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[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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**Original Scientific Paper**

# GENERALIZATION OF A QUADRATIC TRANSFORMATION DUE TO EXTON

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**Abstract.** Exton [Ganita **54** (2003), 13–15] obtained numerous new quadratic transformations involving hypergeometric functions of order two and of higher order by applying various known classical summation theorems to a general transformation formula based on the Bailey transformation. We obtain a generalization of one of the Exton quadratic transformations. The results are derived with the help of a generalization of Dixon's summation theorem for the series ${}_3F_2$ obtained earlier by Lavoie *et al.* Several interesting known as well as new special cases and limiting cases are also given.
**Keywords:** Quadratic transformation, hypergeometric function of order two, generalized classical Dixon's theorem

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## 1. Introduction

The generalized hypergeometric function with $p$ numeratorial and $q$ denominatorial parameters is defined by (see [8, p. 73])

$$
\begin{aligned}
(1.1)\qquad {}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{array};\ z\right] &= {}_pF_q\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q\ ;\ z\right]\\
&= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!},
\end{aligned}
$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number $\alpha$ by

$$
(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & n\in\mathbf{N}=\{1,2,\ldots\},\\ 1, & n=0.\end{cases}
$$

When $q = p$ this series converges for all $|x| < \infty$, but when $q = p-1$ convergence occurs when $|x| < 1$ (unless the series terminates).

It should be remarked here that whenever hypergeometric and generalized hypergeometric functions can be summed in terms of Gamma functions, the results are very important from the application points of view. It should also be noted that summation formulas for ${}_pF_q$ are known for only very restricted arguments and parameters, for example Gauss' two summation theorems, Kummer's summation theorems for the series ${}_2F_1$, and Dixon's, Watson's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$, and others, play an important role in the theory of hypergeometric and generalized hypergeometric functions. The function ${}_pF_q(z)$ has been extensively studied by many authors such as Slater [9] and Exton [2].

By applying various known summation theorems to a general formula based upon Bailey's transformation theorem given in Slater [9] (and re-derived by Kim et al. [4] and written in corrected form), Exton [3] obtained as a special case numerous new general transformation formulas involving hypergeometric functions of order two and of higher order. One of his result is the following transformation formula

(1.2)

$$
\left(\frac{2}{1+\sqrt{1-x}}\right)^{2d-1} {}_2F_1\left[\begin{array}{c}2d-1,\ d-\frac{1}{2}\\ d+\frac{1}{2}\end{array};\frac{x}{\left(x+\sqrt{1-x}\right)^2}\right] = {}_2F_1\left[\begin{array}{c}d-\frac{1}{2},\ d\\ d+\frac{1}{2}\end{array};x\right]
$$

provided $|x| < 1$ and

$$
\left|\frac{x}{\left(x+\sqrt{1-x}\right)^2}\right| < 1.
$$

It is interesting to mention here the very recently Milovanović and Rathie [6] established the generalization of (1.2) by obtaining the following two master formulas for each $i \in \mathbb{N}_0$ viz.

$$(1.3)\qquad \left(\frac{2}{1+\sqrt{1-x}}\right)^{2d-1} {}_2F_1\left[\begin{matrix} 2d+i-1,\ d-\frac{1}{2} \\ d+\frac{1}{2}\end{matrix}\ ;\frac{x}{\left(x+\sqrt{1-x}\right)^2}\right]$$

$$= 2^{-i}\sum_{r=0}^{i}\binom{i}{r}{}_2F_1\left[\begin{matrix} d-\frac{1}{2},\ d+\frac{1}{2}r \\ d+\frac{1}{2}\end{matrix}\ ;x\right]$$

and

$$(1.4)\qquad \left(\frac{2}{1+\sqrt{1-x}}\right)^{2d-1} {}_2F_1\left[\begin{matrix} 2d-i-1,\ d-\frac{1}{2} \\ d+\frac{1}{2}\end{matrix}\ ;\frac{x}{\left(x+\sqrt{1-x}\right)^2}\right]$$

$$= \frac{(-2)^i}{\Gamma(i+1)}\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma\left(d+\frac{1}{2}r\right)}{\Gamma\left(d-i+\frac{1}{2}r\right)}{}_3F_2\left[\begin{matrix} 1,\ d-\frac{1}{2},\ d+\frac{1}{2}r \\ i+1,\ d+\frac{1}{2}\end{matrix}\ ;x\right].$$

Moreover, special cases of the results (1.3) and (1.4) for $i = 0, 1, 2, 3, 4, 5$ are obtained by Pogany and Rathie [7]. In fact, in our present investigation, we shall be concerned with the following interesting transformation formula

$$(1.5)\qquad \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-1,\ b,\ d-\frac{1}{2} \\ 2d-b,\ d+\frac{1}{2}\end{matrix}\ ;\ -\frac{x}{(1+\sqrt{1-x})^2}\right]$$

$$= {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\ d,\ d-b+\frac{1}{2} \\ 2d-b,\ d+\frac{1}{2}\end{matrix}\ ;\ x\right],$$

which is valid for $|x| < 1$ and

$$\left|\frac{x}{(1+\sqrt{1-x})^2}\right| < 1.$$

Moreover, Exton [3] deduced (1.5) from the following more general transformation formula

$$(1.6)\qquad \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_{A+1}F_{H+1}\left[\begin{matrix} (a),\ d-\frac{1}{2} \\ (h),\ d+\frac{1}{2}\end{matrix}\ ;\frac{xy}{(1+\sqrt{1-x})^2}\right]$$

$$= \sum_{m=0}^{\infty}\frac{(d-\frac{1}{2})_m(d)_m}{(2d)_m m!}\,x^m\,{}_{A+1}F_{H+1}\left[\begin{matrix} (a),\ -m \\ (h),\ 2d+m\end{matrix}\ ;\ y\right],$$

which is valid for $|y| < 1$ and

$$\left|\frac{xy}{(1+\sqrt{1-x})^2}\right| < 1.$$

Here, the symbol $(h)$ is a convenient contraction for the sequence of parameters $h_1$, $h_2, \ldots, h_H$ and the Pochhammer symbol $(h)_n$ is defined above.

The aim of this paper is to obtain the following generalization of (1.5) in the form

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} b,\ d-\frac{1}{2},\ 2d-1-i \\ d+\frac{1}{2},\ 2d-b+j \end{matrix};\ -\frac{x}{(1+\sqrt{1-x})^2}\right] \tag{1.7}$$

for integer $i$ satisfying $-3 \leq i \leq 3$ and $j = 0, 1, 2, 3$. For this, we will require the following generalization of Dixon's theorem for the sum of a ${}_3F_2$ of unit argument obtained earlier by Lavoie et al. [5],

$$\begin{aligned} {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b+i,\ 1+a-c+i+j \end{matrix};1\right] &= 2^{-2c+i+j}C_{i,j} \\ &\times\left\{A_{i,j}\frac{\Gamma(\frac{1}{2}a-c+\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(\frac{1}{2}a-b-c+1+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-b+1+[\frac{1}{2}i])}\right. \\ &\left.+B_{i,j}\frac{\Gamma(\frac{1}{2}a-c+1+[\frac{i+j}{2}])\Gamma(\frac{1}{2}a-b-c+\frac{3}{2}+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b+\frac{1}{2}+[\frac{i+1}{2}])}\right\}, \end{aligned} \tag{1.8}$$

Table 1.1: Values of the coefficients $A_{i,j}$

| $i\backslash j$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | $5a-b^2+(a+1)^2$ $-(2a-b+1)(b+c)$ | — | — | — |
| 2 | $\frac{1}{2}(a-1)(a-4)$ $-(b^2-5a+1)$ $-(a-b+1)(b+c)$ | $(b-1)(b-2)$ $-(a-b+1)(a-b-c+3)$ | $\frac{1}{2}(a-c+2)(a-2b-c+5)$ $\times\{(a-c+2)(a-2b+2)$ $-a(c-3)\}$ $-(b-1)(b-2)(c-2)(c-3)$ | — |
| 1 | 1 | $c-a-1$ | $a(a-1)$ $+(b+c-3)(c-2a-1)$ | — |
| 0 | 1 | $-1$ | $\frac{1}{2}\{(a-b-c+1)^2$ $+(c-1)(c-3)-b^2+a\}$ | $c(a-b-c+4)$ $-(a+1)(a+2)$ $-(a-1)(b-1)+3ab$ |
| $-1$ | 1 | 1 | $b+c-1$ | $(c-1)(c-2)$ $-b(a-c+1)$ |
| $-2$ | $\frac{1}{2}(a-1)(a-2b-2)$ $-c(a-b-1)$ | $a-b-1$ | $\frac{1}{2}(a-1)(a-2b-2c)$ $+b(b+c)$ | $(a-b-1)(c-1)$ $-b(b+1)$ |
| $-3$ | $(a-1)$ $\times(a-2b-2c-4)$ $+bc$ | $(a-b-2)$ $\times(a-c-1)$ $-ac$ | $(a-b-1)$ $\times(a-b-2c-2)$ $-bc$ | $b(b+1)$ $+(a-1)(a-b)$ $-c(2a-b-2)$ |

Table 1.2: Values of the coefficients $B_{i,j}$

| $i\backslash j$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | $-a+3b^2-(a+3)^2$ $+(2a-3b+5)(b+c)$ | — | — | — |
| 2 | $-2$ | $(a-b-2c+5)$ $\times(a-b-c+3)$ $-(b-1)(b-2)$ | $-2(a-c+2)$ $\times(a-2b-c+5)$ | — |
| 1 | $-1$ | $a-2b-c+3$ | $(b-1)(b-c+1)$ $-(a-b-c+2)$ $\times(a-b-c+3)$ | — |
| 0 | 0 | 1 | $-2$ | $(a+2)(a+4)-b(2a+5)$ $-3c(a-b-c+4)+3$ |
| $-1$ | 1 | 1 | $-(b-c+1)$ | $(c-1(c-2)$ $+b(a-2b-c+1)$ |
| $-2$ | 2 | $a-b-2c-1$ | 2 | $b(a-2c+2)$ $-(b-c+1)$ $\times(a-b-2c+1)$ |
| $-3$ | $(a-2)$ $\times(a-2b-2c-3)$ $+3bc$ | $(a-b-2)(a-2b-2c-3)$ $+bc$ | $(a-b-2)$ $\times(a-b-2c-1)$ $+bc$ | $(a-1)(a-2)$ $-3b(a-b-2)$ $-c(2a-3b-4)$ |

where

$$C_{i,j} = \frac{\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma\big(b-\frac{1}{2}i-\frac{1}{2}|i|\big)\Gamma\big(c-\frac{1}{2}(i+j+|i+j|)\big)}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\Gamma(b)\Gamma(c)},$$

provided $\mathrm{Re}(a-2b-2c) > -2-2i-j$ with $-3 \le i \le 3$ and $j = 0, 1, 2, 3$.

Here and in what follows, $[x]$ is the greatest integer less than or equal to $x$ and $|x|$ denotes the usual absolute value of $x$. The coefficients $A_{i,j}$ and $B_{i,j}$ are given in Tables 1.1 and 1.2.

Also, if $f_{i,j}$ denotes the ${}_3F_2(1)$ series on the left-hand side of (1.8), the natural symmetry

$$f_{i,j}(a,b,c) = f_{i+j,-j}(a,c,b)$$

makes it possible to extend the result to $j = -1, -2, -3$.

Several interesting cases, including Exton's result, are then deduced as special cases of our main findings. In addition to this, certain known results obtained recently by Pogány and Rathie [7] have also obtained as a limiting case of our main findings. The results derived in this paper are easily established and may be of general interest.

## 2. Extension of Exton's Quadratic Transformation

Here we establish a natural extension of the Exton transformation (1.5) given by the following theorem.

**Theorem 2.1.** *In the domain $\mathcal{D}$ defined by the connected subset*

$$\mathcal{D} = \left\{ x \in \mathbf{C} \;\middle|\; |x| < 1 \wedge \left| \frac{x}{(1+\sqrt{1-x})^2} \right| < 1 \right\},$$

*the following identities*

$$\text{(2.1)} \quad \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} b,\ d-\frac{1}{2},\ 2d-1-i \\ d+\frac{1}{2},\ 2d-b+j \end{matrix};\ -\frac{x}{(1+\sqrt{1-x})^2}\right]$$

$$= \frac{2^i(-1)^{\frac{1}{2}(i+|i|)}\Gamma(d)\Gamma(d+\frac{1}{2})\Gamma(b-\frac{1}{2}(i+j+|i+j|))}{\Gamma(b)\Gamma(d-b+\frac{1}{2}j)\Gamma(d-b+\frac{1}{2}j+\frac{1}{2})} Q_{i,j}(x;b,d),$$

*where*

$$Q_{i,j}(x;b,d) = \sum_{n=0}^{\infty} \frac{n!}{(n+\frac{1}{2}i+\frac{1}{2}|i|)!} \frac{(d)_n(d-\frac{1}{2})_n}{(2d-b+j)_n} \frac{x^n}{n!}$$

$$\times \left\{ A_{i,j} \frac{\Gamma(d-b-\frac{i}{2}+[\frac{i+j+1}{2}])\Gamma(d-b+\frac{i}{2}+\frac{1}{2}+[\frac{j+1}{2}])}{\Gamma(d-\frac{i}{2})\Gamma(d+\frac{1}{2}-\frac{i}{2}+[\frac{i}{2}])} \frac{(d-b+\frac{i}{2}+\frac{1}{2}+[\frac{j+1}{2}])_n}{(d+\frac{1}{2}-\frac{i}{2}+[\frac{i}{2}])_n}\right.$$

$$\left. + B_{i,j} \frac{\Gamma(d-b+\frac{1}{2}-\frac{i}{2}+[\frac{i+j}{2}])\Gamma(d-b+\frac{i}{2}+1+[\frac{j}{2}])}{\Gamma(d-\frac{i}{2}-\frac{1}{2})\Gamma(d-\frac{i}{2}+[\frac{i+1}{2}])} \frac{(d-b+\frac{i}{2}+1+[\frac{j}{2}])_n}{(d-\frac{i}{2}+[\frac{i+1}{2}])_n} \right\},$$

*hold for integer $i$ satisfying $-3 \le i \le 3$ and $j = 0, 1, 2, 3$. As usual $[x]$ denotes the greatest integer less than or equal to $x$ and its modulus is denoted by $|x|$. The coefficients $A_{i,j}$ and $B_{i,j}$ can be obtained from the values of $A_{i,j}$ and $B_{i,j}$ in Tables* 1.1 *and* 1.2 *by changing $a$ to $2d-1-i$, $b$ to $-n$ and $c$ to $b$, respectively.*

*Proof.* We first derive Exton's result (1.6) in an alternative way. Let $\mathcal{S}$ denote the left-hand side of (1.3) and express ${}_{A+1}F_{H+1}$ as a series so that

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n\, ((a))_n\, (d-\frac{1}{2})_n\, x^n\, y^n}{((h))_n\, (d+\frac{1}{2})_n\, 2^{2n}\, n!} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)^{1-2(d+n)}.$$

Use of the well-known result [8, p. 34]

$$\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)^{1-2a} = {}_2F_1\left[\begin{matrix} a-\frac{1}{2},\ a \\ 2a \end{matrix} ;\ x\right],$$

then enables $S$ to be rewritten in the form

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n\, ((a))_n\, (d-\frac{1}{2})_n\, x^n\, y^n}{((h))_n\, (d+\frac{1}{2})_n\, 2^{2n}\, n!}\ {}_2F_1\left[\begin{matrix} d+n-\frac{1}{2},\ d+n \\ 2d+2n \end{matrix} ;\ x\right].$$

Expressing ${}_2F_1$ as a series, we then obtain

$$S = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^n\, ((a))_n\, (d-\frac{1}{2})_n\, (d+n-\frac{1}{2})_m\, (d+n)_m\, x^{n+m}\, y^n}{((h))_n\, (d+\frac{1}{2})_n\, (2d+2n)_m\, 2^{2n}\, n!\, m!}.$$

Changing $m$ to $m-n$ and using the following identities [8, p. 57, Eq. (8)]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k)$$

and

$$(\alpha+k)_{n-k} = \frac{(\alpha)_n}{(\alpha)_k}, \qquad (n-k)! = \frac{(-1)^k\ n!}{(-n)_k},$$

we find after some simplification that

$$S = \sum_{m=0}^{\infty} \frac{(d-\frac{1}{2})_m (d)_m x^m}{(2d)_m m!} \sum_{n=0}^{m} \frac{((a))_n (-m)_n y^n}{((h))_n (2d+m)_n n!}.$$

Finally, summing the inner series as a hypergeometric series, we easily arrive at the right-hand side of (1.6). This completes our proof of (1.6).

Now we are ready to derive our main result (2.1). For this, if we put $A = 2$, $H = 1$, $a_1 = 2d-1-i$, $a_2 = b$, $h_1 = 2d-b-j$ and $y = 1$ in (1.6), we obtain

$$\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-1-i,\ b,\ d-\frac{1}{2} \\ 2d-b+j,\ d+\frac{1}{2} \end{matrix} ;\ -\frac{x}{(1+\sqrt{1-x})^2}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(d-\frac{1}{2})_n (d)_n x^n}{(2d)_n n!}\ {}_3F_2\left[\begin{matrix} 2d-1-i,\ b,\ -n \\ 2d-b+j,\ 2d+n \end{matrix} ;\ 1\right]. \tag{2.2}$$

It is now easy to see that the ${}_3F_2$ on the right-hand side of (2.2) can be evaluated with the help of the generalized Dixon summation theorem in (1.8) by replacing $a$ by $2d-1-i$, $b$ by $-n$ and $c$ by $b$. Then, after a little simplification, we easily arrive at the right-hand side of (2.1). This completes the proof of (2.1). $\square$

## 3. Special Cases

By assigning values to $i$ and $j$ in our main result (2.1), we can obtain a large number of interesting and useful results. However, we shall mention here only a few of them. All these transformations hold in a domain $\mathcal{D}$ defined by the connected subset

$$\mathcal{D} = \left\{ x \in \mathbf{C} \;\middle|\; |x| < 1, \; \left| \frac{x}{(1+\sqrt{1-x})^2} \right| < 1 \right\}.$$

For $i = 0$ and $j = 0$ in (2.1), we obtain

$$(3.1) \qquad \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-1,\, b,\, d-\frac{1}{2} \\ 2d-b,\, d+\frac{1}{2} \end{matrix} ;\; -\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\, d,\, d-b+\frac{1}{2} \\ 2d-b,\, d+\frac{1}{2} \end{matrix} ;\; x\right],$$

which is the result stated in (1.2).

For $i = 0$ and $j = 1$ in (2.1), we obtain

$$(3.2) \qquad \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-1,\, b,\, d-\frac{1}{2} \\ 2d-b+1,\, d+\frac{1}{2} \end{matrix} ;\; -\frac{x}{(1+\sqrt{1-x})^2}\right]$$
$$= \frac{2d-2b+1}{2(1-b)}\, {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\, d,\, d-b+\frac{3}{2} \\ 2d-b+1,\, d+\frac{1}{2} \end{matrix} ;\; x\right] - \frac{2d-1}{2(1-b)}\, {}_2F_1\left[\begin{matrix} d-\frac{1}{2},\; d-b+1 \\ 2d-b+1 \end{matrix} ;\; x\right].$$

For $i = 1$ and $j = 0$ in (2.1), we obtain

$$(3.3) \qquad \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-2,\, b,\, d-\frac{1}{2} \\ 2d-b,\, d+\frac{1}{2} \end{matrix} ;\; -\frac{x}{(1+\sqrt{1-x})^2}\right]$$
$$= \frac{(2d-1)(d-b)}{(1-b)}\, {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\, d-b+1,\, 1 \\ 2d-b,\, 2 \end{matrix} ;\; x\right] - \frac{(d-1)(2d-2b+1)}{(1-b)}\, {}_4F_3\left[\begin{matrix} d,\, d-\frac{1}{2},\, d-b+\frac{3}{2},\, 1 \\ 2d-b,\, d+\frac{1}{2},\, 2 \end{matrix} ;\; x\right].$$

For $i = 1$ and $j = 1$ in (2.1), we obtain

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-2,\ b,\ d-\frac{1}{2} \\ 2d-b+1,\ d+\frac{1}{2}\end{matrix}\ ;\ -\frac{x}{(1+\sqrt{1-x})^2}\right] \tag{3.4}$$

$$= A(b,d)\,{}_3F_2\left[\begin{matrix} d-\frac{1}{2},\ d-b+2,\ 1 \\ 2d-b+1,\ 2\end{matrix}\ ;\ x\right]$$

$$-B(b,d)\,{}_5F_4\left[\begin{matrix} d-\frac{1}{2},\ d,\ d-b+\frac{3}{2},\ d-\frac{1}{2}b+\frac{3}{2},\ 1 \\ 2d-b+1,\ d+\frac{1}{2},\ d-\frac{1}{2}b+\frac{1}{2},\ 2\end{matrix}\ ;\ x\right],$$

where

$$A(b,d) = \frac{(2d-1)(d-b+1)(2d-b-1)}{(b-1)(b-2)}$$

and

$$B(b,d) = \frac{(d-1)(2d-b+1)(2d-2b+1)}{(b-1)(b-2)}.$$

For $i = -1$ and $j = 0$ in (2.1), we obtain

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d,\ b,\ d-\frac{1}{2} \\ 2d-b,\ d+\frac{1}{2}\end{matrix}\ ;\ -\frac{x}{(1+\sqrt{1-x})^2}\right] \tag{3.5}$$

$$= \frac{1}{2}\ {}_2F_1\left[\begin{matrix} d-\frac{1}{2},\ d-b \\ 2d-b\end{matrix}\ ;\ x\right] + \frac{1}{2}\ {}_3F_2\left[\begin{matrix} d,\ d-\frac{1}{2},\ d-b+\frac{1}{2} \\ 2d-b,\ d+\frac{1}{2}\end{matrix}\ ;\ x\right].$$

For $i = -1$ and $j = 1$ in (2.1), we obtain

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d, b, d-\frac{1}{2} \\ 2d-b+1,\ \ d+\frac{1}{2}\end{matrix}\ ;\ -\frac{x}{(1+\sqrt{1-x})^2}\right] \tag{3.6}$$

$$= \frac{1}{2}\,{}_2F_1\left[\begin{matrix} d-\frac{1}{2},\ d-b+1 \\ 2d-b+1\end{matrix}\ ;\ x\right] + \frac{1}{2}\ {}_3F_2\left[\begin{matrix} d,\ d-\frac{1}{2},\ d-b+\frac{1}{2} \\ 2d-b+1,\ d+\frac{1}{2}\end{matrix}\ ;\ x\right].$$

For $i = -2$ and $j = 1$ in (2.1), we obtain

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d+1, b, d-\frac{1}{2} \\ 2d-b+1,\ \ d+\frac{1}{2}\end{matrix}\ ;\ -\frac{x}{(1+\sqrt{1-x})^2}\right] \tag{3.7}$$

$$= \frac{1}{2}\ {}_4F_3\left[\begin{matrix} d,\ d-\frac{1}{2},\ 2d+1,\ d-b+\frac{1}{2} \\ 2d,\ d+\frac{1}{2},\ 2d-b+1\end{matrix}\ ;\ x\right]$$

$$+\frac{1}{2}\ {}_3F_2\left[\begin{matrix} d-b,\ d-\frac{1}{2},\ 2d-2b+1 \\ 2d-b+1,\ 2d-2b\end{matrix}\ ;\ x\right].$$

Similarly other results can also be obtained.

## 4. Interesting Limiting Cases

Here we mention some of the interesting limiting cases of our results. All these transformations hold in the domain $\mathcal{D}$ defined by the connected subset

$$\mathcal{D} = \left\{ x \in \mathbf{C} \;\middle|\; |x| < 1, \left| \frac{x}{(1+\sqrt{1-x})^2} \right| < 1 \right\}.$$

If we let $b \to \infty$ in (3.1) or (3.2), we obtain the following result:

$$\left( \frac{1}{2} + \frac{1}{2}\sqrt{1-x} \right)^{1-2d} {}_2F_1 \left[ \begin{matrix} 2d-1, \; d-\frac{1}{2} \\ d+\frac{1}{2} \end{matrix} \; ; \; \frac{x}{(1+\sqrt{1-x})^2} \right] = {}_2F_1 \left[ \begin{matrix} d-\frac{1}{2}, \; d \\ d+\frac{1}{2} \end{matrix} \; ; \; x \right]. \tag{4.1}$$

If we let $b \to \infty$ in (3.3) or (3.4), we obtain the following result:

$$\left( \frac{1}{2} + \frac{1}{2}\sqrt{1-x} \right)^{1-2d} {}_2F_1 \left[ \begin{matrix} 2d-2, \; d-\frac{1}{2} \\ d+\frac{1}{2} \end{matrix} \; ; \; \frac{x}{(1+\sqrt{1-x})^2} \right] = (2d-1) \; {}_2F_1 \left[ \begin{matrix} d-\frac{1}{2}, 1 \\ 2 \end{matrix} \; ; \; x \right] - 2(d-1) \; {}_3F_2 \left[ \begin{matrix} d-\frac{1}{2}, \; d, \; 1 \\ d+\frac{1}{2}, \; 2 \end{matrix} \; ; \; x \right]. \tag{4.2}$$

If we let $b \to \infty$ in (3.5) or (3.6), we obtain the following result:

$$\left( \frac{1}{2} + \frac{1}{2}\sqrt{1-x} \right)^{1-2d} {}_2F_1 \left[ \begin{matrix} 2d, \; d-\frac{1}{2} \\ d+\frac{1}{2} \end{matrix} \; ; \; \frac{x}{(1+\sqrt{1-x})^2} \right] = \frac{1}{2} \; {}_1F_0 \left[ \begin{matrix} d-\frac{1}{2} \\ - \end{matrix} \; ; \; x \right] + \frac{1}{2} \; {}_2F_1 \left[ \begin{matrix} d-\frac{1}{2}, \; d \\ d+\frac{1}{2} \end{matrix} \; ; \; x \right]. \tag{4.3}$$

If we let $b \to \infty$ in (3.7), we obtain the following result:

$$\left( \frac{1}{2} + \frac{1}{2}\sqrt{1-x} \right)^{1-2d} {}_2F_1 \left[ \begin{matrix} 2d+1, \; d-\frac{1}{2} \\ d+\frac{1}{2} \end{matrix} \; ; \; \frac{x}{(1+\sqrt{1-x})^2} \right] = \frac{1}{2} \; {}_1F_0 \left[ \begin{matrix} d-\frac{1}{2} \\ - \end{matrix} \; ; \; x \right] + \frac{1}{2} \; {}_3F_2 \left[ \begin{matrix} d-\frac{1}{2}, \; d, \; 2d+1 \\ d+\frac{1}{2}, \; 2d \end{matrix} \; ; \; x \right]. \tag{4.4}$$

We remark that the result (4.1) was obtained by Choi and Rathie [1], whereas the results (4.2)–(4.4) were obtained by Pogány and Rathie [7] using a generalization of Kummer's summation theorem. For a remark on the Exton result [3], see the paper by Choi and Rathie [1].

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## REFERENCES

1. J. Choi, A.K. Rathie: *Quadratic transformations involving hypergeometric functions of two and higher order*. East Asian Math. J. **22** (1) (2006), 71–77.
2. H. Exton: *Multiple Hypergeometric Functions*. New York, Halsted, 1976.
3. H. Exton: *Quadratic transformation involving hypergeometric functions of higher order*. Ganita **54** (2003), 13–15.
4. Y.S. Kim, A.K. Rathie, R.B. Paris: *A note on a hypergeometric transformation formula due to Slater with an application*. Mathematica Aeterna **5** (1) (2015), 217–223.
5. J.L. Lavoie, F. Grondin, A.K. Rathie, K. Arora: *Generalizations of Dixon's theorem on the sum of a* ${}_3F_2$. Math. Comput. **62** (1994), 267–276.
6. G. V. Milovanović, A.K. Rathie: *On a quadratic transformation due to Exton and its generalization*. Hacettepe J. Math. Stat. **48** (6) (2019), 1706–1711.
7. T.K. Pogány, A.K. Rathie: *Extension of a quadratic transformation due to Exton*. Appl. Math. Comput. **215** (2009), 423–426.
8. E. D. Rainville: *Special Functions*. The Macmillan Company, New York, 1960.
9. L.J. Slater: *Generalized Hypergeometric Functions*. Cambridge, Cambridge University Press, 1966.

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# OSCULATING-TYPE RULED SURFACES IN THE EUCLEDIAL 3-SPACE

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**Abstract.** In the present paper, a new type of ruled surfaces called osculating-type (OT)-ruled surface is introduced and studied. First, a new orthonormal frame is defined for OT-ruled surfaces. The Gaussian and the mean curvatures of these surfaces are obtained and the conditions for an OT-surface to be flat or minimal are given. Moreover, the Weingarten map of an OT-ruled surface is obtained and the normal curvature, the geodesic curvature and the geodesic torsion of any curve lying on surface are obtained. Finally, some examples related to helices and slant helices are introduced.
**Keywords**:osculating-type ruled surface, minimal surface, geodesic.

## 1. Introduction

In the study of fundamental theory of curves and surfaces, the special ones of these geometric topics have been of significant value because of satisfying some particular conditions. In the curve theory, the most famous one of such special curves is general helix for which the tangent vector of the curve always makes a constant angle with a constant direction. The necessary and sufficient condition for a curve to be a general helix is that the ratio of the second curvature $\tau$ to the first curvature $\kappa$ is constant i.e., $\tau/\kappa$ is constant along the curve [1]. If the principal normal vector of a curve makes a constant angle with a constant direction, then

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that curve is called slant helix and the necessary and sufficient condition for a curve to be a slant helix is that the function $\sigma(s) = \left( \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$ is constant [5].

In the surface theory, the surfaces constructed by the simplest way are important. The well-known example of such surfaces is ruled surface which is generated by a continuous movement of a line along a curve. These surfaces have a wide use in technology and architecture [3]. Furthermore, some special types of these surfaces have particular relationships with helices and slant helices [5, 6, 7, 8]. In [11], Önder considered the notion of "slant helix" for ruled surfaces and defined slant ruled surfaces by the property that the components of the frame along the striction curve of ruled surface make constant angles with fixed lines. He has proved that helices or slant helices are the striction curves of developable slant ruled surfaces. Also, he has defined a new kind of ruled surfaces called general rectifying ruled surface for which the generating line of the surface always lies on the rectifying plane of base curve and he has given many properties of such surfaces [12].

This study introduces a new type of ruled surfaces called osculating-type (OT)-ruled surfaces. First, a new orthonormal frame and new curvatures for OT-ruled surfaces are obtained and many properties of the surface are given by considering the new frame and its curvatures. Later, the Gaussian curvature $K$ and the mean curvature $H$ of OT-ruled surfaces are given. The set of singular points of such surfaces are introduced and some differential equations characterizing special curves lying on the surface are obtained. Finally, some examples related to helix and slant helix are given.

## 2. Preliminaries

A ruled surface in $\mathbb{R}^3$ is constructed by a continuous movement of a straight line along a space curve $\alpha$. For an open interval $I \subset \mathbb{R}$, the parametric equation of a ruled surface is given by $\varphi_{(\alpha,q)}(s,u) : I \times \mathbb{R} \to \mathbb{R}^3$, $\vec{\varphi}_{(\alpha,q)}(s,u) = \vec{\alpha}(s) + u\vec{q}(s)$ where $q : I \to \mathbb{R}^3$, $\|\vec{q}\| = 1$ is called director curve and $\alpha : I \to \mathbb{R}^3$ is called the base curve of the surface $\varphi_{(\alpha,q)}$. The straight lines of the surface defined by $u \to \vec{\alpha}(s) + u\vec{q}(s)$ are called rulings [6]. The ruled surface $\varphi_{(\alpha,q)}$ is called cylindrical if $\vec{q}\,' = 0$ and non-cylindrical otherwise where $\vec{q}\,' = \frac{d\vec{q}}{ds}$ [9]. A curve $c$ lying on $\varphi_{(\alpha,q)}$ with property that $\langle \vec{c}\,', \vec{q}\,' \rangle = 0$ is called striction line of $\varphi_{(\alpha,q)}$. The parametric representation of striction line is given by

$$\vec{c}(s) = \vec{\alpha}(s) - \frac{\langle \vec{c}\,'(s), \vec{q}\,'(s) \rangle}{\langle \vec{q}\,'(s), \vec{q}\,'(s) \rangle}\vec{q}(s) \tag{2.1}$$

The striction line is geometrically important because it is the locus of special points called central points for which considering a common perpendicular between two constructive rulings, the foot of common perpendicular on the main ruling is a central point [9].

The unit surface normal or Gauss map $U$ of the ruled surface $\varphi_{(\alpha,q)}$ is defined by

$$\vec{U}(s,u)=\frac{\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial s}\times\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial u}}{\left\|\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial s}\times\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial u}\right\|}.$$

If $\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial s}\times\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial u}=0$ for some points $(s_0,u_0)\in I\times\mathbb{R}$ then, such points are called singular points of ruled surface $\varphi_{(\alpha,q)}$. Otherwise, they are called regular points. The surface $\varphi_{(\alpha,q)}$ is called developable if the unit surface normal $U$ along any ruling does not change its direction. Otherwise, $\varphi_{(\alpha,q)}$ is called non-developable or skew. A ruled surface $\varphi_{(\alpha,q)}$ is developable if and only if $\det(\vec{\alpha}',\vec{q},\vec{q}')=0$ holds [9].

The unit vectors $\vec{h}=\vec{q}'/\|\vec{q}'\|$ and $\vec{a}=\vec{q}\times\vec{h}$ are called central normal and central tangent of $\varphi_{(\alpha,q)}$, respectively. Then, the orthonormal frame $\left\{\vec{q},\vec{h},\vec{a}\right\}$ is called the Frenet frame of ruled surface $\varphi_{(\alpha,q)}$.

**Definition 2.1.** [11] A ruled surface $\varphi_{(\alpha,q)}$ is called $q$-slant or $a$-slant (resp. $h$-slant) ruled surface if its ruling $\vec{q}$ (resp. central normal $\vec{h}$) always makes a constant angle with a fixed direction.

The first fundamental form $I$ and second fundamental form $II$ of $\varphi_{(\alpha,q)}$ are defined by

$$I=Eds^2+2Fdsdu+Gdu^2,\quad II=Lds^2+2Mdsdu+Ndu^2,$$

respectively, where

$$\begin{aligned}E&=\left\langle\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial s},\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial s}\right\rangle,\quad F=\left\langle\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial s},\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial u}\right\rangle,\\ G&=\left\langle\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial u},\frac{\partial\vec{\varphi}_{(\alpha,q)}}{\partial u}\right\rangle,\end{aligned}\tag{2.2}$$

$$L=\left\langle\frac{\partial^2\vec{\varphi}_{(\alpha,q)}}{\partial s^2},\vec{U}\right\rangle,\quad M=\left\langle\frac{\partial^2\vec{\varphi}_{(\alpha,q)}}{\partial s\partial u},\vec{U}\right\rangle,\quad N=\left\langle\frac{\partial^2\vec{\varphi}_{(\alpha,q)}}{\partial u^2},\vec{U}\right\rangle.\tag{2.3}$$

The Gaussian curvature $K$ and the mean curvature $H$ are defined by

$$K=\frac{LN-M^2}{EG-F^2},\tag{2.4}$$

$$H=\frac{EN-2FM+GL}{2(EG-F^2)}.\tag{2.5}$$

respectively. An arbitrary surface is called minimal if $H=0$ at all points of the surface. Furthermore, a ruled surface is developable (or flat) if and only if $K=0$ [2].

**Theorem 2.1.** *(Catalan Theorem) [4] Among all ruled surfaces except planes only the helicoid and fragments of it are minimal.*

## 3. Osculating-type Ruled Surfaces

In this section, we define the osculating-type ruled surface of a curve $\alpha$ such that the ruling of the surface always lies in the osculating plane of $\alpha$ and also $\alpha$ is the base curve of the surface. Such a surface is defined as follows:

**Definition 3.1.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve in the Euclidean 3-space $\mathbb{E}^3$ with arc-length parameter $s$, curvature $\kappa(s)$, torsion $\tau(s)$ and Frenet frame $\left\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\right\}$. Then, the ruled surface $\varphi_{(\alpha,q_o)} : I \times \mathbb{R} \to \mathbb{R}^3$ given by the parametric form

$$\vec{\varphi}_{(\alpha,q_o)}(s,u) = \vec{\alpha}(s) + u\vec{q}_o(s), \quad \vec{q}_o(s) = \cos\theta\vec{T}(s) + \sin\theta\vec{N}(s) \tag{3.1}$$

is called the osculating-type (OT)-ruled surface of $\alpha$ where $\theta = \theta(s)$ is $C^\infty$-scalar angle function of arc-length parameter $s$ between unit vectors $\vec{q}_o$ and $\vec{T}$. Here, we use the index "$o$" to emphasize that the ruling always lies on the osculating plane $sp\left\{\vec{T}, \vec{N}\right\}$ of base curve $\alpha$.

As we see from equation (3.1), when $\theta(s) = k\pi, (k \in \mathbb{Z})$, for all $s \in I$, the ruling becomes $\vec{q}_o = \pm\vec{T}$ and the OT-ruled surface $\varphi_{(\alpha,q_o)}$ becomes the developable tangent surface $\varphi_{(\alpha,T)}$ of $\alpha$. Similarly, when $\theta(s) = \pi/2 + k\pi, (k \in \mathbb{Z})$ for all $s \in I$, the ruling becomes $\vec{q}_o = \pm\vec{N}$ and the OT-ruled surface $\varphi_{(\alpha,q_o)}$ becomes the principal normal surface $\varphi_{(\alpha,N)}$ of $\alpha$.

**Remark 3.1.** If $\alpha$ is a straight line, then $\varphi_{(\alpha,T)}$ is not a surface, only a line. So, for the case $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$, we always assume that $\alpha$ is not a straight line, i.e., $\kappa \neq 0$.

Considering (3.1) and the fact that the binormal vector $\vec{B}$ of $\alpha$ is perpendicular to $sp\left\{\vec{T}, \vec{N}\right\}$, we get $\left\langle \vec{q}_o, \vec{B} \right\rangle = 0$. Therefore, we can define a unit vector $\vec{r}(s)$ as follows,

$$\vec{r} = \vec{q}_o \times \vec{B} = \sin\theta\,\vec{T} - \cos\theta\vec{N}. \tag{3.2}$$

Then, the frame $\left\{\vec{q}_o, \vec{B}, \vec{r}\right\}$ is an orthonormal moving frame along $\alpha$ on the OT-ruled surface $\varphi_{(\alpha,q_o)}$. From equations (3.1) and (3.2), the relations between that frame and Frenet frame of $\alpha$ are given by $\vec{T} = \cos\theta\,\vec{q}_o + \sin\theta\,\vec{r}$ and $\vec{N} = \sin\theta\,\vec{q}_o - \cos\theta\,\vec{r}$. After some computations, for the derivative formulae of new frame $\left\{\vec{q}_o, \vec{B}, \vec{r}\right\}$, we get

$$\begin{bmatrix} \vec{q}_o' \\ \vec{B}' \\ \vec{r}\,' \end{bmatrix} = \begin{bmatrix} 0 & \mu & -\eta \\ -\mu & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \begin{bmatrix} \vec{q}_o \\ \vec{B} \\ \vec{r} \end{bmatrix}$$

where $\eta(s) = \theta' + \kappa,\ \mu(s) = \tau\sin\theta,\ \xi(s) = \tau\cos\theta$ are called the curvatures of OT-ruled surface $\varphi_{(\alpha,q_o)}$ according to the frame $\left\{\vec{q}_o, \vec{B}, \vec{r}\right\}$. Then, the relationships between the curvatures $\kappa$, $\tau$ of base curve $\alpha$ and the curvatures $\eta$, $\mu$, $\xi$ of OT-ruled surface of OT-ruled surface $\varphi_{(\alpha,q_o)}$ are obtained as $\kappa = \eta - \theta'$, $\tau = \pm\sqrt{\mu^2+\xi^2}$. Now, using these relationships and considering the characterizations for general helix and slant helix, the following theorem is obtained:

**Theorem 3.1.** *For the OT-ruled surface $\varphi_{(\alpha,q_o)}$, we have that*

*(i) $\alpha$ is a plane curve if and only if both $\mu$ and $\xi$ vanish.*

*(ii) $\alpha$ is a general helix if and only if the function $\rho(s) = \pm\frac{\sqrt{\mu^2+\xi^2}}{\eta-\theta'}$ is constant.*

*(iii) $\alpha$ is a slant helix if and only if the function*

$$\sigma(s) = \pm\frac{(\mu\mu' + \xi\xi')(\eta-\theta') - (\mu^2+\xi^2)(\eta'-\theta'')}{\left[(\eta-\theta')^2 + \mu^2 + \xi^2\right](\mu^2+\xi^2)^{1/2}}$$

*is constant.*

Let now consider the special case that the base curve $\alpha$ is a plane curve, i.e., $\tau = 0$. Then, $\alpha$ lies on the osculating plane $sp\left\{\vec{T}, \vec{N}\right\}$ and has constant binormal vector $\vec{B}$. Since, the unit surface normal $\vec{U}$ of OT-ruled surface $\varphi_{(\alpha,q_o)}$ is always perpendicular to both $\vec{q_o}$ and $\vec{T}$, we have that $\vec{U} = \pm\vec{B}$. Then, the OT-ruled surface has a constant unit normal, that is, it is a plane. Conversely, if the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is a plane with constant unit normal $\vec{U}$, since $\vec{U} \perp sp\left\{\vec{q_o}, \vec{T}\right\}$, from (3.1) we get $\vec{U} \perp sp\left\{\vec{T}, \vec{N}\right\}$ which gives $\vec{U} = \pm\vec{B}$ is a constant vector. Then, $\tau = 0$, i.e., $\alpha$ is a plane curve and we have the followings:

**Theorem 3.2.** *The OT-ruled surface $\varphi_{(\alpha,q_o)}$ is a plane if and only if the base curve $\alpha$ is a plane curve.*

Clearly, Theorem 3.4 gives the following corollary:

**Theorem 3.3.** *If $\varphi_{(\alpha,q_o)} \neq \varphi_{(\alpha,T)}$ and $\varphi_{(\alpha,q_o)} \neq \varphi_{(\alpha,N)}$, the followings are equivalent*

*(i) $\alpha$ is a plane curve.* *(ii) The OT-ruled surface $\varphi_{(\alpha,q_o)}$ is a plane.*

*(iii) $\mu = 0$.* *(iv) $\xi = 0$.*

Now, we will give other characterizations and geometric properties of the OT-ruled surfaces.

**Theorem 3.4.** *The set of the singular points of OT-ruled surface $\varphi_{(\alpha,q_o)}$ is given by*

$$S = \{(s_0, u_0) \in I \times \mathbb{R} : \theta(s_0) = k\pi, u_0 = 0, k \in \mathbb{Z}\}.$$

*Proof.* From the partial derivatives of $\vec{\varphi}_{(\alpha,q_o)}(s,u) = \vec{\alpha}(s) + u\vec{q}_o(s)$, we get

$$\frac{\partial \vec{\varphi}_{(\alpha,q_o)}}{\partial s} = \cos\theta\,\vec{q}_o + u\mu\vec{B} + (\sin\theta - u\eta)\vec{r}, \quad \frac{\partial \vec{\varphi}_{(\alpha,q_o)}}{\partial u} = \vec{q}_o. \tag{3.3}$$

Therefore, the direction of surface normal is given by the vector

$$\frac{\partial \vec{\varphi}_{(\alpha,q_o)}}{\partial s} \times \frac{\partial \vec{\varphi}_{(\alpha,q_o)}}{\partial u} = (\sin\theta - u\eta)\vec{B} - u\mu\vec{r}.$$

Then, the OT-ruled surface $\varphi_{(\alpha,q_o)}$ has singular points if and only if the system

$$\begin{cases} \sin\theta - u\eta = 0, \\ u\mu = 0 \end{cases} \tag{3.4}$$

holds. Let now assume that $u = 0$. Then, from the first equality, it follows $\theta(s_0) = k\pi, (k \in \mathbb{Z}, s_0 \in I)$. When this satisfies for all $s \in I$, we have $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$ and the locus of the singular points is the base curve $\alpha$. If $u \neq 0$, from the system (3.4), we get $u(s) = \frac{\sin\theta}{\eta}$ and $\mu = 0$. Since we assume that singular points exist, from Theorem 3.2, we have $\tau \neq 0$. Otherwise, the surface is a plane and regular. Then, $\mu = 0$ implies that $\sin\theta = 0$ which is a contradiction with the assumption that $u \neq 0$. And so, the system (3.4) only holds if and only if $u = 0$, $\theta(s_0) = k\pi, (k \in \mathbb{Z}, s_0 \in I)$. □

Hereafter, for the sake of simplicity, we will take $f = \sin\theta - u\eta$ and $g = u\mu$.

**Proposition 3.1.** *The OT-ruled surface $\varphi_{(\alpha,q_o)}$ is developable if and only if $\varphi_{(\alpha,q_o)}$ is a plane or $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$.*

*Proof.* For the surface $\varphi_{(\alpha,q_o)}$, we have $\det(\vec{\alpha}', \vec{q}_o, \vec{q}_o') = \mu\sin\theta$. Considering Theorem 3.2, we have the desired result. □

**Proposition 3.2.** *Among all OT-ruled surfaces $\varphi_{(\alpha,q_o)}$, only the plane is cylindrical.*

*Proof.* Since a ruled surface is called cylindrical if and only if the direction of the ruling is a constant vector, we get $\vec{q}_o' = 0$ if and only if

$$-\eta\sin\theta\vec{T} + \eta\cos\theta\vec{N} + \tau\sin\theta\vec{B} = 0. \tag{3.5}$$

If $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$, then $\theta(s) = k\pi$ for all $s \in I$ and (3.5) gives $\eta = 0$, which implies that $\kappa = 0$, which is a contradiction with Remark 3.1. If $\varphi_{(\alpha,q_o)} \neq \varphi_{(\alpha,T)}$, then from (3.5) we have $\tau = 0$, $\eta = 0$ which gives $\theta(s) = -\int_0^s \kappa(s)ds$ and Theorem 3.2 gives that $\varphi_{(\alpha,q_o)}$ is a plane. □

Proposition 3.1 and Proposition 3.2 give the following corollary:

**Corollary 3.1.** *If the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is cylindrical, then it is a plane with the parametric form*

$$\vec{\varphi}_{(\alpha,q_o)}(s,u) = \vec{\alpha}(s) + u\left(\cos\left(\int_0^s \kappa(s)ds\right)\vec{T}(s) - \sin\left(\int_0^s \kappa(s)ds\right)\vec{N}(s)\right)$$

**Proposition 3.3.** *The base curve $\alpha$ of the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is also its striction line if and only if $\theta(s) = -\int_0^s \kappa(s)ds$ or $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$.*

*Proof.* The base curve $\alpha$ is the striction line of $\varphi_{(\alpha,q_o)}$ if and only if $\langle\vec{\alpha}', \vec{q}_o'\rangle = 0$. Therefore, we get $\langle\vec{\alpha}', \vec{q}_o'\rangle = -\eta\sin\theta$ which gives the desired result. □

From Proposition 3.3, it is clear that the set of the intersection points of base curve $\alpha$ and striction curve $c$ is $V = S \cup Y$, where $S$ is the set of singular points of $\varphi_{(\alpha,q_o)}$ and

$$Y = \{(s_0, u_0) \in I \times \mathbb{R} : \theta'(s_0) = -\kappa(s_0), u_0 = 0\}.$$

It is clear that the points of $Y$ are non-singular.

Let now investigate the special curves lying on the OT-surface $\varphi_{(\alpha,q_o)}$. The Gauss map (or the unit surface normal) of the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is given by

$$\vec{U}(s,u) = \frac{1}{\sqrt{f^2+g^2}}\left(f\vec{B} - g\,\vec{r}\right). \tag{3.6}$$

Then, for the base curve $\alpha$ we have the followings:

**Theorem 3.5.** *The base curve $\alpha$ is a geodesic on the OT-ruled surface $\varphi_{(\alpha,q_o)}$ if and only if $\alpha$ is a straight line.*

*Proof.* We know that $\alpha$ is a geodesic on $\varphi_{(\alpha,q_o)}$ if and only if the condition

$$\vec{U} \times \vec{\alpha}'' = 0 \tag{3.7}$$

satisfies. Then, by using (3.6), from (3.7) we get

$$\vec{U} \times \vec{\alpha}'' = \frac{1}{\sqrt{f^2+g^2}}\left(-\kappa f\,\vec{T} - g\kappa\sin\theta\,\vec{B}\right) \tag{3.8}$$

and that $\alpha$ is a geodesic curve on $\varphi_{(\alpha,q_o)}$ if and only if the system

$$\begin{cases} \kappa f = 0 \\ g\kappa\sin\theta = 0 \end{cases}$$

holds. If we assume $\varphi_{(\alpha,q_o)} \neq \varphi_{(\alpha,T)}$, from the last system it follows

$$\kappa f = 0,\ \ g\kappa = 0,$$

which gives that $\kappa = 0$, i.e., $\alpha$ is a straight line or the system

$$f = 0, \; g = 0,$$

holds. But for the last system, considering (3.4), it follows that the system has a solution as a curve if and only if $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$ which is a contradiction by the assumption $\varphi_{(\alpha,q_o)} \neq \varphi_{(\alpha,T)}$ and so, we eliminate this case. If $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$, considering Remark 3.1, we should take $\kappa \neq 0$. But for this case, the system gives that $\eta = \kappa = 0$, which is a contradiction. Then we have that $\alpha$ is a geodesic on $\varphi_{(\alpha,q_o)}$ if and only if $\alpha$ is a straight line. □

**Theorem 3.6.** *Let $\alpha$ have non-vanishing curvature $\kappa$. Then, $\alpha$ is an asymptotic curve on the OT-ruled surface $\varphi_{(\alpha,q_o)}$ if and only if one of the followings hold:*

*(i) $\varphi_{(\alpha,q_o)}$ is a plane (ii) $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$ (iii) $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,N)}$.*

*Proof.* $\alpha$ is an asymptotic curve on $\varphi_{(\alpha,q_o)}$ if and only if $\langle \vec{U}, \vec{\alpha}'' \rangle = 0$. Then, we get

$$\left\langle \vec{U}, \vec{\alpha}'' \right\rangle = \frac{u\kappa\tau\cos\theta\sin\theta}{\sqrt{f^2+g^2}} \tag{3.9}$$

From (3.9), we obtain that $\langle \vec{U}, \vec{\alpha}'' \rangle = 0$ if and only if $\tau = 0$ or $\sin\theta = 0$ or $\cos\theta = 0$. □

**Theorem 3.7.** *The base curve $\alpha$ is a line of curvature on the OT-ruled surface $\varphi_{(\alpha,q_o)}$ if and only if $\varphi_{(\alpha,q_o)}$ is a plane.*

*Proof.* The curve $\alpha$ is a line of curvature on the OT-ruled surface $\varphi_{(\alpha,q_o)}$ if and only if $\vec{U}'_\alpha \times \vec{\alpha}' = 0$ holds where $\vec{U}_\alpha$ is the unit surface normal along the curve $\alpha$ and for which we have $\vec{U}_\alpha = \vec{B}$. Then, it follows

$$\vec{U}'_\alpha \times \vec{\alpha}' = -\tau\vec{B} \tag{3.10}$$

The equation (3.10) is equal to zero if and only if $\tau = 0$ and from Theorem 3.2, we have that $\varphi_{(\alpha,q_o)}$ is a plane. □

Now, let us examine first and second fundamental coefficients of the OT-ruled surface $\varphi_{(\alpha,q_o)}$. From (2.2) and (2.3), we get

$$E = f^2 + g^2 + \cos^2\theta, \; F = \cos\theta, \; G = 1 \tag{3.11}$$

$$L = \frac{-(f^2+g^2)\xi + \mu\sin\theta\cos\theta - g^2\left(\frac{f}{g}\right)_s}{\sqrt{f^2+g^2}}, \; M = \frac{\mu\sin\theta}{\sqrt{f^2+g^2}}, \; N = 0 \tag{3.12}$$

By using the fundamental coefficients computed in (3.11) and (3.12), from (2.4) and (2.5) the Gaussian curvature $K$ and the mean curvature $H$ of OT-ruled surface $\varphi_{(\alpha,q_o)}$ are obtained as

$$K = -\frac{\mu^2 \sin^2\theta}{(f^2+g^2)^2}, \quad H = -\frac{(f^2+g^2)\xi + \mu\sin\theta\cos\theta + g^2\left(\frac{f}{g}\right)_s}{2(f^2+g^2)^{3/2}} \tag{3.13}$$

respectively. From (3.13), it follows that $K = 0$ if and only if $\tau = 0$ or $\sin\theta = 0$. This result coincides with Proposition 3.1.

It is clear that if $\tau = 0$, then $H = 0$ and the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is minimal. If $\tau \neq 0$ and $\sin\theta = 0$, then from (3.13) we get $H = \frac{\tau}{2u\eta} \neq 0$ Therefore, in this case, the tangent surface $\varphi_{(\alpha,T)}$ cannot be minimal. Then, followings are obtained:

**Theorem 3.8.** *(i) The OT-ruled surface $\varphi_{(\alpha,q_o)}$ is minimal if and only if the equality*

$$(f^2+g^2)\xi + \mu\sin\theta\cos\theta + g^2\left(\frac{f}{g}\right)_s = 0$$

*satisfies.*

*(ii) If $\tau \neq 0$, there is no minimal tangent surface $\varphi_{(\alpha,T)}$.*

*(iii) The principal normal surface $\varphi_{(\alpha,N)}$ is minimal if and only if $fu\mu' - gf_s = 0$, where $f_s = \partial f/\partial s$.*

Furthermore, considering Catalan Theorem, we have the following corollary:

**Corollary 3.2.** *If the base curve $\alpha$ is not a plane curve, the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is a helicoid if and only if $(f^2+g^2)\xi + \mu\sin\theta\cos\theta + g^2\left(\frac{f}{g}\right)_s = 0$ holds.*

Now, we will consider the special curves lying on an OT-ruled surface $\varphi_{(\alpha,q_o)}$.

Let us consider the tangent space $T_p\varphi_{(\alpha,q_o)}$ and its base $\left\{\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s}, \frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}\right\}$ at a point $p \in \varphi_{(\alpha,q_o)}$. For any tangent vector $\vec{v}_p \in T_p\varphi_{(\alpha,q_o)}$, the Weingarten map of the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is defined by $S_p = -D_p\vec{v} : T_p\varphi_{(\alpha,q_o)} \to T_{\vec{v}_p}S^2$ where $S^2$ is unit sphere with center origin. Therefore, we have

$$\begin{aligned} S_p\left(\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s}\right) &= -D_{\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s}}\vec{U}(s,u), \\ &= A_1(s,u)\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s} + A_2(s,u)\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}, \end{aligned}$$

and

$$\begin{aligned} S_p\left(\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}\right) &= -D_{\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}}\vec{U}(s,u), \\ &= B_1(s,u)\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s} + B_2(s,u)\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}, \end{aligned}$$

where

$$A_1 = \frac{-1}{\left(f^2+g^2\right)^{3/2}}\left[g^2\left(\frac{f}{g}\right)_s + \left(f^2+g^2\right)\xi\right],$$
$$A_2 = \frac{1}{\left(f^2+g^2\right)^{3/2}}\left[\left(f^2+g^2\right)\left(f\mu+g\eta+\xi\cos\theta\right)+g^2\cos\theta\left(\frac{f}{g}\right)_s\right],$$
$$B_1 = \frac{\mu\sin\theta}{\left(f^2+g^2\right)^{3/2}},\ B_2 = \frac{-\mu\cos\theta\sin\theta}{\left(f^2+g^2\right)^{3/2}}.$$

Thus, the matrix form of the Weingarten map can be given by

$$S_p = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \tag{3.14}$$

From (3.14), one can easily compute the Gaussian curvature $K$ and the mean curvature $H$ by considering the equalities $K = \det(S_p)$ and $H = \frac{1}{2}tr(S_p)$ and the results given in (3.13) are obtained. Moreover, from these results, for the parameter curves, we have the following corollary:

**Corollary 3.3.** *(i) The parameter curves $\vec{\varphi}_{(\alpha,q_o)}(s,u_0)$, ($u_0$ is constant) are lines of curvature if and only if $A_2 = 0$ or equivalently, $\left(f^2+g^2\right)\left(f\mu+g\eta+\xi\cos\theta\right)+g^2\cos\theta\left(\frac{f}{g}\right)_s = 0$ holds.*

*(ii) The parameter curves $\vec{\varphi}_{(\alpha,q_o)}(s_0,u)$, ($s_0$ is constant) are lines of curvature if and only if $B_1 = 0$ or equivalently, the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is a plane or $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$.*

Considering the characteristic equation $\det\left(S_p - \lambda \mathrm{I}\right) = 0$, the principal curvatures of OT-ruled surface $\varphi_{(\alpha,q_o)}$ are obtained as

$$\lambda_{1,2} = \frac{A_1 + B_2 \pm \sqrt{\left(A_1 - B_2\right)^2 + 4A_2B_1}}{2}$$

where $I$ is $2\times 2$ unit matrix. Then, the principal directions are obtained as

$$\vec{e}_1 = \frac{1}{B_1}\left(B_1\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s} + kA_2\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}\right),\ \vec{e}_2 = \frac{1}{mA_2}\left(B_1\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s} + mA_2\frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}\right),$$

where $k, m$ are scalar functions such that

$$\frac{\lambda_1 - A_1}{A_2} = \frac{B_1}{\lambda_1 - A_2} = k,\ \frac{\lambda_2 - A_1}{A_2} = \frac{B_1}{\lambda_2 - A_2} = m.$$

Let now $\beta(t) = \varphi_{(\alpha,q_o)}\left(s(t),u(t)\right)$ be a unit speed curve on $\varphi_{(\alpha,q_o)}$ with arc length parameter $t$ and unit tangent vector $\vec{v}_p \in T_p\varphi_{(\alpha,q_o)}$ at the point $\beta(t_o) = p$ on $\varphi_{(\alpha,q_o)}$. The derivative of $\beta$ with respect to $t$ has the form

$$\dot{\vec{\beta}}(t) = \frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial s}\frac{ds}{dt} + \frac{\partial\vec{\varphi}_{(\alpha,q_o)}}{\partial u}\frac{du}{dt}$$

where $\dot{\vec{\beta}} = \frac{d\vec{\beta}}{dt}$. For this tangent vector, we can write

$$\vec{v}_p = C(s,u)\frac{\partial \vec{\varphi}_{(\alpha,q_o)}}{\partial s} + D(s,u)\frac{\partial \vec{\varphi}_{(\alpha,q_o)}}{\partial u} \tag{3.15}$$

where $C, D$ are smooth functions defined by $C(t) = C\left(s(t), u(t)\right) = \frac{ds}{dt} = \dot{s}$ and $D(t) = D\left(s(t), u(t)\right) = \frac{du}{dt} = \dot{u}$. Substituting (3.3) in (3.15), gives

$$\vec{v}_p = \left(C\cos\theta + D\right)\vec{q}_o + Cg\vec{B} + Cf\vec{r}$$

where $\left(C\cos\theta + D\right)^2 + C^2(f^2+g^2) = 1$. Also, by using the linearity of the Weingarten map, we get

$$S_p(\vec{v}_p) = \left[\cos\theta\left(CA_1 + DB_1\right) + \left(CA_2 + DB_2\right)\right]\vec{q}_o + \left(CA_1 + DB_1\right)\left(g\vec{B} + f\vec{r}\right)$$

and so on, the normal curvature $k_n$ in the direction $\vec{v}_p$ is computed as

$$\begin{aligned} k_n(\vec{v}_p) &= \langle S_p(\vec{v}_p), \vec{v}_p\rangle \\ &= C\left[\left(C\cos\theta + D\right)\left(A_1\cos\theta + A_2\right) + \left(CA_1 + DB_1\right)\left(f^2+g^2\right)\right]. \end{aligned} \tag{3.16}$$

Then, from (3.16), we have the following theorem:

**Theorem 3.9.** *The surface curve $\beta(t) = \varphi_{(\alpha,q_o)}\left(s(t), u(t)\right)$ with unit tangent $\vec{v}_p$ is an asymptotic curve if and only if $\beta(t)$ is a ruling or $\left(C\cos\theta + D\right)\left(A_1\cos\theta + A_2\right) + \left(CA_1 + DB_1\right)\left(f^2+g^2\right) = 0$ holds.*

Similarly, the geodesic curvature $\kappa_g$ and the geodesic torsion $\tau_g$ of the curve $\beta(t) = \varphi_{(\alpha,q_o)}\left(s(t), u(t)\right)$ are computed as

$$\begin{aligned} \kappa_g = \frac{1}{\sqrt{f^2+g^2}} &\Big[\left(C\cos\theta + D\right)\Big(-\left(f^2+g^2\right)\dot{C} \\ &+ C\left(\eta f - \mu g\right)\left(2D + \cos\theta\right) - \frac{1}{2}C\left(f^2+g^2\right)_s\Big) \\ &+ Cg\left(\dot{C}g\cos\theta - Cg\theta'\sin\theta - \mu Cg^2 + fC\eta g + \dot{D}g\right) \\ &+ Cf\left(\dot{C}f\cos\theta - Cf\theta'\sin\theta - \mu fgC + Cf^2\eta + \dot{D}f\right)\Big] \end{aligned}$$

and

$$\tau_g = \sqrt{f^2+g^2}\left[C\left(CA_2 + DB_2\right) - D\left(CA_1 + DB_1\right)\right]$$

respectively. Then, we have the followings:

**Theorem 3.10.** *The surface curve $\beta(t) = \varphi_{(\alpha,q_o)}\left(s(t), u(t)\right)$ with unit tangent $\vec{v}_p$ is a geodesic if and only if*

$$\begin{aligned} &\left(C\cos\theta + D\right)\left(-\left(f^2+g^2\right)\dot{C} + C\left(\eta f - \mu g\right)\left(2D + \cos\theta\right) - \frac{1}{2}C\left(f^2+g^2\right)_s\right) \\ &+ Cg\left(\dot{C}g\cos\theta - Cg\theta'\sin\theta - \mu Cg^2 + fC\eta g + \dot{D}g\right) \\ &+ Cf\left(\dot{C}f\cos\theta - Cf\theta'\sin\theta - \mu fgC + Cf^2\eta + \dot{D}f\right) = 0 \end{aligned}$$

*holds.*

Now, we can investigate some special cases:

*Case 1:* Let $\varphi_{(\alpha,q_o)}$ be $\varphi_{(\alpha,T)}$. Then,

$$\begin{aligned}
k_n &= C^2 u\kappa\tau \\
\kappa_g &= C(C+D)\left[u\kappa' + \kappa(2D+1)\right] + u\kappa\left(\dot{C}D - C\dot{D}\right) + C^2u^2\kappa^3 \\
\tau_g &= C\tau(C+D)
\end{aligned}$$

and for the curve $\beta(t) = \varphi_{(\alpha,T)}(s(t),u(t))$, we have followings:

(i) $\beta(t) = \varphi_{(\alpha,T)}(s(t),u(t))$ is an asymptotic curve if and only if $\beta(t)$ is a ruling or $\alpha$ is a plane curve.

(ii) $\beta(t) = \varphi_{(\alpha,T)}(s(t),u(t))$ is a geodesic if and only if

$$C(C+D)\left[u\kappa' + \kappa(2D+1)\right] + u\kappa\left(\dot{C}D - C\dot{D}\right) + C^2u^2\kappa^3 = 0$$

holds.

(iii) $\beta(t) = \varphi_{(\alpha,T)}(s(t),u(t))$ is a line of curvature if and only if one of the followings holds

(a) $\beta(t)$ is a ruling, (b) $\alpha$ is a plane curve, (c) $s(t) = -u(t) + c$, where $c$ is integration constant.

*Case 2*: Let $\varphi_{(\alpha,q_o)}$ be $\varphi_{(\alpha,N)}$. Then,

$$\begin{aligned}
k_n &= \frac{C\left[C\left(u^2\kappa'\tau + (1-u\kappa)u\tau'\right) + 2D\tau\right]}{\sqrt{(1-u\kappa)^2 + u^2\tau^2}} \\
\kappa_g &= \frac{1}{\sqrt{(1-u\kappa)^2 + u^2\tau^2}}\Big[\dot{C}D\left(-\left((1-u\kappa)^2 + u^2\tau^2\right)\right. \\
&\qquad +2CD\left(\kappa - u\left(\kappa^2+\tau^2\right)\right) - C\left((1-u\kappa)\kappa' + u^2\tau\tau'\right)\Big) \\
&\qquad + Cu\tau\left(-\tau Cu^2\tau^2 + C(1-u\kappa)u\kappa\tau + \dot{D}u\tau\right) \\
&\qquad +C(1-u\kappa)\left(-C(1-u\kappa)u\tau^2 + C\kappa(1-u\kappa)^2 + \dot{D}(1-u\kappa)\right)\Big] \\
\tau_g &= C^2\tau - \frac{D\left[Cu\left(u\kappa'\tau + (1-u\kappa)\tau'\right) - D\tau\right]}{\sqrt{(1-u\kappa)^2 + u^2\tau^2}}
\end{aligned}$$

and for the curve $\beta(t) = \varphi_{(\alpha,N)}(s(t),u(t))$ with unit tangent $\vec{v}_p$, we have followings:

(i) $\beta(t) = \varphi_{(\alpha,N)}(s(t),u(t))$ is an asymptotic curve if and only if $\beta(t)$ is a ruling or

$$C\left(u^2\kappa'\tau + (1-u\kappa)u\tau'\right) + 2D\tau = 0$$

holds.

(ii) $\beta(t) = \varphi_{(\alpha,N)}\left(s(t), u(t)\right)$ is a geodesic if and only if

$$\begin{aligned}
&\dot{C}D\left(-\left((1-u\kappa)^2 + u^2\tau^2\right)\right.\\
&\quad +2CD\left(\kappa - u\left(\kappa^2+\tau^2\right)\right) - C\left((1-u\kappa)\kappa' + u^2\tau\tau'\right)\Big)\\
&\quad + Cu\tau\left(-\tau C u^2\tau^2 + C(1-u\kappa)u\kappa\tau + \dot{D}u\tau\right)\\
&\quad + C(1-u\kappa)\left(-C(1-u\kappa)u\tau^2 + C\kappa(1-u\kappa)^2 + \dot{D}(1-u\kappa)\right) = 0
\end{aligned}$$

holds.

(iii) $\beta(t) = \varphi_{(\alpha,N)}\left(s(t), u(t)\right)$ is a line of curvature if and only if

$$\frac{C^2}{D} = \frac{Cu\left(u\kappa'\tau + (1-u\kappa)\tau'\right) - D\tau}{\tau\sqrt{(1-u\kappa)^2 + u^2\tau^2}}$$

holds.

*Case 3*: Let $s = s_0$ be constant. Then, $C = \dot{s} = 0$ and we get that $\vec{v}_p = \vec{q}_o$, i.e., $\beta(t)$ is a ruling. Then, we have followings:

$$k_n = 0, \;\; \kappa_g = 0, \;\; \tau_g = -\frac{D^2\mu\sin\theta}{f^2+g^2}$$

which give us

(i) All rulings are asymptotic.

(ii) All rulings are geodesic.

(iii) The ruling $\beta(t) = \varphi_{(\alpha,q_o)}\left(s_0, u(t)\right)$is a line of curvature if and only if $\mu\sin\theta = 0$ which suggests that either $\varphi_{(\alpha,q_o)} = \varphi_{(\alpha,T)}$ or $\alpha$ is a plane curve.

*Case 4*: Let $u = u_0$ be constant. Then, $D = \dot{u} = 0$ and we have followings:

$$\begin{aligned}
k_n &= \frac{C^2}{\sqrt{f^2+g^2}}\left[(f\mu + g\eta)\cos\theta - g^2\left(\frac{f}{g}\right)_s - \left(f^2+g^2\right)\xi\right]\\
\kappa_g &= \frac{1}{\sqrt{f^2+g^2}}\left[C\cos\theta\left(-\left(f^2+g^2\right)\dot{C}\right.\right.\\
&\qquad \left. +C\cos\theta(\eta f - \mu g) - \frac{1}{2}C\left(f^2+g^2\right)_s\right)\\
&\qquad + Cg\left(\dot{C}g\cos\theta - Cg\theta'\sin\theta - \mu C g^2 + fC\eta g\right)\\
&\qquad \left. +Cf\left(\dot{C}f\cos\theta - Cf\theta'\sin\theta - \mu f g C + Cf^2\eta\right)\right]\\
\tau_g &= \frac{C^2}{f^2+g^2}\left[\left(f^2+g^2\right)(f\mu + g\eta + \xi\cos\theta) + g^2\cos\theta\left(\frac{f}{g}\right)_s\right]
\end{aligned}$$

(i) The parameter curve $\beta(t) = \varphi_{(\alpha,q_o)}(s(t), u_0)$ is an asymptotic curve if and only if

$$(f\mu + g\eta)\cos\theta - g^2\left(\frac{f}{g}\right)_s - \left(f^2+g^2\right)\xi = 0$$

holds.

(ii) The parameter curve $\beta(t) = \varphi_{(\alpha,q_o)}(s(t), u_0)$is a geodesic if and only if

$$\begin{aligned}&C\cos\theta\left(-\left(f^2+g^2\right)\dot{C} + C\cos\theta\left(\eta f - \mu g\right) - \frac{1}{2}C\left(f^2+g^2\right)_s\right)\\&\quad + Cg\left(\dot{C}g\cos\theta - Cg\theta'\sin\theta - \mu C g^2 + fC\eta g\right)\\&\quad + Cf\left(\dot{C}f\cos\theta - Cf\theta'\sin\theta - \mu fgC + Cf^2\eta\right) = 0\end{aligned}$$

holds.

(iii) The parameter curve $\beta(t) = \varphi_{(\alpha,q_o)}(s(t), u_0)$is a line of curvature if and only if

$$\left(f^2+g^2\right)\left(f\mu + g\eta + \xi\cos\theta\right) + g^2\cos\theta\left(\frac{f}{g}\right)_s = 0$$

holds.

*Case 5*: Let $C = \dot{s}$, $D = \dot{u}$ be non-zero constants. Then, the curve has the parametric form $\beta(t) = \varphi_{(\alpha,q_o)}(c_1t + c_2, d_1t + d_2)$ where $c_i, d_i, (i = 1, 2)$ are constants and we have

$$\begin{aligned}k_n &= C\left[\left(C\cos\theta + D\right)\left(A_1\cos\theta + A_2\right) + \left(CA_1 + DB_1\right)\left(f^2+g^2\right)\right]\\ \kappa_g &= \frac{1}{\sqrt{f^2+g^2}}\left[\left(C\cos\theta + D\right)\left(C\left(\eta f - \mu g\right)\left(2D + \cos\theta\right) - \frac{1}{2}C\left(f^2+g^2\right)_s\right)\right.\\&\qquad\left. + Cg\left(-Cg\theta'\sin\theta - \mu Cg^2 + fC\eta g\right) + Cf\left(-Cf\theta'\sin\theta - \mu fgC + Cf^2\eta\right)\right],\\ \tau_g &= \sqrt{f^2+g^2}\left[C\left(CA_2 + DB_2\right) - D\left(CA_1 + DB_1\right)\right]\end{aligned}$$

which give followings:

(i) $\beta(t) = \varphi_{(\alpha,q_o)}(c_1t + c_2, d_1t + d_2)$ is an asymptotic curve if and only if

$$\left(C\cos\theta + D\right)\left(A_1\cos\theta + A_2\right) + \left(CA_1 + DB_1\right)\left(f^2+g^2\right) = 0$$

holds.

(ii) $\beta(t) = \varphi_{(\alpha,q_o)}(c_1t + c_2, d_1t + d_2)$ is a geodesic if and only if

$$\begin{aligned}&\left(C\cos\theta + D\right)\left(+C\left(\eta f - \mu g\right)\left(2D + \cos\theta\right) - \frac{1}{2}C\left(f^2+g^2\right)_s\right)\\&\quad + Cg\left(-Cg\theta'\sin\theta - \mu Cg^2 + fC\eta g\right) + Cf\left(-Cf\theta'\sin\theta - \mu fgC + Cf^2\eta\right) = 0\end{aligned}$$

holds.

(iii) $\beta(t) = \varphi_{(\alpha,q_o)}(c_1 t + c_2, d_1 t + d_2)$ is a line of curvature if and only if

$$\frac{CA_1 + DB_1}{CA_2 + DB_2} = \text{constant}$$

holds.

Let now consider the Frenet frame of a non-cylindrical OT-ruled surface $\varphi_{(\alpha,q_o)}$. Differentiating the ruling $\vec{q}_o = \cos\theta\vec{T} + \sin\theta\vec{N}$, it follows

$$\vec{q}_o{}' = -\eta\sin\theta\,\vec{T} + \eta\cos\theta\,\vec{N} + \tau\sin\theta\,\vec{B} \tag{3.17}$$

Then, the central normal and central tangent vectors of OT-ruled surface $\varphi_{(\alpha,q_o)}$ are computed as

$$\begin{aligned} \vec{h} &= \frac{1}{\sqrt{\eta^2 + \tau^2\sin^2\theta}}\left(-\eta\sin\theta\,\vec{T} + \eta\cos\theta\,\vec{N} + \tau\sin\theta\,\vec{B}\right) \\ \vec{a} &= \frac{1}{\sqrt{\eta^2 + \tau^2\sin^2\theta}}\left(\tau\sin^2\theta\,\vec{T} - \tau\cos\theta\sin\theta\,\vec{N} + \eta\vec{B}\right) \end{aligned} \tag{3.18}$$

respectively. From the equations (3.17) and (3.18), we have following theorem:

**Theorem 3.11.** *For the OT-ruled surface $\varphi_{(\alpha,q_o)}$ the followings are equivalent:*

*(i) The angle between the vectors $\vec{q}_o$ and $\vec{T}$ is given by $\theta = -\int_0^s \kappa ds$.*

*(ii) The central normal vector $\vec{h}$ coincides with the binormal vector $\vec{B}$ of $\alpha$.*

*(iii) The central tangent vector $\vec{a}$ lies in the osculating plane of $\alpha$.*

*Proof.* Let the angle $\theta$ be given by $\theta = -\int_0^s \kappa ds$. Then, we get $\eta = 0$. Thus, the proof is clear from (3.18). □

**Corollary 3.4.** *Let the angle between the vectors $\vec{q}_o$ and $\vec{T}$ is given by $\theta = -\int_0^s \kappa ds$. Then, $\alpha$ is a general helix if and only if the OT-ruled surface $\varphi_{(\alpha,q_o)}$ is an h-slant ruled surface.*

**Theorem 3.12.** *The Frenet frame $\left\{\vec{q}_o, \vec{h}, \vec{a}\right\}$ of OT-ruled surface $\varphi_{(\alpha,q_o)}$ coincides with the Frenet frame $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ of base curve $\alpha$ if and only if $\varphi_{(\alpha,q_o)}$ is the tangent surface $\varphi_{(\alpha,T)}$ of $\alpha$.*

## 4. Examples

**Example 4.1.** Let consider the general helix curve $\alpha_1$ given by the parametrization

$$\vec{\alpha}_1(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$$

For the required Frenet elements of $\alpha_1$, we obtain

$$\begin{aligned}
\vec{T}(s) &= \left(-\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right),\\
\vec{N}(s) &= \left(-\cos\left(\frac{s}{\sqrt{2}}\right), -\sin\left(\frac{s}{\sqrt{2}}\right), 0\right),\\
\kappa(s) &= \frac{1}{2},\ \ \tau(s) = \frac{1}{2}.
\end{aligned}$$

By choosing $\theta(s) = s$, we get

$$\begin{aligned}
\vec{q}_o(s) = \Bigg(&-\frac{1}{\sqrt{2}}\cos(s)\sin\left(\frac{s}{\sqrt{2}}\right) - \sin(s)\cos\left(\frac{s}{\sqrt{2}}\right),\\
&\frac{1}{\sqrt{2}}\cos(s)\cos\left(\frac{s}{\sqrt{2}}\right) - \sin(s)\sin\left(\frac{s}{\sqrt{2}}\right),\ \frac{1}{\sqrt{2}}\cos(s)\Bigg).
\end{aligned}$$

and the OT-ruled surface $\varphi_{1(\alpha_1,q_o)}$ has the parametrization

$$\begin{aligned}
\vec{\varphi}_{1(\alpha_1,q_o)} = \Bigg(&\cos\left(\frac{s}{\sqrt{2}}\right) + u\left(-\frac{1}{\sqrt{2}}\cos(s)\sin\left(\frac{s}{\sqrt{2}}\right) - \sin(s)\cos\left(\frac{s}{\sqrt{2}}\right)\right),\\
&\sin\left(\frac{s}{\sqrt{2}}\right) + u\left(\frac{1}{\sqrt{2}}\cos(s)\cos\left(\frac{s}{\sqrt{2}}\right) - \sin(s)\sin\left(\frac{s}{\sqrt{2}}\right)\right),\\
&\frac{s}{\sqrt{2}} + \frac{1}{\sqrt{2}}u\cos(s)\Bigg).
\end{aligned}$$

From (2.1), the equation of the striction line of OT-ruled surface $\varphi_{1(\alpha_1,q_o)}$ is given by

$$\begin{aligned}
\vec{c}_1(s) = \Bigg(&\frac{\frac{3\sqrt{2}}{2}\sin(2s)\sin\left(\frac{\sqrt{2}}{2}s\right) - \cos\left(\frac{\sqrt{2}}{2}s\right)\left(5\cos^2(s)+4\right)}{\cos^2(s)-10},\\
&-\frac{\frac{3\sqrt{2}}{2}\sin(2s)\cos\left(\frac{\sqrt{2}}{2}s\right) + \sin\left(\frac{\sqrt{2}}{2}s\right)\left(5\cos^2(s)+4\right)}{\cos^2(s)-10},\\
&\frac{\sqrt{2}\left(s\cos^2(s) - 3\sin(2s) - 10s\right)}{2\left(\cos^2(s)-10\right)}\Bigg).
\end{aligned}$$

The curvatures of $\varphi_{1(\alpha_1,q_o)}$ are computed as $\eta(s) = \frac{3}{2}$, $\xi(s) = \frac{1}{2}\cos(s)$, $\mu(s) = \frac{1}{2}\sin(s)$ and the functions $f$ and $g$ are given by $f(s,u) = \sin(s) + \frac{3}{2}u$, $g(s,u) = \frac{1}{2}u\sin(s)$. The graph of $\varphi_{1(\alpha_1,q_o)}$ for the intervals $s \in [0, 3\pi]$, $u \in [-1, 1]$ is given in Figure 4.1. From Proposition 3.3, the base curve $\alpha_1$ (red) and striction line $c_1$ (blue) intersect at the points $\varphi_{1(\alpha_1,q_o)}(0,0)$, $\varphi_{1(\alpha_1,q_o)}(\pi,0)$, $\varphi_{1(\alpha_1,q_o)}(2\pi,0)$, $\varphi_{1(\alpha_1,q_o)}(3\pi,0)$ which are also singular points of $\varphi_{1(\alpha_1,q_o)}$ and shown with black color in Figure 4.1.

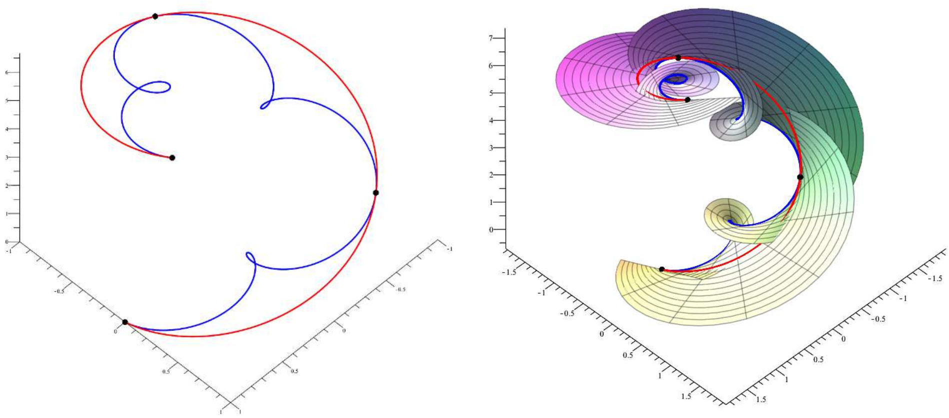

FIG. 4.1: The OT-ruled surface $\varphi_{1(\alpha_1,q_o)}$

**Example 4.2.** Let the curve $\alpha_2$ be given by the parametrization

$$\vec{\alpha}_2(s)=\left(\frac{3}{2}\cos\left(\frac{s}{2}\right)+\frac{1}{6}\cos\left(\frac{3s}{2}\right),\frac{3}{2}\sin\left(\frac{s}{2}\right)+\frac{1}{6}\sin\left(\frac{3s}{2}\right),\sqrt{3}\cos\left(\frac{s}{2}\right)\right)$$

whose required Frenet elements are

$$\begin{aligned}
\vec{T}(s)&=\left(-\frac{3}{4}\sin\left(\frac{s}{2}\right)-\frac{1}{4}\sin\left(\frac{3s}{2}\right),\frac{3}{4}\cos\left(\frac{s}{2}\right)+\frac{1}{4}\cos\left(\frac{3s}{2}\right),-\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right)\right),\\
\vec{N}(s)&=\left(-\frac{\sqrt{3}}{2}\cos(s),-\frac{\sqrt{3}}{2}\sin(s),-\frac{1}{2}\right),\\
\kappa(s)&=\frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right),\ \ \tau(s)=-\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right),
\end{aligned}$$

where we calculate

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'=-\frac{\sqrt{3}}{3}=\text{constant}$$

Therefore, we obtain that $\alpha_2$ is a slant helix. By choosing $\theta(s)=\frac{s}{2}$, we get

$$\begin{aligned}
\vec{q}_o(s)=\Bigg(&-\frac{1}{2}\sin\left(\frac{s}{2}\right)\left(2\cos^2\left(\frac{s}{2}\right)\left(\cos\left(\frac{s}{2}\right)+\sqrt{3}\right)+\cos\left(\frac{s}{2}\right)-\sqrt{3}\right),\\
&\cos\left(\frac{s}{2}\right)\left(\cos^2\left(\frac{s}{2}\right)\left(\sqrt{3}+\cos\left(\frac{s}{2}\right)\right)-\sqrt{3}\right),-\frac{1}{2}\sin\left(\frac{s}{2}\right)\left(\sqrt{3}\cos\left(\frac{s}{2}\right)+1\right)\Bigg).
\end{aligned}$$

Then, the parametrization of the OT-ruled surface $\varphi_{2(\alpha_2,q_o)}$ and its striction line $c_2$ can be written easily by using the equalities (3.1) and (2.1), respectively. The curvatures of that surface are

$$\eta(s)=\frac{1}{2}+\frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right),\ \ \xi(s)=-\frac{\sqrt{3}}{4}\sin(s),\ \ \mu(s)=-\frac{\sqrt{3}}{2}\sin^2\left(\frac{s}{2}\right).$$

Furthermore, the functions $f$ and $g$ are calculated as

$$f(s,u)=\sin\left(\frac{s}{2}\right)+u\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right)\right),\ \ g(s,u)=-\frac{\sqrt{3}}{2}u\sin^2\left(\frac{s}{2}\right).$$

The graph of $\varphi_{2(\alpha_2,q_o)}$ for intervals $s\in[-2\pi,2\pi]$ and $u\in[-1,1]$ is given in Figure 4.2. From Proposition 3.3, the base curve $\alpha_2$ (red) and striction line $c_2$ (blue) intersect at the points

$$p_1=\varphi_{2(\alpha_2,q_o)}(-2\pi,0)=\varphi_{2(\alpha_2,q_o)}(\pi,0),\ \ p_2=\varphi_{2(\alpha_2,q_o)}(0,0)$$
$$p_3=\varphi_{2(\alpha_2,q_o)}\left(2\left(\pi-\arccos\left(\frac{\sqrt{3}}{3}\right)\right),0\right),\ \ p_4=\varphi_{2(\alpha_2,q_o)}\left(2\left(\pi+\arccos\left(\frac{\sqrt{3}}{3}\right)\right),0\right).$$

Here, $p_1,p_2\in S$ are singular points of $\varphi_{2(\alpha_2,q_o)}$ $p_3,p_4\in Y$ are non-singular points which are given black and green in Figure 4.2, respectively.

