UNIVERSITY OF NIŠ

ISSN 0352-9665 (Print)
ISSN 2406-047X (Online)
COBISS.SR-ID 5881090
UDC 51+004

FACTA UNIVERSITATIS

Series
MATHEMATICS AND INFORMATICS
Vol. 39, No 1 (2024)

Scientific Journal **FACTA UNIVERSITATIS**
UNIVERSITY OF NIŠ
Univerzitetski trg 2, 18000 Niš, Republic of Serbia
Phone: +381 18 257 095 Telefax: +381 18 257 950
e-mail: facta@ni.ac.rs http://casopisi.junis.ni.ac.rs/

Scientific Journal FACTA UNIVERSITATIS publishes original high scientific level works in the fields classified accordingly into the following periodical and independent series:

Architecture and Civil Engineering
Automatic Control and Robotics
Economics and Organization
Electronics and Energetics
Law and Politics
Linguistics and Literature
Mathematics and Informatics
Mechanical Engineering
Medicine and Biology
Philosophy, Sociology, Psychology and History
Physical Education and Sport
Physics, Chemistry and Technology
Teaching, Learning and Teacher Education
Visual Arts and Music
Working and Living Environmental Protection

SERIES **MATHEMATICS AND INFORMATICS**

Founded in 1986 by **Gradimir V. Milovanović**, Serbian Academy of Sciences and Arts, and Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia
The cover image taken from http://www.pptbackgrounds.net/binary-code-and-computer-monitors-backgrounds.html.
Publication frequency – one volume, five issues per year.
Published by the University of Niš, Republic of Serbia

This publication was in part supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia
Printed by ATLANTIS DOO, Niš, Republic of Serbia
Circulation 50

ISSN 0352 – 9665 (Print)
ISSN 2406 – 047X (Online)
COBISS.SR-ID 5881090
UDC 51+004

FACTA UNIVERSITATIS

SERIES **MATHEMATICS AND INFORMATICS**
Vol. 39, No 1 (2024)

UNIVERSITY OF NIŠ

INSTRUCTION FOR AUTHORS

The journal Facta Universitatis: Series Mathematics and Informatics publishes original papers of high scientific value in all areas of mathematics and computer science, with a special emphasis on articles in the field of applied mathematics and computer science. Survey articles dealing with interactions between different fields are welcome.

Papers submitted for publication should be concise and written in English. They should be prepared in LaTeX with the style factaMi.cls in accordance with instructions given in the file instructions.tex (see http://casopisi.junis.ni.ac.rs/index.php/FUMathInf/manager/files/styles/facta-style.zip). Under the title, name(s) of the author(s) should be given, at the end of the paper the full name (with official title, institute or company affiliation, etc.) and exact address should appear. Each paper should be accompanied by a brief summary (50–150 words) and by 2010 Mathematics Subject Classification numbers (http://www.ams.org/msc) or 1998 ACM Computing Classification System codes (http://www.acm.org/class). Figures should be prepared in eps format, and footnotes in the text should be avoided if at all possible.

References should be listed alphabetically at the end of the manuscript, in the same way as the following examples (for a book, a paper in a journal, paper in a contributed volume and for an unpublished paper):

[1] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1966.

[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

References should be quoted in the text by giving the corresponding number in square brackets.

Electronic submission. Manuscripts prepared in the above form should be submitted via Electronic editorial system, available at http://casopisi.junis.ni.ac.rs/index.php/FUMathInf/index. Authors are encouraged to check the home page of the journal and to submit manuscripts through the editorial system.

Galley proofs will be sent to the author.

FACTA UNIVERSITATIS

Series

Mathematics and Informatics

Vol. 39, No 1 (2024)

Contents

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 1–20
https://doi.org/10.22190/FUMI210510002G
Original Scientific Paper

NONLOCAL BOUNDARY VALUE PROBLEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION ON THE HALF-LINE

Abdellatif Ghendir Aoun

Faculty of Exact Sciences, Department of Mathematics
University of El-Oued, Algeria, Operators Theory Laboratory (LABTHOP)

ORCID IDs: Abdellatif Ghendir Aoun https://orcid.org/0000-0002-9167-5198

Abstract. This paper aims to investigate a class for nonlocal fractional boundary value problem on an infinite interval due to its importance in provide a powerful tool for mathematical modeling of complex phenomena in science. New existence results are acquired for the given problem by using the Krasnosel'skii's fixed point theorem. Moreover, sufficient conditions are obtained as well as a modified compactness criterion that guarantees the existence of at least one solution. In addition, an illustrative example is given in the final part of the paper.
Keywords: Boundary value problem, infinite interval, fractional differential equation, nonlocal condition, fixed point theorem.

1. Introduction

Fractional calculus is a generalization of classical integer-order calculus and has been studied for more than for several years ago. Unlike integer-order derivatives, the fractional differential equations provide a powerful tool for mathematical modeling of complex phenomena in science, engineering practice and processes in the fields of physics, chemistry, electrical circuits, biology, and so on.

This is the main advantage of fractional differential equations in comparison with classical integer-order models. Further, the concept of nonlocal boundary conditions has been introduced to extend the study of classical boundary value problems. This

Received May 10, 2021, revised: November 23, 2023, accepted: November 27, 2023
Communicated by Hijaz Ahmad
Corresponding Author: Abdellatif Ghendir Aoun. E-mail address: ghendir-aoun-abdellatif@univ-eloued.dz
2010 *Mathematics Subject Classification.* 34B10, 34B15, 34B40.

notion is more precise for describing natural phenomena than the classical notion because additional information is taken into account.

