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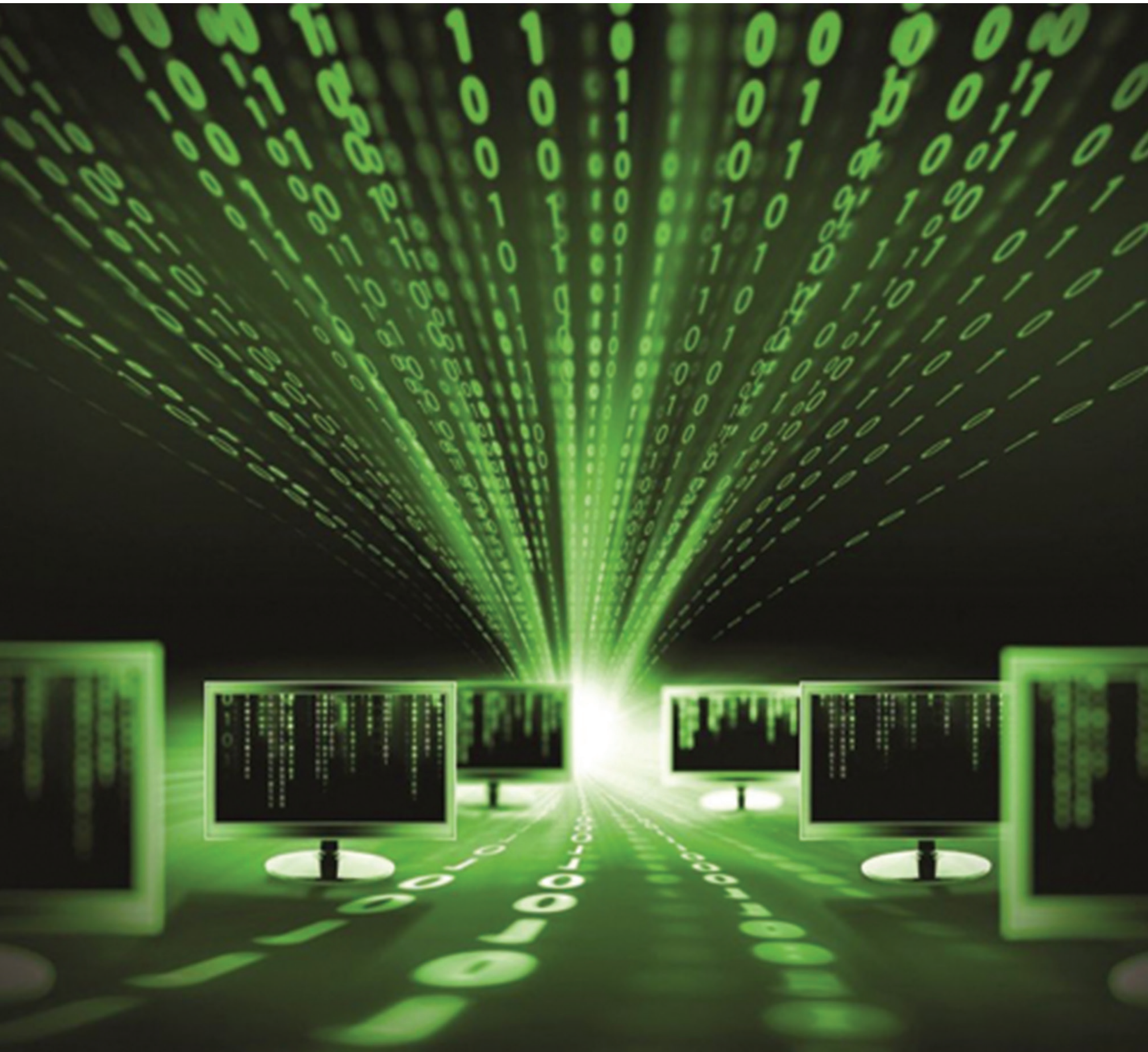

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[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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Original Scientific Paper

FIXED POINT RESULTS IN COMPLEX VALUED METRIC SPACES WITH AN APPLICATION

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Abstract. In this paper, we introduce fixed point theorem for a general contractive condition in complex valued metric spaces. Also, some important corollaries under this contractive condition are obtained. As an application, we find a unique solution for Urysohn integral equations and some illustrative examples are given to support our obtaining results. Our results extend and generalize the results of Azam et al. [2] and some other known results in the literature.
Key words: Single-valued mappings; complex valued metric spaces; common fixed point; nonlinear integral equations.

1. Introduction

A number of articles have been dedicated to the improvement and generalization of Banach contraction mapping principle. There exists various generalizations of the contraction principle, roughly obtained by weakening the contractive properties of the mapping and possibly, by simultaneously giving the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness, or by extending the structure of the space.

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Also, several fixed point theorems are obtained by combining the two ways previously described or by adding supplementary conditions (see, for example, [1, 4, 8, 10, 14, 17, 22]).

The complex valued metric spaces is more general than ordinary metric spaces. According to this concept, a number of articles related to fixed point theory and it's application are presented (see, for example, [3, 5, 6, 7, 9, 11, 12, 13, 15, 16, 18, 19, 20, 21, 23]).

In this paper, we prove some fixed point theorem in complex valued metric spaces under contractive condition for single-valued mappings. Moreover, we give a result of existence and uniqueness for solutions of a nonlinear system of integral equations. Finally, we will give some explained examples to strengthen our results.

2. Preliminaries

In this section, we recall some known notations and definitions that will be used in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:$z_1 z_2$ if and only if $\mathrm{Re}\,(z_1) \leq \mathrm{Re}\,(z_2)$ and $\mathrm{Im}\,(z_1) \leq \mathrm{Im}\,(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

$$\begin{array}{lcl} (C_1)\ \mathrm{Re}\,(z_1) & = & \mathrm{Re}\,(z_2) \text{ and } \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2), \\ (C_2)\ \mathrm{Re}\,(z_1) & < & \mathrm{Re}\,(z_2) \text{ and } \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2), \\ (C_3)\ \mathrm{Re}\,(z_1) & < & \mathrm{Re}\,(z_2) \text{ and } \mathrm{Im}\,(z_1) < \mathrm{Im}\,(z_2), \\ (C_4)\ \mathrm{Re}\,(z_1) & = & \mathrm{Re}\,(z_2) \text{ and } \mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2), \end{array}$$

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C_1), (C_2) and (C_3) is satisfied and we write $z_1 \prec z_2$ if only (C_3) is satisfied.

Definition 2.1. [2] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if the following conditions holds for all $x, y, z \in X$,

(CM_1) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(CM_2) $d(x, y) = d(y, x)$,

(CM_3) $d(x, y) \precsim d(x, z) + d(z, y)$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

For some examples of complex valued metric spaces (see [2, 5, 12, 18]).

Definition 2.2. [2] Let (X, d) be a complex valued metric space. Then

(i) A sequence $\{x_n\}$ in X is said to be converged to $x \in X$ if for every $0 \prec \varepsilon \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec \varepsilon \ \forall n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for every $0 \prec \varepsilon \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec \varepsilon$ for all $n > N$, $m \in N$, Then $\{x_n\}$ is called a Cauchy sequence in (X, d).

(iii) If every Cauchy sequence in X is convergent in X then (X, d) is called a complete complex valued metric space.

Lemma 2.1. [2] *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.2. [2] *Let (X, d) be a complex valued metric space. Then a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.*

3. Main result

We state and prove our first result.

Theorem 3.1. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$ such that*

$$d(Sx, Ty) \precsim \alpha M(x, y), \tag{3.1}$$

for all $x, y \in X$, where $0 < \alpha < 1$ and

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Ty)d(y, Sx)}{1 + d(x, y)}\right\}.$$

Then there exists a unique common fixed point of the pair mappings (S, T).

Proof. Let x_0 be arbitrary point in X and define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \; n = 0, 1, 2, .. \tag{3.2}$$

Then, by (3.1) and (3.2), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \alpha \max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})},\right. \\ &\qquad \left.\frac{d(x_{2n}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{1 + d(x_{2n}, x_{2n+1})}\right\} \\ &\precsim \alpha \max\left\{d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})},\right. \\ &\qquad \left.\frac{d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}\right\} \\ &\precsim \alpha \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

If $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, then

$$d(x_{2n+1}, x_{2n+2}) \precsim \alpha d(x_{2n+1}, x_{2n+2}),$$

This leads to, $\alpha \geq 1$, a contradiction. Therefore

$$d(x_{2n+1}, x_{2n+2}) \precsim \alpha d(x_{2n}, x_{2n+1}). \tag{3.3}$$

Similarly, we can obtain that

$$d(x_{2n+2}, x_{2n+3}) \precsim \alpha d(x_{2n+1}, x_{2n+2}). \tag{3.4}$$

From (3.3) and (3.4) for all $n = 0, 1, 2, ..$, we can write

$$d(x_{n+1}, x_{n+2}) \precsim \alpha d(x_n, x_{n+1}) \precsim ... \precsim \alpha^{n+1} d(x_\circ, x_1).$$

So for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\precsim \left(\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1}\right) d(x_\circ, x_1) \\ &\precsim \left(\frac{\alpha^n}{1-\alpha}\right) d(x_\circ, x_1). \end{aligned}$$

So,

$$|d(x_n, x_m)| \precsim \left(\frac{\alpha^n}{1-\alpha}\right) |d(x_\circ, x_1)| \to 0.$$

As $n \to \infty$, therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, then there exists $u \in X$ such that $x_n \to u$. If S and T are not continuous, it follows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$\begin{aligned} z &\precsim d(u, x_{2k+2}) + d(Su, x_{2k+2}) \\ &\precsim d(u, x_{2k+2}) + d(Su, Tx_{2k+1}) \\ &\precsim d(u, x_{2k+2}) + \alpha \max \left\{ d(u, x_{2k+1}), \frac{d(u, Su) d(x_{2k+1}, Tx_{2k+1})}{1 + d(u, x_{2k+1})}, \right. \\ &\qquad \left. \frac{d(u, Tx_{2k+1}) d(x_{2k+1}, Su)}{1 + d(u, x_{2k+1})} \right\} \\ &\precsim d(u, x_{2k+2}) + \alpha \max \left\{ d(u, x_{2k+1}), \right. \\ &\qquad \left. \frac{d(u, Su) d(x_{2k+1}, x_{2k+2})}{1 + d(u, x_{2k+1})}, \frac{d(u, x_{2k+2}) d(x_{2k+1}, Su)}{1 + d(u, x_{2k+1})} \right\} \\ &\precsim d(u, x_{2k+2}) + \alpha \max\{0, 0, z\} \\ &\precsim d(u, x_{2k+2}) + \alpha z. \end{aligned}$$

This yields,

$$|z| \leq |d(u, x_{2k+2})| + \alpha |z|.$$

That is $\alpha \geq 1$, a contradiction again and hence, $u = Su$. It follows similarly that $u = Tu$.
If S and T are continuous, i.e., the continuity of S, yields

$$u = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Sx_{2n+1} = S \lim_{n\to\infty} x_{2n+1} = Su.$$

Similarly, $u = Tu$. Hence the pair (S, T) has a common fixed point.

For the uniqueness, assume that $v \in X$ is a second common fixed point of S and T. Then

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\precsim \alpha \max\left\{d(u,v), \frac{d(u,Su)d(v,Tv)}{1+d(u,v)}, \frac{d(u,Tv)d(v,Su)}{1+d(u,v)}\right\} \\ &\precsim \alpha d(u,v). \end{aligned}$$

This implies that $u = v$, this completes the proof. $\square$

If we take $S = T$ in the above theorem we have we have the following immediate consequences.

Corollary 3.1. *(X, d) be a complete complex valued metric space and $S : X \to X$ satisfy*

$$d(Sx, Sy) \precsim \alpha M(x, y),$$

for all $x, y \in X$, where $0 < \alpha < 1$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Sy)}{1+d(x,y)}, \frac{d(x,Sy)d(y,Sx)}{1+d(x,y)}\right\}.$$

Then S has a unique fixed point on X.

Corollary 3.2. *Let (X, d) be a complete complex valued metric space and $S : X \to X$ satisfy*

$$d(S^n x, S^n y) \precsim \alpha M(x, y)$$

for all $x, y \in X$, where $0 < \alpha < 1$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,S^n x)d(y,S^n y)}{1+d(x,y)}, \frac{d(x,S^n y)d(y,S^n x)}{1+d(x,y)}\right\}.$$

Then S has a unique fixed point.

Proof. By Corollary 3.1, we obtain $v \in X$ such that

$$S^n v = v.$$

From the fact

$$\begin{aligned} d(Sv,v) &= d(SS^n v, S^n v) = d(S^n Sv, S^n v) \\ &\precsim \alpha \max\left\{d(Sv,v), \frac{d(Sv,S^n Sv)d(v,S^n v)}{1+d(Sv,v)}, \frac{d(Sv,S^n v)d(v,S^n Sv)}{1+d(Sv,v)}\right\} \\ &\precsim \alpha \max\left\{d(Sv,v), \frac{d(Sv,SS^n v)d(v,S^n v)}{1+d(Sv,v)}, \frac{d(Sv,S^n v)d(v,SS^n v)}{1+d(Sv,v)}\right\} \\ &= \alpha d(Sv,v). \end{aligned}$$

The result is follows. $\square$

4. An application to Urysohn integral type equations

In this section, we apply Theorem 3.1 to prove the existence of a unique solution to the following Urysohn integral type equations:

$$\begin{cases} x(t) = h(t) + \int_a^b K_1(t,s,x(s))ds \\ y(t) = h(t) + \int_a^b K_2(t,s,y(s))ds \end{cases} . \tag{4.1}$$

where,

(i) $x(t)$ and $y(t)$ are unknown variables for each $t \in [a,b]$, $a > 0$,

(ii) $h(t)$ is the deterministic free term defined for $t \in [a,b]$,

(iii) $K_1(t,s)$ and $k_2(t,s)$ are deterministic kernels defined for $t,s \in [a,b]$.

Let $X = (C[a,b], \mathbb{R}^n), a > 0$ and $d : X \times X \to \mathbb{R}^n$ defined by

$$d(x,y) = \sup_{t\in[a,b]} \|x(t) - y(t)\|_\infty \sqrt[3]{1+b^3} e^{i\cot^{-1} b},$$

for all $x, y \in X$, $i = \sqrt{-1} \in \mathbb{C}$.

It's obvious that $(C[a,b], \mathbb{R}^n, \|.\|_\infty)$ is a complete complex valued metric space.

Next, we consider a system (4.1) under the following conditions:

(H_1) $h(t) \in X$,

(H_2) $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions satisfying

$$|K_1(t,s,u(s)) - K_1(t,s,v(s))| \precsim \frac{1}{(b-a)e^{ab}} M(u,v),$$

where,

$$M(u,v) = \max\left\{ d(u,v), \frac{d(u,Su)d(v,Tv)}{1+d(u,v)}, \frac{d(u,Tv)d(v,Su)}{1+d(u,v)} \right\}.$$

Next, we state and prove the following theorem:

Theorem 4.1. *$(C[a,b], \mathbb{R}^n, \|.\|_\infty)$ be a complete complex valued metric space, then the system (4.1) under the conditions (H_1) and (H_2) has a unique common solution.*

Proof. For $x, y \in (C[a,b], \mathbb{R}^n)$ and $t \in [a,b]$, we define the continuous mappings $S, T : X \to X$ by

$$\begin{aligned} Sx(t) &= h(t) + \int_a^b K_1(t,s,x(s))ds, \\ Ty(t) &= h(t) + \int_a^b K_2(t,s,y(s))ds. \end{aligned}$$

By this, we have

$$|Sx(t) - Ty(t)| = \int_a^b |K_1(t,s,x(s)) - K_2(t,s,y(s))| \, ds$$

$$
\begin{aligned}
&\precsim \int_a^b \frac{1}{(b-a)e^{ab}} |M(x,y)|\, ds \\
&= \frac{1}{(b-a)e^{ab}} \int_a^b \frac{e^{-i\cot^{-1} b}}{\sqrt[3]{1+b^3}} |M(x,y)| \sqrt[3]{1+b^3} e^{i\cot^{-1} b} ds \\
&\precsim \frac{1}{(b-a)e^{ab}} \frac{e^{-i\cot^{-1} b}}{\sqrt[3]{1+b^3}} \|M(x,y)\|_\infty \int_a^b ds \\
&= \frac{1}{e^{ab}} \frac{e^{-i\cot^{-1} b}}{\sqrt[3]{1+b^3}} \|M(x,y)\|_\infty .
\end{aligned}
$$

This gives,

$$\sqrt[3]{1+b^3}\, |Sx(t) - Ty(t)|\, e^{-i\cot^{-1} b} \precsim \frac{1}{e^{ab}} \|M(x,y)\|_\infty ,$$

or, equivalently

$$\|Sx(t) - Ty(t)\|_\infty \precsim \frac{1}{e^{ab}} \|M(x,y)\|_\infty ,$$

or,

$$d(Sx, Ty) \precsim \alpha M(x,y).$$

So, the condition (3.1) of Theorem 3.1 is satisfied with $0 < \alpha = \frac{1}{e^{ab}} < 1$, Therefore the system (4.1) has a unique common solution on X. □

5. Examples

In this section we present some important examples to support our obtained results.

Example 5.1. Let $X = \mathbb{C}$ be a set of complex number. Define $d^{'} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by

$$d^{'}(z_1, z_2) = d(x_1, x_2) + id(y_1, y_2),$$

for all $z_1, z_2 \in \mathbb{C}$, where $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$. If (X, d) is a complex valued metric space, Then $(X, d^{'})$ is too.

Example 5.2. Let $X = \mathbb{C}$ be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + i(y_1 - y_2)^2},$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Example 5.3. Let $X = [0, \infty)$ define the distance $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = i\,|x - y|\,.$$

It's clearly (X,d) is a complete complex valued metric space. We define the two self-mappings S and T as

$$Sx = 2x^2 - 1, Tx = (2-x)^2.$$

Then the contractive condition (3.1) is satisfied, indeed for $x = \frac{1}{3}$ and $y = 3$, we can write by the simple calculations,

$$d(Sx, Ty) = \frac{16}{9}i.$$

and

$$M(x,y) = \max\left\{\frac{8}{3}i, \frac{-40}{3+8i}, \frac{-68}{9(3+8i)}\right\} = \frac{8}{3}i.$$

So,

$$\frac{16i}{9} \precsim \alpha \frac{8i}{3}.$$

Therefore, the conditions of Theorem 3.1 are verified with $\alpha = \frac{2}{3} < 1$ and $1 \in X$ is a unique common fixed point of S and T.

Example 5.4. Let $X = [0,\infty)$ and $d : X \times X \to \mathbb{C}$ be a mapping defined by

$$d(x,y) = |x-y| + i\,|x-y|\,.$$

Clearly (X,d) is a complete complex valued metric space. Define a self-mapping S by

$$Sx = \frac{2}{\pi}\sin^{-1} x.$$

To verify the contractive condition of Corollary 3.1, we take $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$, one can write by the simple calculations,

$$d(Sx, Sy) \simeq 0.1667(1+i).$$

and

$$M(x,y) \simeq \max\{0.3660(1+i), 0.0483i, 0.1301i\} \simeq 0.3660(1+i).$$

So,

$$0.1667(1+i) \precsim \alpha\ 0.3660(1+i).$$

Therefore, all conditions of corollary 3.1 are satisfied with $\alpha \simeq 0.4555 < 1$ and $1 \in X$ is a unique fixed point of S.

Example 5.5. Let $X = C([0,2],\mathbb{R})$, $b > 0$ and for every $x, y \in X$ let

$$\begin{aligned} N_{xy} &= \max_{t\in[0,2]} |x(t) - y(t)|\,, \\ d(x,y) &= N_{xy}\sqrt[3]{1+b^3}e^{i\cot^{-1} b}. \end{aligned}$$

Define $S : X \to X$ by

$$Sx(t) = 1 + 3\int_0^t u^2 x(u)du,\ t \in [0,2].$$

For every $x, y \in X$, we have

$$\begin{aligned} d(Sx, Sy) &= N_{SxSy}\sqrt[3]{1+b^3}e^{i\cot^{-1}b} = \max_{t\in[0,2]} |Sx(t) - Sy(t)| \sqrt[3]{1+b^3}e^{i\cot^{-1}b} \\ &\precsim 3\int_0^2 \max_{t\in[0,2]} |x(u) - y(u)|\, u^2 \sqrt[3]{1+b^3}e^{i\cot^{-1}b}du \\ &\precsim 8d(x,y). \end{aligned}$$

Similarly,

$$d(S^n x, S^n y) \precsim \frac{8^n}{n!}d(x,y) \precsim \frac{8^n}{n!}M(x,y),$$

where,

$$\frac{8^n}{n!} \simeq \begin{cases} 295.894 & \text{If } n = 10 \\ 26.906 & \text{If } n = 15 \\ 1.185 & \text{If } n = 19 \\ 0.474 & \text{If } n = 20 \end{cases}$$

Thus for $\alpha \simeq 0.474 < 1$, $n = 20$, all conditions of Corollary 3.2 are satisfied and so S has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 1 + 3\int_0^t u^2 x(u)du,\ t \in [0,2],$$

or the differential equation (initial value problem):

$$x^{'}(t) - 3x^2 t = 0,\ t \in [0,2],\ t(0) = 1.$$

Example 5.6. Let $X = C([a,b],\mathbb{R})$ and the following nonlinear integral equation as the form:

$$(5.1)\qquad \begin{cases} x(t) = e^{4it} + \int_a^b \left(\frac{e^{-\frac{1}{4}}}{4\left(t+\frac{is}{1+is}+x(s)\right)} \right) ds \\ y(t) = e^{4it} + \int_a^b \left(\frac{e^{-\frac{1}{4}}}{4\left(t+\frac{is}{1+is}+y(s)\right)} \right) ds \end{cases}.$$

System (5.1) is a particular case of system (4.1), where $h(t) = e^{4it}$ and

$$K_j(t, s, u_j(s)) = \left(\frac{e^{-\frac{1}{4}}}{4\left(t + \frac{is}{1+is} + u_j(s)\right)} \right),\ j = 1, 2.$$

It's obvious that (H_1) is satisfied, for (H_2), we get

$$\begin{aligned} |K_1(t,s,x(s)) - K_2(t,s,y(s))| &= \frac{1}{4}e^{-\frac{1}{4}} \left| \frac{x(s)-y(s)}{\left(t+\frac{is}{1+is}+x(s)\right)\left(t+\frac{is}{1+is}+y(s)\right)} \right| \\ &\precsim \frac{1}{4}e^{-\frac{1}{4}} \left|x(s)-y(s)\right|. \end{aligned}$$

Therefore, (H_2) is hold with $\alpha = \frac{1}{4}e^{-\frac{1}{4}} < 1$ and $M(x,y) = |x(s) - y(s)|$. By Theorem 4.1, the system (5.1) has a unique solution.

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NUMERICAL SOLUTIONS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS VIA LAPLACE TRANSFORM

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Abstract. In this study, solutions of time-space fractional partial differential equations (FPDEs) are obtained by utilizing the Laplace transform iterative method. The utility of the technique is shown by getting numerical solutions of various FPDEs.
Key words: differential equations, Laplace transform, Fractional derivatives and integrals, Functional-differential equations

1. Introduction

Mathematical models by fractional differential equations for various physical phenomena play important roles in all applied sciences such as mathematics physics, biology, dynamical systems, control systems, engineering and so on [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Also, there are various studies on fractional diffusion equations. Exact analytical solutions of heat equations are obtained by using operational method [13]. The existence, uniqueness and regularity of solution of impulsive sub-diffusion equation are established by means of eigenfunction expansion [14]. The anomalous diffusion models with non-singular power-law kernel have been investigated and constructed [15]. Moreover, nonlinear fractional partial differential equations (FPDEs) are employed in modeling various nonlinear phenomena, mainly dealing with memory, and they present a crucial role in technology and science. Taking

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physical knowledge and physical properties of the nonlinear problem into account the exact solution of nonlinear FPDEs can be obtained. This knowledge gives us the idea about how numerical solutions of the nonlinear FPDEs can be constructed by the combination of Daftardar-Jafari method (DJM) and Laplace transform. In this study, Laplace Transform iterative method (LTIM) is extended to obtain solutions for time-space FPDEs. The LTIM method is employed to solve a variety of linear and nonlinear FPDEs. LTIM generally generates an accurate solution of FPDEs, which can be represented in terms of the fractional trigonometric functions or Mittag-Leffler functions. Moreover, it has been shown that semi-analytical methods with Laplace transform need fewer CPU time to compute the solutions of nonlinear fractional models, which are utilized in engineering and applied science. LTIM is a robust method to obtain solutions for distinct types of nonlinear and linear FPDEs. LTIM can decrease the time of calculation as well as error margin of the approximate solution.

2. Preliminaries

In this section, preliminaries, notations and features of the fractional calculus are given [1, 2]. Riemann-Liouville time-fractional integral of a real valued function $u\left(x,t\right)$ is defined as

$$I_t^{\alpha} u\left(x,t\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_0^t \left(t-s\right)^{\alpha-1} u\left(x,s\right) ds \tag{2.1}$$

where $\alpha > 0$ denotes the order of the integral.

α^{th} order the Liouville-Caputo time-fractional derivative operator of $u(x,t)$ is defined as

$$\begin{aligned} \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} &= I_t^{m-\alpha}\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \left(t-y\right)^{m-\alpha-1} \frac{\partial^m u(x,y)}{\partial y^m} dy, & m-1<\alpha<m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \end{cases} \end{aligned} \tag{2.2}$$

Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ Re\left(\alpha\right) > 0,\ z,\beta \in \mathbb{C} \tag{2.3}$$

where α and β are parameters.

The following set of functions has Laplace transformation

$$\left\{ f\left(t\right) | \, \exists M, \tau_1, \tau_2 > 0, \left|f\left(t\right)\right| < M e^{\frac{|t|}{\tau_j}}, if t \in \left(-1\right)^j \times [0,\infty) \right\}$$

and it is defined as

$$L\left[f\left(t\right)\right]=F\left(p\right)=\int_{0}^{\infty}e^{-pt}f\left(t\right)dt \tag{2.4}$$

which has the following property

$$L\left[t^{\alpha}\right]=\int_{0}^{\infty}e^{-pt}t^{\alpha}dt=\Gamma\left(\alpha+1\right)\left(\frac{1}{p}\right)^{\alpha+1},\ Re\left(\alpha\right)>0 \tag{2.5}$$

inverse Laplace inverse transform of $\left(\frac{1}{p}\right)^{n\alpha+1}$ is defined as

$$L^{-1}\left[\left(\frac{1}{p}\right)^{n\alpha+1}\right]=\frac{t^{n\alpha}}{\Gamma\left(n\alpha+1\right)},\ Re\left(\alpha\right)>0 \tag{2.6}$$

where $n>0$ [2].

For α^{th} order the Liouville-Caputo time-fractional derivative of $f\left(x,t\right)$, the Laplace transformation has the following form:

(2.7)
$$L\left[\frac{\partial^{\alpha}f\left(x,t\right)}{\partial t^{\alpha}}\right]=p^{\alpha}L\left[f\left(x,t\right)\right]-\sum_{k=0}^{n-1}\left[p^{\alpha-k-1}\frac{\partial^{k}f\left(x,0\right)}{\partial t^{k}}\right],\ n-1<\alpha\leqslant n,\ n\in\mathbb{N}.$$

3. Methodology

In this section, we take the general time and space FPDE

$$\frac{\partial^{\zeta}f}{\partial t^{\zeta}}=F\left(x,f,\frac{\partial^{\eta}f}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}f}{\partial x^{l\eta}}\right),\ j-1<\zeta\leqslant j,\ i-1<\eta\leqslant i,\ l,j,i\in\mathbb{N} \tag{3.1}$$

along with the initial conditions

$$\frac{\partial^{m}f(x,0)}{\partial t^{m}}=h_{m}(x),\ k=0,1,2,\ldots,j-1, \tag{3.2}$$

into account where $F\left(x,f,\frac{\partial^{\eta}f}{\partial x^{\eta}},\ldots,\frac{\partial^{l\eta}f}{\partial x^{l\eta}}\right)$ could be linear or nonlinear and the function $f=f(x,t)$ is unknown.

Applying the Laplace transform to both sides of Eq. (3.1) and rearranging leads to

$$\begin{aligned} L\left[f\left(x,t\right)\right] = & \sum_{m=0}^{j-1}\left[\left(\frac{1}{p}\right)^{m+1}\frac{\partial^m f\left(x,0\right)}{\partial t^m}\right] \\ & + \left(\frac{1}{p}\right)^{\zeta+1} L\left[F\left(x,f,\frac{\partial^\eta f}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right)\right]. \end{aligned} \tag{3.3}$$

Employing the inverse Laplace transform of Eq. (3.3), we obtain

$$\begin{aligned} f\left(x,t\right) = & L^{-1}\left[\sum_{m=0}^{j-1}\left[\left(\frac{1}{p}\right)^{m+1}\frac{\partial^m f\left(x,0\right)}{\partial t^m}\right]\right] \\ & + L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1} L\left[F\left(x,f,\frac{\partial^\eta f}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right)\right]\right]. \end{aligned} \tag{3.4}$$

Equation (3.4) can be rearranged as

$$f\left(x,t\right) = g\left(x,t\right) + G\left(x,f,\frac{\partial^\eta f}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right), \tag{3.5}$$

where

$$\begin{aligned} & g\left(x,t\right) = L^{-1}\left[\sum_{m=0}^{j-1}\left[\left(\frac{1}{p}\right)^{m+1}\frac{\partial^m f\left(x,0\right)}{\partial t^m}\right]\right] \\ & G\left(x,f,\frac{\partial^\eta f}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right) = L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1} L\left[F\left(x,f,\frac{\partial^\eta f}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} f}{\partial x^{l\eta}}\right)\right]\right] \end{aligned} \tag{3.6}$$

Here G is a nonlinear / linear operator and g is known function. The solution of Eq. (3.5) can be obtained by the DJM introduced by Daftardar-Gejji and Jafari [16]. The solution is represented as an infinite series:

$$f = \sum_{n=0}^{\infty} f_n, \tag{3.7}$$

where the terms f_n are recursively computed. Decomposing the operator G leads to

$$
\begin{aligned}
&G\left(x,\sum_{n=0}^{\infty}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{\infty}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{\infty}f_n\right)}{\partial x^{l\eta}}\right)\\
&=G\left(x,f_0,\frac{\partial^{\eta}f_0}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}f_0}{\partial x^{l\eta}}\right)\\
&+\sum_{c=1}^{\infty}\left(G\left(x,\sum_{n=0}^{c}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{c}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{c}f_n\right)}{\partial x^{l\eta}}\right)\right)\\
&-\sum_{c=1}^{\infty}\left(G\left(x,\sum_{n=0}^{c-1}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{c-1}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{c-1}f_n\right)}{\partial x^{l\eta}}\right)\right)
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,\sum_{n=0}^{\infty}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{\infty}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{\infty}f_n\right)}{\partial x^{l\eta}}\right)\right]\right]\\
&=L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,f_0,\frac{\partial^{\eta}f_0}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}f_0}{\partial x^{l\eta}}\right)\right]\right]\\
&+\sum_{c=1}^{\infty}L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,\sum_{n=0}^{c}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{c}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{c}f_n\right)}{\partial x^{l\eta}}\right)\right]\right]\\
&-\sum_{c=1}^{\infty}L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,\sum_{n=0}^{c-1}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{c-1}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{c-1}f_n\right)}{\partial x^{l\eta}}\right)\right]\right].
\end{aligned}
\tag{3.9}
$$

Using Eqs. (3.7), (3.9) in Eq. (3.5), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}f_n=L^{-1}\left[\sum_{m=0}^{j-1}\left[\left(\frac{1}{p}\right)^{m+1}\frac{\partial^{m}f\left(x,0\right)}{\partial t^{m}}\right]\right]\\
&+L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,f_0,\frac{\partial^{\eta}f_0}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}f_0}{\partial x^{l\eta}}\right)\right]\right]\\
&+\sum_{c=1}^{\infty}L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,\sum_{n=0}^{c}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{c}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{c}f_n\right)}{\partial x^{l\eta}}\right)\right]\right]\\
&-\sum_{c=1}^{\infty}L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}L\left[F\left(x,\sum_{n=0}^{c-1}f_n,\frac{\partial^{\eta}\left(\sum_{n=0}^{c-1}f_n\right)}{\partial x^{\eta}},\dots,\frac{\partial^{l\eta}\left(\sum_{n=0}^{c-1}f_n\right)}{\partial x^{l\eta}}\right)\right]\right].
\end{aligned}
$$

The recurrence relation is defined by as follows:

(3.10)

$$
\begin{aligned}
f_0 &= L^{-1}\left[\sum_{m=0}^{j-1}\left[\left(\frac{1}{p}\right)^{m+1}\frac{\partial^m f(x,0)}{\partial t^m}\right]\right],\\
f_1 &= L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1} L\left[F\left(x, f_0, \frac{\partial^\eta f_0}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} f_0}{\partial x^{l\eta}}\right)\right]\right],\\
f_{r+1} &= L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1} L\left[F\left(x, \sum_{n=0}^{c} f_n, \frac{\partial^\eta \left(\sum_{n=0}^{c} f_n\right)}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} \left(\sum_{n=0}^{c} f_n\right)}{\partial x^{l\eta}}\right)\right]\right]\\
&\quad - L^{-1}\left[\left(\frac{q}{p}\right)^{\zeta+1} L\left[F\left(x, \sum_{n=0}^{c-1} f_n, \frac{\partial^\eta \left(\sum_{n=0}^{c-1} f_n\right)}{\partial x^\eta},\ldots,\frac{\partial^{l\eta} \left(\sum_{n=0}^{c-1} f_n\right)}{\partial x^{l\eta}}\right)\right]\right].
\end{aligned}
$$

The r- term truncated solution of Eqs. (3.1),(3.2) is constructed as $f \approx f_0 + f_1 + \ldots + f_{r-1}$. We lead the reader to [17] for the convergence of DJM.

4. Illustrative Example

Let us consider time and space fractional equation below

$$
\frac{\partial^\zeta f}{\partial t^\zeta} = \left(\frac{\partial^\eta f}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta f}{\partial x^\eta}\right),\ t>0,\ \zeta,\eta\in(0,1], \tag{4.1}
$$

along with the initial condition

$$
u(x,0) = 3 + \frac{5}{2}E_\eta(x^\eta). \tag{4.2}
$$

Let us apply the Laplace transform on both sides of (4.1).

$$
L\left[\frac{\partial^\zeta f}{\partial t^\zeta}\right] = L\left[\left(\frac{\partial^\eta f}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta f}{\partial x^\eta}\right)\right].
$$

By means of the property (2.7), we obtain

$$
L[f(x,t)] = \left(\frac{1}{p}\right) f(x,0) + \left(\frac{1}{p}\right)^{\zeta+1}\left(L\left[\left(\frac{\partial^\eta f}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta f}{\partial x^\eta}\right)\right]\right). \tag{4.3}
$$

Applying the inverse Laplace transform to both sides of Eq. (4.3)

$$f(x,t) = L^{-1}\left[\left(\frac{1}{p}\right) f(x,0)\right] + L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}\left(L\left[\left(\frac{\partial^\eta f}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta f}{\partial x^\eta}\right)\right]\right)\right]$$

is obtained. Using the recurrence relation (3.10)

$$\begin{aligned}
f_0 &= L^{-1}\left[\left(\frac{1}{p}\right) f(x,0)\right] = 3 + \frac{5}{2}E_\eta\left(x^\eta\right),\\
f_1 &= L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}\left(L\left[\left(\frac{\partial^\eta f_0}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta f_0}{\partial x^\eta}\right)\right]\right)\right] = -\frac{15t^\zeta E_\eta\left(x^\eta\right)}{2\Gamma\left(\zeta+1\right)},\\
f_2 &= L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}\left(L\left[\left(\frac{\partial^\eta\left(f_0+f_1\right)}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta\left(f_0+f_1\right)}{\partial x^\eta}\right)\right]\right)\right]\\
&- L^{-1}\left[\left(\frac{1}{p}\right)^{\zeta+1}\left(L\left[\left(\frac{\partial^\eta f_0}{\partial x^\eta}\right)^2 - f\left(\frac{\partial^\eta f_0}{\partial x^\eta}\right)\right]\right)\right] = \frac{45t^{2\zeta} E_\eta\left(x^\eta\right)}{2\Gamma\left(2\zeta+1\right)},\\
f_3 &= -\frac{135t^{3\zeta} E_\eta\left(x^\eta\right)}{2\Gamma\left(3\zeta+1\right)},\\
f_4 &= -\frac{405t^{4\zeta} E_\eta\left(x^\eta\right)}{2\Gamma\left(4\zeta+1\right)}.
\end{aligned}$$

As a result, the series solution of the problem (4.1)-(4.2) is obtained by

$$f(x,t) = f_0 + f_1 + f_2 + f_3 + \ldots = 3 + \left[\frac{5}{2}E_{\zeta}\left(-3t^\zeta\right)\right] E_\eta(x^\eta),$$

which gives the same solution as in [18].

5. Conclusion

LTIM is developed by taking the combination of DJM [16] and Laplace transform. This new approach is convenient for acquiring numerical solutions of time and space FPDEs. Its appicability is illustrated by an example in this study. As a result, the combination of DJM with Laplace transform provides a better and more effective approach than combination of Laplace transformation and homotopy, Sumudu or Adomian polynomials.

The results of this paper can be rewritten easily and trivially for any of the many rather inconsequential parameter and/or variable changes in the integral in Eq. (2.4) which defines the classical Laplace transform.

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PARANORMED SPACES OF ABSOLUTE LUCAS SUMMABLE SERIES AND MATRIX OPERATORS

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Abstract. The aim of this paper is to introduce the absolute series space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ as the set of all series summable by the absolute Lucas method, and to give its topological and algebraic structure such as $FK-$space, duals and Schauder basis. Also, certain matrix operators on this space are characterized.
Keywords: Absolute summability, Lucas numbers, matrix transformations, Maddox space, sequence spaces, bounded operators

1. Introduction

Let ω be the set of all sequences of complex numbers. Any vector subspace of ω is called a *sequence space*. We write c, l_∞, Ψ for the spaces of all convergent, bounded and finite sequences, and also write cs, bs and l_p $(p \geq 1)$ for the spaces of all convergent, bounded, p-absolutely convergent series, respectively.

Let X and Y be two sequence spaces and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. If a series

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv}x_v,$$

converges for all $n \in \mathbb{N} = \{0, 1, 2, ...\}$, then, by $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$. Also, if $Ax = (A_n(x)) \in Y$ for every $x \in X$,

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we say that A defines a matrix transformation from X to Y, and by (X,Y) denote the class of all infinite matrices from X into Y. The set

$$S(X,Y)=\{a=(a_v)\in\omega : ax=(a_kx_k)\in Y \text{ for all } x\in X\}$$

is called the multiplier space of X and Y. According to this notation, the α-, β- and γ- duals of the space X are identified as

$$X^{\alpha}=S(X,l),\ \ X^{\beta}=S(X,cs),\ \ X^{\gamma}=S(X,bs).$$

The concept of the domain of an infinite matrix A in the sequence space X is given by

$$X_A=\{x=(x_n)\in\omega : A(x)\in X\}. \tag{1.1}$$

Using the concept of the matrix domain, several sequence spaces have been introduced and their algebraic, topological structure and matrix transformations have been studied in literature (see [1, 2, 4, 10, 13, 14, 15, 16]).

If $a_{nn}\neq 0$ for all n and $a_{nv}=0$ for $n<v$, then A is called a triangle matrix. The matrix domains of triangles have an important role in literature. For example, if A is a triangle and X is an FK-space, a complete locally convex linear metric space with continuous coordinates $p_n : X\to\mathbb{C}$ defined by $p_n(x)=x_n$ for all $n\in\mathbb{N}$, then the sequence space X_A is also an FK-space [11]. If there exists unique sequence of coefficients (x_k) such that, for each $x\in X$,

$$\lim_{m\to\infty}\sum_{k=0}^{m}x_kb_k=x$$

then, the sequence (b_k) is called the Schauder basis (or briefly basis) for a sequence space X. For instance, the sequence $(e^{(j)})$ is the Schauder basis of the space l_p, where $e^{(j)}$ is the sequence whose only non-zero term is 1 in jth place for each $j\in\mathbb{N}$, [23].

The following result is useful to find a Schauder basis for the matrix domain of a special triangular matrix in a linear metric space.

Lemma 1.1. *([11]). If (b_k) is a Schauder basis of the metric space (X,d), then $(S(b_k))$ is a basis of X_T with respect to the metric d_T given by $d_T(z_1,z_2)=d(Tz_1, Tz_2)$ for all $z_1,z_2\in X_T$, where T is a triangular matrix and S is its inverse.*

The well known space $l(\mu)$ of Maddox is defined by

$$l(\mu)=\left\{x=(x_n) : \sum_{n=1}^{\infty}|x_n|^{\mu_n}<\infty\right\},$$

which is an FK-space with AK with respect to its natural paranorm

$$g(x)=\left(\sum_{n=0}^{\infty}|x_n|^{\mu_n}\right)^{1/M},$$

where $M = \max\{1, \sup_n \mu_n\}$; also it is even a BK-space if $\mu_n \geq 1$ for all n with respect to the norm

$$\|x\| = \inf\left\{\delta > 0 : \sum_{n=0}^{\infty} |x_n/\delta|^{\mu_n} \leq 1\right\},$$

([18, 19, 20]).

Throughout the paper, we suppose that $0 < \inf \mu_n \leq H < \infty$ and μ_n^* is conjugate of μ_n, that is, $1/\mu_n + 1/\mu_n^* = 1$ for $\mu_n > 1$, and $1/\mu_n^* = 0$ for $\mu_n = 1$, for all $n \in \mathbb{N}$.

Let $\sum x_v$ be an infinite series with nth partial sum s_n, (ϕ_n) be a sequence of positive numbers and (μ_n) be a bounded sequence of positive numbers. Then, the series $\sum x_v$ is said to be summable $|A, \phi_n|(\mu)$, if

$$\sum_{n=0}^{\infty} \phi_n^{\mu_n - 1} |\Delta A_n(s)|^{\mu_n} < \infty, \tag{1.2}$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, $A_{-1}(s) = 0$, [6].

Note that, $|A, \phi_n|(\mu)$ includes many well known methods; if A is the matrix of weighted mean $\left(\bar{N}, p_n\right)$ (resp. $\phi_n = P_n/p_n$) with $\mu_n = k$ for all n, then it reduces to the summability $\left|\bar{N}, p_n, \phi_n\right|_k$ [29] (the summability $\left|\bar{N}, p_n\right|_k$ [3]). Also, if we take A as the matrix of Cesàro mean of order $\alpha > -1$ and $\phi_n = n$ with $\mu_n = k$ for all n, then we get the summability $|C, \alpha|_k$ in Flett's notation [5].

In addition to the aforementioned spaces, some absolute series spaces have also been studied in the literature (see [6, 7, 8, 9, 11, 25, 27]).

One of the main purposes of this paper is to define a new series space $\left|\mathcal{L}^{\phi}(r, s)\right|(\mu)$ as the set of all series summable by the absolute Lucas matrix method and investigate its topological and algebraic structures. Also, by means of a given basic lemma, we characterize certain matrix operators on this space.

2. Absolute Lucas Series Space $\left|\mathcal{L}^{\phi}(r, s)\right|(\mu)$

In this section, we will first remind you of some properties of Lucas numbers. The Lucas sequence (L_n) is one of the most interesting number sequence in mathematics and it is named after the mathematician François Edouard Anatole Lucas (1842-1891). The nth Lucas number L_n is given by the Fibonacci recurrence relation with different initial condition such that

$$L_0 = 2,\ L_1 = 1 \ \text{ and } L_n = L_{n-1} + L_{n-2} \ \text{ for } \ n \geq 2,$$

which also has some interesting relations as follows

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3, \ \sum_{k=1}^{n} L_{2k-1} = L_{2n} - 2$$

$$\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1,\ \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2$$
$$L_{n-1}^2 + L_n L_{n-1} - L_n^2 = 5(-1)^{n+1}, n \geq 1$$
$$L_{n-1} L_{n+1} - L_n^2 = 5(-1)^{n+1}, n \geq 1.$$

We refer reader to [17] for other properties of these numbers. In addition to all these features, just like the Fibonacci numbers, the rate of successive Lucas numbers converges to the golden ratio which is one of the most interesting irrational having an important role in number theory, algorithms, network theory, etc.

Recently, using Lucas numbers, the Lucas matrix $\hat{E}(r,s) = (\hat{e}_{nk}(r,s))$ has been defined by

$$\hat{e}_{nk}(r,s) = \begin{cases} s\frac{L_n}{L_{n-1}}, & k = n-1 \\ r\frac{L_{n-1}}{L_n}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

where L_n be the nth Lucas number for every $n \in \mathbb{N}$ and $r, s \in \mathbb{R} \setminus \{0\}$ [15].

We are now ready to establish and study the series space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$. Put the Lucas matrix instead of A in (1.2), then $|A, \phi_n|(\mu)$ summability is reduced to the absolute Lucas summability, i.e.,

$$\sum_{n=0}^{\infty} \phi_n^{\mu_n - 1} |\Delta \hat{E}_n(r,s)|^{\mu_n} < \infty. \tag{2.1}$$

So, we introduce the space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ by the set of all series satisfying the condition (2.1). Also, since (s_n) is the sequence of partial sums of the series $\sum x_k$, it can be written that

$$\begin{aligned}
\hat{E}_n(r,s) = \sum_{v=1}^{n} \hat{e}_{nv}(r,s) s_v &= \sum_{k=1}^{n} x_k \sum_{v=k}^{n} \hat{e}_{nv}(r,s) \\
&= x_n \hat{e}_{nn}(r,s) + \sum_{k=1}^{n-1} (\hat{e}_{nn}(r,s) + \hat{e}_{n,n-1}(r,s)) x_k \\
&= x_n r \frac{L_{n-1}}{L_n} + \sum_{k=1}^{n-1} \left(s\frac{L_n}{L_{n-1}} + r\frac{L_{n-1}}{L_n} \right) x_k \\
&= \sum_{k=1}^{n} l_{nk} x_k
\end{aligned}$$

where $\mathcal{L}(r,s) = (l_{nk})$ is the matrix given by

$$l_{nk} = \begin{cases} r\frac{L_{n-1}}{L_n}, & k = n \\ s\frac{L_n}{L_{n-1}} + r\frac{L_{n-1}}{L_n}, & 1 \leq k \leq n-1 \\ 0, & k > n. \end{cases} \tag{2.2}$$

Hence we get

$$\begin{aligned}\Delta\hat{E}_n(r,s) &= r\frac{L_{n-1}}{L_n}x_n + \left(s\frac{L_n}{L_{n-1}} + r\frac{5(-1)^{n+1}}{L_nL_{n-1}}\right)x_{n-1} \\ &\quad + \sum_{k=1}^{n-2}\frac{5(-1)^n}{L_{n-1}}\left(\frac{s}{L_{n-2}} - \frac{r}{L_n}\right)x_k \\ &= \sum_{k=1}^{n}\xi_{nk}x_k,\end{aligned}$$

where

$$\xi_{nk} = \begin{cases} r\frac{L_{n-1}}{L_n}, & k=n \\ s\frac{L_n}{L_{n-1}} + r\frac{5(-1)^{n+1}}{L_nL_{n-1}}, & k=n-1 \\ \frac{5(-1)^n}{L_{n-1}}\left(\frac{s}{L_{n-2}} - \frac{r}{L_n}\right), & 1\le k<n-2 \\ 0, & k>n. \end{cases}$$

This means that a series $\sum x_k$ is summable by the absolute Lucas method if a sequence $(x_k) \in \left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$, *i.e.*,

$$\left|\mathcal{L}^{\phi}(r,s)\right|(\mu) = \left\{x\in\omega : \left(\phi_n^{1/\mu_n^*}\sum_{k=0}^{n}\xi_{nk}x_k\right)\in l(\mu)\right\}.$$

Note that there is a close relation between this space and the Maddox's space. In fact, according to the concept of domain, it can be redefined by

$$\left|\mathcal{L}^{\phi}(r,s)\right|(\mu) = (l(\mu))_{E^{(\mu)}\circ\mathcal{L}(r,s)}$$

where $E^{(\mu)} = \left(e_{nk}^{(\mu)}\right)$ is given by

$$e_{nk}^{(\mu)} = \begin{cases} \phi_n^{1/\mu^*}, & k=n \\ -\phi_n^{1/\mu^*}, & k=n-1 \\ 0, & k\neq n, n-1. \end{cases} \tag{2.3}$$

Also, note that

$$(E^{(\mu)}\circ\mathcal{L}(r,s))_n(x) = \phi_n^{1/\mu^*}(\mathcal{L}_n(r,s)(x) - \mathcal{L}_{n-1}(r,s)(x)). \tag{2.4}$$

On the other hand, since every triangle matrix has a unique inverse which is also a triangle [30], the matrices $\mathcal{L}(r,s)$ and $E^{(\mu)}$ have unique inverses $\tilde{\mathcal{L}}(r,s) = (\tilde{l}_{nk})$ and $\tilde{E}^{(\mu)} = (\tilde{e}_{nk})$ which we have been computed as

$$\tilde{l}_{nk} = \begin{cases} \frac{1}{r}\frac{L_n}{L_{n-1}}, & k=n \\ \frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_kL_{k-1}}\left(\frac{s}{r}L_n^2 + L_{n-1}^2\right), & 1\le k\le n-1 \\ 0, & k>n, \end{cases} \tag{2.5}$$

$$\tilde{e}_{nk}^{(\mu)} = \begin{cases} \theta_k^{-1/\mu_k^*}, & 1 \le k \le n \\ 0, & k > n, \end{cases} \tag{2.6}$$

respectively.

For the proofs of theorems we require some well known lemmas.

Lemma 2.1. *([12]) Let $\mu = (\mu_v)$ and $\lambda = (\lambda_v)$ be any two bounded sequences of strictly positive numbers.*

(i) If $\mu_v > 1$ for all v, then, $A \in (l(\mu), l)$ if and only if there exists an integer $M > 1$ such that

$$\sup\left\{ \sum_{v=0}^{\infty} \left| \sum_{n\in K} a_{nv} M^{-1} \right|^{\mu_v^*} : K \subset \mathbb{N} \text{ finite} \right\} < \infty. \tag{2.7}$$

(ii) If $\mu_v \le 1$ and $\lambda_v \ge 1$ for all $v \in \mathbb{N}$, then $A \in (l(\mu), l(\lambda))$ if and only if there exists some M such that

$$\sup_v \sum_{n=0}^{\infty} \left| a_{nv} M^{-1/\mu_v} \right|^{\lambda_n} < \infty.$$

(iii) If $\mu_v \le 1$, then, $A \in (l(\mu), c)$ if and only if

$$(a) \lim_n a_{nv} \text{ exists for each } v, \quad (b) \sup_{n,v} |a_{nv}|^{\mu_v} < \infty,$$

and $A \in (l(\mu), l_\infty)$ if (b) holds.

(iv) If $\mu_v > 1$ for all v, then, $A \in (l(\mu), c)$ if and only if

$(a) \lim_n a_{nv}$ exists for each v, (b) there is a number $M > 1$ such that

$$\sup_n \sum_{v=0}^{\infty} \left| a_{nv} M^{-1} \right|^{\mu_v^*} < \infty,$$

and $A \in (l(\mu), l_\infty)$ iff (b) holds.

(v) $A \in (l(\mu), c_0)$ iff $A \in (l(\mu), l_\infty)$ and $\lim_{n\to\infty} a_{nv} = 0$ for every $v \in \mathbb{N}$.

Note that the condition (2.7) has some difficulties in applications. The following lemma presents a more useful and equivalent condition to (2.7).

Lemma 2.2. *([26]) Let (μ_v) be a bounded sequence of positive numbers and $A = (a_{nv})$ be an infinite matrix with complex numbers. If $U_\mu[A] < \infty$ or $L_\mu[A] < \infty$, then*

$$(2C)^{-2}U_\mu[A] \leq L_\mu[A] \leq U_\mu[A],$$

where $C = \max\{1, 2^{H-1}\}$, $H = sup_v \mu_v$,

$$U_\mu[A] = \sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{\mu_v}$$

and

$$L_\mu[A] = \sup\left\{\sum_{v=0}^{\infty}\left|\sum_{n\in K}a_{nv}\right|^{\mu_v} : K \subset \mathbb{N} \text{ finite}\right\}.$$

Lemma 2.3. *[22] Let T be a triangle matrix, and let X, Y be arbitrary subsets of ω. Then, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

Lemma 2.4. *[21, Theorem 3.9] Let X be an FK space with AK, T be a triangle matrix, S be its inverse and Y be an arbitrary subset of ω. Then, we have $A \in (X_T, Y)$ if and only if $\tilde{A} = (\tilde{a}_{nv}) \in (X, Y)$ and $V^{(n)} = (v_{mv}^{(n)}) \in (X, c)$ for all n, where*

$$\tilde{a}_{nv} = \sum_{j=v}^{\infty} a_{nj}s_{jv}; \quad n, v = 0, 1, ...,$$

and

$$v_{mv}^{(n)} = \begin{cases} \sum\limits_{j=v}^{m} a_{nj}s_{jv}, & 0 \leq v \leq m \\ 0, & v > m. \end{cases}$$

We begin with theorems by giving toplogical and algebraic structures of $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$.

Theorem 2.1. *Assume that (ϕ_n) is a sequence of positive numbers and (μ_n) is a bounded sequence of positive numbers.*

(i) The set $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ is a linear space with coordinate-wise addition and scalar multiplication. Moreover, it is an FK-space with respect to the paranorm

$$\|x\|_{|\mathcal{L}^{\phi}(r,s)|(\mu)} = \left\|E^{(\mu)} \circ \mathcal{L}(r,s)(x)\right\|_{l(\mu)},$$

where $M = \max\{1, \sup_n \mu_n\}$.

(ii) The sequence $b^{(j)} = (b_n^{(j)})$ *is a Schauder basis for the space* $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$, *where*

$$b_n^{(j)} = \begin{cases} \phi_j^{\frac{-1}{\mu_j^*}} \left(\frac{1}{r} \frac{L_n}{L_n - 1} + \sum_{k=j}^{n-1} \frac{(-1)^{n-k}}{r} \left(\frac{s}{r}\right)^{n-1-k} \right. & \\ \qquad \left. \cdot \frac{1}{L_k L_{k-1}} \left(\frac{s}{r} L_n^2 + L_{n-1}^2 \right) \right), & 1 \leq j \leq n-1 \\ \phi_n^{-1/\mu_n^*} \frac{1}{r} \frac{L_n}{L_n - 1}, & j = n \\ 0, & j > n, \end{cases}$$

(iii) The space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ *is isometrically isomorphic to* $l(\mu)$, *i.e.*, $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu) \cong l(\mu)$.

Proof. (i) It is a routine verification to prove that $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ is a linear space, so we omit it. Further, since the space $l(\mu)$ is an FK-space and $E^{(\mu)} \circ \mathcal{L}(r,s)$ is a triangle matrix, it follows from Theorem 4.3.2 of [30], $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu) = (l(\mu))_{E^{(\mu)} \circ \mathcal{L}(r,s)}$ is an FK-space.

(ii) It is well-known that the sequence $(e^{(j)})$ is the Schauder basis of the space $l(\mu)$. Also, since $b^{(j)} = \tilde{\mathcal{L}}(r,s)\left(\tilde{E}^{(\mu)}(e^{(j)})\right)$, it is easily seen from Lemma 1.1 that the sequence $\left(b^{(j)}\right)$ is a Schauder basis of the space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$.