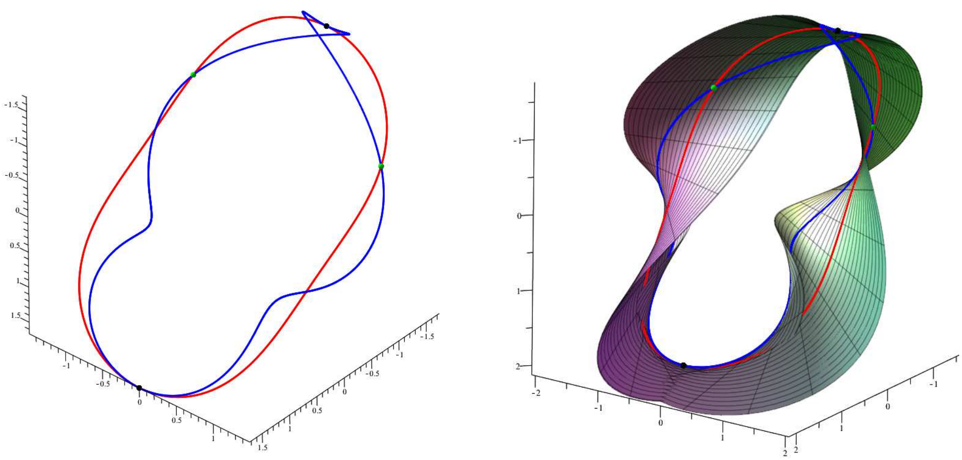

FIG. 4.2: The OT-ruled surface $\varphi_{2(\alpha_2,q_o)}$

**Example 4.3.** Let $\alpha_3$ be given by the parametrization

$$\vec{\alpha}_3(s)=\frac{5\sqrt{26}}{26}\left(\frac{(\sqrt{26}-26)\sin\left(\left(1+\frac{\sqrt{26}}{13}\right)s\right)}{104+8\sqrt{26}}+\frac{(\sqrt{26}+26)\sin\left(\left(1-\frac{\sqrt{26}}{13}\right)s\right)}{-104+8\sqrt{26}}-\frac{1}{2}\sin(s)\ ,\right.$$
$$\frac{(26-\sqrt{26})\cos\left(\left(1+\frac{\sqrt{26}}{13}\right)s\right)}{104+8\sqrt{26}}-\frac{(\sqrt{26}+26)\cos\left(\left(1-\frac{\sqrt{26}}{13}\right)s\right)}{-104+8\sqrt{26}}+\frac{1}{2}\cos(s),$$
$$\left.\frac{5}{4}\cos\left(\frac{\sqrt{26}}{13}s\right)\right)$$

which is a special chosen of general Salkowski curve defined in [10]. The required Frenet elements are

$$\begin{aligned}
\vec{T}(s) = &\left(-\cos(s)\cos\left(\frac{\sqrt{26}}{26}s\right) - \frac{\sqrt{26}}{26}\sin(s)\sin\left(\frac{\sqrt{26}}{26}s\right),\right.\\
&\left.-\sin(s)\cos\left(\frac{\sqrt{26}}{26}s\right) + \frac{\sqrt{26}}{26}\cos(s)\sin\left(\frac{\sqrt{26}}{26}s\right), -\frac{5\sqrt{26}}{26}\sin\left(\frac{\sqrt{26}}{26}s\right)\right)\\
\vec{N}(s) = &\left(\frac{5\sqrt{26}}{26}\sin(s), -\frac{5\sqrt{26}}{26}\cos(s), -\frac{\sqrt{26}}{26}\right),\\
\kappa(s) = &\,1,\ \ \tau(s) = \tan\left(\frac{\sqrt{26}}{26}s\right).
\end{aligned}$$

By choosing $\theta(s) = \frac{s}{\sqrt{26}}$, we get

$$\begin{aligned}
\vec{q}_o(s) = &\left(-\frac{\sqrt{26}}{26}\cos\left(\frac{\sqrt{26}}{26}s\right)\sin(s)\sin\left(\frac{\sqrt{26}}{26}s\right) - \cos(s)\cos^2\left(\frac{\sqrt{26}}{26}s\right) + \frac{5\sqrt{26}}{26}\sin(s)\sin\left(\frac{\sqrt{26}}{26}s\right),\right.\\
&\frac{\sqrt{26}}{26}\cos\left(\frac{\sqrt{26}}{26}s\right)\cos(s)\sin\left(\frac{\sqrt{26}}{26}s\right) - \frac{5\sqrt{26}}{26}\cos(s)\sin\left(\frac{\sqrt{26}}{26}s\right) - \sin(s)\cos^2\left(\frac{\sqrt{26}}{26}s\right),\\
&\left.-\frac{\sqrt{26}}{26}\sin\left(\frac{\sqrt{26}}{26}s\right)\left(5\cos\left(\frac{\sqrt{26}}{26}s\right) + 1\right)\right)
\end{aligned}$$

Then the parametrization of the OT-ruled surface $\varphi_{3(\alpha_3,q_o)}$ and the equation of striction line $c_3$ can be written easily from the equalities (3.1) and (2.1), respectively. This surface has the curvatures

$$\eta(s) = 1 + \frac{\sqrt{26}}{26},\ \ \xi(s) = \sin\left(\frac{\sqrt{26}}{26}s\right),\ \ \mu(s) = \tan\left(\frac{\sqrt{26}}{26}s\right)\sin\left(\frac{\sqrt{26}}{26}s\right),$$

and the functions $f$ and $g$ are calculated as

$$f(s,u) = \sin\left(\frac{\sqrt{26}}{26}s\right) + u\left(1 + \frac{\sqrt{26}}{26}\right),\ \ g(s,u) = u\tan\left(\frac{\sqrt{26}}{26}s\right)\sin\left(\frac{\sqrt{26}}{26}s\right).$$

The graph of $\varphi_{3(\alpha_3,q_o)}$ for intervals $s \in \left[-\frac{\sqrt{26}}{2}\pi, \frac{\sqrt{26}}{2}\pi\right]$ and $u \in [-0.5, 0.5]$ is given in Figure 4.3. Proposition 3.3, the base curve $\alpha_3$ (red) and striction line $c_3$ (blue) intersect at the points $\varphi_{3(\alpha_3,q_o)}\left(-\frac{\sqrt{26}}{2}\pi, 0\right)$, $\varphi_{3(\alpha_3,q_o)}(0,0)$ and $\varphi_{3(\alpha_3,q_o)}\left(\frac{\sqrt{26}}{2}\pi, 0\right)$. All these points are singular points of $\varphi_{3(\alpha_3,q_o)}$ and given by black in Figure 4.3.

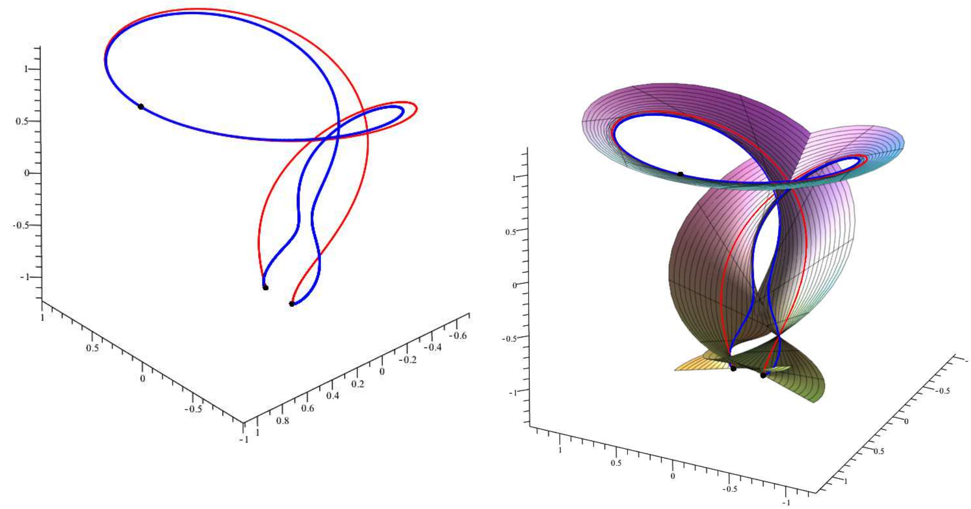

FIG. 4.3: The OT-ruled surface $\varphi_{3(\alpha_3,q_o)}$

## 5. Conclusions

A new type of ruled surfaces has been defined according to the position of the ruling. Taking the ruling on the osculating plane of a curve, these surfaces is defined as osculating type ruled surface or OT-ruled surface. Many properties of such surfaces have been obtained. Of course, this subject can be considered in some other spaces such as Lorentzian space and Galilean space, and properties of OT-ruled surfaces can be given in these spaces according to the characters of base curve and ruling.

## REFERENCES

1. M. Barros: *General helices and a theorem of Lancret.* Proc. Amer. Math. Soc. **125**(5) (1997), 1503–1509.
2. M.P. do Carmo: *Differential Geometry of Curves and Surfaces.* Prentice-Hall, New Jersey, 1976.
3. M. Emmer: *Imagine Math Between Culture and Mathematics.* Springer, 2012 ed. 2012.
4. A.T. Fomenko and A.A. Tuzhilin: *Elements of the geometry and topology of minimal surfaces in three-dimensional space.* American Mathematical Society, Providence, Rhode Island, 2005.

5. S. Izumiya and N. Takeuchi: *New special curves and developable surfaces.* Turkish Journal of Mathematics. **28** (2004), 153–163.
6. S. Izumiya and N. Takeuchi: *Special curves and ruled surfaces.* Beitrage zur Algebra und Geometrie (Contributions to Algebra and Geometry), **44**(1) (2003), 203–212.
7. S. Izumiya and N. Takeuchi: *Geometry of ruled surfaces.* In: Applicable Mathematics in the Golden Age (ed., J.C. Misra), Narosa Publishing House, New Delhi, 2003, pp. 305–338.
8. S. Izumiya, H. Katsumi and T. Yamasaki: *The rectifying developable and the spherical Darboux image of a space curve.* Banach Center Publications, **50** (1999), 137–149.
9. A. Karger, J. Novak: *Space Kinematics and Lie Groups.* STNL Publishers of Technical Lit., Prague, Czechoslovakia, 1978.
10. J. Monterde: *Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion.* Computer Aided Geometric Design, **26** (2009), 271–278.
11. M. Önder: *Slant ruled surfaces.* Transnational Journal of Pure and Applied Mathematics, **1**(1) (2018), 63–82.
12. M. Önder:*Rectifying ruled surfaces.* Kuwait Journal of Science, **47**(4) (2020), 1–11.

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# A STUDY OF THE MATRIX CLASSES $(c_0, c)$ AND $(c_0, c; P)$

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**Abstract.** In this paper, entries of sequences, infinite series and infinite matrices are real or complex numbers. We prove some interesting properties of the matrix classes $(c_0, c)$ and $(c_0, c; P)$.
**Keywords:** First convolution, Banach space, commutative, non-associative algebra, second convolution, groupoid, subgroupoid, ideal.

## 1. Introduction and Preliminaries

We need the following sequence spaces in the sequel:

$$c_0 = \left\{x = \{x_k\} / \lim_{k\to\infty} x_k = 0\right\};$$

$$c = \left\{x = \{x_k\} / \lim_{k\to\infty} x_k \text{ exists}\right\}.$$

We know that $c_0$ and $c$ are Banach spaces under the norm

$$\|x\| = \sup_{k\geq 0} |x_k|, x = \{x_k\} \in c_0 \text{ or } c.$$

Let $A = (a_{nk})$, $n, k = 0, 1, 2, \ldots$ be an infinite matrix. Then we write $A \in (c_0, c)$ if

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n = 0, 1, 2, \ldots$$

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is defined and the sequence $A(x) = \{(Ax)_n\} \in c$, whenever $x = \{x_k\} \in c_0$. $A(x)$ is called the $A$-transform of $x = \{x_k\}$. We write $A \in (c_0, c; P)$ if $A \in (c_0, c)$ and

$$\lim_{n\to\infty} (Ax)_n = \lim_{k\to\infty} x_k = 0, x = \{x_k\} \in c_0.$$

The following results can be easily proved.

**Theorem 1.1.** [2] $A = (a_{nk}) \in (c_0, c)$ *if and only if*

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty; \tag{1.1}$$

*and*

$$\lim_{n\to\infty} a_{nk} = \delta_k \text{ exists}, k = 0, 1, 2, \ldots. \tag{1.2}$$

*Further,* $A \in (c_0, c; P)$ *if and only if (1.1) holds and*

$$\lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, \ldots. \tag{1.3}$$

The following definitions are needed ([1]).

**Definition 1.1.** Given the infinite matrices $A = (a_{nk})$, $B = (b_{nk})$, we define

$$(A * B)_{nk} = \sum_{i=0}^{k} a_{ni} b_{n,k-i}, n, k = 0, 1, 2, \ldots. \tag{1.4}$$

$A * B = ((A * B)_{nk})$ is called the "first convolution" of $A$ and $B$;

$$(A * *B)_{nk} = \frac{1}{k+1} \sum_{i=0}^{k} a_{ni} b_{n,k-i}, n, k = 0, 1, 2, \ldots. \tag{1.5}$$

$A * *B = ((A * *B)_{nk})$ is called the "second convolution" of $A$ and $B$.

## 2. Main Results

We now have

**Theorem 2.1.** $(c_0, c)$ *is a Banach space under the norm*

$$\|A\| = \sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}|, A = (a_{nk}) \in (c_0, c). \tag{2.1}$$

*Proof.* We can check that $\|\cdot\|$, defined by (2.1), is indeed a norm. We will prove that $(c_0, c)$ is complete with respect to the norm defined by (2.1). To this end, let $\{A^{(n)}\}$ be a Cauchy sequence in $(c_0, c)$, where

$$A^{(n)} = (a_{ij}^{(n)}), i, j = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots.$$

Since $\{A^n\}$ is Cauchy, for $\epsilon > 0$, there exists a positive integer $n_0$ such that

$$\|A^{(m)} - A^{(n)}\| < \epsilon, m, n \geq n_0,$$

$$\text{(2.2)} \qquad i.e., \ \sup_{i \geq 0} \sum_{j=0}^{\infty} |a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0.$$

Thus, for all $i, j = 0, 1, 2, \ldots$,

$$\text{(2.3)} \qquad |a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0.$$

So, $\{a_{ij}^{(n)}\}_{n=0}^{\infty}$ is a Cacuhy sequence of real (or complex) numbers. Since the field of real (or complex) numbers is complete,

$$a_{ij}^{(n)} \to a_{ij}, n \to \infty,$$

where $a_{ij}$ is a real (or complex) number, $i, j = 0, 1, 2, \ldots$. Consider the infinite matrix $A = (a_{ij})$. From (2.2), we get, for all $i = 0, 1, 2, \ldots$,

$$\text{(2.4)} \qquad \sum_{j=0}^{J} |a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0, J = 0, 1, 2, \ldots.$$

Now, for all $n \geq n_0$, allowing $m \to \infty$ in (2.4), we get

$$\sum_{j=0}^{J} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, i, J = 0, 1, 2, \ldots,$$

from which we have

$$\sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, i = 0, 1, 2, \ldots,$$

$$\text{(2.5)} \qquad \begin{aligned} &i.e., \quad \sup_{i \geq 0} \sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, \\ &i.e., \quad \|A^{(n)} - A\| \leq \epsilon, n \geq n_0, \\ &i.e., \quad A^{(n)} \to A, n \to \infty. \end{aligned}$$

We now claim that $A \in (c_0, c)$. In view of (2.5),

$$\sup_{i\geq 0}\sum_{j=0}^{\infty}|a_{ij}-a_{ij}^{(n_0)}| \leq \epsilon. \tag{2.6}$$

Since $A^{(n_0)} = (a_{ij}^{(n_0)}) \in (c_0, c)$,

$$\sup_{i\geq 0}\sum_{j=0}^{\infty}|a_{ij}^{(n_0)}| = M < \infty \tag{2.7}$$

and

$$\lim_{i\to\infty} a_{ij}^{(n_0)} = \delta_j^{(n_0)} \text{ exists}, j = 0, 1, 2, \ldots. \tag{2.8}$$

Now, for all $i = 0, 1, 2, \ldots$,

$$\begin{aligned}
\sum_{j=0}^{\infty}|a_{ij}| &= \sum_{j=0}^{\infty}|\{a_{ij}-a_{ij}^{(n_0)}\}+a_{ij}^{(n_0)}| \\
&\leq \sum_{j=0}^{\infty}|a_{ij}-a_{ij}^{(n_0)}|+\sum_{j=0}^{\infty}|a_{ij}^{(n_0)}| \\
&\leq \sup_{i\geq 0}\sum_{j=0}^{\infty}|a_{ij}-a_{ij}^{(n_0)}|+\sup_{i\geq 0}\sum_{j=0}^{\infty}|a_{ij}^{(n_0)}| \\
&\leq \epsilon + M, \text{ using (2.6) and (2.7)} \\
&< \infty,
\end{aligned}$$

so that

$$\sup_{i\geq 0}\sum_{j=0}^{\infty}|a_{ij}| < \infty.$$

Next, we claim that $\{a_{ij}\}_{i=0}^{\infty}$ is a Cauchy sequence of real (or complex) numbers, $j = 0, 1, 2, \ldots$. To this end,

$$\begin{aligned}
|a_{uj}-a_{vj}| &= |\{a_{uj}-a_{uj}^{(n_0)}\}+\{a_{vj}^{(n_0)}-a_{vj}\} \\
&\qquad +\{a_{uj}^{(n_0)}-a_{vj}^{(n_0)}\}| \\
&\leq |a_{uj}-a_{uj}^{(n_0)}|+|a_{vj}^{(n_0)}-a_{vj}| \\
&\qquad +|a_{uj}^{(n_0)}-a_{vj}^{(n_0)}| \\
&\leq 2\epsilon+|a_{uj}^{(n_0)}-a_{vj}^{(n_0)}|, \text{ using (2.6)}. \qquad (2.9)
\end{aligned}$$

Since $\{a_{uj}^{(n_0)}\}_{u=0}^{\infty}$ converges, $A^{(n_0)} \in (c_0, c)$, it is a Cauchy sequence and so, for $\epsilon > 0$, there exists a positive integer $L$ such that

$$|a_{uj}^{(n_0)}-a_{vj}^{(n_0)}| < \epsilon, u, v \geq L. \tag{2.10}$$

In view of (2.9) and (2.10), we have

$$|a_{uj} - a_{vj}| < 2\epsilon + \epsilon, u, v \geq L.$$

Consequently, $\{a_{ij}\}_{i=0}^{\infty}$ is a Cauchy sequence of real (or complex) numbers and so it converges, i.e.,

$$\lim_{i \to \infty} a_{ij} \quad \text{exists}, j = 0, 1, 2, \ldots.$$

Hence $A = (a_{ij}) \in (c_0, c)$, completing the proof of the theorem. □

**Theorem 2.2.** $(c_0, c)$ *is a commutative Banach algebra with identity under the first convolution* $*$.

*Proof.* It suffices to prove closure under $*$ and the submultiplicative property of the norm. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c)$ and $C = (c_{nk}) = A * B$. Now, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} c_{nk} &= (A * B)_{nk} \\ &= \sum_{i=0}^{k} a_{ni} b_{n,k-i} \\ &\to \sum_{i=0}^{k} a_i b_{k-i}, n \to \infty, \end{aligned}$$

where, $\lim\limits_{n\to\infty} a_{nk} = a_k$, $\lim\limits_{n\to\infty} b_{nk} = b_k$, $k = 0, 1, 2, \ldots$.

For $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \sum_{k=0}^{\infty} |c_{nk}| &= \sum_{k=0}^{\infty} \left| \sum_{i=0}^{k} a_{ni} b_{n,k-i} \right| \\ &\leq \sum_{k=0}^{\infty} \sum_{i=0}^{k} |a_{ni}| |b_{n,k-i}| \\ &= \left( \sum_{k=0}^{\infty} |a_{nk}| \right) \left( \sum_{k=0}^{\infty} |b_{nk}| \right) \\ &\leq \left( \sup_{n \geq 0} \sum_{k=0}^{\infty} |a_{nk}| \right) \left( \sup_{n \geq 0} \sum_{k=0}^{\infty} |b_{nk}| \right) \\ &= \|A\| \|B\|, \end{aligned}$$

so that

$$\sup_{n \geq 0} \sum_{k=0}^{\infty} |c_{nk}| \leq \|A\| \|B\|,$$

$$i.e., \|A * B\| \leq \|A\| \|B\|,$$

completing the proof of the theorem. □

**Theorem 2.3.** *$(c_0, c)$ is a Banach space, which is a commutative, non-associative algebra without identity, under the second convolution $**$, with norm defined by (2.1).*

*Proof.* Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c)$. Then

$$(A**B)_{nk} = \frac{1}{k+1}\sum_{i=0}^{k} a_{ni}b_{n,k-i}, \text{ by (1.5).}$$

We first claim that $(c_0, c)$ is closed under the second convolution $**$. For $k = 0, 1, 2, \ldots$,

$$(A**B)_{nk} \to \frac{1}{k+1}\sum_{i=0}^{k} a_i b_{k-i}, n \to \infty,$$

where $\lim_{n\to\infty} a_{nk} = a_k$, $\lim_{n\to\infty} b_{nk} = b_k$, $k = 0, 1, 2, \ldots$.
Also, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
\sum_{k=0}^{\infty} |(A**B)_{nk}| &\leq \sum_{k=0}^{\infty}\sum_{i=0}^{k} |a_{ni}||b_{n,k-i}| \\
&= \left(\sum_{k=0}^{\infty} |a_{nk}|\right)\left(\sum_{k=0}^{\infty} |b_{nk}|\right) \\
&\leq \|A\|\|B\|.
\end{aligned}$$

Thus,

$$\sup_{n\geq 0}\left(\sum_{k=0}^{\infty} |(A**B)_{nk}|\right) \leq \|A\|\|B\|,$$

so that $A**B \in (c_0, c)$ and

$$\|A**B\| \leq \|A\|\|B\|.$$

Commutativity can be easily checked. Non-associativity can be established as follows: Let

$$A = B = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Note that $A, B, C \in (c_0, c)$, using Theorem 1.1. Simple computation shows that

$$((A**B)**C)_{11} = \frac{1}{2}$$

and

$$(A * *(B * *C))_{11} = \frac{1}{4},$$

which proves that

$$(A * *B) * *C \neq A * *(B * *C),$$

i.e., $(c_0, c)$ is non-associative. Again $(c_0, c)$ does not have an identity under $**$. Suppose an identity $E = (e_{nk})$ exists. Then

$$A * *E = A, \text{ for all } A = (a_{nk}) \in (c_0, c).$$

Consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in (c_0, c).$$

Simple computation shows that

$$e_{11} = 1. \tag{2.11}$$

Again, consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in (c_0, c).$$

Again, simple computation shows that

$$e_{11} = 0. \tag{2.12}$$

(2.11) and (2.12) lead to a contradiction, proving that $(c_0, c)$ has no identity. By Theorem 2.1, $(c_0, c)$ is a Banach space under the norm defined by (2.1). This completes the proof of the theorem. □

As noted in ([1], p. 183), the set $S$ of all infinite matrices is a groupoid under the second convolution $**$, i.e., $S$ is closed under $**$. Also $S$ is commutative, non-associative and $S$ has no identity. We now have

**Theorem 2.4.** *$(c_0, c; P)$ is a subgroupoid of $S$ under the second convolution $**$.*

*Proof.* Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c; P)$. Let $C = (c_{nk}) = A * *B$. We already know that $A * *B \in (c_0, c)$.
Now,

$$\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} b_{nk} = 0, k = 0, 1, 2, \dots.$$

$$\begin{aligned} c_{nk} &= \frac{1}{k+1}\left[a_{n0}b_{nk} + a_{n1}b_{n,k-1} + \dots + a_{nk}b_{n0}\right] \\ &\to 0, n \to \infty, k = 0, 1, 2, \dots. \end{aligned}$$

Thus, $A * *B \in (c_0, c; P)$, completing the proof. □

Let $(c_0, c)'$ denote the subclass of $(c_0, c)$ consisting of all $A = (a_{nk}) \in (c_0, c)$ such that

$$a_{nk} \to 0, k \to \infty, n = 0, 1, 2, \ldots.$$

**Theorem 2.5.** *$(c_0, c)'$ is an ideal of $(c_0, c)$ under the second convolution $**$.*

*Proof.* Let $A = (a_{nk}) \in (c_0, c)$ and $B = (b_{nk}) \in (c_0, c)'$. We claim that $A * *B \in (c_0, c)'$. We know that $(c_0, c)$ is commutative under the second convolution $**$. We already know that $A * *B \in (c_0, c)$. Now,

$$(A * *B)_{nk} = \frac{1}{k+1}\left(\sum_{i=0}^{k} a_{ni} b_{n,k-i}\right),$$

$$\begin{aligned} |(A * *B)_{nk}| &\leq \frac{1}{k+1}\left(\sum_{i=0}^{k} |a_{ni}||b_{n,k-i}|\right) \\ &\leq \frac{1}{k+1}\|A\|\|B\| \\ &\to 0, k \to \infty, n = 0, 1, 2, \ldots. \end{aligned}$$

Consequently, $A * *B \in (c_0, c)'$, completing the proof. $\square$

## REFERENCES

1. I. J. Maddox: *Elements of Functional Analysis*, Cambridge, 1977.
2. M. Stieglitz and H. Tietz: *Matrix transformationen von Folgenräumen eine Ergebnisübersicht*. Math. Z. **154** (1977), 1–16.

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# ON PROBABILISTIC $(\epsilon, \lambda)$-LOCAL CONTRACTION MAPPINGS AND A SYSTEM OF INTEGRAL EQUATIONS

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**Abstract.** In this paper, we consider the concept of probabilistic $(\epsilon, \lambda)$-local contraction which is a generalization of probabilistic contraction of Sehgal type, and the concept of probabilistic G-metric space, which is a generalization of the Menger probabilistic metric space. Then we prove some new coupled fixed point theorems for uniformly locally contractive mappings on probabilistic metric spaces. Also, we establish some coupled fixed point theorems for contractive mappings in probabilistic G-metric space. The article includes some examples and an application to a system of integral equations which supports of main results.
**Keywords**: Generalized probabilistic metric space, $(\epsilon, \lambda)$-local contraction, Coupled fixed point, Uniformly locally contractive.

## 1. Introduction

In 1942, Menger [9] developed the theory of metric spaces and proposed a generalization of metric spaces called Menger probabilistic metric spaces (briefly, Menger PM-space). After that, the study of contraction mappings defined on probabilistic metric spaces was initiated by Sehgal [15] and Bharucha-Reid [16]. Then different classes of probabilistic contractions have been defined and probabilistic versions of Banach theorem were stated in [6]. Also, Golet and Hedrea [5] discussed local contractions in probabilistic metric spaces, which were formerly introduced by Cain

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and Kasrie [4]. On the other hand, in 2006, Mustafa and Sims [10] introduced a new version of generalized metric spaces, which is called $G$-metric spaces, and proved some of the fixed point theorems in this space (also, see [2, 11]). In 2014, Zhou et al. [19] defined the probabilistic version of $G$-metric spaces and obtained new fixed point results.

In 2004, Ran and Reurings [14] considered a partial order to the metric space $(X, d)$ and discussed the existence and uniqueness of fixed points for contractive conditions and for the comparable elements of $X$. In 2005, Nieto and Rodríguez-López [12] applied this theory for solving ordinary differential equations. After that, Bhaskar and Lakshmikantham [3] defined coupled fixed point and proved some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces. Also, they studied the existence and uniqueness of a solution to a periodic boundary value problem. For more details on coupled, tripled, and n-tupled fixed point theorems in various metric spaces especially in $G$-metric spaces, we refer to [1, 8, 13, 18] and references therein. On the other hand, Samet and Yazidi [17] introduced the notation of partially ordered $\epsilon$-chainable metric spaces and derived new coupled fixed point theorems for uniformly locally contractive mappings on such spaces.

In the following, we give some preliminary definitions which are needed.

**Definition 1.1.** [6] A function $f : (-\infty, +\infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in\mathbb{R}} f(x) = 0$. In addition if $f(0) = 0$, then $f$ is called a distance distribution function. Furthermore, a distance distribution function $f$ satisfying $\lim_{x\to+\infty} f(x) = 1$ is called a Menger distance distribution function. The set of all Menger distance distribution functions is denoted by $D^+$.

**Definition 1.2.** [6] A triangular norm (abbreviated, $t$-norm) is a binary operation $T$ on $[0, 1]$, which satisfies the conditions: (a) $T$ is associative and commutative; (b) $T$ is continuous; (c) $T(a, 1) = a$ for all $a \in [0, 1]$; (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

**Definition 1.3.** [6] A triangular norm $T$ is said to be of H-type (Hadzić type) if a family of functions $\{T^n(t)\}$ is equicontinuous at $t = 1$; that is, for each $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $t > 1 - \delta$ implies that $T^n(t) > 1 - \epsilon$ $(n \geq 1)$, where $T^n : [0, 1] \longrightarrow [0, 1]$ is defined by $T^1(t) = T(t, t)$ and $T^n(t) = T(t, T^{n-1}(t))$ for $n = 2, 3, \cdots$. Obviously, $T^n(t) \leq t$ for all $n \in \mathbb{N}$ and $t \in [0, 1]$.

**Definition 1.4.** [6] A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, F, T)$, where $X$ is a nonempty set, $T$ is a continuous t-norm and $F$ is a mapping from $X^2$ in to $D^+$ such that if $F_{x,y}$ denotes the value of $F$ at the pair $(x, y)$, then the following conditions hold:

**(PM1)** $F_{x,y}(t) = 1$ for all $t > 0$ if only if $x = y$;

**(PM2)** $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;

**(PM3)** $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Note that Definition 1.4 is the probabilistic version of metric spaces. Also, for notions such as convergent and Cauchy sequences, completeness and examples in Menger PM-space, we refer to [6].

**Definition 1.5.** [5] Let $(X, F, T, \preceq)$ be a partially ordered PM-space. The mapping $f : X^2 \to X$ is called an $(\epsilon, \lambda)$-uniformly local contraction with a constant $k \in (0, 1)$, if $\frac{1}{2}(F_{x,u}(\epsilon) + F_{y,v}(\epsilon)) \geq 1 - \lambda$ for all $t, \epsilon > 0$ and $\lambda \in (0, 1)$ implies that $F_{f(x,y),f(u,v)}(t) \geq \frac{1}{2}(F_{x,u}(\frac{t}{k}) + F_{y,v}(\frac{t}{k}))$ for all $x \succeq u$ and $y \preceq v$.

Under the conditions of Definition 1.5, the set $X$ is called $(\epsilon, \lambda)$-chainable if for all $x, y \in X$ with $x \preceq y$, there exists a finite sequence $x = x_0 \preceq x_1 \preceq \cdots \preceq x_n = y$ such that $F_{x_{i+1},x_i}(\epsilon) > 1 - \lambda$ for $i = 0, 1, \cdots, n - 1$. Also, the finite sequence $x = x_0 \preceq x_1 \preceq \cdots \preceq x_n = y$ is called $(\epsilon, \lambda)$-chain joining $x$ and $y$.

**Definition 1.6.** [19] A Menger probabilistic $G$-metric space (shortly, PGM-space) is a triple $(X, G, T)$, where $X$ is a nonempty set, $T$ is a continuous $t$-norm and $G$ is a mapping from $X^3$ into $D^+$ ($G_{x,y,z}$ denotes the value of $G$ at the point $(x, y, z)$) satisfying the following conditions:

**(PG1)** $G_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;

**(PG2)** $G_{x,x,y}(t) \geq G_{x,y,z}(t)$ for all $x, y \in X$ with $z \neq y$ and $t > 0$;

**(PG3)** $G_{x,y,z}(t) = G_{x,z,y}(t) = G_{y,x,z}(t) = \cdots$ (symmetry in all three variables);

**(PG4)** $G_{x,y,z}(t + s) \geq T(G_{x,a,a}(s), G_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t \geq 0$.

Note that Definition 1.6 is the probabilistic version of generalized metric spaces. Also, for notions such as convergent and Cauchy sequences, completeness, and examples in Menger PGM-space, we refer to [19].

**Definition 1.7.** [19] Let $(X, G, T)$ be a PGM-space and $x_0 \in X$. For any $\epsilon > 0$ and $\delta$ with $0 < \delta < 1$, an $(\epsilon, \delta)$-neighborhood of $x_0$ is the set of all $y \in X$ which $G_{x_0,y,y}(\epsilon) > 1 - \delta$ and $G_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G_{x_0,y,y}(\epsilon) > 1 - \delta, G_{y,x_0,x_0}(\epsilon) > 1 - \delta\}.$$

This means that $N_{x_0}(\epsilon, \delta)$ is the set of all points $y$ in $X$ for which the probability of the distance from $x_0$ to $y$ being less than $\epsilon$ is greater than $1 - \delta$.

**Definition 1.8.** [3] Let $(X, \preceq)$ be a partially ordered set. The mapping $f : X^2 \to X$ is said to be have the mixed monotone property if $f$ is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $f(x_1, y) \preceq f(x_2, y)$ for each $y \in X$, and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $f(x, y_1) \succeq f(x, y_2)$ for each $x \in X$.

**Definition 1.9.** [7] Let $(X, \preceq)$ be an ordered partial metric space. If relation $\sqsubseteq$ is defined on $X^2$ by $(x, y) \sqsubseteq (u, v)$ iff $x \preceq u$ and $y \succeq v$, then $(X^2, \sqsubseteq)$ is an ordered partial metric space.

## 2. Coupled Fixed Point Theorems on Local Contractions in Menger PM-space

In this section, we prove some new coupled fixed point theorems for uniformly locally contractive mappings on probabilistic metric spaces.

**Theorem 2.1.** *Let $(X, F, T, \preceq)$ be a partially ordered complete Menger PM-space with $T$ of Hadzić-type and $f : X^2 \to X$ be a mapping having the mixed monotone property on $X$. Also, suppose that the following conditions are hold:*

1. *$X$ is $(\epsilon, \lambda)$-chainable with respect to the partial order "$\preceq$" on $X$,*
2. *$f$ is continuous,*
3. *$f$ is $(\epsilon, \lambda)$-uniformly locally contractive mapping,*
4. *there exists $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$.*

*Then, $f$ has a coupled fixed point.*

*Proof.* By condition 4, there exists $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$. We define $x_1, y_1 \in X$ as $x_1 = f(x_0, y_0) \succeq x_0$ and $y_1 = f(y_0, x_0) \preceq y_0$. Let $x_2 = f(x_1, y_1)$ and $y_2 = f(y_1, x_1)$. Then we obtain

$$
\begin{aligned}
f^2(x_0, y_0) &= f(f(x_0, y_0), f(y_0, x_0)) = f(x_1, y_1) = x_2, \\
f^2(y_0, x_0) &= f(f(y_0, x_0), f(x_0, y_0)) = f(y_1, x_1) = y_2.
\end{aligned}
$$

Now, the mixed monotone property of $f$ implies that

$$
\begin{aligned}
x_2 &= f^2(x_0, y_0) = f(x_1, y_1) \succeq f(x_0, y_0) = x_1 \succeq x_0, \\
y_2 &= f^2(y_0, x_0) = f(y_1, x_1) \preceq f(y_0, x_0) = y_1 \preceq y_0.
\end{aligned}
$$

Continuing the above procedure, we have

$$
\begin{aligned}
& x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n+1} \preceq \cdots \\
& y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_{n+1} \succeq \cdots
\end{aligned}
$$

for all $n \geq 0$, where

$$
\begin{aligned}
x_{n+1} &= f^{n+1}(x_0, y_0) = f(f^n(x_0, y_0), f^n(y_0, x_0)), \\
y_{n+1} &= f^{n+1}(y_0, x_0) = f(f^n(y_0, x_0), f^n(x_0, y_0)).
\end{aligned}
$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $f$ has a coupled fixed point. Otherwise, let $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$; that is, we assume that either $x_{n+1} = f(x_n, y_n) \neq x_n$ or

$y_{n+1}=f(y_n,x_n)\neq y_n$. Since $X$ is $\epsilon$-chainable, there exists $\alpha_0,\alpha_1,\cdots,\alpha_n\in X$ and $\beta_0,\beta_1,\cdots,\beta_n\in X$ such that

$$x_i=\alpha_0\preceq\alpha_1\preceq\cdots\preceq\alpha_n=x_{i+1}$$
$$y_i=\beta_0\succeq\beta_1\succeq\cdots\succeq\beta_n=y_{i+1}$$

for all $i=1,2,\cdots,n$. Hence, we have $F_{x_i,x_{i+1}}(\epsilon)\geq 1-\lambda$ and $F_{y_i,y_{i+1}}(\epsilon)\geq 1-\lambda$. Using condition 3, we have

$$F_{f(x_i,y_i),f(x_{i+1},y_{i+1})}(t)\geq\frac{1}{2}(F_{x_i,x_{i+1}}(\frac{t}{k})+F_{y_i,y_{i+1}}(\frac{t}{k})).$$

Now, for all $i\geq 0$, one can show by induction that

$$F_{f(x_i,y_i),f(x_{i+1},y_{i+1})}(t)=F_{x_i,x_{i+1}}(t)\geq\frac{1}{2}(F_{x_1,x_0}(\frac{t}{k^i})+F_{y_1,y_0}(\frac{t}{k^i})),$$
$$F_{f(y_i,x_i),f(y_{i+1},x_{i+1})}(t)=F_{y_i,y_{i+1}}(t)\geq\frac{1}{2}(F_{y_1,y_0}(\frac{t}{k^i})+F_{x_1,x_0}(\frac{t}{k^i})).$$

Hence, we have $\frac{1}{2}(F_{x_1,x_0}(\frac{t}{k^i})+F_{y_1,y_0}(\frac{t}{k^i}))\to 1$ and $\frac{1}{2}(F_{y_1,y_0}(\frac{t}{k^i})+F_{x_1,x_0}(\frac{t}{k^i}))\to 1$ as $i\to\infty$, so

$$F_{x_i,x_{i+1}}(t)\geq 1-\lambda \quad\text{and}\quad F_{y_i,y_{i+1}}(t)\geq 1-\lambda \tag{2.1}$$

for all $i\in\mathbb{N}$ and any $t>0$. Now, we show by induction that for any $k\geq 0$, $n\geq 1$ and $t>0$,

$$F_{x_n,x_{n+k}}(t)\geq T^k(F_{x_n,x_{n+1}}(t-\lambda t)). \tag{2.2}$$

For $k=0$, since $T(a,b)$ is a real number, $T^0(a,b)=1$ for all $a,b\in[0,1]$. Hence, $F_{x_n,x_n}(t)=T^0(F_{x_n,x_{n+1}}(t-\lambda t))=1$, which implies that (2.2) holds for $k=0$. Assume that (2.2) holds for some $k\geq 1$. Then, since $T$ is monotone, it follows from **(PM3)** that

$$\begin{aligned}
F_{x_n,x_{n+k+1}}(t) &= F_{x_n,x_{n+k+1}}(t-\lambda t+\lambda t)\\
&\geq T(F_{x_n,x_{n+1}}(t-\lambda t),F_{x_{n+1},x_{n+k+1}}(\lambda t))\\
&\geq T(F_{x_n,x_{n+1}}(t-\lambda t),F_{x_n,x_{n+k}}(\lambda t))\\
&\geq T(F_{x_n,x_{n+1}}(t-\lambda t),T^k(F_{x_n,x_{n+1}}(t-\lambda t)))\\
&= T^{k+1}(F_{x_n,x_{n+1}}(t-\lambda t)). \qquad (2.3)
\end{aligned}$$

Thus, (2.2) is hold. Now, we show that $\{x_n\}$ is a Cauchy sequence in $X$, i.e., $\lim_{m,n\to\infty}F_{x_n,x_m}(t)=1$ for any $t>0$. To this end, by hypothesis of the $t$-norm $T$ is $H$-type we have $\{T^n:n\geq 1\}$ is equicontinuous at 1; that is, there exists $\delta>0$ such that

$$T^n(a)\geq 1-\epsilon \tag{2.4}$$

for all $n\geq 1$ and any $a\in(1-\delta,1]$. On the other hand, it follows from (2.1) that $\lim_{n\to\infty}F_{x_n,x_{n+1}}(t-\lambda t)=1$. Hence, there exists $n_0\in\mathbb{N}$ such that $F_{x_n,x_{n+1}}(t-\lambda t)\in(1-\delta,1]$ for all $n\geq n_0$. By (2.3) and (2.4), we conclude that $F_{x_n,x_{n+k}}(t)>1-\epsilon$

for any $k \geq 1$. This shows $\lim_{n,m\to\infty} F_{x_n,x_m}(t) = 1$ for any $t > 0$; that is $\{x_n\}$ is a Cauchy sequence in $X$. Similarly, $\{y_n\}$ is a Cauchy sequence. Since $X$ is a complete space, there exists $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Now, since $x_{n+1} = f(x_n, y_n)$ and $f$ is continuous, and by taking the limit as $n \to \infty$, we have $f(x, y) = x$. Similarly, $f(y, x) = y$. Thus, $(x, y)$ is a coupled fixed point of $f$. $\square$

**Example 2.1.** Let $X = [0, \infty)$, "$\preceq$" be a partially ordered on $X$ (note that we consider the same ordinary order on real numbers) and $T(a, b) = \min\{a, b\}$. Define $F : X^2 \to D^+$ by $F_{x,y}(t) = 1$ if $x = y$ and otherwise, $F_{x,y}(t) = \exp(-t)$. Clearly, $F$ satisfies in **(PM1)-(PM4)**. Define the mapping $f : X^2 \to X$ by $f(a, b) = ab$. We have

$$F_{f(x,y),f(u,v)}(t) \geq \frac{1}{2}(F_{x,u}(\frac{t}{k}) + F_{y,v}(\frac{t}{k}))$$

for $k \in (0, 1)$. Therefore, $f$ is $(\epsilon, \lambda)$-uniformly locally contractive mapping. Also, $f$ is continuous, $[0, \infty)$ is $(\epsilon, \lambda)$-chainable, and there exists $x_0 = 0$ and $y_0 = 1$ such that $0 = x_0 \preceq f(x_0, y_0) = x_0y_0$ and $1 = y_0 \succeq f(y_0, x_0) = y_0x_0$. Therefore, all the hypothesis of Theorem 2.1 are satisfied and $f$ has a coupled fixed point.

**Theorem 2.2.** *Suppose that the assumptions of Theorem* 2.1 *is true. If we replace the assumption the continuity of $f$ by the following conditions:*

1. *if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preceq x$ for all $n$,*
2. *if a non-increasing sequence $\{y_n\}$ converges to $y \in X$, then $y_n \succeq y$ for all $n$,*

*then $f$ has a coupled fixed point.*

*Proof.* As in the proof of Theorem 2.1, we construct $\{x_n\}$ and $\{y_n\}$. Then, by conditions 1 and 2, we have $x_n \preceq x$ and $y_n \succeq y$ for all $n \geq 0$. Let $x_n = x$ and $y_n = y$ for some $n$. Then, due to the structure of both sequences, we have $x_{n+1} = x$ and $y_{n+1} = y$. Hence, $(x, y)$ is a coupled fixed point. Now, we assume either $x_n \neq x$ or $y_n \neq y$. Since $x_n \to x$ and $y_n \to y$, for given $\epsilon_1, \epsilon_2, \lambda_1, \lambda_2 > 0$, there exists $k_1, k_2 \in \mathbb{N}$ such that $F_{x_{n_1},x}(\epsilon_1) \geq 1-\lambda_1$ and $F_{y_{n_2},y}(\epsilon_2) \geq 1-\lambda_2$ for all $n_1 \geq k_1$ and $n_2 \geq k_2$, respectively. Let $k = \max\{k_1, k_2\}$, $\lambda = \max\{\lambda_1, \lambda_2\}$ and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Then, by conditions 1 and 2, we have $\frac{1}{2}(F_{x_n,x}(\epsilon) + F_{y_n,y}(\epsilon)) \geq 1 - \lambda$ for all $n \geq k$. Since $f$ is $(\epsilon, \lambda)$-uniformly locally contractive, by conditions 1 and 2, we have

$$F_{f(x_n,y_n),f(x,y)}(t) \geq \frac{1}{2}(F_{x_n,x}(\frac{t}{k}) + F_{y_n,y}(\frac{t}{k})).$$

Now, by letting $n \to \infty$ by $x_{n+1} = f(x_n, y_n)$, we have $x = f(x, y)$. Similarly, one can show that $y = f(y, x)$. This completes the proof. $\square$

**Theorem 2.3.** *Adding the following property to the hypotheses of Theorem* 2.1 *(Theorem* 2.2*). Then the coupled fixed point of $f$ is unique.*

*$(H)$ for all $(x, y), (x_1, y_1) \in X^2$, there exists $(z_1, z_2) \in X^2$ such that is comparable with $(x, y)$ and $(x_1, y_1)$.*

*Proof.* Let $(x_1, y_1)$ be another coupled fixed point of $f$. We consider two cases.

Case 1. suppose that $(x, y)$ and $(x_1, y_1)$ are comparable with respect to the partial ordering $\sqsubseteq$ in $X^2$. Without loss of the generality, we can assume that $x \preceq x_1$ and $y \succeq y_1$. Applying the procedure of Theorem 2.1, by $X$ is $(\epsilon, \lambda)$-chainable, we have $F_{x,x_1}(\epsilon) \geq 1 - \lambda$ and $F_{y,y_1}(\epsilon) \geq 1 - \lambda$. Since $f$ is $(\epsilon, \lambda)$-uniformly locally contractive, we have

$$F_{f^n(x,y),f^n(x_1,y_1)}(t) \geq \frac{1}{2}(F_{x,x_1}(\frac{t}{k^n}) + F_{y,y_1}(\frac{t}{k^n}))$$

for all $n \in \mathbb{N}$. Now, by letting $n \to \infty$, we have $x = x_1$. Similarly, $y = y_1$.

Case 2. assume that $(x, y)$ and $(x_1, y_1)$ are not comparable. From $(H)$, there exists $(z_1, z_2) \in X^2$ that is comparable to $(x, y)$ and $(x_1, y_1)$. Without loss of the generality, we can suppose that $x \preceq z_1$ , $y \succeq z_2$, $x_1 \preceq z_1$ and $y_1 \succeq z_2$. Similar to the Case 1, we have

$$F_{f^n(x,y),f^n(z_1,z_2)}(t) \geq \frac{1}{2}(F_{x,z_1}(\frac{t}{k^n}) + F_{y,z_2}(\frac{t}{k^n})),$$

which by letting $n \to \infty$ implies that $\lim_{n\to\infty} f^n(x, y) = \lim_{n\to\infty} f^n(z_1, z_2)$. Similarly, we have $\lim_{n\to\infty} f^n(y, x) = \lim_{n\to\infty} f^n(z_2, z_1)$, $\lim_{n\to\infty} f^n(x_1, y_1) = \lim_{n\to\infty} f^n(z_1, z_2)$ and $\lim_{n\to\infty} f^n(y_1, x_1) = \lim_{n\to\infty} f^n(z_2, z_1)$. Thus, we obtain $F_{x,x_1}(t) = F_{f^n(x,y),f^n(x_1,y_1)}(t)$ and $F_{y,y_1}(t) = F_{f^n(y,x),f^n(y_1,x_1)}(t)$, which by letting $n \to \infty$ implies that $x = x_1$ and $y = y_1$.

Consequently, the coupled fixed point of $f$ is unique in both cases. $\square$

**Theorem 2.4.** *In addition of the hypotheses of Theorem* 2.1 (*Theorem* 2.2), *suppose that every pair of elements of $X$ has an upper or a lower bound in $X$. Then $x = y$.*

*Proof.* Case 1. suppose that $x$ and $y$ are comparable. Without loss of the generality, we can assume that $x \preceq y$ and $y \succeq y$. Then similar to the proof of Theorem 2.3, we have $x = y$

Case 2. suppose $x$ is not comparable to $y$. Then, there exists an upper bound or lower bound of $x$ and $y$; that is, there exists $z \in X$ comparable with $x$ and $y$. For example, we can suppose that $x \preceq z$ and $y \succeq z$. Similar to the proof of Theorem 2.3, we have $(x, y) = (z, z)$. Thus, $x = y$. $\square$

## 3. Coupled Fixed Point Theorems in Menger PGM-spaces

In this section, we establish some coupled fixed point theorems in probabilistic $G$-metric spaces.

**Theorem 3.1.** *Let $(X, G, T, \preceq)$ be a partially ordered complete Menger PGM-space with $T$ of Hadzić-type and $f : X^2 \to X$ be a continuous mapping having the mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$G_{f(x,y),f(u,v),f(w,z)}(t) \geq \frac{1}{2}(G_{x,u,w}(\frac{t}{k}) + G_{y,v,z}(\frac{t}{k})) \tag{3.1}$$

*for all $x, y, z, u, v, w \in X$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$, then $f$ has a coupled fixed point in $X$.*

*Proof.* Construct $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.1. If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $f$ has a coupled fixed point. Otherwise, let $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$; that is, we assume that either $x_{n+1} = f(x_n, y_n) \neq x_n$ or $y_{n+1} = f(y_n, x_n) \neq y_n$. Now, one can show by induction that

$$G_{x_{n+1},x_{n+1},x_n}(t) \geq \frac{1}{2}(G_{x_1,x_1,x_0}(\frac{t}{k^n}) + G_{y_1,y_1,y_0}(\frac{t}{k^n})),$$
$$G_{y_{n+1},y_{n+1},y_n}(t) \geq \frac{1}{2}(G_{y_1,y_1,y_0}(\frac{t}{k^n}) + G_{x_1,x_1,x_0}(\frac{t}{k^n}))$$

for all $n \geq 0$. Since $X$ is a Menger PGM-space, we have

$$\lim_{n\to\infty} G_{x_1,x_1,x_0}(\frac{t}{k^n}) = 1 \quad \text{and} \quad \lim_{n\to\infty} G_{y_1,y_1,y_0}(\frac{t}{k^n}) = 1, \tag{3.2}$$

which imply that

$$\lim_{n\to\infty} G_{x_{n+1},x_{n+1},x_n}(t) = 1 \quad \text{and} \quad \lim_{n\to\infty} G_{y_{n+1},y_{n+1},y_n}(t) = 1$$

for any $t > 0$. Now, by induction, we show that for any $k \geq 0$, $n \geq 1$ and $t > 0$,

$$G_{x_n,x_{n+k},x_{n+k}}(t) \geq T^k(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)). \tag{3.3}$$

For $k = 0$, since $T(a, b)$ is a real number, $T^0(a, b) = 1$ for all $a, b \in [0, 1]$. Hence,

$$G_{x_n,x_n,x_n}(t) \geq T^0(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)),$$

which implies that (3.3) holds for $k = 0$. Assume that (3.3) holds for some $k \geq 1$. Since $T$ is monotone, it follows from **(PG4)** that

$$\begin{aligned}
G_{x_n,x_{n+k+1},x_{n+k+1}}(t) &= G_{x_n,x_{n+k+1},x_{n+k+1}}(t - \lambda t + \lambda t) \\
&\geq T(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), G_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\lambda t)) \\
&\geq T(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), G_{x_n,x_{n+k},x_{n+k}}(t)) \\
&\geq T(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), T^k(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t))) \\
&= T^{k+1}(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)).
\end{aligned}$$

Thus, (3.3) is hold. Now, we show that $\{x_n\}$ is a Cauchy sequence in $X$, i.e., $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$ for all $t > 0$. To this end, we first prove

$$\lim_{n,m\to\infty} G_{x_n,x_m,x_m}(t) = 1$$

for any $t > 0$. By hypothesis of the t-norm T is H-type we have $\{T^n : n \geq 1\}$ is equicontinuous at 1; that is, there exists $\delta > 0$ such that $T^n(a) \geq 1 - \epsilon$ for all $a \in (1-\delta, 1]$, $\epsilon > 0$ and $n \geq 1$. From (3.2), it follows that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t-\lambda t) = 1$. Hence, there exists $n_0 \in \mathbb{N}$ such that $G_{x_n,x_{n+1},x_{n+1}}(t-\lambda t) \in (1-\delta, 1]$ for any $n \geq n_0$. Thus, by (3.2) and (3.3), we conclude that $G_{x_n,x_{n+k},x_{n+k}}(t) > 1 - \epsilon$ for any $k \geq 1$. This shows $\lim_{n,m\to\infty} G_{x_n,x_m,x_m}(t) = 1$ for any $t > 0$, similarly $\lim_{n,l\to\infty} G_{x_n,x_l,x_l}(t) = 1$ for any $t > 0$. By **(PG4)**, we have

$$\begin{aligned}
&G_{x_n,x_m,x_l}(t) \geq T(G_{x_n,x_n,x_m}(\frac{t}{2}), G_{x_n,x_n,x_l}(\frac{t}{2})),\\
&G_{x_n,x_n,x_m}(\frac{t}{2}) \geq T(G_{x_n,x_m,x_m}(\frac{t}{4}), G_{x_n,x_m,x_m}(\frac{t}{4})),\\
&G_{x_n,x_n,x_l}(\frac{t}{2}) \geq T(G_{x_n,x_l,x_l}(\frac{t}{4}), G_{x_n,x_l,x_l}(\frac{t}{4})).
\end{aligned}$$

Therefore, by the continuity of $T$, we conclude that $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$ for any $t > 0$. Hence, $\{x_n\}$ is a Cauchy sequence in $X$. Similarly, $\{y_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Now, we show that $f$ has a coupled fixed point in $X$. From $x_{n+1} = f(x_n, y_n)$, take the limit as $n \to \infty$. Since $f$ is continuous, we have $f(x, y) = x$. Similarly, we have $f(y, x) = y$. $\square$

**Example 3.1.** Consider $X$, "$\preceq$" and $T(a, b)$ as in Example 2.1. Define $G : X^3 \to \mathbb{R}^+$ by

$$G_{x,y,z}(t) = \frac{t}{t + G^*(x, y, z)},$$

where $G^*(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in X$. Clearly, $G$ satisfies in **(PG1)-(PG4)** (see [19]). Define the mapping $f : X^2 \to X$ by $f(x, y) = 1$. Then, for all $t > 0$ and $k \in [0, 1)$, we have

$$G_{f(x,y),f(u,v),f(w,z)}(t) = G_{1,1,1}(t) = 1 \geq \frac{1}{2}(G_{x,u,w}(\frac{t}{k}) + G_{y,v,z}(\frac{t}{k}))$$

for all $x, y, z, u, v, w \in X$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$. Also, there exist $x_0 = 0$ and $y_0 = 1$ such that $0 = x_0 \preceq f(x_0, y_0) = 1$ and $1 = y_0 \succeq f(y_0, x_0) = 1$. Therefore, all the hypothesis of Theorem 3.1 are satisfied. Thus, $f$ has a coupled fixed point.

**Theorem 3.2.** *Assume that the assumptions of Theorem* 3.1 *are hold and replace the assumption the continuity of $f$ by the following conditions:*

1. *if a non-decreasing sequence* $\{x_n\}$ *converges to* $x \in X$*, then* $x_n \preceq x$ *for all* $n$*;*
2. *if a non-increasing sequence* $\{y_n\}$ *converges to* $y \in X$*, then* $y_n \succeq y$ *for all* $n$*.*

*Then* $f$ *has a coupled fixed point.*

*Proof.* As in the proof of Theorem 2.1, we construct $\{x_n\}$ and $\{y_n\}$. Then, by conditions 1 and 2, we have $x_n \preceq x$ and $y_n \succeq y$ for all $n \geq 0$. Let $x_n = x$ and $y_n = y$ for some $n$. Then, due to the structure of both sequences, we have $x_{n+1} = x$ and $y_{n+1} = y$. Hence, $(x, y)$ is a coupled fixed point. Now, we assume that either $x_n \neq x$ or $y_n \neq y$. Then we have

$$\begin{aligned} G_{f(x,y),x,x}(2t) &\geq T(G_{f(x,y),f(x_n,y_n)f(x_n,y_n)}(t), G_{f(x_n,y_n),x,x}(t)) \\ &\geq T(\frac{1}{2}(G_{x,x_n,x_n}(\frac{t}{k}) + G_{y,y_n,y_n}(\frac{t}{k})), G_{x_{n+1},x,x}(t)). \end{aligned}$$

Now, taking $n \to \infty$, we obtain $G_{f(x,y),x,x}(2t) = 1$; that is, $f(x, y) = x$. Similarly, we have $f(y, x) = y$. This completes the proof of the theorem. □

**Theorem 3.3.** *Let* $(X, G, T, \preceq)$ *be a partially ordered complete Menger PGM-space with* $T$ *of Hadzić-type and* $f : X^2 \to X$ *be a continuous mapping having the mixed monotone property on* $X$*, and* $f(x, y) \preceq f(y, x)$ *whenever* $x \preceq y$*. Assume that there exists* $k \in [0, 1)$ *such that*

$$G_{f(x,y),f(u,v),f(w,z)}(t) \geq \frac{1}{2}(G_{x,u,w}(\frac{t}{k}) + G_{y,v,z}(\frac{t}{k}))$$

*for all* $x, y, z, u, v, w \in X$ *with* $x \succeq u \succeq w$ *and* $y \preceq v \preceq z$*, where either* $u \neq w$ *or* $v \neq z$*. If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq y_0$*,* $x_0 \preceq f(x_0, y_0)$ *and* $y_0 \succeq f(y_0, x_0)$*, then* $f$ *has a coupled fixed point in* $X$*.*

*Proof.* By the last assumption of the theorem, there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$. We define $x_1, y_1 \in X$ as $x_1 = f(x_0, y_0) \succeq x_0$ and $y_1 = f(y_0, x_0) \preceq y_0$. Since $x_0 \preceq y_0$ and by another assumption of the theorem, we have $f(x_0, y_0) \preceq f(y_0, x_0)$. Hence, $x_0 \preceq x_1 = f(x_0, y_0) \preceq f(y_0, x_0) = y_1 \preceq y_0$. Continuing the above procedure, we have two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \preceq f(x_n, y_n) = x_{n+1} \preceq y_{n+1} = f(y_n, x_n) \preceq y_n$$

for all $n \geq 0$. Now, if $x_n = y_n = c$ for some $n$, then $c \preceq f(c, c) \preceq f(c, c) \preceq c$. Thus, $c = f(c, c)$ and $(c, c)$ is a coupled fixed point. Hence, we assume that $x_n \preceq y_n$ for all $n \geq 0$. Further, for the same reason as stated in Theorem 3.1, we assume that $(x_n, y_n) \neq (x_{n+1}, y_{n+1})$. Then, for all $n \geq 0$, (3.1) will hold with $x = x_{n+2}, u = x_{n+1}, w = x_n, y = y_n, v = y_{n+1}$ and $z = y_{n+2}$. The rest of the proof is obtained by repeating the same steps as in Theorem 3.1. □

**Theorem 3.4.** *Suppose that the assumptions of Theorem* 3.3 *are true and replace the assumption the continuity of* $f$ *by the following conditions:*

1. *if a non-decreasing sequence* $\{x_n\}$ *converges to* $x \in X$*, then* $x_n \preceq x$ *for all* $n$*;*
2. *if a non-increasing sequence* $\{y_n\}$ *converges to* $y \in X$*, then* $y_n \succeq y$ *for all* $n$*.*

*Then* $f$ *has a coupled fixed point.*

*Proof.* The proof is similar to the proof of Theorem 3.2. □

**Remark 3.1.** (i) All the previous results can be considered if instead "mixed monotone property" we suppose so-called only "monotone property" as in 1 and 2. It is well known that this property has an advantage under the mixed monotone property.

(ii) Some authors think that the notion of coupled fixed point is not still such actual for research. But Soleimani Rad et al. [18] only showed that some of the results in coupled fixed point theory can be obtained from fixed point theory and conversely (also, see [1, 13]).

## 4. Application to a System of Integral Equations

Consider the following system of integral equations:

$$\begin{cases} x(t) = \int_a^b M(t,s)K(s,x(s),y(s))ds \\ \\ y(t) = \int_a^b M(t,s)K(s,y(s),x(s))ds \end{cases} \tag{4.1}$$

for all $t \in I = [a, b]$, where $b > a$, $M \in C(I \times I, [0, \infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Let $C(I, \mathbb{R})$ be the Banach space of all real continuous functions defined on $I$ with the sup norm $||x||_\infty = \max_{t \in I} |x(t)|$ for all $x \in C(I, \mathbb{R})$ and $C(I \times I \times C(I, \mathbb{R}), \mathbb{R})$ be the space of all continuous functions defined on $I \times I \times C(I, \mathbb{R})$ and the induced $G^*$-metric be defined by $G^*(x, y, z) = ||x - y|| + ||x - z|| + ||y - z||$ for all $x, y, z \in C(I, \mathbb{R})$. Now, suppose that $G : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to D^+$ is defined by $G_{x,y,z}(t) = \chi(\frac{t}{2} - G^*(x, y, z))$ for all $x, y, z \in C(I, \mathbb{R})$ and $t > 0$, where

$$\chi(t) = \begin{cases} 0 & \text{if } \ t \le 0, \\ 1 & \text{if } \ t > 0. \end{cases}$$

The space $(C(I, \mathbb{R}), G, T)$ with $T(a, b) = \min\{a, b\}$ is a complete Menger PGM-space. Also, we define the partial order relation " $\preceq$ " on $C(I, \mathbb{R})$ by $x \preceq y$ iff $||x||_\infty \le ||y||_\infty$ for all $x, y \in C(I, \mathbb{R})$. Thus, $(C(I, \mathbb{R}), F, T, \preceq)$ is a partially ordered complete probabilistic $G$-metric space.

**Theorem 4.1.** *Let* $(C(I, \mathbb{R}), G, T, \preceq)$ *be the partially ordered complete probabilistic* $G$*-metric space and* $f : C(I, \mathbb{R}) \times C(I, \mathbb{R}) \to C(I, \mathbb{R})$ *be an operator defined by* $f(x, y)t = \int_a^b M(t, s)K(s, x(s), y(s))ds$*, where* $M \in C(I \times I, [0, \infty))$ *and* $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *are two operators satisfying the following conditions:*

*(i)* $||K||_\infty = \sup\limits_{s \in I,\, x,y \in C(I,\mathbb{R})} |K(s, x(s), y(s))| < \infty$,

*(ii) for all $x, y \in C(I, \mathbb{R})$ and all $t, s \in I$ we have*

$$||K(s, x(s), y(s)) - K(s, u(s), v(s))|| \leq \tfrac{1}{4}(\max|x(s) - u(s)| + \max|y(s) - v(s)|),$$

*(iii)* $\sup\limits_{t \in I} \int_a^b G(t, s)ds < 1.$

*Then, the system of integral equations* (4.1) *has a solution in $C(I, \mathbb{R}) \times C(I, \mathbb{R})$.*

*Proof.* For all $x, y, z \in C(I, \mathbb{R})$, we consider

$$G^*(x, y, z) = \max_{t \in I}(|x(t) - y(t)|) + \max_{t \in I}(|x(t) - z(t)|) + \max_{t \in I}(|y(t) - z(t)|).$$

Therefore, for all $x, y, z, u, v, w \in C(I, \mathbb{R})$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$, we get

$$\begin{aligned}
& G^*(f(x, y), f(u, v), f(w, z)) \\
\leq\ & \max_{t \in I} \int_a^b M(t, s)|K(s, x(s), y(s)) - K(s, u(s), v(s))|ds \\
& + \max_{t \in I} \int_a^b M(t, s)|K(s, x(s), y(s)) - K(s, w(s), z(s))|ds \\
& + \max_{t \in I} \int_a^b M(t, s)|K(s, u(s), v(s)) - K(s, w(s), z(s))|ds \\
\leq\ & \max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|)) \max_{t \in I} \int_a^b M(t, s)ds \\
& + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|)) \max_{t \in I} \int_a^b M(t, s)ds \\
& + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|)) \max_{t \in I} \int_a^b M(t, s)ds \\
\leq\ & \max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|)) \\
& + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|)) \\
& + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|)),
\end{aligned}$$

which implies that

$$\begin{aligned}
G_{f(x,y), f(u,v), f(w,z)}(t) &= \chi(\frac{t}{2} - G^*(f(x, y), f(u, v), f(w, z)) \\
&\geq \chi(\frac{t}{2} - (\max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|)) \\
&\quad + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|))
\end{aligned}$$

$$
\begin{aligned}
& +\max(\frac{1}{4}(|u(s)-w(s)|+|v(s)-z(s)|))) \\
= \;& \chi(\frac{1}{2}(t-\frac{1}{2}(\max(|x(s)-u(s)|+|x(s)-w(s)| \\
& +|u(s)-w(s)|)+\max(|y(s)-v(s)|+|y(s)-z(s)| \\
& +|v(s)-z(s)|)))) \\
\geq \;& \frac{1}{2}\chi(t-(\max(|x(s)-u(s)|+|x(s)-w(s)| \\
& +|u(s)-w(s)|)))+\frac{1}{2}\chi(t-(\max(|y(s)-v(s)| \\
& +|y(s)-z(s)|+|v(s)-z(s)|))) \\
= \;& \frac{1}{2}(G_{x,u,w}(2t)+G_{y,v,z}(2t)).
\end{aligned}
$$

Therefore, all the hypotheses of Theorem 3.1 are held with $k=\frac{1}{2}$ and the operator $f$ has a coupled fixed point which is the solution of the system of the integral equations. □

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## REFERENCES

1. R. P. Agarwal, Z. Kadelburg and S. Radenović: *On coupled fixed point results in asymmetric G-metric spaces.* J. Inequal. Appl. **2013**, 2013:528.
2. R. P. Agarwal, E. Karapinar, D. O'Regan and A. F. Roldan-López-de-Hierro: *Fixed Point Theory in Metric Type Spaces.* Springer, Switzerland, 2015.
3. T. G. Bhaskar and V. Lakshmikantham: *Fixed point theorems in partially ordered metric spaces and applications.* Nonlinear Anal. **65** (2006), 1379–1393.
4. G. L. Cain and R. H. Kasrie: *Fixed and periodic points of local contraction mappings on probabilistic metric spaces.* Math. Syst. Theory. **9** (3) (1975), 289–297.
5. I. Golet and C. Hedrea: *On local contractions in probabilistic metric spaces.* Romai J. **2** (2005), 95–100.
6. O. Hadzic and E. Pap: *Fixed Point Theory in Probabilistic Metric Spaces.* Kluwer Academic, Dordrecht, 2001.
7. J. Harjani, B. López and K. Sadarangani: *Fixed point theorems for mixed monotone operators and applications to integral equations.* Nonlinear Anal. **74** (2011), 1749–1760.

8. Z. Kadelburg, H. K. Nashine and S. Radenović: *Common coupled fixed point results in partially ordered G-metric space*. Bull. Math. Anal. Appl. **4** (2012), 51–63.

9. K. Menger: *Statistical metrics*. Proc. Natl. Acad. Sci. **28** (1942), 535–537.

10. Z. Mustafa and B. Sims: *A new approach to generalized metric spaces*. J. Nonlinear Convex Anal. **7** (2006), 289–297.

11. Z. Mustafa, H. Obiedat and F. Awawdeh: *Some of fixed point theorem for mapping on complete G-metric spaces*. Fixed Point Theory Appl. **2008**, 2008:189870.

12. J. J. Nieto and R. Rodríguez-López: *Contractive mapping theorems in partially ordered sets and applications to ordinary dierential equations*. Order. **22** (2005), 223–239.

13. S. Radenović: *Remarks on some recent coupled coincidence point results in symmetric G-metric space*. J. Oper. **2013**, 2013:290525

14. A. C. M. Ran and M. C. B. Reurings: *A fixed point theorem in partially ordered sets and some applications to matrix equations*. Proc. Amer. Math. Soc. **132** (2004), 1435–1443.

15. V. M. Sehgal: *Some fixed point theorems in functional analysis and probability*. Ph. D. Thesis, Wayne State University, Detroit, 1966.

16. V. M. Sehgal and A. T. Bacharuca-Reid: *Fixed points of contraction mappings on probabilistic metric spaces*. Math. Syst. Theory. **6** (1972), 97–102.

17. B. Samet and H. Yazidi: *Coupled fixed point theorems in partially ordered $\epsilon$-chainable metric spaces*. J. Math. Comput. Sci. **1** (3) (2010), 142–151.

18. G. Soleimani Rad, S. Shukla and H. Rahimi: *Some relations between n-tuple fixed point and fixed point results*. RACSAM. **109** (2015), 471–481.

19. C. Zhou, S. Wang, L. Ćirić and S. Alsulami: *Generalized probabilistic metric spaces and fixed point theorems*. Fixed Point Theory Appl. **2014**, 2014:91.

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# CONSTRUCTION OF OFFSET SURFACES WITH A GIVEN NON-NULL ASYMPTOTIC CURVE

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**Abstract.** In the present work, we study construction of offset surfaces with a given non-null asymptotic curve. Let $\alpha(s)$ be a spacelike or timelike unit speed curve with non-vanishing curvature and $\varphi(s,t)$ be a surface pencil accepting $\alpha(s)$ as a common asymptotic curve. We obtain conditions such that the offset surface possesses the image of $\alpha(s)$ as an asymptotic curve. We validate the method with illustrative examples.
**Keywords**: Ofset surface, Minkowski 3-space, asymptotic curve.

## 1. Introduction

Traditional research on curves and surfaces focuses on to find chracteristic curves, such as geodesic curve, asymptotic curve, and principal curve etc. on a present surface. However, the reverse problem, that is finding surfaces possessing a prescribed curve, is much more interesting. The construction of surfaces with a given characteristic curve is a new research area that attracts the interests of many researchers. The first study of this type of construction conducted by Wang et al. [18]. They presented a method for surfaces accepting a given curve as a common geodesic. Inspired by Wang et al. [18], researchers obtained constraints for a prescribed curve to be a specific curve on constructed surfaces [1 - 3, 8, 10, 16, 17].

Offset surfaces have a great importance among surfaces. An offset surface is a surface at a fixed distance along the unit normal vector field of a given surface.

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An idea of the value of offset surfaces can be realized from the great volume of literature [7, 9, 11, 12, 14, 15]. Moon [12] presented equivolumetric offset surface. Authors in [14] introduced a new algorithm for the efficient and reliable generation of offset surfaces for polygonal meshes. Hermann [9] showed that a base surface and its offset have the same geometric continuum. Güler et al. [8] obtained necessary constraints such that the image curve is a common asymptotic curve on each offset. The properties of offset surfaces have been examined in [7].

Motivated by the increasing importance of surfaces in mathematical physics, and very restricted knowledge about offset surfaces in Minkowski 3-space, we develop the theory of offset surfaces using non-null curves. We present constraints for a non-null curve to be a common asymptotic on an offset surface pencil. In particular, given a surface pencil with a common asymptotic curve, we give conditions such that the image curve is also a common asymptotic on each offset. The method is illustrated with several examples.

## 2. Preliminaries

In this section, we review some notions related with curves and surfaces in Minkowski 3-space.

The real vector space $IR^3$ endowed with the scalar product

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

where $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in IR^3$, is called Minkowski 3-space and denoted by $IR^3_1$.

A vector $X \in IR^3$ is called spacelike, timelike or null if

$$\begin{cases} \langle X, X\rangle > 0 \ \ or \ \ X = 0, \\ \langle X, X\rangle < 0, \\ \langle X, X\rangle = 0 \ \ and \ X \neq 0, \end{cases} \tag{2.2}$$

respectively [5].

The vectoral product of $X$ and $Y$ is defined as [13]

$$X \times Y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_{1}{}_{-}x_1y_2). \tag{2.3}$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha = \alpha(s)$ in Minkowski 3-space, where the vector fields $T$, $N$ and $B$ are called the tangent, the principal normal and the binormal vector field of $\alpha$, respectively.

**Theorem 2.1.** *Let $\alpha = \alpha(s)$ be a spacelike or timelike arclength curve with non vanishing curvature. The Frenet formula of $\alpha$ is given by*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\delta_1\kappa & 0 & \tau \\ 0 & \varepsilon_1\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.4}$$

*where $\langle T,T\rangle = \varepsilon_1$, $\langle N,N\rangle = \delta_1$. Also, we have $B = \varepsilon_1\delta_1 (T \times N)$, $\kappa = \delta_1 \left\langle T^{'}, N\right\rangle$ and $\tau = -\varepsilon_1\delta_1 \left\langle N^{'}, B\right\rangle$. The functions $\kappa$ and $\tau$ are called the curvature and torsion of $\alpha$, respectively.*

If $\alpha(s)$ is a non-null curve on a surface, then we have another frame, the so called Darboux frame $\{T, b, n\}$. Here, $T$ is the unit tangent vector field of $\alpha$, $n$ is the unit normal vector field of the surface and $b$ is a unit vector field given by $b = \varepsilon_1\varepsilon_3 (n \times T)$, where $\langle n,n\rangle = \varepsilon_3$. Because, $T$ is the same in each frame, the other vector fields of these frames lie on the same plane. Thus, we can give the following relation about these frames as:

Let $\varphi$ be a spacelike surface and $\alpha(s)$ a spacelike curve on $\varphi$. We have

$$\begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.5}$$

where $\theta$ is the hyperbolic angle between the vectors $b$ and $N$.

Let $\varphi$ be a timelike surface and $\alpha(s)$ a spacelike or timelike curve on $\varphi$.

1) If $\alpha(s)$ is timelike curve, then

$$\begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.6}$$

where $\theta$ is the angle between the vectors $b$ and $N$.

2) If $\alpha(s)$ is a spacelike curve, then

$$\begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.7}$$

where $\theta$ is the hyperbolic angle between the vectors $b$ and $N$.

Let $\varphi(s,t)$ be a timelike or spacelike surface. We have the following formula for the Darboux frame as

$$\begin{bmatrix} T' \\ b' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_g & \varepsilon_3 k_n \\ -\varepsilon_1 k_g & 0 & \varepsilon_3 \tau_g \\ -\varepsilon_1 k_n & -\varepsilon_2 \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ b \\ n \end{bmatrix}, \tag{2.8}$$

where $\varepsilon_1 = \langle T, T \rangle$, $\varepsilon_2 = \langle b, b \rangle$, $\varepsilon_3 = \langle n, n \rangle$, $b = -\varepsilon_2 (n \times T)$ and $k_g$, $k_n$ and $\tau_g$ are the geodesic curvature, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively [6].

## 3. Construction of surfaces with a non-null asymptotic curve

Let $\alpha(s)$ be a spacelike or timelike arclength curve with nonvanishing curvature. Surfaces passing through $\alpha(s)$ are given by

$$\varphi(s,t) = \alpha(s) + x(s,t) T(s) + y(s,t) N(s) + z(s,t) B(s), \tag{3.1}$$

$A_1 \leq s \leq A_2$, $B_1 \leq t \leq B_2$, where $x(s,t)$, $y(s,t)$ and $z(s,t)$ are $C^2$ marching-scale functions. Assume that $\varphi(s,t_0) = \alpha(s)$ for some $t_0 \in [B_1, B_2]$, so that $\alpha$ becomes a parameter curve on $\varphi(s,t)$.

The normal vector field of $\varphi(s,t)$ is

$$n(s,t) = \frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t} \tag{3.2}$$

and along the curve $\alpha(s)$, one can write it as

$$n(s,t_0) = \phi_1(s,t_0) T(s) + \phi_2(s,t) N(s) + \phi_3(s,t) B(s), \tag{3.3}$$

where

$$\begin{cases} \phi_1(s,t_0) = \left[\frac{\partial z}{\partial s}(s,t_0)\frac{\partial y}{\partial t}(s,t_0) - \frac{\partial y}{\partial s}(s,t_0)\frac{\partial z}{\partial t}(s,t_0)\right]\varepsilon_1, \\ \phi_2(s,t_0) = \left[\left(1+\frac{\partial x}{\partial s}(s,t_0)\right)\frac{\partial z}{\partial t}(s,t_0) - \frac{\partial z}{\partial s}(s,t_0)\frac{\partial x}{\partial t}(s,t_0)\right]\delta_1, \\ \phi_3(s,t_0) = \left[\frac{\partial y}{\partial s}(s,t_0)\frac{\partial x}{\partial t}(s,t_0) - \left(1+\frac{\partial x}{\partial s}(s,t_0)\right)\frac{\partial y}{\partial t}(s,t_0)\right]\delta_2, \end{cases} \tag{3.4}$$

$\varepsilon_1 = \langle T, T \rangle$, $\delta_1 = \langle N, N \rangle$ and $\delta_2 = \langle B, B \rangle$.

**Theorem 3.1.** *A non-null curve $\alpha(s)$ is a common asymptotic curve on the surface pencil $\varphi(s,t)$ [16] if*

$$x(s,t_0) = y(s,t_0) = z(s,t_0) = \frac{\partial z}{\partial t}(s,t_0) \equiv 0. \tag{3.5}$$

To obtain regular surfaces one need $\frac{\partial y}{\partial t}(s,t_0) \neq 0$ as an extra condition.

**Definition 3.1.** Let $\varphi(s,t)$ be a parametric surface with unit normal vector field $\widehat{n}(s,t)$. A parametric offset surface is defined by

$$\overline{\varphi}(s,t) = \varphi(s,t) + r\widehat{n}(s,t), \tag{3.6}$$

r being a non zero real constant [19].

Using Eqn. (3.1) offset surface pencil has the form

$$\overline{\varphi}(s,t) = \alpha(s) + r\widehat{n}(s,t) + x(s,t)T(s) + y(s,t)N(s) + z(s,t)B(s), \tag{3.7}$$

$\beta(s) = \alpha(s) + r\widehat{n}(s,t)$ being the image of $\alpha(s)$ on $\overline{\varphi}(s,t)$.

**Theorem 3.2.** *Let $\alpha(s)$ be a non-null regular curve on the surface pencil $\varphi(s,t)$. Then*

$$\begin{aligned} \overline{k_g}^r &= -\tfrac{1}{v^3}\left[-k_g v^2 - r\varepsilon_3\left(r\tau_g k_n' + \tau_g'(1+r\varepsilon_1 k_n)\right)\right] \\ \overline{k_n}^r &= \tfrac{1}{v^2}\left[k_n(1+r\varepsilon_1 k_n) + r\varepsilon_2\tau_g^2\right] \\ \overline{\tau_g}^r &= -\tfrac{1}{v^2}\left[r\varepsilon_1\varepsilon_2 k_n\tau_g - \varepsilon_2\tau_g(1+r\varepsilon_1 k_n)\right], \end{aligned} \tag{3.8}$$

*for the image curve $\beta(s)$ on the offset surface pencil $\overline{\varphi}(s,t)$, respectively, where*

$$v = \|\beta'(s)\| = \left|(1+r\varepsilon_1 k_n)^2\varepsilon_1 + \varepsilon_2 r^2\tau_g^2\right|^{1/2}, \tag{3.9}$$

*and $k_g$, $k_n$, $\tau_g$ are the geodesic, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively.*

This result also exists in [4] for spacelike surfaces.

**Theorem 3.3.** *Let $\left\{\overline{T}^r, \overline{N}^r, \overline{B}^r\right\}$ be the Frenet frame of the image curve $\beta(s)$ on $\overline{\varphi}(s,t)$ and $\{T, b, n\}$ the Darboux frame of $\alpha(s)$ on $\varphi(s,t)$. Then we have*

$$\begin{cases} \overline{T}^r = \frac{1}{v}\left[(1+r\varepsilon_1 k_n)T + r\varepsilon_2\tau_g b\right] \\ \overline{N}^r = \frac{1}{v^4\sqrt{\left(\overline{k_g}^r\right)^2 - \left(\overline{k_n}^r\right)^2}}\left[-rv^3\tau_g\overline{k_g}^r T + \varepsilon_1 v^3\overline{k_g}^r(1+r\varepsilon_1 k_n)b - \varepsilon_3\overline{k_n}^r v^4 n\right] \\ \overline{B}^r = \frac{1}{v^3\sqrt{\left(\overline{k_g}^r\right)^2 - \left(\overline{k_n}^r\right)^2}}\left[rv^2\tau_g\overline{k_n}^r T - \varepsilon_1 v^2\overline{k_n}^r(1+r\varepsilon_1 k_n)b + v^3\varepsilon_3\overline{k_g}^r n\right], \end{cases} \tag{3.10}$$

*where $v = \|\beta'(s)\| = \left|(1+r\varepsilon_1 k_n)^2\varepsilon_1 + \varepsilon_2 r^2\tau_g^2\right|^{1/2}$, $\overline{k_g}^r$, $\overline{k_n}^r$ are the geodesic curvature and the normal curvature of the image curve $\beta(s)$ and $k_g$, $k_n$, $\tau_g$ are the geodesic, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively.*

Now, suppose that $\alpha(s)$ is a common spacelike asymptotic and parameter curve with timelike binormal on the spacelike surface pencil. Our objective is to find sufficient constraints for the curve $\beta(s)$ to be both an asymptotic curve and parameter curve on the offset surface pencil $\overline{\varphi}(s,t)$.