Recently, several papers have studied questions of existence of solutions for some classes of bvps for fractional differential equations on finite intervals, see, e g., [2, 3, 4, 5, 8, 9, 11, 18, 19] and references therein. Different methods have been employed. However, research works on the existence of multiple solutions for fractional differential equations with nonlocal boundary condition on infinite intervals are few, we refer to [6, 7, 12, 15, 16, 17] and references therein.

In this paper, we will consider the boundary value problem (bvp for short)

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)), & t \in (0, +\infty), \\ u(0) = 0 = D_{0+}^{\alpha-2}u(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0+}^{\alpha-1}u(t) = N(u), \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$ and $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $N : Y \to \mathbb{R}$ are given functions such that Y is a suitable Banach space. D_{0+}^{α} refers to the standard Riemann-Liouville fractional derivative and I_{0+}^{α} is the standard Riemann-Liouville fractional integral.

By using the famous Leray-Schauder Nonlinear Alternative theorem, Y. Gholami [6] obtained an unbounded solution for the following multi-point bvp in unbounded interval

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(t, u(t), u'(t)) = 0, & t \in (0, +\infty), \\ u(0) = u'(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0+}^{\alpha-1}u(t) = \sum\limits_{i=1}^{m} \beta_i D_{0+}^{\alpha-1}u(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $a \in C([0, +\infty), [0, +\infty))$, $0 < \xi_1 < \xi_2 < \ldots < \xi_m < +\infty$, $\beta_i \in \mathbb{R}$ with $\sum\limits_{i=1}^{m} \beta_i < 1$.

In [15], Shen, Zhou and Yang established the existence of positive solutions for the bvp

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), D_{0+}^{\alpha-1}u(t)) = 0, & t \in (0, +\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D_{0+}^{\alpha-1}u(+\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

where $2 < \alpha \leq 3$, $f \in C([0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Gamma(\alpha) - \sum\limits_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} \neq 0$. with a suitable growth condition imposed on the nonlinear term. By using Schauder fixed point theorem, they proved the existence of at least one solution.

Ghanbari, Gholami [7] discussed the existence and multiplicity of positive solutions for a m-point nonlinear fractional bvp on an infinite interval

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda a(t)f(t, u(t)) = 0, & t \in (0, +\infty), \\ u(0) + u'(0) = 0, \quad \lim\limits_{t\to+\infty} D_{0+}^{\alpha-1}u(t) = \sum\limits_{i=1}^{m-2} \beta_i u'(\xi_i), \end{cases}$$

where $2 < \alpha < 3$, $f \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, $a \in C([0,+\infty), [0,+\infty))$, λ is a positive parameter and $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < +\infty$, $\beta_i \in [0,+\infty)$ with $0 < \sum_{i=1}^{m-2} (\alpha-1)\beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$.
Motivated by the above works and by recent studies of nonlocal boundary value problems of fractional order, we consider a more general problem of fractional differential equations of arbitrary order with nonlocal boundary conditions. Precisely, we investigate the problem (1.1).

The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models, in which N is a mapping defined on a proposed space consisting of certain functions which represent the solutions to the problem proposed in this paper. Then we give a model of the function g in the form of a linear combination of the solution at some points in the example proposed in this paper to confirm our results.

The work presented in this paper is a continuation of previous works and is concerned with a bvp of fractional order set on the half-axis. The main difficulty in treating this class of the fractional differential equations is the possible lack of compactness due to the infinite interval. In order to overcome these difficulties, we use a special Banach space in which similar inequalities as finite interval can be established. The main tool used in this paper is Krasnosel'skii' s fixed point theorem (nonlinear alternative). Under a compactness criterion, the existence of solutions is established.

The plan of the paper is as follows. In Section 2, we outline some basic concepts of fractional calculus. We prove some technical lemmas which are needed later in Section 3. Section 4 is devoted to our main existence results. In Section 5, an example of applications is supplied to illustrate our theoretical results.

2. Preliminaries

We start with some definitions and lemmas on the fractional calculus (see [10], [13]).
One of the basic tools of the fractional calculus is the Gamma function which extends the factorial to positive real numbers (and even complex numbers with positive real parts).

Definition 2.1. For $\alpha > 0$, the Euler Gamma function is defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Proposition 2.1. *Let $\alpha > 0$, $p > 0$, $q > 0$ and n a positive integer. Then*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\Gamma(n+1)}, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\Gamma(\alpha+n)=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\Gamma(\alpha).$$

In particular

$$\Gamma(1)=\int_0^{+\infty}e^{-t}dt=1,\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},$$
$$\Gamma(n+1)=n!,\quad \Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}(2n)!}{2^{2n}n!}.$$

Definition 2.2. The fractional integral of order $\alpha>0$ for function h is defined by

$$I_{0^+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)ds,$$

provided the right hand side is point-wise defined on $(0,+\infty)$.

Definition 2.3. For a given function h defined on the interval $[0,+\infty)$, the Riemann-Liouville fractional derivative of order $\alpha>0$ is defined by

$$D_{0^+}^{\alpha}h(t)=\left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{h(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n=[\alpha]+1$.

Lemma 2.1. *([10]) Let $\alpha>0$, then*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+...+c_nt^{\alpha-n},$$

for some $c_i\in\mathbb{R}$, $i=1,2,...,n$, $n=[\alpha]+1$.

