}| &= \frac{1}{\ell_r^{\gamma}}|\{s_{r-1}<k\leqslant k_{r-1}:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\\
&+\frac{1}{\ell_r^{\gamma}}|\{k_r<k\leqslant s_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|+\frac{1}{\ell_r^{\gamma}}|\{k_{r-1}<k\leqslant k_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\\
&\leqslant\frac{k_{r-1}-s_{r-1}}{\ell_r^{\gamma}}+\frac{s_r-k_r}{\ell_r^{\gamma}}+\frac{1}{\ell_r^{\gamma}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\\
&=\frac{\ell_r-h_r}{\ell_r^{\gamma}}+\frac{1}{\ell_r^{\gamma}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\\
&\leqslant\frac{\ell_r-h_r^{\gamma}}{h_r^{\gamma}}+\frac{1}{h_r^{\gamma}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|\\
&\leqslant\left(\frac{\ell_r}{h_r^{\gamma}}-1\right)+\frac{1}{h_r^{\beta}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geqslant\varepsilon\}|
\end{aligned}$$

for all $r\in\mathbb{N}$, where $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$ and $\ell_r=s_r-s_{r-1}$. Thus

$$\begin{aligned}
&\left\{r\in\mathbb{N}:\frac{1}{\ell_r^{\beta}}|\{k\in J_r:|\Delta^{\alpha}x_k-L|\geq\varepsilon\}|\geqslant\delta\right\}\subseteq\\
&\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r^{\beta}}|\{k\in I_r:|\Delta^{\alpha}x_k-L|\geq\varepsilon\}|\geqslant\delta\right\}\in I.
\end{aligned}$$

This implies that $\Delta^{\alpha}(S_{\theta}^{\beta},I)\subseteq\Delta^{\alpha}(S_{\theta'}^{\gamma},I)$.

Theorem 9 Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subseteq J_r$ for all $r\in\mathbb{N}$, β and γ be fixed real numbers such that $0<\beta\leqslant\gamma\leqslant1$ and $0<p<\infty$. Then we have,

i) If (1.3) holds then $\Delta^\alpha(N_{\theta'}^\gamma, p, I) \subset \Delta^\alpha(N_\theta^\beta, p, I)$,

ii) If (1.4) holds and $x \in \Delta^\alpha(\ell_\infty)$ then $\Delta^\alpha(N_\theta^\beta, p, I) \subset \Delta^\alpha(N_{\theta'}^\gamma, p, I)$.

Proof. Omitted.

Theorem 10 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ (for all $r \in \mathbb{N}$), β and γ be fixed real numbers such that $0 < \beta \leqslant \gamma \leqslant 1$ and $0 < p < \infty$. Then,

i) Let (1.3) holds, if a sequence is strongly $\Delta^\alpha(N_{\theta'}^\gamma, p, I)$-summable to L, then it is $\Delta^\alpha(S_\theta^\beta, I)$-statistically convergent to L.

ii) Let (1.4) holds and $x = (x_k)$ be a $\Delta^\alpha-$bounded sequence if $\Delta^\alpha(S_\theta^\beta, I)$-statistically convergent to L, then it is strongly $\Delta^\alpha(N_{\theta'}^\gamma, p, I)-$summable to L.

Proof. i) For any sequence $x = (x_k)$ and $\varepsilon > 0$, we have

$$\begin{aligned}
\sum_{k \in J_r} |\Delta^\alpha x_k - L|^p &= \sum_{\substack{k \in J_r \\ |\Delta^\alpha x_k - L| \geqslant \varepsilon}} |\Delta^\alpha x_k - L|^p + \sum_{\substack{k \in J_r \\ |\Delta^\alpha x_k - L| < \varepsilon}} |\Delta^\alpha x_k - L|^p \\
&\geqslant \sum_{\substack{k \in I_r \\ |\Delta^\alpha x_k - L| \geqslant \varepsilon}} |\Delta^\alpha x_k - L|^p \\
&\geqslant |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \varepsilon^p
\end{aligned}$$

and so that

$$\begin{aligned}
\frac{1}{\ell_r^\gamma} \sum_{k \in J_r} |\Delta^\alpha x_k - L|^p &\geqslant \frac{1}{\ell_r^\gamma} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \varepsilon^p \\
&\geqslant \frac{h_r^\beta}{\ell_r^\gamma} \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \varepsilon^p.
\end{aligned}$$

$$\begin{aligned}
&\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \delta \right\} \subseteq \\
&\subseteq \left\{ r \in \mathbb{N} : \frac{1}{\ell_r^\gamma} \sum_{k \in J_r} |\Delta^\alpha x_k - L|^p \geqslant \frac{h_r^\beta}{\ell_r^\gamma} \delta \varepsilon^p \right\} \in I.
\end{aligned}$$

Hence $x = (x_k)$ is $\Delta^\alpha(S_\theta^\beta, I)-$statistically convergent to L.

ii) Suppose that $\Delta^\alpha(S_\theta^\beta, I)$-statistically convergent to L and $x = (x_k) \in \Delta^\alpha\,(\ell_\infty)$. Then there exists some $M > 0$ such that $|\Delta^\alpha x_k - L| \leqslant M$ for all k. Then for every $\varepsilon > 0$ we may write

$$\frac{1}{\ell_r^\gamma} \sum_{k \in J_r} |\Delta^\alpha x_k - L|^p = \frac{1}{\ell_r^\gamma} \sum_{k \in J_r - I_r} |\Delta^\alpha x_k - L|^p + \frac{1}{\ell_r^\gamma} \sum_{k \in I_r} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r - h_r}{\ell_r^\gamma}\right) M^p + \frac{1}{\ell_r^\gamma} \sum_{k \in I_r} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r - h_r^\gamma}{\ell_r^\gamma}\right) M^p + \frac{1}{\ell_r^\gamma} \sum_{k \in I_r} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) M^p + \frac{1}{h_r^\gamma} \sum_{\substack{k \in I_r \\ |\Delta^\alpha x_k - L| \geqslant \varepsilon}} |\Delta^\alpha x_k - L|^p + \frac{1}{h_r^\gamma} \sum_{\substack{k \in I_r \\ |\Delta^\alpha x_k - L| < \varepsilon}} |\Delta^\alpha x_k - L|^p$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) M^p + \frac{M^p}{h_r^\gamma} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| + \frac{h_r}{h_r^\gamma} \varepsilon^p$$

$$\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) M^p + \frac{M^p}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| + \frac{\ell_r}{h_r^\gamma} \varepsilon^p$$

for all $r \in \mathbb{N}$.

$$\left\{ r \in \mathbb{N} : \frac{1}{\ell_r^\gamma} \sum_{k \in J_r} |\Delta^\alpha x_k - L|^p \geqslant \delta \right\} \subseteq$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| \geqslant \frac{\delta}{M^p} \right\} \in I.$$

Using (1.4) we obtain that $\Delta^\alpha(N_{\theta'}^\gamma, p, I)$-statistically convergent to L, whenever $\Delta^\alpha(S_\theta^\beta, I)$-summable to L.

Definition 3 Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$, α be a proper fraction. The sequence $x = (x_k)$ is said to be $(\Delta^\alpha, I)-$lacunary statistically Cauchy sequence of order β (or $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy) if there is a subsequence $(x_{k'(r)})$ of (x_k) such that $k'(r) \in J_r$ for each $r \in \mathbb{N}$, $x_{k'(r)} \to L(\Delta^\alpha)$ (i.e. $\lim_r |\Delta^\alpha x_{k'(r)} - L| = 0$)

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \varepsilon \right\} \in I$$

for each $\varepsilon > 0$.

Theorem 11 If $x = (x_k)$ is a $\Delta^\alpha(N_\theta^\beta, I)-$summable if and only if it is a $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy sequence.

Proof. Assume that (x_k) is a $\Delta^\alpha(N_\theta^\beta, I)-$summable sequence to L. Then there exists L such that $x_k \to L(\Delta^\alpha(N_\theta^\beta, I)$. Therefore,

$$H_i = \left\{ i \in \mathbb{N} : |\Delta^\alpha x_k - L| < \frac{1}{i} \right\}$$

for each $i \in \mathbb{N}$. Hence for each i, $H_{i+1} \subseteq H_i$ and

$$\left\{ r \in \mathbb{N} : \frac{|H_r \cap J_r|}{h_r^\beta} \geqslant \frac{1}{r} \right\} \in I.$$

We choose k_1, such that $r \geqslant k_1$, then

$$\left\{r \in \mathbb{N} : \frac{|H_1 \cap J_r|}{h_r^\beta} < 1\right\} \notin I.$$

Next we choose $k_2 > k_1$ such that $r > k_2$ implies

$$\left\{r \in \mathbb{N} : \frac{|H_2 \cap J_r|}{h_r^\beta} < 1\right\} \notin I.$$

Proceeding this way, we can choose $k_{p+1} > k_p$ such that $r > k_{p+1}$, implies that $H_{p+1} \cap J_r \neq \emptyset$. Also, we can choose $k'(r) \in H_p \cap J_r$ for each r satisfying $k_p \leqslant r < k_{p+1}$ such that

$$|\Delta^\alpha x_{k'(r)} - L| < \frac{1}{p}.$$

This implies that $x_{k'(r)} \to L(\Delta^\alpha)$. Therefore, for every $\varepsilon > 0$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k,k'(r) \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \varepsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geq \frac{\varepsilon}{2}\right\}$$

$$\cup \left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k'(r) \in J_r} |\Delta^\alpha x_{k'(r)} - L| \geq \frac{\varepsilon}{2}\right\}.$$

Then,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k,k'(r) \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \varepsilon\right\} \in I.$$

Therefore (x_k) is a $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy sequence.

Conversely suppose (x_k) is a $\Delta^\alpha(S_\theta^\beta, I)-$Cauchy sequence. Then for every $\varepsilon > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geq \varepsilon\right\} \subseteq \left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k,k'(r) \in J_r} |\Delta^\alpha (x_k - x_{k'(r)})| \geq \frac{\varepsilon}{2}\right\}$$

$$\cup \left\{r \in \mathbb{N} : \frac{1}{h_r^\beta} \sum_{k'(r) \in J_r} |\Delta^\alpha x_{k'(r)} - L| \geq \frac{\varepsilon}{2}\right\}$$

and so (x_k) is a $(\Delta^\alpha(N_\theta^\beta, I)-$summable sequence to L.

Definition 4 A lacunary sequence $\rho = (\bar{k}(r))$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (\bar{k}(r))$.

Theorem 12 If $\rho = (\bar{k}(r))$ is a lacunary refinement of a lacunary sequence θ and $x_k \to L(\Delta^\alpha(N_\rho^\beta, I))$, then $x_k \to L(\Delta^\alpha(N_\theta^\beta, I))$.

Proof. Suppose that for each J_r of θ contains the points $(\bar{k}_{r,t})_{t=1}^{\nu(r)}$ of ρ such that $k_{r-1} < \bar{k}_{r,1} < \bar{k}_{r,2} < \cdots < \bar{k}_{r,\nu(r)} = k_r$, where $\bar{J}_{r,t} = (\bar{k}_{r,t-1}, \bar{k}_{r,t}]$. For all r and let $\nu(r) \geqslant 1$ this implies $k_r \subseteq (\bar{k}(r))$. Let $(J_j^*)_{j=1}^\infty$ be the sequence of intervals $(\bar{J}_{r,t})$ ordered by increasing right end points. Since $x_k \in L(\Delta^\alpha(N_\rho^\beta, I))$, then for each $\varepsilon > 0$,

$$\left\{ j \in \mathbb{N} : \frac{1}{(h_j^*)^\beta} \sum_{J_j^* \subset J_r} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\} \in I.$$

Also since $h_r = k_r - k_{r-1}$, so $\bar{h}_{r,t} = \bar{k}_{r,t} - \bar{k}_{r,t-1}$. For each $\varepsilon > 0$, we get

$$\left\{ r \in \mathbb{N} : \frac{1}{(h_r)^\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{(h_r)^\beta} \sum_{k \in J_r} \left\{ j \in \mathbb{N} : \frac{1}{(h_j^*)^\beta} \sum_{\substack{J_j^* \subset J_r \\ k \in J_j^*}} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\} \right\}.$$

Therefore $\left\{ r \in \mathbb{N} : (h_r)^{-\beta} \sum_{k \in J_r} |\Delta^\alpha x_k - L| \geqslant \varepsilon \right\} \in I$. Thus $x_k \in (\Delta^\alpha(N_\theta^\beta, I))$.

Theorem 13 Let ψ be set of lacunary sequences.

a) If ψ is closed under arbitrary union, then $\Delta^\alpha(N_\mu^\beta, I) = \bigcap_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$, where $\mu = \bigcup_{\theta \in \psi} \theta$,

b) If ψ closed under arbitrary intersection, then $\Delta^\alpha(N_\tau^\beta, I) = \bigcup_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$, where $\tau = \bigcap_{\theta \in \psi} \theta$,

c) If ψ is closed under union and intersection, then $\Delta^\alpha(N_\mu^\beta, I) \subseteq \Delta^\alpha(N_\theta^\beta, I) \subseteq \Delta^\alpha(N_\tau^\beta, I)$.

Proof. a) By hypothesis, we have $\mu \in \psi$ which is a refinement of each $\theta \in \psi$. Then from Theorem 12, we have if $x_k \in \Delta^\alpha(N_\mu^\beta, I)$ implies that $x_k \in \Delta^\alpha(N_\theta^\beta, I)$. Therefore, for each $\theta \in \psi$, we have $\Delta^\alpha(N_\mu^\beta, I) \subseteq \Delta^\alpha(N_\theta^\beta, I)$. The reverse inclusion is obvious. Hence $\Delta^\alpha(N_\mu^\beta, I) = \bigcap_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$.

b) By part a) and Theorem 12, we have $\Delta^\alpha(N_\tau^\beta, I) = \bigcup_{\theta \in \psi} \Delta^\alpha(N_\theta^\beta, I)$.

c)By part a) and b)we get $\Delta^\alpha(N_\mu^\beta, I) \subseteq \Delta^\alpha(N_\theta^\beta, I) \subseteq \Delta^\alpha(N_\tau^\beta, I)$.

REFERENCES

1. H. Altinok, M. Et, M. and R. Çolak: *Some remarks on generalized sequence space of bounded variation of sequences of fuzzy numbers*. Iran. J. Fuzzy Syst. **11**(5) (2014), 39–46, 109.
2. N. D. Aral and M. Et: *On lacunary statistical convergence of order β of difference sequences of fractional order*. International Conference of Mathematical Sciences, (ICMS 2019), Maltepe University, Istanbul, Turkey.
3. P. Baliarsingh: *Some new difference sequence spaces of fractional order and their dual spaces*. Appl. Math. Comput. **219**(18) (2013), 9737–9742.
4. P. Baliarsingh, U. Kadak and M. Mursaleen: *On statistical convergence of difference sequences of fractional order and related Korovkin type approximation theorems*. Quaest. Math. **41**(8) (2018), 1117–1133.
5. N. L. Braha, H. M. Srivastava and S. A. Mohiuddine: *A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallee Poussin mean*. Appl. Math. Comput. **228** (2014), 162–169.
6. C. Belen and S. A. Mohiuddine: *Generalized weighted statistical convergence and application*. Appl. Math. Comput. **219**(18) (2013), 9821–9826.
7. H. Çakalli: *Lacunary statistical convergence in topological groups*. Indian J. Pure Appl. Math. **26**(2) (1995), 113–119.
8. H. Çakalli, C. G. Aras and A. Sönmez: *Lacunary statistical ward continuity*. AIP Conf. Proc. 1676, 020042 (2015); http://dx.doi.org/10.1063/1.4930468.
9. H. Çakalli and H. Kaplan: *A variation on lacunary statistical quasi Cauchy sequences*. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **66**(2) (2017), 71–79.
10. A. Caserta, G. Di Maio and L. D. R. Kočinac: *Statistical convergence in function spaces*. Abstr. Appl. Anal., 2011, Art. ID 420419, 11 pp.
11. J. S. Connor: *The statistical and strong $p-$Cesaro convergence of sequences*. Analysis **8** (1988), 47-63.
12. M. Çinar, M. Karakaş and M. Et: *On pointwise and uniform statistical convergence of order α for sequences of functions*. Fixed Point Theory And Applications, Article Number: 33, 2013.
13. R. Çolak: *Statistical convergence of order α*. Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub, (2010) 121–129.
14. P. Das, E. Savaş and S. K. Ghosal: *On generalizations of certain summability methods using ideals*. Appl. Math. Lett. **24**(9) (2011), 1509-1514.
15. P. Das and E. Savaş: *On I-statistical and I-lacunary statistical convergence of order α*. Bull. Iranian Math. Soc. **40**(2) (2014), 459-472.
16. M. Et and R. Çolak:*On some generalized difference sequence spaces*. Soochow J. Math. **21**(4) (1995), 377-386.
17. M. Et and F. Nuray: Δ^m-*Statistical convergence*. Indian J. Pure appl. Math. **32**(6) (2001), 961 - 969.
18. M. Et and H. Şengül: *On (Δ^m, I)-Lacunary Statistical Convergence of order α*. J. Math. Analysis **7**(5) (2016), 78-84.

19. M. Et and H. Şengül: *Some Cesaro-Type Summability Spaces of Order alpha and Lacunary Statistical Convergence of Order* α. Filomat **28**(8),1593-1602, 2014.
20. M. Et, R. Çolak and Y. Altin: *Strongly almost summable sequences of order* α. Kuwait J. Sci. **41**(2) (2014), 35–47.
21. M. Et, M. Karakaş and V. Karakaya: *Some geometric properties of a new difference sequence space defined by de la Vallée-Poussin mean.* Appl. Math. Comput. **234** (2014), 237–244.
22. M. Et, M. Mursaleen and M. Işik: *On a class of fuzzy sets defined by Orlicz functions.* Filomat **27**(5) (2013), 789–796.
23. M. Et, B. C. Tripathy and A. J. Dutta: *On pointwise statistical convergence of order* α *of sequences of fuzzy mappings.* Kuwait J. Sci. **41**(3) (2014), 17–30.
24. H. Fast: *Sur la convergence statistique.* Colloq. Math. **2** (1951), 241-244.
25. A. R. Freedman, J. J. Sember and M. Raphael: *Some Cesaro-type summability spaces*, Proc. Lond. Math. Soc. **37**(3) (1978), 508–520.
26. J. Fridy: *On statistical convergence.* Analysis **5** (1985), 301–313.
27. J. Fridy and C. Orhan: *Lacunary statistical convergence.* Pacific J. Math. **160** (1993), 43–51.
28. J. A. Fridy and C. Orhan: *Lacunary statistical summability.* J. Math. Anal. Appl. **173**(2) (1993), 497–504.
29. M. Işik and K. E. Akbaş: *On* $\lambda-$*statistical convergence of order* α *in probability.* J. Inequal. Spec. Funct. **8**(4) (2017), 57–64.
30. M. Işik and K. E. Et: *On lacunary statistical convergence of order* α *in probability.* AIP Conference Proceedings 1676, 020045 (2015); doi: http://dx.doi.org/10.1063/1.4930471.
31. M. Işik and K. E. Akbaş: *On Asymptotically Lacunary Statistical Equivalent Sequences of Order* α *in Probability.* ITM Web of Conferences 13, 01024 (2017). DOI: 10.1051/itmconf/20171301024.
32. U. Kadak: *Generalized lacunary statistical difference sequence spaces of fractional order.* Int. J. Math. Math. Sci. 2015, Art. ID 984283, 6 pp.
33. U. Kadak and S. A. Mohiuddine:*Generalized statistically almost convergence based on the difference operator which includes the* (p,q)*-Gamma function and related approximation theorems.* Results Math. **73**(1) (2018) Article 9.
34. M. Karakaş, M. Et and V. Karakaya: *Some geometric properties of a new difference sequence space involving lacunary sequences.* Acta Math. Sci. Ser. B (Engl. Ed.) **33**(6) (2013), 1711–1720.
35. H. Kizmaz: *On certain sequence spaces.* Canad. Math. Bull. **24**(2) (1981), 169-176.
36. P. Kostyrko, T. Šalát and W. Wilczyński: I*-convergence.* Real Anal. Exchange 26 (2000/2001), 669-686.
37. P. Kostyrko, M. Mačaj, T. Šalát and M. Sleziak: I*-convergence and extremal* I*-limit points.* Math. Slovaca **55**(4) (2005), 443-464.
38. S. A. Mohiuddine and B. A. S. Alamri: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems.* Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM, **113**(3) (2019), 1955–1973.

39. S. A. Mohiuddine, A. Asiri and B. Hazarika: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*. Int. J. Gen. Syst. **48**(5) (2019), 492–506.
40. M. Mursaleen: *λ- statistical convergence*. Math. Slovaca **50**(1) (2012), 111–115.
41. M. Mursaleen and S. A. Mohiuddine: *On ideal convergence in probabilistic normed spaces*. Math. Slovaca **62** (2012), 49–62.
42. L. Nayak, M. Et and P. Baliarsingh: *On certain generalized weighted mean fractional difference sequence spaces*. Proc. Nat. Acad. Sci. India Sect. A **89**(1) (2019), 163–170.
43. T. Šalát, B. C. Tripathy and M. Ziman: *On I-convergence field*. Italian J. Pure Appl. Math. **17** (2005), 45–54.
44. E. Savaş and M. Et: *On $(\Delta_\lambda^m, I)-$statistical convergence of order α*. Period. Math. Hungar. **71**(2) (2015), 135–145.
45. I. J. Schoenberg: *The integrability of certain functions and related summability methods*. Amer. Math. Monthly **66** (1959), 361–375.
46. H. M. Srivastava and M. Et: *Lacunary statistical convergence and strongly lacunary summable functions of order α*. Filomat **31**(6) (2017), 1573–1582.
47. T. Salat: *On statistically convergent sequences of real numbers*. Math. Slovaca **30** (1980), 139–150.
48. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math. **2** (1951), 73–74.
49. H. Şengül and M. Et: *On lacunary statistical convergence of order α*. Acta Mathematica Scientia **34**(2) (2014), 473-482.
50. H. Şengül and M. Et: *On $I-$lacunary statistical convergence of order α of sequences of sets*. Filomat **31**(8) (2017), 2403–2412.
51. B. C. Tripathy and M. Et:*On generalized difference lacunary statistical convergence*. Studia Univ. Babeş-Bolyai Math. **50**(1) (2005), 119–130.
52. B. C. Tripathy, B. Hazarika and B. Choudhary: *Lacunary I-Convergent Sequences*. Kyungpook Math. J. **52** (2012), 473–482.
53. A. Zygmund: *Trigonometric Series*. Cambridge University Press, Cambridge, UK, 1979.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 57-63
https://doi.org/10.22190/FUMI200203005H
Original Scientific Paper

ON GENERALIZED RELATIVE COMMUTATIVITY DEGREE OF FINITE MOUFANG LOOP

Hamideh Hasanzadeh Bashir[1], Ali Iranmanesh[2], Behnam Azizi[3]

[1] **Department of Mathematics, Ahar Branch, Islamic Azad University, Ahar, Iran**
[2] **Faculty of Mathematical Sciences, Department of Mathematics, Tarbiat Modares University**
[3] **Department of Mathematics, Kaleybar Branch, Islamic Azad University, Kaleybar, Iran**

Abstract. For a given element g of a finite group G, the probablility that the commutator of randomly choosen pair elements in G equals g is the relative commutativity degree of g.

In this paper we are interested in studying the relative commutativity degree of the Dihedral group of order $2n$ and the Quaternion group of order 2^n for any $n \geq 3$ and we examine the relative commutativity degree of infinite class of the Moufang Loops of Chein type, $M(G, 2)$.
Keywords. Relative commutativity degree, Moufang loop.

1. Introduction

Every algebraic structure here is non-commutative. A quasi-group is a non-empty set with a binary operation such that for every three elements x, y and z of that, the equation $xy = z$ has a unique solution in this set, whenever two of the three element are specified. A quasi-group with a neutral element is called a loop and following [2, 6, 7, 8] one may see the definition of Moufang loop satisfying four tantamount relators. These loops are of interest because of their appearance in the projective geometry as planes and even they are non-associative, they retain many properties of the groups. During the study of these loops an interesting class introduced by Chein [3, 4, 5] where, for a finite group G and a new element

Received February 3, 2020; accepted June 28, 2020.
Corresponding Author: Ali Iranmanesh, Faculty of Mathematical Sciences, Department of Mathematics, Tarbiat Modares University | E-mail: iranmanesh@modares.ac.ir
2010 *Mathematics Subject Classification.* Primary 11B39; Secondary 20P05, 20N05

$u, (u \notin G)$, the loop $M(G, 2)$ is defined as $M(G, 2) = G \cup Gu$ such that the binary operation in $M(G, 2)$ is defined by:

$$\begin{array}{ll} goh = gh, & \text{if } g, h \in G, \\ go(hu) = (hg)u, & \text{if } g \in G, \quad hu \in Gu, \\ (gu)oh = (gh^{-1})u, & \text{if } gu \in Gu, \quad h \in G, \\ (gu)o(hu) = h^{-1}g, & \text{if } gu, hu \in Gu. \end{array}$$

These loops are studied for their finiteness property in [1, 2]. It is obvious that $M(G, 2)$ is non-associative if and only if the group G is non-abelian. Our next preliminary is the definition of generalized relative commutativity degree. Following [1], for an integer $n \geq 2$, the probability that for two elements x and y of an algebraic structure, $x^n y = yx^n$ holds is called the n^{th}-commutativity degree of the algebraic structure and denoted this probability by $P_n(S)$, for an algebraic structure S.

In what follows we examine $Pr_g(M)$ and $Pr_g(G)$, where for a given group G we give a general relationship between them with $M = M(G, 2)$. Since then we give explicit descriptions for $Pr_g(M)$ in two special cases when G is one of the dihedral group of order $2m$ and the quaternion group of order 2^m, for every $m \geq 3$. Note that these groups are non-abelian and then the loop $M = M(G, 2)$ is non-associative.

2. Main results

For a given element $g \in G$ we define the g-relative commutativity set of G as

$$C_g(G) = \{(x, y) \mid x, y \in G, \quad xyx^{-1}y^{-1} = g\}.$$

This set will be used in computation of $Pr_g(G)$ and we have

$$Pr_g(G) = \frac{|C_g(G)|}{|G|^2}.$$

Also we use the presentations $< a, b \mid a^n = b^2 = (ab)^2 = 1 >$ and $< a, b \mid a^{2^{n-1}} = 1, \quad b^2 = a^{2^{n-2}}, \quad (ab)^2 = 1 >$ for the groups D_{2n} and Q_{2^n}. Our main results are:

Lemma 2.1. *For even values of $n \geq 4$, if $a, b \in D_{2n}$ then*

(i) $[a^i, b] = g$ *if and only if* $[a^i, a^j b] = g$,

(ii) $[a^i, b] = g$ *if and only if* $[a^{\frac{n}{2}+i}, b] = g$,

(iii) $[b, a^i] = g$ *if and only if* $[a^j b, a^i] = g$,

(iv) $[b, a^i] = g$ *if and only if* $[b, a^{\frac{n}{2}+i}] = g$,

(v) $[b, a^i b] = g$ *if and only if* $[b, a^{\frac{n}{2}+i} b] = g$,

(vi) $[a^i b, a^j b] = g$ *if and only if* $[a^{i+1} b, a^{j+1} b] = g$, *where* $g \in D_{2n}$ *and* $(1 \leq i, j \leq n-1)$.

Proof. Let $n \geq 4$ be an even integer. Then by presentation of the group D_{2n} we get:

(i) :

$$\begin{aligned}
[a^i, b] = g &\iff a^i b a^{-i} b^{-1} = g \\
&\iff a^{-2i} b^2 = g \\
&\iff a^{2i} a^{j-j} b^2 = g \\
&\iff a^{i+j} b a^{-i+j} b = g \\
&\iff a^i a^j b a^{-i} a^j b = g \\
&\iff a^i a^j b a^{-i} b^{-1} a^{-j} = g \\
&\iff [a^i, a^j b] = g.
\end{aligned}$$

(ii) :

$$\begin{aligned}
[a^i, b] = g &\iff a^i b a^{-i} b^{-1} = g \\
&\iff a^{2i} b^2 = g \\
&\iff a^{n+2i} b^2 = g \\
&\iff a^{\frac{n}{2}+i} b a^{-\frac{n}{2}-i} b^{-1} = g \\
&\iff [a^{\frac{n}{2}+i}, b] = g.
\end{aligned}$$

The proof in other cases is similar and we omit it. □

Corollary 2.1. *Let* $n \geq 4$ *be an even integer and* $a, b \in D_{2n}$. *For every integers* $0 \leq i, j \leq n-1$ *if* $[a^i b, a^j b] = g$ *then* $g \in \{1, a^2, a^4, \ldots, a^{n-2}\}$.

Theorem 2.1. *For even values of* $n > 3$ *if* $g \in D_{2n}, (g \neq 1)$ *then*

$$Pr_g(D_{2n}) = \frac{3}{2n}$$

where, $g = a^2, a^4, \ldots, a^{n-2}$.

Proof. Let n be an even integer and $G = D_{2n} = A \cup B$ where, $A = \{1, a, \ldots, a^{n-1}\}$ and $B = \{b, ab, \ldots, a^{n-1}b\}$. Clearly, $[a^i, a^j] = 1$, now if $[a^i, b] = g$ then by using [i] in Lemma 2.1 we get there are n pairs $(x, y) \in A \times B$ such that $[x, y] = g$, also by [ii] in Lemma 2.1 we get there are n pairs $(x, y) \in A \times B$ such that $[x, y] = g$. Also, by [iii] and [iv] in Lemma 2.1 we heve there are $2n$ pairs $(x, y) \in B \times A$ such that $[x, y] = g$ and by [v] and [vi] in Lemma 2.1 there are $2n$ pairs $(x, y) \in B \times B$ such that $[x, y] = g$.

Consequently,

$$|C_g(D_{2n})| = 2n + 2n + 2n = 6n,$$

and

$$Pr_g(D_{2n}) = \frac{|C_g(D_{2n})|}{|D_{2n}|^2} = \frac{6n}{4n^2} = \frac{3}{2n}.$$

Lemma 2.2. *For odd values of $n \geq 3$, if $a, b \in D_{2n}$ then*

(i) $[a^i, b] = g$ *if and only if* $[a^i, a^j b] = g$,

(ii) $[b, a^i] = g$ *if and only if* $[a^j b, a^i] = g$,

(iii) $[b, a^i b] = g$ *if and only if* $[b, a^{\frac{n}{2}+i} b] = g$,

where, $g \in D_{2n}$ *and* $(1 \leq i, j \leq n-1)$.

Proof. The proof is similar to the proof of Lemma 2.1. □

Corollary 2.2. *Let $n \geq 4$ be an odd integer and $a, b \in D_{2n}$. For every integers $0 \leq i, j \leq n-1$ if $[a^i b, a^j b] = g$ then $g \in \{1, a, a^2, \ldots, a^{n-1}\}$.*

Theorem 2.2. *For odd values of $n > 3$ if $g \in D_{2n}, (g \neq 1)$ then*

$$Pr_g(D_{2n}) = \frac{3}{4n}$$

where, $g = a, a^2, \ldots, a^{n-1}$.

Proof. Let n be an odd integer and $G = D_{2n} = A \cup B$ where, $A = \{1, a, \ldots, a^{n-1}\}$ and $B = \{b, ab, \ldots, a^{n-1}b\}$. Clearly, $[a^i, a^j] = 1$, now if $[a^i, b] = g$ then by using $[i]$ in Lemma 2.2 we get there are n pairs $(x, y) \in A \times B$ such that $[x, y] = g$. Also, by $[ii]$ in Lemma 2.2 we heve there are n pairs $(x, y) \in B \times A$ such that $[x, y] = g$ and by $[iii]$ in Lemma 2.2 there are n pairs $(x, y) \in B \times B$ such that $[x, y] = g$.

Consequently,

$$|C_g(D_{2n})| = n + n + n = 3n,$$

and

$$Pr_g(D_{2n}) = \frac{|C_g(D_{2n})|}{|D_{2n}|^2} = \frac{3n}{4n^2} = \frac{3}{4n}.$$

□

Lemma 2.3. *For a given element $g \in Q_{2^n}$ and any values of $n \geq 3$, if $a, b \in Q_{2^n}$ and $(1 \leq i, j \leq n-1)$ then*

(i) $[a^i, b] = g$ *if and only if* $[a^i, a^j b] = g$,

(ii) $[a^i, b] = g$ *if and only if* $[a^{\frac{n}{2}+i}, b] = g$,

(iii) $[b, a^i] = g$ *if and only if* $[a^j b, a^i] = g$,

(iv) $[b, a^i] = g$ *if and only if* $[b, a^{\frac{n}{2}+i}] = g$,

(v) $[b, a^i b] = g$ *if and only if* $[b, a^{\frac{n}{2}+i} b] = g$,

(vi) $[a^i b, a^j b] = g$ *if and only if* $[a^{i+1} b, a^{j+1} b] = g$.

Corollary 2.3. *Let $n \geq 3$ be a positive integer and $a, b \in Q_{2^n}$. For every $0 \leq i, j \leq 2^{n-1} - 1$, if $[a^i b, a^j b] = g$ then $g \in \{1, a^2, a^4, \dots, a^{2^{n-1}-2}\}$.*

Theorem 2.3. *For any values of $n \geq 3$ if $g \in Q_{2^n}, (g \neq 1)$ then*

$$Pr_g(Q_{2^n}) = \frac{3}{2^n}$$

where, $g \in \{1, a^2, a^4, \dots, a^{2^{n-1}-2}\}$.

Proof. Let $n \geq 3$ be an even integer and $G = Q_{2^n} = A \cup B$, where $A = \{1, a, \dots, a^{n-1}\}$ and $B = \{b, ab, \dots, a^{n-1}b\}$. Clearly, $[a^i, a^j] = 1$, now if $[a^i, b] = g$ then by using $[i, ii]$ in Lemma 2.3 we get there are 2^{n-1} pairs $(x, y) \in A \times B$ such that $[x, y] = g$. Also, by $[iii, iv]$ in Lemma 2.3 we heve there are 2^{n-1} pairs $(x, y) \in B \times A$ such that $[x, y] = g$ and by $[v, vi]$ in Lemma 2.3 there are 2^{n-1} pairs $(x, y) \in B \times B$ such that $[x, y] = g$. Consequently,

$$|C_g(Q_{2^n})| = 2(2^{n-1}) + 2(2^{n-1}) + 2(2^{n-1}) = 3(2^n),$$

and

$$Pr_g(Q_{2^n}) = \frac{|C_g(Q_{2^n})|}{|Q_{2^n}|^2} = \frac{3(2^n)}{(2^n)^2} = \frac{3}{2^n}.$$

□

Lemma 2.4. *Let G be a finite group of order n, $g \in G$ and $M(G, 2)$ be a finite Moufang loop of order $2n$. we have for all $x, y \in G$:*

(i) $((xu)oy)o((xu)^{-1}oy^{-1}) = g$ *if and only if* $y^{-2} = g$,

(ii) $((xu)o(yu))o((xu)^{-1}o(yu)^{-1}) = g$ *if and only if* $(x^{-1}y)^{-2} = g$.

Proof. By definition of the multiplication in $M(G, 2)$ clearly:

$$(i) \quad \begin{aligned} ((xu)oy)o((xu)^{-1}oy^{-1}) = g &\iff ((xy^{-1})u)o((xy)u) = g \\ &\iff y^{-1}x^{-1}xy^{-1} = g \\ &\iff y^{-2} = g. \end{aligned}$$

$$(ii) \quad \begin{aligned} ((xu)o(yu))o((xu)^{-1}o(yu)^{-1}) = g &\iff (y^{-1}x)o(y^{-1}x) = g \\ &\iff (y^{-1}x)^2 = g \\ &\iff (x^{-1}y)^{-2} = g. \end{aligned}$$

□

Proposition 2.1. *For a given integer $n \geq 2$ and a non-abelian group G,*

$$Pr_g(M) = \frac{1}{4}(Pr_g(G) + \frac{3N_g}{|G|}),$$

where N_g is the number of elements $y \in G$ such that $y^{-2} = g$.

Proof. Let $g \in M = M(G,2)$ and $C_g(M) = \{(x,y) \mid x,y \in G, \quad xyx^{-1}y^{-1} = g\}$. We first note that the multiplication table of the Moufang loop $M(G,2)$ will be as follows:

o	G	Gu
G	$G * G$	$G * Gu$
Gu	$Gu * G$	$Gu * Gu$

Since $Pr_g(M) = \frac{|C_g(M)|}{|M|^2}$. Thus it is sufficient to enumerate $|C_g(M)|$. For every $(x,y) \in M$ we have the following four cases:
Case1: Both $x,y \in G$. Then there are $|C_g(G)|$ distinct ordered pairs $(x,y) \in C_g(M)$ in this case.
Case2: $x \in Gu$ and $y \in G$. Then $x = x_1u$ where $x_1 \in G$. By (i) of Lemma 2.1 we conclude that $y^{-2} = g$, so there are precisely $N_g|Gu| = N_g|G|$ pairs $(x,y) \in C_g(M)$ of this type.
Case3: $x \in G$ and $y \in Gu$. Then $y = y_1u$ where $y_1 \in G$. By using (i) of Lemma 2.1 we get there are $N_g|G|$ distinct pairs in $C_g(M)$ of this type.
Case4: Both $x,y \in Gu$. Then $x = x_1u$ and $y = y_1u$ where $x_1, y_1 \in G$. Using (ii) of Lemma 2.1 we get there are $N_g|G|$ distinct pairs in $C_g(M)$ such that $(x^{-1}y)^{-2} = g$.

Consequently,

$$|C_g(M)| = |C_g(M)| + 3N_g|G|,$$

and so,

$$Pr_g(M) = \frac{|C_g(M)| + 3N_g|G|}{(2|G|)^2} = \frac{1}{4}(Pr_g(G) + \frac{3N_g}{|G|}).$$

□

Proposition 2.2. *Let $M = M(D_{2n},2)$, $n \geq 3$ is a positive integer. Then,*

$$Pr_g(M) = \begin{cases} \frac{3}{8n}(N_g+1), & n \text{ is even,} \\ \frac{3}{16n}(2N_g+1), & n \text{ is odd,} \end{cases}$$

where, N_g is the number of elements $y \in G$ such that $y^{-2} = g$.

Proof. By using Proposition 2.1 and Theorems 2.2 and 2.3 we get

$$Pr_g(M) = \frac{1}{4}(Pr_g(G) + \frac{3N_g}{|G|}) = \begin{cases} \frac{1}{4}(\frac{3}{2n} + \frac{3N_g}{2n}) = \frac{3}{8n}(N_g+1), & n \text{ is even,} \\ \frac{1}{4}(\frac{3}{4n} + \frac{3N_g}{2n}) = \frac{3}{16n}(2N_g+1), & n \text{ is odd,} \end{cases}$$

□

Corollary 2.4. *Let $M = M(D_{2n},2)$, $n \geq 3$ is a positive integer. Then,*

$$Pr_g(M) \leq \frac{15}{48}.$$

Proposition 2.3. *Let $M = M(Q_{2^n}, 2)$, $n \geq$ is an integer . Then*

$$Pr_g(M) = \frac{3}{2^{n+2}}(N_g + 1),$$

where, N_g is the number of elements $y \in G$ such that $y^{-2} = g$.

Proof. The proofs follows by considering the Proposition 2.1:

$$Pr_g(M) = \frac{1}{4}(\frac{3}{2^n} + \frac{3N_g}{2^n}) = \frac{3}{2^{n+2}}(N_g + 1).$$

□

Corollary 2.5. *Let $M = M(Q_{2^n}, 2)$, $n \geq 3$ is an integer . Then*

$$Pr_g(M) \leq \frac{3}{16}.$$

Acknowledgements

The authors would like to thank the referees for a very careful reading of the paper and complete comments and useful suggestions, which improved considerably the presentations of this paper.

REFERENCES

1. K. Ahmadidelir: *On the commutativity degree in finite Moufang loops.* International Journal of Group Theory. **5** (2016), 37-47.
2. B. Azizi and H. Dootie : *Certain numerical results in non-associative structures.* Mathematical Sciences. **13** (2019), 27-32.
3. O. Chein : *Moufang loops of small order I.*Trans American Mathematical Society. **188** (1974), 31-51.
4. O. Chein : *Moufang loops of small order.* Memoirs of the American Mathematical Society. **13** (1978), No, 197, 1-131.
5. O. Chein and A. Rajah : *Possible orders of non-associative Moufang loops.* Comment Mathematics University Carolina. **41**(2000), 237-244.
6. E. G. Goodaire, S. May and M. Raman : *The Moufang loops of order less than 64*, Nova Science Publishers, (1999).
7. P. Lescot : *Isoclinism classes and commutativity degrees of finite groups.* Journal of Algebra. **177** (1995), 847-869.
8. G. P. Naghy and P. Vojtěchovský: *The Moufang loops of order 64 and 81.* Journal of Symbolic Computation. **42**(2007), No. 9, 871-883.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 65-77
https://doi.org/10.22190/FUMI200309006C
Original Scientific Paper

TRIANGULAR $A-$STATISTICAL RELATIVE UNIFORM CONVERGENCE FOR DOUBLE SEQUENCES OF POSITIVE LINEAR OPERATORS

Selin Çınar

Faculty of Science and Arts, Department of Mathematics, Sinop University, 57000 Sinop, Turkey

Abstract. In this paper, we introduce the concept of triangular $A-$statistical relative convergence for double sequences of functions defined on a compact subset of the real two-dimensional space. Based upon this new convergence method, we prove Korovkin-type approximation theorem. Finally, we give some further developments.
Keywords: positive linear operators, the double sequences, regular matrix, triangular A-statistical convergence, Korovkin theorem.

1. Introduction

Classical Bohman-Korovkin theorem is a well known theorem which has an important place in approximation theory ([13], [16], [21]). This theorem establishes the uniform convergence in the space $C[a,b]$ of all continuous real functions defined on the interval $[a,b]$, for a sequence of positive linear operators (L_n), assuming the convergence by the test functions $f_r(s) = s^r$, $r = 0, 1, 2$. Moreover, different finite classes of test functions were studied, in both one and multi-dimensional case. Many mathematicians studied and improved this theory by defining positive linear operators via convergence methods on various function spaces ([1], [4], [5], [6], [15], [18], [20], [23], [24], [25], [26], [34]). In recent years, general versions of Korovkin theorem have been studied, in which a more general notion of convergence is used. One of these convergences is the statistical convergence first introduced by Fast and Steinhaus ([17], [30]). Korovkin type approximation theorems have been first studied via the notion of statistical convergence by Gadjiev and Orhan [19]. For

Received March 9, 2020; accepted May 30, 2020.
Corresponding Author: Selin Çınar, Faculty of Science and Arts, Department of Mathematics, Sinop University, 57000 Sinop, Turkey | E-mail: scinar@sinop.edu.tr
2010 *Mathematics Subject Classification.* 40A35; 41A25, 41A36

double sequences of positive linear operators, statistical convergence and some of its generalizations to convergence generated by summability matrix methods were carried on by Demirci and Dirik ([8], [14]). With the help of these studies, triangular $A-$statistical convergence which is a different kind of statistical convergence was identified by Bardaro et. al. ([2], [3]).

Recently, Demirci and Orhan [11] have defined the statistically relatively uniform convergence by using statistical convergence and the relatively uniform convergence and established its use in the Korovkin-type approximation theory. Also, a type of modular convergence, called relative modular convergence, was introduced in [33] originated by studies in modular spaces and these studies continued ([9], [10], [12]).

Our main aim in this paper is to present a new kind of statistical convergence for double sequence, called triangular $A-$statistical relative uniform convergence. We will compare this new convergence with triangular $A-$statistical convergence and obtain more general results.

Now, we begin with the definitions and notations required for this article.

E. H. Moore [22] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. Then, E. W. Chittenden [7] gave the following definition of relatively uniform converge which is equivalent to the definition given by Moore:

A sequence (f_n) of functions, defined on any compact subset of real space, converges relatively uniformly to a limit function f if there exists a function $\sigma(s)$, such that for every $\varepsilon > 0$ there exists an integer n_ε such that for every $n > n_\varepsilon$ the inequality

$$\left| f_n(s) - f(s) \right| < \varepsilon \left| \sigma(s) \right|$$

holds uniformly in s. The sequence (f_n) is said to converge uniformly relatively to the scale function σ or more briefly relatively uniformly. Similarly, Dirik and Şahin [31] gave the following for double sequences of functions:

A double sequence $(f_{i,j})$ of functions, defined on any compact subset of the real two-dimensional space, converges relatively uniformly to a limit function f if there exists a function $\sigma(s,t)$, called a scale function such that for every $\varepsilon > 0$ there is an integer n_ε such that for every $i, j > n_\varepsilon$ the inequality

$$\left| f_{i,j}(s,t) - f(s,t) \right| < \varepsilon \left| \sigma(s,t) \right|$$

holds uniformly in (s,t). The double sequence $(f_{i,j})$ is said to converge uniformly relatively to scale function σ or more briefly, relatively uniformly.

Let $A = (a_{i,j})$ be a two-dimensional matrix transformation. For a double sequence $x = (x_{i,j})$ of real numbers, we put

$$(Ax)_i := \sum_{j=1}^{\infty} a_{i,j} x_{i,j},$$

if the series is convergent. We will say that A is regular if $\lim Ax = L$ whenever $\lim x = L$. The well-established characterization for regular two-dimensional matrix transformation is known as the Silverman-Toeplitz conditions [32]:

(i) $\|A\| = \sup\limits_{i\epsilon\mathbb{N}} \sum\limits_{j=1}^{\infty} |a_{i,j}| < \infty$

(ii) $\lim\limits_{i} a_{i,j} = 0$ for each $j \in \mathbb{N}$,

(iii) $\lim\limits_{i} \sum\limits_{j=1}^{\infty} a_{i,j} = 1$.

A double sequence $x = (x_{i,j})$ of real numbers, $i, j \in \mathbb{N}$, the set of all positive integers, is said to be convergent in the Pringsheim's sense or $P-$convergent if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{i,j} - L| < \varepsilon$ whenever $i, j > N$ and L is called the Pringsheim limit (denoted by $P-\lim\limits_{i,j} x_{i,j} = L$) [28]. More briefly, we will say that such an x is $P-$convergent to L. A double sequence is said to be bounded if there exists a positive number K such that $|x_{i,j}| \leq K$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Note that in contrast to the case for single sequences, a convergent double sequences need not to be bounded, provided the double sequences converges in Pringsheim's sense for every $(i, j) \in \mathbb{N}^2$.

Let now $A = (a_{n,m,i,j})$ be a four-dimensional matrix and $x = (x_{i,j})$ be a double sequence. Then the double (transformed) sequence, $Ax := \left((Ax)_{n,m}\right)$, is denoted by

$$(Ax)_{n,m} = \sum_{i,j=1,1}^{\infty,\infty} a_{n,m,i,j} x_{i,j},$$

where it is assumed that the summation exists as a Pringsheim limit for each $(n, m) \in \mathbb{N}^2$.

Recall that four-dimensional matrix $A = (a_{n,m,i,j})$ is said to be $RH-$regular if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The Robison-Hamilton conditions (see also [29]) state that a four-dimensional matrix $A = (a_{n,m,i,j})$ is $RH-$regular if and only if

(i) $P - \lim\limits_{n,m} a_{n,m,i,j} = 0$ for each i and j,

(ii) $P - \lim\limits_{n,m} \sum\limits_{i,j}^{\infty,\infty} a_{n,m,i,j} = 1$,

(iii) $P - \lim\limits_{n,m} \sum\limits_{i=1}^{\infty} |a_{n,m,i,j}| = 0$ for each $j \in \mathbb{N}$,

(iv) $P - \lim\limits_{n,m} \sum\limits_{j=1}^{\infty} |a_{n,m,i,j}| = 0$ for each $i \in \mathbb{N}$,

(v) $\sum\limits_{i,j=1,1}^{\infty,\infty} |a_{n,m,i,j}|$ is $P-$convergent for every $(n, m) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that

$$\sum_{i,j>B} |a_{n,m,i,j}| < A$$

for every $(n,m) \in \mathbb{N}^2$.

Let $A = (a_{n,m,i,j})$ be a nonnegative $RH-$regular summability matrix. If $K \subset \mathbb{N}^2$, then the $A-$density of K is denoted by

$$\delta_A^2(K) := P - \lim_{n,m} \sum_{(i,j)\in K} a_{n,m,i,j}$$

provided that the limit on the right-hand side exists in the Pringsheim sense. A real double sequence $x = (x_{i,j})$ is said to be $A-$statistically convergent to L and denoted by $st_{A-}^2\lim_{i,j} x_{i,j} = L$ if, for every $\varepsilon > 0$,

$$P - \lim_{n,m} \sum_{(i,j)\in K(\varepsilon)} a_{n,m,i,j} = 0,$$

where $K(\varepsilon) = \left\{(i,j) \in \mathbb{N}^2 : |x_{i,j} - L| \geq \varepsilon\right\}$. If we take $A = C(1,1)$, then $C(1,1)-$ -statistical convergence coincides with the notion of statistical convergence for double sequences ([27]), where $C(1,1) = (c_{i,j,n,m})$ is the double Cesàro matrix, defined by $c_{i,j,n,m} = 1/ij$ if $1 \leq n \leq i$, $1 \leq m \leq j$ and $c_{i,j,n,m} = 0$ otherwise. We state the set of all $A-$statistically convergent double sequences by st_A^2.

2. Triangular $A-$ Statistical Relative Uniform Convergence

First, we recall some definitions given in [2].

Let $A = (a_{i,j})$ be a nonnegative regular summability matrix, $K \subset \mathbb{N}^2$ be a nonempty set, and for every $i \in \mathbb{N}$, let $K_i := \{j \in \mathbb{N} : (i,j) \in K,\ j \leq i\}$. Triangular $A-$density of K, is given by

$$\delta_A^T(K) := \lim_i \sum_{j\in K_i} a_{i,j},$$

provided that the limit on right-hand side exists in $\mathbb{R}$.

In a similar manner to the natural density, we can give some properties for the triangular $A-$density:

$i)$ if $K_1 \subset K_2$, then $\delta_A^T(K_1) \subset \delta_A^T(K_2)$,

$ii)$ if K has triangular $A-$density, then $\delta_A^T\left(\mathbb{N}^2 \setminus K\right) = 1 - \delta_A^T(K)$.

Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. The double sequence $x = (x_{i,j})$ is triangular $A-$statistically convergent to L provided that for every $\varepsilon > 0$

$$\lim_i \sum_{j\in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \{j \in \mathbb{N} : \ j \leq i, \ \ |x_{i,j} - L| \geq \varepsilon\}$ and this denoted by $st_A^T - \lim_i x_{i,j} = L$. We should note that if we take $A = C_1 := (c_{i,j})$, the Cesàro matrix defined by

$$c_{i,j} := \begin{cases} \frac{1}{i}, & \text{if } 1 \leq j \leq i, \\ 0, & \text{otherwise,} \end{cases}$$

then the triangular $A-$density is called triangular density which is denoted by

$$\delta^T(K) = \lim_i \frac{1}{i} |K_i|$$

where $|K_i|$ be the cardinality of K_i. According to the above definitions triangular $A-$statistical convergent reduces triangular statistical convergent.

Let S is a compact subset of the real two-dimensional space. By $C(S)$ we define the space of all continuous real valued functions on S and $\|f\|_{C(S)}$ denotes the usual supremum norm of f in $C(S)$. Let f and $f_{i,j}$ belong to $C(S)$.

Definition 2.1. Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. A double sequence of fuctions $(f_{i,j})$ is said to triangular $A-$statistically uniformly convergent to f on S provided that for every $\varepsilon > 0$,

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \left\{ j \in \mathbb{N} : j \leq i, \ \sup_{(s,t) \in S} |f_{i,j}(s,t) - f(s,t)| \geq \varepsilon \right\}$. In this case, we write $f_{i,j} \rightrightarrows f \ \left(st_A^T\right)$.

Definition 2.2. Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. $(f_{i,j})$ is said to be triangular $A-$statistically relatively uniformly convergent to f on S if there exists a function $\sigma(s,t)$, $|\sigma(s,t)| > 0$, called a scale function, provided that for every $\varepsilon > 0$,

$$\lim_i \sum_{j \in K_i(\varepsilon)} a_{i,j} = 0,$$

where $K_i(\varepsilon) = \left\{ j \in \mathbb{N} : \ j \leq i, \ \ \sup_{(s,t) \in S} \left| \frac{f_{i,j}(s,t) - f(s,t)}{\sigma(s,t)} \right| \geq \varepsilon \right\}$. In this case, we write $f_{i,j} \rightrightarrows f \ \left(st_A^T, \sigma\right)$.

It will be observed that triangular $A-$statistical uniform convergence is the special case of triangular $A-$statistical relative uniform convergence in which the scale function is a non-zero constant.

Example 2.1. Take $A = C_1$ and $S = [0,1] \times [0,1]$. For each $(i,j) \in \mathbb{N}^2$, define $\gamma_{i,j} : S \to \mathbb{R}$ by

$$\gamma_{i,j}(s,t) = \begin{cases} \frac{2i^2j^2st}{1+i^3j^3s^2t^2}, & i \text{ and } j \text{ are square,} \\ 0, & \text{otherwise.} \end{cases}$$

Since $\| \gamma_{i,j} - \gamma \|_{C(S)} = 1$, this sequence does not triangular statistically uniform convergent to $\gamma = 0$, but triangular statistically relatively uniform convergent to $f = 0$, with a scale function defined by,

$$\sigma(s,t) = \begin{cases} \frac{1}{st}, & \text{if } (s,t) \in (0,1] \times (0,1] \\ 0, & \text{if } s = 0 \text{ or } t = 0 \end{cases}$$

clearly, for every $\varepsilon > 0$,

$$\lim_i \frac{1}{i} \left| \left\{ j \in \mathbb{N} : \ j \leq i, \ \sup_{(s,t)\in S} \left| \frac{\gamma_{i,j}\ (s,t) - \gamma(s,t)}{\sigma(s,t)} \right| \geq \varepsilon \right\} \right| = 0.$$

3. A Korovkin-type approximation theorem

Let L be a linear operator from $C(S)$ into itself and is called positive, if $L(f) \geq 0$, for all $f \geq 0$. Also, we denote the value of $L(f)$ at a point $(s,t) \in S$ by $L(f(u,v); s,t)$ or, briefly, $L(f; s,t)$.

Theorem 3.1. *[2] Let $A = (a_{i,j})$ be a nonnegative regular summability matrix and $(L_{i,j})$ be a double sequence of positive linear operators from $C(S)$ into $C(S)$. Then for every $f \in C(S)$ we have*

$$st_A^T - \lim_i \|L_{i,j}(f) - f\|_{C(S)} = 0 \tag{3.1}$$

if and only if

$$st_A^T - \lim_i \|L_{i,j}(f_r) - f_r\|_{C(S)} = 0 \text{ for every } r = 0,1,2,3, \tag{3.2}$$

where $f_0(s,t) = 1,\ f_1(s,t) = s,\ f_2(s,t) = t,\ f_3(s,t) = s^2 + t^2$.

Now we have the following Korovkin type approximation theorem for triangular $A-$statistical relative convergence that is our main theorem.

Theorem 3.2. *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. Let $(L_{i,j})$ be a double sequence of positive linear operators from $C(S)$ into $C(S)$. Then, for all $f \in C(S)$ we have*

$$L_{i,j}(f) \rightrightarrows f \quad \left(st_A^T, \sigma\right) \tag{3.3}$$

if and only if,

$$L_{i,j}(f_r) \rightrightarrows f_r \ \left(st_A^T, \sigma_r\right) \ \ (\ r = 0,1,2,3), \tag{3.4}$$

where $f_0(s,t) = 1,\ f_1(s,t) = s,\ f_2(s,t) = t,\ f_3(s,t) = s^2 + t^2$ and $\sigma_r(s,t) = \max\{|\sigma_r(s,t)|;\ r = 0,1,2,3\}$, $|\sigma_r(s,t)| > 0$ and $\sigma_r(s,t)$ is unbounded, $r = 0,1,2,3$.

Proof. *Since each $f_r \in C(S)$ $(r = 0, 1, 2, 3)$, (3.3)$\Longrightarrow$(3.4) is obvious. Suppose now that (3.4) holds. By continuity of f on the compact set S, we can write $|f(s,t)| \leq M$ where $M := \|f\|_{C(S)}$. Also, since f is continuous on S, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(u,v) - f(s,t)| < \varepsilon$ for all $(u,v) \in S$ satisfying $|u - s| < \delta$ and $|v - t| < \delta$. Hence, we get*

$$|f(u,v) - f(s,t)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\}. \tag{3.5}$$

Since $L_{i,j}$ is linear and positive, we obtain

$$\begin{aligned}
|L_{i,j}(f;s,t) - f(s,t)| &= |L_{i,j}(f(u,v) - f(s,t);s,t) \\
&\quad - f(s,t)(L_{i,j}(f_0;s,t) - f_0(s,t))| \\
&\leq \left|L_{i,j}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(u-s)^2 + (v-t)^2\right\};s,t\right)\right| \\
&\quad + M|L_{i,j}(f_0;s,t) - f_0(s,t)| \\
&\leq \left(\varepsilon + M + \frac{2M}{\delta^2}\left(A^2 + B^2\right)\right)|L_{i,j}(f_0;s,t) - f_0(s,t)| \\
&\quad + \frac{4M}{\delta^2}A|L_{i,j}(f_1;s,t) - f_1(s,t)| \\
&\quad + \frac{4M}{\delta^2}B|L_{i,j}(f_2;s,t) - f_2(s,t)| \\
&\quad + \frac{2M}{\delta^2}|L_{i,j}(f_3;s,t) - f_3(s,t)| + \varepsilon
\end{aligned}$$

where $A := \max|s|$, $B := \max|t|$. Now we multiply the both-sides of the above inequality by $\frac{1}{|\sigma(s,t)|}$,

$$\begin{aligned}
\left|\frac{L_{i,j}(f;s,t) - f(s,t)}{\sigma(s,t)}\right| &\leq K\left\{\left|\frac{L_{i,j}(f_0;s,t) - f_0(s,t)}{\sigma(s,t)}\right| + \left|\frac{L_{i,j}(f_1;s,t) - f_1(s,t)}{\sigma(s,t)}\right|\right. \\
&\quad \left. + \left|\frac{L_{i,j}(f_2;s,t) - f_2(s,t)}{\sigma(s,t)}\right| + \left|\frac{L_{i,j}(f_3;s,t) - f_3(s,t)}{\sigma(s,t)}\right|\right\} + \frac{\varepsilon}{|\sigma(s,t)|},
\end{aligned} \tag{3.6}$$

where $K = \max\left\{\varepsilon + M + \frac{2M}{\delta^2}\left(A^2 + B^2\right), \frac{4M}{\delta^2}A, \frac{4M}{\delta^2}B, \frac{2M}{\delta^2}\right\}$ and where $\sigma(s,t) = \max\{|\sigma_r(s,t)|; r = 0, 1, 2, 3\}$. Taking the supremum over $(s,t) \in S$, we get

$$\begin{aligned}
\sup_{(s,t)\in S}\left|\frac{L_{i,j}(f;s,t) - f(s,t)}{\sigma(s,t)}\right| &\leq \sup_{(s,t)\in S}\frac{\varepsilon}{|\sigma(s,t)|} + K\left\{\sup_{(s,t)\in S}\left|\frac{L_{i,j}(f_0;s,t) - f_0(s,t)}{\sigma_0(s,t)}\right|\right. \\
&\quad + \sup_{(s,t)\in S}\left|\frac{L_{i,j}(f_1;s,t) - f_1(s,t)}{\sigma_1(s,t)}\right| \\
&\quad + \sup_{(s,t)\in S}\left|\frac{L_{i,j}(f_2;s,t) - f_2(s,t)}{\sigma_2(s,t)}\right| \\
&\quad \left. + \sup_{(s,t)\in S}\left|\frac{L_{i,j}(f_3;s,t) - f_3(s,t)}{\sigma_3(s,t)}\right|\right\}.
\end{aligned} \tag{3.7}$$

Now, for a given $r > 0$, *choose* $\varepsilon > 0$ *such that* $\sup\limits_{(s,t)\in S} \frac{\varepsilon}{|\sigma(s,t)|} < r$. *Then, setting*

$D_i := \left\{ j \in \mathbb{N} : j \leq i, \ \left\| \frac{L_{i,j}(f)-f}{\sigma} \right\|_{C(S)} \geq r \right\}$,

$D_i^r := \left\{ j \in \mathbb{N} \ : \ j \leq i, \ \left\| \frac{L_{i,j}(f_r)-fr}{\sigma_r} \right\|_{C(S)} \geq \frac{r - \sup\limits_{(s,t)\in S} \frac{\varepsilon}{|\sigma(s,t)|}}{4K} \right\}$, $r = 0, 1, 2, 3$. *It is easy to see that*

$$D_i \subseteq \bigcup_{r=0}^{3} D_i^r$$

which gives, for all $i \in \mathbb{N}$, *then*

$$\sum_{j\in D_i} a_{i,j} \leq \sum_{r=0}^{3} \sum_{j\in D_i^r} a_{i,j}.$$

Letting $i \to \infty$ *and using (3.4), we obtain (3.5). The proof is complete.* □

If one replaces the scale function by a non-zero constant, then the Theorem 3.2 reduces to the Theorem 3.1.