Observe that, by Eqn. (3.7), $\beta(s)$ is a parameter curve on each offset.

The necessary and sufficient condition forthe image curve $\beta(s)$ to be an asymptotic curve on the offset surface $\overline{\varphi}(s,t)$ is

$$\left\langle \frac{\partial \overline{n}^r}{\partial s}(s,t_0), \overline{T}^r(s) \right\rangle = 0, \tag{3.11}$$

where $\overline{T}^r(s)$ is the tangent vector field of the image curve $\beta(s)$ and $\overline{n}^r(s,t_0)$ is the unit normal vector field of $\overline{\varphi}(s,t)$ through the image curve. According to [19], we have $\overline{n}^r(s,t_0) = \pm n(s,t_0)$. Now, we have the following equivalent asymptotic requirement

$$\left\langle \frac{\partial n}{\partial s}(s,t_0), \overline{T}^r(s) \right\rangle = 0, \tag{3.12}$$

where $n(s,t_0)$ is the normal vector field of $\varphi(s,t)$. By the asymptotic requirement of $\alpha(s)$, we have

$$n(s,t_0) = \frac{\partial y}{\partial s}(s,t_0)\, B(s). \tag{3.13}$$

With the help of Eqns. (2.4), (2.7), (3.10) and (3.12) we obtain

$$\tau(s)\,\tau_g(s)\,\frac{\partial y}{\partial t}(s,t_0)\, ch\theta(s) = \tau_g(s)\,\frac{\partial^2 y}{\partial s \partial t}(s,t_0)\, sh\theta(s), \tag{3.14}$$

for $\beta(s)$ to be an asymptotic curve on every spacelike offset surface pencil $\overline{\varphi}(s,t)$.

Note that, if $\alpha(s)$ is a line of curvature, i.e $\tau_g(s) \equiv 0$, then Eqn. (3.14) is satisfied and $\beta(s)$ be an asymptotic curve on the spacelike offset surface pencil $\overline{\varphi}(s,t)$.

**Theorem 3.4.** *Let $\varphi(s,t)$ be a spacelike surface pencil with a common spacelike parametric and asymptotic curve $\alpha(s)$ with timelike binormal. The image curve $\beta(s)$ of $\alpha(s)$ is a common asymptotic curve on the spacelike offset surface pencil $\overline{\varphi}(s,t)$, if*

$$\begin{cases} x(s,t_0) = y(s,t_0) = z(s,t_0) \equiv 0. \\ y(s,t) = e^{\int \tau(s)\coth\theta(s)ds} \int \psi(t)\,dt + \xi(s), \end{cases} \tag{3.15}$$

*where $A_1 \le s \le A_2$, $B_1 \le t \le B_2$, $\psi \in C^2$, $\xi \in C^1$.*

*Proof.* Since the $\alpha(s)$ curve is a parameter curve on the surface $\varphi(s,t)$, we have

$$x(s,t_0) = y(s,t_0) = z(s,t_0) \equiv 0.$$

For the image curve $\beta(s)$ of $\alpha(s)$ to be a common asymptotic curve on the spacelike offset surface pencil $\overline{\varphi}(s,t)$, we can use Eqn. (3.12). If Eqns. (3.4), (3.10) and (2.7) are written in Eqn. (3.12), then we obtain a second- order linear partial differential equation with variable coefficients as follows,

$$\tau \cosh\theta \frac{\partial y(s,t_0)}{\partial t} = \sinh\theta \frac{\partial^2 y(s,t_0)}{\partial s \partial t}, \tag{3.16}$$

where since $\alpha(s)$ is an asymptotic on the surface pencil $\varphi(s,t)$, we have $\tau_g \neq 0$.The desired result is obtained from the solution of Eqn. (3.17). □

Now, suppose that $\varphi(s,t)$ is a timelike surface with a common timelike asymptotic curve $\alpha(s)$. Hence, the offset surface $\overline{\varphi}(s,t)$ of $\varphi(s,t)$ is also a timelike surface.

By a similar investigation we obtain the following theorem:

**Theorem 3.5.** *Let $\varphi(s,t)$ be a timelike surface pencil with a common timelike parametric and asymptotic curve $\alpha(s)$ or spacelike parametric and asymptotic curve $\alpha(s)$ with spacelike binormal. The image curve $\beta(s)$ of $\alpha(s)$ is a common asymptotic curve on the timelike offset surface pencil $\overline{\varphi}(s,t)$, if*

$$\begin{cases} x(s,t_0) = y(s,t_0) = z(s,t_0) \equiv 0. \\ y(s,t) = e^{\int \tau(s)\cot\theta(s) ds} \int \psi(t)\, dt + \xi(s), \end{cases} \tag{3.17}$$

*where* $A_1 \leq s \leq A_2, \quad B_1 \leq t \leq B_2,\ \psi \in C^2,\ \xi \in C^1.$

## 4. Examples

### 4.1. Example 1

Unit speed timelike curve $\alpha(s) = \left(\frac{5}{3}s, \frac{4}{9}\cos(3s), \frac{4}{9}\sin(3s)\right)$ has Frenet vector fields as

$$\begin{cases} T(s) = \left(\frac{5}{3}, -\frac{4}{3}\sin(3s), \frac{4}{3}\cos(3s)\right), \\ N(s) = (0, -\cos(3s), -\sin(3s)), \\ B(s) = \left(-\frac{4}{3}, \frac{5}{3}\sin(3s), -\frac{5}{3}\cos(3s)\right), \end{cases}$$

and torsion $\tau(s) \equiv 5$. Choosing $\xi(s) \equiv 0,\ \psi(t) \equiv 1,\ \ t_0 = 0$ and $\theta(s) = \frac{\pi}{4}$ yields $y(s,t) = (t + c_1) e^{5s+c_2}$ and for $c_1 = c_2 = 0$, $y(s,t) = te^{5s}$. Letting $x(s,t) = z(s,t) \equiv 0$ Theorems 3.1 and 3.5 are satisfied. Thus, we obtain the timelike surface

$$\varphi(s,t) = \left(\frac{5}{3}s, \left(\frac{4}{9} - te^{5s}\right)\cos(3s), \left(\frac{4}{9} - te^{5s}\right)\sin(3s)\right),$$

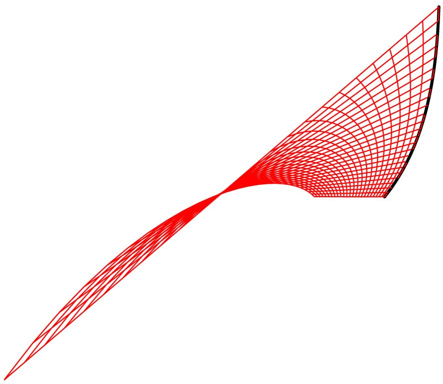

FIG. 4.1: Timelike surface $\varphi(s,t)$ and its asymptotic curve $\alpha(s)$.

$0 \leq s \leq 0.3,\ 0 \leq t \leq 0.2$, accepting $\alpha(s)$ as an asymptotic curve (Figure 4.1).

To obtain the offset surface of $\varphi(s,t)$, first we calculate

$$\widehat{n}(s,t) = \frac{1}{A}\left(4 - 9te^{5s}, 5\sin(3s), -5\cos(3s)\right),$$

where $A = \left|25 - \left(9te^{5s} - 4\right)^2\right|^{\frac{1}{2}}$. Now for $r = 3$, the image curve of $\alpha(s)$ is

$$\begin{aligned} \beta(s) &= \alpha(s) + 3\widehat{n}(s,0) \\ &= \left(\frac{5}{3}s + 4, \frac{4}{9}\cos(3s) + 5\sin(3s), \frac{4}{9}\sin(3s) - 5\cos(3s)\right). \end{aligned}$$

Using Eqn. (3.6), we get the offset timelike surface

$$\begin{aligned} \overline{\varphi}(s,t) = &\left(\frac{5}{3}s - \frac{3\left(9te^{5s} - 4\right)}{A}, \left(\frac{4}{9} - te^{5s}\right)\cos(3s) + \frac{15\sin(3s)}{A},\right. \\ &\left.\left(\frac{4}{9} - te^{5s}\right)\sin(3s) - \frac{15\cos(3s)}{A}\right), \end{aligned}$$

$0 \leq s \leq 0.3,\ 0 \leq t \leq 0.2$, accepting $\beta(s)$ as an asymptotic curve (Figure 4.2).

### 4.2. Example 2

The Frenet vector fields of the spacelike curve $\alpha(s) = \left(\frac{1}{3}\sinh\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}s, \frac{1}{3}\cosh\left(\sqrt{3}s\right)\right)$ with timelike binormal are

$$\left\{ \begin{array}{c} T(s) = \left(\frac{\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right), \\ N(s) = \left(\sinh\left(\sqrt{3}s\right), 0, \cosh\left(\sqrt{3}s\right)\right), \\ B(s) = \left(\frac{2\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right), \end{array} \right.$$

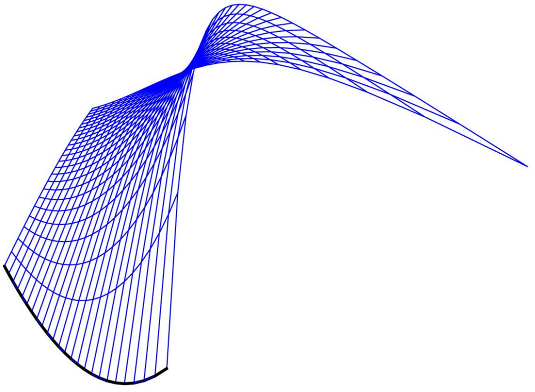

FIG. 4.2: Timelike offset surface $\overline{\varphi}(s,t)$ and its asymptotic curve $\beta(s)$.

and its torsion is $\tau(s) \equiv -2$. Choosing $\xi(s) \equiv 0$, $\psi(t) \equiv 1$, $t_0 = 0$ and $\theta(s) = \coth^{-1}\left(-\frac{1}{2}\right)$ yields $y(s,t) = (t + c_1)\, e^{s+c_2}$ and for $c_1 = c_2 = 0$, $y(s,t) = te^s$. Letting $x(s,t) = z(s,t) \equiv 0$, Theorems 3.1 and 3.4 are satisfied. Thus, we obtain the spacelike surface

$$\varphi(s,t) = \left((3 + te^s)\sinh\frac{s}{4}, \frac{5}{4}s, (3 + te^s)\cosh\frac{s}{4}\right),$$

$0 \le s \le 1$, $-1 \le t \le 1$, accepting $\alpha(s)$ as an asymptotic curve (Figure 4.3).

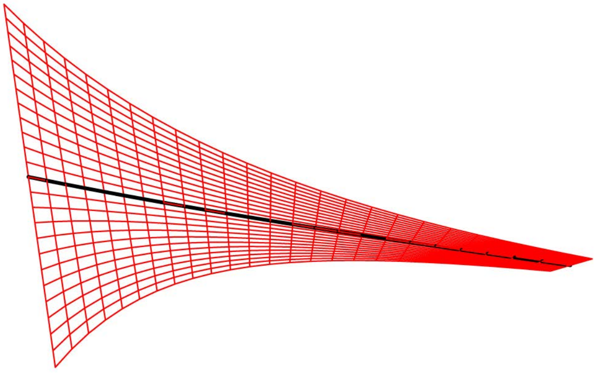

FIG. 4.3: Spacelike surface $\varphi(s,t)$ and its asymptotic curve $\alpha(s)$.

Using Eqn. (3.6), we get the offset spacelike surface

$$\overline{\varphi}(s,t) = \left((3 + te^s)\sinh\frac{s}{4} + \frac{20}{A}\cosh\frac{s}{4}, \frac{5}{4}s + \frac{4\,(te^s + 3)}{A}, (3 + te^s)\cosh\frac{s}{4} + \frac{20}{A}\sinh\frac{s}{4}\right),$$

$0 \le s \le 5$, $0 \le t \le 5$, accepting $\beta(s)$ as an asymptotic curve (Figure 4.4).

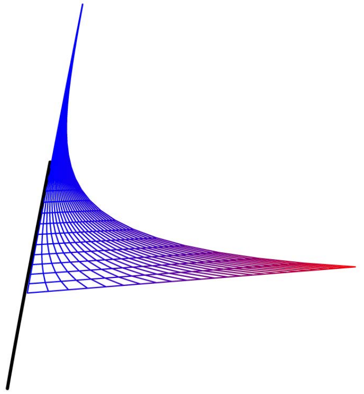

FIG. 4.4: Spacelike offset surface $\overline{\varphi}(s,t)$ and its asymptotic curve $\beta(s)$.

## REFERENCES


1. E. BAYRAM: *Surface pencil with a common adjoint curve*. Turkish Journal of Mathematics **44** (2020), 1649-1659.
2. E. BAYRAM, E. ERGÜN and E. KASAP: *Surface family with a common natural geodesic lift of a spacelike curve with timelike binormal in Minkowski 3-space*. Hagia Sophia Journal of Geometry **1** (2019), 17-23.
3. E. BAYRAM, F. GÜLER and E. KASAP: *Parametric representation of a surface pencil with a common asymptotic curve*. Comput. Aided Design **44** (7) (2012), 637-643.
4. M. ÇİMDİKER and C. EKİCİ: *On the spacelike parallel ruled surfaces with Darboux frame*. Math. Comb. Book Series **2** (2017), 60-69.
5. K. L. DUGGAL and A. BEJMAN: *Ligtlike Submanifolds of Semi Riemannian Manifolds and Applications*. Kluwer Academic Publisher, 1976.
6. C. EKİCİ and A. GÖRGÜLÜ: *Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Minkowski 3-space $E_1^3$*. Turk J Math. **33** (2009), 397-402.
7. R. T. FAROUKI: *The approximation of non-degenerate offset surfaces*. Comput. Aided Geo. Design **3** (1) (1986), 15-43.
8. F. GÜLER, E. BAYRAM and E. KASAP: *Offset surface pencil with a common asymptotic curve*. Int J Geom Methods M **15** (11) (2018), 1850195.
9. T. HERMANN: *On the smootthness of offset surfaces*. Comput. Aided Geo. Design **15** (1998), 529-533.
10. C. Y. LI, R. H. WANG and C. G. ZHU: *Parametric representation of a surface pencil with a common line of curvature*. Comput. Aided Design **43** (9) (2011), 1110-1117.
11. T. MAEKAWA: *An overview of offset curves and surfaces*. Comput. Aided Design **31** (1999), 165-173.
12. H. P. MOON: *Equivolumetric offset surfaces*. Comput. Aided Geo. Design **26** (1) (2009), 17-36.

13. M. Önder and H. H. Uğurlu: *Frenet frames and invariants of timelike ruled surfaces*. Ain Shams Eng. J. **4** (2013), 502-513.

14. D. Pavić and L. Kobbelt: *High-resolution volumetric computation of offset surfaces with feature preservation*. In Computer Graphics Forum **27** (2008), 165-174.

15. B. Pham: *Offset curves and surfaces: a brief survey*. Comput. Aided Design **24** (4) (1992), 223-229.

16. G. Şaffak Atalay, E. Bayram and E. Kasap: *Surface family with a common asymptotic curve in Minkowski 3- space*. J Sci. Art. **2** (2018), 357-68.

17. G. Şaffak Atalay and E. Kasap: *Surfaces family with common null asymptotic*. Applied Mathematics and Computation **26** (2015), 135-139.

18. G. J. Wang, K. Tang and C. L. Tai: *Parametric representation of a surface pencil with a common spatial geodesic*. Comput. Aided Design **36** (2004), 447-459.

19. T. J. Willmore: *An Introduction to Differential Geometry*. Oxford University Press, Delhi, 1959.

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**Original Scientific Paper**

# ON MULTIPLICATIVE LIE $n$-HIGHER DERIVATIONS OF TRIANGULAR ALGEBRAS

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**Abstract.** Let R be a commutative ring with unity, A, B be R-algebras and M be an (A, B)-bimodule. Let $\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a $(n-1)$-torsion free triangular algebra. In this article, we prove that every multiplicative Lie $n$-higher derivation on triangular algebras has the standard form. Also, the main result is applied to some classical examples of triangular algebras such as upper triangular matrix algebras and nest algebras.
**Keywords**: Triangular algebras, Lie type derivation.

## 1. Brief Historical Development

Many authors studied Lie type derivations on several rings and algebras [6,7,10, 12,14–17,19,25]. In most of the cases, authors found that any Lie type derivation has the standard from on that particular ring or algebra under consideration. The first characterization of Lie derivations was obtained by Martindale [17] in 1964 who proved that every Lie derivation on a primitive ring can be written as a sum of derivations and an additive mapping of a ring to its center that maps commutators into zero, i.e, Lie derivation has the standard form.

Moreover, during last few decades, the multiplicative mappings on rings and algebras have been studied by many authors. Martindale [18] established a condition

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on a ring such that multiplicative bijective mappings on this ring are all additive. In particular, he proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Daif [8] studied the additivity of derivable map on a 2-torsion free prime ring containing a nontrivial idempotent. In the year 1978, Miers [19] studied Lie triple derivations of von Neumann algebras and proved that if $M$ is a von Neumann algebra with no central abelian summands then there exists an operator $A \in M$ such that $L(X) = [A, X] + \lambda(X)$ where $\lambda : M \to \mathrm{Z}(M)$ is a linear map which annihilates brackets of operators in $M$. In [7] Cheung initiated the study of Lie derivations of triangular algebras $\mathfrak{T}$ and gave a sufficient condition under which every Lie derivation on $\mathfrak{T}$ is a sum of derivations on $\mathfrak{T}$ and a mapping from $\mathfrak{T}$ to its center $\mathrm{Z}(\mathfrak{T})$. Further, Lie derivations on triangular algebras were studied in [15, 25], whereas the study of Lie triple derivations on triangular algebras can be found in [14, 16]. Yu and Zhang [25] proved that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map from triangular algebra into its center sending commutators to zero. Ji et al. [14] proved the similar result for nonlinear Lie triple derivation of triangular algebras.

Benkovič and Eremita [6] discussed multiplicative Lie $n$-derivations of triangular rings, which in fact, generalized some results on nonlinear Lie (triple) derivations of triangular algebras (see [14, 25]).

Several authors have made important contributions to the related topics see for reference [5, 11, 13, 14, 16, 20, 23–25] where further references can be found. Xiao and Wei [24] considered the case of nonlinear Lie higher derivation on a triangular algebra and they proved that if $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ is a nonlinear Lie higher derivation on a triangular algebra, then $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ is of the standard form, i.e., $\mathrm{L}_r = d_r + \gamma_r$, where $\{d_r\}_{r\in\mathbb{N}}$ is an additive higher derivation and $\{\gamma_r\}_{r\in\mathbb{N}}$ is a functional vanishing on all commutators of triangular algebra. However, much less attention to the study of Lie $n$-higher derivations on operator algebras has been paid. To the best of our knowledge, there are very few articles dealing with Lie $n$-higher derivations on rings and algebras except for [9, 11]. The objective of this article is to describe the structure of multiplicative Lie $n$-higher derivations on triangular algebras.

## 2. Basic Definitions & Preliminaries

Let R be a commutative ring with unity and $\mathrm{Z}(\mathcal{A})$ be the center of an R-algebra $\mathcal{A}$. A map $\mathrm{L} : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a multiplicative derivation (resp. multiplicative Lie derivation) on $\mathcal{A}$ if $\mathrm{L}(ab) = \mathrm{L}(a)b + a\mathrm{L}(b)$ (resp. $\mathrm{L}([a, b]) = [\mathrm{L}(a), b] + [a, \mathrm{L}(b)]$) holds for all $a, b \in \mathcal{A}$. In addition, if L is linear on $\mathcal{A}$, then L is said to be a derivation (resp. Lie derivation) on $\mathcal{A}$.

To explore a more approximate kind of maps. Define a sequence of polynomials:

$$\begin{aligned}
\mathfrak{p}_1(x_1) &= x_1, \\
\mathfrak{p}_2(x_1,x_2) &= [\mathfrak{p}_1(x_1),x_2] = [x_1,x_2], \\
&\vdots \qquad\qquad\qquad \vdots \\
\mathfrak{p}_n(x_1,x_2,\cdots,x_n) &= [\mathfrak{p}_{n-1}(x_1,x_2,\cdots,x_{n-1}),x_n].
\end{aligned}$$

The polynomial $\mathfrak{p}_n(x_1,x_2,\cdots,x_n)$ is called $(n-1)$-th commutator where $n \geq 2$. A map (not necessarily linear) $\mathrm{L}:\mathcal{A}\to\mathcal{A}$ is said to be a multiplicative Lie $n$-derivation on $\mathcal{A}$ if

$$\mathrm{L}(\mathfrak{p}_n(x_1,x_2,\cdots,x_n)) = \sum_{i=1}^{n} \mathfrak{p}_n(x_1,x_2,\cdots,x_{i-1},\mathrm{L}(x_i),x_{i+1},\cdots,x_n)$$

for all $x_1,x_2,\cdots,x_n \in \mathcal{A}$. The concept of Lie $n$-derivation was first introduced by Abdullaev [1] on certain von Neumann algebras. Note that any multiplicative Lie 2-derivation is known as multiplicative Lie derivation and multiplicative Lie 3-derivation is said to be multiplicative Lie triple derivation. Thus multiplicative Lie derivation, multiplicative Lie triple derivation and multiplicative Lie $n$-derivation collectively known as multiplicative Lie type derivations on $\mathcal{A}$.

Apart from these, the concept of derivation were extended to higher derivation. Let us recall the basic facts about higher derivations. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a family of maps $\mathrm{L}_r:\mathcal{A}\to\mathcal{A}$ (not necessarily linear) such that $\mathrm{L}_0 = I_{\mathcal{A}}$. Then $\mathfrak{L}$ is called

1. a multiplicative higher derivation if $\mathrm{L}_r(x_1x_2) = \sum\limits_{i_1+i_2=r} \mathrm{L}_{i_1}(x_1)\mathrm{L}_{i_2}(x_2)$,
2. a multiplicative Lie $n$-higher derivation if

$$\mathrm{L}_r(\mathfrak{p}_n(x_1,x_2,\cdots,x_n)) = \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(x_1),\mathrm{L}_{i_2}(x_2),\cdots,\mathrm{L}_{i_n}(x_n))$$

for all $x_1,x_2,\cdots,x_n \in \mathcal{A}$ and for each $r\in\mathbb{N}$. Note that any multiplicative Lie 2-higher derivation is multiplicative Lie higher derivation and multiplicative Lie 3-higher derivation is multiplicative Lie triple higher derivation. Thus multiplicative Lie higher derivation, multiplicative Lie triple higher derivation and multiplicative Lie $n$-higher derivation collectively known as multiplicative Lie type higher derivations on $\mathcal{A}$. It is easy to observe that every higher derivation is a Lie higher derivation and every Lie higher derivation is a Lie triple higher derivation and so on but the converse need not be true in general.

Note that if $\mathfrak{D} = \{d_r\}_{r\in\mathbb{N}}$ is a higher derivation on $\mathcal{A}$ and for each $r\in\mathbb{N}$, $\mathrm{L}_r = d_r + f_r$ where $f_r:\mathcal{A}\to\mathrm{Z}(\mathcal{A})$ is a linear (resp. nonlinear) mapping, then it is easy to see that $\{\mathrm{L}_r\}_{r\in\mathbb{N}}$ is a Lie $n$-higher derivation (resp. nonlinear Lie $n$-higher derivation) if and only if $f_r(\mathfrak{p}_n(x_1,x_2,\cdots,x_n)) = 0$ for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$. Lie

$n$-higher derivation (resp. nonlinear Lie $n$-higher derivation) of the above kind are called *standard*. The natural problem that one considers in this context is whether or not every Lie $n$-higher derivation (resp. nonlinear Lie $n$-higher derivation) is standard.

Throughout this paper, R will always denote a commutative ring with unity element. Let A and B be unital algebras over R and let M be a unital $(\mathrm{A}, \mathrm{B})$-bimodule ( i.e., $1_{\mathrm{A}} \cdot m = m$ and $m \cdot 1_{\mathrm{B}} = m$ for all $m \in \mathrm{M}$.) which is faithful as a left A-module and also as a right B-module. The R-algebra

$$\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B}) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B} \right\}$$

under the usual matrix operations is called triangular algebra. The center of $\mathfrak{T}$ is

$$\mathrm{Z}(\mathfrak{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb \ \forall \ m \in \mathrm{M} \right\}.$$

Define two natural projections $\pi_{\mathrm{A}} : \mathfrak{T} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathfrak{T} \to \mathrm{B}$ by

$$\pi_{\mathrm{A}}\left( \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right) = a \text{ and } \pi_{\mathrm{B}}\left( \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right) = b.$$

Moreover, $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) \subseteq \mathrm{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) \subseteq \mathrm{Z}(\mathrm{B})$ and there exists a unique algebraic isomorphism $\tau : \pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) \to \pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})), m \in \mathrm{M}$.

Let $1_{\mathrm{A}}$ (resp.$1_{\mathrm{B}}$) be the identity of the algebra A (resp. B) and let $I$ be the unity of triangular algebra $\mathfrak{T}$. Throughout, this paper we shall use the following notions: $p = \begin{bmatrix} 1_{\mathrm{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $q = I - p = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{B}} \end{bmatrix}$ and $\mathrm{A} \cong p\mathfrak{T}p$, $\mathrm{M} \cong p\mathfrak{T}q$, $\mathrm{B} \cong q\mathfrak{T}q$. Thus, $\mathfrak{T} = p\mathfrak{T}p + p\mathfrak{T}q + q\mathfrak{T}q \cong \mathrm{A} + \mathrm{M} + \mathrm{B}$. Also, $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T}))$ and $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T}))$ are isomorphic to $p\mathrm{Z}(\mathfrak{T})p$ and $q\mathrm{Z}(\mathfrak{T})q$ respectively. Then there is an algebra isomorphisms $\tau : p\mathrm{Z}(\mathfrak{T})p \to q\mathrm{Z}(\mathfrak{T})q$ such that $am = m\tau(a)$ for all $m \in p\mathfrak{T}q$.

Let us describe the result which is used subsequently in this article as :

**Lemma 2.1.** *[6, Theorem 5.9] Let $\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a $(n-1)$-torsion free triangular ring. Suppose that $\mathfrak{T}$ satisfies the following conditions:*

1. $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{B})$,
2. $\mathrm{Z}(\mathrm{A}) = \{a \in \mathrm{A} \mid [[a, x], y] = 0 \ \forall \ x, y \in \mathrm{A}\}$
   *or* $\mathrm{Z}(\mathrm{B}) = \{b \in \mathrm{B} \mid [[b, x], y] = 0 \ \forall \ x, y \in \mathrm{B}\}$.

*Then any multiplicative Lie $n$-derivation $\mathrm{L} : \mathfrak{T} \to \mathfrak{T}$ has the standard form.*

## 3. Multiplicative Lie $n$-higher derivation

In this section, we will prove the main result by a series of lemmas. It is clear that every Lie higher derivation is a Lie $n$-higher derivation for $n \geq 3$. Therefore, without loss of generality we assume $n \geq 3$ for convenience and for $n = 2$ we can look into [24].

**Theorem 3.1.** *Let $\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a $(n-1)$-torsion free triangular algebra consisting of unital algebras* $\mathrm{A}, \mathrm{B}$ *and a faithful unital* $(\mathrm{A}, \mathrm{B})$*-bimodule* $\mathrm{M}$. *Suppose that $\mathfrak{T}$ satisfies the following conditions:*

($\star$) $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{B})$,

($\sharp$) $\mathrm{Z}(\mathrm{A}) = \{a \in \mathrm{A} \mid [[a, x], y] = 0 \ \forall \ x, y \in \mathrm{A}\}$
*or* $\mathrm{Z}(\mathrm{B}) = \{b \in \mathrm{B} \mid [[b, x], y] = 0 \ \forall \ x, y \in \mathrm{B}\}$.

*Then every multiplicative Lie $n$-higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ on $\mathfrak{T}$ has the standard form. More precisely, there exists an additive higher derivation $\mathfrak{D} = \{d_r\}_{r\in\mathbb{N}}$ on $\mathfrak{T}$ and a sequence of functionals $\{h_r\}_{r\in\mathbb{N}}$ which annihilates all Lie $n$-product $\mathfrak{p}_n(x_1, x_2, \cdots, x_n)$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $\mathrm{L}_r(x) = d_r(x) + h_r(x)$ for all $x \in \mathfrak{T}$ and $r \in \mathbb{N}$.*

In order to prove our main theorem, we apply an induction method for the component index $r$. For $r = 1$, $\mathrm{L}_1$ is multiplicative Lie $n$-derivation on $\mathfrak{T}$. Hence by Lemma 2.1 it follows that there exists an additive derivation $d_1$ and a functional $h_1$ satisfying $h_1(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $\mathrm{L}_1(x) = d_1(x) + h_1(x)$ for all $x \in \mathfrak{T}$. Moreover, $\mathrm{L}_1$ and $d_1$ satisfy the following properties:

$$\mathbf{C}_1 : \begin{cases} \mathrm{L}_1(0) = 0, & \mathrm{L}_1(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_1(\mathrm{M}) \subseteq \mathrm{M}, & \mathrm{L}_1(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_1(p) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), & \mathrm{L}_1(q) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ d_1(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}, & d_1(\mathrm{B}) \subseteq \mathrm{M} + \mathrm{B}, \\ d_1(\mathrm{M}) \subseteq \mathrm{M}, & d_1(p), d_1(q) \in \mathrm{M}. \end{cases}$$

We assume that the result holds for all $1 < s < r, r \in \mathbb{N}$. Then there exists an additive mapping $d_s$ and a functional $h_s$ satisfying $h_s(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $\mathrm{L}_s(x) = d_s(x) + h_s(x)$ for all $x \in \mathfrak{T}$. Thus the mapping $\mathrm{L}_s$ and $d_s$ satisfy the following properties:

$$\mathbf{C}_s : \begin{cases} \mathrm{L}_s(0) = 0, & \mathrm{L}_s(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_s(\mathrm{M}) \subseteq \mathrm{M}, & \mathrm{L}_s(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_s(p) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), & \mathrm{L}_s(q) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ d_s(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}, & d_s(\mathrm{B}) \subseteq \mathrm{M} + \mathrm{B}, \\ d_s(\mathrm{M}) \subseteq \mathrm{M}, & d_s(p), d_s(q) \in \mathrm{M}. \end{cases}$$

Our aim is to show that above conditions also hold for $r$, it follows from the series of Lemmas:

**Lemma 3.1.** *Let $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a multiplicative Lie $n$-higher derivation on $(n-1)$-torsion free triangular algebra $\mathfrak{T}$. Then $\mathrm{L}_r(0) = 0$, and $\mathrm{L}_r(\mathrm{M}) \subseteq \mathrm{M}$ for each $r \in \mathbb{N}$.*

*Proof.* For each $r \in \mathbb{N}$, $\mathrm{L}_r(0) = 0$ is trivially true. For any $m \in \mathrm{M}$ using conditions $C_s$, we have

$$\begin{aligned}
\mathrm{L}_r(m) &= \mathrm{L}_r(\mathfrak{p}_n(m, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(m), \mathrm{L}_{i_2}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(m), q, \cdots, q) + \mathfrak{p}_n(m, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(m, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n < r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(r), \mathrm{L}_{i_2}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(m), q, \cdots, q) + \mathfrak{p}_n(m, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(m, q, \cdots, \mathrm{L}_r(q)) \\
&= p\mathrm{L}_r(m)q + (n-1)[m, \mathrm{L}_r(q)].
\end{aligned}$$

On multiplying the above equality from left by $p$ and right by $q$, we get $(n-1)[\mathrm{M}, \mathrm{L}_r(q)] = 0$ and hence $\mathrm{L}_r(m) = p\mathrm{L}_r(m)q$. This implies that $\mathrm{L}_r(\mathrm{M}) \subseteq \mathrm{M}$. □

**Lemma 3.2.** *Let $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a multiplicative Lie $n$-higher derivation on $(n-1)$-torsion free triangular algebra $\mathfrak{T}$. Then $\mathrm{L}_r(p), \mathrm{L}_r(q) \in \mathrm{Z}(\mathfrak{T}) + \mathrm{M}$ for each $r \in \mathbb{N}$.*

*Proof.* From the proof of Lemma 3.1, we have seen that $(n-1)[\mathrm{M}, \mathrm{L}_r(q)] = 0$. Since $\mathfrak{T}$ is $(n-1)$-torsion free, we have $[\mathrm{M}, \mathrm{L}_r(q)] = 0$ and hence $p\mathrm{L}_r(q)p + q\mathrm{L}_r(q)q \in \mathrm{Z}(\mathfrak{T})$. Therefore, we have $\mathrm{L}_r(q) \in \mathrm{Z}(\mathfrak{T}) + \mathrm{M}$. Also, for any arbitrary $m \in \mathrm{M}$, we obtain that

$$\begin{aligned}
\mathrm{L}_r(m) &= \mathrm{L}_r(\mathfrak{p}_n(p, m, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(p), \mathrm{L}_{i_2}(m), \mathrm{L}_{i_3}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(p), m, q, \cdots, q) + \mathfrak{p}_n(p, \mathrm{L}_r(m), q, \cdots, q) \\
&\quad + \cdots + \mathfrak{p}_n(p, m, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n < r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(p), \mathrm{L}_{i_2}(m), \mathrm{L}_{i_3}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\mathrm{L}_r(p), m], q, \cdots, q) + \mathfrak{p}_{n-1}([p, \mathrm{L}_r(m)], q, \cdots, q) \\
&= p[\mathrm{L}_r(p), m]q + p[p, \mathrm{L}_r(m)]q.
\end{aligned}$$

Therefore, we get

$$\mathrm{L}_r(m) = p[\mathrm{L}_r(p), m]q + p\mathrm{L}_r(m)q \text{ for all } m \in \mathrm{M}. \tag{3.1}$$

Hence, $p\mathrm{L}_r(m)q = p[\mathrm{L}_r(p), m]q + p\mathrm{L}_r(m)q$, which implies that $[\mathrm{L}_r(p), \mathrm{M}] = 0$. Then $\mathrm{L}_r(p) \in \mathrm{Z}(\mathfrak{T}) + \mathrm{M}$. □

**Lemma 3.3.** *Let $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a multiplicative Lie $n$-higher derivation on $(n-1)$-torsion free triangular algebra $\mathfrak{T}$. Then for any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m \in \mathrm{M}$, the following hold true:*

*1.* $p\mathrm{L}_r(b)p \in \mathrm{Z}(\mathrm{A})$ *and* $q\mathrm{L}_r(a)q \in \mathrm{Z}(\mathrm{B})$,

*2.* $\mathrm{L}_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$ *and* $\mathrm{L}_r(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$

*for each* $r \in \mathbb{N}$.

*Proof.* Let $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$. Using the condition $\mathbf{C}_s$ and the fact that $[a, b] = 0$, we have

$$\begin{aligned}
0 &= \mathrm{L}_r(\mathfrak{p}_n(a, b, m, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(a), \mathrm{L}_{i_2}(b), \mathrm{L}_{i_3}(m), \mathrm{L}_{i_4}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(a), b, m, q, \cdots, q) + \mathfrak{p}_n(a, \mathrm{L}_r(b), m, q, \cdots, q) + \mathfrak{p}_n(a, b, \mathrm{L}_r(m), \cdots, q) \\
&\quad + \mathfrak{p}_n(a, b, m, \mathrm{L}_r(q), q, \cdots, q) + \cdots + \mathfrak{p}_n(a, b, m, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(a), \mathrm{L}_{i_2}(b), \mathrm{L}_{i_3}(m), \mathrm{L}_{i_4}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_{n-2}([[\mathrm{L}_r(a), b], m], q, \cdots, q) + \mathfrak{p}_{n-2}([[a, \mathrm{L}_r(b)], m], q, \cdots, q) \\
&= [[\mathrm{L}_r(a), b], m] + [[a, \mathrm{L}_r(b)], m].
\end{aligned}$$

Hence, $[q\mathrm{L}_r(a)q, b] + [a, p\mathrm{L}_r(b)p] \in \mathrm{Z}(\mathfrak{T})$. Now multiplying from right as well as left side by $p$ and $q$ respectively and on applying the assumptions $(\star)$ and $(\sharp)$, we get

$$p\mathrm{L}_r(b)p \in \mathrm{Z}(\mathrm{A}) \text{ and } q\mathrm{L}_r(a)q \in \mathrm{Z}(\mathrm{B}).$$

Then we obtain

$$\mathrm{L}_r(a) = (p\mathrm{L}_r(a)p - \tau^{-1}(q\mathrm{L}_r(a)q)) + p\mathrm{L}_r(a)q + (\tau^{-1}(q\mathrm{L}_r(a)q) + q\mathrm{L}_r(a)q)$$

and

$$\mathrm{L}_r(b) = (p\mathrm{L}_r(b)p + \tau(p\mathrm{L}_r(b)p)) + p\mathrm{L}_r(b)q + (q\mathrm{L}_r(b)q - \tau(p\mathrm{L}_r(b)p)$$

which gives $\mathrm{L}_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$ and $\mathrm{L}_r(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$. $\square$

**Remark 3.1.** We define $f_{r_1}(a) = q\mathrm{L}_r(a)q$ and $f_{r_2}(b) = p\mathrm{L}_r(b)p$ for any $a \in \mathrm{A},\ b \in \mathrm{B}$. By Lemma 3.3 follows that $f_{r_1} : \mathrm{A} \to q\mathrm{Z}(\mathfrak{T})q$ is a mapping such that $f_{r_1}(\mathfrak{p}_n(\mathrm{A}, \mathrm{A}, \cdots, \mathrm{A})) = 0$ and $f_{r_2} : \mathrm{B} \to p\mathrm{Z}(\mathfrak{T})p$ is a mapping such that $f_{r_2}(\mathfrak{p}_n(\mathrm{B}, \mathrm{B}, \cdots, \mathrm{B})) = 0$. Define the maps $\delta_r : \mathfrak{T} \to \mathfrak{T}$ and $f_r : \mathfrak{T} \to \mathrm{Z}(\mathfrak{T})$ by $\delta_r = \mathrm{L}_r - f_r$ and

$$f_r(x) = f_{r_1}(pxp) + \tau^{-1}(f_{r_1}(pxp)) + f_{r_2}(qxq) + \tau(f_{r_2}(qxq)) \text{ for all } x \in \mathfrak{T}.$$

Obviously, $f_r(\mathrm{M}) = 0$. Hence $\delta_r(\mathrm{M}) = \mathrm{L}_r(\mathrm{M})$. We claim that $f_r(\mathfrak{p}_n(\mathfrak{T}, \mathfrak{T}, \cdots, \mathfrak{T})) = 0$.

Assume $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Since $f_r(x) = f_r(pxp + qxq)$ for each $x \in \mathfrak{T}$, we find that

$$\begin{aligned}
f_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) &= f_r(p(\mathfrak{p}_n(x_1, x_2, \cdots, x_n))p + q(\mathfrak{p}_n(x_1, x_2, \cdots, x_n))q) \\
&= q\mathrm{L}_r(p(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)p)q \\
&\quad + \tau^{-1}(q\mathrm{L}_r(p(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)p)q) \\
&\quad + p\mathrm{L}_r(q(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)q)p \\
&\quad + \tau(p\mathrm{L}_r(q(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)q)p) \\
&= q\mathrm{L}_r(\mathfrak{p}_n(px_1p, px_2p, \cdots, px_np))q \\
&\quad + \tau^{-1}(q\mathrm{L}_r(\mathfrak{p}_n(px_1p, px_2p, \cdots, px_np)q) \\
&\quad + p\mathrm{L}_r(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, qx_nq)p \\
&\quad + \tau(p\mathrm{L}_r(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, qx_nq)p).
\end{aligned}$$

Since

$$\begin{aligned}
& p\mathrm{L}_r(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, qx_nq)p \\
&= p(\mathfrak{p}_n(\mathrm{L}_r(qx_1q), qx_2q, \cdots, qx_nq))p \\
&\quad + p(\mathfrak{p}_n(qx_1q, \mathrm{L}_r(qx_2q), \cdots, qx_nq))p \\
&\quad + p(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, \mathrm{L}_r(qx_nq)))p \\
&\quad + p\left(\sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(qx_1q), \mathrm{L}_{i_2}(qx_2q), \cdots, \mathrm{L}_{i_n}(qx_nq))\right)p \\
&= 0.
\end{aligned}$$

Similarly, $q\mathrm{L}_r(\mathfrak{p}_n(px_1p, px_2p, \cdots, px_np))q = 0$, and hence $f_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Consequently,

$$\begin{aligned}
& \delta_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) \\
&= \mathrm{L}_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(x_1), x_2, \cdots, x_n) + \mathfrak{p}_n(x_1, \mathrm{L}_r(x_2), \cdots, x_n) + \cdots + \mathfrak{p}_n(x_1, x_2, \cdots, \mathrm{L}_r(x_n)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(x_1), \mathrm{L}_{i_2}(x_2), \cdots, \mathrm{L}_{i_n}(x_n)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(x_1) - f_r(x_1), x_2, \cdots, x_n) + \mathfrak{p}_n(x_1, \mathrm{L}_r(x_2) - f_r(x_2), \cdots, x_n) \\
&\quad + \cdots + \mathfrak{p}_n(x_1, x_2, \cdots, \mathrm{L}_r(x_n) - f_r(x_n)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(x_1) - d_{i_1}(x_1), \mathrm{L}_{i_2}(x_2) - d_{i_2}(x_2), \cdots, \mathrm{L}_{i_n}(x_n) - d_{i_n}(x_n)) \\
&= \mathfrak{p}_n(\delta_r(x_1), x_2, \cdots, x_n) + \mathfrak{p}_n(x_1, \delta_r(x_2), \cdots, x_n) + \mathfrak{p}_n(x_1, x_2, \cdots, \delta_r(x_n)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(d_{i_1}(x_1), d_{i_2}(x_2), \cdots, d_{i_n}(x_n))
\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Thus $\{\delta_r\}_{r\in\mathbb{N}}$ is a multiplicative Lie $n$-higher derivation on $\mathfrak{T}$.

Since $\mathfrak{p}_n(p, x, q, \cdots, q) = \mathfrak{p}_n(x, q, q, \cdots, q)$ for all $x \in \mathfrak{T}$, we find that

$$\begin{aligned}
&\mathfrak{p}_n(\mathrm{L}_r(p), x, q, \cdots, q) + \mathfrak{p}_n(p, \mathrm{L}_r(x), q, \cdots, q) + \mathfrak{p}_n(p, x, q, \cdots, \mathrm{L}_r(q)) \\
&\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(p), \mathrm{L}_{i_2}(x), q, \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(x), q, \cdots, q) + \mathfrak{p}_n(x, \mathrm{L}_r(q), \cdots, q) + \mathfrak{p}_n(x, q, \cdots, \mathrm{L}_r(q)) \\
&\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(x), \mathrm{L}_{i_2}(q), \cdots, \mathrm{L}_{i_n}(q)).
\end{aligned}$$

Considering the induction hypothesis, the above equation becomes

$$[\delta_r(p), x] + \sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} [d_{i_1}(p), d_{i_2}(x)] = [x, \delta_r(q)] + \sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} [d_{i_1}(x), d_{i_2}(q)].$$

Note that $d_i$ is additive and $d_i(I) = 0$ for all $0 < i < r$. Thus we arrive $[\delta_r(p), x] = [x, \delta_r(q)]$. That is $\delta_r(p) + \delta_r(q) \in \mathrm{Z}(\mathfrak{T})$. On the other hand, $\delta_r(p) = \mathrm{L}_r(p) - f_r(p) \in \mathrm{M}$ by Lemma 3.2 and $\delta_r(q) \in \mathrm{M}$. By the characterization of the centre of $\mathfrak{T}$, we can calculate that $\delta_r(p) + \delta_r(q) = 0$.

Now from Lemma 3.1 and Lemma 3.2, it is clear that

**Lemma 3.4.** *For $r \in \mathbb{N}$, we have the following:*

1. $\delta_r(0) = 0$,
2. $\delta_r(\mathrm{M}) \subseteq \mathrm{M}$,
3. $\delta_r(p), \delta_r(q) \in \mathrm{M}$ *and* $\delta_r(p) + \delta_r(q) = 0$,
4. $\delta_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}$ *and* $\delta_r(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M}$.

**Lemma 3.5.** *For any $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have*

1. $\delta_r(am) = \delta_r(a)m + a\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} d_{i_1}(a)d_{i_2}(m)$,
2. $\delta_r(mb) = \delta_r(m)b + m\delta_r(b) + \sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} d_{i_1}(m)d_{i_2}(b)$

*for $r \in \mathbb{N}$.*

*Proof.* Using the fact that $\delta_s(q) \in \mathrm{M}$ for all $0 < s \leq r$, we get

$$\begin{aligned}
\delta_r(am) &= \mathrm{L}_r([a,m]) \\
&= \mathrm{L}_r(\mathfrak{p}_n(a,m,q,\cdots,q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a),\delta_{i_2}(m),\delta_{i_3}(q),\cdots,\delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a),m,q,\cdots,q)+\mathfrak{p}_n(a,\delta_r(m),q,\cdots,q) \\
&\quad +\mathfrak{p}_n(a,m,\delta_r(q),q,\cdots,q)+\cdots+\mathfrak{p}_n(a,m,q,\cdots,\delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(d_{i_1}(a),d_{i_2}(m),d_{i_3}(q),\cdots,d_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a),m],q,\cdots,q)+\mathfrak{p}_{n-1}([a,\delta_r(m)],q,\cdots,q) \\
&\quad + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \mathfrak{p}_{n-1}([d_{i_1}(a),d_{i_2}(m)],q,\cdots,q) \\
&= \delta_r(a)m + a\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a)d_{i_2}(m).
\end{aligned}$$

for $a \in \mathrm{A}, m \in \mathrm{M}$. Likewise, $\delta_r(mb) = \delta_r(m)b + m\delta_r(b) + \sum\limits_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(m)d_{i_2}(b)$ for all $b \in \mathrm{B}, m \in \mathrm{M}$. $\square$

**Lemma 3.6.** *For any $a_1, a_2 \in \mathrm{A}$ and $b_1, b_2 \in \mathrm{B}$, we have*

1. $\delta_r(a_1a_2) = \delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum\limits_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)$;
2. $\delta_r(b_1b_2) = \delta_r(b_1)b_2 + b_1\delta_r(b_2) + \sum\limits_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2)$

*for $r \in \mathbb{N}$.*

*Proof.* For any $a_1, a_2 \in \mathrm{A}$ and $m \in \mathrm{M}$.

$$\begin{aligned}
\delta_r(a_1a_2m) &= \delta_r((a_1a_2)m) \\
&= \delta_r(a_1a_2)m + a_1a_2\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1a_2)d_{i_2}(m) \\
&= \delta_r(a_1a_2)m + a_1a_2\delta_r(m) + \sum_{\substack{i_1+i_2+i_3=r \\ 0\leq i_1,i_2<r \\ 0<i_3<r}} d_{i_1}(a_1)d_{i_2}(a_2)d_{i_3}(m).
\end{aligned}$$

On the other way,

$$\begin{aligned}
\delta_r(a_1a_2m) &= \delta_r(a_1(a_2m)) \\
&= \delta_r(a_1)a_2m + a_1\delta_r(a_2m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2m) \\
&= \delta_r(a_1)a_2m + a_1\delta_r(a_2)m + a_1a_2\delta_r(m) \\
&\quad + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)m + \sum_{\substack{i_1+i_2+i_3=r \\ 0\leq i_1,i_2<r \\ 0<i_3<r}} d_{i_1}(a_1)d_{i_2}(a_2)d_{i_3}(m).
\end{aligned}$$

By the condition $\mathbf{C}_s$, the above expression becomes

$$\delta_r(a_1a_2)m = \big\{\delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\big\}m.$$

Since $\delta_r(\mathrm{A}) \subseteq \mathrm{A}+\mathrm{M}$ and M is faithful as left A-module, the above relation implies that

$$\delta_r(a_1a_2)p = \big\{\delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\big\}p. \tag{3.2}$$

Also, $[a_1, q] = 0$ for all $a_1 \in \mathrm{A}$

$$\begin{aligned}
0 &= \mathrm{L}_r(\mathfrak{p}_n(a_1, q, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a_1), \delta_{i_2}(q), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a_1), q, q, \cdots, q) + \mathfrak{p}_n(a_1, \delta_r(q), q, \cdots, q) \\
&\quad + \mathfrak{p}_n(a_1, q, \delta_r(q), q, \cdots, q) + \cdots + \mathfrak{p}_n(a_1, q, q, \cdots, \delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(d_{i_1}(a_1), d_{i_2}(q), d_{i_3}(q), \cdots, d_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a_1), q], q, \cdots, q) + \mathfrak{p}_{n-1}([a_1, \delta_r(q)], q, \cdots, q) \\
&\quad + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \mathfrak{p}_{n-1}([d_{i_1}(a_1), d_{i_2}(q)], q, \cdots, q).
\end{aligned}$$

Since $\delta_r(\mathrm{A}) \subseteq \mathrm{A}+\mathrm{M}, \delta_r(q) \in \mathrm{M}$. The above equation implies that

$$0 = \delta_r(a_1)q + a_1\delta_r(q) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(q) \text{ for all } a_1 \in \mathrm{A}. \tag{3.3}$$

On substituting $a_1$ by $a_2$ and $a_1a_2$ in (3.3) respectively, we get

$$0 = \delta_r(a_2)q + a_2\delta_r(q) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_2)d_{i_2}(q) \tag{3.4}$$

and

$$0 = \delta_r(a_1a_2)q + a_1a_2\delta_r(q) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1a_2)d_{i_2}(q). \tag{3.5}$$

Now left multiplying $a_1$ in (3.4) and combining it with (3.5) gives

$$\delta_r(a_1a_2)q + \sum_{\substack{i_1+i_2+i_3=r\\0<i_1,i_2,i_3<r}} d_{i_1}(a_1)d_{i_2}(a_2)d_{i_3}(q) = a_1\delta_r(a_2)q$$

which implies that

$$\delta_r(a_1a_2)q + \sum_{i_1=1}^{r-1} d_{i_1}(a_1) \sum_{\substack{i_2+i_3=r\\0<i_2,i_3<r}} d_{i_2}(a_2)d_{i_3}(q) = a_1\delta_r(a_2)q.$$

Now using the condition $\mathbf{C}_s$, we find that

$$\delta_r(a_1a_2)q - \sum_{i_1=1}^{r-1} d_{i_1}(a_1)d_{r-i_1}(a_2)q = a_1\delta_r(a_2)q$$

gives us

$$\delta_r(a_1a_2)q = a_1\delta_r(a_2)q + \sum_{i_1=1}^{r-1} d_{i_1}(a_1)d_{r-i_1}(a_2)q.$$

Hence,

$$\delta_r(a_1a_2)q = \Big\{\delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\Big\}q. \tag{3.6}$$

Now adding the (3.2) and (3.6), we have

$$\delta_r(a_1a_2) = \delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2).$$

Similarly, we can obtain that

$$\delta_r(b_1b_2) = \delta_r(b_1)b_2 + b_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2)$$

for all $b_1, b_2 \in \mathrm{B}$. $\square$

**Lemma 3.7.** *For any $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have*

1. $\delta_r(a+m) - \delta_r(a) - \delta_r(m) \in \mathrm{Z}(\mathfrak{T})$;

*2.* $\delta_r(b+m)-\delta_r(b)-\delta_r(m)\in \mathrm{Z}(\mathfrak{T})$

*for* $r\in\mathbb{N}$.

*Proof.* Let $a\in \mathrm{A}$ and $m,m_1\in \mathrm{M}$. Since $[a,m_1]=[a+m,m_1]$, we find

$$\mathrm{L}_r(\mathfrak{p}_n(a,m_1,q,\cdots,q)) = \mathrm{L}_r(\mathfrak{p}_n(a+m,m_1,q,\cdots,q)). \tag{3.7}$$

Using induction hypothesis, Lemma 3.3 and (3.7) reduces to

$$\mathfrak{p}_n(\delta_r(a),m_1,q,\cdots,q)=\mathfrak{p}_n(\delta_r(a+m),m_1,q,\cdots,q).$$

Therefore, $[\delta_r(a),m_1]=[\delta_r(a+m),m_1]$ and hence $[\delta_r(a+m)-\delta_r(a),\mathrm{M}]=0$. Hence, we get that

$$\begin{aligned}&\delta_r(a+m)-\delta_r(a)-p(\delta_r(a+m)-\delta_r(a))q\\ &\quad= p(\delta_r(a+m)-\delta_r(a))p+q(\delta_r(a+m)-\delta_r(a))q\in \mathrm{Z}(\mathfrak{T})\end{aligned} \tag{3.8}$$

for all $a\in \mathrm{A}, m\in \mathrm{M}$. Applying Lemma 3.2, 3.4 and Remark 3.1, we have

$$\begin{aligned}
&p(\delta_r(a+m)-\delta_r(a))q\\
&= [p,\delta_r(a+m)-\delta_r(a)]\\
&= [p,\mathrm{L}_r(a+m)]-[p,\mathrm{L}_r(a)]\\
&= \mathrm{L}_r(\mathfrak{p}_n(p,a+m,q,\cdots,q))-\mathfrak{p}_n(\mathrm{L}_r(p),a+m,q,\cdots,q)\\
&\quad -\mathfrak{p}_n(p,a+m,\mathrm{L}_r(q),\cdots,q)-\cdots-\mathfrak{p}_n(p,a+m,q,\cdots,\mathrm{L}_r(q))\\
&\quad -\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ 0\le i_1,i_2,\cdots,i_n<r}}\mathfrak{p}_n(\delta_{i_1}(p),\delta_{i_2}(a+m),\delta_{i_3}(q),\cdots,\delta_{i_n}(q))\\
&\quad -\mathrm{L}_r(\mathfrak{p}_n(p,a,q,\cdots,q))+\mathfrak{p}_n(\mathrm{L}_r(p),a,q,\cdots,q)\\
&\quad +\mathfrak{p}_n(p,a,\mathrm{L}_r(q),\cdots,q)+\cdots+\mathfrak{p}_n(p,a,q,\cdots,\mathrm{L}_r(q))\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ 0\le i_1,i_2,\cdots,i_n<r}}\mathfrak{p}_n(\delta_{i_1}(p),\delta_{i_2}(a),\delta_{i_3}(q),\cdots,\delta_{i_n}(q))\\
&= \mathrm{L}_r(\mathfrak{p}_n(p,m,q,\cdots,q))\\
&= \mathrm{L}_r(m)=\delta_r(m).
\end{aligned}$$

From (3.8), it follows that $\delta_r(a+m)-\delta_r(a)-\delta_r(m)\in \mathrm{Z}(\mathfrak{T})$ for all $a\in \mathrm{A}, m\in \mathrm{M}$. Similarly, we can prove $\delta_r(b+m)-\delta_r(b)-\delta_r(m)\in \mathrm{Z}(\mathfrak{T})$ for all $b\in \mathrm{B}, m\in \mathrm{M}$. □

**Lemma 3.8.** $\delta_r$ *is additive on* $\mathrm{A},\mathrm{M}$ *and* $\mathrm{B}$.

*Proof.* Using $m_1+m_2=\mathfrak{p}_n(p+m_1, m_2+q, q, \cdots, q)$ and Lemma 3.4, we find that

$$\begin{aligned}
\delta_r(m_1+m_2) &= \mathrm{L}_r(\mathfrak{p}_n(p+m_1, m_2+q, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(p+m_1), \delta_{i_2}(m_2+q), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(p+m_1), m_2+q, q, \cdots, q) \\
&\quad +\mathfrak{p}_n(p+m_1, \delta_r(m_2+q), q, \cdots, q) \\
&\quad +\mathfrak{p}_n(p+m_1, m_2+q, \delta_r(q), q, \cdots, q) \\
&\quad +\cdots+\mathfrak{p}_n(p+m_1, m_2+q, q, \cdots, \delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(p+m_1), \delta_{i_2}(m_2+q), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(p+m_1), m_2+q], q, \cdots, q) \\
&\quad +\mathfrak{p}_{n-1}([p+m_1, \delta_r(m_2+q)], q, \cdots, q) \\
&= \mathfrak{p}_{n-1}([\delta_r(p)+\delta_r(m_1), m_2+q], \cdots, q) \\
&\quad +\mathfrak{p}_{n-1}([p+m_1, \delta_r(m_2)+\delta_r(q)], \cdots, q) \\
&= \delta_r(p)+\delta_r(m_1)+\delta_r(m_2)+\delta_r(q) \\
&= \delta_r(m_1)+\delta_r(m_2)
\end{aligned}$$

for all $m_1, m_2 \in \mathrm{M}$. Now,

$$\begin{aligned}
\delta_r((a_1+a_2)m) &= \delta_r(a_1m)+\delta_r(a_2m) \\
&= \delta_r(a_1)m+a_1\delta_r(m)+\sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} d_{i_1}(a_1)d_{i_2}(m) \\
&\quad +\delta_r(a_2)m+a_2\delta_r(m)+\sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} d_{i_1}(a_2)d_{i_2}(m) \qquad (3.9)
\end{aligned}$$

for all $a_1, a_2 \in \mathrm{A}$ and $m \in \mathrm{M}$. On the other hand,

$$\begin{aligned}
\delta_r((a_1+a_2)m) &= \delta_r(a_1+a_2)m+(a_1+a_2)\delta_r(m) \\
&\quad + \sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} d_{i_1}(a_1+a_2)d_{i_2}(m). \qquad (3.10)
\end{aligned}$$

Combining (3.9), (3.10) and applying condition $\mathbf{C}_s$, we have

$$\delta_r(a_1+a_2)m=\delta_r(a_1)m+\delta_r(a_2)m. \qquad (3.11)$$

Since $\delta_r(\mathrm{A}) \subseteq \mathrm{A}+\mathrm{M}$ and M is faithful as left A. Then (3.11) implies that

$$\delta_r(a_1+a_2)p=\delta_r(a_1)p+\delta_r(a_2)p. \qquad (3.12)$$

Replace $a_1$ for $a_1+a_2$ in (3.3), we get

$$0=\delta_r(a_1+a_2)q+(a_1+a_2)\delta_r(q)+\sum_{\substack{i_1+i_2=r \\ 0<i_1, i_2<r}} d_{i_1}(a_1+a_2)d_{i_2}(q) \qquad (3.13)$$

for all $a_1 \in \mathrm{A}$. Combining (3.13) with (3.3) and (3.4), we obtain

$$\delta_r(a_1 + a_2)q = \delta_r(a_1)q + \delta_r(a_2)q. \tag{3.14}$$

Addition of (3.12) and (3.14) implies that $\delta_r(a_1+a_2) = \delta_r(a_1)+\delta_r(a_2)$ for all $a_1, a_2 \in \mathrm{A}$.

Similarly, we can deduce that $\delta_r(b_1+b_2) = \delta_r(b_1)+\delta_r(b_2)$ for all $b_1, b_2 \in \mathrm{B}$. □

**Lemma 3.9.** $\delta_r(a+m+b) - \delta_r(a) - \delta_r(m) - \delta_r(b) \in \mathrm{Z}(\mathfrak{T})$ *for all* $a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B}$.

*Proof.* Using induction hypothesis and fact $\delta_r(q) \in \mathrm{M}$. On one hand, we have

$$\begin{aligned}
&\mathrm{L}_r(\mathfrak{p}_n(a+m+b), m_1, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a+m+b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a+m+b), m_1, q, \cdots, q) + \mathfrak{p}_n(a+m+b, \delta_r(m_1), q, \cdots, q) \\
&\quad +\mathfrak{p}_n(a+m+b, m_1, \delta_r(q), q, \cdots, q) + \cdots + \mathfrak{p}_n(a+m+b, m_1, q, \cdots, \delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(a+m+b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a+m+b), m_1], q, \cdots, q) + \mathfrak{p}_{n-1}([a+m+b, \delta_r(m_1)], q, \cdots, q) \\
&= [\delta_r(a+m+b), m_1] + [a+m+b, \delta_r(m_1)] \\
&= [\delta_r(a+m+b), m_1] + [a, \delta_r(m_1)] + [b, \delta_r(m_1)]
\end{aligned} \tag{3.15}$$

for all $a \in \mathrm{A}, m, m_1 \in \mathrm{M}, b \in \mathrm{B}$. On the other hand, using Lemma 3.8, we obtain

$$\begin{aligned}
&\mathrm{L}_r(\mathfrak{p}_n(a+m+b, m_1, q, \cdots, q)) \\
&= \mathrm{L}_r([a, m_1] + [b, m_1]) \\
&= \mathrm{L}_r(\mathfrak{p}_n(a, m_1, q, \cdots, q)) + \mathrm{L}_r(\mathfrak{p}_n(b, m_1, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad + \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a), m_1, q, \cdots, q) + \mathfrak{p}_n(a, \delta_r(m_1), q, \cdots, q) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(a), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad +\mathfrak{p}_n(\delta_r(b), m_1, q, \cdots, q) + \mathfrak{p}_n(b, \delta_r(m_1), q, \cdots, q) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a), m_1], q, \cdots, q) + \mathfrak{p}_{n-1}([a, \delta_r(m_1)], q, \cdots, q) \\
&\quad +\mathfrak{p}_{n-1}([\delta_r(b), m_1], q, \cdots, q) + \mathfrak{p}_{n-1}([b, \delta_r(m_1)], q, \cdots, q) \\
&= [\delta_r(a), m_1] + [a, \delta_r(m_1)] + [\delta_r(b), m_1] + [b, \delta_r(m_1)]
\end{aligned} \tag{3.16}$$

for all $a \in \mathrm{A}, m, m_1 \in \mathrm{M}, b \in \mathrm{B}$. Combining (3.15) and (3.16), we get

$$[\delta_r(a+m+b) - \delta_r(a) - \delta_r(b), \mathrm{M}] = 0,$$

which in turn implies that

$$\delta_r(a+m+b) - \delta_r(a) - \delta_r(b) - p(\delta_r(a+m+b) - \delta_r(a) - \delta_r(b))q \in \mathrm{Z}(\mathfrak{T})$$

for all $a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B}$.

$$\begin{aligned}
&p(\delta_r(a+m+b) - \delta_r(a) - \delta_r(b))q \\
&\quad = [p, \delta_r(a+m+b) - \delta_r(a) - \delta_r(b)] \\
&\quad = [p, \mathrm{L}_r(a+m+b)] - [p, \mathrm{L}_r(a)] - [p, \mathrm{L}_r(b)] \\
&\quad = \mathrm{L}_r(\mathfrak{p}_n(p, a+m+b, q, \cdots, q)) - \mathfrak{p}_n(\mathrm{L}_r(p), a+m+b, q, \cdots, q) \\
&\qquad -\mathfrak{p}_n(p, a+m+b, \mathrm{L}_r(q), \cdots, q) - \cdots - \mathfrak{p}_n(p, a+m+b, q, \cdots, \mathrm{L}_r(q)) \\
&\qquad - \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(p), \delta_{i_2}(a+m+b), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\qquad -\mathrm{L}_r(\mathfrak{p}_n(p, a, q, \cdots, q)) + \mathfrak{p}_n(\mathrm{L}_r(p), a, q, \cdots, q) \\
&\qquad +\mathfrak{p}_n(p, a, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(p, a, q, \cdots, \mathrm{L}_r(q)) \\
&\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(p), \delta_{i_2}(a), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\qquad -\mathrm{L}_r(\mathfrak{p}_n(p, b, q, \cdots, q)) + \mathfrak{p}_n(\mathrm{L}_r(p), b, q, \cdots, q) \\
&\qquad +\mathfrak{p}_n(p, b, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(p, b, q, \cdots, \mathrm{L}_r(q)) \\
&\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(p), \delta_{i_2}(b), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad = \mathrm{L}_r(\mathfrak{p}_n(p, m, q, \cdots, q)) \\
&\quad = \mathrm{L}_r(m) = \delta_r(m).
\end{aligned}$$

This leads to $\delta_r(a+m+b) - \delta_r(a) - \delta_r(m) - \delta_r(b) \in \mathrm{Z}(\mathfrak{T})$ for all $a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B}$. $\square$

**Remark 3.2.** Now we establish a mapping $g_r : \mathfrak{T} \to \mathrm{Z}(\mathfrak{T})$ by

$$g_r(x) = \delta_r(x) - \delta_r(pxp) - \delta_r(pxq) - \delta_r(qxq) \text{ for all } x \in \mathfrak{T}.$$

Obviously, $g_r(\mathrm{A}) = g_r(\mathrm{M}) = g_r(\mathrm{B}) = 0$. Observe that $g_r(\mathfrak{p}_n(\mathfrak{T}, \mathfrak{T}, \cdots, \mathfrak{T})) = 0$. Define a mapping $d_r(x) = \delta_r(x) - g_r(x)$ for all $x \in \mathfrak{T}$. It is easy to verify for each $r \in \mathbb{N}$, $d_r$ satisfies $d_r(a+m+b) = d_r(a) + d_r(m) + d_r(b)$. From the definition of $d_r$ and $g_r$, it follows that

$$\mathrm{L}_r = \delta_r + f_r = d_r + g_r + f_r = d_r + h_r, \quad \text{where} \quad h_r = g_r + f_r.$$

*Proof.* [Proof of Theorem 3.1] Suppose $x, y \in \mathfrak{T}$ such that $x = a_1 + m_1 + b_1$ and $y = a_2 + m_2 + b_2$. Then

$$
\begin{aligned}
d_r(x+y) &= d_r\big((a_1+m_1+b_1)+(a_2+m_2+b_2)\big)\\
&= d_r\big((a_1+a_2)+(m_1+m_2)+(b_1+b_2)\big)\\
&= \delta_r(a_1+a_2)+\delta_r(m_1+m_2)+\delta_r(b_1+b_2)\\
&= \delta_r(a_1)+\delta_r(a_2)+\delta_r(m_1)+\delta_r(m_2)+\delta_r(b_1)+\delta_r(b_2)\\
&= d_r(a_1+m_1+b_1)+d_r(a_2+m_2+b_2)\\
&= d_r(x)+d_r(y).
\end{aligned}
$$

By Lemma 3.6 and Lemma 3.7, we have

$$
\begin{aligned}
d_r(xy) &= d_r\big((a_1+m_1+b_1)(a_2+m_2+b_2)\big)\\
&= \delta_r(a_1a_2+a_1m_2+m_1b_2+b_1b_2)\\
&= \delta_r(a_1)a_2+a_1\delta_r(a_2)+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\\
&\quad +\delta_r(a_1)m_2+a_1\delta_r(m_2)+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m_2)\\
&\quad +\delta_r(m_1)b_2+m_1\delta_r(b_2)+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(b_2)\\
&\quad +\delta_r(b_1)b_2+b_1\delta_r(b_2)+\sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2). \qquad (3.17)
\end{aligned}
$$

On the other hand, we have

$$\begin{aligned}
& d_r(x)y + xd_r(y) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(x)d_{i_2}(y)\\
= \quad & (\delta_r(a_1)+\delta_r(m_1)+\delta_r(b_1))y + x(\delta_r(a_2)+\delta_r(m_2)+\delta_r(b_2))\\
& + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m_2)\\
& + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(a_2)\\
& + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(m_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(b_2)\\
& + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(m_2)\\
& + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2). \qquad (3.18)
\end{aligned}$$

By using condition $\mathbf{C}_s$, and from Lemma 3.6, we have

$$\begin{aligned}
& d_r(x)y + xd_r(y) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(x)d_{i_2}(y)\\
= \quad & \delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2) + \delta_r(a_1)m_2 + a_1\delta_r(m_2)\\
& + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m_2) + \delta_r(m_1)b_2 + m_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(b_2)\\
& +\delta_r(b_1)b_2 + b_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2). \qquad (3.19)
\end{aligned}$$

Combining (3.17) and (3.19) ,we get $d_r(xy) = d_r(x)y+xd_r(y)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(x)d_{i_2}(y)$.

This implies that, $\{d_r\}_{r\in\mathbb{N}}$ is an additive higher derivation on $\mathfrak{T}$. Finally, there exists a map $h_r : \mathfrak{T} \to Z(\mathfrak{T})$ such that $h_r(\mathfrak{p}_n(\mathfrak{T},\mathfrak{T},\cdots,\mathfrak{T})) = \mathrm{L}_r(\mathfrak{p}_n(\mathfrak{T},\mathfrak{T},\cdots,\mathfrak{T})) - d_r(\mathfrak{p}_n(\mathfrak{T},\mathfrak{T},\cdots,\mathfrak{T})) = 0$. This completes the proof. □

As a direct consequence of Theorem 3.1, we have the following result:

**Corollary 3.1.** *[3, Theorem 3.1] Let $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ be a 2-torsion free triangular algebra consisting of unital algebras $\mathrm{A},\mathrm{B}$ and a faithful unital $(\mathrm{A},\mathrm{B})$-bimodule $\mathrm{M}$. Suppose that $\mathfrak{T}$ satisfies the following conditions:*

1. $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{B})$,

2. $\mathrm{Z}(\mathrm{A}) = \{a \in \mathrm{A} \mid [[a, x], y] = 0 \ \forall\ x, y \in \mathrm{A}\}$
*or* $\mathrm{Z}(\mathrm{B}) = \{b \in \mathrm{B} \mid [[b, x], y] = 0 \ \forall\ x, y \in \mathrm{B}\}$.

*Then every multiplicative Lie triple higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ on $\mathfrak{T}$ has the standard form. More precisely, there exists an additive higher derivation $\mathfrak{D} = \{d_r\}_{r\in\mathbb{N}}$ on $\mathfrak{T}$ and a sequence of functionals $\{h_r\}_{r\in\mathbb{N}}$ which annihilates all Lie triple product $[[x_1, x_2], x_3]$ for all $x_1, x_2, x_3 \in \mathfrak{T}$ such that $\mathrm{L}_r(x) = d_r(x) + h_r(x)$ for all $x \in \mathfrak{T}$ and $r \in \mathbb{N}$.*

## 4. Applications

In this section, we apply Theorem 3.1 to some triangular and related algebras, such as upper triangular matrix algebras, block upper triangular matrix algebras, nest algebras, incidence algebras.

Since an arbitrary derivation on $\mathcal{T}(\mathcal{N})$ is inner and in view of [23, Proposition 2.6], we know that an arbitrary higher derivation on $\mathcal{T}(\mathcal{N})$ is inner.

**Corollary 4.1.** *Let $X$ be an infinite dimensional Banach space over the real or complex field $\mathbb{F}$ , $\mathcal{N}$ be a nest on $X$ which contains a nontrivial element complemented in $X$ and $\mathcal{T}(\mathcal{N})$ be a nest algebra. Then for every multiplicative Lie $n$-higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$, there exists an inner higher derivation $\{d_r\}_{r\in\mathbb{N}}$ on $\mathcal{T}(\mathcal{N})$ and a sequence of functionals $\{h_r\}_{r\in\mathbb{N}}$ which annihilates all $(n-1)$-th commutators $\mathfrak{p}_n(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N}), \cdots, \mathcal{T}(\mathcal{N}))$ such that $\mathrm{L}_r = d_r + h_r$, where $d_r : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ and $h_r : \mathcal{T}(\mathcal{N}) \to \mathbb{F}I$ for $r \in \mathbb{N}$.*

**Corollary 4.2.** *Let $\mathcal{N}$ be a nest of a Hilbert space $H$ dimension greater than* 2 *and $\mathcal{T}(\mathcal{N})$ be a nontrivial nest algebra. Then for every multiplicative Lie $n$-higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$, there exists an inner higher derivation $\{d_r\}_{r\in\mathbb{N}}$ on $\mathcal{T}(\mathcal{N})$ and a sequence of functionals $\{h_r\}_{r\in\mathbb{N}}$ which annihilates all $(n-1)$-th commutators $\mathfrak{p}_n(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N}), \cdots, \mathcal{T}(\mathcal{N}))$ such that $\mathrm{L}_r = d_r + h_r$, where $d_r : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ and $h_r : \mathcal{T}(\mathcal{N}) \to \mathbb{F}I$ for each $r \in \mathbb{N}$.*

If Hilbert space $H$ is finite dimensional, then nest algebras are upper block triangular matrices algebras [7].

**Corollary 4.3.** *Let* R *be a $(n-1)$-torsion free commutative ring with unity and $B_m^{\bar{k}}(\mathrm{R})(m \geq 3)$ i.e. block upper triangular matrix algebra defined over* R *with $B_m^{\bar{k}}(\mathrm{R}) \neq M_m(\mathrm{R})$. Then for every multiplicative Lie $n$-higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$, there exist an inner higher derivation $\{d_r\}_{r\in\mathbb{N}}$ on $B_m^{\bar{k}}(\mathrm{R})$ and a sequence of functionals $\{h_r\}_{r\in\mathbb{N}}$ which annihilates all $(n-1)$-th commutators $\mathfrak{p}_n(B_m^{\bar{k}}(\mathrm{R}), B_m^{\bar{k}}(\mathrm{R}), \cdots, B_m^{\bar{k}}(\mathrm{R}))$ such that $\mathrm{L}_r = d_r + h_r$, where $d_r : B_m^{\bar{k}}(\mathrm{R}) \to B_m^{\bar{k}}(\mathrm{R})$ and $h_r : B_m^{\bar{k}}(\mathrm{R}) \to \mathrm{R}I$ for each $r \in \mathbb{N}$.*

*Proof.* It can be easily seen that conditions of Theorem 3.1 hold for block upper triangular matrix algebra and since all derivations of $B_m^{\bar{k}}(\mathrm{R})$ are inner. By [23, Proposition 2.6] we arrive at that any higher derivation of $B_m^{\bar{k}}(\mathrm{R})$ is inner. Hence the result follows. □

Note that $T_m(\mathrm{R}) \subseteq B_m^{\bar{k}}(\mathrm{R}) \subseteq M_m(\mathrm{R})(m \geq 3)$ is a proper block upper triangular matrix algebra over a commutative ring R.

**Corollary 4.4.** *Every multiplicative Lie $n$-higher derivation has standard form on upper triangular matrix algebra $T_m(\mathrm{R})$.*

*Incidence algebra.* Let R be a commutative ring with unity. Let $X$ be a finite partially ordered set (poset) with the partial order $\leq$. We define the incidence algebra of $X$ over R as $\mathrm{I}(X, \mathrm{R}) = \{f : X \times X \to \mathrm{R} \mid f(x, y) = 0 \ if \ x \nleq y\}$ with algebraic operation given by

1. $(f + g)(x, y) = f(x, y) + g(x, y)$,
2. $(f \star g)(x, y) = \sum\limits_{x \leq z \leq y} f(x, z)g(z, y)$,
3. $(r.f)(x, y) = r.f(x, y)$

for all $f, g \in \mathrm{I}(X, \mathrm{R}), r \in \mathrm{R}$ and $x, y, z \in X$. Obviously, $f$ is an R-valued function on $\{(x, y) \in X \times X \mid x \leq y\}$. The product $\star$ is usually called convolution in function theory. If $X$ is a partially ordered set (poset) with $n$ elements, then $\mathrm{I}(X, \mathrm{R})$ is isomorphic to a subalgebra of the algebra $M_n(\mathrm{R})$ of square matrices over R with elements $[a_{ij}] \in M_n(\mathrm{R})$ satisfying $a_{ij} = 0$ if $i \nleq j$, for some partial order $\leq$ defined in the partial order set (poset) $\{1, \ldots, n\}$. This shows that $\mathrm{I}(X, \mathrm{R})$ is a triangular algebra.

The incidence algebra of a partially ordered set (poset) $X$ is the algebra of functions from the segments of $X$ into R, which extends the various convolutions in algebras of arithmetic functions. Incidence algebras, in fact, were first considered by Ward [22] as generalized algebras of arithmetic functions. Rota and Stanley [21] developed incidence algebras as the fundamental structures of enumerative combinatorial theory and allied areas of arithmetic function theory. The theory of Möbius functions, including the classical Möbius function of number theory and the combinatorial inclusion-exclusion formula, is established in the context of incidence algebras. For the later, we refer the reader to [21, Sections 2.1 and 3.7].

In the theory of operator algebras, incidence algebras of a finite poset $X$ are referred as bigraph algebras or finite dimensional CSL algebras. If $X$ is connected, then $\mathrm{Z}(\mathrm{I}(X, \mathrm{R})) = \mathrm{R}I$. Clearly, any incidence algebra $\mathrm{I}(X, \mathrm{R})$ is a triangular algebra and hence it satisfies the condition ($\sharp$). Then we have

**Corollary 4.5.** *Let* R *be a $(n-1)$-torsion free commutative ring with unity, $X$ be a connected finite partially ordered set (poset) with the partial order $\leq$ and* $\mathrm{I}(X, \mathrm{R})$ *an incidence algebra of $X$ over* R. *Then every multiplicative Lie $n$-higher derivation has the standard form.*

## 5. For Future Discussions

In this section, we make an attempt to pull out attention of readers towards the obtainable research problem. Let us observe a more general class of maps. Note down the sequence of polynomials:

$$\begin{aligned}
\mathfrak{q}_1(x_1) &= x_1, \\
\mathfrak{q}_2(x_1,x_2) &= \mathfrak{q}_1(x_1)\circ x_2 = x_1\circ x_2, \\
&\vdots \qquad\qquad\quad \vdots \\
\mathfrak{q}_n(x_1,x_2,\cdots,x_n) &= \mathfrak{q}_{n-1}(x_1,x_2,\cdots,x_{n-1})\circ x_n.
\end{aligned}$$

The polynomial $\mathfrak{q}_n(x_1,x_2,\cdots,x_n)$ is called $(n-1)$-th anti-commutator where $n\geq 2$. Let R be a commutative ring with unity and $\mathcal{A}$ be an R-algebra. A map (not necessarily linear) $\mathfrak{J}:\mathcal{A}\to\mathcal{A}$ is said to be a multiplicative Jordan $n$-derivation on $\mathcal{A}$ if

$$\mathfrak{J}(\mathfrak{q}_n(x_1,x_2,\cdots,x_n)) = \sum_{i=1}^{n}\mathfrak{q}_n(x_1,x_2,\cdots,x_{i-1},\mathfrak{J}(x_i),x_{i+1},\cdots,x_n)$$

for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$.

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathfrak{J}=\{\mathfrak{J}_r\}_{r\in\mathbb{N}}$ be a family of maps $\mathfrak{J}_r:\mathcal{A}\to\mathcal{A}$ (not necessarily linear) such that $\mathfrak{J}_0=I_{\mathcal{A}}$. Then $\mathfrak{J}$ is called a multiplicative Jordan $n$-higher derivation if

$$\mathfrak{J}_r(\mathfrak{q}_n(x_1,x_2,\cdots,x_n)) = \sum_{i_1+i_2+\cdots+i_n=r}\mathfrak{q}_n(\mathfrak{J}_{i_1}(x_1),\mathfrak{J}_{i_2}(x_2),\cdots,\mathfrak{J}_{i_n}(x_n))$$

for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$ and for each $r\in\mathbb{N}$. It is easy to see that any multiplicative Jordan 2-higher derivation is a multiplicative Jordan higher derivation and multiplicative Jordan 3-higher derivation is multiplicative Jordan triple higher derivation. Thus multiplicative Jordan higher/Jordan triple higher/$\cdots$/Jordan $n$-higher derivation collectively known as multiplicative Jordan type higher derivations on $\mathcal{A}$. At this point, in view of [2, 4], it is reasonable to raise the following open problem as:

**Problem 5.1.** What is the most general form of multiplicative Jordan type higher derivations on triangular algebras and which constraints are needed to apply on triangular algebras?

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## REFERENCES


1. I. Z. Abdullaev: *$n$-Lie derivations on von Neumann algebra*. Uzbek. Mat. Zh. **5** (1992), no. 6, 3-–9.
2. M. Ashraf and A. Jabeen: *Nonlinear Jordan triple derivable mappings on triangular algebras*. Pac. J. Appl. Math. **7** (2015), no. 4, 225—235.
3. M. Ashraf and A. Jabeen: Contemp. math., Advances in rings and modules, vol. 715, ch. *Nonlinear Lie triple higher derivation on triangular algebras*. pp. 43—57, Amer. Math. Soc., Providence, RI, 2018.
4. M. Ashraf and A. Jabeen: *Nonlinear Jordan triple higher derivable mappings of triangular algebras*. Southeast Asian Bull. Math. **42** (2018), no. 1, 503-–520.
5. M. Ashraf, B. A. Wani and F. Wei: *Multiplicative $*$-Lie triple higher derivations of standard operator algebras*. Quaest. Math. **42** (2019), no. 7, 857-–884.
6. D. Benkovič and D. Eremita: *Multiplicative Lie $n$-derivations of triangular rings*. Linear Algebra Appl. **436** (2012), no. 11, 4223—4240.
7. W. S. Cheung: *Maps on triangular algebras*. Ph.D. dissertation, University of Victoria, 2000.
8. M. Daif: *When is a multiplicative derivation additive?*. Internat. J. Math. Sci. **14** (1991), 615—618.
9. Y. Ding and J. Li: *Lie $n$-higher derivations and Lie $n$-higher derivable mappings*. Bull. Aust. Math. Soc. **96** (2017), no. 2, 223-–232.
10. A. Fošner, F. Wei and Z. Xiao: *Nonlinear Lie-type derivations of von Neumann algebras and related topics*. Colloq. Math. **132** (2013), no. 1, 53—71.
11. D. Han: *Lie type higher derivations on operator algebras*. Bull. Iranian Math. Soc. **40** (2014), no. 5, 1169—1194.
12. A. Jabeen and M. Ashraf: *Nonlinear $*$-Lie derivations on unital algebras*. Beitr. Algebra Geom. **61** (2020), no. 4, 731—746.
13. A. R. Janfada, H. Saidi and M. Mirzavaziri: *Characterization of Lie higher derivations on $C^*$-algebras*. Bull. Iranian Math. Soc. **41** (2015), no. 4, 901—906.
14. P. Ji, R. Liu and Y. Zhao: *Nonlinear Lie triple derivations of triangular algebras*. Linear Multilinear Algebra **60** (2012), no. 10, 1155-–1164.
15. P. Ji and W. Qi: *Characterizations of Lie derivations of triangular algebras*. Linear Algebra Appl. **435** (2011), 1137—1146.
16. J. Li and Q. Shen: *Characterzations of Lie higher and Lie triple derivations of triangular algebra*. J. Korean Math. Soc. **49** (2012), no. 2, 419-–433.
17. W. S. Martindale III: *Lie derivations of primitive rings*. Michigan Math. J. **11** (1964), 183—187.
18. W. S. Martindale III: *When are multiplicative mappings additive?*. Proc. Amer. Math. **21** (1969), 695—698.
19. C. R. Miers: *Lie triple derivations of von Neumann algebras*. Proc. Amer. Math. Soc. **71** (1978), no. 1, 57—61.
20. X. Qi and J. Hou: *Lie higher derivations on nest algebras*. Commun. Math. Res. **26** (2010), no. 2, 131-–143.