Proposition 2.2. *[13] The following composition relations hold:*
(a) $D_{0^+}^{\alpha}I_{0^+}^{\alpha}h(t)=h(t),\quad \alpha>0,\quad h\in L^1[0,+\infty)$.
(b) $D_{0^+}^{\alpha}I_{0^+}^{\gamma}h(t)=I_{0^+}^{\gamma-\alpha}h(t),\quad \gamma>\alpha>0,\quad h\in L^1[0,+\infty)$.
(c) $I_{0^+}^{\alpha}I_{0^+}^{\gamma}h(t)=I_{0^+}^{\alpha+\gamma}h(t),\quad \alpha>0,\quad \gamma>0,\quad h\in L^1[0,+\infty)$.
(d) $D_{0^+}^{\alpha}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}$, for $\lambda>-1$, in particular for $D_{0^+}^{\alpha}t^{\alpha-m}=0$, $m=1,2,\dots,N$, where N is the smallest integer greater than or equal to α.
(e) $I_{0^+}^{\alpha}t^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}t^{\alpha+\lambda},\quad \alpha>0,\quad \lambda>-1$.

The following result is needed to prove our main existence result. This is a nonlinear alternative for Krasnosel'skii' s fixed point theorem [1].

Theorem 2.1. *([1]) Let U be an open set in a closed, convex set C of a Banach space E. Assume $0\in U$, $T(\overline{U})$ bounded and $T:\overline{U}\to C$ is given by $T=T_1+T_2$, where*
$T_1:\overline{U}\to E$ is continuous and completely continuous and $T_2:\overline{U}\to E$ is contraction (i.e., there exists a constant $0<l<1$, such that $\|T_2(x)-T_2(y)\|\leqslant l\|x-y\|$, for all $x,y\in\overline{U}$). Then either,
(a) T has a fixed point in $\overline{U}$, or
(b) There is a point $u\in\partial U$ and $\lambda\in(0,\ 1)$ with $u=\lambda T(u)$.

3. Related Lemmas

Consider the Banach spaces X, Y defined by

$$X = \left\{ u \in C([0, +\infty), \mathbb{R}), \quad \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty \right\}$$

with the norm

$$\|u\|_X = \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}}$$

and

$$\begin{aligned} Y \quad = \quad & \Bigg\{ u \in X,\ D_{0^+}^{\alpha-2}u,\ D_{0^+}^{\alpha-1}u \in C([0, +\infty), \mathbb{R}), \\ & \sup_{t\geq 0} \frac{|D_{0^+}^{\alpha-2}u(t)|}{1+t} < +\infty,\ \sup_{t\geq 0} |D_{0^+}^{\alpha-1}u(t)| < +\infty \Bigg\} \end{aligned}$$

with the norm

$$\|u\|_Y = \max\left\{ \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}},\ \sup_{t\geq 0} \frac{|D_{0^+}^{\alpha-2}u(t)|}{1+t},\ \sup_{t\geq 0} |D_{0^+}^{\alpha-1}u(t)| \right\}.$$

Now, we list some conditions in this paper for convenience:
$(H1)$ The function $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory, i.e., $f(t, u, v, w)$ is Lebesgue measurable in t for all $(u, v, w) \in \mathbb{R}^3$, and continuous in (u, v, w) for a.e. $t \in [0, +\infty)$.
$(H2)$ There exist nonnegative functions $(1 + t^{\alpha-1})\varphi(t)$, $\psi(t)$, $(1 + t)\mu(t)$, $\phi(t) \in L^1[0, +\infty)$ such that
$|f(t, x, y, z)| \leqslant \varphi(t)|x| + \psi(t)|y| + \mu(t)|z| + \phi(t)$ for all $x,\ y,\ z \in \mathbb{R}$ and $t \in [0, +\infty)$.
$(H3)$ There exists a positive constant l such that $0 < l < \Gamma(\alpha)$ and

$$|N(u) - N(v)| \leqslant \frac{l}{\Gamma(\alpha)} \|u - v\|_Y \text{ for all } u,\ v \in Y.$$

$(H4)$ $N(0) = 0$.
$(H5)$ There exists $\rho > 0$ such that

$$\rho > 2 \int_0^{+\infty} \big(\rho((1 + s^{\alpha-1})\varphi(s) + \psi(s) + (1 + s)\mu(s)) + \phi(s)\big)\, ds + \frac{l\rho}{\Gamma(\alpha)}.$$

Lemma 3.1. *Let $h \in L^1[0, +\infty)$, then the bvp*

$$\left\{ \begin{array}{ll} D_{0^+}^{\alpha}u(t) = h(t), & t \in (0, +\infty), \\ u(0) = D_{0^+}^{\alpha-2}u(0) = 0, & \lim\limits_{t\to+\infty} D_{0^+}^{\alpha-1}u(t) = N(u), \end{array} \right. \tag{3.1}$$

has a unique solution given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s) ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} h(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} g(u).$$

Proof. By Lemma 2.1 and from $D^{\alpha}_{0+}u(t) = h(t)$, we have

$$u(t) = I^{\alpha}_{0+}h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \text{ for some constants } c_1,\ c_2,\ c_3 \in \mathbb{R}.$$

So the solution of (3.1) can be written as

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

From $u(0) = 0$ we get

$$t^{\alpha-3}\left(c_1 t^2 + c_2 t + c_3\right) = 0,$$

we known that $c_3 = 0$.
On the other hand, we have

$$\begin{aligned} D^{\alpha-2}_{0+}u(t) &= D^{\alpha-2}_{0+} I^{\alpha}_{0+} h(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \\ &= I^{2}_{0+}h(t) + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) \\ &= \int_0^t (t-s)h(s)ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1). \end{aligned}$$

From $D^{\alpha-2}_{0+}u(0) = 0$ we known that $c_2 = 0$.
Moreover

$$\begin{aligned} D^{\alpha-1}_{0+}u(t) &= D^{\alpha-1}_{0+} I^{\alpha}_{0+} h(t) + c_1\Gamma(\alpha) \\ &= I^{1}_{0+}h(t) + c_1\Gamma(\alpha) \\ &= \int_0^t h(s)ds + c_1\Gamma(\alpha). \end{aligned}$$