(iii) To prove this part, we must show that there exists a linear operator between these spaces which is bijective and norm-preserving. Now, consider the maps $\mathcal{L}(r,s) : \left|\mathcal{L}^{\phi}(r,s)\right|(\mu) \to (l(\mu))_{E^{(\mu)}}$ and $E^{(\mu)} : (l(\mu))_{E^{(\mu)}} \to l(\mu)$ defined by the matrices (2.2) and (2.3). Since these matrices are triangles, the corresponding maps are bijection linear operator. Thus, the composite function $E^{(\mu)} \circ \mathcal{L}(r,s)$ is also a linear bijective operator. Further, by considering

$$\|x\|_{\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)} = \left\| E^{(\mu)} \circ \mathcal{L}(r,s)(x) \right\|_{l(\mu)},$$

one can see that the composite function is norm-preserving. This completes the proof of the theorem. □

At this point, we list the following notations:

$$\eta_{nj} = \frac{1}{r}\frac{L_n}{L_{n-1}} + \sum_{k=j}^{n-1} \frac{(-1)^{n-k}}{r} \left(\frac{s}{r}\right)^{n-1-k} \frac{1}{L_k L_{k-1}} \left(\frac{s}{r} L_n^2 + L_{n-1}^2\right)$$

$$D_1 = \left\{\epsilon \in \omega : \sum_{n=j+1}^{\infty} \eta_{nj}\epsilon_n \text{ exist for all } j\right\}$$

$$D_2 = \left\{\epsilon \in \omega : \exists M > 1, \sup_m \left(\frac{M^{-1/\mu_m^*}}{\phi_m} \left| \frac{1}{r}\frac{L_m}{L_{m-1}}\epsilon_m \right|^{\mu_m^*} + \sum_{j=1}^{m-1} \frac{M^{-1/\mu_j^*}}{\phi_j} \left| \frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j + \sum_{n=j+1}^{m} \eta_{nj}\epsilon_n \right|^{\mu_j^*} \right) < \infty \right\}$$

$$D_3 = \left\{\epsilon \in \omega : \sup_{m,j} \left(\left| \phi_m^{-1/\mu_m^*} \frac{1}{r}\frac{L_m}{L_{m-1}}\epsilon_m \right|^{\mu_m} + \left| \phi_j^{-1/\mu_j^*} \left(\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j + \sum_{n=j+1}^{m} \eta_{nj}\epsilon_n \right) \right|^{\mu_j} \right) < \infty \right\}$$

$$D_4 = \left\{\epsilon \in \omega : \exists M > 1, \sum_{j=1}^{\infty} \frac{M^{-1/\mu_j^*}}{\phi_j} \left(\sum_{n=j+1}^{\infty} |\eta_{nj}\epsilon_n| + \left| \frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j \right| \right)^{\mu_j^*} < \infty \right\}$$

$$D_5 = \left\{\epsilon \in \omega : \exists M > 1, \sup_j \left(M^{-1/\mu_j} \phi_j^{-1/\mu_j^*} \left(\sum_{n=j+1}^{\infty} |\eta_{nj}\epsilon_n| + \left| \frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j \right| \right) \right) < \infty \right\}.$$

Theorem 2.2. *Let $\phi = (\phi_n)$ be a sequence of positive numbers and $\mu = (\mu_n)$ be a bounded sequence of positive numbers.*

(*i*) *If $0 < \mu_n \le 1$ for all $n \in \mathbb{N}$, then*

$$\left\{\left|\mathcal{L}^\phi(r,s)\right|(\mu)\right\}^\alpha = D_5, \ \left\{\left|\mathcal{L}^\phi(r,s)\right|(\mu)\right\}^\beta = D_1 \cap D_3, \ \left\{\left|\mathcal{L}^\phi(r,s)\right|(\mu)\right\}^\gamma = D_3.$$

(*ii*) *If $1 < \mu_n < \infty$ for all $n \in \mathbb{N}$, then*

$$\left\{\left|\mathcal{L}^\phi(r,s)\right|(\mu)\right\}^\alpha = D_4, \ \left\{\left|\mathcal{L}^\phi(r,s)\right|(\mu)\right\}^\beta = D_1 \cap D_2, \ \left\{\left|\mathcal{L}^\phi(r,s)\right|(\mu)\right\}^\gamma = D_2.$$

Proof. Since the proof of the other parts are similar, to avoid repetition we only calculate the β-dual of the space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$. Let $x \in \left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$. Note that $\epsilon \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)\right\}^{\beta}$ if $\epsilon x = (\epsilon_n x_n) \in cs$ for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$. Say $\mathcal{L}(r,s)(x) = y$ and $z = E^{(\mu)}(y)$. Then, since $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu) \simeq l(\mu)$ by Theorem 2.1, $x \in \left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ if $z \in l(\mu)$, and so it is easily seen that

$$\begin{aligned}
\sum_{n=1}^{m} \epsilon_n x_n &= \epsilon_1 x_1 + \sum_{n=2}^{m} \epsilon_n \left(\frac{1}{r} \frac{L_n}{L_{n-1}} y_n \right. \\
&\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^{n-k}}{r} \left(\frac{s}{r}\right)^{n-1-k} \frac{1}{L_k L_{k-1}} \left(\frac{s}{r} L_n^2 + L_{n-1}^2\right) y_k \right) \\
&= \sum_{j=1}^{m} \phi_j^{-1/\mu_j^*} \sum_{n=j}^{m} \epsilon_n \frac{1}{r} \frac{L_n}{L_{n-1}} z_j \\
&\quad + \sum_{j=1}^{m-1} \phi_j^{-1/\mu_j^*} \left(\sum_{n=j+1}^{m} \sum_{k=j}^{n-1} \epsilon_n \frac{(-1)^{n-k}}{r} \left(\frac{s}{r}\right)^{n-1-k} \cdot \right. \\
&\quad \left. \cdot \frac{1}{L_k L_{k-1}} \left(\frac{s}{r} L_n^2 + L_{n-1}^2\right) \right) z_j \\
&= \phi_m^{-1/\mu_m^*} \epsilon_m \frac{1}{r} \frac{L_m}{L_{m-1}} z_m + \sum_{j=1}^{m-1} \phi_j^{-1/\mu_j^*} \left(\epsilon_j \frac{1}{r} \frac{L_j}{L_{j-1}} + \sum_{n=j+1}^{m} \epsilon_n \eta_{nj} \right) z_j \\
&= \sum_{j=1}^{m} b_{mj} z_j
\end{aligned}$$

where $B = (b_{mj})$ is the matrix defined by

$$b_{mj} = \begin{cases} \phi_j^{-1/\mu_j^*} \left(\epsilon_j \frac{1}{r} \frac{L_j}{L_{j-1}} + \sum\limits_{n=j+1}^{m} \epsilon_n \eta_{nj} \right), & 1 \le j \le m-1 \\ \phi_m^{-1/\mu_m^*} \epsilon_m \frac{1}{r} \frac{L_m}{L_{m-1}}, & j = m \\ 0, & j > m. \end{cases}$$

This means that $\epsilon \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)\right\}^{\beta}$ if and only if $B \in (l(\mu), c)$. Thus, by applying Lemma 2.1 to the matrix B, we obtain $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)\right\}^{\beta} = D_1 \cap D_2$, for $1 < \mu_n < \infty$, and $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)\right\}^{\beta} = D_1 \cap D_3$, for $\mu_n \le 1$ $(n = 0, 1, ...)$, which completes the proof. □

The following theorems show that certain matrix transformations on the space $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ correspond to bounded linear operators, and give their characterizations.

Theorem 2.3. *Let $\phi = (\phi_n)$ be a sequence of positive numbers, $\mu = (\mu_n)$ be bounded sequence of positive numbers, $A = (a_{nk})$ be an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$ and $B = (b_{nk})$ be a matrix satisfying the following relation*

$$b_{nk} = \phi_n^{1/\mu_n^*} \sum_{v=0}^{n} \xi_{nv} a_{vk}. \tag{2.8}$$

Then, for any sequence spaces λ, $A \in \left(\lambda, \left|\mathcal{L}^\phi(r,s)\right|(\mu)\right)$ if and only if $B \in (\lambda, l(\mu))$.

Proof. Take $x \in \lambda$. It follows from (2.8) that

$$\sum_{k=0}^{\infty} b_{nk} x_k = \phi_n^{1/\mu_n^*} \sum_{v=0}^{n} \xi_{nv} \sum_{k=0}^{\infty} a_{vk} x_k.$$

By definition of ξ, it is seen immediately that $B_n(x) = \left(E^{(\mu)} \circ \mathcal{L}(r,s)\right)_n (A(x))$ for all $x \in \lambda$. So, it is obtained that $A_n(x) \in \left|\mathcal{L}^\phi(r,s)\right|(\mu)$ whenever $x \in \lambda$ if and only if $B(x) \in l(\mu)$ whenever $x \in \lambda$, which completes the proof of the theorem. □

Theorem 2.4. *Assume that (ϕ_n) and (ψ_n) are sequences of positive numbers, and (μ_n) and (λ_n) are bounded sequences of positive numbers with $\mu_n \leq 1$ and $\lambda_n \geq 1$. Further, let $A = (a_{nk})$ be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$ and $\hat{A}^{(\lambda)} = E^{(\lambda)} \circ \mathcal{L}(r,s) \circ \tilde{A}$, where*

$$\tilde{a}_{nv} = \phi_v^{-1/\mu_v^*} \left(\frac{1}{r} \frac{L_v}{L_{v-1}} a_{nv} + \sum_{j=v+1}^{\infty} a_{nj} \eta_{jv} \right).$$

If $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|(\mu), \left|\mathcal{L}^\psi(r,s)\right|(\lambda)\right)$, then A defines a bounded linear operator L_A such that $L_A(x) = A(x)$ for all $x \in \left|\mathcal{L}^\phi(r,s)\right|(\mu)$, and $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|(\mu),\right.$ $\left.\left|\mathcal{L}^\psi(r,s)\right|(\lambda)\right)$ if and only if there exists an integer $M > 0$ such that, for all n,

$$\sum_{v=j+1}^{\infty} \eta_{vj} a_{nv} \ \textit{exists for all } j, \tag{2.9}$$

$$\begin{aligned} \sup_{m,k} \Bigg\{ & \left| \phi_m^{-1/\mu_m^*} \frac{1}{r} \frac{L_m}{L_{m-1}} a_{nm} \right|^{\mu_m} \\ & + \left(\phi_k^{-1/\mu_k^*} \left| \frac{1}{r} \frac{L_k}{L_{k-1}} a_{nk} + \sum_{j=k+1}^{m} \eta_{jk} a_{nj} \right| \right)^{\mu_k} \Bigg\} < \infty, \end{aligned} \tag{2.10}$$

$$\sup_v \sum_{n=0}^{\infty} \left| M^{-1/\mu_v} \hat{a}_{nv}^{(\lambda)} \right|^{\lambda_n} < \infty. \tag{2.11}$$

Proof. By Theorem 2.1, the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$ and $\left|\mathcal{L}^{\psi}(r,s)\right|(\lambda)$ are FK-spaces. Thus, by Theorem 4.2.8 of [30], L_A is a bounded linear operator.

To prove the second part, take $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), \left|\mathcal{L}^{\psi}(r,s)\right|(\lambda)\right)$. Then, by Lemma 2.4, $\tilde{A} \in \left(l(\mu), \left|\mathcal{L}^{\psi}(r,s)\right|(\lambda)\right)$ and $V^{(n)} \in (l(\mu), c)$, where $V^{(n)}$ is the matrix given by

$$v_{mk}^{(n)} = \begin{cases} \phi_k^{-1/\mu_k^*}\left(a_{nk}\frac{1}{r}\frac{L_k}{L_{k-1}} + \sum\limits_{j=k+1}^{m} a_{nj}\eta_{jk}\right), & 0 \le k \le m-1 \\ \phi_m^{-1/\mu_m^*} a_{nm}\frac{1}{r}\frac{L_m}{L_{m-1}}, & k = m \\ 0, & k > m. \end{cases}$$

Applying the Lemma 2.1 to the matrix $V^{(n)}$, we have the conditions (2.9) and (2.10). Also, for $x \in l(\mu)$, it follows from $\left|\mathcal{L}^{\psi}(r,s)\right|(\lambda) = \{l(\lambda)\}_{E^{(\lambda)}\circ\mathcal{L}(r,s)}$ that $\tilde{A}(x) \in \left|\mathcal{L}^{\psi}(r,s)\right|(\lambda)$ if and only if $\hat{A}^{(\lambda)}(x) = E^{(\lambda)} \circ \mathcal{L}(r,s) \circ \tilde{A}(x) \in l(\lambda)$. This gives that $\tilde{A} \in \left(l(\mu), \left|\mathcal{L}^{\psi}(r,s)\right|(\lambda)\right)$ iff $\hat{A}^{(\lambda)} \in (l(\mu), l(\lambda))$. So, the proof is completed together with Lemma 2.1. □

Theorem 2.5. *Let (ϕ_n) and (ψ_n) be sequences of positive numbers, and (μ_n) be bounded sequence of positive numbers with $\mu_n > 1$. Also, let $A = (a_{nk})$ be an infinite matrix of complex numbers for each $n, k \in \mathbb{N}$. Define the matrix $\hat{A}^{(1)} = E^{(1)} \circ \mathcal{L}(r,s) \circ \tilde{A}$, where $\tilde{A}$ is as in Theorem 2.4. If $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), \left|\mathcal{L}^{\psi}(r,s)\right|\right)$, then A defines a bounded linear operator L_A such that $L_A(x) = A(x)$ for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|(\mu)$. Also, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), \left|\mathcal{L}^{\psi}(r,s)\right|\right)$ if and only if there exists an integer $M > 1$ such that, for all $n \in \mathbb{N}$,*

$$\sum_{v=j+1}^{\infty} \eta_{vj} a_{nv} \text{ exist for all } j, \tag{2.12}$$

$$\begin{aligned} \sup_m \Bigg\{ & \frac{M^{-1/\mu_m^*}}{\phi_m}\left|\frac{1}{r}\frac{L_m}{L_{m-1}} a_{nm}\right|^{\mu_m^*} \\ & + \sum_{j=1}^{m-1} \frac{M^{-1/\mu_j^*}}{\phi_j}\left|\frac{1}{r}\frac{L_j}{L_{j-1}} a_{nj} + \sum_{v=j+1}^{m} \eta_{vj} a_{nv}\right|^{\mu_j^*}\Bigg\} < \infty, \end{aligned} \tag{2.13}$$

$$\sum_{v=1}^{\infty}\left(\sum_{n=1}^{\infty}\left|\hat{a}_{nv} M^{-1}\right|\right)^{\mu_v^*} < \infty. \tag{2.14}$$

Proof. The first part is proved as before. Also, since $\left|\mathcal{L}^{\phi}(r,s)\right|(\mu) = (l(\mu))_{E^{(\mu)}\circ\mathcal{L}(r,s)}$, by Lemma 2.4, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), \left|\mathcal{L}^{\psi}(r,s)\right|\right)$ iff $\tilde{A} \in \left(l(\mu), \left|\mathcal{L}^{\phi}(r,s)\right|\right)$ and $V^{(n)} \in$

$(l(\mu), c)$. Further, by Lemma 2.1, $V^{(n)} \in (l(\mu), c)$ iff the conditions (2.12) and (2.13) hold, and $\tilde{A} \in \left(l(\mu), \left|\mathcal{L}^{\phi}(r,s)\right|\right)$ iff $\hat{A}^{(1)} = E^{(1)} \circ \mathcal{L}(r,s) o \tilde{A} \in (l(\mu), l)$, which completes the proof applying Lemma 2.1 to the matrix $\hat{A}^{(1)}$. □

By following the above lines, we also have the following.

Theorem 2.6. *Let (ϕ_n) be a sequence of positive numbers, and (μ_n) be a bounded sequences of positive numbers. Further let $A = (a_{nk})$ be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$ and Y be arbitrary sequence space. Then, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), Y\right)$ if and only if*

$$V^{(n)} \in (l(\mu), c) \ \textit{for all } n \in \mathbb{N},$$

$$\tilde{A} \in (l(\mu), Y),$$

where the matrices $V^{(n)}$ and $\tilde{A}$ are as in Theorem 2.4.

Now, we list the following notations:

(*i*) $\sup_{n,k} |\tilde{a}_{nk}|^{\mu_k} < \infty.$

(*ii*) There exists $M > 1$ such that $\sup_n \sum_k |M^{-1}\tilde{a}_{nk}|^{\mu_k^*} < \infty.$

(*iii*) $\lim_{n\to\infty} \tilde{a}_{nk} = 0$ for each $k \in \mathbb{N}$.

(*iv*) $\lim_{n\to\infty} \tilde{a}_{nk}$ exists for all $k \in \mathbb{N}$.

(*v*) There exists $M > 1$ such that $\sup_k \sum_{n=0}^{\infty} |M^{-1/\mu_k}\tilde{a}_{nk}| < \infty.$

(*vi*) There exists $M > 1$ such that

$$\sup \left\{ \sum_{k=0}^{\infty} \left| \sum_{n\in K} \tilde{a}_{nk} M^{-1} \right|^{\mu_k^*} : K \subset \mathbb{N} \text{ finite} \right\} < \infty.$$

(*vii*) $\sup_{m,k} |v_{mk}^{(n)}|^{\mu_k} < \infty.$

(*viii*) There exists $M > 1$ such that $\sup_m \sum_k |M^{-1}v_{mk}^{(n)}|^{\mu_k^*} < \infty.$

(*ix*) $\lim_{m\to\infty} v_{mk}^{(n)}$ exists for all $n, k \in \mathbb{N}$.

Thus, by combining our theorems with Lemma 2.1 we obtain the following results:

Theorem 2.7. *The following statements hold:*

1. *If $\mu_n \leq 1$ for all n, then, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), l_\infty\right) \Leftrightarrow$ (i), (vii) and (ix) hold.*

2. *If* $\mu_n > 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), l_\infty\right) \Leftrightarrow$ *(ii), (viii) and (ix) hold.*

3. *If* $\mu_n \leq 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), c\right) \Leftrightarrow$ *(i), (iv), (vii) and (ix) hold.*

4. *If* $\mu_n > 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), c\right) \Leftrightarrow$ *(ii), (iv), (viii) and (ix) hold.*

5. *If* $\mu_n \leq 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), c_0\right) \Leftrightarrow$ *(i), (iii), (vii) and (ix) hold.*

6. *If* $\mu_n > 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), c_0\right) \Leftrightarrow$ *(ii), (iii), (viii) and (ix) hold.*

7. *If* $\mu_n \leq 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), l\right) \Leftrightarrow$ *(v), (vii) and (ix) hold.*

8. *If* $\mu_n > 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), l\right) \Leftrightarrow$ *(vi), (viii) and (ix) hold.*

Also, Theorem 2.7 gives the following.

Corollary 2.1. *Put* $a(n,k) = \sum_{j=0}^{n} a_{jk}$ *instead of* a_{nk} *for all* n, k*.Then,*

1. *If* $\mu_n \leq 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), bs\right) \Leftrightarrow$ *(i), (vii) and (ix) hold.*

2. *If* $\mu_n > 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), bs\right) \Leftrightarrow$ *(ii), (viii) and (ix) hold.*

3. *If* $\mu_n \leq 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), cs\right) \Leftrightarrow$ *(i), (iv), (vii) and (ix) hold.*

4. *If* $\mu_n > 1$ *for all* n*, then,* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), cs\right) \Leftrightarrow$ *(ii), (iv), (viii) and (ix) hold.*

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HOMOTHETIC MOTIONS VIA GENERALIZED BICOMPLEX NUMBERS

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Abstract. In this paper, by using the matrix representation of generalized bicomplex numbers, we have defined the homothetic motions on some hypersurfaces in four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$. Also, for some special cases we have given some examples of homothetic motions in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ and obtained some rotational matrices, too. Therefore, we have investigated some applications about kinematics of generalized bicomplex numbers.
Key words: Bicomplex number, Generalized Bicomplex numbers, Homothetic motion.

1. Introduction

In the middle of the 1800s, several mathematicians discussed the problem of whether a number system extended the field of complex numbers. In 1843, Sir William Rowan Hamilton defined a number system which is called quaternions in four dimensional space. Although quaternions and complex numbers have a lot of similar properties, quaternions are not commutative with respect to multiplication. So, in 1892, a new number system called bicomplex numbers was discovered by Corrado Segre [13]. Unlike quaternions, bicomplex numbers are commutative four dimensional real algebra.

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The set of bicomplex numbers denoted by $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \left\{x = x_1 1 + x_2 i + x_3 j + x_4 ij : i^2 = -1, j^2 = -1, ij = ji, x_k \in \mathbb{R}, 1 \leq k \leq 4\right\}.$$

Any x bicomplex number can be rewritten as $x = z_1 + jz_2$, where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ are complex numbers and j is a different imaginer unit from the imaginer unit i satisfied $j^2 = -1$ and $ij = ji$. Hence, we can perceive a bicomplex number as a complex number whose components are complex numbers. There are some applications of bicomplex numbers on the algebra, geometry and analysis. A first theory of differentiability in $\mathbb{C}_2$ was developed by Price in [12]. Özkaldı Karakuş and Kahraman Aksoyak defined generalized bicomplex numbers and gave some algebraic properties. Also, they showed that some hypersurfaces in four dimensional generalized linear space are Lie groups by using generalized bicomplex number product and obtained Lie algebras of these Lie groups [10].

Kabadayı and Yaylı defined the homothetic motions with the help of bicomplex numbers in $\mathbb{R}^4$ [5]. They showed that this homothetic motion under some conditions holds all of the properties in [14], [15]. Alkaya studied the homothetic motion with bicomplex numbers in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ [1].

In this paper, by using the matrix representation of generalized bicomplex numbers, we shall define the homothetic motions on some hypersurfaces in four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$. Also, for some special cases we shall give some examples of homothetic motions in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ and obtain some rotational matrices, too. Therefore, we shall investigate some applications about kinematics of generalized bicomplex numbers.

2. Preliminaries

In this section we give some basic concepts about generalized bicomplex numbers defined by Özkaldı Karakuş and Kahraman Aksoyak [10].

A generalized bicomplex number x is defined as follows:

$$x = x_1 1 + x_2 i + x_3 j + x_4 ij,$$

where x_k for $1 \leq k \leq 4$ are real numbers and the basis $\{1, i, j, ij\}$ holds $i^2 = -\alpha$, $j^2 = -\beta$, $(ij)^2 = \alpha\beta$, $ij = ji$, $\alpha, \beta \in \mathbb{R}$. The set of generalized bicomplex numbers is denoted by $\mathbb{C}_{\alpha\beta}$. For any two generalized bicomplex numbers $x = x_1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 + y_2 i + y_3 j + y_4 ij$, addition and multiplication are as follows:

$$x + y = (x_1 + y_1) + (x_2 + y_2)\, i + (x_3 + y_3)\, j + (x_4 + y_4)\, ij,$$

$$\begin{aligned} x \cdot y &= (x_1 y_1 - \alpha x_2 y_2 - \beta x_3 y_3 + \alpha\beta x_4 y_4) + (x_1 y_2 + x_2 y_1 - \beta x_3 y_4 - \beta x_4 y_3)\, i \\ &\quad + (x_1 y_3 + x_3 y_1 - \alpha x_2 y_4 - \alpha x_4 y_2)\, j + (x_1 y_4 + x_4 y_1 + x_2 y_3 + x_3 y_2)\, ij \end{aligned} \tag{2.1}$$

and the scalar multiplication of an element in $\mathbb{C}_{\alpha\beta}$ by a real number c is as:

$$cx = cx_1 1 + cx_2 i + cx_3 j + cx_4 ij.$$

Hence, by means of these elementary arithmetic operations on $\mathbb{C}_{\alpha\beta}$, we have two important results. $\mathbb{C}_{\alpha\beta}$ is a four dimensional real vector space with respect to addition and scalar multiplication and it is a commutative real algebra according to generalized bicomplex number product.

Let us consider the following set of matrices

$$Q_{\alpha\beta} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_i \in \mathbb{R},\ 1 \leq i \leq 4 \right\},$$

where the set $Q_{\alpha\beta}$ is a vector space with matrix addition and scalar matrix product and it is an algebra together with matrix product. The algebras $\mathbb{C}_{\alpha\beta}$ and $Q_{\alpha\beta}$ are isomorphic. The isomorphism between two algebras is defined as:

$$h : \mathbb{C}_{\alpha\beta} \to Q_{\alpha\beta},$$

$$h\left(x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

With the help of this isomorphism, any generalized bicomplex number in $\mathbb{C}_{\alpha\beta}$ can be represent by a matrix in $Q_{\alpha\beta}$. Moreover, it is possible to express the generalized bicomplex number product which has been given by (2.1) by matrix product, that is,

$$x \cdot y = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

A generalized bicomplex number can be rewritten as $x = (x_1 + x_2 i) + (x_3 + x_4 i)\, j$. There are three kinds of conjugations for generalized bicomplex numbers. They are given as follows:

$$\begin{aligned} x^{t_1} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_1} = (x_1 - x_2 i) + (x_3 - x_4 i)\, j, \\ x^{t_2} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_2} = (x_1 + x_2 i) - (x_3 + x_4 i)\, j, \\ x^{t_3} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_3} = (x_1 - x_2 i) - (x_3 - x_4 i)\, j, \end{aligned}$$

where x^{t_1}, x^{t_2} and x^{t_3} denote the conjugations of x with respect to i, j and both i and j, respectively. Also we can compute

$$\begin{aligned} x \cdot x^{t_1} &= \left(x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2\right) + 2\left(x_1 x_3 + \alpha x_2 x_4\right) j, \\ x \cdot x^{t_2} &= \left(x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2\right) + 2\left(x_1 x_2 + \beta x_3 x_4\right) i, \\ x \cdot x^{t_3} &= \left(x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2\right) + 2\left(x_1 x_4 - x_2 x_3\right) ij. \end{aligned}$$

3. One Parameter Homothetic Motion

Let the fixed space and the moving space be R_0 and $R.$, respectively. The one-parameter homothetic motion of R_0 with respect to R is denoted by R_0/R. This motion is obtained by the following transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{3.1}$$

in which X_0 and X are the position vectors of the same point in R_0 and R, respectively and $B = hA$. Also, h, A and C are continuously differentiable functions depend on the real parameter t, where $h : I \subset \mathbb{R} \to \mathbb{R}$, $t \to h(t)$ is called homothetic scale of the motion, A is a real quasi-orthogonal matrix that holds $A^T \varepsilon A = \varepsilon$ (ε is a signature matrix according to metric), C is the translation matrix. To avoid the case of affine transformation we suppose that h is not constant and to avoid the cases of pure translation and pure rotation we also assume that $\frac{d(hA)}{dt} \neq 0$ and $\frac{dC}{dt} \neq 0$ [2].

4. Pole Points and Pole Curves of the Homothetic Motion

If we take the derivative of (3.1) with respect to t, we obtain the following equality

$$\dot{X} = \dot{B}X_0 + \dot{C} + B\dot{X}_0,$$

where $\dot{X}$ is the absolute velocity, $\dot{B}X_0 + \dot{C}$ is the sliding velocity and $B\dot{X}_0$ is the relative velocity of the point X_0. The points at which the sliding velocity of the motion vanishes at all time t are called pole points of the motion in R_0. In that case, to determine the pole points of the motion, we solve the following equality

$$\dot{B}X_0 + \dot{C} = 0. \tag{4.1}$$

For more details see[2].

5. Homothetic Motions on Some Hypersurfaces via Generalized Bicomplex Numbers

In this section we have defined the homothetic motions on some hypersurfaces at $\mathbb{R}^4_{\alpha\beta}$ with the help of generalized bicomplex numbers and given some examples about the homothetic motions.

5.1. Homothetic Motion on Hypersurface M_1

Let us consider the hypersurface M_1 as follows:

$$M_1 = \left\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\alpha\beta} : x_1x_3 + \alpha x_2x_4 = 0,\ x \neq 0\right\}.$$

By using generalized bicomplex numbers, M_1 can be rewritten as:

$$M_1 = \left\{x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_{\alpha\beta} : x_1x_3 + \alpha x_2x_4 = 0,\ x \neq 0\right\},$$

or the hypersurface M_1 can be expressed by using the matrix representiation of generalized bicomplex numbers

$$\tilde{M_1} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_1x_3 + \alpha x_2x_4 = 0,\ x \neq 0 \right\},$$

where M_x is the matrix representiation of generalized bicomplex number x. The metric on hypersurface M_1 is defined by $g_1(x,x) = x \cdot x^{t_1} = x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2$ and the norm of any element x on M_1 is defined by $\|x\| = \sqrt{|g_1(x,x)|} = \sqrt{|x \cdot x^{t_1}|}$. This metric is Riemannian or pseudo-Riemannian metric on four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$ and for some special cases, it coincides with four dimensional Euclidean space $\mathbb{R}^4$ or four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$.

Proposition 5.1. *There are following properties about the norm on the hypersurface M_1.*

i) For $x, y \in M_1$, $\|x \cdot y\| = \|x\| \, \|y\|$,

ii) $\|x\|^4 = \det(M_x)$.

Proof. These properties can be easily seen with direct calculations. □

Corollary 5.1. *A unit generalized bicomplex number on the hypersurface M_1 determines a rotation motion.*

Proof. It is obvious from Proposition 5.1. □

Theorem 5.1. *M_1 is a commutative Lie group.*

Proof. The proof can be found in [10]. □

Let us denote the set of unit generalized bicomplex numbers on M_1 by M_1^*. M_1^* is as:

$$\begin{aligned} M_1^* &= \{x \in M_1 : g_1(x,x) = 1\} \\ &= \left\{x \in M_1 : x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2 = 1\right\}. \end{aligned}$$

Theorem 5.2. *M_1^* is Lie subgroup of M_1.*

Proof. The proof can be found in [10]. □

Let γ be a curve on M_1. In that case, it can be expressed as

$$\begin{array}{rcl} \gamma & : & I \subset \mathbb{R} \rightarrow M_1 \\ t & \rightarrow & \gamma(t) = \gamma_1(t) + \gamma_2(t)i + \gamma_3(t)j + \gamma_4(t)ij, \ \gamma_1(t)\gamma_3(t) + \alpha\gamma_2(t)\gamma_4(t) = 0. \end{array}$$

Then the matrix B corresponding to the curve γ is obtained as follows:

$$B = M_{\gamma(t)} = \begin{bmatrix} \gamma_1(t) & -\alpha\gamma_2(t) & -\beta\gamma_3(t) & \alpha\beta\gamma_4(t) \\ \gamma_2(t) & \gamma_1(t) & -\beta\gamma_4(t) & -\beta\gamma_3(t) \\ \gamma_3(t) & -\alpha\gamma_4(t) & \gamma_1(t) & -\alpha\gamma_2(t) \\ \gamma_4(t) & \gamma_3(t) & \gamma_2(t) & \gamma_1(t) \end{bmatrix}. \tag{5.1}$$

Now by using this matrix B, we can define the one parameter motion on M_1 at $\mathbb{R}^4_{\alpha\beta}$.

Definition 5.1. Let R_0 and R be the fixed space and the motional space at $\mathbb{R}^4_{\alpha\beta}$. In that case, the one-parameter motion of R_0 with respect to R is denoted by R_0/R. Then the one-parameter motion on M_1 is defined by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{5.2}$$

where B is the matrix associated with the curve $\gamma(t)$ on the hypersurface M_1, C is the 4×1 real matrix depends on a real parameter t, X and X_0 are the position vectors of any point at $\mathbb{R}^4_{\alpha\beta}$ respectively in R and R_0.

Theorem 5.3. *The equation given by* (5.2) *determines a homothetic motion on M_1.*

Proof. Since the curve γ lies on M_1, it does not pass through the origin. So, the matrix given by (5.1) can be expressed as:

$$B = M_{\gamma(t)} = h \begin{bmatrix} \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} & \frac{-\beta\gamma_3(t)}{h} & \frac{\alpha\beta\gamma_4(t)}{h} \\ \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\beta\gamma_4(t)}{h} & \frac{-\beta\gamma_3(t)}{h} \\ \frac{\gamma_3(t)}{h} & \frac{-\alpha\gamma_4(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} \\ \frac{\gamma_4(t)}{h} & \frac{\gamma_3(t)}{h} & \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} \end{bmatrix} = hA, \tag{5.3}$$

where $h : I \subset \mathbb{R} \rightarrow \mathbb{R}$, $t \rightarrow h(t) = \|\gamma(t)\| = \sqrt{\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2}$. Because of $\gamma(t) \in M_1$, $\gamma_1(t)\gamma_3(t) + \alpha\gamma_2(t)\gamma_4(t) = 0$. By using this equality, we obtain that the matrix A in (5.3) is a real quasi-orthogonal matrix. In that case it satisfies $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, where ε is the signature matrix corresponding to metric g_1 is as:

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\alpha\beta \end{bmatrix}.$$

Hence A, h and C are a real quasi-orthogonal matrix, the homothetic scale of the motion and the translation vector, respectively. So the equation (5.2) is a homothetic motion. $\square$

Remark 5.1. The norm of $\gamma \in \mathbb{R}^4_{\alpha\beta}$ is found as $\|\gamma(t)\| = \sqrt{|\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2|}$. We assume that $\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2 > 0$ in this paper.

Corollary 5.2. *Let $\gamma(t)$ be a curve on M_1^*. Then one-parameter motion on M_1 given by (5.2) is a general motion consists of a rotation and a translation.*

Proof. We assume that $\gamma(t)$ is a curve on M_1^*. Then $\gamma_1^2 + \alpha\gamma_2^2 - \beta\gamma_3^2 - \alpha\beta\gamma_4^2 = 1$. In that case the matrix B given by (5.1) becomes a real-quasi orthogonal matrix, that is, it satisfies $B^T \varepsilon B = \varepsilon$ and $\det B = 1$. This completes the proof. $\square$

Theorem 5.4. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_1. Then the derivative of the matrix B is a real quasi-orthogonal matrix.*

Proof. We suppose that $\gamma(t)$ be a unit velocity curve. Then $\dot{\gamma}_1^2 + \alpha\dot{\gamma}_2^2 - \beta\dot{\gamma}_3^2 - \alpha\beta\dot{\gamma}_4^2 = 1$. Also, since the tangent vector of γ is on M_1, it implies that $\dot{\gamma}_1(t)\dot{\gamma}_3(t) + \alpha\dot{\gamma}_2(t)\dot{\gamma}_4(t) = 0$. Thus $\dot{B}^T \varepsilon \dot{B} = \varepsilon$ and $\det \dot{B} = 1$. $\square$

Theorem 5.5. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_1. Then the motion is a regular motion and it is independent of h.*

Proof. From Theorem 5.4, $\det \dot{B} = 1$ and thus the value of $\det \dot{B}$ is independent of h. $\square$

Theorem 5.6. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_1. Then the pole points of the motion given by (5.2) are $X_0 = -\dot{B}^{-1}\dot{C}$.*

Proof. Since the position vector of the curve γ is on M_1, from Theorem 5.3, the equation (5.2) becomes a homothetic motion. Also, because of $\gamma(t)$ is a unit velocity curve and $\dot{\gamma}(t) \in M_1$, from Theorem 5.4 $\det \dot{B} = 1$ and it implies that there is only one solution of the equation (4.1). Then the pole points of the motion given by (5.2) are obtained as $X_0 = -\dot{B}^{-1}\dot{C}$. $\square$

Corollary 5.3. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_1. The pole point associated with each t- instant in R_0 is the rotation by the matrix $\dot{B}^{-1}$ of the speed vector of translation vector at the opposite direction $\left(-\dot{C}\right)$.*

Proof. From Theorem 5.4, the matrix $\dot{B}$ is a real quasi-orthogonal matrix. Then the matrix $\dot{B}^{-1}$ is quasi-orthogonal matrix, too. This completes the proof. □

Now we will give various examples of the homothetic motions on M_1 according to the situations of real numbers α and β.

Example 5.1. For $\alpha = \beta = 1$, M_1 becomes a hypersurface in $\mathbb{R}^4_2$. Let $\gamma : I \subset \mathbb{R} \to M_1 \subset \mathbb{R}^4_2$ be a curve given by

$$\gamma(t) = h(t) \begin{pmatrix} \cosh(at)\cos(bt) + \cosh(at)\sin(bt)\, i \\ -\sinh(at)\sin(bt)\, j + \sinh(at)\cos(bt)\, ij \end{pmatrix}, \tag{5.4}$$

where a and b are real numbers. By using (5.1) and (5.4), the matrix B associated with the curve γ becomes a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$ in (5.4), then γ is a curve on M_1^* and the matrix B determines a rotation matrix in $\mathbb{R}^4_2$. In (5.4), if we choose as $h(t) = 1$, $a = 0$ and $b = 1$, then we get

$$\gamma(t) = \cos t + i \sin t. \tag{5.5}$$

By using (5.1) and (5.5), we get the matrix B as follows:

$$B = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix},$$

where B is a rotational matrix in $\mathbb{R}^4_2$. Since this curve given by (5.5) is unit speed curve and its tangent vector belongs to M_1, the derivation of the above matrix $\dot{B}$ is a real quasi-orthogonal matrix, too. Then it is a rotational matrix in $\mathbb{R}^4_2$. Similarly, in (5.4) if we take as $h(t) = 1$, $a = 1$ and $b = 0$, then we get

$$\gamma(t) = \cosh t + ij \sinh t. \tag{5.6}$$

By using (5.1) and (5.6), we have the matrix B as follows:

$$B = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & -\sinh t & 0 \\ 0 & -\sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.$$

where B is a rotational matrix in $\mathbb{R}^4_2$.

Example 5.2. For $\alpha = 1$, $\beta = -1$, M_1 is a hypersurface in $\mathbb{R}^4$. Let $\gamma : I \subset \mathbb{R} \to M_1 \subset \mathbb{R}^4$ be a curve given by

$$\gamma(t) = h(t) \begin{pmatrix} \cos(at)\cos(bt) + \cos(at)\sin(bt)\, i \\ -\sin(at)\sin(bt)\, j + \sin(at)\cos(bt)\, ij \end{pmatrix}, \tag{5.7}$$

where a and b are real numbers. By using (5.1) and (5.7), the matrix representation of the curve γ is a homothetic matrix, in here $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, γ becomes a spherical curve on M_1, that is, $\gamma(t) \in M_1^* = M_1 \cap S^3$. The matrix representation of it is a rotation matrix in $\mathbb{R}^4$. Even, if we take as $h(t) = 1$, $a = 0$, $b = 1$, then

$$\gamma(t) = \cos t + i \sin t. \tag{5.8}$$

From (5.1) and (5.8), by determining the matrix representation of the above curve, we obtain

$$B = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}.$$

This matrix is a general rotational matrix in $\mathbb{R}^4$ which is defined by Moore [8]. Also, from Theorem 5.4, $\dot{B}$ is a real orthogonal matrix, too.

Example 5.3. For $\alpha = \beta = -1$, the hypersurface M_1 lies in $\mathbb{R}_2^4$ and the following curve lies on M_1

$$\gamma(t) = h(t) \begin{pmatrix} \cosh(at)\cosh(bt) + \cosh(at)\sinh(bt)\, i \\ +\sinh(at)\sinh(bt)\, j + \cosh(at)\sinh(bt)\, ij \end{pmatrix}, \tag{5.9}$$

in which a and b are real numbers. From (5.1) and (5.9), the matrix B according to the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ lies on M_1^* and the matrix B gives a rotation matrix in $\mathbb{R}_2^4$.

5.2. Homothetic Motion on Hypersurface M_2

Let us consider the hypersurface M_2 as follows:

$$M_2 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\alpha\beta} : x_1x_2 + \beta x_3x_4 = 0,\ x \neq 0 \right\}.$$

By using the generalized bicomplex numbers, M_2 can be rewritten as:

$$M_2 = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_{\alpha\beta} : x_1x_2 + \beta x_3x_4 = 0,\ x \neq 0 \right\},$$

or the hypersurface M_2 can be expressed by using the matrix representiation of generalized bicomplex numbers

$$\tilde{M_2} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_1x_2 + \beta x_3x_4 = 0,\ x \neq 0 \right\},$$

where M_x is the matrix representiation of the generalized bicomplex number x on M_2. The metric on hypersurface M_2 is defined by $g_2(x, x) = x \cdot x^{t_2} = x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2$ and the norm of any element x on M_2 is given by $\|x\| = \sqrt{|g_2(x, x)|} = \sqrt{|x \cdot x^{t_2}|}$. This metric is Riemannian or pseudo-Riemannian metric on four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$ and for some special cases, it coincides with four dimensional Euclidean space $\mathbb{R}^4$ or four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$.

Proposition 5.2. *There are following properties about the norm on the hypersurface M_2.*

i) For $x, y \in M_2$, $\|x \cdot y\| = \|x\| \, \|y\|$,

ii) $\|x\|^4 = \det(M_x)$.

Proof. These properties can be easily seen with directly calculations. □

Corollary 5.4. *A unit generalized bicomplex number on the hypersurface M_2 determines a rotation motion.*

Proof. It is obvious from Proposition 5.2. □

Theorem 5.7. *M_2 is a commutative Lie group.*

Proof. The proof can be found in [10]. □

Let us denote the set of unit generalized bicomplex number on M_2 by M_2^*. M_2^* is given as:

$$\begin{aligned} M_2^* &= \{x \in M_2 : g_2(x, x) = 1\} \\ &= \left\{x \in M_2 : x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2 = 1\right\}. \end{aligned}$$

Theorem 5.8. *M_2^* is Lie subgroup of M_2.*

Proof. The proof can be found in [10]. □

Let γ be a curve on M_2. In that case, it can be expressed as:

$$\begin{aligned} \gamma &: I \subset \mathbb{R} \rightarrow M_2 \\ t &\rightarrow \gamma(t) = \gamma_1(t) + \gamma_2(t)i + \gamma_3(t)j + \gamma_4(t)ij, \ \gamma_1(t)\gamma_2(t) + \beta\gamma_3(t)\gamma_4(t) = 0. \end{aligned}$$

Then the matrix B corresponding to the curve γ is given as follows:

$$B = M_{\gamma(t)} = \begin{bmatrix} \gamma_1(t) & -\alpha\gamma_2(t) & -\beta\gamma_3(t) & \alpha\beta\gamma_4(t) \\ \gamma_2(t) & \gamma_1(t) & -\beta\gamma_4(t) & -\beta\gamma_3(t) \\ \gamma_3(t) & -\alpha\gamma_4(t) & \gamma_1(t) & -\alpha\gamma_2(t) \\ \gamma_4(t) & \gamma_3(t) & \gamma_2(t) & \gamma_1(t) \end{bmatrix}. \tag{5.10}$$

Now by using this matrix B, we can define the one parameter motion on M_2 at $\mathbb{R}^4_{\alpha\beta}$.

Definition 5.2. Let R_0 and R be the fixed space and the motional space at $\mathbb{R}^4_{\alpha\beta}$. In that case, the one-parameter motion of R_0 with respect to R is denoted by R_0/R. Then the one-parameter motion on M_2 is given by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{5.11}$$

where B is the matrix associated with the curve $\gamma(t)$ on the hypersurface M_2, C is the 4×1 real matrix depends on a real parameter t, X and X_0 are the position vectors of any point at $\mathbb{R}^4_{\alpha\beta}$ respectively in R and R_0.

Theorem 5.9. *The equation given by* (5.11) *determines a homothetic motion on* M_2.

Proof. Since the curve γ is on M_2, it does not pass through the origin. So, the matrix given by (5.10) can be expressed as:

$$B = M_{\gamma(t)} = h \begin{bmatrix} \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} & \frac{-\beta\gamma_3(t)}{h} & \frac{\alpha\beta\gamma_4(t)}{h} \\ \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\beta\gamma_4(t)}{h} & \frac{-\beta\gamma_3(t)}{h} \\ \frac{\gamma_3(t)}{h} & \frac{-\alpha\gamma_4(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} \\ \frac{\gamma_4(t)}{h} & \frac{\gamma_3(t)}{h} & \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} \end{bmatrix} = hA, \tag{5.12}$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$, $t \to h(t) = \|\gamma(t)\| = \sqrt{\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2}$. Since $\gamma(t) \in M_2$, $\gamma_1(t)\gamma_2(t) + \beta\gamma_3(t)\gamma_4(t) = 0$. By using this equality, we obtain that the matrix A in (5.12) is a real quasi-orthogonal matrix. In that case it satisfies $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, where ε is the signature matrix corresponding to metric g_2 given by

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\alpha\beta \end{bmatrix}.$$

Hence A, h and C are a real quasi-orthogonal matrix, the homothetic scale of the motion and the translation vector, respectively. So the equation (5.11) determines a homothetic motion. □

Remark 5.2. The norm of the curve $\gamma \in \mathbb{R}^4_{\alpha\beta}$ is found as
$\|\gamma(t)\| = \sqrt{|\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2|}$. We assume that $\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2 > 0$ in this paper.

Corollary 5.5. *Let $\gamma(t)$ be a curve on M_2^*. Then one-parameter motion on M_2 given by (5.11) is a general motion consists of a rotation and a translation.*

Proof. We assume that $\gamma(t)$ is a curve on M_2^*. Then $\gamma_1^2 - \alpha\gamma_2^2 + \beta\gamma_3^2 - \alpha\beta\gamma_4^2 = 1$. In that case the matrix B in (5.11) becomes a real-quasi orthogonal matrix, that is, it satisfies $B^T \varepsilon B = \varepsilon$ and $\det B = 1$. This completes the proof. □

Theorem 5.10. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_2. Then the derivative of the matrix B is a real quasi-orthogonal matrix.*

Proof. We suppose that $\gamma(t)$ be a unit velocity curve then $\dot{\gamma}_1^2 - \alpha\dot{\gamma}_2^2 + \beta\dot{\gamma}_3^2 - \alpha\beta\dot{\gamma}_4^2 = 1$. Also since the tangent vector of the curve γ is on M_2, we have $\dot{\gamma}_1(t)\dot{\gamma}_2(t) + \beta\dot{\gamma}_3(t)\dot{\gamma}_4(t) = 0$. Thus $\dot{B}^T\varepsilon\dot{B} = \varepsilon$ and $\det\dot{B} = 1$. □

Theorem 5.11. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot{\gamma}(t)$ be on M_2. Then the motion is a regular motion and it is independent of h.*

Proof. From Theorem 5.10, $\det\dot{B} = 1$ and thus the value of $\det\dot{B}$ is independent of h. □

Theorem 5.12. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_2. Then the pole points of the motion given by* (5.11) *are $X_0 = -\dot{B}^{-1}\dot{C}$.*

Proof. Since the position vector of the curve γ is on M_2, from Theorem 5.9, the equation (5.11) becomes a homothetic motion. Also, because of $\gamma(t)$ is a unit velocity curve and $\dot{\gamma}(t) \in M_2$, from Theorem 5.10 $\det\dot{B} = 1$ and it implies that there is only one solution of the equation (4.1). Then the pole points of the motion given by (5.11) are found as $X_0 = -\dot{B}^{-1}\dot{C}$. □

Corollary 5.6. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_2. The pole point associated with each t- instant in R_0 is the rotation by the matrix $\dot{B}^{-1}$ of the speed vector of translation vector at the opposite direction $\left(-\dot{C}\right)$.*

Proof. From Theorem 5.10, the matrix $\dot{B}$ is a real quasi-orthogonal matrix. Then the matrix $\dot{B}^{-1}$ is quasi-orthogonal matrix, too. This completes the proof. □

Now we will give various examples of the homothetic motions on M_2 according to the situations of real numbers α and β.

Example 5.4. For $\alpha = \beta = 1$, M_2 becomes a hypersurface in $\mathbb{R}_2^4$. Let $\gamma : I \subset \mathbb{R} \to M_2 \subset \mathbb{R}_2^4$ be a curve as:

$$\gamma(t) = h(t)\left(\begin{array}{c}\cosh(at)\cos(bt) - \sinh(at)\sin(bt)\,i \\ +\cosh(at)\sin(bt)\,j + \sinh(at)\cos(bt)\,ij\end{array}\right), \tag{5.13}$$

where a and b are real numbers. By using (5.10) and (5.13), the matrix B is a homothetic matrix and $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, the matrix B becomes a rotation matrix in $\mathbb{R}_2^4$.

Example 5.5. For $\alpha = -1$, $\beta = 1$, M_2 is a hypersurface in $\mathbb{R}^4$. Let us consider the curve $\gamma : I \subset \mathbb{R} \to M_2 \subset \mathbb{R}^4$ as follows:

$$\gamma(t) = h(t)\left(\begin{array}{c}\cos(at)\cos(bt) - \sin(at)\sin(bt)\,i \\ +\cos(at)\sin(bt)\,j + \sin(at)\cos(bt)\,ij\end{array}\right), \tag{5.14}$$

where a and b are real numbers. From (5.10) and (5.14) we obtain the matrix representation of the curve γ and it determines a homothetic matrix, in here $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, γ becomes a spherical curve on M_2, that is, $\gamma(t) \in M_2^* = M_2 \cap S^3$. From Corollary 5.5, the matrix representation of it is a rotation matrix in $\mathbb{R}^4$.

Example 5.6. For $\alpha = \beta = -1$, the hypersurface M_2 becomes a subset of $\mathbb{R}_2^4$ and the following curve lies on M_2

$$\gamma(t) = h(t) \begin{pmatrix} \cosh(at)\cosh(bt) + \sinh(at)\sinh(bt)\, i \\ + \cosh(at)\sinh(bt)\, j + \sinh(at)\cosh(bt)\, ij \end{pmatrix}, \tag{5.15}$$

in which a and b are real numbers. From (5.10) and (5.15), the matrix B according to the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ lies on M_2^* and the matrix B gives a rotation matrix in $\mathbb{R}_2^4$.

5.3. Homothetic Motion on Hypersurface M_3

Let us consider the hypersurface M_3 as follows:

$$M_3 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\alpha\beta} : x_1x_4 - x_2x_3 = 0,\ x \neq 0 \right\}.$$

By using generalized bicomplex numbers, M_3 can be rewritten as:

$$M_3 = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_{\alpha\beta} : x_1x_4 - x_2x_3 = 0,\ x \neq 0 \right\},$$

or the hypersurface M_3 can be expressed by using the matrix representiation of generalized bicomplex numbers

$$\tilde{M_3} = \left\{ M_x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} : x_1x_4 - x_2x_3 = 0,\ x \neq 0 \right\}.$$

where M_x is the matrix representiation of generalized bicomplex number x on M_3. The metric on hypersurface M_3 is defined by $g_3(x, x) = x \cdot x^{t_3} = x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2$ and the norm of any element x on M_3 is given by $\|x\| = \sqrt{|g_3(x, x)|} = \sqrt{|x \cdot x^{t_3}|}$. This metric is Riemannian or pseudo-Riemannian metric on four dimensional generalized linear space $\mathbb{R}^4_{\alpha\beta}$ and for some special cases, it coincides four dimensional Euclidean space $\mathbb{R}^4$ or four dimensional pseudo-Euclidean space $\mathbb{R}_2^4$.

Proposition 5.3. *There are following properties about the norms on the hypersurface M_3.*

i) For $x, y \in M_3$, $\|x \cdot y\| = \|x\| \, \|y\|$

ii) $\|x\|^4 = \det(M_x)$

Proof. These properties can be easily seen with direct calculations. □

Corollary 5.7. *A unit generalized bicomplex number on the hypersurface M_3 determines a rotation motion.*

Proof. It is obvious from Proposition 5.3. □

Theorem 5.13. *M_3 is a commutative Lie group.*

Proof. The proof can be found in [10]. □

Let us denote the set of unit generalized bicomplex number on M_3 by M_3^*. M_3^* is given as:

$$\begin{aligned} M_3^* &= \{x \in M_3 : g_3(x,x) = 1\} \\ &= \left\{x \in M_3 : x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2 = 1\right\}. \end{aligned}$$

Theorem 5.14. *M_3^* is Lie subgroup of M_3.*

Proof. The proof can be found in [10]. □

Let γ be a curve on M_3. In that case, it can be expressed as

$$\begin{aligned} \gamma \quad &: \quad I \subset \mathbb{R} \to M_3 \\ t \quad &\to \quad \gamma(t) = \gamma_1(t) + \gamma_2(t)i + \gamma_3(t)j + \gamma_4(t)ij,\ \gamma_1(t)\gamma_4(t) - \gamma_2(t)\gamma_3(t) = 0. \end{aligned}$$

Then the matrix B corresponding to the curve γ is given as follows:

$$B = M_{\gamma(t)} = \begin{bmatrix} \gamma_1(t) & -\alpha\gamma_2(t) & -\beta\gamma_3(t) & \alpha\beta\gamma_4(t) \\ \gamma_2(t) & \gamma_1(t) & -\beta\gamma_4(t) & -\beta\gamma_3(t) \\ \gamma_3(t) & -\alpha\gamma_4(t) & \gamma_1(t) & -\alpha\gamma_2(t) \\ \gamma_4(t) & \gamma_3(t) & \gamma_2(t) & \gamma_1(t) \end{bmatrix}. \tag{5.16}$$

Now by using this matrix B, we can define the one parameter motion on M_3 at $\mathbb{R}^4_{\alpha\beta}$.

Definition 5.3. Let R_0 and R be the fixed space and the motional space at $\mathbb{R}^4_{\alpha\beta}$. In that case, the one-parameter motion of R_0 with respect to R is denoted by R_0/R. Then the one-parameter motion on M_3 is given by

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix},$$

or it can be expressed as

$$X = BX_0 + C, \tag{5.17}$$

where B is the matrix associated with the curve $\gamma(t)$ on the hypersurface M_3, C is the 4×1 real matrix depends on a real parameter t, X and X_0 are the position vectors of any point at $\mathbb{R}^4_{\alpha\beta}$ respectively in R and R_0, respectively.

Theorem 5.15. *The equation given by* (5.17) *is a homothetic motion on M_3.*

Proof. Since the curve γ is on M_3, it does not pass through the origin. So, the matrix given by (5.16) can be expressed as:

$$(5.18)\qquad B = M_{\gamma(t)} = h \begin{bmatrix} \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} & \frac{-\beta\gamma_3(t)}{h} & \frac{\alpha\beta\gamma_4(t)}{h} \\ \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\beta\gamma_4(t)}{h} & \frac{-\beta\gamma_3(t)}{h} \\ \frac{\gamma_3(t)}{h} & \frac{-\alpha\gamma_4(t)}{h} & \frac{\gamma_1(t)}{h} & \frac{-\alpha\gamma_2(t)}{h} \\ \frac{\gamma_4(t)}{h} & \frac{\gamma_3(t)}{h} & \frac{\gamma_2(t)}{h} & \frac{\gamma_1(t)}{h} \end{bmatrix} = hA,$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$, $t \to h(t) = \|\gamma(t)\| = \sqrt{\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2}$. Because of $\gamma(t) \in M_3$, we have $\gamma_1(t)\gamma_4(t) - \gamma_2(t)\gamma_3(t) = 0$. By using this equality, we obtain that the matrix A in (5.18) is a real quasi-orthogonal matrix. In that case it satisfies $A^T\varepsilon A = \varepsilon$ and $\det A = 1$, where ε is the signature matrix corresponding to metric g_3 given by

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}.$$

Hence A, h and C are a real quasi-orthogonal matrix, the homothetic scale of the motion and the translation vector, respectively. So the equation (5.17) determines a homothetic motion. □

Remark 5.3. The norm of the curve $\gamma \in \mathbb{R}^4_{\alpha\beta}$ is found as $\|\gamma(t)\| = \sqrt{|\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2|}$. We assume that $\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2 > 0$ in this paper.

Corollary 5.8. *Let $\gamma(t)$ be a curve on M_3^*. Then one-parameter motion on M_3 given by* (5.17) *is a general motion consists of a rotation and a translation.*

Proof. We assume that $\gamma(t)$ is a curve on M_3^*. Then $\gamma_1^2 + \alpha\gamma_2^2 + \beta\gamma_3^2 + \alpha\beta\gamma_4^2 = 1$. In that case the matrix B given by (5.16) becomes a real-quasi orthogonal matrix, that is, it satisfies $B^T\varepsilon B = \varepsilon$ and $\det B = 1$. This completes the proof. □

Theorem 5.16. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot\gamma(t)$ be on M_3. Then the derivative of the matrix B is a real quasi-orthogonal matrix.*

Proof. We suppose that $\gamma(t)$ be a unit velocity curve. Then $\dot\gamma_1^2+\alpha\dot\gamma_2^2+\beta\dot\gamma_3^2+\alpha\beta\dot\gamma_4^2 = 1$. Also since the tangent vector of the curve γ is on M_3, we have $\dot\gamma_1(t)\dot\gamma_4(t) - \dot\gamma_2(t)\dot\gamma_3(t) = 0$. Thus $\dot B^T\varepsilon\dot B = \varepsilon$ and $\det\dot B = 1$. □

Theorem 5.17. *Let $\gamma(t)$ be a unit velocity curve and its tangent vector $\dot\gamma(t)$ be on M_3. Then the motion is a regular motion and it is independent of h.*

Proof. From Theorem 5.16, $\det\dot B = 1$ and thus the value of $\det\dot B$ is independent of h. □

Theorem 5.18. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_3. Then the pole points of the motion given by* (5.17) *are* $X_0 = -\dot{B}^{-1}\dot{C}$.

Proof. Since the position vector of the curve γ is on M_3, from Theorem 5.15, the equation (5.17) is a homothetic motion. Also, because of $\gamma(t)$ is a unit velocity curve and $\dot{\gamma}(t) \in M_3$, from Theorem 5.16 $\det \dot{B} = 1$. Thus the equation (4.1) has only one solution. In that case the pole points of the motion are obtained as $X_0 = -\dot{B}^{-1}\dot{C}$. □

Corollary 5.9. *Let $\gamma(t)$ be a unit velocity curve whose the position vector and tangent vector are on M_3. The pole point associated with each t- instant in R_0 is the rotation by the matrix $\dot{B}^{-1}$ of the speed vector of translation vector at the opposite direction* $\left(-\dot{C}\right)$.

Proof. From Theorem 5.16, the matrix $\dot{B}$ is a real quasi-orthogonal matrix. Then the matrix $\dot{B}^{-1}$ is quasi-orthogonal matrix, too. This completes the proof. □

Now we will give various examples of the homothetic motions on M_3 according to the situations of real numbers α and β.

Example 5.7. For $\alpha = \beta = 1$, M_3 becomes a hypersurface in four dimensional Euclidean space $\mathbb{R}^4$. Let $\gamma : I \subset \mathbb{R} \to M_3 \subset \mathbb{R}^4$ be a curve as:

$$\gamma(t) = h(t) \begin{pmatrix} \cos(at)\cos(bt) + \cos(at)\sin(bt)\, i \\ +\sin(at)\cos(bt)\, j + \sin(at)\sin(bt)\, ij \end{pmatrix}, \tag{5.19}$$

where a and b are real numbers. By using (5.16) and (5.19), the matrix B associated with the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ is a curve on M_3^*. In that case it becomes a spherical curve lies on M_3 and the matrix B becomes a rotation matrix in $\mathbb{R}^4$.

Example 5.8. For $\alpha = 1$, $\beta = -1$, M_3 is a hypersurface in four dimensional Euclidean space $\mathbb{R}^4_2$. Let $\gamma : I \subset \mathbb{R} \to M_3 \subset \mathbb{R}^4_2$ be a curve given by

$$\gamma(t) = h(t) \begin{pmatrix} \cosh(at)\cos(bt) + \cosh(at)\sin(bt)\, i \\ +\sinh(at)\cos(bt)\, j + \sinh(at)\sin(bt)\, ij \end{pmatrix}, \tag{5.20}$$

where a and b are real numbers. By using (5.16) and (5.20), the matrix representation of the curve γ is a homothetic matrix, in here $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, for $h(t) = 1$, γ becomes a spherical curve on M_3, that is, $\gamma(t) \in M_3^*$. The matrix representation of it is a rotation matrix in $\mathbb{R}^4_2$.