21. R. P. Stanley and G. C. Rota: *Enumerative combinatorics.* Cambridge Studies in Advanced Mathematics, vol. 1, Cambridge University Press, 1997.

22. M. Ward: *Arithmetic functions on rings.* Annals of Mathematics **38** (1937), no. 3, 725–732.

23. F. Wei and Z. Xiao: *Higher derivations of triangular algebras and its generalizations.* Linear Algebra Appl. **435** (2011), no. 5, 1034—1054.

24. Z. Xiao and F. Wei: *Nonlinear Lie higher derivations on triangular algebras.* Linear Multilinear Algebra **60** (2012), no. 8, 979-–994.

25. W. Yu and J. Zhang: *Nonlinear Lie derivations of triangular algebras.* Linear Algebra Appl. **432** (2010), no. 11, 2953—2960.

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# A NEW UNBIASED ESTIMATOR OF A MULITPLE LINEAR REGRESSION MODEL OF THE CAPM IN CASE OF MULTICOLLINEARITY

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**Abstract.** In this work we propose an unbiased estimator for a multiple linear regression model of the CAPM in the presence of multicollinearity in the explanatory variables. Multicollinearity is a common problem in empirical Econometrics. The existing methods so far have not dealt with cases of perfect multicollinearity. This new optimization method that belongs to the class of unbiased estimators is suitable for cases with strong or perfect multicollinearity, imposes restrictions of the minimizing matrix and produces small standard errors for the estimated parameters. First, we presented the theoretical background of our approach and next we derive an expression for the covariance matrix of estimated coefficients. As an example, we have estimated the basic linear regression model on Apple Inc expected stock returns and we have examined multivariate extensions of this model in the special case of multicollinearity using the proposed method.
**Keywords** CAPM, Data multicollinearity, Moore-Penrose inverse, MDLUE, Multiple linear regression.

## 1. Introduction

Multicollinearity is a problem that occurs when we estimate linear or generalized linear models and the independent variables in the regression model are highly

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correlated to each other. This situation has as a result unstable estimates of the regression coefficients. The coefficients of the model become very sensitive to small changes in the model. Also, multicollinearity reduces the precision of the estimate coefficients ([1]). In our work we concentrate in cases where we have strong or perfect collinearity between explanatory variables, this means for values of correlation greater than 0.9.

The Capital Asset Pricing Model (CAPM) was introduced by [24] and [15] based on the the work of Markowitz on modern portfolio theory ([17], [16]). The CAPM describes the relationship between expected return and systematic risk for stocks. It is also widely used for pricing risky securities and generating expected returns for assets given the risk of those assets and the cost of the capital. The formula for calculating the CAPM is

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f) \tag{1.1}$$

or else

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i \tag{1.2}$$

where $E(R_i)$ is the expected return of the investment, $R_f$ is the risk-free rate, $\beta_i$ is the systematic risk given by

$$\beta_i = \frac{cov(R_i, R_m)}{\sigma^2(R_m)}, \tag{1.3}$$

and $E(R_m)$ is the expected return of market. The quantity $E(R_m) - R_f$ is the market risk premium. In equation (2), $R_i$ is the return of asset i, $\alpha_i$ is a constant term, $R_m$ refers to the return of the market and $\epsilon_i$ is an error term. The most commonly used estimation method for the CAPM is the ordinary least squares (OLS)([7]).

In [22] the authors derive a multiple linear regression model of the CAPM by examining various explanatory variables that can be added to the basic CAPM for the expected returns on Apple Inc.. Their model, in addition to the market return (S&P500 returns), includes as explanatory variables the average spread and its interaction term with the market return. The average spread is the difference between the daily highest ask price and the lowest bid price divided by the price of the stock at the end of the day.

Various methods have been proposed for dealing with multicollinearity, such as deleting parameters, principal components regression, ridge regression estimation, maximum entropy estimators and shrinkage estimators (e.g. see [23], [13], [20]). The work of [25] introduces the generalized maximum entropy (GME) approach in order to estimate the quantile regression model for CAPM. The OLS method is very sensitive to extreme observations and [7] propose a fuzzy regression method which takes into account possible extreme observations and needs less assumptions from the OLS method. The method that we apply in our work belongs to the class of unbiased estimators, such as the minimum dispersion method (see for example [23]) in contrast to the ridge regression which is a biased estimation method ([26]).

The aim of the current work is to find a purpose for a new unbiased estimator for a multiple regression model of CAPM in case of strong multicollinearity using Linear Algebra techniques. [12] compares through a simulation study various biased and unbiased alternative estimators to the OLS estimator in the case of collinearity. In regression analysis, least squares estimations assume that explanatory variables are not correlated with each other. In the presence of multicollinearity, inference about the coefficients of regression can be difficult due to instability in the coefficients.

In this work we will apply a solution to a minimization problem for a matrix-valued function under linear constraints, in the case of a singular matrix. The theoretical framework of this method is not new and it is based on the paper [19]. Here we adapt and extend this framework by deriving an expression for the covariance matrix of estimated coefficients. Our method differs from others on the restriction of the minimizing matrix to the range of the corresponding quadratic function. In the case of singular positive matrices, many matrix valued functions are investigated using a partial ordering. Using matrix analysis results, we propose this additional relation as a constraint, by taking advantage of the canonical form related to this class of matrices. Moreover, the singularity of the matrix implies the use of the Moore-Penrose inverse matrix, giving us a unique minimal norm solution to the problem.

This paper is organized as follows: In section 2 we present the data and the multiple regression model and define the special case of multicollinearity. Section 3 introduces the proposed estimation technique in case of multicollinearity and we estimate the covariance matrix of estimated coefficients. Section 4 presents the estimation results for the simple CAPM and the multiple regression models of the CAPM. In addition, the proposed method is tested and compared against another known methods in terms of the standard errors of the estimated coefficients. Finally, concluding remarks appear in section 5.

## 2. Data and the Multiple Regression Model

The multiple linear regression model of the CAPM that we use has the following form:

$$E(R_A) = \alpha + \beta_1 R_1 + \beta_2 R_2 + \beta_3 R_3 + \beta_4 E(R_m) + \varepsilon_t \tag{2.1}$$

where $E(R_A)$ are the expected daily returns of the asset and $E(R_m)$ are the expected daily market returns. We remind that the OLS estimator for coefficients of the multiple regression $Y = \alpha + \beta X + \varepsilon$ is given by

$$\hat{\beta} = (X^{'}X)^{-1}X^{'}Y \tag{2.2}$$

In the presence of collinearity the quantity $(X^{'}X)$ is not invertible and the estimation of the variance of the coefficient estimates

$$Var(\hat{\beta}) = \sigma^2(X^{'}X)^{-1} \tag{2.3}$$

is problematic. If the quantity $(X'X)$ is not exactly singular but very close to be non-invertible, then the variance will be large. Moreover, if there is not an exact linear relationship among the predictor variables but they are close to each other, then the matrix $(X'X)$ will be invertible but the inverse matrix will have very large entries, due to the very small value of the determinant. If some of the variables are highly correlated then the matrix $(X'X)$ becomes non-orthogonal and as a result the inversion is unstable. As for the OLS solution of the model, the analysis and interpretation of each of the explanatory variables is difficult (see e.g. [13]). Multicollinearity has several effects in a regression model. For example the high variance of coefficients may reduce the precision of the estimation or the estimated coefficients to have the wrong sign. Also, the estimates of the coefficients may be sensitive to a particular set of the data. In our paper we try to overcome the problem of multicollinearity and find an unbiased solution. Since the problem with multicollinearity in multiple regression has infinite solutions, we will choose among them the minimal norm least squares solution, making use of the Moore-Penrose inverse.

For the basic CAPM model we use daily data of Apple Inc. stock returns (APPLE) and the market returns are the S&P500 daily returns (SP500). In the multiple linear regression model, the observed values are the daily expected stock returns of Apple Inc. (APPLE). The explanatory variables are the S&P500 daily returns (SP500), the opening stock price (OPENP), the semi-sum of opening and lower stock price (OPENLOW) of each day and the closing price (CLOSEP). The data are from January 1, 2007 until June 6, 2014.

Table 1 presents some descriptive statistics for our data. The skewness of the data show that they are approximately symmetric. The distributions of the time series Apple Inc. returns and market S&P500 returns have positive excess kyrtosis and are leptokurtic. Also the distributions of the opening stock price, the semi-sum of opening and lower stock price of each day and the closing price are having thinner tails than those of the normal distribution. Table 2 presents the correlation coef-

| | APPLE | SP500 | OPENP | OPENLOW | CLOSEP |
|---|---|---|---|---|---|
| Maximum | 0.130 | 0.11 | 134.46 | 132.555 | 133 |
| Minimum | -0.197 | -0.095 | 11.341 | 11.438 | 11.171 |
| Mean | 0.0001 | 0.0002 | 56.77 | 56.753 | 56.770 |
| Median | 0.001 | 0.0007 | 51.031 | 51.009 | 56.736 |
| St. Deviation | 0.021 | 0.014 | 123.119 | 35.019 | 35.007 |
| Skewness | -0.448 | -0.315 | 0.482 | 0.481 | 0.481 |
| Kurtosis | 9.725 | 12.511 | 2.059 | 2.057 | 2.057 |
| Range | 0.328 | 0.204 | 123.119 | 121.117 | 121.829 |

Table 2.1: Descriptive Statistics for Data. The data are the Apple Inc. stock returns (APPLE), the S&P 500 daily returns (SP500), the opening stock price (OPENP), the semi-sum of opening and lower stock price (OPEN_LOW) of each day and the closing price (CLOSEP). The data are from January 1, 2007 until June 6, 2014.

ficients of the explanatory variables in the multiple regression model. The results indicate that there is a strong positive relationship between the explanatory variables except for the market S&P500 returns. For the detection of multicollinearity

| | SP500 | OPENP | OPENLOW | CLOSEP |
|---|---|---|---|---|
| SP500 | 1 | 0.014 | 0.018 | 0.022 |
| OPEN | 0.014 | 1 | 0.999 | 0.999 |
| OPENLOW | 0.018 | 0.999 | 1 | 0.999 |
| CLOSEP | 0.022 | 0.999 | 0.999 | 1 |

Table 2.2: Correlation coefficients for the explanatory variables: S&P 500 daily returns (SP500), opening stock price (OPENP), semi-sum of opening and lower stock price (OPENLOW) of each day, closing stock price (CLOSEP). The data are from January 1, 2007 until June 6, 2014.

in regression models there are various diagnostic techniques.

In the following part, we will briefly present two of the basic diagnostic tools for collinearity. The first is the Variance Inflation Factor (VIF) which measures the inflation of the parameter estimates being computed for all the explanatory variables in the regression model ([2]). The VIF is given by

$$VIF = \frac{1}{1-\mathcal{R}_i^2}, i = 1, \ldots, p \tag{2.4}$$

where p is the number of explanatory variables and $\mathcal{R}^2$ is the squared multiple correlation coefficient. The VIF has a lower bound value equal to 1 but no upper bound. Higher values signify that it is difficult to define accurately the contribution of the predictor variable to a regression model. Usually values higher than 10 indicate collinearity. Table 2.3 presents the variance inflation factor (VIF) and condition index results of the explanatory variables for the multiple regression model. From the results it is obvious that there exists high collinearity between the opening stock price, the semi-sum of opening and lower stock price of each day and the closing price. Another measure of collinearity is the condition index. The condition index (CI) is the square root of the ratio of each eigenvalue $\lambda$ to the smallest eigenvalue of X ([6]) and indicates how close the underlying matrix is to a singular matrix. The condition index is defined as

$$C_k = \sqrt{\frac{\lambda}{\lambda_{min}}} \tag{2.5}$$

where $\lambda_{min}$ is the smallest eigenvalue value of $X^{'}X$ . Values between 10 and 30 are a sign of multicollinearity and multicollinearity occurs when the value of the condition indices are greater than 30 ([8]). The results from table 2.3 confirm the existence of collinearity between the explanatory variables except the S&P500 variable.

Table 2.3: Results for the Variance Inflation Factor (VIF) and condition index for the explanatory variables: S&P 500 daily returns (SP500), opening stock price (OPENP), semi-sum of opening and lower stock price (OPENLOW) of each day, closing price (CLOSEP). The data are from January 1, 2007 until June 6, 2014.

| Variable | VIF | Cond. Index |
|---|---|---|
| S&P 500 | 0 1.7323 | 1 |
| OPEN | 5.6830e+14 | 17.330 |
| OPENLOW | 2.2710e+15 | 182.4603 |
| CLOSEP | 5.6740e+14 | 7.8015e+15 |

## 3. Constrained matrix optimization

In this section, we will briefly present the basic concepts of the theoretical background of our matrix constrained optimization (MCO) method, for more information see [19]. As discussed previously, the collinearity of the data makes the quantity $(X^{'}X)$ not invertible (or very close to singular) and the estimation of the variance of the coefficient estimates

$$Var(\hat{\beta}) = \sigma^2 (X^{'}X)^{-1}$$

is problematic. So, a way to tackle the problem is to use a constrained matrix optimization method, making use of the Moore-Penrose inverse matrix.

Suppose that $A \in \mathcal{R}^{n\times n}$ is a square matrix with $\mathcal{N}(A)$ and $\mathcal{R}(A)$ its kernel and its range respectively. Also we denote as $A'$ the transpose of the square matrix $A$. The generalized inverse, also known as the Moore-Penrose inverse of a matrix $A$ is the unique matrix $A^{\dagger}$ satisfying the following four Penrose conditions:

$$AA^{\dagger} = (AA^{\dagger})', \qquad A^{\dagger}A = (A^{\dagger}A)', \qquad AA^{\dagger}A = A, \qquad A^{\dagger}AA^{\dagger} = A^{\dagger}. \tag{3.1}$$

It is easy to see that $AA^{\dagger}$ is the orthogonal projection of $\mathcal{R}^n$ onto $\mathcal{R}(A)$ , denoted by $P_A$, and that $A^{\dagger}A$ is the orthogonal projection of $\mathcal{R}^n$ onto $\mathcal{R}(A^{'})$ noted by $P_{A'}$. It is also well known that $\mathcal{R}(A^{\dagger}) = \mathcal{R}(A^{'})$. For more on the Moore-Penrose inverse, see e.g. [4], [5].

The Moore-Penrose inverse also satisfies the following inequality ([21])

$$\| AA^{\dagger}B - B \|_2 \leq \| AX - B \|_2 \tag{3.2}$$

for all $X$.

We remind that given a matrix $R \in \mathcal{R}^{M\times M}$, minimizing $W'RW$ with

$$W \in \mathcal{R}^{M\times m}$$

means finding a matrix $\hat{W} \in \mathcal{R}^{M\times m}$ such that the $m \times m$ matrix $(W'RW - \hat{W}'R\hat{W})$ is positive semidefinite for all $W \in \mathcal{R}^{M\times m}$. (The Löwner partial ordering for hermitian nonnegative definite matrices, defined as: $A \geq B$, if $A - B$ is positive semidefinite). See e.g. [3], [10].

### 3.1. Matrix Optimization and Linear Regression

Next we assume that t $R \in \mathcal{R}^{M\times M}$ is a positive semidefinite symmetric matrix. The main problem is the minimization of $W'RW, \quad W \in \mathcal{R}^{M\times m}$ under the Löwner ordering, when $W$ satisfies a set of linear constraints :

$$S = \{W \in \mathcal{R}^{M\times m} : C'W = F\}$$

with $C \in \mathcal{R}^{M\times n}, \quad F \in \mathcal{R}^{n\times m}$. As a result, we will find a matrix $\hat{W}$ such that $W'RW \geq \hat{W}'R\hat{W}$ for all $W \in S$.
In [9] and [14] where a similar problem is treated the matrix $R$ is assumed to be positive definite. In our work the matrix $R$ is positive semidefinite (therefore singular). The difference in our method is that the matrix $W$ will also satisfy the relation $\mathcal{R}(W) \subseteq \mathcal{R}(R)$ in order to overcome the singularity of $R$.
In our case the positive semidefinite matrix $R$ is singular, $\mathcal{N}(R) \neq \{0\}$ and therefore we have that $W'RW = 0$ for all matrices $W$ of appropriate dimensions belonging to the set $\mathcal{Z} = \{W : RW = 0\}$ and so, the problem

$$\textit{minimize} \quad W'RW, W \in S \tag{3.3}$$

has many solutions when $\mathcal{S} \cap \mathcal{Z} \neq \emptyset$.
In other words, since the matrix $R$ is symmetric, we have that $\mathcal{R}(R) = \mathcal{R}(R^\dagger)$ and therefore we are looking for the minimum of $W'RW$ under the constraints $C'W = F$ and $\mathcal{R}(W) \subseteq \mathcal{R}(R)$.
From Theorem 1 in [19] we have that the minimizing problem in eq. 10 has the unique solution $\hat{W} = R^\dagger C[C'R^\dagger C]^\dagger F$. In the case now that $S$ is empty then the constraint must be replaced by the equation $C'W = F_1 = P_{\mathcal{R}(C'R^\dagger C)}F$. The following Corollary is a consequence of the previous result:

**Corollary 3.1.** *Let $R \in \mathcal{R}^{M\times M}$ a positive semidefinite symmetric matrix, the matrices $W \in \mathcal{R}^{M\times m}, \quad C \in \mathcal{R}^{M\times n}, \quad F \in \mathcal{R}^{n\times m}$ with $m < M, n < M$, and the equation $C'W = F$. The problem:*

$$\textit{minimize} \quad W'RW, \quad W \in \hat{S}$$

*where $\hat{S} = \{W : C'W = P_{\mathcal{R}(C'R^\dagger C)}F, \textit{such} \quad \textit{that} \quad \mathcal{R}(W) \subseteq \mathcal{R}(R)\}$ has a unique solution among the generalized constrained solutions which is*

$$\hat{W} = R^\dagger C[C'R^\dagger C]^\dagger F$$

In many statistical applications as in our case, the matrix $R$ is equal to $CC'$. In this case, we have the following proposition:

**Proposition 3.1.** *Let $R \in \mathcal{R}^{M\times M}$ to be a positive semidefinite symmetric matrix and the matrices $W \in \mathcal{R}^{M\times m}, \quad C \in \mathcal{R}^{M\times n}, \quad F \in \mathcal{R}^{n\times m}$ with $m < M, n < M$. Also, suppose that $C'W = F$ and the set $S = \{W : C'W = F, \textit{such} \quad \textit{that} \quad \mathcal{R}(W) \subseteq \mathcal{R}(R)\}$ is not empty. Taking as $R = CC'$ then the problem:*

$$\textit{minimize} \quad W'RW, \quad W \in S$$

*has the unique solution*

$$\hat{W} = (C')^{\dagger} F$$

Consider a simple linear model

$$y = C\beta + \epsilon \tag{3.4}$$

where $y \in \mathcal{R}^{m\times 1}$ is a vector of observed data, $C \in \mathcal{R}^{m\times p}$ the matrix of $p$ observed covariates, $\beta \in \mathcal{R}^{p\times 1}$ vector of parameters to be estimated and $\epsilon \in \mathcal{R}^{m\times 1}$ the noise vector. When Rank$(C) = p$, and $C'C$ is nonsingular the inverse $(C'C)^{-1}$ can be computed. Then the Best Linear Unbiased Estimator (BLUE) for $\beta$ is defined as

$$\hat{\beta} = (C'C)^{-1}C'y = W'y \tag{3.5}$$

In the case when the matrix R is singular, then Theorem 1 in [19] can be applied to the problem of computing the Best Linear Unbiased Estimator in linear models. In this specific case, the size of the matrices are:

$$R \in \mathcal{R}^{M\times M}, \quad W \in \mathcal{R}^{M\times m}, \quad C \in \mathcal{R}^{m\times p}, \quad I \in \mathcal{R}^{m\times m}, m < M$$

In the case when the matrix $C$ is of full rank, then $W'$ is the unique left inverse of C, and therefore, Theorem 1 can be applied to find the optimum matrix $\hat{W}$.

There might be cases, however, where there is a deficiency in rank of the design matrix C. That means that Rank$(C) = r < p < m$ and thus $C'C$ is singular. When $m < p$ ordinary least squares method (OLS) cannot be used to estimate $\beta$ in linear model (7). Some methods to overcome this problem have been suggested based on maximum entropy estimation ([11]) or penalized regression ([18]). We will denote any generalized inverse of $C'C$ as $(C'C)^{-}$. In such a case there might be several values of $\beta$ that lead to same values of $C\beta$. In addition, the estimator is not unbiased anymore, since the condition $W'C = I$ does not necessarily hold. However, let $L'\beta$ be linear functions of $\beta$ such that $\mathcal{R}(\text{Ł}) \subset \mathcal{R}(C')$ implying $L = C'A$ for some A. Then if $(C'C)^{-}$ is any generalized inverse of $C'C$ and

$$\hat{\beta} = (C'C)^{-}C'y \tag{3.6}$$

it can be shown ([23] p. 30) that $L'\hat{\beta}$ is the *minimum dispersion unbiased estimator* (MDLUE) of $L'\beta$ with dispersion matrix $\sigma^2 L'(C'C)^{-}L$. In the case when $Rank(C) < p$, then $C$ does not have a left inverse, and therefore the constraint must be slightly modified, as said in Theorem 1 ([19]), since $C'W = I$ does not hold.

Following all the above and using Theorem 1 we will minimize $W'RW$ under the constraint

$$C'W = P_{\mathcal{R}(C'R^{\dagger}C)}, \text{ with } \mathcal{R}(W) \subseteq \mathcal{R}(R)$$

where $P_{\mathcal{R}(C'R^{\dagger}C)}$ is the orthogonal projection on the range of $C'R^{\dagger}C$.

So, from the above discussion and Theorem 1 we have the following Proposition:

**Proposition 3.2.** *Consider a simple linear model*

$$y = C\beta + \epsilon \tag{3.7}$$

*and let $C \in \mathcal{R}^{M\times m}$ the matrix of $m$ observed covariates with $m < M$, $\beta \in \mathcal{R}^{m\times 1}$ the vector of parameters to be estimated , $R \in \mathcal{R}^{M\times M}$ a positive semidefinite symmetric matrix such that $R = E(y - C\beta)(y - C\beta)'$ . Then,*

$$\hat{\beta}_{MCO} = [R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]'y \tag{3.8}$$

*gives a solution which is restricted on a particular set defined by the orthogonal projection, thus giving the minimum dispersion unbiased estimator of any linear combination of $\beta_{MCO}$.*

Moreover, in [23] p. 25, a different way of estimating $\hat{\beta}$ is also presented when $C$ is not of full rank:

$$\hat{\beta}_{RMDLUE} = (C'C)^{-}C'y + (I - (C'C)^{-}C'C)w \tag{3.9}$$

where $(C'C)^{-}$ defined as before and $w$ is an arbitrary vector.
The rationale follows from the fact that the empirical predictor (given as $\hat{y} = C\hat{\beta}$) has the same value for all solutions of $\beta$ that emerge from $C'C\hat{\beta} = C'y$.

In many practical applications estimation of $\beta$ relies on either formulas (3.9) or (3.6). In the next section we will use Proposition 3.2 and hence the result given by eq.(3.8) in order to solve the multicollinearity problem, finding a unique MDLUE solution among the infinite solutions that this problem admits. Our solution gives a model similar to the one found using eq.(3.9) with differences in the coefficients due to the different choice of the unique solution among the infinite ones.
The variance of all the estimated coefficients using the MCO approach is given by

$$\begin{aligned} V(\hat{\beta}_{MCO}) &= V([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]'y) \\ &= ([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]')V(y)([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]')' \\ &= ([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]')\sigma^2[R^{\dagger}C[C'R^{\dagger}C]^{\dagger}] \end{aligned}$$

Also we have that, since $R$ is symmetric, so $(R^{\dagger})' = R^{\dagger}$ :

$$\begin{aligned} ([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]')([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]')' &= ([R^{\dagger}C[C'R^{\dagger}C]^{\dagger}]')(R^{\dagger}C[C'R^{\dagger}C]^{\dagger}) \\ &= ([C'R^{\dagger}C]^{\dagger})'(R^{\dagger}C)'R^{\dagger}C[C'R^{\dagger}C]^{\dagger} \\ &= ([C'R^{\dagger}C]')^{\dagger}C'R^{\dagger}[R^{\dagger}C[C'R^{\dagger}C]^{\dagger} \\ &= [C'R^{\dagger}C]^{\dagger}C'R^{\dagger}R^{\dagger}C[C'R^{\dagger}C]^{\dagger} \end{aligned} \tag{3.10}$$

Denote with $K = C'R^{\dagger}$ then equation (3.10) becomes

$$\begin{aligned} [C'R^{\dagger}C]^{\dagger}C'R^{\dagger}R^{\dagger}C[C'R^{\dagger}C]^{\dagger} &= (KC)^{\dagger}KK'(KC)^{\dagger} \\ &= A^{\dagger}KK'A^{\dagger} \end{aligned} \tag{3.11}$$

where $A = KC$. As a result the variance of the estimated coefficient $\hat{\beta}_{MCO}$ is given by

$$V(\hat{\beta}_{MCO}) = \sigma^2 A^\dagger K K' A^\dagger \tag{3.12}$$

The standard errors are estimated by taking the square root of the diagonal of $V(\hat{\beta}_{MCO})$.

## 4. Estimation Results

In this study, we apply our matrix constrained optimization method to two multivariate regression models. The first (Multiple Regression Model I) contains all ofl the explanatory variables we previous mentioned. The second model (Multiple Regression Model II) excludes the variable with the lowest correlation coefficient, the S&P 500 market returns and as a result we have a model with strong correlation between the regressors, almost equal to one. As said above, in order to compare our method, the regression coefficients have been also estimated using the MDLUE method presented in [23]. Table 4.1 reports the results for the basic CAPM for the Apple Inc. stock returns and the S&P500 expected returns as the market returns. The resulting simple CAPM has the following form

$$E(R_{APPLE}) = 0.0008 + 0.9568(SP500)$$

Table 4.1: Capital Asset Pricing Model (CAPM) coefficients. The table presents the value of the coefficients for the Capital Asset Pricing Model with an intercept and one explanatory variable, the S&P 500.

| Variable | Estimation |
|---|---|
| Intercept | 0.0008 |
| S&P500 | 0.9568 |

Table 4.2 presents the coefficients of the multiple linear regression in case of multicollinearity for the proposed constrained matrix optimization method (MCO, eq. (16)) and the MDLUE proposed by [23] (RMDLUE, eq. (17)) along with their standard errors. The multiple linear regression model under MCO is the following:

$$\begin{aligned} E(R_{APPLE}) = {} & 0.0012 + 0.5671(SP500) - 0.0324(OPENP) \\ & + 0.0005(OPENLOW) + 0.0334(CLOSEP) \end{aligned}$$

We compare the performance of our approach with the RMDLUE method in terms of the standard errors of the estimated regression coefficients. One of the problems of multicollinearity is that affects the standard error of the parameter estimators. For the Multiple Regression Model I the standard errors for the coefficients estimated with the MCO method are smaller in most of the cases than

| MCO | Coefficients | Std Error | t value | p-value |
|---|---|---|---|---|
| Intercept | 0.0012 | 0.0015 | 0.3257 | 0.7447 |
| S&P500 | 0.5671 | 0.0493 | 4.1584 | $< 0.00001$ |
| OPEN | -0.0324 | 0.0015 | -11.9939 | $< 0.00001$ |
| OPENLOW | 0.0005 | 2.6597e-05 | 20.6638 | $< 0.00001$ |
| CLOSEP | 0.0334 | 0.0016 | 12.3642 | $< 0.00001$ |
| RMDLUE | Coefficients | Std Error | t value | p-value |
| Intercept | 0.0015 | 0.7277 | 2.2479 | $< 0.00001$ |
| S&P500 | 0.6885 | 4.7211e-06 | 28.34 | $< 0.00001$ |
| OPEN | -0.1933 | 0.0082 | 18.09 | $< 0.00001$ |
| OPENLOW | 0.3558 | 0.0088 | -14.64 | $< 0.00001$ |
| CLOSEP | -0.1625 | 0.0825 | 24.47 | $< 0.00001$ |

Table 4.2: Multiple Regression Model I.Parameter estimates for the matrix constrained optimization method (MCO) and the minimum dispersion linear unbiased method (RMDLUE) and their standard errors.

those from the RMDLUE approach. The t-value column represents whether the estimated coefficients of the variables in the multiple regression model are statistically significant. Also, in the table we present the p-values, the probability that the variable in the model is not significant. The reported p-values are low, which means that the variables are statistically significant. The results in table 7 for the Multiple Regression Model I show that the relationship between the S&P 500 market returns and the Apple Inc returns is positive and the value of regression coefficient is 0.5671. This means that an increase in the S&P 500 daily market returns lead to an increase in the Apple Inc returns. The same behavior happens between the Apple Inc returns the semi-sum of opening and lower stock price (OPENLOW) of each day and the closing stock price (CLOSEP). In contrast the relationship between the opening stock price (OPENP) and the Apple Inc returns is negative.

Table 4.3 now presents the coefficients of the multiple linear regression in case of multicollinearity for the constrained matrix optimization method (MCO) and the RMDLUE method if we exclude the variable of the stock market returns. The standard errors for the OPEN, OPENLOW and CLOSEP coefficient are smaller than those estimated with the RMDLUE approach. The results for both regression models indicate that the MCO method could be a good alternative when someone wants to obtain estimates with small standard errors and the variable that appear to have strong collinearity are all of interest. As previously, the reported p-values indicate that the variables are statistically significant.

| MCO | Coefficients | Std Error | t value | p-value |
|---|---|---|---|---|
| Intercept | 0.0003 | 0.0015 | 0.0489 | 0.961004 |
| OPEN | -0.0433 | 0.0014 | -12.2014 | $< 0.00001$ |
| OPENLOW | 0.0007 | 3.0788e-05 | 20.9787 | $< 0.00001$ |
| CLOSEP | 0.0488 | 0.0014 | 12.6021 | $< 0.00001$ |
| RMDLUE | Coefficients | Std Error | t value | p-value |
| Intercept | 0.0011 | 6.7648e-04 | 1.3351 | 0.182006 |
| OPEN | -0.1624 | 0.009 | -29.91 | $< 0.00001$ |
| OPENLOW | 0.2850 | 0.619e-05 | 54.12 | $< 0.00001$ |
| CLOSEP | -0.1226 | 0.0078 | 22.57 | $< 0.00001$ |

Table 4.3: Multiple Regression Model II. Parameter estimates for the matrix constrained optimization method (MCO) and the minimum dispersion linear unbiased method (RMDLUE) and their standard errors. The multiple linear regression model excludes the S&P500 factor.

## 5. Concluding Remarks

In this research, we refer to the multicollinearity issue of a multiple regression problem. Various techniques have been proposed in order to overcome this problem such as ridge regression or delete the factors that are collinear. The matrix constrained optimization method that we proposed is an unbiased estimator that can be applied in situations where exists strong or perfect collinearity and we can not delete any collinear factor because this may affect the interpretation of the model results and the factor is important for the analysis. Also, we obtain an expression for the variance-covariance matrix of the estimated coefficients.
The method is applied in a special case of a multiple linear regression model which is an extension of the Capital Asset Pricing Model (CAPM). The matrix constrained optimization method is implemented in two multiple regression models. The difference between these models is that the first includes an explanatory variable with low correlation which in the second model,this factor is excluded. The results are compared with another unbiased linear estimator, the MDLUE, in terms of standard errors of the estimated parameters. We have mentioned that this technique is appropriate when high levels of correlations exist among the regressors, and there is a need for an unbiased estimator. In this case, the solution of deleting the factors with high collinearity may not be feasible because of the importance of the factors in the regression model.

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## REFERENCES


1. M. P. Allen: *Understanding regression analysis*, Springer Science & Business Media,(2004).
2. A. Bager, M. Roman, M. Algedih, B. Mohammed: *Addressing multicollinearity in regression models: a ridge regression application*,(2017).
3. J. K. Baksalary,F. Pukelsheim: *On the Löwner, minus, and star partial orderings of nonnegative definite matrices and their squares* Linear Algebra and its Applications, **151** (1991) 135-141.
4. A. Ben-Israel,T. N. Greville: *Generalized inverses: theory and applications* (Vol. 15).Springer Science & Business Media,(2003).
5. S. L. Campbell,C. P. Meyer: *Generalized Inverses of Linear Transformations*.Dover. New York,(1991).
6. C. F. Dormann, J. Elith, S. Bacher, C. Buchmann, G. Carl, G. Carré, ... S. Lautenbach: *Collinearity: a review of methods to deal with it and a simulation study evaluating their performance. Ecography*, **36** (1) (2013), 27-46.
7. Y. Du,Q. Lu: *Estimate Beta Coefficient of CAPM Based on a Fuzzy Regression with Interactive Coefficients* Journal of Applied Mathematics and Physics, **3** (06) (2015), 664.
8. <. Friendly, E. Kwan: *Where's Waldo? Visualizing collinearity diagnostics* The American Statistician, **63** (1) (2009), 56-65.
9. O. L. Frost: *An algorithm for linearly constrained adaptive array processing*, Proceedings of the IEEE, **60** (8) (1972), 926-935.
10. N. Gaffke,O. Krafft: *Matrix inequalities in the Löwner ordering* Modern Applied Mathematics (1982) 595-622.
11. A. Golan, G. Judge, D. Miller: *Maximum entropy econometrics: Robust estimation with limited data*,(1997).
12. G. Khalaf: *A comparison between biased and unbiased estimators in Ordinary least squares regression*, Journal of Modern Applied Statistical Methods, **12** (2) (2013), 17.
13. G. Khalaf, M. Iguernane: *Multicollinearity and a ridge parameter estimation approach*, Journal of Modern Applied Statistical Methods **15** (2) (2016), 25.
14. O. Krafft: *A matrix optimization problem*, Linear Algebra and its Applications **51** (1983), 137-142.
15. J. Lintner: *Security prices, risk, and maximal gains from diversification*, The journal of finance **20** (4) (1965), 587-615.
16. H. Markowitz: *Portfolio Selection*, The Journal of Finance **7** (1) (1952), 77–91.
17. H. Markowitz: *Portfolio Selection: Efficient Diversification of Investments*, New York: JohnWiley and Sons, Inc.,(1959).
18. B. D. Marx,P. H. Eilers: *Generalized linear regression on sampled signals and curves: a P-spline approach*, Technometrics **41** (1) (1993), 1-13.
19. D. Pappas, A. Perperoglou: *Constrained matrix optimization with applications*, Journal of Applied Mathematics and Computing **40** (1) (2012), 357-369.
20. Q. Paris: *Multicollinearity and maximum entropy estimators*, Economics Bulletin **3** (11) (2001), 1-9.

21. R. Penrose: *A generalized inverse for matrices*, In Mathematical proceedings of the Cambridge philosophical society (Vol. 51, No. 3, pp. 406-413). Cambridge University Press,(1955).
22. S. Qemo,E. Elsaid: *Statistical Modelling of the Capital Asset Pricing Model (CAPM)*, Accounting and Finance Research **7** (146) (2018), 03.
23. R. C. Rao, H. Toutenburg: *Linear Models: Least Squares and Alternatives*, 5-21,(1999).
24. W. F. Sharpe: *Capital asset prices: A theory of market equilibrium under conditions of risk*, The journal of finance, **19** (3) (1964), 425-442.
25. W. Yamaka, K. Autchariyapanitkul,P. Meneejuk, S. Sriboonchitta: *Capital asset pricing model through quantile regression: an entropy approach*, Thai Journal of Mathematics (2017), 53-65.
26. Y. Zakari, S. A. Yau, U. Usman: *Handling multicollinearity; a comparative study of the prediction performance of some methods based on some probability distributions*, Annals. Computer Science Series (2018), 15-21.

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# SOME FIXED POINT RESULTS ON RECTANGULAR $b-$METRIC SPACE

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**Abstract.** In this paper we have obtained some results on a complete rectangular $b-$metric space and these results generalized many existing results in this literature.
**Keywords:** rectangular $b-$metric space.

## 1. Introduction and Preliminaries

The Banach fixed point theorem in metric space has generalized by many researchers in various branches such as cone metric space, $b-$metric space, Generalized metric space, Fuzzy metric space etc. Many researchers such as Tiwary et al.[12], Sarkar et al.([10], [11]), S. Czerwik[3], H. Huang et al.[7], Ding et.al[5], Ozturk[9] and others have worked on Cone Banach Space, $b-$metric space, rectangular $b-$metric space. George et al.[6] have proved some results in rectangular $b-$metric space and have left two open problems for further investigations. Z. D. Mitrović and S. Radenović [8] has given a partial solutions of Reich and Kannan Type contraction in rectangular $b-$metric space. In this paper we have given partial solution of Cirić Type, Cirić almost contraction Type, Hardy Rogers Type contraction condition in rectangular $b-$metric space with some corollaries.

The following definitions are required to prove the main results.

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**Definition 1.1.** [1] Let $X$ be a non-empty set $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}$ is a said to be a $b-$ metric if for a distinct point $u \in X$, different from $x$ and $y$, the following conditions holds:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, u) + d(u, y)]$. $\square$

The pair $(X, d)$ is called a $b-$metric space ( in short bMS) with coefficient $s \geq 1$.

**Definition 1.2.** [6] Let $X$ be a non-empty set $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}$ is a said to be a rectangular $b-$ metric if for all distinct points $u_1, u_2 \in X$, all are different from $x$ and $y$, the following conditions holds:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + d(u_2, y)]$.

The pair $(X, d)$ is called a rectangular $b-$metric space ( in short RbMS) with coefficient $s \geq 1$.

If $s = 1$ then $(X, d)$ is called a rectangular metric space ( in short RMS).

**Definition 1.3.** [6] Let $(X, d)$ be a rectangular $b-$metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$.
Then

i) the sequence $\{x_n\}$ is said to be convergent in $(X, d)$ and converges to $x$ if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;

ii) The sequence $\{x_n\}$ is said to be Cauchy sequence in $(X, d)$ if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n \geq n_0; p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$;

iii) $(X, d)$ is said to be a complete rectangular $b-$metric space if every Cauchy sequence in $X$ converges to some $x \in X$.

R. George et al. [6] has proved the result.

**Theorem 1.1.** *( **[6], Theorem 2.1**) Let $(X, d)$ be a complete rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) < \lambda d(x, y)$$

*for all $x, y \in X$ with $x \neq y$, where $\lambda \in [0, \frac{1}{s}]$. Then $T$ has a unique fixed point.*

## 2. Main Results

Our main resuts are as follows:

**Theorem 2.1.** *Let $(X,d)$ be a complete rectangular $b-$metric space with coefficient $s>1$ and $\{T^i\}$ be a sequence of self-maps satisfying the condition*

$d(T^ix,T^jy)\le \alpha\max\{d(x,y),d(x,T^ix),d(y,T^jy),d(x,T^jy),d(y,T^ix)\}+Ld(y,T^ix)$, *where the constants $\alpha, L\ge 0$ and $\alpha+L<1$. Then the sequence $\{T^i\}$ have unique common fixed point in $X$.*

*Proof. Let $x_0\in X$ be an arbitrary. We construct a sequence for a fixed $i\in\mathbb{N}$ such that $x_n=T^ix_{n-1}$ where $n\in\mathbb{N}$.*

*Let, $d_n=d(x_n,x_{n+1})$ and $d_n^*=d(x_n,x_{n+2})$.*
*Then*

$d(x_n,x_{n+1})=d(T^ix_{n-1},T^jx_n)$
$\le \alpha\max\{d(x_{n-1},x_n),d(x_{n-1},T^ix_{n_1}),d(x_n,T^jx_n),d(x_{n-1},T^jx_n),d(x_n,T^ix_{n-1})\}+$
$Ld(x_n,T^ix_{n-1})$
$\le \alpha\max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),d(x_n,x_n)\}+Ld(x_n,x_n).$

$$\le \alpha\max\{d_{n-1},d_n,d^*_{n-1}\}. \tag{2.1}$$

*Suppose, $\{d_n\}$ is monotone increasing sequence. Then from equation (2.1) we get,*

$$d_n\le\alpha\max\{d_n,d^*_{n-1}\}.$$

*If $d_n>d^*_{n-1}$, then from (2.1) we get, $d_n\le\alpha d_n$ which implies, $1\le\alpha$, a contradiction.*
*Therefore,*

$$d_n\le d^*_{n-1}.$$

*Then from (2.1), we get*

$$d_n\le\alpha d^*_{n-1}\le\alpha^2d^*_{n-2}\le\ldots\le\alpha^nd^*_0$$

*implies, $d_n=0$ as $n\to\infty$. Suppose, $\{d_n\}$ is monotone decreasing sequence. then from (2.1), we get*

$$d_n\le\alpha\max\{d_{n-1},d^*_{n-1}\}. \tag{2.2}$$

*If $d_{n-1}\le d^*_{n-1}$, then from (2.2), we get*

$$d_n=\alpha d^*_{n-1}\le\alpha^2d^*_{n-2}\le\ldots\le\alpha^nd^*_0$$

*implies,*

$$\lim_{n\to\infty}d_n=0.$$

*Again suppose $d^*_{n-1}\le d_{n-1}$, then from (2.2) we have,*

$$d_n = \alpha d_{n-1} \leq \alpha^2 d_{n-2} \leq \ldots \leq \alpha^n d_0$$

*implies,* $\lim_{n\to\infty} d_n = 0$.

*Thus for all cases* $\lim_{n\to\infty} d_n = 0$.

*Now we show*

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \tag{2.3}$$

*holds good by Mathematical Induction on* $p \in \mathbb{N}$.

*Clearly, (2.3) hold for* $p = 1$.
*Suppose it holds for* $p$ *i.e.,* $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$. *So* $\lim_{n\to\infty} d(x_{n+1}, x_{n+p+1}) = 0$.
*We have to show*

$\lim_{n\to\infty} d(x_n, x_{n+p+1}) = 0$.
*Since*

$$d(x_n, x_{n+p+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p}) + d(x_{n+p}, x_{n+p+1})].$$

*Therefore,*

$$\lim_{n\to\infty} d(x_n, x_{n+p+1}) \leq s \lim_{n\to\infty} d(x_{n+1}, x_{n+p}). \tag{2.4}$$

***Case I:*** *If* $p = 2m, m \in \mathbb{N}$. *Then from (2.4) we get,*

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+p+1}) &\leq s \lim_{n\to\infty} d(x_{n+1}, x_{n+2m}) \\
&\leq s^2 \lim_{n\to\infty} d(x_{n+1+1}, x_{n+2m-1}) \\
&\leq s^3 \lim_{n\to\infty} d(x_{n+1+2}, x_{n+2m-2}) \\
&\vdots \\
&\leq s^{m+1} \lim_{n\to\infty} d(x_{n+m}, x_{n+m+1}) \\
&= 0.
\end{aligned}$$

***Case II:*** *If* $p = 2m+1, m \in \mathbb{N}$, *then from (2.4) we get,*

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+2m+1+1}) &\leq s \lim_{n\to\infty} d(x_{n+1}, x_{n+2m+1}) \\
&\leq s^2 \lim_{n\to\infty} d(x_{n+1+1}, x_{n+2m-1}) \\
&\leq s^3 \lim_{n\to\infty} d(x_{n+1+2}, x_{n+2m-2})
\end{aligned}$$

$$\vdots$$

$$\leq s^m \lim_{n\to\infty} d(x_{n+m}, x_{n+m+1})$$

$$= 0.$$

*Thus*

$$\lim_{n\to\infty} d(x_n, x_{n+p+1}) = 0.$$

*Therefore, by Mathematical Induction* $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ *for all* $p \in \mathbb{N}$. *So* $\{x_n\}$ *is a Cauchy sequence. Since* $X$ *is complete, there exists an* $x \in X$ *such that* $\lim_{n\to\infty} x_n = x$. *So* $\lim_{n\to\infty} T^i x_n = \lim_{n\to\infty} x_{n+1} = x$ *i.e.,* $\lim_{n\to\infty} d(T^i x_n, x) = 0$.

*Now*

$$\lim_{n\to\infty} d(T^i x_n, x) \leq \lim_{n\to\infty} s[d(T^i x_n, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x)]$$

$$= s \lim_{n\to\infty} d(T^i x_n, x_{n+1}). \tag{2.5}$$

*Again,*

$$\begin{aligned}
&\lim_{n\to\infty} d(T^i x, x_{n+1})\\
&= \lim_{n\to\infty} d(T^i x, T^j x_n)\\
&\leq \lim_{n\to\infty} \alpha \max\{d(x, x_n), d(x, T^i x), d(x_n, T^j x_n), d(x, T^j x_n), d(x_n, T^i x)\}\\
&\qquad\qquad\qquad\qquad +Ld(x_n, T^i x),
\end{aligned}$$

$$= \alpha \max\{0, \lim_{n\to\infty} d(x, T^i x), 0, 0, \lim_{n\to\infty} d(x_n, T^i x)\} + Ld(x_n, T^i x). \tag{2.6}$$

*If*

$$\lim_{n\to\infty} d(x, T^i x) \leq \lim_{n\to\infty} d(x_n, T^i x)\},$$

*then from above (2.6) we get,*

$$\begin{aligned}
\lim_{n\to\infty} d(T^i x, x_{n+1}) &\leq \lim_{n\to\infty} (\alpha + L) d(x_n, T^i x)\}\\
&\leq \lim_{n\to\infty} (\alpha + L)^2 d(x_{n-1}, T^i x)\}\\
&\vdots\\
&\leq \lim_{n\to\infty} (\alpha + L)^{n+1} d(x_0, T^i x)\}
\end{aligned}$$

*implies,*

$$\lim_{n\to\infty} d(T^i x, x_{n+1}) = 0[\textit{ since } \alpha + L < 1].$$

*Again form (2.5) we get,*

$$\lim_{n\to\infty} d(T^i x, x) \leq \lim_{n\to\infty} sd(T^i x, x_{n+1}) = 0.$$

*Therefore,* $d(T^i x, x) = 0$ *implies,* $T^i x = x$.
*If* $\lim_{n\to\infty} d(T^i x, x_n) \leq \lim_{n\to\infty} d(T^i x, x)$, *then from (2.6) we get,*

$$\lim_{n\to\infty} d(T^i x, x_{n+1}) \leq \lim_{n\to\infty} (\alpha + L)d(T^i x, x)\}.$$

*Therefore from (2.5) we get,*

$$d(T^i x, x) \leq \lim_{n\to\infty} (\alpha + L)d(T^i x, x)\} < d(T^i x, x),$$

*a contradiction.*
*Thus* $x$ *is a common fixed point of* $\{T^i\}$.

*Let,* $y$ *be another common fixed point.*
*Then*

$$\begin{aligned} &d(x, y) = d(T^i x, T^j y) \\ &\leq \alpha \max\{d(x, y), d(x, T^i x), d(y, T^j y), d(x, T^j y), d(y, T^i x)\} + Ld(y, T^i x) \\ &= \alpha \max\{d(x, y), d(x, x), d(y, y), d(x, y), d(y, x)\} + Ld(y, x) \\ &= (\alpha + L)d(x, y) \\ &< d(x, y), \end{aligned}$$

*which is a contradiction.*

*Therefore,* $d(x, y) = 0$ *implies,* $x = y$.
*Hence* $\{T^i\}$ *have unique common fixed point in* $X$. □

**Note:** The theorem is a partial solution of **Open Problem 2** of George et al.[6] another Cirić type [c.f [2]].

**Corollary 2.1.** *Let* $(X, d)$ *be a complete rectangular* $b-$*metric space with coefficient* $s > 1$ *and* $T_1$ *and* $T_2$ *be two self-maps satisfying the condition*

$$d(T_1 x, T_2 y) \leq \alpha \max\{d(x, y), d(x, T_1 x), d(y, T_2 y), d(x, T_2 y), d(y, T_1 x)\} + Ld(y, T_1 x),$$

*where the constants* $\alpha, L \geq 0$ *and* $\alpha + L < 1$. *Then the sequence* $T_1$ *and* $T_2$ *have unique common fixed point in* $X$.

*Proof.* *Putting* $T^i = T_1$ *and* $T^j = T_2$ *in the above* ***Theorem 2.1*** *we get the result.* □

**Corollary 2.2.** *Let* $(X, d)$ *be a complete rectangular* $b-$*metric space with coefficient* $s > 1$ *and* $T$ *be a self-map satisfying the condition*

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} + Ld(y, Tx),$$

*where the constants* $\alpha, L \geq 0$ *and* $\alpha + L < 1$. *Then the sequence* $T$ *have a unique fixed point in* $X$.

*Proof.* Putting $T^i = T^j = T$ in the above ***Theorem 2.1*** we get the desired result. $\square$

**Theorem 2.2.** *Let $(X, d)$ be a complete rectangular b-metric space with coefficient $s > 1$. Let $T : X \to X$ satisfying*

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\}$$

*where $k \in (0, 1)$. Then $T$ has a unique fixed point.*

*Proof.* Let us consider $x_0$ in $X$ as an initial point. Let $\{x_n\}$ be a sequence given by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = Tx_n$ i.e., $x_n = x_{n+1}$, then for all $n \in \mathbb{N}$, $x_n$ is a fixed point of $T$. So we assume that $x_n \neq x_{n+1}$.
Now

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq k \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\qquad \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]\} \\ &\leq k \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\leq k \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}).\} \end{aligned}$$

Suppose $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$. Then from above we get

$$d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1}),$$

which is a contradiction.
Therefore, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. Thus $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. So it converges to a (say).
Then

$$\begin{aligned} a &= \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(Tx_{n-1}, Tx_n) \\ &\leq k \lim_{n\to\infty} \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\qquad \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]\} \\ &= k \lim_{n\to\infty} \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= k \lim_{n\to\infty} d(x_{n-1}, x_n) = k\ a \end{aligned}$$

implies, $a = 0$ i.e., $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$.
Next, we show that $\{x_n\}$ is a Cauchy sequence i.e., $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.
First we suppose that $p =$ odd i.e., $p = 2m + 1, m \in \mathbb{N}$.
Then

$$d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})]$$

$$\leq 2sd(x_n, x_{n+1}) + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})]$$
$$\leq 2sd(x_n, x_{n+1}) + 2s^2 d(x_{n+2}, x_{n+3}) + \ldots + 2s^m \ d(x_{n+2m}, x_{n+2m+1})$$
$$\leq 2s[1 + s + s^2 + \ldots + s^{m-1}]d(x_n, x_{n+1})$$
$$= 2s(\frac{s^{m-1} - 1}{s - 1})d(x_n, x_{n+1}).$$

*Therefore,*

$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ *as* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

*Again suppose* $p =$ *even* $= 2m, m \in \mathbb{N}$.

*Then*

$$d(x_n, x_{n+2m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})]$$
$$\leq 2sd(x_n, x_{n+1}) + 2s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})$$
$$\leq 2sd(x_n, x_{n+1}) + 2s^2 d(x_{n+2}, x_{n+3}) + \ldots + 2s^m \ d(x_{n+2m-1}, x_{n+2m})$$
$$\leq 2s[1 + s + s^2 + \ldots + s^{m-1}]d(x_n, x_{n+1})$$
$$= 2s(\frac{s^{m-1} - 1}{s - 1})d(x_n, x_{n+1}).$$

*Therefore again we get,*

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

*Thus* $\{x_n\}$ *is a Cauchy sequence. Since* $X$ *is a complete space, there exists an* $x \in X$ *such that*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

*Now we show that* $x$ *is a fixed point of* $T$.
*Since*

$$\lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx)$$
$$\leq k \lim_{n\to\infty} \max\{d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{1}{2}[d(x_n, Tx_n) + d(x, Tx)]\}$$
$$\leq k \lim_{n\to\infty} \max\{d(x_n, x), d(x_n, x_{n+1}), d(x, Tx)\}$$
$$\leq k \lim_{n\to\infty} d(x, Tx)$$

*which implies,* $d(x, Tx) = 0$ *i.e.,* $x$ *is a fixed point of* $T$.
*To show the uniqueness, let* $x'$ *be another fixed point of* $T$.
*Then*

$d(x, x') = d(Tx, Tx')$
$\leq k \max\{d(x, x'), d(x, Tx), d(x', Tx'), \frac{1}{2}[d(x, Tx) + d(x', Tx')]\}$
$\leq k \max\{d(x, x'), d(x, x), d(x', x'), \frac{1}{2}[d(x, x) + d(x', x')]\}$
$= kd(x, x')$
*which implies,* $d(x, x') = 0$ *i.e.,* $x$ *is unique.*

*Hence the result.* $\square$

**Note:** This theorem is a partial solution of the **Open Problem 2** of George et al.[6] of Cirić type.

The next theorem is also a partial solution of **Open Problem 2** of George et al.[6] of Hardy-Rogers Type contraction.

**Theorem 2.3.** *Let $(X, d)$ be a complete rectangular $b-$metric space with coefficient $s > 1$. Let $T : X \to X$ be a self-map satisfying the relation*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx) \tag{2.7}$$

*where $\alpha_i \geq 0, \forall i = 1, 2, 3, 4, 5$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < \frac{1}{s}$. Then $T$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$ be an initial approximation. We construct a sequence $\{x_n\}$ in $X$ such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Suppose $d_n(x_n, x_{n+1})$ and $d^*_n(x_n, x_{n+2})$. Then byn the given condition (2.7) we get

$$\begin{aligned} d_n = d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(x_n, Tx_n) + \alpha_4 d(x_{n-1}, Tx_n) \\ &\qquad +\alpha_5 d(x_n, Tx_{n-1}) \\ &= \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + \alpha_4 d(x_{n-1}, x_{n+1}) \\ &\qquad +\alpha_5 d(x_n, x_n) \\ &= (\alpha_1 + \alpha_2) d_{n-1} + \alpha_3 d_n + \alpha_4 d^*_{n-1} \end{aligned}$$

$$\text{implies, } (1 - \alpha_3) d_n \leq (\alpha_1 + \alpha_2) d_{n-1} + \alpha_4 d^*_{n-1}. \tag{2.8}$$

If $d_{n-1} \leq d^*_{n-1}$, then from (2.8) we get,
$(1 - \alpha_3) d_n \leq (\alpha_1 + \alpha_2 + \alpha_4) d^*_{n-1}$
implies,

$$d_n \leq (\frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3}) d^*_{n-1} = k d^*_{n-1} \leq k^2 d^*_{n-2} \leq \quad \ldots \quad \leq k^n d^*_0 \ \ [k = \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3} < 1\,]$$

implies, $d_n \to 0$ as $n \to \infty$.
If $d_{n-1^*} \leq d_{n-1}$, then from (2.8) ,we get

$$(1 - \alpha_3) d_n \leq (\alpha_1 + \alpha_2 + \alpha_4) d_{n-1}$$

implies,

$$d_n \leq (\frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3}) d_{n-1}$$

from which we get as above $d_n \to 0$ as $n \to \infty$.
Now we show that $\{x_n\}$ isa a Cauchy sequence. We show this by Marthematical Induction on $p \in \mathbb{N}$ to established

$$\lim_{n \to \infty} d(x_n, x_{n+p}) = 0. \tag{2.9}$$

*Clearly (2.9) holds for $p = 1$. Suppose it holds for $p$ i.e.,* $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.
*So* $\lim_{n\to\infty} d(x_{n+1}, x_{n+p+1}) = 0$.
*Thus*

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+p+1}) &= \lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+p}) \\
&\leq \lim_{n\to\infty} [\alpha_1 d(x_{n-1}, x_{n+p}) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(x_{n+p}, Tx_{n+p}) \\
&\qquad + \alpha_4 d(x_{n-1}, Tx_{n+p}) + \alpha_5 d(x_{n+p}, Tx_{n-1})] \\
&\leq \lim_{n\to\infty} [\alpha_1 d(x_{n-1}, x_{n+p}) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_{n+p}, x_{n+p+1}) \\
&\qquad + \alpha_4 d(x_{n-1}, x_{n+p+1}) + \alpha_5 d(x_{n+p}, x_n)] \\
&= \lim_{n\to\infty} \alpha_1 d(x_{n-1}, x_{n+p}) + \lim_{n\to\infty} \alpha_4 d(x_{n-1}, x_{n+p+1}) \\
&\leq \lim_{n\to\infty} \alpha_1 s[d(x_{n-1}, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x_{n+p})] \\
&\qquad + \lim_{n\to\infty} \alpha_4 s[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1})] \\
&= \lim_{n\to\infty} \alpha_1 s d^*_{n-1} + \lim_{n\to\infty} \alpha_4 s.0
\end{aligned}$$

$$= \lim_{n\to\infty} s\alpha_1 d^*_{n-1}. \tag{2.10}$$

*Again,*

$$\begin{aligned}
\lim_{n\to\infty} d^*_{n-1} &= \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = \lim_{n\to\infty} d(Tx_{n-2}, Tx_n) \\
&\leq \lim_{n\to\infty} [\alpha_1 d(x_{n-2}, x_n) + \alpha_2 d(x_{n-2}, Tx_{n-2}) + \alpha_3 d(x_n, Tx_n) \\
&\qquad + \alpha_4 d(x_{n-2}, Tx_n) + \alpha_5 d(x_n, Tx_{n-2})] \\
&= \lim_{n\to\infty} [\alpha_1 d(x_{n-2}, x_n) + \alpha_2 d(x_{n-2}, x_{n-1}) + \alpha_3 d(x_n, x_{n+1}) \\
&\qquad + \alpha_4 d(x_{n-2}, x_{n+1}) + \alpha_5 d(x_n, x_{n-1})] \\
&= \lim_{n\to\infty} \alpha_1 d(x_{n-2}, x_n) + \lim_{n\to\infty} \alpha_4 s[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&= \lim_{n\to\infty} \alpha_1 d^*_{n-2} \\
&\leq \lim_{n\to\infty} \alpha_1^2 d^*_{n-3} \\
&\ \ \vdots \\
&\leq \lim_{n\to\infty} \alpha_1^{n-1} d^*_0 \\
&= 0.
\end{aligned}$$

*Thus from (2.10) we get,* $\lim_{n\to\infty} d(x_n, x_{n+p+1}) = 0$.

*Therefore,* $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ *for all* $p \in \mathbb{N}$.
*Thus* $\{x_n\}$ *is a Cauchy sequence in* $X$. *Since* $X$ *is a complete RbMS, there exists*

*an* $x \in x$ *such that* $\lim_{n\to\infty} x_n = x$.
*Now*

$$\begin{aligned}
d(Tx,x) &\le s[d(Tx,x_{n+1}) + d(x_{n+1},x_n) + d(x_n,x)] \\
&= s[d(Tx,Tx_n) + d(x_{n+1},x_n) + d(x_n,x)] \\
&\le s[\alpha_1 d(x,x_n) + \alpha_2 d(x,Tx) + \alpha_3 d(x_n,Tx_n) \\
&\qquad + \alpha_4 d(x,Tx_n) + \alpha_5 d(x_n,Tx) + d(x_{n+1},x_n) + d(x_n,x)]
\end{aligned}$$

$$\begin{aligned}
&= s[\alpha_1 d(x,x_n) + \alpha_2 d(x,Tx) + \alpha_3 d(x_n,x_{n+1}) + \alpha_4 d(x,x_{n+1}) \\
&\qquad + \alpha_5 d(x_n,Tx) + d(x_{n+1},x_n) + d(x_n,x)]. \tag{2.11}
\end{aligned}$$

*Again,*

$$d(x_n,Tx) = d(Tx_{n-1},Tx)$$

$$\le \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},Tx_{n-1}) + \alpha_3 d(x,Tx) + \alpha_4 d(x_{n-1},Tx) + \alpha_5 d(x,Tx_{n-1})$$

$$= \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x,Tx) + \alpha_4 d(x_{n-1},Tx) + \alpha_5 d(x,x_n). \tag{2.12}$$

*Suppose,* $d(x,Tx) \le d(x_{n-1},Tx)$. *Then from (2.12) we get,*

$$d(x_n,Tx) \le \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},x_n) + (\alpha_3+\alpha_4) d(x_{n-1},Tx) + \alpha_5 d(x,x_n)$$

*implies,*

$$\begin{aligned}
\lim_{n\to\infty} d(x_n,Tx) &\le \lim_{n\to\infty} (\alpha_3+\alpha_4) d(x_{n-1},Tx) \\
&\le \lim_{n\to\infty} (\alpha_3+\alpha_4)^2 d(x_{n-2},Tx) \\
&\vdots \\
&\le \lim_{n\to\infty} (\alpha_3+\alpha_4)^n d(x_0,Tx) = 0.
\end{aligned}$$

*Thus from (2.11) we get,*

$$\begin{aligned}
&\lim_{n\to\infty} d(Tx,x) \le s\alpha_2 \lim_{n\to\infty} d(Tx,x) \\
&\textit{implies,}\quad d(Tx,x) = 0 \\
&\textit{implies,}\quad Tx = x.
\end{aligned}$$

*Again suppose,* $d(x_{n-1},Tx) \le d(x,Tx)$. *Then from (2.12) we get,*

$$d(x_n,Tx) \le \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},x_n) + (\alpha_3+\alpha_4) d(x,Tx) + \alpha_5 d(x,x_n).$$

*Therefore,*

$\lim_{n\to\infty} d(x_n, Tx) \leq \lim_{n\to\infty}(\alpha_3 + \alpha_4)d(x, Tx)$.
*From (2.11) we get,*

$$d(Tx, x) \leq s[\alpha_2 d(x, Tx) + \lim_{n\to\infty} \alpha_5 d(x_n, Tx)]$$

$$\leq s\alpha_5(\alpha_3 + \alpha_5)(\alpha_3 + \alpha_4)d(x, Tx)$$

$$\leq s\alpha_5 d(Tx, x)$$

$$\textit{implies,} \quad d(Tx, x) = 0.$$

*Therefore, $x$ a fixed point of $T$.*

*Suppose, $y$ be another fixed point of $T$.*
*Then*

$$d(x, y) = d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)$$

$$= \alpha_1 d(x, y) + \alpha_2 d(x, x) + \alpha_3 d(y, y) + \alpha_4 d(x, y) + \alpha_5 d(y, x)$$

$$= (\alpha_1 + \alpha_4 + \alpha_5)d(x, y),$$

*implies,* $[1 - (\alpha_1 + \alpha_4 + \alpha_5)]d(x, y) = 0$ *i.e.,* $x = y$.

*Thus $x$ is a unique fixed point of $T$.*
*Hence the theorem.* □

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## REFERENCES

1. I.A. Bakhtin: *The contraction principle in quasimetric spaces*, Funct. Anal. **30** (1989), 26–37.
2. V. Berinde: *Some remarks on a fixed point theorem for Cirić-Type almost contraction*, Carpathian J. Math., **25** (2) (2009), 157-162.
3. S. Czerwik: *Contraction mappings in b-metric spaces*, Acta Math. Inform., Univ. Ostrav. **1** (1993), 5–11.
4. H. Ding et al.: *On some fixed point results in b-metric, rectangular and b-rectangular metric spaces*, Arab J Math Sci, **22** (2016) ,151–164.
5. H. Ding, V. Ozturk, S. Radenovic: *On some fixed point results in brectangular metric spaces*, Journal Of Nonlinear Sciences And Applications , **8** (4) (2015), 378-386.
6. R. George, S. Radenović, K.P. Reshma, S. Shukla: *Rectangular b-metric spaces and contraction principle*, J. Nonlinear Sci. Appl. **8** (2015), 1005–1013.

7. H. Huang, G. Deng, Z. Chen, S. Radenović: *On some recent fixed point results for $\alpha$-admissible mappings in $b-$metric spaces*, J. Computational Analysis and applications, **25** (2) (2018), 255-269.

8. Z. D. Mitrović and S. Radenović: *The Banach and Reich contractions in $b_v(s)$metric spaces*, Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam, doi 10.1007/s11784-017-0469-2.

9. V. Ozturk: *Fixed point theorems in b-rectangular metric spaces*, Universal Journal Of Mathematics, **3** (1) (2020), 28-32.

10. K. Sarkar and K. S. Tiwary: *Common Fixed Point Theorems for Weakly Compatible Mappings on Cone Banach Space*, International Journal of Scientific Research in Mathematical and Statistical Sciences, **5** (2) (2018), 75-79.

11. K. Sarkar and K. S. Tiwary: *Fixed point theorem in cone banachspaces*, International Journal of Statistics and Applied Mathematics, 3(4), (2018), 143-146.

12. K. S. Tiwary, K. Sarkar and T. Gain: *Some Common Fixed Point Theorems in B-Metric Spaces*, International Journal of Computational Research and Development, **3** (1) (2018), 128-130.

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# ON MULTIVALUED $\theta$-CONTRACTIONS OF BERINDE TYPE WITH AN APPLICATION TO FRACTIONAL DIFFERENTIAL INCLUSIONS

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**Abstract.** In this paper, we discuss the existence of fixed points for Berinde type multivalued $\theta$- contractions. An example is provided to demonstrate our findings and, as an application, the existence of the solutions for a nonlinear fractional inclusions boundary value problem with integral boundary conditions is given to illustrate the utility of our results.
**Keywords:** fixed point, $\theta$ contraction, $\alpha$-admissible, fractional differential inclusions.

## 1. Introduction and preliminaries

Multivalued fixed point theory has been known some development, starting with the results of Nadler [21], where he proved the existence of multivalued fixed point using the Hausdorff metric, later, some generalizations were given in this way, for example, see [4, 10, 13, 27] and references therein.
Berinde [7] introduced the concept of almost contractions as a generalization to weak contractions notion in the context of single valued mappings, which was later extended to the multivalued case in [8, 9], and some results were obtained using this concept. .
Samet et al. [23] introduced a new concept called $\alpha$-admissible and they obtained some fixed point results for $\alpha - \psi$-contractive mappings, later, some results were

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established in this direction, see for example [2, 14, 15, 20]. Recently, Jleli and Samet [18] introduced $\theta$-contractions type and demonstrated the existence of fixed points for such contractions. It is worth noting here, that a Banach contraction is a particular case of $\theta$ contraction, whereas there are some $\theta$-contractions that are not Banach contraction. Following that, several authors investigated various variants of $\theta$-contraction for single-valued and multivalued mappings, for example, see [1, 11, 12, 28].

In this work, we combine the concept of $\alpha$-admissible mappings with the concept of $\theta$-contractions type in the context of multivalued mappings to demonstrate the existence of a fixed point for such new contractions type in complete metric spaces. Using our main results, we also deduce the existence of a fixed point in partially ordered metric spaces and in metric spaces endowed with a graph. Finally, to demonstrate the significance of the obtained results, we provide an example and an application of the existence of solutions for a fractional differential inclusion.

Denote by $CL(X)$ the family of nonempty and closed subsets of $X$, the family of nonempty, bounded and closed subsets of $X$ is denoted by $CB(X)$ and the family of nonempty and compact subsets of $X$ is denoted by $K(X)$.

Let $(X, d)$ be a metric space, and the Pompeiu-Hausdorff metric is defined as a function $H\colon CL(X) \times CL(X) \to [0, \infty]$ which is defined by:

$$H(A,B) = \begin{cases} \max\left\{\sup\limits_{x\in A} d(x,B),\ \sup\limits_{y\in B} d(y,A)\right\} & \text{if the maximum exists;} \\ \infty, & \text{otherwise,} \end{cases}$$

where $d(a, B) = \inf\{d(a, b)\colon b \in B\}$. Note that, if $A = \{a\}$ (singleton) and $B = \{b\}$, then $H(A, B) = d(a, b)$.

**Lemma 1.1.** *[21] Let $(X, d)$ be a metric space and $A, B \in CL(X)$ with $H(A, B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that $d(a, b) < hH(A, B)$.*

Now, we'll look at some fundamental definitions of $\alpha$-admissibility and $\alpha$-continuity concepts.

**Definition 1.1.** Let $(X, d)$ be a metric space and $\alpha : X \times X \to [0, +\infty)$ be a given mapping. A mapping $T : X \to CL(X)$ is

- $\alpha$-admissible [2], if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ we have $\alpha(y, z) \geq 1$, for all $z \in Ty$.
- $\alpha$-lower semi-continuous [14], if for $x \in X$ and a sequence $\{x_n\}$ in $X$ with $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, implies
$$\lim_{n\to\infty} \inf d(x_n, Tx_n) \geq d(x, Tx).$$

**Definition 1.2.** [18] Let $\Theta$ be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ satisfying:

$(\theta_1)$ : $\theta$ is non decreasing,

$(\theta_2)$ : for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} t_n = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,

$(\theta_3)$ : there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l$.

**Example 1.1.** Let $\theta_i : (0, +\infty) \to (1, +\infty)$, $i \in \{1, 2, 3\}$, defined by:

1. $\theta_1(t) = e^t$.
2. $\theta_2(t) = e^{te^t}$.
3. $\theta_3(t) = e^{\sqrt{x}}$.
4. $\theta_4(t) = e^{\sqrt{t}e^t}$.

Then $\theta_i \in \Theta$, for each $i \in \{1, 2, 3\}$.

Throughout this paper, we will denote by $\Phi$ the set of all continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying:

(1) : $\psi$ is nondecreasing ,

(2) : $\sum_{i=1}^{\infty} \psi^n(t) < \infty$, for all $t \in [0, +\infty)$.

Clearly, if $\psi \in \Psi$, then $\psi(t) < t$, for all $t \in [0, +\infty)$.

## 2. Main results

**Definition 2.1.** Let $(X, d)$ be a metric space and $\alpha : X \times X \to \mathbb{R}$. A mapping $T : X \to CL(X)$ is called a generalized almost $(\alpha, \psi, \theta, k)$ contraction, if there exists a function $\theta \in \Theta$, $\psi \in \Psi$, $L \geq 0$ and $k : (0, \infty) \to [0, 1)$ satisfies $\lim_{t\to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that

$$\theta(H(Tx, Ty)) \leq \Big[\theta(\psi(M(x, y))\Big]^{k(M(x,y))} + LN(x, y), \tag{2.1}$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}$$

and $N(x, y) = min\{d(x, Ty), d(y, Tx)\}$.

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space and $T : X \to K(X)$ be a generalized almost $(\alpha, \psi, \theta, k)$ contraction, with $\theta \in \Theta$. Assume that the following conditions are satisfied:*

1. *$T$ is $\alpha$-admissible.*

2. *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$.*

3. *$T$ is $\alpha$-lower semi-continuous, or $X$ is $\alpha$-regular, that is, for every sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then $T$ has a fixed point.*

*Proof.* From (2) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, then $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$, otherwise $x_1 \in Tx_1$, or, $x_0 = x_1$, which implies $x_1$ is a fixed point and the proof completes. For $H(Tx_0, Tx_1) > 0$ using (2.1) we get:

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1))$$

$$\leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1)} + Ld(x_1, Tx_0) < [\theta(M(x_0, x_1))]^{k(M(x_0,x_1)}.$$

If $d(x_0, x_1) \leq d(x_1, Tx_1)$, we get

$$\theta(d(x_1, Tx_1)) \leq \Big[\theta(\psi(d(x_1, Tx_1)))\Big]^{k(d(x_1,Tx_1))} + LN(x_0, x_1) < \theta(d(x_1, Tx_1),$$

which is a contradiction. Then we have

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1)}.$$

Since $Tx_1$ is compact, then there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) = \theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1))$$

$$\leq \Big[\theta(d(x_0, x_1))\Big]^{k(d(x_0,x_1)} < \theta(d(x_0, x_1)).$$

If $x_1 = x_2$, or $x_2 \in Tx_2$, then $x_2$ is a fixed point. Suppose $x_1 \neq x_2$ and $x_2 \notin Tx_2$, so $H(Tx_2, Tx_1) > 0$ and since $T$ is $\alpha$-admissible we have $\alpha(x_1, x_2) \geq 1$. Using (2.1) we get:

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(M(x_1, x_2)))\Big]^{k(M(x_1,x_2)} + LN(x_1, x_2)$$

$$= \Big[\theta(d(x_1, x_2))\Big]^{k(M(x_1,x_2))}.$$

If $d(x_1, x_2) \leq d(x_2, Tx_2)$, we get

$$\theta(d(x_2, Tx_2)) \leq \Big[\theta(\psi(d(x_2, Tx_2)))\Big]^{k(d(x_2,Tx_2))} + LN(x_1, x_2) < \theta(d(x_2, Tx_2),$$

which is a contradiction. Then we have

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2)}.$$

The compactness of $Tx_2$ implies that there exists $x_3 \in Tx_2$ such that

$$\theta(d(x_2, x_3)) = \theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2))$$

$$\leq \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))} < \theta(d(x_1, x_2)).$$

Continuing in this manner we can construct a sequence $(x_n)$ in $X$, if $x_n = x_{n+1}$ or $x_{n+1} \in Tx_{n+1}$, then $x_{n+1}$ is a fixed point, otherwise we get

$$\theta(d(x_n, Tx_{n+1})) \leq \Big[\theta(\psi(M(x_n, x_{n-1})))\Big]^{k(M(x_n,x_{n-1})} + LN(x_n, x_{n-1}).$$

As the same arguments in previous steps, we get

$$d(x_{n+1}, Tx_{n+1}) \leq d(x_n, x_{n+1}),$$

so we obtain

$$\theta(d(x_n, x_{n+1})) \leq \theta(H(Tx_n, Tx_{n-1})) \leq \Big[\theta(\psi(d(x_n, x_{n-1})))\Big]^{k(d(x_n,x_{n+1})}$$

$$= \Big[\theta(\psi(d(x_n, x_{n-1})))\Big]^{k(d(x_n,x_{n-1}))} < \theta(d(x_n, x_{n-1})).$$

Since $\theta$ is increasing, then the sequence $(d(x_n, x_{n+1}))_n$ is decreasing, further it is bounded at below so it is convergent. On the other hand, $\limsup\limits_{t \to s^+} k(t) < 1$, then there exists $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < \delta$, for all $n \geq n_0$. Thus we have

$$1 < \theta(d(x_n, x_{n+1})) \leq \Big[\theta(d(x_{n_0}, x_{n_0+1}))\Big]^{\delta^{n-n_0}}, \tag{2.2}$$

for all $n \geq n_0$.
Letting $n \to \infty$ in (2.2), we get

$$\lim_{n \to \infty} \theta(d(x_n, x_{n+1})) = 1,$$

By $(\theta_2)$, we infer that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Now, we prove $\{x_n\}$ is a Cauchy sequence, from $(\theta_3)$ there exist $r \in [0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n \to \infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{(d(x_n, x_{n+1})^r} = l.$$

If $l < \infty$, let $2\varepsilon = l$, so from the definition of limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have

$$\varepsilon = l - \varepsilon < \frac{\theta(d(x_n, x_{n+1})) - 1}{(d(x_n, x_{n+1})^r}$$

$$(d(x_n, x_{n+1}))^r < \frac{\theta(d(x_n, x_{n+1})) - 1}{\varepsilon}.$$

Then (2.2) gives

$$n(d(x_n, x_{n+1}))^r < \frac{n(\theta(d(x_0, x_1))^{\delta^{n-n_0}} - 1)}{\varepsilon}. \tag{2.3}$$

In the case where $l = \infty$, let $A$ be an arbitrary positive real number, so from the definition of the limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{(d(x_n, x_{n+1}))^r} > A,$$

which implies that

$$n(d(x_n, x_{n+1}))^r \leq \frac{n(\theta(d(x_0, x_1))^{\delta^{n-n_0}} - 1)}{A}. \tag{2.4}$$

Letting $n \to \infty$ in (2.4)(or in (2.3), we obtain

$$\lim_{n\to\infty} n(d(x_n, x_{n+1}))^r = 0.$$

From the definition of the limit, there exists $n_2 \geq \max\{n_0, n_1\}$ such that for all $n \geq n_2$, we have

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}},$$

This implies

$$\sum_{n=n_2}^{\infty} d(x_n, x_{n+1}) \leq \sum_{1}^{\infty} \frac{1}{n^{\frac{1}{r}}} < \infty.$$

Then $\{x_n\}$ is a Cauchy sequence.
The completness of $(X, d)$ implies that $\{x_n\}$ converges to a some $x \in X$.
Now, we show that $x$ is a fixed point of $T$. In fact, if $T$ is $\alpha$-lower continuous, then for all $n \in \mathbb{N}$ we have

$$0 \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}).$$

Letting $n \to +\infty$, we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

The $\alpha$-lower semi continuity of $T$ implies

$$0 \leq d(x, Tx) < \lim_{n\to\infty} \inf d(x_n, Tx_n) = 0.$$

Hence $d(x, Tx) = 0$ and $x$ is a fixed point of $T$.
If $X$ is regular, so $\alpha(x_n, x) \geq 1$ and $H(Tx_n, Tx) > 0$, by using (2.1) we get

$$1 < \theta(d(x_{n+1}, Tx)) \leq \theta(H(Tx_n, Tx)) < \Big[\theta(d(x_0, x_1))\Big]^{\delta^{n-n_0}}.$$

Letting $n \to +\infty$, we get

$$\lim_{n\to\infty} \theta(d(x_n, Tx)) = 1,$$

so $(\theta_2)$ gives

$$\lim_{n\to\infty} d(x_n, Tx) = 0,$$

which implies that $x \in Tx$. $\square$

**Theorem 2.2.** *Let $(X, d)$ be a complete metric space and let $T : X \to CB(X)$ be a generalized almost $(\alpha, \psi, \theta)$ contraction, with $\theta$ is right continuous. Assume that the following conditions are satisfied:*

$(H_1)$ : *$T$ is $\alpha$-admissible,*

$(H_2)$ : *there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,*

$(H_3)$ : *for every sequence $\{x_n\}$ in $X$ converging to $x \in X$ with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.*

*Then $T$ has a fixed point.*

*Proof.* From $(H_2)$ there are $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, if $x_0 = x_1$, or, $x_1 \in Tx_1$, so $x_1$ is a fixed point. Suppose the contrary, then $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$ and by using (2.1) we get

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(M(x_0, x_1)))\Big]^{k(M(x_0,x_1)} + LN(x_0, x_1)$$

$$< \Big[\theta(M(x_0, x_1))\Big]^{k(M(x_0,x_1)} + LN(x_0, x_1).$$

By right continuity of $\theta$, there exists $h > 1$ such that

$$\theta(hH(Tx_0, Tx_1)) \leq \Big[\theta(\psi(M(x_0, x_1)))\Big]^{k(M(x_0,x_1))} + LN(x_0, x_1).$$

As in proof of Theorem 2.1 we get $M(x_0, x_1) = d(x_0, x_1)$ and $N(x_0, x_1) = 0$, then by using Lemma 1.1, there exist $x_2 \in Tx_1$ and $h_1 > 1$ such that

$$\theta(d(x_1, x_2)) \leq \theta(h_1 H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1))}$$

$$< \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1))} < \theta(d(x_0, x_1)).$$

Since $T$ is $\alpha$-admissible, then $\alpha(x_1, x_2) \geq 1$. Assume that $x_1 \neq x_2$ and $x_2 \in Tx_2$, so $H(Tx_1, Tx_2) \geq d(x_2, Tx_2) > 0$ and using (2.1), we obtain

$$1 < \theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(M(x_1, x_2)))\Big]^{k(M(x_1,x_2))} + LN(x_1, x_2)$$

$$< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))}.$$

As in previous step, we have $M(x_1, x_2) = d(x_1, x_2)$, so we get

$$\theta(d(x_2, Tx_2)) \le \theta(H(Tx_1, Tx_2)) \le \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2))}$$

$$< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))}.$$

Since $\theta$ is right continuous and from Lemma 1.1, there exists $h_2 > 1$ and $x_3 \in Tx_2$ such that

$$\theta(d(x_2, x_3)) \le \theta(h_2 H(Tx_1, Tx_2)) \le \Big[\theta(\psi(d(x_1, x_2)\Big]^{k(d(x_1,x_2))}.$$

$$< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))} < \theta(d(x_1, x_2)).$$

Continuing in this manner, we can construct two sequences $\{x_n\} \subset X$ and $(h_n) \subset (1, \infty)$ such that $x_n \neq x_{n+1}$, $x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) \ge 1$ and

$$1 < \theta(d(x_n, x_{n+1})) \le \theta(h_n H(Tx_{n-1}, Tx_n))$$

$$\le \Big[\theta(d(x_n, x_{n-1}))\Big]^{k(d(x_n,x_{n-1}))} + LN(x_n, Tx_{n-1})$$

$$< \theta(d(x_n, x_{n-1})),$$

which implies that $(d(x_n, x_{n+1}))_n$ is a decreasing sequence and bounded at below, so there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < \delta$, for all $n \ge n_0$. Thus we have

$$1 < \theta(d(x_n, x_{n+1})) < \Big[\theta(d(x_0, x_1))\Big]^{\delta^{n-n_0}}, \tag{2.5}$$

for all $n \ge n_0$.
On taking the limit as $n \to \infty$, we get $\lim\limits_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1$, $(\theta_2)$ gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

The rest of the proof is like in the proof of Theorem 2.1. □

**Corollary 2.1.** *Let $(X, d)$ be a complete metric space, $\alpha: X \times X \to [0, +\infty)$ be a function and $T: X \to K(X)$ (resp $CB(X)$ with $\theta$ is right continuous) be an $\alpha$-admissible multivalued mapping and the following assertions hold:*

*(i) $T$ is $\alpha$-admissible.*

*(ii) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$.*

*(iii) $T$ is $\alpha$-lower semi-continuous, or, for every sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$, for all $n \in \mathbb{N}$.*

(iv) *There exist* $\theta \in \Theta$, $\psi \in \Psi$ *and a function* $k : (0,\infty) \to [0,1)$ *satisfying* $\lim_{t\to s^+} \sup k(t) < 1$ *such for* $x, y \in X$ $\;H(Tx,Ty) > 0$ *implies*

$$\alpha(x,y)\theta\big(H(Tx,Ty)\big) \leq \theta\Big[(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y), \tag{2.6}$$

*where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}$$

*and* $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$.