We now show that our result Theorem 3.2 is stronger than Theorem 3.1.

Example 3.1. Let consider the following Bernstein operators given by

$$B_{i,j}(f; s,t) = \sum_{k=0}^{i} \sum_{p=0}^{j} f\left(\frac{k}{i}, \frac{p}{j}\right) \binom{i}{k} \binom{j}{p} s^k (1-s)^{i-k} t^p (1-t)^{j-p} \tag{3.8}$$

where $(s,t) \in S = [0,1] \times [0,1]$; $f \in C(S)$. Also, observe that

$$\begin{aligned}
B_{i,j}(f_0; s,t) &= f_0(s,t), \\
B_{i,j}(f_1; s,t) &= f_1(s,t), \\
B_{i,j}(f_2; s,t) &= f_2(s,t), \\
B_{i,j}(f_3; s,t) &= f_3(s,t) + \frac{s-s^2}{i} + \frac{t-t^2}{j},
\end{aligned}$$

where $f_0(s,t) = 1$, $f_1(s,t) = s$, $f_2(s,t) = t$ and $f_3(s,t) = s^2 + t^2$. Using these polynomials, we introduce the following positive linear operators on $C(S)$:

(3.9)
$P_{i,j}(f; s,t) = (1 + \gamma_{i,j}(s,t)) B_{i,j}(f; s,t)$, $(s,t) \in S = [0,1] \times [0,1]$ and $f \in C(S)$

where $\gamma_{i,j}(s,t)$ is given in Example 2.1. Now, take $A = C_1$, the Cesàro matrix. Since $\gamma_{i,j} \rightrightarrows \gamma = 0 \ (st_T, \sigma)$, where

$$\sigma(s,t) = \begin{cases} \frac{1}{st}, & if \quad (s,t) \in (0,1] \times (0,1], \\ 0, & if \quad s = 0 \ or \ t = 0. \end{cases}$$

Then, we conclude that

$$P_{i,j}(f_r) \rightrightarrows f_r \ \ (st_T, \sigma) \ \ (\ r = 0, 1, 2, 3).$$

So by our main theorem, Theorem 3.2, we immediately see that

$$P_{i,j}(f) \rightrightarrows f \ \ \ (st_T, \sigma) \ \text{ for } f \in C(S).$$

However, since $(\gamma_{i,j})$ is not triangular statistically uniformly convergent to $\gamma = 0$ on the interval S, we can say that Theorem 3.1 does not work for our operators defined by (3.9).

4. Rates of Triangular $A-$Statistical Relative Uniform Convergence

In this section, using the notion of triangular $A-$statistical relative uniform convergence we study the rate of convergence of positive linear operators with the help of modulus of continuity.

Let $f \in C(S)$. Then the modulus of continuity of f, defined to be

$$w(f;\delta) = \sup\left\{|f(u,v) - f(s,t)| : (u,v), (s,t) \in S \text{ and } \sqrt{(u-s)^2 + (v-t)^2} \leq \delta\right\}$$

for $\delta > 0$.

Then we hold the following result.

Theorem 4.1. *Let $A = (a_{i,j})$ be a nonnegative regular summability matrix. Let $(L_{i,j})$ be a double sequence of positive linear operators acting from $C(S)$ into itself. Assume that the following conditions hold:*

(*a*) $L_{i,j}(f_0) \rightrightarrows f_0 \left(st_A^T, \sigma_0\right)$,

(*b*) $w(f,\delta) \rightrightarrows 0 \left(st_A^T, \sigma_1\right)$, where $\delta := \delta_{i,j} = \sqrt{\|L_{i,j}(\varphi)\|_{C(S)}}$, with $\varphi(u,v) = (u-s)^2 + (v-t)^2$.

Then we have, for all $f \in C(S)$,

$$L_{i,j}(f) \rightrightarrows f \ \ \ \left(st_A^T, \sigma\right)$$

where

$$\sigma(s,t) = \max\{|\sigma_0(s,t)|, \ |\sigma_1(s,t)|, \ |\sigma_0(s,t)\,\sigma_1(s,t)|\},$$

$|\sigma_i(s,t)| > 0$ and $\sigma_i(s,t)$ is unbounded for $i = 0, 1$.

Proof. Let $f \in C(S)$ and $(s,t) \in S$ be fixed. Using linearity and positivity of $L_{i,j}$ we have, for any $(i,j) \in \mathbb{N}^2$ and $\delta > 0$,

$$\begin{aligned}
& \left|L_{i,j}\left(f;s,t\right)-f\left(s,t\right)\right| \\
= \; & \left|L_{i,j}\left(f\left(u,v\right)-f\left(s,t\right);s,t\right)-f\left(s,t\right)\left(L_{i,j}\left(f_0;s,t\right)-f_0\left(s,t\right)\right)\right| \\
\leq \; & L_{i,j}\left(\left|f\left(u,v\right)-f\left(s,t\right)\right|;s,t\right)+M\left|L_{i,j}\left(f_0;s,t\right)-f_0\left(s,t\right)\right| \\
\leq \; & L_{i,j}\left(\left[1+\frac{\sqrt{\left(u-s\right)^2+\left(v-t\right)^2}}{\delta}\right]w\left(f;\delta\right);s,t\right) \\
& +M\left|L_{i,j}\left(f_0;s,t\right)-f_0\left(s,t\right)\right| \\
\leq \; & w\left(f;\delta\right)\left|L_{i,j}\left(f_0;s,t\right)-f_0\left(s,t\right)\right|+\frac{w\left(f;\delta\right)}{\delta^2}L_{i,j}\left(\varphi;s,t\right)+w\left(f;\delta\right) \\
& +M\left|L_{i,j}\left(f_0;s,t\right)-f_0\left(s,t\right)\right|,
\end{aligned}$$

where $M=\left\|f\right\|_{C(S)}$. Taking the supremum over $(s,t)\in S$ in both sides of the above inequality, we obtain, for any $\delta>0$,

$$\begin{aligned}
\left\|\frac{L_{i,j}f-f}{\sigma}\right\|_{C(S)} \leq \; & \frac{w\left(f,\delta_{i,j}\right)}{\left\|\sigma_1\right\|_{C(S)}}\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}+\frac{w\left(f,\delta_{i,j}\right)}{\left\|\sigma_1\right\|_{C(S)}\delta^2}\left\|\frac{L_{i,j}\varphi}{\sigma_1}\right\|_{C(S)} \\
& +\frac{w\left(f,\delta_{i,j}\right)}{\left\|\sigma_1\right\|_{C(S)}}+M\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}.
\end{aligned}$$

Now, if take $\delta:=\delta_{i,j}=\sqrt{\left\|L_{i,j}\left(\varphi\right)\right\|}$, then we may write

$$\left\|\frac{L_{i,j}f-f}{\sigma}\right\|_{C(S)}\leq\frac{w\left(f,\delta\right)}{\left\|\sigma_1\right\|}\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}+2\frac{w\left(f,\delta\right)}{\left\|\sigma_1\right\|}+M\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}$$

and hence,

(4.1)

$$\left\|\frac{L_{i,j}f-f}{\sigma}\right\|_{C(S)}\leq K\left\{\frac{w\left(f,\delta\right)}{\left\|\sigma_1\right\|}\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}+\frac{w\left(f,\delta\right)}{\left\|\sigma_1\right\|}+\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}\right\}$$

where $K=\max\left\{2,M\right\}$. For a given $r>0$, define the following sets:

$$\begin{aligned}
T &: =\left\{j\in\mathbb{N}:j\leq i,\ \left\|\frac{L_{i,j}(f)-f}{\sigma}\right\|_{C(S)}\geq r\right\},\\
T_1 &: =\left\{j\in\mathbb{N}:j\leq i,\frac{w(f,\delta)}{\|\sigma_1\|}\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}\geq\frac{r}{3K}\right\},\\
T_2 &: =\left\{j\in\mathbb{N}:j\leq i,\frac{w(f,\delta)}{\|\sigma_1\|}\geq\frac{r}{3K}\right\},\\
T_3 &: =\left\{j\in\mathbb{N}:j\leq i,\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}\geq\frac{r}{3K}\right\}.
\end{aligned}$$

It follows from (4.1) that

$$T\subset T_1\cup T_2\cup T_3.$$

Also, define the sets:

$$\begin{aligned}
T_4 &: =\left\{j\in\mathbb{N}:j\leq i,\frac{w(f,\delta)}{\|\sigma_1\|}\geq\sqrt{\frac{r}{3K}}\right\},\\
T_5 &: =\left\{j\in\mathbb{N}:j\leq i,\left\|\frac{L_{i,j}f_0-f_0}{\sigma_0}\right\|_{C(S)}\geq\sqrt{\frac{r}{3K}}\right\}.
\end{aligned}$$

Then observe that $T_1\subset T_4\cup T_5$. So we have $T\subset T_2\cup T_3\cup T_4\cup T_5$.

Therefore, using (a) and (b), the proof is complete. $\square$

5. CONCLUSION

If we take $A=C_1=(c_{i,j})$, the Cesàro matrix defined by

$$c_{i,j}:=\begin{cases}\frac{1}{i}, & \text{if } 1\leq j\leq i,\\ 0, & \text{otherwise,}\end{cases}$$

then triangular $A-$statistical relative uniform convergence reduces to the concept of triangular statistical relative convergence. Furthermore, if we take $A=C_1$ and the scale function by a non-zero constant, then triangular $A-$statistical relative uniform convergence reduces to the triangular statistical uniform convergence.

If one replaces the scale function by a non-zero constant, then the triangular $A-$statistical relative uniform convergence reduces to the triangular $A-$statistical uniform convergence.

REFERENCES

1. T. Acar, S. A. Mohiuddine: *statistical* $(C,1)(E,1)$ *summability and Korovkin's theorem.* Filomat **30(2)** (2016), 387-393.

2. C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini, S. Orhan: *Triangular $A-$statistical approximation by double sequences of positive linear operators*. Results Math. **68** (2015), 271-291.
3. C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini, S. Orhan: *Korovkin-type theorems for modular $\Psi - A-$statistical convergence*. J. Funct. Spaces Article ID 160401, **2015** (2015), p. 11.
4. C. Bardaro and I. Mantellini: *Korovkin's theorem in modular spaces*. Commentationes Math. **47** (2007), 239-253.
5. C. Bardaro and I. Mantellini: *A Korovkin Theorem in multivariate modular function spaces*. J. Funct. Spaces Appl. **7** (2009), 105-120.
6. C. Belen and S. A. Mohiuddine: *Generalized weighted statistical convergence and application*. Appl. Math. Comput. **219** (2013), 9821-9826.
7. E. W. Chitten: *On the limit functions of sequences of continuous functions converging relatively uniformly*. Trans. Amer. Math. Soc. **20** (1919), 179-184.
8. K. Demirci, F. Dirik: *A Korovkin type approximation theorem for double sequences of positive linear operators of two variables in $A-$statistical sense*. Bull. Korean Math. Soc. **47(4)** (2010) 825-837.
9. K. Demirci, B. Kolay: *$A-$Statistical Relative Modular Convergence of Positive Linear Operators*. Positivity **21** (2017), 847-863.
10. K. Demirci, S. Orhan: *Statistical approximation by double sequences of positive linear operators on modular spaces*. Positivity **19** (2015) 23-36.
11. K. Demirci, S. Orhan: *Statistically relatively uniform convergence of positive linear operators*. Results. Math. **69** (2016) 359-367.
12. K. Demirci, S. Orhan: *Statistical relative approximation on modular spaces*. Results. Math. **71** (2017) 1167-1184.
13. R. A. DeVore, G. G. Lorentz: *Constructive Approximation (Grund. Math. Wiss. 303)*. Springer Verlag, Berlin, 1993.
14. F. Dirik, K. Demirci: *Korovkin-type approximation theorem for functions of two variables in statistical sense*. Turk. J. Math. **34** (2010) 73-83.
15. K. Donner: *Korovkin Theorems in L^p Spaces*. J. Funct. Anal **42(1)** (1981) 12-28.
16. S. M. Eisenberg: *Korovkin's Theorem*. Bull. Malays. Math. Soc. **2(2)** (1979) 13-29.
17. H. Fast: *Sur la Convergence Statistique*. Colloq. Math. **2** (1951) 241-244.
18. P. Garrancho: *A general Korovkin result under generalized convergence*. Constr. Math. Anal. **2(2)** (2019) 81-88.
19. A. D. Gadjiev, C. Orhan: *Some approximation theorems via statistical convergence*. Rocky Mountain J. Math. **32** (2002) 129-138.
20. U. Kadak, S. A. Mohiuddine: *Generalized statistically almost convergence based on the difference operator which includes the $(p,q)-$gamma function and related approximation theorems*. Results. Math. **73(1)** (2018) 1-31.
21. P. P. Korovkin: *Linear Operators and Approximation Theory*. Hindustan Publ. Co., Delhi, 1960.
22. E. H. Moore: *An Introduction to a Form of General Analysis*. The New Hawen Mathematical Colloquim, Yale University Press, New Hawen, 1910.

23. S. A. Mohiuddine: *Statistical weighted $A-$summability with application to Korovkin's type approximation theorem.* J. Inequal. Appl. **2016(1)** (2016) 1-13.
24. S. A. Mohiuddine, B. A. S. Alamri: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems.* Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM **113(3)** (2019) 1955-1973.
25. S. A. Mohiuddine, A. Asiri, B. Hazarika: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems.* Int. J. Gen. Syst. **48(5)** (2019) 492-506.
26. S. A. Mohiuddine, B. Hazarika, M. A. Alghamdi: *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems.* Filomat **33(14)** (2019) 4549-4560.
27. F. Móricz: *Statistical Convergence of multiple sequences.* Arch. Math. **81(1)** (2003) 82-89.
28. A. Pringsheim: *Zur theorie der zweifach unendlichen zahlenfolgen.* Math. Ann. **53** (1900) 289-321.
29. G. M. Robison: *Divergent double sequences and series.* Amer. Math. Soc. Transl. **28** (1926) 50-57.
30. H. Steinhaus: *Sur la Convergence Ordinaire et la Convergence Asymptotique.* Colloq. Math. **2** (1951) 73-74.
31. P. Şahin, F. Dirik: *Statistical Relative Uniform Convergence of Double Sequence of Positive Linear Operators.* Appl. Math. **17** (2017) 207-220.
32. T. O. Teoplitz: *Über allgemeine lineare Mittelbildungen.* Prace Math. Fiz. **22** (1911) 113-119.
33. B. Yilmaz, K. Demirci, S. Orhan: *Relative Modular Convergence of Positive Linear Operators.* Positivity **20** (2016), 565-577.
34. D. E. Wulbert: *Convergence of Operators and Korovkin's theorem.* J. Approx. Theory **1** (1968) 381-390.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 79-88
https://doi.org/10.22190/FUMI200310007K
Original Scientific Paper

A NEW GENERALIZATION OF M-METRIC SPACE WITH SOME FIXED POINT THEOREMS

Erdal Karapınar[1], Kushal Roy[2] and Mantu Saha[2]

[1]**Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey**
[2]**Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India**

Abstract. In this paper, we introduce a new sequential space as a generalization of $M-$metric spaces and M_b-metric spaces. In this generalized space we define two contractive mappings namely $\mathfrak{m}-$contraction and $\mathfrak{m}-$quasi-contraction and prove some fixed point theorems for such type of mappings. Several illustrative examples have been presented in strengthening the hypothesis of our theorems.
Keywords: M-metric space, Fixed point theory, stability

1. Introduction and Preliminaries

The notion of metric has been generalized in several direction, see e.g. [1, 2, 6, 7, 8, 9]. Among all, we focus on partial metric and $M-$metric. The concept of partial metric space was first introduced by S. Matthews [1] in 1994 as a generalization of usual metric spaces. If (X, p) is a partial metric space then $p(\mu, \mu)$, $\mu \in X$ need not to be zero. Partial metric spaces have vast application potential, in particular, it has been used in the construction of the topological structures in the study of information science, computer science, etc. In 2014, Asadi et al. [2] have extended the notion of the partial metric space: $M-$metric space. The authors [2] proved the Banach contraction principle in the context of the complete $M-$metric space. The definition of $M-$metric is given as follows:

Throughout the manuscript all considered sets are nonempty. Further, the notation X^2 denotes the cross-product of the set X, that is, $X^2 : X \times X$.

Received March 10, 2020; accepted January 9, 2021.
Corresponding Author: Erdal Karapınar, Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey | E-mail: erdal.karapinar@cankaya.edu.tr
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25, 54E50.

Definition 1.1. Let $m : X^2 \to [0, \infty)$ be a function over a set X. Then (X, m) is said to be an $M-$metric space if m satisfies the following conditions:

(m1) $m(p, p) = m(q, q) = m(p, q)$ if and only if $p = q$;

(m2) $m_{pq} \leq m(p, q)$;

(m3) $m(p, q) = m(q, p)$;

(m4) $m(p, q) - m_{pq} \leq (m(p, r) - m_{pr}) + (m(r, q) - m_{rq})$ for all $p, q, r \in X$,

where

$$m_{pq} = \min\{m(p, p), m(q, q)\},$$

and

$$M_{pq} = \max\{m(p, p), m(q, q)\}.$$

It is seen that any partial metric space is an $M-$metric space. In [2] authors have presented an example of $M-$metric that does not form a partial metric.

Example 1.1. Let $X = \{1, 2, 3\}$ and $m : X^2 \to [0, \infty)$ be defined by $m(1, 1) = 1$, $m(2, 2) = 9$, $m(3, 3) = 5$ and

In 2015, Jleli-Samet [5] introduce a new generalization of the notion of metric spaces that involves b-metric and standard metric. Inspired by this work, we characterize the $M-$metric space and observed a new metric space. We present an example to indicate the novelty of this notion. Further, we observe some fixed point results in the setting of this new $M-$metric space.

2. Main results

In this section we introduce a generalized $M-$metric space namely m^*-metric space, as follows:

Let $\mathfrak{m} : X^2 \to [0, \infty)$ be a function such that $\mathfrak{m}_{pq} = \min\{\mathfrak{m}(p, p), \mathfrak{m}(q, q)\}$ and $\mathcal{M}_{pq} = \max\{\mathfrak{m}(p, p), \mathfrak{m}(q, q)\}$. Let us define the set

$$M(\mathfrak{m}, X, p) = \{\{p_n\} \subset X : \lim_{n\to\infty} (\mathfrak{m}(p_n, p) - \mathfrak{m}_{p_n p}) = 0\} \tag{2.1}$$

for all $p \in X$.

Definition 2.1. A function $\mathfrak{m} : X \times X \to [0, \infty)$, over a set X, is called an m^*-metric if the following conditions hold:

($\mathfrak{m}$ 1) $\mathfrak{m}(p, p) = \mathfrak{m}(q, q) = \mathfrak{m}(p, q)$ if and only if $p = q$, $p, q \in X$;

($\mathfrak{m}$ 2) $\mathfrak{m}_{pq} \leq \mathfrak{m}(p, q)$ for all $p, q \in X$;

($\mathfrak{m}$ 3) $\mathfrak{m}(p, q) = \mathfrak{m}(q, p)$ for all $p, q \in X$;

($\mathfrak{m}$ 4) there exists some $b > 0$ such that for any $(p, q) \in X^2$ and $\{p_n\} \in M(\mathfrak{m}, X, p)$ we have

$$\mathfrak{m}(p, q) - \mathfrak{m}_{pq} \leq b \limsup_{n\to\infty} (\mathfrak{m}(p_n, q) - \mathfrak{m}_{p_n q}). \tag{2.2}$$

The pair $(X, \mathfrak{m})$ is called an m^*-metric space.

In the following example we show that the newly defined m^*-metric space is more stronger than $M-$metric space.

Example 2.1. Let $X = \mathbb{N}$ and we define $\mathfrak{m} : X^2 \to \mathbb{R}_+$ by $\mathfrak{m}(n,n) = 1$ for all $n \in \mathbb{N}$, $\mathfrak{m}(1,2) = \mathfrak{m}(2,1) = 4$, $\mathfrak{m}(1,n) = \mathfrak{m}(n,1) = 1 + \frac{1}{n}$ for all $n \geq 3$, $\mathfrak{m}(n,2) = \mathfrak{m}(2,n) = \frac{5}{2}$ for all $n \geq 3$ and $\mathfrak{m}(n,k) = \mathfrak{m}(k,n) = 3$ for any $n,k \notin \{1,2\}$. Here $M(\mathfrak{m},X,1) = \{\{1,1,...\},\{3,4,5,...\}\}$ and for any other $x \in \mathbb{N}$, $M(\mathfrak{m},X,x)$ contains only the constant sequence $\{x,x,...\}$. Then one can easily check that $(X,\mathfrak{m})$ is an m^*-metric space.

Remark 2.1. In 2016, Mlaiki [10] defined M_b-metric space, by replacing the axiom (m4) in Definition 1.1 by

$$(m_b4)\ m_b(p,q) - m_{b_{pq}} \leq s[(m_b(p,r) - m_{b_{pr}}) + (m_b(r,q) - m_{b_{rq}})].$$

Now we show that $(X,\mathfrak{m})$ in Example 2.1 is not an M_b-metric space for any $b > 0$. Here we see that $\mathfrak{m}(n,k) - \mathfrak{m}_{nk} = 2$ for any $n,k \geq 3$. But $b[(\mathfrak{m}(n,1) - \mathfrak{m}_{n1}) + (\mathfrak{m}(1,k) - \mathfrak{m}_{1k})] = b[\frac{1}{n} + \frac{1}{k}] \to 0$ as $n,k \to \infty$ for any $b > 0$. This proves our assertion.

Remark 2.2. (1) Let (X,m) be an $M-$metric space (See Definition 1.1). Clearly m satisfies the conditions ($\mathfrak{m}$ 1), ($\mathfrak{m}$ 2) and ($\mathfrak{m}$ 3). Let $(p,q) \in X^2$ and $\{p_n\} \subset X$ be such that $\lim_{n\to\infty}(m(p_n,p) - m_{p_np}) = 0$ then from condition $(m4)$ we have

$$m(p,q) - m_{pq} \leq (m(p,p_n) - m_{pp_n}) + (m(p_n,q) - m_{p_nq}) \tag{2.3}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ we can easily see that m satisfies the condition ($\mathfrak{m}$ 4). Hence m satisfies all the conditions of m^*-metric and therefore (X,m) is an m^*-metric space.

(2) Let (X,m_b) be an M_b-metric space with coefficient $s \geq 1$. Then it is clear that m_b satisfies the conditions ($\mathfrak{m}$ 1), ($\mathfrak{m}$ 2) and ($\mathfrak{m}$ 3). Let $(p,q) \in X^2$ and $\{p_n\} \subset X$ be such that $\lim_{n\to\infty}(m_b(p_n,p) - m_{b_{p_np}}) = 0$ then from condition (m_b4) we have

$$m_b(p,q) - m_{b_{pq}} \leq s[(m_b(p,p_n) - m_{b_{pp_n}}) + (m_b(p_n,q) - m_{b_{p_nq}})] \tag{2.4}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ it can be easily seen that m_b satisfies the condition ($\mathfrak{m}$ 4). Hence m_b satisfies all the conditions of m^*-metric and therefore (X,m_b) is an m^*-metric space.

Definition 2.2. Let $(X,\mathfrak{m})$ be an m^*-metric space.

(1) A sequence $\{p_n\} \subset X$ is said to be convergent to an element $p \in X$ if $\lim\limits_{n\to\infty}(\mathfrak{m}(p_n,p) - \mathfrak{m}_{p_np}) = 0$ i.e. $\{p_n\} \in M(\mathfrak{m},X,p)$.

(2) A sequence $\{p_n\} \subset X$ is said to be Cauchy if $\lim\limits_{n,k\to\infty}(\mathfrak{m}(p_n,p_k) - \mathfrak{m}_{p_np_k})$ and $\lim\limits_{n,k\to\infty}(\mathcal{M}_{p_np_k} - \mathfrak{m}_{p_np_k})$ exist and finite.

(3) A sequence $\{p_n\} \subset X$ is said to be $0-$Cauchy if $\lim\limits_{n,k\to\infty}(\mathfrak{m}(p_n,p_k) - \mathfrak{m}_{p_np_k}) = 0$ and $\lim\limits_{n,k\to\infty}(\mathcal{M}_{p_np_k} - \mathfrak{m}_{p_np_k}) = 0$.

(4) An m^*-metric space $(X,\mathfrak{m})$ is said to be complete if every Cauchy sequence $\{p_n\} \subset X$ is convergent to some point $z \in X$ with $\lim\limits_{n\to\infty}(\mathcal{M}_{p_nz} - \mathfrak{m}_{p_nz}) = 0$.

Definition 2.3. Let $(X, \mathfrak{m})$ be an m^*-metric space and $T : X \to X$ be a mapping. Then T is said to be continuous at $\varsigma \in X$ if $\{p_n\} \in M(\mathfrak{m}, X, \varsigma)$ implies $\{Tp_n\} \in M(\mathfrak{m}, X, T\varsigma)$.

Proposition 2.1. *Let $(X, \mathfrak{m})$ be an m^*-metric space and $p, q \in X$. If $\{p_n\} \in M(\mathfrak{m}, X, p) \cap M(\mathfrak{m}, X, q)$ then $\mathfrak{m}(p, q) = \mathfrak{m}_{pq}$. Moreover if $\mathfrak{m}(p, p) = \mathfrak{m}(q, q)$ then $p = q$.*

Proof. Since $\{p_n\} \in M(\mathfrak{m}, X, p) \cap M(\mathfrak{m}, X, q)$, we have

$$\mathfrak{m}(p, q) - \mathfrak{m}_{pq} \leq b \limsup_{n\to\infty}(\mathfrak{m}(p_n, q) - \mathfrak{m}_{p_n q}) = 0, \tag{2.5}$$

implying that $\mathfrak{m}(p, q) - \mathfrak{m}_{pq} = 0$ that is $\mathfrak{m}(p, q) = \mathfrak{m}_{pq}$. If $\mathfrak{m}(p, p) = \mathfrak{m}(q, q)$ also, then clearly $p = q$. □

Proposition 2.2. *Let $\{p_n\}$ be a $0-$Cauchy sequence in an m^*-metric space $(X, \mathfrak{m})$. If $\{p_n\}$ has a convergent subsequence $\{p_{n_k}\}$ such that $\{p_{n_k}\} \in M(\mathfrak{m}, X, z)$ then $\{p_n\}$ is also convergent to $z \in X$.*

Proof. Since $\{p_n\}$ is $0-$Cauchy we have $\lim_{n,k\to\infty}(\mathfrak{m}(p_n, p_k) - \mathfrak{m}_{p_n p_k}) = 0$ and $\lim_{n,k\to\infty}(\mathcal{M}_{p_n p_k} - \mathfrak{m}_{p_n p_k}) = 0$. Also it is given that $\lim_{k\to\infty}(\mathfrak{m}(p_{n_k}, z) - \mathfrak{m}_{p_{n_k} z}) = 0$. Now,

$$\mathfrak{m}(p_p, z) - \mathfrak{m}_{p_p z} \leq b \limsup_{k\to\infty}(\mathfrak{m}(p_p, p_{n_k}) - \mathfrak{m}_{p_p p_{n_k}}) \tag{2.6}$$

for all $p \in \mathbb{N}$. Which implies that

$$\lim_{p\to\infty}[\mathfrak{m}(p_p, z) - \mathfrak{m}_{p_p z}] \leq b \lim_{p\to\infty} \limsup_{k\to\infty}(\mathfrak{m}(p_p, p_{n_k}) - \mathfrak{m}_{p_p p_{n_k}}) = 0. \tag{2.7}$$

Therefore $\lim_{p\to\infty}(\mathfrak{m}(p_p, z) - \mathfrak{m}_{p_p z}) = 0$, implying that $\{p_n\}$ is convergent to z. □

3. Topological m^*-metric space

Definition 3.1. Let $(X, \mathfrak{m})$ be an m^*-metric space. The open and closed ball of center at $p \in X$ and radius $t > 0$ in X are defined as follows:

$$\begin{aligned} B^{\mathfrak{m}}(p, t) &= \{q \in X : \mathfrak{m}(p, q) < \mathfrak{m}_{pq} + t\}; \\ B^{\mathfrak{m}}[p, t] &= \{q \in X : \mathfrak{m}(p, q) \leq \mathfrak{m}_{pq} + t\}. \end{aligned} \tag{3.1}$$

Remark 3.1. One can easily check that the collection

$$\tau_{\mathfrak{m}} = \emptyset \cup \{U(\neq \emptyset) \subset X : \text{ for any } p \in U \text{ there exists } t > 0 \text{ such that } B^{\mathfrak{m}}(p, t) \subset U\},$$

forms a topology on X.

Definition 3.2. Let $(X, \mathfrak{m})$ be an m^*-metric space and $\Delta \subset X$. Then Δ is said to be closed if there exists an open set $U \subset X$ such that $\Delta = U^c$.

Proposition 3.1. *Let $(X, \mathfrak{m})$ be an m^*-metric space and $\Delta \subset X$ be closed. Let $\{p_n\} \subset \Delta$ be such that $\{p_n\} \in M(\mathfrak{m}, X, z)$, then $z \in \Delta$.*

Proof. If possible let $z \notin \Delta$. Then $z \in \Delta^c = U$, where U is open. So there exists $t > 0$ such that $B^{\mathfrak{m}}(z, t) \subset U$. Now $\lim_{n\to\infty}(\mathfrak{m}(p_n, z) - \mathfrak{m}_{p_n z}) = 0$ so for $t > 0$ there exists $N \in \mathbb{N}$ such that $\mathfrak{m}(p_n, z) - \mathfrak{m}_{p_n z} < t$ whenever $n \geq N$. Thus $p_n \in B^{\mathfrak{m}}(z, t) \subset U$ for all $n \geq N$, a contradiction. Hence $z \in \Delta$. $\square$

Definition 3.3. Let $(X, \mathfrak{m})$ be an m^*-metric space and $B \subset X$. Then $diam(B) = \sup\{\max\{\mathfrak{m}(p, q) - \mathfrak{m}_{pq}, \mathcal{M}_{pq} - \mathfrak{m}_{pq}\} : p, q \in B\}$.

Definition 3.4. In an m^*-metric space $(X, \mathfrak{m})$, a sequence $\{\Delta_n\}$ of subsets of X is said to be decreasing if $\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset ...$.

Theorem 3.1. *Let $(X, \mathfrak{m})$ be a complete m^*-metric space and $\{\Delta_n\}$ be a decreasing sequence of nonempty closed subsets of X such that $diam(\Delta_n) \to 0$ as $n \to \infty$. Then the intersection $\cap_{n=1}^{\infty} \Delta_n$ contains exactly one point.*

Proof. Let $p_n \in \Delta_n$ be arbitrary for all $n \in \mathbb{N}$. Since $\{\Delta_n\}$ is decreasing, we have $\{p_n, p_{n+1}, ...\} \subset \Delta_n$ for all $n \in \mathbb{N}$.

Now for any $n, p \in \mathbb{N}$ with $n, p \geq k$ we have $\max\{\mathfrak{m}(p_n, p_p) - \mathfrak{m}_{p_n p_p}, \mathcal{M}_{p_n p_p} - \mathfrak{m}_{p_n p_p}\} \leq diam(\Delta_k)$, $k \geq 1$. Let $\epsilon > 0$ be given. Then there exists some $q \in \mathbb{N}$ such that $diam(\Delta_q) < \epsilon$ since $diam(\Delta_n) \to 0$ as $n \to \infty$. From this it follows that $\max\{\mathfrak{m}(p_n, p_p) - \mathfrak{m}_{p_n p_p}, \mathcal{M}_{p_n p_p} - \mathfrak{m}_{p_n p_p}\} < \epsilon$ whenever $n, p \geq q$. Therefore $\{p_n\}$ is Cauchy sequence, more specifically $0-$Cauchy sequence in X. By the completeness of X there exists $z \in X$ such that $\{p_n\} \in M(\mathfrak{m}, X, z)$. Since $\{p_n, p_{n+1}, ...\} \subset \Delta_n$ and Δ_n is closed for each $n \in \mathbb{N}$, using Proposition 3.1 we have $z \in \cap_{n=1}^{\infty} \Delta_n$.

Next we prove the uniqueness of z. Let $q \in \cap_{n=1}^{\infty} \Delta_n$ be another point, then either $m(z, q) > \mathfrak{m}_{zq}$ or $\mathcal{M}_{zq} > \mathfrak{m}_{zq}$. That is $\max\{m(z, q) - \mathfrak{m}_{zq}, \mathcal{M}_{zq} - \mathfrak{m}_{zq}\} > 0$. As $diam(\Delta_n) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$diam(\Delta_n) < \max\{m(z, q) - \mathfrak{m}_{zq}, \mathcal{M}_{zq} - \mathfrak{m}_{zq}\} \leq diam(\Delta_n) \tag{3.2}$$

for all $n \geq N_0$, a contradiction. Hence $\cap_{n=1}^{\infty} \Delta_n = \{z\}$ and this completes the proof of our theorem. $\square$

4. Fixed point results on m^*-metric space

Definition 4.1. Let $(X, \mathfrak{m})$ be an m^*-metric space and $T : X \to X$ be a mapping. Then T is said to be $\mathfrak{m}-$contraction if

$$\mathfrak{m}(Tp, Tq) \leq k\, \mathfrak{m}(p, q) \tag{4.1}$$

for all $p, q \in X$, where $k \in (0, 1)$.

Definition 4.2. Let $(X, \mathfrak{m})$ be an m^*-metric space and $T : X \to X$ be a mapping. Then T is said to be $\mathfrak{m}-$quasi-contraction if

$$\mathfrak{m}(Tp, Tq) \leq k \ \max\{\mathfrak{m}(p, q), \mathfrak{m}(p, Tp), \mathfrak{m}(q, Tq), \mathfrak{m}(Tp, q), \mathfrak{m}(p, Tq)\} \tag{4.2}$$

for all $p, q \in X$ and for some $k \in (0, 1)$.

Now we come to our main fixed point theorems.

Theorem 4.1. *Let $(X, \mathfrak{m})$ be a complete m^*-metric space and $T : X \to X$ be a mapping such that it satisfies the following conditions:*

(1) *T is an $\mathfrak{m}-$contraction;*

(2) *there exists $p_0 \in X$ such that $\delta(\mathfrak{m}, T, p_0) = \sup\{\mathfrak{m}(T^i p_0, T^j p_0) : i, j \geq 1\} < \infty$.*

Then T has a unique fixed point in X.

Proof. Let us define $\delta(\mathfrak{m}, T^{p+1}, p_0) = \sup\{\mathfrak{m}(T^{p+i}p_0, T^{p+j}p_0) : i, j \geq 1\}$ for any $p \geq 0$. Since T satisfies the contractive condition (4.1), we have

$$\begin{aligned} \mathfrak{m}(T^{p+i}p_0, T^{p+j}p_0) &\leq k \ \mathfrak{m}(T^{p-1+i}p_0, T^{p-1+j}p_0) \\ &\leq k \ \delta(\mathfrak{m}, T^p, p_0) \end{aligned} \tag{4.3}$$

for all $i, j, p \geq 1$. From (4.3) it follows that

$$\begin{aligned} \delta(\mathfrak{m}, T^{p+1}, p_0) &\leq k \ \delta(\mathfrak{m}, T^p, p_0) \\ &\cdots \\ &\leq k^p \ \delta(\mathfrak{m}, T, p_0) \end{aligned} \tag{4.4}$$

for all $p \in \mathbb{N}$. As $k \in (0, 1)$ we get $\lim_{p\to\infty} \delta(\mathfrak{m}, T^{p+1}, p_0) = 0$. Therefore $\lim_{n,k\to\infty} \mathfrak{m}(p_n, p_k) = 0$ and $\lim_{n\to\infty} \mathfrak{m}(p_n, p_n) = 0$. Thus $\lim_{n,k\to\infty} (\mathfrak{m}(p_n, p_k) - \mathfrak{m}_{p_n p_k}) = 0$ and $\lim_{n,k\to\infty} (\mathcal{M}_{p_n p_k} - \mathfrak{m}_{p_n p_k}) = 0$. So $\{p_n\}$ is Cauchy sequence in X. By the completeness of X we get some $z \in X$ such that $\{p_n\} \in M(\mathfrak{m}, X, z)$ with $\lim_{n\to\infty} (\mathcal{M}_{p_n z} - \mathfrak{m}_{p_n z}) = 0$. But $\lim_{n\to\infty} \mathfrak{m}_{p_n z} =$
$\lim_{n\to\infty} \min\{\mathfrak{m}(p_n, p_n), \mathfrak{m}(z, z)\} = 0$, follows that $\lim_{n\to\infty} \mathfrak{m}(p_n, z) = 0 = \lim_{n\to\infty} \mathcal{M}_{p_n z}$. Thus

$$\mathfrak{m}(p_{n+1}, Tz) = \mathfrak{m}(Tp_n, Tz) \leq k \ \mathfrak{m}(p_n, z) \to 0 \tag{4.5}$$

as $n \to \infty$. Hence $\lim_{n\to\infty} (\mathfrak{m}(p_{n+1}, Tz) - \mathfrak{m}_{p_{n+1}Tz}) = 0$. Also $\mathfrak{m}(z, z) = 0$ and by the contractive condition (4.1) we get $\mathfrak{m}(Tz, Tz) = 0$.

$$\mathfrak{m}(z, Tz) - \mathfrak{m}_{zTz} = \mathfrak{m}(z, Tz) \leq b \limsup_{n\to\infty} (\mathfrak{m}(z, p_{n+1}) - \mathfrak{m}_{zp_{n+1}}) = 0. \tag{4.6}$$

Therefore it follows that $z = Tz$ and z is a fixed point of T in X.

If z and w are two fixed points of T, then we see that

$$\begin{aligned}\mathfrak{m}(z,w)=\mathfrak{m}(Tz,Tw) &\leq k\ \mathfrak{m}(z,w);\\ \mathfrak{m}(z,z)=\mathfrak{m}(Tz,Tz) &\leq k\ \mathfrak{m}(z,z);\\ \mathfrak{m}(w,w)=\mathfrak{m}(Tw,Tw) &\leq k\ \mathfrak{m}(w,w).\end{aligned}$$

From which it follows that $\mathfrak{m}(z,w)=\mathfrak{m}(z,z)=\mathfrak{m}(w,w)=0$ implies $z=w$ i.e. T has a unique fixed point in X. $\square$

Theorem 4.2. *Let $(X,\mathfrak{m})$ be a complete m^*-metric space and $T:X\to X$ be a mapping such that it satisfies the following conditions:*

(1) *T is an $\mathfrak{m}-$quasi-contraction with $k\in(0,1)\cap(0,\frac{1}{b})$;*

(2) *there exists $p_0\in X$ such that $\delta(\mathfrak{m},T,p_0)=\sup\{\mathfrak{m}(T^ip_0,T^jp_0): i,j\geq 1\}<\infty$.*

Then the Picard iterating sequence $\{T^nx_0\}$ converges to some $u\in X$ which is the unique fixed point of T in X.

Proof. Similar as in Theorem 4.1 we define $\delta(\mathfrak{m},T^{p+1},p_0)=\sup\{\mathfrak{m}(T^{p+i}p_0,T^{p+j}p_0): i,j\geq 1\}$ for any $p\geq 0$. Since T satisfies the contractive condition (4.2), we have

$$\begin{aligned}\mathfrak{m}(T^{p+i}p_0,T^{p+j}p_0) &\leq k\ \max\{\mathfrak{m}(T^{p-1+i}p_0,T^{p-1+j}p_0),\mathfrak{m}(T^{p-1+i}p_0,T^{p+i}p_0),\\ &\qquad \mathfrak{m}(T^{p-1+j}p_0,T^{p+j}p_0),\mathfrak{m}(T^{p-1+i}p_0,T^{p+j}p_0),\\ &\qquad \mathfrak{m}(T^{p-1+j}p_0,T^{p+i}p_0)\}\\ &\leq k\ \delta(\mathfrak{m},T^p,p_0) \qquad (4.7)\end{aligned}$$

for all $i,j,p\geq 1$. By similar calculation as in Theorem 4.1 we deduce that the Picard iterating sequence $\{p_n\}\in M(\mathfrak{m},X,u)$ for some $u\in X$ with $\lim\limits_{n\to\infty}(\mathcal{M}_{p_nu}-\mathfrak{m}_{p_nu})=0$. Therefore we get $\lim\limits_{n\to\infty}\mathfrak{m}(p_n,u)=0$ and $\mathfrak{m}(u,u)=0$.

Now for any fixed $n\in\mathbb{N}$ we have,

$$\begin{aligned}\mathfrak{m}(u,T^np_0)=\mathfrak{m}(u,T^np_0)-\mathfrak{m}_{uT^np_0} &\leq b\limsup_{k\to\infty}(\mathfrak{m}(T^{n+k}p_0,T^np_0)-\mathfrak{m}_{T^{n+k}p_0T^np_0})\\ &= b\limsup_{k\to\infty}\mathfrak{m}(T^{n+k}p_0,T^np_0) \qquad (4.8)\\ &\leq b\ \delta(\mathfrak{m},T^n,p_0)\leq bk^{n-1}\delta(\mathfrak{m},T,p_0).\end{aligned}$$

Now,

$$\begin{aligned}\mathfrak{m}(Tu,T^2p_0) &\leq k\max\{\mathfrak{m}(u,Tp_0),\mathfrak{m}(u,Tu),\mathfrak{m}(Tp_0,T^2p_0),\mathfrak{m}(u,T^2p_0),\mathfrak{m}(Tp_0,Tu)\}\\ &\leq k\max\{b\delta(\mathfrak{m},T,p_0),\mathfrak{m}(u,Tu),\delta(\mathfrak{m},T,p_0),bk\delta(\mathfrak{m},T,p_0),\mathfrak{m}(Tp_0,Tu)\} \qquad (4.9)\\ &= k\max\{b\delta(\mathfrak{m},T,p_0),\mathfrak{m}(u,Tu),\delta(\mathfrak{m},T,p_0),\mathfrak{m}(Tp_0,Tu)\}.\end{aligned}$$

Also,

$$\mathfrak{m}(Tu,T^3p_0)\leq$$

$$
\begin{aligned}
&\leq k\max\{\mathfrak{m}(u,T^2p_0),\mathfrak{m}(u,Tu),\mathfrak{m}(T^2p_0,T^3p_0),\mathfrak{m}(u,T^3p_0),\mathfrak{m}(T^2p_0,Tu)\}\\
(4.10)\quad &\leq k\max\{bk\delta(\mathfrak{m},T,p_0),\mathfrak{m}(u,Tu),\delta(\mathfrak{m},T^2,p_0),bk^2\delta(\mathfrak{m},T,p_0),\mathfrak{m}(T^2p_0,Tu)\}\\
&\leq k\max\{bk\delta(\mathfrak{m},T,p_0),\mathfrak{m}(u,Tu),k\delta(\mathfrak{m},T,p_0),\mathfrak{m}(T^2p_0,Tu)\}\\
&\leq k\max\{bk\delta(\mathfrak{m},T,p_0),\mathfrak{m}(u,Tu),k\delta(\mathfrak{m},T,p_0),k\mathfrak{m}(Tp_0,Tu)\}.
\end{aligned}
$$

Proceeding in a similar way for every $n \geq 1$ we get,

$$
\begin{aligned}
(4.11)\qquad & \mathfrak{m}(Tu,T^{n+1}p_0)\leq\\
\leq\ & \max\{bk^n\delta(\mathfrak{m},T,p_0),k\mathfrak{m}(u,Tu),k^n\delta(\mathfrak{m},T,p_0),k^n\mathfrak{m}(Tp_0,Tu)\}.
\end{aligned}
$$

From (4.11) it follows that $\limsup_{n\to\infty}\mathfrak{m}(Tu,T^{n+1}p_0)\leq k\ \mathfrak{m}(u,Tu)$. Thus we have,

$$
\begin{aligned}
\mathfrak{m}(u,Tu)=\mathfrak{m}(u,Tu)-\mathfrak{m}_{uTu} &\leq b\limsup_{n\to\infty}(\mathfrak{m}(T^{n+1}p_0,Tu)-\mathfrak{m}_{T^{n+1}p_0Tu})\\
(4.12)\qquad &= b\limsup_{n\to\infty}\mathfrak{m}(T^{n+1}p_0,Tu)\\
&\leq bk\ \mathfrak{m}(u,Tu).
\end{aligned}
$$

From the inequality (4.12) it clear that $\mathfrak{m}(u,Tu)=0$. Since T satisfies the contractive condition (4.2), we get $\mathfrak{m}(Tu,Tu)=0$. Therefore $Tu=u$ and u is a fixed point of T.

If w is a fixed point of T in X, then we get

$$
\begin{aligned}
\mathfrak{m}(u,w)=\mathfrak{m}(Tu,Tw) &\leq k\ \max\{\mathfrak{m}(u,w),\mathfrak{m}(u,u),\mathfrak{m}(w,w)\};\\
\mathfrak{m}(u,u)=\mathfrak{m}(Tu,Tu) &\leq k\ \mathfrak{m}(u,u);\\
\mathfrak{m}(w,w)=\mathfrak{m}(Tw,Tw) &\leq k\ \mathfrak{m}(w,w).
\end{aligned}
$$

From which it follows that $\mathfrak{m}(u,w)=\mathfrak{m}(u,u)=\mathfrak{m}(w,w)=0$ implies $u=w$ i.e. $u=w$. $\square$

Example 4.1. Let $X=\{1,2,3\}$ and we define $\mathfrak{m}: X\times X\to[0,\infty)$ as $\mathfrak{m}(1,1)=1$, $\mathfrak{m}(2,2)=2$, $\mathfrak{m}(3,3)=0$ and

Example 4.2. Let $X=\{1,2,3\}$ and we define $\mathfrak{m}: X\times X\to[0,\infty)$ as $\mathfrak{m}(1,1)=2$, $\mathfrak{m}(2,2)=1$, $\mathfrak{m}(3,3)=0$ and

Corollary 4.1. *The conclusion of Theorem 4.2 can be made also by using the following contractive conditions instead of contractive condition (4.2):*

(*a*) $\mathfrak{m}(Tp,Tq)\leq\alpha[\mathfrak{m}(p,Tp)+\mathfrak{m}(q,Tq)]$, $\alpha\in(0,\frac{1}{2})$;

(*b*) $\mathfrak{m}(Tp,Tq)\leq\beta[\mathfrak{m}(p,Tq)+\mathfrak{m}(q,Tp)]$, $\beta\in(0,\frac{1}{2})$;

(*c*) $\mathfrak{m}(Tp,Tq)\leq\xi[\mathfrak{m}(p,q)+\mathfrak{m}(p,Tp)+\mathfrak{m}(q,Tq)]$, $\xi\in(0,\frac{1}{3})$;

(*d*) $\mathfrak{m}(Tp,Tq)\leq\omega[\mathfrak{m}(p,q)+\mathfrak{m}(p,Tq)+\mathfrak{m}(Tp,q)]$, $\omega\in(0,\frac{1}{3})$;

(*e*) $\mathfrak{m}(Tp,Tq)\leq p\mathfrak{m}(p,q)+q\mathfrak{m}(p,Tp)+r\mathfrak{m}(q,Tq)+s\mathfrak{m}(p,Tq)+t\mathfrak{m}(Tp,q)$, $p,q,r,s,t\in(0,1)$ *with* $p+q+r+s+t<1$;

(f) $\mathfrak{m}(Tp,Tq) \leq \gamma \max\{\mathfrak{m}(p,q), \mathfrak{m}(p,Tp), \mathfrak{m}(q,Tq)\}$, $\gamma \in (0,1)$;

(g) $\mathfrak{m}(Tp,Tq) \leq \eta \max\{\mathfrak{m}(p,q), \mathfrak{m}(p,Tq), \mathfrak{m}(Tp,q)\}$, $\eta \in (0,1)$;

(h) $\mathfrak{m}(Tp,Tq) \leq \zeta \max\{\mathfrak{m}(p,q), \mathfrak{m}(p,Tp), \mathfrak{m}(q,Tq), \dfrac{\mathfrak{m}(p,Tq)+\mathfrak{m}(Tp,q)}{2}\}$, $\zeta \in (0,1)$.

5. Application to the stability of fixed point problem

In this section, we will discuss Hyers-Ulam stability of fixed points of mappings. For more details on Hyers-Ulam stability of functional equations and its applications on fixed point problems one can refer to [4], [11] and [12].

Let $(X,\mathfrak{m})$ be an m^*-metric space and $T : X \to X$ be a given mapping. Let us consider the fixed point equation

$$Tp = p, \ \mathfrak{m}(p,p) = 0 \tag{5.1}$$

and the inequality

$$\mathfrak{m}(Tq,q) - \mathfrak{m}_{Tq\ q} < \epsilon \tag{5.2}$$

for any $\epsilon > 0$.

Definition 5.1. The fixed point problem (5.1) is said to be Hyers-Ulam stable if there exists an element $c > 0$ such that for each $\epsilon > 0$ and an $\epsilon-$solution (A solution of (5.2)) $v \in X$ there exists a solution $u \in X$ of the fixed point equation (5.1) such that $\mathfrak{m}(u,v) < c\epsilon$.

Theorem 5.1. *Let $(X,\mathfrak{m})$ be a complete M_b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping such that T satisfies all the conditions of Theorem 4.1 with the Lipschitz constant $k \in (0, \frac{1}{s})$. Then the fixed point equation of T is Hyers-Ulam stable.*

Proof. Since any M_b-metric space is m^*-metric space, from Theorem 4.1 we see that T has a unique fixed point u in X with $\mathfrak{m}(u,u) = 0$ that is the fixed point equation (5.1) of T has a unique solution. Let $\epsilon > 0$ be arbitrary and v be an ϵ-solution of T. Then

$$\begin{aligned}
\mathfrak{m}(u,v) &= \mathfrak{m}(u,v) - \mathfrak{m}_{uv} \\
&\leq s[(\mathfrak{m}(u,Tv) - \mathfrak{m}_{uTv}) + (\mathfrak{m}(Tv,v) - \mathfrak{m}_{Tv\ v})] \\
&= s[\mathfrak{m}(Tu,Tv) + (\mathfrak{m}(Tv,v) - \mathfrak{m}_{Tv\ v})] \\
&\leq s[k\mathfrak{m}(u,v) + (\mathfrak{m}(Tv,v) - \mathfrak{m}_{Tv\ v})] \qquad (5.3)
\end{aligned}$$

This implies $\mathfrak{m}(u,v) \leq \frac{s}{1-sk}(\mathfrak{m}(Tv,v) - \mathfrak{m}_{Tv\ v}) < \frac{s}{1-sk}\epsilon$. Therefore the fixed point equation of T is Hyers-Ulam stable. □

Acknowledgements

The second author acknowledges financial support awarded by the Council of Scientific and Industrial Research, New Delhi, India, through research fellowship for carrying out research work leading to the preparation of this manuscript.

REFERENCES

1. S. Matthews: *Partial metric topology.* Ann. N. Y. Acad. Sci., **728** (1994), 183-197.
2. M. Asadi, E. Karapinar and P. Salimi: *New extension of $p-$metric spaces with some fixed-point results on $M-$metric spaces.* Journal of Inequalities and Applications 2014, 2014:18.
3. I. A. Bakhtin: *The contraction principle in quasimetric spaces.* Func. An., Ulianowsk, Gos. Ped. Ins., **30** (1989), 26-37.
4. S. M. Jung: *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis.* Springer Optimization and Its Applications, **48** (2011).
5. M. Jleli, B. Samet: *A generalized metric space and related fixed point theorems.* Fixed Point Theory Appl. 2015, **14** (2015).
6. M. A. Alghamdi, S. Gulyaz-Ozyurt, E. Karapinar: *A Note on Extended Z-Contraction.* Mathematics 2020, 8, 195.
7. E. Karapinar, C. Chifu: *Results in wt-Distance over b-Metric Spaces.* Mathematics 2020, 8, 220.
8. C. Chifu, E. Karapinar, G. Petrusel: *Fixed point results in ε-chainable complete b-metric spaces.* Fixed Point Theory, **21**(2) (2020), 453-464.
9. A. Fulga, E. Karapinar, G. Petrusel: *On Hybrid Contractions in the Context of Quasi-Metric Spaces.* Mathematics 2020, 8, 675.
10. N. Mlaiki, A. Zarrad, N. Souayah, A. Mukheimer and T. Abdeljawed: *Fixed point theorems in M_b-metric spaces.* Journal of Mathematical Analysis, **7**(5) (2016), Pages 1-9.
11. K. Roy and M. Saha: *Fixed points of mappings over a locally convex topological vector space and Ulam-Hyers stability of fixed point problems.* Novi Sad J. Math., **50**(1) (2020), 99-112.
12. K. Roy and M. Saha: *On fixed points of Ćirić-type contractive mappings over a C^*-algebra valued metric space and Hyers-Ulam stability of fixed point problems.* J. Adv. Math. Stud., **12**(3) (2019), 350-363.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 89-100
https://doi.org/10.22190/FUMI200311008Y

Original Scientific Paper

APPROXIMATION PROPERTIES OF MODIFIED GAUSS-WEIERSTRASS INTEGRAL OPERATORS IN EXPONENTIAL WEIGHTED L_p SPACES

Başar Yılmaz

Kirikkale University, Faculty of Science and Arts, Department of Mathematics, 71450 Kiriklale, Turkey

Abstract. In this paper, we deal with modified Gauss-Weierstrass integral operators from exponentially weighted spaces $L_{p,a}(\mathbb{R})$ into $L_{p,2a}(\mathbb{R})$. We give the rate of convergence in terms of weighted modulus of continuity. Moreover, we prove weighted approximation of functions belonging to the space $L_{p,a}(\mathbb{R})$ by these operators with the help of a Korovkin type theorem. Finally, we give pointwise approximation of such functions by these operators at generalized Lebesgue points.
Keywords: Gauss-Weierstrass operators, Korovkin type theorem, exponential weighted spaces

1. Introduction

The well-known Gauss-Weierstrass singular integral operators are given by

$$(W_n f)(x) := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(x-t)\, e^{-nt^2} dt, \qquad x \in \mathbb{R},\ n \in \mathbb{N},$$

where the function f is selected such that the integrals are finite. These operators were extensively studied by many researchers [3],[4],[5],[7] and [14]. Some approximation problems including Voronovskaya type theorem and quantitative type results have been investigated in L_p and weighted L_p spaces in [9].

In [8], Agratini et al. considered a generalization of the Gauss-Weierstrass singular integral operators defined by

Received March 11, 2020; accepted December 2, 2020.
Corresponding Author: Başar Yılmaz, Kirikkale University, Faculty of Science and Arts, Department of Mathematics, 71450 Kiriklale, Turkey | E-mail: basaryilmaz77@yahoo.com
2010 *Mathematics Subject Classification.* Primary 41A36,41A35 Secondary 41A25

$$(1.1)\qquad (W_n^* f)(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(\beta_n(x) + t)\, e^{-nt^2} dt, \quad x \in (-\infty, \infty),\ n \in \mathbb{N},$$

where

$$(1.2)\qquad \beta_n(x) = x - \frac{a}{2n}, \quad n \geq 1,\ a > 0.$$

These operators reproduce not only e_0, where $e_0(t) = 1,\ t \in \mathbb{R}$, but also certain exponential functions. In that work, the authors studied these operators in the polynomial weighted continuous functions spaces. They also proved that these operators have better approximation properties than the classical ones. The linear positive operators preserving exponential functions in approximation theory have been intensively studied (see [10],[11],[12],[13],[15],[16] and [17]).

In this paper, we consider the operators W_n^* in the setting of large classes of exponential weighted L_p spaces. Firstly, we show that these operators act from the exponential weighted $L_{p,a}(\mathbb{R})$ space into $L_{p,2a}(\mathbb{R})$, which will be defined below. Then, we get quantitative results for the rate of convergence by the operators in terms of weighted L_p modulus of continuity. Similar result is also given for the derivates of the operators. Furthermore, we obtain weighted approximation by the operators using a weighted Korovkin type theorem. Finally, we investigate a pointwise convergence result by the operators at generalized Lebesgue points.

Below, we recall the definition of exponential weighted space $L_{p,a}(\mathbb{R})$.

Let $a > 0$ and $1 \leq p < \infty$ be fixed,

$$\nu_a(x) = e^{-ax^2} \text{ for } x \in \mathbb{R},$$

and let $L_{p,a}(\mathbb{R})$ be the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which $\nu_a f$ is Lebesgue integrable with p-th power over $\mathbb{R}$, where $1 \leq p < \infty$, and uniformly continuous and bounded on $\mathbb{R}$. The norm in $L_{p,a}(\mathbb{R})$ is defined by

$$(1.3)\qquad \|f\|_{p,a} = \|f(\cdot)\|_{p,a} = \left(\int_{-\infty}^{\infty} |\nu_a(x) f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty$$

(see [6]).

As usual, for $f \in L_{p,a}(\mathbb{R})$ the weighted modulus of continuity is defined as

$$(1.4)\qquad \omega(f; L_{p,a}(\mathbb{R}); t) := \sup_{|h| \leq t} \|\Delta_h f(\cdot)\|_{p,a} \text{ for } t \geq 0,$$

where

$$\Delta_h f(x) := f(x+h) - f(x).$$

The above ω has the following properties:

$$\omega\left(f; L_{p,a}\left(\mathbb{R}\right); t_1\right) \leq \omega\left(f; L_{p,a}\left(\mathbb{R}\right); t_2\right) \text{ for } 0 \leq t_1 < t_2, \tag{1.5}$$

$$\omega\left(f; L_{p,a}\left(\mathbb{R}\right); \lambda t\right) \leq \left(1+\lambda\right) e^{a(\lambda t)^2} \omega\left(f; L_{p,a}\left(\mathbb{R}\right); t\right) \text{ for } \lambda, t \geq 0, \tag{1.6}$$

$$\lim_{t \to 0^+} \omega\left(f; L_{p,a}\left(\mathbb{R}\right); t\right) = 0 \tag{1.7}$$

for every $f \in L_{p,a}\left(\mathbb{R}\right)$ (see[1]).

2. Auxiliary results

In this part, we shall give some fundamental properties of the generalized Gauss-Weierstrass integral operators W_n^* in the spaces $L_{p,2\alpha}\left(\mathbb{R}\right)$. Lemma 2.1 can be obtained by elementary calculations.

Lemma 2.1. *The equality*

$$\int_0^\infty x^p e^{-2ax^2 p} dx = \frac{1}{2^{\frac{p+3}{2}}} \frac{\Gamma\left(\frac{p+1}{2}\right)}{(ap)^{\frac{p+1}{2}}},$$

where Γ *is the Gamma function and holds for every* $p \in [1, \infty)$ *and* $a > 0$.