Example 5.9. For $\alpha = \beta = -1$, the hypersurface M_3 is in four dimensional pseudo-Euclidean space $\mathbb{R}^4_2$ and the following curve lies on M_3

$$\gamma(t) = h(t) \begin{pmatrix} \cosh(at)\cosh(bt) + \cosh(at)\sinh(bt)\, i \\ +\sinh(at)\cosh(bt)\, j + \sinh(at)\sinh(bt)\, ij \end{pmatrix}, \tag{5.21}$$

in which a and b are real numbers. From (5.16) and (5.21), the matrix B according to the curve γ is a homothetic matrix, where $h : I \subset \mathbb{R} \to \mathbb{R}$ is a homothetic scale. Also, if we take as $h(t) = 1$, then γ lies on M_3^* and the matrix B gives a rotation matrix in $\mathbb{R}^4_2$.

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ON THREE-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In the present paper, we study three-dimensional trans-Sasakian manifolds admitting the Schouten-van Kampen connection. Also, we have proved some results on ϕ-projectively flat, ξ-projectively flat and ξ-concircularly flat three-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection. Locally ϕ-symmetric trans-Sasakian manifolds of dimension three have been studied with respect to Schouten-van Kampen connection. Finally, we construct an example of a three-dimensional trans-Sasakian manifold admitting Schouten-van Kampen connection which verifies Theorem 4.1. and Theorem 5.2.
Key words: General geometric structures on manifolds, Schouten-van Kampen connection, Special Riemannian manifolds

1. Introduction

The Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection. Solov'ev investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection ([18], [19], [20], [21]). In 2014, Olszak studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure [17]. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection. Recently, G. Ghosh [10], Yildiz [26], Nagaraja [15] and D. L. Kiran Kumar [12] have studied the

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Schouten-van Kampen connection in Sasakian manifolds, f-Kenmotsu manifolds and Kenmotsu manifolds respectively.

A transformation of an n-dimensional differentiable manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [27], [13]. A concircular transformation is always a conformal transformation [13]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor $\mathbb{W}$ with respect to Levi-Civita connection. It is defined by [27], [28]

$$\mathbb{W}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where $X, Y, Z \in \chi(M), R$ and r are the curvature tensor and the scalar curvature with respect to the Levi-Civita connection.
The concircular curvature tensor $\tilde{\mathbb{W}}$ with respect to the Schouten-van Kampen connection is defined by

$$\tilde{\mathbb{W}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.2}$$

where $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and the scalar curvature with respect to the Schouten-van Kampen connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In 1985, a new class of n-dimensional almost contact manifold namely trans-Sasakian manifold was introduced by J. A. Oubina [16] and further study about the local structures of trans-Sasakian manifolds was carried by J. C. Marrero [14]. Trans-Sasakian manifolds of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are, called the cosymplectic, α-Sasakian and β-Kenmotsu respectively ([2], [11]). In particular, if $\alpha = 0, \beta = 1; \alpha = 1, \beta = 0$; then a trans-Sasakian manifold becomes Kenmotsu and Sasakian manifolds respectively. Hence, trans-Sasakian structures give a large class of generalized Quasi-Sasakian structures. It has been proven that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha-$Sasakian and $\beta-$Kenmotsu manifold. Three-dimnesional trans-Sasakian manifolds with different restrictions on curvature and smooth functions α, β are studied in ([7], [8], [5], [6]).

In the present paper, we have studied three-dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection.

The present paper is organized as follows: After the introduction in Section 1, we give some required preliminaries in Section 2. Section 3 is devoted to the study of the curvature tensor, the Ricci tensor, scalar curvature of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. Section 4 is devoted to the study of ξ-projectively and ϕ-projectively flat trans-Sasakian manifolds of dimension three with respect to the Schouten-van Kampen connection. In this section, we have proved that a three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection is ξ-projectively flat if and only if the scalar curvature of the manifold vanishes. In Section 5, we study ξ-concircularly flat trans-Sasakian manifold of dimension three admitting Schouten-van Kampen connection. In the next section, we study locally ϕ-symmetric trans-Sasakian manifolds of dimensional three with respect to Schouten-van Kampen connection. In Section 7, we study Weyl ξ-conformally flat in three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection. In the last section, we construct an example of a three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection to support the results obtained in Section 4 and Section 5.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in T(M)$ [1]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$.

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called trans-Sasakian structure [16] if $(M \times R, J, G)$ belongs to the class W_4 [9], where J is the almost complex structure on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \eta(X) d/dt),$$

for all vector fields X on M, a smooth function f on $M \times R$ and the product metric G on $M \times R$. This may be expressed by the condition [3]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for smooth functions α and β on M. Here ∇ is Levi-Civita connection on M. We say M as the trans-Sasakian manifold of type (α, β). From (2.5) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X\eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

In a three-dimensional trans-Sasakian manifold following relations hold [7], [8]:

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$\begin{aligned} S(X,Y) &= \{\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\}g(X,Y) \\ &\quad -\{\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y) - \{Y\beta + (\phi X)\alpha\}\eta(Y), \end{aligned} \tag{2.9}$$

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) \\ &\quad -g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ &\quad -\eta(X)(\phi\mathrm{grad}\alpha - \mathrm{grad}\beta) + (X\beta + (\phi X)\alpha)\xi] \\ &\quad +g(X,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ &\quad -\eta(Y)(\phi\mathrm{grad}\alpha - \mathrm{grad}\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\ &\quad -[(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ &\quad +[(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ &\quad +(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.10}$$

where S is the Ricci tensor of type $(0,2)$, and r is the scalar curvature of the manifold M with respect to Levi-Civita connection.

From here after we consider α and β are constants, then the above relations become

$$\begin{aligned} R(X,Y)Z &= \{\frac{r}{2} - (\alpha^2 - \beta^2)\}[g(Y,Z)X - g(X,Z)Y] \\ &\quad +\{\frac{r}{2} - (\alpha^2 - \beta^2)\}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ &\quad +\{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X], \end{aligned} \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= \{\frac{r}{2} - (\alpha^2 - \beta^2)\}g(X,Y) \\ &\quad -\{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y), \end{aligned} \tag{2.12}$$

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.13}$$

$$QX = \{\frac{r}{2} - (\alpha^2 - \beta^2)\}X - \{\frac{r}{2} - 3(\alpha^2 - \beta^2)\}\eta(X)\xi, \tag{2.14}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{2.15}$$

$$R(\xi, X)Y = 2(\alpha^2 - \beta^2)(g(X,Y)\xi - \eta(Y)X). \tag{2.16}$$

From (2.8) it follows that if α and β are constants, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic.

3. Curvature tensor of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection

For an almost contact metric manifold M, the Schouten-van Kampen connection $\tilde{\nabla}$ is given by [17]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{3.1}$$

Let M be a three-dimensional trans-Sasakian manifold. Then from above equation we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)\phi X) - g(\phi X, Y)\xi\} + \beta\{g(X,Y)\xi - \eta(Y)X\}. \tag{3.2}$$

We define the curvature tensor $\tilde{R}$ of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{3.3}$$

In view of (3.2) and (3.3) we obtain

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha^2\{g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi\} \\ &\quad + \beta^2\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.4}$$

Taking inner product in both sides of (3.4) with W, we have

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \alpha^2\{g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \\ &\quad + g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z) \\ &\quad - g(Y,Z)\eta(X)\eta(W) + g(X,Z)\eta(Y)\eta(W)\} \\ &\quad + \beta^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned} \tag{3.5}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z, W)$.

Taking a frame field from (3.5), we get

$$\tilde{S}(Y,Z) = S(Y,Z) + 2\beta^2 g(Y,Z) - 2\alpha^2\eta(Y)\eta(Z). \tag{3.6}$$

From above equation we have

$$\tilde{Q}Y = QY + +2\beta^2 Y - 2\alpha^2\eta(Y)\xi. \tag{3.7}$$

Again putting $Y = Z = e_i$ $(i = 1, 2, 3)$ and taking summation over i in (3.6), we obtain

$$\tilde{r} = r - 2\alpha^2 + 6\beta^2, \tag{3.8}$$

where $\tilde{r}$ and r are the scalar curvatures of the Schouten-van Kampen connection $(\tilde{\nabla})$ and Levi-Civita connection (∇) respectively.

Hence we have the following :

Proposition 3.1. A three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection following statements are equivalent
(a) The curvature tensor $\tilde{R}$ is given by (3.4),
(b) The Ricci tensor $\tilde{S}$ is given by (3.6),
(c) $\tilde{r} = r - 2\alpha^2 + 6\beta^2$,
(d) The Ricci tensor $\tilde{S}$ is symmetric,
provided α and β are constants.

4. ξ-Projectively and ϕ-projectively flat trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

In this section, we study projectively flat three-dimensional trans-Sasakian manifold M with respect to the Schouten-van Kampen connection. In a three-dimensional trans-Sasakian manifold, the projective curvature tensor with respect to the Schouten-van Kampen connection is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \tag{4.1}$$

Definition 4.1. A three-dimensional trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ξ-projectively flat if

$$\tilde{P}(X,Y)\xi = 0,$$

for all vector fields X, Y on M. This notion was first defined by Tripathi and Dwivedi [22]. If $\tilde{P}(X,Y)\xi = 0$, just holds for X, Y orthogonal to ξ, we call such a manifold a horizontal ξ-projectively flat manifold.

Using (3.4) in (4.1) we have

$$\begin{aligned}\tilde{P}(X,Y)Z &= R(X,Y)Z+\alpha^2\{g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\\ &\quad +\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\\ &\quad -g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi\}\\ &\quad +\beta^2\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad -\frac{1}{2}\{\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y\}. \end{aligned} \tag{4.2}$$

Putting $Z=\xi$ and using (2.1), (2.3), (2.15) and (3.6) in (4.2), we get

$$\tilde{P}(X,Y)\xi=0. \tag{4.3}$$

Thus we can state the following:

Theorem 4.1. A three-dimensional trans-Sasakian manifold is ξ-projectively flat with respect to the Schouten-van Kampen connection provided α and β are constants.

Again putting (3.6) in (4.2) we get

$$\begin{aligned}\tilde{P}(X,Y)Z &= P(X,Y)Z+\alpha^2\{g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\\ &\quad -g(Y,Z)\eta(X)\xi+g(X,Z)\eta(Y)\xi\}. \end{aligned} \tag{4.4}$$

Putting $Z=\xi$ in (4.4) and using (2.1) and (2.3), it follows that

$$\tilde{P}(X,Y)\xi=P(X,Y)\xi. \tag{4.5}$$

In view of above discussion we state the following theorem:

Theorem 4.2. A three-dimensional trans-Sasakian manifold is ξ-projectively flat with respect to the Schouten-van Kampen connection if and only if the manifold is ξ-projectively flat with respect to the Levi-Civita connection provided α and β are constants.

Definition 4.2. A trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ϕ-projectively flat if

$$\phi^2\tilde{P}(\phi X,\phi Y)\phi Z=0.$$

It can be easily seen that $\phi^2\tilde{P}(\phi X,\phi Y)\phi Z=0$ holds if and only if

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W)=0, \tag{4.6}$$

for $X,Y,Z,W\in T(M)$.

Using (4.1) and (4.6), ϕ-projectively flat means

$$\begin{aligned} g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{2}\{\tilde{S}(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-\tilde{S}(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{4.7}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (4.7) and summing up with respect to i, we have

$$\begin{aligned} \sum_{i=1}^{2} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \frac{1}{2}\sum_{i=1}^{2}\{\tilde{S}(\phi Y,\phi Z)g(\phi e_i,\phi e_i) \\ &-\tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\}. \end{aligned} \tag{4.8}$$

Using (2.1), (2.2), (2.3) and (3.5) it can be easily verified that

$$\begin{aligned} \sum_{i=1}^{2} g(\tilde{R}(\phi e_i,\phi Y)\phi Z,\phi e_i) &= \sum_{i=1}^{2} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) \\ &+(\alpha^2+\beta^2)g(Y,Z)+(\beta^2-3\alpha^2)\eta(Y)\eta(Z) \\ &= S(\phi Y,\phi Z)+(\alpha^2+\beta^2)g(Y,Z) \qquad (4.9)\\ &+(\beta^2-3\alpha^2)\eta(Y)\eta(Z). \qquad (4.10) \end{aligned}$$

$$\sum_{i=1}^{2} g(\phi e_i,\phi e_i) = 2. \tag{4.11}$$

$$\sum_{i=1}^{2} \tilde{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i) = \tilde{S}(\phi Y,\phi Z). \tag{4.12}$$

Using (4.9), (4.10) and (4.11), the equation (4.8) becomes

$$\tilde{S}(\phi Y,\phi Z) = 2\{S(\phi Y,\phi Z)+(\alpha^2+\beta^2)g(Y,Z)+(\beta^2-3\alpha^2)\eta(Y)\eta(Z)\}. \tag{4.13}$$

Using (3.6) in (4.12), we get

$$S(\phi Y,\phi Z) = -2\alpha^2 g(Y,Z)+2(3\alpha^2-\beta^2)\eta(Y)\eta(Z). \tag{4.14}$$

Putting $Y = \phi Y$ and $Z = \phi Z$ in (4.13) and using (2.1) (2.2) and (2.13), we obtain

$$S(Y,Z) = -2\alpha^2 g(Y,Z)+2(2\alpha^2-\beta^2)\eta(Y)\eta(Z). \tag{4.15}$$

Conversely, let S be of the form (4.14), then obviously

$$g(\tilde{P}(\phi X,\phi Y)\phi Z,\phi W) = 0.$$

Thus we can state the following:

Theorem 4.3. A three-dimensional trans-Sasakian manifold admitting the Schouten-van Kampen connection is ϕ-projectively flat if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection provided α, β are constants with $\beta \neq \pm\sqrt{2}\alpha, (\alpha \neq 0)$.

5. ξ-Concircularly flat trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

Definition 5.1. A trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is said to be ξ-concircularly flat if

$$\tilde{\mathbb{W}}(X,Y)\xi = 0, \tag{5.1}$$

for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Theorem 5.1. A three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection is horizontally ξ-concircularly flat if and only if the manifold with respect to the Levi-Civita connection is also ξ-concircular flat provided α, β are constants.

Proof. Combining (1.1),(1.2) and using (3.4), (3.6) (3.8), we get

$$\begin{aligned}\tilde{\mathbb{W}}(X,Y)Z \;&= \mathbb{W}(X,Y)Z + \alpha^2\{g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &\qquad -g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ &\qquad -\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\}.\end{aligned} \tag{5.2}$$

Putting $Z = \xi$ in (5.2) we get

$$\tilde{\mathbb{W}}(X,Y)\xi = \mathbb{W}(X,Y)\xi + \frac{2\alpha^2}{3}\{\eta(X)Y - \eta(Y)X\}. \tag{5.3}$$

From (5.3), implies that

$$\tilde{\mathbb{W}}(X,Y)\xi = \mathbb{W}(X,Y)\xi; \quad \text{for all } X, Y \text{orthogonal to } \xi. \tag{5.4}$$

Hence the proof of theorem is complete.

Theorem 5.2. A three-dimensional trans-Sasakian manifold is ξ-concircularly flat with respect to the Schouten-van Kampen connection if and only if the scalar curvature $\tilde{r}$ is zero, provided α and β are constants.

Proof. Putting $Z = \xi$ in (1.2) and using (2.1), (2.3), (2.3), (2.15) and (3.4), we have

$$\tilde{\mathbb{W}}(X,Y)\xi = -\frac{\tilde{r}}{6}\{\eta(Y)X - \eta(X)Y\}. \tag{5.5}$$

Thus the theorem is proved.

6. Locally ϕ-symmetric trans-Sasakian manifolds with respect to the Schouten-van Kampen connection

Definition 6.1. A trans-Sasakian manifold M with respect to the Schouten-van Kampen connection is called to be locally ϕ-symmetric if

$$\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z = 0, \tag{6.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ on M. This notion was introduced by Takahashi [24], for Sasakian manifolds.

We know that

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= \tilde{\nabla}_W(\tilde{R}(X,Y)Z) - \tilde{R}(\tilde{\nabla}_W X,Y)Z \\ &\quad -R(X,\tilde{\nabla}_W Y)Z - \tilde{R}(X,Y)\tilde{\nabla}_W Z.\end{aligned} \tag{6.2}$$

By virtue of (3.1), above equation is reduced to

$$\begin{aligned}(\tilde{\nabla}_W\tilde{R})(X,Y)Z &= (\nabla_W\tilde{R})(X,Y)Z + \eta(X)\tilde{R}(\nabla_W\xi,Y)Z + (\nabla_W\eta)(X)\tilde{R}(\xi,Y)Z \\ &\quad +\eta(Y)\tilde{R}(X,\nabla_W\xi)Z + (\nabla_W\eta)(Y)\tilde{R}(X,\xi)Z \\ &\quad +\eta(Z)\tilde{R}(X,Y)\nabla_W Z + (\nabla_W\eta)(Z)\tilde{R}(X,Y)\xi.\end{aligned} \tag{6.3}$$

Now differentiating (3.4) with respect to W, using (2.1), (2.2), (2.3), (2.5) and (2.7) we obtain

$$\begin{aligned}(\nabla_W\tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z \\ &\quad +\alpha^3[\{g(X,Y)g(\phi Y,Z) - g(W,Y)g(\phi X,Z)\}\xi \\ &\quad +\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}W] \\ &\quad +\alpha^2\beta[\{g(\phi W,X)g(\phi Y,Z) - g(\phi W,Y)g(\phi X,Z)\}\xi \\ &\quad +\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}\phi W] \\ &\quad +(\alpha^2-\beta^2)[\{\alpha(g(\phi W,Y)X - g(\phi W,X)Y) \\ &\quad -\beta^2(g(\phi W,\phi Y)X + g(\phi W,\phi X)Y)\}\eta(Z) \\ &\quad +(\beta g(\phi W,\phi Z) - \alpha g(\phi W,Z))(\eta(X)Y - \eta(Y)X)] \\ &\quad +\alpha^2(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))(-\alpha\phi W + \beta(W - \eta(W)\xi)) \\ &\quad -\alpha^2[-\alpha(g(Y,Z)g(\phi W,X) + g(X,Z)g(\phi W,Y)) \\ &\quad +\beta(g(Y,Z)g(\phi W,\phi X) + g(X,Z)g(\phi W,\phi Y))]\xi \\ &\quad +\beta^2[\{-\alpha(g(W,\phi Z)\eta(Y) + g(W,\phi Y)\eta(Z)) \\ &\quad -\beta(g(\phi W,\phi Z)\eta(Y) + g(\phi W,\phi Y)\eta(Z))\}X \\ &\quad +\{\alpha(g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)) \\ &\quad -\beta(g(\phi W,\phi Z)\eta(X) + g(\phi W,\phi X)\eta(Z))\}Y].\end{aligned} \tag{6.4}$$

Using (6.4) in (6.3) we have

$$
\begin{aligned}
(\tilde{\nabla}_W \tilde{R})(X,Y)Z \;=\;& (\nabla_W R)(X,Y)Z \\
&+\alpha^3[\{g(X,Y)g(\phi Y,Z) - g(W,Y)g(\phi X,Z)\}\xi \\
&+\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}W] \\
&+\alpha^2\beta[\{g(\phi W,X)g(\phi Y,Z) - g(\phi W,Y)g(\phi X,Z)\}\xi \\
&+\{g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\}\phi W] \\
&+(\alpha^2-\beta^2)[\{\alpha(g(\phi W,Y)X - g(\phi W,X)Y) \\
&-\beta^2(g(\phi W,\phi Y)X + g(\phi W,\phi X)Y)\}\eta(Z) \\
&+(\beta g(\phi W,\phi Z) - \alpha g(\phi W,Z))(\eta(X)Y - \eta(Y)X)] \\
&+\alpha^2(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))(-\alpha\phi W + \beta(W-\eta(W)\xi)) \\
&-\alpha^2[-\alpha(g(Y,Z)g(\phi W,X) + g(X,Z)g(\phi W,Y)) \\
&+\beta(g(Y,Z)g(\phi W,\phi X) + g(X,Z)g(\phi W,\phi Y))]\xi \\
&+\beta^2[\{-\alpha(g(W,\phi Z)\eta(Y) + g(W,\phi Y)\eta(Z)) \\
&-\beta(g(\phi W,\phi Z)\eta(Y) + g(\phi W,\phi Y)\eta(Z))\}X \\
&+\{\alpha(g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)) \\
&-\beta(g(\phi W,\phi Z)\eta(X) + g(\phi W,\phi X)\eta(Z))\}Y] \\
&+\eta(X)\tilde{R}(\nabla_W\xi,Y)Z + (\nabla_W\eta)(X)\tilde{R}(\xi,Y)Z \\
&+\eta(Y)\tilde{R}(X,\nabla_W\xi)Z + (\nabla_W\eta)(Y)\tilde{R}(X,\xi)Z \\
&+\eta(Z)\tilde{R}(X,Y)\nabla_W Z + (\nabla_W\eta)(Z)\tilde{R}(X,Y)\xi.
\end{aligned}
\tag{6.5}
$$

Now applying ϕ^2 on both sides of (6.5) and taking X, Y, Z, W are orthogonal to ξ and using (2.1), (2.3) we get from above equation

$$
\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z. \tag{6.6}
$$

Hence we can state the following:

Theorem 6.1. A three-dimensional trans-Sasakian manifold is locally ϕ-symmetry with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ if and only if the manifold is also locally ϕ-symmetry with respect to the Levi-Civita connection ∇ provided α, β are constants.

U. C. De and Avijit Sarkar [7] have proved that a trans-Sasakian manifold is locally ϕ-symmetry if and only if the scalar curvature is constant provided α, β are constants.

In view of above result we can state the following:

Theorem 6.2. A three-dimensional trans-Sasakian manifold is locally ϕ-symmetric with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ if and only if the scalar curvature is constant, provided α, β are constants.

7. Weyl conformally flat trans-Sasakian manifold with respect to Schouten-van Kampen connection

The Weyl conformal curvature tensor $\tilde{C}$ of type (1,3) of M, an $n-$dimensional trans-Sasakian manifolds with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ is given by [23]

$$\begin{aligned} \tilde{C}(X,Y)Z &= \tilde{R}(X,Y)Z - \frac{1}{n-2}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X \\ &\quad -g(X,Z)\tilde{Q}Y] + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{7.1}$$

where $\tilde{Q}$ is the Ricci operator with respect to the Schouten-van Kampen connection.

Let us consider that a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection is Weyl conformally flat, that is $\tilde{C} = 0$. Then from (7.1), we get

$$\begin{aligned} \tilde{R}(X,Y)Z &= [\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X \\ &\quad -g(X,Z)\tilde{Q}Y] - \frac{\tilde{r}}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{7.2}$$

Let us take inner product of the equation (7.2) with W. Then we get

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) &= [\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W) + g(Y,Z)g(\tilde{Q}X,W) \\ &\quad -g(X,Z)g(\tilde{Q}Y,W)] - \frac{\tilde{r}}{2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{7.3}$$

Using (2.1), (2.3), (3.5)-(3.8), we get

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) &= [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)g(QX,W) \\ &\quad -g(X,Z)g(QY,W)] - \frac{r-2\alpha^2}{2}[g(Y,Z)g(X,W) \qquad (7.4) \\ &\quad -g(X,Z)g(Y,W)] \\ &\quad -\alpha^2[g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W) \\ &\quad -g(Y,W)\eta(X)\eta(Z) + g(X,W)\eta(Y)\eta(Z) \\ &\quad +g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)]. \qquad (7.5) \end{aligned}$$

Putting $X = W = \xi$ in (7.4) and using (2.1) and (2.3), we get

$$\begin{aligned} g(\tilde{R}(\xi,Y)Z,\xi) &= [S(Y,Z) - S(\xi,Z)\eta(Y) + g(Y,Z)S(\xi,\xi) \\ &\quad -\eta(Z)S(Y,\xi)] - \frac{r}{2}[g(Y,Z) - \eta(Z)\eta(Y)], \end{aligned} \tag{7.6}$$

where $g(QY, Z) = S(Y, Z)$.

Now, using (2.13) and (2.16), we get

$$S(Y,Z) = \frac{r}{2}g(Y,Z) + [6(\alpha^2 - \beta^2) - \frac{r}{2}]\eta(Y)\eta(Z). \tag{7.7}$$

Therefore

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = \frac{r}{2}$ and $b = [6(\alpha^2 - \beta^2) - \frac{r}{2}]$.

This shows that the manifold M is an η-Einstein manifold.

Thus we can state the following:

Theorem 7.1. A three-dimensional Weyl conformally flat trans-Sasakian manifold with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ is an η-Einstein manifold provided α, β are constants with $\alpha \neq \beta$.

8. Example of a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen Connection

In this section, we wanted to construct an example of a three-dimensional trans-Sasakian manifold with respect to Schouten-van Kampen connection.

We have considered the three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \qquad e_2 = e^{-z}(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \qquad e_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \qquad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \qquad \phi^2 Z = -Z + \eta(Z)e_3, \qquad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_3] = e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{8.1}$$

By Koszul formula

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From above we see that the manifold satisfies (2.6) for $\alpha = 0$, $\beta = 1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0,1)$. With the help of the above results it can be verified that

$$\begin{array}{lll} R(e_1,e_2)e_3 = 0, & R(e_2,e_3)e_3 = -e_2, & R(e_1,e_3)e_3 = -e_1, \\ R(e_1,e_2)e_2 = -e_1, & R(e_2,e_3)e_2 = e_3, & R(e_1,e_3)e_2 = 0, \\ R(e_1,e_2)e_1 = e_2, & R(e_2,e_3)e_1 = 0, & R(e_1,e_3)e_1 = e_3. \end{array}$$

Now we consider the Schouten-Van Kampen connection to this example.

Using (3.2) and above result we have

$$\begin{array}{lll} \tilde{\nabla}_{e_1} e_3 = (1-\beta)e_1 + \alpha e_2, & \tilde{\nabla}_{e_1} e_2 = -\alpha e_3, & \tilde{\nabla}_{e_1} e_1 = (\beta - 1)e_3, \\ \tilde{\nabla}_{e_2} e_3 = -\alpha e_1 + (1-\beta)e_2 & \tilde{\nabla}_{e_2} e_2 = (\beta - 1)e_3, & \tilde{\nabla}_{e_2} e_1 = 0, \\ \tilde{\nabla}_{e_3} e_3 = 0 & \tilde{\nabla}_{e_3} e_2 = -\beta e_2, & \tilde{\nabla}_{e_3} e_1 = -\beta e_1. \end{array}$$

Using (3.4) we get

$$\begin{array}{ll} \tilde{R}(e_1,e_2)e_3 = 0, & \tilde{R}(e_2,e_3)e_3 = (\beta^2 - \alpha^2 - 1)e_2, \\ \tilde{R}(e_1,e_3)e_3 = (\beta^2 - \alpha^2 - 1)e_1, & \tilde{R}(e_1,e_2)e_2 = \alpha^2 e_2 + (\beta^2 + \alpha^2 - 1)e_1, \\ \tilde{R}(e_2,e_3)e_2 = (-\beta^2 + \alpha^2 + 1)e_3, & \tilde{R}(e_1,e_3)e_2 = 0, \\ \tilde{R}(e_1,e_2)e_1 = (1 - \beta^2 - \alpha^2)e_2, & \tilde{R}(e_2,e_3)e_1 = 0, \\ \tilde{R}(e_1,e_3)e_1 = (1 + \alpha^2 - \beta^2)e_3. & \end{array}$$

From the above expressions of the curvature tensor we obtain

$$S(e_1,e_1) = \sum_{i=1}^{3} g(R(e_i,e_1)e_1,e_i) = -2.$$

Similarly, we have

$$S(e_2,e_2) = -2 \quad \text{and} \quad S(e_3,e_3) = -2.$$

$$\tilde{S}(e_1, e_2) = \tilde{S}(e_2, e_2) = 2(\beta^2 - 1) \quad \tilde{S}(e_3, e_3) = 2(\beta^2 - \alpha^2 - 1).$$
$$r = -6 \quad \tilde{r} = 6\beta^2 - 2\alpha^2 - 6.$$

From above we see that $\tilde{r} = 0$ for $\alpha = 0, \beta = 1$. Therefore, the manifold under consideration satisfies the Theorem 5.2.
Using (4.1) and above relations, we get

$$P(e_1, e_2)e_3 = P(e_1, e_3)e_3 = P(e_2, e_3)e_3 = 0,$$

$$\tilde{P}(e_1, e_2)e_3 = \tilde{P}(e_1, e_3)e_3 = \tilde{P}(e_2, e_3)e_3 = 0.$$

Therefore, the manifold will be ξ-projectively flat on a three-dimensional trans-Sasakian manifold with respect to the Schouten-van Kampen connection which varifies the Theorem 4.1.

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SOME RESULT ON LIE IDEALS WITH SYMMETRIC REVERSE BI-DERIVATIONS IN SEMIPRIME RINGS I

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Abstract. Let R be a semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. In the present paper, we shall prove that R is commutative ring if any one of the following holds: i) $d(U) = (0)$, ii)$d(U) \subset Z$, iii)$[d(x), y] \in Z$, iv)$d(x)oy \in Z$, v)$d([x, y]) \pm [d(x), y] \in Z$, vi)$d(x \circ y) \pm (d(x) \circ y) \in Z$, vii)$d([x, y]) \pm d(x) \circ y \in Z$ viii)$d(x \circ y) \pm [d(x), y] \in Z$, ix)$d(x) \circ y \pm [d(y), x] \in Z$, x)$d([x, y]) - (d(x) \circ y) - [d(y), x] \in Z$ xi)$[d(x), y] \pm [g(y), x] \in Z$, for all $x, y \in U$, where $G : R \times R \to R$ is symmetric reverse bi-derivation such that g is the trace of G.z

Key words: Lie ideals, bi-derivations, actions of Lie algebras

1. Introduction

Throughout the paper, R will represent an associative ring with center Z. A ring R is said to be prime if $xRy = (0)$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = (0)$ implies that $x = 0$, where $x, y \in R$. A prime ring is obviously semiprime. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy stands for the commutator $xy + yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$, $r \in R$. U is called a square-closed Lie ideal of R if U is a Lie ideal and $u^2 \in U$ for all $u \in U$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ is said to be a reverse derivation

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if $d(xy) = d(y)x + yd(x)$ holds for all $x, y \in R$. A mapping $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x, x)$ for all $x \in R$.It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x + y) = d(x) + d(y) + 2D(x, y)$, for all $x, y \in R$. If $D(.,.)$ is bi-additive and satisfies the identities

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z),$$

for all $x, y, z \in R$. Then $D(.,.)$ is called a symmetric bi-derivation. If $D(.,.)$ is reverse bi-additive and satisfies the identity

$$D(xy, z) = D(y, z)x + yD(x, z)$$

and

$$D(x, yz) = D(x, z)y + zD(x, y).$$

Then $D(.,.)$ is called a symmetric reverse bi-derivation.

The study of commuting mappings was initiaded by a well-known theorem due to Posner [7] which stetes that the existence of a nonzero commuting derivation on a prime ring R implies that R is commutative. A number of authors have extended the Posner's theorem in several ways. The notion of additive commuting mapping is closely connected with the notion of bi-derivation. Every additive commuting mapping $F : R \to R$ gives rise to a bi-derivation on R. Namely, linearizing $[F(x), x] = 0$, we get $[F(x), y] = [x, F(y)]$ and we note that the map $(x, y) \longmapsto [F(x), y]$ is a bi-derivation. The concept of bi-derivation was introduced by Maksa in [5]. It is shown in [6] that symmetric bi-derivations are related to general solution of some functional equations. Some results concerning symmetric bi-derivations in prime rings can found in [9] and [10].

In [4], Herstein showed that if R is a prime ring of characteristic different from two and d is a nonzero derivation such that $d(R) \subset Z$, then R must be commutative. Bergen et al. proved the following results in [2]: Let R be a prime ring of characteristic different from 2, U a nonzero Lie ideal of R and d a nonzero derivation. If $d(U) \subset Z$, then $U \subset Z$. Several authors investigated this result for a prime ring admitting derivation or generalized derivation.

Many authors investigated the commutativity of prime or semiprime rings satisfying certain functional identities involving derivation or generalized derivation. In this paper, we extend some well known these results concerning of Lie ideals in semiprime rings to a reverse bi-derivations. Throughout the present paper, we shall make use of the following basic identities without any specific mentioning:

i) $[x, yz] = y[x, z] + [x, y]z$
ii) $[xy, z] = [x, z]y + x[y, z]$
iii) $xyoz = (xoz)y + x[y, z] = x(yoz) - [x, z]y$
iv) $xoyz = y(xoz) + [x, y]z = (xoy)z + y[z, x]$.

1.1. Results

Lemma 1.1. *[1, Theorem 1.3] Let R be a $2-$ torsion free semiprime ring and U a noncentral Lie ideal of R such that $u^2 \in U$ for all $x \in U$. Then there exists a nonzero ideal I of R such that $I \subseteq U$.*

Lemma 1.2. *[3, Lemma 2 (b)] If R is a semiprime ring, then the center of a nonzero ideal of R is contained in the center of R.*

Lemma 1.3. *[8, Theorem 2.1] Let R be a semiprime ring, I a nonzero two-sided ideal of R and $a \in R$ such that $axa = 0$ for all $x \in I$, then $a = 0$.*

Lemma 1.4. *Let R be a semiprime ring. If a nonzero ideal of R is in the center of R, then R is a commutative ring.*

Proof. By the hypothesis, we get

$$[x, r] = 0, \text{ for all } x \in I, r \in R.$$

Replacing x by $sx, s \in R$ in this equation and using this equation, we obtain that

$$[s, r]x = 0, \text{ for all } x \in I, r \in R.$$

Thus, $[R, R]I = (0)$. Multiplying this equation on the right by $[R, R]$, we have $[R, R]I[R, R] = (0)$. By Lemma 3, we conclude that R is a commutative ring. The proof is completed. □

Theorem 1.1. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(U) = (0)$, then $D = 0$.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we have

$$d(x) = 0, \text{ for all } x \in I.$$

Replacing x by $x + y$, $y \in I$ in this equation and using the hypothesis, we get

$$2D(x, y) = 0, \text{ for all } x, y \in I.$$

Since R is $2-$torsion free, we have

$$D(x, y) = 0, \text{ for all } x, y \in I.$$

Taking x by xr, $r \in R$ in the above equation and using this equation, we obtain that

$$D(r, y)x = 0, \text{ for all } x, y \in I, r \in R.$$

Replacing y by $ys, s \in R$, we have

$$D(r,s)yx = 0, \text{ for all } x, y \in I, r, s \in R.$$

Multiplying this equation on the right by $D(r,s)y$, we get

$$D(r,s)yID(r,s)y = 0, \text{ for all } y \in I, r, s \in R.$$

By Lemma 3, we arrive at

$$D(r,s)y = 0, \text{ for all } y \in I, r, s \in R.$$

Again, multiplying this equation on the right by $D(r,s)$, we find that

$$D(r,s)yD(r,s) = 0, \text{ for all } y \in I, r, s \in R.$$

Using Lemma 3 in the above equation, we get $D = 0$. The proof is completed. $\square$

Theorem 1.2. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(U) \subset Z$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. By our hypothesis, we get

$$d(x) \in Z, \text{ for all } x \in I.$$

Replacing x by $x + y$, $y \in I$ in above expression, we get

$$d(x) + d(y) + 2D(x,y) \in Z, \text{ for all } x, y \in I.$$

Using the hypothesis and R is $2-$torsion free, we have

$$D(x,y) \in Z, \text{ for all } x, y \in I. \tag{1.1}$$

Commuting this term with $r, r \in R$, we get

$$[D(x,y), r] = 0, \text{ for all } x, y \in I, r \in R.$$

Taking x by $xs, s \in R$ in the last equation, we obtain that

$$[sD(x,y) + D(s,y)x, r] = 0, \text{ for all } x, y \in I, r, s \in R.$$

Using equation (2.1), we get

$$[s,r]D(x,y) + D(s,y)[x,r] + [D(s,y), r]x = 0, \text{ for all } x, y \in I, r, s \in R.$$

Replacing s by x in the last equation, we get

$$[x,r]D(x,y) + D(x,y)[x,r] + [D(x,y), r]x = 0, \text{ for all } x, y \in I, r, s \in R.$$

Applying equation (2.1), we see that

$$2[x,r]D(x,y)=0, \text{ for all } x,y\in I, r\in R.$$

Since R is 2−torsion free, we get

$$[x,r]D(x,y)=0, \text{ for all } x,y\in I, r\in R.$$

Using $D(x,y)\in Z$, we have

$$[x,r]tD(x,y)=0, \text{ for all } x,y\in I, r,t\in R.$$

Taking y by x, we have

$$[x,r]td(x)=0, \text{ for all } x,\in I, r,t\in R.$$

Replacing r by $d(x)$ in this equation, we find that

$$[x,d(x)]td(x)=0, \text{ for all } x,\in I, t\in R. \tag{1.2}$$

Multiplying this equation on the right by x, we get

$$[x,d(x)]td(x)x=0, \text{ for all } x,\in I, t\in R.$$

Taking t by tx in equation (2.2), we find that

$$[x,d(x)]txd(x)=0, \text{ for all } x,\in I, t\in R.$$

Subtracting two last equations, we arrive at

$$[x,d(x)]t[x,d(x)]=0, \text{ for all } x,\in I, t\in R.$$

Since R is semiprime ring, we obtain that d is commuting on I. The proof is completed. □

Theorem 1.3. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D: R\times R\to R$ a symmetric reverse bi-derivation and d be the trace of D. If $[d(x),y]\in Z$, for all $x,y\in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I\subseteq U$. By the hypothesis, we get

$$[d(x),y]\in Z, \text{ for all } x,y\in I.$$

Replacing y by yz in the hypothesis, we have

$$[d(x),y]z+y[d(x),z]\in Z, \text{ for all } x,y,z\in I.$$

Commuting this term with $r, r\in R$, we get

$$[[d(x),y]z+y[d(x),z],r]=0,$$

and so,

$$[d(x),y][z,r]+[y,r][d(x),z]=0 \text{ for all } x,y,z\in I, r\in R.$$

Replacing r by z in the last equation, we obtain that

$$[y,z][d(x),z]=0, \text{ for all } x,y,z\in I.$$

Taking y by $ty, t\in R$ in above equation, we see that

$$[t,z]y[d(x),z]=0 \text{ for all } x,y,z\in I, r\in R.$$

Replacing t by $d(x)$, we get

$$[d(x),z]y[d(x),z]=0, \text{ for all } x,y,z\in I.$$

By Lemma 3, we have

$$[d(x),z]=0, \text{ for all } x,z\in I.$$

Using Lemma 2, we obtain that $d(x)\in Z$, for all $x\in I$. We conclude that d is commuting on I by Theorem 2. $\square$

Theorem 1.4. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D:R\times R\to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(x)\circ y\in Z$, for all $x,y\in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I\subseteq U$. By the hypothesis, we get

$$d(x)\circ y\in Z, \text{ for all } x,y\in I.$$

Taking y by yz in the last expression, we obtain that

$$y(d(x)\circ z)+[d(x),y]z\in Z, \text{ for all } x,y,z\in I.$$

Commuting this term with $r, r\in R$, we see that

$$(1.3)\quad [d(x),y][z,r]+[y,r](d(x)\circ z)+[[d(x),y],r]z=0, \text{ for all } x,y,z\in I, r\in R.$$

Taking z by $zt, t\in R$ in the above equation, we get

$$[d(x),y][z,r]t+[d(x),y]z[t,r]+[y,r](d(x)\circ z)t+[y,r]z[t,d(x)]+[[d(x),y],r]zt=0.$$

Using equation (2.3), we get

$$[d(x),y]z[t,r]+[y,r]z[t,d(x)]=0, \text{ for all } x,y,z\in I, r,t\in R.$$

Replacing t by $d(x)$ and r by y, we have

$$[d(x),y]z[d(x),y]=0, \text{ for all } x,y,z\in I.$$

By Lemma 3, we have

$$[d(x),z]=0, \text{ for all } x,z\in I.$$

We conclude that d is commuting on I by Theorem 2 and Lemma 2. $\square$

Theorem 1.5. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d([x,y]) \pm [d(x),y] \in Z$, for all $x,y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we have

$$d([x,y]) \pm [d(x),y] \in Z, \quad \text{for all } x,y \in I.$$

Writting y by $y+z, z \in I$, we have

$$d([x,y]) + d([x,z]) + 2D([x,y],[x,z]) \pm [d(x),y] \pm [d(x),z] \in Z$$

By the hypothesis, we get

$$2D([x,y],[x,z]) \in Z.$$

Since R is $2-$torsion free, we see that

$$D([x,y],[x,z]) \in Z, \text{ for all } x,y \in I.$$

Replacing y by z in the last expression, we get

$$D([x,y],[x,y]) \in Z, \text{ for all } x,y \in I.$$

That is,

$$d([x,y]) \in Z, \quad \text{for all } x,y \in I.$$

By the hypothesis, we have

$$[d(x),y] \in Z, \quad \text{for all } x,y \in I.$$

By Theorem 3, we conclude that d is commuting on I. $\square$

Theorem 1.6. *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(x \circ y) \pm (d(x) \circ y) \in Z$, for all $x,y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We have

$$d(x \circ y) \pm (d(x) \circ y) \in Z, \quad \text{for all } x,y \in I.$$

Taking y by $y+z, z \in I$, we get

$$d(x \circ y) + d(x \circ z) + 2D(x \circ y, x \circ z) \pm d(x) \circ y \pm d(x) \circ z \in Z$$

By the hypothesis and since R is $2-$torsion free, we have

$$D(x \circ y, x \circ z) \in Z.$$

Replacing y by z in the this expression, we get

$$D(x \circ y, x \circ y) \in Z, \text{ for all } x, y \in I,$$

and so,

$$d(x \circ y) \in Z, \quad \text{for all } x, y \in I.$$

Using the hypothesis, we obtain that

$$d(x) \circ y \in Z, \quad \text{for all } x, y \in I.$$

We see that d is commuting on I by Theorem 4. $\square$

Theorem 1.7. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d\left([x, y]\right) \pm d(x) \circ y \in Z$, for all $x, y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We have

$$d\left([x, y]\right) \pm \left(d(x) \circ y\right) \in Z, \quad \text{for all } x, y \in I.$$

Taking y by $y + z, z \in I$ in the hypothesis, we get

$$d\left([x, y]\right) + d([x, z]) + 2D([x, y], [x, z]) \pm d(x) \circ y \pm d(x) \circ z \in Z$$

Using the hypothesis and since R is $2-$torsion free, we find that

$$D([x, y], [x, z]) \in Z.$$

Writting y by z in the above expression, we see that

$$D([x, y], [x, y]) \in Z, \text{ for all } x, y \in I.$$

That is,

$$d\left([x, y]\right) \in Z, \quad \text{for all } x, y \in I.$$

By the hypothesis, we have

$$d(x) \circ y \in Z, \quad \text{for all } x, y \in I.$$

By Theorem 4, we obtain that d is commuting on I. $\square$

Theorem 1.8. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d\left(x \circ y\right) \pm [d(x), y] \in Z$, for all $x, y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We get

$$d(x \circ y) \pm [d(x), y] \in Z, \quad \text{for all } x, y \in I.$$

Replacing y by $y + z, z \in I$, we get

$$d(x \circ y) + d(x \circ z) + 2D(x \circ y, x \circ z) \pm [d(x), y] \pm [d(x), z] \in Z.$$

Using the hypothesis and R is $2-$torsion free, we have

$$D(x \circ y, x \circ z) \in Z.$$

Writting y by z in the last expression, we get

$$D(x \circ y, x \circ y) \in Z, \text{ for all } x, y \in I,$$

and so, $d(x \circ y) \in Z$, for all $x, y \in I$. Using the hypothesis, we have

$$[d(x), y] \in Z, \quad \text{for all } x, y \in I.$$

We conclude that d is commuting on I by Theorem 3. □

Theorem 1.9. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(x) \circ y \pm [d(y), x] \in Z$, for all $x, y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We have

$$d(x) \circ y \pm [d(y), x] \in Z, \quad \text{for all } x, y \in I.$$

Replacing y by $y + z, z \in I$, we get

$$d(x) \circ y + d(x) \circ z \pm 2[D(y, z), x] \pm [d(y), x] \pm [d(z), x] \in Z$$

Applying the hypothesis, we see that

$$2[D(y, z), x] \in Z.$$

Since R is $2-$torsion free, we find that

$$[D(y, z), x] \in Z.$$

Replacing z by y in this expression, we get

$$[D(y, y), x] \in Z, \text{ for all } x, y \in I$$

and so, $[d(y), x] \in Z$, for all $x, y \in I$. Using the hypothesis, we have

$$d(x) \circ y \in Z, \quad \text{for all } x, y \in I.$$

By Theorem 4, we conclude that d is commuting on I. □

Theorem 1.10. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R\times R \to R$, $G : R\times R \to R$ two symmetric reverse bi-derivations where d is the trace of D and g is the trace of G. If $[d(x), y] \pm [g(y), x] \in Z$, for all $x, y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We get

$$[d(x), y] \pm [g(y), x] \in Z, \quad \text{for all } x, y \in I.$$

Taking y by $y + z, z \in I$, we get

$$[d(x), y] + [d(x), z] \pm [g(y), x] \pm [g(z), x] \pm 2[G(y, z), x] \in Z$$

Since R is $2-$torsion free and using the hypothesis , we obtain

$$[G(y, z), x] \in Z.$$

Replacing y by z in the above expression, we have

$$[G(y, y), x] \in Z, \text{ for all } x, y \in I.$$

That is,

$$[g(y), x] \in Z, \quad \text{for all } x, y \in I.$$

We see that d is commuting on I by Theorem 3. □

Theorem 1.11. *Let R be a $2-$torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d([x, y]) - (d(x) \circ y) - [d(y), x] \in Z$, for all $x, y \in U$, then d is commuting on I where I is nonzero ideal of R.*

Proof. By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we obtain that

$$d([x, y]) - (d(x) \circ y) - [d(y), x] \in Z, \quad \text{for all } x, y \in I.$$

Taking y by $y + z, z \in I$, we get

$$\begin{aligned} &d([x, y]) + d([x, z]) + 2D([x, y], [x, z]) \\ &\qquad - d(x) \circ y - d(x) \circ z - [d(y), x] - [d(z), x] - 2[D(y, z), x] \in Z. \end{aligned}$$

Using the hypothesis, we have

$$D([x, y], [x, z]) - [D(y, z), x] \in Z.$$

Replacing y by z in this expression, we see that

$$D([x, y], [x, y]) - [D(y, y), x] \in Z, \text{ for all } x, y \in I.$$

That is,

$$d([x,y]) - [d(y),x] \in Z, \quad \text{for all } x,y \in I.$$

Hence we can write

$$d([x,y]) - [d(y),x] - d(y) \circ x + d(y) \circ x \in Z$$

and using the hypothesis, we get

$$d(y) \circ x \in Z, \quad \text{for all } x,y \in I.$$

By Theorem 4, we conclude that d is commuting on I. $\square$

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A NONLINEAR F-CONTRACTION FORM OF SADOVSKII'S FIXED POINT THEOREM AND ITS APPLIACTION TO A FUNCTIONAL INTEGRAL EQUATION OF VOLTERRA TYPE

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Abstract. In this paper, we give a nonlinear F-contraction form of the Sadovskii fixed point theorem and we also investigate the existence of solutions for a functional integral equation of Volterra type.
Key words: Fixed-point theorems, nonlinear contraction, functional integral equations

1. Introduction and preliminaries

Let F be an increasing real-valued function on $(0, \infty)$ with $\lim_{t\to 0} F(t) = -\infty$, and $\varphi : (0, \infty) \to (0, \infty)$ be a function such that $\liminf_{s\to t^+} \varphi(s) > 0$ for any $t \geq 0$. A self-mapping T on a metric space (X, d) is said to be a (φ, F)-contraction (or a nonlinear F-contraction) if

$$\varphi(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y)),$$

for all $x, y \in X$ provided $Tx \neq Ty$.

Nonlinear F-contractions were considered from a new viewpoint in [13] and the author showed, among other results, that in complete metric spaces these self-mappings have unique fixed points and also under a suitable condition connecting F and φ, a nonlinear F-contraction is condensing with respect to the Hausdorff

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measure of noncompactness; for more information on nonlinear F-contractions see [9] and related references in [13].

The Schauder fixed point theorem states that any continuous self-mapping defined on a nonempty convex and compact subset of a Banach space must have a fixed point and it plays a crucial role in topological fixed point theory. In 1955, Darbo [8] presented a fixed point theorem in terms of the measure of noncompactness (the notion was first defined by Kuratowski [10]) in Banach spaces which generalized the Schauder fixed point theorem. A slight generalization of Darbo's theorem is the Sadovskii fixed point theorem [12]: If Ω is a nonempty bounded closed convex subset of a Banach X and if γ is a measure of noncompactness and $T : \Omega \to \Omega$ is a continuous mapping satisfying one of the following conditions:
(1) (Darbo [8]) T is a k-set contraction, i.e., there exists $k \in [0, 1)$ such that for any set $C \subset \Omega$,

$$\gamma(T(C)) \leq k\gamma(C);$$

(2) (Sadovskii [12]) T is a γ-condensing mapping, i.e., for any set $C \subset \Omega$ with positive measure of noncompactness,

$$\gamma(T(C)) < \gamma(C),$$

then T has a fixed point.

Some generalizations of these fixed point theorems can be found for example in [1, 7, 11] and the references therein.

In this paper, we give a nonlinear F-contraction form of the Sadovskii fixed point theorem (Theorem 1.1) and then inspired by the main result in [13] (and [1]) we give an application to solving a functional integral equation of Volterra type (Theorem 2.1).

We first give some preliminaries which will be needed in this paper.

Measures of noncompactness serve as useful tools in the theory of operator equations in Banach spaces and are used in the theory of functional equations, ordinary and partial differential equations, integral and integro-differential equations, etc (see e.g., [2, 3, 4, 5]). In this section, we give a generalization of the Sadovskii fixed point theorem in terms of a general notion of the measure of noncompactness and nonlinear F-contractions.

For the convenience of the reader we recall some basic notations and definitions; see [2, 3, 4, 5]. Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_0^+ = [0, \infty)$. In a Banach space E, the symbols $\overline{X}$ and $\operatorname{conv} X$ stand for the closure and closed convex hull of the subset X of E, respectively. Denote by $\mathfrak{M}_E$ the family of all nonempty bounded subsets of E and by $\mathfrak{N}_E$ the family consisting of all nonempty relatively compact subsets of E.

Definition 1.1. A measure of noncompactness in the Banach space E is a mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_0^+$ which satisfies the following conditions:
(MN1) the family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;
(MN2) $\mu(X) \leq \mu(Y)$ if $X \subset Y$;

(MN3) $\mu(X) = \mu(\overline{X})$, for all $X \in \mathfrak{M}_E$;
(MN4) $\mu(X) = \mu(\text{conv}X)$, for all $X \in \mathfrak{M}_E$;
(MN5) $\mu(tX + (1-t)Y) \leq t\mu(X) + (1-t)\mu(Y)$, for all $X, Y \in \mathfrak{M}_E$ and $t \in [0,1]$;
(MN6) If $\{X_n\}$ is a decreasing sequence of closed and nonempty sets of $\mathfrak{M}_E$ with $\lim_{n\to\infty} \mu(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n$ is nonempty.

The following result is a generalization of the Sadovskii fixed point theorem using F-contractions.

Theorem 1.1. *Let Ω be a nonempty bounded closed convex subset of a Banach space E and $T : \Omega \to \Omega$ be a continuous mapping which satisfies*

$$\varphi(\mu(X)) + F(\mu(TX)) \leq F(\mu(X)),$$

for any nonempty subset X of Ω with $\mu(TX) \neq 0$ and $\mu(X) \neq 0$, where μ is an arbitrary measure of noncompactness and $F : (0,\infty) \to \mathbb{R}$ and $\varphi : (0,\infty) \to (0,\infty)$ are functions such that

$$\forall (t_n) \in (0,\infty)^{\mathbb{N}} \quad (F(t_n) \to -\infty \Rightarrow t_n \to 0) \tag{1.1}$$

and

$$\liminf_{s\to t^+} \varphi(s) > 0, \quad (t \geq 0). \tag{1.2}$$

Then T has a fixed point.

Proof. Define a sequence (Ω_n) inductively as follows:

$$\Omega_0 = \Omega \quad \text{and} \quad \Omega_n = \text{conv}\, T\Omega_{n-1}, \quad n \geq 1.$$

We may assume, without loss of generality, that $\mu(\Omega_n) > 0$, for every $n \geq 1$, for otherwise, by Schauder's fixed point theorem T has a fixed point.

Observe that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, which means that the sequence $(\mu(\Omega_n))$ is decreasing and therefore is convergent to some nonnegative real number. From (1.2), there exist a $c > 0$ and $n_1 \in \mathbb{N}$ such that $\varphi(\mu(\Omega_n)) > c$ for all $n \geq n_1$. Therefore, we have

$$\begin{aligned}
F(\mu(\Omega_n)) &\leq F(\mu(\Omega_{n-1})) - \varphi(\mu(\Omega_{n-1})) \\
&\vdots \\
&\leq F(\mu(\Omega_0)) - \sum_{i=0}^{n-1} \varphi(\mu(\Omega_i)) \\
&= F(\mu(\Omega_0)) - \sum_{i=0}^{n_1-1} \varphi(\mu(\Omega_i)) - \sum_{i=n_1}^{n-1} \varphi(\mu(\Omega_i)) \\
&< F(\mu(\Omega_0)) - (n - n_1)c,
\end{aligned}$$

for all $n > n_1$. We get $F(\mu(\Omega_n)) \to -\infty$ as $n \to \infty$ and hence by (1.1), $\mu(\Omega_n) \to 0$.

Since (Ω_n) is a decreasing sequence of closed and nonempty sets of $\mathfrak{M}_E$ with $\lim_{n\to\infty} \mu(\Omega_n) = 0$, by (MN1) and (MN6) of Definition 1.1 we have that $\Omega_\infty = \cap_{n=0}^{\infty}\Omega_n$ is a nonempty compact convex subset of Ω. Moreover, T maps Ω_∞ into itself. Now, applying the Schauder fixed point theorem we infer that T has a fixed point in Ω_∞. □

Note that in the setting of Banach spaces Theorem 2.1 in [13] can be deduced from Theorem 1.1. Furthermore, notice that putting $F(x) = \ln x$ in Theorem 1.1 we have the Sadovskii fixed point theorem. In fact, we have

$$\varphi(\mu(X)) + \ln(\mu(TX)) \leq \ln(\mu(X))$$

and therefore

$$\mu(TX) \leq \frac{1}{e^{\varphi(\mu(X))}}\mu(X) < \mu(X).$$

Also, Theorem 1.1 can be easily generalized in the spirit of [11, Theorem 5]. Indeed, let Φ denote the set of all functions $f : {\mathbb{R}^+}^2 \to \mathbb{R}$ satisfying

$$\forall (t_n) \in (0,\infty)^{\mathbb{N}} \ (\forall n \in \mathbb{N}\ (t_{n+1} \leq t_n \wedge f(t_{n+1}, t_n) \geq 0) \Rightarrow t_n \to 0)\,.$$

Observe that $f \in \Phi$ such that $f(x, y) = -F(x)+F(y)-\varphi(y)$, where F and φ satisfy (1.1) and (1.2), respectively. Then, we clearly have the following:

Theorem 1.2. *Let μ be a measure of noncompactness on a Banach space E. Assume that Ω is a nonempty bounded closed and convex subset of E and $T : \Omega \to \Omega$ is a continuous mapping such that*

$$\exists f \in \Phi \forall X \subset \Omega\ (\overline{X} \textit{ is noncompact} \Rightarrow f(\mu(TX), \mu(X)) \geq 0).$$

Then, T has a fixed point.

2. Application to a functional integral equation of Volterra type

In this section, we follow the terminology and notations used in [6] unless otherwise specified. Consider the Banach space $BC(\mathbb{R}_0^+)$ consisting of all bounded and continuous real-valued functions on the nonnegative real numbers $\mathbb{R}_0^+$ equipped with the norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\},$$

where $x \in BC(\mathbb{R}_0^+)$. Let X be a nonempty bounded subset of $BC(\mathbb{R}_0^+)$ and $L > 0$. For $x \in X$ and $\varepsilon > 0$, the modules of continuity of the function x on the interval $[0, L]$, denoted by $w^L(x, \varepsilon)$, is defined as

$$w^L(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, L], |t - s| \leq \varepsilon\}.$$

Moreover, let

$$\begin{aligned} w^L(X,\varepsilon) &= \sup\{w^L(x,\varepsilon) : x \in X\} \\ w_0^L(X) &= \lim_{\varepsilon\to 0} w^L(X,\varepsilon) \\ w_0(X) &= \lim_{L\to\infty} w_0^L(X). \end{aligned}$$

Also, for a fixed number $t \in \mathbb{R}_0^+$ let

$$X(t) = \{x(t) : x \in X\}.$$

Define a function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_0^+)}$ as

$$\mu(X) = w_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t), \tag{2.1}$$

where $\operatorname{diam} X(t)$ is defined as

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

The function μ defined above is a measure of noncompactness in the Banach space $BC(\mathbb{R}_0^+)$ (see [4, 6]).

Let $\mathcal{F}$ denote the set of all functions $F : (0,\infty) \to \mathbb{R}$ which satisfy the following conditions:
(F1) For all $t_1, t_2 > 0$, $t_1 > t_2$ implies $F(t_1) > F(t_2)$;
(F2) For any sequence $(t_n) \subset (0,\infty)$, $t_n \to 0$ if and only if $F(t_n) \to -\infty$.