From $\lim_{t\to+\infty} D^{\alpha-1}_{0+}u(t) = N(u)$, we get $c_1 = \frac{1}{\Gamma(\alpha)}N(u) - \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} h(s)ds$.
Therefore, the unique solution of fractional bvp (3.1) is

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} h(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} N(u).$$

□

Now, define the following operators $T_1,\ T_2,\ T$ on Y by

$$\begin{aligned} (T_1u)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} h(s)ds, \\ (T_2u)(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} N(u), \\ (Tu)(t) &= (T_1u)(t) + (T_2u)(t). \end{aligned}$$

Bvp (1.1) has a solution u if and only if u solves the operator equation $u = Tu$. We will prove the existence of a fixed point of T. For this we verify that the operator T satisfies all conditions of Theorem 2.1.
Since the Arzela-Ascoli theorem fails to work in the space Y, we need a modified compactness criterion to prove T_1 is compact.

Lemma 3.2. *([14]) Let $Z = \{u \in Y,\ \|u\|_Y < l\}$ such that $l > 0$, $Z_1 = \{\frac{u(t)}{1+t^{\alpha-1}},\ u \in Z\}$,*
$Z_2 = \{D_{0^+}^{\alpha-1}u(t),\ u \in Z\}$ and $Z_3 = \left\{\frac{D_{0^+}^{\alpha-2}u(t)}{1+t},\ u \in Z\right\}$. Then Z is relatively compact on Y if Z_1, Z_2 and Z_3 are equicontinuous on any compact intervals of $[0,+\infty)$ and are equiconvergent at infinity.

Definition 3.1. Z_1, Z_2 and Z_3 are called equiconvergent at infinity if and only if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon,\ \left|D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)\right| < \varepsilon \text{ and}$$
$$\left|\frac{D_{0^+}^{\alpha-2}u(t_1)}{1+t_1} - \frac{D_{0^+}^{\alpha-2}u(t_2)}{1+t_2}\right| < \varepsilon,$$

for any $t_1,\ t_2 > \delta$ and $u \in Z$.

Let $\Omega_r = \{u \in Y,\quad \|u\|_Y < r\},\ (r > 0)$ be the open ball of radius r in Y.

Lemma 3.3. *If $(H1)-(H4)$ hold, then $T(\overline{\Omega}_r)$ is a bounded set.*

Proof. We have

$$\begin{aligned}
&\sup_{t\geq 0}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\left(\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right. \\
&+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}|N(u)|\right) \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds+\frac{lr}{\Gamma(\alpha)}\right).
\end{aligned}$$

In addition

$$\begin{aligned}
&\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \\
\leqslant\ & 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)| \\
\leqslant\ & 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds+\frac{lr}{\Gamma(\alpha)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| \\
&\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)| \\
&\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds+\frac{lr}{\Gamma(\alpha)}.
\end{aligned}$$

So

$$\|Tu\|_Y<+\infty,\ \text{for}\ u\in\overline{\Omega}_r.$$

□

Lemma 3.4. *If* $(H1)$, $(H2)$ *hold, then* $T_1:\overline{\Omega}_r\to Y$ *is completely continuous.*

Proof. We firstly verify that the set $T_1(\overline{\Omega}_r)$ is bounded.
By definition of the operator T_1 we have that, for any $u\in\overline{\Omega}_r$,

$$\begin{aligned}
\left|\frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| &\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right. \\
&\quad\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right) \\
&\leqslant \frac{2}{\Gamma(\alpha)}\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

In addition

$$\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_1u(t)\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

Also

$$\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right| &\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

Hence

$$\|T_1u\|_Y\leqslant 2\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds,\ \text{for}\ u\in\overline{\Omega}_r.$$

Now, we divide the proof into three steps.
- Step 1: We show that T_1 is continuous.
Let $u_n \to u$ as $n \to +\infty$ in $\overline{\Omega}_r$, we have

$$\begin{aligned}
&\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s))|ds \\
&+\frac{2}{\Gamma(\alpha)}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & \frac{2}{\Gamma(\alpha)}\int_0^{+\infty} (\|u_n\|_Y((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds \\
&+\frac{2}{\Gamma(\alpha)}\int_0^{+\infty} (\|u\|_Y((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds.
\end{aligned}$$

So

$$\begin{aligned}
&\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \\
\leqslant\ & \frac{4}{\Gamma(\alpha)}\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

Using the continuity of f, we obtain that

$$|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))| \to 0, \text{ as } n \to +\infty,$$

which implies

$$\|T_1u_n - T_1u\|_X = \sup_{t\geq 0}\left|\frac{(T_1u_n)(t)}{1+t^{\alpha-1}} - \frac{(T_1u)(t)}{1+t^{\alpha-1}}\right| \to 0,$$

uniformly as $n \to +\infty$.
Moreover

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}T_1u_n(t) - D_{0^+}^{\alpha-1}T_1u(t)\right| \\
\leqslant\ & 2\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s)) \\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & 4\left(\int_0^{+\infty} (r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

Also

$$\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u_n(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right|\\
\leqslant\ & 2\int_0^{+\infty}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s),D_{0^+}^{\alpha-2}u_n(s))\\
&-f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
\leqslant\ & 4\left(\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}$$

Using again the continuity of f, we get

$$\sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}T_1u_n(t)-D_{0^+}^{\alpha-1}T_1u(t)\right|\to 0,\ \sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}T_1u_n(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_1u(t)}{1+t}\right|\to 0,$$

uniformly as $n\to+\infty$.
We conclude

$$\|T_1u_n-T_1u\|_Y\to 0,\ \text{uniformly as } n\to+\infty,\ \text{as claimed.}$$