Lemma 2.2. *(see [8]) If* W_n^*, $n \geq 1$, *are the operators given by (1.1), then for each integer* $j \geq 0$, $e_j\left(t\right) = t^j$ $t \in \mathbb{R}$, *we have*

$$\left(W_n^* e_j\right)(x) = \beta_n^j(x) + \sum_{s=0}^{\lfloor j/2 \rfloor} \frac{(2s-1)!}{(2n)^s} \binom{j}{2s} \beta_n^{j-2s}\left(x\right), \quad p \geq 2 \quad, x \in \mathbb{R}.$$

Also, as particular cases, we have

$$W_n^* e_0 = 1, \; W_n^* e_1 = \beta_n\left(x\right), \; W_n^* e_2 = \beta_n^2\left(x\right) + \frac{1}{2n}.$$

This formula shows that $W_n^* f$ $(n > 2a, \; a > 0)$ is a sequence of linear positive operators from $L_{p,a}\left(\mathbb{R}\right)$ into $L_{p,2a}\left(\mathbb{R}\right)$.

Lemma 2.3. *If* $f \in L_{p,\alpha}\left(\mathbb{R}\right)$, *with* $1 \leq p < \infty$, *then for* $n > 2a$, *we have*

$$\left\|W_n^* f\right\|_{p,2a} \leq M_n \left\|f\right\|_{p,a}, \tag{2.1}$$

where

$$M_n = \sqrt{\frac{\pi}{n-2a}} e^{\frac{a^3}{2n^2-4an}}.$$

Proof. In view of the definition of the operators W_n^*, we can write

$$\begin{aligned}\|W_n^*f\|_{p,2a} &= \left(\int_{-\infty}^{\infty}\left|e^{-2ax^2}\left(W_n^*f\right)(x)\right|^p dx\right)^{1/p}\\ &= \left(\int_{-\infty}^{\infty} e^{-2ax^2p}\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f\left(\beta_n(x)+t\right)e^{-nt^2}dt\right|^p dx\right)^{1/p}.\end{aligned}$$

By a generalization of Minkowski's inequality and making use of substitution $\beta_n(x)+t=u$, the above formula reduces to

$$\begin{aligned}\|W_n^*f\|_{p,2a} &\le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}\left(e^{-nt^2p}\int_{-\infty}^{\infty}|f(u)|^p e^{-2a\left(u+\frac{a}{2n}-t\right)^2p}du\right)^{1/p}dt\\ &\le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}\left(e^{-nt^2p}\int_{-\infty}^{\infty}|f(u)|^p e^{-au^2p}e^{2a\left(\frac{a}{2n}-t\right)^2p}du\right)^{1/p}dt\\ &\le \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2}e^{2a\left(\frac{a}{2n}-t\right)^2}\left(\int_{-\infty}^{\infty}|f(u)|^p e^{-au^2p}du\right)^{1/p}dt\\ &= \|f\|_{p,a}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-nt^2+2a\left(\frac{a}{2n}-t\right)^2}dt\\ &= \|f\|_{p,a}\sqrt{\frac{\pi}{n-2a}}e^{\frac{a^3}{2n^2-4an}}.\end{aligned}$$

Thus, the proof of Lemma 2.3 is completed. □

3. Approximation theorems

Firstly, we shall prove rate of convergence by the operators (1.1) of functions belonging to $L_{p,\alpha}(\mathbb{R})$.

Theorem 3.1. *If $f\in L_{p,a}(\mathbb{R})$, then we have*

$$\|W_n^*f-f\|_{p,2a}\le\omega\left(f;L_{p,2a};\frac{a}{2n}\right)+\omega\left(f;L_{p,2a};\frac{1}{\sqrt{n}}\right)\left(\sqrt{\frac{n}{n-a}}\right)$$

for $n>a$.

Proof. From (1.1), (1.3) and the Minkowski inequality, we get

$$\|W_n^*f-f\|_{p,2a}$$

$$= \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2} \left(\left(W_n^* f \right)(x) - f(x) \right) \right|^p dx \right)^{1/p}$$

$$= \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2 p} \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \left(f\left(\beta_n(x) + t \right) - f(x) \right) e^{-nt^2} dt \right|^p dx \right)^{1/p}$$

$$\leq \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \left(\int_{-\infty}^{\infty} e^{-2ax^2 p} \left| f\left(\beta_n(x) + t \right) - f(x) + f(x+t) - f(x+t) \right|^p dx \right)^{1/p} dt$$

$$\leq \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2} \left(f\left(\beta_n(x) + t \right) - f(x+t) \right) \right|^p dx \right)^{1/p} dt$$

$$+ \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2} \left(f(x+t) - f(x) \right) \right|^p dx \right)^{1/p} dt$$

$$\leq \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \left\| f\left(\beta_n(\cdot) + t \right) - f(\cdot + t) \right\|_{p,2a} dt +$$

$$+ \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \left\| f(\cdot + t) - f(\cdot) \right\|_{p,2a} dt.$$

Then by (1.4), we obtain

$$\left\| W_n^* f - f \right\|_{p,2a} \leq \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \omega\left(f; L_{p,2a}; \frac{a}{2n} \right) dt + \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \omega\left(f; L_{p,2a}; t \right) dt$$

and from (1.6), we have

$$\begin{aligned} \left\| W_n^* f - f \right\|_{p,2a} &\leq \omega\left(f; L_{p,2a}; \frac{a}{2n} \right) + \omega\left(f; L_{p,2a}; \frac{1}{\sqrt{n}} \right) \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \left(1 + \sqrt{n} t \right) e^{-t^2(n-a)} dt \\ &= \omega\left(f; L_{p,2a}; \frac{a}{2n} \right) + \omega\left(f; L_{p,2a}; \frac{1}{\sqrt{n}} \right) \left(\sqrt{\frac{n}{n-a}} \right) \quad , \quad n > a. \end{aligned}$$

□

Also, the following theorem is obvious from the formula

$$\left(W_n^* f \right)^{(r)}(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f^{(r)}\left(\beta_n(x) + t \right) e^{-nt^2} dt.$$

Theorem 3.2. *If $f^{(r)} \in L_{p,a}(\mathbb{R})$ with fixed $a > 0$ and $r \in \mathbb{N}$, then we have*

$$\left\| W_n^* f^{(r)} - f^{(r)} \right\|_{p,2a} \leq \omega\left(f; L_{p,2a}; \frac{a}{2n}\right) + \omega\left(f; L_{p,2a}; \frac{1}{\sqrt{n}}\right)\left(\sqrt{\frac{n}{n-a}}\right)$$

for $n > a$.

Let ω be a positive continuous function on the whole real axis satisfying the condition

$$\int_{\mathbb{R}} t^{2p} \omega(t)\, dt < \infty,$$

where $p \in [1, \infty)$ is fixed. Let also $L_{p,\omega}(\mathbb{R})$ denote the linear space of measurable, p-absolutely integrable functions on $\mathbb{R}$ with respect to the weight function ω, that is

$$L_{p,\omega}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R};\ \|f\|_{p,\omega} := \left(\int_{\mathbb{R}} |f(t)|^p \omega(t)\, dt \right)^{\frac{1}{p}} < \infty \right\}.$$

In [2], the authors obtained the following weighted Korovkin type approximation theorem for any function $f \in L_{p,\omega}(\mathbb{R})$,

Theorem 3.3. *(see [2]) Let $(L_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of positive linear operators from $L_{p,\omega}(\mathbb{R})$ into $L_{p,\omega}(\mathbb{R})$, satisfying the conditions*

$$\lim_{n\to\infty} \|L_n e_j - e_j\|_{p,\omega} = 0,\ \ j = 0, 1, 2.$$

Then for every $f \in L_{p,\omega}(\mathbb{R})$, we have

$$\lim_{n\to\infty} \|L_n f - f\|_{p,\omega} = 0.$$

Our aim is to study the weighted approximation by the sequence of operators W_n^* in the norm $L_{p,2a}(\mathbb{R})$. We consider a weight commonly used in defining spaces of functions with exponential growth. If we choose $\omega(x) = e^{-2ax^2 p}$, $x \in \mathbb{R}$, we can give the following theorem.

Theorem 3.4. *If $f \in L_{p,a}(\mathbb{R})$, then we have*

$$\lim_{n\to\infty} \|W_n^* f - f\|_{p,2a} = 0.$$

Proof. According to Theorem 3.3, for the proof, it is sufficient to show that the conditions

$$\lim_{n\to\infty} \|W_n^* e_j - e_j\|_{p,2a} = 0, \qquad j = 0, 1, 2 \tag{3.1}$$

are satisfied. Since $W_n^* e_0 = 1$ the first condition of (3.1) is fulfilled for $j = 0$. Considering Lemma 2.2, we have

$$\begin{aligned}
\|W_n^* e_1 - e_1\|_{p,2a} &= \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2} \left[\left(W_n^* e_1 \right)(x) - x \right] \right|^p dx \right)^{1/p} \\
&= \frac{a}{2n} \left(2 \int_0^{\infty} e^{-2ax^2 p} dx \right)^{1/p} .
\end{aligned}$$

Then, we get

$$\|W_n^* e_1 - e_1\|_{p,2a} = \frac{a}{n} 2^{\frac{1}{p}-1} \left(\sqrt{\frac{\pi}{2ap}} \right)^{1/p},$$

and the second condition of (3.1) holds for $j = 1$ as $n \to \infty$. Finally, from Lemma 2.2, we obtain

$$\begin{aligned}
\|W_n^* e_2 - e_2\|_{p,2a} &= \left(\int_{-\infty}^{\infty} \left| e^{-2ax^2 p} \left[\left(W_n^* e_2 \right)(x) - x^2 \right] \right|^p dx \right)^{1/p} \\
&= \left(\int_{-\infty}^{\infty} e^{-2ax^2 p} \left| \frac{a^2}{4n^2} + \frac{1}{2n} - \frac{ax}{n} \right|^p dx \right)^{1/p} .
\end{aligned}$$

From triangle inequality

$$\begin{aligned}
\|W_n^* e_2 - e_2\|_{p,2a} \leq 2^{1+1/p} \left(\frac{a^2}{4n^2} + \frac{1}{2n} \right) \left(\int_0^{\infty} e^{-2ax^2 p} dx \right)^{1/p} + \\
+ 2^{1+1/p} \frac{a}{n} \left(\int_0^{\infty} e^{-2ax^2 p} x^p dx \right)^{1/p},
\end{aligned}$$

and using Lemma 2.1, we get

$$\|W_n^* e_2 - e_2\|_{p,2a} = \left(\frac{a^2}{4n^2} + \frac{1}{2n} \right) 2^{1+1/p} \left(\sqrt{\frac{\pi}{2pa}} \right)^{1/p} + 2^p \frac{a}{n} \frac{1}{2^{\frac{p+1}{2p}}} \frac{\Gamma\left(\frac{p+1}{2}\right)^{1/p}}{(ap)^{\frac{p+1}{2p}}}$$

and the third condition of (3.1) holds for $j = 2$ as $n \to \infty$. Thus, the proof is completed. □

Here, we give a pointwise convergence result at the points called as *generalized weighted p-Lebesgue point* which is consistent with exponential weighted space $L_{p,a}(\mathbb{R})$.

Theorem 3.5. *If x is a generalized weighted p-Lebesgue point of the function $f \in L_{p,a}(\mathbb{R})$, i.e.; for $x \in \mathbb{R}$ the condition*

$$\lim_{h\to 0}\left(\frac{1}{h}\int_0^h \left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right|^p dt\right)^{\frac{1}{p}} = 0, \tag{3.2}$$

holds, where β_n and α are given by (1.2), then we have

$$\lim_{n\to\infty} W_n^*(f;x) = f(x). \tag{3.3}$$

Proof. We observe that

$$\begin{aligned} W_n^*(f;x) &= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f(\beta_n(x)+t)\,e^{-nt^2}dt, \quad x\in(-\infty,\infty),\ n\in\mathbb{N} \\ &= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{0} f(\beta_n(x)+t)\,e^{-nt^2}dt + \sqrt{\frac{n}{\pi}}\int_{0}^{\infty} f(\beta_n(x)+t)\,e^{-nt^2}dt \\ &= \sqrt{\frac{n}{\pi}}\int_{0}^{\infty} \left[f(\beta_n(x)+t)+f(\beta_n(x)-t)\right]e^{-nt^2}dt. \end{aligned}$$

Hence by the fact $\sqrt{\frac{n}{\pi}}\int\limits_{-\infty}^{\infty} e^{-nt^2}dt = 1$, we get

$$\begin{aligned} W_n^*(f;x)-f(x) &= \sqrt{\frac{n}{\pi}}\int_{0}^{\infty}\left[f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)\right]e^{-nt^2}dt \\ &= \sqrt{\frac{n}{\pi}}\int_{0}^{\infty}\left(\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right)e^{-t^2(n-2a)}dt. \end{aligned}$$

Since $f \in L_{p,a}(-\infty,\infty)$, $1 \le p < \infty$ and if $\frac{1}{p}+\frac{1}{p'} = 1$, then by Hölder's inequality

$$\begin{aligned} |W_n^*(f;x)-f(x)| \le &\int_{0}^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right| \\ &\times\left(\sqrt{\frac{n}{\pi}}e^{-t^2(n-2a)}\right)^{\frac{1}{p}}\left(\sqrt{\frac{n}{\pi}}e^{-t^2(n-2a)}\right)^{\frac{1}{p'}}dt \end{aligned}$$

$$\le \left(\sqrt{\frac{n}{\pi}}\int_{0}^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right|^p e^{-t^2(n-2a)}dt\right)^{\frac{1}{p}} \tag{3.4}$$

$$\times\left(\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}e^{-t^2(n-2a)}dt\right)^{\frac{1}{p'}}.$$

Since the integral at the last row is convergent for all $n > 2a$, we have

$$|W_n^*(f;x)-f(x)|^p \leq$$
$$\sqrt{\frac{n}{\pi}}\int_0^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2\alpha t^2}}\right|^p e^{-t^2(n-2a)}dt.$$

Let

$$F(t) := \int_0^t \left|\frac{f(\beta_n(x)+\xi)+f(\beta_n(x)-\xi)-2f(x)}{e^{2a\xi^2}}\right|^p d\xi. \tag{3.5}$$

Then

$$dF(t) = \left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p dt.$$

Suppose that x is a generalized p-Lebesgue point of the function f. According to conditions (3.2) and (3.5), we shall write

$$\lim_{h\to 0}\frac{F(h)}{h}=0.$$

In this case, for every $\varepsilon > 0$ there exist a $\delta > 0$ such that when

$$F(h) \leq \frac{\varepsilon}{B}h \tag{3.6}$$

for all $h \leq \delta$. Let

$$B = \left(\frac{\delta e^{-\delta^2(n-2a)}2\sqrt{(n-2a)}+1}{2\sqrt{(n-2a)}}\right)\sqrt{n}. \tag{3.7}$$

We can split the right-hand side of the last inequality into two parts:

$$\begin{aligned}
|W_n^*(f;x)-f(x)|^p &\leq \sqrt{\frac{n}{\pi}}\int_0^{\delta}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p e^{-t^2(n-2a)}dt \\
&\quad+\sqrt{\frac{n}{\pi}}\int_{\delta}^{\infty}\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p e^{-t^2(n-2a)}dt \\
&= I_1+I_2.
\end{aligned}$$

To complete the proof, we have to show that

$$\lim_{n\to\infty} I_1 = \lim_{n\to\infty} I_2 = 0.$$

We consider I_1. Using integration by parts and (3.6), we find that

$$\begin{aligned}
I_1 &= \sqrt{\frac{n}{\pi}}\int_0^\delta e^{-t^2(n-2a)}dF(t) \\
&= \sqrt{\frac{n}{\pi}}\; e^{-t^2(n-2a)}F(t)\Big|_0^\delta + 2\sqrt{\frac{n}{\pi}}\int_0^\delta tF(t)(n-2a)e^{-t^2(n-2a)}dt \\
&\leq \sqrt{\frac{n}{\pi}}e^{-\delta^2(n-2a)}F(\delta) + 2\sqrt{\frac{n}{\pi}}(n-2a)\int_0^\delta tF(t)e^{-t^2(n-2a)}dt \\
&= \sqrt{\frac{n}{\pi}}e^{-\delta^2(n-2a)}F(\delta) + 2\sqrt{\frac{n}{\pi}}\frac{\varepsilon}{B}(n-2a)\int_0^\delta t^2e^{-t^2(n-2a)}dt \\
&\leq \sqrt{\frac{n}{\pi}}e^{-\delta^2(n-2a)}\frac{\varepsilon}{B}\delta + 2\sqrt{\frac{n}{\pi}}\frac{\varepsilon}{B}(n-2a)\frac{\Gamma(\frac{3}{2})}{2(n-2a)^{\frac{3}{2}}} \qquad n-2a>0 \\
&= \sqrt{n}e^{-\delta^2(n-2a)}\frac{\varepsilon}{B}\delta + \frac{\sqrt{n}}{2}\frac{\varepsilon}{B}\frac{1}{\sqrt{(n-2a)}} \\
&\leq \frac{\varepsilon}{B}\sqrt{n}\left(\frac{2\delta e^{-\delta^2(n-2a)}\sqrt{(n-2a)}+1}{2\sqrt{(n-2a)}}\right).
\end{aligned}$$

Using (3.7), we have, for all $\varepsilon > 0$,

$$I_1 < \varepsilon.$$

For I_2, we can easily see that

$$\begin{aligned}
&\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p \\
\leq\;& 2^p\left(\left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)}{e^{2at^2}}\right|^p + 2^p\left|\frac{f(x)}{e^{2\alpha t^2}}\right|^p\right) \\
=\;& 2^{2p}\left(\left|\frac{f(\beta_n(x)+t)}{e^{2at^2}}\right|^p + \left|\frac{f(\beta_n(x)-t)}{e^{2at^2}}\right|^p + \left|\frac{f(x)}{e^{2at^2}}\right|^p\right).
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
&\sqrt{\frac{n}{\pi}}\int_\delta^\infty \left|\frac{f(\beta_n(x)+t)+f(\beta_n(x)-t)-2f(x)}{e^{2at^2}}\right|^p e^{-2t^2(2n-a)}dt \\
\leq\;& \sqrt{\frac{n}{\pi}}e^{-\delta^2(n+\alpha)}2^{2p}\left(\int_\delta^\infty\left|\frac{f(\beta_n(x)+t)}{e^{2at^2}}\right|^p dt + \int_\delta^\infty\left|\frac{f(\beta_n(x)-t)}{e^{2at^2}}\right|^p dt\right)
\end{aligned}$$

$$+\sqrt{\frac{n}{\pi}}2^{2p}\left|f(x)\right|^{p}\int\limits_{\delta}^{\infty}e^{-t^{2}(n+2a(p-1))}dt.$$

Making use of the substitutions

$$\beta_{n}\left(x\right)+t=u,\ \beta_{n}\left(x\right)-t=w \text{ and } v=t^{2}\left(n+2a(p-1)\right), \tag{3.8}$$

to the above integrals, respectively, we get

$$\int\limits_{-\infty}^{\infty}\left|\frac{f\left(u\right)}{e^{2a(u+\frac{a}{2n}-x)^{2}}}\right|^{p}du\leq\int\limits_{-\infty}^{\infty}\frac{\left|f\left(u\right)\right|^{p}}{e^{\alpha u^{2}p+2\alpha(\frac{a}{2n}-x)^{2}p}}du=\left\|f\right\|_{p,a}^{p}e^{-2ap\left(\frac{a}{2n}-x\right)^{2}}\leq\left\|f\right\|_{p,a}^{p} \tag{3.9}$$

from (3.8) and (3.9), we can write the following inequality.

$$\begin{aligned}\left|W_{n}^{*}\left(f;x\right)-f(x)\right|^{p} &\leq \sqrt{\frac{n}{\pi}}e^{-\delta^{2}(2n-a)}2^{2p+1}\left\|f\right\|_{p,a}^{p}\\ &+\sqrt{\frac{n}{\pi}}\frac{2^{2p-1}\left|f(x)\right|^{p}}{\left(n+2a(p-1)\right)}\int\limits_{\delta^{2}n+2a(p-1)}^{\infty}\frac{1}{\sqrt{\frac{v}{n+2a(p-1)}}}e^{-v}dv.\end{aligned}$$

We get that

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi}}e^{-t^{2}(2n-a)}=0 \quad\text{and}\quad \lim_{n\to\infty}\int\limits_{\delta^{2}(n+2a(p-1))}^{\infty}\frac{1}{\sqrt{\frac{v}{n+2a(p-1)}}}e^{-v}dv=0. \tag{3.10}$$

If we take the limit of both sides of the last inequality, we find

$$\lim_{n\to\infty}I_{2}=0$$

by (3.10). Therefore, for a large n, we obtain

$$\left|W_{n}^{*}\left(f;x\right)-f(x)\right|<\varepsilon$$

and the proof is completed. □

REFERENCES

1. Ditzian, Z., Totik, V., Moduli of Smoothness, Springer-Verlag, New-York, 1987.
2. Gadjiev A. D., Aral A., Weighted Lp-approximation with positive linear operators on unbounded sets, Appl. Math. Lett., 20 (10), (2007), 1046–1051
3. Mohapatra, R.N., Rodriguez, R.S. :On the rate of convergence of singular integrals for Hölders continuous functions. Math. Nachr. 149, (1990), 117-124
4. Butzer, P.L., Nessel, R.J. Fourier Analysis and Approximation, I,II; Birkhauser, Basel and Academic Press, New York, (1971)

5. Anastassiou, G. A.,Aral, A., On Gauss-Weierstrass type integral operators. Demonstratio Mathematica 43(4), (2010), 841–849
6. Deeba, E., Mohapatra, R.N., Rodriguez, R. S., On the degree of approximation of some singular integrals, Rend. Mat. Appl. (7), (1988), 345-355
7. Aydın, D., Aral A., Some Approximation properties of Complex q-Gauss-Weierstrass type integral operators in the unit disk, Annals of Oradea Universty-Fascicola Mathematica,(2013)Tom XX (1) 155-168
8. Agratini, O. Aral, A., Deniz E., On two classes of approximation processes of integral type. Positivity. 3(21), (2017), 1189-1199
9. Rempulska, L. and Tomczak, K., On some properties of the Picard operators, Arch. Math.(Brno), 45, (2009), 125-135
10. T. Acar, A. Aral, D. Cardenas-Morales, P. Garrancho, Szasz-Mirakyan type operators which fix exponentials, Results in Mathematics, 72 (3), 2017, 1393–140.
11. A. Aral, D.Cardenas-Morales and P. Garrancho, Bernstein-type operators that reproduce exponential functions,J. of Math. Ineq., V: 12, No:3 (2018), 861–872
12. T. Acar, A. Aral, H. Gonska, On Szász-Mirakyan operators preserving e^{2ax}, $a > 0$, Mediterr. J. Math. 14(1), (2017). https://doi.org/10.1007/s00009-016-0804-7
13. V. Gupta, A. Aral, A note on Szász-Mirakyan-Kantorovich type operators preserving e^{-x}, Positivity, 22(2), (2018), 415–423, 2018.
14. G. Krech, I. Krech, On Some Bivariate Gauss-Weierstrass Operators, Constr. Math. Anal., 2 (2) (2019), 57-63.
15. M. Bodur, O. G. Yılmaz, A. Aral, Approximation by Baskakov-Szász-Stancu Operators Preserving Exponential Functions, Constr. Math. Anal., 1 (1) (2018), 1-8.
16. T. Acar, M. C. Montano, P. Garrancho, V. Leonessa, On Bernstein-Chlodovsky operators preserving e^{-2x}, Bulletin of the Belgian Mathematical Society–Simon Stevin, 26(5) (2019), 681-698
17. F. Ozsarac, T. Acar, Reconstruction of Baskakov operators preserving some exponential functions, Mathematical Methods in the Applied Sciences, 42 (16), 2019, 5124-5132.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 101-117
https://doi.org/10.22190/FUMI200322009T
Original Scientific Paper

ON THE EMBEDDING OF GROUPS AND DESIGNS IN A DIFFERENCE BLOCK DESIGN

Maryam Tale Masouleh[1], Ali Iranmanesh[1] and Henk Koppelaar[2]

[1]**Faculty of Mathematical Sciences, Department of Mathematics, Tarbiat Modares University, P. O. Box 14115-137, Tehran**
[2] **Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Computer Science, Delft University of Technology, Delft, Netherlands**

Abstract. A *difference BIBD* is a balanced incomplete block design on a group, which is constructed by transferring a regular perfect difference system by a subgroup of its point set. There is an obvious bijection between these BIBDs and some copies of their point sets as two sets. In this paper, we investigate the algebraic structure of these block designs by defining a group-isomorphism between them and their point sets. It has been done by defining some relations between the independent-graphs of difference BIBDs and some Cayley graphs of their point sets. It has been shown that some Cayley graphs are embedded in the independent-graph of difference BIBDs as a spanning subgraphs. In order to find these relations, we find out a configuration ordering on these BIBDs, also we have also obtained some results about the classification of these BIBDs. This paper deals with difference BIBDs with even numbers of points.
Keywords. Balanced incomplete block design, sub-design, independent graph, Cayley graph, dihedral groups, configuration ordering.

1. Introduction

Let G be a finite group of order ν ($|G| = \nu$) and k, λ be two integers, where k is less than ν. A $t-(\nu, k, \lambda)$-*balanced incomplete block design* is an ordered pair (G, β) such that β is a family of k-subsets of G, named *blocks*, and every t elements of G do appear in exactly λ blocks. For simplicity of notation, we write (ν, k, λ)-BIBD (and some times (ν, k, λ)-block design) instead of $2-(\nu, k, \lambda)$-balanced incomplete

Received March 22, 2020; accepted June 25, 2020.
Corresponding Author: Ali Iranmanesh, Faculty of Mathematical Sciences, Department of Mathematics, Tarbiat Modares University, P. O. Box 14115-137, Tehran, Iran | E-mail: iranmanesh@modarest.ac.ir
2010 *Mathematics Subject Classification.* Primary 05B05, 05C45 ; Secondary 05C51, 05B30.

block design. it will be called a *trivial block design*, when $k = \nu$. Suppose that D is a (ν, k, λ)-BIBD. A *sub-design* $D' : (\nu, k, \lambda')$-BIBD of D is such that every block of D' is a block of D and this is denoted by $D' \leq D$. Two block designs are isomorphic if there exists a bijection between the point sets such that blocks are mapped onto blocks. The embedding of a (family of) block designs into others are studied in [7, 11, 12, 14, 22, 23].Also there are some papers about the embedding of some block designs into some other mathematical objects like graphs [16], groups [1], surfaces or some applied mathematical concepts like (security of) coding, the mutually orthogonal [24], fast name retrieval in databases (named hashing) used for example in airports [5] or in social media [8]. A simple graph Γ is an ordered pair $(V(\Gamma), E(\Gamma))$ consisting of a set $V(\Gamma)$ of vertices and a set $E(\Gamma)$, disjoint from $V(\Gamma)$, of edges, together with an incidence function $\rho\Gamma$ that associates with each edge of Γ an unordered pair of vertices of Γ. A path P_n is a simple graph with n vertices whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are non adjacent otherwise. A cycle C_n is a P_n such that the first and the last vertices are adjacent.

The aim of this paper is to bring together two areas in which a family of BIBDs have the same structure of groups. The first area is some of the block designs, whereas some graphs depend on them. The second area is the structure of groups as graphs.

Let $B_1, B_2, \ldots, B_c$ be k-subsets of G. For a finite group G, the *difference* of two elements of the group, say x and y, is defined as xy^{-1}. Let $\Delta\beta$ denote the list of all possible differences between two blocks of β; $\Delta\beta = \{xy^{-1} | x \in B, y \in B', B, B' \in \beta\}$. Let $\mathcal{S} = \{B_1, B_2, \ldots, B_c\}$ be a subset of β. If every element of G does appear exactly λ times in $\Delta\mathcal{S}$, then $\mathcal{S}$ is called a (ν, k, λ)-*regular perfect difference system.* This naming is in agreement with [21]. In the notation of [17], every element of this list is called an *initial block* and we will follow this notation. To shorten notation, we continue to write (ν, k, λ)-d-system, for a (ν, k, λ)-regular perfect difference system and only d-system if there is no confusion. When a d-system has only one initial block, this block is well-known as a *difference set.* For a treatment of a more general case we refer the reader to [3, 19]. The methods of constructing a d-system has been noted by many researchers. The best general reference and the classical work here is [6, 17].

Let θ and y be two elements of the group G, the *transference* of y by θ is equal to θy and is denoted by y^θ. The presentation $B = [y_1 \quad y_2 \quad \ldots \quad y_k]$ of a block B is used instead of $B = \{y_1, y_2, \ldots, y_k\}$ to avoid confusion with the set notation and to mention that a block is different from a usual subset of G. Assume that A is a subset of G, maybe equals to G. A *transference* of block B by a set A, is the set $B^A := \{B^\theta | \theta \in A\}$, where $B^\theta = [{y_1}^\theta \quad {y_2}^\theta \quad \ldots \quad {y_k}^\theta]$. The transferring of a (ν, k, λ)-difference system by its point set is a well known way to construct a BIBD, which is called a (ν, k, λ)-*difference block design* or a (ν, k, λ)-*d BIBD* or briefly *d-BIBD*, when it will cause no confusion. Also we can do this transferring by a subgroup of G. It is easy to see and it is also well known that there is a bijection between a d-BIBD and some copies of G as its point set (or some copies of one of its subgroups as a set, which is done the transferring of the d-system). The question, which arises here is: "Is this bijection a group-isomorphism?" In other words, does a d-BIBD have an algebraic structure as G (or its subgroup) or some copies of it? Or this bijection is only a one-to-one function? Our view point sheds some new light on classification of d-BIBDs, finding the existence of some d-BIBDs and have a regular creatures, which are in math. In this paper, we investigate this problem and we can see that d-BIBDs have the same algebraic properties as their point set (Corollary 3.3). Also, as another result of proving the Lemma 3.1 and Lemma 3.2, we have the ordering of these designs, which can be applied to some groups. In fact, the d-BIBD inherited the algebraic structure of the point set. The corollary gains interest if we realize that it works for a bigger family of BIBDs. So we can see the extension of this method in Section 4. In the end, our theorems provide a natural and intrinsic characterization of these BIBDs (Theorem 4.1). These results can be applied to all d-BIBDs, as some corollaries, which are omitted in this paper. We can see a near view of these results about the automorphism of d-BIBDs in [19].

2. Notation and Preliminaries

Let $D := (G, \beta)$ be a (ν, k, λ)-BIBD, b be the size of β and r be the number of blocks with one point of G appearing in them. It is well known that

$$b = \frac{\vartheta(\vartheta - 1)\lambda}{k(k-1)}, \tag{2.1}$$

$$r = \frac{(\vartheta - 1)\lambda}{k - 1}. \tag{2.2}$$

Let $B = [y_1 \quad y_2 \quad \dots \quad y_k]$ be a block in β. As it was said in the introduction, a *difference list* of a difference system on G is the list

$$\Delta B := \{Y_{i,j} := y_i {y_j}^{-1} | 1 \leq i, j \leq k\}.$$

We want to use the *transferring of difference system* for building a BIBD on a non-Abelian group. According to this method, we can use this method for non-Abelian groups by fixing the direction of the group action from the left (or right) (For example, for a set $B = \{x, y, z\} \subseteq G$, for every $\theta \in G$; $\theta B = \{\theta x, \theta y, \theta z\}$) as is done in [15]. This method will be denoted by *LTDS* (Left Transferring Difference System). The right and the left action have the same results up to isomorphism by a simple isomorphism function. We follow the LTDS on a non-Abelian group. From now on, all block designs are built by the LTDS unless it is mentioned.

The following theorem is useful about the structure of subgroups of D_{2n}, where D_{2n} is dihedral group of order $2n$, i.e., $D_{2n} = \langle a, b | a^n = b^2 = 1;\ bab = a^{-1} \rangle$.

Theorem 2.1. [9, Theorem 2.3] *If N is any proper normal subgroup of D_{2n}, then $\frac{D_{2n}}{N}$ is a dihedral group.*

Theorem 2.2. [9, Theorem 3.1] *Every subgroup of D_{2n} is cyclic or dihedral. A complete listing of the subgroups is as follows:*

1. *$\langle a^d \rangle$, where $d|n$, with index $2d$.*
2. *$\langle a^d, a^i b \rangle$, where $d|n$ and $0 \le i \le d-1$, with index d.*

Every subgroup of D_{2n} occurs exactly once in this listing.

By Theorem 2.2 and [9], for every two dihedral groups, D_{2n} and D_{2m}, either one is a subgroup of the other or both of them are subgroups of D_{2w}, where $w = lcm(2n, 2m)$. Assume that Γ and Υ are two graphs with vertex sets $V(\Gamma)$ and $V(\Upsilon)$, respectively. The *adjoint* of Γ and Υ, denoted by $\Gamma \vee \Upsilon$, is a graph with vertex set $V(\Gamma) \cup V(\Upsilon)$ and edge set $E(\Gamma) \cup E(\Upsilon)$. The *Cartesian product* of Γ and Υ denoted by $(\Gamma \odot \Upsilon)$ is a graph such that its vertex set is the Cartesian product of $V(\Gamma)$ and $V(\Upsilon)$ and any two vertices (u, u') and (v, v') are adjacent in $\Gamma \odot \Upsilon$, if and only if either $u = v$ and u' is adjacent with v' in Υ, or $u' = v'$ and u is adjacent with v in Γ. Let G be a group and S be a self-inverse and unit-free subset of it. The *Cayley graph $Cay(G, S)$* is a graph with vertex set G such that two vertices x and y are connected by an edge if and only if $xy^{-1} \in S$. It's well known that $Cay(G, S)$ is connected if and only if S is a generator of G, see [13].

Theorem 2.3. [4] *Let $C_1 = Cay(G, S_1)$ and $C_2 = Cay(H, S_2)$ be two Cayley graphs on groups G and H, respectively. Then the Cartesian product $C_1 \odot C_2$ is the Cayley graph $C = Cay(G \times H, S)$, where $S = \{(x, 1), (1, y) | x \in S_1; y \in S_2\}$ and $G \times H$ is the direct product of the groups G and H.*

Let s be an integer and d_1, d_2 be two integers less than s. A Toeplitz graph $T_s\langle d_1, d_2\rangle$ is a graph with $\{1, 2, \ldots, s\}$ as its vertex set and two integers f and f' from $\{1, 2, \ldots, s\}$ are adjacent if and only if $|f - f'| \in \{d_1, d_2\}$.

Theorem 2.4. [18, Theorem 2] *$T_n\langle d_1, d_2\rangle$ decomposes into exactly $gcd(d_1, d_2)$ connected and isomorphic components.*

Remark 2.1. Now let G be a cyclic group of order n as $\langle a \rangle$. Also d_1 and d_2 are two integers less than n. A Toeplitz graph $T_G\langle a^{d_1}, a^{d_2}\rangle$ is a graph with G as its vertex set and two elements f and f' of G are adjacent if and only if $ff'^{-1} \in \{a^{d_1}, a^{d_2}\}$ or $f^{-1}f' \in \{a^{d_1}, a^{d_2}\}$. As it is mentioned in [20], this graph is equal to the Cayley graph $Cay(G, \{a^{d_1}, a^{d_2}\})$. Note that there can be only one parameter for Toeplitz graphs.

3. The even difference block designs

One of the objectives in this section is to investigate difference block designs with point sets of even size. Furthermore, for doing this, we need some relations of these block designs, which can be found by algebraic properties of the point sets. In this section, we illustrate construction of a block design on dihedral groups by the LTDS method.

We are going to construct a block design $(2n, k, \lambda)$-BIBD on the point set D_{2n}, by the LTDS method. Let $\langle a \rangle = \{1 = a^0, a, a^2, \ldots, a^{n-1}\}$ be a cyclic group of order n.

Assume that there is a d-system with c blocks $B_1, B_2, \ldots, B_c$ for a pair $\{k, \lambda\}$. Transfer these blocks by $\langle a \rangle$ from the left. There is a $(2n, k, \lambda)$-BIBD by the LTDS. From now on, all block designs are supposed that are constructed by the LTDS on a d-system with c blocks. The set B_i^H will be called a family of blocks related to B_i, where B_i is an initial block, H is a subgroup of G and $1 \leq i \leq c$.

Example 3.1. The $(10, 3, 2)$- block design on D_{10} with d-system:

$$B_1 = [b \;\; ab \;\; a^2b], \quad B_2 = [a \;\; a^4 \;\; b], \quad B_3 = [a^2 \;\; a^3 \;\; b],$$

$$B_4 = [a \;\; a^4 \;\; a^2b], \quad B_5 = [a^2 \;\; a^3 \;\; b], \quad B_6 = [1 \;\; b \;\; a^2b];$$

is a set of blocks as follow:

$$\begin{array}{llllll}
[b \;\; ab \;\; a^2b] & \to & [ab \;\; a^2b \;\; a^3b] & [a^2b \;\; a^3b \;\; a^4b] & [a^3b \;\; a^4b \;\; b] & [a^4b \;\; b \;\; ab] \\
[a \;\; a^4 \;\; b] & \to & [a^2 \;\; 1 \;\; ab] & [a^3 \;\; a \;\; a^2b] & [a^4 \;\; a^2 \;\; a^3b] & [1 \;\; a^3 \;\; a^4b] \\
[a^2 \;\; a^3 \;\; b] & \to & [a^3 \;\; a^4 \;\; ab] & [a^4 \;\; 1 \;\; a^2b] & [1 \;\; a \;\; a^3b] & [a \;\; a^2 \;\; a^4b] \\
[a \;\; a^4 \;\; a^2b] & \to & [a^2 \;\; 1 \;\; a^3b] & [a^3 \;\; a \;\; a^4b] & [a^4 \;\; a^2 \;\; b] & [1 \;\; a^3 \;\; ab] \\
[a^2 \;\; a^3 \;\; a^2b] & \to & [a^3 \;\; a^4 \;\; a^3b] & [a^4 \;\; 1 \;\; a^4b] & [1 \;\; a \;\; b] & [a \;\; a^2 \;\; ab] \\
[1 \;\; b \;\; a^2b] & \to & [a \;\; ab \;\; a^3b] & [a^2 \;\; a^2b \;\; a^4b] & [a^3 \;\; a^3b \;\; b] & [a^4 \;\; a^4b \;\; ab].
\end{array}$$

It's clear that if we multiply b $(b \in D_{2n})$ into families of blocks, then we have these families with the new names again. So by multiplying b into initial blocks and transferring them, again we have a block design with the same parameters but different initial blocks. On the other hand, we know that the union of two block designs $D : (\nu, k, \lambda_1)$ and $D' : (\nu, k, \lambda_2)$ is a block design $\bar{D} : (\nu, k, \lambda_1 + \lambda_2)$. By the above notations, if the initial blocks of block design D are transferred by D_{2n}, which is the union of $\langle a \rangle$ and $\langle a \rangle b$, then there is a block design $\bar{D}$ with the same point set and the same block sizes but with a different parameter λ. By these, we can conclude the following remarks:

Remark 3.1. The difference block design $D_1 : (2n, k, \lambda)$ is isomorphic to a subdesign of d-BIBD $D_2 : (2n, k, 2\lambda)$.

A difference block design on even points will be called the even block design.

Remark 3.2. Assume that $B = [y_1 \;\; y_2 \;\; \ldots \;\; y_k]$ and $B' = [y'_1 \;\; y'_2 \;\; \ldots \;\; y'_k]$ are two arbitrary blocks of an even block design. Suppose that they do not have any common elements. By a simple calculation, we can see that B^a and B'^a are disjoint, too.

3.1. Even difference block design as a finite group

In this section, we will provide some lemmas that we need for proving the main result. Also we can present the structure of even d-BIBDs by these theorems. The initial blocks are denoted by $B_1, B_2, \ldots, B_c$. In this case, for any initial block, the order of the points is arbitrary and fixed. Throughout this section, consider $G = D_{2n}$ as the point set and $D = (D_{2n}, \beta)$ is a (ν, k, λ)-block design with $n \geq 5$. By Remark 3.1, apply the LTDS method with transferring of d-system by D_{2n}, unless it is mentioned and also λ is an even number.

Definition 3.1. Assume that A and B are two sets such that $B \subseteq P(A)$, where $P(A)$ is the power set of A. The *independent graph* of B is a graph with vertex set B and two vertices are connected by an edge if and only if they are disjoint. The independent graph is denoted by $IG(A, B)$.

Base on design theory, the independent graph of a BIBD is a graph whose vertices are the blocks of this BIBD and two blocks are adjacent if and only if they are disjoint as two sets. The independent graph has its blocks as the vertices.

Lemma 3.1. *Let B_i be an initial block of design $D = (D_{2n}, \beta)$, where $k < \frac{n}{3}$. Then the independent graph $IG(D_{2n}, {B_i}^{D_{2n}})$ is Hamiltonian.*

Proof. It is sufficient to show that $IG(D_{2n}, {B_i}^{D_{2n}})$ has a spanning sub-graph, which is Hamiltonian. Let $d_s := q$ $(1 \leq s \leq \kappa = \binom{2}{k})$ if and only if there are x and y in B_i with $xy^{-1} = a^q b^\varepsilon$ $(\varepsilon = 0, 1)$. And define $\Delta\bar{B}_i := \{d_1, d_2, \dots, d_\kappa\}$. Note that $\Delta\bar{B}_i$ is a set of integers modulo n. The proof will be divided into two cases:

Case 1. Assume that there is an integer such that is relatively prime to n and belongs to $\{1, 2, \dots, n\} \setminus \Delta B_i$. Let d be the minimum integer with this property. So by our assumption on d, it is clear that the set $B = \{B_{i,h} := {B_i}^{a^{hd}}\}_{h=1,2,\dots,n}$ is a sequence of blocks such that $B_{i,h}$ and $B_{i,h+1}$ are independent for every h and $h+1$ modulo n. This means that $IG(D_{2n}, B)$ is a cycle C_n, by Remark 2.1. The lemma is proved for ${B_i}^H$ with $H = \langle a \rangle$. Subtitute $\langle a \rangle$ and B_i for $b\langle a \rangle$ and B_i^a, respectively. There will be another cycle, by applying some of the above methods. By Remark 3.2, it is sufficient to find a block from ${B_i}^{\langle a \rangle b}$, which is disjoint from B_i. If B_i is a subset of $\langle a \rangle$ or $\langle a \rangle b$, then ${B_i}^b$ is disjoint from B_i and so we find a common edge. Assume that B_i contains the elements $x \in \langle a \rangle$ and $y \in \langle a \rangle b$. Without loss of generality, suppose that $x = by$ such that $x = a^s b^\epsilon$ and $y = a^j b^\delta$, where $s, j \in \{1, 2, \dots, n\}$, $\epsilon, \delta \in \{0, 1\}$ and $\epsilon \neq \delta$. By multiplying b by y for obtaining $x = by$, we have $a^{s+j-n} b^{\epsilon-\delta-1} = 1$. It shows that $s + j \equiv 0 \pmod{n}$. If there is another element of B_i, which is equal to x after transferring by b, then it has to be equal to y, which is impossible. Now we can consider another case, ${B_i}^{ab}$. Let z be another element of B_i. By the above discussion, we know that $ax \neq ay$. If $ax = abz$ $(ax = y)$, $x = bz$ but $z \neq y$. If there are pairwise blocks with common element(s), due to the condition $k < \frac{n}{3}$, there is at least one block, which is disjoint from B_i, say B'. There is $j \in \{1, 2, \dots, n\}$ such that $B' = {B_i}^{a^j b}$ (Fig.1). By Remark 3.2, the set $\{\{{B_i}^{a^s}, {B_i}^{a^{s+j}b}\} | 1 \leq s \leq n\}$ is a subset of edges of graph $IG(D_{2n}, {B_i}^{D_{2n}})$. Clearly, this graph is Hamiltonian and the proof is complete in this case.

Case 2. Assume that there is not any element of the set $\{1, 2, \dots, n\} \setminus \Delta\bar{B}_i$, where these integers are relatively prime to n. Let d be the minimum element of the set $\{1, 2, \dots, n\} \setminus \Delta\bar{B}_i$ and $g := gcd(n, d)$. Remark 2.1 and our choosing integer d, show that the transferring of B by a^d, forms a new graph. This graph includes g isomorphic components, which every component is a spanning sub-graph of $IG(D_{2n}, {B_i}^{D_{2n}})$. Hence, the sequence $\{B_{i,h} := {B_i}^{a^{hd}} \mid h = 1, 2, \dots, n\}$ forms g isomorphic components by this transferring. Every component contains a spanning

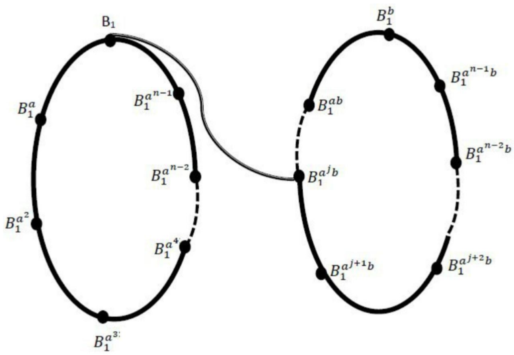


FIG. 3.1: The existence of an edge between two components of $IG(D_{2n}, B_i^{D_{2n}})$.

cycle. So it is sufficient to find some edges between these components to obtain a Hamiltonian cycle. Let d' be an integer in $\{1, 2, \ldots, n\} \setminus \{\Delta\bar{B}_i \cup \{d\}\}$, so there exists another element d' due to the condition $k < \frac{n}{3}$. Moreover, there is an edge $\{B_i, {B_i}^{a^{d'}}\}$ because d' is not in $\Delta\bar{B}$. On the other hand, every component is built by transferring the blocks by a^d and for every vertex of $IG(D_{2n}, {B_i}^{D_{2n}})$, this result can be applied by Remark 3.2, and hence the components are adjacent with more than one edge. Since every component contains a spanning cycle, so does $IG(D_{2n}, {B_i}^{D_{2n}})$. $\square$

As a consequence of the above Lemma, we have the following corollary.

Corollary 3.1. *Let B_i be an initial block of design $D = (D_{2n}, \beta)$ and d be as in the proof of Lemma 3.1 with, $k < \frac{n}{3}$:*

1. *The graph $IG(D_{2n}, {B_i}^{D_{2n}})$ has a spanning sub-graph, which is isomorphic to $Cay(D_{2n}, \{a^d, a^{-d}, b\})$;*

2. *There is a bijection between ${B_i}^{D_{2n}}$ and D_{2n}.*

Lemma 3.2. *For a difference block design D, if $k < \frac{n}{3}$, there is a cycle with c vertices as a sub-graph of $IG(D_{2n}, \beta)$ containing one and only one vertex from every ${B_i}^{D_{2n}}$, for $1 \leq i \leq c$.*

Proof. Assume that there is a d-system of size c equal to $\{B_1, B_2, \ldots, B_c\}$ with c different difference lists $\Delta B_1, \Delta B_2, \ldots, \Delta B_c$, respectively. At first, we have to find an edge, which has B_1 as one of its vertices. Choose an element B_i ($2 \leq i \leq c$) among all initial blocks. We continue the proof into two cases: either $B_1 \cap B_i = \phi$ or $B_1 \cap B_i \neq \phi$. In the first case, there is an edge between these blocks and we should go to the next step. But by the second case, we need to find a block from ${B_i}^{D_{2n}}$, which is disjoint from B_i. For doing this, we need defining the new sets. Put $\Delta B_{1,i} = \Delta B_1 \cup \Delta B_i$ and $\Delta\bar{B}_{1,i} := \{d \mid d \in \{1, 2, \ldots, n\}; a^d \in \Delta B_{1,i} \textit{ or } a^i b \in \Delta B_{1,i}\}$. Choose an integer belongs to $\{1, 2, \ldots, n\} \setminus \Delta\bar{B}_{1,i}$ and denote it by $d_{1,i}$.

Let $B'_i := (B_i)^{a^{d_{1,i}}}$. We can see that either $B_1 \cap B_2 = \phi$ or $B_1 \cap B'_i \neq \phi$. In the first case, we have the edge that we are looking for that. Suppose that there is at least one common element between B_1 and B'_i. Then there are two cases:

1. There is no integer $d \in \Delta\bar{B}_1$ such that $d|d_{1,i}$ or $d_{1,i}|d$. So by transferring B_i by $a^{d_{1,i}}$, we obtain the intended block.

2. There exists an integer $d \in \Delta\bar{B}_1$ such that $d|d_{1,i}$ (or $d_{1,i}|d$). In this case, by transferring B_i by $a^{d_{1,i}}$ repeatedly, we will achieve some blocks, which are not disjoint from B_1. But $|B_1| = k$ and $k < \frac{n}{3}$, so we can choose another d from $\Delta\bar{B}_{1,i}$. It allows us to look for this block (a block disjoint from B_1) between the vertices of other components in $IG(D_{2n}, {B_i}^{\langle a\rangle})$. Note that by transferring B_i by $a^{d_{1,i}}$ for $gcd(d_{1,i}, n)$ times, if there is not a disjoint block from B_1, then we should choose another d from $\Delta\bar{B}_{1,i}$. Now we find an edge between two blocks from two different families ${B_1}^{D_{2n}}$ and ${B_i}^{D_{2n}}$. Continue process for the gained block (B'_i) and the remaining initial blocks. Go on, until finding a path with c vertices. Denote the last block of this path by B'_j ($P_c : B_1, B'_i, \ldots, B'_q, B'_j$). The proof falls into two cases: either $B_1 \cap B'_j = \phi$ or $B_1 \cap B'_j \neq \phi$. In the first case, there is a cycle, which we were searching for. In other case, we need a restoration during the last step, when we are finding the last block (B'_j). We want to find a transference of block B_j such that is disjoint from B_1 and B'_q. Consider the value $d_{1,q,j}$ instead of $d_{q,j}$ to continue. This integer exists because of the condition $k < \frac{n}{3}$. By the above discussion, the transferring B_j by the $(d_{1,q,j})^{th}$ power of a, gives us a block, which is disjoint from these three blocks. The cycle is completed now and so is the proof (Figure 2). $\square$

Remark 3.2 guarantees that the transferring of the cycle, which is obtained in Lemma 3.2, forms n isomorphic cycles in graph $IG(D_{2n}, \beta)$. Let S_i be the set $\{a^d, a^{-d}, b\}$, which is a subset of D_{2n} as it is mentioned in the proof of Lemma 3.1, for every B_i ($1 \le i \le c$). Also, we have seen that the graph $Cay(D_{2n}, S_i)$, for $1 \le i \le c$, is isomorphic to a spanning sub-graph of $IG(D_{2n}, \beta)$ for a difference block design (D_{2n}, β), with $k < \frac{n}{3}$. The following corollary is an immediate consequence of the previous lemma.

Corollary 3.2. *The Cartesian product of $\bigvee Cay(D_{2n}, S_i)$ and C_c, is isomorphic to a spanning sub-graph of $IG(D_{2n}, \beta)$, with $k < \frac{n}{3}$.*

To embed an even d-BIBD into a finite group we use difference systems and graph theory to find a relation between an even difference block design and a finite group. By definition, it is clear that the Cartesian product of two Hamiltonian graphs is a Hamiltonian graph. So by Corollary 3.2, we have a relation between groups and difference block designs. By the method of constructing the even d-BIBDs, the size of d-system, c is equal to $\frac{b}{2n}$. So by Corollary 3.1, Lemma 3.2 and Theorem 2.3, we have the following theorem.

Theorem 3.1. *Assume that there is a difference block design $D : (2n, k, \lambda)$-BIBD, with $k < \frac{n}{3}$ and an even number λ, such that its d-system is transferred by D_{2n}.*

Let $c = \frac{(2n-1)\lambda}{k(k-1)}$. Then there is a bijection between the group $D_{2n} \times Z_c$ and the block design D.

The bijection of previous theorem is:

$$\begin{array}{rccc} \Phi: & \beta & \longrightarrow & D_{2n} \times Z_c \\ & B & \longmapsto & (a^j b^\epsilon, i) \end{array}$$

where B is the transference of B_i by $a^j b^\epsilon$ ($j \in \{1, 2, \ldots, n\}$). There is not any action defined between the elements of β. The action between any pair of blocks of β, say B and B', will be defined as follows:

$$B \odot B' = \Phi(\Phi^{-1}(B)\ \Phi^{-1}(B')).$$

Corollary 3.3. *Under the above assumption, $f := \Phi^{-1}$ is an isomorphism from $D_{2n} \times Z_c$ onto β.*

We know that symmetric BIBDs are isomorphic to their point sets. Also, by definition, in these symmetric difference block designs $c = 1$ and this isomorphism is comparable with the above corollary.

Now suppose that λ is an odd number and equal to $2L + 1$, where L is a positive integer. We saw in the proof of Lemma 3.1 that there are two cycles due to transference by $\langle a \rangle$ and $\langle a \rangle b$, which are connected by n edges. For odd λ's, the transference can only be the group $\langle a \rangle$ ($\langle a \rangle b$), by the LTDS method. By Theorem 2.4, $IG(D_{2n}, {B_i}^{\langle a \rangle})$ ($IG(D_{2n}, {B_i}^{\langle a \rangle b})$) is a cycle or a union of isomorphic cycles. We now apply the above argument again, with D_{2n} or d copies of D_g replaced by C_{2n} or d copies of C_g, respectively, where d is as mentioned in proof of Lemma 3.1.

Corollary 3.4. *Suppose that there is a difference block design $D : (2n, k, \lambda)$-BIBD with $k < \frac{n}{3}$, where λ is an even or an odd integer such that its d-system is transferred by $\langle a \rangle$ of order n. Let $c = \frac{(2n-1)\lambda}{k(k-1)}$. Then there is an isomorphism between the group $\langle a \rangle \times Z_c$ and the difference block design D.*

Remark 3.3. Assume that there is a difference block design $D : (2n, k, \lambda)$-BIBD with $k < \frac{n}{3}$ such that the d-system is transferred by H, where H is a subgroup of D_{2n}. Let $c = \frac{2n(2n-1)\lambda}{k(k-1)|H|}$. Then there is a bijection between group $H \times Z_c$ and the difference block design D.

Remark 3.4. Let H be a subgroup of D_{2n} of order n, which is not cyclic and suppose that $4|n$. By Theorem 2.1, $\frac{D_{2n}}{\langle b \rangle} \cong \langle a \rangle$ and $\langle b \rangle \cong Z_2$. Hence doing the LTDS method with transferring by H, is a block design isomorphic to a difference block design on the point set D_{2n}, which is achieved by transferring via D_n.

3.2. Configurations and the ordering

In this section, we will see that the Hamiltonian cycle, which is achieved in the proofs of Lemmas 3.1 and 3.2, gives us the ordering of even designs.

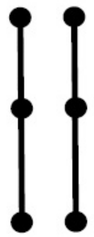


FIG. 3.2: Configuration A_1.

Definition 3.2. A *ν-configuration* is a collection of b lines (or subsets) having the property that every t-element subset is contained in at most λ lines. And a *(ν, l)-configuration* is a configuration of p points on l lines.

Definition 3.3. Let $D = (V, \beta)$ be a BIBD with $|\beta| = b$. Let C be configuration on l blocks. A *Configuration ordering* (or a C-ordering) for D is a list of the blocks of D, $B_0, B_1, \ldots, B_{b-1}$, with the property that $B_i, B_{i+1}, \ldots, B_{i+l-1} \equiv C$ holds for all $0 \leq i \leq b - l$. If $B_i, B_{i+1}, \ldots, B_{I+l-1} \equiv C$ holds for all $0 \leq i \leq b - 1$, with subscript addition performed modulo b, then the ordering is called *$C - cyclic$*.

Let A_1 be a configuration as is mentioned in [10](part 4.1.1) and is shown in Figure 1.

Theorem 3.2. [2] *The existence of an A_1-cyclic ordering is equivalent to existence of a Hamiltonian cyclic in the independent graph of the block design.*

It is known that the Cartesian product of two Hamiltonian graphs is Hamiltonian. So by these two lemmas (3.1 and 3.2) and Theorems 3.2, we have the following theorem:

Theorem 3.3. *Every even block design with $k < \frac{\nu}{6}$, has the A_1-ordering.*

Remark 3.5. According to the Corollary 3.3, there is the A_1-ordering on the point sets of even block designs with $k < \frac{\nu}{6}$, by Theorem 3.3.

3.3. Even difference sub-designs and subgroups

We have seen that some copies of dihedral groups are embedded into an even d-BIBD with $k < \frac{n}{3}$. To find the relation between these BIBDs, we construct d-BIBDs by the LTDS method and find an isomorphism between these BIBDs and dihedral groups. Throughout this section, suppose that $k < \frac{n}{3}$ and H is a subgroup of group G.

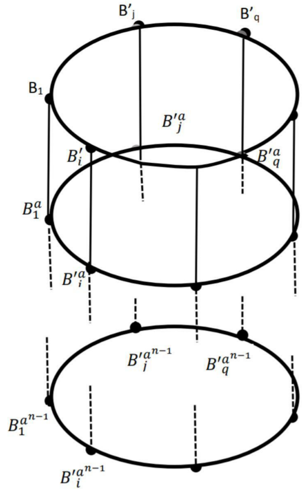


FIG. 3.3: The regular relations between cycles.

Definition 3.4. Let G be a group and H be a subgroup of G. A design-group $DH(G,k,\lambda)$ is an even difference block design D with parameters k and λ with point set G Also $\beta = \{{B_i}^H | 1 \leq i \leq c\}$, where H is a subgroup of G. A design-group $DK(G,k,\lambda')$ is a subdesign-group of $DH(G,k,\lambda)$ if K is a subgroup of H and even difference block design $(G,k,\lambda')^K$ is a sub-design of $(G,k,\lambda)^H$ as an even difference block design.

We can look at a design-group, both as a group and as a block design, simultaneously. Let $D_1 := DD_{2n}(D_{2n},k,\lambda_1)$, $D_2 := DH(D_{2n},k,\lambda_2)$ and B be an arbitrary block of D_2 and $H = \langle a \rangle$. Note that the order of a is n. Therefore, the blocks of D_2 are blocks of D_1 too. Because of the transference of its d-system by $\langle a \rangle$, it only has $c\ O(a)(= cn)$ blocks in D_1. For every $y_j \in B_i$, $\{{y_j}^H\}$ is a cosset of H and is closed under the product of H in itself. On the other hand, because of the same c's, for D_1 and D_2, the number of initial blocks is equal to c, but the number of θ's is different. By calculating the number c for both of them, if $k > 3$, then according to equation (1), we obtain

$$c = \frac{2n(2n-1)\lambda_1}{k(k-1)2n}, \qquad c = \frac{2n(2n-1)\lambda_2}{k(k-1)n}.$$

So we have $\frac{\lambda_1}{2} = \lambda_2$ and we can have Remark 3.1 as bellow by the new view of even block designs:

Proposition 3.1. *Let $DD_{2n}(D_{2n},k,\lambda)$ be an even design-group with even λ and $H = \langle a \rangle$, then $DH(D_{2n},k,\frac{\lambda}{2})$ is a subdesign-group of $DD_{2n}(D_{2n},k,\lambda)$.*

In fact, by changing the set of H (the set, which the transferring of the LTDS method is done by that), we can control the possible values of λ. The last proposition shows us that for the point set D_{2n}, if the d-system is transferred by the maximal subgroup $\langle a\rangle$, then we have a block design again. Now, what can we say about other subgroups of a point set?