Suppose that two functions f and g satisfy the following conditions:

(i) $f : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $t \mapsto f(t,0)$ is also an element of the Banach space $BC(\mathbb{R}_0^+)$, $f(0,0) = 0$, and $c \le f(0,2c)$ for each $c \ge 0$;

(ii) If $F \in \mathcal{F}$ is continuous and $\varphi : (0,\infty) \to (0,\infty)$ is a nonincreasing continuous function such that

$$\lim_{s\to t^+} \varphi(s) > 0, \quad (t \in \mathbb{R}_0^+)$$

(or equivalently, $\liminf_{s\to t^+} \varphi(s) > 0$), then

$$F(|f(s,u)-f(s,u_1)|+|f(t,w)-f(t,w_1)|+f(0,c)) \le (F-\varphi)(|u-u_1|+|w-w_1|+c),$$

where $c, s, t \in \mathbb{R}_0^+$ and $u, u_1, w, w_1 \in \mathbb{R}$ with

$$|f(s,u)-f(s,u_1)|+|f(t,w)-f(t,w_1)|+f(0,c) > 0 \text{ and } |u-u_1|+|w-w_1|+c > 0.$$

Note that under the above assumptions we, in particular, have

$$F(|f(s,u) - f(s,u_1)|) < F(|u - u_1|)$$

and consequently

$$|f(s,u) - f(s,u_1| \leq |u - u_1|;$$

(iii) $g : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $\alpha, \beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\lim_{t\to\infty} \alpha(t) \int_0^t \beta(s)ds = 0 \tag{2.2}$$

and

$$|g(t,s,x)| \leq \alpha(t)\beta(s)$$

for any $x \in \mathbb{R}$ and $t, s \in \mathbb{R}_0^+$ with $s \leq t$;

(iv) With

$$\kappa = \sup\left\{ f(t,0) + \alpha(t) \int_0^t \beta(s)ds : t \geq 0 \right\},$$

we have $0 \leq \kappa < \infty$ and there exists a positive real number r_0 such that

$$(F - \varphi)(r + 2\kappa) \leq F(r).$$

Example 2.1. The following functions satisfy conditions (i)-(iv):

- $f : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ defined by $f(s,t) = |\sin(s)| + \frac{t}{2}$;
- $F \in \mathcal{F}$ defined by $F(t) = -\frac{1}{t}$;
- $\varphi : (0, \infty) \to (0, \infty)$ defined by $\varphi(t) = \frac{1}{t}$;
- $g : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ defined by $g(t,s,x) = se^{-t} \frac{h_1(x)}{1+|h_2(x)|}$, where $h_1, h_2 \in BC(\mathbb{R})$ with $|h_1(x)| \leq 1 + |h_2(x)|$, for all $x \in \mathbb{R}$;
- $\alpha, \beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $\alpha(t) = e^{-t},\ \beta(s) = s$.

Now we give an application of Theorem 1.1 where we show that a functional equation of Volterra type has a solution.

Theorem 2.1. *The integral equation of Volterra type*

$$x(t) = f(t, x(t)) + \int_0^t g(t,s,x(s))ds, \quad (t \in \mathbb{R}_0^+) \tag{2.3}$$

where f and g are functions satisfying conditions (i)-(iv) has a solution in the Banach space $BC(\mathbb{R}_0^+)$.

Proof. We will use some ideas from [6]. Define a mapping T on the Banach space $BC(\mathbb{R}_0^+)$ as follows:

$$(Tx)(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \quad (t \in \mathbb{R}_0^+).$$

It is easy to see that for any $x \in BC(\mathbb{R}_0^+)$, the function Tx is a real-valued continuous function on $\mathbb{R}_0^+$. In addition, for any function $x \in BC(\mathbb{R}_0^+)$ and $t \geq 0$ we have

$$\begin{aligned}
F(|(Tx)(t)|) &= F\left(\left|f(t,x(t)) + \int_0^t g(t,s,x(s))ds\right|\right) \\
&\leq F\left(|f(t,x(t)) - f(t,0)| + \int_0^t |g(t,s,x(s))|\,ds + f(t,0)\right) \\
&\leq F\left(|f(t,x(t)) - f(t,0)| + \kappa\right) \\
&\leq F\left(|f(t,x(t)) - f(t,0)| + f(0,2\kappa)\right) \\
&\leq (F-\varphi)\left(|x(t))| + 2\kappa\right) \\
&\leq (F-\varphi)(\|x\| + 2\kappa).
\end{aligned}$$

From assumption (iv), choose $r_0 > 0$ such that $(F-\varphi)(r_0+2\kappa) \leq F(r_0)$. Then, in particular, we have

$$F(|(Tx)(t)|) \leq (F-\varphi)(|x(t)|+2\kappa) \leq (F-\varphi)(\|x\|+2\kappa) \leq (F-\varphi)(r_0+2\kappa) \leq F(r_0),$$

when $\|x\| \leq r_0$ and since F is increasing we get $|(Tx)(t)| \leq r_0$ and finally since $t \geq 0$ was arbitrary, we have $\|Tx\| \leq r_0$. Therefore T maps the closed ball $B_{r_0} = \{x \in BC(\mathbb{R}_0^+) : \|x\| \leq r_0\}$ into itself. (Notice also, since

$$F(|(Tx)(t)|) \leq F\left(|f(t,x(t)) - f(t,0)| + \kappa\right) \leq F\left(|x(t))| + \kappa\right) \leq F(\|x\| + \kappa),$$

Tx is bounded for any $x \in BC(\mathbb{R}_0^+)$.)

Now, we show that the self-mapping T is continuous on the ball B_{r_0}. Let $\varepsilon > 0$ be given. Suppose that $x, y \in B_{r_0}$ such that $\|x - y\| < \varepsilon$. We have

$$\begin{aligned}
&F\left(|(Tx)(t) - (Ty)(t)|\right) \\
&\quad \leq F\left(|f(t,x(t)) - f(t,y(t))| + \left|\textstyle\int_0^t g(t,s,x(s))ds - \int_0^t g(t,s,y(s))ds\right|\right) \\
&\quad \leq F\left(|f(t,x(t)) - f(t,y(t))| + \left|\textstyle\int_0^t g(t,s,x(s))ds\right| + \left|\textstyle\int_0^t g(t,s,y(s))ds\right|\right) \\
&\quad \leq F\left(|f(t,x(t)) - f(t,y(t))| + 2\alpha(t)\textstyle\int_0^t \beta(s)ds\right),
\end{aligned}$$

for any $t \geq 0$. From assumption (iii), there exists a number $L > 0$ such that

$$2\alpha(t)\int_0^t \beta(s)ds < \varepsilon,$$

for each $t \geq L$. Thus, for an arbitrary $t \geq L$ we obtain

$$\begin{aligned} F\left(|(Tx)(t) - (Ty)(t)|\right) &\leq F\left(|f(t, x(t)) - f(t, y(t))| + \varepsilon\right) \\ &\leq F(|x(t)) - y(t))| + \varepsilon) \\ &< F(2\varepsilon), \end{aligned}$$

and since F is increasing, we obtain

$$|(Tx)(t) - (Ty)(t)| \leq 2\varepsilon, \quad (t \geq L). \tag{2.4}$$

Since the function g is uniformly continuous on the set $[0, L] \times [0, L] \times [-r_0, r_0]$, we have $\lim_{\varepsilon \to 0} w^L(g, \varepsilon) = 0$, where

$$w^L(g, \varepsilon) = \sup\{|g(t, s, x) - g(t, s, y) : t, s \in [0, L],\ x, y \in [-r_0, r_0],\ |x - y| \leq \varepsilon\}.$$

On the other hand, for an arbitrary fixed $t \in [0, L]$, we have

$$\begin{aligned} &F\left(|(Tx)(t) - (Ty)(t)|\right) \\ &\quad \leq F\left(|f(t, x(t)) - f(t, y(t))| + |\textstyle\int_0^t g(t, s, x(s))ds - \int_0^t g(t, s, y(s))ds|\right) \\ &\quad \leq F\left(|f(t, x(t)) - f(t, y(t))| + \textstyle\int_0^L w^L(g, \varepsilon)ds\right) \\ &\quad = F\left(|f(t, x(t)) - f(t, y(t))| + Lw^L(g, \varepsilon)\right) \\ &\quad \leq F\left(|x(t) - y(t)| + Lw^L(g, \varepsilon)\right), \end{aligned}$$

and since F is increasing we obtain

$$|(Tx)(t) - (Ty)(t)| \leq |x(t) - y(t)| + Lw^L(g, \varepsilon), \tag{2.5}$$

for any $t \in [0, L]$. Finally, the continuity of T on the ball B_{r_0} is obtained from (2.4) and (2.5).

Let X be an arbitrary nonempty subset of the closed ball B_{r_0}. Let $\varepsilon > 0$, $L > 0$, $z, w \in X$, and $t' > 0$ be fixed. Choose $t, s \in [0, L]$ with $|t - s| \leq \varepsilon$. Without loss of generality, we assume that $s < t$. Then for any $x, y \in X$ we have

$$\begin{aligned} &F(|(Tx)(t) - (Tx)(s)| + |(Tz)(t') - (Tw)(t')|) \\ &\quad \leq F(|f(t, x(t)) + \textstyle\int_0^t g(t, \tau, x(\tau))d\tau - f(s, x(s)) - \int_0^s g(s, \tau, x(\tau))d\tau| \\ &\qquad + |f(t', z(t')) + \textstyle\int_0^{t'} g(t', s, z(s))ds - f(t', w(t')) - \int_0^{t'} g(t', s, w(s))ds|) \\ &\quad \leq F(|f(t, x(t)) - f(s, x(t))| + |f(s, x(t)) - f(s, x(s))| + |\textstyle\int_0^t g(t, \tau, x(\tau))d\tau \\ &\qquad - \textstyle\int_0^t g(s, \tau, x(\tau))d\tau| + |\int_s^t g(s, \tau, x(\tau))d\tau| + |f(t', z(t')) - f(t', w(t'))| \\ &\qquad + |\textstyle\int_0^{t'} g(t', s, z(s))ds - \int_0^{t'} g(t', s, w(s))ds|) \\ &\quad \leq F(w_1^L(f, \varepsilon) + |f(s, x(t)) - f(s, x(s))| + Lw_1^L(g, \varepsilon) + \textstyle\int_s^t |g(t, \tau, x(\tau))|d\tau \\ &\qquad + |f(t', z(t')) - f(t', w(t'))| + \textstyle\int_0^{t'} |g(t', s, z(s)) - g(t', s, w(s))|ds) \\ &\quad \leq F(w_1^L(f, \varepsilon) + |f(t, x(t)) - f(s, x(s))| + Lw_1^L(g, \varepsilon) + \varepsilon \sup\{\alpha(s)\beta(t) : t, s \in [0, L]\} \\ &\qquad + |f(t', z(t')) - f(t', w(t')))| + 2\alpha(t') \textstyle\int_0^{t'} \beta(s)ds)) \end{aligned}$$

$$\leq F(|f(t,x(t)) - f(s,x(s))| + |f(t',z(t')) - f(t',w(t'))| + \eta^L_{f,g}(\varepsilon) + \gamma(t')),$$

where

$$w_1^L(f,\varepsilon) = \sup\{|f(t,x) - f(s,x)| : t,s \in [0,L],\, x \in [-r_0,r_0],\, |t-s| \leq \varepsilon\}$$

$$w_1^L(g,\varepsilon) = \sup\{|g(t,\tau,x) - g(s,\tau,x)| : t,s,\tau \in [0,L],\, x \in [-r_0,r_0],\, |t-s| \leq \varepsilon\},$$

and

$$\eta^L_{f,g}(\varepsilon) = w_1^L(f,\varepsilon) + Lw_1^L(g,\varepsilon) + \varepsilon \sup\{\alpha(s)\beta(t) : t,s \in [0,L]\},$$

$$\gamma(t') = 2\alpha(t') \int_0^{t'} \beta(s)ds.$$

Notice that since f and g are uniformly continuous on the sets $[0,L] \times [-r_0,r_0]$ and $[0,L] \times [0,L] \times [-r_0,r_0]$, respectively, we have $w_1^L(f,\varepsilon) \to 0$ and $w_1^L(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. In addition, since α and β are continuous functions on $\mathbb{R}_0^+$, we have

$$\sup\{\alpha(s)\beta(t) : t,s \in [0,L]\} < \infty.$$

Note also that, from (2.2), we have $\gamma(t') \to 0$ as $t' \to \infty$. Since, from assumption (iv),

$$\eta^L_{f,g}(\varepsilon) + \gamma(t')) \leq f\left(0, 2\left(\eta^L_{f,g}(\varepsilon) + \gamma(t')\right)\right),$$

using assumption (iii) we obtain

$$\begin{aligned} &F\left(|(Tx)(t) - (Tx)(s)| + |(Tz)(t') - (Tw)(t')|\right) \\ &\quad \leq (F-\varphi)(|x(t) - x(s)| + |z(t') - w(t')| + 2\,\eta^L_{f,g}(\varepsilon) + 2\,\gamma(t')). \end{aligned}$$

Now, taking the supremum of the previous inequality as $t,s \in [0,L]$ with $|t-s| \leq \varepsilon$ we get

$$\begin{aligned} &F(w^L(Tx,\varepsilon) + |(Tz)(t') - (Tw)(t')|) \\ &\quad \leq (F-\varphi)(w^L(x,\varepsilon)) + |z(t') - w(t')| + 2\,\eta^L_{f,g}(\varepsilon) + 2\,\gamma(t')). \end{aligned}$$

Taking the supremum as $z,w \in X$, we have

$$F\left(w^L(TX,\varepsilon) + \operatorname{diam}(TX)(t')\right) \leq (F-\varphi)(w^L(X,\varepsilon)) + \operatorname{diam} X(t') + 2\,\eta^L_{f,g}(\varepsilon) + 2\,\gamma(t')).$$

Letting first $\varepsilon \to 0$ and then $L \to \infty$ we get

$$F\left(w_0(TX) + \operatorname{diam}(TX)(t')\right) \leq (F-\varphi)\left(w_0(X) + \operatorname{diam} X(t') + 2\,\gamma(t')\right).$$

Finally, taking the limit superior as $t' \to \infty$ we obtain

$$F\left(w_0(TX) + \limsup_{t'\to\infty} \operatorname{diam}(TX)(t')\right) \leq (F-\varphi)\left(w_0(X) + \limsup_{t'\to\infty} \operatorname{diam} X(t')\right).$$

This can be restated as

$$F(\mu(TX)) \leq F(\mu(X)) - \varphi(\mu(X)),$$

in which μ is the measure of noncompactness given in (2.1). Now, Theorem 1.1 gives the desired result. $\square$

Example 2.2. Consider the functions given in Example 2.1. An easy application of Theorem 2.1 shows that the integral equation

$$x(t) = 2|\sin(t)| + \int_0^t \frac{2se^{-t}\sin^3 x(s)}{1+\cos^2 x(s)} ds, \quad (t \in \mathbb{R}_0^+)$$

has a solution in the Banach space $BC(\mathbb{R}_0^+)$.

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LINEAR DIFFERENTIAL POLYNOMIALS WEIGHTED-SHARING A SET OF ROOTS OF UNITY

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Abstract. In this paper, we study the uniqueness of linear differential polynomials of meromorphic functions when they share a set of roots of unity. Our results shall generalize recent results.
Key words: Meromorphic function, Sharing set, Linear Differential polynomial, Uniqueness.

1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna Theory of meromorphic functions as explained in [5, 13, 14]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = \circ(T(r, f))$ as $r \to \infty,\ r \notin E$. A meromorphic function a is said to be small with respect to f if $T(r, a) = S(r, f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers.

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For any two non-constant meromorphic functions f and g, and $a \in S(f) \cap S(g)$, we say that f and g share a IM(CM) provided that $f-a$ and $g-a$ have the same zeros ignoring(counting) multiplicities.

During the last few decades the uniqueness theory of entire or meromorphic functions has developed as an active sub-field of the value distribution theory. The main interest of the uniqueness theory is to determine an entire or meromorphic function uniquely satisfying some necessary conditions.

In 1997 Yang and Hua [6] studied the unicity problem for meromorphic functions and differential monomials of the form $f^n f^{(1)}$, when they share only one value. S. S. Bhoosnurmath and R. S. Dyavanal [3] extended the Yang-hua's results to the case of $(f^n)^{(k)}$.

Definition 1.1. Let f be a non-constant meromorphic function. An expression of the form

$$P[f] = \sum_{k=1}^{u} a_k \prod_{j=0}^{p} \left(f^{(j)}\right)^{l_{kj}} \tag{1.1}$$

where $a_k \in S(f)$ for $k = 1, 2, ..., u$ and l_{kj} are non-negative integers for $k = 1,\ 2, ..., u;\ j = 0,\ 1,\ 2, ..., p$ and $d = \sum_{j=0}^{p} l_{kj}$, for $k = 1, 2, ..., u$, is called a homogeneous differential polynomial of degree d generated by f.

In 2019, Bhoosnurmath, Chakrabarty and Srivastava [4] proved that for a non-constant homogeneous differential polynomial $P[f]$, the equation $P[f] = 1$ has infinitely many zeros.

To state the result we need the following definition.

Definition 1.2. For a meromorphic function f and a set $S \subseteq \mathbb{C}$, we define $E_f(S) = \bigcup_{a\in S}\{z|f(z)-a = 0\}$, counting multiplicities; $\overline{E}_f(S) = \bigcup_{a\in S}\{z|f(z)-a = 0\}$, ignoring multiplicities. If $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$, then we say that f and g share S CM (IM). Evidently, if S contains only one element then it coincides with the usual definition of CM (respectively IM) shared values.

Recently in 2018 V. H. An and H. H. Khoai [7] have proved the following uniqueness theorem of meromorphic functions.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions. Let k, d, n be three positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share S CM, then one of the following holds:*
1. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.
2. $f = tg$ for some $t \in \mathbb{C}$ such that $t^{nd} = 1$.

Question 1.1. *Regarding Theorem 1.1, a natural question to asked: Can CM be replace by IM keeping the same conclusion?*

In 2020 Dilip et al. answered the above question positively and proved the following theorem.

Theorem 1.2. *[11] Let f and g be two non-constant meromorphic functions. Let k, d, n be three positive integers with $n > 2k + \frac{8k+14}{d}$, $d \geq 2$ and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share S IM, then one of the following holds:*
1. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.
2. $f = tg$ for some $t \in \mathbb{C}$ such that $t^{nd} = 1$.

Now we recall the notion of weighted sharing which appeared in the literature in ([8, 9]) as this definition paves the way for future discussions as far as relaxation of sharing is concerned. In the following definition, we shall explain this notion.

Definition 1.3. [8, 9]. Let l be a non-negative integer or infinity and $a \in S(f)$. We denote by $E_l(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f, g share the function a with weight l. We write f and g share (a, l) to mean that f and g share the function a with weight l. Since $E_l(a, f) = E_l(a, g)$ implies that $E_s(a, f) = E_s(a, g)$ for any integer $s\,(0 \leq s < l)$, if f, g share (a, l), then f, g share (a, s), $(0 \leq s < l)$. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.4. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and l be a non-negative integer or ∞. We denote by $E_f(S, l)$ the set $E_f(S, l) = \bigcup_{a \in S} E_l(a, f)$. We say that f and g share the set S with weight l if $E_f(S, l) = E_g(S, l)$.

Definition 1.5. Let f be a non-constant meromorphic function. Then we denote by $L(f)$ a differential polynomial of the following form: $L(f) = f^{(k)}$ for $k = 1,\ 2,\ 3$ and $L(f) = \sum_{j=1}^{k-3} a_j f^{(j)} + f^{(k)}$ for $k \geq 4$, where $a_1,\ a_2, \ldots\ldots, a_{k-3}$ are constants.

In 2020 Lahiri et al proved the following theorem which improved and generalized Theorem 1.1.

Theorem 1.3. *[10] Let f and g be two non-constant meromorphic functions sharing $(\infty, 0)$ and k, d, n be three positive integers with $n > 2k + \frac{2k+8}{d}$, $d \geq 2$. Let $S = \{a \in \mathbb{C} : a^d = 1\}$. If $L(f^n)$ and $L(g^n)$ share $(S, 2)$ then one of the following holds:*
1. $L(f^n) = hL(g^n)$ for some $h \in \mathbb{C}$ such that $h^d = 1$.
2. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that

$$(c_1c_2)^n \left\{ A \sum_{j=1}^{k-3} a_j (nc)^j + (nc)^k \right\} + \left\{ A \sum_{j=1}^{k-3} a_j (-nc)^j + (-nc)^k \right\} = h$$

and $h^d = 1$, and $A = 0$ if $k = 1, 2, 3$ and $A = 1$ if $k \geq 4$.

In this paper, we shall prove the following result:

Theorem 1.4. *Let f and g be two non-constant meromorphic functions sharing $(\infty, 0)$. Let $k\ (\geq 1)$, $l\ (\geq 0)$, $d\ (\geq 2)$, $n\ (\geq 1)$ be integers and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $L(f^n)$ and $L(g^n)$ share (S, l) with one of the following conditions:*
(i) $l \geq 2$ and

$$n > 2k + \frac{2k+8}{d}, \tag{1.2}$$

(ii) $l = 1$ and

$$n > 2k + \frac{3k+9}{d}, \tag{1.3}$$

(ii) $l = 0$ and

$$n > 2k + \frac{8k+14}{d}, \tag{1.4}$$

then one of the following holds:
1. $L(f^n) = hL(g^n)$, where $h^d = 1$;
2. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three non-zero constants such that

$$(c_1c_2)^n \left\{ A \sum_{j=1}^{k-3} a_j (nc)^j + (nc)^k \right\} + \left\{ A \sum_{j=1}^{k-3} a_j (-nc)^j + (-nc)^k \right\} = h$$

and $h^d = 1$, and $A = 0$ if $k = 1, 2, 3$ and $A = 1$ if $k \geq 4$.

Corollary 1.1. *Let f and g be two non-constant entire functions. Let $k\ (\geq 1)$, $l\ (\geq 0)$, $d\ (\geq 2)$, $n\ (\geq 1)$ be integers and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $L(f^n)$ and $L(g^n)$ share (S, l) with one of the following conditions:*
(i) $l \geq 2$ and

$$n > 2k + \frac{2k+4}{d},$$

(ii) $l = 1$ and

$$n > 2k + \frac{5k+9}{2d},$$

(ii) $l = 0$ and

$$n > 2k + \frac{5k+7}{d},$$

then one of the following holds:
1. $L(f^n) = hL(g^n)$, where $h^d = 1$;

2. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three non-zero constants such that*

$$(c_1c_2)^n \left\{ A\sum_{j=1}^{k-3} a_j(nc)^j + (nc)^k \right\} + \left\{ A\sum_{j=1}^{k-3} a_j(-nc)^j + (-nc)^k \right\} = h$$

and $h^d = 1$, *and* $A = 0$ *if* $k = 1, 2, 3$ *and* $A = 1$ *if* $k \geq 4$.

Corollary 1.2. *Let* f *and* g *be two non-constant meromorphic functions sharing* $(\infty, 0)$. *Let* k (≥ 1), l (≥ 0), d (≥ 2), n (≥ 1) *be integers and* $S = \{a \in \mathbb{C} : a^d = 1\}$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* (S, l) *with one of the following conditions:*
(i) $l \geq 2$ *and*

$$n > \max\{3, 2k + \frac{2k+8}{d}\},$$

(ii) $l = 1$ *and*

$$n > \max\{3, 2k + \frac{3k+9}{d}\},$$

(ii) $l = 0$ *and*

$$n > \max\{3, 2k + \frac{8k+14}{d}\},$$

then one of the following holds:
1. $f = \omega g$, *where* $\omega^{nd} = 1$;
2. $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three non-zero constants such that* $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$.

2. Lemmas

In this section we present some lemmas which will needed in the sequel. Let F and G be non-constant meromorphic functions and H be another function which is defined as follows:

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right). \tag{2.1}$$

Lemma 2.1. *[12, 14] Let* f *be a non-constant meromorphic function and let* $a_0, a_1, \ldots, a_n (\not\equiv 0)$ *be small functions with respect to* f. *Then*

$$T\big(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_0\big) = nT(r, f) + S(r, f).$$

Lemma 2.2. *[5] Let* f *be a non-constant meromorphic function and let* k *be a positive integer. Then*

$$T(r, L(f)) \leq (k+1)T(r, f) + S(r, f).$$

Lemma 2.3. *[10] Let f be a non-constant meromorphic function and k, n be positive integers with $n \geq k+2$, $a \in \mathbb{C} \setminus \{0\}$. Then*

$$\frac{n-k-2}{n+k} T(r,f) \leq \overline{N}\left(r, \frac{1}{L(f^n)-a}\right) + S(r,f).$$

Lemma 2.4. *[10] Let f be a non-constant meromorphic function and k, n be positive integers with $n > 2k$. Then*

$$(i)\ (n-2k)T(r,f) + kN(r,f) + N\left(r, \frac{f^{n-k}}{L(f^n)}\right) \leq T\big(r, L(f^n)\big) + S(r,f).$$

$$(ii)\ N\left(r, \frac{f^{n-k}}{L(f^n)}\right) \leq kT(r,f) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.5. *[10] Let f and g be two non-constant meromorphic functions sharing $(\infty, 0)$ and k, n be integers with $n \geq k+1$. If $L(f^n).L(g^n) = h$, $h \in \mathbb{C} \setminus \{0\}$. Then $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where*

$$(c_1 c_2)^n \left\{ A \sum_{j=1}^{k-3} a_j (nc)^j + (nc)^k \right\} + \left\{ A \sum_{j=1}^{k-3} a_j (-nc)^j + (-nc)^k \right\} = h$$

and $h^d = 1$ and $A = 0$ if $k = 1, 2, 3$ and $A = 1$ if $k \geq 4$.

Lemma 2.6. *[1] If F and G be non-constant meromorphic functions sharing $(1,1)$ then*
$2\overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1}) + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) - \overline{N}_{F>2}(r, \frac{1}{G-1}) \leq N(r, \frac{1}{G-1}) - \overline{N}(r, \frac{1}{G-1})$ $+ S(r,F) + S(r,G)$.

Lemma 2.7. *[1] If F and G be non-constant meromorphic functions sharing $(1,1)$ then*
$\overline{N}_{F>2}(r, \frac{1}{G-1}) \leq \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}(r, \frac{1}{F}) - \frac{1}{2}N_0(r, \frac{1}{F^{(1)}})$.

Lemma 2.8. *[2] If F and G be non-constant meromorphic functions sharing $(1,0)$ then $\overline{N}_L(r, \frac{1}{F-1}) \leq \overline{N}(r,F) + \overline{N}(r, \frac{1}{F}) + S(r,F)$.*

Lemma 2.9. *Let l be a non-negative integer or infinity. F and G be non-constant meromorphic functions sharing $(1,l)$ and H as defined in (2.1). If $H \not\equiv 0$, then*
(i) If $l \geq 2$, then

$$T(r,F) \leq 2\overline{N}(r,F) + 2\overline{N}(r,G) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r,F) + S(r,G).$$

(ii) If $l = 1$, then

$$T(r,F) \leq \tfrac{5}{2}\overline{N}(r,F) + 2\overline{N}(r,G) + N_2(r, \tfrac{1}{F}) + N_2(r, \tfrac{1}{G}) + \tfrac{1}{2}\overline{N}(r, \tfrac{1}{F}) + S(r,F) + S(r,G).$$

(iii) If $l=0$, then

$$T(r,F) \leq 4\overline{N}(r,F) + 3\overline{N}(r,G) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G})$$
$$+2\overline{N}(r,\tfrac{1}{F} + \overline{N}(r,\tfrac{1}{G}) + S(r,F) + S(r,G).$$

The same inequality holds for $T(r,G)$.

Proof. By second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) + T(r,G) &\leq \overline{N}(r,F) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{F-1}) + \overline{N}(r,G) + \overline{N}(r,\tfrac{1}{G}) \\ &+\overline{N}(r,\tfrac{1}{G-1}) - N_0(r,\tfrac{1}{F^{(1)}}) - N_0(r,\tfrac{1}{G^{(1)}}) + S(r,F) + S(r,G), \end{aligned} \tag{2.2}$$

where $N_0(r,\frac{1}{F^{(1)}})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ which are not the zeros of F and $F-1$. Similarly defined $N_0(r,\frac{1}{G^{(1)}})$.
We consider the following cases:
Case 1: $l \geq 1$. Then from (2.1) we have

$$\begin{aligned} N_E^{1)}\left(r,\tfrac{1}{F-1}\right) &\leq N(r,\tfrac{1}{H}) \leq T(r,H) + O(1) \leq N(r,H) + S(r,F) + S(r,G) \\ &\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}(r,G) + \overline{N}_{(2}(r,\tfrac{1}{G}) + \overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1}) \\ &+N_0(r,\tfrac{1}{F^{(1)}}) + N_0(r,\tfrac{1}{G^{(1)}}) + S(r,F) + S(r,G), \end{aligned}$$

and so

$$\begin{aligned} \overline{N}(r,\tfrac{1}{F-1}) + \overline{N}(r,\tfrac{1}{G-1}) &= N_E^{1)}\left(r,\tfrac{1}{F-1}\right) + \overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1}) \\ &+\overline{N}_E^{(2}\left(r,\tfrac{1}{F-1}\right) + \overline{N}(r,\tfrac{1}{G-1}) \leq \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_{(2}(r,\tfrac{1}{F}) \\ &+\overline{N}_{(2}(r,\tfrac{1}{G}) + 2\overline{N}_L(r,\tfrac{1}{F-1}) + 2\overline{N}_L(r,\tfrac{1}{G-1}) + \overline{N}_E^{(2}\left(r,\tfrac{1}{F-1}\right) \\ &+\overline{N}(r,\tfrac{1}{G-1}) + N_0(r,\tfrac{1}{F^{(1)}}) + N_0(r,\tfrac{1}{G^{(1)}}) + S(r,F) + S(r,G). \end{aligned} \tag{2.3}$$

Subcase 1.1: $l=1$. Using Lemmas 2.6 and 2.7 we get

$$\begin{aligned} 2\overline{N}_L(r,\tfrac{1}{F-1}) &+ 2\overline{N}_L(r,\tfrac{1}{G-1}) + \overline{N}_E^{(2}\left(r,\tfrac{1}{F-1}\right) + \overline{N}(r,\tfrac{1}{G-1}) \leq N(r,\tfrac{1}{G-1}) \\ &+\overline{N}_{F>2}(r,\tfrac{1}{G-1}) \leq N(r,\tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r,F) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) - \tfrac{1}{2}N_0(r,\tfrac{1}{F^{(1)}}) \\ &+S(r,F) + S(r,G). \end{aligned} \tag{2.4}$$

Thus from (2.3) and (2.4) we have

$$\begin{aligned} \overline{N}(r,\tfrac{1}{F-1}) + \overline{N}(r,\tfrac{1}{G-1}) &\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}(r,G) + \overline{N}_{(2}(r,\tfrac{1}{G}) \\ &+N(r,\tfrac{1}{G-1}) + \tfrac{1}{2}\overline{N}(r,F) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) + \tfrac{1}{2}N_0(r,\tfrac{1}{F^{(1)}}) + N_0(r,\tfrac{1}{G^{(1)}}) \\ &+S(r,F) + S(r,G). \end{aligned} \tag{2.5}$$

Now we deduce from (2.2) and (2.5) that

$$T(r,F) \le 2\overline{N}(r,F) + 2\overline{N}(r,G) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G}) + \overline{N}_{(2}(r,\tfrac{1}{F})$$
$$+\overline{N}_{(2}(r,\tfrac{1}{G}) + \tfrac{1}{2}\overline{N}(r,F) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) + S(r,F) + S(r,G) \le \tfrac{5}{2}\overline{N}(r,F)$$
$$+2\overline{N}(r,G) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) + S(r,F) + S(r,G).$$

Subcase 1.2: $l \ge 2$. For this case we have

$$2\overline{N}_L(r,\tfrac{1}{F-1}) + 2\overline{N}_L(r,\tfrac{1}{G-1}) + \overline{N}_E^{(2}\left(r,\tfrac{1}{F-1}\right) + \overline{N}(r,\tfrac{1}{G-1})$$
$$\le N(r,\tfrac{1}{G-1}) + S(r,F) + S(r,G). \tag{2.6}$$

From (2.2), (2.3) and (2.6), we get

$$T(r,F) \le 2\overline{N}(r,F) + 2\overline{N}(r,G) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G}) + \overline{N}_{(2}(r,\tfrac{1}{F})$$
$$+\overline{N}_{(2}(r,\tfrac{1}{G}) + S(r,F) + S(r,G) \le 2\overline{N}(r,F) + 2\overline{N}(r,G) + N_2(r,\tfrac{1}{F})$$
$$+N_2(r,\tfrac{1}{G}) + S(r,F) + S(r,G).$$

Case 2: $l = 0$. Then we have

$$N_E^{1)}(r,\frac{1}{F-1}) = N_E^{1)}(r,\frac{1}{G-1}) + S(r,G),$$
$$\overline{N}_E^{(2}(r,\frac{1}{F-1}) = \overline{N}_E^{(2}(r,\frac{1}{G-1}) + S(r,G).$$

From (2.1) we have

$$\overline{N}(r,\tfrac{1}{F-1}) + \overline{N}(r,\tfrac{1}{G-1}) = N_E^{1)}(r,\tfrac{1}{F-1}) + \overline{N}_E^{(2}(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{F-1})$$
$$+\overline{N}_L(r,\tfrac{1}{G-1}) + \overline{N}(r,\tfrac{1}{G-1}) \le N_E^{1)}(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{F-1}) + N(r,\tfrac{1}{G-1})$$
$$\le \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}_{(2}(r,\tfrac{1}{G}) + 2\overline{N}_L(r,\tfrac{1}{F-1}) + \overline{N}_L(r,\tfrac{1}{G-1})$$
$$+N(r,\tfrac{1}{G-1}) + N_0(r,\tfrac{1}{F^{(1)}}) + N_0(r,\tfrac{1}{G^{(1)}}) + S(r,F) + S(r,G). \tag{2.7}$$

By (2.7) and Lemma 2.8 we get from (2.2)

$$T(r,F) \le 2\overline{N}(r,F) + 2\overline{N}(r,G) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G}) + \overline{N}_{(2}(r,\tfrac{1}{F}) + \overline{N}_{(2}(r,\tfrac{1}{G})$$
$$+2\overline{N}(r,F) + 2\overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,G) + \overline{N}(r,\tfrac{1}{G}) + S(r,F) + S(r,G) \le 4\overline{N}(r,F)$$
$$+3\overline{N}(r,G) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + 2\overline{N}(r,\tfrac{1}{F} + \overline{N}(r,\tfrac{1}{G}) + S(r,F) + S(r,G).$$

□

3. Proof of Main the Theorem

Proof of **Theorem 1.4**:

Proof. Let
$$F = \{L(f^n)\}^d \text{ and } G = \{L(g^n)\}^d.$$

Since $n \geq k+3$, from Lemma 2.3 with the value 1, it implies that $L(f^n) = 1$ has infinitely many solutions. So $\overline{E}_{L(f^n)}(S) \neq \phi$. Similarly $\overline{E}_{L(g^n)}(S) \neq \phi$. Also by the hypothesis F, G share $(1, l)$.
By Lemmas 2.1, 2.2 and 2.4, we get

$$\begin{aligned}(n-2k)T(r,f) &\leq T(r, L(f^n)) + S(r,f) \leq (k+1)T(r,f^n) + S(r,f)\\ &\leq (k+1)nT(r,f) + S(r,f)\end{aligned}$$

and
$$\begin{aligned}(n-2k)T(r,g) &\leq T(r, L(g^n)) + S(r,g) \leq (k+1)T(r,g^n) + S(r,g)\\ &\leq (k+1)nT(r,g) + S(r,g).\end{aligned}$$

Also we have

$$S(r,F) = S(r, L(f^n)) = S(r,f) \tag{3.1}$$

and

$$S(r,G) = S(r, L(g^n)) = S(r,g). \tag{3.2}$$

Now, if a is a zero of $L(f^n)$, then $F(a) = 0$ with multiplicity ≥ 2. By (ii) of Lemma 2.4 we get

$$\begin{aligned}N_2(r, \tfrac{1}{F}) &= 2\overline{N}\left(r, \tfrac{1}{L(f^n)}\right) \leq 2\overline{N}\left(r, \tfrac{1}{f^{n-k}}\right) + 2N\left(r, \tfrac{f^{n-k}}{L(f^n)}\right)\\ &\leq 2\overline{N}\left(r, \tfrac{1}{f}\right) + 2N\left(r, \tfrac{f^{n-k}}{L(f^n)}\right) \leq 2T(r,f) + 2N\left(r, \tfrac{f^{n-k}}{L(f^n)}\right)\\ S(r,f) &\leq 2T(r,f) + 2kT(r,f) + 2k\overline{N}(r,f) + S(r,f)\\ &= 2(k+1)T(r,f) + 2k\overline{N}(r,f) + S(r,f).\end{aligned} \tag{3.3}$$

$$\begin{aligned}N_2(r, \tfrac{1}{G}) &= 2\overline{N}\left(r, \tfrac{1}{L(g^n)}\right) \leq 2T(r,g) + 2N\left(r, \tfrac{g^{n-k}}{L(g^n)}\right) + S(r,g)\\ &\leq 2(k+1)T(r,g) + 2k\overline{N}(r,g) + S(r,g).\end{aligned} \tag{3.4}$$

Case 1: $H \not\equiv 0$. Then by Lemma 2.9 we get following subcases:
Subcase 1.1: If $l \geq 2$, then

$$\begin{aligned}T(r,F) \leq N_2(r, \tfrac{1}{F}) + N_2(r, \tfrac{1}{G}) + 2\overline{N}(r,F) + 2\overline{N}(r,G)\\ +S(r,F) + S(r,G).\end{aligned} \tag{3.5}$$

Using (3.1)–(3.4) in (3.5) we get

$$\begin{aligned}T(r, \{L(f^n)\}^d) \leq (2k+4)T(r,f) + 2k\overline{N}(r,f) + 4T(r,g) + 2N\left(r, \tfrac{g^{n-k}}{L(g^n)}\right)\\ +S(r,f) + S(r,g).\end{aligned} \tag{3.6}$$

Similarly,

$$T(r,\{L(g^n)\}^d) \le (2k+4)T(r,g) + 2k\overline{N}(r,g) + 4T(r,f) + 2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)$$
$$+S(r,f)+S(r,g). \tag{3.7}$$

Adding (3.6) and (3.7) we obtain

$$T(r,\{L(f^n)\}^d) + T(r,\{L(g^n)\}^d) \le (2k+8)\{T(r,f)+T(r,g)\} + 2k\{\overline{N}(r,f)$$
$$+\overline{N}(r,g)\} + 2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) + 2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f)+S(r,g). \tag{3.8}$$

By Lemma 2.4 we get

$$d\left\{(n-2k)T(r,f) + kN(r,f) + N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)\right\}$$
$$\le T(r,\{L(f^n)\}^d) + S(r,f) \tag{3.9}$$

and

$$d\left\{(n-2k)T(r,g) + kN(r,g) + N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right)\right\}$$
$$\le T(r,\{L(g^n)\}^d) + S(r,g). \tag{3.10}$$

Combining (3.9), (3.10) and using (3.8) we get

$$d(n-2k)\{T(r,f)+T(r,g)\} + dk\{N(r,f)+N(r,g)\} + dN\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)$$
$$+dN\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) \le (2k+8)\{T(r,f)+T(r,g)\} + 2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)$$
$$+2k\{\overline{N}(r,f)+\overline{N}(r,g)\} + 2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f)+S(r,g). \tag{3.11}$$

Since $d \ge 2$ we have

$$dN\left(r,\frac{f^{n-k}}{L(f^n)}\right) \ge 2N\left(r,\frac{f^{n-k}}{L(f^n)}\right), \tag{3.12}$$

$$dN\left(r,\frac{g^{n-k}}{L(g^n)}\right) \ge 2N\left(r,\frac{g^{n-k}}{L(g^n)}\right) \tag{3.13}$$

and

$$dk\{N(r,f)+N(r,g)\} \ge 2k\{\overline{N}(r,f)+\overline{N}(r,g)\}. \tag{3.14}$$

Using (3.12)–(3.14) we get from (3.11)

$$d(n-2k)\{T(r,f)+T(r,g)\} \le (2k+8)\{T(r,f)+T(r,g)\} + S(r,f)+S(r,g).$$

Therefore $d(n-2k) \le 2k+8 \Rightarrow n \le 2k + \frac{2k+8}{d}$, which contradicts (1.2).

Subcase 1.2: $l = 1$, then

$$T(r,F) \le \tfrac{5}{2}\overline{N}(r,F) + 2\overline{N}(r,G) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + \tfrac{1}{2}\overline{N}(r,\tfrac{1}{F}) + S(r,F) + S(r,G). \tag{3.15}$$

Using (3.1)–(3.4) in (3.15) we get

$$T(r,\{L(f^n)\}^d) \le (3k+5)T(r,f) + 2k\overline{N}(r,f) + 4T(r,g) + 2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f) + S(r,g). \tag{3.16}$$

Similarly,

$$T(r,\{L(g^n)\}^d) \le (3k+5)T(r,g) + 2k\overline{N}(r,g) + 4T(r,f) + 2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) + S(r,f) + S(r,g). \tag{3.17}$$

Adding (3.16) and (3.17) we obtain

$$T(r,\{L(f^n)\}^d) + T(r,\{L(g^n)\}^d) \le (3k+9)\{T(r,f)+T(r,g)\} + 2k\{\overline{N}(r,f) + \overline{N}(r,g)\} + 2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) + 2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f) + S(r,g). \tag{3.18}$$

Combining (3.9), (3.10) and using (3.18) we get

$$\begin{aligned} d(n-2k)\{T(r,f)+T(r,g)\} &+ dk\{N(r,f)+N(r,g)\} + dN\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) \\ &+ dN\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) \le (3k+9)\{T(r,f)+T(r,g)\} + 2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) \\ &+ 2k\{\overline{N}(r,f)+\overline{N}(r,g)\} + 2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.19}$$

Using (3.12)–(3.14) we have from (3.19)

$$d(n-2k)\{T(r,f)+T(r,g)\} \le (3k+9)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).$$

Therefore $d(n-2k) \le 3k+9 \Rightarrow n \le 2k + \frac{3k+9}{d}$, which contradicts (1.3).

Subcase 1.3: $l = 0$, then

$$T(r,F) \le 4\overline{N}(r,F) + 3\overline{N}(r,G) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + 2\overline{N}(r,\tfrac{1}{F}) + \overline{N}(r,\tfrac{1}{G}) + S(r,F) + S(r,G). \tag{3.20}$$

Using (3.1)–(3.4) in (3.20) we get

$$T(r,\{L(f^n)\}^d) \le (6k+8)T(r,f) + 2k\overline{N}(r,f) + (2k+6)T(r,g) + 2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f) + S(r,g). \tag{3.21}$$

Similarly,

$$\begin{aligned} T(r,\{L(g^n)\}^d) &\le (6k+8)T(r,g)+2k\overline{N}(r,g)+(2k+6)T(r,f) \\ &+2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.22}$$

Adding (3.21) and (3.22) we obtain

$$\begin{aligned} T(r,\{L(f^n)\}^d)+T(r,\{L(g^n)\}^d) &\le (8k+14)\{T(r,f)+T(r,g)\}+2k\{\overline{N}(r,f) \\ +\overline{N}(r,g)\}+2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)&+2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.23}$$

By (3.9), (3.10) and (3.23) we get

$$\begin{aligned} d(n-2k)\{T(r,f)+T(r,g)\}+dk\{N(r,f)+N(r,g)\}+dN\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) \\ +dN\left(r,\tfrac{g^{n-k}}{L(g^n)}\right)\le(8k+14)\{T(r,f)+T(r,g)\}+2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right) \\ +2k\{\overline{N}(r,f)+\overline{N}(r,g)\}+2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right)+S(r,f)+S(r,g). \end{aligned} \tag{3.24}$$

Using (3.12)–(3.14) in (3.24) we obtain

$$d(n-2k)\{T(r,f)+T(r,g)\}\le(8k+14)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

Therefore $d(n-2k)\le 8k+14\Rightarrow n\le 2k+\frac{8k+14}{d}$, which contradicts (1.4).
Case 2: $H\equiv 0$. Integrating twice we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B,$$

where A ($\neq 0$) and B are constants.
Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)} \tag{3.25}$$

and

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{3.26}$$

Next we consider the following three subcases :
Subcase 2.1: $B\neq 0,-1$. Then from (3.26) we have

$$\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)=\overline{N}(r,F).$$

By Nevanlinna second fundamental theorem

$$\begin{aligned} T(r,G) &\le \overline{N}(r,G)+\overline{N}(r,\tfrac{1}{G})+\overline{N}\left(r,\tfrac{1}{G-\frac{B+1}{B}}\right)+S(r,G) \\ &\le \overline{N}(r,G)+N_2(r,\tfrac{1}{G})+\overline{N}(r,F)+S(r,G), \end{aligned}$$

$$\Rightarrow T(r,\{L(g^n)\}^d) \le 2T(r,g) + 2N\left(r, \tfrac{g^{n-k}}{L(g^n)}\right) + \overline{N}(r,g)$$
$$+\overline{N}(r,f) + S(r,f) + S(r,g). \tag{3.27}$$

If $A-B-1 \neq 0$, then it follows from (3.25) that

$$\overline{N}\left(r, \frac{1}{F - \frac{-A+B+1}{B+1}}\right) = \overline{N}(r, \frac{1}{G}).$$

Again by Nevanlinna second fundamental theorem we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}(r,\tfrac{1}{F}) + \overline{N}\left(r, \tfrac{1}{F - \frac{-A+B+1}{B+1}}\right) + S(r,F) \\ &\le \overline{N}(r,F) + N_2(r,\tfrac{1}{F}) + N_2(r,\tfrac{1}{G}) + S(r,F) + S(r,G), \end{aligned}$$

$$\Rightarrow T(r,\{L(f^n)\}^d) \le \overline{N}(r,f) + 2T(r,f) + 2N\left(r, \tfrac{f^{n-k}}{L(f^n)}\right)$$
$$+2k\overline{N}(r,g) + 2(k+1)T(r,g) + S(r,f) + S(r,g). \tag{3.28}$$

Combining (3.9), (3.10) and using (3.27), (3.28) we get

$$\begin{aligned} &d(n-2k)\{T(r,f)+T(r,g)\} + dk\{N(r,f)+N(r,g)\} + dN\left(r, \tfrac{f^{n-k}}{L(f^n)}\right) \\ &+dN\left(r, \tfrac{g^{n-k}}{L(g^n)}\right) \le 2T(r,f) + (2k+5)T(r,g) + 2N\left(r, \tfrac{f^{n-k}}{L(f^n)}\right) \\ &+2k\{\overline{N}(r,f)+\overline{N}(r,g)\} + 2N\left(r, \tfrac{g^{n-k}}{L(g^n)}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.29}$$

By using (3.12)–(3.14) in (3.29), we obtain

$$(n-2k-\frac{2}{d})T(r,f) + (n-2k-\frac{2k+5}{d})T(r,g) \le S(r,f) + S(r,g),$$

$$\Rightarrow (n-2k-\frac{2k+5}{d})\{T(r,f)+T(r,g)\} \le S(r,f) + S(r,g),$$

which contradict our assumptions (1.2)–(1.4).

Therefore $A-B-1=0$. Then by (3.25)

$$\overline{N}\left(r, \frac{1}{F+\frac{1}{B}}\right) = \overline{N}(r,G).$$

By Nevanlinna second fundamental theorem and Lemma 2.4 we get

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r, \frac{1}{F+\frac{1}{B}}\right) + S(r,F) \\ &\le \overline{N}(r,F) + N_2(r,\frac{1}{F}) + \overline{N}(r,G) + S(r,F), \end{aligned}$$

$$\Rightarrow T(r,\{L(f^n)\}^d) \leq \overline{N}(r,f)+2T(r,f)+2N\left(r,\frac{f^{n-k}}{L(f^n)}\right)+\overline{N}(r,g)$$
$$+ S(r,f)+S(r,g). \tag{3.30}$$

Adding (3.9), (3.10) and using (3.27), (3.30) we get

$$d(n-2k)\{T(r,f)+T(r,g)\}+dk\{N(r,f)+N(r,g)\}$$
$$+dN\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)+dN\left(r,\tfrac{g^{n-k}}{L(g^n)}\right)\leq 2T(r,f)+2T(r,g)+2N\left(r,\tfrac{f^{n-k}}{L(f^n)}\right)$$
$$+2\{\overline{N}(r,f)+\overline{N}(r,g)\}+2N\left(r,\tfrac{g^{n-k}}{L(g^n)}\right)+S(r,f)+S(r,g). \tag{3.31}$$

By using (3.12)–(3.14) we get from 3.31

$$(n-2k-\frac{2}{d})\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which again violate assumptions (1.2)–(1.4).

Subcase 2.2: $B=-1$. Then

$$G=\frac{A}{A+1-F}$$

and

$$F=\frac{(1+A)G-A}{G}.$$

If $A+1\neq 0$,

$$\overline{N}\left(r,\frac{1}{F-(A+1)}\right)=\overline{N}(r,G),$$
$$\overline{N}\left(r,\frac{1}{G-\frac{A}{A+1}}\right)=\overline{N}(r,\frac{1}{F}).$$

By similar argument as Subcase 2.1 we get a contradiction.
Therefore $A+1=0$ then $FG=1\Rightarrow\{L(f^n)\}^d.\{L(g^n)\}^d=1$. Thus we get $L(f^n).L(g^n)=h$, where $h^d=1$. Then by Lemma 2.5 we get possibility 2. of the Theorem.

Subcase 2.3: $B=0$. Then (3.25) and (3.26) gives $G=\frac{F+A-1}{A}$ and $F=AG+1-A$
If $A-1\neq 0$, $\overline{N}\left(r,\frac{1}{A-1+F}\right)=\overline{N}(r,\frac{1}{G})$ and $\overline{N}\left(r,\frac{1}{G-\frac{A-1}{A}}\right)=\overline{N}(r,\frac{1}{F})$. Proceeding similarly as in Subcase 2.1 we get a contradiction.
Therefore, $A-1=0$ then $F\equiv G$ i.e., $L(f^n)=hL(g^n)$ for some $h\in\mathbb{C}$ such that $h^d=1$. This completes the proof. □

Proof of **Corollary 1.2**:

Proof. Putting $L(f^n) = (f^n)^{(k)}$ and $L(g^n) = (g^n)^{(k)}$ we get the following cases from Theorem 1.4.
Case A. $(f^n)^{(k)} = h(g^n)^{(k)}$ where $h^d = 1$.
By Case I of Theorem 1.2 of [10] we get the possibility 1.
Case B. $(f^n)^{(k)}.(g^n)^{(k)} = h$ where $h^d = 1$.
For $L(f^n) = (f^n)^{(k)}$ and $L(g^n) = (g^n)^{(k)}$ in Lemma 2.5 we get the possibility 2.

□

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FIXED POINTS OF GENERALIZED (α, ψ, φ)-CONTRACTIVE MAPS AND PROPERTY(P) IN S-METRIC SPACES

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Abstract. In this paper, we have introduced generalized (α, ψ, φ)-contractive maps and proved the existence and uniqueness of fixed points in complete S-metric spaces. We have also proved that these maps satisfy property (P). The results presented in this paper extend several well known comparable results in metric and G-metric spaces. We have provided an example in support of our result.

Keywords: S-metric space, property(P), generalized contractive maps, fixed points

1. Introduction and Preliminaries

Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in various areas. Finding the existence of fixed points of a self map by considering more general ambient spaces is an interesting aspect. In this course of development, some authors have tried to give generalizations of metric spaces in various ways. In 2005, Mustafa and Sims [13] introduced a new structure of metric spaces which are called G-metric spaces as a generalization of metric spaces to develop and introduce new concepts on contraction maps and proved the existence of fixed points of various mappings in this new space. For more works on G-metric spaces, we refer [3, 14, 21]. In 2007, Sedghi [18] introduced D^*-metric spaces which is a probable modification of the definition

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of D-metric spaces introduced by Dhage [7] and proved some basic properties of D^*-metric spaces [17, 18]. In 2012, Sedghi, Shobe and Aliouche [19] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. Sedghi, Shobe and Aliouche [19] asserted that S-metric space is a generalization of G-metric space. But, very recently Dung, Hieu and Radojevic [8] have verified by example (Example 2.1 and Example 2.2) that S-metric space is not a generalization of G- metric space or vice versa. Therefore, the classes of G-metric spaces and S- metric spaces are different. Recent papers dealing with fixed point theorems for mappings satisfying certain contractive conditions on S-metric spaces can be referred in [1, 2, 8, 12, 15, 16, 20].

Now we provide some preliminaries and basic definitions which we use throughout this paper. We start with a G- metric spaces introduced by Mustafa and Sims [13].

Definition 1.1. [13] Let X be a non-empty set, $G : X^3 \to [0, \infty)$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(z, x, y) = ...$ (symmetry in all three variables),

(G5) $G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric (G-metric) and the pair (X, G) is called a G-metric space.

Definition 1.2. [11] A mapping $\psi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies: (i) ψ is continuous (ii) ψ non-decreasing and (iii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions by Ψ.

We denote $\Phi = \{\varphi : [0, \infty) \to [0, \infty)\ \varphi$ is continuous and non-decreasing$\}$.

Remark 1.1. [4] If $\psi \in \Psi$ and $\varphi \in \mathit{\Phi}$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then $\varphi(0) = 0$. Therefore $\varphi \in \Psi$.

Definition 1.3. [9] Let X be a non-empty set and T be a self map of X. We denote the set of all fixed points of T by $F(T)$, where $F(T) \neq \emptyset$. Then, T is said to satisfy property (P) if $F(T) = F(T^n)$ for all $n \in \mathbb{N}$.

Here we note that even though, a map $f : X \to X$ has a unique fixed point, it may not have property (P).

In [4] Bousselsal et.al proved the existence and uniqueness of fixed points and property (P) in G-metric spaces.

Theorem 1.1. [4] *Let (X, G) be a complete G-metric space and $f : X \to X$ be a mapping. If there exists $\psi \in \Psi$ and $\varphi \in \Phi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, such that*

$$\psi(G(fx, fy, fz)) \leqslant \varphi\big(\max\big\{G(x, y, y), G(x, fx, fx), G(y, fy, fy), \tag{1.1}$$

$$G(z, fz, fz), \alpha G(fx, fx, y)+(1-\alpha)G(fy, fy, z), \beta G(x, fx, fx)+(1-\beta)G(y, fy, fy)\big\}\big)$$

for all $x, y, z \in X$, where $\alpha, \beta \in (0, 1)$.
Then f has a unique fixed point (say u) and f is G-continuous at u. Further, f has property (P).

Note: In view of Remark 1.1, we can choose $\varphi \in \Psi$ in Theorem 1.1.

Remark 1.2.

$$\begin{aligned}\text{Since } \max\big\{&G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \alpha G(fx, fx, y)\\ &+(1-\alpha)G(fy, fy, z), \beta G(x, fx, fx) + (1-\beta)G(y, fy, fy)\big\}\\ = \max\big\{&G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \alpha G(fx, fx, y)\\ &+(1-\alpha)G(fy, fy, z)\big\}\end{aligned}$$

so that we need not consider the β terms in the inequality (1.1).

In 2012, Sedghi, Shobe and Aliouche [19] introduced S-metric spaces as follows:

Definition 1.4. [19] Let X be a non-empty set. An $S-$metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$

(S1) $S(x, y, z) \geqslant 0$,

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$ and

(S3) $S(x, y, z) \leqslant S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 1.1. (Example 2.4 [19]). Let (X, d) be a metric space. Define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then S is an S-metric on X. This S-metric is called the S-metric induced by the metric d.

Example 1.2. (Example 1.9 [8]). Let $X = \mathbb{R}$ and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is an S-metric space.

Example 1.3. (Example 2.2 [8]). There exists an S-metric which is not a G-metric. Let (X, S) be the S-metric space in Example 1.2. We have
$S(1, 0, 2) = |0 + 2 - 2| + |0 - 2| = 2, S(2, 0, 1) = |0 + 1 - 4| + |0 - 1| = 4$.
Then $S(1, 0, 2) \neq S(2, 0, 1)$. So that $(G4)$ fails. Hence S is not a G-metric.

Example 1.4. (Example 2.1, [8]). There exists a G-metric which is not an S-metric. Let $X = \{a, b\}$. Define $G : X^3 \to [0, \infty)$ by $G(a, a, a) = G(b, b, b) = 0$, $G(a, b, b) = 2$, $G(a, a, b) = 1$ and extend G to all X^3 by using $(G4)$. Then G is a G-metric but not an S-metric. Since $2 = G(a, b, b) \nleq 1 = G(a, a, b) + G(b, b, b) + G(b, b, b)$. This shows that G is not an S-metric on X.

Remark 1.3. From Example 1.3 and Example 1.4, we can conclude that the class of S-metrics and the class of G-metrics are distinct.

The following lemmas are very useful in our subsequent discussions in proving our main results.

Lemma 1.1. [19] *In an S-metric space, we have* $S(x, x, y) = S(y, y, x)$.

Lemma 1.2. [8] *Let (X, S) be an S-metric space. Then*
(i) $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ *and*
(ii) $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$.

Definition 1.5. [19] Let (X, S) be an S-metric space. We define the following:

(i) A sequence $\{x_n\}$ in X converge to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geqslant n_0$.

(iii) The S-metric space (X, S) is said to be complete if each Cauchy sequence in x is convergent.

Definition 1.6. [12] Let (X, S) and (Y, S') be two S-metric spaces. Then the function $f : X \to Y$ is S-continuous at $x \in X$ if it is S-sequentially continuous at x, that is, whenever $\{x_n\}$ is S-convergent to x, we have $f(x_n)$ is S'-convergent to $f(x)$.