*Then* $T$ *has a fixed point.*

*Proof.* Let $x, y \in X$ be such that $\alpha(x,y) \geq 1$ and $H(Tx,Ty) > 0$. So from (2.7) we get

$$\begin{aligned}\theta\big(H(Tx,Ty)\big) &\leq \alpha(x,y)\theta\big(H(Tx,Ty)\big) \\ &\leq \theta\Big[(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y),\end{aligned}$$

which implies that the inequality (2.1) holds. Thus, the rest of proof is like in the proof of Theorem 2.2 (resp. Theorem 2.1). $\square$

If $\alpha(x,y) = 1$, for all $x, y \in X$, we get the following corollary.

**Corollary 2.2.** *Let* $(X,d)$ *be a complete metric space and* $T\colon X \to K(X)$ *(resp.* $CB(X)$ *with* $\theta$ *is right continuous) be a multivalued mapping such that there exists* $\theta \in \Theta$, $\psi \in \Psi$ *and a function* $k : (0,\infty) \to [0,1)$ *satisfying* $\lim_{t\to s^+} \sup k(t) < 1$ *for all* $s \in (0,\infty)$ *such that*

$$\theta\big(H(Tx,Ty)\big) \leq \theta\Big[(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y), \tag{2.7}$$

*for* $x, y \in X$ *with* $H(Tx,Ty) > 0$ *where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}$$

*and* $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$. *Then* $T$ *has a fixed point in* $X$.

**Example 2.1.** Let $X = \{1,2,3\}$ and $d(x,y) = |x-y|$. Define $T\colon X \to CB(X)$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \{1\}, & x \in \{1,2\} \\ \{2\}, & x = 3 \end{cases}$$

and $\alpha(x,y) = e^{|x-y|}$. Taking $\theta(t) = e^t$, $\psi(t) = \frac{4}{5}t$ and $k(t) = \frac{1}{2}$.
Now, we show that the contractive condition holds.
For $x, y \in X$, we have $|x-y| \geq 0$, which implies $e^{|x-y|} \geq 1$. Then $T$ is $\alpha$-admissible.
On other hand, $H(Tx,Ty) > 0$ and $\alpha(x,y) \geq 1$ for all $(x,y) \in \{(1,3),(3,1),(2,3),(3,2)\}$.
Then we have the following cases:

1. for $x = 1$ and $y = 3$, we have

$$H(T1, T3) = 1, \quad d(1,3) = 2, \quad \psi(d(1,3)) = \frac{8}{5} \quad \text{and} \quad d(3, T1) = 2,$$

then

$$\begin{aligned} e = e^{H(T1,T3)} &< (e^{\psi(d(1,3))})^{\frac{1}{2}} + d(3, T1) \\ &= e^{\frac{4}{5}} + 2. \end{aligned}$$

2. For $x = 2$ and $y = 3$, we have

$$H(T2, T3) = 1, \quad d(2,3) = 1, \quad \psi(d(1,3)) = \frac{4}{5} \quad \text{and} \quad d(3, T2) = 2,$$

then

$$\begin{aligned} e = e^{H(T2,T3)} &< (e^{\psi(d(1,3))})^{\frac{1}{2}} + d(3, T2) \\ &= e^{\frac{2}{5}} + 2. \end{aligned}$$

There exists $x_0 = 2$ and $x_1 = 1 \in Tx_0$ such that $\alpha(2,1) \geq 1$.

It is clear that $T$ is $\alpha$- lower semi continuous. Consequently, all conditions of Theorem 2.1 are satisfied. Then $T$ has a fixed point which is 1.

## 3. Fixed point on partially ordered metric spaces

Now, we give an existence theorem of fixed point in a partially order metric space, by using the results provided in previous section.

**Theorem 3.1.** *Let $(X, \preceq, d)$ be a complete ordered metric space and $T\colon X \to CB(X)$ be a multivalued mapping. Assume that the following assertions hold:*

1. *For each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$;*
2. *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.*
3. *For every nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$, we have $x_n \preceq x$, for all $n \in \mathbb{N}$.*
4. *There exists a right continuous function $\theta \in \Theta$, $\psi \in \Psi$ and $k : (0, \infty) \to [0, 1)$ satisfies $\lim\limits_{t \to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that*

$$\theta(H(Tx, Ty)) \leq \Big[\theta(\psi(M(x,y)))\Big]^{k(M(x,y)} + LN(x,y), \tag{3.1}$$

*for all $x, y \in X$ with $x \preceq y$ and $H(Tx, Ty) > 0$, where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))$$

*and $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$.*

*Then $T$ has a fixed point.*

*Proof.* Define $\alpha: X \times X \to [0,+\infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (1), for each $x \in X$ and $y \in Tx$ with $x \preceq y$, i.e., $\alpha(x,y) = 1 \geq 1$, we have $z \preceq y$, for all $z \in Ty$, i.e., $\alpha(x,y) = 1 \geq 1$. Thus $T$ is $\alpha$-admissible.
From (2), there exit $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$, i.e., $\alpha(x_0, x_1) = 1 \geq 1$.
Condition (3) implies $\alpha$- lower semi continuity of $T$, or regularity of $X$.
From (4), for $x \preceq y$, we have $\alpha(x,y) = 1 \geq 1$ then the inequality (2.1) holds, which implies that $T$ is a generalized almost $(\alpha, \psi, \theta, k)$ contraction. □

## 4. Fixed point on metric spaces endowed with a graph

In this section, as a consequence of our main results, we present an existence theorem of fixed point for a multivalued mapping in a metric space $X$, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph $G$ such that the set $V(G)$ of its vertices coincides with $X$ and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x,x) : x \in X\}$. We assume $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$.

**Theorem 4.1.** *Let $(X,d)$ be a complete metric space endowed with a graph $G$ and $T: X \to CB(X)$ be a multivalued mapping. Assume that the following conditions are satisfied:*

1. *For each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$;*

2. *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*

3. *$T$ is $G$-lower semi-continuous, that is, for $x \in X$ and a sequence $\{x_n\}$ in $X$ with*
   *$\lim_{n\to\infty} d(x_n, x) = 0$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, implies*
   $$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx)$$
   *or, for every sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;*

4. *There exists a right continuous function $\theta \in \Theta$, $\psi \in \Psi$ and $k : (0,\infty) \to [0,1)$ satisfing $\lim_{t\to s^+} \sup k(t) < 1$ for all $s \in (0,\infty)$ such that*
   $$\theta(H(Tx,Ty)) \leq \Big[\theta(\psi(M(x,y)))\Big]^{k(M(x,y)} + LN(x,y), \tag{4.1}$$
   *for all $x, y \in X$ with $(x,y) \in E(G)$ and $H(Tx,Ty) > 0$, where*
   $$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))$$
   *and $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$.*

*Then $T$ has a fixed point.*

*Proof.* This result is a direct consequence of results of Theorem 2.1 by taking the function $\alpha\colon X \times X \to [0, +\infty)$ defined by:

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

□

## 5. Application to fractional differential inclusions

Consider the following boundary value problem of fractional order differential inclusion with boundary integral conditions:

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)),\ 0 \le t \le 1,\ 1 < q \le 2 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 h(s) g(s, x(s)) ds \end{cases} \tag{5.1}$$

where ${}^cD^q$ , $1 < q \le 2$ is the Caputo fractional derivative, $F$, $g$, and $h$ are given continuous functions, where
$F : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{K}(\mathbb{R})$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, $h \in L^1([0,1])$, $a + b > 0$, $\frac{a}{a+b} < q - 1$ and $h_0 = \|h\|_{L^1}$.
Denote by $X = \mathcal{C}([0,1], \mathbb{R})$ the Banach space of continuous functions $x : [0,1] \longrightarrow \mathbb{R}$, with the supermum norm

$$\| x \|_\infty = \sup\{\| x(t) \|,\ \ t \in I = [0,1]\}.$$

$X$ can be endowed with the partial order relationship $\preceq$, that is, for all $x, y \in X$ $x \preceq y$ if and only if $x(t) \le y(t)$, so $(X, d_\infty, \preceq)$ is a complete order metric space.
$x$ is a solution of problem (5.1) if there exists $v(t) \in F(t, x(t)))$, for all $t \in I$ such that

$$\begin{cases} {}^cD^q x(t) = v(t),\ 0 \le t \le 1,\ 1 < q \le 2 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 h(s) g(s)) ds \end{cases} \tag{5.2}$$

**Lemma 5.1.** *Let $1 < q \le 2$ and $v \in \mathcal{AC}(I, \mathbb{R}) = \{v : I \to \mathbb{R}, f \text{ is absolutely continuous}\}$. A function $x$ is a solution of (5.2) if and only if it is a solution of the integral equation:*

$$x(t) = \int_0^1 G(t,s) v(s) ds + \frac{at+b}{a+b} \int_0^1 h(s) g(s) ds,$$

*where $G$ is the Green function given by*

$$G(t,s) = \begin{cases} \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)}, & s \le t \\ \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)}, & t \le s. \end{cases} \tag{5.3}$$

*Proof.* The problem (5.2) can be reduced to an equivalent integral equation:

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + c_0 + c_1 t,$$

for some constants $c_0, c_1 \in X$.
Using the boundary conditions on (5.2), we get

$$ac_0 - bc_1 = 0,$$

$$\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + c_0 + c_1 = \int_0^1 h(s)g(s)ds.$$

Therefore

$$c_0 = \frac{b}{a+b} \left[ \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} g(s, x(s))ds + \int_0^1 h(s)g(s, x(s))ds \right].$$

$$c_1 = \frac{a}{a+b} \left[ \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s, x(s))ds \right].$$

It means that

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + \frac{b}{a+b} \left[ \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s, x(s))ds \right] \\ &\quad + \frac{at}{a+b} \left[ \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s, x(s))ds \right] \\ &= \int_0^t \left[ \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)} \right] v(s)ds + \int_t^1 \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} v(s)ds \\ &\quad + \frac{at+b}{a+b} \int_0^1 h(s)g(s, x(s))ds = \int_0^1 G(t,s)v(s)ds + \frac{at+b}{a+b} \int_0^1 h(s)g(s)ds. \end{aligned}$$

□

Moreover, we have

$$\begin{aligned} \int_0^1 G(t,s)ds &= \frac{(1}{\Gamma(q)} \Big[ \int_0^t (t-s)^{q-1} ds + \frac{at+b}{a+b} \int_0^1 (1-s)^{q-1} ds \Big] \\ &\leq \frac{1}{\Gamma(q+1)} t^q + \frac{1}{\Gamma(q+1)} \leq \frac{2}{\Gamma(q+1)}. \end{aligned}$$

Define a set valued mapping

$$Tx_1(t) = \{z \in X, z(t) = \int_0^1 G(t,s)v(s)ds + \frac{at+b}{a+b} \int_0^1 h(s)g(s, x_1(s)ds\}.$$

The problem (5.1) has a solution if and only if $T$ has a fixed point. Assume that the following assumptions hold:

- $(A_1)$ : For each $x_1 \in X$ and $x_2 \in Tx_1$ with $x_1 \preceq x_2$ we have $x_2 \preceq x_3$ for all $x_3 \in Tx_2$.
- $(A_2)$ : There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.
- $(A_3)$ : There exists $K > 0$ and $L > 0$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$H(F(t, x_1(t)) - F(t, x_2(t))) \leq K|x_1 - x_2|)$$

and

$$|g(t, x_1(t)) - g(t, x_2(t))| \leq L|x_1 - x_2|,$$

with $k_0 = \frac{2K}{\Gamma(q+1)} + h_0 L < \frac{1}{2}$.

**Theorem 5.1.** *Under the assumptions $(A_1)$–$(A_3)$ the problem (5.1) has a solution in $X$.*

*Proof.* Since $F$ is continuous, it has a selection, i,e., there exists a continuous function $v_1 \in F(t, x_1(t))$ such that $Tx_1$ is nonempty and has compact values.
Let $x_1, x_2 \in X$ and $z_1 \in Tx_1$, then there exists $v_1 \in F(t, x_1(t))$ such that

$$z_1(t) = \int_0^1 G(t,s)v_1(s)ds + \frac{at+b}{a+b}\int_0^1 h(s)g(s, x_1(s))ds.$$

Then by using $(A_2)$, we get

$$d(v_1, Fx_2) = \inf_{u \in Fx_2} |v_1 - u| \leq H(F(t, x_1(t)) - F(t, x_2(t)))$$

$$\leq K\|x_1 - x_2\|,$$

the compactness of $F(t, x_2(t))$ implies that there exists $u^* \in F(t, x_2(t))$ such that

$$d(v_1, Fx_2) = |v_1 - u^*| \leq K|x_1 - x_2|.$$

Define an operator $P(t) = \{u^* \in \mathbb{R}, |u_1(t) - u^*| \leq K|x_1(t) - x_2(t)|\}$. Clearly $P \cap F(t, x_2(t))$ is continuous, so it has a selection $v_2$ such that

$$|u_1 - u_2| \leq K|x_1 - x_2|.$$

Define

$$z_2 = \int_0^1 G(t,s)u_2(s)ds + \frac{at+b}{a+b}\int_0^1 h(s)g(s, x_2(s)ds.$$

For all $t \in I$, we have

$$|z_1 - z_2| \leq \int_0^1 |G(t,s)||u_1 - u_2|ds + \frac{at+b}{a+b}\int_0^1 |h(s)||g(s, x_1(s)) - g(s, x_2(s))|ds$$

$$\leq K|x_1 - x_2|) \int_0^1 |G(t,s)|ds + \frac{at+b}{a+b} h_0 L |x_1(s) - x_2(s)|$$

$$\leq (\frac{2K}{\Gamma(q+1)} + h_0 L)|x_1 - x_2| = k_0 |x_1 - x_2|$$

Then, we have

$$\sup_{z_1 \in Tx_1} \left[ \inf_{z_2 \in Tx_2} |z_1 - z_2| \right] \leq k_0 \|x_1 - x_2\|.$$

Hence, by interchanging the role of $x_1$ and $x_2$ we obtain

$$H(Tx_1, Tx_2) \leq k_0 |x_1 - x_2|).$$

On taking the exponential of two sides, we get

$$e^{H(Tx_1,Tx_2)} \leq (e^{2k_0|x_1 - x_2|})^{\frac{1}{2}}$$

$$\leq e^{k_0|x_1 - x_2|} + d(x_2, Tx_1).$$

If $\{x_n\}$ is a nondecreasing sequence in $X$ which converges to $x \in X$, so for all $t \in I$ and $n \in \mathbb{N}$ we have $x_n(t) \leq x(t)$, which implies that $x$ is an upper bound for all terms $x_n$ (see [22]), then $x_n \preceq x$.
Consequently, all the conditions of Theorem 3.1 are satisfied, with $\theta(t) = e^t$, $\psi(t) = 2k_0 t$ and $k(t) = k_0$.
Hence, $T$ has a fixed point which is a solution of the problem (5.1). $\square$

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## REFERENCES

1. J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho and Y.O. Yang: *Fixed point results for generalized $\theta$-contractions*, J. Nonlinear Sci. Appl., **10** (2017), 2350-2358.
2. H. Asl, J, Rezapour and S, Shahzad: *On fixed points of $\alpha - \psi$-contractive multifunctions*, Fixed Point Theory Appl. 2012, Article ID 212 (2012).
3. G. V. R. Babu, M. L. Sandhy and M. V. R. Kameshwari: *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math, **24** (2008), 8-12.
4. S. Beloul: *Common fixed point theorems for multi-valued contractions satisfying generalized condition(B) on partial metric spaces*, Facta Univ Nis Ser. Math. Inform., **30** (5) (2015), 555-566.
5. S. Beloul: *A Common Fixed Point Theorem For Generalized Almost Contractions In Metric-Like Spaces*, Appl. Maths. E - Notes. **18** (2018), 27-139.
6. V. Berinde: *Approximating fixed points of weak $\varphi$-contractions using the Picard iteration*, Fixed Point Theory, (2003), 131-142.

7. V. Berinde: *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin Heidelberg, 2007.

8. V. Berinde and M. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl., **326** (2007), 772-782.

9. V. Berinde: *Some remarks on a fixed point theorem for Ćiric-type almost contractions*, Carpath. J. Math. **25** (2009), 157-162.

10. L. B. Ćirić]: *Multi-valued nonlinear contraction mappings*, Nonlinear Anal. (2009), 2716-2723.

11. M. Cosentino and P. Vetro: *Fixed point results for F-contractive mappings of Hardy-Rogers-type*, Filomat **28** (4) (2014), 715-722.

12. G. Durmaz: *Some theorems for a new type of multivalued contractive maps on metric space*, Turkish J. Math., **41** (2017), 1092-1100.

13. N.Hussain, P. Salimi and A. Latif: *Fixed point results for single and set-valued $\alpha-\eta-\psi$-contractive mappings*, Fixed Point Theory Appl. 2013, Article ID 212 (2013).

14. I. Iqbal and N.Hussain: *Fixed point results for generalized multivalued nonlinear F-contractions*, J. Nonlinear Sci. Appl. **9** (2016), 5870-5893.

15. H. Isik and C. Ionescu: *New type of multivalued contractions with related results and applications* U.P.B. Sci. Bull., Series A, **80**(2) (2018), 13-22.

16. H. Kaddouri, H. Isik and S. Beloul: *On new extensions of F-contraction with an application to integral inclusions*, U.P.B. Sci. Bull., Series A, **81** (3) (2019), 31-42.

17. J. Jachymski: *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359-1373.

18. M. Jleli and B. Samet: *A new generalization of the Banach contraction principle*, J. Inequal. Appl. **38** (2014), 8 pp.

19. M. A. Kutbi, W. Sintunavarat: *On new fixed point results for $(\alpha, \psi, \xi)$-contractive multi-valued mappings on $\alpha$-complete metric spaces and their consequences*, Fixed Point Theory Appl., 2015 (2015), 15 pages.

20. B. Mohammadi, S. Rezapour and N. Shahzad: *Some results on fixed points of $\alpha-\psi$ -Ćirić generalized multifunctions*, Fixed Point Theory Appl., Art. No. 24 (2013).

21. S.B. Nadler: *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475-488.

22. J.J. Nieto and R. Rodríguez-López: *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, **22** (2005), 223-239 .

23. B. Samet, C. Vetro and P. Vetro: *Fixed point theorems for $\alpha-\psi$ -contractive type mappings*, Nonlinear Analysis, **75** (4) (2012), 2154-2165.

24. C. Shiau, K.K. Tan and C.S. Wong: *Quasi-nonexpansive multi-valued maps and selections*, Fund. Math. **87** (1975), 109-119.

25. M. Sgroi and C. Vetro: *Multi-valued F-contractions and the solution of certain mappings and integral equations*, Filomat, **27** (7), (2013), 1259-1268.

26. A. Tomar, S. Beloul, R. Sharma and Sh. Upadhyay: *Common fixed point theorems via generalized condition (B) in quasi-partial metric space and applications*, Demonst. maths journal **50** (2017), 278-298.

27. M. Usman Ali and T. Kamran: *Multivalued F-Contractions and Related Fixed Point Theorems with an Application*, Filomat **30** (14) (2016), 3779-3793.

28. F. Vetro: *A generalization of Nadler fixed point theorem*, Carpathian J. Math. **31** (3) (2015), 403-410.

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**Original Scientific Paper**

# KUELBS-STEADMAN SPACES ON SEPARABLE BANACH SPACES

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**Abstract.** The purpose of this paper is to construct a new class of separable Banach spaces $\mathbb{K}^p[\mathbb{B}]$, $1 \leq p \leq \infty$ on separable Banach space $\mathbb{B}$. Each of these spaces contain the $\mathcal{L}^p[\mathbb{B}]$ spaces. These spaces are of interest because they also contain the Henstock-Kurzweil integrable functions on $\mathbb{B}$.
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## 1. Introduction and Preliminaries

T.L. Gill and T. Myers [5] introduced a new theory of Lebesgue measure on $\mathbb{R}^\infty$; the construction of which is virtually the same as the development of Lebesgue measure on $\mathbb{R}^n$. This theory can be useful in formulating a new class of spaces which will provide a Banach Space structure for Henstock-Kurzweil (HK) integrable functions. This later integral is interesting because it generalizes the Lebesgue, Bochner and Pettis integrals see for instance [6, 8, 9, 12, 16, 18]. However, fly in the ointment of HK-integrable function space is not naturally Banach space (see [1, 2, 6, 7, 9, 10, 12, 13, 15] references therein). In [20], Yeong broach a clue of wind up about the drawback, pointing about canonical construction. Gill and Zachary [3, 4], introduced a new class of Banach spaces $KS^p[\Omega], \forall\ 1 \leq p \leq \infty$ (Kuelbs-Steadman spaces) and $\Omega \subset \mathbb{R}^n$ which are canonical spaces (also see [11]). These spaces are separable and contain the corresponding $\mathcal{L}^p$ spaces as dense, continuous,

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compact embedding. They wanted to find these spaces containing Denjoy integrable function, also additive measures. They found that these spaces are perfect for distinctly vibrating functions that occur in quantum theory and non linear analysis.

Throughout the paper, we assume $J = [-\frac{1}{2}, \frac{1}{2}]$. We denote by $\mathcal{L}^1,\ \mathcal{L}^p$ the classical Lebesgue spaces. Our study focused on the main class of Banach spaces $\mathbb{K}^p[\mathbb{B}_J^\infty],\ 1 \le p \le \infty$. These spaces contain the HK-integrable functions, the $\mathcal{L}^p[\mathbb{B}_J^\infty]$ spaces, $1 \le p \le \infty$ as continuous dense and compact embedding.

**Definition 1.1.** [15] A function $f : [a, b] \to \mathbb{R}$ is Henstock integrable (HK integrable) if there exists a function $F : [a, b] \to \mathbb{R}$ and for every $\epsilon > 0$ there is a function $\delta(t) > 0$ such that for any $\delta-$fine partition $\mathsf{D} = \{[u, v], t\}$ of $I_0 = [a, b]$, we have

$$\left\| \sum [f(t)(v-u) - F(u,v)] \right\| < \epsilon,$$

where the sum $\sum$ is run over $\mathsf{D} = \{([u, v], t)\}$ and $F(u, v) = F(v) - F(u)$. We write $H \int_{I_0} f = F(I_0)$.

**Definition 1.2.** [5] If $\mathbb{A}_n = \mathbb{A} \times J_n$ and $\mathbb{B}_n = \mathbb{B} \times J_n$ ($n^{th}$ box of order sets in $\mathbb{R}^\infty$). We consider

1. $\mathbb{A}_n \cup \mathbb{B}_n = (\mathbb{A} \cup \mathbb{B}) \times J_n$;
2. $\mathbb{A}_n \cap \mathbb{B}_n = (\mathbb{A} \cap \mathbb{B}) \times J_n$;
3. $\mathbb{B}_n^c = \mathbb{B}^c \times J_n$.

**Definition 1.3.** [4] Assume $\mathbb{R}_J^n = \mathbb{R}^n \times J_n$. If $T$ is a linear transformation on $\mathbb{R}^n$ and $\mathbb{A}_n = \mathbb{A} \times J_n$, then $T_J$ on $\mathbb{R}_J^n$ is denoted by $T_J[\mathbb{A}_n] = T[\mathbb{A}]$. We denote $B[\mathbb{R}_J^n]$ to be the Borel $\sigma-$algebra for $\mathbb{R}_J^n$, where the topology on $\mathbb{R}_J^n$ is define via the class of open sets $D_n = \{U \times J_n : U \text{ is open in } \mathbb{R}_J^n\}$. For any $\mathbb{A} \in B[\mathbb{R}^n]$, we define $\mu_{\mathbb{B}}(\mathbb{A}_n)$ on $\mathbb{R}_J^n$ by product measure $\mu_\infty(\mathbb{A}_n) = \mu\mathbb{A}_n(\mathbb{A}) \times \Pi_{i=n+1}^\infty \mu_J(J) = \mu\mathbb{A}_n(\mathbb{A})$.

Clearly $\mu_{\mathbb{R}}(.)$ is a measure on $B[\mathbb{R}_J^n]$, which is equivalent to $n$-dimensional Lebesgue measure on $\mathbb{R}_J^n$. The measure $\mu_{\mathbb{R}}(.)$ is both translationally and rotationally invariant on $(\mathbb{R}_J^n, B[\mathbb{R}_J^n])$ for each $n \in \mathbb{N}$. Recollecting the theory on $\mathbb{R}_J^n$ that completely paralleis that on $\mathbb{R}^n$. Since $\mathbb{R}_J^n \subset \mathbb{R}_J^{n+1}$, we have an increasing sequence, so we define $\widehat{\mathbb{R}}_J^\infty = \lim_{n\to\infty} \mathbb{R}_J^n = \bigcup_{k=1}^\infty \mathbb{R}_J^k$. Suppose $X_1 = \widehat{\mathbb{R}}_J^\infty$ and $\tau_1$ is the topology induced by the class of open sets $D \subset X_1$ such that $D = \bigcup_{n=1}^\infty D_n = \bigcup_{n=1}^\infty \{U \times J_n : U \text{ is open in } \mathbb{R}^n\}$. Suppose $X_2 = \mathbb{R}^\infty \setminus \widehat{\mathbb{R}}_J^\infty$ and $\tau_2$ is the discrete topology on $X_2$ induced by the discrete metric so that, for $x, y \in X_2,\ x \ne y,\ d_2(x, y) = 1$ and for $x = y\ d_2(x, y) = 0$

**Definition 1.4.** [4] Let $(\mathbb{R}_J^\infty, \tau)$ be the co-product $(X_1, \tau_1) \otimes (X_2, \tau_2)$ of $(X_1, \tau_1)$ and $(X_2, \tau_2)$. Every open set in $(\mathbb{R}_J^\infty, \tau)$ is the disjoint union of two open sets $G_1 \cup G_2$ with $G_1$ in $(X_1, \tau_1)$ and $G_2$ in $(X_2, \tau_2)$.

As a result $\mathbb{R}^\infty_J = \mathbb{R}^\infty$ as sets. However, since every point in $X_2$ is open and closed in $\mathbb{B}^\infty_J$ and no point is open and closed in $\mathbb{R}^\infty$. So, $\mathbb{R}^\infty_J \neq \mathbb{R}^\infty$ as topological spaces. It was shown in [5] that it can be extended the measure $\mu_{\mathbb{R}}(.)$ to $\mathbb{R}^\infty$.

Similarly, if $B[\mathbb{R}^n_J]$ is the Borel $\sigma$-algebra for $\mathbb{R}^n_J$, then $B[\mathbb{R}^n_J] \subset B[\mathbb{R}^{n+1}_J]$ defined by

$$\widehat{B}[\mathbb{R}^\infty_J] = \lim_{n\to\infty} B[\mathbb{R}^n_J] = \bigcup_{k=1}^{\infty} B[\mathbb{R}^k_J].$$

Suppose $B[\mathbb{R}^\infty_J]$ is the smallest $\sigma$-algebra restraining $\widehat{R}[\mathbb{R}^\infty_J] \cup P(\mathbb{R}^\infty \setminus \bigcup_{k=1}^{\infty} [\mathbb{R}^k_J])$, where $P(.)$ is the power set. It is obvious that the class $B[\mathbb{R}^\infty_J]$ coincides with the Borel $\sigma-$algebra generated by the $\tau-$topology on $\mathbb{R}^\infty_J$.

### 1.1. Measurable functions

We consider measurable function on $\mathbb{R}^\infty_J$ as follows. Suppose $x = (x_1, x_2, \ldots) \in \mathbb{B}^\infty_j$, $J_n = \Pi^\infty_{k=n+1}[-\frac{1}{2}, \frac{1}{2}]$ and $h_n(\widehat{x}) = \chi_{J_n}(\widehat{x})$, where $\widehat{x} = (x_i)^\infty_{i=n+1}$.

**Definition 1.5.** [4] Suppose $M^n$ represents the class of measurable functions on $\mathbb{R}^n$. On condition that $x \in \mathbb{R}^\infty_j$ and $f^n \in M^n$, suppose $\overline{x} = (x_i)^n_{i=1}$ and define an essentially docile measurable function of order $n$( or $e_n-$ docile ) on $\mathbb{B}^\infty_j$ by

$$f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}).$$

We suppose $M^n_J = \{f(x)\ : f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}),\ x \in \mathbb{R}^\infty_J\}$ is the class of all $e_n-$ docile function.

**Definition 1.6.** A function $f : \mathbb{R}^\infty_J \to \mathbb{R}$ is said to be measurable, written $f \in M_J$, if there is a sequence $\{f_n \in M^n_J\}$ of $e_n-$ docile functions, such that

$$\lim_{n\to\infty} f_n(x) \to f(x)\ \mu_\infty - (a.e.).$$

This definition highlights our requirement that all functions on infinite dimensional space must be constructively defined as (essentially) finite dimensional limits. The existence of functions satisfying above definition is not obvious. So, we have the following theorem.

**Theorem 1.1.** (Existence) *Suppose that $f : \mathbb{R}^\infty_J \to (-\infty, \infty)$ and $f^{-1}(a) \in B[\mathbb{R}^\infty_j]$ for all $a \in B[\mathbb{R}]$ then there exists a family of functions $\{f_n\},\ f_n \in M^n_J$ such that $f_n(x) \to f(x), \mu_\infty - (a.e.)$*

**Remark 1.1.** Recalling that any set $A$, of non zero measure is concentrated in $X_1$ that is $\mu_\infty(A) = \mu_\infty(A \cap X_1)$ also follows that the essential support of the limit function $f(x)$ in Definition 1.6, i.e., $\{x :\ f(x) \neq 0\}$ is concentrated in $\mathbb{R}^n_J$ for some $N$.

### 1.2. Integration theory on $\mathbb{R}_J^\infty$

We deal with integration on $\mathbb{R}_J^\infty$ by using the known properties of integration on $\mathbb{R}_J^n$. This approach has the advantages that all the theorems for Lebesgue measure apply. Let $\mathcal{L}^1[\mathbb{R}_J^n]$ be the class of integrable functions on $\mathbb{R}_J^n$. Since $\mathcal{L}^1[\mathbb{R}_J^n] \subset \mathcal{L}^1[\mathbb{R}_J^{n+1}]$, we define $\mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty] = \bigcup_{n=1}^{\infty} \mathcal{L}^1[\mathbb{R}_J^n]$.

**Definition 1.7.** A measurable function $f$ is said to be in $\mathcal{L}^1[\mathbb{R}_J^\infty]$ if there is a Cauchy-sequence $\{f_n\} \subset \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ with $f_n \in \mathcal{L}^1[\mathbb{R}_J^n]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x), \ \mu_\infty - (a.e.).$$

**Theorem 1.2.** $\mathcal{L}^1[\mathbb{R}_J^\infty] = \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$.

*Proof.* We know that $\mathcal{L}^1[\mathbb{R}_J^n] \subset \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ for all $n$. It needs only to prove that $\mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ is closed. Suppose $f$ is the limit point of $\mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ ($f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$). On condition that $f = 0$ then the result is proved. So we consider $f \neq 0$. On condition that $a_f$ is the support of $f$, then $\mu_{\mathbb{R}}(A_f) = \mu_\infty(A_f \cap X_1)$. Thus $A_f \cup X_1 \subset \mathbb{R}_J^n$ for some $N$. This means that there is a function $g \in \mathcal{L}^1[\mathbb{R}_J^{N+1}]$ with $\mu_\infty(\{x : \ f(x) \neq g(x)\}) = 0$. So, $f(x) = g(x)-$a.e. as $\mathcal{L}^1[\mathbb{R}_J^n]$ is a set of equivalence classes. So, $\mathcal{L}^1[\mathbb{R}_J^\infty] = \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$. $\square$

**Definition 1.8.** On condition that $f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$, we define the integral of $f$ by

$$\int_{\mathbb{R}_J^\infty} f(x) d\mu_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_J^\infty} f_n(x) d\mu_\infty(x),$$

where $\{f_n\} \subset \mathcal{L}^1[\mathbb{R}_J^\infty]$ is any Cauchy sequence converging to $f(x)-$a.e.

**Theorem 1.3.** *On condition that $f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$ then the above integral exists and all theorems that are true for $f \in \mathcal{L}^1[\mathbb{R}_J^n]$, also hold for $f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$.*

## 2. Class of $\mathbb{B}_j^\infty$, where $\mathbb{B}$ is a separable Banach space

As an application of $\mathbb{R}_J^\infty$, we can construct $\mathbb{B}_J^\infty$, where $\mathbb{B}$ is separable Banach space. The important fact is we can construct the measure $\mu_{\mathbb{B}}$ on separable Banach space $\mathbb{B}$ in similar fashion of $\mu_\infty$ of $\mathbb{R}^\infty$. Recalling a sequence $(e_n) \in \mathbb{B}$ is called a Schauder basis (S-basis) for $\mathbb{B}$, On condition that $||e_n||_{\mathbb{B}} = 1$ and for each $x \in \mathbb{B}$, there is a unique sequence $(x_n)$ of scalars such that

$$x = \lim_{k\to\infty} \sum_{n=1}^{k} x_n e_n = \sum_{n=1}^{\infty} x_n e_n.$$

Any sequence $(x_n)$ of scalars associated with a $x \in \mathbb{B}$, $\lim_{n\to\infty} x_n = 0$. Suppose

$$j_k = \left[\frac{-1}{2\ln(k+1)}, \frac{1}{2\ln(k+1)}\right]$$

and

$$j^n = \Pi_{k=n+1}^{\infty} j_k, \; j = \Pi_{k=1}^{\infty} J_k.$$

Suppose $\{e_k\}$ is an S-basis for $\mathbb{B}$, and suppose $x = \sum_{n=1}^{\infty} x_n e_n$, from $\mathcal{P}_n(x) = \sum_{k=1}^{n} x_k e_k$ and $\mathcal{Q}_n x = (x_1, x_2, \dots, x_n)$, we define $\mathbb{B}_j^n$ as follows

$$\mathbb{B}_j^n = \{\mathcal{Q}_n x : x \in \mathbb{B}\} \times j^n$$

with norm

$$||(x_k)||_{\mathbb{B}_j^n} = \max_{1\le k\le n} \left\| \sum_{i=1}^{k} x_i e_i \right\|_{\mathbb{B}} = \max_{1\le k\le n} ||\mathcal{P}(x)||_{\mathbb{B}}.$$

As $\mathbb{B}_j^n \subset \mathbb{B}_j^{n+1}$ so we can set $\mathbb{B}_j^{\infty} = \bigcup_{n=1}^{\infty} \mathbb{B}_j^n$ and $\mathbb{B}_j$ is a subset of $\mathbb{B}_j^{\infty}$. We set $\mathbb{B}_j$ as $\mathbb{B}_j = \left\{(x_1, x_2, \dots) : \sum_{k=1}^{\infty} x_k e_k \in \mathbb{B}\right\}$ and norm on $\mathbb{B}_j$ by

$$||x||_{\mathbb{B}_j} = \sup_n \|\mathcal{P}_n(x)\|_{\mathbb{B}} = ||x||_{\mathbb{B}}.$$

On condition that we consider $\mathbb{B}[\mathbb{B}_j^{\infty}]$ as the smallest $\sigma$-algebra restraining $\mathbb{B}_j^{\infty}$ and define $\mathbb{B}[\mathbb{B}_j] = \mathbb{B}[\mathbb{B}_j^{\infty}] \cap \mathbb{B}_J$ then by a known result

$$||x||_{\mathbb{B}} = \sup_n \left\| \sum_{k=1}^{n} x_k e_k \right\|_{\mathbb{B}} \tag{2.1}$$

is an equivalent norm on $\mathbb{B}$.

**Proposition 2.1.** *[4] When $\mathbb{B}$ carries the equivalent norm (2.1), the operator*

$$T : (\mathbb{B}, ||.||_{\mathbb{B}}) \to (\mathbb{B}_j, ||.||_{\mathbb{B}_j})$$

*denoted by $T(x) = (x_k)$ is an isometric isomorphism from $\mathbb{B}$ onto $\mathbb{B}_j$.*

This shows that every Banach space with an S-basis has a natural embedding in $\mathbb{B}_j^{\infty}$. So, we call $\mathbb{B}_j$ the canonical representation of $\mathbb{B}$ in $\mathbb{B}_j^{\infty}$. With $\mathbb{B}[\mathbb{B}_j] = \mathbb{B}_j \cap \mathbb{B}[\mathbb{B}_j^{\infty}]$ we define $\sigma-$algebra generated by $\mathbb{B}$ and associated with $\mathbb{B}[\mathbb{B}_j]$ by

$$\mathbb{B}_j[\mathbb{B}] = \{T^{-1}(A) \; |A \in \mathbb{B}[\mathbb{B}_j]\} = T^{-1}\{\mathbb{B}[\mathbb{B}_j]\}.$$

Since $\mu_{\mathbb{B}}(A_j^n) = 0$ for $A_j^n \in \mathbb{B}[\mathbb{B}_j^n]$ with $A_j^n$ compact, we see $\mu_{\mathbb{B}}(\mathbb{B}_j^n) = 0, \; n \in \mathbb{N}$. So, $\mu_{\mathbb{B}}(\mathbb{B}_j) = 0$ for every Banach space with an S-basis. Thus the restriction of $\mu_{\mathbb{B}}$ to $\mathbb{B}_j$ will not induce a non trivial measure on $\mathbb{B}$.

**Definition 2.1.** [4, 19] We define $\overline{v}_k, \overline{\mu}_k$ on $A \in B[\mathbb{R}]$ by

$$\overline{v}_k(A) = \frac{\mu(A)}{\mu(j_k)}, \overline{\mu}_k(A) = \frac{\mu(A \cap j_k)}{\mu(j_k)}$$

and for an elementary set $A = \pi_{k=1}^{\infty} A_k \in B[\mathbb{B}_j^n]$, define $\overline{V}_j^n$ by

$$\overline{V}_j^n = \pi_{k=1}^{n} \overline{v}_k(A) \times \pi_{k=n+1}^{\infty} \overline{\mu}_k(A).$$

Let $V_j^n$ denote the Lebesgue extension of $\overline{V}_j^n$ to all $\mathbb{B}_j^n$ and $V_j(A) = \lim_{n\to\infty} V_j^n(A)$, $\forall A \in B[\mathbb{B}_j]$. We adopt a variation of method developed by Yamasaki [19], to define $V_j^n$ to the Lebesgue extension of $\overline{V}_j^n$ for all $\mathbb{B}_j^n$ and define $V_j(\mathbb{B}) = \lim_{n\to\infty} V_j^n(\mathbb{B}), \forall \mathbb{B} \in \mathbb{B}[\mathbb{B}_j]$.

**Remark 2.1.** Let $\mathbb{B}_j$ be the image of $\mathbb{B}$ in $\mathbb{B}_j^{\infty}$, which can be endued with a norm via $||u_1, u_2, \ldots, u_n||_{\mathbb{B}_j} = ||u||_{\mathbb{B}}$.

### 2.1. Integration theory on $\mathbb{B}_j^{\infty}$

In this section, we study the integration on a separable Banach space $\mathbb{B}_j^{\infty}$ with an $S-$basis. Recalling $\mu_{\mathbb{B}}$ restricted to $B[\mathbb{B}_j^n]$ is equivalent to $\mu\mathbb{A}_n$. Assume that the integral restricted to $B[\mathbb{B}_j^n]$ is the integral on $\mathbb{R}^n$. Suppose $f : \mathbb{B} \to [0, \infty]$ is a measurable function and suppose $\mu_{\mathbb{B}}$ is constructed using the family $\{j_k\}$. If $\{j_n\} \subset M$ is an increasing family of non negative simple functions with $j_n \in M_j^n$, for each $n$ and $\lim_{n\to\infty} j_n(x) = f(x)$, $\mu_{\mathbb{B}}-$a.e. We consider the integral of $f$ over $\mathbb{B}_j^{\infty}$ by

$$\int_{\mathbb{B}_j^{\infty}} f(x) d\mu_{\mathbb{B}} = \lim_{n\to\infty} \int_{\mathbb{B}_j^{\infty}} \left[ j_n(x) \prod_{i=1}^{n} \mu(j_i) \right] d\mu_{\mathbb{B}}(x).$$

Suppose $\mathcal{L}^1[\mathbb{B}_j^n]$ is the class of integrable functions on $\mathbb{B}_j^n$. Since $\mathcal{L}^1[\mathbb{B}_j^n] \subset \mathcal{L}^1[\mathbb{B}_j^{n+1}]$, we define $\mathcal{L}^1[\widehat{\mathbb{B}}_j^n] = \bigcup_{n=1}^{\infty} \mathcal{L}^1[\mathbb{B}_j^n]$.

1. We say that a measurable function $f \in \mathcal{L}^1[\mathbb{B}_j^{\infty}]$ if there exists a Cauchy sequence $\{f_m\} \subset \mathcal{L}^1[\widehat{\mathbb{B}}_j^{\infty}]$, such that

$$\lim_{m\to\infty} \int_{\mathbb{B}_j^{\infty}} |f_m(x) - f(x)| d\mu_{\mathbb{B}}(x) = 0.$$

That is a measurable function $f \in \mathcal{L}^1[\mathbb{B}_j^{\infty}]$ if there exists a Cauchy sequence $\{f_m\} \subset \mathcal{L}^1[\widehat{\mathbb{B}}_j^{\infty}]$, with $f_m \in \mathcal{L}^1[\mathbb{B}_j^n]$ and

$$\lim_{m\to\infty} f_m(x) = f(x), \ \mu_{\mathbb{B}} - (a.e.).$$

2. We say that a measurable function $f \in C_0[\mathbb{B}_j^\infty]$, the space of continuous functions that vanish at infinity, if there exists a Cauchy sequence $\{f_m\} \subset C_0[\widehat{\mathbb{B}}_j^\infty]$, such that

$$\lim_{m\to\infty} \int_{\mathbb{B}_j^\infty} \sup_{x\in\mathbb{B}_j^\infty} |f_m(x) - f(x)| d\mu_{\mathbb{B}}(x) = 0.$$

**Theorem 2.1.** $\mathcal{L}^1[\widehat{\mathbb{B}}_j^\infty] = \mathcal{L}^1[\mathbb{B}_j^\infty]$.

**Definition 2.2.** If $f \in \mathcal{L}^1[\mathbb{B}_j^\infty]$, we define the integral of $f$ by

$$\lim_{m\to\infty} \int_{\mathbb{B}_j^\infty} f_m(x) d\mu_{\mathbb{B}}(x) = \int_{\mathbb{B}_j^\infty} f(x) d\mu_{\mathbb{B}}(x), \mu_{\mathbb{B}} - (a.e.),$$

where $\{f_m\} \subset \mathcal{L}^1[\mathbb{B}_j^\infty]$ is any Cauchy sequence converging to $f(x)-$a.e.

**Theorem 2.2.** *If $f \in L^1[\mathbb{B}_j^\infty]$, then the above integral exists and all theorems that are true for $f \in \mathcal{L}^1[\mathbb{B}_j^n]$, also hold for $f \in \mathcal{L}^1[\mathbb{B}_j^\infty]$.*

**Lemma 2.1.** (Kuelbs Lemma) *[11] Let $\mathbb{B}$ be a separable Banach space. Then there exists a separable Hilbert space such that $\mathbb{B} \hookrightarrow H$ is a continuous dense embedding.*

## 3. The Kuelbs-Steadman space $\mathbb{K}^p[\mathbb{B}]$

In this section, we study the Kuelbs-Steadman space $\mathbb{K}^p[\mathbb{B}]$, where $\mathbb{B}$ is a separable Banach space. We proceed for the construction of the canonical space $\mathbb{K}^p[\mathbb{B}_J^\infty]$. Suppose $\mathbb{B}_j^n$ is a separable Banach space with $S-$basis, $\mathbb{K}^p[\widehat{\mathbb{B}}_j^n] = \bigcup_{k=1}^{\infty} \mathbb{K}^p[\mathbb{B}_j^k]$, and $C_0[\widehat{\mathbb{B}}_j^n] = \bigcup_{n=1}^{\infty} C_0[\mathbb{B}_j^n]$.

**Definition 3.1.** A measurable function $f$ is said to be in $\mathbb{K}^p[\mathbb{B}_j^n]$ if there exists a Cauchy sequence $\{f_m\} \subset \mathbb{K}^p[\widehat{\mathbb{B}}_j^n]$, with $f_m \in \mathbb{K}^p[\widehat{\mathbb{B}}_j^n]$ such that

$$\lim_{m\to\infty} f_m(x) = f(x), \ \mu_{\mathbb{B}} - (a.e.)$$

**Theorem 3.1.** $\mathbb{K}^p[\widehat{\mathbb{B}}_j^n] = \mathbb{K}^p[\mathbb{B}_j^n]$.

**Definition 3.2.** Let $f \in \mathbb{K}^p[\mathbb{B}_j^n]$. The integral of $f$ is defined by

$$\lim_{m\to\infty} \int_{\mathbb{B}_j^n} f_m(x) d\mu_{\mathbb{B}}(x) = \int_{\mathbb{B}} f(x) d\mu_{\mathbb{B}}(x), \ \mu_{\mathbb{B}} - (a.e.),$$

where $\{f_m\} \subset \mathbb{K}^p[\mathbb{B}_j^n]$ is any Cauchy sequence converging to $f(x)-$a.e.

**Theorem 3.2.** *If $f \in \mathbb{K}^p[\mathbb{B}_j^n]$, then the above integral exists.*

### 3.1. The construction of $\mathbb{K}^p[\mathbb{B}_j^n]$

We start with $\mathcal{L}^1[\mathbb{B}_j^n]$, picking a countable dense set of sequences $\{\mathcal{E}_n(x)\}_{n=1}^{\infty}$ on the unit ball of $\mathcal{L}^1[\mathbb{B}_j^n]$ and assume $\{\mathcal{E}_n^*\}_{n=1}^{\infty}$ is any corresponding set of duality mapping in $\mathcal{L}^{\infty}[\mathbb{B}]$, also on condition that $\mathbb{B}$ is $\mathcal{L}^1[\mathbb{B}_j^n]$, using Kuelbs Lemma, it is clear that the Hilbert space $\mathbb{K}^2[\mathbb{B}_j^n]$ will contain some non absolute integrable functions. From [17], we confirm that the non absolute integral is Henstock-Kurzweil integral (HK). Let $\overline{\mathcal{E}}_k(x)$ be the characteristic function of $\mathbb{B}_k$, so that $\overline{\mathcal{E}}_k(x) \in \mathcal{L}^p[\mathcal{B}_j^n] \cap \mathcal{L}^{\infty}[\mathbb{B}_j^n]$ for $1 \le p < \infty$. Define $F_k(f)$ on $\mathcal{L}^1[\mathbb{B}_j^n]$ by

$$F_k(f) = \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x).$$

Since each $\mathbb{B}_k$ is a cube with sides parallel to the co-ordinate axes, $F_k(.)$ is well defined for all HK-integrable functions, and is a bounded linear functional on $\mathcal{L}^p[\mathbb{B}_j^n]$ for $1 \le p \le \infty$. Let $\mathbf{b}_k > 0$ be such that $\sum_{k=1}^{\infty} \mathbf{b}_k = 1$ and denote the inner product $(.)$ on $\mathcal{L}^1[\mathbb{B}_j^n]$ by

$$(f,g) = \sum_{k=1}^{\infty} \mathbf{b}_k \left[\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x)\right] \left[\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(y) g(y) d\mu_{\mathbb{B}}(y)\right]^c .$$

The completion of $\mathcal{L}^1[\mathbb{B}_j^n]$ in the inner product is the space $\mathbb{K}^2[\mathbb{B}_j^n]$. We can see directly that $\mathbb{K}^2[\mathbb{B}_j^n]$ contains the HK-integrable functions. We call the completion of $\mathcal{L}^1[\mathbb{B}_j^n]$ with the above inner product, the Kuelbs-Steadman space $\mathbb{K}^2[\mathbb{B}_j^n]$.

**Theorem 3.3.** *The space $\mathbb{K}^2[\mathbb{B}_j^n]$ contains $\mathcal{L}^p[\mathbb{B}_j^n]$ (for each $p$, $1 \le p < \infty$) as a dense subspace.*

*Proof.* We know $\mathbb{K}^2[\mathbb{B}_j^n]$ contains $\mathcal{L}^1[\mathbb{B}_j^n]$ densely. Thus we need only to show $L^q[\mathbb{B}_j^n] \subset \mathbb{K}^2[\mathbb{B}_j^n]$ for $q \neq 1$. Suppose $f \in L^q[\mathbb{B}_j^n]$ and $q < \infty$. Since $|\mathcal{E}(x)| = \mathcal{E}(x) \le 1$ and $|\mathcal{E}(x)|^q \le \mathcal{E}(x)$, we have

$$\begin{aligned} ||f||_{\mathbb{K}^2} &= \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left| \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x) \right|^{\frac{2q}{q}}\right]^{\frac{1}{2}} \\ &\le \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left( \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x) \right)^{\frac{2}{q}}\right]^{\frac{1}{2}} \\ &\le \sup_k \left( \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x) \right)^{\frac{1}{q}} \le ||f||_q . \end{aligned}$$

Therefore $f \in \mathbb{K}^2[\mathbb{B}_j^n]$. $\square$

We can construct the norm of $\mathbb{K}^p[\mathbb{B}^n_j]$, which is defined by

$$||f||_{\mathbb{K}^p[\mathbb{B}^n_j]} = \begin{cases} \left(\sum\limits_{k=1}^{\infty} \mathbf{b}_k \left|\int_{\mathbb{B}^n_j} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathcal{B}}(x)\right|^p\right)^{\frac{1}{p}}, \text{ for } 1 \le p < \infty; \\ \sup\limits_{k \ge 1} \left|\int_{\mathbb{B}^n_j} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathcal{B}}(x)\right|, \text{ for } p = \infty \end{cases}$$

It is easy to see that $||.||_{\mathbb{K}^p[\mathbb{B}^n_j]}$ is a norm on $\mathcal{L}^p[\mathbb{B}^n_j]$. If $\mathbb{K}^p[\mathbb{B}]$ is the completion of $\mathcal{L}^p[\mathbb{B}]$ with respect to this norm, we have the following theorem.

**Theorem 3.4.** *For each $q$, $1 \le q < \infty$, $L^q[\mathbb{B}^n_j] \hookrightarrow \mathbb{K}^p[\mathbb{B}^n_j]$ is a densely continuous embedding.*

*Proof.* We know from Theorem 3.3, and by the construction of $\mathbb{K}^p[\mathbb{B}^n_j]$ contains $\mathcal{L}^p[\mathbb{B}^n_j]$ densely. Thus we need only to show $L^q[\mathbb{B}^n_j] \subset \mathbb{K}^p[\mathbb{B}^n_j]$ for $q \neq p$. Suppose $f \in L^q[\mathbb{B}^n_j]$ and $q < \infty$. Since $|\mathcal{E}(x)| = \mathcal{E}(x) \le 1$ and $|\mathcal{E}(x)|^q \le \mathcal{E}(x)$, we have

$$\begin{aligned} ||f||_{\mathbb{K}^p} &= \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left|\int_{\mathbb{B}^n_j} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x)\right|^{\frac{qp}{q}}\right]^{\frac{1}{p}} \\ &\le \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left(\int_{\mathbb{B}^n_j} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x)\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} \\ &\le \sup_k \left(\int_{\mathbb{B}} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x)\right)^{\frac{1}{q}} \le ||f||_q. \end{aligned}$$

Therefore $f \in \mathbb{K}^p[\mathbb{B}^n_j]$. □

**Corollary 3.1.** $\mathcal{L}^\infty[\mathbb{B}^n_j] \subset \mathbb{K}^p[\mathbb{B}^n_j]$.

**Theorem 3.5.** $\mathbb{C}_c[\mathbb{B}^n_j]$ *is dense in* $\mathbb{K}^2[\mathbb{B}^n_j]$.

*Proof.* As $\mathbb{C}_c[\mathbb{B}^n_j]$ is dense in $\mathcal{L}^p[\mathbb{B}^n_j]$ and $\mathcal{L}^p[\mathbb{B}^n_j]$ is densely contained in $\mathbb{K}^2[\mathbb{B}^n_j]$, the conclusion follows. □

**Remark 3.1.** As Hölder and generalized Hölder inequalities for $\mathcal{L}^p[\mathbb{B}^n_j]$ are valid for $1 \le p < \infty$ (see [4, P. 83]). As $\mathbb{K}^p[\mathbb{B}^n_j]$ is completion of $\mathcal{L}^p[\mathbb{B}^n_j]$, the Hölder and generalized Hölder inequalities hold in $\mathbb{K}^p[\mathbb{B}^n_j]$ for $1 \le p < \infty$.

**Theorem 3.6.** (The Minkowski Inequality) *Suppose $1 \le p < \infty$ and $f, g \in \mathbb{K}^p[\mathbb{B}^n_j]$. Then $f + g \in \mathbb{K}^p[\mathbb{B}^n_j]$ and*

$$||f+g||_{\mathbb{K}^p[\mathbb{B}^n_j]} \le ||f||_{\mathbb{K}^p[\mathbb{B}^n_j]} + ||g||_{\mathbb{K}^p[\mathbb{B}^n_j]}.$$

*Proof.* The proof follows from the Lemma 2 of [14]. $\square$

**Theorem 3.7.** *For $1 \leq p \leq \infty$, we have*

1. *If $f_n \to f$ weakly in $\mathcal{L}^p[\mathbb{B}_j^n]$, then $f_n \to f$ strongly in $\mathbb{K}^p[\mathbb{B}_j^n]$.*
2. *If $1 < p < \infty$, then $\mathbb{K}^p[\mathbb{B}_j^n]$ is uniformly convex.*
3. *If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual space of $\mathbb{K}^p[\mathbb{B}_j^n]$ is $\mathbb{K}^q[\mathbb{B}_j^n]$.*
4. *$\mathbb{K}^\infty[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$, for $1 \leq p < \infty$.*

*Proof.* (1) If $\{f_n\}$ is weakly convergence sequence in $\mathcal{L}^p[\mathbb{B}_j^n]$ with limit $f$. Then $\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)[f_n(x) - f(x)]d\mu_{\mathbb{B}}(x) \to 0$ for each $k$.
Now for $\{f_n\} \in \mathbb{K}^p[\mathbb{B}_j^n]$ we find the following:

$$\lim_{n\to\infty} \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)[f_n(x) - f(x)]d\mu_{\mathbb{B}}(x) \to 0.$$

So, $\{f_n\}$ is converges strongly in $\mathbb{K}^p[\mathbb{B}_j^n]$.
(2) We know $\mathcal{L}^p[\mathbb{B}_j^n]$ is uniformly convex and that is dense and compactly embedded in $\mathbb{K}^q[\mathbb{B}_j^n]$ for all $q,\ 1 \leq q \leq \infty$. So, $\bigcup\limits_{n=1}^{\infty} \mathcal{L}^p[\mathbb{B}_j^n]$ is uniformly convex for each $n$ and that is dense and compactly embedded in $\bigcup\limits_{n=1}^{\infty} \mathbb{K}^q[\mathbb{B}_j^n]$ for all $q,\ 1 \leq q \leq \infty$. However $\mathcal{L}^p[\widehat{\mathbb{B}_j^n}] = \bigcup\limits_{n=1}^{\infty} \mathcal{L}^p[\mathbb{B}_j^n]$. That is $\mathcal{L}^p[\widehat{\mathbb{B}_j^n}]$ is uniformly convex, dense and compactly embedded in $\mathbb{K}^q[\widehat{\mathbb{B}_j^n}]$ for all $q,\ 1 \leq q \leq \infty$ as $\mathbb{K}^q[\mathbb{B}_j^n]$ is the closure of $\mathbb{K}^q[\widehat{\mathbb{B}_j^n}]$. Therefore $\mathbb{K}^q[\mathbb{B}_j^n]$ is uniformly convex.
(3) From (2), that $\mathbb{K}^p[\mathbb{B}_j^n]$ is reflexive for $1 < p < \infty$ as

$$\{\mathbb{K}^p[\mathbb{B}_j^n]\}^* = \mathbb{K}^q[\mathbb{B}_j^n],\ \frac{1}{p} + \frac{1}{q} = 1,\ \forall n$$

and

$$\mathbb{K}^p[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_J^{n+1}],\ \forall n\ \Rightarrow \bigcup_{n=1}^{\infty}\{\mathbb{K}^p[\mathbb{B}_j^n]\}^* = \bigcup_{n=1}^{\infty} \mathbb{K}^q[\mathbb{B}_j^n],\ \frac{1}{p} + \frac{1}{q} = 1.$$

Since each $f \in \mathbb{K}^p[\mathbb{B}_j^n]$ is the limit of a sequence $\{f_n\} \subset \mathbb{K}^p[\widehat{\mathbb{B}_j^n}] = \bigcup\limits_{n=1}^{\infty} \mathbb{K}^p[\mathbb{B}_j^n]$, we see that $\{\mathbb{K}^p[\mathbb{B}_j^n]\}^* = \mathbb{K}^q[\mathbb{B}_j^n]$, for $\frac{1}{p} + \frac{1}{q} = 1$.
(4) Suppose $f \in \mathbb{K}^\infty[\mathbb{B}_j^n]$. This implies $|\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)f(x)d\mu_{\mathbb{B}}(x)|$ is uniformly bounded for all $k$. It follows that $|\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)f(x)d\mu_{\mathbb{B}}(x)|^p$ is uniformly bounded for all $1 \leq p < \infty$. It is clear from the definition of $\mathbb{K}^p[\mathbb{B}_j^n]$ that

$$\left[\sum \left|\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)f(x)d\mu_{\mathbb{B}}(x)\right|^p\right]^{\frac{1}{p}} \quad \leq \quad M||f||_{\mathbb{K}^p[\mathbb{B}_j^n]} < \infty.$$

So, $f \in \mathbb{K}^p[\mathbb{B}_j^n]$. This completes the proof. $\square$

**Theorem 3.8.** *$C_c^\infty[\mathbb{B}_j^n]$ is a dense subset of $\mathbb{BK}^2[\mathbb{B}_j^n]$.*

*Proof.* As $C_c^\infty[\mathbb{B}_j^n]$ is dense in $\mathcal{L}^p[\mathbb{B}_j^n], \forall p$. Moreover $\mathcal{L}^p[\mathbb{B}_j^n]$ is a dense subset of $\mathbb{K}^2[\mathbb{B}_j^n]$. So, $C_c^\infty[\mathbb{B}_j^n]$ is a dense subset of $\mathbb{K}^2[\mathbb{B}_j^n]$. $\square$

**Corollary 3.2.** *The embedding $C_0^\infty[\mathbb{B}_j^n] \hookrightarrow \mathbb{K}^p[\mathbb{B}_j^n]$ is dense.*

**Remark 3.2.** Since $\mathcal{L}^1[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$ and $\mathbb{K}^p[\mathbb{B}_j^n]$ is reflexive for $1 < p < \infty$. We see the second dual $\{\mathcal{L}^1[\mathbb{B}_j^n]\}^{**} = \mathfrak{M}[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$, where $\mathfrak{M}[\mathbb{B}_j^n]$ is the space of bounded finitely additive set functions define on the Borel sets $B[\mathbb{B}_j^n]$.

### 3.2. The family of $\mathbb{K}^p[\mathbb{B}_j^\infty]$

We can now construct the spaces $\mathbb{K}^p[\mathbb{B}_j^\infty]$, $1 \le p \le \infty$, using the same approach that led to $\mathcal{L}^1[\mathbb{B}_j^\infty]$. Since $\mathbb{K}^p[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^{n+1}]$. We define $\mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty] = \bigcup_{n=1}^\infty \mathbb{K}^p[\mathbb{B}_j^n]$.

**Definition 3.3.** A measurable function $f$ is said to be in $\mathbb{K}^p[\mathbb{B}_j^\infty]$, for $1 \le p \le \infty$, if there is a Cauchy sequence $\{f_n\} \subset \mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty]$ with $f_n \in \mathbb{K}^p[\mathbb{B}_j^n]$ such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ $\mu_{\mathbb{B}}$-a.e.

The functions in $\mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty]$ differ from functions in its closure $\mathbb{K}^p[\mathbb{B}_j^\infty]$, by sets of measure zero.

**Theorem 3.9.** $\mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty] = \mathbb{K}^p[\mathbb{B}_j^\infty]$.

**Definition 3.4.** If $f \in \mathbb{K}^p[\mathcal{B}_j^\infty]$, we define the integral of $f$ by

$$\int_{\mathbb{B}_j^\infty} f(x)d\mu_{\mathbb{B}}(x) = \lim_{n\to\infty} \int_{\mathbb{B}_j^n} f_n(x)d\mu_{\mathbb{B}}(x),$$

where $f_n \in \mathbb{K}^p[\mathbb{B}_j^n]$ is any Cauchy sequence convergerging to $f(x)$.

**Theorem 3.10.** *If $f \in \mathbb{K}^p[\mathbb{B}_j^\infty]$, then the integral of $f$ define in Definition 3.4 exists and is unique for every $f \in \mathbb{K}^p[\mathbb{B}_j^\infty]$.*

*Proof.* If in the consideration of the family of functions $\{f_n\}$ is Cauchy, it follows: On condition that the integral exists, it is unique. For existence considering $f(x) \geqslant 0$ with standard argument with the assumption of increasing sequence so that the integral exists. The general case now follows by the standard decomposition. $\square$

**Theorem 3.11.** *If $f \in \mathbb{K}^p[\mathbb{B}_j^\infty]$, then all theorems that are true for $f \in \mathbb{K}^p[\mathbb{B}_j^n]$, also hold for $f \in \mathbb{K}^p[\mathbb{B}_j^\infty$.]*

**Theorem 3.12.** *$\mathbb{K}^p[\mathbb{B}_j^n]$ and $\mathbb{K}^p[\mathbb{B}_j]$ are equivalent spaces.*

*Proof.* Let $\mathbb{B}_j^n$ is a separable Banach space, $T$ maps $\mathbb{B}_j^n$ onto $\mathbb{B}_j \subset \mathbb{B}_j^\infty$, where $T$ is an isometric isomorphism so that $\mathbb{B}_j$ is an embedding of $\mathbb{B}_j^n$ into $\mathbb{R}_J^\infty$. This is how we able to define a Lebesgue integral on $\mathbb{B}_j^n$ using $\mathbb{B}_j$ and $T^{-1}$. Thus $\mathbb{K}^p[\mathbb{B}_j^n]$ and $\mathbb{K}^p[\mathbb{B}_j]$ are not different spaces. $\square$

**Theorem 3.13.** *$\mathbb{K}^p[\mathbb{B}_j^\infty]$ can be embedded into $\mathbb{K}^p[\mathbb{R}_J^\infty]$ as a closed subspace.*

*Proof.* As every separable Banach space can be embedded in $\mathbb{R}_J^\infty$ as a closed subspace containing $\mathbb{B}_j^\infty$. So, $\mathbb{K}^p[\mathbb{B}_j^\infty] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as a closed subspace. That is $\mathbb{K}^p[\bigcup_{n=1}^\infty \mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as a closed subspace. So, $\mathbb{K}^p[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as a closed subspace. Finally we can conclude that $\mathbb{K}^p[\mathbb{B}_J^\infty] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as closed subspace. $\square$

### 3.3. Feynman path integral

The properties of $\mathbb{K}^2[\mathbb{B}_J^\infty]$ derived earlier suggests that it may be a better replacement of $\mathcal{L}^2[\mathbb{B}_J^\infty]$ in the study of the Path Integral formulation of quantum theory developed by Feynman. We see that position operator have closed densely define extensions to $\mathbb{K}^2[\mathbb{B}_J^\infty]$. Further Fourier and convolution insure that all of the Schrödinger and Heisenberg theories have a faithful representation on $\mathbb{K}^2[\mathbb{B}_J^\infty]$. Since $\mathbb{K}^2[\mathbb{B}_J^\infty]$ contains the space of measures, it follows that all the approximating sequences for Dirac measure convergent strongly in $\mathbb{K}^2[\mathbb{B}_J^\infty]$.

## REFERENCES


1. A. Alexiewicz, *Linear functionals on Denjoy-integrable functions*. Colloq. Math. **1**(1948) 289–293.
2. R. O. Davies, Z. Schuss, *A proof that Henstock's integral includes Lebesgue's*. Lond. Math. Soc. **2**(1970) 561–562.
3. T. L. Gill, *A new class of Banach spaces*. Journal of Physics A Mathematical and Theoretical, 2008
4. T. L. Gill, W. W. Zachary, *Functional Analysis and The Feynman Operator Calculus*. Springer International Publishing Switzerland, 2016.
5. T. L. Gill, T. Myers, *Constructive Analysis on Banach spaces*. Real Analysis Exchange **44**(1)(2019) 1–36.
6. R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron and Henstock*. Graduate Studies in Mathematics Vol 4, American Mathematical Society, 1994.
7. R. Henstock, *The General Theory of Integration*. Clarendon Press, Oxford 1991.
8. R. Henstock, *The General Theory of Integration*. Oxford mathematical monographs, Clarendo Press, Oxford 1991.
9. R. Henstock, *Lectures on the Theory of Integration*. World Scientific condition thatic, Singapore, 1988.

10. R. Henstock, *A Riemann-type integral of Lebesgue power*. Canadian J. Math. **20** (1968) 79-87.

11. J. Kuelbs, *Gaussian measures on a Banach space*. J. Funct. Anal. **5**(1970) 354–367.

12. J. Kurzweil, *Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces*. World Scientific condition thatic, Singapore, 2000.

13. J. Kurzweil, *Nichtabsolut konvergente Integrale*. Teubner-Texte zür Mathematik, Band 26, Teubner Verlagsgesellschaft, Leipzig, 1980.

14. L. Maligranda, *A simple proof of the Hölder and Minkowski Inequality*. The American Mathematical Monthly, **102**(3)(1995) 256–259.

15. C. Schwartz, *Introduction to Gauge Integral*. World Scientific Publisher, 2004.

16. S. Schwabik, Ye Guoju: *Topics in Banach Spaces Integration, Series in Real Analysis*, **10** World Scientific, Singapore (2005).

17. V. Steadman, *Theory of operators on Banach spaces*. Ph. D. thesis, Howard University, 1988.

18. B. S. Thomson, *Theory of Integral.* Classical Real Analysis.com, 2013.

19. Y. Yamasaki: *Measures on infinite dimensional spaces*. Series in Pure Mathematics, **5** World ScientOn condition thatic Publishing Co. Pvt. Ltd 1985.

20. L. Tuo-Yeong: *Henstock-Kurweil Integration on Euclidean Spaces*. Series in Real Analysis **12**, World Scientific, New Jersey, 2011.

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# ON CARTAN NULL BERTRAND CURVES IN MINKOWSKI 3-SPACE

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**Abstract.** In this paper, we consider Cartan null Bertrand curves in Minkowski 3-space. Since the principal normal vector of a null curve is a spacelike vector, the Bertrand mate curve of a null curve can be a timelike curve and a spacelike curve with spacelike principal normal. We give the necessary and sufficient conditions for these cases to be Bertrand curves and we also give the related examples.
**Keywords**: Bertrand curve, Minkowski 3-space, Cartan null curve, non-null curve.

## 1. Introduction

In the theory of curves in Euclidean space, one of the important and interesting problem is characterization of a regular curve. In the solution of the problem, the curvature functions $\kappa_1$ (or $\varkappa$) and $\kappa_2$ (or $\tau$) of a regular curve have an effective role. For example: if $\kappa_1 = 0 = \kappa_2$, then the curve is a geodesic or if $\kappa_1 = \text{constant} \neq 0$ and $\kappa_2 = 0$, then the curve is a circle with radius $(1/\kappa_1)$, etc. Another way in the solution of the problem is the relationship between the Frenet vectors and Frenet planes of the curves ([8],[13]). *Mannheim* curves is an interesting examples for such classification. If there exists a corresponding relationship between the space curves $\alpha$ and $\beta$ such that, at the corresponding points of the curves, the principal normal lines of $\alpha$ coincides with the binormal lines of $\beta$, then $\alpha$ is called a Mannheim curve, $\beta$ is called Mannheim partner curve of $\alpha$. Mannheim partner curves was studied by *Liu* and *Wang* (see [10]) in Euclidean 3-space and Minkowski 3-space.

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Another interesting example is *Bertrand* curves. A Bertrand curve is a curve in the Euclidean space such that its principal normal is the principal normal of the second curve ([3],[18]). The study of this kind of curves has been extended to many other ambient spaces. In [12], Pears studied this problem for curves in the $n$-dimensional Euclidean space $\mathbb{E}^n$, $n > 3$, and showed that a Bertrand curve in $\mathbb{E}^n$ must belong to a three-dimensional subspace $\mathbb{E}^3 \subset \mathbb{E}^n$. This result is restated by Matsuda and Yorozu [11]. They proved that there was not any special Bertrand curves in $\mathbb{E}^n$ ($n > 3$) and defined a new kind, which is called $(1, 3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1], [2], [6], [7], [14], [15]) as well as in Euclidean space. In addition, $(1, 3)$-type Bertrand curves were studied in semi-Euclidean 4-space with index 2 ([16],).

Following [17], in this paper, we consider Cartan null Bertrand curves in Minkowski 3-space. Since the principal normal vector of a null curve is a spacelike vector, the Bertrand mate curve of a null curve can be a null curve, a timelike curve and a spacelike curve with spacelike principal normal. The case where the Bertrand mate curve is a null curve, were studied in [2]. Thus, we give the necessary and sufficient conditions for other cases to be Bertrand curves and we also give the related examples.

## 2. Preliminaries

The Minkowski space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1 \backslash \{0\}$ can be *spacelike* if $g(v, v) > 0$ , *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector $v$ is given by $||v|| = \sqrt{|g(v, v)|}$, and two vectors $v$ and $w$ are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([9]). Spacelike curve in $\mathbb{E}^3_1$ is called *pseudo null curve* if its principal normal vector $N$ is null [4]. A null curve $\alpha$ is parameterized by pseudo-arc $s$ if $g(\alpha''(s), \alpha''(s)) = 1$. Also null curve is called null Cartan curve if it is parameterized by pseudo-arc function. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([4]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve $\alpha$ in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of $\alpha$, the Frenet equations have the following forms.

**Case I.** If $\alpha$ is a non-null curve, the Frenet equations are given by ([9]):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $k_1$ and $k_2$ are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T,T) = \epsilon_1 = \pm 1, g(N,N) = \epsilon_2 \ = \pm 1, g(B,B) = \epsilon_3 = \pm 1$$

and

$$g(T,N) = g(T,B) = g(N,B) = 0.$$

**Case II.** If $\alpha$ is a null Cartan curve, the Cartan equations are given by ([4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where the first curvature $k_1 = 0$ if $\alpha$ is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, g(N,N) = g(T,B) = 1.$$

## 3. Cartan Null Bertrand curves in Minkowski 3-space

In this section, we consider the Cartan null Bertrand curves in $\mathbb{E}^3_1$. We get the necessary and sufficient conditions for the Cartan null curves to be Bertrand curves in $\mathbb{E}^3_1$ and we also give the related examples.

**Definition 3.1.** A Cartan null curve $\alpha : I \to \mathbb{E}^3_1$ with $\kappa_1(s) \neq 0$ is a Bertrand curve if there is a curve $\alpha^* : I^* \to \mathbb{E}^3_1$ such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ at $s \in I$, $s^* \in I^*$ are equal. In this case, $\alpha^*(s^*)$ is the Bertrand mate of $\alpha(s)$.

Let $\beta : I \to \mathbb{E}^3_1$ be a Cartan null Bertrand curve in $\mathbb{E}^3_1$ with the Frenet frame $\{T, N, B\}$ and the curvatures $\kappa_1, \kappa_2$, and $\beta^* : I \to \mathbb{E}^3_1$ be a Bertrand mate curve of $\beta$ with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures $\kappa_1^*, \kappa_2^*$.

**Theorem 3.1.** *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a Cartan null curve parametrized by pseudo arc parameter with curvatures $\kappa_1 \neq 0, \kappa_2$. Then the curve $\beta$ is a Bertrand curve with Bertrand mate $\beta^*$ if and only if one of the following conditions holds:*
*$(i)$ there exists constant real numbers $\lambda$ and $h$ satisfying*

$$h < 0, \quad 1 + \lambda\kappa_2 = -h\lambda\kappa_1, \quad \kappa_2 - h\kappa_1 \neq 0, \quad \kappa_2 + h\kappa_1 \neq 0. \tag{3.1}$$

*In this case, $\beta^*$ is a timelike curve in $\mathbb{E}_1^3$.*

*$(ii)$ there exists constant real numbers $\lambda$ and $h$ satisfying*

$$h > 0, \quad 1 + \lambda\kappa_2 = -h\lambda\kappa_1, \quad \kappa_2 - h\kappa_1 \neq 0, \quad \kappa_2 + h\kappa_1 \neq 0. \tag{3.2}$$

*In this case, $\beta^*$ is a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$.*

*Proof.* Assume that $\beta$ is a Cartan null Bertrand curve parametrized by pseudo arc parameter $s$ with $\kappa_1 \neq 0, \kappa_2$ and the curve $\beta^*$ is the Bertrand mate curve of the curve $\beta$ parametrized by with arc-length or pseudo arc $s^*$.

$(i)$ Let $\beta^*$ be a timelike curve. Then, we can write the curve $\beta^*$ as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s) \tag{3.3}$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty$−function on $I$. Differentiating (3.3) with respect to $s$ and using (2.1),(2.2), we get

$$T^* f' = (1 + \lambda\kappa_2)T + \lambda' N - \lambda\kappa_1 B. \tag{3.4}$$

By taking the scalar product of (3.4) with $N$, we have

$$\lambda' = 0. \tag{3.5}$$

Substituting (3.5) in (3.4), we find

$$T^* f' = (1 + \lambda\kappa_2)T - \lambda\kappa_1 B. \tag{3.6}$$

By taking the scalar product of (3.6) with itself, we obtain

$$(f')^2 = 2\lambda\kappa_1(1 + \lambda\kappa_2). \tag{3.7}$$

If we denote

$$\delta = \frac{1 + \lambda\kappa_2}{f'} \quad \text{and} \quad \gamma = \frac{-\lambda\kappa_1}{f'}, \tag{3.8}$$

we get

$$T^* = \delta T + \gamma B. \tag{3.9}$$

Differentiating (3.9) with respect to $s$ and using (2.1),(2.2), we find

$$f'\kappa_1^* N^* = \delta' T + (\delta\kappa_1 - \gamma\kappa_2) N + \gamma' B. \tag{3.10}$$

By taking the scalar product of (3.10) with itself, we get

$$\delta' = 0 \quad \text{and} \quad \gamma' = 0. \tag{3.11}$$

Since $\gamma \neq 0$, we have $1 + \lambda\kappa_2 = -h\lambda\kappa_1$ where $h = \delta/\gamma$. Substituting (3.11) in (3.10) , we find

$$f'\kappa_1^* N^* = (\delta\kappa_1 - \gamma\kappa_2) N \tag{3.12}$$

By taking the scalar product of (3.12) with itself, using (3.7) and (3.8), we have

$$(f')^2 (\kappa_1^*)^2 = -\frac{(\kappa_2 - h\kappa_1)^2}{2h} \tag{3.13}$$

where $\kappa_2 - h\kappa_1 \neq 0$ and $h < 0$. If we put $v = \frac{\delta\kappa_1 - \gamma\kappa_2}{f'\kappa_1^*}$, we get

$$N^* = vN. \tag{3.14}$$

Differentiating (3.14) with respect to $s$ and using (2.1),(2.2), we find

$$f'\kappa_2^* B^* = v\kappa_2 T - v\kappa_1 B - f'\kappa_1^* T^* \tag{3.15}$$

where $v' = 0$. Rewriting (3.15) by using (3.6) , we get

$$f'\kappa_2^* B^* = P(s)\, T + Q(s)\, B$$

where

$$\begin{aligned} P(s) &= \frac{\lambda\kappa_1 (\kappa_2 - h\kappa_1)(\kappa_2 + h\kappa_1)}{2(f')^2\kappa_1^*}, \\ Q(s) &= \frac{-\lambda\kappa_1 (\kappa_2 - h\kappa_1)(\kappa_2 + h\kappa_1)}{2h(f')^2\kappa_1^*} \end{aligned}$$

which implies that $\kappa_2 + h\kappa_1 \neq 0$.

Conversely, assume that $\beta$ is a Cartan null curve parametrized by pseudo arc parameter $s$ with $\kappa_1 \neq 0, \kappa_2$ and the conditions of (3.1) holds for constant real numbers $\lambda$ and $h$. Then, we can define a curve $\beta^*$ as

$$\beta^*(s^*) = \beta(s) + \lambda N(s). \tag{3.16}$$

Differentiating (3.16) with respect to $s$ and using (2.2), we find

$$\frac{d\beta^*}{ds} = -\lambda\kappa_1 \{hT + B\} \tag{3.17}$$

which leads to that

$$f' = \sqrt{\left| g\left(\frac{d\beta^*}{ds}, \frac{d\beta^*}{ds}\right)\right|} = m_1\lambda\kappa_1\sqrt{-2h}$$

where $m_1 = \pm 1$ such that $m_1\lambda\kappa_1 > 0$. Rewriting (3.17), we obtain

$$T^* = \frac{-m_1}{\sqrt{-2h}}\{hT + B\}, \quad g(T^*, T^*) = -1. \tag{3.18}$$

Differentiating (3.18) with respect to $s$ and using (2.2), we get

$$\frac{dT^*}{ds^*} = \frac{m_1 (\kappa_2 - h\kappa_1)}{f'\sqrt{-2h}} N$$

which causes that

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{m_2\left(\kappa_2 - h\kappa_1\right)}{f'\sqrt{-2h}} \tag{3.19}$$

where $m_2 = \pm 1$ such that $m_2\left(\kappa_2 - h\kappa_1\right) > 0$. Now, we can find $N^*$ as

$$N^* = m_1 m_2 N, \quad g\left(N^*, N^*\right) = 1. \tag{3.20}$$

Differentiating (3.20) with respect to $s$, using (3.18) and (3.19), we get

$$\frac{dN^*}{ds^*} - \kappa_1^* T^* = \frac{m_1 m_2\left(\kappa_2 + h\kappa_1\right)}{2hf'}\left\{hT - B\right\}$$

which bring about that

$$\kappa_2^* = \frac{m_3\left(\kappa_2 + h\kappa_1\right)}{f'\sqrt{-2h}},$$

where $m_3 = \pm 1$ such that $m_3\left(\kappa_2 + h\kappa_1\right) > 0$. Lastly, we define $B^*$ as

$$B^* = \frac{m_1 m_2 m_3\sqrt{-2h}}{2}\left\{T - \frac{1}{h}B\right\}, \quad g\left(B^*, B^*\right) = 1.$$

Then $\beta^*$ is a timelike curve and the Bertrand mate curve of $\beta$. Thus $\beta$ is a Bertrand curve.

$(ii)$ Let $\beta^*$ be a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$. Then the proof can be done similarly to $(i)$. $\square$

In the following results, the relationships between the Frenet vectors and curvature functions of Cartan Null Bertrand Curve and its Bertrand Mate curve are given

**Corollary 3.1.** *Let $\beta : I \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve with the Frenet frame $\{T, N, B\}$ and the curvatures $\kappa_1, \kappa_2$, and $\beta^* : I \to \mathbb{E}_1^3$ be a spacelike Bertrand mate curve with spacelike principal normal of $\beta$ with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures $\kappa_1^*, \kappa_2^*$. Then the curvatures of $\beta$ and $\beta^*$ satisfy the relations*

$$\begin{aligned}
\kappa_1^* &= \frac{\lambda(\kappa_2 - h)}{(f')^2}, \\
\kappa_2^* &= \frac{1}{(f')^3}\sqrt{-2\left(h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2\left(f'\right)^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) + \left(f'\right)^2\right)}
\end{aligned}$$

*and the corresponding frames of $\beta$ and $\beta^*$ are related by*

$$\begin{aligned}
T^* &= \left(\frac{h\lambda}{f'}\right)T - \left(\frac{\lambda}{f'}\right)B, \\
N^* &= N,
\end{aligned}$$

$$B^* = \left(\frac{h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2}{\sqrt{-2\left(h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) + (f')^2\right)}}\right) T + \left(\frac{\lambda(\lambda\kappa_2 - h\lambda) + (f')^2}{\sqrt{-2\left(h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) + (f')^2\right)}}\right) B$$

*where* $(f')^2 = 2\lambda^2 h$ *and* $1 + \lambda\kappa_2 = -h\lambda$, $h > 0$, $\lambda \neq 0$.

**Corollary 3.2.** *Let* $\beta : I \to \mathbb{E}_1^3$ *be a Cartan null Bertrand curve with the Frenet frame* $\{T, N, B\}$ *and the curvatures* $\kappa_1, \kappa_2$, *and* $\beta^* : I \to \mathbb{E}_1^3$ *be a timelike Bertrand mate curve of* $\beta$ *with the Frenet frame* $\{T^*, N^*, B^*\}$ *and the curvatures* $\kappa_1^*, \kappa_2^*$. *Then the curvatures of* $\beta$ *and* $\beta^*$ *satisfy the relations*

$$\kappa_1^* = \frac{\lambda(\kappa_2 - h)}{(f')^2},$$

$$\kappa_2^* = \frac{1}{(f')^3}\sqrt{2\left(h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) - (f')^2\right)}$$

*and the corresponding frames of* $\beta$ *and* $\beta^*$ *are related by*

$$T^* = \left(\frac{-h\lambda}{f'}\right) T - \left(\frac{\lambda}{f'}\right) B,$$

$$N^* = N,$$

$$B^* = \left(\frac{h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2}{\sqrt{2\left(h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) - (f')^2\right)}}\right) T + \left(\frac{\lambda(\lambda\kappa_2 - h\lambda) - (f')^2}{\sqrt{2\left(h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) - (f')^2\right)}}\right) B$$

*where* $(f')^2 = -2\lambda^2 h$ *and* $1 + \lambda\kappa_2 = -h\lambda$, $h < 0$, $\lambda \neq 0$.