Theorem 3.4. *Let D_1 and D_2 be two difference block designs as $DH_1(D_{2n}, k, \lambda_1)$ and $DH_2(D_{2n}, k, \lambda_2)$, respectively with the same difference systems such that $\lambda_2 \leq \lambda_1$ and k is odd.The even design-group D_1 can be embedded into the even design-group D_2 if and only if H_1 can be embedded into H_2 as a subgroup.*

Proof. By Theorem 2.2, it is sufficient to check two types of subgroups; $\langle a^d\rangle$ and $\langle a^d, a^i b\rangle$ ($d|n$ and $i = 0, 1, \ldots, d-1$). Let D be a design-group $DD_{2n}(D_{2n}, k, \lambda)$ such that $\lambda_1 \leq \lambda$. We will prove the theorem for difference designs D and D_1,so it is true for every pair of designs. Let B be an initial block of D. The proof naturally falls into the following two cases:

Case 1: Let $\theta \in \langle a^d\rangle$. This case is illustrated before in Proposition 3.1, for $d = 1$. The block B of D_1 is an element of $B^{\langle a\rangle}$. So $B^{\langle a^d\rangle}$ is a subset of $B^{D_{2n}}$. Every difference block design has an independent graph. By the proof of Lemma 3.1, to obtain a new difference block design, the deletion of blocks from β has to follow a rule: Choose all ${B_i}^{a^d}$ and ${B_i}^{a^d b}$ such that B_i is an initial block and there is a subgroup of G of order d. So if D_1 is a sub-design of D, then it means that H_1 can be embedded into D_{2n} as a subgroup. Conversely, if $H_1 < D_{2n}$, then by the above facts about the choosing of blocks and existence of the independent graph, $D_1 < D$ as a difference block design.

Case 2: Let $\theta \in \langle a^d, a^i b\rangle$. For every initial block B, the set $B^{\langle a^d\rangle}$ is a subset of $B^{\langle a\rangle}$, which was studied in Case 1, and all remaining blocks in this set are in common with $B^{\langle a\rangle b}$. Again by Lemma 3.1, $B^{\langle a\rangle b}$ and $B^{\langle a\rangle}$ are disjoint, for odd k only. All blocks of $B^{\langle a\rangle b}$ are in common between two design-groups and others were studied in the previous case. The proof of the converse is similar to the above procedure. □

4. The second method and main result

Sometimes there is no way to construct a BIBD on a point set with a special parameter. In these situations, there is a well known method: Assume that we want to have a BIBD on $(n+1)$ points. By this method, at first we construct a BIBD on group G with n elements and then we add an extra point, named ∞ such that $x\infty = \infty$, for every $x \in G$. So for having a difference BIBD, it is sufficient to add block(s) to the difference system such that the achieved system has the parameters, which we want. In this paper, this method will be called the LTDSE method (the LTDS method with an Extra element). There are numbers of general examples of the LTDSE method in [17]. From now on, a d-BIBD which is based on the LTDS method will be called a type 1 BIBD and a d-BIBD, which is based on the LTDSE method will be called a type 2 BIBD. Note that the initial blocks of a type 2 BIBD are partitioned into two classes: a class of blocks including the extra variety (∞), which is of the size r ($\{B'_1, B'_2, \ldots, B'_q\}$) and the other class includes the remaining blocks ($\{B_1, B_2, \ldots, B_p\}$). All facts about the transference of d-system $\{B_1, B_2, \ldots, B_p\}$ in Lemma 3.1 and Corollary 3.1, are applied for the first class of blocks of type 2 BIBDs.

Every type 1 BIBD on even points is comparable to a type 1 BIBD on D_{2n}, and every type 2 BIBD on odd points is comparable to a type 2 BIBD on even points with an extra variety. For having this generalization on the point sets, at the end replace the integers by elements of dihedral group as below:

$$2i-1(1 \le i \le n) \longmapsto a^i \qquad 2i(1 \le i \le n) \longmapsto a^i b.$$

Assume that D is a d-BIBD. A natural question that arises here is "How can we find out whether 4it is a type 1 BIBD or a type 2 BIBD?" From the previous sections, we know that replacing a subgroup of G, say H in the LTDSE, leads us to a d-BIBD on $mn+1$ points with λ less than the λ of the d-BIBD, which is constructed by transferring its difference system by G. We have seen the relations between even d-BIBDs with point sets of similar size. But what can we say about two even d-BIBDs with point sets of different sizes? What can we say about d-BIBDs in a general case?

Suppose that we can not see and detect the extra variety ∞ in blocks. Let $\Delta\beta$ be the list of all differences from the blocks of β and $|\Delta\beta| = \psi\lambda + R$ (where $0 \le R < \lambda$). It is clear that $\frac{R}{\binom{k}{2}}$ is the number of initial blocks, which are including the adjoined variety. First, we need some notations and observations:

Fact 1. By a review of the concept of the LTDS and the LTDSE methods, it is easy to see that ψ is the size of the set, which transfers a d-system to obtain the d-BIBD and $\psi|b$. So we obtain the size of H, a subgroup of the point set that transfers d-system.

Fact 2. By doing the LTDS method and the LTDSE method, any element of $\Delta\beta$ is repeated symmetrically $\psi\lambda$ times for type 1 BIBD, and at least $(\psi\lambda+1)$ times for type 2 BIBD, By the definition of differences. Therefore the d-BIBD is of type 1 if $R = 0$ and otherwise, it is a type 2 BIBD.

Fact 3. Assume that B_i is a block of β and ${B_i}^{(H)} := \{B \in \beta | \Delta B_i = \Delta B\}$. If $R = 0$, then $|{B_i}^{(H)}| = \psi$; otherwise, delete the blocks of ${B_i}^{(H)}$, which includes the element ν.

Fact 4. Assume that $d, d' \in \Delta B_i$. To find the elements, which transfer B_i to build ${B_i}^{(H)}$, define $B(d,d') := \{(x,y,z,)|xy^{-1} = d, yz^{-1} = d'; d \ne d'\}$. Then H is equal to the set $\{x'x^{-1}, y'y^{-1}, z'z^{-1} | \ \forall (x,y,z), (x',y',z') \in B_i\}$. From now on, all block designs are even difference block designs or a type 2 BIBD on an odd number of points.

According to Fact 4, we have a subgroup of the point set H, for every block design D, which is the transferrer for that d-BIBD; have a look at Theorem 2.2.

Theorem 4.1. *Let $D_1 : (\nu_1, k, \lambda_1)$ and $D_2 : (\nu_2, k, \lambda_2)$ be two difference block designs based on the LTDS or the LTDSE method. Let $c_i = \lfloor \frac{|\delta\beta_i|}{\lambda_i} \rfloor$ be the number of initial block(s) of the difference system of D_i and p_i, q_i are the numbers of initial blocks without ∞ and including ∞ in D_i, respectively as introduced in beginning of this section, for $i = 1, 2$.*

1. *Suppose that $|\Delta\beta_1| - \lfloor \frac{|\Delta\beta_1|}{\lambda_1} \rfloor = 0$ or $|\Delta\beta_2| - \lfloor \frac{|\Delta\beta_2|}{\lambda_2} \rfloor = 0$ and $\lfloor \frac{|\Delta\beta_2|}{\lambda_2} \rfloor$ is a multplied of $\lfloor \frac{|\Delta\beta_1|}{\lambda_1} \rfloor$. If $c_{p_1} \leq c_{p_2}$, then $D_1 \leq D_2$, otherwise $D_1 \leq \lceil \frac{c_2}{c_1} \rceil$ copies of D_2.*
2. *Suppose that $|\Delta\beta_i| - \lfloor \frac{|\Delta\beta_i|}{\lambda_i} \rfloor \neq 0$ and $\lfloor \frac{|\Delta\beta_2|}{\lambda_2} \rfloor$ is a multplied of $\lfloor \frac{|\Delta\beta_1|}{\lambda_1} \rfloor$. If $c_{p_1} \leq c_{p_2}$ and $c_{q_1} \leq c_{q_2}$, then $D_1 \leq D_2$, otherwise $D_1 \leq \lceil \frac{c_2}{c_1} \rceil$ copies of D_2.*

Proof. Let $|\Delta\beta_i| = \psi_i\lambda_i + R_i$ such that $0 \leq R_i < \lambda_i$ and $c_i = \frac{b_i}{\psi_i}$, where $i \in \{1, 2\}$. We follow the proof in two steps. We first compare an arbitrary family of each design according to their H_i ($B_i^{(H_i)}$, for a block B_i of a design) and then compare the number of these families, which is equal to c_i. The numbers ν and μ can be odd or even. As we have mentioned above, we illustrate difference block designs on even points for type 1 BIBDs and on odd points for type 2 BIBDs. Let $\nu = 2n + j$ and $\mu = 2m + j'$, where $j, j' \in \{0, 1\}$. The numbers $2n$ and $2m$, where $m, n \in \mathcal{N}$, are called the even parts of the ν and μ, respectively. Assume that $w = gcd(2n, 2m)$ and $\nu \leq \mu$. So D_{2n} and D_{2m} are subgroups of D_w. If $\nu = \mu$ and the designs are type 1 BIBDs, then Theorem 3.4 lipids the relation between them. If $R_i \neq 0$ (for difference block design on $2L + 1$ points for a natural number L), then the initial blocks are of two types, as we have seen: the blocks including the adjoined variety $\beta_q := \{B'_1, \ldots, B'_q\}$ and the blocks without the adjoined variety $\beta_p := \{B_1, \ldots, B_p\}$. For that q initial blocks, suppose that the new set of blocks is achieved by deleting the adjoined variety: $\beta''_q := \{B''_1, \ldots, B''_q\}$ such that $B''_j = B'_j \setminus \{\infty\}$, where $j \in \{1, \ldots, q\}$. Note that maybe we do not see the adjoined variety as ∞, so there is an element out of the D_{2L} in blocks and also it is equal to $2L + 1$, where $2L$ is the even part of the size of the point set. Also from now on, during the proof, i belongs to $\{1, 2\}$. According to the proof of Lemma 3.1, we are allowed to delete the blocks of every family by jumping d steps, as it is mentioned there, unless the remaining blocks can not form a difference block design. This rule is the base of our work.

Step 1: Choose a block from each of the difference designs. Note, when we choose a block from β_i, it is an initial block (the representative of its family). In the next choice, we can choose another initial block by choosing every block, which doesn't have the same list of differences to the first block. As the first choice, assume that B_i is an initial block of D_i. By Theorem 3.4, if D_1 and D_2 are type 1 BIBDs, then $B_1 \subseteq B_2$ if and only if $H_1 \leq H_2$. If m, n are odd and ψ_1, ψ_2 have the same parity, then both of the H_i's are either cyclic groups or dihedral groups, by Theorem 2.2. Therefore, $B_1 \subseteq B_2$ if and only if $\psi_1 \leq \psi_2$. We need to know the relations between H_1 and H_2. There is an algorithm to know that H_i is a cyclic subgroup or a dihedral subgroup of D_ω. By the above Facts, H_i is known. There are some cases:

1. $H \leq \langle a \rangle$ such that $O(a) = w$;

2. $H \leq \langle a \rangle b$ such that $O(a) = w$ and $Ord(b) = 2$.

3. Otherwise, H is a dihedral group ($|H|$ is even) such that $\frac{|H_i|}{2} | \; Ord(a)$.

Note that $\langle a \rangle b$ and $\langle a \rangle$ are isomorphic sets and one is a cosset of another in D_w.

1'. If both of the H_i's are of the form of case 1 (or 2), then $B_1^{(H)} < B_2^{(H)}$ if and only if $\psi_1 | \psi_2$.

2'. If H_1 is of the form of case 1 or 2 and H_2 is of the form 3, then $B_1^{(H)} < B_2^{(H)}$ if and only if $\psi_1 | \; \frac{\psi_2}{2}$ (it yields $H_1 \leq H_2$).

3'. If both of the H_1 and H_2 are of the form 3, then $B_1^{(H)} < B_2^{(H)}$ if and only if $\psi_1 | \psi_2$.

Now we are ready to illustrate the second step:

Step 2: Assume that ${B_1}^{(H_1)}$ is a subgroup of ${B_2}^{(H_2)}$ up to isomorphism. If $c_1 \leq c_2$, then $D_1 \leq D_2$. Otherwise, D_1 needs $\lceil \frac{c_2}{c_1} \rceil$ copies of D_2 to be embedded into, because $D_1 \cong H_1 \times Z_{c_1}$ and $D_2 \cong H_2 \times Z_{c_2}$. Also if $H_1 \leq H_2$, then it has to have the case $Z_{c_1} \leq Z_{c_2}$, base on group theory (and $Z_{c_1} \subseteq Z_{c_2}$). But due to the condition $c_2 < c_1$, it is impossible unless there are $\lceil \frac{c_2}{c_1} \rceil$ copies of D_2.

We study the case $R_1 = R_2 = 0$ in steps 1 and 2. If $R_1 = 0$ and $R_2 \neq 0$, we apply the same manner as steps 1 and 2 on β_{p_1} of D_1 and β_{p_2} of D_2. If $R_1, R_2 \neq 0$, then we apply the same arguments in steps 1 and 2 on β_{p_1} of D_1 and β_{P_2} of D_2 after that we do that on β''_{q_1} of D_1 and β''_{q_2} of D_2. $\square$

5. Conclusion

As the first step, we find the algebraic structure of difference block design on a dihedral group (D_{2n} with arbitrary n) as its point set. We did that by finding the relation between its independent-graph and the Cayley graph of dihedral group with S, which is introduce during the proof of lemmas (Corollary 3.4). Due to our method to find this relation, we can prove that there exists a configuration ordering on these difference block designs (Theorem 3.3). Though the method of finding initial blocks can be from [17] or a lot of other references. At the end, we investigate these block designs, when they have an extra point. We present a method to recognize, when they are with an extra point or with the odd points. And finally, we can classify the big family of difference block designs, by presenting the Theorem 4.1.

Acknowledgement

The authors would like to thank the anonymous referee for careful reading and the helpful comments improving this paper. This research of the first and the second authors was funded by a grant from INSF, Iran National Science Foundation [96004816].

REFERENCES

1. A. Caggegi and G. Falcone and M. Pavone: *On the additivity of block designs.* J. Algebraic Combin., **45**(1) (2017), 271-294.
2. C. J. Colbourn and A. Rosa: *Triple systems.* OxfordMathematical Monographs, The Clarendon Press Oxford University Press: New York, USA, 1999.
3. C. J. Colbourn and J. H. Dinitz: *Handbook of combinatorial designs, Second edition.* Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC: Boca Raton, FL, 2007.
4. D. Cvetkovic and P.Rowlinson and S. Simic: *An introduction to the theory of graph spectra.* London Mathematical Society Student Texts, **75**; Cambridge University Press: Cambridge, Britannia, 2010.
5. D. R. Stinson and R. Wei and L. Zhu: *New constructions for perfect Hash families and related structures using combinatorial designs and codes.* J. Combin. Des., **8**(2000), No. 3, 189-200.
6. G. Ge and M. Ying and S. Xianwei: *Perfect difference families, perfect difference matrices, and related combinatorial structures.* J. Combin. Des., **18** (20180), No. 6, 65-78.
7. G. Lo Faro and A. Tripodi: *Embeddings of λ fold Kite systems, $\lambda \geq 2$.* Australas.J. Combin., **36** (2006), 143-150.
8. J. Beel and B. Gipp: *Google Scholar's Ranking Algorithm: An Introductory Overview.* Proceedings of the 12th International Conference on Scientometrics and Informetrics. **1**(2009), 230-241.
9. K. Conrad: *Dihedral Groups.* Retrieved from: *http://www.math.uconn.edu/kconrad/blurbs/grouptheory/dihedral.pdf.*
10. M. Dewar and B. Stevens: *Ordering block designs, Gray code, Universal cycles and configuration orderings.* CMS Books in Mathematics/Ouvrages de Mathematiques de la SMC; Springer: New York, USA, 2012.
11. M. Gionfriddo and S. Milici: *Minimum embedding of a K_3-design into a balanced incomplete block design of index $\lambda \geq 2$* Ars Combin., **112** (2013), 279-291.
12. M. Meszka and A. Rosa: *Embedding Steiner systems into Steiner systems* $S(2; 4, \nu)$. Discrete Math., **274** (2004), No. 1-3 199-212.
13. PH. Zieschang: *Cayley graphs of finite groups.* J. Algebra, **118** (1988), No. 2, 447-454.
14. P. J. Dukes and T. Feng and A. C. H. Ling: *A finite embedding theorem for partial Steiner 3-designs.* Finite Fields Appl., **33** (2015), 29-36.

15. R. A. Bailey: *Association Schemes Designed experiments Algebra and Combinatorics. Cambridge University Pres: New York, USA, 2004.*

16. *R. A Baily and P. J. Cameron:* Using graphs to find the best block designs, Topics in structural graph theory. Encyclopedia Math. Appl., Cambridge Univ. Press, Cambridge, **1**47 (2013).

17. R. C. Bose: *On the construction of balanced incomplete block designs.* Ann. Eugenics, **9** (1939), 353-399.

18. R. Euler: *Coloring planar Toeplitz graphs and the stable set polytope.* 6th International Conference on Graph Theory, Discrete Math., **276** (2004), No. 1-3, 183-200.

19. R. Fuji-Hara and Y. Miao and S. Shinohara: *Complete sets of disjoint difference families and their applications. Experimental design and related combinatorics.* J. Statist. Plann. Inference, **106** (2002), No. 1-2, 87-103.

20. S. Bau: *A generalization of the concept of Toeplitz graphs.* Mong. Math. J. , **15** (2011), 54-61.

21. S. Chisaki: *difference systems of sets and their construction.* Graduate School of Science and Technology; Tokyo University of Science: Tokyo, Japan, 2016.

22. S. Kucukcifci and G. Quattrocchi and B. R. Smith and E. S. Yzici: *On regular embedding of H-designs into G-designs.* Util. Math, **92** (2013), 97-127.

23. S. Milici: *Maximum embedding of an* $E_2(\nu\omega, 4, 1)$ *into an* $H(\nu, 4, \lambda)$. Util. Math., **93** (2014), 383-391.

24. W. Ogataa and K. Kurosawa and D. R. Stinson and H. Saido: *New combinatorial designs and their applications to authentication codes and secret sharing Schemes.* Discrete Math., **279** (2004), No. 1-3, 383-405.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 119-123
https://doi.org/10.22190/FUMI200322010H

Original Scientific Paper

ON THE AUTOMORPHISMS GROUP OF FINITE POWER GRAPHS

Asma Hamzeh

Property and Casualty (Non-life) Insurance Research Group, Insurance Research Center, Tehran, Iran

Abstract. The power graph of a group G is the graph with vertex set G, having an edge joining x and y whenever one is a power of the other. The purpose of this paper is to study the automorphism groups of the power graphs of infinite groups.
Keywords: Power graphs, infinite groups, automorphism group.

1. Introduction

In this paper, we consider only simple undirected graphs. Let Γ and Δ be two graphs. These graphs are said to be *isomorphic* and write $\Gamma \cong \Delta$, if there exists a bijection Φ from $V(\Gamma)$ to $V(\Delta)$ such that $uv \in E(\Gamma)$ if and only if $\Phi(u)\Phi(v) \in E(\Delta)$. An isomorphism from Γ to itself is called an *automorphism* of Γ. We denote the set of all automorphisms on Γ as $\mathrm{Aut}(\Gamma)$. A (vertex) *coloring* of Γ is a mapping $c : V(\Gamma) \to S$, where S is the set of colors. The vertices assigned to a given color form a color class. If $|S| = k$, we say that c is a k-coloring (often we use $S = \{1, \ldots, k\}$). A coloring for Γ is proper if adjacent vertices have different colors and it is k-colorable if Γ has a proper k-coloring. The *chromatic number* $\chi(\Gamma)$ is the least number k such that Γ is k-colorable. Obviously, $\chi(\Gamma)$ exists as assigning distinct colors to vertices yield a proper $|V(\Gamma)|$-coloring. An optimal coloring of Γ is a $\chi(\Gamma)$-coloring and Γ is called k-chromatic if $\chi(\Gamma) = k$. Finally, Γ is *planar* if it has no subdivision of the $K_{3,3}$ and K_5.

Suppose G and H are two groups. The group G is called a *torsion group* if the order of elements of G is finite. The *free product* $G * H$ of groups G and H is the set of elements of the form $g_1h_1g_2h_2 \ldots g_rh_r$, where $g_i \in G$ and $h_i \in H$, with g_1 and

Received March 22, 2020; accepted June 22, 2020.
Corresponding Author: Asma Hamzeh, Property and Casualty (Non-life) Insurance Research Group, Insurance Research Center, Tehran, Iran | E-mail: hamze2006@yahoo.com
2010 *Mathematics Subject Classification.* Primary 05C25; Secondary 05C50.

h_r possibly equal to the identity elements of G and H. The *semidirect product* of a group G by a group H is a group T containing G and H, where H is normal in T and $G \cap H = \{1\}$ and is denoted $G \rtimes H$. The *symmetric group* S_n of degree n is the group of all permutations on n symbols. The *cyclic group* of order n is denoted by Z_n and for prime number p, the *Prüfer p-group* is denoted by $Z(p^\infty)$.

Graphs associated to algebraic constructions are significant, because they have valuable applications in mathematics and computer science (see, for example, the survey [12] and the monographs [13, 14]). The *power graph* of a group G is the graph with vertex set G, having an edge joining x and y whenever one is a power of the other. The concept of a power graph was first introduced and considered in [11] in the case of groups. Also, they in this paper described the structure of the power graphs of all finite abelian groups. For semigroups, it was first investigated in [10], and then in [8, 9]. Chakrabarty, Ghosh, and Sen in [4] characterized the class of semigroups S for which $\mathcal{P}(S)$ is connected or complete. As a consequence, they proved that $\mathcal{P}(G)$ is connected for any finite group G and $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m. In [3] Cameron and Ghosh proved that non-isomorphic finite groups may have isomorphic power graphs, but about abelian groups with isomorphic power graphs, the groups are also isomorphic. In [5] Doostabadi, Erfanian, and Jafarzadeh obtained some results on the power graph of infinite groups. The power graphs were also investigated in [2, 17, 15]. In [7], the automorphism groups of the power graph in general for finite groups are computed. Also, Feng et al. [6], computed the full automorphism group of the power graph of a finite group. In [1] Abawajy, Kelarev, and Chowdhury gave a survey of all results on the power graphs of groups and semigroups obtained in the literature.

The purpose of this paper is to study the power graphs of infinite groups and their automorphism groups.

2. Main Results

Let p be a fixed prime number. An infinite group T is called a *Tarski Monster group* for p if every nontrivial subgroup (i.e. Every subgroup other than 1 and G itself) has p elements. The group G is necessarily finitely generated. It is generated by every two non-commuting elements. It is simple. The Tarski groups were first constructed by Olshanskii in 1979. Olshanskii showed in fact that there are continuum-many non-isomorphic Tarski Monster groups for each prime $p > 10^{75}$ [18, 19].

Let A be an abelian group. The *generalized dihedral group $Dih(A)$* is the semidirect product $A \rtimes Z_2$, where Z_2 is the cyclic group of order 2, and the generator of Z_2 maps elements of A to their inverses. If A is cyclic, then $Dih(A)$ is called a *dihedral group.* The *finite dihedral group $Dih(Z_n)$* is commonly denoted by D_n or D_{2n}. The *infinite dihedral group $Dih(Z)$* is denoted by D_∞ and is isomorphic to the free product $Z_2 * Z_2$ of two cyclic groups of order 2.

First, we state two theorems for the automorphism group of graphs.

Theorem 2.1. *Let Γ be a graph, then* $\mathrm{Aut}(n\Gamma) = \mathrm{Aut}(\Gamma) \wr S_n$.

Theorem 2.2. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be distinct graphs and*

$$\Gamma = n_1\Gamma_1 \bigcup n_2\Gamma_2 \bigcup \ldots \bigcup n_r\Gamma_r.$$

Then

$$\mathrm{Aut}(\Gamma) = (\mathrm{Aut}(\Gamma_1) \wr S_{n_1}) \times \ldots \times (\mathrm{Aut}(\Gamma_r) \wr S_{n_r}).$$

In the next theorem, we obtain the automorphism group of the power graph of $(Z \times Z \times \ldots \times Z, +)$, where Z is a set of integer numbers.

Theorem 2.3. *The automorphism group of the power graph of $(Z \times Z \times \ldots \times Z, +)$ is isomorphic to*

$$\mathrm{Aut}(\mathcal{P}(Z \times Z \times \ldots \times Z, +)) = (S_P \times S_P \times \ldots \times S_P) \wr S_2.$$

Where P denotes the set of all prime numbers.

Proof. Consider the group Z under addition. Two integers a and b of Z are adjacent in the power graph if and only if $a \mid b$ or $b \mid a$. This graph has three components containing $\{0\}$, Z^+, and Z^-, where Z^+ and Z^- are the set of all positive and negative integers, respectively. On the other hand, any integer n is adjacent to all positive multiples of n and there is no edge connecting a positive and a negative number. The components containing all positive and all negative integers is denoted by H_1 and H_2 and it can be easily seen that $H_1 \cong H_2$. We now calculate the automorphism group of subgraph H_1. By power graph structure of Z, $\mathrm{Aut}(H_1)$ is isomorphic to the automorphism group of the partially ordered set $L = (N, |)$, that N is set of the natural numbers. This group is isomorphic to the symmetric group S_P. Therefore, $\mathrm{Aut}(\mathcal{P}(Z)) = S_P \wr S_2$.

We now apply induction and Theorems 2.1 and 2.2. Since $\mathrm{Aut}(L \times L \times \ldots \times L) \cong S_P \times S_P \times \ldots \times S_P$, so,

$$\mathrm{Aut}(\mathcal{P}(Z \times Z \times \ldots \times Z, +)) = (S_P \times S_P \times \ldots \times S_P) \wr S_2.$$

This completes the proof. $\square$

By [16, Proposition 7], $\mathcal{P}(D_{2n})$ is a union of $\mathcal{P}(Z_n)$ and n copies of K_2 that share the identity element of D_{2n} and by [15, Corollary 2.4], the automorphism group of the power graph D_{2n}, if n is a prime power, then is equal to $S_{n-1} \times S_n$ and otherwise, is equal to $S_n \times \prod_{d|\varphi(n)} S_{\varphi(d)}$ that φ is Euler's totient function. In the next two theorems, we compute power graphs and automorphism groups of these graphs for generalized dihedral group $Dih(A)$ and D_∞.

Theorem 2.4. *For the dihedral groups $Dih(A)$ and D_∞,*

$$\begin{aligned} \mathrm{Aut}(\mathcal{P}(Dih(A))) &= \mathrm{Aut}(\mathcal{P}(A)) \times S_{|A|}, \\ \mathrm{Aut}(\mathcal{P}(D_\infty)) &= (S_P \wr S_2) \times S_L. \end{aligned}$$

where $L = (N, |)$.

Proof. In the generalized dihedral group $Dih(A)$, for an abelian group A, all elements outside A have order two. So, according to the power graph structure, the power graph of $Dih(A)$, is the union of a copy of $\mathcal{P}(A)$ and $|A|$ copies of K_2 that share in identity. So $\mathrm{Aut}(\mathcal{P}(Dih(A))) = \mathrm{Aut}(\mathcal{P}(A)) \times S_{|A|}$. In this case, if A is a finite abelian group, then the structure of the power graph of it described in [11].

Consider the infinite dihedral group $D_\infty =< r, s|s^2 = 1, srs = r^{-1} >$. The power graph of this group is a union of a copy of $\mathcal{P}(Z)$ and infinite copies of K_2 that share in identity. Thus, by Theorem 2.3, $\mathrm{Aut}(\mathcal{P}(D_\infty)) = (S_P \wr S_2) \times S_L$. □

For the sake of completeness, we mention here two important results which are crucial in our investigation of the power graphs of the infinite groups.

Theorem 2.5. *[5] Let G be an infinite group. Then $\mathcal{P}(G)$ is complete if and only if $G \cong Z(p^\infty)$ for some prime p.*

By previous theorem $\mathrm{Aut}(Z(p^\infty)) \cong S_N$.

Theorem 2.6. *[5] Let G be a group. Then $\mathcal{P}(G)$ is planar if and only if G is a torsion group and $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.*

By previous theorem for the Tarski Monster group T, $\mathcal{P}(T)$ is planar.

Theorem 2.7. *For the Tarski Monster group T,* $\mathrm{Aut}(\mathcal{P}(T)) = S_p \wr S_N$. *Also* $\chi(\mathcal{P}((T)) = p$.

Proof. The power graph of the Tarski Monster group T for the prime number p is a union infinite copy of K_p that share in identity. So by Theorem 2.1, the automorphism group of this graph is $\mathrm{Aut}(\mathcal{P}(T)) = S_p \wr S_N$. Also, according to power graph structure of the Tarski Monster group, $\chi(\mathcal{P}((T)) = p$. □

Direct product of graphs $\Gamma_1(V_1, E_1), \dots, \Gamma_n(V_n, E_n)$ is graph $\Gamma(V, E)$, where $V = V_1 \times \dots \times V_n$ and E is the set of all pairs $((a_1, \dots, a_n), (b_1, \dots, b_n))$ such that $(a_1, \dots, a_n) \neq (b_1, \dots, b_n)$ and $(a_i, b_i) \in E_i \cup \triangle(V_i)$ for all $1 \leq i \leq n$[11].

Theorem 2.8. *[11] If group G is a direct product of the p_i-primary components of itself, that p_i's are pairwise distinct primes, then the power graph of G is the direct product of the power graphs of the p_i-primary component.*

Corollary 2.1. *We know that $Q/Z = \bigoplus_{p\in P} Z(p^\infty)$ that $Z(p^\infty)$'s are p-primary components of Q/Z. By theorem previous and Theorem 2.5, the power graph of group Q/Z is a direct product of the power graphs of complete graphs.*

Acknowledgment. The author would like to thank the referee for the valuable suggestions and comments.

REFERENCES

1. J. Abawajy, A. V. Kelarev and M. Chowdhury: *Power graphs: a survey.* Electronic J. Graph Theory and Applications **1** (2013), (2) 125–147.
2. A. R. Ashrafi and B. Soleimani: *Normal edge-transitive and 12-arc-transitive Cayley graphs on non-abelian groups of order $2pq$, $p > q$ are odd primes*. Int. J. Group Theory **5** (2016), 1-8.
3. P. J. Cameron and S. Ghosh: *The power graph of a finite group*. Discrete Math. **311** (2011), 1220–1222.
4. I. Chakrabarty, S. Ghosh and M. K. Sen: *Undirected power graphs of semigroups*. Semigroup Forum **78** (2009), 410–426.
5. A. Doostabadi, A. Erfanian and A. Jafarzadeh: *Some results on the power graphs of finite groups*. Science Asia **41** (2015), 73–78.
6. M. Feng, X. Ma and K. Wang: *The full automorphism group of the power (di)graph of a finite group*. European J. Combin. **52** (2016), 197–206.
7. A. Hamzeh and A. R. Ashrafi: *Automorphism group of supergraphs of the power graph of a finite group*. European J. Combin. **60** (2017), 82–88.
8. A. V. Kelarev and S. J. Quinn: *A combinatorial property and power graphs of semigroups*. Comment. Math. Univ. Carolin. **45** (1) (2004), 1–7.
9. A. V. Kelarev and S. J. Quinn: *Directed graphs and combinatorial properties of semigroups*. J. Algebra **251** (1) (2002), 16–26.
10. A. V. Kelarev, S. J. Quinn and R. Smolíková: *Power graphs and semigroups of matrices*. Bull. Austral. Math. Soc. **63** (2) (2001), 341–344.
11. A. V. Kelarev and S. J. Quinn: *A combinatorial property and power graphs of groups*. Contributions to General Algebra **12** (Vienna, 1999), 229–235, Heyn, Klagenfurt, 2000.
12. A. V. Kelarev, J. Ryan and J. Yearwood: *Cayley graphs as classifiers for data mining: the in uence of asymmetries*. Discrete Math. **309** (2009), 5360-5369.
13. A. V. Kelarev: *Ring constructions and applications*, World Scientific, NJ, 2002.
14. A. V. Kelarev: *Graph algebras and automata*, Marcel Dekker, New York, 2003.
15. Z. Mehranian, A. Gholami and A. R. Ashrafi: *A note on the power graph of a finite group*. Int. J. Group Theory **5** (1) (2016), 1–10.
16. M. Mirzargar, A. R. Ashrafi and M. J. Nadjafi-Arani: *On the power graph of a finite group*. Filomat **26** (2012), 1201–1208.
17. M. Shaker and M. ali Iranmanesh: *On groups with specified quotient power graphs*. Int. J. Group Theory **5** (2016), 49-60.
18. A. Yu. Olshanskii: *An infinite group with subgroups of prime orders*. Math. USSR Izv. **16** (1981), 279-289; translation of Izvestia Akad. Nauk SSSR Ser. Matem. **44** (1980), 309-321.
19. A. Yu. Olshanskii: *Groups of bounded period with subgroups of prime order*. Algebra and Logic **21** (1983), 369-418; translation of Algebra i Logika **21** (1982), 553-618.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 125-141
https://doi.org/10.22190/FUMI200325011A

Original Scientific Paper

APPROXIMATION PROPERTIES OF MODIFIED BASKAKOV GAMMA OPERATORS

Seda Arpaguş and Ali Olgun

Kırıkkale University, Department of Mathematics, 71450 Kırıkkale, Turkey

Abstract. In this paper, we have studied an approximation properties of modified Baskakov-Gamma operator. Using Korovkin type theorem, firste we gave the approximation properties of this operator. Secondly, we computed the rate of convergence of this operator by means of the modulus of continuity and we gave an approximation properties of weighted spaces. Finally, we studied the Voronovskaya type theorem of this operator.

1. Introduction

The Baskakov operators and their connections with different branches of analysis such as convex and numerical analysis have been studied intensively.

In 1957, V.A. Baskakov defined the well _known Baskakov operators as follows[22];

$$B_n\left(f;x\right)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{\left(1+x\right)^{n+k}} \qquad x\geq 0, n\in\mathbb{N}.$$

Later, many authors studied the approximation properties and gave many generalizations of these operators [1] ,[5],[10],[11],[15],[16],[17],[25],[26].Recently İnce İlarslan et al.[12]discussed some approximation properties of (p,q)-Baskakov-Kantoro -vich operators. Some authors studied the approximation properties Szasz type generalization[21].

Received March 25, 2020; accepted June 1, 2020.
Corresponding Author: Ali Olgun, Kırıkkale University, Department of Mathematics, 71450 Kırıkkale, Turkey | E-mail: aliolgun71@gmail.com
2010 *Mathematics Subject Classification.* Primary 41A10; Secondary 41A36, 41A25

In 1998, V. Miheşan cosnructed and studied the convergence properties a generalization of the Baskakov operators as follows[24]:

$$B_n^a(f;x)=\sum_{k=0}^{\infty}e^{-\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right)\qquad x\geq 0,n\in\mathbb{N}, \tag{1.1}$$

where $P_k(x;a)=\sum\limits_{i=0}^{k}\binom{k}{i}(x)_i\,a^{k-i}$.

In [6], Wafi and Khatoon examined the convergence features of the integral type modification of the operators (1.1)

$$V_n^a(f;x)=n\sum_{k=0}^{\infty}e^{\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt\qquad x\geq 0,n\in\mathbb{N}. \tag{1.2}$$

In 2010, Erençin and Başcanbaz-Tunca [2] identified a more general version of these operators, with the help of sequence, and examined the convergence features.

In 2011, Erençin constructed a Durrmeyer type modification of generalized Baskakov operators (1.1) as follows

(1.3)

$$L_n^{\alpha}(f;x)=e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{1}{B(k+1,n)}\int\limits_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}f(t)\,dt\quad;\quad x\geq 0$$

and studied some approximations properties[3].In (2012), Krech and Malejki investigated a modified type this operators[13].

In 2014, Erençin and Büyükdurakoğlu extended the operator (1.2) as

$$K_n(f;x)=e^{-\frac{a_nx}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{b_n}{d_n-c_n}\int\limits_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}}f(t)\,dt\quad;\ x\geq 0,n\in\mathbb{N},$$

which is a more general version of the operators and examined the convergence features in weighted spaces[1].

In 2017, N. Rao and A. Wafi [8] defined as follows

$$L_{n,a}^{\alpha,\beta}(f,x)=e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k+\alpha}{n+\beta}\right)$$

and examined the convergence features of Stancu variant the operator of (1.2) .

In 2015, Goyal and Agrawal examined the convergence features of bivariate generalization of operators L_n^{α} given by (1.3)[15].

Gamma operator is identified as[7]

$$G_n(f,x) = \int_0^\infty \frac{x^{n+1}}{n!} e^{-xy} y^n f\left(\frac{n}{y}\right) dy \qquad x \in (0,\infty), n \in \mathbb{N}.$$

In 2011, L. Rempulska and M. Skorupka extended the modified version of Gamma operator as follows

$$G_{n,p}(f,x) = \int_0^\infty \frac{x^{n+1}}{n!} e^{-xy} y^n F_p\left(x, \frac{n}{y}\right) dy$$

and investigated the approximation properties for differentiable functions in polynomial weighted spaces[14].

Different modification of this operator were examined[19],[20],[21].

In 2014, R. Malejki and E. Wachnicki[18] constructed integral type modification the operators B_n^α given by(1.1)as follows:

$$M_n^{\alpha,a}(f,x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^\infty \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{1}{\Gamma(\alpha+k+1)} \int_0^\infty e^{-ns} (ns)^{\alpha+k} f(s)\, ds$$

and studied approximation properties of the operator. In (2015), E. Pandey and S.P. Mishra investigated a differet type this operators[9].

In 2016, I. Krech and R. Malejki[13] defined a multivariate version of the operators $M_n^{\alpha,a}$.

In this paper, we give a new generalization consisting of the linear combination of Baskakov- Gamma operators.

2. Constructions of the Operators

Let $x \in (0,\infty)$, $n \in \mathbb{N}$, $0 < \alpha < \beta$ and f be defined on the space $C_B(0,\infty)$ of all continuous bounded functions. We define the operator as follows:

$$(2.1) \quad S_{n,a}^{\alpha,\beta}(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^\infty \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^\infty \frac{x^{n+1}}{n!} y^n e^{-xy} f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) dy$$

where $a > 0$ is a constant and

$$P_k(n,a) = \sum_{i=0}^k \binom{k}{i} (n)_i\, a^{k-i}$$

with $(n)_0 = 1, (n)_i = n(n+1)(n+2)...(n+i-1); i \geq 1$denotes Pochammer Symbol. With the help of derivatives, $e^{-\frac{ax}{1+x}} \sum\limits_{k=0}^{\infty} \frac{P_{k+1}(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} = n(1+x) + a$ and

$e^{-\frac{ax}{1+x}} \sum\limits_{k=0}^{\infty} \frac{P_{k+2}(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} = n(n+1)(1+x)^2 + 2an(1+x) + a^2$ can be easily proved.

3. Auxiliary results

Lemma 3.1. *For the operators (2.1), we have*

$$\begin{aligned}
S_{n,a}^{\alpha,\beta}(1;x) &= 1, \\
S_{n,a}^{\alpha,\beta}(t;x) &= \tfrac{nx+\alpha}{n+\beta} \tfrac{ax}{(1+x)(n+\beta)}, \\
S_{n,a}^{\alpha,\beta}(t^2;x) &= \tfrac{n^2(1+n)x^2}{(n+\beta)^2(n-1)} + \tfrac{\left[2an^2(1+x)+\alpha^2 n\right]}{(n+\beta)^2(n-1)} \tfrac{x^2}{(1+x)^2} \\
&\quad \times \tfrac{\left\{\left[n^2+2\alpha n(n-1)\right](1+x)+\left[an+2a\alpha(n-1)\right]\right\}}{(n+\beta)^2(n-1)} \tfrac{x}{1+x} + \tfrac{\alpha^2}{(n+\beta)^2}.
\end{aligned}$$

Proof. Using the operator (2.1), , it follows

$$S_{n,a}^{\alpha,\beta}(1;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} dy$$

If we say $xy = t$ then it follows

$$S_{n,a}^{\alpha,\beta}(1;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} = 1,$$

which proves the first result.

For $f(t) = t$ we have

$$\begin{aligned}
&S_{n,a}^{\alpha,\beta}(t;x) \\
&= e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \tfrac{P_k(n,a)}{k!} \tfrac{x^k}{(1+x)^{n+k}} \tfrac{x^{n+1}}{n!} \tfrac{1}{n+\beta} \left[n \int_0^{\infty} y^n e^{-xy} \tfrac{k}{xy} dy + \alpha \int_0^{\infty} y^n e^{-xy} dy\right] \\
&= \tfrac{1}{n+\beta} \tfrac{x}{1+x} \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \tfrac{P_{k+1}(n,a)}{k!} \tfrac{x^k}{(1+x)^{n+k}}\right) + \tfrac{\alpha}{n+\beta} = \tfrac{nx+\alpha}{n+\beta} + \tfrac{ax}{(1+x)(n+\beta)}.
\end{aligned}$$

For $f(t) = t^2$, it follows

$$S_{n,a}^{\alpha,\beta}(t^2;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right)^2 dy$$

$$\begin{aligned}
&= \frac{e^{-\frac{ax}{1+x}}}{(n+\beta)^2}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^{k+n+1}}{(1+x)^{n+k}n!}\\
&\quad\times\left[n^2\int_0^{\infty}y^n e^{-xy}\frac{k^2}{x^2y^2}dy+2\alpha n\int_0^{\infty}y^n e^{-xy}\frac{k}{xy}dy+\alpha^2\int_0^{\infty}y^n e^{-xy}dy\right]\\
&= \frac{n}{(n-1)(n+\beta)^2}\frac{x^2}{(1+x)^2}\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_{k+2}(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\right)\\
&\quad+\frac{n}{(n-1)(n+\beta)^2}\frac{x}{1+x}\times\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_{k+1}(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\right)\\
&\quad+\frac{2\alpha}{(n+\beta)^2}\frac{x}{1+x}\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_{k+1}(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\right)+\frac{\alpha^2}{(n+\beta)^2}\\
&= \frac{n^2(1+n)x^2}{(n+\beta)^2(n-1)}+\frac{\left[2an^2(1+x)+\alpha^2 n\right]}{(n+\beta)^2(n-1)}\frac{x^2}{(1+x)^2}\\
&\quad+\frac{\left\{\left[n^2+2\alpha n(n-1)\right](1+x)+\left[an+2a\alpha(n-1)\right]\right\}}{(n+\beta)^2(n-1)}\frac{x}{1+x}+\frac{\alpha^2}{(n+\beta)^2},
\end{aligned}$$

which completes the proof.

□

$S_{n,a}^{\alpha,\beta}\left(t^3;x\right)$ and $S_n^a\left(t^4;x\right)$ can be proved in a similarly way that of the proof of Lemma 3.1.

Theorem 3.1. *Let $f\in C_B(0,\infty)$, $x\in(0,\infty)$ and $n\in\mathbb{N}$. Then we have*

$$\lim_{n\to\infty}\left(S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right)=0.$$

Proof. Proof is clear that by Lemma 3.1. □

Lemma 3.2. *For the operators (2.1),*

$$S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)\le M^*\frac{x^2+x+1}{(n+\beta)^2}$$

where $M_i=(n,a,\beta,\alpha)$, $i=1,2,...;\quad M^*=\max(M_i)$.

Proof. From linearity of the operator (2.1) and Lemma3.1, since $\frac{x^s}{(1+x)^l}\leqq x^s$ for all $x\geqq 0,\quad l<s\quad(l,s=1,2,3,4)$, we can write

$$\begin{aligned}
S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) &\le \frac{1}{(n+\beta)^2}\left(\frac{n\left(n+\alpha^2+2an+n^2\right)-(n-1)(n+\beta)(n+2\alpha-\beta)}{n-1}\right)x^2\\
&\quad+\frac{1}{(n+\beta)^2}\left(\frac{(n-2\alpha+2n\alpha)(a+n)-2\alpha(n-1)(n+\beta)}{n-1}\right)x+\frac{\alpha^2}{(n+\beta)^2}\\
&= \frac{x^2}{(n+\beta)^2}M_1+\frac{x}{(n+\beta)^2}M_2+\frac{1}{(n+\beta)^2}M_3\\
&\le M^*\frac{x^2+x+1}{(n+\beta)^2}.
\end{aligned}$$

□

4. Rates of Convergence

We can show the approximation of the operator with the help of the modulus of continuity.

Theorem 4.1. *Let $x \in (0,\infty)$, $n \in \mathbb{N}$ and $f \in C_B$, then we have*

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq M^{**} w\left(f; \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}\right).$$

Proof. By the definition of the operators (2.1) and properties of modulus of continuity, we may write

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq \\
&\leq e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left| f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) - f(x)\right| dy
\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} w\left(f; \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right|\right) dy \\
&= w(f,\delta) + e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \frac{1}{\delta} w(f;\delta) \left(\int_0^{\infty} y^n e^{-xy} \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right| dy\right)
\end{aligned}$$

By applying the Cauchy-Schwarz inequality two times succesively to the right side, we get

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq w(f,\delta)$$

$$\begin{aligned}
&+\frac{1}{\delta} w(f;\delta) \left\{ \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^{\infty} \frac{x^{n+1}}{n!} y^n e^{-xy} \left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right)^2 dy \right)^{\frac{1}{2}} \right. \\
&\left. \times \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_0^{\infty} \frac{x^{n+1}}{n!} y^n e^{-xy} dy \right)^{\frac{1}{2}} \right\} \\
&\leq w(f,\delta) + \frac{1}{\delta} w(f;\delta) \sqrt{M^*} \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}
\end{aligned}$$

If we take $\delta = \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}$, then it follows

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq M^{**} w\left(f; \sqrt{\frac{x^2+x+1}{(n+\beta)^2}}\right).$$

which ends the proof where

$$M^{**} = 1 + \sqrt{M^*}.$$

□

Let $C_B(0,\infty)$ denote the space of real valued continuous and bounded functions on the interval $(0,\infty)$, with the norm

$$\|f\| = \sup_{0\le x<\infty} |f(x)|.$$

For every $\delta > 0$, Peetre's K- functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2(0,\infty)} \{\|f-g\| + \delta \|g''\|\}$$

where

$$C_B^2(0,\infty) = \{g \in C_B(0,\infty) : g', g'' \in C_B(0,\infty)\}.$$

There exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C w_2\left(f;\sqrt{\delta}\right) \tag{4.1}$$

holds where w_2 is the second order modulus of smoothness of f, defined by

$$w(f;\delta) = \sup_{0<h\le\delta} \sup_{0<x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Now, we consider the following $\hat{S}_{n,a}^{\alpha,\beta}(f;x)$ by means of operator $S_{n,a}^{\alpha,\beta}$

$$\hat{S}_{n,a}^{\alpha,\beta}(f;x) = S_{n,a}^{\alpha,\beta}(f;x) - f\left(\frac{ax + (1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) + f(x). \tag{4.2}$$

Then, the following Lemma can be given.

Lemma 4.1. *Let $g \in C_B^2(0,\infty)$. Then we have*

$$\left|\hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x)\right| \le \delta_n(x)\|g''\|$$

where

$$\delta_n(x) = S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) + \left(\frac{ax + (1+x)(\alpha - x\beta)}{(1+x)(n+\beta)}\right)^2.$$

Proof. For the operators $\hat{S}_{n,a}^{\alpha,\beta}(f;x)$, we get

$$\begin{aligned}\hat{S}_{n,a}^{\alpha,\beta}(t-x;x) &= S_{n,a}^{\alpha,\beta}(t-x;x) - \left(\tfrac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)\\ &= S_{n,a}^{\alpha,\beta}(t;x) - xS_{n,a}^{\alpha,\beta}(1;x) - S_{n,a}^{\alpha,\beta}(t;x) + xS_{n,a}^{\alpha,\beta}(1;x) = 0.\end{aligned}$$

□

Let $g \in C_B^2(0,\infty)$ and $x \in (0,\infty)$. By Taylor's formula of g, we may write

$$g(t) - g(x) = (t-x)\, g\prime(x) + \int_x^t (t-u)\, g''(u)\, du \qquad ; \qquad t \in [0,\infty).$$

If we apply the operator $\hat{S}_{n,a}^{\alpha,\beta}$ to this equality, we obtain

$$\hat{S}_{n,a}^{\alpha,\beta}(g(t)-g(x);x) = g'(x)\, \hat{S}_{n,a}^{\alpha,\beta}((t-x);x) + \hat{S}_{n,a}^{\alpha,\beta}\left(\int_x^t (t-x)\, g''(u)\, du; x\right)$$

$$\begin{aligned}
&= \hat{S}_{n,a}^{\alpha,\beta}\left(\int_x^t (t-x)\, g''(u)\, du; x\right) \\
&= S_{n,a}^{\alpha,\beta}\left(\int_x^t (t-x)\, g''(u)\, du; x\right) - \\
&- \left(\int_x^{\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}} \left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)} - u\right) g''(u)\, du; x\right) \\
&\quad + \int_x^x (x-u)\, du.
\end{aligned}$$

By using the following inequality

$$\left|\int_x^t (t-u)\, g''(u)\, du\right| \leq (t-x)^2 \left\|g''(u)\right\|$$

we can write

$$\int_x^{\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}} \left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)} - u\right) g''(u)\, du \leq \left(\tfrac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)^2 \left\|g''(u)\right\|.$$

In wiev of this inequality, we can conclude that

$$\begin{aligned}
\left|\hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x)\right| &\leq \left\{S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) + \left(\tfrac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)^2\right\} \left\|g''\right\| \\
&= \delta_n(x) \left\|g''\right\|.
\end{aligned}$$

Theorem 4.2. *Let $f \in C_B(0,\infty)$. For all $x \in (0,\infty)$, there exists a constant $B > 0$ such that*

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \leq B w_2\left(f; \sqrt{\delta_n(x)}\right) + w\left(f; \frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)$$

where

$$\delta_n(x) = S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) + \left(\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)^2.$$

Proof. For the operators $\hat{S}_{n,a}^{\alpha,\beta}$, we write

$$\hat{S}_{n,a}^{\alpha,\beta}(f;x) - f(x) = \hat{S}_{n,a}^{\alpha,\beta}(f-g;x) + (f-g)(x) + \hat{S}_{n,a}^{\alpha,\beta}(g-g(x);x) \tag{4.3}$$

from the equality(4.1), it follows

$$\begin{aligned} S_{n,a}^{\alpha,\beta}(f;x) - f\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) + f(x) - f(x) &= \hat{S}_{n,a}^{\alpha,\beta}(f-g;x) + (f-g)(x) \\ &\quad + \hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| \le{}& \left|\hat{S}_{n,a}^{\alpha,\beta}(f-g;x)\right| + |(f-g)(x)| \\ &+ \left|\hat{S}_{n,a}^{\alpha,\beta}(g;x) - g(x)\right| + \left|f\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) - f(x)\right|. \end{aligned}$$

By taking the supremum of $\hat{S}_{n,a}^{\alpha,\beta}$operators, we get

$$\begin{aligned} \left|\hat{S}_{n,a}^{\alpha,\beta}(f;x)\right| &= \left|S_{n,a}^{\alpha,\beta}(f;x) - f\left(\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) + f(x)\right| \\ &\le \left|S_{n,a}^{\alpha,\beta}(f;x)\right| + 2\|f\| \\ &\le 3\|f\|. \end{aligned}$$

Now if equality (4.3) is replaced by inequality (4.4), we have

$$\begin{aligned} |S_n^a(f;x) - f(x)| &\le 4\|f-g\| + \left|\hat{S}_n^a(g;x) - g(x)\right| \\ &\quad + \left|f\left(\frac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\right) - f(x)\right| \end{aligned}$$

from Lemma4.1 we obtain

$$\begin{aligned} \left|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right| &\le 4\{\|f-g\| + \delta_n(x)\|g''\|\} \\ &\quad + w\left(f;\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right). \end{aligned}$$

By taking the infimum for all $g \in C_B^2(0,\infty)$ on the right-hand side of the last inequality and considering (4.1), we get that

$$\begin{aligned}
\left|S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right| &\leq 4K_2(f;\delta_n)+w\left(f;\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)\\
&\leq 4Cw_2\left(f;\sqrt{\delta_n}\right)+w\left(f;\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right)\\
&= Bw_2\left(f;\sqrt{\delta_n}\right)+w\left(f;\frac{ax+(1+x)(\alpha-x\beta)}{(1+x)(n+\beta)}\right),
\end{aligned}$$

which completes the proof. □

Theorem 4.3. *Let $0<\gamma\leq 1$ and $f\in C_B(0,\infty)$. Then if $f\in Lip_M(\gamma)$, that is, the inequality*

$$|f(t)-f(x)|\leq M|t-x|^{\gamma}, \qquad x,t\in(0,\infty)$$

holds, then for each $x\in(0,\infty)$ we have

$$\left|S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right|\leq M\delta_n^{\frac{\gamma}{2}}(x)$$

where
$\delta_n=S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)$ and $M>0$ is a constant.

Proof. Let $f\in C_B(0,\infty)\cap Lip_M(\gamma)$. By the linearity and monotonicity of the $S_{n,a}^{\alpha,\beta}$ operators, we get

$$\begin{aligned}
\left|S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right| &\leq S_{n,a}^{\alpha,\beta}(|f(t)-f(x)|;x)\\
&\leq MS_{n,a}^{\alpha,\beta}(|t-x|^{\gamma};x)\\
&= M\sum_{k=0}^{\infty}e^{-\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{x^{n+1}}{n!}\int_0^{\infty}y^n e^{-xy}\left|\frac{\frac{kn}{xy}+\alpha}{n+\beta}-x\right|^{\gamma}dy.
\end{aligned}$$

By applying the Hölder inequality two times succesively to the right side with $p=\frac{2}{\gamma}, q=\frac{2}{2-\gamma}$, we obtain

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right|\\
\leq\ & M\left(e^{-\frac{ax}{1+x}}\sum_{k=0}^{\infty}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}\frac{x^{n+1}}{n!}\int_0^{\infty}y^n e^{-xy}\left|\frac{\frac{kn}{xy}+\alpha}{n+\beta}-x\right|^2dy\right)^{\frac{\gamma}{2}}\\
\leq\ & MS_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right)^{\frac{\gamma}{2}}\\
=\ & M\delta_n^{\frac{\gamma}{2}}(x),
\end{aligned}$$

which is the desired result.

□

5. Weighted Approximation Properties

Firstly, we give some definitions and theorem

Let $\rho(x) = 1 + x^2$ and $B_\rho[0,\infty)$ denote the space of all functions having the property

$$|f(x)| \leq M_f \rho(x)$$

where $x \in [0,\infty)$ and M_f is a positive constant on f functions. The norm on $B_\rho[0,\infty)$ is defined as follows

$$\|f\|_\rho = \sup_{0\leq x<\infty} \frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$ and $C_\rho^0[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = 0.$$

The basic theorem for approximation of weighted spaces is given by Gadjiev in[4].

Theorem 5.1. *Let $\{A_n\}$ be a sequence of positive linear operators defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$,and satisfying the conditions*

$$\lim_{n\to\infty} \|A_n(t^v;x) - x^v\|_\rho = 0, \qquad v = 0,1,2.$$

Then for any $f \in C_\rho^0[0,\infty)$,

$$\lim_{n\to\infty} \|A_n(f;x) - f(x)\|_\rho = 0.$$

It is shown in [4] that, a sequence of linear positive operators A_n is defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$\|A_n(\rho;x)\|_\rho \leq M_\rho$$

where M_ρ is a positive constant.

Theorem 5.2. *Let $\left\{S_{n,a}^{\alpha,\beta}\right\}$ be the sequence of positive linear operators. For each $f \in C_\rho^0(0,\infty)$, we have*

$$\lim_{n\to\infty} \left\|S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right\|_\rho = 0.$$

Proof. Using Lemma3.2, we get

$$\begin{aligned}\sup_{0\leq x<\infty}\tfrac{\left|S_{n,a}^{\alpha,\beta}(\rho;x)\right|}{1+x^2} &= \sup_{0\leq x<\infty}\tfrac{\left|S_{n,a}^{\alpha,\beta}\left(1+t^2;x\right)\right|}{1+x^2}\\ &\leq 1+\tfrac{n\left(a+2n+\alpha^2+2an+n^2\right)+\alpha(n-1)(2a+2n+\alpha)}{(n+\beta)^2(n-1)}.\end{aligned}$$

There exists a positive contant D such that for each n and $\alpha, a, \beta<\infty$

$$\frac{n\left(a+2n+\alpha^2+2an+n^2\right)+\alpha\left(n-1\right)\left(2a+2n+\alpha\right)}{\left(n+\beta\right)^2\left(n-1\right)}<D.$$

Hence we may write

$$\sup_{0\leq x<\infty}\frac{\left|S_{n,a}^{\alpha,\beta}\left(\rho;x\right)\right|}{1+x^2}=\left\|S_{n,a}^{\alpha,\beta}\left(\rho;x\right)\right\|_\rho\leq 1+D.$$

which shows that $\{S_n^a\}$ is a sequence of positive linear operators defined from $C_\rho^0(0,\infty)$ to $B_\rho(0,\infty)$.

For $v=0$, it is clear that

$$\left\|S_{n,a}^{\alpha,\beta}\left(1;x\right)-1\right\|_\rho=0.$$

For $v=1$, we have

$$\begin{aligned}\left\|S_{n,a}^{\alpha,\beta}\left(t;x\right)-x\right\|_\rho &= \sup_{0\leq x<\infty}\tfrac{\left|S_{n,a}^{\alpha,\beta}(t;x)-x\right|}{1+x^2}\\ &= \sup_{0\leq x<\infty}\left|\tfrac{ax+(1+x)(nx+\alpha)}{(1+x)(n+\beta)}\tfrac{1}{1+x^2}-\tfrac{x}{1+x^2}\right|\\ &\leq \left|\tfrac{A}{n+\beta}\right|\end{aligned}$$

holds. Similarly, for $v=2$, we get

$$\begin{aligned}\left\|S_{n,a}^{\alpha,\beta}\left(t^2;x\right)-x^2\right\|_\rho &\leq \sup_{0\leq x<\infty}\tfrac{\left|S_{n,a}^{\alpha,\beta}\left(t^2;x\right)-x^2\right|}{1+x^2}\\ &\leq \left|\tfrac{n\left(a+2n+\alpha^2+2an+n^2\right)+\alpha(n-1)(2a+2n+\alpha)}{(n+\beta)^2(n-1)}-1\right|\\ &= \left|\tfrac{B}{(n+\beta)^2(n-1)}\right|.\end{aligned}$$

As a result, we obtain

$$\lim_{n\to\infty}\left\|S_{n,a}^{\alpha,\beta}\left(t^v;x\right)-x^v\right\|_\rho=0,\quad v=0,1,2.$$

Thus, the proof is completed.