Lemma 1.3. [19] *Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.*

Lemma 1.4. [19] *Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that* $\lim\limits_{n\to\infty} x_n = x$ *and* $\lim\limits_{n\to\infty} y_n = y$, *then* $\lim\limits_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1.5. [1] *Any S-metric space is a Hausdorff space.*

In 2012, Sedghi [19] proved an analogue of Banach's contraction principle in S-metric space.

Definition 1.7. [19] Let (X,S) be an S-metric space. A map $f:X\to X$ is said to be an S-contraction if there exists a constant $0\leqslant\lambda<1$ such that $S(f(x),f(x),f(y))\leq\lambda S(x,x,y)$ for all $x,y\in X$.

Theorem 1.2. [19] *Let (X,S) be a complete $S-$metric space and $f:X\to X$ be a contraction. Then f has a unique fixed point $u\in X$. Furthermore, for any $x\in X$ we have $\lim\limits_{n\to\infty}f^n(x)=u$ with $S(f^n(x),f^n(x),u)\leq\frac{2\lambda^n}{1-\lambda}(Sx,x,f(x))$.*

We now introduce the following definition.

Definition 1.8. Let (X,S) be an S-metric space. Let $f:X\to X$ be a self map of X. If there exists $\alpha\in(0,1)$ and $\psi,\varphi\in\Psi$ such that

$$\begin{aligned}(1.2)\qquad \psi(S(fx,fy,fz))\leqslant\varphi\Big(&\max\big\{S(x,y,z),S(x,x,fx),S(y,y,fy),\\&S(z,z,fz),\alpha S(fx,fx,y)+(1-\alpha)S(fy,fy,z)\big\}\Big)\end{aligned}$$

for all $x,y,z\in X$. Then we say that f is a generalized (α,ψ,φ)-contractive map on X.

Remark 1.4. We note that S-contraction map is a generalized (α,ψ,φ)-contraction map with $\psi(t)=t$, for all $t\geqslant0$ and $\varphi(t)=\lambda t$, for all $t\geqslant0$ where λ is an S-contraction constant. But its converse is not true (Example 3.1). Thus the class of S-contraction map is a proper subset of the class of all generalized (α,ψ,φ)-contraction map.

Hence we study the existence of fixed points of generalized (α,ψ,φ)-contractions in S-metric spaces.

2. Main Results

We start this section with following lemma which is useful in proving our main results.

Lemma 2.1. *Let (X,S) be an S-metric space and $\{x_n\}$ be a sequence in X such that*

$$(2.1)\qquad \lim_{n\to\infty}S(x_n,x_n,x_{n+1})=0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k>n_k>k$ such that

$$(2.2)\qquad S(x_{m_k},x_{m_k},x_{n_k})\geqslant\epsilon,\ S(x_{m_k-1},x_{m_k-1},x_{n_k})<\epsilon \text{ and}$$

(*i*) $\lim\limits_{k\to\infty}S(x_{m_k},x_{m_k},x_{n_k})=\epsilon$,

(*ii*) $\lim\limits_{k\to\infty}S(x_{m_k},x_{m_k},x_{n_k-1})=\epsilon$,

(iii) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon.$

Proof. Let $\{x_n\} \subset X$ be not Cauchy. Then there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon. \tag{2.3}$$

We choose m_k, the least positive integer satisfying (2.3). Then $m_k > n_k > k$ with $S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon$ and $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$. Hence (2.2) holds.

From (2.2), we have

$$\epsilon \leqslant S(x_{m_k}, x_{m_k}, x_{n_k}). \tag{2.4}$$

On taking the lower limit in (2.4), we get

$$\epsilon \leq \liminf_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}). \tag{2.5}$$

By Lemma 1.2, we have

$$\begin{aligned} S(x_{m_k}, x_{m_k}, x_{n_k}) &\leqslant 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}) \\ &= 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \\ &< 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \epsilon. \end{aligned} \tag{2.6}$$

From (2.4), (2.6) and on taking the upper limit as $k \to \infty$, we have

$$\limsup_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon. \tag{2.7}$$

From (2.5) and (2.7), we obtain

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon. \tag{2.8}$$

Hence (i) is proved.

Again, from (2.2), by Lemma 1.1 and Lemma 1.2, we have

$$\begin{aligned} \epsilon &\leq S(x_{m_k}, x_{m_k}, x_{n_k}) = S(x_{n_k}, x_{n_k}, x_{m_k}) \\ &\leqslant 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \\ &= 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}). \end{aligned} \tag{2.9}$$

From (2.9) and on taking the upper limit as $k \to \infty$, we obtain

$$\epsilon \leq \limsup_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}). \tag{2.10}$$

Once again, by Lemma 1.2 and (2.3), we get

$$S(x_{m_k}, x_{m_k}, x_{n_k-1}) = S(x_{n_k-1}, x_{n_k-1}, x_{m_k})$$

$$
\begin{aligned}
&\leqslant 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}) \\
&= 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{m_k}, x_{m_k}, x_{n_k}). \qquad (2.11)
\end{aligned}
$$

Now, on taking the upper limit as $k \to \infty$ in (2.11), we obtain

$$\limsup_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) \leqslant \epsilon. \tag{2.12}$$

By (2.10) and (2.12), we get

$$\limsup_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \epsilon. \tag{2.13}$$

From (2.9), we obtain

$$S(x_{m_k}, x_{m_k}, x_{n_k-1}) \geq \epsilon - 2S(x_{n_k}, x_{n_k}, x_{n_k-1}). \tag{2.14}$$

Hence on taking the lower limit as $k \to \infty$ in (2.14), we get

$$\epsilon \leq \liminf_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}). \tag{2.15}$$

Therefore from (2.13) and (2.15), we obtain

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) = \epsilon. \tag{2.16}$$

So, (ii) is proved.

Again, from (2.2), by Lemma 1.1 and Lemma 1.2, we have

$$
\begin{aligned}
\epsilon &\leq S(x_{m_k}, x_{m_k}, x_{n_k}) = S(x_{n_k}, x_{n_k}, x_{m_k}) \\
&\leqslant 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \\
&= 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}) \\
&\leq 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}). \qquad (2.17)
\end{aligned}
$$

From (2.17) and on taking the upper limit as $k \to \infty$, we obtain

$$\epsilon \leq \limsup_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}). \tag{2.18}$$

Again, by Lemma 1.1 and Lemma 1.2, we have

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}). \tag{2.19}$$

On taking the upper limit as $k \to \infty$ in (2.19) and by using (2.16), we have

$$\limsup_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \leq \epsilon. \tag{2.20}$$

From (2.17), we obtain

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) \geq \epsilon - 2S(x_{m_k}, x_{m_k}, x_{m_k-1}). \tag{2.21}$$

Hence on taking the lower limit as $k \to \infty$ in (2.21), we get

$$\epsilon \leq \liminf_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}). \tag{2.22}$$

Therefore by combining (2.18), (2.20) and (2.22), we obtain

$$\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{2.23}$$

So, (iii) is proved. Hence the lemma follows. □

In the following we prove the main result of this paper.

Theorem 2.1. *Let (X, S) be a complete S-metric space and let f be a generalized (α, ψ, φ)-contractive map. If there exists $\psi, \varphi \in \Psi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$, then f has a unique fixed point (say u) and f is S-continuous at u.*

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and we are through.

Now, we assume that $x_n \neq x_{n+1}$ for all n. By (1.2) and substituting $x = y = x_{n-1}, z = x_n$, we have

$$\begin{aligned}
\psi(S(x_n, x_n, x_{n+1})) = \psi(S(fx_{n-1}, fx_{n-1}, fx_n)) &\leqslant \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n),\\
&S(x_{n-1}, x_{n-1}, fx_{n-1}), S(x_{n-1}, x_{n-1}, fx_n), S(x_n, x_n, fx_n),\\
&\alpha S(fx_{n-1}, fx_{n-1}, x_{n-1}) + (1-\alpha)S(fx_{n-1}, fx_{n-1}, x_n)\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n),\\
&S(x_n, x_n, x_{n+1}), \alpha S(x_n, x_n, x_{n-1}) + (1-\alpha)S(x_n, x_n, x_n)\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \alpha S(x_n, x_n, x_{n-1})\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\big\}\big). \qquad (2.24)
\end{aligned}$$

If $S(x_n, x_n, x_{n+1}) > S(x_{n-1}, x_{n-1}, x_n)$, then (2.24) becomes

$$\psi(S(x_n, x_n, x_{n+1})) \leqslant \varphi(S(x_n, x_n, x_{n+1})) < \psi(S(x_n, x_n, x_{n+1})), \tag{2.25}$$

a contradiction. Hence $S(x_{n-1}, x_{n-1}, x_n)$ is the maximum. Therefore

$$\psi(S(x_n, x_n, x_{n+1})) \leqslant \varphi\big(S(x_{n-1}, x_{n-1}, x_n)\big) < \psi(S(x_{n-1}, x_{n-1}, x_n)). \tag{2.26}$$

By using the property of ψ, φ and from (2.26), we obtain

$$S(x_n, x_n, x_{n+1}) \leqslant S(x_{n-1}, x_{n-1}, x_n) \text{ for all } n.$$

Hence $\{S(x_n, x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers. Then there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = r. \tag{2.27}$$

On letting $n \to \infty$ in (2.26) and using (2.27), we have $\psi(r) \leqslant \varphi(r) < \psi(r)$, a contradiction. Hence $r = 0$.

We now prove that $\{x_n\}$ is an S-Cauchy sequence. If possible $\{x_n\}$ is not S-Cauchy. By Lemma 2.1, there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that $S(x_{m_k}, x_{m_k}, x_{n_k}) \geqslant \epsilon$, $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$ and the identities (i)-(ii) of Lemma 2.1.
Putting $x = y = x_{m_k-1}, z = x_{n_k-1}$ and applying (1.2), we get

$$\begin{aligned}
\psi(S(x_{m_k}, x_{m_k}, x_{n_k})) &= \psi(S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k-1}))\\
&\leqslant \varphi\big(\max\big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}),\\
&\quad S(x_{m_k-1}, x_{m_k-1}, fx_{m_k-1}), S(x_{n_k-1}, x_{n_k-1}, fx_{n_k-1}),\\
&\quad \alpha S(fx_{m_k-1}, fx_{m_k-1}, x_{m_k-1}) + (1-\alpha)S(fx_{m_k-1}, fx_{m_k-1}, x_{n_k-1})\big\}\big),\\
&= \varphi\big(\max\big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k}, x_{m_k}, x_{m_k-1}),\\
&\quad S(x_{m_k}, x_{m_k}, x_{m_k-1}), S(x_{n_k}, x_{n_k}, x_{n_k-1}),\\
&\quad \alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\big\}\big)\\
&= \varphi\big(\max\big\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}), S(x_{m_k}, x_{m_k}, x_{m_k-1}), S(x_{n_k}, x_{n_k}, x_{n_k-1}),\\
&\quad \alpha S(x_{m_k}, x_{m_k}, x_{m_k-1}) + (1-\alpha)S(x_{m_k}, x_{m_k}, x_{n_k-1})\big\}\big).
\end{aligned} \tag{2.28}$$

On letting $k \to \infty$ in (2.28), using (2.27) and Lemma 2.1, we obtain

$$\psi(\epsilon) \leqslant \varphi\big(\max\big\{\epsilon,\ 0,\ 0,\ (1-\alpha)\epsilon\big\}\big) = \varphi(\epsilon) < \psi(\epsilon),$$

a contradiction.
Hence $\{x_n\}$ is an S-Cauchy sequence. Since (X, S) is complete, there exists $u \in X$ such that $x_n \to u$.

We now show that u is a fixed point of f. Here by Lemma 1.4, we note that

$$\lim_{n\to\infty} S(x_n, x_n, fu) = S(u, u, fu).$$

Suppose that $f(u) \neq u$ and we consider

$$\begin{aligned}
\psi(S(fu, fu, x_n)) = \psi(S(fu, fu, fx_{n-1})) &\leqslant \varphi\big(\max\big\{S(u, u, x_{n-1}), S(u, u, fu), S(u, u, fu),\\
&\quad S(x_{n-1}, x_{n-1}, fx_{n-1}), \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1})\big\}\big)\\
&= \varphi\big(\max\big\{S(u, u, x_{n-1}), S(u, u, fu), S(x_{n-1}, x_{n-1}, x_n),\\
&\quad \alpha S(fu, fu, u) + (1-\alpha)S(fu, fu, x_{n-1}), S(fu, fu, u)\big\}\big).
\end{aligned} \tag{2.29}$$

On letting $n \to \infty$ in (2.29), we have

$$\psi(S(fu, fu, u)) \leqslant \varphi\big(\max\big\{S(u, u, u), S(u, u, fu), S(u, u, u),$$

$$\alpha S(fu,fu,u)+(1-\alpha)S(fu,fu,u)\}\big)=\varphi(S(fu,fu,u))<\psi(S(fu,fu,u)),$$

a contradiction. Hence $fu=u$.

Next we prove uniqueness of fixed point. Suppose u and v are two distinct fixed points of f. Now, we consider

$$\begin{aligned}\psi(S(u,u,v)) &= \psi(S(fu,fu,fv))\\ &\leqslant \varphi\big(\max\big\{(S(u,u,v),S(u,u,fu),S(u,u,fu),S(v,v,fv),\\ &\quad \alpha S(fu,fu,u)+(1-\alpha)S(fu,fu,v)\big\}\big)\\ &= \varphi\big(\max\big\{S(u,u,v),(1-\alpha)S(u,u,v)\big\}\big)\\ &= \varphi(S(u,u,v))<\psi(S(u,u,v)),\end{aligned}$$

a contradiction. Therefore $u=v$.

Finally we prove that f is S-continuous at u. Let $\{x_n\}$ be a sequence in X such that $x_n\to u$ as $n\to\infty$. We show that $fx_n\to fu$ as $n\to\infty$. For this purpose, we consider

$$\begin{aligned}\psi(S(u,u,fx_n)) &= \psi(S(fu,fu,fx_n))\\ &\leqslant \varphi\big(\max\big\{S(u,u,x_n),S(u,u,fu),S(u,u,fu),S(x_n,x_n,fx_n),\\ &\quad \alpha S(fu,fu,u)+(1-\alpha)S(fu,fu,x_n)\big\}\big),\\ &= \varphi\big(\max\big\{S(u,u,x_n),S(u,u,u),S(x_n,x_n,x_{n+1}),\\ &= \alpha S(u,u,u)+(1-\alpha)S(u,u,x_n)\big\}\big).\end{aligned}\tag{2.30}$$

By taking the limit on both sides of (2.30), and using the continuity of φ, we have $\lim\limits_{n\to\infty}\psi(S(fu,fu,fx_n))=0$.
By the continuity of ψ, we have $\psi(\lim\limits_{n\to\infty}S(fu,fu,fx_n))=0$.
i.e., $\psi(\lim\limits_{n\to\infty}S(fx_n,fx_n,fu)=0$ (by Lemma 1.1).
Again, by property of ψ we have $\lim\limits_{n\to\infty}S(fx_n,fx_n,fu))=0$.
Hence by the definition of continuity of f, it follows that $fx_n\to fu$ as $n\to\infty$. Therefore, f is S-continuous at u. $\square$

Theorem 2.2. *Under the hypotheses of Theorem 2.1 f has Property (P).*

Proof. In view of the proof of Theorem 2.1, f has a fixed point. Therefore $F(f^n)\neq\emptyset$. Now, we fix $n>1$ and assume that $u\in F(f^n)$. That is $f^nu=u$. We show that $u\in F(f)$. Assume $f(u)\neq u$, we consider

$$\begin{aligned}\psi(S(u,u,fu)) &= \psi(S(f^nu,f^nu,f^{n+1}u))=\psi(S(ff^{n-1}u,ff^{n-1}u,ff^nu))\\ &\leqslant \varphi\big(\max\big\{S(f^{n-1}u,f^{n-1}u,f^nu),S(f^{n-1}u,f^{n-1}u,ff^{n-1}u),\\ &\quad S(f^{n-1}u,f^{n-1}u,ff^{n-1}u),S(f^nu,f^nu,ff^nu),\\ &\quad \alpha S(ff^{n-1}u,ff^{n-1}u,f^{n-1}u)\end{aligned}$$

$$\begin{aligned}
& +(1-\alpha)S(ff^{n-1}u, ff^{n-1}u, f^n u)\}\big), \\
= \;& \varphi\big(\max\{S(f^{n-1}u, f^{n-1}u, u), S(u,u,f^{n-1}u), S(u,u,f^{n-1}u), \\
& S(u,u,fu), \alpha S(u,u,f^{n-1}u) + (1-\alpha)S(u,u,u)\}\big), \\
= \;& \varphi\big(\max\{S(u,u,f^{n-1}u), S(u,u,fu)\}\big).
\end{aligned} \tag{2.31}$$

If $S(u,u,fu)$ is the maximum, then from (2.31), we have
$\psi(S(u,u,fu)) \leqslant \varphi(S(fu,fu,u)) = \varphi(S(u,u,fu)) < \psi(S(u,u,fu))$,
a contradiction. Consequently, $S(u,u,f^{n-1}u)$ is the maximum. Therefore, from (2.31) and Lemma 1.1, we obtain

$$\begin{aligned}
\psi(S(u,u,fu)) \quad & = \psi(S(f^n u, f^n u, f^{n+1}u)) \leqslant \varphi(S(u,u,f^{n-1}u)) \\
& = \varphi(S(f^n u, f^n u, f^{n-1}u)) < \psi(S(f^n u, f^n u, f^{n-1}u)) \\
& = \psi(S(f^{n-1}u, f^{n-1}u, f^n u)).
\end{aligned} \tag{2.32}$$

Since ψ is non decreasing, from (2.32), it follows that

$$S(f^n u, f^n u, f^{n+1}u) \leqslant S(f^{n-1}u, f^{n-1}u, f^n u).$$

Hence $\{S(f^n u, f^n u, f^{n+1}u)\}$ is a decreasing sequence of positive real numbers. Then, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} S(f^n u, f^n u, f^{n+1}u) = r. \tag{2.33}$$

On letting $n \to \infty$ in (2.32) and using (2.33), we get $\psi(r) \leqslant \varphi(r) < \psi(r)$, a contradiction. Therefore $r = 0$.
Hence $\psi(S(u,u,fu)) = \lim_{n\to\infty}(\psi(S(f^n u, f^n u, f^{n+1}u)) = 0$. That is, $fu = u$. Therefore, $u \in F(f)$. Hence f has property (P). $\square$

In Section 3 we draw some corollaries from our results and provide a supportive example.

3. Corollaries and an Example

If ψ is the identity mapping on $[0,\infty)$ in Theorem 2.1, we have the following
.

Corollary 3.1. *Let (X,S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exists $\alpha \in (0,1)$, $\varphi \in \Psi$ satisfying $\varphi(t) < t$ for $t > 0$ such that*

$$\begin{aligned}
S(fx,fy,fz) \leqslant \varphi\big(\max\{& S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \\
& \alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\}\big),
\end{aligned}$$

for all $x,y,z \in X$. Then f has a unique fixed point $(\text{say } u)$ and f is S-continuous at u.

Here we observe that the φ that is used in the inequality (3.1) is a Boyd-Wong [5] type contraction.

Corollary 3.2. *Let (X, S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist $\lambda, \alpha \in (0,1)$, such that*

$$\begin{aligned} S(fx, fy, fz) \leqslant \lambda \max\big\{&S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz),\\ &\alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\big\}, \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$. Then f has a unique fixed point (say u) and f is S-continuous at u.

Proof: By choosing $\varphi(t) = \lambda t$, for all $t \geqslant 0$ in Corollary 3.1, then the conclusion follows.

Corollary 3.3. *Let (X, S) be a complete $S-$metric space and $f : X \to X$ be a mapping. Assume there exist a constant $0 \leqslant \lambda < 1$, $\alpha \in (0,1), \psi$, such that $S(fx, fy, fz) \leq \lambda S(x, y, z)$ for all $x, y, z \in X$. Then f has a unique fixed point $u \in X$.*

If $\alpha = \frac{1}{2}$ in the inequality (1.2), we have the following corollary.

Corollary 3.4. *Let (X, S) be a complete S-metric space and $f : X \to X$ be a mapping. Assume that there exist $\psi, \varphi \in \Psi$ satisfying $\varphi(t) < \psi(t)$ for all $t > 0$ such that*

$$\begin{aligned} \psi(S(fx, fy, fz)) \leqslant \varphi\big(\max\big\{&S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz),\\ &\tfrac{1}{2}[S(fx,fx,y) + S(fy,fy,z)]\big\}\big), \end{aligned} \tag{3.2}$$

for all $x, y, z \in X$. Then f has a unique fixed point (say u) and f is S-continuous at u.

In the following, we provide an example in support of our result.

$$\text{Let } M_\alpha(x,y,z) = \max\big\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz),$$

$$\alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\big\}.$$

Example 3.1. Let $X = [0, \frac{7}{4}]$. We define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = |x - z| + |y - z|$ [19] and $f : X \to X$ by

$$f(x) = \begin{cases} \frac{7}{4} - x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{x+1}{2} & \text{if } x \in (\frac{1}{2}, \frac{7}{4}]. \end{cases}$$

We define $\psi,\ \varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{t}{2} \quad \text{for all } \ t \geqslant 0\text{and} \qquad \varphi(t) = \begin{cases} \frac{t}{2} - \frac{t^2}{4} & \text{if } t \in [0, \frac{1}{2}] \\ \frac{t}{2} - \frac{1}{16} & \text{if } t \geqslant \frac{1}{2}. \end{cases}$$

We now show that f satisfies the inequality (1.2).

<u>*Case (i)*</u>: Let $x, y, z \in [0, \frac{1}{2}]$. Here we consider

$$\psi(S(fx, fy, fz)) = \psi|\frac{7}{4} - x - (\frac{7}{4} - z)| + |\frac{7}{4} - y - (\frac{7}{4} - z)| = \psi\big(|z - x| + |z - y|\big)$$
$$= \frac{1}{2}(|z - x| + |z - y|) \leq \frac{1}{2} \leq |2x - \frac{7}{4}| - \frac{1}{16}$$
$$= \varphi(S(x, x, fx)) \leq \varphi\big(M_\alpha(x, y, z)\big).$$

<u>*Case*</u> (ii): Let $x, y, z \in (\frac{1}{2}, \frac{7}{4}]$.

<u>*Sub-case*</u> (i): $|x - z| + |y - z| \in [0, \frac{1}{2}]$. Therefore

$$\psi(S(fx, fy, fz)) = \psi|\frac{x+1}{2} - \frac{z+1}{2}| + |\frac{y+1}{2} - \frac{z+1}{2}| = \psi\big(\frac{1}{2}(|x - z| + |y - z|)\big)$$
$$= \frac{1}{4}(|x-z|+|y-z|) \leq \frac{1}{2}(|x-z|+|y-z|) - \frac{1}{4}(|x-z|+|y-z|)^2$$
$$= \varphi(S(x, y, z)) \leq \varphi\big(M_\alpha(x, y, z)\big).$$

<u>*Sub-case*</u> (ii): $|x - z| + |y - z| \geqslant \frac{1}{2}$. In this case

$$\psi(S(fx, fy, fz)) = \psi|\frac{x+1}{2} - \frac{z+1}{2}| + |\frac{y+1}{2} - \frac{z+1}{2}| = \psi\big(\frac{1}{2}(|x-z|+|y-z|)\big)$$
$$= \frac{1}{4}(|x - z| + |y - z|) \leq \frac{1}{2}(|x - z| + |y - z|) - \frac{1}{16}$$
$$= \varphi(S(x, y, z)) \leq \varphi\big(M_\alpha(x, y, z)\big).$$

<u>*Case*</u> (iii): Let $z \in [0, \frac{1}{2}]$ and $x, y \in (\frac{1}{2}, \frac{7}{4}]$.

$$\psi(S(fx, fy, fz)) = \psi\big(S(\tfrac{x+1}{2}, \tfrac{y+1}{2}, \tfrac{7}{4} - z)\big)$$
$$= \psi\big(|\tfrac{x+1}{2} - (\tfrac{7}{4} - z)| + |\tfrac{y+1}{2} - (\tfrac{7}{4} - z)|\big)$$
$$\tfrac{1}{2}\big(|\tfrac{x}{2} + z - \tfrac{5}{4}| + |\tfrac{y}{2} + z - \tfrac{5}{4}|\big) \leq |2z - \tfrac{7}{4}| - \tfrac{1}{16} = \tfrac{27}{16} - 2z$$
$$= \varphi(S(z, z, fz)) \leq \varphi\big(M_\alpha(x, y, z)\big).$$

<u>*Case*</u> (iv): Let $x, y \in [0, \frac{1}{2}]$ and $z \in (\frac{1}{2}, \frac{7}{4}]$.

$$\psi(S(fx, fy, fz)) = \psi\big(S(\tfrac{7}{4} - x, \tfrac{7}{4} - y, \tfrac{z+1}{2})\big) = \psi\big(|\tfrac{7}{4} - x - \tfrac{z+1}{2}| + |\tfrac{7}{4} - y - \tfrac{z+1}{2}|\big)$$
$$= \psi\big(|\tfrac{5}{4} - x - \tfrac{z}{2}| + |\tfrac{5}{4} - y - \tfrac{z}{2}|\big) = \tfrac{1}{2}\big(|\tfrac{5}{4} - x - \tfrac{z}{2}| + |\tfrac{5}{4} - y - \tfrac{z}{2}|\big) \leq 1$$
$$\leq |2x - \tfrac{7}{4}| - \tfrac{1}{16} = \tfrac{27}{16} - 2x = \varphi(S(x, x, fx)) \leq \varphi\big(M_\alpha(x, y, z)\big).$$

<u>*Case*</u> (v): Let $z, y \in [0, \frac{1}{2}]$ and $x \in (\frac{1}{2}, \frac{7}{4}]$.

$$\psi(S(fx, fy, fz)) = \psi\big(S(\tfrac{x+1}{2}, \tfrac{7}{4} - y, \tfrac{7}{4} - z)\big)$$
$$= \psi\big(|\tfrac{x+1}{2} - (\tfrac{7}{4} - z)| + |\tfrac{7}{4} - y - (\tfrac{7}{4} - z)|\big)$$
$$= \tfrac{1}{2}\big(|\tfrac{x}{2} + z - \tfrac{5}{4}| + |z - y|\big) \leq \tfrac{5}{16} \leq |2z - \tfrac{7}{4}| - \tfrac{1}{16} = \tfrac{27}{16} - 2z$$
$$= \varphi(S(z, z, fz)) \leq \varphi\big(M_\alpha(x, y, z)\big).$$

Hence f, ψ, φ satisfy all the hypotheses of Theorem 2.1 and f has a unique fixed point $u = 1$.

4. Summary

In our result, the concept generalized (α, ψ, φ)-contractive map was introduced with the proof of the existence and uniqueness of fixed points in complete S-metric spaces. The new idea, property (P), was also introduced and we proved that these maps satisfy property (P). The results presented in this paper extend several well known comparable results in metric and G-metric spaces. We derived corollaries and provided an example to show the validity of our result.

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ON A CERTAIN TRANSFORMATION IN ALMOST CONTACT METRIC MANIFOLDS

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Abstract. In this work, we have investigated a new deformation of almost contact metric manifolds. New relations between classes of 3-dimensional almost contact metric have been discovered. Several concrete examples are discussed.
Key words: Sasakian manifolds; Kenmotsu manifolds; cosymplectic manifolds.

1. Introduction

The lifts of geometrical objects, functions, vector fields, 1-forms etc., on any manifold have important role in differential geometry. For example, they are used to define the different geometric structures. Manifolds equipped with certain differential-geometric structures possess rich geometric structures and such manifolds have been studied widely in differential geometry. Indeed, almost contact manifolds and relations between such manifolds have been studied extensively by many authors.

The construction of almost contact metric structures (Sasakian, Kenmotsu, cosymplectic, etc.) from other almost contact metric structures on a given $(2n+1)$-dimensional manifold M, in general, a non-trivial problem. The more interesting and well-known results correspond with the 3-dimensional case.

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The notion of a D-homothetic deformation on a contact metric manifold was introduced by Tanno [12]. Next, A. Sharfuddin and S. I. Hussain [11] gave a study on conformal transformation of almost contact structures.

In 1992, J. C. Marrero [7] proved that with certain deformation, we can get a trans-Sasakian structure starting from a Sasakian one. In [1], generalized D-conformal deformations are applied to trans-Sasakian manifolds where the covariant derivatives of the deformed metric is evaluated under the condition that the functions used in deformation depend only on the direction of the characteristic vector field of the trans-Sasakian structure. Other similar deformations are studied in [2, 3, 6].

Recently, Özdemir et al. [9], investigated the generalized D-conformal deformations of nearly K-cosymplectic, quasi-Sasakian and β-Kenmotsu manifolds. They analyzed how the class of almost contact metric structures changes.There exists several type of deformations of almost contact metric structures. These different known deformations are mainly based on a deformation of the Riemannian metric.

The present paper deals with the deformation of the structural tensor φ and metric tensor g at the same time, which allows us to define new relations between almost contact metric structures. The paper is organized in the following way.
Section 2 is devoted to the background of the structures which will be used in the subsequent sections to make the paper self-contained. In Section 3, we have introduced a new deformation of almost contact metric structures using a function and a 1-form and prove some basic properties. In Section 4, we focused on the case of three-dimensional geometric structures and have shown how to construct some basic structures with concrete examples. In the last Section, we constructed the examples of almost contact manifolds starting from another class of examples of almost contact manifolds, based on the three types (Sasakian, Kenmotsu, cosymplectic).

2. Preliminaries

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exists on M, a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1,\ \varphi^2(X) = -X + \eta(X)\xi \quad and \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$ [13].

Such manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M.
On the other hand, the almost contact metric structure of M is said to be normal if

$$N^{(1)}(X,Y) = [\varphi,\varphi](X,Y) + 2\mathrm{d}\eta(X,Y)\xi = 0, \tag{2.2}$$

for any X, Y on M, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

In [10], the author proves that (φ, ξ, η, g) is trans-Sasakian structure if and only if it is normal and

$$d\eta = \alpha\phi, \qquad d\phi = 2\beta\eta \wedge \phi, \tag{2.3}$$

where d denotes the exterior derivative, $\alpha = \frac{1}{2n}\delta\phi(\xi)$, $\beta = \frac{1}{2n}\mathrm{div}\xi$ and δ is the codifferential of g.
It is well known that the trans-Sasakian condition may be expressed as an almost contact metric structure satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big) + \beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big). \tag{2.4}$$

From this formula, one can easily obtain

$$\nabla_X \xi = -\alpha\varphi X - \beta\varphi^2 X, \tag{2.5}$$

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) + \beta g(\varphi X, \varphi Y). \tag{2.6}$$

It is clear that a trans-Sasakian manifold of type $(1,0)$ is a Sasakian manifold and a trans-Sasakian manifold of type $(0,1)$ is a Kenmotsu manifold. A trans-Sasakian manifold of type $(0,0)$ is called a cosymplectic manifold. More generally, a trans-Sasakian structure (φ, ξ, η, g) on M is said to be

$$\begin{cases} (a): \ \alpha - Sasaki \quad if \quad \beta = 0, \\ (b): \ \beta - Kenmotsu \quad if \quad \alpha = 0, \\ (c): \ Cosymplectic \quad if \quad \alpha = \beta = 0, \end{cases} \tag{2.7}$$

where α and β are two functions.

The relations between trans-Sasakian, α-Sasakian and β-Kenmotsu structures were discussed by Marrero [7].

Proposition 2.1. [7] *A trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.*

For more background on almost contact metric manifolds, we recommend the reference [4] and [5].

3. Deformation of almost contact metric structures

Let (φ, ξ, η, g) be an almost contact metric structure on M^{2n+1}. For any X, Y on M, we mean a change of structure tensors of the form

$$\tilde{\varphi}X = \varphi X + \theta(\varphi X)\xi, \quad \tilde{\xi} = \xi, \quad \tilde{\eta} = \eta - \theta, \quad \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = fg(\varphi X, \varphi Y), \tag{3.1}$$

where θ is a 1-form orthogonal to η and f a positive function on M.

Proposition 3.1. *The structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric structure.*

Proof. The proof follows by a usual calculation, by using (2.1). $\square$

In particular, if $\theta = 0$ then we get

$$\tilde{g} = fg + (1-f)\eta \otimes \eta$$

and this deformation was studied by Marrero [7].

Remark 3.1. In this new deformation we required the orthogonality between θ and η. But, if we take $\theta = (1-h)\eta$ with h a function on M, we get

$$g = fg + (h^2 - f)\eta \otimes \eta.$$

This deformation appeared in [1]. In addition, if $f = 1$ then we have D-isometric deformation [3], but for $h = f$ we get the deformation of Blair [6] and for $h = f = a$ where a is a positive constant, we obtain D-homothetic deformation [12].

We denote the tensor field of type $(1,2)$ by $\tilde{N}^{(1)}$ on M defined for any X, Y on M by

$$\tilde{N}^{(1)}(X,Y) = [\tilde{\varphi}, \tilde{\varphi}](X,Y) + 2\mathrm{d}\tilde{\eta}(X,Y)\xi,$$

where

$$[\tilde{\varphi}, \tilde{\varphi}](X,Y) = \tilde{\varphi}^2[X,Y] + [\tilde{\varphi}X, \tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X, Y] - \tilde{\varphi}[X, \tilde{\varphi}Y].$$

By long direct calculation, using (3.1) one can get

$$\begin{aligned} \tilde{N}^{(1)}(X,Y) &= N^{(1)}(X,Y) + \theta(N^{(1)}(X,Y))\xi \\ &- \theta(\varphi X)\Big(N^{(3)}(Y) + \theta\big(N^{(3)}(Y)\big)\xi\Big) - \theta(\varphi Y)\Big(N^{(3)}(X) + \theta\big(N^{(3)}(X)\big)\xi\Big) \\ &+ 2\mathrm{d}\theta\big(\tilde{\varphi}X, \tilde{\varphi}Y\big)\xi - 2\mathrm{d}\theta(X,Y)\xi \end{aligned} \tag{3.2}$$

with $N^{(3)}$ is a tensor field on M given by

$$N^{(3)}(X) = \big(L_\xi \varphi\big)(X) = \varphi[X,\xi] - [\varphi X, \xi],$$

where L_ξ denotes the Lie derivative with respect to the vector field ξ.

Proposition 3.2. *Let (φ, ξ, η, g) be a normal almost contact metric structure on M. The almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is normal if and only if*

$$\mathrm{d}\theta\big(\varphi X, \varphi Y\big) = \mathrm{d}\theta(X,Y),$$

for any $X, Y \in \Gamma(TM)$.

Proof. Firstly, we have

$$N^{(1)}(X,Y) = 0 \Rightarrow N^{(1)}(\varphi X, \xi) = [\xi, \varphi X] - \varphi[\xi, X] = N^{(3)}(X) = 0.$$

So, if (φ, ξ, η, g) is normal then from (3.2), we obtain

$$\tilde{N}^{(1)}(X,Y) = 2\mathrm{d}\theta\big(\tilde{\varphi}X, \tilde{\varphi}Y\big)\xi - 2\mathrm{d}\theta(X,Y)\xi. \tag{3.3}$$

Suppose that

$$\mathrm{d}\theta\big(\varphi X, \varphi Y\big) = \mathrm{d}\theta(X, Y).$$

For $Y = \xi$ we get for any X on M,

$$\mathrm{d}\theta(X, \xi) = 0. \tag{3.4}$$

Applying (3.4) and (2.1) in (3.3) we obtain $\tilde{N}^{(1)}(X, Y) = 0$.
For the inverse, suppose that $\tilde{N}^{(1)} = 0$ and taking $Y = \xi$ we obtain for any X on M,

$$\mathrm{d}\theta(X, \xi) = 0. \tag{3.5}$$

Applying (3.5) in (3.3) we get

$$\mathrm{d}\theta\big(\varphi X, \varphi Y\big) = \mathrm{d}\theta(X, Y).$$

□

Corollary 3.1. *Let (φ, ξ, η, g) be a normal almost contact metric structure on M. $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is normal if one of the following four conditions is satisfied*

$$\theta = 0, \qquad \theta = \mathrm{d}h, \qquad \mathrm{d}\theta = 0, \qquad \mathrm{d}\theta = \sigma\phi,$$

where h, σ are two functions on M.

Using the Koszul formula for the Levi-Civita connection of a Riemannian metric, one can obtain the following:

Proposition 3.3. *Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of g and $\tilde{g}$ respectively. For any X and Y on M, we have the relation:*

$$\begin{aligned}
\tilde{g}(\tilde{\nabla}_X Y, Z) &= \tilde{g}(\nabla_X Y, Z) + \frac{1}{2}\big(X(f)g(Y,Z) + Y(f)g(X,Z) - Z(f)g(X,Y)\big) \\
&- f\Big(\frac{1}{2}\big((\nabla_X \eta)Y + (\nabla_Y \eta)X\big)\eta(Z) + d\eta(X,Z)\eta(Y) + d\eta(Y,Z)\eta(X)\Big) \\
&+ \frac{1}{2}\big((\nabla_X \tilde{\eta})Y + (\nabla_Y \tilde{\eta})X\big)\tilde{\eta}(Z) + d\tilde{\eta}(X,Z)\tilde{\eta}(Y) + d\tilde{\eta}(Y,Z)\tilde{\eta}(X).
\end{aligned}$$

4. Application to three dimensional geometric structures

In the remaining part of the paper, we focus on the case of 3-dimensional normal almost contact metric manifold. Let us mention here an important result of Olszak [8], which states that any normal almost contact metric structure is trans-Sasakian structure of type (α, β), where $2\alpha = tr(\varphi\nabla\xi)$ and $2\beta = \mathrm{div}\xi$.

This is what leads us to consider (φ, ξ, η, g) a trans-Sasakian structure of type $(\alpha,\ \beta)$ i.e., we have

$$\mathrm{d}\eta = \alpha\phi, \qquad \mathrm{d}\phi = 2\beta\eta \wedge \phi.$$

In this section, we shall apply the new deformation on trans-Sasakian manifold. Since the expression of connection $\tilde{\nabla}$ is not easy, we prefer to use in our study the first and second fundamental forms.

Firstly, the fundamental 2-form $\tilde{\phi}$ of $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is

$$\tilde{\phi}(X, Y) = \tilde{g}(X, \tilde{\varphi}Y),$$

One can easily obtain

$$\tilde{\phi} = f\phi \tag{4.1}$$

and hence

$$\left\{ \begin{array}{l} \tilde{\eta} = \eta - \theta \\ \tilde{\phi} = f\phi \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \mathrm{d}\tilde{\eta} = \mathrm{d}\eta - \mathrm{d}\theta \\ \mathrm{d}\tilde{\phi} = \big(\mathrm{d}(\ln f) + 2\beta\eta\big) \wedge \tilde{\phi}. \end{array} \right. \tag{4.2}$$

Lemma 4.1. *For any* 3*-dimensional almost contact metric manifold* $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$*, we have*

$$\mathrm{d}(\ln f) \wedge \tilde{\phi} = \xi(\ln f)\tilde{\eta} \wedge \tilde{\phi}. \tag{4.3}$$

Proof. Let $\{\tilde{e}_0 = \xi, \tilde{e}_1, \tilde{e}_2\}$ be the frame of vector fields and $\{\tilde{\theta}^0 = \tilde{\eta}, \tilde{\theta}^1, \tilde{\theta}^2\}$ be the dual frame of differential 1-forms on M. Then,

$$\tilde{\phi} = 2\tilde{e}_2 \wedge \tilde{e}_1,$$

and

$$\mathrm{d}(\ln f) = \xi(\ln f)\tilde{\eta} + \tilde{\theta}^1(\ln f)\tilde{e}_1 + \tilde{\theta}^2(\ln f)\tilde{e}_2.$$

Thus

$$\mathrm{d}(\ln f) \wedge \tilde{\phi} = \xi(\ln f)\tilde{\eta} \wedge \tilde{\phi}.$$

□

From (4.2) and Lemma 4.1, we get

$$\left\{ \begin{array}{l} \mathrm{d}\eta = \alpha\phi \\ \mathrm{d}\phi = 2\beta\eta \wedge \phi \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \mathrm{d}\tilde{\eta} = \frac{\alpha}{f}\tilde{\phi} - \mathrm{d}\theta \\ \mathrm{d}\tilde{\phi} = 2\Big(\beta + \frac{1}{2}\xi(\ln f)\Big)\tilde{\eta} \wedge \tilde{\phi} + 2\beta\theta \wedge \tilde{\phi}. \end{array} \right. \tag{4.4}$$

We will discuss the different new structures according to the four cases indicated in the Corollary 3.1.

First case: For $\theta = 0$, (4.4) lead to us the following result:

Proposition 4.1. (φ, ξ, η, g) *is a trans-Sasakian of type* (α, β) *if and only if* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is a trans-Sasakian of type* $\big(\frac{\alpha}{f}, \beta + \frac{1}{2}\xi(\ln f)\big)$.

Remark 4.1. for $(\alpha, \beta) = (1, 0)$, we can see immediately that the Proposition 4.2 of Marrero [7] is a particular case.

Remark 4.2. In this case, we can not get a Sasakian structure starting from a Kenmosu structure or vice versa.

Second case: For $\theta = \mathrm{d}h$, (4.4) becomes:

$$(4.5)\begin{cases} \mathrm{d}\eta = \alpha\phi \\ \mathrm{d}\phi = 2\beta\eta \wedge \phi \end{cases} \Leftrightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \frac{\alpha}{f}\tilde{\phi} \\ \mathrm{d}\tilde{\phi} = 2\Big(\beta\big(1 + \xi(h)\big) + \frac{1}{2}\xi(\ln f)\Big)\tilde{\eta} \wedge \tilde{\phi}. \end{cases}$$

Proposition 4.2. *(φ, ξ, η, g) is a trans-Sasakian of type (α, β) if and only if $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a trans-Sasakian of type $\Big(\frac{\alpha}{f}, \beta\big(1 + \xi(h)\big) + \frac{1}{2}\xi(\ln f)\Big)$.*

Third case: For $\mathrm{d}\theta = 0$, (4.4) becomes:

$$(4.6)\begin{cases} \mathrm{d}\eta = \alpha\phi \\ \mathrm{d}\phi = 2\beta\eta \wedge \phi \end{cases} \Leftrightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \frac{\alpha}{f}\tilde{\phi} \\ \mathrm{d}\tilde{\phi} = 2\Big(\beta + \frac{1}{2}\xi(\ln f)\Big)\tilde{\eta} \wedge \tilde{\phi} + 2\beta\theta \wedge \tilde{\phi}. \end{cases}$$

Proposition 4.3. *(φ, ξ, η, g) is an α-Sasakian if and only if $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a trans-Sasakian of type $\Big(\frac{\alpha}{f}, \frac{1}{2}\xi(\ln f)\Big)$.*

Fourth case: For $\mathrm{d}\theta = \sigma\phi$, (4.4) becomes:

$$(4.7)\begin{cases} \mathrm{d}\eta = \alpha\phi \\ \mathrm{d}\phi = 2\beta\eta \wedge \phi \end{cases} \Leftrightarrow \begin{cases} \mathrm{d}\tilde{\eta} = \frac{1}{f}(\alpha - \sigma)\tilde{\phi} \\ \mathrm{d}\tilde{\phi} = 2\Big(\beta + \frac{1}{2}\xi(\ln f)\Big)\tilde{\eta} \wedge \tilde{\phi} + \frac{2}{\sigma}\beta f\theta \wedge \mathrm{d}\theta. \end{cases}$$

Proposition 4.4.

1) *For $\theta \wedge \mathrm{d}\theta = 0$, (φ, ξ, η, g) is a trans-Sasakian of type of type (α, β) if and only if $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a trans-Sasakian of type $\Big(\frac{1}{f}(\alpha - \sigma), \beta + \frac{1}{2}\xi(\ln f)\Big)$.*

2) *(φ, ξ, η, g) is an α-Sasakian if and only if $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a trans-Sasakian of type $\Big(\frac{1}{f}(\alpha - \sigma), \frac{1}{2}\xi(\ln f)\Big)$.*

Remark 4.3. Unlike the previous cases, this case is very interesting because we can get Sasakian structure starting from a Kenmotsu structure and vice versa (see Examples 5.1 and 5.2).

5. A class of examples

For this construction, we rely on our Example in [2]. We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by (x, y, z) and define a symmetric tensor field g by

$$g = \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix}$$

where ρ and τ are functions on $\mathbb{R}^3$ such that $\rho \neq 0$ everywhere.
Further, we define an almost contact metric (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-\tau, 0, 1).$$

The fundamental 1-form η and the 2-form ϕ can be expressed as,

$$\eta = dz - \tau dx \qquad \textit{and} \qquad \phi = -2\rho^2 dx \wedge dy,$$

and hence

$$\begin{aligned} \mathrm{d}\eta &= \tau_2\, dx \wedge dy + \tau_3\, dx \wedge dz \\ \mathrm{d}\phi &= -4\rho_3 \rho\, dx \wedge dy \wedge dz, \end{aligned}$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.

We know that the components of the Nijenhuis tensor N_φ in (2.2) can be written as,

$$N^i_{kj} = \varphi^l_k (\partial_l \varphi^i_j - \partial_j \varphi^i_l) - \varphi^l_j (\partial_l \varphi^i_k - \partial_k \varphi^i_l) + \eta_k (\partial_j \xi^i) - \eta_j (\partial_k \xi^i),$$

where the indices i, j, k and l run over the range $1, 2, 3$, then by a direct computation we can verify that

$$N^i_{kj} = 0, \qquad \forall i, j, k.$$

implying that the structure (φ, ξ, η, g) is normal. From (2.7), the structure (φ, ξ, η, g) is a:

(1) Sasaki when $\tau_2 = -2\rho^2$ and $\rho_3 = \tau_3 = 0$,

(2) Cosymplectic when $\rho_3 = 0$ and $\tau_2 = \tau_3 = 0$,

(3) Kenmotsu when $\rho_3 = \rho$ and $\tau_2 = \tau_3 = 0$.

Since θ is a 1-form orthogonal to η, i.e. $\theta(\xi) = 0$ then θ has the following form

$$\theta = a\ dx + b\ dy,$$

where a and b are two functions on $\mathbb{R}^3$. Under these data and use (2.1), one can get

$$\tilde{g} = \begin{pmatrix} f\rho^2 + (a+\tau)^2 & b(a+\tau) & -(a+\tau) \\ b(a+\tau) & f\rho^2 + b^2 & -b \\ -(a+\tau) & -b & 1 \end{pmatrix}$$

and

$$\tilde{\varphi} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ b & -(a+\tau) & 0 \end{pmatrix}, \quad \tilde{\xi} = \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \tilde{\eta} = \big(-(a+\tau), -b, 1 \big).$$

Using the above cases, we get the following:

(1): Let (φ, ξ, η, g) be a Sasakian structure

- (a): If $\mathrm{d}\theta = 0$ and $f = 1$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian structure.
- (b): If $\mathrm{d}\theta = \phi$ and $f_3 = 2f$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu structure.
- (c): If $\mathrm{d}\theta = \phi$ and $f_3 = 0$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a cosymplectic structure.

(2): Let (φ, ξ, η, g) be a Kenmotsu structure

- (a): If $\mathrm{d}\theta = -f\phi$ and $f_3 = -2f$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian structure.
- (b): If $\mathrm{d}\theta = 0$ and $f_3 = 0$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu structure.
- (c): If $\mathrm{d}\theta = \phi$ and $f_3 = -2f$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a cosymplectic structure.

(3): Let (φ, ξ, η, g) be a cosymplectic structure

- (a): If $\mathrm{d}\theta = -\phi$ and $f = 1$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian structure.
- (b): If $\mathrm{d}\theta = 0$ and $f_3 = 2f$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu structure.
- (c): If $\mathrm{d}\theta = \phi$ and $f_3 = 0$, then $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a cosymplectic structure.

By using the above cases, we can discuss other classes of well-known almost contact metric structures.

Given the importance of Remark 4.3 and from the above examples, we will extract non-trivial examples in the following:

Example 5.1. (From Kenmotsu to Sasaki)
Taking $\rho = \mathrm{e}^z$ and $\tau = x$, we get

$$g = \begin{pmatrix} x^2 + \mathrm{e}^{2z} & 0 & -x \\ 0 & \mathrm{e}^{2z} & 0 \\ -x & 0 & 1 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -x & 0 \end{pmatrix},$$

$$\xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-x, 0, 1).$$

It can be checked that (φ, ξ, η, g) is a Kenmotsu structure.
Taking $\sigma = -f = -\mathrm{e}^{-2z}$, we obtain $\mathrm{d}\theta = 2dx \wedge dy$, which implies

$$\theta = adx + bdy, \quad with \quad b_1 - a_2 = 2, \quad and \quad a_3 = b_3 = 0,$$

where $a_i = \frac{\partial a}{\partial x_i}$ and $b_i = \frac{\partial b}{\partial x_i}$.
Notice that there is an infinite number of solutions for θ. We will continue with the following particular solution $\theta = 2xdy$. So, we get

$$\tilde{g} = \begin{pmatrix} 1+x^2 & 2x^2 & -x \\ 2x^2 & 1+4x^2 & -2x \\ -x & -2x & 1 \end{pmatrix}, \; \tilde{\varphi} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 2x & -x & 0 \end{pmatrix},$$

$$\tilde{\xi} = \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{\eta} = \eta = (-x, -2x, 1).$$

Finally, we can verify that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian structure.

Example 5.2. (From Sasaki to Kenmotsu)
Now, taking $\rho = \mathrm{e}^x$ and $\tau = -y\mathrm{e}^{2x}$, we get

$$g = \mathrm{e}^{2x} \begin{pmatrix} 1+4y^2\mathrm{e}^{2x} & 0 & 2y \\ 0 & 1 & 0 \\ 2y & 0 & \mathrm{e}^{-2x} \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 2y\mathrm{e}^{2x} & 0 \end{pmatrix},$$

$$\xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (2y\mathrm{e}^{2x}, 0, 1).$$

It can be checked that (φ, ξ, η, g) is a Sasakian structure.
Taking $\sigma = 1$ and $f = \mathrm{e}^{2z}$, we obtain $\mathrm{d}\theta = -2\mathrm{e}^{2x} dx \wedge dy$. So, we have numerious choices for θ. Let's take $\theta = 2y\mathrm{e}^{2x}dx$, we get

$$\tilde{g} = \begin{pmatrix} \mathrm{e}^{2(x+z)} & 0 & 0 \\ 0 & \mathrm{e}^{2(x+z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \tilde{\varphi} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\xi} = \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{\eta} = dz.$$

Finally, we can verify that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu structure.

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SOME FIXED POINT THEOREMS VIA CYCLIC CONTRACTIVE CONDITIONS IN S-METRIC SPACES

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Abstract. We present some fixed point theorems for mappings which satisfy certain cyclic contractive conditions in the setting of S-metric spaces. The results presented in this paper generalize or improve many existing fixed point theorems in the literature. At the end of the paper, we give some examples to demonstrate our results.
Key words: Fixed point, cyclic contraction, S-metric space.

1. Introduction

In the field of fixed point theory, to find the solution of fixed point problems, the contractive conditions on ambient functions play a significant role. The most fundamental result in metric fixed point theory is Banach Contraction Principle ([4]).

Let (X, d) be a complete metric space and let $T\colon X \to X$ be a self-mapping. If there exists $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq k\, d(x, y), \tag{1.1}$$

for all $x, y \in X$, then T has a unique fixed point $u \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\} \subset X$ defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$, is convergent to the fixed point u. Inequality (1.1) also implies the continuity of T.

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Over the years, due to its importance and applications in different fields of science, several authors generalized the well-known Banach Contraction Principle by introducing a new ambient space or a contractive condition. It is no surprise that there is a great number of generalizations of this fundamental result.

Cyclic representation and cyclic contraction were introduced by Kirk et al. [14] in metric spaces and investigated the existence of proximity points and fixed points for cyclic contraction mappings and further used by several authors to obtain various fixed point results for not necessary continuous mappings in different spaces (see, e.g., [3, 6, 9, 11, 12, 13, 16, 17, 18, 19] and others).

Sedghi et al. [24] introduced the notion of S-metric spaces that generalized G-metric spaces and D^*-metric spaces. In [24] the authors proved some properties of S-metric spaces. They also obtained some fixed point theorems in the setting of S-metric spaces for a self-map.

Gupta [9] introduced the concept of cyclic contraction in S-metric spaces and proved some fixed theorems in the said spaces which are proper generalizations of the results of Sedghi et al. [24].

In this paper, we establish some fixed point theorems for cyclic contractive mappings in the setting of S-metric spaces. Our results generalize or improve several existing fixed point theorems in the literature.

2. Preliminaries

The notion of cyclic contraction is as follows:

Definition 2.1. ([14]) Let X be a nonempty set, $m \in \mathbb{N}$ and let $f\colon X \to X$ be a self-mapping. Then $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f if

a) A_i, $i = 1, 2, \ldots, m$ are nonempty subsets of X;

b) $f(A_1) \subset A_2, f(A_2) \subset A_3, \ldots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Kirk et al. [14] proved the following fixed point result via cyclic contraction which is one of the extraordinary generalizations of the Banach's contraction principle.

Theorem 2.1. ([14]) *Let (X, d) be a complete metric space, $f\colon X \to X$ and let $X = \cup_{i=1}^{m} A_i$ be a cyclic representation of X with respect to f. Suppose that f satisfies the following condition:*

$$d(fx, fy) \leq \psi(d(x, y)), \tag{2.1}$$

for all $x \in A_i$, $y \in A_{i+1}$, $i \in \{1, 2, \ldots, m\}$, where $A_{m+1} = A_1$ and $\psi\colon [0, \infty) \to [0, \infty)$ is a function, upper semi-continuous from the right and $0 \leq \psi(t) < t$ for $t > 0$. Then f has a fixed point $z \in \cap_{i=1}^{m} A_i$.

In 2010, Păcurar and Rus [17] introduced the following notion of cyclic weaker φ-contraction.

Definition 2.2. ([17]) Let (X, d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be closed nonempty subsets of X and $X = \cup_{i=1}^m A_i$. An operator $f \colon X \to X$ is called a cyclic weaker φ-contraction if

1) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;

2) there exists a continuous, nondecreasing function $\varphi \colon [0, 1) \to [0, 1)$ with $\varphi(t) > 0$ for $t \in (0, 1)$ and $\varphi(0) = 0$ such that

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \tag{2.2}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$.

They proved the following result.

Theorem 2.2. ([17]) *Suppose f is a cyclic weaker φ-contraction on a complete metric space (X, d). Then f has a fixed point $z \in \cap_{i=1}^m A_i$.*

We need the following definitions and lemmas in the sequel.

Definition 2.3. ([24]) Let X be a nonempty set and $S \colon X^3 \to [0, \infty)$ be a function satisfying the following conditions for all x, y, z, $t \in X$:

$(S1)$ $S(x, y, z) = 0$ if and only if $x = y = z$;

$(S2)$ $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.

Then the function S is called an S-metric on X and the pair (X, S) is called an S-metric space or simply SMS.

Example 2.1. ([24]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.

Example 2.2. ([24]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Example 2.3. ([25]) Let $X = \mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Definition 2.4. ([24]) Let (X, S) be an S-metric space.

$(a1)$ A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

$(a2)$ A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

$(a3)$ The S-metric space (X, S) is called complete if every Cauchy sequence in X is convergent in X.

Definition 2.5. ([24]) Let (X,S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \le L < 1$ such that

$$S(Tx,Ty,Tz) \le L\,S(x,y,z) \tag{2.3}$$

for all $x,y,z \in X$. If the S-metric space (X,S) is complete then the mapping defined as above has a unique fixed point.

Every S-metric on X defines a metric d_S on X by

$$d_S = S(x,x,y) + S(y,y,x) \ \ \forall\, x,y \in X. \tag{2.4}$$

Let τ be the set of all subsets A of X with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x,r) \subset A$. Then τ is a topology on X. Also, a nonempty subset A in the S-metric space (X,S) is S-closed if $\bar{A} = A$.

Lemma 2.1. ([24, Lemma 2.5]) *In an S-metric space, we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Lemma 2.2. ([24, Lemma 2.12]) *Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ converges to x, that is, $x_n \to x$ as $n \to \infty$ and the sequence $\{y_n\}$ converges to y, that is, $y_n \to y$ as $n \to \infty$, then the sequence $\{S(x_n,x_n,y_n)\}$ converges to $S(x,x,y)$, that is, $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

Lemma 2.3. ([9, Lemma 8]) *Let (X,S) be an S-metric space and A is a nonempty subset of X. Then A is said to be S-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

3. Main Result

In this section, we shall prove some fixed point theorems via certain cyclic contractive conditions in the setting of complete S-metric spaces.

First of all, we shall denote Ψ the set of functions $\psi\colon [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) ψ is continuous; (Ψ_2) $\psi(t) < t$ for all $t > 0$.

Obviously, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) \le t$ for all $t \ge 0$.

Now, we introduce the notion of cyclic generalized g_ψ-contraction in S-metric space as follows.

Definition 3.1. Let (X,S) be a S-metric space. Let m be a positive integer, $A_1, A_2, \dots, A_m$ be nonempty subsets of X and $Y = \cup_{i=1}^m A_i$. An operator $g\colon Y \to Y$ is a cyclic generalized g_ψ-contraction for some $\psi \in \Psi$, if

(**I**) $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to g;

(**II**) there exists $b_1, b_2 \in [0,1)$ with $b_1 + b_2 < 1$ such that for all $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1,2,\ldots,m$ (with $A_{m+1} = A_1$)

$$S(gx,gy,gz) \le b_1\, \mathbf{n}(x,y,z) + b_2\, \mathbf{N}(x,y,z), \tag{3.1}$$

where

$$\mathbf{n}(x,y,z) = \psi\Big(S(gz,gz,z)\frac{1+S(gy,gy,y)}{1+S(x,y,z)}\Big),$$

and

$$\begin{aligned}\mathbf{N}(x,y,z) \quad &= \quad \max\Big\{\psi(S(x,y,z)), \psi(S(gx,gx,x)), \psi(S(gy,gy,y)),\\ &\qquad\qquad \psi\big(\frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)]\big)\Big\}.\end{aligned}$$

Now, we are in a position to prove our main result.