**Remark 3.1.** It can easily be proved that a Cartan null curve has no pseudo null Bertrand mate in $\mathbb{E}_1^3$.

**Example 3.1.** Let us consider a Cartan null curve in $\mathbb{E}_1^3$ parametrized by

$$\beta(s) = (\sinh s, \cosh s, s)$$

with

$$\begin{aligned}
T(s) &= (\cosh s, \sinh s, 1)\,, \\
N(s) &= (\sinh s, \cosh s, 0)\,, \\
B(s) &= \left(-\tfrac{\cosh s}{2}, -\tfrac{\sinh s}{2}, \tfrac{1}{2}\right) \\
\kappa_1(s) &= 1 \quad \text{and} \quad \kappa_2(s) = 1/2.
\end{aligned}$$

If we take $h = -3/2$ and $\lambda = 1$ in $(i)$ of theorem 3.1, then we get the curve $\beta^*$ as follows:

$$\beta^*(s) = \beta(s) + N(s) = (2\sinh s, 2\cosh s, s)$$

By straight calculations, we get

$$\begin{aligned}
T^*(s) &= \left(\tfrac{2\cosh s}{\sqrt{3}}, \tfrac{2\sinh s}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \\
N^*(s) &= (\sinh s, \cosh s, 0)\,, \\
B^*(s) &= \left(-\tfrac{\cosh s}{\sqrt{3}}, -\tfrac{\sinh s}{\sqrt{3}}, -\tfrac{2}{\sqrt{3}}\right), \\
\kappa_1^*(s) &= 2/3, \quad \kappa_2^*(s) = 1/3.
\end{aligned}$$

It can be easily seen that the curve $\beta^*$ is a timelike Bertrand mate curve of the curve $\beta$.

**Example 3.2.** For the same Cartan null curve $\beta$ in Example 1, if we take $h = 3/2$ and $\lambda = -1/2$ in $(ii)$ of theorem 3.1, then we get the curve $\beta^*$ as follows:

$$\beta^*(s) = \beta(s) - \frac{1}{2}N(s) = \left(\frac{\sinh s}{2}, \frac{\cosh s}{2}, s\right)$$

By straight calculations, we get

$$\begin{aligned}
T^*(s) &= \left(\tfrac{\cosh s}{\sqrt{3}}, \tfrac{\sinh s}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}\right), \\
N^*(s) &= (\sinh s, \cosh s, 0)\,, \\
B^*(s) &= \left(-\tfrac{2\cosh s}{\sqrt{3}}, -\tfrac{2\sinh s}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right), \\
\kappa_1^*(s) &= 2/3, \quad \kappa_2^*(s) = 4/3.
\end{aligned}$$

It can be easily seen that the curve $\beta^*$ is a spacelike Bertrand mate curve of the curve $\beta$.

In the graphic below, the curves given in Example 3.1 and Example 3.2 are illustrated together.

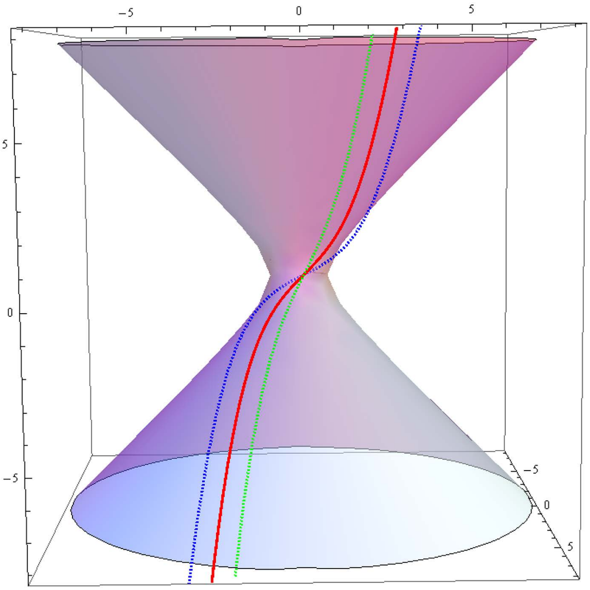


Fig. 3.1: Cartan null Bertrand curve $\beta$ (red) and its spacelike (blue) and timelike (green) Bertrand mates curves in $\mathbb{E}_1^3$

## REFERENCES


1. H. Balgetir, M. Bektaş and M. Ergüt: *Bertrand curves for non-null curves in 3-dimensional Lorentzian space.* Hadronic J. **27** (2) (2004), 229-236.
2. H. Balgetir, M. Bektaş and J. Inoguchi: *Null Bertrand curves in Minkowski 3-space and their characterizations.* Note Mat. **23** (1) (2004/05), 7-13.
3. J. M. Bertrand:, *Mémoire sur la théorie des courbes á double courbure.* Comptes Rendus. **36**, (1850).
4. W. B. Bonnor: *Null curves in a Minkowski space-time.* Tensor **20** (1969), 229-242.
5. W. B. Bonnor: *Curves with null normals in Minkowski space-time.* In: A random walk in relativity and cosmology, Wiley Easten Limited (1985), 33-47.
6. N. Ekmekci and K. İlarslan: *On Bertrand curves and their characterization.* Differ. Geom. Dyn. Syst. **3** (2) (2001), 17-24.
7. D. H. Jin: *Null Bertrand curves in a Lorentz manifold.* J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **15** (3) (2008), 209-215.
8. S. Özkaldi Karakuş, K. İlarslan and Y. Yayli: *A new approach for characterization of curve couples in Euclidean 3-space.* Honam Mathematical J. **36** (1) (2014), 113-129.
9. W. Kuhnel: *Differential geometry: curves-surfaces-manifolds.* Braunschweig, Wiesbaden, (1999).
10. H. Liu and F. Wang: *Mannheim partner curves in 3-space.* Journal of Geometry. **88** (2008), 120-126.

11. H. Matsuda and S. Yorozu: *Notes on Bertrand curves.*Yokohama Math. J. **50** (1-2) (2003), 41-58.
12. L. R. Pears: *Bertrand curves in Riemannian space.* J. London Math. Soc. Volumes **1-10** (2) (1935), 180-183 .
13. A. Uçum, K. İlarslan and S. Özkaldi Karakuş: *On curve couples with joint lightlike Frenet planes in Minkowski* 3*-space.* Acta Univ. Apulensis Math. Inform. **41** (2015), 111-129.
14. A. Uçum, K. İlarslan and M. Sakaki: *On* $(1,3)$*-Cartan Null Bertrand curves in Semi-Euclidean* 4*-Space with index* 2. J. Geom. **107** (2016), 579-591.
15. A. Uçum, O. Keçilioğlu and K. İlarslan: *Generalized Pseudo Null Bertrand curves in Semi-Euclidean* 4*-Space with index* 2. Rend. Circ. Mat. Palermo, II. Ser, **65** (2016), 459-472.
16. A. Uçum, O. Keçilioğlu and K. İlarslan: *Generalized Bertrand curves with spacelike* $(1,3)$*-normal plane in Minkowski space-time.* Turk J Math. **40** (2016), 487-505.
17. A. Uçum and K. İlarslan: *On timelike Bertrand Curves in Minkowski 3-space.* Honam Mathematical J. **38(3)** (2016), 467-477.
18. B. Saint Venant: *Mémoire sur les lignes courbes non planes.* Journal de l'Ecole Polytechnique. **18** (1845), 1-76.

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# A FIXED POINT THEOREM FOR $F$-CONTRACTION MAPPINGS IN PARTIALLY ORDERED BANACH SPACES

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**Abstract.** In this paper, we first introduce a new notion of an $F$-contraction mapping, also we establish a fixed point theorem for such mappings in partially ordered Banach spaces. Moreover, two examples are represented to show the compatibility of our results.
**Keywords:** F-Contraction; Fixed point; Partially ordered.

## 1. Introduction and Preliminaries

$F$-contractions were introduced initially by Wardowski [24]. Indeed, Wardowski [24] extended the Banach Contraction Principle and proved some fixed-point results for $F$-contraction mappings. Since then, several authors proved many fixed point results for $F$-contraction mappings (refer to [1, 4, 5, 8, 11, 12, 13, 15, 19, 21, 22, 25]).

Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:
$(F_1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, +\infty)$ with $\alpha < \beta$ we have $F(\alpha) < F(\beta)$,
$(F_2)$ for each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to+\infty} \alpha_n = 0 \;\; if \, and \, only \, if \;\; \lim_{n\to+\infty} F(\alpha_n) = -\infty;$$

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$(F_3)$ there exists $k \in (0, +\infty)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.
Let $F_1(\alpha) = ln(\alpha), F_2(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_3(\alpha) = \alpha + ln(\alpha)$ for $\alpha > 0$, then $F_1, F_2, F_3 \in \mathcal{F}$. A mapping $T : X \to X$ is called an $F$-contraction if there exists $\tau > 0$ and $F \in \mathcal{F}$ shch that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

holds for all $x, y \in X$ with $d(Tx, Ty) > 0$. From $(F_1)$ and (1.1), we can easily see that any $F$-contraction is a contractive mapping. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be givin by $F(\alpha) = ln\, \alpha$. By (1.1), we obtain

$$d(Tx, Ty) \leq e^{-\tau} d(x, y),$$

for all $x, y \in X$ and $d(Tx, Ty) > 0$. Let $F(\alpha) = \alpha + ln\, \alpha$ for $\alpha > 0$. From (1.1), we get

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty) - d(x,y)} \leq e^{-\tau},$$

for any $x, y \in X$ and $d(Tx, Ty) > 0$. Wardowski [24] proved the following fixed point theorem.

**Theorem 1.1.** *[24] Let $(X, d)$ be a complete metric space and let $T : X \to X$ be an $F$-contraction. Then $T$ has a fixed point $x^*$ and for an arbitrary point $x \in X$, the sequence $\{T^n x\}$ is convergent to $x^*$.*

Let $X$ be an ordered normed space, i.e., a vector space over the real equipped with a partial order $\preccurlyeq$ and a norm $||.||$. For every $\alpha \geq 0$ and $x, y, z \in X$ with $x \preccurlyeq y$ one has that $x + z \preccurlyeq y + z$ and $\alpha x \preccurlyeq \alpha y$. Two elements $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A self-mapping $T$ on $X$ is called non-decreasing if $Tx \preccurlyeq Ty$ whenever $x \preccurlyeq y$ for all $x, y \in X$.

Ran and Reurings [18] initiate the fixed point theory in the metric spaces equipped with a partial order relation. Thereafter, several authors obtained many fixed point results in ordered metric space (see [2, 3, 6, 7, 10, 16, 17, 23] and references therein).

**Definition 1.1.** [9] Let $E$ be a Banach space. A subset $P$ of $E$ is called cone if the following conditions are satisfied:
1) $P$ is nonempty closed set and $P \neq \{\theta\}$, where $\theta$ denotes the zero element in $E$;
2) if $x, y \in P$ and $a, b \in \mathbb{R}, a, b \geq 0$, then $ax + by \in P$;
3) if $x \in P$ and $-x \in P$, then $x = \theta$.

Let $P \subseteq E$ be a cone. We define a partial ordering $\preccurlyeq$ with respect to $P$ by $x \preccurlyeq y$ if and only if $y - x \in P$. A cone $P$ is called normal if there is a number $L > 0$ such that

$$\theta \preccurlyeq x \preccurlyeq y \textit{ implies } ||x|| \leq L||y||,$$

for all $x, y \in E$. The least positive number $L$ satisfying the above inequality is called the normal constant of $P$.

**Definition 1.2.** [14, 20] A set $P \subseteq E$ is said to be a lattice under the partial ordering $\preccurlyeq$, if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in P$.

**Lemma 1.1.** *[9] A cone $P$ in a normed space $(E, ||.||)$ is normal if and only if there exists a norm $||.||_1$ on $E$, equivalent to the given norm $||.||$, such that the cone $P$ is monotone w.r.t. $||.||_1$.*

**Lemma 1.2.** *[9] Let $E$ be a real Banach space, $P$ be a normal cone and $\{x_{n_k}\}$ be a subsequence converging to $p$ of monotone sequence $\{x_n\}$. Then $\{x_n\}$ converges to $p$. Also if $\{x_n\}_{n\in\mathbb{N}}$ is an increasing(decreasing) sequence, then $x_n \preccurlyeq p (p \preccurlyeq x_n)$ for all $n \in \mathbb{N}$.*

## 2. Main results

In this section, we prove a fixed point result in partially ordered Banach spaces. Let $E$ be a partially ordered Banach space, $P$ be a normal cone and the partial order $\preccurlyeq$ on $E$ be induced by the cone $P$. We denote by $\mathcal{F}$, the set of all functions $F : P - \{\theta\} \to \mathbb{R}$ that satisfying the following conditions:
$(F_1')$ $F$ is strictly increasing, i.e., for all $x, y \in P$ such that $x \prec y, F(x) < F(y)$ or $x \preccurlyeq y$ and $x \neq y$ yields $F(x) \leq F(y)$ and $F(x) \neq F(y)$.
$(F_2')$ For each sequence $\{x_n\}$ in $P$,

$$\lim_{n\to+\infty} x_n = \theta \ \ if\, and\, only\, if \ \ \lim_{n\to+\infty} F(x_n) = -\infty.$$

$(F_3')$ There exists $k \in (0, +\infty)$ such that $\lim_{x\to\theta} ||x||^k F(x) = 0$.
Our new result is the following:

**Theorem 2.1.** *Let $X \subseteq E$ be a closed set, $P \subseteq X$ and let $T : X \to X$ be a self-mapping on $X$. Suppose that the following hypotheses hold:*
*(i) $X$ is a lattice;*
*(ii) $T$ is a decreasing operator, i.e., $x \preccurlyeq y$ implies $Tx \succcurlyeq Ty$;*
*(iii) there exsits $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\tau + F(Tu - Tv) \leq F(v - u), \tag{2.1}$$

*for all $u, v \in X$, where $u \preccurlyeq v$ and $Tu \neq Tv$. Then, $T$ has a unique fixed point $p \in X$.*

*Proof.* Let $x_0 \in X$ be arbitrary. If $Tx_0 = x_0$ the proof is finished, that is $T$ has a fixed point $x_0$. Let $Tx_0 \neq x_0$ and we consider the following two case.
Case1. Let $x_0$ is comparable with $Tx_0$. Without loss of generality, we suppose that $x_0 \prec Tx_0$. Since $T$ is decreasing, we get $Tx_0 \succcurlyeq T^2x_0$. We can easily check that $T^2$ is increasing. From (2.1), we have

$$\tau + F(Tx_0 - T^2x_0) \leq F(Tx_0 - x_0).$$

Then, we get

$$F(Tx_0 - T^2x_0) \leq F(Tx_0 - x_0).$$

Since, $F$ is strictly increasing, we get

$$Tx_0 - T^2x_0 \quad \preccurlyeq Tx_0 - x_0.$$

Then, we have

$$x_0 \preccurlyeq T^2x_0. \tag{2.2}$$

Using (2.1), we obtain

$$\begin{aligned} \tau + F(T^2v - T^2u) &\leq F(Tu - Tv) \\ &\leq F(v-u) - \tau \\ &< F(v-u), \end{aligned} \tag{2.3}$$

for all $u, v \in X$, where $u \prec v$ or $u \preccurlyeq v$ and $u \neq v$. Let $Sx = T^2x$ for all $x \in X$. Then, from (2.3), we have

$$\tau + F(Sv - Su) \leq F(v-u), \tag{2.4}$$

for all $u, v \in X$, where $u \prec v$ or $u \preccurlyeq v, u \neq v$ and $F \in \mathcal{F}$. Also, from (2.2) we have $x_0 \preccurlyeq Sx_0$. Now, we show that $S$ has a unique fixed point. Consider the iterated sequence $\{x_n\}$, where $x_{n+1} = Sx_n$ for $n = 0, 1, 2, \ldots$. Since $S$ is increasing, we have $x_{n+1} \preccurlyeq x_n$ for all $n = 0, 1, 2, \ldots$. Using (2.4), we have

$$F(x_{n+1} - x_n) \leq F(x_n - x_{n-1}) - \tau \leq \ldots \leq F(x_1 - x_0) - n\tau. \tag{2.5}$$

Letting $n \to +\infty$ above inequality, we obtain

$$\lim_{n\to+\infty} F(x_{n+1} - x_n) = -\infty.$$

Using $F_2'$, we get $\alpha_n = x_{n+1} - x_n \to \theta$ as $n \to +\infty$. This implies that

$$\lim_{n\to+\infty} ||\alpha_n|| = 0. \tag{2.6}$$

From $(F_3')$, there exsits $k \in (0,1)$ such that

$$\lim_{n\to+\infty} ||\alpha_n||^k F(\alpha_n) = 0. \tag{2.7}$$

From, (2.5) we have

$$(||\alpha_n||^k F(\alpha_n) - ||\alpha_n||^k F(\alpha_0)) \leq ||\alpha_n||^k (F(\alpha_0) - n\tau) - ||\alpha_n||^k F(\alpha_0) = -||\alpha_n||^k n\tau \leq 0.$$

Using (2.6) and (2.7) and letting $n \to +\infty$ in above inequality, we get

$$\lim_{n\to+\infty} n||\alpha_n||^k = 0. \tag{2.8}$$

It follows from (2.8), there exists $N \in \mathbb{N}$, such that

$$||\alpha_n|| \leq \frac{1}{n^{\frac{1}{k}}}, \tag{2.9}$$

for all $n > N$. Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose $m, n \in \mathbb{N}$ and $m > n > N$.

$$||x_m - x_n|| \leq ||\alpha_{m-1}|| + ||\alpha_{m-2}|| + \ldots + ||\alpha_n|| \leq \sum_{i=n}^{+\infty} ||\alpha_i|| \leq \sum_{i=n}^{+\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Then $||x_m - x_n|| \to 0$ as $m, n \to +\infty$, which implies $\{x_n\}$ is a Cauchy sequence. Since $X$ is closed, then there exists point $p$ in $X$ such that $\lim_{n\to+\infty} x_n = p$. Using Lemma 1.2, we get $x_n \preccurlyeq p$ for all $n \in \mathbb{N}$. From (2.4), we have

$$F(Sx_n - Sp) \leq F(x_n - p) - \tau \leq F(x_n - p).$$

Since $F$ is strictly increasing, we have

$$Sx_n - Sp \prec x_n - p, \tag{2.10}$$

for all $n \in \mathbb{N}$. From Lemma (1.1) exists a norm $||.||_1$ such that is equivalent with $||.||$ and

$$||Sx_n - Sp||_1 \leq ||x_n - p||_1, \tag{2.11}$$

for all $n \in \mathbb{N}$. Using (2.11), we obtain

$$\begin{aligned} ||p - Sp||_1 &\leq ||p - x_{n+1}||_1 + ||x_{n+1} - Sp||_1 \\ &\leq ||p - x_{n+1}||_1 + ||x_n - p||_1, \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in above inequality, we get $||p - Sp||_1 = 0$, which implies $Sp = p$. To see the uniqueness of the fixed point, let us consider $p$ and $q$ be two distinct fixed points of $S$, that is, $Sp = p \neq q = Sq$. If $q$ comparable with $p$, without loss of generality, we suppose that $q \preccurlyeq p$. Then, by (2.4), we obtain

$$\tau \leq F(p - q) - F(Sp - Sq) = 0, \tag{2.12}$$

which is a contradiction. Now, suppose $p$ is not comparable to $q$. Since $X$ is a lattice, there exists $r \in X$ such that $r = inf\{p, q\}$, which implies $r \preccurlyeq p$ and $r \preccurlyeq q$. Since $S$ is increasing, we have $S^n r \preccurlyeq S^n p$ and $S^n r \preccurlyeq S^n q$. Using (2.4) we obtain,

$$F(p - S^n r) = F(S^n p - S^n r) \leq F(S^{n-1} p - S^{n-1} r) - \tau \leq \ldots \leq F(p - r) - n\tau,$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in above inequality, we have $\lim_{n\to+\infty} F(p - S^n r) = -\infty$ that together with $(F_2')$ gives $lim_{n\to+\infty}(p - S^n r) = \theta$. This implies that $\lim_{n\to+\infty} S^n r = p$. Similarly, $\lim_{n\to+\infty} S^n r = q$. So, $p = q$ that is $S$ has a unique fixed point $p$. Now, we show that the unique fixed point of $S$ is also the unique fixed point of $T$. Since $S$ has a fixed point $p$, we have

$$S(Tp) = T^2(Tp) = T(T^2p) = T(Sp) = Tp. \tag{2.13}$$

From the uniqueness of the fixed point of $S$, we know $Tp = p$.
Case2. Suppose $x_0$ is not comparable to $Tx_0$. Since $X$ is a lattice, there exists $y \in X$ such that $y = inf\{x_0, Tx_0\}$, which implies $y \preccurlyeq x_0$ and $y \preccurlyeq Tx_0$. Since $T$ is decreasing, we have $Tx_0 \preccurlyeq Ty$, which implies $y \preccurlyeq Ty$. Similarly to the proof of case 1, we can show $T$ has a unique fixed point.

□

**Example 2.1.** Let $E = R \times R$ endowed with the norm $||.||_1$ which is defined as follows $||(x_1, x_2)||_1 = |x_1| + |x_2|, x_1, x_2 \in \mathbb{R}$. Also, we define a partial order on $\mathbb{R}^2$ as follows

$$(a, b) \preccurlyeq (c, d) \ if\, and\, only\, if \ a \leq c, b \leq d.$$

Then $(X, ||.||, \preccurlyeq)$ is a partially ordered Banach space. Suppose $X = [0, +\infty) \times [0, +\infty)$, $P = \{(\alpha, 0) : \alpha \geq 0\}$ and $F : P - \{\theta\} \to \mathbb{R}$ by $F\alpha = ln\alpha$. Define $T = (T_1, T_2)$ where $T_i : [0, +\infty) \to \mathbb{R}, i = 1, 2$ and $T_1(a) = e^{-\tau}\frac{-a}{1+a}, T_2(b) = e^{-\tau}\frac{2}{1+b}$,

$$T(a, b) = (T_1(a), T_2(b)) = (e^{-\tau}\frac{-a}{1+a}, e^{-\tau}\frac{2}{1+b}),$$

for all $a, b \in [0, +\infty)$ where $\tau > 0$. It is clear that both $T_i, i = 1, 2$ are strictly decreasing, so, $T$ is decreasing. We show that $T$ is $F$-contraction. Indeed, let $u = (x_1, y_1) \preccurlyeq v = (x_2, y_2)$, we have

$$\begin{aligned}
Tu - Tv &= e^{-\tau}(\frac{-x_1}{2+x_1}, \frac{2}{1+y_1}) - e^{-\tau}(\frac{-x_2}{2+x_2}, \frac{2}{1+y_2}) \\
&= e^{-\tau}(\frac{-2x_1 - x_1x_2 + 2x_2 + x_1x_2}{4 + 2x_1 + 2x_2 + x_1x_2}, \frac{2 + 2y_2 - 2 - 2y_1}{1 + y_2 + y_1 + y_1y_2}) \\
&\leq e^{-\tau}(x_2 - x_1, y_2 - y_1) \\
&= e^{-\tau}(v - u).
\end{aligned}$$

Which implies that

$$\tau + ln(Tu - Tv) \leq ln(v - u).$$

Then, all the conditions of Theorem 2.1 are satisfied and so $T$ has a unique fixed point $(0, \frac{-1+\sqrt{1+8e^{-\tau}}}{2})$, where $\tau$ is given.

**Example 2.2.** Let $E = \mathbb{R}, X = [0, +\infty), P = [0, +\infty)$ and $F : P\backslash \{0\} \to \mathbb{R}$ with $F(r) = -\frac{1}{r}$. Define the mapping $T : X \to X$ by $Tx = \frac{1}{1+x}$. . It is clear that the all conditions of Theorem 2.1 are satisfied. The condition (2.1) is true i.e. exists $\tau > 0$ such that

$$\tau + F(Tu - Tv) \leq F(v - u).$$

Indeed, for $v > u$, we obtain

$$
\begin{aligned}
F(v-u) - F(Tu - Tv) &= -\frac{1}{v-u} + \frac{1}{\frac{1}{1+u} - \frac{1}{1+v}} \\
&= -\frac{1}{v-u} + \frac{(1+v)(1+u)}{v-u} \\
&= -\frac{1}{v-u} + \frac{1+u+v+vu}{v-u} \\
&= \frac{u+v+vu}{v-u} \\
&\geq \frac{u+v}{v-u} \\
&\geq \frac{v-u}{v-u} = 1.
\end{aligned}
$$

Hence, for any $\tau \in (0,1]$, we have

$$\tau + F(Tu - Tv) \leq F(v-u).$$

Thus, $T$ has a unique fixed point $u_0 = \frac{\sqrt{5}-1}{2}$.

## REFERENCES


1. M. Abbas, T. Nazir, T. L. Aleksić and S. Radenović: *Common fixed points of set-valued $F$-contraction mappings on domain of sets endowed with directed graph.* Comput. Appl. Math. **36**(4) (2017), 1607-1622.
2. M. Abtahi, Z. Kadelburg and S. Radenović: *Fixed points and coupled fixed points in partially ordered $\nu$-generalized metric spaces.* Appl. Gen. Topol. **19**(2) (2018), 189-201.
3. M. Afkhami: *The cozero-divisor graph of partially ordered sets.* Southeast Asian Bull. Math. **44**(2) (2020), 297-306.
4. I. Altun, G. Minak and H. Dag: *Multivalued $F$-contractions on complete metric spaces.* J. Nonlinear Convex Anal. **16** (2015), 659-666.
5. I. Altun, G. Minak and M. Olgun: *Fixed points of multivalued nonlinear $F$-contractions on complete metric spaces.* Nonlinear Anal. Model. Control **21**(2) (2016), 201-210.
6. A. L. Ansari, V. Gupta and N. Mani: *$C$-class functions on some coupled fixed point theorems in partially ordered $S$-metric spaces.* Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **68**(2) (2019), 1694-1708.
7. K. Bouzkoura and S. Benkaddour: *Some Common Fixed Point Theorems in Partially Ordered Sets.* J. Appl. Math. (2020), Art. ID 4707962, 5 pp.
8. L. Chen, S. Huang, C. Li and Y. Zhao: *Several Fixed-Point Theorems for $F$-Contractions in Complete Branciari $b$-Metric Spaces and Applications.* J. Funct. Spaces (2020), Art. ID 7963242, 10 pp.
9. D. Guo, Y. J. Cho and J. Zhu: *Partial ordering methods in nonlinear problems.* Nova Science Publishers, Inc., Hauppauge, NY, 2004.

10. V. Gupta, G. Jungck and N. Mani: *Some novel fixed point theorems in partially ordered metric spaces*. AIMS Math. **5**(5) (2020), 4444-4452.

11. A. Hussain, H. Al-Sulami, N. Hussain and H. Farooq: *Newly fixed disc results using advanced contractions on $F$-metric space*. J. Appl. Anal. Comput. **10**(6) (2020), 2313-2322.

12. N. Hussain, A. Latif, I. Iqbal and M. A. Kutbi: *Fixed point results for multivalued $F$-contractions with application to integral and matrix equations*. J. Nonlinear Convex Anal. **20**(11) (2019), 2297-2311.

13. Z. Kadelburg and S. Radenović: *Notes on some recent papers concerning $F$-contractions in b-metric spaces*. Constr. Math. Anal. **1**(2) (2018), 108-112.

14. X. Lin and Z. Zhao: *Sign-changing solution for a third-order boundary-value problem in ordered Banach space with lattice structure*. Bound. Value Probl. **2014** (2014), 132, 10 pp.

15. S. K. Mohanta and S. Patra: *Coincidence points for graph preserving generalized almost $F-G$-contractions in b-metric spaces*. Nonlinear Stud. **27**(4) (2020), 897-914.

16. J. J. Nieto and R. R. Lopez: *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*. Order **22**(3) (2005), 223-239.

17. J. J. Nieto and R. R. Lopez: *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*. Acta Math. Sin. (Engl. Ser.) **23**(12) (2007), 2205-2212.

18. A. C. M. Ran and M. C. B. Reurings: *A fixed point theorem in partially ordered sets and some applications to matrix equations*. Proc. Amer. Math. Soc. **132**(5) (2004), 1435-1443.

19. M. Shoaib, M. Sarwar and P. Kumam: *Multi-valued fixed point theorem via $F$-contraction of Nadler type and application to functional and integral equations*. Bol. Soc. Parana. Mat. (3), **39**(4) (2021), 83-95.

20. J. Sun and X. Liu: *Computation of topological degree in ordered Banach spaces with lattice structure and its application to superlinear differential equations*. J. Math. Anal. Appl. **348**(2) (2008), 927-937.

21. A. Taheri and A. P. Farajzadeh: *A new generalization of $\alpha$-type almost-$F$-contractions and $\alpha$-type $F$-Suzuki contractions in metric spaces and their fixed point theorems*. Carpathian Math. Publ. **11**(2) (2019), 475-492.

22. A. Tomar and R. Sharma: *Almost $\alpha$-Hardy-Rogers-$F$-contractions and their applications*. Armen. J. Math. **11**(11) (2019), 19 pp.

23. C. Wang, J. Mao and Z. Zhao: *A fixed-point theorem for ordered contraction-type decreasing operators in Banach space with lattice structure*. J. Funct. Spaces 2020, Art. ID 3527430, 7 pp.

24. D. Wardowski: *Fixed points of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. **2012**(94) (2012), 6 pp.

25. D. Wardowski and N. V. Dung: *Fixed points of $F$-weak contractions on complete metric spaces*. Demonstr. Math. **47**(1) (2014), 146-155.

26. *D. Allen*: Relations between the local and global structure of fnite semigroups. Ph. D. Thesis, University of California, Berkeley, 1968.

27. P. Erdős: *On the distribution of the roots of orthogonal polynomials.* In: Proceedings of a Conference on Constructive Theory of Functions (G. Alexits, S. B. Steckhin, eds.), Akademiai Kiado, Budapest, 1972, 145–150.

28. A. Ostrowski: *Solution of Equations and Systems of Equations.* Academic Press, New York, 1966.

29. E. B. Saff and R. S. Varga: *On incomplete polynomials II.* Pacific J. Math. **92** (1981), 161–172.

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# SOME PROPERTIES OF BOUNDED TRI-LINEAR MAPS

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**Abstract.** Let $X, Y, Z$ and $W$ be normed spaces and $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping. In this manuscript, we introduce the topological centers of bounded tri-linear mapping and we invistagate their properties. We study the relationships between weakly compactenss of bounded linear mappings and regularity of bounded tri-linear mappings. We extend some factorization property for bounded tri-linear mappings. We also establish the relations between regularity and factorization property of bounded tri-linear mappings.
**Keywords**: Arens product, Module action, Factors, Topological center and Tri-linear mappings

## 1. Introduction

Let $X, Y, Z$ and $W$ be normed spaces and $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping. One of the natural extensions of $f$ can be derived by the following procedure:

1. $f^* : W^* \times X \times Y \longrightarrow Z^*$, given by $\langle f^*(w^*, x, y), z\rangle = \langle w^*, f(x, y, z)\rangle$, where $x \in X, y \in Y, z \in Z, w^* \in W^*$.

   The map $f^*$ is a bounded tri-linear mapping and is called the adjoint of $f$.

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2. $f^{**} = (f^*)^* : Z^{**}\times W^*\times X \longrightarrow Y^*$, given by $\langle f^{**}(z^{**}, w^*, x), y\rangle = \langle z^{**}, f^*(w^*, x, y)\rangle$, where $x \in X, y \in Y, z^{**} \in Z^{**}, w^* \in W^*$.

3. $f^{***} = (f^{**})^* : Y^{**} \times Z^{**} \times W^* \longrightarrow X^*$, given by $\langle f^{***}(y^{**}, z^{**}, w^*), x\rangle = \langle y^{**}, f^{**}(z^{**}, w^*, x)\rangle$, where $x \in X, y^{**} \in Y^{**}, z^{**} \in Z^{**}, w^* \in W^*$.

4. $f^{****} = (f^{***})^* : X^{**} \times Y^{**} \times Z^{**} \longrightarrow W^{**}$, given by $\langle f^{****}(x^{**}, y^{**}, z^{**}), w^*\rangle = \langle x^{**}, f^{***}(y^{**}, z^{**}, w^*)\rangle$, where $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{**} \in Z^{**}, w^* \in W^*$.

Now let $f^r : Z \times Y \times X \longrightarrow W$ be the flip of $f$ defined by $f^r(z, y, x) = f(x, y, z)$, whenever $x \in X, y \in Y$ and $z \in Z$. Then $f^r$ is a bounded tri-linear map and it may be extended as above to $f^{r****} : Z^{**}\times Y^{**}\times X^{**} \longrightarrow W^{**}$. When $f^{****}$ and $f^{r****r}$ are equal, then $f$ is called regular. Regularity of $f$ is equvalent to the following

$$w^* - \lim_\alpha w^* - \lim_\beta w^* - \lim_\gamma f(x_\alpha, y_\beta, z_\gamma) = w^* - \lim_\gamma w^* - \lim_\beta w^* - \lim_\alpha f(x_\alpha, y_\beta, z_\gamma),$$

where $\{x_\alpha\} \subset X, \{y_\beta\} \subset Y$ and $\{z_\gamma\} \subset Z$ and convergence to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the $w^*-$topologies, respectively. A bounded tri-linear mapping $f : X\times Y\times Z \longrightarrow W$ is regular whenever at least two of $X, Y$ or $Z$ are reflexive, see [19] and [20]. Also, we have naturally six different Aron-Berner extensions to the bidual spaces based on six different elements in S3 and compeletly regularity should be defined accordingly to the equalities of all these six Aron-Berner extensions, see [13].

**Example 1.1.** Let $G$ be an infinite, compact Hausdorff group and let $1 < p < \infty$. By [9, pp 54], we know that $L^p(G) * L^1(G) \subset L^p(G)$, where

$$(k * g)(x) = \int_G k(y)g(y^{-1}x)dy, \qquad (x \in G, k \in L^p(G), g \in L^1(G)).$$

On the other hand, since the Banach space $L^p(G)$ is reflexive, the bounded tri-linear mapping

$$f : L^p(G) \times L^1(G) \times L^p(G) \longrightarrow L^p(G)$$

defined by $f(k, g, h) = (k*g)*h$, is regular for every $k, h \in L^p(G)$ and $g \in L^1(G)$, see [20].

A bounded bilinear(resp. tri-linear) mapping $m : X \times Y \longrightarrow Z$(resp. $f : X \times Y \times Z \longrightarrow W$) is said to be factor if is surjective, that is, $m(X \times Y) = Z$(resp. $f(X \times Y \times Z) = W$), see [5].

For a discussion of Arens regularity for Banach algebras and bounded bilinear maps, see [1], [2], [11], [12] and [18]. For example, every $C^*$-algebra is Arens regular, see [4]. Also $L^1(G)$ is Arens regular if and only if G is finite,[21].
The left topological center of $m$ may be defined as follows:

$$Z_l(m) = \{x^{**} \in X^{**} : y^{**} \longrightarrow m^{***}(x^{**}, y^{**}) \; is\ weak^* - to - weak^* - continuous\}.$$

Also the right topological center of turns out to be

$$Z_r(m) = \{y^{**} \in Y^{**} : x^{**} \longrightarrow m^{r***r}(x^{**}, y^{**}) \; is\ weak^*-to-weak^*-continuous\}.$$

The subject of topological centers has been investigated in [6], [7] and [16]. In [14], Lau and Ulger gave several significant results related to the topological centers of certain dual algebras. In [11], authors extend some problems from Arens regularity and Banach algebras to module actions. They also extend the definitions of the left and right multiplier for module actions, see [10] and [12].

Let $A$ be a Banach algebra, and let $\pi : A \times A \longrightarrow A$ denote the product of A, so that $\pi(a,b) = ab$ for every $a, b \in A$. The Banach algebra $A$ is Arens regular whenever the map $\pi$ is Arens regular. The first and second Arens products, denoted by $\Box$ and $\Diamond$ respectively, are definded by

$$a^{**}\Box b^{**} = \pi^{***}(a^{**}, b^{**}) \ \ , \ \ a^{**}\Diamond b^{**} = \pi^{r****r}(a^{**}, b^{**}) \ \ , \ \ (a^{**}, b^{**} \in A^{**}).$$

## 2. Module actions for bounded tri-linear maps

Let $(\pi_1, X, \pi_2)$ be a Banach $A$-module and let $\pi_1 : A \times X \longrightarrow X$ and $\pi_2 : X \times A \longrightarrow X$ be the left and right module actions of $A$ on $X$, respectively. If $(\pi_1, X)$ (resp. $(X, \pi_2)$) is a left (resp. right) Banach $A$-module of $A$ on $X$, then $(X^*, \pi_1^*)$(resp. $(\pi_2^{r*r}, X^*)$) is a right (resp. left) Banach $A$-module and $(\pi_2^{r*r}, X^*, \pi_1^*)$ is the dual Banach $A$-module of $(\pi_1, X, \pi_2)$. We note also that $(\pi_1^{***}, X^{**}, \pi_2^{***})$ is a Banach $(A^{**}, \Box)$-module with module actions $\pi_1^{***} : A^{**} \times X^{**} \longrightarrow X^{**}$ and $\pi_2^{***} : X^{**} \times A^{**} \longrightarrow X^{**}$. Similary, $(\pi_1^{r****r}, X^{**}, \pi_2^{r****r})$ is a Banach $(A^{**}, \Diamond)$-module with module actions $\pi_1^{r****r} : A^{**} \times X^{**} \longrightarrow X^{**}$ and $\pi_2^{r****r} : X^{**} \times A^{**} \longrightarrow X^{**}$. If we continue dualizing we shall reach $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$ and $(\pi_2^{r*****r}, X^{***}, \pi_1^{r****r*})$ are the dual Banach $(A^{**}, \Box)$-module and Banach $(A^{**}, \Diamond)$-module of $(\pi_1^{***}, X^{**}, \pi_2^{***})$ and $(\pi_1^{r****r}, X^{**}, \pi_2^{r****r})$, respectively (see [15]). In [8], Eshaghi Gordji and Fillali show that if a Banach algebra $A$ has a bounded left (or right) approximate identity, then the left (or right) module action of $A$ on $A^*$ is Arens regular if and only if $A$ is reflexive.

We commence with the following definition for bounded tri-linear mapping.

**Definition 2.1.** Let $X$ be a Banach space, $A$ be a Banach algebra and $\Omega_1 : A \times A \times X \longrightarrow X$ be a bounded tri-linear map. Then the pair $(\Omega_1, X)$ is said to be a left Banach $A-$module when

$$\Omega_1(\pi(a,b), \pi(c,d), x) = \Omega_1(a, b, \Omega_1(c, d, x)),$$

for each $a, b, c, d \in A$ and $x \in X$. A right Banach $A-$module can be defined similarly. Let $\Omega_2 : X \times A \times A \longrightarrow X$ be a bounded tri-linear map. Then the pair $(X, \Omega_2)$ is said to be a right Banach $A-$module when

$$\Omega_2(x, \pi(a,b), \pi(c,d)) = \Omega_2(\Omega_2(x, a, b), c, d).$$

A triple $(\Omega_1, X, \Omega_2)$ is said to be a Banach $A-$module when $(\Omega_1, X)$ and $(X, \Omega_2)$ are left and right Banach $A-$modules respectively, also

$$\Omega_2(\Omega_1(a, b, x), c, d) = \Omega_1(a, b, \Omega_2(x, c, d)).$$

**Lemma 2.1.** *If $(\Omega_1, X, \Omega_2)$ is a Banach $A-$module, then $(\Omega_2^{r*r}, X^*, \Omega_1^*)$ is a Banach $A-$module.*

*Proof.* Since the pair $(X, \Omega_2)$ is a right Banach $A-$module, thus for every $a, b, c, d \in A$, $x \in X$ and $x^* \in X^*$ we have

$$\begin{aligned}
&\langle \Omega_2^{r*r}(\pi(a,b), \pi(c,d), x^*), x\rangle = \langle \Omega_2^{r*}(x^*, \pi(c,d), \pi(a,b)), x\rangle \\
&= \langle x^*, \Omega_2^r(\pi(c,d), \pi(a,b), x)\rangle = \langle x^*, \Omega_2(x, \pi(a,b), \pi(c,d))\rangle \\
&= \langle x^*, \Omega_2(\Omega_2(x,a,b), c, d)\rangle = \langle x^*, \Omega_2^r(d, c, \Omega_2(x,a,b))\rangle \\
&= \langle \Omega_2^{r*}(x^*, d, c), \Omega_2(x,a,b)\rangle = \langle \Omega_2^{r*r}(c,d,x^*), \Omega_2^r(b,a,x)\rangle \\
&= \langle \Omega_2^{r*}(\Omega_2^{r*r}(c,d,x^*), b, a), x\rangle = \langle \Omega_2^{r*r}(a, b, \Omega_2^{r*r}(c,d,x^*)), x\rangle.
\end{aligned}$$

Therefore $\Omega_2^{r*r}(\pi(a,b), \pi(c,d), x^*) = \Omega_2^{r*r}(a, b, \Omega_2^{r*r}(c,d,x^*))$, so $(\Omega_2^{r*r}, X)$ is a left Banach $A-$module. In the other hands, $(\Omega_1, X)$ is a left Banach $A-$module, thus we have

$$\begin{aligned}
&\langle \Omega_1^*(x^*, \pi(a,b), \pi(c,d)), x\rangle = \langle x^*, \Omega_1(\pi(a,b), \pi(c,d), x)\rangle \\
&= \langle x^*, \Omega_1(a, b, \Omega_1(c,d,x))\rangle = \langle \Omega_1^*(x^*, a, b), \Omega_1(c,d,x)\rangle \\
&= \langle \Omega_1^*(\Omega_1^*(x^*, a, b), c, d), x\rangle.
\end{aligned}$$

It follows that $(X, \Omega_1^*)$ is a right Banach $A-$module. Finally, we show that

$$\Omega_1^*(\Omega_2^{r*r}(a,b,x^*), c, d) = \Omega_2^{r*r}(a, b, \Omega_1^*(x^*, c, d)).$$

For every $x \in X$ we have

$$\begin{aligned}
&\langle \Omega_1^*(\Omega_2^{r*r}(a,b,x^*), c, d), x\rangle = \langle \Omega_2^{r*r}(a,b,x^*), \Omega_1(c,d,x)\rangle \\
&= \langle \Omega_2^{r*}(x^*, b, a), \Omega_1(c,d,x)\rangle = \langle x^*, \Omega_2^r(b, a, \Omega_1(c,d,x))\rangle \\
&= \langle x^*, \Omega_2(\Omega_1(c,d,x), a, b)\rangle = \langle x^*, \Omega_1(c, d, \Omega_2(x,a,b))\rangle \\
&= \langle \Omega_1^*(x^*, c, d), \Omega_2(x,a,b)\rangle = \langle \Omega_1^*(x^*, c, d), \Omega_2^r(b,a,x)\rangle \\
&= \langle \Omega_2^{r*}(\Omega_1^*(x^*, c, d), b, a), x\rangle = \langle \Omega_2^{r*r}(a, b, \Omega_1^*(x^*, c, d)), x\rangle.
\end{aligned}$$

Thus $(\Omega_2^{r*r}, X^*, \Omega_1^*)$ is a Banach $A-$module. $\square$

**Theorem 2.1.** *Let $(\Omega_1, X, \Omega_2)$ be a Banach $A-$module, then*

1. *The triple $(\Omega_1^{****}, X^{**}, \Omega_2^{****})$ is a Banach $(A^{**}, \Box, \Box)-$module.*

2. *The triple $(\Omega_1^{r****r}, X^{**}, \Omega_2^{r****r})$ is a Banach $(A^{**}, \Diamond, \Diamond)-$module.*

*Proof.* We prove only (1), the other part has the same argument. Let $\{a_\alpha\}, \{b_\beta\}, \{c_\gamma\}$ and $\{d_\theta\}$ are nets in $A$ which converge to $a^{**}, b^{**}, c^{**}$ and $d^{**} \in A^{**}$ in the

$w^*-$topologies, respectively. Then by lemma 2.1 for every $x^* \in X^*$ we have

$$
\begin{aligned}
&\langle \Omega_1^{****}(a^{**}, b^{**}, \Omega_1^{****}(c^{**}, d^{**}, x^{**})), x^* \rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, \Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*) \rangle \\
&= \lim_\alpha \langle \Omega_1^{***}(b^{**}, \Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*), a_\alpha \rangle \\
&= \lim_\alpha \langle b^{**}, \Omega_1^{**}(\Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*, a_\alpha) \rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{**}(\Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*, a_\alpha), b_\beta \rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{****}(c^{**}, d^{**}, x^{**}), \Omega_1^{*}(x^*, a_\alpha, b_\beta) \rangle \\
&= \lim_\alpha \lim_\beta \langle c^{**}, \Omega_1^{***}(d^{**}, x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta)) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \Omega_1^{***}(d^{**}, x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta)), c_\gamma \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle d^{**}, \Omega_1^{**}(x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta), c_\gamma) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle \Omega_1^{**}(x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta), c_\gamma), d_\tau \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle x^{**}, \Omega_1^{*}(\Omega_1^{*}(x^*, a_\alpha, b_\beta), c_\gamma, d_\tau) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle x^{**}, \Omega_1^{*}(x^*, \pi(a_\alpha, b_\beta), \pi(c_\gamma, d_\tau)) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)), \pi(c_\gamma, d_\tau) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle \pi^*(\Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)), c_\gamma), d_\tau \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle d^{**}, \pi^*(\Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)), c_\gamma) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \pi^{**}(d^{**}, \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta))), c_\gamma \rangle \\
&= \lim_\alpha \lim_\beta \langle c^{**}, \pi^{**}(d^{**}, \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta))) \rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^{***}(c^{**}, d^{**}), \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)) \rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*), \pi(a_\alpha, b_\beta) \rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^*(\Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*), a_\alpha), b_\beta \rangle \\
&= \lim_\alpha \langle b^{**}, \pi^*(\Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*), a_\alpha) \rangle \\
&= \lim_\alpha \langle \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*)), a_\alpha \rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*)) \rangle \\
&= \langle \pi^{***}(a^{**}, b^{**}), \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*) \rangle \\
&= \langle \Omega_1^{****}(\pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**}), x^{**}), x^* \rangle.
\end{aligned}
$$

Thus $(\Omega_1^{****}, X^{**})$ is a left Banach $(A^{**}, \Box, \Box)-$module. Now we show that the pair $(X^{**}, \Omega_2^{****})$ is a right Banach $(A^{**}, \Box, \Box)-$module. Let $\{x_\eta\}$ be a net in $X$ which converge to $x^{**} \in X^{**}$ in the $w^*-$topologies. The pair $(X, \Omega_2)$ is a right Banach

$A-$module, so we have

$$
\begin{aligned}
&\langle \Omega_2^{****}(\Omega_2^{****}(x^{**},a^{**},b^{**}),c^{**},d^{**}),x^*\rangle\\
&=\langle \Omega_2^{****}(x^{**},a^{**},b^{**}),\Omega_2^{***}(c^{**},d^{**},x^*)\rangle\\
&=\langle x^{**},\Omega_2^{***}(a^{**},b^{**},\Omega_2^{***}(c^{**},d^{**},x^*))\rangle\\
&=\lim_{\eta}\langle \Omega_2^{***}(a^{**},b^{**},\Omega_2^{***}(c^{**},d^{**},x^*)),x_\eta\rangle\\
&=\lim_{\eta}\langle a^{**},\Omega_2^{**}(b^{**},\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\langle \Omega_2^{**}(b^{**},\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta),a_\alpha\rangle\\
&=\lim_{\eta}\lim_{\alpha}\langle b^{**},\Omega_2^{*}(\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta,a_\alpha)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle \Omega_2^{*}(\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta,a_\alpha),b_\beta\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle \Omega_2^{***}(c^{**},d^{**},x^*),\Omega_2(x_\eta,a_\alpha,b_\beta)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle c^{**},\Omega_2^{**}(d^{**},x^*,\Omega_2(x_\eta,a_\alpha,b_\beta))\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\langle \Omega_2^{**}(d^{**},x^*,\Omega_2(x_\eta,a_\alpha,b_\beta)),c_\gamma\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\langle d^{**},\Omega_2^{*}(x^*,\Omega_2(x_\eta,a_\alpha,b_\beta),c_\gamma)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\lim_{\tau}\langle \Omega_2^{*}(x^*,\Omega_2(x_\eta,a_\alpha,b_\beta),c_\gamma),d_\tau\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\lim_{\tau}\langle x^*,\Omega_2(\Omega_2(x_\eta,a_\alpha,b_\beta),c_\gamma,d_\tau)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\lim_{\tau}\langle x^*,\Omega_2(x_\eta,\pi(a_\alpha,b_\beta),\pi(c_\gamma,d_\tau))\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\lim_{\tau}\langle \Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),\pi(c_\gamma,d_\tau)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\lim_{\tau}\langle \pi^*(\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),c_\gamma),d_\tau\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\langle d^{**},\pi^*(\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),c_\gamma)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\lim_{\gamma}\langle \pi^{**}(d^{**},\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),c_\gamma\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle c^{**},\pi^{**}(d^{**},\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)))\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle \pi^{***}(c^{**},d^{**}),\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta))\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle \Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta),\pi(a_\alpha,b_\beta)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\lim_{\beta}\langle \pi^*(\Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta),a_\alpha),b_\beta\rangle\\
&=\lim_{\eta}\lim_{\alpha}\langle b^{**},\pi^*(\Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta),a_\alpha)\rangle\\
&=\lim_{\eta}\lim_{\alpha}\langle \pi^{**}(b^{**},\Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta)),a_\alpha\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\eta} \langle a^{**}, \pi^{**}(b^{**}, \Omega_2^{**}(\pi^{***}(c^{**}, d^{**}), x^{*}, x_{\eta}))\rangle \\
&= \lim_{\eta} \langle \pi^{***}(a^{**}, b^{**}), \Omega_2^{**}(\pi^{***}(c^{**}, d^{**}), x^{*}, x_{\eta})\rangle \\
&= \lim_{\eta} \langle \Omega_2^{***}(\pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**}), x^{*}), x_{\eta}\rangle \\
&= \langle x^{**}, \Omega_2^{***}(\pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**}), x^{*})\rangle \\
&= \langle \Omega_2^{****}(x^{**}, \pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**})), x^{*}\rangle.
\end{aligned}
$$

Finally, we show that

$$
\Omega_2^{****}(\Omega_1^{****}(a^{**}, b^{**}, x^{**}), c^{**}, d^{**}) = \Omega_1^{****}(a^{**}, b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**})).
$$

Next we have

$$
\begin{aligned}
&\langle \Omega_2^{****}(\Omega_1^{****}(a^{**}, b^{**}, x^{**}), c^{**}, d^{**}), x^{*}\rangle \\
&= \langle \Omega_1^{****}(a^{**}, b^{**}, x^{**}), \Omega_2^{***}(c^{**}, d^{**}, x^{*})\rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^{*}))\rangle \\
&= \lim_{\alpha} \langle \Omega_1^{***}(b^{**}, x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^{*})), a_{\alpha}\rangle \\
&= \lim_{\alpha} \langle b^{**}, \Omega_1^{**}(x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^{*}), a_{\alpha})\rangle \\
&= \lim_{\alpha} \lim_{\beta} \langle \Omega_1^{**}(x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^{*}), a_{\alpha}), b_{\beta}\rangle \\
&= \lim_{\alpha} \lim_{\beta} \langle x^{**}, \Omega_1^{*}(\Omega_2^{***}(c^{**}, d^{**}, x^{*}), a_{\alpha}, b_{\beta})\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \langle \Omega_1^{*}(\Omega_2^{***}(c^{**}, d^{**}, x^{*}), a_{\alpha}, b_{\beta}), x_{\eta}\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \langle \Omega_2^{***}(c^{**}, d^{**}, x^{*}), \Omega_1(a_{\alpha}, b_{\beta}, x_{\eta})\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \langle c^{**}, \Omega_2^{**}(d^{**}, x^{*}, \Omega_1(a_{\alpha}, b_{\beta}, x_{\eta}))\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \lim_{\gamma} \langle \Omega_2^{**}(d^{**}, x^{*}, \Omega_1(a_{\alpha}, b_{\beta}, x_{\eta})), c_{\gamma}\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \lim_{\gamma} \langle d^{**}, \Omega_2^{*}(x^{*}, \Omega_1(a_{\alpha}, b_{\beta}, x_{\eta}), c_{\gamma})\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \lim_{\gamma} \lim_{\tau} \langle \Omega_2^{*}(x^{*}, \Omega_1(a_{\alpha}, b_{\beta}, x_{\eta}), c_{\gamma}), d_{\tau}\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \lim_{\gamma} \lim_{\tau} \langle x^{*}, \Omega_2(\Omega_1(a_{\alpha}, b_{\beta}, x_{\eta}), c_{\gamma}, d_{\tau})\rangle \\
&= \lim_{\alpha} \lim_{\beta} \lim_{\eta} \lim_{\gamma} \lim_{\tau} \langle x^{*}, \Omega_1(a_{\alpha}, b_{\beta}, \Omega_2(x_{\eta}, c_{\gamma}, d_{\tau}))\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\lim_{\gamma}\lim_{\tau}\langle \Omega_1^*(x^*, a_\alpha, b_\beta), \Omega_2(x_\eta, c_\gamma, d_\tau)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\lim_{\gamma}\langle d^{**}, \Omega_2^*(\Omega_1^*(x^*, a_\alpha, b_\beta), x_\eta, c_\gamma)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\lim_{\gamma}\langle \Omega_2^{**}(d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta), x_\eta), c_\gamma\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\langle c^{**}, \Omega_2^{**}(d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta), x_\eta)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\langle \Omega_2^{***}(c^{**}, d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta)), x_\eta\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle x^{**}, \Omega_2^{***}(c^{**}, d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta))\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle \Omega_2^{****}(x^{**}, c^{**}, d^{**}), \Omega_1^*(x^*, a_\alpha, b_\beta)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle \Omega_1^{**}(\Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*, a_\alpha), b_\beta\rangle \\
&= \lim_{\alpha}\langle b^{**}, \Omega_1^{**}(\Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*, a_\alpha)\rangle \\
&= \lim_{\alpha}\langle \Omega_1^{***}(b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*), a_\alpha\rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*)\rangle \\
&= \langle \Omega_1^{****}(a^{**}, b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**})), x^*\rangle.
\end{aligned}
$$

as claimed. $\square$

**Example 2.1.** Let $A$ be a Banach algebra, for any $a, b \in A$ the symbol $[a, b] = ab - ba$ stands for multiplicative commutator of $a$ and $b$. Let $M_{n\times n}(C)$ be the Banach algebra of all $n \times n$ matrices. We define

$$A = \{\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C})|\ u, v \in \mathbb{C}\}, \quad X = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in M_{2\times 2}(\mathbb{C})|\ x, y, z \in \mathbb{C}\}.$$

Now let $\Omega_1 : A \times A \times X \longrightarrow X$ to be the bounded tri-linear map given by

$$\Omega_1(a, b, x) = -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, abx] \quad , \quad (a, b \in A, \quad x \in X).$$

For every $a = \begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} u_3 & v_3 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} u_4 & v_4 \\ 0 & 0 \end{pmatrix} \in A$ and $x =$

$\begin{pmatrix} x_1 & y_1 \\ 0 & z_1 \end{pmatrix} \in X$, we have

$$\Omega_1(\pi(a,b),\pi(c,d),x) = \Omega_1(\begin{pmatrix} u_1u_2 & u_1v_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_3u_4 & u_3v_4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1 & y_1 \\ 0 & z_1 \end{pmatrix})$$
$$= -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_1u_2u_3u_4x_1 & u_1u_2u_3u_4y_1 + u_1u_2u_3v_4z_1 \\ 0 & 0 \end{pmatrix}]$$
$$= \begin{pmatrix} 0 & u_1u_2u_3u_4x_1 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & u_1u_2u_3u_4x_1 \\ 0 & 0 \end{pmatrix}$$
$$= -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u_1u_2u_3u_4x_1 \\ 0 & 0 \end{pmatrix}] = \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u_3u_4x_1 \\ 0 & 0 \end{pmatrix})$$
$$= \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, (-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & u_3u_4x_1 \\ 0 & 0 \end{pmatrix}))$$
$$= \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_3u_4x_1 & u_3u_4y_1 + u_3v_4z_1 \\ 0 & 0 \end{pmatrix}])$$
$$= \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, \Omega_1(\begin{pmatrix} u_3 & v_3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_4 & v_4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1 & y_1 \\ 0 & z_1 \end{pmatrix}))$$
$$= \Omega_1(a,b,\Omega_1(c,d,x)),$$

Therefore $(\Omega_1, X)$ is a left Banach $A-$module.

**Theorem 2.2.** *Let $a,b,c,d \in A$, $x^* \in X^*$, $x^{**} \in X^{**}$ and $b^{**}, c^{**} \in A^{**}$. Then*

1. *If $(\Omega_1, X)$ is a left Banach $A-$module, then*
$$\Omega_1^{***}(b^{**}, \Omega_1^{****}(c,d,x^{**}), x^*) = \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*)),$$
2. *If $(X, \Omega_2)$ is a right Banach $A-$module, then*
$$\Omega_2^{r****r}(x^*, \Omega_2^{r*****r}(x^{**}, a, b), c^{**}) = \pi^{r***}(c^{**}, \Omega_2^{r****r}(x^*, x^{**}, \pi^{***}(a,b)).$$

*Proof.* (1) Since the pair $(\Omega_1, X)$ is a left Banach $A-$module, thus for every $x \in X$ we have

$$\langle \Omega_1^*(x^*, \pi(a,b), \pi(c,d)), x\rangle = \langle x^*, \Omega_1(\pi(a,b), \pi(c,d), x)\rangle$$
$$= \langle x^*, \Omega_1(a,b,\Omega_1(c,d,x))\rangle = \langle \Omega_1^*(x^*,a,b), \Omega_1(c,d,x)\rangle$$
$$= \langle \Omega_1^*(\Omega_1^*(x^*,a,b),c,d), x\rangle.$$

Hence $\Omega_1^*(x^*, \pi(a,b), \pi(c,d)) = \Omega_1^*(\Omega_1^*(x^*,a,b),c,d)$, which implies that

$$\langle \pi^*(\Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*), a), b\rangle = \langle \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*), \pi(a,b)\rangle$$
$$= \langle \pi^{***}(c,d), \Omega_1^{**}(x^{**}, x^*, \pi(a,b))\rangle = \langle c, \pi^{**}(d, \Omega_1^{**}(x^{**}, x^*, \pi(a,b)))\rangle$$
$$= \langle d, \pi^*(\Omega_1^{**}(x^{**}, x^*, \pi(a,b)), c)\rangle = \langle \Omega_1^{**}(x^{**}, x^*, \pi(a,b)), \pi(c,d)\rangle$$
$$= \langle x^{**}, \Omega_1^*(x^*, \pi(a,b), \pi(c,d))\rangle = \langle x^{**}, \Omega_1^*(\Omega_1^*(x^*,a,b),c,d)\rangle$$
$$= \langle \Omega_1^{**}(x^{**}, \Omega_1^*(x^*,a,b),c), d\rangle = \langle \Omega_1^{***}(d, x^{**}, \Omega_1^*(x^*,a,b)), c\rangle$$
$$= \langle \Omega_1^{***}(c,d,x^{**}), \Omega_1^*(x^*,a,b)\rangle = \langle \Omega_1^{**}(\Omega_1^{***}(c,d,x^{**}), x^*, a), b\rangle.$$

Thus $\pi^*(\Omega_1^{***}(\pi^{***}(c,d),x^{**},x^*),a) = \Omega_1^{**}(\Omega_1^{***}(c,d,x^{**}),x^*,a)$. Finally, we have

$$\begin{aligned}\langle\Omega_1^{***}(b^{**},\Omega_1^{****}(c,d,x^{**}),x^*),a\rangle &= \langle b^{**},\Omega_1^{**}(\Omega_1^{****}(c,d,x^{**}),x^*,a\rangle\\ &= \langle b^{**},\pi^*(\Omega_1^{***}(\pi^{***}(c,d),x^{**},x^*),a)\rangle\\ &= \langle\pi^{**}(b^{**},\Omega_1^{***}(\pi^{***}(c,d),x^{**},x^*)),a\rangle.\end{aligned}$$

A similar argument applies for (2). □

## 3. Topological centers of bounded tri-linear maps

In this section, we shall investigate the topological centers of bounded tri-linear maps. The main definition of this section is as follows.

**Definition 3.1.** Let $f : X\times Y\times Z \longrightarrow W$ be a bounded tri-linear map. We define the topological centers of $f$ by

$Z_l^1(f) = \{x^{**} \in X^{**}|\ y^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})\ \textit{is weak}^* - \textit{to} - \textit{weak}^* - \textit{continuous}\}$,

$Z_l^2(f) = \{x^{**} \in X^{**}|\ z^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})\ \textit{is weak}^* - \textit{to} - \textit{weak}^* - \textit{continuous}\}$,

$Z_r^1(f) = \{z^{**} \in Z^{**}|\ x^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})\ \textit{is weak}^* - \textit{to} - \textit{weak}^* - \textit{continuous}\}$,

$Z_r^2(f) = \{z^{**} \in Z^{**}|\ y^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})\ \textit{is weak}^* - \textit{to} - \textit{weak}^* - \textit{continuous}\}$,

$Z_c^1(f) = \{y^{**} \in Y^{**}|\ x^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})\ \textit{is weak}^* - \textit{to} - \textit{weak}^* - \textit{continuous}\}$.

$Z_c^2(f) = \{y^{**} \in Y^{**}|\ z^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})\ \textit{is weak}^* - \textit{to} - \textit{weak}^* - \textit{continuous}\}$.

**Lemma 3.1.** *For a bounded tri-linear map $f : X\times Y\times Z \longrightarrow W$, we have*

1. *The map $f^{****}$ is the extension of $f$ such that $x^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})$ is weak$^*-$weak$^*$ continuous for each $y^{**}\in Y^{**}$ and $z^{**}\in Z^{**}$.*
2. *The map $f^{****}$ is the extension of $f$ such that $y^{**} \longrightarrow f^{****}(x,y^{**},z^{**})$ is weak$^*-$weak$^*$ continuous for each $x\in X$ and $z^{**}\in Z^{**}$.*
3. *The map $f^{****}$ is the extension of $f$ such that $z^{**} \longrightarrow f^{****}(x,y,z^{**})$ is weak$^*-$weak$^*$ continuous for each $x\in X$ and $y\in Y$.*
4. *The map $f^{r*****r}$ is the extension of $f$ such that $z^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})$ is weak$^*-$weak$^*$ continuous for each $x^{**}\in X^{**}$ and $y^{**}\in Y^{**}$.*
5. *The map $f^{r*****r}$ is the extension of $f$ such that $x^{**} \longrightarrow f^{r*****r}(x^{**},y,z)$ is weak$^*-$weak$^*$ continuous for each $y\in Y$ and $z\in Z$.*

6. *The map* $f^{r*****r}$ *is the extension of* $f$ *such that* $y^{**} \longrightarrow f^{r*****r}(x^{**}, y^{**}, z)$ *is weak*$^*-$*weak*$^*$ *continuous for each* $x^{**} \in X^{**}$ *and* $z \in Z$.

*Proof.* See [19] and [20]. $\square$

As immediate consequences, we give the next Theorem.

**Theorem 3.1.** *If* $f : X \times Y \times Z \longrightarrow W$ *is a bounded tri-linear map, then* $X \subseteq Z_l^1(f)$ *and* $Z \subseteq Z_r^2(f)$.

The mapping $f^{****}$ is the extension of $f$ such that $x^{**} \longrightarrow f^{****}(x^{**}, y^{**}, z^{**})$ from $X^{**}$ into $W^{**}$ is weak$^*-$ to $-$ weak$^*$ continuous for every $y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$, hence for first right topological center of $f$ we have

$Z_r^1(f) \supseteq \{z^{**} \in Z^{**} | f^{r*****r}(x^{**}, y^{**}, z^{**}) = f^{****}(x^{**}, y^{**}, z^{**}),\ for\ every\ x^{**} \in X^{**}, y^{**} \in Y^{**}\}$.

The mapping $f^{r*****r}$ is the extension of $f$ such that $z^{**} \longrightarrow f^{r*****r}(x^{**}, y^{**}, z^{**})$ from $Z^{**}$ into $W^{**}$ is weak$^*-$ to $-$ weak$^*$ continuous for every $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$, hence for second left topological center of $f$ we have

$Z_l^2(f) \supseteq \{x^{**} \in X^{**} | f^{r*****r}(x^{**}, y^{**}, z^{**}) = f^{****}(x^{**}, y^{**}, z^{**}),\ for\ every\ y^{**} \in Y^{**}, z^{**} \in Z^{**}\}$.

**Example 3.1.** Let $G$ be a finite locally compact Hausdorff group. Then

$$f : L^1(G) \times L^1(G) \times L^1(G) \longrightarrow L^1(G)$$

defined by $f(k, g, h) = k * g * h$, is regular for every $k, g$ and $h \in L^1(G)$. So $L^1(G) \subseteq Z_r^1(f)$.

**Theorem 3.2.** *Let* $A$ *be a Banach algebra. Then*

1. *If* $(\Omega_1, X)$ *is a left Banach* $A-$*module and* $\Omega_1^{***}, \pi^{***}(A, A)$ *are factors, then* $Z_l^1(\Omega_1) \subseteq Z_l(\pi)$.

2. *If* $(X, \Omega_2)$ *is a right Banach* $A-$*module and* $\Omega_2^{r***r}, \pi^{***}(A, A)$ *are factors, then* $Z_r^2(\Omega_2) \subseteq Z_r(\pi)$.

*Proof.* We prove only (1), the other one has the same argument. Let $a^{**} \in Z_l^1(\Omega_1)$, we show that $a^{**} \in Z_l(\pi)$. Let $\{b_\alpha^{**}\}$ be a net in $A^{**}$ which converges to $b^{**} \in A^{**}$ in the $w^*-$topologies. We must show that $\pi^{***}(a^{**}, b_\alpha^{**})$ converges to $\pi^{***}(a^{**}, b^{**})$ in the $w^*-$topologies. Let $a^* \in A^*$, since $\Omega_1^{***}$ factors, so there exists $x^* \in X^*, x^{**} \in X^{**}$ and $c^{**} \in A^{**}$ such that $a^* = \Omega_1^{***}(c^{**}, x^{**}, x^*)$. In the other hands $\pi^{***}(A, A)$ factors, thus there exists $c, d \in A$ such that $\pi^{***}(c, d) = c^{**}$. Because $a^{**} \in Z_l^1(\Omega_1)$ thus $\Omega_1^{****}(a^{**}, b_\alpha^{**}, x^{**})$ converges to $\Omega_1^{****}(a^{**}, b^{**}, x^{**})$ in the $w^*-$topologies.

In partiqular $\Omega_1^{****}(a^{**}, b_\alpha^{**}, \Omega_1^{****}(c,d,x^{**}))$ converges to $\Omega_1^{****}(a^{**}, b^{**}, \Omega_1^{****}(c, d, x^{**}))$ in the $w^*-$topologies. Now by Theorem 2.2, we have

$$\begin{aligned}
\lim_\alpha \langle \pi^{***}(a^{**}, b_\alpha^{**}), a^*\rangle &= \lim_\alpha \langle \pi^{***}(a^{**}, b_\alpha^{**}), \Omega_1^{***}(c^{**}, x^{**}, x^*)\rangle \\
&= \lim_\alpha \langle \pi^{***}(a^{**}, b_\alpha^{**}), \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*)\rangle \\
&= \lim_\alpha \langle a^{**}, \pi^{**}(b_\alpha^{**}, \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*))\rangle \\
&= \lim_\alpha \langle a^{**}, \Omega_1^{***}(b_\alpha^{**}, \Omega_1^{****}(c,d,x^{**}), x^*)\rangle \\
&= \lim_\alpha \langle \Omega_1^{****}(a^{**}, b_\alpha^{**}, \Omega_1^{****}(c,d,x^{**}), x^*\rangle \\
&= \langle \Omega_1^{****}(a^{**}, b^{**}, \Omega_1^{****}(c,d,x^{**}), x^*\rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, \Omega_1^{****}(c,d,x^{**}), x^*)\rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*))\rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, \Omega_1^{***}(c^{**}, x^{**}, x^*))\rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, a^*)\rangle \\
&= \langle \pi^{***}(a^{**}, b^{**}), a^*\rangle.
\end{aligned}$$

Therefore $\pi^{***}(a^{**}, b_\alpha^{**})$ converges to $\pi^{***}(a^{**}, b^{**})$ in the $w^*-$topologies, as required. □

**Theorem 3.3.** *Let $A$ be a Banach algebra and $\Omega : A \times A \times A \longrightarrow A$ be a bounded tri-linear mapping. Then for every $a \in A, a^* \in A^*$ and $a^{**} \in A^{**}$,*

1. *If $A$ has a bounded right approximate identity and bounded linear map $T : A^* \longrightarrow A^*$ given by $T(a^*) = \pi^{**}(a^{**}, a^*)$ is weakly compactenss, then $\Omega$ is regular.*
2. *If $A$ has a bounded left approximate identity and bounded linear map $T : A \longrightarrow A^*$ given by $T(a) = \pi^{r*r*}(a^{**}, a)$ is weakly compactenss, then $\Omega$ is regular.*

*Proof.* We only prove (1). Let $T$ be weakly compact, then $T^{**}(A^{***}) \subseteq A^*$. On the other hand, a direct verification reveals that $T^{**}(A^{***}) = \pi^{*****}(A^{**}, A^{***})$. Thus $\pi^{*****}(A^*, A^{***}) \subseteq A^*$. Now let $a^{**}, b^{**} \in A^{**}$, $a^{***} \in A^{***}$ and let $\{a_\alpha\}$, $\{a_\beta^*\}$ be nets in $A$ and $A^*$ which convergence to $a^{**}$, $a^{***}$ in the $w^*-$topologies, respectively. Then we have

$$\begin{aligned}
\langle \pi^{*r****r}(a^{***}, a^{**}), b^{**}\rangle &= \langle \pi^{*r****}(a^{**}, a^{***}), b^{**}\rangle = \langle a^{**}, \pi^{*r**}(a^{***}, b^{**})\rangle \\
&= \lim_\alpha \langle \pi^{*r**}(a^{***}, b^{**}), a_\alpha\rangle = \lim_\alpha \langle a^{***}, \pi^{*r*}(b^{**}, a_\alpha)\rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^{*r*}(b^{**}, a_\alpha), a_\beta^*\rangle = \lim_\alpha \lim_\beta \langle b^{**}, \pi^{*r}(a_\alpha, a_\beta^*)\rangle \\
&= \lim_\alpha \lim_\beta \langle b^{**}, \pi^*(a_\beta^*, a_\alpha)\rangle = \lim_\alpha \lim_\beta \langle \pi^{**}(b^{**}, a_\beta^*), a_\alpha\rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^{***}(a_\alpha, b^{**}), a_\beta^*\rangle = \lim_\alpha \langle a^{***}, \pi^{***}(a_\alpha, b^{**})\rangle \\
&= \lim_\alpha \langle \pi^{****}(a^{***}, a_\alpha), b^{**}\rangle = \lim_\alpha \langle \pi^{*****}(b^{**}, a^{***}), a_\alpha\rangle \\
&= \langle a^{**}, \pi^{*****}(b^{**}, a^{***})\rangle = \langle \pi^{****}(a^{***}, a^{**}), b^{**}\rangle.
\end{aligned}$$

Therefore $\pi^*$ is Arens regular. It follows that $A$ is reflexive, see [8, Theorem 2.1]. Thus $\Omega$ is regular. $\square$

## 4. Factors of bounded tri-linear mapping

We commence with the following definition.

**Definition 4.1.** Let $X, Y, Z, S_1, S_2$ and $S_3$ be normed spaces, $f : X \times Y \times Z \longrightarrow W$ and $g : S_1 \times S_2 \times S_3 \longrightarrow W$ be bounded tri-linear mappings. Then we say that $f$ factors through $g$ by bounded linear mappings $h_1 : X \longrightarrow S_1$, $h_2 : Y \longrightarrow S_2$ and $h_3 : Z \longrightarrow S_3$, if $f(x, y, z) = g(h_1(x), h_2(y), h_3(z))$.

The following theorem gives some necessary and sufficient conditions under which for factorization of the first and second extension of a bouneded tri-linear mappings.

**Theorem 4.1.** *Let $f : X \times Y \times Z \longrightarrow W$ and $g : S_1 \times S_2 \times S_3 \longrightarrow W$ be bounded tri-linear mapping. Then*

1. *The map $f$ factors through $g$ if and only if $f^{****}$ factors through $g^{****}$,*
2. *The map $f$ factors through $g$ if and only if $f^{r****r}$ factors through $g^{r****r}$.*

*Proof.* (1) Let $f$ factor through $g$ by bounded linear mappings $h_1 : X \longrightarrow S_1, h_2 : Y \longrightarrow S_2$ and $h_3 : Z \longrightarrow S_3$, then $f(x, y, z) = g(h_1(x), h_2(y), h_3(z))$ for every $x \in X, y \in Y$ and $z \in Z$. Let $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ be nets in $X, Y$ and $Z$ which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the $w^*-$topologies, respectively. Then for every $w^* \in W^*$ we have

$$\begin{aligned}
\langle f^{****}(x^{**}, y^{**}, z^{**}), w^* \rangle &= \lim_\alpha \lim_\beta \lim_\gamma \langle w^*, f(x_\alpha, y_\beta, z_\gamma) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle w^*, g(h_1(x_\alpha), h_2(y_\beta), h_3(z_\gamma)) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle g^*(w^*, h_1(x_\alpha), h_2(y_\beta)), h_3(z_\gamma) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle h_3^*(g^*(w^*, h_1(x_\alpha), h_2(y_\beta))), z_\gamma \rangle \\
&= \lim_\alpha \lim_\beta \langle z^{**}, h_3^*(g^*(w^*, h_1(x_\alpha), h_2(y_\beta))) \rangle \\
&= \lim_\alpha \lim_\beta \langle h_3^{**}(z^{**}), g^*(w^*, h_1(x_\alpha), h_2(y_\beta)) \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_\alpha \lim_\beta \langle g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha)), h_2(y_\beta)\rangle \\
&= \lim_\alpha \lim_\beta \langle h_2^*(g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha))), y_\beta\rangle \\
&= \lim_\alpha \langle y^{**}, h_2^*(g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha)))\rangle \\
&= \lim_\alpha \langle h_2^{**}(y^{**}), g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha))\rangle \\
&= \lim_\alpha \langle g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*), h_1(x_\alpha)\rangle \\
&= \lim_\alpha \langle h_1^*(g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*)), x_\alpha\rangle \\
&= \langle x^{**}, h_1^*(g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*))\rangle \\
&= \langle h_1^{**}(x^{**}), g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*)\rangle \\
&= \langle g^{****}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})), w^*\rangle.
\end{aligned}
$$

Therefore $f^{****}$ factors through $g^{****}$.

Conversely, suppose that $f^{****}$ factors through $g^{****}$, thus

$$f^{****}(x^{**}, y^{**}, z^{**}) = g^{****}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})),$$

in particular, for $x \in X, y \in Y$ and $z \in Z$ we have

$$f^{****}(x, y, z) = g^{****}(h_1^{**}(x), h_2^{**}(y), h_3^{**}(z)).$$

Then for every $w^* \in W^*$ we have

$$
\begin{aligned}
&\langle w^*, f(x,y,z)\rangle = \langle f^*(w^*, x, y), z\rangle \\
&= \langle f^{**}(z, w^*, x), y\rangle = \langle f^{***}(y, z, w^*), x\rangle \\
&= \langle f^{****}(x, y, z), w^*\rangle = \langle g^{****}(h_1^{**}(x), h_2^{**}(y), h_3^{**}(z)), w^*\rangle \\
&= \langle h_1^{**}(x), g^{***}(h_2^{**}(y), h_3^{**}(z), w^*)\rangle = \langle x, h_1^*(g^{***}(h_2^{**}(y), h_3^{**}(z), w^*))\rangle \\
&= \langle g^{***}(h_2^{**}(y), h_3^{**}(z), w^*), h_1(x)\rangle = \langle h_2^{**}(y), g^{**}(h_3^{**}(z), w^*, h_1(x))\rangle \\
&= \langle y, h_2^*(g^{**}(h_3^{**}(z), w^*, h_1(x)))\rangle = \langle g^{**}(h_3^{**}(z), w^*, h_1(x)), h_2(y)\rangle \\
&= \langle h_3^{**}(z), g^*(w^*, h_1(x), h_2(y))\rangle = \langle z, h_3^*(g^*(w^*, h_1(x), h_2(y)))\rangle \\
&= \langle g^*(w^*, h_1(x), h_2(y)), h_3(z)\rangle = \langle w^*, g(h_1(x), h_2(y)), h_3(z))\rangle.
\end{aligned}
$$

It follows that $f$ factors through $g$ and proof follows.

(2) The proof is similar to (1). □

**Corollary 4.1.** *Let $f : X \times Y \times Z \longrightarrow W$ and $g : S_1 \times S_2 \times S_3 \longrightarrow W$ be bounded tri-linear map and let $f$ factors through $g$. If $g$ is regular then $f$ is also regular.*

*Proof.* Let $g$ be regular then $g^{****} = g^{r****r}$. Since the $f$ factors through $g$ then for every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ we have

$$
\begin{aligned}
f^{****}(x^{**}, y^{**}, z^{**}) &= g^{****}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})) \\
&= g^{r****r}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})) \\
&= f^{r****r}(x^{**}, y^{**}, z^{**}).
\end{aligned}
$$

Therefore $f^{****} = f^{r****r}$, as claimed. □

## 5. Approximate identity and Factorization properties

Let $X$ be a Banach space, $A$ and $B$ be Banach algebras with bounded left approximate identitis $\{e_\alpha\}$ and $\{e_\beta\}$, respectively. Then a bounded tri-linear mapping $K_1 : A \times B \times X \longrightarrow X$ is said to be left approximately unital if

$$w^* - \lim_\beta w^* - \lim_\alpha K_1(e_\alpha, e_\beta, x) = x,$$

and $K_1$ is said left unital if there exists $e_1 \in A$ and $e_2 \in B$ such that $K_1(e_1, e_2, x) = x$, for every $x \in X$. Similarly, bounded tri-linear mapping $K_2 : X \times B \times A \longrightarrow X$ is said to be right approximately unital if

$$w^* - \lim_\beta w^* - \lim_\alpha K_1(x, e_\beta, e_\alpha) = x,$$

and $K_2$ is also said to be right unital if $K_2(x, e_2, e_1) = x$.

**Lemma 5.1.** *Let $X$ be a Banach space, $A$ and $B$ be Banach algebras. Then bounded tri-linear mapping*

1. *$K_1 : A \times B \times X \longrightarrow X$ is left approximately unital if and only if $K_1^{r*****r} : A^{**} \times B^{**} \times X^{**} \longrightarrow X^{**}$ is left unital.*
2. *$K_2 : X \times B \times A \longrightarrow X$ is right approximately unital if and only if $K_2^{****} : X^{**} \times B^{**} \times A^{**} \longrightarrow X^{**}$ is right unital.*

*Proof.* We prove only (1), the other part has the same argument. Let $K_1$ be a left approximately unital. Thus there exists bounded left approximate identitys $\{e_\alpha\} \subseteq A$ and $\{e_\beta\} \subseteq B$ such that

$$w^* - \lim_\beta w^* - \lim_\alpha K_1(e_\alpha, e_\beta, x) = x,$$

for every $x \in X$. Let $\{e_\alpha\}$ and $\{e_\beta\}$ converge to $e_1^{**} \in A^{**}$ and $e_2^{**} \in B^{**}$ in the $w^*-$topologies, respectively. On the other hand, for every $x^{**} \in X^{**}$, let $\{x_\gamma\} \subseteq X$ converge to $x^{**}$ in the $w^*-$topologies, then we have

$$\begin{aligned}
&\langle K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}), x^*\rangle = \langle K_1^{r*****}(x^{**}, e_2^{**}, e_1^{**}), x^*\rangle \\
&= \langle x^{**}, K_1^{r****}(e_2^{**}, e_1^{**}, x^*)\rangle = \lim_\gamma \langle K_1^{r****}(e_2^{**}, e_1^{**}, x^*), x_\gamma\rangle \\
&= \lim_\gamma \langle e_2^{**}, K_1^{r**}(e_1^{**}, x^*, x_\gamma)\rangle = \lim_\gamma \lim_\beta \langle K_1^{r**}(e_1^{**}, x^*, x_\gamma), e_\beta\rangle \\
&= \lim_\gamma \lim_\beta \langle e_1^{**}, K_1^{r*}(x^*, x_\gamma, e_\beta)\rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle K_1^{r*}(x^*, x_\gamma, e_\beta), e_\alpha\rangle \\
&= \lim_\gamma \lim_\beta \lim_\alpha \langle x^*, K_1^{r}(x_\gamma, e_\beta, e_\alpha)\rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle x^*, K_1(e_\alpha, e_\beta, x_\gamma)\rangle \\
&= \lim_\gamma \langle x^*, x_\gamma\rangle = \langle x^{**}, x^*\rangle.
\end{aligned}$$

Therefore $K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = x^{**}$. It follows that $K_1^{r*****r}$ is left unital.