- Step 2: We show that $T_1:\overline{\Omega}_r\to X$ is relatively compact.
According to the above $T_1(\overline{\Omega}_r)$ is uniformly bounded. We show that functions from $\left\{\frac{T_1\overline{\Omega}_r}{1+t^\alpha}\right\}$ and functions from $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equicontinuous on any compact intervals of $[0,+\infty)$.
Let $I\subset[0,+\infty)$ be a compact interval, then for any $t_1,\ t_2\in I$ such that $t_1<t_2$, and for $u\in\overline{\Omega}_r$, we have

$$\begin{aligned}
&\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|\\
=\ &\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&-\int_0^{+\infty}\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\left.+\int_0^{+\infty}\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\right.\\
&\left.-\int_0^{t_2}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^{t_2}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\left.-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\left.+\int_0^{+\infty}\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\
&+\int_0^{t_2}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&\left.+\int_0^{+\infty}\frac{|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right.\\
+\ &\int_0^{t_2}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\\
+\ &\left.\int_0^{+\infty}\frac{|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\right).
\end{aligned}
$$

The last term converges to 0 uniformly as $|t_1-t_2|\to 0$.
Moreover

$$
\begin{aligned}
&\left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right|\\
=\ &\left|\int_0^{t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\left.-\int_0^{t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right|\\
\leqslant\ &\int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds,
\end{aligned}
$$

which converges to 0 uniformly as $|t_1-t_2|\to 0$. Also

$$
\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right|\\
=\ &\left|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&\left.+\frac{t_2-t_1}{(1+t_1)(1+t_2)}\int_0^{+\infty}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \left|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\quad\left.-\int_0^{t_2}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\quad+\left|\int_0^{t_2}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\
&\quad\left.-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
&\quad+\frac{|t_2-t_1|}{(1+t_1)(1+t_2)}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds.\\
&\leqslant \int_{t_1}^{t_2}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s))ds\\
&\quad+\frac{2|t_1-t_2|}{(1+t_1)(1+t_2)}\int_0^{+\infty}(r((1+s^{\alpha-1})\varphi(s)+\psi(s)\\
&\quad+(1+s)\mu(s))+\phi(s))ds,
\end{aligned}
$$

which converges to 0 uniformly as $|t_1-t_2|\to 0$.
Then, for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$
\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|<\varepsilon,\ \left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right|<\varepsilon
$$

and

$$
\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right|<\varepsilon,
$$

for all $u\in\overline{\Omega}_r$, if $|t_1-t_2|<\delta$, $t_1,\ t_2\in I$.
Showing that, the functions belonging to $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$ and the functions belonging to $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and to $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are locally equicontinuous on $[0,+\infty)$.

- Step 3: We show that the functions from $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$, $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equiconvergent at infinity.

For any $u\in\overline{\Omega}_r$, we have

$$
\int_0^{+\infty}\left|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))\right|ds<+\infty.
$$

Considering condition $(H2)$, for given $\varepsilon>0$, there exists a constant $L>0$ such that

$$
\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds<\varepsilon.
$$

On the other hand, since $\lim_{t\to+\infty}\frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and $\lim_{t\to+\infty}\frac{t-L}{1+t}=1$, there exists a constant $\delta>L>0$ such that for any $t_1,\ t_2\geq\delta$ and $0\leq s\leq L$, we have

$$\begin{aligned}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right| &= \left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-1+1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\\ &\leqslant \left|1-\frac{(t_1-L)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|+\left|1-\frac{(t_2-L)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|<\varepsilon\end{aligned}$$

and

$$\begin{aligned}\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right| &= \left|\frac{t_1-s}{1+t_1}-1+1-\frac{t_2-s}{1+t_2}\right|\\ &\leqslant \left|1-\frac{t_1-L}{1+t_1}\right|+\left|1-\frac{t_2-L}{1+t_2}\right|<\varepsilon.\end{aligned}$$

Thus, for any $t_1,\ t_2\geq\delta>L>0$, we get

$$\begin{aligned}&\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|\\ =\ &\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right.\\ &-\int_0^{+\infty}\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\ &-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\ &\left.+\int_0^{+\infty}\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|.\end{aligned}$$

So

$$\begin{aligned}&\left|\frac{(T_1u)(t_1)}{1+t_1^{\alpha-1}}-\frac{(T_1u)(t_2)}{1+t_2^{\alpha-1}}\right|\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_0^L\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\ &+\int_L^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\ &+\int_L^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\ &\left.+2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right)\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_0^L\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^L |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&+4\int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\Bigg)\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\Bigg(\sup_{s\in[0,L],\,u\in\overline{\Omega}_r} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L\varepsilon\\
&+2\sup_{s\in[0,L],\,u\in\overline{\Omega}_r} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L+4\varepsilon\Bigg).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&\left|D_{0^+}^{\alpha-1}T_1u(t_1)-D_{0^+}^{\alpha-1}T_1u(t_2)\right|\\
=\ &\left|\int_{t_1}^{t_2} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\right|\\
\leqslant\ &\int_L^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds<\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right|\\
=\ &\Bigg|\int_0^{t_1}\frac{t_1-s}{1+t_1}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&-\int_0^{t_2}\frac{t_2-s}{1+t_2}f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&-\frac{t_1}{1+t_1}\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\\
&+\frac{t_2}{1+t_2}\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))ds\Bigg|.
\end{aligned}
$$