Theorem 3.1. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that $g\colon Y \to Y$ is a cyclic generalized g_ψ-contraction mapping, for some $\psi \in \Psi$. Then g has a unique fixed point. Moreover, the fixed point of g belongs to $\cap_{i=1}^m A_i$.*

Proof. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in X by $x_{n+1} = gx_n$, $n = 0,1,2,\ldots$. We shall prove that

$$\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_{n+2}) = 0. \tag{3.2}$$

If for some k, we have $\lim_{k\to\infty} S(x_{k+1},x_{k+1},x_{k+2}) = 0$, then equation (3.2) follows immediately. So, we can assume that $S(x_{n+1},x_{n+1},x_{n+2}) > 0$ for all n. From the condition (**I**), we observe that for all n, there exists $i = i_n \in \{1,2,\ldots,m\}$ such that $(x_{n+1},x_{n+1},x_{n+2}) \in A_i \times A_i \times A_{i+1}$. Then from condition (**II**) and using Lemma 2.1, we have

$$\begin{aligned}S(x_{n+1},x_{n+1},x_{n+2}) \quad &= \quad S(gx_n,gx_n,gx_{n+1})\\ &\le \quad b_1\, \mathbf{n}(x_n,x_n,x_{n+1}) + b_2\, \mathbf{N}(x_n,x_n,x_{n+1}), \quad n = 1,2,\ldots.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}\mathbf{n}(x_n,x_n,x_{n+1}) \quad &= \quad \psi\Big(S(x_{n+1},x_{n+1},x_{n+2})\frac{1+S(x_n,x_n,x_{n+1})}{1+S(x_n,x_n,x_{n+1})}\Big)\\ &= \quad \psi(S(x_{n+1},x_{n+1},x_{n+2})),\end{aligned}$$

and

$$\mathbf{N}(x_n,x_n,x_{n+1}) = \max\Big\{\psi(S(x_n,x_n,x_{n+1}), \psi\big(\frac{1}{2}S(x_n,x_n,x_{n+2})\big)\Big\}.$$

• If $\mathbf{N}(x_n, x_n, x_{n+1}) = \psi(S(x_n, x_n, x_{n+1})$, we obtain from (3.3) and the property of ψ that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_{n+2}) &\leq b_1\, \psi(S(x_{n+1}, x_{n+1}, x_{n+2})) + b_2\, \psi(S(x_n, x_n, x_{n+1}) \\ &< b_1\, S(x_{n+1}, x_{n+1}, x_{n+2}) + b_2\, S(x_n, x_n, x_{n+1}), \end{aligned}$$

that is,

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \Big(\frac{b_2}{1-b_1}\Big)\, \psi(S(x_n, x_n, x_{n+1}). \tag{3.4}$$

• If $\mathbf{N}(x_n, x_n, x_{n+1}) = \psi\big(\frac{1}{2}S(x_n, x_n, x_{n+2})\big)$, we obtain from (3.3) and the property of ψ that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_{n+2}) &\leq b_1\, \psi(S(x_{n+1}, x_{n+1}, x_{n+2})) + b_2\, \psi\big(\frac{1}{2}S(x_n, x_n, x_{n+2})\big) \\ &< b_1\, S(x_{n+1}, x_{n+1}, x_{n+2}) + b_2\, \frac{1}{2}S(x_n, x_n, x_{n+2}). \qquad (3.5) \end{aligned}$$

By $(S2)$ and Lemma 2.1, we have

$$\begin{aligned} S(x_n, x_n, x_{n+2}) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+2}, x_{n+2}, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}). \end{aligned}$$

Therefore, we have

$$\frac{1}{2}S(x_n, x_n, x_{n+2}) \leq S(x_n, x_n, x_{n+1}) + \frac{1}{2}S(x_{n+1}, x_{n+1}, x_{n+2}). \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_{n+2}) &\leq b_1\, S(x_{n+1}, x_{n+1}, x_{n+2}) + b_2\, [S(x_n, x_n, x_{n+1}) \\ &\quad +\frac{1}{2}S(x_{n+1}, x_{n+1}, x_{n+2})], \end{aligned}$$

that is,

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \Big(\frac{2b_2}{2-2b_1-b_2}\Big)\, [S(x_n, x_n, x_{n+1}). \tag{3.7}$$

Define $\mu = \max\{\frac{b_2}{1-b_1}, \frac{2b_2}{2-2b_1-b_2}\Big\} < 1$ and let $Q_{n+1} = S(x_{n+1}, x_{n+1}, x_{n+2})$ and $Q_n = S(x_n, x_n, x_{n+1})$. Consequently, it can be concluded that

$$Q_{n+1} \leq \mu\, Q_n \leq \mu^2\, Q_{n-1} \leq \ldots \leq \mu^{n+1}\, Q_0. \tag{3.8}$$

Therefore, since $0 \leq \mu < 1$, taking the limit as $n \to \infty$, we have $S(x_{n+1}, x_{n+1}, x_{n+2}) \to 0$, which is (3.2).

Thus for all $n < m$, by using $(S2)$, Lemma 2.1 and equation (3.8), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1}) \\ &= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\ &\dots \\ &\leq 2[\mu^n + \ldots + \mu^{m-1}]S(x_0, x_0, x_1) \\ &\leq \Big(\frac{2\mu^n}{1-\mu}\Big)S(x_0, x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$, we get

$$S(x_n, x_n, x_m) \to 0,$$

since $0 < \mu < 1$. Thus, we have $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S-metric space (X, S). Since Y is closed in (X, S), then (Y, S) is also complete and there exists $u \in Y = \cup_{i=1}^m A_i$. Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = 1, 2, \ldots, m$. Hence in each A_i, $i = 1, 2, \ldots, m$, we can construct a subsequence of $\{x_n\}$ that converges to u. Using that each A_i, $i = 1, 2, \ldots, m$, is closed, we conclude that $u \in \cup_{i=1}^m A_i$ and thus $\cup_{i=1}^m A_i \neq \emptyset$.

Now, we shall prove that u is a fixed point of g (which is possible since u belongs to each A_i). Indeed, since $u \in \cup_{i=1}^m A_i$, so for all n, there exists $i(n) \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i(n)}$, using (**II**) and Lemma 2.1, we obtain

$$\begin{aligned} S(x_{n+1}, x_{n+1}, gu) &= S(gx_n, gx_n, gu) \\ &\leq b_1\, \mathbf{n}(x_n, x_n, u) + b_2\, \mathbf{N}(x_n, x_n, u), \end{aligned} \tag{3.9}$$

for all n. On the other hand, we have

$$\mathbf{n}(x_n, x_n, u) = \psi\Big(S(gu, gu, u)\frac{1 + S(x_{n+1}, x_{n+1}, x_n)}{1 + S(x_n, x_n, u)}\Big),$$

on letting $n \to +\infty$ and using the continuity of ψ, condition $(S1)$ and Lemma 2.1, we obtain that

$$\mathbf{n}(x_n, x_n, u) \to \psi(S(u, u, gu)),$$

and

$$\begin{aligned} \mathbf{N}(x_n, x_n, u) &= \max\Big\{\psi(S(x_n, x_n, u)), \psi(S(gx_n, gx_n, x_n)), \psi(S(gx_n, gx_n, x_n)), \\ &\qquad \psi(\frac{1}{2}[S(gx_n, gx_n, u) + S(gu, gu, x_n)])\Big\} \\ &= \max\Big\{\psi(S(x_n, x_n, u)), \psi(S(x_{n+1}, x_{n+1}, x_n)), \psi(S(x_{n+1}, x_{n+1}, x_n)), \\ &\qquad \psi(\frac{1}{2}[S(x_{n+1}, x_{n+1}, u) + S(gu, gu, x_n)])\Big\}. \end{aligned}$$

On letting $n \to +\infty$ and using the continuity of ψ, condition $(S1)$ and Lemma 2.1, we obtain that

$$\mathbf{N}(x_n, x_n, u) \to \psi\Big(\frac{S(u,u,gu)}{2}\Big).$$

On letting $n \to +\infty$ in (3.9) and using (3.10) and (3.10), we obtain

$$\begin{aligned} S(u,u,gu) &\leq b_1\,\psi(S(u,u,gu)) + b_2\,\psi\Big(\frac{S(u,u,gu)}{2}\Big) \\ &\leq b_1\,\psi(S(u,u,gu)) + b_2\,\psi(S(u,u,gu)). \end{aligned}$$

Suppose that $S(u,u,gu) > 0$. In this case, using condition (Ψ_2), we get

$$\begin{aligned} S(u,u,gu) &< b_1\,S(u,u,gu) + b_2\,S(u,u,gu) \\ &= (b_1+b_2)S(u,u,gu) < S(u,u,gu), \ \text{ since } b_1+b_2<1, \end{aligned}$$

which is a contradiction. Hence $S(u,u,gu) = 0$. Thus, $gu = u$. This shows that u is a fixed point of g.

Finally, we prove that u is the unique fixed point of g. Assume that v is another fixed point of g, that is, $gv = v$ with $v \neq u$. From condition (**I**), this implies that $v \in \cup_{i=1}^{m} A_i$. Now, we apply condition (**II**) for $x = y = u$ and $z = v$, we obtain

$$\begin{aligned} S(u,u,v) &= S(gu,gu,gv) \\ &\leq b_1\,\mathbf{n}(u,u,v) + b_2\,\mathbf{N}(u,u,v), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \mathbf{n}(u,u,v) &= \psi\Big(S(gv,gv,v)\frac{1+S(gu,gu,u)}{1+S(u,u,v)}\Big) \\ &= \psi\Big(S(v,v,v)\frac{1+S(u,u,u)}{1+S(u,u,v)}\Big). \end{aligned}$$

Using the property of ψ and condition $(S1)$, we get

$$\mathbf{n}(u,u,v) \to 0, \tag{3.11}$$

and

$$\begin{aligned} \mathbf{N}(u,u,v) &= \max\Big\{\psi(S(u,u,v)), \psi(S(gu,gu,u)), \psi(S(gu,gu,u)), \\ &\qquad \psi\Big(\frac{1}{2}[S(gu,gu,v)+S(gv,gv,u)]\Big)\Big\} \\ &= \max\Big\{\psi(S(u,u,v)), \psi(S(u,u,u)), \psi(S(u,u,u)), \\ &\qquad \psi\Big(\frac{1}{2}[S(u,u,v)+S(v,v,u)]\Big)\Big\}. \end{aligned}$$

Using Lemma 2.1, condition $(S1)$ and the property of ψ, we get

$$\mathbf{N}(u,u,v) \to S(u,u,v). \tag{3.12}$$

If $S(u,u,v) > 0$, from equations (3.10), (3.11) and (3.12), we get

$$S(u,u,v) \leq b_2\, S(u,u,v) < S(u,u,v), \tag{3.13}$$

which is a contradiction. Hence, $S(u,u,v) = 0$, that is, $u = v$. Thus we have proved the uniqueness of the fixed point. This completes the proof. □

Next, we derive some fixed point theorems from Theorem 3.1.

If we take $m = 1$ and $A_1 = X$ in Theorem 3.1, then we obtain immediately the following result.

Corollary 3.1. *Let (X,S) be a complete S-metric space and $g\colon X \to X$ satisfies the following condition: there exists $b_1, b_2 \in [0,1)$ with $b_1 + b_2 < 1$ and some $\psi \in \Psi$ such that*

$$\begin{aligned} S(gx,gy,gz) \quad \leq \quad & b_1\,\psi\Big(S(gz,gz,z)\frac{1+S(gy,gy,y)}{1+S(x,y,z)}\Big) \\ & + b_2 \max\Big\{\psi(S(x,y,z)), \psi(S(gx,gx,x)), \psi(S(gy,gy,y)), \\ & \qquad \psi\big(\frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)]\big)\Big\}, \end{aligned}$$

for all $x,y,z \in X$. Then g has a unique fixed point.

Remark 3.1. Corollary 3.1 extends and generalizes many existing fixed point theorems in the literature to the setting of complete S-metric spaces (see, [7, 12]).

Corollary 3.2. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y = \cup_{i=1}^m A_i$ and $g\colon Y \to Y$. Suppose that there exists a nondecreasing function $\psi \in \Psi$ such that:*

$(h1)$ $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to g;

$(h2)$ there exist $b_1, b_2 \in [0,1)$ with $b_1 + b_2 < 1$ such that for all $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1,2,\ldots,m$ (with $A_{m+1} = A_1$),

$$\begin{aligned} S(gx,gy,gz) \quad \leq \quad & b_1\,\psi\Big(S(gz,gz,z)\frac{1+S(gy,gy,y)}{1+S(x,y,z)}\Big) \\ & + b_2\,\psi\Big(\max\Big\{S(x,y,z), S(gx,gx,x), S(gy,gy,y), \\ & \qquad \frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)]\Big\}\Big), \end{aligned} \tag{3.14}$$

for all $x,y,z \in X$. Then g has a unique fixed point. Moreover, the fixed point of g belongs to $\cap_{i=1}^m A_i$.

Proof. It follows from Theorem 3.1 by taking that if $\psi \in \Psi$ is a nondecreasing function, we have

$$\begin{aligned} \mathbf{N}(x,y,z) \quad = \quad & \psi\Big(\max\Big\{S(x,y,z), S(gx,gx,x), S(gy,gy,y), \\ & \qquad \frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)]\Big\}\Big). \end{aligned}$$

□

Remark 3.2. It is clear that the conclusions of the Corollary 3.2 remain valid if in condition (3.14), the second term of the right-hand side is replaced by one of the following terms:

$$b_2\,\psi(S(x,y,z));\;\; b_2\,\psi\bigg(\frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)]\bigg);$$

$$b_2\,\max\Big\{\psi(S(gx,gx,x)),\psi(S(gy,gy,y))\Big\};$$

$$\text{or }\; b_2\,\max\Big\{\psi(S(x,y,z)),\psi(S(gx,gx,x)),\psi(S(gy,gy,y))\Big\}.$$

Corollary 3.3. *Let (X,S) be a complete S-metric space, $m\in\mathbb{N}$, $A_1,A_2,\dots,A_m$ be nonempty closed subsets of X, $Y=\cup_{i=1}^m A_i$ and $g\colon Y\to Y$. Suppose that there exist five positive constants d_j, $j=1,2,3,4,5$ with $\sum_{j=1}^5 d_j<1$ such that:*

$(h1)$ $Y=\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to g;

$(h2)$ for all $(x,y,z)\in A_i\times A_i\times A_{i+1}$, $i=1,2,\dots,m$ (with $A_{m+1}=A_1$),

$$\begin{aligned} S(gx,gy,gz) \;\;&\le\;\; d_1\Big(S(gz,gz,z)\frac{1+S(gy,gy,y)}{1+S(x,y,z)}\Big)+d_2\,S(x,y,z)\\ &\quad +d_3\,S(gx,gx,x)+d_4\,S(gy,gy,y)\\ &\quad +d_5\,\frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)], \end{aligned}\tag{3.15}$$

for all $x,y,z\in X$. Then g has a unique fixed point. Moreover, the fixed point of g belongs to $\cap_{i=1}^m A_i$.

Proof. It follows from Theorem 3.1 with $\psi(t)=(d_1+d_2+d_3+d_4+d_5)t$. □

As special case we obtain S-metric space versions of Banach ([4]), Kannan ([10]) and Chatterjea ([5]) fixed point results (relation (1), (4) and (11) in [23]) in the cyclic variant from Corollary 3.3.

Corollary 3.4. *Let (X,S) be a complete S-metric space, $m\in\mathbb{N}$, $A_1,A_2,\dots,A_m$ be nonempty closed subsets of X and $Y=\cup_{i=1}^m A_i$. Let $g\colon Y\to Y$ be such that:*

(1) $Y=\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to g;

(2) there exists $\delta\in[0,1)$ such that one of the following conditions hold for all $(x,y,z)\in A_i\times A_i\times A_{i+1}$, $i=1,2,\dots,m$ (with $A_{m+1}=A_1$),

$$S(gx,gy,gz)\le\delta\,S(x,y,z),$$

$$S(gx,gy,gz)\le\frac{\delta}{2}\,[S(x,x,gx)+S(y,y,gy)],$$

$$S(gx,gy,gz)\le\frac{\delta}{2}\,[S(x,x,gy)+S(y,y,gx)],$$

for all $x,y,z\in X$. Then g has a unique fixed point $u\in Y$.

Proof. It follows from Corollary 3.3 by taking (1) $d_2 = \delta$ and $d_1 = d_3 = d_4 = d_5 = 0$, (2) $d_3 = d_4 = \frac{\delta}{2}$ and $d_1 = d_2 = d_5 = 0$, and (3) $d_5 = \delta$ and $d_1 = d_2 = d_3 = d_4 = 0$. □

If we take $b_1 = 0$, $b_2 = 1$ and $\max\Big\{\psi(S(x,y,z)), \psi(S(gx,gx,x)), \psi(S(gy,gy,y)), \psi\big(\frac{1}{2}[S(gx,gx,z)+S(gz,gz,x)]\big)\Big\} = \psi(S(x,y,z))$ in the Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.5. *Let (X,S) be a complete S-metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y = \cup_{i=1}^m A_i$, $g\colon Y \to Y$ an operator and $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to g. Suppose that g satisfies the following condition: for any $(x,y,z) \in A_i \times A_i \times A_{i+1}$, $i = 1,2,\ldots,m$ with $A_{m+1} = A_1$,*

$$S(gx,gy,gz) \le \psi(S(x,y,z)).$$

Then g has a unique fixed point. Moreover, the fixed point of g belongs to $\cap_{i=1}^m A_i$.

Remark 3.3. Corollary 3.4 extends the corresponding result of Kirk et al. [14] to the setting of S-metric space.

If we take $A_1 = A_2 = \ldots = A_m = X$ and $\psi(t) = k\,t$, where $0 < k < 1$ in the Corollary 3.4, then we obtain the following result.

Corollary 3.6. ([24]) *Let (X,S) be a complete S-metric space and $g\colon X \to X$ be a mapping such that for any $x,y,z \in X$,*

$$S(gx,gy,gz) \le k\,S(x,y,z),$$

where $0 < k < 1$. Then g has a unique fixed point in X.

Remark 3.4. Corollary 3.5 also extends the well-known Banach fixed point theorem [4] form complete metric space to the setting of complete S-metric space.

Now, we give some examples in support of our results.

Example 3.1. Let $X = [0,1]$ and $g\colon X \to X$ be given by $g(x) = \frac{x}{8}$. Let $A_1 = [0,\frac{1}{2}]$ and $A_1 = [\frac{1}{2},1]$. Define the function $S\colon X^3 \to [0,\infty)$ by $S(x,y,z) = \max\{x,y,z\}$ for all for all $x,y,z \in X$, then S is an S-metric on X. Now, define the function $\psi\colon [0,+\infty) \to [0,+\infty)$ by $\psi(t) = \frac{t}{2}$, $t \in [0,1]$. Then ψ has the properties mentioned in Corollary 3.5. Let $x \ge y \ge z$ for all $x,y,z \in X$. It is clear that $X = \cup_{i=1}^2 A_i$ is a cyclic representation of X with respect to g.

(1) Now, consider the inequality of Corollary 3.5, we have

$$\begin{aligned} S(gx,gy,gz) &= S\Big(\frac{x}{8},\frac{y}{8},\frac{z}{8}\Big) \\ &= \max\Big\{\frac{x}{8},\frac{y}{8},\frac{z}{8}\Big\} \\ &= \frac{x}{8} \leq \psi(S(x,y,z)) = \psi(\max\{x,y,z\}) \\ &= \psi(x) = \frac{x}{2} \end{aligned}$$

or

$$\frac{1}{8} \leq \frac{1}{2},$$

which is true. Thus, all the conditions of Corollary 3.5 are satisfied and $u = \frac{1}{2} \in \cup_{i=1}^{2} A_i$ is a unique fixed point of g.

(2) Again, consider the inequality of Corollary 3.6, we have

$$\begin{aligned} S(gx,gy,gz) &= S\Big(\frac{x}{8},\frac{y}{8},\frac{z}{8}\Big) \\ &= \max\Big\{\frac{x}{8},\frac{y}{8},\frac{z}{8}\Big\} \\ &= \frac{x}{8} \leq k\,S(x,y,z) = k\,max\{x,y,z\} \\ &= k\,x, \end{aligned}$$

or

$$k \geq \frac{1}{8}.$$

If we take $0 < k < 1$, then all the conditions of Corollary 3.6 are satisfied and $u = 0 \in X$ is a unique fixed point of g.

Example 3.2. Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x,y,z\}, & \text{if otherwise,} \end{cases}$$

for all $x,y,z \in X$. Then (X,S) is a complete S-metric space. Suppose $A_1 = [0,1]$, $A_2 = [0,\frac{1}{2}]$ and $Y = \cup_{i=1}^{2} A_i$. Consider the mapping $g\colon Y \to Y$ such that $g(x) = \frac{x^2}{2(1+x)}$ for all $x \in Y$. It is clear that $Y = \cup_{i=1}^{2} A_i$ is a cyclic representation of X with respect to g. Let us suppose that $\psi\colon [0,+\infty) \to [0,+\infty)$ be such that $\psi(t) = \frac{t^2}{1+t}$, $t \in [0,1]$. Then ψ has the properties mentioned in Theorem 3.1. Moreover, the mapping g is a cyclic representation of Y with respect to g. Without loss of generality, we assume that $x \geq y \geq z$ for all $x,y,z \in Y$. Then

$$\begin{aligned} S(gx,gy,gz) &= \max\{gx,gy,gz\} \\ &= \max\Big\{\frac{x^2}{2(1+x)},\frac{y^2}{2(1+y)},\frac{z^2}{2(1+z)}\Big\} \\ &= \frac{x^2}{2(1+x)}, \end{aligned}$$

$$S(x,y,z) = \max\{x,y,z\} = x.$$

On the other hand,

$$\begin{aligned}
\mathbf{N}(x,y,z) &= \max\Big\{\psi(S(x,y,z)), \psi\Big(S\Big(\frac{x^2}{2(1+x)}, \frac{x^2}{2(1+x)}, x\Big)\Big), \\
&\qquad \psi\Big(S\Big(\frac{y^2}{2(1+y)}, \frac{y^2}{2(1+y)}, y\Big)\Big), \\
&\qquad \psi\Big(\frac{1}{2}\Big[S\Big(\frac{x^2}{2(1+x)}, \frac{x^2}{2(1+x)}, z\Big) + S\Big(\frac{z^2}{2(1+z)}, \frac{z^2}{2(1+z)}, x\Big)\Big]\Big)\Big\} \\
&= \max\Big\{\psi(x), \psi(x), \psi(y), \psi\Big(\frac{1}{2}\Big[\max\Big\{\frac{x^2}{2(1+x)}, z\Big\} + x\Big]\Big)\Big\} \\
&= \psi(x).
\end{aligned}$$

(Since it was used that the function ψ is increasing and since $x \geq z$, $x \geq \frac{x^2}{2(1+x)}$, that $\frac{1}{2}\Big[\max\Big\{\frac{x^2}{2(1+x)}, z\Big\} + x\Big] \leq x$.)

Hence in this case

$$S(gx, gy, gz) \leq \frac{1}{2}\mathbf{N}(x,y,z)$$

is satisfied for $b_1 = 0$. Thus, the condition (**II**) holds for $b_1 = 0$ and $b_2 = \frac{1}{2}$.

Hence, all conditions of Theorem 3.1 are satisfied (with $m = 2$) and so g has a unique fixed point which is in this case is $u = 0 \in \cap_{i=1}^{2} A_i$.

Example 3.3. Let $X = [0,1]$ and $S\colon X^3 \to \mathbb{R}_+$ be given by

$$S(x,y,z) = \begin{cases} |x-z| + |y-z|, & \text{if } x,y,z \in [0,1) \\ 1, & \text{if } x = 1 \text{ or } y = 1 \text{ or } z = 1, \end{cases}$$

for all $x,y,z \in X$. Then (X,S) is a complete S-metric space.

If a mapping $g\colon X \to X$ is given by

$$g(x) = \begin{cases} 1/2, & \text{if } x,y,z \in [0,1) \\ 1/6, & \text{if } x = y = z = 1, \end{cases}$$

and $A_1 = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{2}, 1]$, then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to g. Now, define the function $\psi\colon [0,\infty) \to [0,1)$ and $\psi(t) = \frac{3t}{4}$, $t \in [0,1]$. Then ψ has the properties mentioned in Corollary 3.5. Moreover, the mapping g is a cyclic representation of Y with respect to g. Without loss of generality, we assume that $x \geq y \geq z$ for all $x,y,z \in X$. Indeed, consider the following cases.

Case I: If $x,y \in [0,\frac{1}{2}]$, $z \in [\frac{1}{2}, 1)$ or $z \in [0,\frac{1}{2}]$, $x,y \in [\frac{1}{2}, 1)$. Then

$$\begin{aligned}
S(gx, gy, gz) &= S\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\Big) = 0 \\
&\leq \psi(S(x,y,z)).
\end{aligned}$$

Thus, the inequality of Corollary 3.5 is trivially satisfied.

Case II: If $x, y \in [0, \frac{1}{2}]$ and $z = 1$. Then

$$S(gx, gy, gz) = S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{6}\right) = \frac{2}{3},$$

$$S(x, y, z) = 1,$$

and

$$\psi(S(x, y, z)) = \frac{3}{4}.$$

Consequently,

$$\begin{aligned} S(gx, gy, gz) &= \frac{2}{3} \leq \psi(S(x, y, z)) \\ &= \frac{3}{4}, \end{aligned}$$

which is true. Thus, all the conditions of Corollary 3.5 are satisfied.

Case III: If $x, z \in [0, \frac{1}{2}]$ and $y = 1$. Then

$$S(gx, gy, gz) = S\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right) = \frac{1}{3},$$

$$S(x, y, z) = 1,$$

and

$$\psi(S(x, y, z)) = \frac{3}{4}.$$

Consequently,

$$S(gx, gy, gz) = \frac{1}{3} \leq \psi(S(x, y, z)) = \frac{3}{4},$$

which is true. Thus, all the conditions of Corollary 3.5 are satisfied.

Case IV: If $y, z \in [0, \frac{1}{2}]$ and $x = 1$. Then

$$S(gx, gy, gz) = S\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3},$$

$$S(x, y, z) = 1,$$

and

$$\psi(S(x, y, z)) = \frac{3}{4}.$$

Consequently,

$$S(gx, gy, gz) = \frac{1}{3} \leq \psi(S(x, y, z)) = \frac{3}{4},$$

which is true. Thus, all the conditions of Corollary 3.5 are satisfied.

Considering all the above cases, we conclude that the inequality used in Corollary 3.5 remains valid for ψ and g constructed in the above example and consequently by applying Corollary 3.5, g has a unique fixed point (which is $u = \frac{1}{2} \in A_1 \cap A_2$).

4. Application to well posedness fixed point problem

The notion of well posedness of a fixed point problem has generated much interest to several mathematicians, for example [1, 2, 8, 15, 20, 21, 22]. Here, we study well posedness of a fixed point problem of mappings in Theorem 3.1.

Definition 4.1. ([8]) Let (X, d) be a metric space and $g\colon X \to X$ be a mapping. The fixed point problem of g is said to be well-posed if

(i) g has a unique fixed point u in X;

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(gx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, u) = 0$.

Now, we generalize the above notion in S-metric space.

Definition 4.2. Let (X, S) be a S-metric space and $g\colon X \to X$ be a mapping. The fixed point problem of g is said to be well-posed if

(i) g has a unique fixed point u in X;

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} S(gx_n, gx_n, x_n) = 0 = \lim_{n\to\infty} S(x_n, x_n, gx_n)$, we have $\lim_{n\to\infty} S(x_n, x_n, u) = 0 = \lim_{n\to\infty} S(u, u, x_n)$.

Concerning the well-posedness of the fixed point problem in a S-metric space satisfying the conditions of Theorem 3.1, we have the following result.

Theorem 4.1. *Let $g\colon Y \to Y$ be a self mapping as in Theorem 3.1. Then the fixed point problem for g is well posed.*

Proof. From Theorem 3.1, we know that g has a unique fixed point, say, $u \in Y$. Let $\{x_n\} \subset Y$ be a sequence in Y such that $\lim_{n\to\infty} S(x_n, x_n, gx_n) = 0 = \lim_{n\to\infty} S(gx_n, gx_n, x_n)$. Then using $(S1)$, Lemma 2.1, condition (**II**) and the property of ψ, we have

$$\begin{aligned} S(x_n, x_n, u) &\leq 2S(x_n, x_n, gx_n) + S(u, u, gx_n) \\ &= 2S(x_n, x_n, gx_n) + S(gx_n, gx_n, gu) \\ &\leq 2S(x_n, x_n, gx_n) + b_1\, \mathbf{n}(x_n, x_n, u) \\ &\quad +b_2\, \mathbf{N}(x_n, x_n, u), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \mathbf{n}(x_n, x_n, u) &= \psi\Big(S(gu, gu, u)\frac{1 + S(gx_n, gx_n, x_n)}{1 + S(x_n, x_n, u)}\Big) \\ &= \psi\Big(S(u, u, u)\frac{1 + S(gx_n, gx_n, x_n)}{1 + S(x_n, x_n, u)}\Big) = 0, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \mathbf{N}(x_n, x_n, u) &= \max\Big\{\psi(S(x_n, x_n, u)), \psi(S(gx_n, gx_n, x_n)), \psi(S(gx_n, gx_n, x_n)), \\ &\qquad \psi\Big(\frac{1}{2}[S(gx_n, gx_n, u) + S(gu, gu, x_n)]\Big)\Big\} \\ &= \max\Big\{\psi(S(x_n, x_n, u)), \psi(S(gx_n, gx_n, x_n)), \psi(S(gx_n, gx_n, x_n)), \\ &\qquad \psi\Big(\frac{1}{2}[S(gx_n, gx_n, u) + S(u, u, x_n)]\Big)\Big\} \\ &= \max\Big\{\psi(S(x_n, x_n, u)), \psi(S(gx_n, gx_n, x_n)), \psi(S(gx_n, gx_n, x_n)), \\ &\qquad \psi\Big(\frac{1}{2}[2S(gx_n, gx_n, x_n) + 2S(x_n, x_n, u)]\Big)\Big\} \\ &= \max\Big\{\psi(S(x_n, x_n, u)), \psi(S(gx_n, gx_n, x_n)), \psi(S(gx_n, gx_n, x_n)), \\ &\qquad \psi\big(S(gx_n, gx_n, x_n) + S(x_n, x_n, u)\big)\Big\} \\ &= \psi(S(gx_n, gx_n, x_n)). \end{aligned} \tag{4.3}$$

From equations (4.1)-(4.3), we obtain

$$S(x_n, x_n, u) \le 2S(x_n, x_n, gx_n) + b_2\, \psi(S(x_n, x_n, u)). \tag{4.4}$$

Using the property of ψ in equation (4.4), we obtain

$$S(x_n, x_n, u) < 2S(x_n, x_n, gx_n) + b_2\, S(x_n, x_n, u),$$

taking the limit as $n \to \infty$ in the above inequality, we get $S(x_n, x_n, u) \to 0$ as $n \to \infty$ since $b_2 < 1$, which is equivalent to saying that $x_n \to u$ as $n \to \infty$. This completes the proof. □

5. Conclusion

In this paper, we prove some fixed point theorems for generalized g_ψ-cyclic contractions in the setting of complete S-metric spaces. Also we give some examples in support of our results. The results presented in this paper extend, generalize and improve several fixed point results in the literature (see, e.g., [11, 12, 16, 17, 24] and many others) to the setting of complete S-metric spaces.

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SOME CURVATURE PROPERTIES ON PARACONTACT METRIC (k, μ)-MANIFOLDS WITH RESPECT TO THE SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. The object of the present paper is to characterize paracontact metric (k, μ)-manifolds satisfying certain semisymmetry curvature conditions with respect to the Schouten-van Kampen connection.
Key words: Paracontact metric $(k; \mu)$-manifolds; Schouten-van Kampen connection; h-projective semisymmetric; ϕ-projective semisymmetric.

1. Introduction

Paracontact metric structures have been introduced in [5], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds have been studied by many authors in the recent years, particularly since the appearance of [19]. An important class among paracontact metric manifolds is that of the (k, μ)-manifolds, which satisfies the nullity condition [2]

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{1.1}$$

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for all X, Y vector fields on M, where k and μ are constants and $h = \frac{1}{2}\mathcal{L}_\xi \phi$. This class includes the para-Sasakian manifolds [5, 19], the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all X, Y [20].

Among the geometric properties of manifolds symmetry is an important one. From the local point view it was introduced by Shirokov as a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where ∇ is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was introduced by Cartan in 1927. A manifold is called semisymmetric if the curvature tensor R satisfies $R(X,Y) \cdot R = 0$, where $R(X,Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Semisymmetric manifolds were locally classified by Szabó [16]. Also in [17] and [18], Yildiz and De studied h-Weyl semisymmetric, ϕ-Weyl semisymmetric, h-projectively semisymmetric and ϕ-projectively semisymmetric non-Sasakian (k,μ)-contact metric manifolds and paracontact metric (k,μ)-manifolds respectively. Recently Mandal and De have studied certain curvature conditions on paracontact (k,μ)-spaces [6].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional semi-Riemannian manifold with metric g. The Ricci operator Q of (M,g) is defined by $g(QX,Y) = S(X,Y)$, where S denotes the Ricci tensor of type $(0,2)$ on M. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.2}$$

for all $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor.

In fact M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

A paracontact metric (k,μ)-manifold is said to be an Einstein manifold if the Ricci tensor satisfies $S = \lambda_1 g$, and an η-Einstein manifold if the Ricci tensor satisfies $S = \lambda_1 g + \lambda_2 \eta \otimes \eta$, where λ_1 and λ_2 are constants.

On the other hand, the Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [1, 4, 10]. Solov'ev has investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [12, 13, 14, 15]. Then Olszak has studied the Schouten-van Kampen connection to adapted to an almost (para) contact metric structure [8]. He has characterized some classes of almost (para) contact metric manifolds with the Schouten-van Kampen connection and he has finded certain curvature properties of this connection on these manifolds

In the present paper we have studied certain curvature properties of a paracontact metric (k,μ)-space. The outline of the article goes as follows: After introduction, in Section 2, we recall basic facts which we will need throughout the paper. Section 3 deals with some basic results of paracontact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution with respect to the Schouten-van Kampen connection. In section 4, we characterize paracontact metric (k,μ)-manifolds satisfying some semisymmetry curvature conditions. We prove that a h-projectively semisymmetric and ϕ-projectively semisymmetric paracontact metric (k,μ)-manifold with respect to the Schouten-van Kampen connection is an η-Einstein manifold with respect to the Levi-Civita connection, respectively. In the all cases we assume that $k \neq -1$.

2. Preliminaries

An $(2n+1)$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi)=1,\ \ \phi^2 = I - \eta\otimes\xi,$

(ii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D}=\ker(\eta)$, i.e. the ±1-eigendistributions, $\mathcal{D}^{\pm}=\mathcal{D}_\phi(\pm1)$ of ϕ have equal dimension n.

From the definition it follows that $\phi\xi=0$, $\eta\circ\phi=0$ and the endomorphism ϕ has rank $2n$. The Nijenhius torsion tensor field $[\phi,\phi]$ is given by

$$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]. \tag{2.1}$$

When the tensor field $N_\phi=[\phi,\phi]-2d\eta\otimes\xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{2.2}$$

for all $X,Y\in\Gamma(TM)$, then we say that (M,ϕ,ξ,η,g) is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1,n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1,\dots,X_n,Y_1,\dots,Y_n,\xi\}$, such that $g(X_i,X_j)=\delta_{ij}$, $g(Y_i,Y_j)=-\delta_{ij}$, $g(X_i,Y_j)=0$, $g(\xi,X_i)=g(\xi,Y_j)=0$, and $Y_i=\phi X_i$, for any $i,j\in\{1,\dots,n\}$. Such basis is called a ϕ-basis.

We can now define the fundamental form of the almost paracontact metric manifold by $\theta(X,Y)=g(X,\phi Y)$. If $d\eta(X,Y)=g(X,\phi Y)$, then (M,ϕ,ξ,η,g) is said to be paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator $h=\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [19] that h anti-commutes with ϕ and satisfies $h\xi = 0$, $trh = trh\phi = 0$ and

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.3}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) - g(hX, \phi Y), \tag{2.4}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g). Let R be Riemannian curvature operator

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.5}$$

Moreover $h = 0$ if and only if ξ is Killing vector field. In this case (M, ϕ, ξ, η, g) is said to be a K-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Also, in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{2.6}$$

3. Paracontact metric (k, μ)-manifolds with respect to the Schouten-van Kampen connection

Let (M, ϕ, ξ, η, g) be a paracontact manifold. The (k, μ)-nullity distribution of a (M, ϕ, ξ, η, g) for the pair (k, μ) is a distribution

$$\begin{aligned} N(k,\mu) \quad &: \quad p \to N_p(k,\mu) \\ &= \left\{ \begin{array}{c} Z \in T_pM \mid R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y) \\ +\mu(g(Y,Z)hX - g(X,Z)hY) \end{array} \right\}, \end{aligned} \tag{3.1}$$

for some real constants k and μ. If the characteristic vector field ξ belongs to the (k, μ)-nullity distribution we have (3.1). [2] is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (the condition (3.1), for some real numbers k and μ).

Lemma 3.1. [2] *Let M be a paracontact metric (k, μ)-manifold of dimension $2n+1$. Then the following holds:*

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X \quad = \quad & -(1+k)(2g(X,\phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X) \\ & +(1-\mu)(\eta(X)\phi hY - \eta(Y)\phi hX), \end{aligned} \tag{3.2}$$

$$\begin{aligned} (\nabla_X \phi h)Y - (\nabla_Y \phi h)X \quad = \quad & (1+k)(\eta(X)Y - \eta(Y)X) \\ & +(\mu - 1)(\eta(X)hY - \eta(Y)hX), \end{aligned} \tag{3.3}$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi + g(hX, Y)\xi + \eta(Y)X - \eta(X)Y, \qquad k \neq -1, \tag{3.4}$$

for any vector fields X, Y on M.

Lemma 3.2. [2] *In any $(2n+1)$-dimensional paracontact metric (k,μ)-manifold (M,ϕ,ξ,η,g) such that $k \neq -1$, the Ricci operator Q is given by*

$$Q = (2(1-n)+n\mu)I + (2(n-1)+\mu)h + (2(n-1)+n(2k-\mu))\eta \otimes \xi. \tag{3.5}$$

On the other hand, we have two naturally defined distribution in the tangent bundle TM of M as follows:

$$H = ker\eta, \qquad V = span\{\xi\}. \tag{3.6}$$

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition allows one to define the Schoutenvan Kampen connection $\widetilde{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\widetilde{\nabla}$ on an almost (para) contact metric manifold with respect to Levi-Civita connection ∇ is defined by [12]

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{3.7}$$

Thus with the help of the Schouten-van Kampen connection (3.7), many properties of some geometric objects connected with the distributions H, V can be characterized [12, 13, 14]. For example g, ξ and η are parallel with respect to $\widetilde{\nabla}$, that is, $\widetilde{\nabla}\xi = 0, \widetilde{\nabla}g = 0, \widetilde{\nabla}\eta = 0$. Also, the torsion $\widetilde{T}$ of $\widetilde{\nabla}$ is defined by

$$\widetilde{T}(X,Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2d\eta(X,Y)\xi. \tag{3.8}$$

Now we consider a paracontact metric (k,μ)-manifold with respect to the Schouten-van Kampen connection. Firstly, using (2.3) and (2.4) in (3.7), we get

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\phi X - \eta(Y)\phi hX + g(X,\phi Y)\xi - g(hX,\phi Y)\xi. \tag{3.9}$$

Let R and $\widetilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\widetilde{\nabla}$,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \qquad \widetilde{R}(X,Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}. \tag{3.10}$$

If we substitute equation (3.7) in the definition of the Riemannian curvature tensor

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z. \tag{3.11}$$

Using (3.9) in (3.11), we have

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \widetilde{\nabla}_X(\nabla_Y Z - \eta(Z)\phi Y - \eta(Z)\phi hY \\ &\quad +g(Y,\phi Z)\xi - g(hY,\phi Z)\xi) \\ &\quad -\widetilde{\nabla}_Y(\nabla_X Z - \eta(Z)\phi X - \eta(Z)\phi hX \\ &\quad +g(X,\phi Z)\xi - g(hX,\phi Z)\xi) \\ &\quad -(\nabla_{[X,Y]}Z + \eta(Z)\phi[X,Y] - \eta(Z)\phi h[X,Y] \\ &\quad +g([X,Y],\phi Z)\xi - g(h[X,Y],\phi Z)\xi). \end{aligned} \tag{3.12}$$

Using (3.2), (3.3) and (3.4) in (3.12), we obtain the following formula connecting R and $\widetilde{R}$ on M

$$
\begin{aligned}
\widetilde{R}(X,Y)Z \quad = \quad & R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + g(hY,\phi Z)\phi X \\
& -g(hX,\phi Z)\phi Y + g(Y,\phi Z)\phi hX - g(X,\phi Z)\phi hY \\
& +g(hX,\phi Z)\phi hY - g(hY,\phi Z)\phi hX \\
& +(k+1)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \\
& +k(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\
& +(\mu-1)(g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi) \\
& +\mu(\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX).
\end{aligned} \tag{3.13}
$$

Now taking the inner product in (3.13) with a vector field W, we have

$$
\begin{aligned}
g(\widetilde{R}(X,Y)Z,W) \quad = \quad & g(R(X,Y)Z,W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\
& +g(hY,\phi Z)g(\phi X,W) - g(hX,\phi Z)g(\phi Y,W) \\
& +g(Y,\phi Z)g(\phi hX,W) - g(X,\phi Z)g(\phi hY,W) \\
& +g(hX,\phi Z)g(\phi hY,W) - g(hY,\phi Z)g(\phi hX,W) \\
& +(k+1)(g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\
& +k(g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)) \\
& +(\mu-1)(g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)) \\
& +\mu(g(hY,W)\eta(X)\eta(Z) - g(hX,W)\eta(Y)\eta(Z)).
\end{aligned} \tag{3.14}
$$

If we take $X = W = e_i$, $\{i = 1, ..., 2n+1\}$, in (3.14), where$\{ei\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$
\begin{aligned}
\widetilde{S}(Y,Z) \quad = \quad & S(Y,Z) - (k+2)g(Y,Z) \\
& +(k+2-2nk)\eta(Y)\eta(Z) - (\mu-1)g(hY,Z),
\end{aligned} \tag{3.15}
$$

where $\widetilde{S}$ and S denote the Ricci tensor of the connections $\widetilde{\nabla}$ and ∇, respectively. As a consequence of (3.15), we get for the Ricci operator $\widetilde{Q}$

$$
\widetilde{Q}Y = QY - (k+2)Y + (k+2-2nk)\eta(Y)\xi - (\mu-1)hY, \tag{3.16}
$$

Also if we take $Y = Z = e_i$, $\{i = 1, ..., 2n+1\}$, in (3.16), we get

$$
\widetilde{r} = r - 4n(k+1), \tag{3.17}
$$

where $\widetilde{r}$ and r denote the scalar curvatures of the connections $\widetilde{\nabla}$ and ∇, respectively.

4. Some semisymmetry curvature conditions on paracontact metric (k,μ)-manifolds

In this section we study some semisymmetry curvature conditions on paracontact metric (k,μ)-manifolds with respect to the Schouten-van Kampen connection. Firstly we give the following:

Definition 4.1. A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be h-projectively *semisymmetric if*

$$P(X,Y)\cdot h = 0, \tag{4.1}$$

holds on M.

Let M be a h-projectively semisymmetric paracontact metric (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then above equation is equivalent to

$$\widetilde{P}(X,Y)hZ - h\widetilde{P}(X,Y)Z = 0. \tag{4.2}$$

for any $X, Y, Z \in \chi(M)$. Thus we write

$$\begin{aligned}
&\widetilde{R}(X,Y)hZ - h\widetilde{R}(X,Y)Z \\
&-\frac{1}{2n}\{\widetilde{S}(Y,hZ)X - \widetilde{S}(X,hZ)Y - \widetilde{S}(Y,Z)hX + \widetilde{S}(X,Z)hY\} = 0.
\end{aligned} \tag{4.3}$$

Using (3.13) in (4.3), we have

$$\begin{aligned}
&R(X,Y)hZ - hR(X,Y)Z + g(X,\phi hZ)\phi Y - g(Y,\phi hZ)\phi X \\
&-g(hY,h\phi Z)\phi X + g(hX,h\phi Z)\phi Y + g(Y,\phi hZ)\phi hX \\
&-g(X,\phi hZ)\phi hY - g(hX,h\phi Z)\phi hY + g(hY,h\phi Z)\phi hX \\
&+(k+1)\{g(X,hZ)\eta(Y)\xi - g(Y,hZ)\eta(X)\xi\} \\
&+(\mu-1)\{g(hX,hZ)\eta(Y)\xi - g(hY,hZ)\eta(X)\xi\} \\
&-g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X - g(hY,\phi Z)h\phi X \\
&+g(hX,\phi Z)h\phi Y - g(Y,\phi Z)h\phi hX + g(X,\phi Z)h\phi hY \\
&-g(hX,\phi Z)h\phi hY + g(hY,\phi Z)h\phi hX \\
&-k\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\} \\
&-\mu\{\eta(X)\eta(Z)h^2Y - \eta(Y)\eta(Z)h^2X\} \\
&-\frac{1}{2n}\{S(Y,hZ)X - S(X,hZ)Y - S(Y,Z)hX + S(X,Z)hY \\
&-(k+2)[g(Y,hZ)X - g(X,hZ)Y + g(Y,Z)hX - g(X,Z)hY] \\
&+(\mu-1)[g(hX,hZ)Y - g(hY,hZ)X + g(hY,Z)hX - g(hX,Z)hY] \\
&+(k+2-2nk)[\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX]\} = 0.
\end{aligned} \tag{4.4}$$

Yıldız and De [18] proved that

$$\begin{aligned}
R(X,Y)hZ - hR(X,Y)Z \;=\;& \mu(k+1)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
&+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \\
&+k\{g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi \\
&+\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\
&+g(\phi Y,Z)\phi hX - g(\phi X,Z)\phi hY\} \\
&+(\mu+k)\{g(\phi hX,Z)\phi Y - g(\phi hY,Z)\phi X\} \\
&+2\mu g(\phi X,Y)\phi hZ.
\end{aligned} \tag{4.5}$$

Again using (4.5) in (4.4), we get

$$
\begin{aligned}
&\mu(k+1)\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}\\
&+k\{g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi + \eta(X)\eta(Z)hY\\
&-\eta(Y)\eta(Z)hX - g(\phi Y,Z)h\phi X + g(\phi X,Z)h\phi Y\}\\
&-(\mu+k)\{g(h\phi X,Z)\phi Y - g(h\phi Y,Z)\phi X\}\\
&-2\mu g(\phi X,Y)h\phi Z - g(X,h\phi Z)\phi Y + g(Y,h\phi Z)\phi X\\
&-(k+1)[g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y - g(X,\phi Z)h\phi Y\\
&+g(Y,\phi Z)h\phi X - g(X,hZ)\eta(Y)\xi + g(Y,hZ)\eta(X)\xi]\\
&+g(Y,h\phi Z)h\phi X - g(X,h\phi Z)h\phi Y\\
&+(\mu-1)(k+1)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
&-g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X - g(hY,\phi Z)h\phi X + g(hX,\phi Z)h\phi Y\\
&+(k+1)[g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y + g(hX,\phi Z)\phi Y - g(hY,\phi Z)\phi X]\\
&-k\{\eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX\} - \mu(k+1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}\\
&-\frac{1}{2n}\{S(Y,hZ)X - S(X,hZ)Y - S(Y,Z)hX + S(X,Z)hY\\
&-(k+2)[g(Y,hZ)X - g(X,hZ)Y + g(X,Z)hY - g(Y,Z)hX]\\
&+(\mu-1)(k+1)[g(X,Z)Y - \eta(X)\eta(Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X]\\
&-(k+2-2nk)[\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY]\\
&+(\mu-1)[g(hY,Z)hX + g(hX,Z)hY] = 0,
\end{aligned}
\tag{4.6}
$$

which gives to

$$
\begin{aligned}
&\mu\{g(h\phi Y,Z)g(\phi X,W) - g(h\phi X,Z)g(\phi Y,W) + 2(X,\phi Y)g(h\phi Z,W)\}\\
&+(k+1)\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)\}\\
&+g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)\\
&-\frac{1}{2n}\{S(Y,hZ)g(X,W) - S(X,hZ)g(Y,W)\\
&+S(X,Z)g(hY,W) - S(Y,Z)g(hX,W)\\
&-(k+2)[g(Y,hZ)g(X,W) - g(X,hZ)g(Y,W)\\
&+g(X,Z)g(hY,W) - g(Y,Z)g(hX,W)]\\
&-(\mu-1)(k+1)[g(Y,Z)g(X,W) - g(X,W)\eta(Y)\eta(Z)\\
&+g(Y,W)\eta(X)\eta(Z) - g(X,Z)g(Y,W)]\\
&-(k+2-2nk)[g(hX,W)\eta(Y)\eta(Z) - g(hY,W)\eta(X)\eta(Z)]\\
&+(\mu-1)[g(hY,Z)g(hX,W) + g(hX,Z)g(hY,W)] = 0.
\end{aligned}
\tag{4.7}
$$

Putting $X = W = e_i$ in (4.7), we get

$$
\begin{aligned}
&\mu(k+1)g(hZ,Y) + \mu(k+1)\{g(Y,Z) - \eta(Y)\eta(Z)\} - g(hY,Z)\\
&-\frac{1}{2n}\{(2n+1)[S(hY,Z) - (k+2)g(Y,hZ)
\end{aligned}
$$

$$\begin{aligned}(4.8)\qquad & -(\mu-1)(k+1)(g(Y,Z)-\eta(Y)\eta(Z))]\\ & +(k+2)g(Y,hZ)+2(\mu-1)(k+1)[g(Y,Z)-\eta(Y)\eta(Z)]\\ & -(k+2)g(hY,Z)\}=0.\end{aligned}$$

Again putting $Y = hY$ in (4.8) and using $h^2 = (k+1)\phi^2$, we obtain

$$\begin{aligned} & (k+1)\{[2n\mu(k+1)-2n+(2n+1)(k+2)]g(Y,Z)\\ & -[2n\mu(k+1)-2n+(2n+1)(k+2)-(2n+1)2nk]\eta(Y)\eta(Z)\\ (4.9)\qquad & +[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]g(hY,Z)\\ & -(2n+1)S(Y,Z)\}=0.\end{aligned}$$

As well known that

$$\begin{aligned} g(hY,Z) &= \frac{1}{(2(n-1)+\mu)}S(Y,Z)-\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(Y,Z)\\ (4.10)\qquad & \quad -\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z).\end{aligned}$$

Hence using (4.10) in (4.9), we get

$$\begin{aligned} & (k+1)\{[2n\mu(k+1)-2n+(2n+1)(k+2)]g(Y,Z)\\ & -[2n\mu(k+1)-2n+(2n+1)(k+2)-(2n+1)2nk]\eta(Y)\eta(Z)\\ (4.11)\qquad & +[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\{\frac{1}{(2(n-1)+\mu)}S(Y,Z)\\ & -\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(Y,Z)-\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z)\}\\ & -(2n+1)S(Y,Z)\}=0.\end{aligned}$$

Hence one can write

$$(4.12)\qquad S(Y,Z)=\frac{A_1}{A}g(Y,Z)+\frac{A_2}{A}\eta(Y)\eta(Z),$$

where

$$\begin{aligned} A_1 &= 2n\mu(k+1)-2n+(2n+1)(k+2)\\ &\quad -[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\frac{(2(1-n)+n\mu)}{2(n-1)+\mu},\\ A_2 &= -2n\mu(k+1)+2n+(2n+1)(k+2)+(2n+1)2nk\\ &\quad -[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu},\\ A &= 2n+1-[2n\mu+(2n+1)(\mu-1)-2(\mu-1)]\frac{1}{(2(n-1)+\mu)}.\end{aligned}$$

Therefore from (4.12) it follows that the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 4.1. *Let M be a $(2n+1)$-dimensional h-projectively semisymmetric paracontact (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection provided $\mu \neq 2(1-n)$.*

Definition 4.2. A semi-Riemannian manifold $(M^{2n+1}, g), n > 1$, is said to be ϕ-projectively *semisymmetric if*

$$P(X,Y)\cdot\phi = 0 = 0, \tag{4.13}$$

holds on M for all $X, Y \in \chi(M)$.