Conversely, suppose that $K_1^{r*****r}$ is left unital. So there exists $e_1^{**} \in A^{**}$ and $e_2^{**} \in b^{**}$ such that $K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = x^{**}$ for every $x^{**} \in X^{**}$. Now let $\{e_\alpha\}, \{e_\beta\}$ and $\{x_\gamma\}$ be nets in $A, B$ and $X$ converging to $e_1^{**}, e_2^{**}$ and $x^{**}$ in the $w^*-$topologies, respectively. Thus

$$\begin{aligned} w^* - \lim_\gamma w^* - \lim_\beta w^* - \lim_\alpha K_1(e_\alpha, e_\beta, x_\gamma) &= K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) \\ &= x^{**} = w^* - \lim_\gamma x_\gamma. \end{aligned}$$

Therefore $K_1$ is left approximately unital and proof follows. □

**Remark 5.1.** It should be remarked that in contrast to the situation occurring for $K_1^{r*****r}$ and $K_2^{****}$ in the above lemma, $K_1^{****}$ and $K_2^{r*****r}$ are not necessarily left and right unital respectively, in general.

**Theorem 5.1.** *Suppose $X, S$ are Banach spaces and $A, B$ are Banach algebras.*

1. *Let $K_1 : A \times B \times X \longrightarrow X$ be left approximately unital and factors through $g_r : A \times B \times S \longrightarrow X$ from rigth by $h : X \longrightarrow S$. If $h$ is weakly compactenss, then $X$ is reflexive.*

2. *Let $K_2 : X \times B \times A \longrightarrow X$ be right approximately unital and factors through $g_l : S \times B \times A \longrightarrow X$ from left by $h : X \longrightarrow S$. If $h$ is weakly compactenss, then $X$ is reflexive.*

*Proof.* We only give the proof for (1). Since $K_1$ is left approximately unital, there exists $e_1^{**} \in A^{**}$ and $e_2^{**} \in B^{**}$ such that

$$K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = x^{**},$$

for every $x^{**} \in X^{**}$. On the other hand, the bounded tri-linear mapping $K_1$ factors through $g_r$ from right, so by Theorem 4.1, $K_1^{r*****r}$ factors through $g_r^{r*****r}$ from right. Thus

$$K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = g_r^{r*****r}(e_1^{**}, e_2^{**}, h_3^{**}(x^{**})).$$

Then for every $x^{***} \in X^{***}$ we have

$$\begin{aligned} \langle x^{***}, x^{**} \rangle &= \langle x^{***}, K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) \rangle \\ &= \langle x^{***}, g_r^{r*****r}(e_1^{**}, e_2^{**}, h^{**}(x^{**})) \rangle \\ &= \langle g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**}), h^{**}(x^{**}) \rangle \\ &= \langle h^{***}(g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**})), x^{**} \rangle. \end{aligned}$$

Therefore $x^{***} = h^{***}(g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**}))$. The weak compactness of $h$ implies that $h^{***}(S^{***}) \subseteq X^*$. In particular $h^{***}(g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**})) \subseteq X^*$, that is, $X^*$ is reflexive. So $X$ is reflexive. □

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## REFERENCES


1. R. Arens: *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc, **2** (1951) 839–848.
2. N. Arikan: *A simple condition ensuring the Arens regularity of bilinear mappings*, Proc. Amer. Math. Soc, **84** (1982) 525–532.
3. S. Barootkoob, S. Mohammadzadeh and H. R. E. Vishki: *Topological centers of certain Banach module actions*, Bull. Iranian. Math. Soc, **35** (2) (2009), 25–36.
4. P. Civin and B. Yood: *The second conjugate space of a Banach algebra as an algebra*, Pacific. J. Math, **11** (3) (1961), 847–870.
5. H. G. Dales: *Banach algebras and automatic continuity*, London. Math. Soc. Monographs, **24**, Clarendon Press, Oxford (2000).
6. H. G. Dales and A. T. Lau: *The second duals of Beurling algebras*,Mem. Amer. Math. Soc, **177**(2005) No. 836.
7. H. G. Dales, A. T. Lau and D. Strauss: *Banach algebras on semigroups and on their compactifications*, Mem. Amer. Math. Soc, **205** (2010), No. 966.
8. M. Eshaghi Gordji and M. Filali: *Arens regularity of module actions*, Studia Math, **181** (3) (2007), 237–254.
9. G. B. Folland: *A course in abstract harmonic analysis*, 1-276, Crc Press, (1995).
10. K. Haghnejad Azar: *Arens regularity and factorization property*, J. Sci. Kharazmi University, **13** (2) (2013), 321–336.
11. K. Haghnejad Azar and A. Riazi: *Arens regularity of bilinear forms and unital Banach module space*, Bull. Iranian. Math. Soc, **40** (2) (2014), 505–520.
12. K. Haghnejad Azar and A. Riazi: *Topological centers of the n-th dual of module actions*, Bull. Iranian. Math. Soc, **38** (1) (2012), 1–16.
13. A. A. Khosravi, H. R. E. Vishki and A. M. Peralta: *Aron–Berner extensions of triple maps with application to the bidual of Jordan Banach triple systems*, Linear Algebra Appl. **580** (2019), 436–463 .
14. A. T. Lau and A. Űlger: *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc, **348** (3) (1996), 1191–1212.
15. S. Mohamadzadeh and H. R. E. Vishki: *Arens regularity of module actions and the second adjoint of a drivation*, Bull. Austral. Mat. Soc, **77** (2008), 465–476.
16. M. Neufang: *On a conjecture by Ghahramani-Lau and related problems concerning topological centres*, J. Funct. Anal, **224** (1) (2005), 217–229
17. A. Űlger: *Weakly compact bilinear forms and Arens regularity*, Proc. Amer. Math. Soc, **101** (4) (1987), 697–704.
18. A. Sheikhali, A. Sheikhali and N. Akhlaghi: *Arens regularity of Banach module actions and the strongly irregular property*, J. Math. Computer. Sci, **13** (1) (2014), 41–46.
19. A. Sheikhali, K. Haghnejad Azar and A. Ebadian: *Close-to-regularity of bounded tri-linear maps* , Global Analysis and Discrete Mathematics, **6** (1) (2021), 33–39.
20. A. Sheikhali, A. Ebadian and K. Haghnejad Azar: *Regularity of bounded tri-linear maps and the fourth adjoint of a tri-derivation*, Global Analysis and Discrete Mathematics, **5** (1) (2020), 51–65.

21. N. J. Young: *The irregularity of multiplication in group algebras*, Quart. J. Math. Oxford, **24** (1973), 59–62.

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# EKELAND'S VARIATIONAL PRINCIPLE IN $S^{JS}$-METRIC SPACES

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**Abstract.** We prove Ekeland's variational principle in $S^{JS}$ - metric spaces. A generalization of Caristi fixed point theorem on $S^{JS}$ - metric spaces is obtained as a consequence.
**Keywords**:Ekeland's variational principle; $S^{JS}$- metric space; fixed point

## 1. Introduction

In his classic paper Ekeland [7] proved a theorem (Ekeland's variational principle) that asserts that there exists nearly optimal solutions to some optimization problems. Ekeland's variational principle can be applied when the lower level set of a minimization problems is not compact, so that the Bolzano–Weierstrass theorem cannot be used. Ekeland's principle relies on Cantor intersection theorem and axiom of choice. Ekeland's principle also leads to an elegant proof of the famous Caristi fixed point theorem [5]. For further generalizations and applications of Ekeland's variational principle we refere to [2, 8, 9, 11] and their references. Recently Beg et al. [1, 12, 13] introduced a very general notion of $S^{JS}$ - metric spaces (see preliminaries) which does not satisfy the triangle inequality and symmetry, and obtained several interesting results with examples. In fact $b$ - metric spaces [6], $S_b$- metric spaces [14], $JS$- metric spaces [10], and partial metric spaces [4] are special cases

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of $S^{JS}$ - metric spaces. The aim of this paper is to prove a variant of Ekeland's variational principle in $S^{JS}$ - metric spaces and then derive Caristi fixed point theorem as an application. The results above generalize/extend several results from the existing literature.

## 2. Preliminaries

In this section, we first give the notion of $S^{JS}$- metric space $(X, J)$, due to [1], some notations and terminology and a lemma to use in next section.

Let $X$ be a nonempty set and $J : X^3 \to [0, \infty]$ be a function. We define the set

$$S(J, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} J(x, x, x_n) = 0\},$$

for all $x \in X$. If J satisfies

$(i)$ $J(x, y, z) = 0$ implies $x = y = z$ for any $x, y, z \in X$;

$(ii)$ there exists some $s > 0$ such that for any $(x, y, z) \in X^3$ and $\{z_n\} \in S(J, X, z)$, we have

$$J(x, y, z) \leq s \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n)),$$

then the pair $(X, J)$ is called an $S^{JS}$- metric space (with coefficient $s$). Several known examples of $S^{JS}$- metric spaces are given in [1] and [13], we give another examples of $S^{JS}$- metric spaces in the below.

**Example 2.1.** Let $X = \overline{\mathbb{R}}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = \exp(|x|) + \exp(|y|) + \exp(|z|) - 3$ for all $x, y, z \in X$, then clearly $(J_1)$ is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. For any $\{z_n\} \in S(J, X, 0)$, we see that

$$J(x, y, 0) \leq h \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n)),$$

where $h \geqslant \frac{1}{2}$, for all $x, y \in X$. Then condition $(J_2)$ is also satisfied. So $J$ is an $S^{JS}$-metric. It is a non-symmetric $S^{JS}$-metric space.

**Example 2.2.** Let $X = \overline{\mathbb{R}}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x-y|+|y|+2|z|$ for all $x, y, z \in X$, then clearly $(J_1)$ is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. If $z = 0$ then for any sequence $\{z_n\} \in S(J, X, 0)$, we get

$$J(x, y, 0) = |x - y| + |y| \leqslant |x| + 2|y| \leqslant 2(|x| + |y|) = 2 \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n)),$$

for all $x, y \in X$. Therefore, the condition $(J_2)$ is satisfied and $J$ is an $S^{JS}$-metric on $X$. It is a non-symmetric $S^{JS}$-metric space.

In an $S^{JS}$- metric space $(X, J)$, a sequence $\{x_n\} \subset X$ is said to be convergent to an element $x \in X$ if $\{x_n\} \in S(J, X, x)$. A sequence $\{x_n\} \subset X$ is said to be Cauchy if $\lim_{n,m\to\infty} J(x_n, x_n, x_m) = 0$.

Space $(X, J)$ is said to be complete if every Cauchy sequence in $X$ is convergent. Open ball of center $x \in X$ and radius $r > 0$ in $X$ is defined as follows:

$$B_J(x, r) = \{y \in X : J(x, x, y) < r\}.$$

A nonempty subset $U$ of $X$, with the property that for any $x \in U$ there exists $r > 0$ such that $B_J(x, r) \subset U$ is called an open set. A subset $B$ of $X$ is called closed if $B^c$ is open.

**Lemma 2.1.** *[1][Cantor's Intersection Theorem] Every complete $S^{JS}$- metric space has Cantor's intersection property.*

## 3. Ekeland's variational principle

**Definition 3.1.** In an $S^{JS}$-metric space $(X, J)$, a mapping $\psi : X \to \overline{\mathbb{R}}$ is said to be lower semi-continuous at $t_0 \in X$ if for any $\epsilon > 0$ there exits some $\delta_\epsilon > 0$ such that $\psi(t_0) < \psi(t) + \epsilon$ for all $t \in B_J(t_0, \delta_\epsilon)$.

**Definition 3.2.** Let $(X, J)$ be an $S^{JS}$-metric space and $\{A_n\}$ be a decreasing sequence of nonempty subsets of $X$. Then $\{A_n\}$ is said to have vanishing diameter property ($vd-$property) if for each $i \in \mathbb{N}$ there exists some fixed $a_i \in A_i$ such that $J(x, x, a_i) \leq J(a_i, a_i, a_i) + r_i$ for all $x \in A_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

**Definition 3.3.** An $S^{JS}$-metric space $(X, J)$ is said to have vanishing diameter property if for any decreasing sequence of nonempty subsets $\{A_n\}$ of $X$ with $vd-$property we have $diam(A_n) \to 0$ as $n \to \infty$.

We now establish Ekeland's variational principle in an $S^{JS}$-metric space. Let us denote $d_J(x, y) = J(x, x, y)$ for all $x, y \in X$.

**Theorem 3.1.** *Let $(X, J)$ be a complete $S^{JS}$-metric space with coefficient $s > 1$, such that $d_J$ is continuous in both variables, $\sup\{J(x, x, x) : x \in X\} < \infty$ and $X$ has vanishing diameter property. Now let, $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \epsilon \tag{3.1}$$

*there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:*

*(i) $x_n \to x_\epsilon$ as $n \to \infty$,*

*(ii) For all $n \geq 1$,*

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) \leq \frac{\epsilon}{2^n}$$

*(iii) For all $x \neq x_\epsilon$,*

$$f(x)+\sum_{n=0}^{\infty}\frac{1}{s^n}J(x,x,x_n) > f(x_\epsilon)+\sum_{n=0}^{\infty}\frac{1}{s^n}J(x_\epsilon,x_\epsilon,x_n)$$

*(iv)*

$$\begin{aligned} f(x_\epsilon)+\sum_{n=0}^{\infty}\frac{1}{s^n}J(x_\epsilon,x_\epsilon,x_n) &\leq f(x_0)+\sum_{n=0}^{\infty}\frac{1}{s^n}J(x_n,x_n,x_n) \\ &\leq \inf_{x\in X} f(x)+\epsilon+\sum_{n=0}^{\infty}\frac{1}{s^n}J(x_n,x_n,x_n). \end{aligned}$$

*Proof.* Consider the set

$$S_f(x_0)=\{x\in X : f(x)+d_J(x,x_0)\leq f(x_0)+d_J(x_0,x_0)\}.$$

Since $x_0 \in S_f(x_0)$ then $S_f(x_0)$ is nonempty. Let $\{z_n\} \subset S_f(x_0)$ be such that $\{z_n\}$ converges to some $z \in X$. Then $f(z_n)+d_J(z_n,x_0) \leq f(x_0)+d_J(x_0,x_0)$ for all $n \in \mathbb{N}$. Now $f$ is lower semi-continuous at $z \in X$, so for any $\epsilon_1 > 0$, $f(z) < f(t) + \frac{\epsilon_1}{2}$ for all $t \in B_J(z,\delta_{\epsilon_1})$ for $\delta_{\epsilon_1} > 0$. Also $\{z_n\}$ converges to some $z$, so there exists $N_1 \geq 1$ such that $z_n \in B_J(z,\delta_{\epsilon_1})$ for all $n \geq N_1$. Therefore $f(z) < f(z_n)+\frac{\epsilon_1}{2}$ for all $n \geq N_1$. Now continuity of $d_J$ implies that $d_J(z_n,x_0) \to d_J(z,x_0)$ as $n \to \infty$. Thus for all $n \geq N_2$

$$d_J(z,x_0)-\frac{\epsilon_1}{2} < d_J(z_n,x_0) < d_J(z,x_0)+\frac{\epsilon_1}{2}.$$

Therefore, for all $n \geq N = \max\{N_1, N_2\}$ we get,

$$\begin{aligned} f(z)+d_J(z,x_0) &< f(z_n)+d_J(z_n,x_0)+\epsilon_1 \forall n \geqslant N \\ &\leq f(x_0)+d_J(x_0,x_0)+\epsilon_1. \end{aligned} \tag{3.2}$$

Since $\epsilon_1 > 0$ is arbitrary, thus $f(z) + d_J(z,x_0) \leq f(x_0) + d_J(x_0,x_0)$. Therefore $z \in S_f(x_0)$. Hence $S_f(x_0)$ is closed. Also for any $y \in S_f(x_0)$ we get

$$\begin{aligned} d_J(y,x_0)-d_J(x_0,x_0) &\leq f(x_0)-f(y) \\ &\leq f(x_0)-\inf_{x\in X} f(x) \leq \epsilon. \end{aligned} \tag{3.3}$$

We choose $x_1 \in S_f(x_0)$ such that $f(x_1)+d_J(x_1,x_0) \leq \inf_{x\in S_f(x_0)}\{f(x)+d_J(x,x_0)\}+\frac{\epsilon}{2s}$ and let

$$S_f(x_1) = \{x\in X : f(x)+d_J(x,x_0)+\frac{1}{s}d_J(x,x_1) \leq f(x_1)+d_J(x_1,x_0)+\frac{1}{s}d_J(x_1,x_1)\}. \tag{3.4}$$

Thus $x_1 \in S_f(x_1)$ and in a similar way as above we can prove that $S_f(x_1)$ is also closed.

Inductively, we can suppose that $x_{n-1} \in S_f(x_{n-2})$ (for $n > 2$) was already chosen and we consider

$$S_f(x_{n-1}) = \{x \in S_f(x_{n-2}) : f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i) \leq f(x_{n-1}) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x_{n-1}, x_i)\}. \tag{3.5}$$

Let us choose $x_n \in S_f(x_{n-1})$ such that

$$f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x_n, x_i) \leq \inf_{x \in S_f(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i)\} + \frac{\epsilon}{2^n s^n}$$

and we define the set

$$S_f(x_n) = \{x \in S_f(x_{n-1}) : f(x) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x, x_i) \leq f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\}. \tag{3.6}$$

Clearly $x_n \in S_f(x_n)$ and $S_f(x_n)$ is also closed. Now for each $y \in S_f(x_n)$ we get

$$\begin{aligned} \frac{1}{s^n} d_J(y, x_n) &\leq \{f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\} - \{f(y) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(y, x_i)\} \\ &\leq \{f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\} - \inf_{x \in S_f(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i)\} \\ &\leq \frac{1}{s^n} d_J(x_n, x_n) + \frac{\epsilon}{2^n s^n}. \end{aligned} \tag{3.7}$$

Therefore, for any $y \in S_f(x_n)$ we have

$$d_J(y, x_n) - d_J(x_n, x_n) \leq \frac{\epsilon}{2^n} \forall n \in \mathbb{N}.$$

Thus the decreasing sequence of nonempty closed subsets $\{S_f(x_n)\}_{n \geq 0}$ has $vd-$property. Since $X$ has $vd-$ property therefore $diam(S_f(x_n)) \to 0$ as $n \to \infty$. Thus by Cantor's intersection theorem (See Lemma 2.1) we have $\cap_{n=0}^{\infty} S_f(x_n) = \{x_\epsilon\}$.

Now $d_J(x_\epsilon, x_n) \leq diam(S_f(x_n)) \to 0$ as $n \to \infty$ and we have $x_n \to x_\epsilon$ as $n \to \infty$. From (3.7) we see that

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) \leq \frac{\epsilon}{2^n} \forall n \in \mathbb{N}.$$

Now

$$\begin{aligned} f(x_1) + d_J(x_1, x_0) &\leq f(x_0) + d_J(x_0, x_0), \\ f(x_2) + d_J(x_2, x_0) + \frac{1}{s} d_J(x_2, x_1) &\leq f(x_1) + d_J(x_1, x_0) + \frac{1}{s} d_J(x_1, x_1) \end{aligned}$$

$$
\begin{aligned}
&\leq f(x_0) + d_J(x_0,x_0) + \frac{1}{s}d_J(x_1,x_1)\\
&\dots\\
f(x_m) + \sum_{i=0}^{m-1}\frac{1}{s^i}d_J(x_m,x_i) &\leq f(x_0) + \sum_{i=0}^{m-1}\frac{1}{s^i}d_J(x_i,x_i) \forall m > 1.
\end{aligned}
\tag{3.8}
$$

Also $x_\epsilon \in S_f(x_q)$ for all $q \in \mathbb{N}$, therefore

$$
\begin{aligned}
f(x_\epsilon) + \sum_{i=0}^{q}\frac{1}{s^i}d_J(x_\epsilon,x_i) &\leq f(x_q) + \sum_{i=0}^{q}\frac{1}{s^i}d_J(x_q,x_i)\\
&\leq f(x_0) + \sum_{i=0}^{q}\frac{1}{s^i}d_J(x_i,x_i)\forall q \geq 1,
\end{aligned}
\tag{3.9}
$$

which in turn implies that

$$
\begin{aligned}
f(x_\epsilon) + \sum_{i=0}^{\infty}\frac{1}{s^i}d_J(x_\epsilon,x_i) &\leq f(x_0) + \sum_{i=0}^{\infty}\frac{1}{s^i}d_J(x_i,x_i)\\
&\leq \inf_{x\in X} f(x) + \epsilon + \sum_{i=0}^{\infty}\frac{1}{s^i}d_J(x_i,x_i).
\end{aligned}
\tag{3.10}
$$

Moreover for all $x \neq x_\epsilon$, we have $x \notin \cap_{n=0}^{\infty} S_f(x_n)$ and thus there exists $m \in \mathbb{N}$ such that $x \notin S_f(x_m)$. So $x \notin S_f(x_q)$ for all $q \geq m$. Therefore,

$$
\begin{aligned}
f(x) + \sum_{i=0}^{q}\frac{1}{s^i}d_J(x,x_i) &> f(x_q) + \sum_{i=0}^{q}\frac{1}{s^i}d_J(x_q,x_i)\\
&\geq f(x_\epsilon) + \sum_{i=0}^{q}\frac{1}{s^i}d_J(x_\epsilon,x_i)\forall q \geq m.
\end{aligned}
\tag{3.11}
$$

Hence we see that

$$
f(x) + \sum_{i=0}^{\infty}\frac{1}{s^i}d_J(x,x_i) > f(x_\epsilon) + \sum_{i=0}^{\infty}\frac{1}{s^i}d_J(x_\epsilon,x_i).
$$

□

**Example 3.1.** Let us consider $X = (-\infty, +\infty)$ and let $J : X^3 \to [0,\infty]$ be defined as $J(x,y,z) = |x-y|^2 + |y-z|^2$ for all $x,y,z \in X$. Then $(X,J)$ is an $S^{JS}$-metric space for $s = 3$. Here $d_J(x,y) = |x-y|^2$, which is continuous in both the variables and $\sup\{J(x,x,x) : x \in X\} = 0$. Now we show that $X$ has vanishing diameter property.

Let $\{E_n\}$ be a decreasing sequence of nonempty subsets of $X$ such that it has $vd-$property. Then for any $i \in \mathbb{N}$ there exists some fixed $e_i \in E_i$ such that $J(x,x,e_i) = |x-e_i|^2 \leq J(e_i,e_i,e_i) + r_i = r_i$ for all $x \in E_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

Let $x^{(i)}, y^{(i)}, z^{(i)} \in E_i$ be arbitrary. Then

$$\begin{aligned} J(x^{(i)}, y^{(i)}, z^{(i)}) &= |x^{(i)} - y^{(i)}|^2 + |y^{(i)} - z^{(i)}|^2 \\ &\leqslant 2[|x^{(i)} - e_i|^2 + |y^{(i)} - e_i|^2] + 2[|y^{(i)} - e_i|^2 + |z^{(i)} - e_i|^2] \\ &= 2[|x^{(i)} - e_i|^2 + 2|y^{(i)} - e_i|^2] + |z^{(i)} - e_i|^2] \\ &\leqslant 8r_i \to 0 \end{aligned}$$

as $i \to \infty$. This implies $diam(A_i) \leq 8r_i$. Since this is true for all $i \in \mathbb{N}$ we get $diam(A_i) \to 0$ as $r_i \to \infty$. Thus $(X, J)$ has vanishing diameter property.

Let $f : X \to \overline{\mathbb{R}}$ be defined as $f(x) = e^{|x|} + x^2 + 4|x|$ for all $x \in X$. Then $f$ is continuous and lower bounded. Let us take $\epsilon > 0$ as arbitrary and choose $x_0 \in X$ which satisfies $f(x_0) \leq \inf_{x \in X} f(x) + \epsilon$. Now let us consider $x_\epsilon = 0$, if $x_0 = 0$ then we have to choose $x_n = 0$ for all $n \geq 1$ and clearly Theorem 3.1 follows immediately. Now if $x_0 \neq 0$ then we choose $x_n = \sqrt{\frac{\epsilon}{Kr^n}}$, where $K \geq 1$ and $r > 2$ are chosen in such a way that

$$\epsilon \leq \min\{\frac{K(3r-1)}{3r}[f(x_0) - 1], K\}.$$

Then we have

$(i)$ $x_n \to x_\epsilon$ as $n \to \infty$,

$(ii)$ For all $n \geq 1$,

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) = |x_\epsilon - x_n|^2 = \frac{\epsilon}{Kr^n} < \frac{\epsilon}{2^n}$$

$(iii)$ For all $x \neq x_\epsilon$,

$$\begin{aligned} & f(x) + \textstyle\sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) \\ = \quad & e^{|x|} + x^2 + 4|x| + \textstyle\sum_{n=0}^{\infty} \frac{1}{3^n} |x - \sqrt{\frac{\epsilon}{Kr^n}}|^2 \\ = \quad & e^{|x|} + x^2 + 4|x| + \tfrac{3}{2}x^2 - 2\sqrt{\tfrac{\epsilon}{K}} \frac{3r^{\frac{1}{2}}}{3r^{\frac{1}{2}} - 1} x + \frac{\epsilon}{K} \frac{3r}{3r-1} \\ \geqslant \quad & e^{|x|} + x^2 + 4|x| + \tfrac{3}{2}x^2 - 2\frac{3r^{\frac{1}{2}}}{3r^{\frac{1}{2}} - 1} x + \frac{\epsilon}{K} \frac{3r}{3r-1} \\ > \quad & 1 + \frac{\epsilon}{K} \frac{3r}{3r-1} = f(x_\epsilon) + \textstyle\sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n). \end{aligned}$$

$(iv)$

$$\begin{aligned} f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) &= 1 + \frac{\epsilon}{K} \frac{3r}{3r-1} \leqslant f(x_0) \\ &= f(x_0) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n) \\ &\leq \inf_{x \in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n). \end{aligned}$$

Next we have the following consequence of Ekeland's variational principle in $S^{JS}$-metric spaces.

**Corollary 3.1.** *Let $(X, J)$ be a complete $S^{JS}$-metric space with coefficient $s > 1$, such that $d_J$ is continuous in both variables,* $\sup\{J(x, x, x) : x \in X\} < \infty$ *and $X$ has vanishing diameter property. Now let, $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $\epsilon > 0$ there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:*

*(i) $x_n \to x_\epsilon$ as $n \to \infty$,*

*(ii) $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) \geq f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n)$ for every $x \in X$,*

*(iii) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) \leq \inf_{x\in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n)$.*

As an application of Theorem 3.1 we now prove Caristi's fixed point theorem in the context of $S^{JS}$-metric spaces.

**Theorem 3.2.** *Let $(X, J)$ be a complete $S^{JS}$-metric space with coefficient $s > 1$, such that $d_J$ is continuous in both variables,* $\sup\{J(x, x, x) : x \in X\} < \infty$ *and $X$ has vanishing diameter property. Let $T : X \to X$ be an operator for which there exists a lower semi-continuous mapping, proper and lower bounded mapping $f : X \to \overline{\mathbb{R}}$ such that*

$$J(u, u, v) + sJ(u, u, Tu) \geq J(Tu, Tu, v) \tag{3.12}$$

*and*

$$\frac{s^2}{s-1} J(u, u, Tu) \leq f(u) - f(Tu) \forall u, v \in X. \tag{3.13}$$

*Then $T$ has at least one fixed point in $X$.*

*Proof.* Let us assume that for all $x \in X$, $Tx \neq x$. Using Corollary 3.1 for $f$, we obtain that for each $\epsilon > 0$ there exists a sequence $\{x_n\} \subset X$ such that $x_n \to x_\epsilon$ as $n \to \infty$ and

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) \forall x \neq x_\epsilon.$$

If in the above inequality, we put $x = T(x_\epsilon)$ then, since $T(x_\epsilon) \neq x_\epsilon$, we get that

$$\begin{aligned} f(x_\epsilon) - f(Tx_\epsilon) &< \sum_{n=0}^{\infty} \frac{1}{s^n} [d_J(Tx_\epsilon, x_n) - d_J(x_\epsilon, x_n)] \\ &< \sum_{n=0}^{\infty} \frac{1}{s^n} s d_J(x_\epsilon, Tx_\epsilon) (5.15) \\ &= s \sum_{n=0}^{\infty} \frac{1}{s^n} d_J(x_\epsilon, Tx_\epsilon) \\ &= \frac{s^2}{s-1} d_J(x_\epsilon, Tx_\epsilon). \qquad (3.14) \end{aligned}$$

Also from (3.13) we get $\frac{s^2}{s-1}d_J(x_\epsilon, Tx_\epsilon) \leq f(x_\epsilon) - f(Tx_\epsilon)$, a contradiction. Therefore there exists at least one $x^* \in X$ such that $Tx^* = x^*$. □

**Definition 3.4.** [14] Let $X$ be a nonempty set and $s \geq 1$ be a given number. Also let a function $S_b : X^3 \to [0, \infty)$ satisfy the following conditions, for each $x, y, z, w \in X$ :

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S_b(x, y, z) \leq s[S_b(x, x, w) + S_b(y, y, w) + S_b(z, z, w)]$.

The pair $(X, S_b)$ is called an $S_b$-metric space.

Souayah and Mlaiki [14, Theorem 2.4] follows from our Theorem 3.1 as an immediate corollary.

**Corollary 3.2.** *Let $(X, S_b)$ be a complete $S_b$-metric space with coefficient $s > 1$, such that the $S_b$-metric is continuous and $f : X \to \overline{\mathbb{R}}$ is a lower semi-continuous, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \epsilon, \tag{3.15}$$

*there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:*

*(i) $x_n \to x_\epsilon$ as $n \to \infty$,*

*(ii) $S_b(x_\epsilon, x_\epsilon, x_n) \leq \frac{\epsilon}{2^n}$ for all $n \geq 1$,*

*(iii) $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n)$ for every $x \neq x_\epsilon$,*

*(iv) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \leq f(x_0) \leq \inf_{x \in X} f(x) + \epsilon$.*

*Proof.* Let $\{A_n\}$ be a decreasing sequence of nonempty subsets of $X$ such that it has $vd-$property. Then for each $i \in \mathbb{N}$ there exists some fixed $a_i \in A_i$ such that $S_b(x, x, a_i) \leq S_b(a_i, a_i, a_i) + r_i = r_i$ for all $x \in A_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

Let $x^{(i)}, y^{(i)}, z^{(i)} \in A_i$ be arbitrary. Then

$$\begin{aligned} S_b(x^{(i)}, y^{(i)}, z^{(i)}) &\leq s[S_b(x^{(i)}, x^{(i)}, a_i) + S_b(y^{(i)}, y^{(i)}, a_i) + S_b(z^{(i)}, z^{(i)}, a_i)] \\ &\leq 3sr_i. \end{aligned} \tag{3.16}$$

It implies $diam(A_i) \leq 3sr_i$. Since this is true for all $i \in \mathbb{N}$ we get $diam(A_i) \to 0$ as $r_i \to \infty$. Thus $(X, S_b)$ has vanishing diameter property. Therefore all the conditions of Theorem 3.1 are satisfied and the result follows immediately. □

**Corollary 3.3.** *Let $(X, S_b)$ be a complete $S_b$-metric space with coefficient $s > 1$, such that the $S_b$-metric is continuous and let $T : X \to X$ be an operator for which* *there exists a lower semi-continuous, proper and lower bounded mapping* $f : X \to \overline{\mathbb{R}}$*, such that:*

$$S_b(u,u,v) + sS_b(u,u,Tu) \geq S_b(Tu,Tu,v) \tag{3.17}$$

*and*

$$\frac{s^2}{s-1} S_b(u,u,Tu) \leq f(u) - f(Tu) \forall u, v \in X. \tag{3.18}$$

*Then* $T$ *has at least one fixed point in* $X$*.*

*Proof.* Using Theorem 3.2 and Corollary 3.2 we get the required proof. □

**Remark 3.1.** [3, Theorem 2.2] is a particular case of our Theorem 3.1.

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## REFERENCES

1. I. Beg, K. Roy and M. Saha: *$S^{JS}$- metric and topological spaces,*. J. Math. Extension, **15**(4) (2021), 1–16.
2. J. M. Borwein and Q. J. Zhu: *Techniques of Variational Analysis, Springer*, 2005.
3. M. Bota, A. Molnar and C. Varga: *On Ekeland's variation principle in b-metric spaces,* Fixed Point Theory, **12**(2) (2011), 21–28.
4. M. Bukatin and R. Kopperman, S. Mathews and H. Pajoohesh: *Partial metric spaces,* Amer. Math. Mon., **116**(8) (2009), 708–718.
5. J. Caristi: *Fixed point theorems for mappings satisfying inwardness conditions,* Trans. Amer. Math. Soc., **215** (1976), 241–251.
6. S. Czerwik: *Contraction mappings in b- metric spaces,* Acta Math. Inform. Univ. Ostrav., **1** (1993), 5–11.
7. I. Ekeland: *On the variational principle,* J. Math. Anal. Appl., **47** (1974), 324–353
8. A. P. Farajzadeh, S. Plubtieng and A. Hoseinpour: *A generalization of Ekeland's variational principle by using the $\tau$-distance with its applications,* J. Inequalities and Appl., **1** (2017), 1–7.
9. E. Hashemi and R. Saadati: *Ekeland's variational principle and minimization Takahashi's theorem in generalized metric spaces,* Mathematics, **6**(6) (2018), 93.

10. M. Jleli and B. Samet: *A generalized metric space and related fixed point theorems*, Fixed Point Theory and Applications, (2015), doi:10.1186/s13663-015-0312-7.

11. I. Meghea: *Ekeland Variational Principle with Generalizations and Variants*, Old City Publishing, 2009.

12. K. Roy, I. Beg and M. Saha: *Sequentially compact $S^{JS}$- metric spaces,* Commun. Optim. Theory 2020, (2020), Article ID 4.

13. K. Roy, M. Saha and I. Beg: *Fixed point of contractive mappings of integral type over an $S^{JS}$- metric space,* Tamkang J. Math., **52**(2) (2021), 267–280.

14. N. Souayah and N. Mlaiki: *A fixed point theorem in $S_b$-metric spaces,* J. Math. Computer Sci., **16** (2016), 131–139.

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# ON THE GEOMETRIC STRUCTURES OF GENERALIZED $(k, \mu)$-SPACE FORMS

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**Abstract.** In this paper, the geometric structures of generalized $(k, \mu)$-space forms and their quasi-umbilical hypersurface are analyzed. First $\xi$-$Q$ and conformally flat generalized $(k, \mu)$-space form are investigated and shown that a conformally flat generalized $(k, \mu)$-space form is Sasakian. Next, we prove that a generalized $(k, \mu)$-space form satisfying Ricci pseudosymmetry and $Q$-Ricci pseudosymmetry conditions is $\eta$-Einstein. We obtain the condition under which a quasi-umbilical hypersurface of a generalized $(k, \mu)$-space form is a generalized quasi Einstein hypersurface. Also $\xi$-sectional curvature of a quasi-umbilical hypersurface of generalized $(k, \mu)$-space form is obtained. Finally, the results obtained are verified by constructing an example of 3-dimensional generalized $(k, \mu)$-space form.
**Keywords**:$(k, \mu)$-space form, Q curvature, Hypersurface, Sasakian, $\eta$-Einstein.

## 1. Introduction

The curvature tensor $R$ of the Riemannian manifold mostly determines the nature of the manifold and the sectional curvature of the manifold completely determines the curvature tensor $R$. A Riemannian manifold having a constant sectional curvature $c$ is known as real space-form. The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector $X$ orthogonal to $\xi$ is called a $\phi$-sectional curvature. If the $\phi$-sectional curvature of a Sasakian manifold is constant, then it is called Sasakian space form. Alegre et al. [2] introduced the notion of generalized Sasakian

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space forms and gave many examples of it. Throughout the years, many geometers [3, 4, 13, 15, 16, 17] focused on generalized Sasakian space forms under different geometric conditions.

Blair et al. [5] introduced the notion of $(k,\mu)$-contact metric manifolds. Following this, Koufogiorgos [23] introduced and studied $(k,\mu)$ space forms. The $(k,\mu)$ space forms are studied by [1, 14, 23, 30]. Carriazo et al. [8] introduced generalized $(k,\mu)$ space form which generalizes the notion of $(k,\mu)$ space forms. An almost contact metric manifold $(M^{2n+1},\phi,\xi,g,\eta)$ is said to be a generalized $(k,\mu)$ space form if there exists differentiable functions $f_1,f_2,f_3,f_4,f_5,f_6$ on the manifold whose curvature tensor $R$ is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.1}$$

where $R_1,R_2,R_3,R_4,R_5,R_6$ are the following tensors:

$$\begin{aligned}
R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y,\\
R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,\\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,\\
R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,\\
R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,\\
R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,
\end{aligned}$$

for any $X,Y,Z \in \chi(M)$. Here, $h$ is a symmetric tensor given by $2h = \mathcal{L}_\xi\phi$, where $\mathcal{L}$ is Lie derivative. In particular, for $f_4 = f_5 = f_6 = 0$ it reduces to the generalized Sasakian space form [2]. It is obvious that $(k,\mu)$ space form is an example of generalized $(k,\mu)$ space form when

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - k, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1-\mu$$

are constants. In [8], the author studied generalized $(k,\mu)$ space forms in contact metric and Trans-Sasakian manifolds. Carriazo and Molina [9] studied $D_\alpha$-homothetic deformations of generalized $(k,\mu)$-space forms and found that deformed spaces are again generalized $(k,\mu)$-space forms in dimension 3, but not in general. In recent years, many geometers studied generalized $(k,\mu)$-space forms under several conditions [21, 28, 22, 20, 27, 29].

In [26], Mantica and Suh introduced and studied $Q$ curvature tensor. In a $(2n+1)$-dimensional Riemannian manifold $(M,g)$, the $Q$ curvature tensor is given by

$$Q(X,Y)Z = R(X,Y)Z - \frac{v}{2n}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{1.2}$$

for any $X,Y,Z \in \chi(M)$ and $v$ is an arbitrary scalar function on $M$. If $v = \frac{r}{2n+1}$, then $Q$ curvature tensor reduces to concircular curvature tensor [32]. In [13], De and Majhi studied $Q$ curvature tensor in a generalized Sasakian space form.

One of the most important curvature tensors for analyzing the intrinsic properties of Riemannian manifold is the conformal curvature tensor introduced by Yano and Kon [33]. This curvature is invariant under conformal transformation. The conformal curvature $C$ of type (1,3) on a $(2n+1)$-dimensional Riemannian manifold $(M, g), n > 1$, is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)PX \\ -g(X,Z)PY\big] + \frac{r}{2n(2n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \end{aligned} \tag{1.3}$$

where $R, S, P, r$ denote the Riemannian curvature tensor, the Ricci tensor, Ricci-operator and the scalar curvature of the manifold respectively. Kim [25] studied conformally flat generalized Sasakian space forms. De and Majhi [15] studied $\phi$-conformal semisymmetric generalized Sasakian space forms.

Cartan [10] first initiated and completely classified complete simply connected locally symmetric spaces. A Riemannian manifold is said to be locally symmetric if the curvature tensor satisfies $\nabla R = 0$. The notion of local symmetry is weakened by many authors throughout the years. One such notion is pseudosymmetric spaces introduced by Deszcz [19]. It should be noted that pseudosymmetric spaces introduced by Deszcz is different from those introduced by Chaki [11]. In [31], authors obtained the necessary and sufficient condition for a Chaki pseudosymmetric manifold to be Deszcz pseudosymmetric. De and Samui [14] studied Ricci pseudosymmetric $(k, \mu)$-contact space forms and show that it is an $\eta$-Einstein manifold.

The authors in [14], studied quasi-umbilical hypersurface on $(k, \mu)$-space forms. A hypersurface $(\widetilde{M}^{2n+1}, \tilde{g})$ of a Riemannian manifold $M^{2n+1}$ is called quasi-umbilical [12] if its second fundamental tensor has the form

$$H_\rho(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y), \tag{1.4}$$

where $\omega$ is the 1-form, $\alpha, \beta$ are scalars and the vector field corresponding to the 1-form $\omega$ is a unit vector field. Here, the second fundamental tensor $H_\rho$ is defined by $H_\rho(X,Y) = \widetilde{g}(A_\rho, Y)$, where $A$ is (1,1) tensor and $\rho$ is the unit normal vector field and $X, Y$ are tangent vector fields.
A Riemannian manifold is called a generalized quasi-Einstein manifold [18] if its Ricci tensor $S$ satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\lambda(X)\lambda(Y),$$

where $a, b$ and $c$ are non-zero scalars and $\eta, \lambda$ are 1-forms. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

The paper is organized as follows: After preliminaries, $\xi$-$Q$ and conformally flat generalized $(k,\mu)$-space forms are investigated in section 3. Next in section 4, it is shown that $Q$-Ricci pseudosymmetric and Ricci pseudosymmetric generalized $(k,\mu)$-space forms are $\eta$-Einstein under certain conditions. Moreover, conformal Ricci pseudosymmetric generalized $(k,\mu)$-space forms are studied. In section 5, quasi-umbilical hypersurface of generalized $(k,\mu)$-space form are investigated and shown that it is a generalized quasi Einstein hypersurface. Also $\xi$-sectional curvature of a quasi-umbilical hypersurface of generalized $(k,\mu)$-space form is obtained. Finally, the obtained results are verified by using an example of a 3-dimensional generalized $(k,\mu)$-space form.

## 2. Preliminaries

In this section, we highlight some of the formulae and statements which will be used later in our studies.

A $(2n+1)$-dimensional smooth manifold $M$ is said to be a contact metric manifold if there exists a global 1-form $\eta$, known as the contact form, such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$ and there exists a unit vector field $\xi$, called the Reeb vector field, corresponding to 1-form $\eta$ such that $d\eta(\xi,\cdot)=0$, a $(1,1)$ tensor field $\phi$ and Riemannian metric $g$ such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\phi Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie-algebra of all vector fields on $M$. The metric $g$ is called the associate metric and the structure $(\phi,\xi,\eta,g)$ is called contact metric structure. A Riemannian manifold $M$ together with contact structure $(\phi,\xi,\eta,g)$ is called contact metric manifold. It follows from (2.1) that

$$\begin{aligned} \phi(\xi) = 0, \quad \eta\cdot\phi = 0, \quad g(X,\phi Y) = -g(\phi X, Y), \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.2}$$

for any $X, Y \in \chi(M)$. Further we define two self-adjoint operators $h$ and $l$ by $h = \frac{1}{2}(\mathcal{L}_\xi \phi)$ and $l = R(\cdot,\xi)\xi$ respectively, where $R$ is the Riemannian curvature of $M$. These operators satisfy

$$h\xi = l\xi = 0, \quad h\phi + \phi h = 0, \quad Tr.h = Tr.h\phi = 0. \tag{2.3}$$

Here, "Tr." denotes trace. When unit vector $\xi$ is Killing (i.e. $h=0$ or $Tr.l = 2n$) then contact metric manifold is called $K$-contact. A contact structure is said to be normal if the almost complex structure $J$ on $M \times \mathbb{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where $t$ is the coordinate of $\mathbb{R}$ and $f$ is a real function on $M \times \mathbb{R}$, is integrable. A normal contact metric manifold is called Sasakian. A Sasakian manifold is $K$-contact but the converse is true only in dimension 3. The $(k,\mu)$-nullity distribution of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) = \{Z \in \chi(M) : R(X,Y)Z = k\{g(Y,Z)X \\ -g(X,Z)Y\} + \mu\{g(Y,Z)hX - g(X,Z)hY\}\}, \end{aligned}$$

for any $X,Y,Z \in \chi(M)$ and real numbers $k$ and $\mu$. A contact metric manifold $M$ with $\xi \in N(k,\mu)$ is called a $(k,\mu)$-contact metric manifold.
In a generalized $(k,\mu)$-space form $(M^{2n+1},g)$ the following relations hold [2]:

$$\begin{aligned} R(X,Y)\xi &= (f_1-f_3)\{\eta(Y)X-\eta(X)Y\} \\ &+ (f_4-f_6)\{\eta(Y)hX-\eta(X)hY\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} PX &= (2nf_1+3f_2-f_3)X-(3f_2+(2n-1)f_3)\eta(X)\xi \\ &+ ((2n-1)f_4-f_6)hX, \end{aligned} \tag{2.5}$$

$$r = 2n\{(2n+1)f_1+3f_2-2f_3\}, \tag{2.6}$$

$$S(\phi X,\phi Y) = S(X,Y)-2n(f_1-f_3)\eta(X)\eta(Y). \tag{2.7}$$

where, $R,S,P,r$ are respectively the curvature tensor of type (1,3), the Ricci tensor, the Ricci operator i.e. $g(PX,Y)=S(X,Y)$, for any $X,Y \in \chi(M)$ and the scalar curvature of the manifold respectively.

## 3. Flatness of generalized $(k,\mu)$-space form

De and Samui [14] studied conformally flat $(k,\mu)$ space form and De and Majhi [13] analyzed $\xi$-$Q$ flatness of generalized Sasakian space form. Generalizing the results obtained, in this section we studied $\xi$-$Q$ flat and conformally flat generalized $(k,\mu)$-space form.

### 3.1. $\xi$-$Q$ flat generalized $(k,\mu)$-space form

**Definition 3.1.** A generalized $(k,\mu)$-space form $(M^{2n+1},g)$, is said to be $\xi$-$Q$ flat if $Q(X,Y)\xi=0$, for any $X,Y \in \chi(M)$ on $M$.

We have, from (1.2)

$$Q(X,Y)\xi = R(X,Y)\xi - \frac{v}{2n}\big[\eta(Y)X-\eta(X)Y\big], \tag{3.1}$$

for any $X,Y \in \chi(M)$. Using (2.4) in (3.1) we get

$$\begin{aligned} Q(X,Y)\xi &= \big(f_1-f_3-\frac{v}{2n}\big)\big[\eta(Y)X-\eta(X)Y\big] \\ &+ (f_4-f_6)[\eta(Y)hX-\eta(X)hY]. \end{aligned} \tag{3.2}$$

Suppose non-Sasakian generalized $(k,\mu)$-space form is $\xi - Q$ flat. Then from (3.2) we get

$$(3.3)\quad \big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY] = 0.$$

Taking $X = \phi X$ in (3.3), we obtain

$$(3.4)\qquad \Big\{(f_1 - f_3 - \frac{v}{2n})\phi X + (f_4 - f_6)h\phi X\Big\}\eta(Y) = 0.$$

Since $\eta(Y) \neq 0$ and taking inner product with $U$ in (3.4) gives

$$(3.5)\qquad \big(f_1 - f_3 - \frac{v}{2n}\big)g(\phi X, U) + (f_4 - f_6)g(\phi X, hU) = 0.$$

Since $g(\phi X, U) \neq 0$ and $g(\phi X, hU) \neq 0$, we see that $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$. Conversely, taking $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$, and putting these values in (3.2) gives $Q(X,Y)\xi = 0$ and hence $M$ is $\xi - Q$ flat. Therefore, we can state the following:

**Theorem 3.1.** *A non-Sasakian generalized $(k,\mu)$-space form $(M^{2n+1}, g)$, is $\xi$-$Q$ flat if and only if $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$.*

In particular, if $v = \frac{r}{2n+1}$ then $Q$ tensor reduces to concircular curvature tensor. Making use of (2.6) in the forgoing equation gives $v = \frac{2n\{(2n+1)f_1 + 3f_2 - 2f_3\}}{2n+1}$. In regard of Theorem 3.1, for $\xi$-concircularly flat we obtain $f_3 = \frac{3f_2}{1-2n}$ and hence we can state the following corollary:

**Corollary 3.1.** *A non-Sasakian generalized $(k,\mu)$-space form $(M^{2n+1}, g)$, is $\xi$-concircularly flat if and only if $f_3 = \frac{3f_2}{1-2n}$ and $f_4 = f_6$.*

We can easily see that Theorem 3.1 and Corollary 3.1 obtained by the geometers in [13], are particular cases of Theorem 3.1 and Corollary 3.1 respectively for $f_4 = f_5 = f_6 = 0$.

Substituting the values, $f_4 - f_6 = \mu$ and $f_1 - f_3 = k$ in Theorem 3.1, we obtained the following corollary:

**Corollary 3.2.** *A $(k,\mu)$-space form $(M^{2n+1}, g)$, is $\xi$-$Q$ flat if and only if $k = \frac{v}{2n}$ and $\mu = 0$.*

### 3.2. Conformally flat generalized $(k,\mu)$-space form

**Definition 3.2.** A generalized $(k,\mu)$-space form $(M^{2n+1}, g), n > 1$, is said to be conformally flat if $C(X,Y)Z = 0$, for any $X, Y, Z \in \chi(M)$ on $M$.

Suppose generalized $(k, \mu)$-space form is conformally flat. Then from (1.3), we get

$$
\begin{aligned}
R(X,Y)Z - \frac{1}{2n-1}\big\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)PX - g(X,Z)PY\big\} \\
+\frac{r}{2n(2n-1)}\big\{g(Y,Z)X - g(X,Z)Y\big\} = 0. \qquad (3.6)
\end{aligned}
$$

In consequence of taking $X = \xi$ in (3.6) and using (2.1), (2.4) and (2.5). Eq.(3.6) becomes

$$
\begin{aligned}
(f_1 - f_3)\{g(Y,Z)\xi - \eta(Z)Y\} + (f_4 - f_6)\{g(hY,Z)\xi - \eta(Z)hY\} \\
-\frac{1}{2n-1}\big\{S(Y,Z)\xi - 2n(f_1 - f_3)\eta(Z)Y + 2n(f_1 - f_3)g(Y,Z)\xi \\
-\eta(Z)PY\big\} + \frac{r}{2n(2n-1)}\big\{g(Y,Z)\xi - \eta(Z)Y\big\} = 0. \qquad (3.7)
\end{aligned}
$$

Putting $Z = \phi Z$ in (3.7) and making use of (2.4), (2.5) and (2.6) results in the following

$$
2(n+1)f_6 g(hY, \phi Z) = 0. \qquad (3.8)
$$

This shows that either $f_6 = 0$ or $\phi h = 0$. In the second case, from (2.1) we have $h = 0$. Therefore, we can state the following:

**Theorem 3.2.** *A generalized $(k, \mu)$-space form $(M^{2n+1}, g), n > 1$, is conformally flat, then either $f_6 = 0$ or $M$ is Sasakian.*

**Corollary 3.3.** *A $(k, \mu)$-space form $(M^{2n+1}, g), n > 1$, is conformally flat, then $\mu = 1$ or $M$ is Sasakian.*

## 4. Pseudosymmetric generalized $(k, \mu)$-space form

In this section certain pseudo symmetry such as Ricci pseudo symmetry, $Q$-Ricci pseudo symmetry and conformal Ricci pseudo symmetry in the context of generalized $(k, \mu)$-space form are studied. First, we review an important definition

**Definition 4.1.** [19, 31] A Riemannian manifold $(M, g), n \geq 1$, admitting a $(0, k)$-tensor field $T$ is said to be $T$-pseudosymmetric if $R \cdot T$ and $D(g, T)$ are linearly dependent, i.e., $R \cdot T = L_T D(g, T)$ holds on the set $U_T = \{x \in M : D(g, T) \neq 0$ at $x\}$, where $L_T$ is some function on $U_T$.

In particular, if $R \cdot R = L_R D(g, R)$ and $R \cdot S = L_S D(g, S)$ then the manifold is called pseudosymmetric and Ricci pseudosymmetric respectively. Moreover, if $L_R = 0$ ( resp., $L_S = 0$) then pseudosymmetric (resp., Ricci pseudosymmetric) reduces to semisymmetric (resp., Ricci semisymmetric) introduced by Cartan in 1946.

### 4.1. Ricci pseudosymmetric generalized $(k, \mu)$-space form

**Definition 4.2.** A generalized $(k, \mu)$-space form $(M^{2n+1}, g)$, is said to be Ricci pseudosymmetric if its Ricci curvature satisfies the following relation,

$$R \cdot S = f_{S_2} D(g, S),$$

holds on the set $U_{S_2} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where $f_{S_2}$ is some function on $U_{S_2}$.

Suppose a generalized $(k, \mu)$-space form $(M^{2n+1}, g)$, is Ricci pseudosymmetric i.e.,

$$R \cdot S = f_{S_2} D(g, S),$$

which can be written as

$$\begin{aligned} S(R(X,Y)U,V) + S(U,R(X,Y)V) = -f_s\big[&S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - &S(U,X)g(Y,V)\big] \end{aligned} \tag{4.1}$$

Taking $X = U = \xi$ in (4.1) and using (2.4), (2.5) and (2.7), we get

$$\begin{aligned} \big(f_3 - f_1 + f_{S_2}\big)S(Y,V) + \big[2n(f_1 - f_3)(f_1 - f_3 - f_{S_2}) - (k-1)(f_4 \\ -f_6)((2n-1)f_4 - f_6)\big]g(Y,V) - (k-1)(f_4 - f_6)\big((2n-1)f_4 \\ -f_6\big)\eta(Y)\eta(V) + (f_4 - f_6)\big((1-2n)f_3 - 3f_2\big)g(hY,V) = 0. \end{aligned} \tag{4.2}$$

Considering $f_{S_2} \neq f_1 - f_3$ and further taking $(1-2n)f_3 - 3f_2 = 0$ in (4.2), the manifold is $\eta$-Einstein. Hence we can state the following:

**Theorem 4.1.** *A Ricci pseudosymmetric generalized $(k, \mu)$-space form $(M^{2n+1}, g)$, with $f_{S_2} \neq f_1 - f_3$, is $\eta$-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

If $f_{S_2} = 0$, then Ricci pseudosymmetric generalized $(k, \mu)$-space form reduces to Ricci semisymmetric generalized $(k, \mu)$-space form. In view of Theorem (4.1) we obtain the following:

**Corollary 4.1.** *A Ricci semisymmetric generalized $(k, \mu)$-space form $(M^{2n+1}, g)$, with $f_1 - f_3 \neq 0$ is $\eta$-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

### 4.2. $Q$-Ricci pseudosymmetric generalized $(k, \mu)$-space form

**Definition 4.3.** A generalized $(k, \mu)$-space form $(M^{2n+1}, g)$, is said to be $Q$-Ricci pseudosymmetric if

$$Q \cdot S = f_{S_3} D(g, S),$$

holds on the set $U_{S_3} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where $f_{S_3}$ is any function on $U_{S_3}$.

Proceeding similarly as in Theorem 4.1, one can easily obtain the following relation:

**Theorem 4.2.** *A Q-Ricci pseudosymmetric generalized* $(k, \mu)$*-space form* $(M^{2n+1}, g)$*, with* $f_{S_3} \neq f_3 - f_1 - \frac{v}{2n}$ *is* $\eta$*-Einstein manifold if* $f_3 = \frac{3f_2}{1-2n}$.

Taking $f_{S_3} = 0$ in Theorem 4.2, we easily obtain the following:

**Corollary 4.2.** *A Q-Ricci semisymmetric generalized* $(k, \mu)$*-space form* $(M^{2n+1}, g)$*, with* $f_3 - f_1 \neq \frac{v}{2n}$ *is* $\eta$*-Einstein manifold if* $f_3 = \frac{3f_2}{1-2n}$.

### 4.3. Conformal Ricci pseudosymmetric generalized $(k, \mu)$-space form

**Definition 4.4.** A generalized $(k, \mu)$-space form $(M^{2n+1}, g), n > 1$, is said to be conformal Ricci pseudosymmetric if

$$C \cdot S = f_{S_4} D(g, S),$$

holds on the set $U_{S_4} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where $f_{S_4}$ is any function on $U_{S_4}$.

Suppose a generalized $(k, \mu)$-space form is conformal Ricci pseudosymmetric. Then, we have

$$\begin{aligned} S(C(X,Y)U,V) + S(U, C(X,Y)V) = -f_{S_4}\big[S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big]. \end{aligned} \tag{4.3}$$

Taking $X = U = \xi$ and $f_4 = f_6$ in (4.3) and making use of (1.3),(2.1) and (2.5), we obtain

$$\begin{aligned} S^2(Y,V) = \big(4nf_1 + 3f_2 - (2n+1)f_3 + 2n(2n-1)f_{S_4}\big)S(Y,V) \\ -(2n-1)f_{S_4}\eta(Y)\eta(V) - \big(2nf_1 + 3f_2 - f_3\big)g(Y,V). \end{aligned} \tag{4.4}$$

Thus, we can state the following:

**Theorem 4.3.** *If a generalized* $(k, \mu)$*-space form* $(M^{2n+1}, g), n > 1$*, is conformal Ricci pseudosymmetric with* $f_4 = f_6$*, then the relation(4.4) holds.*

## 5. Quasi-umbilical hypersurface of generalized $(k, \mu)$-space form

Let us consider a quasi-umbilical hypersurface $\widetilde{M}$ of a generalized $(k, \mu)$-space form. From Gauss [12], for any vector fields $X, Y, Z, W$ tangent to the hypersurface we have

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) - g(H(X,W), H(X,Z)) \\ &+ g(H(X,Z), H(Y,W)), \end{aligned} \tag{5.1}$$

where, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\widetilde{R}(X,Y,Z,W) = g(\widetilde{R}(X,Y)Z,W)$. Here, $H$ is the second fundamental tensor of $\widetilde{M}$ given by

$$H(X,Y) = \alpha g(X,Y)\rho + \beta\omega(X)\omega(Y)\rho, \tag{5.2}$$

where, $\rho$ is the only unit normal vector field. Here, $\omega$ is the 1-form, the vector field corresponding to the 1-form $\omega$ is a unit vector field and $\alpha, \beta$ are scalars.
Using (5.2) in (5.1), we obtain the following result

$$\begin{aligned}
&f_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + f_2\big[g(X,\phi Z)g(\phi Y,W)\\
&-g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\big] + f_3\big[\eta(X)\eta(Z)g(Y,W)\\
&-\eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\big]\\
&+f_4\big[g(Y,Z)g(hX,W) - g(Y,Z)g(hY,W) + g(hY,Z)g(X,W)\\
&-g(hX,Z)g(Y,W)\big] + f_5\big[g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W)\\
&+g(\phi hX,Z)g(\phi hY,W) - g(\phi hY,Z)g(\phi hX,W)\big] + f_6\big[\eta(X)\eta(Z)g(hY,W)\\
&-\eta(Y)\eta(Z)g(hX,W) + g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)\big]\\
&= \widetilde{R}(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z)\\
&-\alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z) + \alpha\beta g(Y,W)\omega(X)\omega(Z)\\
&+\alpha\beta g(X,Z)\omega(Y)\omega(W).
\end{aligned} \tag{5.3}$$

Contracting over $X$ and $W$ in (5.3), we obtain

$$\begin{aligned}
\widetilde{S}(Y,Z) &= \big(2nf_1 + 3f_2 - f_3 + 2n\alpha^2 + \alpha\beta\big)g(Y,Z)\\
&-\big(3f_2 + (2n+1)f_3\big)\eta(Y)\eta(Z) + \big((2n-1)f_4 - f_6\big)g(hY,Z)\\
&+\alpha\beta(2n-1)\omega(Y)\omega(Z).
\end{aligned} \tag{5.4}$$

Hence, we can state the following:

**Theorem 5.1.** *A quasi-umbilical hypersurface of a generalized $(k,\mu)$-space form is a generalized quasi Einstein hypersurface, provided $f_4 = \frac{f_6}{2n-1}$*

In particular, for a $(k,\mu)$-space form, the above Theorem 5.1 reduces to the following:

**Theorem 5.2.** [14] *A quasi-umbilical hypersurface of a $(k,\mu)$-contact space form is a generalized quasi-Einstein hypersurface, provided $\mu = 2 - 2n$.*

**Corollary 5.1.** *A quasi-umbilical hypersurface of a generalized Sasakian space form is a generalized quasi-Einstein hypersurface.*

For any vector fields $X, Y$, the tensor field $K(X,Y) = \widetilde{R}(X,Y,Y,X)$ is called the sectional curvature of $\widetilde{M}$ given by the sectional plane $\{X,Y\}$. The sectional curvature $K(X,\xi)$ of a sectional plane spanned by $\xi$ and vector field $X$ orthogonal to $\xi$ is called the $\xi$-sectional curvature of $\widetilde{M}$.

**Theorem 5.3.** *A $\xi$-sectional curvature of a quasi-umbilical hypersurface of generalized $(k, \mu)$-space form is given by*

$$K(X, \xi) = \big(f_1 - f_3 + \alpha^2\big) g(\phi X, \phi X) + (f_4 - f_6) g(hX, X) \\ +\alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

*Proof.* Taking $W = X$ and $Z = Y$ in (5.3) results in following

$$\begin{aligned} & f_1\big[g(Y, Y)g(X, X) - g(X, Y)g(Y, X)\big] + f_2\big[g(X, \phi Y)g(\phi Y, X) \\ & -g(Y, \phi Y)g(\phi X, X) + 2g(X, \phi Y)g(\phi Y, X)\big] + f_3\big[\eta(X)\eta(Y)g(X, Y) \\ & -\eta(Y)\eta(Y)g(X, X) - g(X, Y)\eta(X)\eta(Y) - g(Y, Y)\eta(X)\eta(X)\big] \\ & +f_4\big[g(Y, Y)g(hX, X) - g(X, Y)g(hY, X) + g(hY, Y)g(X, X) \\ & -g(hX, Y)g(Y, X)\big] + f_5\big[g(hY, Y)g(hX, X) - g(hX, Y)g(hY, X) \\ & +g(\phi hX, Y)g(\phi hY, X) - g(\phi hY, Y)g(\phi hX, X)\big] + f_6\big[\eta(x)\eta(Y)g(hY, X) \\ & -\eta(Y)\eta(Y)g(hX, X) + g(hX, Y)\eta(Y)\eta(X) - g(hY, Y)\eta(X)\eta(X)\big] \\ & = K(X, Y) - \alpha^2 g(X, X)g(Y, Y) - \alpha\beta g(X, X)\omega(Y)\omega(Y) \\ & -\alpha\beta g(Y, Y)\omega(X)\omega(X) + \alpha^2 g(X, Y)g(X, Y) + \alpha\beta g(X, Y)\omega(X)\omega(Y) \\ & +\alpha\beta g(X, Y)\omega(Y)\omega(X). \end{aligned} \tag{5.5}$$

Putting $Y = \xi$ in (5.5) gives

$$K(X, \xi) = \big(f_1 - f_3 + \alpha^2\big) g(\phi X, \phi X) + (f_4 - f_6) g(hX, X) \\ +\alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

This completes the proof. □

## 6. Examples of generalized $(k, \mu)$-space forms

Now we will show the validity of obtained result by considering an example of a generalized $(k, \mu)$-space form of dimension 3. Koufogiorgos and Tsichlias [24] constructed an example of generalized $(k, \mu)$-space of dimension 3 which was later shown by Carriazo et al. [8] to be a contact metric generalized $(k, \mu)$-space form $M^3(f_1, 0, f_3, f_4, 0, 0)$ with non-constant $f_1, f_3, f_4$.

*Example 6.1:* Let $M^3$ be the manifold $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 \neq 0\}$ where $(x_1, x_2, x_3)$ are standard coordinates on $\mathbb{R}^3$. Consider the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = -2x_2x_3\frac{\partial}{\partial x_1} + \frac{2x_1}{x_3^2}\frac{\partial}{\partial x_2} - \frac{1}{x_3^2}\frac{\partial}{\partial x_3}, \quad e_3 = \frac{1}{x_3}\frac{\partial}{\partial x_2},$$

are linearly independent at each point of $M$ and are related by

$$[e_1, e_2] = \frac{2}{x_3^2}e_3, \quad [e_2, e_3] = 2e_1 + \frac{1}{x_3^3}e_3, \quad [e_3, e_1] = 0.$$

Let $g$ be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}, i, j = 1, 2, 3$ and $\eta$ be the 1-form defined by $\eta(X) = g(X, e_1)$ for any $X$ on $M$. Also, let $\phi$ be the $(1,1)$-tensor field defined by $\phi e_1 = 0, \quad \phi e_2 = e_3 \quad \phi e_3 = -e_2$. Therefore, $(\phi, e_1, \eta, g)$ defines a contact metric structure on $M$. Put $\lambda = \frac{1}{x_3^2}, k = 1 - \frac{1}{x_3^4}$ and $\mu = 2(1 - \frac{1}{x_3^2})$, then symmetric tensor $h$ satisfies $he_1 = 0, he_2 = \lambda e_2, he_3 = -\lambda e_3$. The non-vanishing components of the Riemannian curvature are as follows:

$$\begin{aligned} R(e_1, e_2)e_1 &= -(k + \lambda\mu)e_2, \quad R(e_1, e_2)e_2 = (k + \lambda\mu)e_1, \\ R(e_1, e_3)e_1 &= (-k + \lambda\mu)e_3, \quad R(e - 1, e_3)e_3 = (k - \lambda\mu)e_1, \\ R(e_2, e_3)e_2 &= (k + \mu - 2\lambda^3)e_3, \quad R(e_2, e_3)e_3 = -(k + \mu - 2\lambda^3)e_2. \end{aligned}$$

Therefore, $M$ is a generalized $(k, \mu)$-space with $k, \mu$ not constant. As a contact metric generalized $(k, \mu)$-space is a generalized $(k, \mu)$-space form with $k = f_1 - f_3$ and $\mu = f_4 - f_6$ (Theorem 4.1, [8]), the manifold under consideration is a generalized $(k, \mu)$-space form $M^3(f_1, 0, f_3, f_4, 0, 0)$ where

$$\begin{aligned} f_1 &= -3 + \frac{2}{x_3^2} + \frac{1}{x_3^4} + \frac{2}{x_3^6}, \\ f_3 &= -4 + \frac{2}{x_3^2} + \frac{2}{x_3^4} + \frac{2}{x_3^6} \\ f_4 &= 2(1 - \frac{1}{x_3^2}). \end{aligned}$$

Next we obtain the non-vanishing components of $Q$-curvature tensor for arbitrary function $v$ as follows:

$$\begin{aligned} Q(e_1, e_2)e_1 &= -(k + \lambda\mu - \frac{v}{2})e_2, \quad Q(e_1, e_2)e_2 = (k + \lambda\mu - \frac{v}{2})e_1, \\ Q(e_1, e_3)e_1 &= (-k + \lambda\mu + \frac{v}{2})e_3, \quad Q(e_1, e_3)e_3 = (k - \lambda\mu - \frac{v}{2})e_1, \\ Q(e_2, e_3)e_2 &= (k + \mu - 2\lambda^3 + \frac{v}{2})e_3, \quad Q(e_2, e_3)e_3 = -(k + \mu - 2\lambda^3 + \frac{v}{2})e_2. \end{aligned}$$

From the above equations we see that $Q(X, Y)e_1 = 0$ for all $X, Y$ on $M$ if and only if $v = 2(1 - \frac{1}{x_3^4})$ and $x_3^2 = 1$. Hence, Theorem 3.1 is verified.

*Example 6.2:* In [2], it was shown that the warped product $\mathbb{R} \times_f \mathbb{C}^m$ with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

is a generalized Sasakian space form. Since every generalized Sasakian space form is a particular case of generalized $(k, \mu)$-space form, $\mathbb{R} \times_f \mathbb{C}^m$ with $f_1, f_2, f_3$ define as above and $f_4 = f_5 = f_6 = 0$ is a generalized $(k, \mu)$-space form.

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## REFERENCES

1. A. Akbar and A. Sarkar: *Some curvature properties of $(k, \mu)$-contact space forms*. Malaya J. Mat. **3**(1) (2015), 45–50.
2. P. Alegre, D. E. Blair and A. Carriazo: *Generalized Sasakian space forms*. Israel J. Math. **141** (2004), 157–183.
3. P. Alegre and A. Carriazo: *Submanifolds of generalized Sasakian space forms*. Taiwanese J. Math. **13** (2009), 923–911.
4. P. Alegre and A. Carriazo: *Structures on generalized Sasakian space forms*. Diff. Geom. Appl. **26** (2008), 656–666.
5. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou: *Contact Metric Manifolds satisfying a nullity condition*. Israel J. Math. **19**(1-3) (1995), 189–214.
6. D. E. Blair: *Contact manifolds in Riemannian geometry*. Lecture note in Mathematics, 509. Springer-Verlag, Berlin, New-York, 1976.
7. D. E. Blair: *Riemannian geometry of contact and sympletic manifolds*. Birkhauser Boston, 2002.
8. A. Carriazo, V. M. Molina and M. M. Tripathi: *Generalized $(k, \mu)$ space forms*. Mediterr. J. Math. **10** (2013), 175–196.
9. A. Carriazo and V. M. Molina: *Generalized $(k, \mu)$-space forms and $D_\alpha$-homothetic deformations*. Balkan J. Geo. Appl. **16**(1) (2011), 37–47.
10. E. Cartan: *Sur une classe remarquable d'espaces de Riemann*. Bull. Soc. Math. France **54** (1926), 214–264.
11. M. C. Chaki: *On pseudosymmetric manifolds*. An. Stiint. Univ. AL.I. Cuza din Iasi Sect. I-a Math. N. S. **33**(1) (1987), 53–58.
12. B. Y. Chen: *Geometry of submanifolds*. Marcel Dekker, Inc. New York, 1973.
13. U. C. De and P. Majhi: *On the Q curvature tensor of a generalized sasakian-space-form*. Krag. J. Math. **13**(3) (2019), 333–319.
14. U. C. De and S. Samui: *The structure of some classes of $(k, \mu)$-contact space forms*. Diff. Geom.-Dynamical System, **18** (2016), 1–13.
15. U. C. De and P. Majhi: *$\phi$-semisymmetric generalized Sasakian space-form*. Arab J. Math. Sci. **21** (2015), 170–178.
16. U. C. De and A. Sarkar: *Some results on generalized Sasakian space forms*. Thai J. Math. **8** (2010), 1–10.
17. U. C. De and A. Yildiz: *Certain curvature conditions on generalized Sasakian space forms*. Quaest. Math. **38**(1) (2015), 195–504.

18. U. C. De and G. C. Ghosh: *On generalized quasi Einstein manifolds.* Kyungpook Math. J. **44** (2001), 607–615.

19. R. Deszcz: *On pseudosymmetric spaces.* Bull. Belg. Math. Soc., Ser. A, **44** (1992), 1–34.

20. M. Faghfouri and N. Ghaffarzadeh: *On doubly warped product submanifolds of generalized $(k, \mu)$-space forms.* Afrika Mat. **26**(7-8) (2015), 1443–1455.

21. S. K. Hui, S. Uddin and P. Mandal: *Submanifolds of generalized $(k, \mu)$-space-forms.* Per. Math. Hung. **77** (2018), 329–339.

22. D. L. Kumar and H. G. Nagaraja: *Second Order Parallel Tensor and Ricci solitons on Generalized $(k, \mu)$-Space forms.* Math. Adv. Pure Appl. Sci. **2**(1) (2019), 1–7.

23. T. Koufogiorgos: *Contact Riemannian manifolds with constant $\phi$-sectional curvature.* Tokyo J. Math. **20** (1997), 13–22.

24. T. Koufogiorgos and C. Tsichlias: *On the existence of a new class of contact metric manifolds.* Canad. Math. Bull. **20**(1) (2000), 400–447.

25. U. K. Kim: *Conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space forms.* Note Mat. **25** (2006), 55–67.

26. C. A. Mantica and Y. J. Suh: *Pseudo Q-symmetric Riemannian manifolds.* Int. J. Geom. Methods Mod. Phys. **10**(5) (2013), 1–25.

27. B. Shammukha and V. Venkatesha: *Projective curvature tensor on generalized $(k, \mu)$-space forms.* Italian J. Pure Appl. Math. **12** (2019), 810–850.

28. G. S. Shivaprasanna: *Some results on generalized $(k, \mu)$ space-forms.* Int. J. sci. Engg. Appl. Sci. **2**(7) (2016), 184–191.

29. G. S. Shivaprasanna, Y. B. Maralabhavi and G. Somashekhara: *On Semi-Symmetric Metric Connection in a Generalized $(k, \mu)$ Space Forms.* Int. J. Math. Trends Tech. **9**(3) (2014), 173–188.

30. S. Shashidhar and H. G. Nagaraja: *$\eta$-Einstein $(k, \mu)$-space forms.* J. Info. Math. Sci. **7**(2) (2015), 109–120.

31. A. A. Shaikh, R. Deszcz, M. Hotlos and J. Jelwicki: *On pseudosymmetric manifolds.* Publicationes Mathematicae, **86**(3-4) (2015), 433–456.

32. K. Yano: *Concircular geometry I. Concircular transformation.* Proc. Imp. Acad. Tokyo, **16** (1910), 195–200.

33. K. Yano and M. Kon: *Structures on manifolds.* Series in Pure Math. Vol 3, World Scientific Publ. Co., Singapore, 1984.

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# ON THE STRECH CURVATURE OF HOMOGENEOUS FINSLER METRICS

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**Abstract.** In this paper, we study the stretch curvature of homogeneous Finsler manifolds. First, we prove that every homogeneous Finsler metric has relatively isotropic stretch curvature if and only if it is a Landsberg metric. It follows that every weakly Berwald homogeneous metric has relatively isotropic stretch curvature if and only if it is a Berwald metric. We show that a homogeneous metric of non-zero scalar flag curvature has relatively isotropic stretch curvature if and only if it is a Riemannian metric of constant sectional curvature. It turns out that a homogeneous $(\alpha, \beta)$-metric with relatively isotropic stretch curvature is a Berwald metric. Also, it follows that a homogeneous spherically symmetric metric with relatively isotropic stretch curvature reduces to a Riemannian metric. Finally, we prove that every homogeneous stretch-recurrent metric is a Landsberg metric.
**Keywords**: Strech metric, Landsberg metric, Berwald metric, (fi; fl)-metric, homogeneous metric.

## 1. Introduction

In [7], Deng-Hou proved that the group of isometries of a Finsler manifold $(M.F)$, denoted by $I(M, F)$, is a Lie transformation group of the underlying manifold that can be used to study homogeneous Finsler manifolds. This important result opens

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an interesting window to generalize the concept of homogeneous Riemannian manifold to homogeneous Finsler manifold. An $n$-dimensional Finsler manifold $(M, F)$ is called a homogeneous Finsler manifold if the group $I(M, F)$ acts transitively on the manifold $M$.

A Finsler metric $F$ on a manifold $M$ is called a Berwald metric if its spray coefficients $G^i$ are quadratic in $y \in T_xM$ for all $x \in M$. The important described characteristic of a Berwald space is that all its tangent spaces are linearly isometric to a common Minkowski space. For a Landsberg space, all its tangent spaces are isometric to a common Minkowski space. Thus every Berwald space is a Landsberg space. However, it has been one of the longest-standing problems in Finsler geometry whether there exists a Landsberg space that is not a Berwald space. In [35], Xu-Deng conjectured that every homogeneous Landsberg space must be a Berwald space.

In 1924, at the annual meeting of the Mathematical Society of Germany in Innsbruck, Berwald defined of the stretch curvature as a generalization of Landsberg curvature and denoted it by $\mathbf{T}$ [3]. He published the stretch curvature in 1925 on the first of his main papers [5]. He showed that $\mathbf{T} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. In his lecture at the International Congress of Mathematics, Bologna, 1928, he introduced a series of special classes of Finsler metrics, such as Landsberg metrics and stretch metrics [2]. He proved that for two-dimensional stretch metrics, the total curvature (curvature integral) $\int\int \mathbf{R}\sqrt{g}dx^1dx^2$ can be defined, which means the integrand is a function of position alone, where $\mathbf{R}$ is the Underhill curvature. Then, this curvature has been investigated by Shibata in [21] and Matsumoto in [10]. Matsumoto denoted this curvature by $\mathbf{\Sigma}$. We have the following big picture.

$$\{\text{Berwald metrics}\} \subseteq \{\text{Landsberg metrics}\} \subseteq \{\text{Stretch metrics}\}.$$

Let $(M, F)$ be a Finsler manifold. Then $F$ is called a relatively isotropic stretch metric if its stretch curvature is given by

$$\Sigma_{ijkl} = cF(C_{ijk|l} - C_{ijl|k}), \tag{1.1}$$

where $c = c(x)$ is a scalar function on $M$, and "$|$" denotes the horizontal covariant derivative with respect to the Berwald connection of $F$. In this case, $(M, F)$ is called a relatively isotropic stretch manifold.

**Example 1.1.** A Finsler metric $F$ satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $\mathbb{B}^n(1)$ is defined by

$$F(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - < x, y >^2)}}{1 - |x|^2} + \frac{< x, y >}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n(1) \simeq \mathbb{R}^n, \tag{1.2}$$

where $<, >$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. It follows from $G^i = \frac{1}{2}Fy^i$ that $F$ satisfies (1.1) with $c = -1$.

**Example 1.2.** For $y \in T_x\mathbb{B}^n(1) \simeq \mathbb{R}^n$, let us define

$$F_a(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - < x,y >^2)}}{1-|x|^2} + \frac{< x,y >}{1-|x|^2} + \frac{< a,y >}{1+ < a,x >}, \tag{1.3}$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$. For $a \neq 0$, it is easy to see that $F_a$ is a locally projectively flat Finsler metric with negative constant flag curvature. It follows that $F$ is a relatively isotropic stretch metric with $c = -1$.

In this paper, we prove the following.

**Theorem 1.1.** *Every homogeneous Finsler metric on a manifold $M$ has relatively isotropic stretch curvature if and only if it is a Landsberg metric.*

In [27], Tayebi-Najafi proved that every homogeneous Landsberg surface is Riemannian or locally Minkowskian spaces. Then by Theorem 1.1, we conclude that every homogeneous Finsler surface of relatively isotropic stretch curvature is Riemannian or locally Minkowskian spaces.

There is another important quantity defined by the spray of a Finsler metric $F$. Taking a trace of Berwald curvature implies the mean Berwald curvature $\mathbf{E}$. A Finsler metric $F$ is said to be weakly Berwaldian if $\mathbf{E} = 0$.

**Corollary 1.1.** *Every weakly Berwald homogeneous Finsler metric on a manifold $M$ has relatively isotropic stretch curvature if and only if it is a Berwald metric.*

Douglas curvature is a non-Riemannian projectively invariant constructed from the Berwald curvature. The notion of Douglas curvature was proposed by Bácsó-Matsumoto as a generalization of Berwald curvature [1]. The Douglas curvature vanishes for Riemannian spaces; therefore, it plays a prominent role only outside the Riemannian world. Finsler metrics with $\mathbf{D} = 0$ are called Douglas metrics.

**Corollary 1.2.** *Every Douglas homogeneous Finsler metric on a manifold $M$ has relatively isotropic stretch curvature if and only if it is a Berwald metric.*

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, first introduced by L. Berwald [2][5]. For a Finsler manifold $(M, F)$, the flag curvature is a function $\mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. A Finsler metric $F$ is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of flags $P$ associated with any fixed flagpole $y$. Finsler metrics of scalar flag curvature are the natural extension of Riemannian metrics of isotropic sectional curvature (of constant sectional curvature in dimension $n \geq 3$ by the Schur Lemma). One of the central problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature.

**Corollary 1.3.** *Let $(M, F)$ be a homogeneous Finsler metric of dimension $n \geq 3$. Suppose that $F$ has non-zero scalar flag curvature. Then $F$ has relatively isotropic stretch curvature if and only if it is a Riemannian metric of constant sectional curvature.*

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with a certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $M$ (see [26], [31] and [32]).

**Corollary 1.4.** *Every homogeneous $(\alpha, \beta)$-metric on a manifold $M$ has relatively isotropic stretch curvature if and only if it is a Berwald metric.*

A Finsler metric $F = F(x, y)$ on a domain $\Omega \subseteq \mathbb{R}^n$ is called spherically symmetric metric if it is invariant under any rotation in $\mathbb{R}^n$. Indeed, the class of spherically symmetric metrics in the Finsler setting was first introduced by S.F. Rutz, who studied the spherically symmetric Finsler metrics in 4-dimensional space-time and generalized the classic Birkhoff theorem in general relativity to the Finsler case [17]. According to the equation of Killing fields, there exists a positive function $\phi$ depending on two variables so that $F$ can be written as

$$F = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big),$$

where $x$ is a point in the domain $\Omega$, $y$ is a tangent vector at the point $x$. There are classical Finsler metrics which are spherically symmetric, such as Funk metric, Berwald's metric, Bryant's metric, etc, (see [14] for more details). .

**Corollary 1.5.** *Every homogeneous spherically symmetric Finsler metric on a manifold $M$ has relatively isotropic stretch curvature if and only if it is a Riemannian metric.*

A homogeneous Finsler manifold $(M, F)$ is said to be stretch-recurrent or $\mathbf{\Sigma}$-recurrent if its stretch curvature satisfies following

$$\Sigma_{ijkl|s}y^s = \Psi\Sigma_{ijkl}, \tag{1.4}$$

where $\Psi$ is a non-zero smooth function on $TM_0$ satisfying $\Psi(x, ty) = t\Psi(x, y)$ for all positive real number $t$ and $(x, y) \in TM_0$. It is easy to see that every stretch metric and then Landsberg metric is a $\Sigma$-recurrent metric. However, the converse is not valid in general. Here, we prove that every homogeneous $\Sigma$-recurrent Finsler metric is a Landsberg metric.

**Theorem 1.2.** *Any homogeneous $\Sigma$-recurrent metric is a Landsberg metric.*

## 2. Preliminaries

Let $(M, F)$ be an $n$-dimensional Finsler manifold. The fundamental tensor $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ of $F$ is defined by following

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Given a Finsler manifold $(M, F)$, then a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are scalar functions on $TM_0$ given by

$$G^i := \frac{1}{4}g^{ij}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^j}y^k - \frac{\partial[F^2]}{\partial x^j}\right\}, \quad y \in T_xM. \tag{2.1}$$

The $\mathbf{G}$ is called the spray associated to $(M, F)$.

For a non-zero vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature. $F$ is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

Define the mean of Berwald curvature by $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u, v) := \frac{1}{2}\sum_{i=1}^{n} g^{ij}(y)g_y\Big(\mathbf{B}_y(u, v, e_i), e_j\Big). \tag{2.2}$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in TM\setminus\{0\}}$ is called the *mean Berwald curvature* or *E-curvature*. In local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y)u^i v^j$, where

$$E_{ij} := \frac{1}{2}B^m{}_{mij}.$$

By definition, $\mathbf{E}_y(u,v)$ is symmetric in $u$ and $v$ and we have $\mathbf{E}_y(y,v) = 0$. The quantity $\mathbf{E}$ is called the mean Berwald curvature. $F$ is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$.