So

$$
\begin{aligned}
&\left|\frac{D_{0^+}^{\alpha-2}T_1u(t_1)}{1+t_1}-\frac{D_{0^+}^{\alpha-2}T_1u(t_2)}{1+t_2}\right|\\
\leqslant\ &\int_0^L\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&+\int_L^{t_1}\frac{t_1-s}{1+t_1}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\\
&+\int_L^{t_2}\frac{t_2-s}{1+t_2}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
& +2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & \int_0^L\left|\frac{t_1-s}{1+t_1}-\frac{t_2-s}{1+t_2}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
& +2\int_0^L|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
& +4\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
\leqslant\ & \sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L\varepsilon \\
& +2\sup_{s\in[0,L],\,u\in\overline{\Omega}_r}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|L+4\varepsilon.
\end{aligned}
$$

Which yield that the functions from $\{\frac{T_1\overline{\Omega}_r}{1+t^{\alpha-1}}\}$, $\{D_{0^+}^{\alpha-1}T_1\overline{\Omega}_r\}$ and from $\left\{\frac{D_{0^+}^{\alpha-2}T_1\overline{\Omega}_r}{1+t}\right\}$ are equiconvergent at infinity. According to Lemma 3.2, it follows that $T_1(\overline{\Omega}_r)$ is relatively compact, ending the proof of the Lemma. □

Lemma 3.5. *If $(H3)$ holds. Then $T_2:\overline{\Omega}_r\to Y$ is a contraction mapping.*

Proof. We have

$$
\begin{aligned}
\left|\frac{T_2u(t)}{1+t^{\alpha-1}}-\frac{T_2v(t)}{1+t^{\alpha-1}}\right| &\leqslant \frac{1}{\Gamma(\alpha)}\left|\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\right||N(u)-N(v)| \\
&\leqslant \frac{1}{\Gamma(\alpha)}|N(u)-N(v)| \\
&\leqslant \frac{l}{(\Gamma(\alpha))^2}\|u-v\|_Y.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\left|D_{0^+}^{\alpha-1}T_2u(t)-D_{0^+}^{\alpha-1}T_2v(t)\right| &= |N(u)-N(v)| \\
&\leqslant \frac{l}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}
$$

Also

$$
\begin{aligned}
\left|\frac{D_{0^+}^{\alpha-2}T_2u(t)}{1+t}-\frac{D_{0^+}^{\alpha-2}T_2v(t)}{1+t}\right| &= \left|\frac{t}{1+t}(N(u)-N(v))\right| \\
&\leqslant \frac{l}{\Gamma(\alpha)}\|u-v\|_Y.
\end{aligned}
$$

We conclude

$$
\|T_2u-T_2v\|_Y \leqslant \frac{l}{\Gamma(\alpha)}\|u-v\|_Y.
$$

From $(H3)$, we infer that T_2 is a contraction mapping. □

4. Main results

Theorem 4.1. *If assumptions* $(H1)-(H5)$ *hold, then the problem (1.1) has at least one solution.*

Proof. Consider the parameterized bvp

$$(4.1)\quad \begin{cases} D_{0^+}^{\alpha}u(t)=\lambda f(t,u(t),D_{0^+}^{\alpha-1}u(t),D_{0^+}^{\alpha-2}u(t)), & t\in(0,+\infty),\\ u(0)=D_{0^+}^{\alpha-2}u(0)=0, \quad \lim\limits_{t\to+\infty}D_{0^+}^{\alpha-1}u(t)=\lambda N(u), \end{cases}$$

for $\lambda\in(0\ ,\ 1)$.
Solving problem (4.1) is equivalent to solving the fixed point of equation $u=\lambda Tu$.
Let

$$\Omega_\rho=\{u\in Y,\quad \|u\|_Y<\rho\}.$$

From Lemma 3.3, the set $T(\overline{\Omega}_\rho)$ is bounded and by Lemma 3.4, the operator $T_1:\overline{\Omega}_\rho\to Y$ is completely continuous, while Lemma 3.5 implies that the operator $T_2:\overline{\Omega}_\rho\to Y$ is contractive. So it remains to prove that $u\neq\lambda Tu$ for $u\in\partial\Omega_\rho$ and $\lambda\in(0\ ,\ 1)$.
Arguing by contradiction, if there exists $u\in\partial\Omega_\rho$ with $u=\lambda Tu$, then for $\lambda\in(0\ ,\ 1)$ we have

$$\begin{aligned} &\sup_{t\geq0}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|\\ \leqslant\ &\sup_{t\geq0}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right|\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds\right.\\ &\left.+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}|N(u)|\right).\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)-N(0)|+|N(0)|\right). \end{aligned}$$

So

$$\begin{aligned} &\sup_{t\geq0}\left|\frac{u(t)}{1+t^{\alpha-1}}\right|\\ \leqslant\ &\frac{1}{\Gamma(\alpha)}\left(2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}\right). \end{aligned}$$

In addition

$$\begin{aligned} &\sup_{t\geq0}\left|D_{0^+}^{\alpha-1}u(t)\right|\\ =\ &\sup_{t\geq0}\left|\lambda D_{0^+}^{\alpha-1}Tu(t)\right| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_{t\geq 0}\left|D_{0^+}^{\alpha-1}Tu(t)\right| \\
&\leqslant 2\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+|N(u)| \\
&\leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}u(t)}{1+t}\right| \\
=\ &\sup_{t\geq 0}\left|\lambda\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| \\
\leqslant\ &\sup_{t\geq 0}\left|\frac{D_{0^+}^{\alpha-2}Tu(t)}{1+t}\right| \\
\leqslant\ &\int_0^t\frac{t-s}{1+t}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds \\
&+\frac{t}{1+t}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s),D_{0^+}^{\alpha-2}u(s))|ds+\frac{t^{\alpha-1}}{1+t^{\alpha-1}}|g(u)| \\
\leqslant\ &2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}.
\end{aligned}
$$