□

Theorem 5.3. *Let $x \in (0,\infty)$, $n \in \mathbb{N}$ and $f \in C_B$. For the operators*

$$S_{n,a}^{\alpha,\beta}(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) dy$$

and

$$L_{n,a}^{\alpha,\beta}(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k+\alpha}{n+\beta}\right),$$

the inequality

$$\left|S_{n,a}^{\alpha,\beta}(f;x) - L_{n,a}^{\alpha,\beta}(f;x)\right| \leq w(f;\delta)\,\varphi(x)$$

is holds true, where

$$\varphi(x) = \left(1 + \frac{1}{\delta}\sqrt{\frac{n(n+1)+2an+a^2}{(n-1)(n+\beta)^2}x^2 + \frac{n+a}{(n-1)(n+\beta)^2}x}\right)$$

and

$$\delta = \sqrt{\frac{n(n+1)+2an+a^2}{(n-1)(n+\beta)^2}x^2 + \frac{n+a}{(n-1)(n+\beta)^2}x}.$$

Proof. From the definition and properties of modulus of continuity, we have

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x) - L_{n,a}^{\alpha,\beta}(f;x)\right| \\
&\leq e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left| f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}\right) - f\left(\frac{k+\alpha}{n+\beta}\right)\right| dy \\
&\leq w(f,\delta) + \tfrac{1}{\delta} w(f,\delta)\, e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \left[\int_0^{\infty} y^n e^{-xy} \left|\frac{\frac{kn}{xy}+\alpha}{n+\beta} - \frac{k+\alpha}{n+\beta}\right|\right] dy.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality two times succesively to the right side, we get

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x) - L_{n,a}^{\alpha,\beta}(f;x)\right| \\
&\leq w(f,\delta) + \tfrac{1}{\delta} w(f,\delta) \left\{ \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} \left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - \frac{k+\alpha}{n+\beta}\right)^2 dy \right)^{\frac{1}{2}} \right. \\
&\left. \times \left(e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^{\infty} y^n e^{-xy} dy \right)^{\frac{1}{2}} \right\} \\
&= w(f,\delta) + \tfrac{1}{\delta} w(f,\delta) \sqrt{S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - \frac{k+\alpha}{n+\beta}\right)^2; x\right)}.
\end{aligned}$$

If we calculate the $S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}-\frac{k+\alpha}{n+\beta}\right)^{2};x\right)$, we show that

$$\begin{aligned}
&S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}-\frac{k+\alpha}{n+\beta}\right)^{2};x\right)=S_{n,a}^{\alpha,\beta}\left(\frac{k^{2}\left(\frac{n}{xy}-1\right)^{2}}{(n+\beta)^{2}};x\right)\\
&=\frac{1}{(n-1)(n+\beta)^{2}}\left[\left(n(n+1)x^{2}+2an\frac{x^{2}}{(1+x)}+\frac{a^{2}x^{2}}{(1+x)}+nx+\frac{ax}{(1+x)}\right)\right]\\
&\leq\frac{n(n+1)+2an+a^{2}}{(n-1)(n+\beta)^{2}}x^{2}+\frac{n+a}{(n-1)(n+\beta)^{2}}x,
\end{aligned}$$

from which, it follows

$$\lim_{n\to\infty}S_{n,a}^{\alpha,\beta}\left(\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta}-\frac{k+\alpha}{n+\beta}\right)^{2};x\right)=0.$$

Thus, we have

$$\begin{aligned}
&\left|S_{n,a}^{\alpha,\beta}(f;x)-L_{n,a}^{\alpha,\beta}(f;x)\right|\\
\leq\ &w(f,\delta)+\frac{1}{\delta}w(f,\delta)\sqrt{\frac{n(n+1)+2an+a^{2}}{(n-1)(n+\beta)^{2}}x^{2}+\frac{n+a}{(n-1)(n+\beta)^{2}}x}\\
\leq\ &w(f,\delta)\varphi(x).
\end{aligned}$$

□

6. Voronovskaya Type Theorem

Lemma 6.1. *For the operators $S_{n,a}^{\alpha,\beta}(f;x)$ defined (2.1), we have*

$$S_{n,a}^{\alpha,\beta}(t-x;x)=\frac{\alpha-\beta x}{n+\beta}+\frac{ax}{(n+\beta)(1+x)}.$$

$$\begin{aligned}
S_{n,a}^{\alpha,\beta}\left((t-x)^{2};x\right)=&\frac{2n^{2}+n\beta^{2}-\beta^{2}}{(n+\beta)^{2}(n-1)}x^{2}+\frac{2a(n+\beta-n\beta)}{(n+\beta)^{2}(n-1)}\frac{x^{2}}{1+x}+\frac{a^{2}n}{(n+\beta)^{2}(n-1)}\frac{x^{2}}{(1+x)^{2}}\\
&+\frac{\left(n^{2}-2\alpha\beta n+2\alpha\beta\right)}{(n+\beta)^{2}(n-1)}x+\frac{an+2a\alpha(n-1)}{(n+\beta)^{2}(n-1)}\frac{x}{1+x}+\frac{\alpha^{2}}{(n+\beta)^{2}}.
\end{aligned}$$

Proof. By using the definition of $S_{n,a}^{\alpha,\beta}$, it can be proved easily. □

Theorem 6.1. *Let $a,x>0$, $0\leq\alpha\leq\beta$ and $n\in N$. For $f\in C^{2}(0,\infty)$ and bounded, we have*

$$\lim_{n\to\infty}(n+\beta)\left[S_{n,a}^{\alpha,\beta}(f;x)-f(x)\right]=\left(\alpha-\beta x+\frac{ax}{1+x}\right)f'(x)+\frac{2x^{2}+x}{2}f''(x).$$

Proof. Let $x, t \in (0,\infty)$, $f \in C^2(0,\infty)$. By Taylor's formula for f, we have

$$f(t) = f(x) + (t-x) f'(x) + \frac{(t-x)^2}{2!} f''(x) + (t-x)^2 \phi(t;x) \tag{6.1}$$

where the function $\phi(t;x) \in C[0,\infty)$ and $\lim_{t\to x} \phi(t;x) = 0$. By applying the operator $S_{n,a}^{\alpha,\beta}$ to the both sides of (6.1), we have

(6.2)

$$\begin{aligned} S_{n,a}^{\alpha,\beta} f(t) = \quad & f(x) S_{n,a}^{\alpha,\beta}(1;x) + f'(x) S_{n,a}^{\alpha,\beta}(t-x;x) + \tfrac{f''(x)}{2!} S_{n,a}^{\alpha,\beta}\left((t-x)^2;x\right) \\ & + S_{n,a}^{\alpha,\beta}\left((t-x)^2 \phi(t;x);x\right). \end{aligned}$$

According to Lemma6.1, the equality (6.2) can be written as follows

$$\begin{aligned} &(n+\beta)\left[S_{n,a}^{\alpha,\beta}(f;x) - f(x)\right] = (n+\beta)\left[\tfrac{\alpha-\beta x}{n+\beta} + \tfrac{ax}{(n+\beta)(1+x)}\right] f'(x) \\ &+ (n+\beta)\left[\tfrac{2n^2+n\beta^2-\beta^2}{(n+\beta)^2(n-1)} x^2 + \tfrac{2a(n+\beta-n\beta)}{(n+\beta)^2(n-1)}\tfrac{x^2}{1+x} + \tfrac{a^2 n}{(n+\beta)^2(n-1)}\tfrac{x^2}{(1+x)^2}\right. \\ &\left. + \tfrac{\left(n^2-2\alpha\beta n+2\alpha\beta\right)}{(n+\beta)^2(n-1)} x + \tfrac{an+2a\alpha(n-1)}{(n+\beta)^2(n-1)}\tfrac{x}{1+x} + \tfrac{\alpha^2}{(n+\beta)^2}\right] \tfrac{f''(x)}{2!} + S_{n,a}^{\alpha,\beta}\left((t-x)^2\phi(t;x);x\right), \end{aligned}$$

where

$$S_{n,a}^{\alpha,\beta}\left((t-x)^2 \phi(t;x);x\right) =$$

$$= e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{x^{n+1}}{n!} \int_0^\infty y^n e^{-xy} f\left(\frac{\frac{kn}{xy}+\alpha}{n+\beta} - x\right)^2 \phi(t;x)\, dy.$$

By applying the Cauchy-Schwarz inequality two times succesively to the right side, we get

(6.3)

$$(n+\beta) S_{n,a}^{\alpha,\beta}\left((t-x)^2\phi(t;x);x\right) \le \sqrt{(n+\beta)^2 S_{n,a}^{\alpha,\beta}\left((t-x)^4;x\right)}\sqrt{S_{n,a}^{\alpha,\beta}\left(\phi^2(t;x);x\right)}.$$

From Lemma 6.1, we have $S_{n,a}^{\alpha,\beta}\left((t-x)^4;x\right) = O(n^{-2})$. Thus, we get

$$\lim_{n\to\infty} (n+\beta)^2 S_{n,a}^{\alpha,\beta}\left((t-x)^4;x\right) = 12x^4 + 12x^3 + 3x^2. \tag{6.4}$$

On the other hand, since $\phi(t;x) \in C[0,\infty)$ and $\lim_{t\to x}\phi(t;x) = 0$, then we conclude

$$\lim_{n\to\infty} S_{n,a}^{\alpha,\beta}\left(\phi^2(t;x);x\right) = \phi^2(x;x) = 0. \tag{6.5}$$

Hence, we get from (6.3), (6.4) and (6.5) that

$$\lim_{n\to\infty}(n+\beta)\,S_{n,a}^{\alpha,\beta}\left((t-x)^2\,\phi\,(t;x)\,;x\right)=0$$

and then, we find

$$\lim_{n\to\infty}(n+\beta)\left[S_{n,a}^{\alpha,\beta}\,(f;x)-f\,(x)\right]=\left(\alpha-\beta x+\frac{ax}{1+x}\right)f'\,(x)+\frac{2x^2+x}{2}f''\,(x)$$

which completed the proof. [1] □

REFERENCES

1. A. Erencin, B. Sevim, *A modification of generalized Baskakov Kantorovich operators*, Stud. Univ. Babeş-Bolyai Math.59 (2014), 3, 351–364.
2. A.Erencin, G. Bascanbaz-Tunca, *Approximation properties of a class of linear positive operators in weighted spaces*, C.R. Acad. Bulgare Sci., 63(2010), no. 10, 1397-1404.
3. A. Erencin, *Durrmeyer type modification of generalized Baskakov operators*, Appl. Math. Comput. 218 (2011) 4384-4390.
4. A.D.Gadjiev, *Theorems of the type of P. P. Korovkin's theorems*, Math. Zametki, 20 (1976), no. 5-6, 781-786(in Russian), Math. Notes, 20(1976), no. 5-6, 995-998 (Endl. Trans.)
5. A. Kajla, On the Bézier Variant of the Srivastava-Gupta Operators, Constr. Math. Anal.,(2)(2018), 99-107
6. A.Wafi , S. Khatoon, *The Voronovskaya theorem for generalized Baskakov-Kantorovich operators in polynomial weight spaces*, Mat. Vesnik, 57(2005), no. 3-4, 87-94.
7. A. Lupas, and M. Müller, *Approximationseigenschaften der Gammaoperatoren*, Math. Zeitschr. 98(1967), 208–226.
8. A. Wafi, N. Rao, *Stancu-Variant of Generalized Baskakov Opera*tors, Filomat 31(2015) (9), 2625–2632.
9. E. Pandey, R.K. Mishra, S.P. Pandey, *Approximation Properties of Some Modified Summation-Integral Type Operator,* (IJSCE) ISSN: , Vol.5 Issue-1, (2015), 2231-2307.
10. F.L. Cao, C.M. Ding, *Lp approximation by multivariate Baskakov–Kantorovich operators,* J. Math. Anal. 27 (1978), pp. 127-142.
11. F.L. Cao, C.M. Ding, Z.B. Xu, *On multivariate Baskakov operator*, J. Math. Anal. Appl. 348 (2008) 856–861.
12. H. G. İnce İlarslan, T. Acar, Approximation by bivariate (p,q)-Baskakov-Kantorovich operators, Georgian Mathematical Journal, 25 (3), 2018, 397-407.
13. I. Krech, R. Malejki.*Approximation of functions of several variables by the Baskakov-Durrmeyer type operators,* Technical Transactions FundamentalSciences.(2016) DOI:2353737XCT.16.142.5753, 98-105

14. L. Rempulska, M. Skorupka, *Approximation properties of modified gamma operators*, Integral Transforms and Special Functions, Vol.18, No. 9, (2007), 653-662.
15. M. Goyal, P. N. Agrawal, *Bivariate generalized Baskakov-Kantorovich operators*, (2015)arXiv preprint arXiv:1505.06071
16. M. Bodur, O.G. Yılmaz, A. Aral, Approximation by Baskakov-Szasz-Stancu Operators Preserving Exponential Functions, Constr. Math. Anal., 1 (1) (2018), 88-98
17. M. Murselan, M. Nasurizzaman, Approximation of Modified Jakimovski- Leviatan-Beta type operators, Constr. Math. Anal. 1 (2) (2018) 88-98
18. R. Malejki, E. Wachnicki, *On the Baskakov-Durrmeyer type operators*, Comment. Math. 54.1(2014), 39–49.
19. S. N. Deveci, T. Acar, O. Alagöz, Approximation by Gamma type Operators, Mathematical Methods in the Pplied Sciences, 43 (5), (2020), 2772-2782
20. S.Guo, Q.Qiulan, *On pointwise estimate for gamma operators*. Approximation Theory and its Applications, (2002), 18(3): 93–98.
21. T. Acar, V. Gupta and A. Aral, Rate of Convergence for Generalized Szasz Operators, Bulletin of Mathematical Science, 1 (1), 99-113 (2011)
22. V.A. Baskakov, *An example of a sequence of linear positive operators in the spaces of continuous functions*, Dokl. Acad. Nauk SSSR 113 (1957) 249-251
23. V.Totik, *The gamma operators in Lp spaces*. Publicationes Mathematicae (Debrecen), (1985), 32: 43–55.
24. V. Mihesan, *Uniform approximation with positive linear operators generated by generalized Baskakov method*, Au-tomat. Comput. Appl. Math. 7 (1998) 34-37.
25. Z. Walczak, *Baskakov type operators,* Rocky Mountain J. Math., 39 (3) (2009), pp. 981-993.
26. Z. Walczak, V. Gupta, *A note on the convergence of Baskakov type operators*, Appl. Math. Comput. 202 (2008), no. 1, 370–375.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 143-156
https://doi.org/10.22190/FUMI200420012P

Original Scientific Paper

A MIXED (NONLINEAR) INAR(1) MODEL

Predrag M. Popović

Faculty of Civil Engineering and Architecture, Department of Mathematics, Informatics and Physics, 18000 Niš, Serbia

Abstract. The paper introduces a new autoregressive model of order one for time series of counts. The model is comprised of a linear as well as nonlinear autoregressive component. These two components are governed by random coefficients. The autoregression is achieved by using the negative binomial thinning operator. The method of moments and the conditional maximum likelihood method are discussed for the parameter estimation. The practicality of the model is presented on a real data set.
Keywords: Time series of counts, Negative binomial thinning operator, Linear model, Nonlinear model.

1. Introduction

In the past few decades, time series modeling has been drawing a lot of attention to researchers as well as practitioners. Understanding the dependence and the evolution of an observed series is an important task. A significant contribution in this field is modeling time series of counts. Time series of counts arises in many real-life situations. For example, number of infected persons, number of stock transactions, number of spaces, number of committed crimes, etc. Studding of these types of time series started after the introduction of the thinning operator in [15]. Some of the first integer-valued autoregressive (INAR) models based on the thinning parameter are presented in [11], [1], [2]. These models experienced various modifications regarding their structure, the definition of thinning operator and the dimensionality. A comprehensive review of INAR models can be found in [16] and [14]. The extension to bivariate INAR models can be found in [7], [9], [8].

Received April 20, 2020; accepted May 28, 2020.
Corresponding Author: Predrag M. Popović, Faculty of Civil Engineering and Architecture, Department of Mathematics, Informatics and Physics, 18000 Niš, Serbia | E-mail: popovicpredrag@yahoo.com
2010 *Mathematics Subject Classification.* Primary 62M10

In general, the INAR models are composed of the survival and the innovation process. The survival process is an autoregressive component, which is defined through the thinning operator. Some of the most exploited thinning operators are the binomial thinning operator introduced in [15], and the negative binomial thinning operator introduced in [13]. The autoregressive component, usually named the survival process, is of the form

$$\alpha \circ X = \sum_{i=1}^{X} W_i,$$

where W_i is a counting sequence. And the major drawback of the autoregressive models is that they put too much or too little weight on thier previous value when predicting the next one. Some of the solutions to this problem were given in [5] where the thinning parameter α is governed by an external process. The generalization of this model was discussed in [6] and [4]. Some other modifications of the autoregressive dependence are based on introducing a bilinear autoregressive component, [3]. Also, there are autoregressive INAR modes that are dealing with the excess number of zeros and ones [12].

The aim of this paper is to introduce a model whose autoregressive part is comprised of a linear as well as a nonlinear component. The nonlinear component is defined through the current state of the innovation process. The idea for that lies in the fact that the survival process might depend on the innovation process. For example, if we have a lot of new specimens of some population, probably the environment conditions are adequate for that species so the survival rate will be higher. Random coefficients determine whether the autoregressive component is linear or not. The linear, as well as the nonlinear component, are defined through the negative binomial thinning operator. Even though the model has this complex definition of the autoregressive component, the conditional expectation can be determined. This fact increases the practical aspect of the model, since the one-step-ahead prediction is possible. Also, the model is proved to be stationary.

The next section gives us the definition of the model. In Section 3. the main properties of the model are derived. Section 4. proposes two methods for the parameter estimation, whose efficiency are tested in Section 5. Section 6. discusses the practical aspect of the model. The concluding remarks are given in Section 7.

2. Model definition

In this section, we introduce the Mixed nonlinear INAR(1) model (MNLINAR(1)) in a general form, without specifying a distribution of the innovation process. For such a model, we prove the existence and the strict stationarity. Also, the main properties of the model are derived.

Let $\{X_t\}$ be a non-negative integer-valued time series. Then, the MNLINAR(1) model is defined as follows:

$$X_t = \begin{cases} \alpha * X_{t-1} + \varepsilon_t, & \text{w.p. } p \\ \alpha * (X_{t-1}\varepsilon_t) + \varepsilon_t, & \text{w.p. } 1-p \end{cases} \tag{2.1}$$

where the negative binomial thinning operator is defined as $\alpha * X = \sum_{i=1}^{X} W_i$, where $\{W_t\}$ is independent identically distributed random variables with geometric marginal distribution $Geom(\alpha/(1+\alpha))$, whose probability mass function is $P(W_i = w) = \frac{\alpha^w}{(1+\alpha)^{w+1}}$. The counting sequence $\{W_t\}$ are independent of $\{X_t\}$ and $\{\varepsilon_t\}$. Further, the random variable ε_t is independent of X_s for $s < t$.

As we can see, the MNLINAR(1) model evolves as a linear model with probability p and as a nonlinear model with probability $1-p$. So, the model can be expressed with random variables U_t and V_t where $P(U_t = \alpha, V_t = 0) = 1 - P(U_t = 0, V_t = \alpha) = p$. Than the MNLINAR(1) model is defined as

$$X_t = U_t * X_{t-1} + V_t * (X_{t-1}\varepsilon_t) + \varepsilon_t. \tag{2.2}$$

Theorem 2.1. *There exist a unique strictly stationary bivariate time series $\{X_t\}$ that satisfies equation (2.2), when $\alpha(p + (1-p)\lambda) < 1$, $\alpha^2(p + (1-p)E(\varepsilon_t^2)) < 1$ and $E(\varepsilon_t^2) < \infty$, where λ stands for $E(\varepsilon_t)$.*

Proof. Let us introduce a series $\{X_t^{(n)}\}$ in the following way:

$$X_t^{(n)} = \begin{cases} 0, & n < 0 \\ \varepsilon_t, & n = 0 \\ U_{(t)} * X_{t-1}^{(n-1)} + V_{(t)} * (X_{t-1}^{(n-1)}\varepsilon_t) + \varepsilon_t, & n > 0 \end{cases}.$$

Here, notations $U_{(t)}$ and $V_{(t)}$ implies that the counting series that figure in $U_{(t)} * X^{(n)}$ are fixed at time t for all n. Now, we define the Hilbert space $L^2(\Omega, \mathcal{F}, P) = \{X : E(X^2) < \infty\}$, where the measure between two random variables is defined as $E(XY)$. The idea is to prove that $\{X_t^{(n)}\}$ is strictly stationary, and then to show that $\{X_t^{(n)}\}$ is a Cauchy sequence that belongs to just defined L^2 space.

Using the same approach as in [3], it can be proved that the series $\{X_t^{(n)}\}$ is strictly stationary, so we omit that proof here.

To show that $X_t^{(n)}$ belong to the above defined Hilbert space, we need to prove that $E(X_t^{(n)})^2 < \infty$. For $n \leq 0$ it obviously holds, thus let us focus on $n > 0$. We obtain the following equation:

$$
\begin{aligned}
&E\left(X_t^{(n)}\right)^2 = pE(\alpha * X_{t-1}^{(n-1)} + \varepsilon_t)^2 + (1-p)E(\alpha * (X_{t-1}^{(n-1)}\varepsilon_t) + \varepsilon_t) \\
&= pE((\alpha * X_{t-1}^{(n-1)})^2 + 2\alpha * X_{t-1}^{(n-1)}\varepsilon_t + \varepsilon_t^2) \\
&\qquad +(1-p)E((\alpha * (X_{t-1}^{(n-1)}\varepsilon_t))^2 + 2\alpha * (X_{t-1}^{(n-1)}\varepsilon_t)\varepsilon_t + \varepsilon_t^2) \\
&= p\Big[\alpha^2 E(X_{t-1}^{(n-1)})^2 + \alpha(1+\alpha)E(X_{t-1}^{(n-1)}) \\
&\qquad +2\alpha E(X_{t-1}^{(n-1)})E(\varepsilon_t) + E(\varepsilon_t^2)\Big] \\
&+(1-p)\Big[\alpha E(X_{t-1}^{(n-1)}\varepsilon_t)^2 + \alpha(1+\alpha)E(X_{t-1}^{(n-1)}\varepsilon_t) \\
&\qquad +2\alpha E(X_{t-1}^{(n-1)}\varepsilon_t^2) + E(\varepsilon_t^2)\Big] = \dots \\
&= E\left(X_{t-1}^{(n-1)}\right)^2 \left(p\alpha^2 + (1-p)\alpha^2 E(\varepsilon_t^2) + E(X_{t-1}^{(n-1)})\right) \\
&\ \cdot \left[\alpha(1+\alpha)\left(p + (1-p)E(\varepsilon_t)\right) + 2\alpha\left(pE(\varepsilon_t) + (1-p)E(\varepsilon_t^2)\right)\right] + E(\varepsilon_t^2).
\end{aligned}
$$

Since the series $\{X_t^{(n)}\}$ is strictly stationary, it follows that $E\left(X_t^{(n)}\right)^2 < \infty$ if $1-\alpha^2(p+(1-p)E(\varepsilon_t^2)) > 0$, which is satisfied by the condition of the theorem. In the above derivation, we used some known properties of the negative binomial thinning operator which can be found in [13].

Now, let us prove that $\{X_t^{(n)}\}$ is a Cauchy sequence. Notice that equation (2.3) holds if and only if the sequence $\{X_t^{(n)}\}$ is non-decreasing.

$$
X_t^{(n)} - X_t^{(n-1)} = U_{(t)} * (X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}) + V_{(t)} * ((X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)})\varepsilon_t) \tag{2.3}
$$

To show that the sequence is non-decreasing we use mathematical induction. Notice that

$$
X_t^{(1)} = U_{(t)} * X_{t-1}^{(0)} + V_{(t)} * (X_{t-1}^{(0)}\varepsilon_t) + \varepsilon_t \geq \varepsilon_t = X_t^{(0)}.
$$

Suppose that $X_t^{(k)} > X_t^{(k-1)}$ for some k and let's prove it for $k+1$.

$$
\begin{aligned}
X_t^{(k)} &= U_{(t)} * X_{t-1}^{(k-1)} + V_{(t)} * (X_{t-1}^{(k-1)}\varepsilon_t) + \varepsilon_t \leq \\
&\leq U_{(t)} * X_{t-1}^{(k)} + V_{(t)} * (X_{t-1}^{(k)}\varepsilon_t) + \varepsilon_t = X_t^{(k+1)}.
\end{aligned}
$$

So, $\{X_t^{(n)}\}$ is non-decreasing and equation (2.3) holds. Taking expectation of the both sides of equation (2.3) we obtain that

$$\begin{aligned}
&E(X_t^{(n)} - X_t^{(n-1)}) \\
&= p\alpha E(X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}) + (1-p)\alpha E[(X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)})\varepsilon_t] \\
&= p\alpha E(X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}) + (1-p)\alpha\lambda E(X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}) \\
&= (p\alpha + (1-p)\alpha\lambda)E(X_{t-1}^{(n-1)} - X_{t-1}^{(n-2)}) = \dots \\
&= (p\alpha + (1-p)\alpha\lambda)^{n-1} E(X_{t-1}^{(1)} - X_{t-1}^{(0)}) \\
&\qquad + (p\alpha + (1-p)\alpha\lambda)^{n} E(\varepsilon_t^2).
\end{aligned}$$

We can conclude that

$$E(X_t^{(n)} - X_t^{(n-1)}) \xrightarrow[n\to\infty]{} 0 \Longleftrightarrow p\alpha + (1-p)\alpha\lambda < 1.$$

Thus, $\{X_t^{(n)}\}$ is a Cauchy sequence in the above-defined Hilbert space which implies that the Cauchy sequence converges, i.e. $\lim_{n\to\infty} X_t^{(n)} = X_t$. Since the series $\{X_t^{(n)}\}$ is strictly stationary it follows that its limit is strictly stationary as well.

The uniqueness of the solution of equation (2.2) can be proved using the same approach as [3], so we omit it here. □

3. Properties of the model

In this section, we derive the most important properties of the MNLINAR(1) model, including the first and the second moments as well as the conditional expectation and the conditional probability mass function.

From the model definition given by equation (2.1), and the properties of the negative binomial thinning operator we obtain

$$E(X_t) = \alpha p E(X_{t-1}) + \alpha(1-p)E(X_{t-1})E(\varepsilon_t) + E(\varepsilon_t).$$

Having in mind that $\{X_t\}$ is a strictly stationary process, and relying on the conditions of Theorem 2.1, it follows that

$$E(X_t) = \frac{E(\varepsilon_t)}{1-\alpha(p+(1-p)E(\varepsilon_t))}. \tag{3.1}$$

For the derivation of the second moment we use the same extensive technique as in Theorem (2.1), so we only notice that

$$\begin{aligned}
&E\left(X_t^2\right) = E\left(X_{t-1}\right)^2 (p\alpha^2 + (1-p)\alpha^2 E(\varepsilon_t^2) \\
&+E(X_{t-1})\left[\alpha(1+\alpha)(p+(1-p)E(\varepsilon_t)) + 2\alpha(pE(\varepsilon_t) + (1-p)E(\varepsilon_t^2))\right] + E(\varepsilon_t^2).
\end{aligned}$$

Under the conditions of Theorem (2.1), it follows that

$$\begin{aligned}
&E\left(X_t^2\right) = \\
&\frac{E(X_{t-1})\left[\alpha(1+\alpha)(p+(1-p)E(\varepsilon_t))+2\alpha(pE(\varepsilon_t) + (1-p)E(\varepsilon_t^2))\right]+E(\varepsilon_t^2)}{1-\alpha^2(p+(1-p)E(\varepsilon_t^2))}.
\end{aligned} \tag{3.2}$$

Further, let us pay the attention on the expected value of the product X_tX_{t-k}. It is equal to

$$\begin{aligned}&E(X_tX_{t-k}) = pE((\alpha * X_{t-1}+\varepsilon_t)X_{t-k}) + (1-p)E((\alpha*(X_{t-1}\varepsilon_t)+\varepsilon_t)X_{t-k})\\&= \alpha pE(X_{t-1}X_{t-k}) + \alpha(1-p)E(X_{t-1}X_{t-k})E(\varepsilon_t) + E(\varepsilon_t)E(X_{t-k})\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))E(X_{t-1}X_{t-k}) + E(\varepsilon_t)E(X_{t-k}) = \dots\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))^kE(X_{t-k}^2) + E(X_{t-k})E(\varepsilon_t)\sum_{j=0}^{k-1}(\alpha p+\alpha(1-p)E(\varepsilon_t))^j.\end{aligned}$$

It can be notice that, under the conditions of Theorem 2.1,

$$E(X_tX_{t-k}) \xrightarrow[k\to\infty]{} \frac{E(X_{t-k})E(\varepsilon_t)}{1-\alpha(p+(1-p)E(\varepsilon_t))}.$$

Substituting $E(\varepsilon_t)$ by using equation (3.1), we obtain

$$E(X_tX_{t-k}) \xrightarrow[k\to\infty]{} E(X_t)E(X_{t-k}).$$

For further discussion, it will be particularly important the case when $k = 1$, so let us notice that

$$\begin{aligned}E(X_tX_{t-1}) &= (\alpha p+\alpha(1-p)E(\varepsilon_t))E(X_{t-1}^2) + E(X_{t-1})E(\varepsilon_t)\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))E(X_t^2) + E(X_t)E(\varepsilon_t).\end{aligned} \tag{3.3}$$

But the autocorrelation structure of the series $\{X_t\}$ would be much easier to observe through the autocovariance function directly. Namely, we obtain the following:

$$\begin{aligned}&Cov(X_t,X_{t-k}) = E(X_tX_{t-k}) - E(X_t)E(X_{t-k})\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))E(X_{t-1}X_{t-k}) + E(\varepsilon_t)E(X_{t-k})\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))Cov(X_{t-1},X_{t-k})\\&+E(\varepsilon_t)E(X_{t-k}) + (\alpha p+\alpha(1-p)E(\varepsilon_t))E(X_{t-1})E(X_{t-k}) - E(X_t)E(X_{t-k})\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))Cov(X_{t-1},X_{t-k}) = \dots\\&= (\alpha p+\alpha(1-p)E(\varepsilon_t))^kCov(X_{t-k},X_{t-k}) = (\alpha p+\alpha(1-p)E(\varepsilon_t))^kVar(X_t).\end{aligned}$$

The above equation follows from the property of the negative binomial thinning operator, which can be found in Lemma 3 of [13]. Now, it can be easily concluded that, under assumption of Theorem 2.1, the autocorrelation tends to zero when k tends to infinity.

Regarding the practicality of the MNLINAR(1) model, the most important aspect of the model is the ability to predict forthcoming values of a modeled series. Unlike some other nonlinear models ([3], [10]), for the MNLINAR(1) model the conditional expectation can be derived as

$$\begin{aligned}E(X_t|X_{t-1}) &= pE(\alpha * X_{t-1}+\varepsilon_t|X_{t-1}) + (1-p)E(\alpha*(X_{t-1}\varepsilon_t)+\varepsilon_t|X_{t-1})\\&= p(\alpha X_{t-1}+E(\varepsilon_t)) + (1-p)(\alpha E(X_{t-1}\varepsilon_t|X_{t-1}) + E(\varepsilon_t))\\&= \alpha(p+(1-p)E(\varepsilon_t))X_{t-1} + E(\varepsilon_t).\end{aligned}$$

Finally, we focus on the conditional probability mass function, where we focus on the one-step-ahead conditional probability.

$$
\begin{aligned}
P(X_t = x|X_{t-1} = u) &= pP(\alpha * X_{t-1} + \varepsilon_t = x|X_{t-1} = u) \\
&\quad + (1-p)P(\alpha * (X_{t-1}\varepsilon_t) + \varepsilon_t = x|X_{t-1} = u) \\
&= p\sum_{i=0}^{x} P(\alpha * X_{t-1} = i|X_{t-1} = u)P(\varepsilon_t = x-i) \\
&\quad + (1-p)\sum_{i=0}^{x} P(\alpha * (X_{t-1}\varepsilon_t) = i|X_{t-1} = u, \varepsilon_t = x-i)P(\varepsilon_t = x-i) \\
&= p\sum_{i=0}^{x} P(\alpha * u = i)P(\varepsilon_t = x-i) \\
&\quad + (1-p)\sum_{i=0}^{x} P(\alpha * (u(x-i)) = i)P(\varepsilon_t = x-i) \\
&= p\sum_{i=0}^{x} P(N = i)P(\varepsilon_t = x-i) \\
&\quad + (1-p)\sum_{i=0}^{x} P(M = i)P(\varepsilon_t = x-i), \qquad (3.4)
\end{aligned}
$$

where N and M are random variables with negative binomial distribution with parameters (α, u) and $(\alpha, u(x-i))$, respectively.

3.1. Specification of the innovation process

So far, we have not specified the marginal distribution of the innovation process $\{\varepsilon_t\}$. And as we could notice, that didn't affect the derivation of the MNLINAR(1) model properties. In order to complete the definition of the MNLINAR(1) model, we introduce the assumption about the distribution of ε_t. In the succeeding sections, we assume that ε_t follows the geometric distribution with parameter $\lambda/(1+\lambda)$. The corresponding probability mass function is equal to $P(\varepsilon_t = k) = \frac{\lambda^k}{(1+\lambda)^{k+1}}$. Notice that this model can be easily adjusted for a different type of series by introducing different distributions of the innovation process.

4. Parameter estimation

In this section, we propose two methods for the estimation of unknown parameters of the MNLINAR(1) model. First, we discuss in detail the method of moments, and then the conditional maximum likelihood method. At the end, we test the efficiency of the presented methods on simulated data sets.

4.1. Method of moments

Assume that we have a realization of the series given by the equation (2.1) of length N. Then, for the given series $\{X_1, X_2, \ldots, X_N\}$, first sample moment is denoted as $\overline{X}_N$, the second sample moment as $\overline{X^2}_N$ and $E(X_k X_{k-1})$ as γ.

Since we have that $E(\varepsilon_t) = \lambda$ from equation (3.1) we obtain the estimate parameter λ as

$$\lambda = \frac{\overline{X}_N(1-\alpha p)}{1+\alpha(1-p)\overline{X}_N}. \tag{4.1}$$

Then, we can easily solve equation (3.3) for λ, since it is a linear equation with respect to λ.

$$\lambda = \frac{\gamma - \alpha p \overline{X^2}_N}{\alpha(1-p)\overline{X^2}_N + \overline{X}_N}. \tag{4.2}$$

The left sides of equations (4.1) and (4.2) are equal, so it follows that

$$\frac{\overline{X}_N(1-\alpha p)}{1+\alpha(1-p)\overline{X}_N} = \frac{\gamma - \alpha p \overline{X^2}_N}{\alpha(1-p)\overline{X^2}_N + \overline{X}_N}.$$

After some algebraic transformations, we can solve the above equation for α, where we obtain

$$\begin{aligned} \alpha &= \frac{\gamma - \left(\overline{X}_N\right)^2}{(1-p)(\overline{X^2}_N - \gamma)\overline{X}_N + p(\overline{X^2}_N - \left(\overline{X}_N\right)^2)} \\ &= \frac{C_x}{(1-p)(\overline{X^2}_N - \gamma)\overline{X}_N + pD_x}. \end{aligned} \tag{4.3}$$

where C_x is the sample lag-one covariance, and D_x is the sample variance. Further, since the equation (3.2) is liner with respect to p, the estimate of parameter p we obtain from equation (3.2) as

$$p = \frac{\overline{X^2}_N - \alpha^2 \overline{X^2}_N E(\varepsilon_t^2) - \alpha(1+\alpha)\lambda \overline{X}_N - 2\alpha \overline{X}_N E(\varepsilon_t^2) - E(\varepsilon_t^2)}{\alpha^2 \overline{X^2}_N (1 - E(\varepsilon_t^2)) + \alpha \overline{X}_N [(1+\alpha)(1 - E(\varepsilon_t) + 2(E(\varepsilon_t) - E(\varepsilon_t^2)]} \tag{4.4}$$

Note that under the assumption introduced in Subsection 3.1. we have $E(\varepsilon_t^2) = \lambda(2\lambda + 1)$.

The system of equations (4.1), (4.3) and (4.4) cannot be solved analytically. Thus, we apply the following numerical procedure. For a given p_0, we can calculate α_0 from equation (4.3), and then with these two values we get λ_0 from equation (4.1). From equation (4.4) we obtain p_1. We repeat the procedure until $|p_{k+1} - p_k| + |\lambda_{k+1} - \lambda_k| + |\alpha_{k+1} - \alpha_k| < \delta$, where δ is set to be a sufficiently small value.

4.2. Conditional maximum likelihood

For the given series $\{X_1, X_2, \ldots, X_N\}$, we estimate the parameters of the MNLINAR(1) model using the conditional maximum likelihood method (CML). The likelihood function that we maximize here, is actually the log-likelihood function determined through equation (3.4). Let us denote the set of parameters of the MNLINAR(1) model as vector $\boldsymbol{\theta} = (\alpha, p, \lambda)$. Then, the estimate of the vector $\boldsymbol{\theta}$ is obtained as

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} L(\boldsymbol{\theta}), \tag{4.5}$$

where

$$\begin{aligned} L(\boldsymbol{\theta}) &= \sum_{i=2}^{N} \ln P(X_i = x_i | X_{i-1} = x_{i-1}) \\ &= \sum_{i=2}^{N} \ln \Bigg[\frac{p\lambda^{x_i}}{(1+\alpha)^{x_{i-1}}(1+\lambda)^{x_i+1}} \sum_{j=0}^{x_i} \binom{j + x_{i-1} - 1}{x_{i-1} - 1} \left(\frac{\alpha(1+\lambda)}{\lambda(1+\alpha)} \right)^j \\ &+ \frac{(1-p)\lambda^{x_i}}{(1+\alpha)^{x_{i-1}x_i}(1+\lambda)^{x_i+1}} \sum_{j=0}^{x_i} \binom{j + x_{i-1}(x_i - j) - 1}{x_{i-1}(x_i - j) - 1} \left(\frac{\alpha(1+\lambda)}{\lambda(1+\alpha)^{1-x_{i-1}}} \right)^j \Bigg]. \end{aligned}$$

Since this maximization procedure cannot be done analytically, some numerical approach must be applied. For that purpose, we use built-in functions of the program language `R`.

5. Simulation

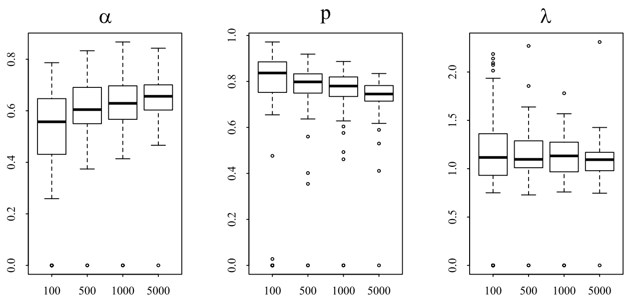
In this section, by using the Monte Carlo method, we generate time series according to equation (2.1). We conduct this procedure using different sets of parameters that figure in the MNLINAR(1) model. On these simulated series we test the efficiency of the MM and CML methods described in the previous section. The efficiency of the proposed methods is measured with respect to the bias and the standard deviation of the obtained estimates.

We have chosen four sets of parameters, considering conditions of Theorem 2.1. The following parameter were used for the simulation purpose: a) $\alpha = 0.7$, $p = 0.7$, $\lambda = 1$; b) $\alpha = 0.3$, $p = 0.3$, $\lambda = 2$; c) $\alpha = 0.5$, $p = 0.9$, $\lambda = 3$; d) $\alpha = 0.1$, $p = 0.9$, $\lambda = 7$. The estimates obtained by the MM and CML methods are given in Table 8.1 (the table can be found in Appendix).

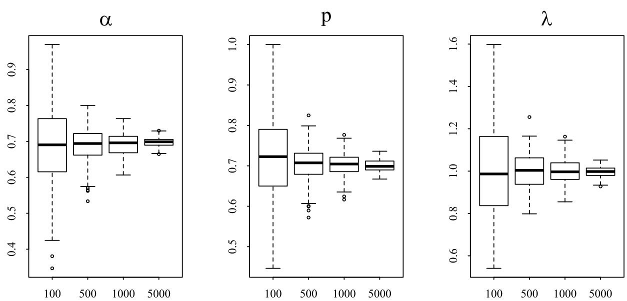
According to the results presented in Table 8.1, we can conclude that both methods converge to the true value of parameters. Also, the standard error of estimates is reducing with the increase of the sample size. It should be noticed that the MM method is not very accurate estimates when the length of a sample is 100 and even 500. But for samples whose length is 1000 or 5000, the estimates are quite adequate. On the other side, CML demonstrates remarkable precision even for samples of length 100.

The MM method is conducted through the iterative procedure described in Subsection 4.1. The maximal number of iterations was set to be 100, and the estimation procedure is very fast. The method usually converges in less than 100 iterations.

The CML method is based on the numerical maximization of the function given by equation (4.5). The numerical procedure is obtained using `nlm` function of the programming language `R`. It doesn't take too much of computation time except for the samples of length 5000.



(a) The method of moments estimates.



(b) The conditional maximum likelihood method estimates.

FIG. 5.1: The box plots of estimates for the set of parameters $\alpha = 0.7$, $p = 0.7$, $\lambda = 1$, obtained by the method of moments (upper) and the conditional maximum likelihood method (lower).

For the MM method, approximately, one of ten estimates is outside the feasible range. On the other side, the CML method had only a few estimates outside the feasible range, and only for the case when the length of the series was 100. The distribution of the estimates for the parameter set a) is given in Figure 5.1. Also, from Figure 5.1 we can notice the convergence of the estimates toward the true values.

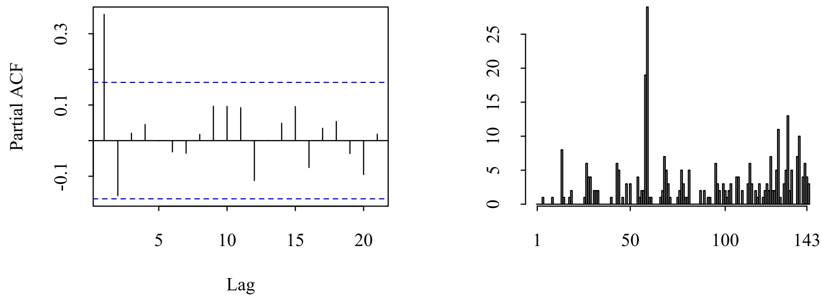


FIG. 6.1: The partial autocorrelation function and the bar plot for DRUG series.

6. Real data example

In this section, we will demonstrate a practical aspect of the MNLINAR(1) model. We test the ability of this model to capture and predict values of an observed time series. Our goal is not to compare the MNLINAR(1) with all known models, but to see what is the effect of having a model with the linear as well as the nonlinear component, in comparison with the models that have only linear (named LINAR(1) model which is actually the model presented in [13]) or only nonlinear component (named NLINAR(1) model). The criteria for the goodness-of-fit are going to be the root mean square error (RMS), the Akaike information criterion and the Bayesian information criterion (BIC).

For this test, we use time series of criminal records, collected by the Pittsburgh police station number 2206. The data can be found on the link `http://www.forecastingprinciples.com/`. We focus on series of monthly drug offenses (DRUG) that took place between January 1990 and December 2001. There are 144 observed values, whose mean value is 2.1 and the standard deviation 12.9. The bar plot of the series is presented in Figure 6.1. In Figure 6.1 there is also the partial autocorrelation diagram. Although the MNLINAR(1) model is not a standard autocorrelated model, it has some properties of the autocorrelated model of order one. Figure 6.1 shows that the observed series is autocorrelated on lag one.

The results obtained from the three tested models are presented in Table 6.1. As we can see that by introducing the linear and the nonlinear component, we have reduced the one-step ahead prediction error, while also reduced the values of AIC and BIC measures. Having in mind that the MNLINAR(1) model has one more parameter than the other two models, and considering values of AIC and BIC, we can conclude that the best fit of the observed series is provided by the MNLINAR(1) model.

Table 6.1: Estimated parameters, standard errors of the estimates, the root mean square error for one step ahead prediction, AIC and BIC values for MNLINAR(1) , LINAR(1) and NLINAR(1) models.

Model	Estimates	RMS	AIC	BIC
MNLINAR(1)	$\hat{\alpha} = 0.174$ (0.068) $\hat{p} = 0.395$ (0.204) $\hat{\lambda} = 1.49$ (0.211)	3.37	548.17	557.08
LINAR(1)	$\hat{\alpha} = 0.044$ (0.037) $\hat{\lambda} = 2.033$ (0.343)	3.52	565.06	573.97
NLINAR(1)	$\hat{\alpha} = 0.108$ (0.075) $\hat{\lambda} = 1.615$ (0.192)	3.42	551.15	560.06

7. Conclusion

The model discussed in this paper is the INAR model of order one. Although it is not a pure autoregressive model, it still preserves some of the autoregressive properties. The survival component is composed of linear and nonlinear processes, both defined through the negative binomial thinning operator, while the innovation component is driven by the geometrical marginal distribution. The method of moments and the conditional maximum likelihood method are presented for the estimation of the model parameters. While the method of moments showed to be unreliable for small samples, the conditional maximum likelihood provides very accurate estimates for all testes samples. The practicality of the model was discussed on a real data set, where the surplus of having both linear and nonlinear components was demonstrated.

Some further modifications of the model can be based on choosing different thinning operators or different marginal distribution of the innovation process. Both components of the model can be adjusted in order to better model an observed series.

REFERENCES

1. Al-Osh, M.A., Alzaid, A.A. (1987) First-order integer-valued autoregressive (INAR (1)) process, Journal of Time Series Analysis 8, 261–275.
2. Alzaid, A., Al-Osh, M. (1988) First-order integer-valued autoregressive (INAR (1)) process: distributional and regression properties, Statistica Neerlandica 42, 53–61.
3. Doukhan, P., Latour, A., Oraichi, D. (2006) A simple integer-valued bilinear time series model, Advances in Applied Probability 38, 559–578.
4. Laketa, P.N., Nastić, A.S. (2019) Conditional least squares estimation of the parameters of higher order Random environment INAR models, Facta Universitatis, Series: Mathematics and Informatics 34, 525–535.
5. Nastić, A.S., Laketa, P.N., Ristić, M.M. (2016) Random environment integer-valued autoregressive process, Journal of Time Series Analysis 37, 267–287.

6. Nastić, A.S., Laketa, P.N., Ristić, M.M. (2019) Random environment INAR models of higher order, REVSTAT-Statistical Journal 17, 35–65.
7. Pedeli, X., Karlis, D. (2013) Some properties of multivariate INAR (1) processes, Computational Statistics & Data Analysis 67, 213–225.
8. Popović, P. (2015) Random coefficient bivariate INAR (1) process, Facta Universitatis, Series: Mathematics and Informatics 30, 263–280.
9. Popović, P.M., Ristić, M.M., Nastić, A.S. (2016) A geometric bivariate time series with different marginal parameters, Statistical Papers 57, 731–753.
10. Popović, P.M., Bakouch, H.S. (2020) A bivariate integer-valued bilinear autoregressive model with random coefficients, Statistical Papers 61, 1819–1840.
11. McKenzie, E. (1986) Autoregressive moving-average processes with negative-binomial and geometric marginal distributions, Advances in Applied Probability 18, 679–705.
12. Qi, X., Li, Q., Zhu, F. (2019) Modeling time series of count with excess zeros and ones based on INAR (1) model with zero-and-one inflated Poisson innovations, Journal of Computational and Applied Mathematics 346, 572–590.
13. Ristić, M.M., Bakouch, H.S., Nastić, A.S. (2009) A new geometric first-order integer-valued autoregressive (NGINAR (1)) process, Journal of Statistical Planning and Inference 139, 2218–2226.
14. Scotto, M.G., Weiß, C.H., Gouveia, S. (2015) Thinning-based models in the analysis of integer-valued time series: a review, Statistical Modelling 15, 590–618.
15. Steutel, F.W., van Harn, K. (1979) Discrete analogues of self-decomposability and stability, The Annals of Probability, 893–899.
16. Weiß, C.H. (2008) Thinning operations for modeling time series of counts—a survey, AStA Advances in Statistical Analysis 92, 319–341.

8. Appendix

Table 8.1: The bias and the standard errors of the estimates obtained by the method of moments and the conditional maximum likelihood method.

	MM			CML		
N	α	p	λ	α	p	λ
a) $\alpha = 0.7$, $p = 0.7$, $\lambda = 1$						
100	-0.039	-0.195	-0.168	-0.007	-0.004	0.004
	0.141	0.209	0.355	0.094	0.151	0.307
500	-0.069	-0.127	-0.156	-0.002	0.018	0.017
	0.127	0.157	0.312	0.035	0.084	0.125
1000	-0.057	-0.154	-0.10	-0.003	0.007	0.012
	0.121	0.183	0.187	0.026	0.062	0.094
5000	-0.034	-0.053	-0.07	-0.001	-0.002	0.008
	0.104	0.164	0.048	0.01	0.025	0.039
b) $\alpha = 0.3$, $p = 0.3$, $\lambda = 2$						
100	0.159	-0.134	-0.189	0.035	-0.021	-0.032
	0.14	0.118	0.548	0.114	0.127	0.24
500	0.087	-0.088	-0.163	0.011	0.005	-0.007
	0.112	0.075	0.218	0.05	0.05	0.091
1000	0.058	-0.07	-0.111	0.006	0.004	-0.005
	0.103	0.087	0.202	0.03	0.032	0.061
5000	0.034	-0.045	-0.07	0.003	0.002	0.007
	0.08	0.051	0.153	0.01	0.022	0.011
c) $\alpha = 0.5$, $p = 0.9$, $\lambda = 3$						
100	0.027	-0.063	-0.634	-0.002	0.023	-0.113
	0.098	0.033	0.973	0.048	0.238	1.279
500	0.032	-0.047	-0.533	0.001	0.002	-0.094
	0.066	0.03	0.545	0.019	0.087	0.498
1000	0.017	-0.044	-0.366	0.001	-0.002	-0.081
	0.05	0.023	0.289	0.013	0.055	0.324
5000	0.014	-0.031	-0.259	-0.002	-0.007	0.005
	0.04	0.02	0.265	0.02	0.021	0.019
d) $\alpha = 0.1$, $p = 0.9$, $\lambda = 7$						
100	-0.033	-0.063	0.185	-0.003	0.001	-0.065
	0.062	0.034	0.87	0.07	0.069	0.58
500	-0.023	-0.034	0.103	0.001	0.003	-0.026
	0.048	0.033	0.477	0.029	0.025	0.241
1000	-0.013	-0.028	0.066	-0.001	-0.001	-0.011
	0.04	0.036	0.423	0.022	0.02	0.181
5000	-0.001	-0.008	-0.018	-0.001	-0.001	0.007
	0.02	0.025	0.231	0.019	0.011	0.059

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 157-163
https://doi.org/10.22190/FUMI200420013K

Original Scientific Paper

PARALLELISM OF DISTRIBUTIONS AND GEODESICS ON $F(\pm a^2, \pm b^2)$-STRUCTURE LAGRANGIAN MANIFOLD

Mohammad Nazrul Islam Khan[1] and Lovejoy S. Das[2]

[1]**Department of Computer Engineering, College of Computer, Qassim University, Buraydah, Saudia Arabia**
[2]**Department of Mathematics, College of Computer, Kent State University, New Philadelphia, OH 44663, U.S.A.**

Abstract. This paper deals with the Lagrange vertical structure on the vertical tangent space $T_V(N)$ endowed with a non-zero (1,1) tensor field F_v satisfying $(F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2) = 0$. The similar structure on the horizontal subspace $T_H(N)$ and on $T(N)$ is investigated if the $F(\pm a^2, \pm b^2)$-structure on $T_V(N)$ is given. Furthermore, we have proved some theorems and obtained conditions under which the distribution P and Q are ∇-parallel, $\bar{\nabla}$ anti half parallel when $\nabla = \bar{\nabla}$. Finally, certain theorems on geodesics on the Lagrange manifold are established.
Keywords: Distribution, Parallelism, Geodesic, Almost product structure.

1. Introduction

Let M and N be two differentiable manifolds of dimension n and $2n$ respectively and (N, π, M) be vector bundle with $\pi(N) = M$. The local coordinate systems $(x^1, x^2, ..., x^n)$ about x in M and $(y^1, y^2, ..., y^n)$ about y in N. Let $(x^i, y^\alpha), 1 \leq i \leq n, 1 \leq \alpha \leq n$ be system of local coordinates in the open set $\pi^{-1}(U)$ and called induced coordinates in $\pi^{-1}(U)$, where U is a coordinate neighborhood in M. Let $T_p(N)$ be tangent space and $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}\right\}$ canonical basis for $T_p(N)$ such that $p \in \pi^{-1}(U)$ and it is also denoted by $\{\partial_i, \partial_\alpha\}$ where $\partial_i = \frac{\partial}{\partial x^i}$. If (x^h, x^{α^1}) be coordinates of a point in the interesting region $\pi^{-1}(U) \cap \pi^{-1}(U)$, then [2, 6]

$$x^{i^1} = x^{i^1}(x^i), \tag{1.1}$$

Received April 20, 2020; accepted December 5, 2020.
Corresponding Author: Mohammad Nazrul Islam Khan, Department of Computer Engineering, College of Computer, Qassim University, Buraydah, Saudia Arabia, | E-mail: m.nazrul@edu.qu.sa, mnazrul@rediffmail.com
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C22

$$y^{\alpha^1} = \frac{\partial x^{\alpha^1}}{\partial x^\alpha} y^\alpha, \tag{1.2}$$

and another canonical basis in the intersecting region are given by

$$\partial_{i^1} = \frac{\partial x^i}{\partial x^{i^1}} \partial_i \tag{1.3}$$

$$\partial_{\alpha^1} = \frac{\partial y^\alpha}{\partial y^{\alpha^1}} \partial_\alpha. \tag{1.4}$$

The tangent space of N is denoted by $T(N)$ and spanned by $\{\partial_i, \partial_\alpha\}$ and its subspaces by $T_V(N)$ and $T_H(N)$ spanned by $\{\partial_\alpha\}$ and $\{\partial_i\}$ respectively [8]. Then we have,

$$dim T_V(N) = dim T_H(N) = n. \tag{1.5}$$

The Riemannian material structure on $T(N)$ is given by

$$G = g_{ij}(x^i, y^\alpha) dx^i \otimes dx^j + g_{ab}(x^i, y^\alpha) \delta y^\alpha \otimes \delta y^b, \tag{1.6}$$

where $g_{ij}(x^i, y^\alpha) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2}\partial_a \partial_b L(x^i, y^\alpha)$ and $L(x^i, y^\alpha)$ denotes the Lagrange function. The manifold referred as Lagrangian manifold [2].

Let X be an element of $T(N)$, then

$$X = \bar{X}^i \partial_i + X^\alpha \partial_\alpha. \tag{1.7}$$

The automorphism $J : \chi(T(N)) \to \chi(T(N))$ given as

$$JX = \bar{X}^i \partial_i + X^\alpha \partial_\alpha \tag{1.8}$$

is a natural almost product structure on $T(N)$ that is $J^2 = I, I$ denotes the identity operator. The projection morphisms of $T(N)$ onto $T_V(N)$ and $T_H(N)$ denoted by v and h respectively, then we have

$$J_0 h = v_0 J. \tag{1.9}$$

2. The $F(\pm a^2, \pm b^2)$-structure

Let $T_V(N)$ be the vertical space and F_v a non-zero tensor field of type (1,1) satisfying [10]

$$(F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2) = 0, \tag{2.1}$$

where a, b are real or complex constants, then the vertical space $T_V(N)$ admits $F(\pm a^2, \pm b^2)$-structure. The rank $(F_v) = r$ and such structure is called Lagrange vertical structure on $T_V(N)$.

Theorem 2.1. *Let $T_V(N)$ be a vertical space ad F_v Lagrange vertical structure on $T_V(N)$. Then the structure define on the subspace $T_H(N)$ with respect to almost product strcture of $T(N)$.*

Proof: Suppose that

$$F_h = JF_vJ, \tag{2.2}$$

then F_h is a tensor field of type (1,1) on $T_H(N)$, where J is an almost product structure on $T(N)$.

Apply F_h on both sides we get

$$F_h^2 = (JF_vJ)(JF_vJ) = JF_v^2J,$$

$$F_h^3 = JF_v^3J$$

and so on.

In the view of equation (2.1), we have

$$\begin{aligned}(2.3) \qquad & (F_h^2 - a^2)(F_h^2 + a^2)(F_h^2 - b^2)(F_h^2 + b^2) \\ &= J((F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2))J \\ &= 0,\end{aligned}$$

Hence, F_h gives $F(\pm a^2, \pm b^2)$-structure on $T_H(N)$.

Theorem 2.2. *Let $T_V(N)$ be a vertical space ad F_v Lagrange vertical structure on $T_V(N)$. Then the similar structure define on the enveloping space $T(N)$ by using projection morphism of $T(N)$.*

Proof: In the view of Theorem (2.1), the projection morphisms of $T_V(N)$ and $T_H(N)$ on $T(N)$ denoted by v and h respectively then we have

$$F = F_vh + F_vv \tag{2.4}$$

As $hv = vh = 0$ and $h^2 = h, v^2 = v$, we obtain

$$F^2 = F_h^2h + F_v^2v$$

Now,

$$\begin{aligned} & (F^2 - a^2)(F^2 + a^2)(F^2 - b^2)(F^2 + b^2) \\ &= (F_h^2 - a^2)(F_h^2 + a^2)(F_h^2 - b^2)(F_h^2 + b^2)h \\ (2.5) \qquad & +(F_v^2 - a^2)(F_v^2 + a^2)(F_v^2 - b^2)(F_v^2 + b^2)v\end{aligned}$$

By theorem 2.1, we have

$$(F^2 - a^2)(F^2 + a^2)(F^2 - b^2)(F^2 + b^2) = 0.$$

As $rank(F_v) = rank(F_h) = r$,

Hence, $rank(F) = 2r$.

Let us define tensor fields p and q of type (1,1) on $T(N)$ with $F(\pm a^2, \pm b^2)$-structure of rank $2r$ as follows

$$p = \frac{(F^2 + a^2)(F^2 - a^2)}{b^4 - a^4}$$

$$q = \frac{(F^2 + b^2)(F^2 - b^2)}{a^4 - b^4} \tag{2.6}$$

Then it is easy to show that

$$p^2 = p, \quad q^2 = q, \quad pq = qp = 0, \quad p + q = I. \tag{2.7}$$

This implies that p and q are complementary projection operators [4, 5, 7].

3. Parallelism of distributions

Suppose that N be Lagrangian manifold with $F(\pm a^2, \pm b^2)$-structure on $T(N)$ and let P and Q complementary distributions corresponding to complementary projection operators p and q respectively. The linear connection $\bar{\nabla}$ and $\tilde{\nabla}$ are given by [2]

$$\bar{\nabla}_X Y = p\nabla_X(pY) + q\nabla_X(qY) \tag{3.1}$$

and

$$\tilde{\nabla}_X Y = p\nabla_{pX}(pY) + q\nabla_{qX}(qY) + p[qX, pY] + q[pX, qY]. \tag{3.2}$$

We have the following definitions [3, 6]:

∇-parallel: The distribution P is said ∇-parallel if $\forall X \in P, Y \in T(N)$ implies that $\nabla_Y X \in P$.

∇-half parallel: The distribution P is said ∇-half parallel if $\forall X \in P, Y \in T(N)$, $(\Delta F)(X, Y) \in P$ where

$$(\Delta F)(X, Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y(FX) \tag{3.3}$$

∇-anti half parallel: The distribution P is said ∇-anti half parallel if for all $X \in P, Y \in T(N), (\Delta F)(X, Y) \in Q$.

Theorem 3.1. *On the $F(\pm a^2, \pm b^2)$-structure manifold, the complementary distributions namely P and Q are $\bar{\nabla}$-parallel and $\tilde{\nabla}$-parallel.*

Proof: By using the equations (3.1), (3.2) and $pq = qp = 0, \quad q^2 = q$, we obtain

$$q\bar{\nabla}_X Y = q\nabla_X(qY)$$

If $Y \in P, qY = 0$ so $q\bar{\nabla}_X Y = 0 \to \bar{\nabla}_X Y = 0$, as $qY = 0$ because Y is an element of P.

This implies that $\bar{\nabla}_X Y \in P$.

Thus, $\forall Y \in P, \forall X \in T(N) \Rightarrow \bar{\nabla}_X Y \in P$.

Hence P is $\bar{\nabla}$-parallel.

In a similar way $\forall X \in T(N), \forall Y \in P$

$\tilde{\nabla}_X Y = q\nabla_{qX}(qY) + q[pX, qY] = 0$ as $qY = 0$.

So $\tilde{\nabla}_X Y \in P$.

Thus P is $\tilde{\nabla}$-parallel.

In a similar way, it can be shown that distribution Q is $\bar{\nabla}$ as well as $\tilde{\nabla}$ parallel.

Theorem 3.2. *On the $F(\pm a^2, \pm b^2)$-structure manifold, the complementary distributions namely P and Q are ∇-parallel iff $\bar{\nabla} = \tilde{\nabla}$.*

Proof: Let distributions P and Q are ∇-parallel. By definition of ∇-parallel, we have

$$q\nabla_X(pY) = 0, \quad p\nabla_X(qY) = 0.$$

where X and Y are elements of $T(N)$.