Let M be a ϕ-projectively semisymmetric paracontact metric (k,μ)-manifold $(k \neq -1)$ with respect to the Schouten-van Kampen connection. Then above equation is equivalent to

$$\widetilde{P}(X,Y)\phi Z - \phi\widetilde{P}(X,Y)Z = 0, \tag{4.14}$$

for any $X, Y, Z, W \in \chi(M)$. Thus we have

$$\begin{aligned}&\widetilde{R}(X,Y)\phi Z - \phi\widetilde{R}(X,Y)Z \\ &-\frac{1}{2n}\{\widetilde{S}(Y,\phi Z)X - \widetilde{S}(X,\phi Z)Y - \widetilde{S}(Y,Z)\phi X + \widetilde{S}(X,Z)\phi Y\} = 0,\end{aligned} \tag{4.15}$$

Using (3.13) in (4.15), we get

$$\begin{aligned}&R(X,Y)\phi Z - \phi R(X,Y)Z + g(X,Z)\phi Y - \eta(X)\eta(Z)\phi Y \\ &-g(Y,Z)\phi X + \eta(Y)\eta(Z)\phi X + g(hY,Z)\phi X - g(hX,Z)\phi Y \\ &+g(Y,Z)\phi hX - \eta(Y)\eta(Z)\phi hX - g(X,Z)\phi hY + \eta(X)\eta(Z)\phi hY \\ &+g(hX,Z)\phi hY - g(hY,Z)\phi hX \\ &+(k+1)\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\} \\ &+(\mu-1)\{g(hX,\phi Z)\eta(Y)\xi - g(hY,\phi Z)\eta(X)\xi\} \\ &-g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X - g(Y,\phi Z)\eta(X)\xi \\ &-g(hY,\phi Z)X + g(hY,\phi Z)\eta(X)\xi + g(hX,\phi Z)Y - g(hX,\phi Z)\eta(Y)\xi \\ &-g(Y,\phi Z)hX + g(X,\phi Z)hY - g(hX,\phi Z)hY + g(hY,\phi Z)hX \\ &-k\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\} - \mu\{\eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX\} \\ &-\frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y - S(Y,Z)\phi X + S(X,Z)\phi Y \\ &-(k+2)[g(Y,\phi Z)X - g(X,\phi Z)Y + g(X,Z)\phi Y - g(Y,Z)\phi X] \\ &-(\mu-1)[g(hY,\phi Z)X - g(hX,\phi Z)Y] \\ &-(k+2-2nk)[\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y] \\ &+(\mu-1)[g(hY,Z)\phi X - g(hX,Z)\phi Y]\} = 0.\end{aligned} \tag{4.16}$$

In [18], Yıldız and De proved that

$$
\begin{aligned}
R(X,Y)\phi Z-\phi R(X,Y)Z &= g(X,\phi Z)Y-g(Y,\phi Z)X+g(Y,Z)\phi X\\
&\quad -g(X,Z)\phi Y-g(X,\phi Z)hY+g(Y,\phi Z)hX\\
&\quad +g(hY,\phi Z)X-g(hX,\phi Z)Y-g(Y,Z)\phi hX\\
&\quad +g(X,Z)\phi hY-g(hY,Z)\phi X+g(hX,Z)\phi Y\\
&\quad +\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)hX-g(hX,\phi Z)hY-g(hY,Z)\phi hX\\
&\quad +g(hX,Z)\phi hY\}-\frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,Z)\phi hY-g(hY,Z)\phi hX\\
&\quad +g(hY,\phi Z)hX-g(hX,\phi Z)hY\}\\
&\quad +(k+1)\{g(\phi X,Z)\eta(Y)\xi-g(\phi Y,Z)\eta(X)\xi\\
&\quad +\eta(X)\eta(Z)\phi Y-\eta(Y)\eta(Z)\phi X\}\\
&\quad +(\mu-1)\{g(\phi hX,Z)\eta(Y)\xi-g(\phi hY,Z)\eta(X)\xi\\
&\quad +\eta(X)\eta(Z)\phi hY-\eta(Y)\eta(Z)\phi hX\}.
\end{aligned}
\tag{4.17}
$$

Using (4.17) in (4.16), we obtain

$$
\begin{aligned}
&g(hX,Z)g(\phi hY,W)-g(hY,Z)g(\phi hX,W)+\eta(X)\eta(Z)g(\phi hY,W)\\
&-\eta(Y)\eta(Z)g(\phi hX,W)+g(X,\phi Z)\eta(Y)\eta(W)-g(Y,\phi Z)\eta(X)\eta(W)\\
&+g(hY,\phi Z)g(hX,W)-g(hX,\phi Z)g(hY,W)\\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(hY,\phi Z)g(hX,W)-g(hX,\phi Z)g(hY,W)\\
&-g(hY,Z)g(\phi hX,W)+g(hX,Z)g(\phi hY,W)\}\\
&-\frac{-k+\frac{\mu}{2}}{k+1}\{g(hX,Z)g(\phi hY,W)-g(hY,Z)g(\phi hX,W)\\
&+g(hY,\phi Z)g(hX,W)-g(hX,\phi Z)g(hY,W)\}\\
&+(\mu-1)\{g(hX,\phi Z)\eta(Y)\xi-g(hY,\phi Z)\eta(X)\xi\}\\
&-\frac{1}{2n}\{S(Y,\phi Z)g(X,W)-S(X,\phi Z)g(Y,W)+S(X,Z)g(\phi Y,W)\\
&-S(Y,Z)g(\phi X,W)-(k+2)[g(Y,\phi Z)g(X,W)-g(X,\phi Z)g(Y,W)\\
&+g(X,Z)g(\phi Y,W)-g(Y,Z)g(\phi X,W)]+(\mu-1)[g(hY,Z)g(\phi X,W)\\
&-g(hX,Z)g(\phi Y,W)+g(hY,\phi Z)g(X,W)-g(hX,\phi Z)g(Y,W)]\\
&-(k+2-2nk)[\eta(Y)\eta(Z)g(\phi X,W)-\eta(X)\eta(Z)g(\phi Y,W)]\}=0,
\end{aligned}
\tag{4.18}
$$

If we put $Y=\phi Y$ in (4.18), we have

$$
\begin{aligned}
&g(h\phi Y,Z)g(hX,\phi W)-g(X,hZ)g(h\phi^2Y,W)-g(h\phi^2Y,W)\eta(X)\eta(Z)\\
&-g(\phi Y,\phi Z)\eta(X)\eta(W)+g(h\phi Y,\phi Z)g(hX,W)-g(X,h\phi Z)g(h\phi Y,W)\\
&+\frac{-1-\frac{\mu}{2}}{k+1}\{g(h\phi Y,\phi Z)g(hX,W)-g(X,h\phi Z)g(h\phi Y,W)
\end{aligned}
$$

$$
\begin{aligned}
&+g(h\phi Y,Z)g(hX,\phi W)-g(X,hZ)g(h\phi^2Y,W)\}\\
&-\frac{-k+\frac{\mu}{2}}{k+1}\{-g(X,hZ)g(h\phi^2Y,W)+g(h\phi Y,Z)g(hX,\phi W)\\
&-g(\phi h\phi Y,Z)g(hX,W)-g(X,h\phi Z)g(h\phi Y,W)\}\\
&-(\mu-1)g(h\phi Y,\phi Z)\eta(X)\eta(W)\\
&-\frac{1}{2n}\{S(\phi Y,\phi Z)g(X,W)-S(X,\phi Z)g(\phi Y,W)+S(X,Z)g(\phi^2Y,W)\\
&-S(\phi Y,Z)g(\phi X,W)-(k+2)[g(\phi Y,\phi Z)g(X,W)-g(X,\phi Z)g(\phi Y,W)\\
&+g(X,Z)g(\phi^2Y,W)-g(\phi Y,Z)g(\phi X,W)]+(\mu-1)[g(h\phi Y,Z)g(\phi X,W)\\
&-g(hX,Z)g(\phi^2Y,W)+g(h\phi Y,\phi Z)g(X,W)-g(hX,\phi Z)g(\phi Y,W)]\\
&+(k+2-2nk)\eta(X)\eta(Z)g(\phi^2Y,W)\}=0.
\end{aligned}
\tag{4.19}
$$

Putting $X=W=e_i$, $\{i=1,...,2n+1\}$, in (4.19), we obtain

$$
\begin{aligned}
S(Y,Z) &= \frac{2n}{2n-1}[\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\}g(Y,Z)\\
&+\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\}\eta(Y)\eta(Z)\\
&-(\mu-1)\{1+\frac{2n-1}{2n}\}g(hY,Z)].
\end{aligned}
\tag{4.20}
$$

Using (4.10) in (4.20), we obtain

$$
\begin{aligned}
S(Y,Z) &= \frac{2n}{2n-1}[\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\}g(Y,Z)\\
&-\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\}\eta(Y)\eta(Z)\\
&-\{(\mu-1)(1+\frac{2n-1}{2n})\}\{\frac{1}{(2(n-1)+\mu)}S(Y,Z)\\
&-\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}g(Y,Z)-\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z)\}],
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\{1+[(\mu-1)(1+\frac{2n-1}{2n})][\frac{1}{(2(n-1)+\mu)}]\}S(Y,Z)\\
=\ &\{\frac{2n}{2n-1}\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\}\\
&+\{(\mu-1)(1+\frac{2n-1}{2n})\}\{\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}\}g(Y,Z)\\
&-\{\frac{2n}{2n-1}\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\}\\
&+\{(\mu-1)(1+\frac{2n-1}{2n})\}\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}\eta(Y)\eta(Z).
\end{aligned}
\tag{4.21}
$$

Hence one can write

$$S(Y,Z)=\frac{B_1}{B}g(Y,Z)+\frac{B_2}{B}\eta(Y)\eta(Z), \tag{4.22}$$

where

$$\begin{aligned}
B_1 &= \frac{2n}{2n-1}\{1+2k-\mu+\frac{(2n-1)(k+2)}{2n}\} \\
&+\{(\mu-1)(1+\frac{2n-1}{2n})\}\{\frac{(2(1-n)+n\mu)}{2(n-1)+\mu}\}, \\
B_2 &= -\{\frac{2n}{2n-1}\{-1-2k+\mu-\frac{(2n-1)(k+2)}{2n}+(2n-1)k\} \\
&+\{(\mu-1)(1+\frac{2n-1}{2n})\}\frac{(2(n-1)+n(2k-\mu))}{2(n-1)+\mu}, \\
B &= 1+(\mu-1)(1+\frac{2n-1}{2n})\frac{1}{(2(n-1)+\mu)}.
\end{aligned}$$

Therefore from (4.22) it follows that the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 4.2. *Let M be a $(2n+1)$-dimensional ϕ-projectively semisymmetric paracontact (k,μ)-manifold $(k\neq -1)$ with respect to the Schouten-van Kampen connection. Then the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection provided $\mu \neq 2(1-n)$.*

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THE STATISTICAL MULTIPLICATIVE ORDER CONVERGENCE IN RIESZ ALGEBRAS

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Abstract. The statistically multiplicative convergence in Riesz algebras was studied and investigated with respect to the solid topology. In the present paper, the statistical convergence with the multiplication in Riesz algebras is introduced by developing topology-free techniques using the order convergence in vector lattices. Moreover, we give some relations with the other kinds of convergences such as the order statistical convergence, the mo-convergence, and the order convergence.
Key words: Statistical convergence, Statistical mo-convergence, Order convergence, Order statical convergence, Riesz algebra, Riesz spaces, f-algebra

1. Introduction and Preliminaries

Steinhaus introduced the concept of statistical convergence in [15] that is a generalization of the convergence of real sequences. Another important concept of functional analysis is vector lattice (or, Riesz spaces) which was introduced by F. Riesz [13]. We refer the reader for applications of Riesz spaces to [1, 2, 3, 4, 5, 18]. We aim to combine concepts of the order and the statistical convergence, and the multiplicative on Riesz algebras, and so, we introduce the convergence on Riesz algebras without topological structure.

For the statistical convergence, the natural density of subsets of $\mathbb{N}$ has critical points. Take a subset B in $\mathbb{N}$. Then the unique limit $\lim\limits_{n\to\infty} \frac{1}{n}\left|\{k \leq n : k \in A\}\right|$ is

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said to be *the natural density* of B whenever it exists. Also, we abbreviate it as $\delta(A)$. Now, take a sequence (x_n) of reel numbers. If, for a given $\varepsilon > 0$, the limit

$$\lim_{n\to\infty} \frac{1}{n}\big|\{k : n \geq k,\ |x_n - x| > \varepsilon\}\big| = 0.$$

exists then it is called that (x_n) statistical converges to x. Several applications and generalizations about the statistical convergence have been investigated by several authors (cf. [3, 7, 8, 11, 16, 17]). In this paper, we abbreviate the cardinality of subsets in the vertical bar.

Let "$\leq$" be an order relation on a reel vector space E. Then E is called *ordered vector space* if $\beta x \leq \beta y$ and $x + z \leq y + z$ hold in E for all $\beta \in \mathbb{R}_+$ and $z \in E$ whenever $x \leq y$. Let E be an order vector space. Then it is said to be *vector lattice* or *Riesz space* if, for every pair $x, y \in E$, we have

$$x \vee y = \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}$$

in E. Moreover, a Riesz space is called σ-order or σ-*Dedekind complete* whenever each countable and bounded above subset has a supremum. Take an element x in a vector lattice E. Then $x^+ := x \vee 0$ is *the positive part*, $x^- := (-x) \vee 0$ is *the negative part*, and $|x| := x \vee (-x)$ is the *module* of x. So, in this paper, we use the vertical bar $|\cdot|$ of elements for the module of the given elements. Some works on Riesz spaces with statistical convergence have done. For example, a characterization of statistical convergence was introduced by Ercan in [8], and Aydın introduced the statistical convergence with unbounded order convergence [3]. The crucial point in Riesz spaces is the order convergence. Thus, we continue with its definition.

Definition 1.1. The order convergence of a sequence (x_n) to an element x in a Riesz space E defined as follows:

(i) There exists another sequence (y_n) in E such that $\inf y_n = 0$ and $y_n \downarrow$ in E (i.e., $y_n \downarrow 0$);

(ii) $|x_n - x| \leqslant y_n$ for each $n \in \mathbb{N}$.

Next, we turn our attention to Riesz algebras. If, for an associative algebraic vector lattice E, $x \cdot y \in E_+$ holds for every $x, y \in E_+$ then E is called a *Riesz algebra* (or, shortly, *l-algebra*). Also, if $x \cdot y = y \cdot x$ holds for all pair $x, y \in E$ then E is said to be *commutative*. For much more information on l-algebras, we refer [2, 6, 10, 12, 18]. Aydın and Et introduced the statistical convergence on Riesz algebra with the solid topology [7].

Definition 1.2. Let E be a Riesz algebra. Then it is called

(1) *d-algebra* if $(x \wedge y) \cdot u = (x \cdot u) \wedge (y \cdot u)$ and $u \cdot (x \wedge y) = (u \cdot x) \wedge (u \cdot y)$ hold for each $x, y \in E$ and $u \in E_+$;

(2) *unital* if E has a multiplicative unit.

(3) *f-algebra* whenever we have $y \wedge (u \cdot x) = 0$ and $y \wedge (x \cdot u) = 0$ for all $y \wedge x = 0$, $x, y \in E$ and $u \in E_+$.

It is clear that $u \cdot y \leq u \cdot x$ holds in Riesz algebras for elements $y \leq x$ and for all positive vector u. Remind that if $\frac{1}{n}x \downarrow 0$ holds for any positive vector x in a Riesz space E then it is said to be *Archimedean Riesz space*. By considering [18, Thm.140.10], one can see that each Archimedean f-algebra has the commutative property. In the works (cf. [4, 5, 6, 10, 12]), the reader can find more features and some kinds of convergences in l-algebras.

Example 1.1. Consider the set of orthomorphisms on a Riesz space E

$$Orth(E) := \{\pi \in L_b(E) : x \perp y \text{ implies } \pi x \perp y\}.$$

That is, $|\pi x| \wedge |y| = 0$ whenever $|x| \wedge |y| = 0$ in E. Now, let's take E as an σ-Dedekind complete Riesz space. Then, by using [12, Thm.15.4], we have $\mathrm{Orth}(E)$ is an σ-Dedekind complete, and also, $\mathrm{Orth}(E)$ is an unital f-algebra.

For much more examples of Riesz algebras see for example [6, 10, 12]. In this paper, unless otherwise, we assume that all Riesz spaces are Archimedean and all multiplications are commutative.

2. The statistical *mo*-convergence

We define the statistical convergence in Riesz algebras with respect to multiplicative order convergence in this section. To give this notion, we use the statistical monotonicity for real sequences that was introduced by Salat in [14]. We take the following notions from [4] and [16].

Definition 2.1.

(a) Let (x_n) be a sequence in a Riesz algebra E. Then it is called *multiplicative order convergent* to $x \in E$ whenever $u \cdot |x_\alpha - x| \xrightarrow{o} 0$ for every $u \in E_+$. Abbreviated as $x_\alpha \xrightarrow{\mathrm{mo}} x$.

(b) Let (q_n) be a sequence in a Riesz space E. Then it is called *statistical monotone convergent* to $x \in E$ if there exists a subset J in $\mathbb{N}$ with $\delta(J) = 1$ and $(q_{n_k})_k \downarrow x$. It is abbreviated as $q_n \downarrow^{st} x$.

(c) A sequence (x_n) is said to be *statistical order converges* to x in a vector lattice E if there are a subset $\delta(J) = 1$ and a sequence $y_n \downarrow^{\mathrm{st}} 0$ with $|x_n - x| \leq y_n$ for all $n \in J$.

We give a basic observation in the following result.

Lemma 2.1. *Every order convergent monotone sequence is statistical monotone convergent in vector lattices.*

Proof. Take an order convergent sequence $x_n \xrightarrow{o} x$ in a Riesz space E such that $x_n \downarrow$ (i.e., $x_n \downarrow x$). Now, we can choose J in Definition 2.1(b) as $\mathbb{N}$. Then we have $\delta(J) = 1$ and $x_n \downarrow x$ on J. So, we obtain the desired, $x_n \downarrow^{\text{st}} x$, result. □

Now, motivated from above definitions, we give the following crucial notion.

Definition 2.2. Let E be an l-algebra and (x_n) be a sequence in E. Then (x_n) is called *statistical multiplicative order convergent* (or, *statistical mo-convergent*, shortly) to $x \in E$ if, for each positive element $u \in E_+$, there exists a subset J of the natural numbers with $\delta(J) = 1$ and a sequence $q_n \downarrow^{st} 0$ such that

$$|x_{n_j} - x| \cdot u \leqslant q_{n_j}$$

for all $n_j \in J$. We abbreviate it as $x_n \xrightarrow{\text{st-mo}} x$.

It can be seen that $x_n \xrightarrow{\text{st-mo}} x$ if, for each $u \in E_+$, there exists a sequence $q_n \downarrow^{\text{st}} 0$ such that the natural density of the set $\{n \in \mathbb{N} : |x_n - x| \cdot u \nleqslant q_n\}$ is equal to zero.

Proposition 2.1. *The mo-convergence implies the statistical mo-convergence in l-algebras.*

Proof. Assume that a sequence (x_n) is *mo*-convergent to x in an l-algebra E. Let's fix $u \in E_+$. Then, following from Definition 2.1(a), we have $|x_n - x| \cdot u \xrightarrow{o} 0$. Thus, there is a sequence $y_n \downarrow 0$ in E such that $|x_n - x| \cdot u \leqslant y_n$ holds for all $n \in \mathbb{N}$. So, by applying Lemma 2.1, we obtain $y_n \downarrow^{\text{st}} 0$. Since $u \in E_+$ is arbitrary, if we take the subset J as $\mathbb{N}$ then we get the desired result, $x_n \xrightarrow{\text{st-mo}} x$. □

It is known that the order convergence does not imply the *mo*-convergence in l-algebras because l-algebras do not have the infinite distributive property, i.e., if $\inf(A)$ exists and positive for any subset A of an l-algebra E then the infimum of the subset $u \cdot A$ exists and $\inf(u \cdot A) = u \cdot \inf(A)$ for every $u \in E_+$ (see, [4, p.2] and [6, Thm.12]). By the way, the order and the statistical order convergences do not imply the statistical *mo*-convergent, in general. But, we have a positive implication in the following work.

Theorem 2.1. *If (x_n) in a d-algebra is order or statistical order convergent sequence then it is statistical mo-convergent to their order or statistical order limit points.*

Proof. Assume that (x_n) statistical order converges to x in a d-algebra E. We show that (x_n) is statistical *mo*-convergent to x. Similarly, one can show the other case. Following from Definition 2.1(c), there exists a sequence $q_n \downarrow^{st} 0$ and a subset J of the natural numbers with $\delta(J) = 1$ such that $|x_{n_j} - x| \leqslant q_{n_j}$ for all $n_j \in J$. On the other hand, there is a subset $\delta(K) = 1$, and also, (q_{n_k}) is decreasing to zero because of $q_n \downarrow^{st} 0$. Next, consider the set $M = J \cap K$. Hence, following from the inequality $\delta(J) + \delta(K) \leq 1 + \delta(J \cap K)$, we have $\delta(M) = 1$. As a result, we obtain that $|x_{n_m} - x| \leqslant q_{n_m} \downarrow 0$. Therefore, we get $|x_{n_m} - x| \cdot u \leqslant (q_{n_m} \cdot u) \downarrow 0$ for all $u \in E_+$ because every d-algebra having infinite distributive properties; see [6, Thm.12.]. Thus, for every $u \in E_+$, we can obtain a sequence $w_n = (q_n \cdot u) \downarrow^{\text{st}} 0$, and also, $|x_n - x| \cdot u \leqslant w_n$ holds on M, i.e., we get $x_n \xrightarrow{\text{st-mo}} x$. $\square$

In the following result, we give a partial answer for the converse implication of Theorem 2.1

Proposition 2.2. *Every statistical mo-convergent sequence in an unital f-algebra is statistical order convergent to its statistical mo-limit.*

Proof. Let (x_n) be a statistical mo-convergent sequence in an unital f-algebra E with the multiplicative unit e. Then there exists a sequence $q_n \downarrow^{\text{st}} 0$ such that the natural density of the subset $\{n \in \mathbb{N} : |x_n - x| \cdot u \nleq y_n, \forall u \in E_+\}$ is equal to zero. By applying [18, Thm.142.1(v)], in view of $e = e \cdot e = e^2 \geq 0$, one clearly can obtain that unit element is positive in E. Thus, in a special case, we can take $u = e \in E_+$. Then we have

$$\delta\big(\{n : |x_n - x| \nleq y_n\}\big) = \delta\big(\{n : |x_n - x| \cdot e \nleq y_n\}\big) = 0.$$

Therefore, we obtain that (x_n) statistical order converges to x. $\square$

3. Main Results of the Statistical mo-Convergence

In this section, we give the main results and properties of the statistical mo-convergence. First of all, to mention the uniqueness of the statistical mo-limit, we need the notion of semiprime l-algebra. Consider an element x in a Riesz algebra E with $x^n = 0$ for some natural numbers $n \in \mathbb{N}$ then it is said to be a *nilpotent element*. Moreover, if the only nilpotent element of a Riesz algebra E is zero element then E is called *semiprime* (cf., [9, 10, 12, 18]).

Lemma 3.1. *Let (x_n) be a sequence of nilpotent elements of an f-algebra E. If $x_n \xrightarrow{\text{st-mo}} x$ then x is a nilpotent element of E.*

Proof. Suppose $x_n \xrightarrow{\text{st-mo}} x$. Fix a positive element $u \in E_+$. Then there exists a sequence $q_n \downarrow^{st} 0$ and a subset $\delta(J) = 1$ such that $|x_{n_j} - x| \cdot u \leqslant q_{n_j}$ for all $n_j \in J$. Now, following from [12, Prop.10.2(iii)] and [18, Thm.142.1(ii)], we have

$$q_{n_j} \geq |x_{n_j} - x| \cdot u = |x_{n_j} \cdot u - x \cdot u| = |x \cdot u|$$

because (x_n) consists of nilpotent elements. Thus, we obtain $|x \cdot u| = 0$, i.e., we have $x \cdot u = 0$ for every $u \in X_+$ because of $q_{n_j} \downarrow 0$. Then $x \cdot y = 0$ for each $y \in E$ because of $y = y^+ - y^-$ and $y^+, y^- \in E_+$. Therefore by using [9, p.157], one can see that x is also a nilpotent element. $\square$

Proposition 3.1. *The limit of a statistically mo-convergent sequence is uniquely determined in semiprime f-algebras.*

Proof. Suppose that (x_n) is a statistically *mo*-convergent to x and y sequence in a semiprime f-algebra E. Fix $u \in E_+$. Then there exists sequences $q_n \downarrow^{st} 0$ and $p_n \downarrow^{st} 0$, and subsets J and K of the natural numbers with $\delta(J) = \delta(K) = 1$ such that $|x_{n_j} - x| \cdot u \leqslant q_{n_j}$ and $|x_{n_k} - y| \cdot u \leqslant p_{n_k}$ for all $n_j \in J$ and $n_k \in K$. Choose $M = J \cap K$. Thus, we have $\delta(M) = 1$, $|x_{n_m} - x| \cdot u \leqslant q_{n_m}$ and $|x_{n_m} - y| \cdot u \leqslant p_{n_m}$ for every $n_m \in M$. Now, it follows that

$$|x - y| \cdot u \leq |x_{n_m} - x| \cdot u + |x_{n_m} - y| \cdot u$$

satisfies for every $m \in \mathbb{N}$. Thus, we obtain $|x - y| \cdot u = 0$. Since u is arbitrary, one can see that $|x - y|$ is a nilpotent element in E (cf. [9, p.157]). Therefore, we get $|x - y| = 0$, i.e., we have $x = y$ because of E is semiprime. $\square$

Next, we give several results that are parallel to some kinds of statistical convergence such as [3, Thm.2.2.] and [1, Thm.2.17.].

Theorem 3.1. *If $x_n \xrightarrow{\text{st-mo}} x$ and $y_n \xrightarrow{\text{st-mo}} y$ in an l-algebra E then the following holds:*

(i) *The lattice operations are statistical mo-order continuous;*

(ii) $x_n \xrightarrow{\text{st-mo}} x$ *iff* $(x_n - x) \xrightarrow{\text{st-mo}} 0$ *iff* $|x_n - x| \xrightarrow{\text{st-mo}} 0$;

(iii) *The statistical mo-limit is linear;*

(iv) $x_{n_k} \xrightarrow{\text{st-mo}} x$ *for any subsequence* (x_{n_k}) *of* (x_n);

(v) *E_+ that is the positive cone of E is closed under the statistical mo-convergence whenever E is semiprime f-algebra.*

Proof. *(i)* It is enough to show that $(x_n \vee y_n)$ statistical *mo*-converges to $x \vee y$. Take fixed $u \in E_+$. Since $x_n \xrightarrow{\text{st-mo}} x$ and $y_n \xrightarrow{\text{st-mo}} y$, by the same argument in the proof of Proposition 3.1, there exists a subset of the natural numbers with $\delta(M) = 1$ and sequences $q_n \downarrow^{st} 0$ and $p_n \downarrow^{st} 0$ such that $|x_{n_m} - x| \cdot u \leqslant q_{n_m}$ and $|x_{n_m} - y| \cdot u \leqslant p_{n_m}$ for every $n_m \in M$. By using [2, Thm.1.2(2)], we have

$$|x_{n_m} \vee y_{n_m} - x \vee y| \cdot u \leq |x_{n_m} - x| \cdot u + |y_{n_m} - y| \cdot u \leq q_{n_m} + p_{n_m}$$

for each $m \in \mathbb{N}$. Hence, if we denote a sequence $r_n := q_n + p_n$ then we have $|x_{n_m} \vee y_{n_m} - x \vee y| \cdot u \leq r_{n_m}$ and $r_n \downarrow^{\text{st}} 0$. Hence, we obtain $x_n \vee y_n \xrightarrow{\text{st-mo}} x \vee y$ in E.

One can get *(ii)* and *(iv)* directly from the definition of the statistical *mo*-convergence. Also, *(iii)* is similar to *(i)*.

(v) Suppose that (x_n) is non-negative and statistical mo-converges to $x \in E$. It follows from (i) that $x_n = x_n^+ \xrightarrow{\text{st-mo}} x^+$, and also, following from Proposition 3.1, we obtain $x = x^+$. So, we get the desired, $x \in E_+$, result. □

In the following result, we give a positive answer for the converse of Theorem 2.1.

Proposition 3.2. *Ever monotone statistical mo-convergent sequence in a semiprime f-algebra order converges to its statistical mo-limit.*

Proof. Suppose that a sequence (x_n) in a semiprime f-algebra E is increasing and statistical mo-convergent to $x \in E$. It is enough to show $x_n \uparrow x$. Let's fix an index n_0. It is clear that $x_n - x_{n_0} \in X_+$ for each $n \geqslant n_0$. Now, by using linearity of statistical mo-limit, we have $x_n - x_{n_0} \xrightarrow{\text{st-mo}} x - x_{n_0}$. Since $x_n - x_{n_0} \in E_+$, by applying Theorem 3.1(v), we can obtain $x - x_{n_0} \in E_+$, i.e., $x \geq x_{n_0}$. Thus, x is an upper bound of (x_n) because x_{n_0} is arbitrary. Take another upper bound y of (x_n), i.e., $y \geq x_n$ for all n. Then we obtain $y - x_n \xrightarrow{\text{st-mo}} y - x \in E_+$, or equivalently, we get $y \geqslant x$. Thus, $x_n \uparrow x$. □

Proposition 3.3. *If $0 \leq y_n \leq x_n$ holds for every natural number $n \in \mathbb{N}$ and $x_n \xrightarrow{\text{st-mo}} 0$ in an l-algebra E then we have $y_n \xrightarrow{\text{st-mo}} 0$ in E.*

Proof. Fix $u \in E_+$. Since $x_n \xrightarrow{\text{st-mo}} 0$, there exist a subset $\delta(J) = 1$ and a sequence $q_n \downarrow^{\text{st}} 0$ such that $x_{n_j} \cdot u \leqslant q_{n_j}$ for every $n_j \in J$. So, we have $0 \leq y_{n_j} \leq x_{n_j}$, and so, following from the inequality $y_{n_j} \cdot u \leq x_{n_j} \cdot u$ for all j, we obtain the desired, $y_n \xrightarrow{\text{st-mo}} 0$, result. □

Recall that every order convergent sequence in a d-algebra is statistical mo-convergent (see, Theorem 2.1). But, for the general case, we give the following notions.

Definition 3.1. Assume (x_n) is a sequence in a Riesz algebra E. Then

(a) (x_n) in E is called *statistical mo-Cauchy* whenever the sequence $(x_n - x_m)_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ statistical mo-converges to 0;

(b) E is said to be *statistical mo-complete* whenever each statistical mo-Cauchy sequence is statistical mo-convergent;

(c) E is called *statistical mo-continuous* whenever every order convergent sequence is statistical mo-convergent.

Proposition 3.4. *The following statements are equivalent for arbitrary Riesz algebra E.*

(i) E is statistical mo-continuous;

(ii) $x_n \downarrow 0$ *in* X *implies* $x_n \xrightarrow{\text{st-mo}} 0$.

Proof. We show the implication $(ii) \Rightarrow (i)$ because the converse is trivial. Take a sequence $x_n \xrightarrow{o} x$ in E. Thus, there exists a sequence $y_n \downarrow 0$ in E such that $|x_n - x| \leq y_n$ for every $n \in \mathbb{N}$. Moreover, by using (ii), we have $y_n \xrightarrow{\text{st-mo}} 0$ because of $y_n \downarrow 0$. So, it follows from Proposition 3.3 that $|x_\alpha - x|$ is also statistical mo-converges to zero. Therefore, by considering Theorem 3.1(ii), we have $x_n \xrightarrow{\text{st-mo}} x$. □

Theorem 3.2. *Let E be a statistical mo-continuous and mo-complete semiprime f-algebra. Then E is σ-order complete.*

Proof. Consider a sequence $0 \leq x_n \uparrow \leq x$ in E. Thus, by considering [1, Lem.1.39.], it is enough to show the existence of $\sup x_n$. Now, by [2, Lem.4.8.], we have a new sequence (y_n) in E with $(y_n - x_n) \downarrow 0$. Then it follows from Proposition 3.4 that $(y_n - x_n) \xrightarrow{\text{st-mo}} 0$ because E is statistical mo-continuous. Next, by considering the linearity of statistical mo-limit, Proposition 3.3 and the following inequality

$$|x_n - x_m| \leq |x_n - y_n| + |y_n - x_m|,$$

we obtain that the sequence (x_n) is a statistical mo-Cauchy. Thus, there is some $x \in E$ such that $x_n \xrightarrow{\text{st-mo}} x$ because E is statistical mo-complete. Now, by applying Proposition 3.2, since we have $x_n \uparrow x$, we obtain the σ-order completeness of E. □

Proposition 3.5. *If every increasing order bounded sequence in a semiprime f-algebra E is statistical mo-convergent then E is statistical mo-continuous.*

Proof. Suppose $x_n \downarrow 0$. So, we show that it is statistical mo-convergent to 0. Let's fix an index n_0 and consider a sequence $y_n := x_{n_0} - x_n$ for $n \geqslant n_0$. It is clear that $0 \leq y_n \uparrow \leqslant x_{n_0}$. Therefore, we see that (y_n) is increasing and order bounded sequence. Thus, by our assumption, one can say that (y_n) is statistical mo-convergent to some $y \in E$. Since (y_n) is increasing and statistical mo-convergent, Proposition 3.2 gives the following equality

$$y = \sup_{n \geqslant n_0} y_n = \sup_{n \geqslant n_0} (x_{n_0} - x_n) = x_{n_0}.$$

Therefore, we have $y_n = x_{n_0} - x_n \xrightarrow{\text{st-mo}} x_{n_0}$, or $x_n \xrightarrow{\text{st-mo}} 0$. So by Proposition 3.4, E is statistical mo-continuous. □

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η-RICCI SOLITONS ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract. The objective of present research article is to investigate the geometric properties of η-Ricci solitons on Lorentzian para-Kenmotsu manifolds. In this manner, we consider η-Ricci solitons on Lorentzian para-Kenmotsu manifolds satisfying $R{\cdot}S = 0$. Further, we obtain results for η-Ricci solitons on Lorentzian para-Kenmotsu manifolds with quasi-conformally flat property. Moreover, we get results for η-Ricci solitons in Lorentzian para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor, η-quasi-conformally semi-symmetric, η-Ricci symmetric and quasi-conformally Ricci semi-symmetric. At last, we construct an example of a such manifold which justify the existence of proper η-Ricci solitons.
Key words: η-Ricci solitons; Lorentzian Para-Kenmotsu manifolds; Codazzi type of Ricci tensor; Cyclic parallel Ricci tensor; quasi-conformal curvature tensor.

1. Introduction

In 1982, Hamilton [17] introduced the notion of the Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics

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on a Riemannian manifold

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) [6]. A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field (called the potential vector field), and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where S is a Ricci tensor of M and $\pounds_V$ denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively [18]. A Ricci soliton with V zero is reduced to Einstein equation. Metrics satisfying (1.2) is interesting and useful in physics and is often referred as quasi-Einstein [22, 23]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contributions in this direction is due to Friedmann [7], who discusses some aspects of it. Ricci solitons were introduced in Riemannian geometry [17], as the self-similar solutions of the Ricci flow, and play an important role in understanding its singularities. Ricci solitons have been studied in many contexts by several authors such as [24, 25, 26, 19, 21] and many others.

As a generalization of Ricci soliton, the notion of η-Ricci soliton introduced by J. T. Cho and M. Kimura [9], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [6]. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, and λ are μ constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g.

In particular, if $\mu = 0$, then the notion of η-Ricci solitons (g, V, λ, μ) reduces to the notion of Ricci solitons (g, V, λ). If $\mu \neq 0$, then the η-Ricci solitons are called proper η-Ricci solitons. We refer to [28, 14, 10] and references therein for a survey and further references on the geometry of Ricci solitons on pseudo-Riemannian manifolds.

Recently an η-Ricci soliton has been studied by several authors such as [3, 5, 16, 20, 8, 4] and they found many interesting geometric properties.

These above results motivated me to study η-Ricci solitons on Lorentzian para-Kenmotsu manifolds satisfying certain curvature conditions. The paper is organized in the following way. In Section 2, we give a brief introduction of an Lorentzian para-Kenmotsu manifold. Section 3 deals with the study of Ricci solitons and η-Ricci solitons in Lorentzian para-Kenmotsu manifolds. In Section 4, we study η-Ricci solitons on Lorentzian para-Kenmotsu manifolds satisfying $R \cdot S = 0$. η-Ricci solitons on quasi-conformally flat Lorentzian para-Kenmotsu manifolds have been studied in Section 5. In Section 6, we study η-Ricci solitons in Lorentzian para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. Section 7 is devoted to the study of φ-quasi-conformally semi-symmetric η-Ricci solitons on Lorentzian para-Kenmotsu manifolds. Beside these we study η-Ricci solitons on φ-Ricci symmetric Lorentzian para-Kenmotsu manifolds. Also η-Ricci solitons on conformally Ricci semi-symmetric Lorentzian para-Kenmotsu manifolds has been considered. Finally, we construct a 3-dimensional example of a Lorentzian para-Kenmotsu manifold which admits an η-Ricci soliton.

2. Preliminaries

An n-dimensional differential manifold M is a Lorentzian metric manifold. If it admits a $(1,1)$-tensor field φ, contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g, which satisfy [15]:

$$\varphi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

which implies

$$(a)\ \varphi\xi = 0, \quad (b)\ \ \eta(\varphi X) = 0, \quad (c)\ \ rank(\varphi) = n-1. \tag{2.2}$$

Then M admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

and M is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). In this case, we have

$$(a)\ \ g(X,\xi) = \eta(X), \quad (b)\ \ \nabla_X \xi = \varphi X, \tag{2.4}$$

$$(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

If we put

$$\Omega(X,Y) = g(X,\varphi Y) = g(\varphi X, Y) = \Omega(Y,X), \tag{2.6}$$

for any vector fields X and Y, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field.

Recently, Haseeb and Prasad define a new manifold called Lorentzian para-Kenmostu manifold [1].

Definition 2.1. A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmostu manifold if

$$(\nabla_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.7}$$

for any vector fields X, Y on M [1].

In the Lorentzian para-Kenmostu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)(Y) = -g(X,Y) - \eta(X)\eta(Y). \tag{2.9}$$

Also in an Lorentzian para-Kenmostu manifold M, the following relations hold:

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{2.10}$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.11}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.13}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.15}$$

$$Q\xi = (n-1)\xi, \tag{2.16}$$

for any X, Y, Z on M.

Definition 2.2. The quasi-conformal curvature tensor W in a Lorentzian para-Kenmotsu manifold M is defined by

$$\begin{aligned} W(X,Y)Z &= aR(X,Y)Z + b(S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY) \\ &\quad - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{2.17}$$

where a and b are constants such that $ab \neq 0$ and R, S, Q, and r are the Riemannian curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $g(QX,Y) = S(X,Y)$, and the scalar curvature of the manifold, respectively (see [12]).

If $a = 1$ and $b = -\frac{1}{(n-2)}$, then (2.17) takes the form

$$\begin{aligned} W(X,Y)Z &= R(X,Y) - \frac{1}{(n-2)}(S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY) \\ &\quad - \frac{r}{(n-1)(n-2)}(g(Y,Z)X - g(X,Z)Y) \\ &= C(X,Y)Z, \end{aligned} \tag{2.18}$$

where C is the conformal curvature tensor [13, 11]. Thus the conformal curvature tensor C is a particular case of a quasi-conformal curvature tensor W. The manifold is said to be quasi-conformally flat if W vanishes identically on M.

Definition 2.3. If (M, V, λ, μ) is an η-Ricci soliton, then the 1-form ξ is said to be a potential vector field.

3. Ricci and η-Ricci solitons on Lorentzian para-Kenmotsu manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an paracontact metric manifolds. Consider the equation

$$\pounds_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where $\pounds_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $\pounds_\xi g$ in terms of the Levi-Civita connection ∇, we have:

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any $X, Y \in \chi(M)$, or equivalent:

$$S(X, Y) = (1 - \lambda)g(X, Y) + (1 - \mu)\eta(X)\eta(Y), \tag{3.3}$$

for any $X, Y \in \chi(M)$.

The above equation yields

$$S(X, \xi) = (\mu - \lambda)\eta(X), \tag{3.4}$$

$$QX = (1 - \lambda)X + (1 - \mu)\eta(X). \tag{3.5}$$

Comparing (2.15) with (3.4), we get

$$\mu - \lambda = n - 1. \tag{3.6}$$

The data (g, ξ, λ, μ) which satisfy the equation (3.1) is said to be an η-Ricci soliton on M [9].

Thus, we can state the following theorem:

Theorem 3.1. *Let M be an n-dimensional Lorentzian para-Kenmotsu manifold. If the manifold admits an η-Ricci soliton (g, ξ, λ, μ), then M an η-Einstein manifold of the form (3.3), and the scalars λ and μ are related by $\mu - \lambda = n - 1$.*

In particular, if we take $\mu = 0$ in (3.3) and (3.6), then we obtain $S(X, Y) = (1 - \lambda)g(X, Y) + \eta(X)\eta(Y)$, and $\lambda = 1 - n$, respectively. Thus, we have

Corollary 3.1. *Let M be an n-dimensional Lorentzian para-Kenmotsu manifold. If the manifold admits a Ricci soliton (g, ξ, λ), then M is an η-Einstein manifold and the manifold is expanding or shrinking according to the vector field ξ being spacelike or timelike.*

4. η-Ricci solitons on Lorentzian para-Kenmotsu manifolds satisfying $R \cdot S = 0$

In this section we are going to study, an n-dimensional Lorentzian para-Kenmotsu manifold admitting an η-Ricci soliton satisfies $R \cdot S = 0$, which implies

$$(R(X,Y) \cdot S)(Z,W) = 0. \tag{4.1}$$

From (4.1), we have

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{4.2}$$

Putting $X = \xi$ in (4.2), we obtain

$$S(R(\xi,Y)Z,W) + S(Z,R(\xi,Y)W) = 0. \tag{4.3}$$

Replacing the expression of S from (3.3) and from the symmetries of R, we find

$$(\mu - 1)[\eta(Y)g(X,Z) + \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{4.4}$$

taking $Z = \xi$ in (4.4), we have

$$(\mu - 1)[g(X,Y) + \eta(X)\eta(Y)] = 0, \tag{4.5}$$

from which it follows that $\mu = 1$. From the relation (3.6), we get $\lambda = (2 - n)$.

Thus, we can state the following theorem:

Theorem 4.1. *Let (g, ξ, λ, μ) be an n-dimensional Lorentzian para-Kenmotsu manifold admitting a proper η-Ricci soliton satisfies $R \cdot S = 0$, then $\mu = 1$ and $\lambda = (2-n)$.*

From the above theorem we get:

Corollary 4.1. *On a Lorentzian para-Kenmotsu manifold M satisfying $R \cdot S = 0$, there is no Ricci soliton with the potential vector field ξ.*

5. η-Ricci solitons on quasi-conformally flat Lorentzian para-Kenmotsu manifolds

In this constituent we review an η-Ricci solitons on quasi-conformally flat Lorentzian para-Kenmotsu manifolds.

Let us assume that the manifold M admitting η-Ricci solitons is quasi-conformally flat, that is, $W = 0$. Then, from (2.17) it follows that

$$\begin{aligned} R(X,Y)Z &= -\frac{b}{a}(S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY) \\ &\quad - \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{5.1}$$

Taking the inner product of (5.1) with ξ and using (2.4), (3.3) and (3.4), we get

$$\eta(R(X,Y)Z) = \left[\frac{\lambda b}{a} - \frac{nb}{a} + \frac{r}{an}\left(\frac{a}{n-1} + 2b\right)\right] \cdot [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{5.2}$$

By virtue of (2.11) and (5.3), we obtain

$$\left[\frac{\lambda b}{a} - \frac{nb}{a} + \frac{r}{an}\left(\frac{a}{n-1} + 2b\right) - 1\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0. \tag{5.3}$$

Now, putting $X = \xi$ in the last equation, we find

$$\left[\frac{\lambda b}{a} - \frac{nb}{a} + \frac{r}{an}\left(\frac{a}{n-1} + 2b\right) - 1\right][g(Y,Z) + \eta(Y)\eta(Z)] = 0, \tag{5.4}$$

from which it follows that $\lambda = \left[n + \frac{a}{b} - \frac{r}{bn}\left(\frac{a}{n-1} + 2b\right)\right]$.

From the relation (3.6), we obtain $\mu = \left[2n + \frac{a}{b} - \frac{r}{bn}\left(\frac{a}{n-1} + 2b\right) - 1\right]$.

Thus, we can state the following theorem:

Theorem 5.1. *A quasi-conformally flat Lorentzian para-Kenmotsu manifold admits a proper η-Ricci solitons with $\lambda = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(n - \frac{2r}{n}\right)$ and $\mu = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(2n - \frac{2r}{n} - 1\right)$.*

As a corollary of this theorem we have:

Corollary 5.1. *On a Lorentzian para-Kenmotsu manifold satisfying $W = 0$, there is no Ricci soliton with the potential vector field ξ.*

6. η-Ricci solitons on Lorentzian para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor

In this section we consider η-Ricci solitons in Lorentzian para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. Gray [2] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor.

Definition 6.1. A Lorentzian para-Kenmotsu manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z). \tag{6.1}$$

for all X, Y, Z on M.

Taking covariant derivative of (3.3) along Z, we get

$$(\nabla_Z S)(X,Y) = (1-\mu)[(\nabla_Z \eta)(X)\eta(Y) + \eta(X)(\nabla_Z \eta)(Y)], \tag{6.2}$$

by virtue of (2.9) and (6.2), we obtain

$$(\nabla_Z S)(X,Y) = (\mu-1)[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{6.3}$$

By hypothesis the Ricci tensor S is of Coddazi type. Then

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z). \tag{6.4}$$

Making use of (6.3), (6.4) takes the form

$$(\mu-1)[g(Z,Y)\eta(X) - g(X,Y)\eta(Z)] = 0. \tag{6.5}$$

Substituting $Z = \xi$ in (6.5), we find

$$(\mu-1)[\eta(Y)\eta(X) + g(X,Y)] = 0, \tag{6.6}$$

from which it follows that $\mu = 1$. From the relation (3.6), we obtain $\lambda = (2-n)$.

Thus, we can state the following theorem:

Theorem 6.1. *Let (g,ξ,λ,μ) be a proper η-Ricci soliton in an n-dimensional Lorentzian para-Kenmotsu manifold. If the manifold has Ricci tensor of Codazzi type, then $\mu = 1$ and $\lambda = (2-n)$.*

From the above theorem we have:

Corollary 6.1. *A Lorentzian para-Kenmotsu manifold Ricci tensor is of Codazzi type does not admit Ricci solitons with potential vector field ξ.*

Definition 6.2. A Lorentzian para-Kenmotsu manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition [18]

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{6.7}$$

for all X, Y, Z on M.

Let (g,ξ,λ,μ) be an η-Ricci soliton in an n-dimensional Lorentzian para-Kenmotsu manifold and the manifold has cyclic parallel Ricci tensor, then (6.7) holds. Taking covariant derivative of (3.3) and making use of (2.9), we obtain

$$(\nabla_Z S)(X,Y) = (\mu-1)[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{6.8}$$

Similarly, we have

$$(\nabla_X S)(Y,Z) = (\mu-1)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)], \tag{6.9}$$

and

$$(\nabla_Y S)(Z,X) = (\mu-1)[g(Y,Z)\eta(X) + g(Y,X)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{6.10}$$

By using (6.8)–(6.10) in (6.10), we find

$$(\mu-1)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(Z,X)\eta(Y) + 3\eta(X)\eta(Y)\eta(Z)] = 0. \tag{6.11}$$

Now, putting $Z = \xi$ in (6.11), we have

$$2(\mu-1)[g(X,Y) + \eta(X)\eta(Y)] = 0, \tag{6.12}$$

from which it follows that $\mu = 1$. From the relation (3.6), we obtain $\lambda = (2-n)$.

Thus, we can state the following theorem:

Theorem 6.2. *Let (g, ξ, λ, μ) be a proper η-Ricci soliton in an n-dimensional Lorentzian para-Kenmotsu manifold. If the manifold has cyclic parallel Ricci tensor, then $\mu = 1$ and $\lambda = (2-n)$.*

From the above theorem we get:

Corollary 6.2. *A Lorentzian para-Kenmotsu manifold has cyclic parallel Ricci tensor does not admit Ricci solitons with potential vector field ξ.*

7. η-Ricci solitons on φ-quasi-conformally semi-symmetric Lorentzian para-Kenmotsu manifolds

This section is devoted to the study of φ-quasi-conformally semi-symmetric η-Ricci solitons on Lorentzian para-Kenmotsu manifolds. Then, we have

$$W \cdot \varphi = 0, \tag{7.1}$$

from which it follows that

$$W(X,Y)\varphi Z - \varphi(W(X,Y)Z) = 0. \tag{7.2}$$

Taking $Z = \xi$ in (7.2), we have

$$\varphi(W(X,Y)\xi) = 0. \tag{7.3}$$

Now, replacing $Z = \xi$ in (2.17) and using (2.2), (2.12), (2.15) and (2.16), we obtain

$$W(X,Y)\xi = \left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[\eta(Y)X - \eta(X)Y]. \tag{7.4}$$

In view of (7.3) and (7.4), we get

$$\begin{aligned}\varphi(W(X,Y)\xi) &= \left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[\eta(Y)\varphi X - \eta(X)\varphi Y] \\ &= 0. \end{aligned}\tag{7.5}$$

Now, substituting X by φX in (7.5), we find

$$\left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)\eta(Y)\varphi^2 X = 0. \tag{7.6}$$

By virtue of (2.1) and (7.7), we have

$$\left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[X + \eta(X)\xi] = 0. \tag{7.7}$$

Taking the inner product of (7.7) with respect to U, we find

$$\left(a + nb - \frac{ra}{n(n-1)} - \frac{2rb}{n} - b\lambda\right)[g(X,U) + \eta(X)\eta(U)] = 0, \tag{7.8}$$

from which it follows that $\lambda = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(n - \frac{2r}{n}\right)$. From the relation (3.6), we obtain $\mu = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(2n - \frac{2r}{n} - 1\right)$.

Thus, we can state the following theorem:

Theorem 7.1. *A φ-conformally semi-symmetric Lorentzian para-Kenmotsu manifold admits a proper η-Ricci solitons with* $\lambda = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(n - \frac{2r}{n}\right)$ *and* $\mu = \frac{a}{b}\left(1 - \frac{r}{n(n-1)}\right) + \left(2n - \frac{2r}{n} - 1\right)$.

As a corollary of this theorem we have:

Corollary 7.1. *On a Lorentzian para-Kenmotsu manifold satisfying $W \cdot \varphi = 0$, there is no Ricci soliton with the potential vector field ξ.*

8. η-Ricci solitons on φ-Ricci symmetric Lorentzian para-Kenmotsu manifolds

In this section, we study φ-Ricci symmetric η-Ricci soliton on Lorentzian para-Kenmotsu manifolds.

Definition 8.1. A Lorentzian para-Kenmotsu manifold is said to be φ-Ricci symmetric if

$$\varphi^2(\nabla_X Q)Y = 0, \tag{8.1}$$

holds for all smooth vector fields X, Y.

If X, Y are orthogonal to ξ, then the manifold is said to be locally φ-Ricci symmetric. It is well-known that φ-symmetric implies φ-Ricci symmetric, but the converse, is not, in general true. φ-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [27].

We know that, the Ricci tensor for an η-Ricci soliton on Lorentzian para-Kenmotsu manifold is given by

$$S(X,Y) = (1-\lambda)g(X,Y) + (1-\mu)\eta(X)\eta(Y). \tag{8.2}$$

Then it follows that

$$QY = (1-\lambda)Y + (1-\mu)\eta(Y)\xi, \tag{8.3}$$

for all smooth vector fields Y. Taking covariant derivative of above equation with respect to X, we have

$$\begin{aligned}(\nabla_X Q)Y &= \nabla_X QY - Q(\nabla_X Y) \\ &= (1-\mu)[(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi].\end{aligned} \tag{8.4}$$

Using (2.8) and (2.9),we get

$$(\nabla_X Q)Y = (\mu - 1)[g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi]. \tag{8.5}$$

Applying φ^2 on both sides of the above equation, we find

$$\varphi^2(\nabla_X Q)Y = (\mu-1)\eta(Y)\varphi^2 X. \tag{8.6}$$

From (8.1) and (8.6), we obtain

$$(\mu-1)\eta(Y)\varphi^2 X = 0, \tag{8.7}$$

by virtue of (2.1) and (8.7), we find

$$(\mu-1)[X + \eta(X)\xi] = 0. \tag{8.8}$$

Taking inner product of (8.8) with respect to U, we get

$$(\mu-1)[g(X,U) + \eta(X)\eta(U)] = 0, \tag{8.9}$$

from which it follows that $\mu = 1$. From the relation (3.6), we obtain $\lambda = (2-n)$.

Thus, we can state the following theorem:

Theorem 8.1. *On φ-Ricci symmetric Lorentzian para-Kenmotsu manifold admits a proper η-Ricci soliton with $\mu = 1$ and $\lambda = (2-n)$.*

From the above theorem we get:

Corollary 8.1. *On a φ-Ricci symmetric Lorentzian para-Kenmotsu manifold, there is no Ricci solitons with potential vector field ξ.*

9. η-Ricci solitons on quasi-conformally Ricci semi-symmetric Lorentzian para-Kenmotsu manifolds

In this section, we study η-Ricci solitons on conformally Ricci semi-symmetric Lorentzian para-Kenmotsu manifolds, that is,

$$W \cdot S = 0, \tag{9.1}$$

which implies

$$(W(X, Y) \cdot S)(Z, U) = 0. \tag{9.2}$$

From (9.2), we get

$$S(W(X, Y)Z, U) + S(Z, W(X, Y)U) = 0. \tag{9.3}$$

Using (3.3) in (9.3), we have

$$\begin{aligned} & (1-\lambda)g(W(X,Y)Z,U) + (1-\mu)\eta(W(X,Y)Z)\eta(U) \\ & +(1-\lambda)g(Z,W(X,Y)U) + (1-\mu)(W(X,Y)U)\eta(Z) \\ = \; & 0. \end{aligned} \tag{9.4}$$

Taking $X = U = \xi$ in (9.4), we get

$$\begin{aligned} & (1-\lambda)g(W(\xi,Y)Z,\xi) - (1-\mu)\eta(W(\xi,Y)Z) + \\ & (1-\lambda)g(Z,W(\xi,Y)\xi) + (1-\mu)(W(\xi,Y)\xi)\eta(Z) \\ = \; & 0. \end{aligned} \tag{9.5}$$

From (2.18), we obtain

$$W(\xi, Y)\xi) = \left[a + nb - b\lambda - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right][Y + \eta(Y)\xi]. \tag{9.6}$$

By virtue of (9.6), we get

$$\begin{aligned} \eta(W(\xi,Y)Z) &= g(W(\xi,Y)Z,\xi) \\ &= -g(W(\xi,Y)\xi,Z) \\ &= -\left[a + nb - b\lambda - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right][g(Y,Z) + \eta(Y)\eta(Z)]. \end{aligned} \tag{9.7}$$

Also, from (9.7), we find

$$\eta(W(\xi, Y)\xi) = 0. \tag{9.8}$$

Making use of (9.6), (9.7) and (9.8) in (9.5), we have

$$(1-\mu)\left[a + nb - b\lambda - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right][g(Y,Z) + \eta(Y)\eta(Z)] = 0, \tag{9.9}$$

either

$$1-\mu=0, \ or \ \left[a+nb-b\lambda-\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right]=0, \tag{9.10}$$

either $\mu=1$ or $\left[a+nb-b\lambda-\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right]=0$,

Case I : if $\mu=1$ then from (3.6), we find $\lambda=2-n$.

Case II : if $\left[a+nb-b\lambda-\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right]=0$,

implies that $\lambda=\frac{a}{b}\left(1-\frac{r}{n(n-1)}\right)+\left(n-\frac{2r}{n}\right)$. From the relation (3.6), we obtain $\mu=\frac{a}{b}\left(1-\frac{r}{n(n-1)}\right)+\left(2n-\frac{2r}{n}-1\right)$.

Thus, we can state the following theorem:

Theorem 9.1. *If a conformally Ricci semi-symmetric Lorentzian para-Kenmotsu manifold admits a proper η-Ricci soliton, then $\mu=1$ and $\lambda=2-n$ or $\lambda=\frac{a}{b}\left(1-\frac{r}{n(n-1)}\right)+\left(n-\frac{2r}{n}\right)$ and $\mu=\frac{a}{b}\left(1-\frac{r}{n(n-1)}\right)+\left(2n-\frac{2r}{n}-1\right)$.*

From the above theorem we have the following:

Corollary 9.1. *On conformally Ricci semi-symmetric Lorentzian para-Kenmotsu manifold, there is no Ricci solitons with potential vector field ξ.*

10. Example

Now, we consider the 3-dimensional manifold

$$M=\{(x,y,z)\in R^3 : z\neq 0,\} \tag{10.1}$$

where x,y,z are the standard coordinates in R^3.

The vector fields

$$e_1=z\frac{\partial}{\partial x}, e_2=z\frac{\partial}{\partial y}, \texttt{and} \ \ e_3=z\frac{\partial}{\partial z}=\xi, \tag{10.2}$$

are linearly independent at each point of M and α is constant.

Let g be the Lorentzian metric defined by

$$\begin{aligned} g(e_1,e_3) &= g(e_2,e_3)=g(e_1,e_2)=0, \\ g(e_1,e_1) &= g(e_2,e_2)=1, g(e_3,e_3)=-1. \end{aligned}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,e_3)=g(Z,\xi), \tag{10.3}$$

for any vector field Z on M.

Let φ be the $(1,1)$-tensor field defined by

$$\varphi(e_1) = -e_2,\ \varphi(e_2) = -e_1,\ \varphi(e_3) = 0. \tag{10.4}$$

Then, using the linearity of φ and g, we have

$$\begin{aligned} \eta(e_3) &= -1, \varphi^2 Z = Z + \eta(Z)e_3, \\ g(\varphi Z, \varphi W) &= g(Z,W) + \eta(Z)\eta(W), \end{aligned}$$

for any vector field Z, W on M.

It is easy to see that

$$\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = -1. \tag{10.5}$$

Thus for $e_3 = \xi$, the structure (φ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M. [1]

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0, [e_1, e_3] = -e_1, [e_2, e_3] = -e_2. \tag{10.6}$$

Using Koszul's formula for Levi-Civita connection ∇ with respect to g, i.e.,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) \\ &\quad +g(Z,[X,Y]), \end{aligned}$$

One can easily calculate

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1, \tag{10.7}$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2, \tag{10.8}$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \tag{10.9}$$

From the above calculations, we see that the manifold under consideration satisfies ∇, i.e.,

$$\nabla_Z \xi = -Z - \eta(Z)\xi, \texttt{and } (\nabla_Z \varphi)W = -g(\varphi Z, W)\xi - \eta(W)\varphi Z. \tag{10.10}$$

Also, the Riemannian curvature tensor R is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{10.11}$$

Then

$$R(e_1, e_2)e_2 = e_1,\ R(e_1, e_3)e_3 = -e_1,\ R(e_1, e_2)e_1 = -e_2, \tag{10.12}$$

$$R(e_2, e_3)e_1 = 0,\ R(e_1, e_3)e_2 = 0,\ R(e_1, e_2)e_3 = 0, \tag{10.13}$$

$$R(e_2, e_3)e_3 = -e_2,\ R(e_1, e_3)e_1 = -e_3,\ R(e_2, e_3)e_2 = -e_3. \tag{10.14}$$

Then, the Ricci tensor S is given by

$$S(e_1, e_1) = S(e_2, e_2) = 2, \quad S(e_3, e_3) = -2. \tag{10.15}$$

From (3.5), we obtain $S(e_1, e_1) = S(e_2, e_2) = 1 - \lambda$ and $S(e_3, e_3) = \lambda - \mu$, therefore $\lambda = -1$ and $\mu = 1$. The data (g, ξ, λ, μ) for $\lambda = -1$
and $\mu = 1$ defines an η-Ricci soliton on the Lorentzian para Kenmotsu manifold M.

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ON GENERALIZED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES VIA IDEALS IN INTUITIONISTIC FUZZY NORMED SPACES

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Abstract. In this paper, we have given $\mathcal{I}_2$-lacunary statistical convergence and strongly $\mathcal{I}_2$-lacunary convergence with regards to the intuitionistic fuzzy norm (μ, v), investigate their relationships, and make some observations about these classes. Also, we have examined the relation between these two new methods and the relation between $\mathcal{I}_2$-statistical convergence in the corresponding intuitionistic fuzzy normed space.
Key words: ideal; $I2$-lacunary statistical convergence; intuitionistic fuzzy normed space; banach space; strongly convergence.

1. Introduction

Statistical convergence of a real number sequence was firstly originated by Fast [15]. It became a notable topic in summability theory after the work of Fridy [16] and Šalát [50]. This concept was constracted to the double sequences by Mursaleen and Edely [43]. Some beneficial results on this topic can be found in [6, 22, 24, 36, 37, 38, 39, 40, 54].

Theory of $\mathcal{I}$-convergence of sequences in a metric space was given by Kostyrko et al. [32]. Other investigations and applications of ideals can be found in the study Das and Ghosal [8], Das et al. [9] and Savaş and Das [51]. Belen et al. [5] generalized the notions of statistical convergence, (λ, μ)-statistical convergence, (V, λ, μ) summability and $(C, 1, 1)$ summability for a double sequence via ideals. The other studies of this concept were examined by [21, 33, 41, 42, 44, 45, 47].

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Using lacunary sequence, Fridy and Orhan [17] examined the concept of lacunary statistical convergence. Afterwards, it was developed by Fridy and Orhan [18], Li [35], Mursaleen and Mohiuddine [46], Bakery [3]. Çakan and Altay [7] provided multidimensional analogues of the results presented by Fridy and Orhan [17]. Lacunary ideal convergence of real sequences was inquiried by Tripathy et al. [55].