So

$$
\|u\|_Y \leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}
$$

and thus

$$
\rho \leqslant 2\int_0^{+\infty}\big(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\big)\,ds+\frac{l\rho}{\Gamma(\alpha)}.
$$

This implies that

$$
\frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s)+\psi(s)+(1+s)\mu(s))+\phi(s)\right)ds+\frac{l\rho}{\Gamma(\alpha)}} \leqslant 1,
$$

contradicting condition $(H5)$. With theorem 2.1 we conclude that bvp (1.1) has at least one solution. □

5. Example

Example 5.1. Consider the bvp on infinite interval

$$(5.1)\quad \begin{cases} D_{0^+}^{\frac{5}{2}}u(t) = \frac{e^{-30t}}{1+\sqrt{t^3}}u(t) + \frac{D_{0^+}^{\frac{3}{2}}u(t)}{(50+t)^2} + \frac{D_{0^+}^{\frac{1}{2}}u(t)}{50(1+t)^3} + e^{-t}, \quad t \in (0,+\infty), \\ u(0) = D_{0^+}^{\frac{1}{2}}u(0) = 0, \quad \lim_{t\to+\infty} D_{0^+}^{\frac{3}{2}}u(t) = \frac{1}{10}u(1) + \frac{1}{20}u(4). \end{cases}$$

In this case, $\alpha = \frac{5}{2}$, $\Gamma(\frac{5}{2}) \approx 1.329340388$, $N(u) = \frac{1}{10}u(1) + \frac{1}{20}u(4)$, it's mean $c_1 = \frac{1}{10}$, $c_2 = \frac{1}{20}$, $\xi_1 = 1$, $\xi_2 = 4$.
We will apply Theorem 4.1 to show that problem (5.1) has at least a solution.
Let

$$f(t,x,y,z) = \frac{e^{-30t}}{1+\sqrt{t^3}}x + \frac{y}{(50+t)^2} + \frac{z}{50(1+t)^3} + e^{-t}.$$

Choose

$$\rho > \frac{600\Gamma(\frac{5}{2})}{266\Gamma(\frac{5}{2}) - 195}.$$

Then
$(H1)$ $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \times\mathbb{R}$ is Carathéodory.
$(H2)$ $|f(t,x,y,z)| \leqslant \frac{e^{-30t}}{1+\sqrt{t^3}}|x| + \frac{1}{(50+t)^2}|y| + \frac{1}{50(1+t)^3}|z| + e^{-t}$. So we may take

$$\varphi(t) = \frac{e^{-30t}}{1+\sqrt{t^3}},\ \psi(t) = \frac{1}{(50+t)^2},\ \mu(t) = \frac{1}{50(1+t)^3},\ \phi(t) = e^{-t}$$

and note that $(1+\sqrt{t^3})\varphi(t)$, $\psi(t)$, $(1+t)\mu(t)$, $\phi(t) \in L^1[0,+\infty)$ such that

$$\begin{aligned} \int_0^{+\infty} \left(1+s^{\frac{3}{2}}\right)\varphi(s)ds &= \frac{1}{30},\ \int_0^{+\infty}\psi(s)ds = \frac{1}{50}, \\ \int_0^{+\infty}(1+s)\mu(s)ds &= \frac{1}{50},\ \int_0^{+\infty}\phi(s)ds = 1. \end{aligned}$$

$(H3)$ Choose $l = c_1(1+\sqrt{\xi_1^3}) + c_2(1+\sqrt{\xi_2^3}) = \frac{9}{10}$ verify $0 < l < \Gamma(\frac{5}{2})$ with $|N(u)-N(v)| \leqslant \frac{l}{\Gamma(\frac{5}{2})}\|u-v\|_Y$ for all $u,\ v \in Y$.
$(H4)$ $N(0) = 0$.
$(H5)$

$$\begin{aligned} & \frac{\rho}{2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s) + \psi(s) + (1+s)\mu(s)) + \phi(s)\right)ds + \frac{l\rho}{\Gamma(\alpha)}} \\ = & \frac{\rho}{\frac{34\Gamma(\frac{5}{2})+195}{300\Gamma(\frac{5}{2})}\rho + 2} \\ = & \frac{300\Gamma(\frac{5}{2})}{34\Gamma(\frac{5}{2}) + 195 + \frac{600\Gamma(\frac{5}{2})}{\rho}} \\ > & \ 1. \end{aligned}$$

Which implies

$$\rho > 2\int_0^{+\infty}\left(\rho((1+s^{\alpha-1})\varphi(s) + \psi(s) + (1+s)\mu(s)) + \phi(s)\right)ds + \frac{l\rho}{\Gamma(\alpha)}.$$

Hence, all conditions of Theorem 4.1 are satisfied, we deduce that the bvp (5.1) has at least one solution.

6. Conclusion

In this work, we considered a class of fractional differential equation with nonlocal boundary conditions on an infinite interval. With the aid of the Krasnosel'skii's fixed point theorem, we have obtained existence results for the proposed problem in this paper. An example was presented to illustrate the main results. The boundary value problem of fractional differential equations on an infinite interval have been widely discussed in recent years. The examples of this is establishing the existence of solutions for fractional differential equations with multi-point boundary conditions, as well as the existence of positive solutions for fractional boundary value problem on an infinite interval.

Acknowledgements

The authors thank the anonymous referees for their thorough review.