Using equation (2.7), we get

$$\nabla_X(pY) = p\nabla_X(pY) \tag{3.4}$$

and

$$\nabla_X(qY) = q\nabla_X(qY) \tag{3.5}$$

Thus

$$\nabla_X Y = p\nabla_X(pY) + q\nabla_X(qY) = \bar{\nabla}_X Y.$$

This shows that $\nabla = \bar{\nabla}$.

The converse of the theorem showed easily.

Theorem 3.3. *On the $F(\pm a^2, \pm b^2)$-structure manifold N, the complementary distribution M is $\bar{\nabla}$-anti half parallel if*

$$q\bar{\nabla}_Y(FX) = q\nabla_{FX}qY.$$

where X is an element of Q and Y element of $T(N)$.

Proof: Let $\bar{\nabla}$ be linear connection on N. Then by using equations (3.3) and (2.7), we obtain

$$q(\Delta F)(X, Y) = q\bar{\nabla}_Y FX - q\bar{\nabla}_{FX} Y, \quad as \quad qF = Fq = 0. \tag{3.6}$$

Making use of the equation (3.1), the obtained equation is

$$\bar{\nabla}_{FX} Y = p\nabla_{FX}(pY) + q\nabla_{FX}(qY)$$

operating q on both sides of above equation and using $pq = 0, q^2 = q$, we get

$$q\bar{\nabla}_{FX} Y = q\nabla_{FX}(qY)$$

and
$$q(\Delta F)(X,Y) = q\bar{\nabla}_Y FX - q\bar{\nabla}_{FX}Y,$$
as $(\Delta F)(X,Y) \in P$ so $q(\Delta F)(X,Y) = 0$.

Hence,
$$q\bar{\nabla}_Y(FX) = q\nabla_{FX}(qY),$$
This completes the proof.

3.1. Geodesics on the Lagrangian manifold

Let T be tangent to the curve γ in N. The curve γ is said the geodesic concernig to the connection ∇ if $\nabla_T T$ [6].

Theorem 3.4. *A curve γ is said to be geodesic concerning to connection $\bar{\nabla}$ if the vector fields $\nabla_T T - \nabla_T(qT) \in Q$ and $\nabla_T(qT) \in P$.*

Proof: The curve γ is said to be geodesic concerning to the connection $\bar{\nabla}$, we have $\bar{\nabla}_T T = 0$.

In the view of the equation (3.1), $\bar{\nabla}_T T = 0$ becomes

$$p\nabla_T(pT) + q\nabla_T(qT) = 0, \tag{3.7}$$

Using the equation (2.7), the equation (3.7) becomes
$$p\nabla_T(I-q)T + q\nabla_T(qT) = 0$$
or
$$p\nabla_T T - p\nabla_T(qT) + q\nabla_T(qT) = 0.$$
or
$$p(\nabla_T T - \nabla_T(qT)) \;\; and \;\; q\nabla_T(qT) = 0.$$

Hence, $\nabla_T T - \nabla_T(qT) \in Q$ and $\nabla_T(qT) \in P$.

This completes the proof.

Theorem 3.5. *The tensor fields p and q of type (1,1) are always covariantly constants concerning to connection $\bar{\nabla}$.*

Proof: Let X and Y be elements of $T(N)$, then

$$(\bar{\nabla}_X p)(Y) = \bar{\nabla}_X(pY) - p\bar{\nabla}_X Y. \tag{3.8}$$

From equation (3.1), we have

$$(\bar{\nabla}_X p)(Y) = p\nabla_X(p^2Y) + q\nabla_X(qpY) - p\{p\nabla_X pY + q\nabla_X qY)\}$$

Using the properties $p^2 = p, q^2 = q, pq = qp = 0$, we have
$$(\bar{\nabla}_X p)(Y) = p\nabla_X(pY) - p\nabla_X pY = 0.$$

This shows that p is covariantly constant. In similar way, q is covariantly constant can be proved easily.

REFERENCES

1. Yano, K., Ishihara, S.: Tangent and Cotangent bundles, Mercel Dekker, Inc., New York, 1973.
2. Nikic, J.: *Distribution's parallelism and geodesics in the* $F(3,\epsilon)$*-structure Lagrangian manifold.* Novi Sad J. Math. 27(2), 117-125 (1997).
3. Nikic, J., Comic, I: *The Recurrent and Metric Connection andf-structures in Gauge Spaces of Second Order*, FACTA UNIVERSITATIS Series: Mechanics, Automatic Control and Robotics, 5(1), 91-98, (2006).
4. Yano, K.: *On a structure defined by a tensor field* f *of type (1,1) satisfying* $F^3 + f = 0$. Tensor N. S. 14, 99-109 (1963).
5. Das, Lovejoy, Nivas, R.: *On a differentiable manifold with* $[F_1, F_2](K+1,1)$*-structure.* Tensor N. S. 65, 29-35 (2004).
6. Singh, A., Pandey, A.K. and Khare, S.: *Parallelism of distributions and geodesics on* $F(2K+S;S)$*-structure Lagrangian manifolds*, International Journal of Contemporary Mathematical Sciences, 9(11), 515-522 (2014).
7. Khan, M. N. I., Jun, J. B.: *Lorentzian Almost r-para-contact Structure in Tangent Bundle*, Journal of the Chungcheong Mathematical Society 27 (1), 29-34 (2014).
8. Das, Lovejoy: *Fiberings on almost* r*-contact manifolds.* Publications Mathematicae, Debrecen, Hongrie. 43(1-2), 1-7 (1993).
9. Das, Lovejoy: *On CR-structure and* F*-structure satisfying* $F^K + (-)^{K+1}F = 0$,. Rocky Mountain Journal of Mathematics. 36(3), 885-892 (2006).
10. Das, Lovejoy, Nivas, R. and Saxena, M.: *A structure defined by a tensor field of type (1,1) satisfying* $(f^2 - a^2)(f^2 + a^2)(f^2 - b^2)(f^2 + b^2) = 0$. Tensor N.S., 65, 36-41 (2004).
11. Das, Lovejoy, Nivas, R.: *On certain structures defined on the tangent bundle.* Rocky Mountain Journal of Mathematics. 36(6), 1857-1866 (2006).

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 165-177
https://doi.org/10.22190/FUMI200421014K

Original Scientific Paper

(ω, c)- PSEUDO ALMOST PERIODIC FUNCTIONS, (ω, c)- PSEUDO ALMOST AUTOMORPHIC FUNCTIONS AND APPLICATIONS

Mohammed Taha Khalladi[1], Marko Kostić[2], Abdelkader Rahmani[3] and Daniel Velinov[4]

[1]**Department of Mathematics and Computer Sciences, University of Adrar, Adrar, Algeria**
[2]**Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6 73, 21125 Novi Sad, Serbia**
[3]**Laboratory of Mathematics, Modeling and Applications (LaMMA), University of Adrar, Adrar, Algeria**
[4]**Faculty of Civil Engineering, Ss. Cyril and Methodius University, Partizanski Odredi,24, P.O. box 560, 1000 Skopje, North Macedonia**

Abstract. In this paper, we introduce the classes of (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions. These collections include (ω, c)-pseudo periodic functions, pseudo almost periodic functions and their automorphic analogues. We present an application to the abstract semilinear first-order Cauchy inclusions in Banach spaces.
Keywords: (ω, c)-Pseudo almost periodic functions, (ω, c)-pseudo almost automorphic functions, abstract semilinear Cauchy inclusions

1. Introduction and Preliminaries

The theory of almost periodic functions and almost automorphic functions is an attractive field of investigation, which has a significant role in the qualitative theory of ordinary and partial differential equations, physics, mathematical biology and control theory.

The classes of (ω, c)-periodic functions and (ω, c)-pseudo periodic functions were introduced by Alvarez, Gómez, Pinto in [3] and Alvarez, Castillo, Pinto in [4],

Received April 21, 2020; accepted June 21, 2020.
Corresponding Author: Marko Kostić, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6 73, 21125 Novi Sad, Serbia | E-mail: marco.s@verat.net
2010 *Mathematics Subject Classification.* Primary 34C25; Secondary 42A75, 43A60

motivated by some known results regarding the qualitative properties of solutions to the Mathieu linear second-order differential equation

$$y''(t) + [a - 2q\cos 2t]y(t) = 0,$$

arising in seasonally forced population dynamics. The authors of [3] have analyzed the existence and uniqueness of mild (ω, c)-periodic solutions to the abstract semilinear integro-differential equation

$$D^{\gamma}_{t,+}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \quad t \in \mathbb{R},$$

where A is a closed linear operator, $a \in L^1([0,\infty))$ is a scalar-valued kernel and $f(\cdot,\cdot)$ satisfies some Lipschitz type conditions. Further on, Alvarez, Castillo and Pinto have analyzed in [4] the existence and uniqueness of mild (ω, c)-pseudo periodic solutions to the abstract semilinear differential equation of the first order:

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A generates a strongly continuous semigroup. The authors have proved the existence of positive (ω, c)-pseudo periodic solutions to the Lasota-Wazewska equation with (ω, c)-pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geq 0.$$

This equation describes the survival of red blood cells in the blood of an animal. (ω, c)-Pseudo periodic functions can be also solutions of the time varying impulsive differential equations and the linear delayed equations; for further information about applications of (ω, c)-pseudo periodic functions, we refer the reader to [8] and references cited therein.

In our recent paper [8], we have introduced and analyzed various generalizations of the concept of (ω, c)-periodicity. Among others, we have defined and analyzed the classes of (asymptotically) (ω, c)-almost periodic functions and (asymptotically) (ω, c)-almost automorphic functions. The main aim of this paper is to analyze the classes of (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions by taking into consideration the class of pseudo ergodic components introduced by C. Zhang [13]. We introduce two new types of (ω, c)-pseudo ergodic components and two new classes of (ω, c)-almost periodic ((ω, c)-almost automorphic) functions. It is our strong belief that these classes of functions will attract the attention of our readers and serve for some new applications in the theory of abstract differential equations soon.

The organization of paper is briefly described as follows. After recalling the basic definitions from the theory of almost periodic functions and almost automorphic functions in Subsection 1.2, we introduce the classes of (ω, c, i)-almost periodic functions, resp. (ω, c, i)-almost automorphic functions, and (ω, c, i)-pseudo ergodic vanishing components in Definition 2.2 and Definition 2.3; in Definition

2.4, we introduce the notion of an (ω, c)-pseudo almost periodic function, resp. an (ω, c)-pseudo almost automorphic function, and the notion of a two-parameter (ω, c, i)-pseudo almost periodic function, resp. two-parameter (ω, c, i)-pseudo almost automorphic function ($i = 1, 2$). After that, we clarify some basic results about the class of (ω, c)-pseudo almost periodic functions, resp. (ω, c)-pseudo almost automorphic functions, depending on one variable. Subsection 2.1 investigates composition principles for introduced classes and Section 3 provides an interesting application in the qualitative analysis of (ω, c)-pseudo almost periodic solutions of the abstract semilinear Cauchy inclusions of the first order.

We use the standard notation throughout the paper. Let $I = \mathbb{R}$ or $I = [0, \infty)$; unless stated otherwise, we will always assume henceforth that $f : I \to E$ is a continuous function. By $C(I : E)$, $C_b(I : E)$ and $C_0(I : E)$ we denote the vector spaces consisting of all continuous functions $f : I \to E$, all bounded continuous functions $f : I \to E$ and all bounded continuous functions $f : I \to E$ satisfying that $\lim_{|t| \to +\infty} \|f(t)\| = 0$. As is well known, $C_b(I : E)$ and $C_0(I : E)$ are Banach spaces equipped with the sup-norm, denoted by $\| \cdot \|_\infty$. If X is also a complex Banach space, then by $L(E, X)$ we denote the space consisting of all bounded continuous mappings from E into X; $L(E) \equiv L(E, E)$. The principal branches are always used for taking the powers of complex numbers.

1.1. Almost Periodic Functions, Almost Automorphic Functions and Their Generalizations

Let $I = [0, \infty)$ or $I = \mathbb{R}$. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if $\|f(t + \tau) - f(t)\| \leq \epsilon, \quad t \in I$. The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$. The vector space consisting of all almost periodic functions is denoted by $AP(I : E)$.

Let $f : \mathbb{R} \to E$ be continuous. Then it is said that $f(\cdot)$ is almost automorphic if and only if for every real sequence (b_n) there exists a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to E$ such that $\lim_{n \to \infty} f(t + a_n) = g(t)$ and $\lim_{n \to \infty} g(t - a_n) = f(t)$, pointwise for $t \in \mathbb{R}$. The space consisting of all almost automorphic functions will be denoted by $AA(\mathbb{R} : E)$.

A function $f : I \times X \to E$ is called almost periodic if and only if $f(\cdot, \cdot)$ is bounded continuous as well as for every $\epsilon > 0$ and every compact $K \subseteq X$ there exists $l(\epsilon, K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\epsilon, K)$ contains a number τ with the property that $\|f(t + \tau, x) - f(t, x)\| \leq \epsilon$ for all $t \in I$, $x \in K$. The collection of such functions will be denoted by $AP(I \times X : E)$. Observe that we require the boundedness of function $f(\cdot, \cdot)$ a priori, which is not the common case in the existing literature. This is also not the case in the usual definition of an almost automorphic function depending on two variables, given as follows. A continuous function $F : \mathbb{R} \times X \to E$ is said to be almost automorphic if and only if for every sequence of real numbers (s_n') there exists a subsequence (s_n) such that $G(t,x) := \lim_{n\to\infty} F(t+s_n,x)$ is well defined for each $t \in \mathbb{R}$ and $x \in X$, and $\lim_{n\to\infty} G(t-s_n,x) = F(t,x)$ for each $t \in \mathbb{R}$ and $x \in X$. The vector space consisting of such functions will be denoted by $AA(\mathbb{R} \times X : E)$.

By $PAP_0(\mathbb{R} : E)$ we denote the space consisting of all bounded continuous functions $\Phi : \mathbb{R} \to E$ such that $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\Phi(s)\|\, ds = 0$. For example, it is well known that $f \in PAP_0(\mathbb{R} : \mathbb{C})$ if and only if $f \cdot f \in PAP_0(\mathbb{R} : \mathbb{C})$. Moreover, let us define

$$f(t) := \frac{1}{2t}\int_{-t}^{t} s|\sin s|^{s^N}\, ds, \quad t \in \mathbb{R},$$

where $N > 6$. From [1, Example p. 1143] we know that $\lim_{t\to+\infty} f(t) = 0$ and therefore $\cdot|\sin\cdot|^{\cdot^N} \in PAP_0(\mathbb{R} : \mathbb{C})$ for $N > 6$.

By $PAP_0(\mathbb{R} \times X : E)$ we denote the space consisting of all continuous functions $\Phi : \mathbb{R} \times X \to E$ such that $\{\Phi(t,x) : t \in \mathbb{R}\}$ is bounded for all $x \in X$, and $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|\Phi(s,x)\|\, ds = 0$, uniformly in bounded sets of X. A function $f \in C_b(\mathbb{R} : X)$ is said to be pseudo-almost periodic, resp. pseudo-almost automorphic, if and only if it admits a decomposition $f(t) = g(t)+q(t)$, $t \in \mathbb{R}$, where $g \in AP(\mathbb{R} : E)$, resp. $g \in AA(\mathbb{R} : E)$, and $q \in PAP_0(\mathbb{R} : E)$. The parts $g(\cdot)$ and $q(\cdot)$ are called the almost periodic part of $f(\cdot)$, resp. the almost automorphic part of $f(\cdot)$, and the ergodic perturbation of $f(\cdot)$. The vector space consisting of such functions is denoted by $PAP(\mathbb{R} : E)$, resp. $PAA(\mathbb{R} : E)$; the sup-norm turns $PAP(\mathbb{R} : E)$, resp. $PAA(\mathbb{R} : E)$, into a Banach space ([13]).

For more details about almost periodic type functions and almost automorphic type functions, we refer the reader to the research monographs [5, 6, 7, 9, 12].

2. (ω, c)-Pseudo Almost Periodic Functions and (ω, c)-Pseudo Almost Automorphic Functions

Unless specified otherwise, in the remainder of paper we will always assume that $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$. The following definition has been recently introduced in [8].

Definition 2.1. It is said that a continuous function $f : I \to E$ is (ω, c)-almost periodic, resp. (ω, c)-almost automorphic, if and only if the function $f_{\omega,c}(\cdot)$, defined by $f_{\omega,c}(t) := c^{-(t/\omega)}f(t)$, $t \in I$, is almost periodic, resp. almost automorphic. By $AP_{\omega,c}(I : E)$, resp. $AA_{\omega,c}(I : E)$, we denote the space consisting of all (ω, c)-almost periodic functions, resp. all (ω, c)-almost automorphic functions.

Let us recall that $AP_{\omega,c}(I : E)$, resp. $AA_{\omega,c}(I : E)$, is a vector space with the usual operations of addition of functions and pointwise multiplication of functions with scalars ([8]). Furthermore, the space $AP_{\omega,c}(I : E)$, resp. $AA_{\omega,c}(I : E)$, equipped with the norm $\|\cdot\|_{\omega,c}$, where

$$\|f\|_{\omega,c} := \sup_{t\in I}\left\|c^{-\frac{t}{\omega}}f(t)\right\|,$$

is a Banach space.

With the exception of consideration preceding Definition 2.3, in the remainder of paper we will deal with the interval $I = \mathbb{R}$, only. Let us recall the (ω, c)-mean of a function $h : \mathbb{R} \to E$ is introduced in [4] by

$$\mathcal{M}_{\omega,c}(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c^{-\sigma/\omega} h(\sigma)\, d\sigma,$$

whenever the limit exists. For example, for $h_1(t) = c^{t/\omega}$ and $h_2(t) = c^{t/\omega} e^{it}$, we have that $\mathcal{M}_{\omega,c}(h_1) = 1$ and $\mathcal{M}_{\omega,c}(h_2) = 0$. Furthermore, $\mathcal{M}_{\omega,c}$ is a linear and continuous operator. Indeed, if $c^{-t/\omega} h_n(t) \to c^{-t/\omega} h(t)$ uniformly as $n \to \infty$, then $\mathcal{M}_{\omega,c}(h_n) \to \mathcal{M}_{\omega,c}(h)$ as $n \to \infty$.

Remark 2.1. If $h(\cdot)$ is (ω, c)-almost periodic in the sense of Definition 2.1, then the mean $\mathcal{M}_{\omega,c}(h)$ always exists, because the function $c^{-(\cdot/\omega)} f(\cdot)$ is almost periodic and the usual mean value of any almost periodic function exists.

In this paper, we will use the space

$$PAP_{0;\omega,c}(\mathbb{R} : E) := \big\{h \in C(\mathbb{R} : E)\ ;\ c^{-\cdot/\omega} h(\cdot) \in PAP_0(\mathbb{R} : E)\big\}.$$

A function $h(\cdot)$ is said to be c-ergodic if and only if belongs to this space. Therefore, the ergodic space of Zhang ([13]) can be recovered by plugging $c = 1$ in the above definition.

Furthermore, we will use the following two types of (ω, c)-pseudo ergodic components:

Definition 2.2. Let $c \in \mathbb{C}\backslash\{0\}$ and $\omega > 0$.

(i) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 1)$-pseudo ergodic vanishing if and only if $c^{-t/\omega} f(t, \cdot) \in PAP_0(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $PAP_{0;\omega,c,1}(\mathbb{R} \times X : E)$.

(ii) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 2)$-pseudo ergodic vanishing if and only if $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in PAP_0(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $PAP_{0;\omega,c,2}(\mathbb{R} \times X : E)$.

Similarly, we will use two different types of (ω, c)-almost periodic functions, resp. (ω, c)-almost automorphic functions, depending on two variables (albeit some composition principles for two-parameter (ω, c)-almost periodic functions have been clarified in [8], we have not explicitly defined the notion of a two-parameter (ω, c)-almost periodic function there; the notion introduced in Definition 2.3 should not be mistakenly identified with the notion of an (ω, c)-almost periodic function of type 1 (type 2), introduced and analyzed in [8, Section 3]).

Definition 2.3. Let $c \in \mathbb{C}\backslash\{0\}$, $\omega > 0$ and $i = 1, 2$.

(i) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 1)$-almost periodic, resp. $(\omega, c, 1)$-almost automorphic, if and only if $c^{-t/\omega} f(t, \cdot) \in AP(\mathbb{R} \times X : E)$, resp. $c^{-t/\omega} f(t, \cdot) \in AA(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $AP_{\omega,c,1}(\mathbb{R} \times X : E)$, resp. $AA_{\omega,c,1}(\mathbb{R} \times X : E)$.

(ii) A function $f \in C(\mathbb{R} \times X : E)$ is said to be $(\omega, c, 2)$-almost periodic, resp. $(\omega, c, 2)$-almost automorphic, if and only if $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in AP(\mathbb{R} \times X : E)$, resp. $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in AA(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $AP_{\omega,c,2}(\mathbb{R} \times X : E)$, resp. $AA_{\omega,c,2}(\mathbb{R} \times X : E)$.

In [8], we have analyzed the classes of asymptotically (ω, c)-almost periodic functions, resp. asymptotically (ω, c)-almost automorphic functions, defined on the non-negative real axis by adding the usual ergodic components from the space $C_0([0, \infty) : E)$ to the principal components, which are (ω, c)-almost periodic functions, resp. (ω, c)-almost automorphic functions. In order to stay consistent with the notion introduced in [4, Definition 2.5], we will slightly change the approach obeyed in [8] and use the following notion in case $I = \mathbb{R}$:

Definition 2.4. Let $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and $i = 1, 2$.

(i) A function $f \in C(\mathbb{R} : E)$ is said to be (ω, c)-pseudo almost periodic, resp. (ω, c)-pseudo almost automorphic, if and only if it admits a decomposition $f(t) = g(t) + h(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is (ω, c)-almost periodic, resp. (ω, c)-almost automorphic, and $h \in PAP_{0;\omega,c}(\mathbb{R} : E)$. The space of all such functions will be denoted by $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$.

(ii) A function $f(\cdot, \cdot) \in C(\mathbb{R} \times X : E)$ is said to be (ω, c, i)-pseudo almost periodic, resp. (ω, c, i)-pseudo almost automorphic, if and only if it admits a decomposition $f(t, x) = g(t, x) + h(t, x)$, $t \in \mathbb{R}$, $x \in X$, where $g(\cdot, \cdot)$ is (ω, c, i)-almost periodic, resp. (ω, c, i)-almost automorphic, and $h(\cdot, \cdot) \in PAP_{0;\omega,i}(\mathbb{R} \times X : E)$. The space of all such functions will be denoted by $PAP_{\omega,c,i}(\mathbb{R} \times X : E)$, resp. $PAA_{\omega,c,i}(\mathbb{R} \times X : E)$.

For simplicity, we will not consider here the class of (ω, c)-pseudo compactly almost automorphic functions; for some applications of compactly almost automorphic functions, the reader may consult the article [2] by Ait Dads, Boudchich, Es-sebbar and references cited therein.

Theorem 2.1. *Let $f \in C(\mathbb{R} : E)$. Then $f(\cdot)$ is (ω, c)-pseudo almost periodic, resp. (ω, c)-pseudo almost automorphic, if and only if:*

$$f(t) \equiv c^{\wedge}(t) u(t), \quad \text{with } c^{\wedge}(t) \equiv c^{t/\omega},\ u \in PAP(\mathbb{R} : E), \tag{2.1}$$

resp.

$$f(t) \equiv c^{\wedge}(t) u(t), \quad \text{with } c^{\wedge}(t) \equiv c^{t/\omega},\ u \in PAA(\mathbb{R} : E).$$

Proof. We will consider only (ω, c)-pseudo almost periodic functions for simplicity. It is clear that if $f(\cdot)$ satisfies (2.1), then $f(\cdot)$ is an (ω, c)-pseudo almost periodic function. In order to show the converse statement, let $f \in PAP_{\omega,c}(\mathbb{R} : E)$. Then there exists $g \in AP_{\omega,c}(\mathbb{R} : E)$ and $PAP_{0;\omega,c}(\mathbb{R} : E)$ such that $f = g + h$. Therefore,

$$u(t) = c^{-t/\omega} g(t) + c^{-t/\omega} h(t) = F_1(t) + F_2(t), \quad t \in \mathbb{R}.$$

So, $u(t)$ is written as a sum of $F_1(\cdot)$ which is almost periodic and $F_2(\cdot)$ which belongs to $PAP_{0;\omega,c}(\mathbb{R} : E)$. $\square$

Remark 2.2. Let us note that the decompositions given in Definition 2.4 are unique; see also [4, Remark 2.9]. The proof of this simple fact can be left to the interested readers.

It can be simply shown that:

(i) We have $f + g \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f + g \in PAA_{\omega,c}(\mathbb{R} : E)$, and $\alpha h \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $\alpha h \in PAA_{\omega,c}(\mathbb{R} : E)$, provided $f,\ g,\ h \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f,\ g,\ h \in PAA_{\omega,c}(\mathbb{R} : E)$, and $\alpha \in \mathbb{C}$.

(ii) If $\tau \in \mathbb{R}$ and $f \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : E)$, then $f_\tau(\cdot) \equiv f(\cdot + \tau) \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f_\tau(\cdot) \in PAA_{\omega,c}(\mathbb{R} : E)$.

Now we would like to endow the introduced space of (ω, c)-pseudo almost periodic functions, resp. (ω, c)-pseudo almost automorphic functions, with a certain norm.

Proposition 2.1. *The space $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$, equipped with the norm $\|\cdot\|_{\omega,c}$ is a Banach space.*

Proof. We will consider the space $PAP_{\omega,c}(\mathbb{R} : E)$, only. Let (f_n) be a Cauchy sequence in $PAP_{\omega,c}(\mathbb{R} : E)$. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m,\ n \geq N$, we have

$$\|f_n - f_m\|_{\omega,c} < \epsilon.$$

Since $f_m,\ f_n \in PAP_{\omega,c}(\mathbb{R} : E)$, Theorem 2.1 implies that there exists $u_m,\ u_n \in PAP(\mathbb{R} : E)$ such that $f_m(t) \equiv c^\wedge(t) u_m(t)$ and $f_n(t) \equiv c^\wedge(t) u_n(t)$ for all $t \in \mathbb{R}$. Now, for $m,\ n \geq N$ we have $\|u_m - u_n\|_\infty \leq \|f_n - f_m\|_{\omega,c} < \epsilon$. It follows that (u_n) is a Cauchy sequence in $PAP(\mathbb{R} : E)$. Since $PAP(\mathbb{R} : E)$ is complete, there exists $u \in PAP(\mathbb{R} : E)$ such that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Let us define $f(t) := c^\wedge(t) u(t)$, $t \in \mathbb{R}$. We claim that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Indeed, $\|f_n - f\|_{\omega,c} = \sup_{t\in\mathbb{R}} \|u_n(t) - u(t)\| \to 0 \quad (n \to \infty)$. Hence, $PAP_{\omega,c}(\mathbb{R} : E)$ is a Banach space with the norm $\|\cdot\|_{\omega,c}$. $\square$

Lemma 2.1. ([4]) *Assume that $k^\sim(\cdot) := c^\wedge(-\cdot)k(\cdot) \in L^1(\mathbb{R})$. Then $h \in PAP_{0;\omega,c}(\mathbb{R} : E)$ implies that $k * h \in PAP_{0;\omega,c}(\mathbb{R} : E)$.*

Theorem 2.2. *Let $f \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : E)$, with $f(\cdot) = c^{\wedge}(\cdot)p(\cdot)$, $p \in PAP(\mathbb{R} : E)$, resp. $p \in PAA(\mathbb{R} : E)$. If for some $k(\cdot)$ we have that $k^{\sim}(\cdot) := c^{\wedge}(-\cdot)k(\cdot) \in L^1(\mathbb{R})$, then*

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds = c^{\wedge}(t)\,(k^{\sim} * p)\,(t), \quad t \in \mathbb{R}.$$

In particular, $k * f \in PAP_{\omega,c}(\mathbb{R} : E)$, resp. $k * f \in PAA_{\omega,c}(\mathbb{R} : E)$.

Proof. As before, we will consider the space $PAP_{\omega,c}(\mathbb{R} : E)$ only, because the proof is quite analogous for the space $PAA_{\omega,c}(\mathbb{R} : E)$. Since $p \in PAP(\mathbb{R} : E)$, we have that there exists $p_1 \in AP(\mathbb{R} : E)$ and $p_2 \in PAP_0(\mathbb{R} : E)$ such that $p = p_1 + p_2$. Then $f = f_1 + f_2$, where $f_1(\cdot) = c^{\wedge}(\cdot)p_1(\cdot) \in AP_{\omega,c}(\mathbb{R} : E)$ and $f_2(\cdot) = c^{\wedge}(\cdot)p_1(\cdot) \in PAP_{0;\omega,c}(\mathbb{R} : E)$. For every $t \in \mathbb{R}$, we have

$$\begin{aligned}(k * f)(t) &= \int_{-\infty}^{\infty} k(t-s)f(s)\,ds \\ &= \int_{-\infty}^{\infty} k(t-s)f_1(s)\,ds + \int_{-\infty}^{\infty} k(t-s)f_2(s)\,ds \\ &= (k * f_1)\,(t) + (k * f_2)\,(t) =: I_1(t) + I_2(t).\end{aligned}$$

We have that $I_1 \in AP_{\omega,c}(\mathbb{R} : E)$; see [8]. Next, by Lemma 2.1, we have that $I_2 \in PAP_{0;\omega,c}(\mathbb{R} : E)$. Moreover, by definition of $f(\cdot)$, we have $(k * f)(\cdot) = c^{\wedge}(\cdot)\,(k^{\sim} * p)\,(\cdot)$ so that $k * f \in PAP_{\omega,c}(\mathbb{R} : E)$. □

Example 2.1. ([12]) Let us consider the heat equation $u_t(x,t) = u_{xx}(x,t)$, $t > 0$, $x \in \mathbb{R}$, with the initial value condition $u(x,0) = f(x)$. Let $u(x,t)$ be a regular solution satisfying the initial value condition. It is well known that

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\,ds, \quad t > 0,\ x \in \mathbb{R}.$$

Fix $t_0 > 0$ and assume that $f(\cdot)$ is an (ω, c)-pseudo almost periodic function. Then, by Theorem 2.2, the solution $u(x,t_0)$ is (ω, c)-pseudo almost periodic with respect to x.

2.1. Composition principles

In this subsection, we will use two lemmae. The first one is a slight extension of the well known result of H.-X. Li, F.-L. Huang and J.-Y. Li [10, Theorem 2.1], clarified recently in [9, Lemma 2.12.2]:

Lemma 3.1. *Let $f \in PAP(\mathbb{R} \times X : E)$ and $u \in PAP(\mathbb{R} : X)$. Then the mapping $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ belongs to the space $PAP(\mathbb{R} : E)$ provided that the following conditions hold:*

(i) *The set* $\{f(t,x) : t \in \mathbb{R},\ x \in B\}$ *is bounded for every bounded subset* $B \subseteq X$.

(ii) $f(t,x)$ *is uniformly continuous in each bounded subset of* X *uniformly in* $t \in \mathbb{R}$. *That is, for any* $\epsilon > 0$ *and* $B \subseteq X$ *bounded, there exists* $\delta > 0$ *such that* $x,\ y \in B$ *and* $\|x - y\| \leq \delta$ *imply* $\|f(t,x) - f(t,y)\| \leq \epsilon$ *for all* $t \in \mathbb{R}$.

The second lemma is the following slight extension of the composition principle established by J. Liang et al. in [11, Theorem 2.4]:

Lemma 3.2. (see [9, Theorem 3.2.4]) *Suppose that* $f = g + \phi \in PAA(\mathbb{R} \times X : E)$ *with* $g \in AA(\mathbb{R} \times X : E)$, $\phi \in PAP_0(\mathbb{R} \times X : E)$ *and the following holds:*

(i) *the mapping* $(t,x) \mapsto g(t,x)$ *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$;

(ii) *the mapping* $(t,x) \mapsto \phi(t,x)$ *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$.

Then for each $u \in PAA(\mathbb{R} : X)$ *one has* $f(\cdot, u(\cdot)) \in PAA(\mathbb{R} : E)$.

For simplicity, we will not consider Stepanov p-almost periodic functions and Stepanov p-almost automorphic functions depending on two variables here (see [8, Section 3] for some composition principles for Stepanov (p,ω,c)-almost periodic functions).

Suppose now that a continuous function $g : \mathbb{R} \times X \to E$ satisfies $g(t+\omega, x) = cg(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$, resp. $g(t+\omega, cx) = cg(t,x)$ for all $t \in \mathbb{R}$ and $x \in X$. Define the functions

$$G_1(t,x) := c^{-\frac{t}{\omega}} g(t,x), \quad t \in \mathbb{R},\ x \in X \tag{2.2}$$

and

$$G_2(t,x) := c^{-\frac{t}{\omega}} g\big(t, c^{t/\omega} x\big), \quad t \in \mathbb{R},\ x \in X. \tag{2.3}$$

Then, for every $t \in \mathbb{R}$ and $x \in X$, we have

$$G_1(t+\omega, x) = c^{-\frac{t+\omega}{\omega}} g(t+\omega, x) = c^{-\frac{t+\omega}{\omega}} cg(t+\omega, x) = c^{-\frac{t}{\omega}} g(t,x) = G_1(t,x)$$

and

$$\begin{aligned} G_2(t+\omega, x) &= c^{-\frac{t+\omega}{\omega}} g\Big(t+\omega, c^{\frac{t+\omega}{\omega}} x\Big) = c^{-\frac{t+\omega}{\omega}} cg\big(t, c^{t/\omega} x\big) \\ &= c^{-t/\omega} g\big(t, c^{t/\omega} x\big) = G_2(t,x). \end{aligned}$$

In both cases, the function $G_i(\cdot,\cdot)$ is ω-periodic in time variable $(i = 1, 2)$. Furthermore, if the requirements of [4, Theorem 2.24] hold (case $i = 2$), then condition (i) of Lemma 3.2 holds with the function $g(\cdot,\cdot)$ replaced therein with the function $G_2(\cdot,\cdot)$, and condition (ii) of Lemma 3.2 holds with the function $\phi(\cdot,\cdot)$ replaced therein with the function $h_2(t,\cdot) \equiv c^{-t/\omega} h(t, c^{t/\omega}\cdot)$, $t \in \mathbb{R}$. Furthermore, $G_2 \in AA(\mathbb{R} \times X : E)$ and

$h_2 \in PAP_0(\mathbb{R} \times X : E)$ so that repeating verbatim the arguments used in the proof of [11, Theorem 2.4] with appealing to [3, Theorem 2.11] in place of [11, Lemma 2.2] immediately yields a much simpler proof of [4, Theorem 2.24]. Furthermore, the statement of [3, Theorem 2.11] can be formulated for continuous functions which maps the space $\mathbb{R} \times X$ into E; in other words, we can use two different pivot spaces X and E. Keeping in mind this observation, we can immediately clarify an extension of [4, Theorem 2.24] in this context (the interested reader may try to reexamine [4, Theorem 2.25] for (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions). Furthermore, using Lemma 3.2 we can immediately clarify the following result:

Proposition 3.1.

(i) *Suppose that* $f = g+\phi$ *with* $g \in AA_{\omega,c,1}(\mathbb{R}\times X : E)$, $\phi \in PAP_{0;\omega,c,1}(\mathbb{R}\times X : E)$ *and the following holds:*

  (a) *the mapping* $(t, x) \mapsto G_1(t, x)$ *given by (2.2) is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$;

  (b) *the mapping* $(t, x) \mapsto \phi_1(t, x)$ *given by (2.2), with the function* $g(\cdot, \cdot)$ *replaced therein with the function* $\phi(\cdot, \cdot)$, *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$.

  Then for each $u \in PAA(\mathbb{R} : X)$ *one has* $f(\cdot, u(\cdot)) \in PAA_{\omega,c}(\mathbb{R} : E)$.

(ii) *Suppose that* $f = g+\phi$ *with* $g \in AA_{\omega,c,2}(\mathbb{R}\times X : E)$, $\phi \in PAP_{0;\omega,c,2}(\mathbb{R}\times X : E)$ *and the following holds:*

  (c) *the mapping* $(t, x) \mapsto G_2(t, x)$ *given by (2.2) is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$;

  (d) *the mapping* $(t, x) \mapsto \phi_2(t, x)$ *given by (2.2), with the function* $g(\cdot, \cdot)$ *replaced therein with the function* $\phi(\cdot, \cdot)$, *is uniformly continuous in any bounded subset* $B \subseteq X$ *uniformly for* $t \in \mathbb{R}$.

  Then for each $u \in PAA_{\omega,c}(\mathbb{R} : X)$ *one has* $f(\cdot, u(\cdot)) \in PAA_{\omega,c}(\mathbb{R} : E)$.

Concerning possible applications of Lemma 3.1, we can immediately clarify the following result:

Proposition 3.2.

(i) *Let* $f \in PAP_{\omega,c,1}(\mathbb{R} \times X : E)$ *and* $u \in PAP(\mathbb{R} : X)$. *Then the mapping* $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ *belongs to the space* $PAP_{\omega,c}(\mathbb{R} : E)$ *provided that the following conditions hold:*

  (a) *The set* $\{c^{-t/\omega} f(t, x) : t \in \mathbb{R},\ x \in B\}$ *is bounded for every bounded subset* $B \subseteq X$.

(b) *$c^{-t/\omega}f(t,x)$ is uniformly continuous in each bounded subset of X uniformly in $t \in \mathbb{R}$.*

(ii) *Let $f \in PAP_{\omega,c,2}(\mathbb{R} \times X : E)$ and $u \in PAP_{\omega,c}(\mathbb{R} : X)$. Then the mapping $t \mapsto f(t,u(t))$, $t \in \mathbb{R}$ belongs to the space $PAP_{\omega,c}(\mathbb{R} : E)$ provided that the following conditions hold:*

(a) *The set $\{c^{-t/\omega}f(t,c^{t/\omega}x) : t \in \mathbb{R},\ x \in B\}$ is bounded for every bounded subset $B \subseteq X$.*

(b) *$c^{-t/\omega}f(t,c^{t/\omega}x)$ is uniformly continuous in each bounded subset of X uniformly in $t \in \mathbb{R}$.*

3. An Application to the Abstract Semilinear Cauchy Inclusions in Banach Spaces

Consider the semilinear fractional Cauchy inclusion

$$D_{t,+}^{\gamma}u(t) \in \mathcal{A}u(t) + f(t,u(t)),\ t \in \mathbb{R}, \tag{3.1}$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1]$, $f : \mathbb{R} \to E$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator in E satisfying the condition

(P) There exists finite constants $a,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -a\big(|\operatorname{Im}\lambda| + 1\big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

Then there exists a finite constant $M_0 > 0$ such that the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(E)$ generated by $\mathcal{A}$ satisfies the estimate $\|T(t)\| \leq M_0 e^{-at}t^{\beta-1}$, $t > 0$; cf. [9] for more details. By a mild solution of problem (3.1), we mean any continuous function $t \mapsto u(t)$, $t \in \mathbb{R}$ satisfying

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s,u(s))\,ds, \quad t \in \mathbb{R}.$$

We will use the following auxiliary result:

Lemma 4.1. (see the proof of [9, Lemma 2.12.3]) *Suppose that $f : \mathbb{R} \to E$ is pseudo-almost periodic (pseudo-almost automorphic) and $(R(t))_{t>0} \subseteq L(E,X)$ is a strongly continuous operator family satisfying that $\|R(t)\| \leq Me^{-bt}t^{\beta-1}$, $t > 0$ for some finite numbers $M \geq 1$, $b > 0$ and $\beta \in (0,1]$. Then the function $F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds$, $t \in \mathbb{R}$ is well-defined and pseudo-almost periodic (pseudo-almost* *automorphic).*

Suppose now that

$$0 < M_0/(a + (\ln |c|/\omega)) < 1 \tag{3.2}$$

and define the mapping

$$Pu : PAP_{\omega,c}(\mathbb{R} : E) \to PAP_{\omega,c}(\mathbb{R} : E), \text{ resp. } Pu : PAA_{\omega,c}(\mathbb{R} : E) \to PAA_{\omega,c}(\mathbb{R} : E),$$

by

$$(Pu)(t) := \int_{-\infty}^{t} T(t-s)f(s,u(s))\,ds, \quad t \in \mathbb{R}.$$

If the mapping $f(\cdot,\cdot)$ satisfies the requirements of Proposition 3.2(ii), resp. Proposition 3.1(ii), then we have that the mapping $f(\cdot,u(\cdot))$ belongs to the class $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$. Using the decomposition

$$\int_{-\infty}^{t} T(t-s)f(s,u(s))\,ds = \int_{-\infty}^{t} \Bigl[c^{-\frac{t-s}{\omega}} T(t-s)\Bigr]\Bigl[c^{-\frac{s}{\omega}} f(s,u(s))\Bigr]\,ds, \quad t \in \mathbb{R},$$

the estimate (3.2) yields that the mapping $t \mapsto \int_{-\infty}^{t} T(t-s)f(s,u(s))\,ds$, $t \in \mathbb{R}$ belongs to the class $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$. Hence, the mapping $P(\cdot)$ is well defined. Using a simple calculation, we get that (see also Proposition 3.1):

$$\|Pu\|_{\omega,c} \le \frac{M_0}{a + (\ln |c|/\omega)} \|Pu\|_{\omega,c}, \quad u \in PAP_{\omega,c}(\mathbb{R} : E) \;\; \bigl[u \in PAA_{\omega,c}(\mathbb{R} : E)\bigr].$$

Applying the Banach contraction principle, we get that the mapping $P(\cdot)$ has a unique fixed point, so that there exists a unique solution of the abstract semilinear Cauchy inclusion (3.1) which belongs to the class $PAP_{\omega,c}(\mathbb{R} : E)$, resp. $PAA_{\omega,c}(\mathbb{R} : E)$.

Acknowledgements

This research is partially supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia and Bilateral project between MANU and SANU.

REFERENCES

1. E. Ait Dads, K. Ezzinbi, O. Arino: *Pseudo almost periodic solutions for some differential equations in a Banach space.* Nonlinear Anal. **28** (1997), 1145–1155.
2. E. H. Ait Dads, F. Boudchich, B. Es-sebbar: *Compact almost automorphic solutions for some nonlinear integral equations with time-dependent and state-dependent delay.* Advances Diff. Equ. (2017) 2017:**307** doi 10.1186/s13662-017-1364-2.

3. E. Alvarez, A. Gómez, M. Pinto: *(ω, c)-Periodic functions and mild solution to abstract fractional integro-differential equations.* Electron. J. Qual. Theory Differ. Equ. **16** (2018), 1–8.
4. E. Alvarez, S. Castillo, M. Pinto: *(ω, c)-Pseudo periodic functioins, first order Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells.* Bound. Value Probl. **106** (2019), 1–20.
5. T. Diagana: *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces.* Springer, New York, 2013.
6. A. M. Fink: *Almost Periodic Differential Equations.* Springer-Verlag, Berlin, 1974.
7. G. M. N'Guérékata: *Almost Automorphic and Almost Periodic Functions in Abstract Spaces.* Kluwer Acad. Publ., Dordrecht, 2001.
8. M. T. Khalladi, M. Kostić, A. Rahmani, D. Velinov: *(ω, c)-Almost periodic type functions and applications.* Filomat, submitted.
9. M. Kostić: *Almost Periodic and Almost Automorphic Type Solutions to Integro-Differential Equations.* W. de Gruyter, Berlin, 2019.
10. H.-X. Li, F.-L. Huang, J.-Y. Li: *Composition of pseudo almost-periodic functions and semilinear differential equations.* J. Math. Anal. Appl. **255** (2001), 436–446.
11. J. Liang, J. Zhang, T.-J. Xiao: *Composition of pseudo-almost automorphic and asymptotically almost automorphic functions.* J. Math. Anal. Appl. **340** (2008), 1493–1499.
12. S. Zaidman: *Almost-Periodic Functions in Abstract Spaces.* Pitman Research Notes in Math., Vol. **126**, Pitman, Boston, 1985.
13. C. Zhang: *Pseudo almost periodic solutions of some differential equations.* J. Math. Anal. Appl. **181** (1994), 62–76.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 179-189
https://doi.org/10.22190/FUMI200804015G

Original Scientific Paper

PROPERTIES OF A NEW SUBCLASS OF ANALYTIC FUNCTION ASSOCIATED TO RAFID - OPERATOR AND q-DERIVATIVE

Mohammad Hassn Golmohammadi and Shahram Najafzadeh

Faculty of Mathematical Sciences, Department of Pure Mathematics,
Payame Noor University, P. O. Bax: 19395 - 3697, Tehran, Iran

Abstract. In this article, we introduce a new subclass of analytic functions, using the exponent operators of Rafid and q-derivative. The coefficient estimates, extreme points, convex linear combination, radii of starlikeness, convexity and finally integral have been investigated.

Keywords: Rafid - operator, q-derivative, q-integral, univalent function, coefficient bound, convex set, partial sum.

1. Introduction

The theory of univalent functions can be described by using the theory of the q-calculus. In recent years, such q-calculus as the q-integral and q-derivative have been used to construct several subclasses of analytic functions [1, 6, 11, 12]. The theory of q-analysis has motivated the researchers owing to many branches of mathematics and physics. For example, in the areas of special functions, q-difference, q-integral equations, optimal control problems, q-difference, q-integral equations, q-transform analysis and in quantum physics see for instance, [7, 8, 10, 14].

The main subject of the present paper is to introduce and investigate a new subclass of analytic functions in the open unit disk U by using the operators Rafid and q-derivative. Let $\mathcal{A}$ denote the class of functions $f(z)$ in the form of:

$$f(z) = z + \sum_{k=2}^{+\infty} a_k z^k, \tag{1.1}$$

Received August 4, 2020; accepted November 26, 2020.
Corresponding Author: Mohammad Hassan Golmohammadi, Faculty of Mathematical Sciences, Department of Pure Mathematics, Payame Noor University, P. O. Bax: 19395 - 3697, Tehran, Iran | E-mail: golmohamadi@pnu.ac.ir
2010 *Mathematics Subject Classification.* Primary 20D15; Secondary 20F14

which are analytic in the punctured unit disk

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $f(z) \in \mathcal{A}$, the q- derivative, $0 < q < 1$, of $f(z)$ is defined by Gasper and Rahman [5].

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & (z \neq 0) \\ \\ f'(0) & (z = 0). \end{cases} \tag{1.2}$$

where $z \in U$ and $0 < q < 1$.
Let $T(p)$ be the class of all p -valent functions of the form

$$f(z) = z^p - \sum_{n=p+1}^{+\infty} a_n z^n \quad a_n \geq 0, \tag{1.3}$$

which are analytic in the punctured unit disk

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

If $f \in T(p)$ is given by Equation (1.3) and $g \in T(p)$ is given by

$$g(z) = z^p - \sum_{n=p+1}^{+\infty} b_n z^n \qquad b_n \geq 0, \tag{1.4}$$

then the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z^p - \sum_{n=p+1}^{+\infty} a_n b_n z^n = (g * f)(z). \tag{1.5}$$

From Equation (1.2) for a function $f(z)$ given by Equation(1.3) we get

$$D_q f(z) = [p]_q z^{p-1} - \sum_{n=p+1}^{\infty} [n]_q a_n z^{p-1} \quad , \quad z \in U, \tag{1.6}$$

where

$$[p]_q := \frac{1-q^p}{1-q} = 1 + q + q^2 + \cdots + q^{p-1},$$

and

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Also $[p]_q \to p$ and $[n]_q \to n$ as $q \to \overline{1}$. So we conclude that

$$\lim_{q \to \overline{1}} D_q f(z) = f'(z) \quad , \quad z \in U,$$

see also [13].

Waggas and Rafid defined the Rafid -operator of a function $f(z) = z-\sum_{n=2}^{+\infty} a_n z^n$ by

$$R_\mu^\theta(f(z)) = z - \sum_{n=2}^{+\infty} \frac{(1-\mu)^{n-1}\Gamma(\theta, n)}{\Gamma(\theta+1)} a_n z^n. \tag{1.7}$$

See for instance, [2, 3, 4]).
By using Rafid and q-derivative operators, we define the $R_\mu^\theta D_q(f(z))$ for a function $f \in T(p)$ as follows:

Definition 1.1. The Rafid -operator of $f \in T(p)$, is denoted by $R_\mu^\theta D_q$ and defined as following:

$$R_\mu^\theta D_q(f(z)) = \frac{z}{[p]_q(1-\mu)^{p+\theta+1}\Gamma(p+\theta+1)} \int_0^{+\infty} t^{\theta-1} e^{-(\frac{t}{1-\mu})} D_q(f(zt))dt \tag{1.8}$$

Then it is easy to deduce the series representation of the function $R_\mu^\theta(f(z))$ as following:

$$\begin{aligned} R_\mu^\theta D_q(f(z)) &= z^p - \sum_{n=p+1}^{+\infty} \frac{[n]_q(1-\mu)^{n-p}\Gamma(n+\theta+1)}{[p]_q\Gamma(p+\theta+1)} a_n z^n \\ &= z^p - \sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta) a_n z^n \end{aligned} \tag{1.9}$$

where

$$M(n,p,q,\mu,\theta) = \frac{[n]_q(1-\mu)^{n-p}\Gamma(n+\theta+1)}{[p]_q\Gamma(p+\theta+1)}. \tag{1.10}$$

We now define a new subclass $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ of analytic functions of $T(p)$ by using the operators Rafid and q-derivative. Let $f(z) \in T(p)$ is said to be in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ if and only if it satisfies the inequality:

$$\left| \frac{\lambda z^2 (R_\mu^\theta(D_q(f*g)(z)))'' + z(R_\mu^\theta(D_q(f*g)(z)))'}{z(R_\mu^\theta(D_q(f*g)(z)))' + (1-\lambda)(R_\mu^\theta(D_q(f*g)(z)))} - (1-\beta) \right| \le \alpha. \tag{1.11}$$

Here, $0 < q < 1, 0 \le \lambda < 1, 0 \le \alpha \le 1, 0 \le \mu < 1, 0 \le \theta \le 1$ and $\beta < 1$.

2. Main Results

Unless otherwise mentioned, we suppose throughout this paper that $0 < q < 1, 0 \le \lambda < 1, 0 \le \alpha \le 1, 0 \le \mu < 1, 0 \le \theta \le 1$ and $\beta < 1$. First we state coefficient estimates on the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$.

Theorem 2.1. *Let $f(z) \in T(p)$, then $f(z) \in T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$ if and only if*

$$(2.1)\quad \sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta)a_n b_n \leq 1-2\lambda.$$

Proof. Suppose $f(z)$ difined by Equation(1.3) *and $f(z) \in T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$, then Equation* (1.11) *holds true, we have*

$$\Big| \frac{[(2-\beta)\lambda + 2\beta - 1]z^p}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n}$$
$$- \frac{[n(1-n)+\beta-1]\lambda + 1 - (1+n)\beta]\sum_{n=p+1}^{+\infty} n(n-1)M(n,p,q,\mu,\theta)a_n b_n z^n}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n} \Big| < \alpha.$$

Since $Re(z) \leq |z|$ for all z,

$$Re\Big\{ \frac{[(2-\beta)\lambda + 2\beta - 1]z^p}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n}$$
$$- \frac{[n(1-n)+\beta-1]\lambda + 1 - (1+n)\beta]\sum_{n=p+1}^{+\infty} n(n-1)M(n,p,q,\mu,\theta)a_n b_n z^n}{(2-\lambda)z^p - (n-\lambda+1)\sum_{n=p+1}^{+\infty} M(n,p,q,\mu,\theta)a_n b_n z^n} \Big\} < \alpha.$$

By letting $z \to \overline{1}$ through real values, we have

$$\sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta)a_n b_n \leq 1-2\lambda.$$

Conversely, let Equation (2.1) *holds true, it is enough to show that*

$$X(f) = \Big| \lambda z^2 (R_\mu^\theta(D_q(f*g)(z)))'' + z(R_\mu^\theta(D_q(f*g)(z)))'$$
$$-(1-\beta)[z(R_\mu^\theta(D_q(f*g)(z)))' + (1-\lambda)(R_\mu^\theta(D_q(f*g)(z)))] \Big|$$
$$-\alpha \Big| z(R_\mu^\theta(D_q(f*g)(z)))' + (1-\lambda)(R_\mu^\theta(D_q(f*g)(z))) \Big| \leq 0$$

But for $0 < |z| = r < 1$ we have

$$X(f) = \Big| [(2-\beta\lambda[z^p - \sum_{n=p+1}^{+\infty} n(n-1)M(n,p,q,\mu,\theta)a_n b_n z^n]$$
$$+ z^p - \sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n$$
$$-(1-\beta)([z^p - \sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n]$$
$$+(1-\lambda)[z^p - \sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n]) \Big|$$
$$-\alpha([z^p - \sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n]$$
$$+(1-\lambda)[z^p - \sum_{n=p+1}^{+\infty} nM(n,p,q,\mu,\theta)a_n b_n z^n]) \Big|$$
$$\leq \sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big] M(n,p,q,\mu,\theta)|a_n||b_n|r^n$$
$$-(1-2\lambda).$$

Since the above inequality holds for all r $(0 < r < 1)$, *by letting* $r \to \overline{1}$ *and using Equation*(2.1) *we obtain* $X(f) \leq 0$. *This completes the proof.* $\square$

Corollary 2.1. *If function* $f(z)$ *of the form Equation* (1.3) *belongs to* $T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$ *then*

$$a_n \leq \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)b_n}$$

where

$$M(n,p,q,\mu,\theta) = \frac{[n]_q(1-\mu)^{n-p}\Gamma(n+\theta+1)}{[p]_q\Gamma(p+\theta+1)}, \quad n \geq p+1.$$

With the equality for the function

$$f(z) = z^p - \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)b_n}z^p$$

Next we obtain extreme points and convex linear combination property for $f(z)$ belongs to $T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$.

Theorem 2.2. *The function* $f(z)$ *of the form Equation* (1.3) *belongs to* $T_{p,q}R(\lambda, \alpha, \beta, \mu, \theta)$ *if and only if it can be expressed by*

$$f(z) = \sigma_1 f_1(z) + \sum_{n=p+1}^{\infty} \sigma_n f_n(z)\ , \qquad \sigma_n \geq 1\ , \qquad \sigma_1 + \sum_{n=p+1}^{\infty} \sigma_n = 1$$

where

$$f_1(z) = z^p,$$

$$f_n(z) = \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)b_n}z^k,$$
$$(n \geq p+1).$$

Proof. *Let*

$$\begin{aligned} f(z) &= \sigma_1 f_1(z) + \textstyle\sum_{n=p+1}^{\infty} \sigma_n f_n(z) \\ &= \sigma_1 f_1(z) + \textstyle\sum_{n=p+1}^{\infty} \sigma_n \left[z^n - \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)b_n}\right]z^n \\ &= z^p - \textstyle\sum_{n=p+1}^{\infty} \frac{1-2\lambda}{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)b_n}\sigma_n z^n. \end{aligned}$$

Now apply Theorem 2.1 to conclude that $f(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. Conversely, if $f(z)$ given by Equation (1.3) *belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, by letting*

$$\sigma_1 = 1 - \sum_{n=p+1}^{+\infty} \sigma_n,$$

where

$$\sigma_k = \frac{\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)b_n}{1-2\lambda}a_n \qquad (n \geq p+1).$$

we conclude the required result. □

Theorem 2.3. *Let for $t = 1,2,\cdots,k$, $f_t(z) = z^p - \sum_{n=p+1}^{+\infty} a_{n,t}z^n$ belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, then $F(z) = \sum_{t=1}^{k}\sigma_t f_t(z)$ is also in the same class, where $\sum_{t=1}^{k}\sigma_t = 1$. Hence $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ is a convex set.*

Proof. *According to Theorem 2.1 for every $t = 1,2,\cdots,k$ we have*

$$\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_{n,t}b_n \leq 1-2\lambda.$$

But

$$\begin{aligned} F(z) &= \sum_{t=1}^{k}\sigma_t f_t(z) \\ &= \sum_{t=1}^{k}\sigma_t\left(z^p - \sum_{n=p+1}^{\infty} a_{n,t}z^n\right) \\ &= z^p\sum_{t=1}^{k}\sigma_t - \sum_{n=p+1}^{\infty}\left(\sum_{t=1}^{k}\sigma_t a_{n,t}\right)z^n \\ &= z^p - \sum_{n=p+1}^{\infty}\big(\sum_{t=1}^{k}\sigma_t a_{n,t}\big)z^n. \end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]\\
&\times M(n,p,q,\mu,\theta)a_{n,t}b_n\big(\sum_{n=1}^{m}\sigma_n a_{k,n}\big)\\
=\;&\sum_{n=1}^{m}\sigma_n\Big(\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]\\
&\times M(n,p,q,\mu,\theta)a_{n,t}b_n\\
\leq\;&\sum_{t=1}^{k}\sigma_t\big(1-2\lambda\big)=(1-2\lambda)\sum_{t=1}^{k}\sigma_t=1-2\lambda,
\end{aligned}$$

by Theorem 2.1 the proof is complete. □

3. Radii of close−to−convexity, starlikeness and convexity

In this section we obtain radii of close−to−convexity, starlikeness , convexity and investigate about partial sum property.

In the proof of next theorem, we need the Alexander's Theorem. This theorem states that if f is an analytic function in the unit disk and normalized by $f(0)=f^{'}(0)-1=0$, then $f(z)$ is convex if and only if $zf^{'}(z)$ is starlike.

Theorem 3.1. *Let $f(z)$ of the form Equation* (1.3) *belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ then*

(i) $f(z)$ is $p-$valently close−to−convex of order γ in $|z|<R_1$, where $0\leq\gamma<p$ and

$$R_1=\inf_n\Big\{\frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{n(1-2\lambda)}\Big\}^{\frac{1}{n-p}},$$

(ii) $f(z)$ is $p-$valently starlike of order γ in $|z|<R_2$, where $0\leq\gamma<p$ and

$$R_2=\inf_n\Big\{\frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{(n-\gamma)(1-2\lambda)}\Big\}^{\frac{1}{n-p}},$$

(iii) $f(z)$ is $p-$valently convex of order γ in $|z|<R_3$, where $0\leq\gamma<p$ and

$$R_3=\inf_n\Big\{\frac{p(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{n(n-\gamma)(1-2\lambda)}\Big\}^{\frac{1}{n-p}}.$$

Proof. (i) *For close−to−convexity it is enough to show that*

$$\Big|\tfrac{zf^{'}}{z^{p-1}}-p\Big|<p-\gamma,$$

but

$$\left|\frac{zf'}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} - \sum_{n=p+1}^{+\infty} na_n|z|^n - pz^{p-1}}{z^{p-1}}\right| \leq \sum_{n=p+1}^{+\infty} na_n|z|^{n-p} \leq p - \gamma,$$

or $\sum_{n=p+1}^{+\infty} \frac{n}{p-\gamma} a_n|z|^{n-p} \leq 1$. *By using Equation* (2.1) *we obtain*

$$\begin{aligned}&\sum_{n=p+1}^{+\infty} \frac{n}{p-\gamma} a_n|z|^{n-p}\\ &\leq \sum_{k=1}^{+\infty} \frac{n(1-2\lambda)|z|^{n-p}}{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]}\\ &\times \frac{1}{M(n,p,q,\mu,\theta)a_nb_n} \leq 1.\end{aligned}$$

So, it is enough to suppose

$$\begin{aligned}&|z|^{n-p}\\ &\leq \frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{n(1-2\lambda)},\end{aligned}$$

which completes the case (*i*).