Fuzzy set theory has become an important working area after the study of Zadeh [56]. Atanassov [1] investigated intuitionistic fuzzy set; this concept was utilized by Atanassov et al. [2] in the study of decision-making problems. The idea of an intuitionistic fuzzy metric space was put forward by Park [48]. In [19], it was shown that the topology generated by every IF-metric coincides with the topology generated by its F-metric. Hence, the definition of an IF-metric space needed some refinement, in the light of having independent results. In [34], motivated by Park's definition of an IF-metric, the authors defined an IF-normed spaces (IFNS for shortly) and then investigated, among other results, the fundamental theorems: open mapping, closed graph and uniform boundedness in IFNS. Several studies of the convergence of sequences in some normed linear spaces with a fuzzy setting might be revealed by the research of [10, 11, 12, 13, 14, 23, 25, 26, 27, 28, 29, 30, 31, 52, 53].

Let us start with fundamental definitions from the literature.

The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{m \le n : m \in K\}|,$$

where $|m \le n : m \in K|$ denotes the number of elements of K not exceeding m.

A number sequence $x = (x_m)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{m \le n : |x_m - L| \ge \varepsilon\}| = 0,$$

i.e.,

$$|x_m - L| < \varepsilon \quad (a.a.\ m) \tag{1.1}$$

In this case we write $st - \lim x_m = L$. For example, define $x_m = 1$ if m is a square and $x_m = 0$ otherwise. Then, $|\{m \le n : x_m \ne 0\}| \le \sqrt{n}$, so $st - \lim x_m = 0$. Note that we could have assigned any values whatsoever to x_m when m is a square, and we would still have $st - \lim x_m = 0$. But x is neither convergent nor bounded. It is clear that if the inequality in (1.1) holds for all but finitely many m, then $\lim x_m = L$. Statistical convergence is a natural generalization of ordinary convergence. It follows that $\lim x_m = L$ implies $st - \lim x_m = L$, so statistical convergence may be considered as a regular summability method. The sequence that converges statistically need not be convergent and also need not be bounded.

A double sequence $x = (x_{mn})$ has Pringsheim limit L (denoted by $P - \lim = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ whenever

$m, n > N$. We shall describe such an (x_{mn}) more briefly as "P-convergent." In Pringsheim convergence the row-index m and the column-index n tend to infinity independently from each other [49].

The essential deficiency of this kind of convergence is that a convergent sequence does not require to be bounded. Hardy [20] defined the concept of regular sense, does not have this hortcoming, for double sequence. In regular convergence, both the row-index and the column-index of the double sequence need to be convergent besides the convergent in Pringsheim's sense.

The notion of Cesàro summable double sequences was given by [40]. Note that if a bounded sequence (x_{mn}) is statistically convergent then it is also Cesàro summable but not conversely.

Let $(x_{mn}) = (-1)^m$, $\forall n$; then $\lim_{p,r} \sum\limits_{m=1}^{p} \sum\limits_{n=1}^{r} x_{mn} = 0$, but obviously x is not statistically convergent.

The convergence of double sequences play an important role not only in pure mathematics but also in other branches of science involving computer science, biological science and dynamical systems. Also, the double sequence can be use in convergence of double trigonometric series and in the opening series of double functions and in the making differential solution.

In the wake of the study of ideal convergence defined by Kostyrko et al. [32], there has been comprehensive research to discover applications and summability studies of the classical theories.

Let $\emptyset \neq S$ be a set, and then a non empty class $\mathcal{I} \subseteq P(S)$ is said to be an *ideal* on S iff (i) $\emptyset \in \mathcal{I}$, (ii) $\mathcal{I}$ is additive under union, (iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we find $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \emptyset$ and $S \notin \mathcal{I}$. A non-empty family of sets $\mathcal{F}$ is called *filter* on S iff (i) $\emptyset \notin \mathcal{F}$, (ii) for each $A, B \in \mathcal{F}$ we get $A \cap B \in \mathcal{F}$, (iii) for every $A \in \mathcal{F}$ and each $B \supseteq A$, we obtain $B \in \mathcal{F}$. Relationship between ideal and filter is given as follows:

$$\mathcal{F}(\mathcal{I}) = \{K \subset S : K^c \in \mathcal{I}\},$$

where $K^c = S - K$.

A non-trivial ideal $\mathcal{I}$ is (i) an *admissible ideal* on S iff it contains all singletons.

A sequence (x_m) is said to be ideal convergent to L if for every $\varepsilon > 0$, i.e.

$$A(\varepsilon) = \{m \in \mathbb{N} : |x_m - L| \geq \varepsilon\} \in \mathcal{I}.$$

Taking $\mathcal{I} = \mathcal{I}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ indicates the asymptotic density of set A. If $\mathcal{I}_\delta$ is a non-trivial admissible ideal then ideal convergence coincides with statistical convergence.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$, and l_∞^2 as the space of all bounded double sequences.

A double sequence $\overline{\theta} = \theta_{us} = \{(k_u, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_u) and (l_s) such that

$$k_0 = 0,\ h_u = k_u - k_{u-1} \to \infty \ \ and \ \ l_0 = 0,\ \overline{h}_s = l_s - l_{s-1} \to \infty, \quad u, s \to \infty.$$

We will use the following notation $k_{us} := k_u l_s$, $h_{us} := h_u \overline{h}_s$ and θ_{us} is determined by

$$J_{us} := \{(k,l) : k_{u-1} < k \leq k_u \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_u := \frac{k_u}{k_{u-1}},\ \overline{q}_s := \frac{l_s}{l_{s-1}} \text{ and } q_{us} := q_u \overline{q}_s.$$

Throughout the paper, by $\theta_2 = \theta_{us} = \{(k_u, l_s)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

A double sequence $x = \{x_{mn}\}$ of numbers is said to be $\mathcal{I}_2$-lacunary statistical convergent or $S_{\theta_2}(\mathcal{I}_2)$-convergent to L, if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_u \overline{h}_s} \left|\{(m,n) \in J_{us} : |x_{mn} - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{mn} \to L\,(S_{\theta_2}(\mathcal{I}_2))$ or $S_{\theta_2}(\mathcal{I}_2)\text{-}\lim\limits_{m,n\to\infty} x_{mn} = L$.

The concept of IFNS was given by Lael and Nourouzi [34]. In order to have a different topology from the topology generated by the F-norm μ, the condition $\mu + v \leq 1$ was omitted from Park's definition.

The triplicate (X, μ, v) is said to be an IF-normed space if X is a real vector space, and μ, v are F-sets on $X \times F$ satisfying the following conditions for every $x, y \in X$ and $t, s \in \mathbb{R}$:

(a) $\mu(x,t) = 0$ for all non-positive real number t,

(b) $\mu(x,t) = 1$ for all $t \in \mathbb{R}^+$ iff $x = 0$,

(c) $\mu(cx,t) = \mu\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,

(d) $\mu(x+y, s+t) \geq \min\{\mu(x,t), \mu(y,s)\}$,

(e) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,

(f) $v(x,t) = 1$ for all non-positive real number t,

(g) $v(x,t) = 0$ for all $t \in \mathbb{R}^+$ iff $x = 0$,

(h) $v(cx,t) = v\left(x, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,

$(\imath)$ $\max\{v(x,t), v(y,s)\} \geq v(x+y, t+s)$,

(i) $\lim_{t\to\infty} v(x,t) = 0$ and $\lim_{t\to 0} v(x,t) = 1$.

In this case, we will call (μ, v) an IF-norm on X. In addition, (X, μ) is called an F-normed space.

In this study, we deal with the relation between these two new methods and with relations between $\mathcal{I}_2$-lacunary statistical convergence and strongly $\mathcal{I}_2$-lacunary convergence introduced by the author in an IFNS. Also, we examine the relation between the $\mathcal{I}_2$-lacunary statistical convergence and $\mathcal{I}_2$-statistical convergence in an IFNS.

2. Main Results

Definition 2.1. Let $(X, \mu, v, *, \Theta)$ be an IFNS, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. A sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-statistically convergent to $\xi \in X$ with regards to the IFN (μ, v), and is demonstrated by $S(\mathcal{I}_2)^{(\mu,v)} - \lim x = \xi$ or $x_{k,j} \overset{(\mu,v)}{\rightarrow} \xi\,(S(\mathcal{I}_2))$, if for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\{k \le m, j \le n : \mu(x_{kj} - \xi, t) \le 1 - \varepsilon \text{ or } \nu(x_{kj} - \xi, t) \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

Definition 2.2. A sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-lacunary statistically convergent to $\xi \in X$ with regards to the IFN (μ, v), and is demonstrated by $S_\theta(\mathcal{I}_2)^{(\mu,v)} - \lim x = \xi$ or $x_{k,j} \overset{(\mu,v)}{\rightarrow} \xi\,(S_\theta(\mathcal{I}_2))$, if for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\left|\{(k,j) \in J_{ru} : \mu(x_{kj} - \xi, t) \le 1 - \varepsilon \text{ or } \nu(x_{kj} - \xi, t) \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}_2.$$

Definition 2.3. A sequence $x = (x_{kj})$ is said to be strongly $\mathcal{I}_2$-lacunary convergent to ξ or $N_\theta(\mathcal{I}_2)$-convergent to $\xi \in X$ with regards to the IFN (μ, v) and is denoted by $x_{kj} \overset{(\mu,v)}{\rightarrow} \xi\,(N_\theta(\mathcal{I}_2))$, if for every $\delta > 0$ and $t > 0$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}} \mu(x_{kj} - \xi, t) \le 1 - \delta \text{ or } \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in J_{ru}} \nu(x_{kj} - \xi, t) \ge \delta\right\} \in \mathcal{I}_2.$$

Theorem 2.1. *Let $(X, \mu, v, *, \Theta)$ be an IFNS, θ be a double lacunary sequence, $\mathcal{I}_2$ be a strongly admissible ideal in $\mathbb{N}$, and $x = (x_{jk}) \in X$, then*

(i) (a) If $x_{kj} \overset{(\mu,v)}{\rightarrow} \xi\,(N_\theta(\mathcal{I}_2))$ then $x_{kj} \overset{(\mu,v)}{\rightarrow} \xi\,(S_\theta(\mathcal{I}_2))$.

(b) If $x \in l^2_\infty(X)$, the space of all bounded sequences of X and $x_{kj} \overset{(\mu,v)}{\rightarrow} \xi\,(S_\theta(\mathcal{I}_2))$ then $x_{kj} \overset{(\mu,v)}{\rightarrow} \xi\,(N_\theta(\mathcal{I}_2))$.

(ii) $S_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l^2_\infty(X) = N_\theta(\mathcal{I}_2)^{(\mu,v)} \cap l^2_\infty(X)$.

Proof. $(i)-(a)$. By hypothesis, for every $\varepsilon>0$, $\delta>0$ and $t>0$, let $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(N_{\theta}\left(\mathcal{I}_{2}\right)\right)$. Then we can write

$$\sum_{(k,j)\in J_{ru}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)$$

$$\geq\sum_{\substack{(k,j)\in J_{ru}\\ \mu(x_{kj}-\xi,t)\leq 1-\varepsilon\text{ or }\nu(x_{kj}-\xi,t)\geq\varepsilon}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)$$

$$\geq\varepsilon.\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|.$$

Then observe that

$$\frac{1}{h_r\overline{h_u}}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|\geq\delta$$

and

$$\frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\mu\left(x_{kj}-\xi,t\right)\leq(1-\varepsilon)\,\delta\text{ or }\frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\delta,$$

which implies

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h_u}}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|\geq\delta\right\}$$

$$\subset\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h_u}}\{\sum_{(k,j)\in J_{ru}}\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon$$
$$\text{or}\sum_{(k,j)\in J_{ru}}\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\}\geq\varepsilon\delta\}.$$

Since $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(N_{\theta}\left(\mathcal{I}_{2}\right)\right)$, we immediately see that $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(S_{\theta}\left(\mathcal{I}_{2}\right)\right)$.

$(i)-(b)$. We assume that $x_{kj}\overset{(\mu,v)}{\rightarrow}\xi\left(S_{\theta}\left(\mathcal{I}_{2}\right)\right)$ and $x\in l_{\infty}^{2}(X)$. The inequalities $\mu\left(x_{kj}-\xi,t\right)\geq 1-M$ or $\nu\left(x_{kj}-\xi,t\right)\leq M$ hold for all k,j. Let $\varepsilon>0$ be given. Then we have

$$\tfrac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)$$

$$=\tfrac{1}{h_r\overline{h_u}}\sum_{\substack{(k,j)\in J_{ru}\\ \mu(x_{kj}-\xi,t)\leq 1-\varepsilon\text{ or }\nu(x_{kj}-\xi,t)\geq\varepsilon}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)$$

$$+\tfrac{1}{h_r\overline{h_u}}\sum_{\substack{(k,j)\in J_{ru}\\ \mu(x_{kj}-\xi,t)>1-\varepsilon\text{ or }\nu(x_{kj}-\xi,t)<\varepsilon}}\left(\mu\left(x_{kj}-\xi,t\right)\text{ or }\nu\left(x_{kj}-\xi,t\right)\right)$$

$$\leq\tfrac{M}{h_r\overline{h_u}}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-\xi,t\right)\leq 1-\varepsilon\text{ or }\nu\left(x_{kj}-\xi,t\right)\geq\varepsilon\right\}\right|+\varepsilon.$$

Note that

$$A_{\mu,v}(\varepsilon,t)=\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h_u}}|\{(k,j)\in J_{ru}:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|\ge\frac{\varepsilon}{M}\}$$

belongs to $\mathcal{I}_2$. If $r\in(A_{\mu,v}(\varepsilon,t))^c$ then we have

$$\frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\mu(x_{kj}-\xi,t)>1-2\varepsilon \text{ or } \frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\nu(x_{kj}-\xi,t)<2\varepsilon.$$

Now

$$T_{\mu,v}(\varepsilon,t)=\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\mu(x_{kj}-\xi,t)\le 1-2\varepsilon \text{ or } \frac{1}{h_r\overline{h_u}}\sum_{(k,j)\in J_{ru}}\nu(x_{kj}-\xi,t)\ge 2\varepsilon\}.$$

Hence, $T_{\mu,v}(\varepsilon,t)\subseteq A_{\mu,v}(\varepsilon,t)$ and so, by the definition of an ideal, $T_{\mu,v}(\varepsilon,t)\in\mathcal{I}_2$. Therefore, we conclude that $x_{kj}\overset{(\mu,v)}{\to}\xi(N_\theta(\mathcal{I}_2))$.

(ii) This readily follows from $(i)-(a)$ and $(i)-(b)$. $\square$

Theorem 2.2. *Let $(X,\mu,v,*,\Theta)$ be an IFNS. If θ be a double lacunary sequence with $\liminf_r q_r>1$, $\liminf_u q_u>1$ then*

$$x_{kj}\overset{(\mu,v)}{\to}\xi(S(\mathcal{I}_2))\Rightarrow x_{kj}\overset{(\mu,v)}{\to}\xi(S_\theta(\mathcal{I}_2)).$$

Proof. Suppose first that $\liminf_r q_r>1$, $\liminf_u q_u>1$ then there exists a $\alpha,\beta>0$ such that $q_r\ge 1+\alpha$, $q_u>1+\beta$ for sufficiently large r,u, which implies that

$$\frac{h_r\overline{h_u}}{k_{ru}}\ge\frac{\alpha\beta}{(1+\alpha)(1+\beta)}.$$

If $x_{kj}\overset{(\mu,v)}{\to}\xi(S(\mathcal{I}_2))$, then for every $\varepsilon>0$ and for sufficiently large r,u, we have

$$\frac{1}{k_{ru}}|\{k\le k_r,j\le j_u:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|$$

$$\ge\frac{1}{k_{ru}}|\{(k,j)\in J_{ru}:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|$$

$$\ge\frac{\alpha\beta}{(1+\alpha)(1+\beta)}\left(\frac{1}{h_r\overline{h_u}}|\{(k,j)\in J_{ru}:\mu(x_{kj}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t)\ge\varepsilon\}|\right)$$

Then for any $\delta > 0$, we get

$$\Big\{(r,u) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{h_r \overline{h}_u} \left|\{(k,j) \in J_{ru} : \mu\left(x_{kj} - \xi, t\right) \leq 1-\varepsilon \text{ or } \nu\left(x_{kj} - \xi, t\right) \geq \varepsilon\}\right| \geq \delta\Big\}$$

$$\subseteq \{(r,u) \in \mathbb{N}\times\mathbb{N} : \tfrac{1}{k_{ru}} |\{k \leq k_r,\ j \leq j_u : \mu\left(x_{kj} - \xi, t\right) \leq 1-\varepsilon \\ \text{or } \nu\left(x_{kj} - \xi, t\right) \geq \varepsilon\}| \geq \tfrac{\delta\alpha\beta}{(1+\alpha)(1+\beta)}\}.$$

If $x_{kj} \overset{(\mu,v)}{\to} \xi\left(S\left(\mathcal{I}_2\right)\right)$ then the set on the right-hand side belongs to $\mathcal{I}_2$ and so the set on the left-hand side belongs to $\mathcal{I}_2$. This shows that $x_{kj} \overset{(\mu,v)}{\to} \xi\left(S_\theta\left(\mathcal{I}_2\right)\right)$. □

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F\left(\mathcal{I}_2\right)$, $\bigcup\{n : k_{r-1} < n \leq k_r, r \in C\} \in F\left(\mathcal{I}_2\right)$.

Theorem 2.3. *Let $(X,\mu,v,*,\Theta)$ be an IFNS. If θ be a double lacunary sequence with $\limsup_r q_r < \infty$, $\limsup_u q_u < \infty$ then*

$$x_{kj} \overset{(\mu,v)}{\to} \xi\left(S_\theta\left(\mathcal{I}_2\right)\right) \Longrightarrow x_{jk} \overset{(\mu,v)}{\to} \xi\left(S\left(\mathcal{I}_2\right)\right).$$

Proof. If $\limsup_r q_r < \infty$, $\limsup_u q_u < \infty$ then without any loss of generality we can assume that there exists a M, $N > 0$ such that $q_r < M$ and $q_u < N$ for all r, u. Suppose that $x_{kj} \overset{(\mu,v)}{\to} \xi\left(S_\theta\left(\mathcal{I}_2\right)\right)$, and let

$$C_{ru} := \left|\{(k,j) \in J_{ru} : \mu\left(x_{kj} - \xi, t\right) \leq 1-\varepsilon \text{ or } \nu\left(x_{kj} - \xi, t\right) \geq \varepsilon\}\right|.$$

Since $x_{kj} \overset{(\mu,v)}{\to} \xi\left(S_\theta\left(\mathcal{I}_2\right)\right)$, it follows that for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \overline{h}_u} |\{(k,j) \in J_{ru} : \mu\left(x_{kj} - \xi, t\right) \leq 1-\varepsilon \\ \text{or } \nu\left(x_{kj} - \xi, t\right) \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}_2.$$

Hence, we can choose a positive integers $u_0, s_0 \in \mathbb{N}$ such that

$$\frac{C_{ru}}{h_r \overline{h}_u} < \delta, \text{ for all } r > r_0, u > u_0.$$

Now let

$$K := \max\{C_{ru} : 1 \leq r \leq r_0, 1 \leq u \leq u_0\}$$

and let t and v be any integers satisfying $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$. Then, we have

$$\frac{1}{tv}\left|\{k \leq t,\, j \leq v : \mu(x_{kj}-\xi,t) \leq 1-\varepsilon \text{ or } \quad \nu(x_{kj}-\xi,t) \geq \varepsilon\ \}\right|$$

$$\leq \frac{1}{k_{r-1}j_{u-1}}\left|\{k \leq k_r,\, j \leq j_u : \mu(x_{kj}-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t) \geq \varepsilon\}\right|$$

$$\leq \frac{1}{k_{r-1}j_{u-1}}\left(C_{11}+C_{12}+C_{21}+C_{22}+...+C_{r_0u_0}+...C_{ru}\right)$$

$$\leq \tfrac{K}{k_{r-1}j_{u-1}}.r_0u_0+\tfrac{1}{k_{r-1}j_{u-1}}\left(h_{r_0}\overline{h}_{u_{r0}+1}\tfrac{C_{r_0,u_0+1}}{h_{r_0}\overline{h}_{u_0+1}}+h_{r_{0+1}}\overline{h}_{u_0}\tfrac{C_{r_{0+1},u_0}}{h_{r_{0+1}}\overline{h}_{u_0}}+\dots\right.$$
$$\left.+h_r\overline{h}_u\tfrac{C_{ru}}{h_r\overline{h}_u}\right)$$

$$\leq \tfrac{r_0u_0.K}{k_{r-1}j_{u-1}}+\tfrac{1}{k_{r-1}j_{u-1}}\left(\sup_{r>r_0,u>u_0}\tfrac{C_{ru}}{h_r\overline{h}_u}\right)\left(h_{r_0}\overline{h}_{u_0+1}+h_{r_{0+1}}\overline{h}_{u_0}+...+h_r\overline{h}_u\right)$$

$$\leq \tfrac{r_0u_0.K}{k_{r-1}j_{u-1}}+\varepsilon.\tfrac{\left(k_r-k_{r_0}\right)\left(j_u-j_{u-1}\right)}{k_{u-1}j_{s-1}}$$

$$\leq \tfrac{u_0s_0.K}{k_{r-1}j_{u-1}}+\varepsilon.q_u.q_s \leq \tfrac{r_0u_0.K}{k_{r-1}j_{u-1}}+\varepsilon.M.N$$

Since $k_{r-1}j_u \to \infty$ as $t, v \to \infty$, it follows that

$$\frac{1}{tv}\left|\{k \leq t,\, j \leq v : \mu(x_{kj}-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t) \geq \varepsilon\ \}\right| \to 0$$

and consequently for any $\delta_1 > 0$, the set

$$\left\{(t,v) \in \mathbb{N}\times\mathbb{N} : \frac{1}{tv}\left|\{k \leq t, j \leq v : \mu(x_{kj}-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_{kj}-\xi,t) \geq \varepsilon\}\right|\right\} \in \mathcal{I}_2.$$

This shows that $x_{jk} \overset{(\mu,v)}{\to} \xi\left(S\left(\mathcal{I}_2\right)\right)$. □

Combining Theorem 2.2 and Theorem 2.3 we have

Theorem 2.4. *Let θ be a strongly lacunary sequence. IFNS. If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty,$ and $1 < \liminf_u q_u \leq \limsup_u q_u < \infty$ then*

$$x_{kj} \overset{(\mu,v)}{\to} \xi\left(S_\theta\left(\mathcal{I}_2\right)\right) \Leftrightarrow x_{kj} \overset{(\mu,v)}{\to} \xi\left(S\left(\mathcal{I}_2\right)\right).$$

Proof. This readily follows from Theorem 2.2 and Theorem 2.3. □

Theorem 2.5. *Let $(X,\mu,v,*,\Theta)$ be an IFNS such that $\frac{1}{4}\varepsilon_{mn}\Theta\frac{1}{4}\varepsilon_{mn} < \frac{1}{2}\varepsilon_{mn}$ and $\left(1-\frac{1}{4}\varepsilon_{mn}\right) * \left(1-\frac{1}{4}\varepsilon_{mn}\right) > 1-\frac{1}{2}\varepsilon_{mn}$. If X is a Banach space then $S_\theta\left(\mathcal{I}_2\right)^{(\mu,v)} \cap l^2_\infty(X)$ is a closed subset of $l^2_\infty(X)$.*

Proof. We first assume that $(x^{mn}) = \left(x_{kj}^{mn}\right)$ be a convergent sequence in $S_\theta\left(\mathcal{I}_2\right)^{(\mu,v)} \cap l_\infty^2(X)$. Suppose $x^{(mn)}$ convergent to x. It is clear $x \in l_\infty^2(X)$. We need to show that $x \in S_\theta\left(\mathcal{I}_2\right)^{(\mu,v)} \cap l_\infty^2(X)$. Since $x^{mn} \in S_\theta\left(\mathcal{I}_2\right)^{(\mu,v)} \cap l_\infty^2(X)$ there exists real numbers L_{mn} such that

$$x_{kj}^{mn} \overset{(\mu,v)}{\to} L_{mn}\left(S_\theta\left(\mathcal{I}_2\right)\right) \text{ for } m,n = 1,2,3,...$$

Take a double sequence $\{\varepsilon_{mn}\}$ of strictly decreasing positive numbers converging to zero. Then for every $m,n = 1,2,3,...$ there is positive N_{mn} such that if $m,n \geq N_{mn}$ then $\sup_{m,n} \nu\left(x - x^{mn}, t\right) \leq \frac{\varepsilon_{mn}}{4}$. Without loss of generality assume that $N_{mn} = mn$ and choose a $\delta > 0$ such that $\delta < \frac{1}{5}$. Now set

$$A_{\mu,v}\left(\varepsilon_{mn}, t\right) = \left\{ \begin{array}{c} (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h_u}} \left|\{(k,j) \in J_{ru} : \right. \\ \mu\left(x_{kj}^{mn} - L_{mn}, t\right) \leq 1 - \frac{\varepsilon_{mn}}{4} \text{ or} \\ \left. \nu\left(x_{kj}^{mn} - L_{mn}, t\right) \geq \frac{\varepsilon_{mn}}{4} \}\right| < \delta \end{array} \right\}$$

belongs to $F\left(\mathcal{I}_2\right)$ and

$$B_{\mu,v}\left(\varepsilon_{m+1,n+1}, t\right) = \left\{ \begin{array}{c} (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h_u}} \left|\{(k,j) \in J_{ru} : \right. \\ \mu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \leq 1 - \frac{\varepsilon_{m+1,n+1}}{4} \text{ or} \\ \left. \nu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \geq \frac{\varepsilon_{m+1,n+1}}{4} \}\right| < \delta \end{array} \right\}$$

belongs to $F\left(\mathcal{I}_2\right)$. Since $A_{\mu,v}\left(\varepsilon_{mn}, t\right) \cap B_{\mu,v}\left(\varepsilon_{m+1,n+1}, t\right) \in F\left(\mathcal{I}_2\right)$ and $\varnothing \notin F\left(\mathcal{I}_2\right)$, we can choose $(r,u) \in A_{\mu,v}\left(\varepsilon_{mn}, t\right) \cap B_{\mu,v}\left(\varepsilon_{m+1,n+1}, t\right)$. Then

$$\begin{aligned} &\frac{1}{h_r \overline{h_u}} \left|\{(k,j) \in J_{ru} : \mu\left(x_{kj}^{mn} - L_{mn}, t\right) \leq 1 - \tfrac{\varepsilon_{mn}}{4} \text{ or } \nu\left(x_{kj}^{mn} - L_{mn}, t\right) \geq \tfrac{\varepsilon_{mn}}{4} \right. \\ &\vee \mu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \leq 1 - \tfrac{\varepsilon_{m+1,n+1}}{4} \text{ or } \nu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) \\ &\left. \geq \tfrac{\varepsilon_{m+1,n+1}}{4} \}\right| \leq 2\delta < 1. \end{aligned}$$

Since $h_r \overline{h_u} \to \infty$ and $A_{\mu,v}\left(\varepsilon_{mn}, t\right) \cap B_{\mu,v}\left(\varepsilon_{m+1,n+1}, t\right) \in F\left(\mathcal{I}_2\right)$ is finite, we can choose the above r,u so that $h_r \overline{h_u} > 5$. Hence there must exist a $(k,j) \in J_{ru}$ for which we have simultaneously, $\mu\left(x_{kj}^{mn} - L_{mn}, t\right) > 1 - \frac{\varepsilon_{mn}}{4}$ or $\nu\left(x_{kj}^{mn} - L_{mn}, t\right) < \frac{\varepsilon_{mn}}{4}$ and $\mu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) > 1 - \frac{\varepsilon_{mn}}{4}$ or $\nu\left(x_{kj}^{m+1,n+1} - L_{m+1,n+1}, t\right) < \frac{\varepsilon_{mn}}{4}$. For a given $\varepsilon_{mn} > 0$ choose $\frac{\varepsilon_{mn}}{2}$ such that $\left(1 - \frac{1}{2}\varepsilon_{mn}\right) * \left(1 - \frac{1}{2}\varepsilon_{mn}\right) > 1 - \varepsilon_{mn}$ and $\frac{1}{2}\varepsilon_{mn} \Theta \frac{1}{2}\varepsilon_{mn} < \varepsilon_{mn}$. Then it follows that

$$\nu\left(L_{mn} - x_{kj}^{mn}, \frac{t}{2}\right) \Theta \nu\left(L_{m+1,n+1} - x_{kj}^{m+1,n+1}, \frac{t}{2}\right) \leq \frac{\varepsilon_{mn}}{4} \Theta \frac{\varepsilon_{mn}}{4} < \frac{\varepsilon_{mn}}{2}$$

and

$$\begin{aligned} \nu\left(x_{kj}^{mn} - x_{kj}^{m+1,n+1}, t\right) &\leq \sup_{m,n} \nu\left(x - x^{mn}, \tfrac{t}{2}\right) \Theta \sup_{m,n} \nu\left(x - x^{m+1,n+1}, \tfrac{t}{2}\right) \\ &\leq \tfrac{\varepsilon_{mn}}{4} \Theta \tfrac{\varepsilon_{mn}}{4} < \tfrac{\varepsilon_{mn}}{2}. \end{aligned}$$

Hence, we have

$$\begin{aligned}\nu\left(L_{mn}-L_{m+1,n+1},t\right) &\leq \Big[\nu\left(L_{mn}-x_{kj}^{mn},\tfrac{t}{3}\right)\Theta\nu\left(x_{kj}^{m+1,n+1}-L_{m+1,n+1},\tfrac{t}{3}\right)\\ &\quad\Theta\nu\left(x_{kj}^{mn}-x_{kj}^{m+1,n+1},\tfrac{t}{3}\right)\Big]\\ &\leq \tfrac{\varepsilon_{mn}}{2}\Theta\tfrac{\varepsilon_{mn}}{2}<\varepsilon_{mn}\end{aligned}$$

and similarly $\mu\left(L_{mn}-L_{m+1,n+1},t\right)>1-\varepsilon_{mn}$. This implies that $\{L_{mn}\}_{m,n\in\mathbb{N}}$ is a Cauchy sequence in X and let $L_{mn}\to L\in X$ as $m,n\to\infty$. We shall prove that $x\overset{(\mu,v)}{\to}L_{mn}\left(S_\theta\left(\mathcal{I}_2\right)\right)$. For anay $\varepsilon>0$ and $t>0$, choose $(m,n)\in\mathbb{N}\times\mathbb{N}$ such that $\varepsilon_{mn}<\frac{1}{4}\varepsilon$, $\sup_{m,n}\nu\left(x-x^{mn},t\right)<\frac{1}{4}\varepsilon$, $\nu\left(L_{mn}-L,t\right)>1-\frac{1}{4}\varepsilon$ or $\nu\left(L_{mn}-L,t\right)<\frac{1}{4}\varepsilon$. Now since

$$\begin{aligned}&\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\nu\left(x_{kj}-L,t\right)\geq\varepsilon\right\}\right|\\ &\quad\leq\tfrac{1}{h_r\overline{h}_u}\Big|\Big\{(k,j)\in J_{ru}:\nu\left(x_{kj}-x_{kj}^{mn},\tfrac{t}{3}\right)\Theta\\ &\quad\Big[\nu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\Theta\nu\left(L_{mn}-L,\tfrac{t}{3}\right)\Big]\geq\varepsilon\Big\}\Big|\\ &\quad\leq\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\nu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\geq\tfrac{\varepsilon}{2}\right\}\right|\end{aligned}$$

and similarly

$$\begin{aligned}&\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}-L,t\right)\leq1-\varepsilon\right\}\right|\\ &\quad>\tfrac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in J_{ru}:\mu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\leq1-\tfrac{\varepsilon}{2}\right\}\right|.\end{aligned}$$

It follows that

$$\begin{aligned}&\Big\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h}_u}\left|\{(k,j)\in J_{ru}:\mu\left(x_{kj}-L,t\right)\leq1-\varepsilon\right.\\ &\qquad\left.\text{or }\nu\left(x_{kj}-L,t\right)\geq\varepsilon\}\right|\geq\delta\}\\ &\subset\Big\{(r,u)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_r\overline{h}_u}\Big|\Big\{(k,j)\in J_{ru}:\mu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\leq\\ &\qquad1-\tfrac{\varepsilon}{2}\text{ or }\nu\left(x_{kj}^{mn}-L_{mn},\tfrac{t}{3}\right)\geq\tfrac{\varepsilon}{2}\Big\}\Big|\geq\delta\Big\}\end{aligned}$$

for any given $\delta>0$. Hence we have $x\overset{(\mu,v)}{\to}L_{mn}\left(S_\theta\left(\mathcal{I}_2\right)\right)$. $\square$

3. Conclusion

In this paper we introduce the notions of $\mathcal{I}_2$-lacunary statistical convergence and strongly $\mathcal{I}_2$-lacunary convergence with respect to the IFN (μ,v), investigate their relationship, and make some observations about these classes. Our study of $\mathcal{I}_2$-statistical convergence and $\mathcal{I}_2$-lacunary statistical convergence of sequences in IFN spaces also provides a tool to deal with convergence problems of sequences of fuzzy real numbers. These results can be used to study the convergence problems of sequences of fuzzy numbers having a chaotic pattern in IFN spaces.

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A NOTE ON SOME SYSTEMS OF GENERALIZED SYLVESTER EQUATIONS *

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Abstract. In this paper, we study two systems of generalized Sylvester operator equations. We derive necessary and sufficient conditions for the existence of a solution and provide the general form of a solution. We extend some recent resuts to more general settings.
Key words: Sylvester equations, generalized inverses, Matrix equations and identities

1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of operator A, respectively. The identity operator is always denoted by I. If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ has a closed range, then there exists unique operator $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying the following equations

$$(1)\ AXA = A \quad (2)\ XAX = X \quad (3)\ (AX)^* = AX \quad (4)\ (XA)^* = XA.$$

Such operator is called the Moore-Penrose inverse of an operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ which is denoted by $A^\dagger$. If $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfies the equation (1), i.e. $AXA = A$, then X is an inner generalized inverse of A, and is usually denoted by A^-. For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ there exists a Moore-Penrose inverse, if and only if there exists its

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inner generalized inverse if and only if $\mathcal{R}(A)$ is closed. In this case, we say that A is regular. Furthermore, L_A and R_A stand for two projections $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$. induced by A, respectively.

In this paper, we study two systems of generalized Sylvester operator equations

$$A_1X_1 - X_2B_1 = C_1, \quad A_2X_3 - X_2B_2 = C_2, \tag{1.1}$$

where $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F},\mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{M},\mathcal{K})$, $B_2 \in \mathcal{B}(\mathcal{L},\mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L},\mathcal{K})$, and

$$A_1X_1 - X_2B_1 = C_1, \quad A_2X_2 - X_3B_2 = C_2, \tag{1.2}$$

where $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F},\mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{K},\mathcal{M})$, $B_2 \in \mathcal{B}(\mathcal{G},\mathcal{N})$, $C_2 \in \mathcal{B}(\mathcal{G},\mathcal{M})$.

Systems of such type of matrix equations have been considered by many authors [3, 4, 5, 6, 7]. In this pape,r we extended recent results [7] on systems of quaternion matrix equations to infinite dimensional settings and provide much simpler proofs to existing conditions.

2. Main results

The following two lemmas play a key role in this paper:

Lemma 2.1. [1] *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{F},\mathcal{G})$ and $C \in \mathcal{B}(\mathcal{F},\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Then the operator equation*

$$AXB = C$$

is consistent if and only if

$$AA^{-}CB^{-}B = C,$$

for some A^{-} and B^{-}, in which case the general solution is given by

$$X = A^{-}CB^{-} + Y - A^{-}AYBB^{-},$$

for arbitrary $Y \in \mathcal{B}(\mathcal{G},\mathcal{H})$.

Lemma 2.2. [2] *Let E, F, G, D, N, M be Banach spaces. Let $A_1 \in \mathcal{B}(F,E)$, $A_2 \in \mathcal{B}(F,N)$, $B_1 \in \mathcal{B}(D,G)$, $B_2 \in \mathcal{B}(M,G)$ and*

$$T := (I_G - B_1B_1^{-})B_2 \quad \textit{and} \quad S := A_2(I_F - A_1^{-}A_1)$$

be all regular. Moreover, let $A_1A_1^{-}C_1B_1^{-}B_1 = C_1$ and $A_2A_2^{-}C_2B_2^{-}B_2 = C_2$. Then the equations

$$A_1XB_1 = C_1 \quad \textit{and} \quad A_2XB_2 = C_2$$

have a common solution if and only if

$$(I_N - SS^{-})C_2(I_M - T^{-}T) = (I_N - SS^{-})A_2A_1^{-}C_1B_1^{-}B_2(I_M - T^{-}T).$$

In this case, the general common solution is given by

$$\begin{aligned} X &= (A_1^- C_1 - (I_F - A_1^- A_1)S^-(A_2 A_1^- C_1 - W))B_1^-(I_G - B_2 T^-(I_G - B_1 B_1^-)) \\ &\quad +((I_F - (I_F - A_1^- A_1)S^- A_2)A_1^- V + (I_F - A_1^- A_1)S^- C_2)T^-(I_G - B_1 B_1^-) \\ &\quad +Z - (A_1^- A_1 + (I_F - A_1^- A_1)S^- S)Z(B_1 B_1^- + TT^-(I_G - B_1 B_1^-)), \end{aligned}$$

where

$$\begin{aligned} V &= C_1 B_1^- B_2(I_M - T^- T) + A_1 A_2^-(I_N - SS^-)C_2 T^- T + A_1 A_1^- Q T^- T \\ &\quad -A_1 A_2^-(I_N - SS^-)A_2 A_1^- Q T^- T, \end{aligned}$$

$$\begin{aligned} W &= (I_N - SS^-)A_2 A_1^- C_1 + SS^- C_2(I_M - T^- T)B_2^- B_1 + SS^- P B_1^- B_1 \\ &\quad -SS^- P B_1^- B_2(I_M - T^- T)B_2^- B_1, \end{aligned}$$

in which P,Q, Z *are arbitrary elements of* $\mathcal{B}(D,N)$, $\mathcal{B}(M,E)$ *and* $\mathcal{B}(G,F)$, *respectively.*

Note that in the preceding lemmas, in the solvability conditions and formulas for general solutions, arbitrary inner generalized inverses can be replaced by the Moore-Penrose inverse. For example, in Lemma 2.1, if

$$AA^- CB^- B = C$$

holds for some A^- and B^-, then

$$AA^\dagger CB^\dagger B = AA^\dagger(AA^- CB^- B)B^\dagger B = AA^- CB^- B = C.$$

Conversly, if

$$AA^\dagger CB^\dagger B = C$$

holds, then for arbitrary A^- and B^- it follows

$$AA^- CB^- B = AA^-(AA^\dagger CB^\dagger B)B^- B = AA^\dagger CB^\dagger B = C.$$

So, for A^- and B^- in the solvability conditions and formulas for general solutions, we can choose exactly $A^\dagger$ and $B^\dagger$, respectively.

Theorem 2.1. *Let* $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F},\mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{M},\mathcal{K})$, $B_2 \in \mathcal{B}(\mathcal{L},\mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L},\mathcal{K})$ *be such that* A_1, A_2, B_1, B_2, S *and* T *are all regular. Put*

$$\begin{aligned} &T = (I - B_1 B_1^\dagger)B_2, \quad S = (I - A_2 A_2^\dagger)A_1 A_1^\dagger, \\ &C = (I - A_2 A_2^\dagger)(C_2 - (I - A_1 A_1^\dagger)C_1 B_1^\dagger B_2)(I - T^\dagger T). \end{aligned}$$

The following statements are equivalent:

(i) *The system* (1.1) *is consistent;*

(ii) $R_{A_1}C_1L_{B_1} = 0$, $R_{A_2}C_2L_{B_2} = 0$, $R_SC = 0$;

(iii) $R_{A_1}C_1L_{B_1} = 0$, $C(I-(B_2L_T)^\dagger(B_2L_T)) = 0$, $R_SC = 0$.

In this case, the general solution to the system (1.1) *is given by*

$$X_1 = A_1^\dagger S^\dagger(R_{A_1}C_1+W)B_1^\dagger B_1 + A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + L_{A_1}R,$$

$$\begin{aligned} X_2 = &\left(-R_{A_1}C_1 + S^\dagger(R_{A_1}C_1+W)\right)B_1^\dagger(I-B_2T^\dagger) \\ &+ \left((I-S^\dagger)R_{A_1}V - S^\dagger C_2\right)T^\dagger + Z - (I-A_1A_1^\dagger+S^\dagger S)Z(B_1B_1^\dagger+TT^\dagger), \end{aligned}$$

$$\begin{aligned} X_3 = &A_2^\dagger\left(-R_{A_1}C_1 - S^\dagger(R_{A_1}C_1+W)\right)B_1^\dagger B_2L_T \\ &+ A_2^\dagger\left((I-S^\dagger)R_{A_1}V + S^\dagger C_2\right)T^\dagger B_2 \\ &+ A_2^\dagger ZB_2 - A_2^\dagger(I-A_1A_1^\dagger+S^\dagger S)Z(B_1B_1^\dagger B_2+T) + A_2^\dagger C_2 + L_{A_2}Y, \end{aligned}$$

where

$$\begin{aligned} V = &-R_{A_1}C_1B_1^\dagger B_2L_T - R_{A_1}R_{A_2}R_SR_{A_2}C_2T^\dagger T \\ &+ R_{A_1}QT^\dagger T - R_{A_1}R_{A_2}R_SR_{A_2}R_{A_1}QT^\dagger T \end{aligned}$$

and

$$\begin{aligned} W = &-R_SR_{A_2}R_{A_1}C_1 - SS^\dagger C_2L_TB_2^\dagger B_1 \\ &+ SS^\dagger PB_1^\dagger B_1 - SS^\dagger PB_1^\dagger B_2L_TB_2^\dagger B_1, \end{aligned}$$

where P, Q, R and Y are arbitrary elements of $\mathcal{B}(\mathcal{F},\mathcal{K})$, $\mathcal{B}(\mathcal{G},\mathcal{K})$, $\mathcal{B}(\mathcal{F},\mathcal{H})$ and $\mathcal{B}(\mathcal{L},\mathcal{K})$, respectively.

Proof. *(i)* $\Rightarrow$ *(ii)*: Since the system (1.1) is consistent, there exists $X_2 \in \mathcal{B}(\mathcal{G},\mathcal{K})$ such that equations

$$\begin{aligned} A_1X_1 - X_2B_1 &= C_1 \\ A_2X_3 - X_2B_2 &= C_2 \end{aligned}$$

are solvable for X_1 and X_3, respectively. According to Lemma 2.1 equation

$$A_1X_1 - X_2B_1 = C_1$$

is solvable for X_1 if and only if

$$(I-A_1A_1^\dagger)(C_1+X_2B_2) = 0, \tag{2.1}$$

and equation

$$A_2X_3 - X_2B_2 = C_2$$

is solvable for X_2 if and only if

$$(I - A_2A_2^\dagger)(C_2 + X_2B_2) = 0. \tag{2.2}$$

So, from (2.1) and (2.2) it follows that equations

$$\begin{aligned} (I - A_1A_1^\dagger)X_2B_1 &= -(I - A_1A_1^\dagger)C_1, \\ (I - A_2A_2^\dagger)X_2B_2 &= -(I - A_2A_2^\dagger)C_2 \end{aligned} \tag{2.3}$$

have a common solution. From Lemma 2.1 and Lemma 2.2 system (2.3) is consistent if and only if

$$\begin{aligned} &(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \\ &(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0, \\ &(I - SS^\dagger)C = 0. \end{aligned}$$

$(ii) \Rightarrow (i)$: If (ii) holds, then by Lemma 2.2 it follows that system (2.3) is consistent. Let $X_2 \in \mathcal{B}(G, K)$ be the solution to the system (2.3) and let $X_1 = A_1^\dagger(X_2B_1 + C_1)$ and $X_3 = A_2^\dagger(X_2B_2 + C_2)$. Then it is easy to see that such X_1, X_2 and X_3 satisfy (1.1).

$(ii) \Rightarrow (iii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.4}$$

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0 \tag{2.5}$$

and

$$(I - SS^\dagger)C = 0 \tag{2.6}$$

hold. From (2.6) we get

$$\begin{aligned} & C(I - (B_2L_T)^\dagger(B_2L_T)) \\ = \; & C(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & (I - A_2A_2^\dagger)C_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ & -(I - A_2A_2^\dagger)(I - A_1A_1^\dagger)C_1B_1^\dagger B_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & (I - A_2A_2^\dagger)C_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & (I - A_2A_2^\dagger)C_2B_2^\dagger B_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & 0. \end{aligned}$$

$(iii) \Rightarrow (ii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.7}$$

$$C(I-(B_2(I-T^\dagger T))^\dagger(B_2(I-T^\dagger T)))=0 \tag{2.8}$$

and

$$(I-SS^\dagger)C=0 \tag{2.9}$$

hold. From (2.8) we get

$$\begin{aligned}
& R_{A_2}C_2(I-T^\dagger T)(I-(B_2(I-T^\dagger T))^\dagger(B_2(I-T^\dagger T))) \\
= \ & R_{A_2}R_{A_1}C_1B_1^\dagger B_2(I-T^\dagger T)L_{B_2(I-T^\dagger T)} \\
= \ & 0. \qquad (2.10)
\end{aligned}$$

Note that

$$\begin{aligned}
& (I-T^\dagger T)L_{B_2} \\
= \ & (I-((I-B_1B_1^\dagger)B_2)^\dagger(I-B_1B_1^\dagger)B_2)(I-B_2^\dagger B_2) \\
= \ & I-B_2^\dagger B_2 \\
= \ & L_{B_2}, \qquad (2.11)
\end{aligned}$$

so from (2.11) and (2.10) we get

$$\begin{aligned}
& R_{A_2}C_2L_{B_2} \\
= \ & R_{A_2}C_2(I-T^\dagger T)L_{B_2} \\
= \ & R_{A_2}C_2(I-T^\dagger T)(B_2(I-T^\dagger T))^\dagger B_2(I-T^\dagger T)L_{B_2} \\
= \ & R_{A_2}C_2(I-T^\dagger T)(B_2(I-T^\dagger T))^\dagger(I-T^\dagger R_{B_1})B_2L_{B_2} \\
= \ & 0.
\end{aligned}$$

Suppose that system (1.1) is consistent.

Since $X_2\in\mathcal{B}(G,K)$ is a solution to (1.1) if and only if it satisfies (2.3), its general form, according to Lemma 2.2, is given by

$$\begin{aligned}
X_2 = & \left(-R_{A_1}C_1+S^\dagger(R_{A_1}C_1+W)\right)B_1^\dagger(I-B_2T^\dagger) \\
& +\left((I-S^\dagger)R_{A_1}V-S^\dagger C_2\right)T^\dagger \\
& +Z-(I-A_1A_1^\dagger+S^\dagger S)Z(B_1B_1^\dagger+TT^\dagger),
\end{aligned}$$

where Z is an arbitrary element of $\mathcal{B}(\mathcal{G},\mathcal{K})$, and

$$\begin{aligned}
V = & -R_{A_1}C_1B_1^\dagger B_2L_T-R_{A_1}R_{A_2}R_SR_{A_2}C_2T^\dagger T \\
& +R_{A_1}QT^\dagger T-R_{A_1}R_{A_2}R_SR_{A_2}R_{A_1}QT^\dagger T
\end{aligned}$$

and

$$\begin{aligned}
W = & -R_SR_{A_2}R_{A_1}C_1-SS^\dagger C_2L_TB_2^\dagger B_1 \\
& +SS^\dagger PB_1^\dagger B_1-SS^\dagger PB_1^\dagger B_2L_TB_2^\dagger B_1,
\end{aligned}$$

where P and Q are arbitrary elements of $\mathcal{B}(\mathcal{F},\mathcal{K})$ and $\mathcal{B}(\mathcal{G},\mathcal{K})$.

From the first equation in (1.1) we have

$$A_1X_1 = X_2B_1 + C_1,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_1 &= A_1^\dagger(X_2B_1 + C_1) + L_{A_1}R \\ &= A_1^\dagger S^\dagger(R_{A_1}C_1 + W)B_1^\dagger B_1 + A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + L_{A_1}R, \end{aligned}$$

where R is an arbitrary element of $\mathcal{B}(\mathcal{F},\mathcal{H})$.

From the second equation in (1.1) we have

$$A_2X_3 = X_2B_2 + C_2,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_3 &= A_2^\dagger(X_2B_2 + C_2) + L_{A_2}Y \\ &= A_2^\dagger\left(-R_{A_1}C_1 - S^\dagger(R_{A_1}C_1 + W)\right)B_1^\dagger B_2L_T \\ &\quad + A_2^\dagger\left((I - S^\dagger)R_{A_1}V + S^\dagger C_2\right)T^\dagger B_2 \\ &\quad + A_2^\dagger ZB_2 - A_2^\dagger(I - A_1A_1^\dagger + S^\dagger S)Z(B_1B_1^\dagger B_2 + T) + A_2^\dagger C_2 + L_{A_2}Y, \end{aligned}$$

where Y is an arbitrary element of $\mathcal{B}(\mathcal{L},\mathcal{K})$. $\square$

Theorem 2.2. *Let $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{M},\mathcal{L})$, $C_1 \in \mathcal{B}(\mathcal{M},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{K},\mathcal{N})$, $B_2 \in \mathcal{B}(\mathcal{L},\mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L},\mathcal{N})$ be such that A_1, A_2, B_1, B_2, S and T are all regular. Put*

$$\begin{aligned} &T = (I - B_1B_1^\dagger)(I - B_2^\dagger B_2), \quad S = A_2A_1A_1^\dagger, \\ &C = (I - (A_2A_1)(A_2A_1)^\dagger)(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)(I - B_2^\dagger B_2). \end{aligned}$$

The following statements are equivalent:

(i) The system (1.2) *is consistent;*

(ii) $R_{A_1}C_1L_{B_1} = 0$, $R_{A_2}C_2L_{B_2} = 0$, $CL_T = 0$;

(iii) $R_{A_1}C_1L_{B_1} = 0$, $(I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)C = 0$, $CL_T = 0$.

In this case, the general solution to the system (1.2) *is given by*

$$\begin{aligned} X_1 = A_1^\dagger S^\dagger A_2R_{A_1}C_1 &+ A_1^\dagger S^\dagger WB_1^\dagger B_1 + A_1^\dagger(I - S^\dagger)VB_1 \\ &+ A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + R_{A_1}R, \end{aligned}$$

$$\begin{aligned} X_2 = \big(-R_{A_1}C_1 &+ S^\dagger(A_2R_{A_1}C_1 + W)\big)B_1^\dagger(I - T^\dagger) \\ &+ \left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right)T^\dagger \\ &+ Z - (R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger), \end{aligned}$$

$$\begin{aligned}X_3 = A_2 &\left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right)B_1^\dagger(I - T^\dagger)B_2^\dagger \\ &+ A_2\left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right)T^\dagger B_2^\dagger \\ &+ A_2ZB_2^\dagger - A_2(R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger)B_2^\dagger - C_2B_2^\dagger + YR_{B_2},\end{aligned}$$

where

$$V = -R_{A_1}C_1B_1^\dagger L_{B_2}L_T + R_{A_1}QT^\dagger T - R_{A_1}A_2^\dagger R_SA_2R_{A_1}QT^\dagger T$$

and

$$W = -R_SA_2R_{A_1}C_1 + SS^\dagger C_2L_{B_2}B_1 + SS^\dagger PB_1^\dagger B_1 - SS^\dagger PB_1^\dagger L_{B_2}B_1$$

with P,Q, Z *and* Y *arbitrary elements of* $\mathcal{B}(\mathcal{F},\mathcal{K})$, $\mathcal{B}(\mathcal{N},\mathcal{K})$, $\mathcal{B}(\mathcal{G},\mathcal{K})$, *and* $\mathcal{B}(\mathcal{N},\mathcal{M})$, *respectively.*

Proof. $(i) \Rightarrow (ii)$: Since the system (1.1) is consistent, there exists $X_2 \in \mathcal{B}(\mathcal{G},\mathcal{K})$ such that equations

$$\begin{aligned}A_1X_1 - X_2B_1 &= C_1\\ A_2X_2 - X_3B_2 &= C_2\end{aligned}$$

are solvable for X_1 and X_3, respectively. According to Lemma 2.1 equation

$$A_1X_1 - X_2B_1 = C_1 \tag{2.12}$$

is solvable for X_1 if and only if

$$(I - A_1A_1^\dagger)(C_1 + X_2B_2) = 0 \tag{2.13}$$

and equation

$$A_2X_2 - X_3B_2 = C_2 \tag{2.14}$$

is solvable for X_3 if and only if

$$(A_2X_2 - C_2)(I - B_2^\dagger B_2) = 0. \tag{2.15}$$

So, from (2.13) and (2.15) it follows that equations

$$\begin{aligned}(I - A_1A_1^\dagger)X_2B_1 &= -(I - A_1A_1^\dagger)C_1,\\ A_2X_2(I - B_2^\dagger B_2) &= C_2(I - B_2^\dagger B_2)\end{aligned} \tag{2.16}$$

have a common solution. From Lemma 2.1 and Lemma 2.2 system (2.16) is consistent if and only if

$$\begin{aligned}&(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0,\\ &(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0,\\ &C'(I - T^\dagger T) = 0,\end{aligned}$$

where

$$C' = (I - SS^\dagger)(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)(I - B_2^\dagger B_2).$$

Note that condition

$$C'(I - T^\dagger T) = 0 \tag{2.17}$$

is equivalent to

$$C(I - T^\dagger T) = 0, \tag{2.18}$$

since (2.17) implies

$$\begin{aligned} & C(I - T^\dagger T) \\ = \; & R_{A_2A_1}(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\ = \; & R_{A_2A_1}SS^\dagger(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\ = \; & R_{A_2A_1}A_2A_1A_1^\dagger S^\dagger(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\ = \; & 0, \end{aligned}$$

and (2.18) implies

$$\begin{aligned} & C'(I - T^\dagger T) \\ = \; & R_S(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\ = \; & R_S(A_2A_1)(A_2A_1)^\dagger(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\ = \; & (I - (A_2A_1A_1^\dagger)(A_2A_1A_1^\dagger)^\dagger)(A_2A_1)(A_2A_1)^\dagger(C_2 + A_2(I - A_1A_1^-)C_1B_1^-)L_{B_2}L_T \\ = \; & 0. \end{aligned}$$

I follows that

$$\begin{aligned} & (I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \\ & (I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0, \\ & C(I - T^\dagger T) = 0. \end{aligned}$$

$(ii) \Rightarrow (i)$: If (ii) holds, then by Lemma 2.2 it follows that system (2.16) is consistent. Let $X_2 \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ be the solution to the system (2.16) and let $X_1 = A_1^\dagger(X_2B_1 + C_1)$ and $X_3 = (A_2X_2 - C_2)B_2^\dagger$. Then it is easy to see that such X_1, X_2 and X_3 satisfy (1.2).

$(ii) \Rightarrow (iii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.19}$$

$$(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0 \tag{2.20}$$

and

$$C(I - T^\dagger T) = 0. \tag{2.21}$$

From (2.20) we obtain

$$\begin{aligned}
& (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)C \\
= \;& (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2} \\
= \;& (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}C_2L_{B_2} \\
& +(I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}A_2(I - A_1A_1^\dagger)C_1B_1^\dagger L_{B_2} \\
= \;& (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}A_2A_2^\dagger C_2L_{B_2} \\
= \;& 0.
\end{aligned}$$

$(ii) \Rightarrow (iii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.22}$$

$$(I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)C = 0 \tag{2.23}$$

and

$$C(I - T^\dagger T) = 0. \tag{2.24}$$

From (2.23) we get

$$\begin{aligned}
& (I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) \\
= \;& (I - A_2A_2^\dagger)C \\
= \;& (I - A_2A_2^\dagger)R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger C \\
= \;& 0.
\end{aligned}$$

Suppose that system (1.2) is consistent. Since $X_2 \in \mathcal{B}(G, K)$ is a solution to (1.2) if and only if it is solution to (2.16), its general form, according to Lemma 2.2, is given by

$$\begin{aligned}
X_2 = & \left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right) B_1^\dagger(I - T^\dagger) \\
& + \left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right) T^\dagger \\
& + Z - (R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger),
\end{aligned}$$

where

$$V = -R_{A_1}C_1B_1^\dagger L_{B_2}L_T + R_{A_1}QT^\dagger T - R_{A_1}A_2^\dagger R_SA_2R_{A_1}QT^\dagger T$$

and

$$W = -R_SA_2R_{A_1}C_1 + SS^\dagger C_2L_{B_2}B_1 + SS^\dagger PB_1^\dagger B_1 - SS^\dagger PB_1^\dagger L_{B_2}B_1$$

with P, Q, Z arbitrary elements of $\mathcal{B}(\mathcal{F},\mathcal{M})$, $\mathcal{B}(\mathcal{G},\mathcal{K})$ and $\mathcal{B}(\mathcal{G},\mathcal{K})$, respectively.

From the first equation in (1.2) we have

$$A_1X_1 = X_2B_1 + C_1,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_1 &= A_1^\dagger(X_2B_1 + C_1) + L_{A_1}R \\ &= A_1^\dagger S^\dagger(A_2R_{A_1}C_1 + W)B_1^\dagger B_1 + A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + L_{A_1}R, \end{aligned}$$

where R is an arbitrary element of $\mathcal{B}(\mathcal{F},\mathcal{H})$.

From the second equation in (1.2) we have

$$X_3B_2 = A_2X_2 - C_2,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_3 &= (A_2X_2 - C_2)B_2^\dagger + YR_{B_2} \\ &= A_2\left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right)B_1^\dagger(I - T^\dagger)B_2^\dagger \\ &\quad + A_2\left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right)T^\dagger B_2^\dagger \\ &\quad + A_2ZB_2^\dagger - A_2(R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger)B_2^\dagger - C_2B_2^\dagger + YR_{B_2}, \end{aligned}$$

where Y is an arbitrary element of $\mathcal{B}(\mathcal{N},\mathcal{M})$. □

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