For non-zero vector $y \in T_xM_0$, define $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\Big[G^i - \frac{2}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big]. \tag{2.3}$$

$\mathbf{D}$ is called the Douglas curvature. $F$ is called a Douglas metric if $\mathbf{D} = \mathbf{0}$ [1]. By definition, it follows that the Douglas tensor $\mathbf{D}_y$ is symmetric trilinear form and has the following properties

$$\mathbf{D}_y(y,u,v) = 0, \qquad trace(\mathbf{D}_y) = 0.$$

According to (2.3), the Douglas tensor can be written as follows

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\Big\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\Big\}.$$

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w), y\big).$$

$F$ is called a Landsberg metric if $\mathbf{L}_y = 0$. By definition, every Berwald metric is a Landsberg metric.

For $y \in T_xM_0$, define the stretch curvature $\mathbf{\Sigma}_y : T_xM \times T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u,v,w,z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}), \tag{2.4}$$

and "|" denotes the horizontal derivation with respect to the Berwald connection of $F$. A Finsler metric is said to be a stretch metric if $\mathbf{\Sigma} = 0$.

The second variation of geodesics gives rise to a family of linear maps $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y$, $\forall\lambda > 0$ which is defined by

$$\mathbf{R}_y(u) := R^i{}_k(y)u^k\frac{\partial}{\partial x^i},$$

where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j\partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

$\mathbf{R}_y$ is called the Riemann curvature in the direction $y$.

For a flag $P := \mathrm{span}\{y, u\} \subset T_xM$ with flagpole $y$, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{2.5}$$

The flag curvature $\mathbf{K}(x, y, P)$ is a function of tangent planes $P = \mathrm{span}\{y, v\} \subset T_xM$. A Finsler metric $F$ is of scalar flag curvature if $\mathbf{K} = \mathbf{K}(x, y)$ is independent of flag $P$ (see [23], [24] and [25]).

## 3. Proof of Theorem 1.1

Every two points of a homogeneous Finsler manifold map to each other by an isometry. Then, the norm of arbitrary tensor of a homogeneous Finsler manifold is a constant function on the underlying manifold. Thus the norm of an arbitrary tensor of a homogeneous Finsler space is bounded. This fact is proved in [28].

**Lemma 3.1.** ([28]) *Let $(M, F)$ be a homogeneous Finsler manifold. Then the norm of an arbitrary tensor of $F$ which is invariant under every isometry of $F$ is bounded.*

We define the norm of the Landsberg curvature at $x \in M$ by

$$\|\mathbf{L}\|_x := \sup_{y,u,v,w \in T_xM\setminus\{0\}} \frac{F(y)|\mathbf{L}_y(u, v, w)|}{\sqrt{\mathbf{g}_y(u, u)\mathbf{g}_y(v, v)\mathbf{g}_y(w, w)}}.$$

We showed that the Landsberg curvature of homogeneous Finsler metric $F$ is bounded.

**Lemma 3.2.** ([28]) *Let $(M, F)$ be a homogeneous Finsler manifold. Then the Landsberg curvature of $F$ is bounded.*

In order to prove Theorem 1.1, we need the following.

**Theorem 3.1.** ([29]) Homogeneous Finsler manifolds are complete.

Now, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1**: Let $p$ be an arbitrary point of manifold $M$, and $y, u, v, w \in T_pM$. Let $c : (-\infty, \infty) \to M$ is the unit speed geodesic passing from $p$ and

$$\frac{dc}{dt}(0) = y.$$

If $U(t), V(t)$ and $W(t)$ are the parallel vector fields along $c$ with

$$U(0) = u, \quad V(0) = v \quad W(0) = w.$$

Let us put

$$\mathbf{L}(t) = \mathbf{L}\big(U(t), V(t), W(t)\big),$$
$$\mathbf{L}'(t) = \mathbf{L}'_{\dot{c}}\big(U(t), V(t), W(t)\big).$$

Contracting (1.1) with $y^l$ implies that

$$L_{ijk|l}y^l = cFC_{ijl|k}y^l. \tag{3.1}$$

By definition, we have

$$L_{ijk} = C_{ijl|k}y^l. \tag{3.2}$$

From (3.1) and (3.2), we have

$$L_{ijk|l}y^l = cFL_{ijk}. \tag{3.3}$$

According to the definition, (3.3) yields the following ODE

$$\mathbf{L}'(t) = c\mathbf{L}(t), \tag{3.4}$$

which its general solution is

$$\mathbf{L}(t) = e^{ct}\mathbf{L}(0). \tag{3.5}$$

Using $||\mathbf{L}|| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we get

$$\mathbf{L}(0) = \mathbf{L}(u, v, w) = 0.$$

So $\mathbf{L} = 0$, i.e., $(M, F)$ is a Landsberg manifold. $\Box$

**Proof of Corollary 1.1**: In [6], Crampin showed that every Landsberg metric with vanishing mean Berwald curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. $\Box$

**Proof of Corollary 1.2**: Let $(M, F)$ be a Douglas manifold of dimension $n$. Suppose that $F$ has vanishing Landsberg curvature. In [4], Berwald proved that every 2-dimensional Douglas metric with vanishing Landsberg curvature is a Berwald metric. In 1984, Izumi pointed out that the Berwald theorem must be true for the higher dimensions [8]. In [1], Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. $\Box$

**Proof of Corollary 1.3**: According to by Theorem 1.1, $F$ is a Landsberg metric. In [16], Numata proved that every Landsberg metric of non-zero scalar flag curvature is a Riemannian metric of constant sectional curvature. This completes the proof. $\Box$

**Proof of Corollary 1.4**: In [19], Shen proved that an $(\alpha, \beta)$-metric with vanishing Landsberg curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. $\square$

**Proof of Corollary 1.5**: In [14], Mo-Zhou classified the spherically symmetric Finsler metrics in $\mathbb{R}^n$ with Landsberg type and found some exceptional almost regular metrics which do not belong to Berwald type. They proved that every regular spherically symmetric Finsler metric in $\mathbb{R}^n$ is a Berwald metric. Then they proved that all of Berwaldian spherically symmetric Finsler metrics are Riemannian. Then by Theorem 1.1, we get the proof. $\square$

## 4. Stretch-Recurrent Homogeneous Metrics

In this section, we are going to prove Theorem 1.2.

**Proof of Theorem 1.2**: We know that $(M, F)$ is homogeneous, and the scalar function $\Psi$ is invariant under the isometries of $F$. In general, if a continuous function $f : TM_0 \to \mathbb{R}$ is invariant under isometries of $(M, F)$ and also is positively homogeneous of degree zero with respect to directions, then $f$ is a bounded function. Thus, $f := \Psi/F$ is bounded and, by definition, is everywhere non-zero. Since $M$ is connected, the range of $\Psi/F$ is an interval, say $(c_1, c_2) \subseteq \mathbb{R}$, which does not contain zero. Without loss of generality, suppose that $c_1 > 0$. Thus, we have

$$c_1 F(x, y) \leq \Psi(x, y) \leq c_2 F(x, y), \quad \forall (x, y) \in TM_0. \tag{4.1}$$

For $y \in T_x M$, let $c = c(t)$ be the unit speed geodesic of $(M, F)$ with $\dot{c}(0) = y$ and $c(0) = x$. Suppose $X = X(t)$, $Y = Y(t)$, $Z = Z(t)$ and $W = W(t)$ are parallel vector fields along the geodesic $c$. Define $\mathbf{\Sigma}(t)$ as follows

$$\mathbf{\Sigma}(t) = \mathbf{\Sigma}_{\dot{c}}\Big(X(t), Y(t), Z(t), W(t)\Big). \tag{4.2}$$

Thus, the restriction of (1.4) to the canonical lift of $c$, i.e., $(c, \dot{c})$ becomes

$$\mathbf{\Sigma}'(t) = \Psi(t)\mathbf{\Sigma}(t). \tag{4.3}$$

For simplicity, we have used the following nomination:

$$\Psi(t) := \Psi\big(c(t), \dot{c}(t)\big).$$

By (4.3), we get

$$\mathbf{\Sigma}(t) = e^{\int_0^t \Psi(s)\ ds}\,\mathbf{\Sigma}(0) \tag{4.4}$$

It follows from (4.1) and $F(c(t), \dot{c}(t)) = 1$ that

$$e^{c_1 t} \leq e^{\int_0^t \Psi(s)\ ds} \leq e^{c_2 t}, \quad \forall t > 0. \tag{4.5}$$

The stretch tensor of any homogeneous metric is a bounded tensor. Let $\mathbf{\Sigma}(0) \neq 0$. By Theorem 3.1, $M$ is complete, and the parameter $t$ takes all the values in $(-\infty, +\infty)$. Letting $t \to \infty$, we conclude that the norm of $\mathbf{\Sigma}(t)$ is unbounded which arises a contradiction. Therefore, we get

$$\mathbf{\Sigma}(0) = 0,$$

and $F$ reduces to a stretch metric. On the other hand, in [28] it is proved that every homogenous stretch metric is a Landsberg metric. This completes the proof. □

## REFERENCES


1. S. Bácsó and M. Matsumoto: *Reduction theorems of certain Douglas spaces to Berwald spaces*, Publ. Math. Debrecen. **62**(2003), 315-324.
2. L. Berwald: *Parallelubertragung in allgemeinen Räumen*, Atti. Cong. Internal. Math. Bologna. **4**(1928), 263-270.
3. L. Berwald: *Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung*, Jber. Deutsch. Math.-Verein, **34**(1925), 213-220.
4. L. Berwald: *On Finsler and Cartan geometries, III. Two-dimensional Finsler spaces with rectilinear extremals*, Ann. of Math. **42**(1941), 84-112.
5. L. Berwald: *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z. **25**(1926), 40-73.
6. M. Crampin: *A condition for a Landsberg space to be Berwaldian*, Publ. Math. Debrecen, **8111**(2018), 1-13.
7. S. Deng and Z. Hou: *The group of isometries of a Finsler space*, Pacific. J. Math, **207**(1) (2002), 149-155.
8. H. Izumi: *Some problems in special Finsler spaces*, Symp. on Finsler Geom. at Kinosaki, Nov. 14, 1984.
9. D. Latifi and A. Razavi: *On homogeneous Finsler spaces*, Rep. Math. Phys. **57**(2006), 357-366.
10. M. Matsumoto: *An improvment proof of Numata and Shibata's theorem on Finsler spaces of scalar curvature*, Publ. Math. Debrecen, **64**(2004), 489-500.
11. M. Matsumoto: *Remarks on Berwald and Landsberg spaces*, Contemp. Math., **196**(1996), 79-82.
12. M. Matsumoto and S. Hōjō: *A conclusive theorem for C-reducible Finsler spaces*, Tensor. N. S. **32**(1978), 225-230.
13. V.S. Matveev: *On "Regular Landsberg metrics are always Berwald" by Z. I. Szabó*, Balkan J. Geom. Appl. **14**(2) (2009), 50-52.
14. X. Mo and L. Zhou: *The curvatures of spherically symmetric Finsler metrics in* $R^n$, arXiv:1202.4543.
15. B. Najafi and A. Tayebi: *Weakly stretch Finsler metrics*, Publ. Math. Debrecen. **91**(2017), 441-454.
16. S. Numata: *On Landsberg spaces of scalar curvature*, J. Korean. Math. Soc. **12**(1975), 97-100.

17. S.F. Rutz: *Symmetry in Finsler spaces*, Contemp. Math. **196**(1996), 289-170.

18. Z. Shen: *On R-quadratic Finsler spaces*, Publ. Math. Debrecen. **58**(2001), 263-274.

19. Z. Shen: *On a class of Landsberg metrics in Finsler geometry*, Canadian. J. Math. **61**(2009), 1357-1374.

20. Z. Shen: *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

21. C. Shibata: *On Finsler spaces with Kropina metrics*, Report. Math. **13**(1978), 117-128.

22. Z. I. Szabó: *All regular Landsberg metrics are Berwald*, Ann. Glob. Anal. Geom. **34**(2008), 381-386; correction, ibid, **35**(2009), 227-230.

23. T. Tabatabeifar, B. Najafi and A. Tayebi: *On weighted projective Ricci curvature in Finsler geometry*, Math. Slovaca. **71**(2021), 183-198.

24. A. Tayebi: *On generalized 4-th root metrics of isotropic scalar curvature*, Math. Slovaca. **68**(2018), 907-928.

25. A. Tayebi: *On 4-th root Finsler metrics of isotropic scalar curvature*, Math. Slovaca. **70**(2020), 161-172.

26. A. Tayebi and M. Barzagari: *Generalized Berwald spaces with $(\alpha, \beta)$-metrics*, Indagationes Mathematicae. **27**(2016), 670-683.

27. A. Tayebi and B. Najafi: *On homogeneous Landsberg surfaces*, J. Geom. Phys. **168** (2021), 104314.

28. A. Tayebi and B. Najafi: *A class of homogeneous Finsler metrics*, J. Geom. Phys. **140**(2019), 265-270.

29. A. Tayebi and B. Najafi: *On homogeneous isotropic Berwald metrics*, European. J. Math. **7**(2021), 404-415.

30. A. Tayebi and H. Sadeghi: *On Cartan torsion of Finsler metrics*, Publ. Math. Debrecen. **82**(2013), 461-471.

31. A. Tayebi and H. Sadeghi: *Generalized P-reducible $(\alpha, \beta)$-metrics with vanishing S-curvature*, Ann. Polon. Math. **114**(1) (2015), 67-79.

32. A. Tayebi and H. Sadeghi: *On generalized Douglas-Weyl $(\alpha, \beta)$-metrics*, Acta Mathematica Sinica, English Series, **31**(2015), 1611-1620.

33. A. Tayebi and H. Sadeghi: *On a class of stretch metrics in Finsler geometry*, Arabian Journal of Mathematics, (2018), accepted.

34. A. Tayebi and T. Tabatabeifar: *Dougals-Randers manifolds with vanishing stretch tensor*, Publ. Math. Debrecen. **86**(2015), 423-432.

35. M. Xu and S. Deng: *The Landsberg equation of a Finsler space*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) XXII (2021) 3117.

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# ON THE BI-$P$-HARMONIC MAPS AND THE CONFORMAL MAPS

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**Abstract.** The objective of this paper is to study the bi-$p$-harmonicity of a conformal maps. We establish necessary and sufficient condition for a conformal map to be bi-$p$-harmonic and we construct several examples of this type of maps.
**Keywords**: $p$-harmonic map, bi-$p$-harmonic map, conformal map.

## 1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds. Then $\phi$ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, $\phi$ is harmonic if it satisfies the associated Euler-Lagrange equations given as follows:

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of $\phi$. The map $\phi$ is said to be biharmonic if it is a critical point of the bi-energy functional:

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

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The biharmonicity of $\phi$ is characterized by the following equation:

$$\tau_2(\phi) = -Tr_g\left(\nabla^\phi\right)^2\tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where $\nabla^\phi$ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on $M$, then

$$Tr_g\left(\nabla^\phi\right)^2\tau(\phi) = \left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla^M_{e_i}e_i}\right)\tau(\phi).$$

We will call the operator $\tau_2(\phi)$, the bi-tension field of the map $\phi$. A generalization of harmonic and biharmonic maps, $p$-harmonic and bi-$p$-harmonic maps are defined as follows : Let $p\geq 2$, the $p$-energy functional of $\phi$ is defined by

$$E_p(\phi) = \frac{1}{p}\int_M |d\phi|^p dv_g.$$

$\phi$ is said to be $p$-harmonic if it is a critical point of the $p$-energy functional (with respect to any variation of compact support). Equivalently, $\phi$ is $p$-harmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_p(\phi) = |d\phi|^{p-2}\left\{\tau(\phi) + (p-2)\,d\phi\left(grad\ln|d\phi|\right)\right\} = 0,$$

$\tau_p(\phi)$ is called the $p$-tension field of $\phi$, one can refer to [1], [12] and [15] for more details on $p$-harmonic maps. The bi-$p$-energy of $\phi$ is defined by (see [4]) :

$$E_{2,p}(\phi) = \frac{1}{2}\int_M |\tau_p(\phi)|^2 dv_g.$$

Equivalently, $\phi$ is bi-$p$-harmonic if it satisfies the following equation:

$$\begin{aligned}\tau_{2,p}(\phi) = &-Tr_g\nabla^\phi|d\phi|^{p-2}\nabla^\phi\tau_p(\phi) - |d\phi|^{p-2}Tr_g R^N(\tau_p(\phi), d\phi)\,d\phi\\ &-(p-2)\,Tr_g\nabla^\phi\left(\langle\nabla\tau_p(\phi), d\phi\rangle|d\phi|^{p-4}d\phi\right) = 0,\end{aligned} \tag{1.1}$$

where

$$Tr_g\nabla^\phi|d\phi|^{p-2}\nabla^\phi\tau_p(\phi) = \nabla^\phi_{e_i}|d\phi|^{p-2}\nabla^\phi_{e_i}\tau_p(\phi) - |d\phi|^{p-2}\nabla^\phi_{\nabla_{e_i}e_i}\tau_p(\phi)$$

and

$$\begin{aligned}Tr_g\nabla^\phi\left(\langle\nabla\tau_p(\phi), d\phi\rangle|d\phi|^{p-4}d\phi\right) = &\nabla^\phi_{e_i}|d\phi|^{p-4}\langle\nabla\tau_p(\phi), d\phi\rangle d\phi(e_i)\\ &-|d\phi|^{p-4}\langle\nabla\tau_p(\phi), d\phi\rangle d\phi(\nabla_{e_i}e_i).\end{aligned}$$

$\tau_{2,p}(\phi)$ is called the bi-$p$-tension of $\phi$. Following Jiang's notion (see [9]), we define stress bi-$p$-energy tensor associated to the bi-$p$-energy functionals by varying the functionals with respect to the metric on the domain (see [11]). For any $X, Y \in \Gamma(TM)$, we have

$$
\begin{aligned}
S_{2,p}(\phi)(X,Y) = {} & \frac{1}{2}|\tau_p(\phi)|^2 g(X,Y) + |d\phi|^{p-2}\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle g(X,Y) \\
& - |d\phi|^{p-2}\left\{h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right) + h\left(d\phi(Y), \nabla^{\phi}_{X}\tau_p(\phi)\right)\right\} \\
& - (p-2)|d\phi|^{p-4}\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle h(d\phi(X), d\phi(Y)).
\end{aligned} \tag{1.2}
$$

The stress bi-$p$-energy tensor of $\phi$ satisfies the following relationship

$$
divS_{2,p}(\phi) = -h(\tau_{2,p}(\phi), d\phi).
$$

The notion of bi-$p$-harmonic maps was introduced by A.M.Cherif [4] where he gave the Euler-Lagrange equations associated with the bi-$p$-energy and he proved a Liouville type theorem for this class of maps. It is important to recall that the $p$-biharmonic maps are the critical points of the $p$-bi-energy functional

$$
E_{p,2}(\phi) = \frac{1}{p}\int_M |\tau(\phi)|^p dv_g,
$$

and this type of maps was studied in [3], [5] and [8]. This paper is a continuation of Cherif's work [4] on bi-$p$-harmonic maps where we study the bi-$p$-harmonicity of a conformal map $\phi : (M^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$, we calculate $\tau_{2,p}(\phi)$ and we prove that any conformal map is bi-$p$-harmonic if and only if the gradient of its dilation satisfies a certain second-order elliptic partial differential equation. From these results, we construct new examples of bi-$p$-harmonic maps.

## 2. The main results

In the first we give the relation between $\tau_{2,p}(\phi)$ and $\tau_p(\phi)$.

**Proposition 2.1.** *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map, then the relation between $\tau_{2,p}(\phi)$ and $\tau_p(\phi)$ is given by the following equation*

$$
\begin{aligned}
\tau_{2,p}(\phi) = {} & -|d\phi|^{p-2}\left(Tr_g\left(\nabla^{\phi}\right)^2\tau_p(\phi) + Tr_g R^N(\tau_p(\phi), d\phi)d\phi\right) \\
& + (p-2)|d\phi|^{p-4}\langle\nabla\tau_p(\phi), d\phi\rangle d\phi\left(grad\left(\ln|d\phi|^2\right)\right) \\
& - (p-2)|d\phi|^{p-4}d\phi(grad\langle\nabla\tau_p(\phi), d\phi\rangle) \\
& - (p-2)|d\phi|^{-2}\langle\nabla\tau_p(\phi), d\phi\rangle\tau_p(\phi) \\
& - \frac{(p-2)}{2}|d\phi|^{p-2}\nabla^{\phi}_{grad\left(\ln|d\phi|^2\right)}\tau_p(\phi).
\end{aligned} \tag{2.1}
$$

**Proof of Proposition 2.1.** Let us choose $\{e_i\}_{1\leq i\leq m}$ to be an orthonormal frame on $(M, g)$. By definition, we have

$$
\begin{aligned}
\tau_{2,p}(\phi) = {} & -Tr_g\nabla^{\phi}|d\phi|^{p-2}\nabla^{\phi}\tau_p(\phi) - |d\phi|^{p-2}Tr_g R^N(\tau_p(\phi), d\phi)d\phi \\
& - (p-2)Tr_g\nabla^{\phi}\left(\langle\nabla\tau_p(\phi), d\phi\rangle|d\phi|^{p-4}d\phi\right).
\end{aligned} \tag{2.2}
$$

For the term $Tr_g \nabla^{\phi} |d\phi|^{p-2} \nabla^{\phi} \tau_p(\phi)$, we obtain

$$Tr_g \nabla^{\phi} |d\phi|^{p-2} \nabla^{\phi} \tau_p(\phi) = \nabla^{\phi}_{e_i} |d\phi|^{p-2} \nabla^{\phi}_{e_i} \tau_p(\phi) - |d\phi|^{p-2} \nabla^{\phi}_{\nabla_{e_i} e_i} \tau_p(\phi),$$

a simple calculation gives us

$$\begin{aligned}\nabla^{\phi}_{e_i} |d\phi|^{p-2} \nabla^{\phi}_{e_i} \tau_p(\phi) &= |d\phi|^{p-2} \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} \tau_p(\phi) + e_i \left( |d\phi|^{p-2} \right) \nabla^{\phi}_{e_i} \tau_p(\phi) \\ &= |d\phi|^{p-2} \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} \tau_p(\phi) + \frac{(p-2)}{2} |d\phi|^{p-2} \nabla_{grad\left(\ln|d\phi|^2\right)} \tau_p(\phi),\end{aligned}$$

then

$$\begin{aligned}Tr_g \nabla^{\phi} |d\phi|^{p-2} \nabla^{\phi} \tau_p(\phi) &= |d\phi|^{p-2} Tr_g \left(\nabla^{\phi}\right)^2 \tau_p(\phi) \\ &+ \frac{(p-2)}{2} |d\phi|^{p-2} \nabla^{\phi}_{grad\left(\ln|d\phi|^2\right)} \tau_p(\phi).\end{aligned} \tag{2.3}$$

We will develop the term $Tr_g \nabla^{\phi} \left( \langle \nabla \tau_p(\phi), d\phi \rangle |d\phi|^{p-4} d\phi \right)$, we have

$$\begin{aligned}&Tr_g \nabla^{\phi} \left( \langle \nabla \tau_p(\phi), d\phi \rangle |d\phi|^{p-4} d\phi \right) \\ &= \nabla^{\phi}_{e_i} |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle d\phi(e_i) - |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle d\phi(\nabla_{e_i} e_i) \\ &= |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle \nabla^{\phi}_{e_i} d\phi(e_i) + e_i \left( |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle \right) d\phi(e_i) \\ &- |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle d\phi(\nabla_{e_i} e_i) \\ &= |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle \nabla^{\phi}_{e_i} d\phi(e_i) - |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle d\phi(\nabla_{e_i} e_i) \\ &+ |d\phi|^{p-4} e_i \left( \langle \nabla \tau_p(\phi), d\phi \rangle \right) d\phi(e_i) + \langle \nabla \tau_p(\phi), d\phi \rangle e_i \left( |d\phi|^{p-4} \right) d\phi(e_i) \\ &= |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle \tau(\phi) + |d\phi|^{p-4} d\phi \left( grad \langle \nabla \tau_p(\phi), d\phi \rangle \right) \\ &+ \frac{p-4}{2} |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle d\phi \left( grad \left( \ln |d\phi|^2 \right) \right).\end{aligned}$$

Using the fact that

$$\tau(\phi) = |d\phi|^{-p+2} \tau_p(\phi) - \frac{(p-2)}{2} d\phi \left( grad \left( \ln |d\phi|^2 \right) \right),$$

it follows that

$$\begin{aligned}Tr_g \nabla^{\phi} \left( \langle \nabla \tau_p(\phi), d\phi \rangle |d\phi|^{p-4} d\phi \right) &= |d\phi|^{-2} \langle \nabla \tau_p(\phi), d\phi \rangle \tau_p(\phi) \\ &+ |d\phi|^{p-4} d\phi \left( grad \langle \nabla \tau_p(\phi), d\phi \rangle \right) \\ &- |d\phi|^{p-4} \langle \nabla \tau_p(\phi), d\phi \rangle d\phi \left( grad \left( \ln |d\phi|^2 \right) \right).\end{aligned} \tag{2.4}$$

By replacing (2.3) and (2.4) in (2.2), we deduce that

$$\begin{aligned}\tau_{2,p}(\phi) =& -|d\phi|^{p-2}\left(Tr_g\left(\nabla^{\phi}\right)^2\tau_p(\phi)+Tr_gR^N\left(\tau_p(\phi),d\phi\right)d\phi\right)\\ &+(p-2)|d\phi|^{p-4}\langle\nabla\tau_p(\phi),d\phi\rangle d\phi\left(grad\left(\ln|d\phi|^2\right)\right)\\ &-(p-2)|d\phi|^{p-4}d\phi\left(grad\langle\nabla\tau_p(\phi),d\phi\rangle\right)\\ &-(p-2)|d\phi|^{-2}\langle\nabla\tau_p(\phi),d\phi\rangle\tau_p(\phi)\\ &-\frac{(p-2)}{2}|d\phi|^{p-2}\nabla^{\phi}_{grad\left(\ln|d\phi|^2\right)}\tau_p(\phi).\end{aligned}$$

**Theorem 2.1.** *Let $\phi:(M^n,g)\longrightarrow(N^n,h)$ $(n\geq 3)$ be a conformal map of dilation $\lambda$ , then the bi-p-tension of $\phi$ is given by*

$$\tau_{2,p}(\phi)=(n-p)\,n^{p-3}\lambda^{2p-4}d\phi\left(H(\lambda,n,p)\right),$$

*where*

$$\begin{aligned}H(\lambda,n,p)=&\,(n+p-2)\,grad\left(\Delta\ln\lambda\right)\\ &-\frac{\left(n^2-5np+4n-2p^2+8p-8\right)}{2}grad\left(|grad\ln\lambda|^2\right)\\ &-(p-1)\left(n^2-3np+4n-2p^2+8p-8\right)|grad\ln\lambda|^2\,grad\ln\lambda\\ &-2\left(n-p^2+3p-2\right)(\Delta\ln\lambda)\,grad\ln\lambda+2nRicci\left(grad\ln\lambda\right).\end{aligned}$$

**Lemma 2.1.** *Let $\phi:(M^m,g)\longrightarrow(N^n,h)$ be a smooth map. For any vector filed $X$ and for any smooth function $f$ on $M$, we have*

$$Tr_g\left(\nabla^{\phi}\right)^2 fd\phi(X)=fTr_g\left(\nabla^{\phi}\right)^2 d\phi(X)+2\nabla^{\phi}_{gradf}d\phi(X)+(\Delta f)\,d\phi(X).$$

**Proof of Theorem 2.1.** The fact that the map $\phi$ is conformal of dilation $\lambda$ gives us

$$\tau(\phi)=(2-n)\,d\phi\left(grad\ln\lambda\right),\qquad |d\phi|^2=n\lambda^2,\qquad |d\phi|^{p-2}=n^{\frac{p-2}{2}}\lambda^{p-2}$$

and

$$grad\left(\ln|d\phi|^2\right)=2grad\ln\lambda.$$

Then

$$\tau_p(\phi)=(p-n)\,n^{\frac{p-2}{2}}\lambda^{p-2}d\phi\left(grad\ln\lambda\right).$$

By replacing the expression of $\tau_p(\phi)$ in (2.1), we obtain

$$\begin{aligned}\tau_{2,p}(\phi)=&-(p-n)\,n^{p-2}\lambda^{p-2}Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right)\\ &-(p-n)\,n^{p-2}\lambda^{p-2}Tr_gR^N\left(\lambda^{p-2}d\phi\left(grad\ln\lambda\right),d\phi\right)d\phi\\ &-(p-2)(p-n)\,n^{p-2}\lambda^{p-2}\nabla^{\phi}_{grad\ln\lambda}\lambda^{p-2}d\phi\left(grad\ln\lambda\right)\\ &-(p-2)(p-n)^2n^{p-3}\lambda^{p-4}\left\langle\nabla\lambda^{p-2}d\phi\left(grad\ln\lambda\right),d\phi\right\rangle d\phi\left(grad\ln\lambda\right)\\ &-(p-2)(p-n)\,n^{p-3}\lambda^{p-4}d\phi\left(grad\left\langle\nabla\lambda^{p-2}d\phi\left(grad\ln\lambda\right),d\phi\right\rangle\right)\\ &+2(p-2)(p-n)\,n^{p-3}\lambda^{p-4}\left\langle\nabla\lambda^{p-2}d\phi\left(grad\ln\lambda\right),d\phi\right\rangle d\phi\left(grad\ln\lambda\right).\end{aligned}\tag{2.5}$$

We will simplify the terms of this last equation.
For the term $Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right)$, we have

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right) &= \lambda^{p-2}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\lambda\right)\\&+2\nabla^{\phi}_{grad\lambda^{p-2}}d\phi\left(grad\ln\lambda\right)\\&+\left(\Delta\lambda^{p-2}\right)d\phi\left(grad\ln\lambda\right).\end{aligned}$$

The fact that $\phi$ is conformal gives us (see [13])

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\lambda\right) &= d\phi\left(grad\Delta\ln\lambda\right)+2d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right)\\&-(n-2)\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\&-\left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right)+d\phi\left(Ricci\left(grad\ln\lambda\right)\right)\end{aligned}$$

and

$$\begin{aligned}2\nabla^{\phi}_{grad\lambda^{p-2}}d\phi\left(grad\ln\lambda\right) &= 2\left(p-2\right)\lambda^{p-2}\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\&+\left(p-2\right)\lambda^{p-2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right).\end{aligned}$$

A simple calculation gives

$$\Delta\lambda^{p-2}=\left(p-2\right)\lambda^{p-2}\left(\Delta\ln\lambda+\left(p-2\right)\left|grad\ln\lambda\right|^2\right),$$

then

$$\begin{aligned}Tr_g\left(\nabla^{\phi}\right)^2\lambda^{p-2}d\phi\left(grad\ln\lambda\right) &= \lambda^{p-2}d\phi\left(grad\Delta\ln\lambda\right)\\&+p\lambda^{p-2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right)\\&-\left(n-p^2+2p-2\right)\lambda^{p-2}\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\&+\left(p-3\right)\lambda^{p-2}\left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right)\\&+\lambda^{p-2}d\phi\left(Ricci\left(grad\ln\lambda\right)\right).\end{aligned}\tag{2.6}$$

The fact that $\phi$ conformal also gives us the following formulas (see [13])

$$\begin{aligned}Tr_gR^N\left(d\phi\left(grad\ln\lambda\right),d\phi\right)d\phi &= -\frac{n-2}{2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right)\\&-\left(\Delta\ln\lambda\right)d\phi\left(grad\ln\lambda\right)\\&+d\phi\left(Ricci\left(grad\ln\lambda\right)\right)\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}\nabla^{\phi}_{grad\ln\lambda}\lambda^{p-2}d\phi\left(grad\ln\lambda\right) &= \left(p-1\right)\lambda^{p-2}\left|grad\ln\lambda\right|^2 d\phi\left(grad\ln\lambda\right)\\&+\frac{1}{2}\lambda^{p-2}d\phi\left(grad\left(\left|grad\ln\lambda\right|^2\right)\right).\end{aligned}\tag{2.8}$$

For the term $\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle$, we have

$$\begin{aligned}\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle &= Tr_g h\left(\nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right) \\ &= h\left(\nabla_{e_i} \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\left(e_i\right)\right) \\ &= \lambda^{p-2} h\left(\nabla_{e_i} d\phi\left(grad \ln \lambda\right), d\phi\left(e_i\right)\right) \\ &+ e_i\left(\lambda^{p-2}\right) h\left(d\phi\left(grad \ln \lambda\right), d\phi\left(e_i\right)\right) \\ &= \lambda^{p-2}\left(\lambda^2 \Delta \ln \lambda + n\lambda^2 \left|grad \ln \lambda\right|^2\right) \\ &+ (p-2)\lambda^{p-2}\lambda^2 \left|grad \ln \lambda\right|^2.\end{aligned}$$

Then

$$\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle = \lambda^p\left(\Delta \ln \lambda + (n+p-2)\left|grad \ln \lambda\right|^2\right). \tag{2.9}$$

Finally, using the following formulas

$$grad\left(\lambda^p\left(\Delta \ln \lambda\right)\right) = \lambda^p grad \Delta \ln \lambda + p\lambda^p\left(\Delta \ln \lambda\right) grad \ln \lambda$$

and

$$grad\left(\lambda^p \left|grad \ln \lambda\right|^2\right) = \lambda^p grad\left(\left|grad \ln \lambda\right|^2\right) + p\lambda^p \left|grad \ln \lambda\right|^2 grad \ln \lambda,$$

we obtain

$$\begin{aligned}grad\left\langle \nabla \lambda^{p-2} d\phi\left(grad \ln \lambda\right), d\phi\right\rangle &= \lambda^p grad \Delta \ln \lambda + p\lambda^p\left(\Delta \ln \lambda\right) grad \ln \lambda \\ &+ \lambda^p (n+p-2)\, grad\left(\left|grad \ln \lambda\right|^2\right) \\ &+ p(n+p-2)\lambda^p \left|grad \ln \lambda\right|^2 grad \ln \lambda\end{aligned} \tag{2.10}$$

If we replace (2.6), (2.7), (2.8),(2.9) and (2.10) in (2.5), we conclude that

$$\tau_{2,p}(\phi) = (n-p)\, n^{p-3}\lambda^{2p-4} d\phi\left(H(\lambda, n, p)\right),$$

where

$$\begin{aligned}H(\lambda, n, p) &= (n+p-2)\, grad\left(\Delta \ln \lambda\right) \\ &- \frac{\left(n^2 - 5np + 4n - 2p^2 + 8p - 8\right)}{2} grad\left(\left|grad \ln \lambda\right|^2\right) \\ &- (p-1)\left(n^2 - 3np + 4n - 2p^2 + 8p - 8\right)\left|grad \ln \lambda\right|^2 grad \ln \lambda \\ &- 2\left(n - p^2 + 3p - 2\right)\left(\Delta \ln \lambda\right) grad \ln \lambda + 2nRicci\left(grad \ln \lambda\right).\end{aligned}$$

**Theorem 2.2.** *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation $\lambda$ , then $\phi$ is bi-p-harmonic if and only if*

$$\begin{aligned}&(n+p-2)\, grad\left(\Delta \ln \lambda\right) - \frac{\left(n^2 - 5np + 4n - 2p^2 + 8p - 8\right)}{2} grad\left(\left|grad \ln \lambda\right|^2\right) \\ &- (p-1)\left(n^2 - 3np + 4n - 2p^2 + 8p - 8\right)\left|grad \ln \lambda\right|^2 grad \ln \lambda \\ &- 2\left(n - p^2 + 3p - 2\right)\left(\Delta \ln \lambda\right) grad \ln \lambda + 2nRicci\left(grad \ln \lambda\right) = 0.\end{aligned}$$

If we consider a conformal map $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ where we suppose that the dilation $\lambda$ is radial, then the bi-$p$-harmonicity of $\phi$ is equivalent to an ordinary differential equation.

**Corollary 2.1.** *Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation $\lambda$ where we suppose that the dilation $\lambda$ is radial $(\lambda = \lambda(r), r = |x|)$. By setting $\beta = (\ln \lambda)'$, we get (see [13])*

$$grad \ln \lambda = \beta \frac{\partial}{\partial r}, \qquad |grad \ln \lambda|^2 = \beta^2, \qquad grad\left(|grad \ln \lambda|^2\right) = 2\beta\beta' \frac{\partial}{\partial r}$$

*and*

$$\Delta \ln \lambda = \beta' + \frac{n-1}{r}\beta, \qquad grad \Delta \ln \lambda = \left(\beta'' + \frac{n-1}{r}\beta' - \frac{n-1}{r^2}\beta\right)\frac{\partial}{\partial r}.$$

*Using Theorem 2.2, we deduce that $\phi$ is bi-p-harmonic if and only if $\beta$ satisfies the following differential equation :*

(2.11)

$$\begin{aligned}&(n+p-2)\beta'' - \left(n^2 - 5np + 6n - 4p^2 + 14p - 12\right)\beta\beta' + \frac{(n+p-2)(n-1)}{r}\beta' \\ &- \frac{(n+p-2)(n-1)}{r^2}\beta + \frac{2\left(p^2 - 3p - n + 2\right)(n-1)}{r}\beta^2 \\ &+ (p-1)\left(-n^2 + 3np - 4n + 2p^2 - 8p + 8\right)\beta^3 = 0.\end{aligned}$$

To solve equation (2.11) , we will study two types of solutions. In the first case, we look at the solutions which are written in the form $\beta = \frac{a}{r}, a \in \mathbb{R}^*$, we obtain the following result.

**Corollary 2.2.** *Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$ $(n \geq 3)$ be a conformal map of dilation $\lambda$ where we suppose that $(\ln \lambda)' = \beta = \frac{a}{r}, a \in \mathbb{R}^*$. Then $\phi$ is bi-p-harmonic if and only if $a$ is solution of the following algebraic equation :*

(2.12)

$$\begin{aligned}&a^2n^2p - a^2n^2 - 3a^2np^2 + 7a^2np - 4a^2n - 2a^2p^3 + 10a^2p^2 - 16a^2p + 8a^2 + an^2 \\ &- 2anp^2 + 11anp - 12an + 6ap^2 - 20ap + 16a + 2n^2 + 2np - 8n - 4p + 8 = 0.\end{aligned}$$

**Remark 2.1.** Equation (2.12) leads us to two types of solutions

1.
$$a = -\frac{2(n-2)\left(n + \sqrt{n(17n-16)}\right)}{(3n^2 - 6n + 4)\sqrt{n(17n-16)} - 13n^3 + 42n^2 - 28n}$$

   and
$$p = \frac{1}{4}\sqrt{n(17n-16)} - \frac{3}{4}n + 2,$$

   where $n \geq 3$.

2.
$$a=\frac{A\left(n,p\right)-12n-20p-2np^{2}+11np+n^{2}+6p^{2}+16}{8n+32p+6np^{2}-2n^{2}p-14np+2n^{2}-20p^{2}+4p^{3}-16}$$
or
$$a=-\frac{A\left(n,p\right)+12n+20p+2np^{2}-11np-n^{2}-6p^{2}-16}{8n+32p+6np^{2}-2n^{2}p-14np+2n^{2}-20p^{2}+4p^{3}-16},$$
where
$$A\left(n,p\right)=\sqrt{\begin{array}{c}4\left(n-1\right)^{2}p^{4}-4\left(n-1\right)\left(n-4\right)p^{3}+\left(12n^{3}-35n^{2}+8n+16\right)p^{2}\\ -2n\left(4n^{3}-3n^{2}-16n+16\right)p+n^{2}\left(3n-4\right)^{2}\end{array}}$$
and
$$p\neq\frac{1}{4}\sqrt{n\left(17n-16\right)}-\frac{3}{4}n+2$$

Remark 2.1 allows us to study the following examples. The examples to be cited correspond to the cases where $a=-2$ and $a=-1$.

**Example 2.1.** We consider the inversion $\phi:\mathbb{R}^{n}\backslash\{0\}\longrightarrow\mathbb{R}^{n}\backslash\{0\}$ $(n\geq 3)$ defined by $\phi\left(x\right)=\frac{x}{|x|^{2}}$. $\phi$ is a conformal map with dilation $\lambda=\frac{1}{r^{2}}$. We deduce that $\phi$ is bi-$p$-harmonic if and only if
$$p=-\frac{1}{2}n+\frac{1}{4}\sqrt{-20n+12n^{2}+9}+\frac{5}{4},\qquad n\geq 4$$
or
$$p=-\frac{3}{4}n+\frac{1}{4}\sqrt{n\left(17n-16\right)}+2,\qquad n\geq 3.$$

**Example 2.2.** Let $\phi:\mathbb{R}^{n}\backslash\{0\}\longrightarrow mathbbR\times S^{n-1}$ given in polar coordinates by
$$\phi\left(r\theta\right)=\left(\ln r,\theta\right),\qquad r>0,\qquad \theta\in S^{n-1}\subset\mathbb{R}^{n}.$$
$\phi$ is a conformal map with dilation $\lambda=\frac{1}{r}$. We conclude that $\phi$ is bi-$p$-harmonic if and only if
$$p=\frac{n}{2},\qquad n\geq 4$$
or
$$p=-\frac{3}{4}n+\frac{1}{4}\sqrt{n\left(17n-16\right)}+2,\qquad n\geq 3.$$

As a second particular case, we will look for the solutions of the form $\beta=\frac{a}{1+r^{2}}, a\in\mathbb{R}^{*}$.

**Corollary 2.3.** *Let $\phi:(\mathbb{R}^{n},g)\longrightarrow(N^{n},h)$ $(n\geq 3)$ be a conformal map of dilation $\lambda$ where we suppose that $\left(\ln\lambda\right)'=\beta=\frac{a}{1+r^{2}}, a\in\mathbb{R}^{*}$. Then $\phi$ is bi-p-harmonic if and only if a is solution of the following system:*
$$\left\{\begin{array}{c}n^{5}p+2n^{5}-3n^{4}p^{2}-6n^{4}p-4n^{4}+n^{3}p^{3}+6n^{3}p^{2}\\ +14n^{3}p-4n^{3}+3n^{2}p^{4}-6n^{2}p^{3}-12n^{2}p^{2}+4n^{2}p\\ +8n^{2}-2np^{5}+4np^{4}-2np^{3}+16np^{2}-24np\\ +4p^{4}-16p^{3}+16p^{2}=0\\ \textit{and}\\ 3an^{2}-2anp^{2}+anp-2ap^{2}+8ap-8a+2n^{2}+2np+4p-8=0\end{array}\right.\tag{2.13}$$

**Remark 2.2.** To solve this system, we distingue three cases

1.
$$p = n, \qquad a = \frac{2}{n-2}, \qquad n \geq 3.$$
In this case, the conformal map is $n$-harmonic so bi-$n$-harmonic.

2.
$$p = \frac{n}{2}, \qquad a = \frac{6n-8}{n^2-8n+8}, \qquad n \geq 4.$$
Then $\phi$ is bi-$p$-harmonic non-$p$-harmonic.

3.
$$p = \frac{1}{2n}\left(\sqrt{-16n+4n^2+8n^3+n^4+4}+n^2+2\right)$$
and
$$a = -\frac{2n^2+2np+4p-8}{3n^2-2np^2+np-2p^2+8p-8}, \qquad n \geq 3.$$
Then $\phi$ is bi-$p$-harmonic non-$p$-harmonic.

As the last result of this paper, we calculate the stress bi-$p$-energy tensor for a conformal map.

**Theorem 2.3.** *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map of dilation $\lambda$, then we have*

$$\begin{aligned} & S_{2,p}(\phi)(X,Y) \\ & = \frac{p-n}{2} n^{p-3} \lambda^{2p-2} \left( n(n+p-4) - 2(p-2)^2 \right) |grad \ln \lambda|^2 g(X,Y) \\ & + (p-n)(n-p+2) n^{p-3} \lambda^{2p-2} (\Delta \ln \lambda) g(X,Y) \\ & - 2(p-n) n^{p-2} \lambda^{2p-2} \left( \nabla d \ln \lambda (X,Y) - (p-2) X(\ln \lambda) Y(\ln \lambda) \right), \end{aligned} \tag{2.14}$$

*and the trace of $S_{2,p}(\phi)$ is given by*

$$\begin{aligned} & Tr_g S_{2,p}(\phi) \\ & = \frac{p-n}{2} n^{p-2} \lambda^{2p-2} \left( n(n+p-4) - 2(p-2)(p-4) \right) |grad \ln \lambda|^2 \\ & - (p-n)^2 n^{p-2} \lambda^{2p-2} (\Delta \ln \lambda). \end{aligned} \tag{2.15}$$

By using the fact that

$$\Delta \lambda^k = k \lambda^k \left( \Delta \ln \lambda + k |grad \ln \lambda|^2 \right),$$

we obtain the following corollary :

**Corollary 2.4.** *Let $\phi : (M^n, g) \longrightarrow (N^n, h)$ be a conformal map of dilation $\lambda$ where $n \neq p$, then*

$$Tr_g S_{2,p}(\phi) = -(p-n)^2 n^{p-2} \lambda^{2p-2} T(\lambda),$$

*where*

$$T(\lambda) = \Delta \ln \lambda + \frac{n(n+p-4) - 2(p-2)(p-4)}{2(n-p)} |grad \ln \lambda|^2$$

*and*

$$Tr_g S_{2,p}(\phi) = 0 \text{ if and only if the function } \lambda^{\frac{n(n+p-4)-2(p-2)(p-4)}{2(n-p)}} \text{ is harmonic.}$$

**Remark 2.3.** Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$, $(n \neq p)$ be a conformal map of dilation $\lambda$ where we suppose that the dilation $\lambda$ is radial. By setting $\beta = (\ln \lambda)'$, we deduce that the trace of $S_{2,p}(\phi)$ is zero if if and only if $\beta$ satisfies the following differential equation :

$$\beta' + \frac{n-1}{r}\beta + \frac{n+2p-4}{2}\beta^2 = 0. \tag{2.16}$$

The general solution of this equation is given by :

$$\beta = \begin{cases} \frac{2(n-2)}{A(n-2)r^{n-1} - (n+2p-4)r}, & n \neq 2, \quad A \in \mathbb{R} \\ \frac{2}{(n+2p-4)r\ln r + Ar}, & n = 2, \quad A \in \mathbb{R}. \end{cases}$$

**Remark 2.4.** Let $\phi : (\mathbb{R}^n, g) \longrightarrow (N^n, h)$, $(n \neq p, \quad n \neq 2)$ be a conformal map of dilation $\lambda$ where we suppose that the dilation $\lambda$ is radial. we will look for the solutions of the form $\beta = \frac{a}{r}, a \in \mathbb{R}^*$. we deduce that the trace of $S_{2,p}(\phi)$ is zero if if and only if

$$a = -\frac{2(n-2)}{n+2p-4}, \qquad n + 2p - 4 \neq 0. \tag{2.17}$$

For example, if we consider the conformal map $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R} \times S^{n-1}$ given in polar coordinates by $\phi(r\theta) = (\ln r, \theta)$, we conclude that for this map $\phi$ the trace of $S_{2,p}(\phi)$ is zero if if and only if $n = 2p$.

**Proof of Theorem 2.3.** Let us choose $\{e_i\}_{1 \leq i \leq n}$ to be an orthonormal frame on $(M, g)$. By definition, we have

$$\begin{aligned} S_{2,p}(\phi)(X,Y) &= \frac{1}{2}|\tau_p(\phi)|^2 g(X,Y) + |d\phi|^{p-2} \left\langle d\phi, \nabla^\phi \tau_p(\phi) \right\rangle g(X,Y) \\ &- |d\phi|^{p-2} h\left(d\phi(X), \nabla^\phi_Y \tau_p(\phi)\right) - |d\phi|^{p-2} h\left(d\phi(Y), \nabla^\phi_X \tau_p(\phi)\right) \\ &- (p-2)|d\phi|^{p-4} \left\langle d\phi, \nabla^\phi \tau_p(\phi) \right\rangle h(d\phi(X), d\phi(Y)). \end{aligned} \tag{2.18}$$

Using the fact that

$$\tau_p(\phi) = (p-n)\, n^{\frac{p-2}{2}} \lambda^{p-2} d\phi(grad \ln \lambda),$$

we obtain

$$|\tau_p(\phi)|^2 = (p-n)^2 n^{p-2} \lambda^{2p-2} |grad \ln \lambda|^2. \tag{2.19}$$

For the term $\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle$, we have

$$\begin{aligned}\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle &= h\left(d\phi(e_i), \nabla^{\phi}_{e_i}\tau_p(\phi)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}} h\left(d\phi(e_i), \nabla^{\phi}_{e_i}\lambda^{p-2} d\phi(grad \ln\lambda)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}} \lambda^{p-2} h\left(d\phi(e_i), \nabla^{\phi}_{e_i} d\phi(grad \ln\lambda)\right)\\ &+ (p-n)\, n^{\frac{p-2}{2}} e_i\left(\lambda^{p-2}\right) h\left(d\phi(e_i), d\phi(grad \ln\lambda)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\left(\Delta\ln\lambda + n\left|grad\ln\lambda\right|^2\right)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}} \lambda^{p}\left|grad\ln\lambda\right|^2.\end{aligned}$$

It follows that

$$\left\langle d\phi, \nabla^{\phi}\tau_p(\phi)\right\rangle = (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\left(\Delta\ln\lambda + (n+p-2)\left|grad\ln\lambda\right|^2\right). \tag{2.20}$$

It remains to simplify $h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right)$ and $h\left(d\phi(Y), \nabla^{\phi}_{X}\tau_p(\phi)\right)$, we have

$$\begin{aligned}h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right) &= (p-n)\, n^{\frac{p-2}{2}} h\left(d\phi(X), \nabla^{\phi}_{Y}\lambda^{p-2} d\phi(grad\ln\lambda)\right)\\ &= (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\nabla d\ln\lambda(X,Y)\\ &+ (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}} \lambda^{p} X(\ln\lambda)\, Y(\ln\lambda),\end{aligned}$$

which gives us

$$\begin{aligned}h\left(d\phi(X), \nabla^{\phi}_{Y}\tau_p(\phi)\right) &= (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\nabla d\ln\lambda(X,Y)\\ &+ (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}} \lambda^{p} X(\ln\lambda)\, Y(\ln\lambda).\end{aligned} \tag{2.21}$$

A similar calculation gives

$$\begin{aligned}h\left(d\phi(Y), \nabla^{\phi}_{X}\tau_p(\phi)\right) &= (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\nabla d\ln\lambda(X,Y)\\ &+ (p-n)\, n^{\frac{p-2}{2}} \lambda^{p}\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(p-2)\, n^{\frac{p-2}{2}} \lambda^{p} X(\ln\lambda)\, Y(\ln\lambda).\end{aligned} \tag{2.22}$$

By substituting (2.19), (2.20), (2.21) and (2.22) in (2.18) and using the fact that

$$|d\phi|^{p-2} = n^{\frac{p-2}{2}}\lambda^{p-2}, \quad |d\phi|^{p-4} = n^{\frac{p-4}{2}}\lambda^{p-4},$$

we deduce that

$$\begin{aligned}&S_{2,p}(\phi)(X,Y)\\ &= \frac{p-n}{2} n^{p-3}\lambda^{2p-2}\left(n(n+p-4) - 2(p-2)^2\right)\left|grad\ln\lambda\right|^2 g(X,Y)\\ &+ (p-n)(n-p+2)\, n^{p-3}\lambda^{2p-2}(\Delta\ln\lambda)\, g(X,Y)\\ &- 2(p-n)\, n^{p-2}\lambda^{2p-2}\left(\nabla d\ln\lambda(X,Y) - (p-2)\, X(\ln\lambda)\, Y(\ln\lambda)\right).\end{aligned}$$

To complete the proof, let's calculate the trace of $S_{2,p}(\phi)$, we have

$$\begin{aligned}
&Tr_g S_{2,p}(\phi) = S_{2,p}(\phi)(e_i, e_i) \\
&= \frac{p-n}{2} n^{p-3} \lambda^{2p-2} \left( n(n+p-4) - 2(p-2)^2 \right) |grad \ln \lambda|^2 g(e_i, e_i) \\
&+ (p-n)(n-p+2) n^{p-3} \lambda^{2p-2} (\Delta \ln \lambda) g(e_i, e_i) \\
&- 2(p-n) n^{p-2} \lambda^{2p-2} (\nabla d \ln \lambda (e_i, e_i) - (p-2) e_i (\ln \lambda) e_i (\ln \lambda)),
\end{aligned}$$

then

$$\begin{aligned}
Tr_g S_{2,p}(\phi) = &\frac{p-n}{2} n^{p-2} \lambda^{2p-2} (n(n+p-4) - 2(p-2)(p-4)) |grad \ln \lambda|^2 \\
&- (p-n)^2 n^{p-2} \lambda^{2p-2} (\Delta \ln \lambda).
\end{aligned}$$

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## REFERENCES

1. D. Allen: *Relations between the local and global structure of fnite semigroups.* Ph. D. Thesis, University of California, Berkeley, 1968.
2. P. Erdős: *On the distribution of the roots of orthogonal polynomials.* In: Proceedings of a Conference on Constructive Theory of Functions (G. Alexits, S. B. Steckhin, eds.), Akademiai Kiado, Budapest, 1972, pp. 145–150.
3. A. Ostrowski: *Solution of Equations and Systems of Equations.* Academic Press, New York, 1966.
4. E. B. Saff and R. S. Varga: *On incomplete polynomials II.* Pacific J. Math. **92** (1981), 161–172.
5. M. A. Aprodu : *Some remarks on p-harmonic maps.* Stud. Cercet. Mat., **50** (1998), 297–305.
6. P. Baird and S. Gudmundsson: *p-harmonic maps and minimal submanifolds.* Mathematische Annalen, **294** (1992), 611–624.
7. X. Cao and Y. Luo: *On p-biharmonic submanifolds in non-positively curved manifolds.* Kodai Math. J., **39** (2016), 567–578.
8. A. M. Cherif: *On the p-harmonic and p-biharmonic maps.* Journal of Geometry, **109** (41) (2018), 1–11.
9. D. Djebbouri and S. Ouakkas: *Some results of the f-biharmonic maps and applications.* Arab. J. Math., **24** (1) (2018), 70–81.
10. S. Dragomir and A. Tommasoli: *On p-harmonic maps into spheres.* Tsukuba J. Math., **35** (2) (2011), 161–167.
11. A. Fardoun: *On equivariant p-harmonic maps.* Ann. Inst. Henri Poincar e, **15** (1) (1998), 25–72.

12. Y. Han and Y. Luo: *Several results concerning nonexistence of proper p-biharmonic maps and Liouville type theorems*. Journal of Elliptic and Parabolic Equations, **6** (2020), 409–426.

13. G. Y. Jiang: *2-harmonic maps and their first and second variational formulas*. Chin. Ann. Math. Ser., **A7** (1986), 389–402.

14. S. Kawai: *A p-harmonic maps and convex functions*. Geometriae Dedicata, **74** (1999), 261–265.

15. K. Mouffoki and A. M. Cherif: *On the p-biharmonic submanifolds and stress p-bienergy tensors*. arXiv : 2104.12562, 2021.

16. N. Nakauchi and S. Takakuwa: *A remark on p-harmonic maps*. Nonlinear Analysis, Theory, Methods and Applications, **25** (2) (1995), 169–185.

17. S. Ouakkas and D. Djebbouri: *Conformal Maps, Biharmonic Maps, and the Warped Product*. Mathematics, **15** (4) (2016), doi : 10.3390/math 4010015.

18. Y. L. Ou: *p-Harmonic morphisms, biharmonic morphisms, and non-harmonic biharmonic maps*. Journal of Geometry and Physics, **56** (2006), 358–374.

19. Z. Wei: *Some results on p-harmonic maps and exponentially harmonic maps between Finsler manifolds*. Appl. Math. J. Chinese Univ., **25** (2) (2010), 236–242.

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# SOME RESULTS ON MIXED SUPER QUASI-EINSTEIN MANIFOLDS SATISFYING CERTAIN VECTOR FIELDS

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**Abstract.** The objective of this paper is to discuss various properties of mixed super quasi-Einstein manifolds admitting certain vector fields. We analyze the behaviour of $MS(QE)_n$ satisfying Codazzi type of Ricci tensor. We have also constructed a non-trivial example related to mixed super quasi-Einstein manifolds.
**Keywords:** Mixed super quasi-Einstein manifolds, pseudo quasi-Einstein manifold, Codazzi type of Ricci tensor, cyclic parallel Ricci tensor, Killing vector field, concurrent vector field.

## 1. Introduction

An $n$-dimensional semi-Riemannian or Riemannian manifold $(M^n, g)$ $(n > 2)$, is called an Einstein manifold if its Ricci tensor $S$ satisfies the criteria

$$S = \frac{r}{n} g, \tag{1.1}$$

where $r$ denotes the scalar curvature of $(M^n, g)$. We can also say an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M.C. Chaki and R.K. Maity [5]. A non-flat Riemannian manifold $(M^n, g)$, $(n \geq 3)$ is a quasi-Einstein manifold if its Ricci tensor $S$ satisfies the criteria

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.2}$$

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and is not identically zero, where $a$, $b$ are scalars, $b \neq 0$ and $A$ is a non-zero 1-form such that

$$g(X,U) = A(X),$$

for all vector field $X$. $U$ being a unit vector field.
Here $a$ and $b$ are called the associated scalars, $A$ is called the associated 1-form and $U$ is called the generator of the manifold. Such an $n$-dimensional manifold is denoted by $(QE)_n$. The quasi-Einstein manifolds have also been studied by De and Ghosh [7], Bejan [1], De and De [6], Han, De and Zhao [15] and many others. Quasi-Einstein manifolds have been generalized by many authors in several ways such as generalized quasi-Einstein manifolds [3, 9, 11, 23], $N(K)$-quasi Einstein manifolds [17, 24], super quasi-Einstein manifolds [4, 10, 19] etc.

Chaki [4] introduced the notion of a super quasi-Einstein manifold. His work suggested a non-flat Riemannian or semi-Riemannian manifold $(M^n, g)$ $(n > 2)$ is called a super quasi-Einstein manifold if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) \\ &+ c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.3}$$

where $a$, $b$, $c$, $d$ are scalars in which $b \neq 0$, $c \neq 0$ $d \neq 0$ and $A$, $B$ are non-zero 1-forms such that

$$g(X,U) = A(X), \;\; g(X,V) = B(X),$$

where $U$, $V$ are mutually orthogonal unit vector fields, $D$ is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0,$$

for all $X$. In that case $a$, $b$, $c$, $d$ are called the associated scalars, $A$, $B$ are called the associated main and auxiliary 1-forms, $U$, $V$ are called the main and auxiliary generators of the manifold and $D$ is called the associated tensor of the manifold. Such an $n$-dimensional manifold is denoted by $S(QE)_n$.

In [2], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of mixed super quasi-Einstein manifolds. Their work suggested that a non-flat Riemannian manifold $(M^n, g)$, $(n \geq 3)$ is said to be mixed super quasi-Einstein manifold if its Ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &+ d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y), \end{aligned} \tag{1.4}$$

where $a$, $b$, $c$, $d$, $e$ are scalars on $(M^n, g)$ of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and $A$, $B$ are two non-zero 1-forms such that

$$g(X,U) = A(X), \;\; g(X,V) = B(X), \tag{1.5}$$

$U$, $V$ being unit vector fields which are orthogonal, $D$ is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \tag{1.6}$$

for all $X$. Here $a$, $b$, $c$, $d$, $e$ are called the associated scalars, $A$, $B$ are called the associated main and auxiliary 1-forms, $U$, $V$ are called the main and auxiliary generators of the manifold and $D$ is called the associated tensor of the manifold. If $c = 0$, then the manifold becomes $S(QE)_n$. This type of manifold is denoted by the symbol $MS(QE)_n$. If $c = d = 0$, then the manifold is reduced to a pseudo quasi-Einstein manifold which was studied by Shaikh [22].

On the other hand, Gray [14] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class $A$ consists of all Riemannian manifolds whose Ricci tensor $S$ is a Codazzi type tensor, i.e.,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class B contains all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold $(M^n, g)$ $(n > 2)$ is called a generalized Ricci recurrent manifold [8] if its Ricci tensor $S$ of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \gamma(X) S(Y,Z) + \delta(X) g(Y,Z),$$

where $\gamma(X)$ and $\delta(X)$ are non-zero 1-forms such that $\gamma(X) = g(X,\rho)$ and $\delta(X) = g(X,\mu)$; $\rho$ and $\mu$ being associated vector fields of the 1-forms $\gamma$ and $\delta$, respectively. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold [20].

After studying and analyzing various papers [12, 13, 18], we got motivation to work in this area. Recently in the paper [16], we have studied generalized Quasi-Einstein manifolds satisfying certain vector fields. In the present work we have tried to develop a new concept. This paper is organized as follows: After introduction in Section 2, we have studied that if the generators $U$ and $V$ of a $MS(QE)_n$ are Killing vector fields, then the manifold satisfies cyclic parallel Ricci tensor if and only if the associated tensor $D$ is cyclic parallel. Section 3 is concerned with $MS(QE)_n$ satisfying Codazzi type of Ricci tensor. In the next two sections, we have studied $MS(QE)_n$ with generators $U$ and $V$ both as concurrent and recurrent vector fields. Finally the existence of $MS(QE)_n$ is shown by constructing non-trivial example.

## 2. The generators $U$ and $V$ as Killing vector fields

In this section we consider the generators $U$ and $V$ of the manifold are Killing vector fields.

**Theorem 2.1.** *If the generators of a $MS\,(QE)_n$ are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor if and only if the associated tensor $D$ is cyclic parallel.*

*Proof.* Let us assume that the generators $U$ and $V$ of the manifold are Killing vector fields. Then we have

$$(\pounds_U g)(X,Y) = 0 \tag{2.1}$$

and

$$(\pounds_V g)(X,Y) = 0, \tag{2.2}$$

where $\pounds$ denotes the Lie derivative.
From (2.1) and (2.2), we get

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0 \tag{2.3}$$

and

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0. \tag{2.4}$$

Since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$ and $g(\nabla_X V, Y) = (\nabla_X B)(Y)$.
Thus from (2.3) and (2.4) we obtain

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \tag{2.5}$$

and

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0, \tag{2.6}$$

for all $X$, $Y$.
Similarly, we have

$$(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0, \tag{2.7}$$

$$(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0, \tag{2.8}$$

$$(\nabla_X B)(Z) + (\nabla_Z B)(X) = 0, \tag{2.9}$$

$$(\nabla_Z B)(Y) + (\nabla_Y B)(Z) = 0, \tag{2.10}$$

for all $X$, $Y$, $Z$.
We assume that the associated scalars are constants. Then from (1.4) we have

$$\begin{aligned}(\nabla_Z S)(X,Y) &= b\left[(\nabla_Z A)(X)\,A(Y) + A(X)\,(\nabla_Z A)(Y)\right]\\ &\quad + c\left[(\nabla_Z B)(X)\,B(Y) + B(X)\,(\nabla_Z B)(Y)\right]\\ &\quad + d\left[(\nabla_Z A)(X)\,B(Y) + A(X)\,(\nabla_Z B)(Y)\right.\\ &\quad + \left.(\nabla_Z A)(Y)\,B(X) + A(Y)\,(\nabla_Z B)(X)\right]\\ &\quad + e\,(\nabla_Z D)(X,Y).\end{aligned} \tag{2.11}$$

Using (2.11), we get

$$
\begin{aligned}
&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = b\,[\{(\nabla_X A)(Y) \\
&+\ (\nabla_Y A)(X)\}\, A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}\, A(Y) \\
&+\ \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}\, A(X)] + c\,[\{(\nabla_X B)(Y) \\
&+\ (\nabla_Y B)(X)\}\, B(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}\, B(Y) \\
&+\ \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}\, B(X)] + d\,[\{(\nabla_X B)(Y) \\
&+\ (\nabla_Y B)(X)\}\, A(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}\, A(Y) \\
&+\ \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}\, A(X) + \{(\nabla_X A)(Y) \\
&+\ (\nabla_Y A)(X)\}\, B(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}\, B(Y) \\
&+\ \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}\, B(X)] + e\,[(\nabla_X D)(Y,Z) \\
&+\ (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)]\,. \qquad (2.12)
\end{aligned}
$$

Using the equations (2.5) - (2.10) in (2.12), we get

$$
\begin{aligned}
(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = e\,[(\nabla_X D)(Y,Z) \\
+ (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)]\,.
\end{aligned}
$$

Thus the proof of theorem is completed. □

## 3. $MS\,(QE)_n$ admits Codazzi type of Ricci tensor

We know that a Riemannian or semi-Riemannian manifold satisfies Codazzi type of Ricci tensor if its Ricci tensor $S$ satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)\,, \qquad (3.1)$$

for all $X$, $Y$, $Z$.

**Theorem 3.1.** *If a $MS\,(QE)_n$ admits the Codazzi type of Ricci tensor with the associated tensor $D$ satisfying the relation $(\nabla_X D)(Y,V) = (\nabla_Y D)(V,X)$, then either $d = \pm\sqrt{bc}$ or the associated 1-forms $A$ and $B$ are closed.*

*Proof.* Using (2.11) and (3.1), we obtain

$$
\begin{aligned}
&b\,[(\nabla_X A)(Y)\, A(Z) + A(Y)\,(\nabla_X A)(Z)] + c\,[(\nabla_X B)(Y)\, B(Z) \\
&+ B(Y)\,(\nabla_X B)(Z)] + d\,[(\nabla_X A)(Y)\, B(Z) + A(Y)\,(\nabla_X B)(Z) \\
&+\ (\nabla_X A)(Z)\, B(Y) + A(Z)\,(\nabla_X B)(Y)] + e\,(\nabla_X D)(Y,Z) \\
&- b\,[(\nabla_Y A)(Z)\, A(X) + A(Z)\,(\nabla_Y A)(X)] - c\,[(\nabla_Y B)(Z)\, B(X) \\
&+ B(Z)\,(\nabla_Y B)(X)] - d\,[(\nabla_Y A)(Z)\, B(X) + A(Z)\,(\nabla_Y B)(X) \\
&+\ (\nabla_Y A)(X)\, B(Z) + A(X)\,(\nabla_Y B)(Z)] - e\,(\nabla_Y D)(Z,X) = 0. \qquad (3.2)
\end{aligned}
$$

Putting $Z = U$ in (3.2) and using $(\nabla_X A)(U) = 0$, we have

$$b\,[(\nabla_X A)(Y) - (\nabla_Y A)(X)] + d\,[(\nabla_X B)(Y) - (\nabla_Y B)(X)] = 0,$$

i.e.,

$$bdA(X,Y) = -d dB(X,Y). \tag{3.3}$$

Similarly, putting $Z = V$ in (3.2) and using $(\nabla_X B)(V) = 0$, we have

$$c[(\nabla_X B)(Y) - (\nabla_Y B)(X)] + d[(\nabla_X A)(Y) - (\nabla_Y A)(X)]$$
$$+e[(\nabla_X D)(Y,V) - (\nabla_Y D)(V,X)] = 0,$$

i.e.,

$$c\mathbf{d}B(X,Y) + d\mathbf{d}A(X,Y) + e[(\nabla_X D)(Y,V) - (\nabla_Y D)(V,X)] = 0. \tag{3.4}$$

If $(\nabla_X D)(Y,V) = (\nabla_Y D)(V,X)$, then from the equations (3.3) and (3.4) we get either

$$d = \pm\sqrt{bc}$$

or

$$\mathbf{d}A(X,Y) = 0$$

and

$$\mathbf{d}B(X,Y) = 0.$$

Thus, we complete the proof. $\square$

**Theorem 3.2.** *If a $MS(QE)_n$ admits the Codazzi type of Ricci tensor with the associated tensor $D$ satisfying the condition $(\nabla_V D)(Y,V) = (\nabla_Y D)(V,V)$, then the integral curves of the parallel vector fields $U$ and $V$ are geodesics.*

*Proof.* Putting $X = Z = U$ in (3.2), we get

$$b(\nabla_U A)(Y) + d(\nabla_U B)(Y) = 0,$$

which means that

$$bg(\nabla_U U, Y) + dg(\nabla_U V, Y) = 0. \tag{3.5}$$

Similarly, putting $X = Z = V$ in (3.2), we get

$$c(\nabla_V B)(Y) + d(\nabla_V A)(Y) + e[(\nabla_V D)(Y,V) - (\nabla_Y D)(V,V)] = 0,$$

i.e.,

$$cg(\nabla_V V, Y) + dg(\nabla_V U, Y) + e[(\nabla_V D)(Y,V) - (\nabla_Y D)(V,V)] = 0. \tag{3.6}$$

If $U$, $V$ are parallel vector fields, then $\nabla_U V = 0 = \nabla_V U$.
We assume that $(\nabla_V D)(Y,V) = (\nabla_Y D)(V,V)$. So from (3.5) and (3.6), we obtain

$$g(\nabla_U U, Y) = 0, \text{ for all } \mathrm{Y}, \text{ i.e., } \nabla_\mathrm{U}\mathrm{U} = 0$$

and

$$g(\nabla_V V, Y) = 0, \text{ for all } \mathrm{Y}, \text{ i.e., } \nabla_\mathrm{V}\mathrm{V} = 0.$$

Thus the theorem is proved. $\square$

## 4. The generators $U$ and $V$ as concurrent vector fields

A vector field $\xi$ is called concurrent if [21]

$$\nabla_X \xi = \rho X, \tag{4.1}$$

where $\rho$ is a non-zero constant. If $\rho = 0$, then the vector field reduces to a parallel vector field.

**Theorem 4.1.** *If the associated vector fields of a $MS\,(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a pseudo quasi-Einstein manifold.*

*Proof.* We consider the vector fields $U$ and $V$ corresponding to the associated 1-forms $A$ and $B$ respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X,Y) \tag{4.2}$$

and

$$(\nabla_X B)(Y) = \beta g(X,Y), \tag{4.3}$$

where $\alpha$ and $\beta$ are non-zero constants.
Using (4.2) and (4.3) in (2.11), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= b\left[\alpha g(Z,X)A(Y) + \alpha g(Z,Y)A(X)\right] + c\left[\beta g(Z,X)B(Y)\right. \\ &\quad \left.+ \beta g(Z,Y)B(X)\right] + d\left[\alpha g(Z,X)B(Y) + \beta g(Z,Y)A(X)\right. \\ &\quad \left.+ \alpha g(Z,Y)B(X) + \beta g(Z,X)A(Y)\right] + e(\nabla_Z D)(X,Y).\end{aligned} \tag{4.4}$$

Contracting (4.4) over $X$ and $Y$, we obtain

$$dr(Z) = 2\left[(b\alpha + d\beta)A(Z) + (c\beta + d\alpha)B(Z)\right], \tag{4.5}$$

where $r$ is the scalar curvature of the manifold.
In a $MS\,(QE)_n$ if the associated scalars $a$, $b$, $c$, $d$ and $e$ are constants, then contracting (1.4) over $X$ and $Y$ we get

$$r = an + b + c,$$

which implies that the scalar curvature $r$ is constant, i.e., $dr(X) = 0$, for all $X$. Thus equation (4.5) gives

$$(b\alpha + d\beta)A(Z) + (c\beta + d\alpha)B(Z) = 0. \tag{4.6}$$

Since $\alpha$ and $\beta$ are non-zero constants, using (4.6) in (1.4), we finally get

$$S(X,Y) = ag(X,Y) + \left[b + c\left(\frac{b\alpha + d\beta}{c\beta + d\alpha}\right)^2 - 2d\left(\frac{b\alpha + d\beta}{c\beta + d\alpha}\right)\right]A(X)A(Y) + eD(X,Y).$$

Thus the manifold reduces to a pseudo quasi-Einstein manifold. □

## 5. The generators $U$ and $V$ as recurrent vector fields

**Definition 5.1.** A non-flat Riemannian or semi-Riemannian manifold $(M^n, g)$ $(n > 2)$ will be called a pseudo generalized Ricci recurrent manifold if its Ricci tensor $S$ of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \beta(X) S(Y,Z) + \gamma(X) g(Y,Z) + \delta(X) D(Y,Z),$$

where $\beta(X)$, $\gamma(X)$ and $\delta(X)$ are non-zero 1-forms such that

$$\beta(X) = g(X,\xi_1), \quad \gamma(X) = g(X,\xi_2), \quad \delta(X) = g(X,\xi_3);$$

$\xi_1$, $\xi_2$ and $\xi_3$ are associated vector fields of the 1-forms $\beta$, $\gamma$ and $\delta$ respectively, $D$ is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,\xi_1) = 0,$$

for all $X$.

**Theorem 5.1.** *If the generators of a $MS(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence and the associated scalars are constants with an additional condition that $D$ is covariant constant, then the manifold is a pseudo generalized Ricci recurrent manifold.*

*Proof.* A vector field $\xi$ corresponding to the associated 1-form $\eta$ is said to be recurrent if [21]

$$(\nabla_X \eta)(Y) = \psi(X) \eta(Y), \tag{5.1}$$

where $\psi$ is a non-zero 1-form.

Here, we consider the generators $U$ and $V$ corresponding to the associated 1-forms $A$ and $B$ as recurrent. Then we have

$$(\nabla_X A)(Y) = \lambda(X) A(Y) \tag{5.2}$$

and

$$(\nabla_X B)(Y) = \mu(X) B(Y), \tag{5.3}$$

where $\lambda$ and $\mu$ are non-zero 1-forms.
Using (5.2) and (5.3) in (2.11), we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2b\lambda(Z) A(X) A(Y) + 2c\mu(Z) B(X) B(Y) \\ &+ d[\lambda(Z) + \mu(Z)][A(X) B(Y) + A(Y) B(X)] \\ &+ e(\nabla_Z D)(X,Y).\end{aligned} \tag{5.4}$$

We assume that the 1-forms $\lambda$ and $\mu$ are equal, i.e.,

$$\lambda(Z) = \mu(Z), \tag{5.5}$$

for all $Z$. From the equations (5.4) and (5.5), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2\lambda(Z)\left[bA(X)A(Y)+cB(X)B(Y)\right.\\ &\quad \left.+d\{A(X)B(Y)+A(Y)B(X)\}\right]\\ &\quad +e(\nabla_Z D)(X,Y).\end{aligned} \tag{5.6}$$

Using (1.4) and (5.6), we obtain

$$(\nabla_Z S)(X,Y)=\alpha_1(Z)S(X,Y)+\alpha_2(Z)g(X,Y)+\alpha_3(Z)D(X,Y)+e(\nabla_Z D)(X,Y),$$

where $\alpha_1(Z)=2\lambda(Z)$, $\alpha_2(Z)=-2a\lambda(Z)$ and $\alpha_3(Z)=-2e\lambda(Z)$.
So the proof is complete. □

## 6. Example of $MS(QE)_4$

In this section, we prove the existence of $MS(QE)_4$ by constructing a non-trivial concrete example.

Let $\left(x^1,x^2,\ldots,x^n\right)\in\mathbb{R}^n$, where $\mathbb{R}^n$ is an $n$-dimensional real number space. We consider a Riemannian metric $g$ on $\mathbb{R}^4=\left(x^1,x^2,x^3,x^4\right)$, by

$$ds^2=g_{ij}dx^idx^j=\left(dx^1\right)^2+\left(x^1\right)^2\left(dx^2\right)^2+\left(x^2\right)^2\left(dx^3\right)^2+\left(dx^4\right)^2, \tag{6.1}$$

where $i,j=1,2,3,4$. Using (6.1), we see the non-vanishing components of Riemannian metric are

$$g_{11}=1,\ \ g_{22}=\left(x^1\right)^2,\ \ g_{33}=\left(x^2\right)^2,\ \ g_{44}=1 \tag{6.2}$$

and its associated components are

$$g^{11}=1,\ \ g^{22}=\frac{1}{\left(x^1\right)^2},\ \ g^{33}=\frac{1}{\left(x^2\right)^2},\ \ g^{44}=1. \tag{6.3}$$

Using (6.2) and (6.3), we can calculate that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22}=-x^1,\ \ \Gamma^2_{33}=-\frac{x^2}{\left(x^1\right)^2},\ \ \Gamma^2_{12}=\frac{1}{x^1},\ \ \Gamma^3_{23}=\frac{1}{x^2},\ \ R_{1332}=-\frac{x^2}{x^1},\ \ S_{12}=-\frac{1}{x^1x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature $r$ of the resulting manifold $\left(\mathbb{R}^4,g\right)$ is zero. We shall show that $\left(\mathbb{R}^4,g\right)$ is a $MS(QE)_4$.
Let us consider the associated scalars as follows:

$$a=\frac{1}{x^1\left(x^2\right)^2},\ \ b=\frac{1}{\left(x^2\right)^3},\ \ c=-\frac{1}{x^2},\ \ d=\frac{1}{x^1},\ \ e=-\frac{1}{\left(x^1\right)^2x^2}. \tag{6.4}$$

We choose the 1-form as follows:

$$A_i(x)=\begin{cases}x^1, & \text{when } i=2\\ 0, & \text{otherwise}\end{cases} \tag{6.5}$$

and

$$B_i(x) = \begin{cases} x^2, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{6.6}$$

at any point $x \in \mathbb{R}^4$.
We take the associated tensor as follows:

$$D_{ij}(x) = \begin{cases} 1, & \text{when } i = j = 1, 3 \\ -2, & \text{when } i = j = 2 \\ x^1, & \text{when } i = 1, j = 2 \\ 0, & \text{otherwise} \end{cases} \tag{6.7}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 + d\left[A_1B_2 + A_2B_1\right] + eD_{12}, \tag{6.8}$$

since, for the other cases (1.4) holds trivially.
From the equations (6.4), (6.5), (6.6), (6.7) and (6.8) we get

$$\begin{aligned} \text{Right hand side of (6.8)} &= ag_{12} + bA_1A_2 + cB_1B_2 + d\left[A_1B_2 + A_2B_1\right] + eD_{12} \\ &= \frac{1}{x^1\left(x^2\right)^2} \cdot 0 + \frac{1}{\left(x^2\right)^3} \cdot 0 \cdot x^1 + \left(-\frac{1}{x^2}\right) \cdot 0 \cdot 0 \\ &+ \frac{1}{x^1}\left[0 + x^1 \cdot 0\right] + \left(-\frac{1}{\left(x^1\right)^2 x^2}\right) \cdot x^1 \\ &= -\frac{1}{x^1x^2} = S_{12}. \end{aligned}$$

Clearly, the trace of the $(0, 2)$ tensor $D$ is zero.
We shall now show that the 1-forms $A_i$ and $B_i$ are unit and also they are orthogonal. Here,

$$g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MS\left(QE\right)_4$.

## REFERENCES


1. C. L. Bejan: *Characterization of quasi-Einstein manifolds.* An. Stiint. Univ. Al. I. Cuza Iasi. Mat.(NS) **53** (2007), 67–72.
2. A. Bhattacharyya, M. Tarafdar and D. Debnath: *On mixed super quasi-Einstein manifolds.* Differential Geometry–Dynamical Systems **10** (2008), 44–57.
3. M. C. Chaki: *On generalized quasi-Einstein manifolds.* Publ. Math. Debrecen **58** (2001), 683–691.

4. M. C. Chaki: *On super quasi Einstein manifolds*. Publicationes Mathematicae Debrecen **64** (2004), 481–488.

5. M. C. Chaki and R. K. Maity: *On quasi-Einstein manifolds*. Publ. Math. Debrecen **57** (2000), 297–306.

6. U. C. De and B. K. De: *On quasi Einstein manifolds*. Communications of the Korean Mathematical Society **23** (2008), 413–420.

7. U. C. De and G. C. Ghosh: *On quasi Einstein manifolds*. Periodica Mathematica Hungarica **48** (2004), 223–231.

8. U. C. De, N. Guha and D. Kamilya: *On generalized Ricci-recurrent manifolds*. Tensor(N.S.) **56** (1995), 312—317.

9. U. C. De and S. Mallick: *On generalized quasi-Einstein manifolds admitting certain vector fields*. Filomat **29** (2015), 599–609.

10. P. Debnath and A. Konar: *On super quasi Einstein manifold*. Publications de l'Institut Mathematique **89** (2011), 95–104.

11. Y. Deng: *A note on generalized quasi-Einstein manifolds*. Mathematische Nachrichten **288** (2015), 1122–1126.

12. S. Dey, B. Pal and A. Bhattacharyya: *A Non-Flat Riemannian Manifold Admitting Certain Vectors Fields*. Journal of Dynamical Systems and Geometric Theories **17** (2019), 221–237.

13. F. Fu, Y. Han and P. Zhao: *Geometric and physical characteristics of mixed super quasi-Einstein manifolds*. International Journal of Geometric Methods in Modern Physics **16** (2019), 1950104.

14. A. Gray: *Einstein-like manifolds which are not Einstein*. Geom. Dedicata **7** (1978), 259–280.

15. Y. Han, A. De and P. Zhao: *On a semi-quasi-Einstein manifold*. Journal of Geometry and Physics **155** (2020), 103739.

16. D. Hazra: *A study on generalized Quasi-Einstein manifolds satisfying certain vector fields*. Bull. Cal. Math. Soc. **113** (2021), 73—80.

17. A. A. Hosseinzadeh and A. Behzadi: *Some results on N(k)-quasi Einstein manifolds*. Journal of Interdisciplinary Mathematics **23** (2020), 1065–1075.

18. S. Mallick: *Super Quasi-Einstein Manifolds with Applications to General Relativity*. Kyungpook Mathematical Journal **58** (2018), 361–375.

19. C. Özgür: *On some classes of super quasi-Einstein manifolds*. Chaos, Solitons Fractals **40** (2009), 1156–1161.

20. E. M. Patterson: *Some Theorems on Ricci-Recurrent Spaces*. Journal of the London Mathematical Society **27** (1952), 287–295.

21. J. A. Schouten: *Ricci-Calculus*. Springer, Berlin, 1954.

22. A. A. Shaikh: *On pseudo quasi-Einstein manifolds*. Periodica Mathematica Hungarica **59** (2009), 119–146.

23. S. Sular and C. Özgür: *Characterizations of generalized quasi-Einstein manifolds*. An. St. Univ. Ovidius Constanta **20** (2012), 407–416.

24. M. M. Tripathi and J. S. Kim: *On N(k)-quasi Einstein manifolds*. Communications of the Korean Mathematical Society **22** (2007), 411–417.

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