(*ii*) *For starlikeness it is enough to show that* $\left|\frac{zf'}{f} - p\right| < p - \gamma$.
But

$$\left|\frac{zf'}{f} - p\right| = \left|\frac{\sum_{n=p+1}^{+\infty}(n-p)a_nz^n}{z^p - \sum_{n=p+1}^{+\infty} a_nz^n}\right| \leq \frac{\sum_{n=p+1}^{+\infty}(n-p)a_n|z|^{n-p}}{1-\sum_{n=p+1}^{+\infty} a_n|z|^{n-p}} \leq p-\gamma.$$

Therefore,

$$\sum_{n=p+1}^{+\infty}(n-p)a_n|z|^{n-p} \leq (p-\gamma)(1-\sum_{n=p+1}^{+\infty} a_n|z|^{n-p}),$$

or

$$\sum_{n=p+1}^{+\infty} \frac{n-\gamma}{p-\gamma} a_n|z|^{n-p} \leq 1.$$

Now by Equation (2.1), *we obtain*

$$\begin{aligned}&\sum_{n=p+1}^{+\infty} \frac{n-\gamma}{p-\gamma} a_n|z|^{n-p}\\ &\leq \sum_{n=p+1}^{+\infty} \frac{(n-\gamma)(1-2\lambda)|z|^{n-p}}{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]}\\ &\times \frac{1}{M(n,p,q,\mu,\theta)a_nb_n} \leq 1.\end{aligned}$$

So, it is enough to suppose

$$|z|^{n-p} \leq \frac{(p-\gamma)\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_nb_n}{(n-\gamma)(1-2\lambda)}.$$

Hence we get the required result.

(iii) For convexity, by Alexander's Theorem and by applying an easy calculation, we reach the required result. Hence the result. □

Theorem 3.2. *The class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ is a convex set.*

Proof. Let $f(z) = z^p - \sum_{n=p+1}^{+\infty} a_n z^n$ and $g(z) = z^p - \sum_{n=p+1}^{+\infty} b_n z^n$, be in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. For $t \in (0,1)$, it is enough to show that the function $h(z) = (1-t)f(z) + tg(z)$ is in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. Since $h(z) = z^p - \sum_{n=p+1}^{+\infty}((1-t)a_n + tb_n)z^n$,

$$\sum_{n=p+1}^{+\infty}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda+(n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)((1-t)a_n+tb_n)b_n \leq (1-2\lambda)$$

and so $h(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$. □

Corollary 3.1. *Let $f_k(z)$, $1 \leq k \leq m$, defined by $f_k(z) = z^p - \sum_{n=p+1}^{+\infty} a_{n,k} z^n$ be in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, then the function $F(z) = \sum_{k=1}^{m} c_k f_k(z)$ is also in $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, where $\sum_{k=1}^{m} c_k = 1$.*

4. Integral operators on $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$

In this section we investigate properties of functions in the class $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, involving the familiar operator $F_c(z)$.

Theorem 4.1. *If $f(z) = z^p - \sum_{n=p+1}^{+\infty} a_n z^n$ belongs to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$, then the function $F_c(z)$ defined by $F_c(z) = \frac{c+p}{z^c}\int_0^1 t^c f(tz)dt$, $c \geq 1$, is also in $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$.*

Proof. Since $f(z)$ belong to $T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$,

$$\begin{aligned}
F_c(z) &= \frac{p+c}{z^c}\int_0^z t^{c-1}[z^p - \sum_{n=p+1}^{+\infty} a_n t^n]dt \quad , \qquad c>1,\\
&= \frac{p+c}{z^c}\int_0^z [t^{p+c-1} - \sum_{n=p+1}^{+\infty} a_n t^{n+c-1}]dt\\
&= \frac{p+c}{z^c}[\frac{1}{p+c}t^{p+c} - \sum_{n=p+1}^{+\infty} a_n \frac{1}{n+c}t^{n+c}]_0^z\\
&= \frac{p+c}{z^c}[\frac{1}{p+c}z^{p+c} - \sum_{n=p+1}^{+\infty} a_n \frac{1}{n+c}z^{n+c}]\\
&= z^p - \sum_{n=p+1}^{+\infty} \frac{p+c}{n+c} a_n z^n
\end{aligned}$$

Since $\frac{p+c}{n+c} < 1$,

$$\begin{aligned}
&\sum_{n=p+1}^{+\infty} \frac{p+c}{n+c}\Big[(n(n-1)-(\alpha+\beta)+1)\lambda + (n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_n b_n\\
&\le \sum_{n=p+1}^{+\infty} \Big[(n(n-1)-(\alpha+\beta)+1)\lambda + (n+1)(\alpha+\beta)-1\Big]M(n,p,q,\mu,\theta)a_n b_n\\
&\le (1-2\lambda).
\end{aligned}$$

Hence the result. $\square$

Corollary 4.1. *If $f(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$ and $F_c(z)$ is defined as $F_c(z) = c\int_0^1 v^c f(vz)\,dv$, $c \ge 1$. Then $F_c(z) \in T_{p,q}R(\lambda,\alpha,\beta,\mu,\theta)$.*

REFERENCES

1. E. S. Aqlan: *Some problems connected with geometric function theory*, Ph.D. Thesis, Pune University, 2004.
2. W. G. Atshan and R. H. Buti: *Fractional calculus of a class of univalent functions with negative coefficients defined by Hadamard product with Rafid-operator*, European J. Pure Appl. Math. 4(2) (2011), 162-173.
3. A. H. El-Qadeem and S. K. Al-Ghazal: *On subclasse of p-valent functions with negative coefficients defined by using Rafid operator*, Comput. Math. Appl. (2019).

4. A. H. El-Qadeem and S. K. Al-ghazal: *Some aspect of certain two subclass of analytic functions with negative coefficients defined by using Rafid operator*, arXiv:1904.07913v1.

5. G. Gasper and M. Rahman: *Basic Hypergeometric series*, Cambridge University Press, Cambridge, 1990.

6. F. H. Jackson: *On q−functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, 46 (2) (1909), 253-281.

7. F. H. Jackson: *On q-definite integrals*, Q. J. Pure Appl. Math., 41 (1910), 193-203.

8. F. H. Jackson: *q-Difference equations*, Am. J. Math., 32 (1910), 305-314.

9. A. E. Livingston: *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., 17 (1966), 99- 107.

10. J. L. Liu: *The Noor integral and strongly starlike functions*, J. Math. Anal. Appl. 261 (2001), 441-447.

11. R. J. Libera: *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., 16 (1969), 755-758.

12. Z. S. I. Gasper: *Linear sequential q-difference equations of fractional order*, Fract. Calc. Appl. Anal., 12 (2) (2009), 159–178.

13. A. O. Mostafa, H. M. Aouf and T. Bulboaca: *Convolution conditions for subelasse of mermorphic functions of complex order associated with basic Bessel functions*, J. Egyptian Math. Soc 25 (2017), 286-290.

14. T. M. Seoudy and M. K. Aouf: *Convolution properties for certain classes of analytic functions defined by q-derivative operator*, Abstr. Appl. Anal., 2014 (2014), Art. ID 846719, 1-7.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 191-204
https://doi.org/10.22190/FUMI201118016I
Original Scientific Paper

MISSING DATA SAMPLES: SYSTEMATIZATION AND CONDUCTING METHODS-A REVIEW

Ivana D. Ilić[1], Jelena M. Višnjić[1], Branislav M. Randjelović[2] and Vojislav V. Mitić[2]

[1]**Faculty of Medical Sciences, Department of Mathematics and Informatics, Bulevar dr. Zorana Djindjića 81, 18 000 Niš, Serbia**
[2]**Faculty of Electronic Engineering, Department of Mathematics and Informatics, Aleksandra Medvedeva 14, 18 000 Niš, Serbia**

Abstract. This paper investigates the phenomenon of the incomplete data samples by analyzing their structure and also resolves the necessary procedures regularly used in missing data analysis. The research gives a crucial perceptive of the techniques and mechanisms needed in dealing with missing data issues in general. The motivation for writing this brief overview of the topic lies in the fact that statistical researchers inevitably meet missing data in their analysis. The authors examine the applicability of regular approaches for handling the missing data situations. Based on several previously published results, the authors provide an example of the incomplete data sample model that can be implemented when confronting with specific missing data patterns.

Keywords: Missing data, EM algorithm, Listwise deletion, Missing data analysis.

1. Introduction

One important issue which affects almost all datasets, despite major advances in the design and collection of data is the incompleteness. This situation appears when no data value is stored for some feature or an attribute in the dataset. The incompleteness may occur for different reasons. For instance, missing data in a survey may arise when there are no data for a respondent or when some variables for a respondent are unknown because of refusal to provide or failure to collect the

Received November 18, 2020; accepted December 14, 2020.
Corresponding Author: Ivana D. Ilić, Faculty of Medical Sciences, Department of Mathematics and Informatics, Bulevar dr. Zorana Djindjića 81, 18 000 Niš, Serbia | E-mail: ivanailicc3@gmail.com
2010 *Mathematics Subject Classification.* Primary 62D10; Secondary 62D05

response. Also, missing data may occur if the data collection was not done properly or if the mistakes were made with the data entry caused by the researchers themselves. Nevertheless, the problem of an adequate conduction of missing data remains, regardless of whether missing data result from a participant disintegration, a nonresponse item, or an irregular availability of respondents. See [10] or [20] for a summarization of these questions. In addition, we must point out a significant difference between "the item nonresponse" and "the unit nonresponse". The item nonresponse situation indicates that the respondent skipped one or more questions in the analysis. On the other hand, the unit nonresponse appears when the respondent refused to cooperate and consequently, all the resulting data are missing for this respondent. Trough the existing literature we conclude that the methods used for the item nonresponse and the unit nonresponse have been completely different.

In the last few years many articles devoted to the problem where practical missing data issues are discussed have appeared in various domains such as: economy, politics, biomedical research, social sciences, medicine and engineering. Giannone et al.(see [8]) developed a formal method for evaluating gross domestic product (GDP) growth using the large datasets with missing observations monitored by central banks. Schumacher and Breitung (see [34]) used a novel real-time dataset with missing values for the German economy in the empirical application of forecasting the GDP growth. For more practical applications with incomplete samples in various domains see for instance: [16] and [22] in economy and finance, [9], [3], [17] and [26] in biomedical field, [5] in social sciences and [29] in astrophysics.

The prosperity of the missing data procedures available to scientists often produces uncertainty regarding to the choice of the eventual implemented method. Our purpose is to discuss the applicability of general methods for dealing with missing data and to review current advances associated with specific missing data techniques. An additional intention of this paper is to propose a mathematical model (Chapter 4) that can be used in certain missing data situations under specified conditions.

2. An overview of the missing data classification

The task of classification of the data incompleteness type is a complex phenomena and its attaintment depends upon several factors that need to be taken under consideration. In the results obtained in [27] each data has certain likelihood of being missing. Based on that assumption he classified the incomplete sample problems into three categories. The data are said to be missing completely at random (MCAR) if the probability of being missing is the same for all cases. This practically means that the reasons of the data missingness are unrelated to the data, meaning that the missingness has nothing to do with the person being questioned. For example, a questionnaire might be lost in the post, or a blood sample might be ruined in the laboratory for an unknown reason, so that certain portion of the data will be missing simply because of some bad coincidences. An example which describes clearly this type of the data is when we take a random sample of a population. In this situation, each member of the population has an equal chance of being included in the sample. So, the unobserved data of members in the population that were not included in the sample are MCAR. Basically, we may conclude that the points that are missing in the MCAR case present a random subset of the data. There is no systematic mechanism that makes some data more likely to be missing than others. Although in the MCAR pattern we may consequently neglect many of the difficulties that come about the data are missing, we must have in mind that the MCAR model is a bit rare in the real life statistical researches. If we denote a full matrix of the data in the analysis with $\mathbb{X}$, it is obvious that it can be written in the form $\mathbb{X} = \{X, \widetilde{X}\}$, where X are the observed and $\widetilde{X}$ the missing data. Let us define R as a matrix with the identical dimensions as $\mathbb{X}$ where:

$$R_{i,j} = \begin{cases} 1, & \text{if the data is missing} \\ 0, & \text{otherwise.} \end{cases}$$

Now, mathematical simplification of MCAR data type can be formulated as:

$$P(R \big| X, \widetilde{X}) = P(R),$$

meaning that the probability of the realization of R matrix will not depend neither on the observed nor on the unobserved data.

The second structure of the incompleteness is missing at random (MAR) and it covers much wider class of the statistical survey settlements. In this case, the probability of being missing is the same only within groups defined by the observed data. As an example of this situation is the case of a survey where only younger people have missing values measuring IQ. This fact indicates that the probability of missing data referring to IQ is clearly related to age. Another example might be the missing answers considering the body weight only in the women's respondents, so that we may consequently conclude that in this case missingness is related to sex. Such data obviously are not MCAR. But, if however, we know the sex of the respondents and if we can assume MCAR within the particular gender, then the data are MAR. Another example of MAR is when we take a sample from a population, where the probability of the data being inserted depends on some known property. Basically, missing data are missing at random (MAR) when the likelihood of missing data on a variable depends on some other measured variable in the model, but not to the value of the variable with missing values itself. Nevertheless, the assumption that the pattern is MAR is in practice very difficult to prove, so it is crucial to implement the correlates of missingness into the chosen missing data procedure in order to reduce bias and enhance the chances of satisfying the MAR assumption. Definitely, MAR is more general situation and therefore more realistic than MCAR. The largest number of the modern incomplete data tools generally start from the MAR hypothesis. Mathematically reduced, this data type can be express as follows:

$$P(R \big| X, \widetilde{X}) = P(R \big| X),$$

meaning that the realization of the R matrix will depend on the observed data only.

The third concept is called missing not at random (MNAR), although in the literature we can often notice the term NMAR (not missing at random) for the same model. MNAR indicates that the data likelihood of being missing differs for some unknown reasons. The fact is that in this particular case the missing values on a variable are dependent on the values of that variable itself, even after controlling all other variables. MNAR is the most complicated case for the researches. Approaches to overcome the MNAR situation are to reveal more detailes about the causes of the missingness or to carry out what-if analyses in order to evaluate the measure of subtleness of the results. The example which illustrates this type of the data is when the answers refer to IQ are missing only at the respondents with low IQ. Another illustration of this structure is that when the survey participants with serious depression are more likely to refuse to fulfill the answers referring to the depression severity. More, in public opinion research the MNAR appears when persons having infirm opinions answer less frequently. The difficulty with the MNAR structure is that it is unfeasible to prove that outcomes are MNAR without recognize the values that are missing. So, the trouble lies in the fact that the data incompleteness is totally related to the unobserved data, meaning to the incidences or components that are not evaluated and registered by the researcher.

The differences between these structures that are firmly described in [27] are crucial for realize why some techniques will offer better results against the others. His basic hypothesis lays in the fact that the researcher needs to provide the conditions under which a missing data method can produce valid statistical interpretations. Basic methods settle only the restrictive and sometimes implausible MCAR premise. Therefore, in this case we must have in mind that there is a substantial probability of obtaining biased estimates. Mostly, missing data are neither MCAR nor MNAR. Instead, the probability that an observation is missing commonly depends on information for that subject that is present, meaning that the reason for missingness is based on other observed respondent characteristics. This situation defines obviously the MAR model. For the additional description and comparison of the three basic patterns of the missing data see [33].

In order to illustrate an example taken from the real data, we used the result [18] given by Lai, who created the regression line and predict the voting intention by using peoples' age. Please see Figure 2.1 of scatter plots for the comparison of different types of the missing data. The model that we define in the Chapter 4 can be implemented on the MCAR type of the data.

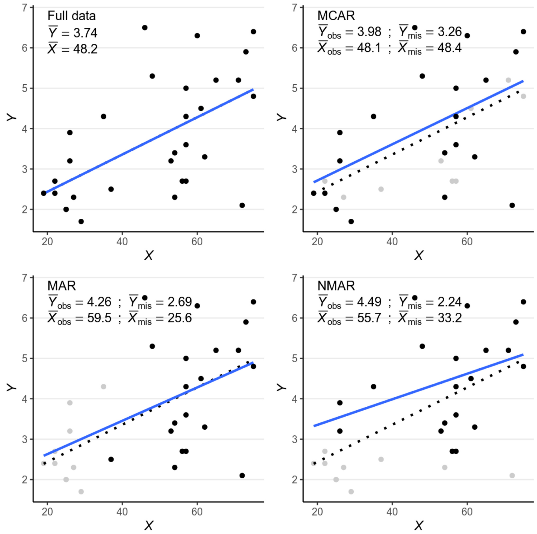


FIG. 2.1: Scatter plots of different types of missing data

3. The analysis of the incomplete sample regulation techniques

The crucial strategy in dealing with the missing data problem is to apply the data analysis techniques which are robust to the deviations caused by the incompleteness of the data set. This robustness of the technique practically means that there exists reliance that some smooth and tolerable violations of the premises and starting hypothesis will result in almost no bias or misinterpretation in the resulting outcomes based on the population under analysis. On the other hand, it needs to be pointed out that it is not achievable to use such methods in every situation. That is why a large number of different handling procedures for the missing data issues has been established.

According to [10], the methods for dealing with missing values can be evaluated by three means: it should yield to an unbiased parameter estimate, one should be able to obtain reasonable estimates of the standard error of confidence intervals and it should have good statistical power. Traditional missing data methods such as complete case analysis often produce bias and inaccurate conclusions. Similar problems extend to single imputation techniques commonly thought of as improvements over complete case methods. Research demonstrates that procedures such as multiple imputation, which incorporate uncertainty into estimates for missing data, often provide significant improvements over traditional methods.

Generally, the most commonly used procedures can be divided into three main groups which are explained thoroughly in next paragraphs: Deletion methods, Single Imputation Methods and Multiple Imputation methods.

3.1. Deletion methods

Listwise deletion stands for the basic method in overcoming the possible complications caused by the incompleteness of the data set. This procedure is also called the Complete-case analysis. The conducting mechanism simply ignores all the cases which obtain one or more missing values recognizing the variables that are under examination and it is an inevitable part of many statistical softwares such as STATA, SPS, SAS etc.

The advantage of the listwise deletion method is its reliability, accuracy and its availability. Under hypothesis of MCAR data type, the listwise deletion produces the standard errors and significance levels absolutely acceptable referring to the reduced subset of data. But, we must note that these values are often higher when implement this technique using all possible data.

In real life situations various challenges occur. For instance, when the number of variables is huge and when more than a half of the original sample is obscured and vanished. More, dealing with structures that are not MCAR, the listwise deletion can severely bias the evalution of means, regression coefficients and correlations. It is showed in the study of Little and Rubin (see [20]) that the bias of the estimated mean grows together with the disparity among means of the observed and missing variables. Also, the bias grows with the higher percentage of the data that are missing. Interesting investigation on the subject was performed by Schafer and Graham (see [33]), where the bias of the complete-case analysis under MAR and MNAR premises was analyzed. It is important to imply that there are settlements in which listwise deletion can give better estimates than even the most refined and smooth statistical mechanisms. Miettinen (see [21]) indicates that this method states for the only access that guaranties that no bias is possible under any conditions. If we go further trough literature, Enders (see [7]) claims that in most settlements, the discommodities of listwise deletion far exceed its conveniences. Schafer and Graham (see [33]) show that only if the incompleteness problem can be solved by eliminating only a small part of the sample, then the technique may be solidly efficient. Vach (see [35]) claims that "there exists something like a critical missing rate up to which missing values are not too dangerous".

Another method, known as the Pairwise deletion (often called the available-case analysis) tries to improve the waist data problem of listwise deletion. In listwise deletion a case is ignored from a survey for the reason that it consists of one or more missing values within the variables under analysis. Pairwise deletion appears in the situations when statistical method accepts cases that involve some missing data. The technique cannot include the specific variable with a missing value into analysis, but it can still exploits the incomplete case when investigating other variables with complete values. The advantage of this procedure is that it increments a power of the survey. On the other hand, it has certain deficiencies. It presumes that the incomplete sample is MCAR.

The illustration for understanding the mechanism of the method of pairwise deletion is to take a dataset having following variables: age, gender, education, income, and political affiliation. For each case in the dataset, the values of some of the variables are more likely to be missing than others depending on the surveyee's sensitiveness to the survey questions. Let's say we are interested in knowing if there is a correlation between age and political affiliation. Using pairwise deletion, any given case may contribute to certain analysis but not to others, depending on whether the needed data are available. Hence for our analysis in this example, all cases with available data on age and political affiliation will be included regardless of the missing values for other variables like gender, income or education. The pairwise deletion is an alternative to the listwise deletion to mitigate the loss of data.

3.2. Imputation methods

Other routine way that is frequently practiced among the statisticians is imputation. This method basically replaces the missing values with certain estimated values and then it analysis the complete data set such that it treats the imputed estimates as the original observed values. The procedures for the best choice of these estimates differ and in this paragraph we describe the most exploited imputations that are used in surveys. The imputation procedures are divided in two groups: single imputation methods and multiple imputation methods.

3.2.1. Single imputation

In single imputation, missing values are replaced by a value defined by a certain rule. For example, Mean imputation is a smooth and simple method which evaluates the mean of the observed values for the particular variable in all cases that are not missing. Conceivably, the preference of this technique is that it retains the same sample size and the same mean. On the other hand, mean substitution reduces the variation of analyzed scores and this reduction in separate variables is proportional to the number of missing data. Further, mean substitution may significantly transform the values of correlations. The regression imputation is a procedure which utilizes the values of other variables in order to forecast the missing values in a variable. That is achieved by applying a regression model. Usually the regression model is structured by using the observed data and eventually related to the regression weights the missing values are projected and restored.

Next example of the single imputation is the Hot-deck imputation, the technique which inserts a missing value from a randomly selected similar data set. The part of the expression "deck" suggests that the contributed values arrive from the identical set as the initial data-set. The term "hot" in the above phrase is for the reason of data being instantly employed.

On the contrary to the last method, the cold-deck imputation chooses contributors data belonging to a different data-set. It is a term for a technique that fills a missing values with values from some outward origin, such as some previous similar survey. According to the above explanation, the reason for the expression "cold-deck" is evident.

3.2.2. Multiple imputation

Multiple imputation methods use the distribution of the observed data in order to estimate multiple values that catch the oscillations around the true value. The idea of multiple imputation (MI) was first introduced by Rubin (see [28]), in which each missing value is replaced with $m > 1$ simulated values prior to analysis. In multiple imputation, there are three operational steps: imputation or fill-in phase, the analysis phase and pooling phase. First phase constitute the complete data set by filling in the missing values with the estimated values (using some of the convenient statistical methods). This process of fill-in repeats several times. The analysis phase, studies each of the obtained complete data sets by using a suitable statistical method. Finally, in the third step the parameter estimates resulted from each of the considered data set are then connected and analyzed so that the best conclusions can are accomplished. Final phase aggregates all the results and reveals the best summary estimate of the missing data. Clearly, it is obvious that the method of multiple imputation is more unbiased that the single imputation method, because of the use of multiple sets. That way we kind of "washing" out the coincidences that might occur. The disadvantage of this approach is the greater expanse of time and effort comparing to single imputation.

The most familiar and widely exploited model-based method is the EM algorithm described thoroughly by Dempster et al (see [4]). Also, high influential articles given by Rubin (see [27]) and Little and Rubin (see [19]), gave the formulation of EM algorithm and the dominated framework for dealing with missing data. Many examples of EM algorithm were provided by Little and Rubin (see [20]) and Schafer (see [30]).This iterative technique involves the expectation (E-part) and the maximization (M-part). It replaces missing data with estimated values, evaluates the parameters, repeatedly estimates the missing values, re-estimates the parameters and iterates until convergence (see [20]). Over the repetitions until convergence, we conclusively obtain the missing values.

To simplify this approach, let us assume that the complete data-set consists of $\mathbb{X} = \{X, \widetilde{X}\}$ but that only X is observed. The complete-data log likelihood function is then denoted by $l(\theta; X, \widetilde{X})$ where θ is the unknown parameter vector for which we need to find the MLE (which is based on EM algorithm). Further, let $t = 1, 2, ...$ represents all parameters of distribution and $f_{\theta_t}(X)$ and $f_{\theta_t}(\widetilde{X})$ are the assumed probability distributions at t-th iteration. First, the E-part is activated and evaluates the expected value of $l(\theta; X, \widetilde{X})$ given the observed data X and the current iteration parameter estimate θ.

Principally, we define

$$Q(\theta; \theta_t) := E[l(\theta; X, \widetilde{X})|X, \theta_t] = \int l(\theta; X, \widetilde{X}) p(\widetilde{x}|X, \theta_t) dx, \tag{3.1}$$

where $p(\cdot|X, \theta_t)$ is the conditional density of $\widetilde{X}$ given the observed data X and assuming $\theta = \theta_t$.

Next, the M-part of the analysis starts and it maximizes the expectation (3.1) over θ. That is we put:

$$\theta^t := \max_{\{\theta\}} Q(\theta; \theta_t).$$

We then set $\theta_t = \theta^t$. The two steps are iterated until the sequence of θ^t converges.

Recent work implies that multiple imputation and specialized modeling procedures offer universal methods for handling the missing data. It is proven that they perform fine over many types of missing data structures. There are different EM algorithms for different applications. Although this method provides excellent parameter estimates, EM is not particularly good for hypothesis testing.

Nevertheless, the development of informational technology and the advances in relevant statistical software make these methods available to the researchers in various fields. For example, multiple imputation procedures under the normal model are implemented in Schafer's NORM program [30]. Detailed, step-by-step instructions for running NORM are available in [12] (also see [11], [31], [32]). ML methods, often called FIML (full information maximum likelihood) methods deal with the missing data, do parameter estimation, and estimate standard errors all in a single step. Available software for running this procedure are AMOS: [1], LISREL: [15]; also see Mplus: [24]; and Mx: [25]. Basically, in 1987. Little and Rubin published their classical book *Statistical Analysis With Missing Data* (see [19]), and they established the groundwork for missing data software to be developed over the next 20 years and beyond. See also [13] for recent review of software handling missing data.

4. Mathematical model generated for the MCAR type of data

Let $X_1, X_2, \ldots$ be independent identically distributed random variables and let us assume that only observations at certain points are available. Denote the observed random variables among $\{X_1, \ldots, X_n\}$ by $\widetilde{X}_1$, ..., $\widetilde{X}_{M_n}$. Here the random variable M_n represents the number of the registrated random variables among the first n terms of the sequence (X_n). Incomplete sample may be obtained, for example, if every term of (X_n) is observed with probability p, independently of other terms, and in this case M_n is binomial random variable. This refers to MCAR type of missing data distribution. Now, let:

$$E(X_j) = m, \quad D(X_j) = \sigma^2 \quad \textit{and} \quad S(n) = \sum_{j=1}^{M_n} \widetilde{X}_j.$$

We obtain the following results straightforward:

$$\begin{aligned}
E(S(n)) &= \sum_{k=0}^{\infty} E\left(S(n) \middle| M_n = k\right) \cdot P\{M_n = k\} \\
&= \sum_{k=0}^{\infty} E\left(\sum_{j=1}^{M_n} \widetilde{X}_j \middle| M_n = k\right) \cdot P\{M_n = k\} \\
&= \sum_{k=0}^{\infty} E\left(\sum_{j=1}^{k} \widetilde{X}_j\right) \cdot P\{M_n = k\} = \sum_{k=0}^{\infty} k \cdot m \cdot P\{M_n = k\}.
\end{aligned}$$

Conclusively we have:

$$E\left(S(n)\right) = m \cdot E\left(M_n\right) = E(X_1) \cdot E\left(M_n\right). \tag{4.1}$$

Further we have that:

$$\begin{aligned}
D\left(S(n)\right) &= E\left(S(n)\right)^2 - \left(E(S(n)\right)^2 = E\left(S(n)\right)^2 - m^2\left(E\left(M_n\right)\right)^2 \\
&= E\left(\sum_{j=1}^{M_n} \widetilde{X}_j\right)^2 - m^2\left(E\left(M_n\right)\right)^2 \\
&= \sum_{k=0}^{\infty} E\Bigg\{\left(\sum_{j=1}^{M_n} \widetilde{X}_j\right)^2 \Big| M_n = k\Bigg\} \cdot P\{M_n = k\} - m^2\left(E\left(M_n\right)\right)^2 \\
&= \sum_{k=0}^{\infty} E\left(\sum_{j=1}^{k} \widetilde{X}_j\right)^2 \cdot P\{M_n = k\} - m^2\left(E\left(M_n\right)\right)^2 \\
&= \sum_{k=0}^{\infty} \left\{D\left(\sum_{j=1}^{k} \widetilde{X}_j\right) + \left(\sum_{j=1}^{k} E(\widetilde{X}_j)\right)^2\right\} \cdot P\{M_n = k\} - m^2\left(E\left(M_n\right)\right)^2 \\
&= \sum_{k=0}^{\infty} (k\sigma^2 + k^2 m^2) \cdot P\{M_n = k\} - m^2\left(E\left(M_n\right)\right)^2 \\
&= \sigma^2 E(M_n) + m^2 E(M_n)^2 - m^2\left(E\left(M_n\right)\right)^2 \\
&= \sigma^2 E(M_n) + m^2 D(M_n).
\end{aligned}$$

Since we assumed that $X_1, X_2, \ldots$ are identically distributed, the last equality we can write as:

$$D\left(S(n)\right) = D(X_1)E(M_n) + E(X_1)^2 D(M_n). \tag{4.2}$$

If M_n has a binomial distribution with parameters n and p where p is the probability of a successful outcome, i.e the probability of a variable to be observed. If we put $q = 1 - p$ the probability of failure, that is the probability of a variable to be missing we have the equations (4.1) and (4.2) written in the form:

$$E\left(S(n)\right) = mnp$$

and

$$D\left(S(n)\right) = \sigma^2 np + m^2 npq = np(\sigma^2 + m^2 + q).$$

Further, it is possible to extend the application of the proposed model in the case when the observed random variables are determined by a general point process and when only conditions on M_n are imposed. It may be interesting to see the implementation of the proposed mathematical model based on a strictly stationary sequence of random variables $(X_n)_{n\geqslant 1}$ with "short range" dependence. This problem was considered and analyzed by Mladenovic and Piterbarg (see [23]) where consistency of Hill's estimator was proved. The main presumed condition in the paper means that the finite dimensional distributions of (X_n) are invariant under shifts and the dependence between observations from (X_n) becomes weaker as time separation becomes larger. More, under additional conditions this model of incompleteness was considered by Ilic and Mladenovic (see [14]), where the asymptotic behavior of the Pareto index estimator, proposed by Bacro and Brito (see [2]), was analyzed. Also it it can be proved that in the case when the number of observed variables M_n has the binomial distribution the sequence $\widetilde{X}_1$, ..., $\widetilde{X}_{M_n}$ of observed variables is asymptotically stationary (according to the definition from [6]). The proposed model can be used for various practical situations where more thorough theoretical tool is necessary in order to describe the incompleteness of the data. It can be interesting for the researchers in this area for the mathematical establishment of certain incomplete structures in the surveys.

Finally, we give the necessary conditions that are used in the above research papers in order to enhance the mathematical approach in confronting with the missing data in stationary sequences.

Assumption A. The sequence $X_1, X_2, \ldots$ does not depend on M_n and

$$\frac{M_n}{n} \xrightarrow{p} c_0 > 0 \quad \text{as } n \to +\infty.$$

Suppose β_n is a sequence of real numbers such that

$$\lim_{n\to\infty} \beta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_n}{n} = 0.$$

Let

$$K_n = \left[\frac{M_n}{\beta_n}\right] \quad \text{and } B_n = \begin{cases} 0, & M_n = 0 \\ \frac{K_n}{M_n}, & M_n \geq 1 \end{cases}$$

where the floor function [.] denotes the largest previous integer. Define $\widetilde{Y}_i = (\ln \widetilde{X}_i - \ln F^{-1}(1-B_n))_+$ and $\widetilde{Y}_i^{\zeta} = I\left\{\ln \widetilde{X}_i - \ln F^{-1}(1-B_n) > \frac{\zeta}{\sqrt{K_n}}\right\}$ where $\zeta \in R$.

Assumption B. For any $h \in N$ and $\theta \in R$

$$Var\left\{\sum_{j=1}^{h}\Big((\widetilde{Y}_{j+k} - E\widetilde{Y}_{j+k}) + \theta(\widetilde{Y}^{\zeta}_{j+k} - E\widetilde{Y}^{\zeta}_{j+k})\Big)\right\}$$

does not depend on k.

Remark 4.1. In the case when the number of observed variables M_n has the binomial distribution both the **Assumption A** and **Assumption B** are satisfied. In this case the sequence $\widetilde{X}_1$, ..., $\widetilde{X}_{M_n}$ of observed variables is asymptotically stationary, according to the definition from [6].

5. Conclusion

Missing data is an intermittent issue in many areas such as: market research, database analysis, social analysis, medical research and generally in survey research. Even a small percent of missing data can produce significant problems in the statistical analysis possibly leading to wrong conclusions. The purpose of this article is to identify the problem, to recognize the missing data pattern and to choose the proper methodology for dealing with the incomplete sample. Further intention of this paper is to indicate the possibility of the potential implementation of the proposed mathematical formulation in statistical researches having the MCAR data structure. Prospective research will undeniably derive further improvements and expansions of the proposed mathematical models and practical techniques in order to achieve higher efficiency in situations in which missing data appear.

Acknowledgements

Authors were supported in part by Ministry of Education, Science and Technological Development, Republic of Serbia (Grant No. 174025, 174007, III-43007, TR-32012, 144)

REFERENCES

1. J. L. Arbuckle and W. Wothke: *Amos 4.0 User Guide.* Smallwaters, Chicago IL, 1999.
2. J. N. Bacro and M. Brito: *Strong limiting behaviour of a simple tail Pareto-index estimator.* Statistics and Decisions, **3** (1993) 133–134.
3. S. R. Browning and B. R. Browning: *Rapid and Accurate Haplotype Phasing and Missing-Data Inference for Whole-Genome Association Studies By Use of Localized Haplotype Clustering.* The American Journal of Human Genetics, **81** (2007) 1084–1097.
4. A. P. Dempster, N. M. Laird, D. B. Rubin: *Maximum likelihood from incomplete data via the EM algorithm (with discussion).* J. R. Stat. Soc., **B39** (1977) 1–38.
5. S. Dilmaghani, H. Isaacc, P. Soonthornnonda, E. Christensen, C. H. Ronald: *Harmonic Analysis of Environmental Time Series with Missing Data or Irregular Sample Spacing.* Environ. Sci. Technol., **41** (2007) 7030–7038.
6. W. Dunsmuir and P. M. Robinson: *Asymptotic theory for time series containing and amplitude modulated observations.* The Indian Jornal of Statistics, Series A, **43(3)** (1981) 260–281.
7. C. K. Enders: *Missing not at random models for latent curve analyses.* Manuscript submitted for publication, 2010.
8. D. Giannone, L. Reichlin, D. Small: *Nowcasting: The real-time informational content of macroeconomic data.* Journal of Monetary Economics No., **55** (2008) 665–676.

9. M. Gottschalk, M. E. Tinetti, D. I. Baker, M. King, E. M. Terrence, D. Acampora, L. Leo-Summers, H. G. Allore : *Effect of Dissemination of Evidence in Reducing Injuries from Falls.* New England Journal of Medicine, **359** (2008) 252–261.
10. J W. Graham: *Missing data analysis: Making it work in the real world.* Annu. Rev. Psychol., **60** (2009) 549–576.
11. J. W. Graham, S. M. Hofer: *Multiple imputation in multivariate research.* In Modeling Longitudinal and Multiple-Group Data: Practical Issues, Applied Approaches, and Specific Examples, Eds. T. D. Little,K. U. Schnabel, J. Baumert. Hillsdale, NJ: Erlbaum, 2000, pp. 201–218.
12. J. W. Graham, P. E. Cumsille, E. Elek-Fisk: *Methods for handling missing data.* In Research Methods in Psychology, Eds. J. A. Schinka, W. F. Velicer, (pp. 87–114). Volume 2 of Handbook of Psychology, Ed. IB Weiner. New York:Wiley 2003.
13. N. J. Horton, K. P. Kleinman: *Much ado about nothing: a comparison of missing data methods and software to fit incomplete data regression models.* Am. Stat. **61** (2007) 79–90.
14. I. Ilić and P. Mladenović: *On tail index estimation using a sample with missing observations.* Statistics and Probability Letters, **82** (2012) 949–958.
15. K. G. Jöreskog and D. Sörbom: *LISREL 8 Users Reference Guide.* Sci. Software, Chicago, 1996.
16. P. Kline and A. Santos: *Sensitivity to Missing Data Assumptions: Theory and An Evaluation of the U.S. Wage Structure.* In: Proceedings of a 2010 Seoul Summer Economics Conference, 2010.
17. L. Koopman, J. M. G. van der Heiden, D. E. Grobbee and M. M. Rovers: *Comparison of Methods of Handling Missing Data in Individual Patient Data Meta-analyses: An Empirical Example on Antibiotics in Children with Acute Otitis Media.* American Journal of Epidemiology, **167(5)** (2007) 540-545.
18. M. Lai: Course Handouts for Bayesian Data Analysis Class, 2020.
19. R. J. A. Little and D. B. Rubin: *Statistical Analysis with Missing Data.* Wiley, New York, 1987.
20. R. J. A. Little and D. B. Rubin: *Statistical Analysis with Missing Data.* Wiley, New York, 2n ed., 2002.
21. O. S. Miettinen: *Theoretical Epidemiology: Principles of Occurence Research in Medicine.* John Wiley and Sons, New York, 1985.
22. P. Mladenović and Z. Petrovic: *Cagan's paradox and money demand in hyperinflation:Revisited at daily frequency.* Journal of International Money and Finance, **29** (2010) 1369–1384.
23. P. Mladenović and V. Piterbarg: *On estimation of the exponent of regular variation using a sample with missing observations.* Statist. Prob. Letters. **78** (2008) 327–335.
24. L. K. Muthén and B. O. V. Muthén: *Mplus User's Guide.* CA: Muthén and Muthén, Los Angeles, 4th ed., 2007.
25. M. C. Neale, S. M. Boker, G. Xie, H. H. Maes: *Mx: Statistical Modeling.* Virginia Commonwealth Univ. Dept. Psychiatry., Richmond, 5th ed., 1999.

26. F. J. Prevosti and M. A. Chemisquy: *The impact of missing data on real morphological phylogenies: influence of the number and distribution of missing entries.* Cladistics, **26** (2009) 326–339.

27. D. B. Rubin: *Inference and missing data.* Biometrika, **63** (1976) 581–592.

28. D. B. Rubin: *Multiple Imputation for Nonresponse in Surveys.* Wiley, New York, 1987.

29. R. J. Ryder and G. K. Nicholls: *Missing data in a stochastic Dollo model for cognate data, and its application to the dating of Proto-Indo-European.* The Smithsonian/NASA Astrophysics Data System,eprint arXiv:0908.1735, 2009.

30. J. L. Schafer: *Analysis of Incomplete Multivariate Data.* Chapman and Hall, New York, 1997.

31. J. L. Schafer: *Multiple imputation: a primer.* Stat. Methods Med. Res. **8** (1999) 3–15.

32. J. L. Schafer, M. K. Olsen: *Multiple imputation for multivariate missing data problems: a data analyst's perspective.* Multivar. Behav. Res., **33** (1998) 545–571.

33. J. L. Schafer, J. W. Graham: *Missing data: our view of the state of the art.* Psychol. Methods, **7** (2002) 147–177.

34. C. Schumacher, J. Breitung: *Real-time forecasting of German GDP based on a large factor model with monthly and quarterly data.* International Journal of Forecasting, **24** (2008) 386–398.

35. W. Vach: *Logistic Regression with Missing Values in the Covariates.* Springer, New York, 1994.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 1 (2021), 205-236
https://doi.org/10.22190/FUMI201203017P

Original Scientific Paper

COMPUTER TOOLS FOR SOLVING MATHEMATICAL PROBLEMS: A REVIEW

Ivan Petković[1] and Đorđe Herceg[2]

[1]University of Niš, Faculty of Electronic Engineering, Department of Computer Science, 18000 Niš, Serbia
[2]University of Novi Sad, Faculty of Science, Department of Mathematics and Informatics, 21000 Novi Sad, Serbia

Abstract. The rapid development of digital computer hardware and software has had a dramatic influence on mathematics, and vice versa. The advanced hardware and modern sophisticated software such as computer visualization, symbolic computation, computer-assisted proofs, multi-precision arithmetic and powerful libraries, have provided resolution to many open problems, very difficult mathematical problems, and discovering new patterns and relationships, far beyond a human capability. In the first part of the paper, we give a short review of some typical mathematical problems solved by computer tools. In the second part we present some new original contributions, such as an intriguing consequence of the presence of roundoff errors, distribution of zeros of random polynomials, dynamic study of zero-finding methods, a new three-point family of methods for solving nonlinear equations and two algorithms for the inclusion of a simple complex zero of a polynomial.
Keywords: Experimental mathematics, computer graphics, symbolic computation, visualization of iterative processes, interval arithmetic, roundoff error.

1. Introduction

The advance of digital computer hardware and software, circa 1970, has had a remarkable impact on almost every part of scientific disciplines such as mathematics, engineering disciplines, physics, chemistry, communication, biology, education, astronomy, geology, banking, business, insurance, health care, social science, as well

Received December 3, 2020; accepted February 20, 2021.
Corresponding Author: Ivan Petković, University of Niš, Faculty of Electronic Engineering, Department of Computer Science, 18000 Niš, Serbia | E-mail: ivan.petkovic@elfak.ni.ac.rs
2010 *Mathematics Subject Classification.* Primary Computer mathematics; Secondary computer graphics, symbolic computation; interval arithmetic, roundoff error.

as many other fields of human activities. At present, computers are playing an increasingly central role in mathematics; they have found the application in almost every branch of mathematics. Many practical problems are solved by numerical methods of various types, for instance, simulating dynamical systems and determining their global properties, or calculating approximate solutions to nonlinear equations where no closed-form solution is available. Symbolic computation, a part of computer algebraic systems, is manipulating, simplifying, factorizing, and expanding complicated expressions that contain variables and non-numerical values. This powerful tool is very useful for solving very difficult mathematical problems producing, in addition, *exact computation.* Graphical representations can visualize complex objects to a good extent and thereby comprehend their properties, see [1].

For a long time (many decades and even centuries) a lot of mathematical problems remained unsolved. Simply, "paper-and-pencil methods", human-memory limitation, impossibility to handle lengthy expressions, primitive computer tools (logarithmic tables, abacus, slide rule) and other objective obstacles, were insufficient to solve them. These problems resisted until the digital computer era emerged. In this paper, we present a short review of some typical mathematical problems solved by computer tools (Section 2) and some new original contributions (Section 3).

2. Computers in mathematical research - a review

First applications of computers in mathematics were restricted to the calculation of complicated numerical expressions and the verification of some particular mathematical identities, relations and other issues. The *brutal force* of computers was used to suggest or test general claims and to pose hypotheses based on a finite number of patterns.

Let us mention some well-known examples concerned with the application of computers. The first calculation of the number π happened in 1949, when the outstanding scientist John von Neumann and his team used a room-sized digital computer with vacuum tubes ENIAC (Electronic Numerical Integrator And Computer) to compute 2 037 digits of π. The time of calculation: 70 hours. Computer-assisted proof of the four-color theorem, given by Appel and Haken in 1977, is a typical example where brute force combinatorial enumeration played an essential role in solving this 125 years old open problem (posed by F. Francis Guthrie in 1852). A similar combinatorial enumeration method (combined with interval arithmetic) was used in Thomas Hales's proof of the Kepler conjecture (posed in 1611), which asserts that the optimal density of packing equal spheres is achieved by the familiar face-centered cubic packing (see, e.g., [2], [3], and pretty interesting, on the markets where oranges are packed).

Today, computers are employed in mathematical research in a number of ways; one of the simplest ways is the implementation of *proof-by-exhaustion*: posting a proof so that a statement is valid for a large but finite number of cases and then check all the cases by a suitable program using a computer. More sophisticated use of computers is to discover and analyze interesting patterns in data, which then serve to state conjectures. Helping to find conjectures is the first step, a proper advance is a rigorous proof of them.

Extensive development of computer algebra systems (briefly CAS), such as *Mathematica* and *Maple*, provides very fast manipulations with complex mathematical expressions, a work beyond human ability. Can you check that the sequence "0123456789" appears in the decimal expansion of π? Using a computer, Yasumasa Kanada of the University of Tokyo found in 1997 that this sequence begins at position 17 387 594 880. Advance versions of CAS deliver new improvements and very powerful algorithms. Evaluating the infinite product

$$\prod_{n=2}^{\infty} \frac{n^4-1}{n^4+1},$$

Mathematica 6 (issued 2007) gives the result involving the Gamma function. *Mathematica* 10 (2014) delivers the answer directly:

$$\prod_{n=2}^{\infty} \frac{n^4-1}{n^4+1} = \frac{\pi \sinh \pi}{\cosh(\pi\sqrt{2}) - \cos(\pi\sqrt{2})}.$$

Both tasks are obviously missions impossible for humans.

Symbolic computation, embedded in computer algebra systems like *Mathematica* or *Maple*, was a great advance in manipulating very complicated expressions of more variables. Suitable algorithms implemented on current powerful computers can solve problems whose answers are algebraic expressions tens or thousands of terms long. David Bailey, a mathematician and computers scientist at Lawrence Berkeley National Laboratory and one of the world leaders in experimental mathematics, said: *"The computer can then simplify this to five or 10 terms. Not only could a human not have done that, they certainly could not have done it without errors."* In & 3.5 we will show how to construct new iterative methods for solving nonlinear equations and determine the order of convergence by using symbolic computation in CAS *Mathematica*. Besides, CAS provides a powerful computer visualization of data, which is a very useful tool in helping us understand the behavior of iterative processes, as shown in & 3.5.

2.1. Short list of mathematical problems solved by computer

Below we give a list of theorems proved (completely or partially) with the help of computer programs. It is assumed that this list is far from being exhaustive.

- Archimedes' cattle problem, **1965** (the most famous ancient Diophantine equation), was solved by H. C. Williams, R. A. German and C. R. Zarnke [4] using computers).

- Euler's wrong hypothesis, **1966**. In 1769 Euler stated that there is no nth degree which can be sum of less than n nth degrees of natural numbers. In 1966 L. L. Lander and T. R. Parker found by computer the counterexample for $n = 5$ in the form of identity $27^5 + 84^5 + 110^5 + 133^5 = 144^5$.

- Four color theorem, **1976**. The four-color theorem states that any map in a plane (divided into contiguous regions) can be colored using no more than four colors so that no two adjacent regions have the same color. The theorem was proved by Kenneth Appel and Wolfgang Haken (published in [5], [6]) by inspecting reduced graph configurations by a computer program. Widely accepted proof of the four-color theorem was given in 2008 by Georges Gonthier with general-purpose theorem-proving software [7].

- Perfect squared square of the lowest order, **1978**. The task is tilling one integral square using only other integral squares of different sizes. In 1978, using a computer program, the Dutch computer scientist A. J. W. Duijvestijn found the perfect squared square of lowest order consisting of 21 smaller squares.

- Mitchell Feigenbaum's universality conjecture in non-linear dynamics, **1982** (proved by O. E. Lanford using rigorous computer arithmetic);

- The non-existence of a finite projective plane of order 10, **1989** (proved by C. W. H. Lam, L. Thiel and S. Swiercz).

- Problems solved by interval arithmetic **1993+**: Kepler's conjecture [8] (partially applied), the existence of eigenvalues of the Sturm-Liouville problem [9], the bound of Feigenbaum constant [10], the double bubble conjecture [11], verification of chaos [12], [13], Lorenz attractor [14], etc.

- BBP (Borwein, Bailey, Plouffe) formula for π, **1996** (published in [15]):

  $$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\Big(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\Big).$$

  BBP formula is revolutionary and fascinating since provides the determination, for example, the one-billionth hexadecimal digit (or the four billionth binary digit) of π without needing to compute any of the previous digits. Practical BBP algorithm for computing the requested *individual digit* of π was described in [16, pp. 121–125].

- Robbins conjecture, **1996**: All *Robbins algebra*, supplied with a single *binary operation* denoted by $\vee$ (OR) and a single *unary operation* denoted by $\neg$ (NEGATION) are *Boolean algebras.* This conjecture was proved by W. McCune in 1996.

- Kepler conjecture (from 1611) on the most density package of identical spheres in three-dimensional Euclidean space, **2000**. The measure of the *density* $\delta = V_s/V_c$ is the total volume V_s of all packed spheres divided by the total volume V_c of the container in the form of a cube assuming that the cube edge is infinitely large. In 2000 Tomas Hales completed the solution proving that the so-called *face-centered cubic packing* has the maximum density $\delta_{max} = \pi/\sqrt{18}$, just as Kepler assumed (see the book [3, pp. 137–147] for details). Hales' proof, published in [8], combines methods from the theory of global optimization, linear programming and interval arithmetic.

- Lorenz attractor, **2002**, known as 14th of Smale's problem. It is the solution of Lorenz's system that describes chaotic behavior. Its existence was shown by W. Tucker [14] using validated interval arithmetic and normal forms; he also proved that Lorenz attractor is so-called *strange attractor*. Lorenz attractor appears in fluid dynamics and illustrates the phenomenon now known as the *butterfly effect* which demonstrates sensitive dependence on initial conditions.
- NP-hardness of minimum-weight triangulation. The minimum-weight triangulation problem belongs to computational geometry and computer science that asks for the minimal sum of the length of perimeters which make a triangulation (subdivision by triangles) of a given polygon or the convex hull. In 2008 W. Mulzer and G. Rote proved that this problem is NP-hard.
- Optimal solutions for Rubik's Cube can be obtained in at most 20 face moves starting from arbitrary initial position, **2010** (computer-assisted proof was given by T. Rokicki, H. Koceimba, M. Davidson, J. Dethridge).
- The primality test of very large natural numbers and the factorization of very large numbers, **1949+**. Many cryptographic protocols are based on the difficulty of factoring large composite integers. At present, the largest prime number is $2^{82589933} - 1$ having 24 862 048 decimal digits (found by Laroche et al. in December of 2018).

Although the computer solution of the four-color theorem (1976) and the Kepler's conjecture (2006) attracted considerable attention in mathematics, the proofs were not accepted by all mathematicians who made a serious objection that the presented computer-assisted proofs (better to say, the program codes) were not verifiable for a human by hand. Their reaction with many arguments against Hales' computer-assisted proof was justified; for illustration, Hales' computer program consisted of 40 000 lines. Fortunately, these two stories had a happy ending. As mentioned above, in 2008 G. Gonthier [7] delivered widely accepted proof of the four-color theorem using general-purpose theorem-proving software. Hales started in 2003 with a project named FlysPecK (F, P and K standing for Formal Proof of Kepler) aiming to come up with a formal proof of the Kepler conjecture that can be checked by automated proof verifying software. After 14 years Hales and his team finished this challenging but very difficult project; their formal proof was published in the journal *Forum of Mathematics* in 2017.

2.2. Interval arithmetic and self-validated method

An important use of computers in proving mathematical hypotheses and problems, known as self-validating numerics, is a special kind of computation that preserves strong mathematical rigor. This approach uses interval arithmetic which provides the enclosure, control, and propagation of roundoff and truncation errors of the executed calculation. The fruitful feature of interval arithmetic is the *inclusion principle* (essentially meaning *subset property*) which assures that the results of computations or solutions of the posed mathematical problems are enclosed by the set-valued output. In this way, it is possible to calculate upper and lower bounds on the sets of solutions. Therefore, self-validating numerical methods deliver true results.

The described very useful property has provided the application of interval arithmetic not only in mathematics but also in many scientific disciplines where the control of a true result is of primary interest. Some of the mathematical problems solved by self-validated methods is listed above. Note that German scientist Siegfried M. Rump (Technische Universität Hamburg) created a special software INTLAB, based on Matlab, intended for the implementation of interval arithmetic for solving a huge number of mathematical problems [17].

Note that Professor Urlich Kulisch, one of the greatest world experts in the field of computer architecture and interval arithmetic, claims that further advance in computer technology and software will lead to the weird situation that the accuracy of results obtained by a computer can only be verified with the help of a computer (again!) and interval arithmetic that would control the intermediate results at every step, see his monograph *Computer Arithmetic and Validity* [18].

2.3. Experimental mathematics

A relatively new approach to mathematics that makes use of advanced and powerful computing technology to investigate mathematical objects and identify properties and patterns is called *experimental mathematics*, the term introduced by J. Borwein, D. Bailey, R, Girgensohn and their contributors, see, e.g., the books [16], [19], [20]. Experimental mathematics, a growing branch of applied mathematics, provides computational methodologies of doing mathematics that include the use of computations for the following activities quoted in [16]:

(1) Gaining insight and intuition.

(2) Discovering new patterns and relationships.

(3) Using graphical displays to suggest underlying mathematical principles.

(4) Testing and especially falsifying conjectures.

(5) Exploring a possible result to see if it is worth a formal proof.

(6) Suggesting approaches for formal proof.

(7) Replacing lengthy hand derivations with computer-based derivations.

(8) Confirming analytically derived results.

In the book *Mathematics by Experiments* [16], J. Borwain and D. Bailey, the world-leading experts in experimental mathematics, gave the list of things computers do better than humans. We cite their list below:

- High precision integer and floating-point arithmetic;

- Symbolic computation for algebraic and calculus manipulations;
- Formal power-series manipulation;
- Changing representations, e.g., continued fraction expansions, partial fraction expansions, Padé approximations;
- Recursion solving (e.g., `Rsolve` in *Mathematica*);
- Integer relation algorithms, e.g., the PSLQ algorithm;
- Creative telescoping (e.g., the Gosper and Wilf-Zeilberger methods) for proving summation identities;
- Iterative approximations to continuous functions;
- Identification of functions based on graph characteristics;
- Graphics and visualization methods.

"Some of the algorithms involved in this list have had the great influence on the development and practice of science and engineering", wrote Dongarra and Sullivan in [21], and added often cited sentence: *"Great algorithms are the poetry of computation."*

2.4. Computer-assisted proofs

Attempts have been also made in the area of Artificial Intelligence research to create new proofs of mathematical theorems using machine reasoning techniques. A computer-assisted proof or automated theorem prover are relatively recent notions which mean that a mathematical proof has been generated (at least partially) by computer. The majority of computer-aided proofs of mathematical theorems up to now were the simple application of *proofs-by-exhaustion* of all items of the problem (*brute force, backtrack algorithms*), for example, in searching for counterexamples of hypotheses in Number theory or solutions of problems having a huge outcomes/configurations. In contrast to the exhaustion method, *interactive proof assistants* most frequently gives human-readable proofs which can be checked for correctness; hence it is considerably preferable. The third type, sometimes named *a proper* computer-aided proofs, is completely based on sets of axioms and logical statements of computer software and gives reliable and correct results. More details devoted to computer-assisted proofs can be found on the link `https://en.wikipedia.org/wiki/Computer-assisted_proof`.

As examples of important achievements in the field of computer-assisted proofs, let us mention theorem-proving packages and algorithms of Wilf-Zeilberger's type. Theorem-proving package methods, such as Microsoft's Z3 Theorem Prover (now available under MIT Open Source), can either verify certain types of statements or find a counterexample demonstrating that a statement is false. The Wilf-Zeilberger method (invented by Doron Zeilberger and Herbert Wilf in 1990) can perform symbolic computations working with variables instead of numbers to produce exact results in a general form free of roundoff errors.

2.5. Computer visualization

The development of high quality computer visualization enables entirely new and remarkable insights into a wide variety of mathematical concepts and objects. Today researchers are able to study the geometric aspects of many mathematical and engineering disciplines. Computer graphics have become powerful tools for discovering new properties on various topics of mathematics and constructing new very efficient algorithms. Undoubtedly, computer visualization delivers modern and novel perspectives of some mathematical topics yielding a new dimension and a deep insight into properties and behavior of many mathematical processes, as well as various processes and phenomena in physics, biology, chemistry, and other scientific disciplines.

As one illustration of high sophistication of computer visualization, we present Tupper's astounding formula

$$\frac{1}{2} < \left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x\rfloor - \mathrm{mod}(\lfloor y\rfloor, 17)}, 2\right)\right\rfloor.$$

published in 2001. Here $\lfloor x\rfloor$ denotes the floor function (the greatest integer part of a number x) and $\mathrm{mod}(a, m)$ is the remainder in dividing the integer a by the integer m (the mod function). The area of graphics is determined by $0 \leqslant x \leqslant 105$ and $k \leqslant y \leqslant k + 16$ where k is the natural number with 543 digits

960 939 379 918 958 884 971 672 962 127 852 754 715 004 339 660 129 306 651 505
519 271 702 802 395 266 424 689 642 842 174 350 718 121 267 153 782 770 623 355
993 237 280 874 144 307 891 325 963 941 337 723 487 857 735 749 823 926 629 715
517 173 716 995 165 232 890 538 221 612 403 238 855 866 184 013 235 585 136 048
828 693 337 902 491 454 229 288 667 081 096 184 496 091 705 183 454 067 827 731
551 705 405 381 627 380 967 602 565 685 016 981 482 083 418 783 163 849 115 590
225 610 003 652 351 370 343 874 461 848 378 737 238 198 224 849 863 465 033 159
410 054 974 700 593 138 339 226 497 249 461 751 545 728 366 702 369 745 461 014
655 997 933 798 537 483 143 786 841 806 593 422 227 898 388 722 980 000 748 404
719

Using Tupper's formula, a simple program in CAS *Mathematica*

```
ArrayPlot[Table[Boole[1/2 < Floor[Mod[Floor[y/17] 2^(-17 Floor[x]-
Mod[Floor[y], 17]), 2]]], {y,n,n+16},{x,105,-2,-1}],
PixelConstrained -> True, Frame -> False, ImageSize -> 400]
```

gives the *self-referential* "plot" presented in the figure below.

In fact, Tupper demonstrated a method of decoding a bitmap stored in the constant k; k is a simple monochrome bitmap image of the formula treated as a binary number and multiplied by 17. Note that Tapper's approach is a general-purpose method to draw any other image.

2.6. Symbolic computation

Symbolic computation, a part of Computer algebra serving as a bridge between Mathematics and Computer science, is handling non-numerical values. Symbolic computation is widely used to experiment in mathematics and to study and design formulas, algorithms and software that are used in numerical programs. Computer algebra systems that perform symbolic calculations contain a lot of routines to carry out many operations, like polynomial factorization, solving nonlinear equations, manipulation with very complicated expressions. They are also capable to expand or simplify mathematical expressions with symbols, or differentiate or integrate them, etc. It should be emphasized that, contrary to numerical computation, symbolic computation produces *exact computation* with expressions containing variables that are manipulated as symbols.

As an illustration of the use of symbolic computation we present everyday practical problem posed by George Polya, a Stanford professor, in *American Mathematical Monthly* article (1956). *In how many ways can you make change for a dollar?* We modify Polya's task and consider Serbian currency assuming that there are 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000, 2000 and 5000 coins or banknotes. Hence:

In how many ways can you make change for a banknote of 5000 *Serbian dinars?*

Problems of this type are solved by generating functions. Let P_k be the number of all possible ways of changes. The problem reduces to the generating function (the Serbian currency case)

$$\sum_{k=1}^{\infty} P_k x^k = \frac{1}{(1-x^1)(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})\cdots(1-x^{2000})(1-x^{5000})}.$$

To find P_{5000} it is necessary to develop the expression on the right-hand side into geometric series and sum all coefficients standing next to x^{5000}. Using a *Mathematica* command

```
Series[1/((1-x)*(1-x^2)*(1-x^5)* (1-x^(10))*(1-x^(20))*(1-x^(50))
*(1-x^(100))*(1-x^(200))*(1-x^(500))*(1-x^(1000))
*(1-x^(2000)*(1-x^(5000),{x,0,5000}]
```

computer calculates $P_{5000} = 23\,303\,034\,594\,532$. It is impossible for a human to determine such a huge number. In the case of US currency, one obtains $P_{100} = 292$, which is reachable for a human so that Polya's task had a sense in 1956.

Symbolic computation has successfully substituted lengthy manual calculation with computer-based computation and manipulation. In this paper, we concentrate in & 3.5 on methods and procedures for the construction, analysis and practical application of algorithms for solving nonlinear equations with the support of symbolic computation. We emphasize that the construction of presented root-solvers is most likely impossible without the use of this specific computer software.

2.7. Computer-assisted proofs: how much can we trust computers

The use of computers in mathematics is undoubtedly widespread and in unstoppable expansion. Many mathematicians have turned to numerical experiments, symbolic computation, computer visualization and other computer methods as their main tools for mathematical investigation. In that way, they have achieved extraordinary results. However, ignoring these advances, a number of researchers often underestimate the role of computers in mathematics. In some cases, their skepticism cannot be fully disregarded since there are some specialized fields in mathematics that do not need the use of computers. Recall that, without using a computer, Andrew Wiles solved the famous Fermat last theorem (stated by Fermat in 1637) in 1995, Grigori Perelman presented a proof of Poincaré's conjecture, one of the most important open problems in topology, through three papers made available in 2002 and 2003 on arXiv of Cornell University. The proof of the Riemann hypothesis on the locations of zeros of the Riemann zeta function (posed in 1859) has not yet been given. Many mathematicians believe that the Riemann hypothesis, one of the most important open problems in mathematics, will be proved by a human using an analytical method, not by computer tools.

It seems that another kind of disputable question is more serious. Today, in search for the exact result or ultimate truth, mathematicians, philosophers and computer scientists (among them, Turing, Voevodsky, Avigard, Teleman, Kim, Mancosu, Hanke), ask: *"How much can we trust computers, whether computer-assisted proofs have the mathematical sense, is it possible to verify so many logical steps, how to evaluate the reliability of the data, how to check that the computer source program is perfectly accurate, whether the researcher can fully believe in the perfect work of hardware, what if there is a bug?, etc."* Errors of this kind could be sometimes avoided by using different programming languages, different compilers, and different computer hardware. For instance, this approach was applied to Gonthier's proof of the four-color theorem, see [7].

Professor Jonathan Hanke, a number theorist and skilled programmer at the Princeton University, is quite careful; he is focused on developing and implementing algorithms to solve concrete problems in programming language Python. To his opinion, software should never be trusted; it should be checked. Besides, in Hanke's opinion, the only way to avoid false results is to use computers in the proofs of theorems step by step very carefully, using special tests applied to separated sections (small or large, as needed) of a global program with unmistakable logic.

The science of program proving was a formally accepted field of computer science. Program proving, model checking, theorem solving – this is the terminology occupying the research space of computer science devoted to making sure programs work correctly. Computer programs analyze, check and inspect key situations and outcomes by sophistical algorithms, and verify the validity of the theorem using the data collected passing through this process. David Bailey, a mathematician and computers scientist at Lawrence Berkeley National Laboratory (now at University of California, Davis), one of the world leaders in experimental mathematics, said: *"The time when someone can do real, publishable mathematics completely without the aid of a computer is coming to a close. Or if you do, you are going to be restricted into some specialized realms."*

Doron Zeilberger (1950–), a world-renowned Israeli professor at Rutgers University, the winner of prestigious awards such as Ford Award, Steele Prize, Euler Medal and Robbins Prize, does not share the mentioned view of his sceptical colleagues. He said: *"Contemporary mathematics is becoming significantly complicated, making further progress more difficult. In many mathematical disciplines computers are so much incorporated that only at the frontiers of some research areas of mathematics, human proofs still exist."*. Note that Zeilberger writes his own code using a computer algebra system *Maple* and believes computers are overtaking humans in their ability to discover new mathematics. See the link `www.wired.com/2013/03/computers=and=math/`.

Searching for ultimate truth in mathematics, the Field medallist Vladimir Voevodsky (1966–2017) (Institute for Advanced Studies in Princeton) posed the question: *"How do mathematicians know that something they prove is actually true?"* Similarly, as Zeilberger and Bailey, he comprehended that increasing the complexity of mathematics could be resolved only by the computer since a human brain could not keep up a huge amount of data and manipulate with them. To resolve this very hard problem, Voevodsky started, as the leader of a team, a long-term extraordinary project to create fundamentally new computer tools to confirm the accuracy of proofs. For this purpose, Voevodsky and his team have united different research fields, such as homotopy theory, mathematical logic, and the theory of programming languages, to make computer-verified proofs.

We end this section with an interesting story that tells how much Zeilberger believes in computer-assisted proofs and other computer tools for solving mathematical problems. Some thirty years ago several mysterious but excellent research papers (77 in total) appeared in a short period in the renowned mathematical journals (co)-authored by Shalosh B. Ekhad; in addition, notable Rutgers University (New Jersey) was marked as the affiliation. Curious mathematicians have tried to learn anything about the personality of Ekhad for three reasons; this name was fully unknown in the mathematical literature, nobody has ever seen him, and there was no Professor Ekhad employed at Rutgers University. The Israeli mathematician Doron Zeilberger from Rutgers University (the affiliation was correct) resolved the mystery admitting that Shalosh B. Ekhad is not a person but his computer. In Hebrew the words "Shalosh and "Ekhad" mean THREE and ONE respectively, and "three B one" refers to the AT&T 3B1, the first computer that he had been using in his work. Wishing to emphasize the great importance of computers to his research, Zeilberger cited Shalosh B. Ekhad as his co-author of scientific papers.

3. Computers in mathematical research - authors contributions

In this section we present some illustrative mathematical problems of dual nature; they belong to numerical mathematics but also to computer science (roundoff error analysis). An unexpected but interesting behavior of iteration procedure, arising as a consequence of the presence of roundoff errors, are discussed in Section 2.& 3.1 and & 3.2. Strange distribution of zeros of algebraic polynomials with random coefficients is demonstrated by two examples in & 3.3. In & 3.4 we present the dynamic study of root-finding methods by basins of attractions and point to useful benefits of visualization and associate data. A new three-point weighted family of iterative methods for approximating solutions of nonlinear equations is the subject of & 3.5. The derivation of the method and its convergence analysis are performed using symbolic computation. This study deals with very complicated and lengthly expressions (consists of 200 and more outcome lines) so that the construction and analysis of the proposed family is far beyond human capability. Two self-validated iterative methods for the inclusion of a simple zero of a given polynomial are presented and numerically tested in & 3.6.

3.1. Strange recurrent relation and roundoff errors

This example originally constructed in [22], inspired by Kahan's recurrent relations, presents in illustrative way the influence of roundoff error to the accuracy of result of computation. Let us calculate the members of the sequence $\{x_k\}$ in floating-point arithmetic of double or quadruple precision using the recurrent relation

$$\left\{\begin{aligned} x_0 &= 1, \\ x_1 &= -5, \\ x_{k+1} &= 207 - \frac{1412}{x_k} + \frac{2400}{x_{k-1}x_k}. \end{aligned}\right. \tag{3.1}$$

After a certain number of iterative steps, we observe that x_k approaches 200, see Figure 3.1. However, using methods for solving difference equations we find the general solution of the recurrent relation (3.1) in the form

$$x_k = \frac{200^{k+1}a + 4^{k+1}b + 3^{k+1}}{200^k a + 4^k b + 3^k},$$

where a and b are arbitrary constants. For the given initial values $x_0 = 1$ and $x_1 = -5$ one obtains $a = 0,\ b = -2/3$, so that the above formula reduces to the simple form

$$x_k = \frac{-\frac{2}{3}\cdot 4^{k+1} + 3^{k+1}}{-\frac{2}{3}\cdot 4^k + 3^k} = 4 - \frac{1}{1 - \frac{2}{3}\left(\frac{4}{3}\right)^k}. \tag{3.2}$$

From (3.2) it is clear that $x_k \to 4$ when $k \to \infty$.

Incorrect result ($x_k \to 200$) is the consequence of roundoff error during calculation. Namely, the application of floating-point arithmetic does not calculate the theoretical value $a = 0$ but $a = \eta \neq 0$ and $b = -2/3 + \varepsilon$, where η and ε are of the order of machine-precision, say 10^{-16}. In this way, instead of (3.2), we have

$$\hat{x}_k \approx \frac{200^{k+1}\eta + (-\frac{2}{3} + \varepsilon) \cdot 4^{k+1} + 3^{k+1}}{200^k \eta + (-\frac{2}{3} + \varepsilon) \cdot 4^k + 3^k} = \varphi(k, \eta, \varepsilon) + 200.$$

where

$$\varphi(k, \eta, \varepsilon) = \frac{-49(3\varepsilon - 2) \cdot 4^{k+1} - 197 \cdot 3^{k+1}}{3\eta \cdot 200^k + (3\varepsilon - 2) \cdot 4^k + 3^{k+1}}.$$

Since $\varphi(k, \eta, \varepsilon) \to 0$ when $k \to \infty$ independently on the value of η and ε (but having in mind that both are of the order of machine-precision), one obtains $\hat{x}_k \to 200$. Observe that if $\eta = \varepsilon = 0$, then $\varphi(k, \eta, \varepsilon) \to -196$ and $\hat{x}_k \to 4$.

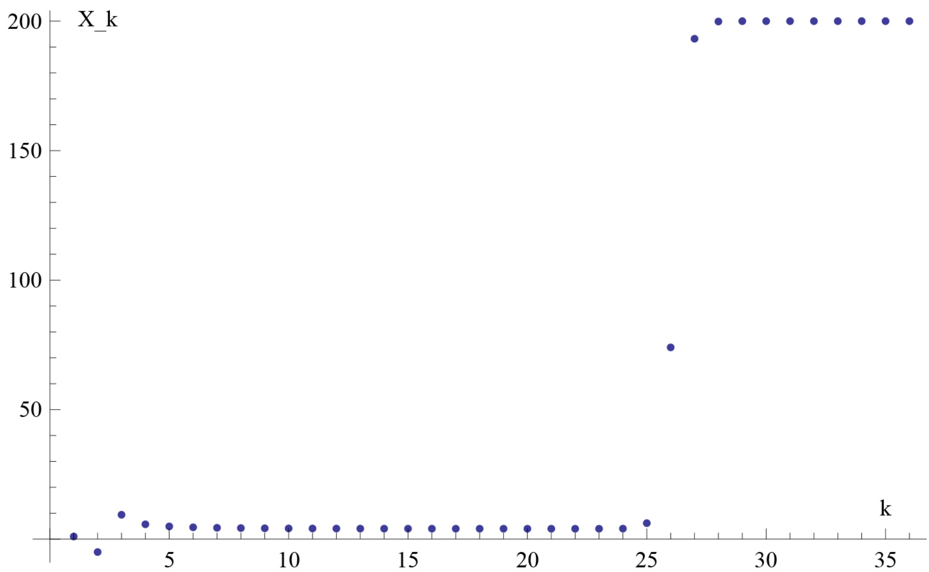


FIG. 3.1: Convergence of the sequence (3.1) to (incorrect) limit 200; double-precision arithmetic was employed.

Calculating x_k by (3.1) in double -precision arithmetic and using the termination criterion $|x_k - x_{k-1}| < \tau$, we have found that the iterative computation breaks when $k = 36$ dealing with $\tau = 10^{-12}$. First 36 iterations and the values of x_k are given in Figure 3.1. From this figure, we observe that, in the beginning, approximations of x_k approach the exact limit $x_\infty = 4$ but do not reach the required precision. The minimal error is $x_{23} - 4 \approx 3.55 \times 10^{-3}$. Then a very steep jump appears for $k = 25$ and through few steps x_k approaches (incorrect) limit 200. This jump, in fact, arises due to the presence of roundoff errors η and ε which make that the function $\varphi(k, \eta, \varepsilon)$ *acquires* a vertical asymptote. Note that this asymptote does not appear for $k \geqslant 2$ if $\eta = \varepsilon = 0$.

3.2. Power method for dominant eigenvalue – the benefit of roundoff

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. λ_1 is called the **dominant eigenvalue** or **spectral radius** of A if

$$|\lambda_1| > |\lambda_i| \quad (i = 2, \ldots, n).$$

The eigenvector corresponding to λ_1 is called **dominant eigenvector** of A. In the literature, the spectral radius is most frequently denoted by $\rho(A)$.

The **power method** is an iterative method which is often applied for approximating spectral radius of a given matrix A. Scaling version of the power method can be presented in the following algorithmic form:

$$(3.3) \begin{cases} \text{1. } \textbf{Choose} \text{ starting non-zero vector } \boldsymbol{y}_0 = \{y_{1,0}, \ldots, y_{n,0}\}; \\ \text{2. } \textbf{For } k = 1, 2, \ldots \ \textbf{calculate} \\ \qquad \boldsymbol{z}_k = A\,\boldsymbol{y}_{k-1}, \quad \boldsymbol{y}_k = \boldsymbol{z}_k / \alpha_k, \\ \quad \text{where } \alpha_k \text{ is the coordinate of the vector } \boldsymbol{z}_k \text{ with the largest moduli.} \\ \text{3. } \textbf{Finish} \text{ the iterative process when the stopping criterion is fulfilled.} \end{cases}$$

Note that

$$\alpha_k \to \lambda_1 \quad \text{and} \quad \boldsymbol{y}_k \to \frac{\boldsymbol{x}_1}{\| \boldsymbol{x}_1 \|_\infty},$$

where $\boldsymbol{x}_1$ is the dominant eigenvector that correspond to the dominant eigenvalue λ_1. The value α_k is taken to be the approximation of the spectral radius $\lambda_1 = \rho(A)$.

In practical problems, the presence of roundoff error can often cause inaccurate results. Opposite to the previous request, in the application of the power method roundoff errors can play a positive role, as mentioned by Higham [39]. Such a situation is demonstrated by the following example.

Example 3.1. Let us determine the approximative value of the spectral radius of the matrix

$$A = \begin{bmatrix} 0.5 & -0.8 & 0.3 \\ -0.6 & 0.8 & -0.2 \\ 0.24 & 0.67 & -0.91 \end{bmatrix}$$

using the presented power method with scaling. First of all, note that the power method (3.3) applied in single precision fails if we take $\boldsymbol{y}_1 = \{1, 1, 1\}$ since in the next step it produces the zero vector. Hence, there is no indication of the wanted dominant eigenvalue. However, executing the first step in double-precision arithmetic, we get

$$\boldsymbol{y}_1 = A \cdot \{1, 1, 1\} = \{5.55112 \times 10^{-17}, 5.55112 \times 10^{-17}, 0.\}\,.$$

The presence of roundoff errors produces $\boldsymbol{y}_1 \neq \mathbf{0}$.. Applying the power method (8), after 18 iterations we obtain $\alpha_{18} = 1.30818$ and take this value as an approximation of the spectral radius. The spectral radius of A with 15 correct decimal digits is $\rho(A) = 1.308114998551363$, which means that the approximation α_{18} has 5 significant decimal digits.

3.3. Distribution of zeros of random polynomials

Algebraic polynomials whose coefficients are random numbers are of great importance since they appear in various problems of physics, engineering and economics such as filtering theory, spectral analysis of random matrices, statistical communication, regression curves in statistics, characteristic equations of random matrices, the study of random difference equations, the analysis of capital and investment in mathematical economics. etc. For these reasons, a number of books and papers have been devoted to the study of random polynomials, see, e.g., [23] and [24]. Working in the area of *Experimental Mathematics*, we have used graphical methods to visualize an important theorem on distribution of zeros and pose a conjecture of the symmetry of complex zeros od random polynomials.

Example 3.2. Denote a sequence of independently identically distributed (real or complex) valued random variables with $\{c_k\}_{k=0}^{\infty}$. Let

$$F_n(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$$

be a random polynomial of degree n with the zeros $\zeta_1, \zeta_2, \ldots, \zeta_n$ of F_n. Furthermore, for a, b $(0 \leqslant a \leqslant b < \infty)$ introduce a probability measure on the complex plane $R_n(a, b) = N_n(\{z : a \leqslant |\zeta_i| \leqslant b\})$, where $N_n(\cdot)$ denotes the number of zeros that belong to the ring $\{z : a \leqslant |\zeta_i| \leqslant b\}$ in the complex plane. Then R_n/n defines the *empirical distribution* of zeros of F_n. If for any $\delta \in (0, 1)$ define $a = 1 - \delta$, $b = 1 + \delta$, then δ is called *delta measure* in the empirical distribution.

The following theorem has been proved in [24]:

Theorem 3.1. *If and only if*

$$\mathbf{E}\, log(1 + |c_0|) < \infty,$$

then the sequence of the empirical distributions R_n/n converges to the delta measure at 1 *almost surely, that is,*

$$\frac{1}{n} R_n(1 - \delta, 1 + \delta) \xrightarrow{\mathbf{P}} 1, \quad n \to \infty$$

holds for any $\delta \in (0, 1)$.

In the above theorem, $\mathbf{E}$ is mathematical expectation while the denotation $\xrightarrow{\mathbf{P}}$ denotes so-called **convergence in probability**. This theorem asserts that, under some weak constraints on the coefficients of a random polynomial, *almost all its zeros* "concentrate uniformly" close to the unit circle with high probability.

Using graphical tools of *Mathematica* we have tested a random polynomial of degree 2000 with random coefficients belonging to the interval $[-2, 2]$. From Figure 3.2 we observe that almost all zeros are located in the ring $\{z \,|\, 1 - \delta < |z| < 1 + \delta\}$ where $\delta \approx 0.01$, which empirically confirms Theorem 3.1 to a good extent.

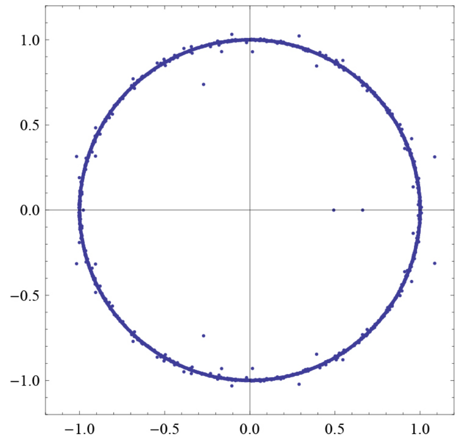


FIG. 3.2: Location of zeros of random polynomials

Example 3.3. Several years ago, on a web page concerning the distribution of zeros of polynomials, the following question appeared: *Given ten or more thousand polynomials of degree* $n \in [n_1, n_2]$ *with the leading coefficient* 1 *while the remaining coefficients are chosen randomly from the set* $\{-1, +1\}$*. Mark the location of each zero by a small circle in the complex point in such a way that the different zeros of the selected polynomial are colored by different colors. Does the plotted figure possess some specific properties?*

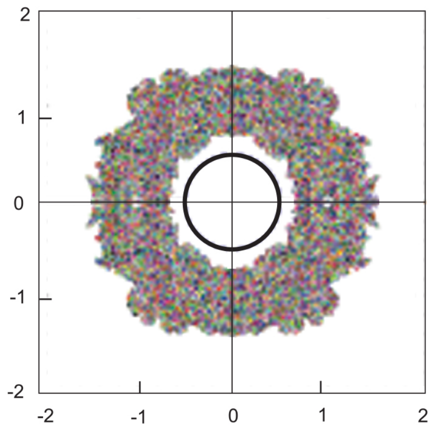


FIG. 3.3: Distribution of zeros of a random polynomial of degree $n \in [10, 18]$

We have taken the range [10,18] for the polynomial degrees and plotted the location of zeros of 50 000 random polynomials. The generated figure, plotted by using the program BWH in *Mathematica* and presented in Figure 3.3, is of “bagel-with-handle” (BWH) form.

BWH PROGRAM (*Mathematica*)

```
Clear[koef]; koef := Sign[-0.5 + Random[]]; Clear[genP];
genP[n_] := Sum[koef x^k, {k, 0, n}]; Clear[solP];
solP[p_] := Map[x /.  # &, Solve[p == 0, x] // Flatten ] // N;
Clear[graP];
graP[s_] := ListPlot[Map[{Re[#], Im[#]} &, s], Frame -> True,
AspectRatio -> 1, PlotRange -> {{-2, 2}, {-2, 2}},
PlotStyle -> {RGBColor[Random[], Random[], Random[]],
PointSize[0.01]}]; t = Table[genP[RandomInteger[10,18]],{50000}];
s = Map[solP, t]; g = Map[graP, s]; Show[g]
```

From Figure 3.3 we observe that the generated picture BWH is entirely symmetric to any straight line L_α passing through the origin, where $\alpha \in (0, \pi)$ is the angle related to the positive direction of abscissa axes. More precisely, any pair of the boundaries ∂A and ∂B of exterior parts A and B bounded by two lines L_α and L_β are of the same shape. The same is valid for two corresponding interior boundaries. To the authors' hypothesis, the presented symmetry arises following the low of large numbers, an important theorem in Probability theory. This theorem asserts that the average of the results of performing the same experiment a large number of times approaches the expected value, as the case in our experiments. This effect is known in the statistics when dealing with very large randomly chosen numbers with uniform distribution.

The second characteristic of our BWH figure is the existence of an empty space (hole) inside BWH. What is the size of this hole? More generally, what are the bounds of the zeros of the considered polynomials

$$\begin{aligned} P_{n,m}(z) &= a_0^{(m)} z^n + a_1^{(m)} z^{n-1} + \cdots + a_{n-1}^{(m)} z + a_n^{(m)}, \\ & a_k^{(m)} \in \{-1, +1\}, \quad n \in [10, 18], \quad m = 1, 2, \ldots, 50\,000\,? \end{aligned} \tag{3.4}$$

Let $\zeta_{1,k}, \ldots, \zeta_{n,k}$ be the zero of the polynomial $P_{n,m}$. According to Henrici's result [25, p. 457], all zeros of $P_{n,m}$ are contained in the disk centered at the origin and with radius R determined as

$$\rho = 2 \max_{1 \leqslant j \leqslant n} \left| \frac{a_j^{(m)}}{a_0^{(m)}} \right|^{1/j}. \tag{3.5}$$

Note that this result holds for polynomials with arbitrary coefficients. According to (3.4) and (3.5) we find $|\zeta_{j,k}| \leqslant \rho = 2$. Substituting $y = 1/z$ in (3.4) and applying again (3.5), we determine the lower bound $|\zeta_{j,k}| \geqslant \frac{1}{2}$. Therefore,

$$\frac{1}{2} \leqslant |\zeta_{j,k}| \leqslant 2 \quad \text{for any } j \in \{1, \ldots, n\} \ \vee \ k \in \{1, \ldots, 50\,000\}.$$

According to the last inequalities, we conclude that all zeros of all 50 000 polynomials lie in the disk $\{0; 2\}$ and there is "no zero" in the hole containing the disk $\{0; 0.5\}$.

Finally, if we deal with the polynomials of relatively low degree (as in the presented example), we can observe the holes on the real axis at the points -1 and 1, see Figure 3.3. We also see that there is no complex zeros in these holes. This effect was noted in the book [20] but without discussion and explanation.

3.4. Dynamic study of root-finding methods

One of the most challenging tasks in the area of iterative methods for solving nonlinear equations is to detect the best algorithm or at least the group of best algorithms. For a long time, the comparative studies of root-finding algorithms were based on comparisons of (i) the number of iterations needed to provide the required accuracy of produced approximations to the solutions, (ii) the convergence rate, (iii) the number of function evaluations per iteration, and (iv) the computational costs of compared algorithms often measured by the consumed CPU time required to fulfill the given stopping criterion. All of the mentioned criteria suffer from the disadvantage consisting of the request for ideal conditions; namely, they are usable only if the chosen initial approximation to the wanted zero of a given function is sufficiently good to provide the convergence, which is difficult to achieve in practice. Even in those cases when it is possible, the rank of compared methods is not reliable since the convergence behavior of root-finding methods depends in a complicated and unpredictable way on the starting points.

The growing development of computer hardware and computer graphics at the end of the twentieth century has provided the significant advance of a new methodology for the visual study of convergence behavior of root-finding methods. It turned out that a realistic quality study of root-finding methods and their reliable ranking can be successfully accomplished by plotting the basins of attraction for the methods. Basins of attractions are the sets of points in the complex plane which simulate the convergence to the zeros of a given function by applying the iterative process. They are of great benefit since offer essential information and insight into the basic features of a considered iterative method such as its convergence behavior and domain of convergence. Also, we can apply basins of attraction to analyze the computational advantages of one iteration function against another and to rank root-solvers within a class of iteration functions, which is of interest for the user to decide which iteration method is preferable for solving a concrete problem.

Definition 3.1. *Let f be a given sufficiently many times differentiable function in some complex domain $R \subseteq \mathbb{C}$ with simple or multiple zeros $\alpha_1, \alpha_2, \ldots, \alpha_\lambda \in S$, and a (convergent) root-finding iteration defined by*

$$z_{k+1} = g(z_k) \quad (k = 0, 1, 2, \ldots),$$

the basin of attraction for the zero α_i is defined as follows:

$$\mathcal{B}_{f,g}(\alpha_i) = \{\zeta \in R \,|\, \text{the iteration } z_{k+1} = g(z_k) \text{ with } z_0 = \zeta \text{ converges to } \alpha_i\}.$$

The dynamic study for the comparison of root-finding algorithms for simple zeros is based on basins of attraction for a given method and a given example. It was launched by Stewart [26] and Varona [27] and continued in the works of Amat et al. [28]–[30], Scott et al. [31], Chun and Neta [32], [33], Neta et al. [34], Argyros and Magreñan [35], Kalantari [36], I. Petković and Neta [37], I. Petković and Đ. Herceg [38] and others.

In the case of algebraic polynomials, the basin of attraction for a given rectangle R with sides parallel to coordinate axes is plotted in the following way. Let (a_1, b_1) be the lower left vertex and (a_2, b_2) the upper right vertex (a_2, b_2) of this rectangle. Using computer algebra system *Mathematica* by the statement

```
CountRoots[P[z],{z,a1+I*b1,a2+I*b2}]
```

we determine the number of zeros of $P(z)$ inside the rectangle R. Analyzing convergence behavior for all zeros of a polynomial of degree n, the rectangle R must be taken so that the outcome N_P of the above statement is n (= number of polynomial zeros). Otherwise, we continue with the enlargement of the size of the rectangle R until $N_P = n$ is satisfied.

The considered method is tested on the $m_1 \times m_2$ equally spaced points in the rectangle $R = \{a_1, b_1\} \times \{a_2, b_2\}$ (forming an equidistant lattice L_R) centered at the origin. At the beginning we define the limit number of iterations IT; if the iterative process, starting from an initial point $z_0 \in L_R$, does not satisfy the given stopping criterion in $\leqslant IT$ iterations, then this starting point is proclaimed "divergent." For each basin we record the CPU time in seconds for all $m_1 \times m_2$ points, average number of iterations (for all points of the lattice L_R) required to satisfy the stopping criterion $|z_k - \alpha| < \tau$ (τ defines the accuracy of approximations, say, $\tau = 10^{-5}$ or $\tau = 10^{-6}$) and the number of black (divergent) points for each method and each example. We associate exactly one color to each attraction basin of a root following two rules: 1) each basin will have a different color and 2) the shading is darker if the number of iterations is higher. Starting points which do not fulfill the stopping criterion after IT iterations are colored black.

The basin of attraction is a kind of computer visualization that provides visual insight into convergence behavior of a root-finding method but it also delivers some valuable qualitative data such as the CPU execution time, the average number of iterations and function evaluations per point, and the number of "divergent" points. These data are most frequently sufficient for deeper insight into the behavior of an iterative method and its domain of convergence from the point of view of dynamical systems. Obviously, a method is better if the consumed CPU time, the average number of iterations and function evaluations per point, and the number of "divergent" points are smaller. It is desirable that the number of divergent point is 0, which points to global convergence of the method.

Convergence behavior of any method can also be estimated to a certain extent according to the shape of basins of attraction for the tested example. It is preferable that the basins of attraction for the zeros have as large as possible unvaried contiguous areas, separated by the boundaries that have (approximately) straight-line form. As small as possible blobs and fractals on the boundaries also point to good convergence properties.

To demonstrate the dynamic study of two iterative methods for finding simple zeros, we give three examples. In all examples we have used an equidistant lattice made of 360 000 points, that is, the resolution is 600×600, the permitted number of iterations is $IT = 40$, and the stopping criterion has been given by $|z_k - \alpha| < 10^{-6} = \tau$.

We emphasize that the dynamic study by basins of attraction is most frequently used in comparative study of different methods of the same order of convergence. Since the main goal of this section is only the presentation of a graphical method for the analysis of the quality of particular methods, any comparative study is beyond our consideration.

We have considered the well-known Halley's method of the third order

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \cdot \frac{1}{1 - \dfrac{f''(x_k)f(x_k)}{2f'(x_k)^2}} \quad (k = 0, 1, 2, \ldots), \tag{3.6}$$

and the three-point-method of order eight

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \dfrac{1}{1 - \frac{2f(y_k)}{f(x_k)}} \cdot \dfrac{f(y_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \dfrac{f[z_k, y_k]}{f[z_k, x_k]} \cdot \dfrac{f(z_k)}{2f[z_k, y_k] - f[z_k, x_k]} \end{cases} \quad (k = 0, 1, 2, \ldots), \tag{3.7}$$

proposed by Sharma and Arora in [40]. It is not difficult to show (see [41]) that this method is a special case of the family of three-point methods constructed in [42]. An extensive investigation presented in [43] and [41] shown that the method (3.7) possesses the best convergence characteristics among three-point methods of the (maximal) order eight in the class of algebraic polynomials.

Example 3.4. We have plotted two basins of attraction applying the methods (3.6) and (3.7) to the polynomial

$$P_1(z) = z^5 - 1$$

and the square $R = \{z = x + iy \mid -3 \leqslant x \leqslant 3, -3 \leqslant y \leqslant 3\}$. The basins are given in Figures 3.4 and Figure 3.5.

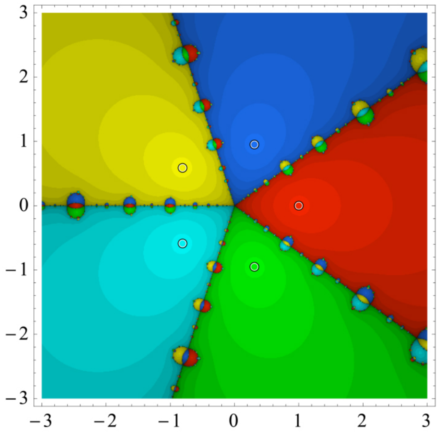


FIG. 3.4: Halley's method (3.6)

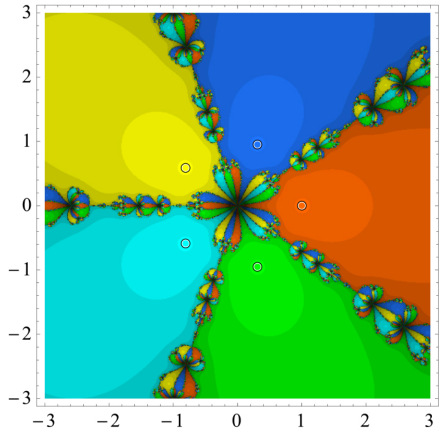


FIG. 3.5: Three-point method (3.7)

Plotting these two basins of attraction we have recorded the following useful data:

	Halley's method (3.6)	Three-point method (3.7)
divergent point	12	21
average number of iterations	5.20	3.45
CPU time (in sec)	58.16	35.37

According to the above data, we conclude that both methods diverge for less than 0.006% starting points, which is rather satisfactory. The three-point method (3.7) reaches the stopping criterion using only 3.45 iterations (in average) against 5.30 for Halley's method and consumes 35.37 seconds for all 360 000 starting points, which is considerably less than Halley's method (58.16 seconds). Regarding the shapes of basins, we observe that in both cases particular basins are of large unvaried size. However, the basins of Halley's method (3.6) has a mild advantage since their boundaries are almost straight lines and contain only a few small blobs, while the boundaries of basins of the method (3.7) have not only larger blobs but also fractal parts.

Example 3.5. We have plotted the basin of attraction for the method (3.7) applied to the polynomial

$$\begin{aligned} P_2(z) = z^{15} &- 1.94409z^{14} - 1.89382z^{13} - 0.00444z^{12} - 0.51467z^{11} - 0.77406z^{10} \\ &-1.80464z^9 + 1.18177z^8 + 0.36718z^7 + 1.31631z^6 - 1.061788z^5 \\ &+1.43835z^4 1.86766z^3 + 0.53726z^2 + 1.72913z - 0.08069, \end{aligned}$$

whose zeros are contained in the square $R = \{z = x+iy \mid -4 \leqslant x \leqslant 4, -4 \leqslant y \leqslant 4\}$. This polynomial has random coefficients (except the leading coefficient) belonging to the interval $[-2, 2]$.

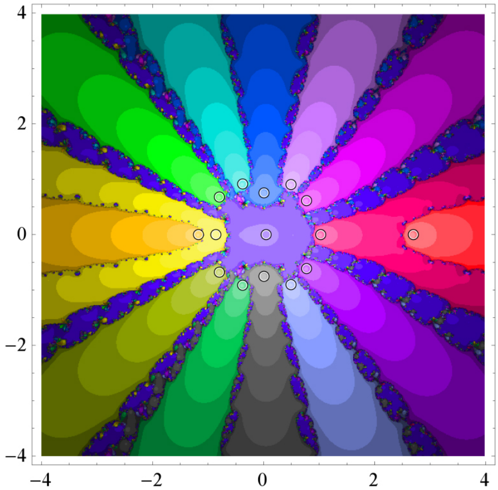


FIG. 3.6: The basins of attraction for the method (3.7) applied to $P_2(z)$.

The basin of attraction for all 15 zeros is presented in Figure 3.6. Small circles mark the location of zeros of the polynomial $P_2(z)$. Considering all 360 000 points we have recorded:

- 0 divergent points,
- the average number of iterations = 6.44,
- the CPU time = 298.8 sec.

The fact that the number of divergent points is 0 points to the global convergence of the method (3.7). However, the boundaries of particular basins are not straight lines but strips (corresponding to some other zeros). This undesirable phenomenon is typical for random polynomials of a high degree, which can lead to certain problems when choosing initial approximations.

Example 3.6. The three-point method (3.7) has been applied to the polynomial

$$P_3(z) = \prod_{m=1}^{13} (z - m).$$

of Wilkinson's type. It is well-known that polynomials of this form are ill-conditioned, causing that many root-finding methods work with big efforts in solving this class of polynomials. However, the applied method (3.7) showed very good convergence behavior, which is evident from the basins of attraction presented in Figure 3.7: particular basins have large unvaried contiguous areas with regular boundaries free of fractal parts and with very small blobs. The associated data are given below:

- 0 divergent points (excellent outcome),
- the average iteration = 4.69,
- the CPU time = 422.7 sec.

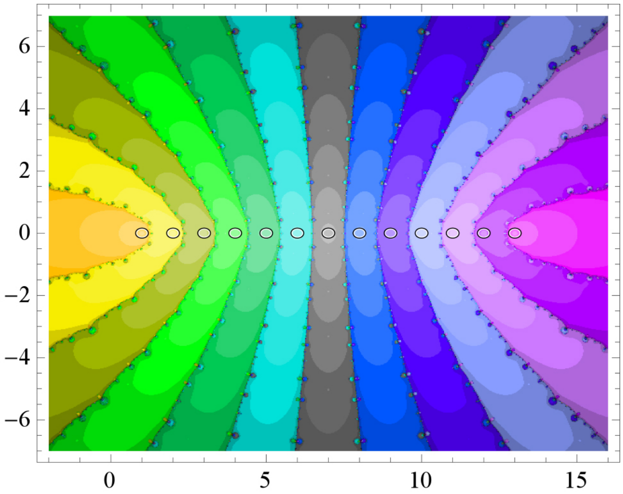


FIG. 3.7: The basins of attraction for the method (3.7) applied to $P_3(z)$.

3.5. On a new three-point weighted method for simple zeros

We start from three-point iterative scheme

$$\begin{cases} N(x_k) = \dfrac{f(x_k)}{f'(x_k)}, \\ y_k = x_k - N(x_k), \\ z_k = y_k - u_k H(u_k) N(x_k), \\ x_{k+1} = z_k - w_k(2w_k+1)P(u_k)Q(v_k)N(x_k), \end{cases} \tag{3.8}$$

where

$$u_k = \frac{f(y_k)}{f(x_k)}, \quad v_k = \frac{f(z_k)}{f(y_k)}, \quad w_k = u_k v_k.$$

We omit the iteration index k and define the errors

$$\varepsilon = x - \alpha, \quad \varepsilon_y = y - \alpha, \quad \varepsilon_z = z - \alpha, \quad \hat{\varepsilon} = \hat{x} - \alpha,$$

where $\hat{x}$ is a new approximation x_{k+1}. Introduce

$$c_r = \frac{f^{(r)}(\alpha)}{r! f'(\alpha)} \quad (r = 1, 2, \ldots).$$

We will use the following development of the function f about the zero α

$$f(x)=f'(\alpha)\Big(1+c_1\varepsilon+c_2\varepsilon^2+c_3\varepsilon^3+c_4\varepsilon^4+c_5\varepsilon^5+c_6\varepsilon^6+c_7\varepsilon^7+c_8\varepsilon^8+O(\varepsilon^9)\Big),$$

and a program in *Mathematica*. As usual, in finding the weight functions H, P and Q, we represent these functions by their Taylor's series at the neighborhood of $u=0$ (for H and P), and $v=0$ (for Q):

$$\begin{aligned} H(u)&=H(0)+H'(0)u+\frac{H''(0)}{2}u^2+\frac{H'''(0)}{6}u^3+\cdots,\\ P(u)&=P(0)+P'(0)u+\frac{P''(0)}{2}u^2+\frac{P'''(0)}{6}u^3+\cdots,\\ Q(v)&=Q(0)+Q'(0)v+\frac{Q''(0)}{2}v^2+\frac{Q'''(0)}{6}v^3+\cdots. \end{aligned}$$

The coefficients of Taylor's developments of the weight functions P and Q are determined using an interactive approach by combining the program realized in *Mathematica* (two parts) and the annihilation of coefficients standing at ε of lower degree. For simplicity, we write $H_0=H(0)$, $H_1=H'(0)$, $Q_3=Q'''(0)$, etc, and

$\mathtt{fa}=f'(\alpha)$, $\mathtt{fx}=f(x)$, $\mathtt{fy}=f(y)$, $\mathtt{fz}=f(z)$, $\mathtt{fx1}=f'(x)$, $\mathtt{newt}=f(x)/f'(x)$, $\mathtt{e}=\varepsilon$, $\mathtt{ey}=\varepsilon_y$, $\mathtt{ez}=\varepsilon_z$, $\mathtt{e1}=\hat{\varepsilon}$.

PART I (*Mathematica*)

```
fxx = 1+c1*e+c2*e^2+c3*e^3+c4*e^4 +c5*e^5+c6*e^6+c7*e^7+ c8*e^8;
fx = fa*e*fxx; fx1 = D[fx, e]; newt = Series[fx/fx1,{e, 0, 8}];
ey = e - newt; fy = fa*ey (1+1*ey+2*ey^2+c3*ey^3+c4*ey^4);
u = fy*Series[1/fx, {e, 0, 8}];
H = H0+H1*u+H2/2*u^2+H3/6*u^3;
ez = Series[ey - u*newt*H // FullSimplify, {e, 0, 8}]
```

This program gives

$$e_z=(c_1-c_1H_0)e^2+(-2c_2(-1+H_0)+c_1^2(-2+4H_0-H_1))e^3+O(e^4)$$

To annihilate coefficients by e^2 and e^3, it is necessary and sufficient to take

$$H_0=1,\quad H_1=2,\quad H_2 \text{ and } H_3 \text{ arbitrary},$$

which gives

$$e_z=(-c_1c_2+c_1^3(5-H_2/2))e^4+O(e^5).$$

The part II of the program uses previously found entries and serves for finding additional conditions which provide optimal order eight.

PART II - CONTINUATION (*Mathematica*)

```
fz = fa*ez*(1+c1*ez+c2*ez^2); v = fz*Series[1/fy,{e, 0, 8}];
P = P0+P1*u+P2/2*u^2+P3/6*u^3;
Q = Q0+Q1*v;
e1 = Series[ez-u*v*P*G*(2u*v+1)*newt,{e,0,8}]//FullSimplify
```

The error $\hat{\varepsilon} = \hat{x} - \alpha$ $(= e1)$ is given in the form

$$\varepsilon_1 = \sum_{r=4}^{8} T_r \varepsilon^r + O\big(\varepsilon^9\big)$$

From the conditions $T_4 = 0,\ T_5 = 0,\ T_6 = 0,\ T_7 = 0$, we find the following relations for finding the required coefficients:

$$\begin{aligned}
&H(0) = 1, \quad H'(0) = 2,\\
&P'(0) = 2P(0), \quad P''(0) = P(0)(2 + H''(0)), \quad P'''(0) = P(0)(H''(0) + 6H''(0) - 24),\\
&Q(0) = Q'(0) = \frac{1}{P(0)}.
\end{aligned}$$

A natural choice $P(0) = 1$ gives

$$\begin{aligned}
&H(0) = 1, \quad H(0) = 2,\\
&P(0) = 1, \quad P'(0) = 2, \quad P''(0) = 2 + H''(0), \quad p'''(0) = H'''(0) + 6H'(0) - 24,\\
&Q(0) = Q'(0) = 1.
\end{aligned} \tag{3.9}$$

In this way we have proved the following assertion.

Theorem 3.2. *If the initial approximation x_0 is sufficiently close to the zero α of f and the conditions* (3.9) *are valid, then the order of the three-point family* (3.8) *is eight.*

Kung-Traub hypotheses [44] assert that as high as possible order of convergence of the n-point method that uses $n+1$ function evaluations per iteration is 2^n. Such methods are called *optimal methods*. Therefore, according to this hypothesis and Theorem 3.2, the three-point iterative method (3.8) is optimal.

3.6. Iterative method for the inclusion of a simple complex zero

R. E. Moore, the founder of Interval analysis, introduced in his monograph [45] the interval version of Newton's method, often called Moore-Newton's method. Let f be a differentiable function on a real interval Ω and let $X_0 = [\underline{x}_0, \overline{x}_0] \subset \Omega$ be a real interval containing a simple real zero η of f. An interval extension $F'(X)$

over the interval X is a real interval such that $F'(X) \supseteq \bar{f}(X) = \{x \,|\, x \in X\}$. Moore-Newton's method is defined by

$$X_{k+1} = \Big\{m(X_k) - \frac{m(X_k)}{F'(X_k)}\Big\} \cap X_k \quad (k = 0, 1, \ldots), \tag{3.10}$$

where $m(X_k) = \frac{1}{2}\big([\underline{x}_k, \overline{x}_k]\big)$ is the midpoint of the interval X_k. It is obvious that this method will be defined if $0 \notin F'(X_k)$ in every iteration.

Moore-Newton's method (3.10) can be applied only for enclosing real zeros, which is a serious disadvantage. Here we present two simple algorithms for finding a simple complex zero ζ of a given algebraic polynomial P that produces a disk $\{c; r\} := \{z \,|\, |z - c| \leqslant r\}$ in the complex plane such that $|c - \zeta| < r$. In this way, these methods provide the upper error bound (given by the radius r) of the approximation c to the desired complex zero ζ. Recall that the inversion of a disk $\{c; r\}$ not containing 0 (that is, $|c| > r$ holds) is defined in [46] by

$$\{c; r\}^{-1} = \Big\{\frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2}\Big\}.$$

Finding initial approximation to the sought zero of a function, sufficiently close to this zero to provide guaranteed convergence, is an equally important task as the construction of an efficient iterative method. This topic is beyond the main subject of this paper and it will not be considered here. Instead, we cite the paper [47] and the master thesis [48] where a composed search-subdividing algorithm for the localization of all complex zeros of algebraic polynomials has been presented with the help of CAS *Mathematica*. This algorithm produces arbitrary small inclusion squares, each of which contains one and only one zero, and calculate the multiplicity of these zeros. It can be of benefit for iterative methods implemented in ordinary complex arithmetic and complex interval arithmetic, discussed in what follows.

Algorithm 1. Let $Z_0 = \{a; R\} = \{z_0; \rho\}$ be the disk that contains one and only one zero ζ of a polynomial P of degree n. The following iterative method was proposed in [49]:

$$Z_{k+1} = z_k - \frac{1}{\{c_k; \rho_k\}} = \Big\{z_k - \frac{\bar{c}_k}{|c_k|^2 - \rho_k^2}; \frac{\rho_k}{|c_k|^2 - \rho_k^2}\Big\} \quad (k = 0, 1, \ldots), \tag{3.11}$$

$$c_k = \frac{P'(z_k)}{P(z_k)} - \frac{(n-1)(\bar{z}_k - \bar{a})}{R^2 - |z_k - a|^2}, \quad \rho_k = \frac{(n-1)R}{R^2 - |z_k - a|^2}, \quad (k > 0). \tag{3.12}$$

The stopping criterion was given by $|P(c_k)| < \tau$, where τ is, say, 10^{-16} or 10^{-33}.

Considering the formulas (3.11) and (3.12) we observe two drawbacks of Algorithm 1. To avoid the division by a zero-interval in (3.11) (which produces a disk of infinity large radius) and negative radius (formula (3.12)), it is necessary to satisfy two conditions in each iteration

$$\text{(i)} \;\; |c_k| > \rho_k, \quad \text{(ii)} \;\; R > |z_k - a|.$$

Regarding (i) we conclude that c_k must be reasonably large and hence, $|P(z_k)|$ should be rather small. Therefore, z_k should be a very good approximation to the zero ζ. Most frequently this is not the case at the beginning of any iterative process so that the first iterations are very critical. To resolve this inconvenient situation the only way is to choose the center a of the initial inclusion disk $\{a; R\}$ very close to the sought zero ζ, which is rather strong requirement (the first drawback). From this discussion there follows $z_k \approx a$ so that $\rho_k \approx (n-1)/R$. The choice of small R increases ρ_k (see (3.12)) so that the validity of inequality (i) may be endangered. Therefore, contrary to the usual request for as small as possible radius of initial inclusion disk, in the case of Algorithm 1 the radius R should be relatively large. Consequently, in this way the inequality (ii) will be ensured. On the other hand, a large R can lead to an undesired enclosure of other zeros of P (next to the zero ζ). It follows that the choice of R has to be refined, sometimes by trial and error method (the second drawback).

Example 3.7. Using Algorithm 1, determine sufficiently small disk that contains the zero $\zeta = 2i$ of the polynomial

$$P(z) = z^9 + 3zt^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300,$$

starting from the inclusion disk $Z_0 = \{0.1 + 2.1i; 1.7\}$ and setting $\tau = 10^{-33}$. The locations of all zeros of P and initial disks Z_0 (containing the sought zero $\zeta = 2i$) are displayed in Figure 3.8.

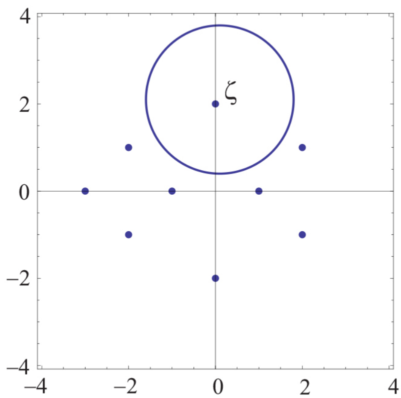


FIG. 3.8: The locations of all zeros of P and initial disks Z_0

We have used CAS *Mathematica* and multi-precision arithmetic (40 significant decimal digits). The following inclusion disks have been obtained:

$$Z_1 = \{0.00473 + 1.97173\,i\,;\,0.0856\}$$
$$Z_2 = \{-0.00128 + 2.00249\,i\,;\,0.00456\ldots\}$$
$$Z_3 = \{-1.7 \times 10^{-5} + 2.00000697\,i\,;\,3.70 \times 10^{-5}\}$$

$$
\begin{aligned}
Z_4 &= \{-4.33 \times 10^{-10} + 1.99999999931\, i\,;\ 1.6 \times 10^{-9}\}\\
Z_5 &= \{1.25 \times 10^{-18} + 2.00000000000000000096\, i\,;\ 3.11 \times 10^{-18}\}\\
Z_6 &= \{5.95 \times 10^{-36} + 2.\underbrace{0000000000 \cdots 0000000000}_{\text{thirty six } 0}74\, i\,;\ 1.18 \times 10^{-35}\}
\end{aligned}
$$

Algorithm 2. We present a combined method for approximate a simple zero of a given polynomial. This method possesses a low computation cost since it uses Newton's method in ordinary complex arithmetic in all iterations except the last one, where a very simple procedure is applied to provide the upper error bound which is involved in the following theorem due to Laguerre (see, e.g., [25, pp. 466–468]):

Theorem 3.3. *Let z be an arbitrary complex number and let P be a given algebraic polynomial. Then the disk $D = \{z; n|P(z)/P'(z)|\}$ contains at least one zero of P.*

The disk D is usually called *Laguerre's disk*. As in the case of Algorithm 1, Algorithm 2 also requires sufficiently good initial approximation z_0 to the zero.

1° step: *Starting from z_0, apply Newton's iteration*

$$z_{k+1} = z_k - \frac{P(z_k)}{P'(z_k)}$$

for $k = 1, 2, ..., K$, where K is the iteration index of the approximation z_K that fulfils the stopping criterion given in the form $|P(z_K)| < \tau$.

2° step: *We use the last approximation z_K obtained in the first step and, using Laguerre's disk defined in Theorem* 3.3*, calculate the inclusion disk*

$$Z_k = \Big\{ z_7; n\Big|\frac{P(z_7)}{P'(z_7)}\Big|\Big\}.$$

The upper error bound is determined by the radius $r = n|P(z_K)/P'(z_K)|$.

Example 3.8. Using Algorithm 2, determine the inclusion disk for the zero $\zeta = 2i$ of the polynomial P given in Example 3.7. In contrast to Algorithm 1, the initial approximation z_0 need not to be very close to ζ and we have chosen $z_0 = 0.2 + 2.3i$. As in Example 3.7, we have used CAS *Mathematica*, multi-precision arithmetic (40 significant decimal digits) and $\tau = 10^{-33}$. First, we have applied Newton's method until the fulfilment of the stopping criterion $|P(z_K)| < 10^{-33}$ and obtained

$$
\begin{aligned}
z_1 &= 0.11848 + 2.10232\, i\\
z_2 &= 0.03978 + 2.00577\, i\\
z_3 &= 0.00151 + 1.99709\, i\\
z_4 &= -1.97 \times 10^{-5} + 2.0000106\, i
\end{aligned}
$$

$$
\begin{aligned}
z_5 &= -9.04 \times 10^{-10} + 1.99999999933\, i \\
z_6 &= 2.26 \times 10^{-18} + 1.99999999999999999983\, i \\
z_7 &= -4.14 \times 10^{-37} + 1.\underbrace{9999999999999999999999999999999999}_{\text{thirty four 9}}89\, i.
\end{aligned}
$$

Since $|P(z_7)| < 10^{-33}$ we have stopped Newton's method and calculated the radius

$$
r = n\left|\frac{P(z_7)}{P'(z_7)}\right| = 9.62 \times 10^{-35}.
$$

Hence, the inclusion disk containing the zero $\zeta = 2i$ is given by

$$
Z_7 = \{-4.14 \times 10^{-37} + 1.\underbrace{9999999999999999999999999999999999}_{\text{thirty four 9}}89\, i\,;\, 9.62 \times 10^{-35}\}.
$$

Considering the results of Examples 3.7 and 3.8, we observe that the upper error bounds are very small and of the same order. Algorithm 1 finished the iterative process through 6 iterations, while Algorithm 2 requested one more. However, the computational cost of Algorithm 2 is considerably less than the cost of Algorithm 1.

Acknowledgement. The authors thank the reviewer for helpful suggestions and comments that improve the presentation of the paper. The authors were supported in part by the Serbian Ministry of Education and Science under Grants 174022 and 174030.

REFERENCES

1. P. Mancosu: *The Philosophy of Mathematical Practice*, Oxford University Press, 2008.
2. T. C Hales: *Cannonballs and honeycombs.* Notices Amer. Math. Soc. **47** (April 2000), 440–449.
3. M. S. Petković: *Famous Puzzles of Great Mathematicians.* American Mathematical Society, 2008.
4. H. C. Williams, R. A. German and C. R. Zarnke: *Solution of the cattle problem of Archimedes.* Math. Comp. **10** (1965), 671–674.
5. K. Appel and W. Haken: *Every planar map is four colorable, Part I, Discharging.* Illinois J. Math. **21** (1977), 429–490.
6. K. Appel and W. Haken, *Every planar map is four colorable, Part I, Reducibility.* Illinois J. Math. **21** (1977), 491–567.
7. G. Gonthier: *Formal proof – The Four-Color Theorem.* Notices Amer. Math. Soc. **55** (Dec. 2008), 1382–1393.
8. T. C. Hales: *A proof of the Kepler conjecture, (unabridged).* Discrete Comput. Geom. **36** (2006), 1–265.

9. B. M. Brown, D. K. R. McCormack and A. Zettl: *On the existence of an eigenvalue below the essential spectrum.* Proc. R. Soc. Lond. **455** (1999), 2229–2234.
10. J.-P. Eckmann and P. Wittwer: *Computer Methods and Borel Summability Applied Feigenbaum's Equation.* Lecture Notes in Physics, Vol. 227, Springer Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.
11. J. Hass, M. Hutchings and R. Schlafy: *The Double Bubble Conjecture.* Elec. Res. Announcement of the Am. Math. Soc. **1** (1995), 98–102.
12. K. Mischaikow and M. Mrozek: *Chaos in the Lorenz equations: A computer assisted proof. Part II: Details.* Math. Comput. **67** (1998), 1023–1046.
13. A. Neumaier and Th. Rage: *Rigorous chaos verification in discrete dynamical systems.* Physica D, **67** (1993), 327–346.
14. W. Tucker: *A rigorous ODE solver and Smale's 14th problem.* Found. Comput. Math. **2** (2002), 53–117.
15. D. Bailey, P. Borwein and S. Plouffe: *On the rapid computation of various polylogarithmic constants.* Math. Comp. **66** (1997), 903–913.
16. J. Borwein and D. Bailey: *Mathematics by Experiment.* A K Peters, Wellesley, Massachusetts, 2008.
17. S.M. Rump: *INTLAB - INTerval LABoratory.* In: Proceeding of the Conference on Developement in Reliable Computing (Scendes, T., ed.), Kluwer Academic Publishers, 1999, pp. 77–104. `http://www.ti3.tu-harburg.de/rump/intlab/index.html.`
18. U. Kulisch: *Computer Arithmetic and Validity.* Walter de Gruyter, Berlin-New York, 2008.
19. J. Borwein, D. Bailey and R. Girgensohn: *Experimentation in Mathematics.* A K Peters, Natick, Massachusetts, 2004.
20. D. Bailey, J. M. Borwein, N. J. Calcin, R. Girgensohn, D. R. Luke and V. H. Moll: *Experimental Mathematics in Action.* A K Peters, Wellesley, Massachusetts, 2007.
21. J. Dongarra and F. Sullivan: *The top 10 algorithms.* Computing in Science and Engineering, Jan./Feb (2000), 22–23.
22. I. Petković: *Analysis of Processing and Computing Iterations by Applying Modern Computer Arithmetics.* Ph. D. Thesis, Faculty of Electronic Engineering, University of Niš, Niš, 2012.
23. A.T. Bharucha-Reid and M. Sambandham, *Random Polynomials.* Academic Press, 1986.
24. I. Ibragimov and D. Zaporozhets: *On distribution of zeros of random polynomials in complex plane.* arXiv:1102.3517v1 [math.PR] 17 Feb 2011.]
25. P. Henrici: *Applied and Computational Complex Analysis, Vol. I. John Wiley and Sons.* New York, 1974.
26. B. D. Stewart: *Attractor Basins of Various Root-finding Methods.* Master Thesis, Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, June 2001.
27. J. L. Varona: *Graphic and numerical comparison between iterative methods.* Math. Intelligencer **24** (2002), 37–46.
28. S. Amat, S. Busquier and S. Plaza, *Review of some iterative root-finding methods from a dynamical point of view.* Scientia **10** (2004), 3–35.

29. S. Amat, S. Busquier and S. Plaza: *Dynamics of a family of third-order iterative methods that do not require using second derivatives.* Appl. Math. Comput. **154** (2004), 735–746.

30. S. Amat, S. Busquier and S. Plaza, *Dynamics of the King and Jarratt iterations.* Aequ. Math. **69** (2005), 212–223.

31. M. Scott, B. Neta and C. Chun: *Basin attractors for various methods.* Appl. Math. Comput. **218** (2011), 2584–2599.

32. C. Chun and B. Neta: *Basins of attraction for Zhou-Chen-Song fourth order family of methods for multiple roots.* Math. Comput. Simul. **109** (2015), 74–91.

33. C. Chun and B. Neta: *Basins of attraction for several third order methods to find multiple roots of nonlinear equations.* Appl. Math. Comput. **268** (2015), 129–137.

34. B. Neta, M. Scott and C. Chun: *Basin of attractions for several methods to find simple roots of nonlinear equations.* Appl. Math. Comput. **218** (2012), 10548–10556.

35. I. K. Argyros and Á. A. Magreñán: *n the convergence of an optimal fourth-order family of methods and its dynamics.* Appl. Math. Comput. **252** (2015), 336–346.

36. B. Kalantari: *Polynomial Root-Finding and Polynomiography.* World Scientific, New Jersey, 2009.

37. I. Petković and B. Neta: *On an application of symbolic computation and computer graphics to root-finders: The case of multiple roots of unknown multiplicity.* J. Comput. Appl. Math. **308** (2016), 215–230.

38. I. Petković and Đ. Herceg: *Symbolic computation and computer graphics as tools for developing and studying new root-finding methods.* Appl. Math. Comput. **295** (2017), 95–113.

39. N. Higham: *Accuracy and Stability of Numerical Algorithms.* SIAM, Philadelpha, 2002.

40. J.R. Sharma and H. Arora: *A new family of optimal eighth order methods with dynamics for nonlinear equations.* Appl. Math. Comput. **273** (2016), 924–933.

41. I. Petković and Đ. Herceg: *Computers in mathematical research: the study of three-point root-finding methods.* Numer. Algor. doi.org/10.1007/s11075-019-00796-6 (online version).

42. J. Džunić, M.S. Petković and L.D. Petković: *A family of optimal three-point methods for solving nonlinear equations using two parametric functions.* Appl. Math. Comput. **217** (2011), 7612–7619.

43. C. Chun and B. Neta: *Comparative study of eighth-order methods for finding simple roots of nonlinear equations.* Numer. Algor. **74** (2017), 1169–1201.

44. H.T. Kung, J.F. Traub: *Optimal order of one-point and multipoint iteration.* Journal of the ACM **21** (1974), 643–651.

45. R.E. Moore: *Interval Analysis.* Prentice Hall, Englewood Cliff, New Jersey, 1966.

46. I. Gargantini, P. Henrici: *Circular arithmetic and the determination of polynomial zeros.* Numer. Math. **18** (1972), 305–320.

47. Đ. Herceg: *An algorithm for localization of polynomial zeros.* In: Proc. VIII International Conf. on Logic and Computer Science (R. Tošić, Z. Budimac, eds.), Institute of Mathematics, University of Novi Sad, Novi Sad, 1997, pp. 67–75.

48. Đ. Herceg: *Computer Implementation and Analysis of Iterative Methods for Solving Nonlinear Equations* (in Serbian). Master Thesis, Faculty of Science, University of Novi Sad, Novi Sad, 1997.

49. M.S. Petković: *Some interval iterations for finding a zero of a polynomial with error bounds*. Comput. Math. Appls. **14** (1987), 479–495.

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51+004

FACTA Universitatis. Series, Mathematics and informatics / editors-in-chief Dragana S. Cvetković-Ilić, Marko D. Petković. - 1986, no. 1- . - Niš : University of Niš, 1986- (Niš: Unigraf-X-Copy). - 24 cm

Dostupno i na: http://casopisi.junis.ni.ac.rs/index.php/FUMathInf. -
Pet puta godišnje. - Drugo izdanje na drugom medijumu: Facta Universitatis.
Series: Mathematics and Informatics (Online) = ISSN 2406-047X
ISSN 0352-9665 = Facta Universitatis. Series: Mathematics and informatics
COBISS.SR-ID 5881090