REFERENCES

1. R. P. Agarwal, M. Meehan and D. O'Regan: *Fixed point theory and applications.* Cambridge Tracts in Mathematics **141**. Cambridge University Press, Cambridge, 2001.
2. I. Ahmad, H. Ahmad, Mustafa INC, S. W. Yao and B. Almohsen: *Application of local meshless method for the solution of two term time fractional-order multi-dimensional PDE arising in heat and mass transfer* Thermal Science **24(1)** (2020), 95-105.
3. H. Ahmad, T. A. Khan, I. Ahmad, P. S. Stanimirović and Y. M. Chu: *A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations.* Results in Physics **19** (2020) 103462.
4. I. Ahmad, Siraj-ul-Islam, Mehnaz and S. Zaman: *Local meshless differential quadrature collocation method for time-fractional PDEs.* Discrete and Continuous Dynamical Systems-S **13(10)** (2020), 2641-2654.
5. Z. Bai and Y. Zhang: *The existence of solutions for a fractional multi-point boundary value problem.* Computers and Mathematics with Applications **60** (2010), 2364-2372.
6. Y. Gholami: *Existence of an unbouded solution for multi-point boundary value problems of fractional differential equations on an infinite domain, class of Riemann-Liouville fractional differential equations.* Fractional Differential Calculus **4(2)** (2014), 125-136.
7. K. Ghanbari and Y. Gholami: *Existence and multiplicity of positive solutions for m-point nonlinear fractional differential equations on the half-line.* Electronic Journal of Differential Equations **2012(238)** (2012), 1-15.
8. J. He, X. Zhang, L. Liu, Y. Wu and Y. Gui: *Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions.* Boundary Value Problems (2018) 2018:189.
9. R. A. Khan and H. Khan: *On existence of solution for multi-points boundary value problem.* Journal of Fractional Calculus and Applications **5** (2014), 121-132.

10. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204. Elsevier Science B. V., Amsterdam, 2006.

11. B. Li, S. Sun, Y. Li and P. Zhao: *Multi-point boundary value problems for a class of Riemann-Liouville fractional differential equations*. Advances in Difference Equations 2014, 2014:151.

12. S. Liang and J. Zhang: *Existence of multiple positive solutions for m-point fractional boundary value problems of on an infinite interval*. Mathematical and Computer Modelling **54** (2011) 1334-1346.

13. K. Oldham and J. Spanier: *The fractional calculus*. Theory and applications of differentiation and integration to arbitrary order, (AP, 1974)(T)(ISBN 0-12-525550-0)(240s)-MCat.

14. X. Su and S. Zhang: *Unbounded solutions to a boundary value problem of fractional order on the half-line*. Computers and Mathematics with Applications **61** (2011), 1079-1087.

15. C. Shen, H. Zhou and L. Yang: *On the existence of solution to a boundary value problem of fractional differential equation on the infinite interval*. Boundary Value Problems (2015), 2015:241.

16. P. Thiramanus, S.K. Ntouyas and J. Tariboon: *Existence of solutions for Riemann-Liouville fractional differential equations with nonlocal Erdelyi-Kober integral boundary conditions on the half-line*. Boundary Value Problems (2015), 2015:196.

17. G. Wang: *Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval*. Applied Mathematics Letters, 47 (2015) 1-7.

18. Y. Yang and F. Meng: *Positive solutions for nonlocal boundary value problems of fractional differential equation*. WSEAS TRANSACTIONS on MATHEMATICS, 12 (2013) 1154-1163.

19. X. Zhang and Q. Zhong: *Multiple positive solutions for nonlocal boundary value problems of singular fractional differential equations*. Boundary Value Problems (2016) 2016:65.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 21–31
https://doi.org/10.22190/FUMI220322001E
Original Scientific Papere

ONE-SIDED GENERALIZED $(\alpha, \beta)-$REVERSE DERIVATIONS OF ASSOCIATIVE RINGS

Ayşe Engin and Neşet Aydın

Department of Mathematics, Faculty of Science
Çanakkale Onsekiz Mart University, 17100 Çanakkale, Turkiye

ORCID IDs: Ayşe Engin https://orcid.org/0000-0002-1626-5498
Neşet Aydın https://orcid.org/0000-0002-7193-3399

Abstract. In this paper, we introduce the notion of the one-sided generalized (α, β)-reverse derivation of a ring R. Let R be a semiprime ring, ϱ be a non-zero ideal of R, α be an epimorphism of ϱ, β be a homomorphism of ϱ (α be a homomorphism of ϱ, β be an epimorphism of ϱ) and $\gamma : \varrho \to R$ be a non-zero (α, β)-reverse derivation. We show that there exists $F : \varrho \to R$, an l-generalized (α, β)-reverse derivation (an r-generalized (α, β)-reverse derivation) associated with γ iff $F(\varrho), \gamma(\varrho) \subset C_R(\varrho)$ and F is an r-generalized (β, α)-derivation (an l-generalized (β, α)-derivation) associated with (β, α)-derivation γ on ϱ. This theorem generalized the results of A. Aboubakr and S. Gonzalez proved in [1, Theorem 3.1, and Theorem 3.2].
Keywords: Semiprime ring, prime ring, one-sided generalized $(\alpha, \beta)-$reverse derivation, $(\alpha, \beta)-$reverse derivation.

1. Introduction

Throughout the paper, R is an associative ring with Z, which the center of R denotes. Recall that a ring R is prime if for any $r_1, r_2 \in R$, $r_1 R r_2 = (0)$ implies $r_1 = 0$ or $r_2 = 0$, and is a semiprime in case $r_1 \in R$, $r_1 R r_1 = (0)$ implies $r_1 = 0$. For $r_1, r_2 \in R$, $[r_1, r_2]$ denotes the element $r_1 r_2 - r_2 r_1$. The symbol $[r_1, r_2]$ stands for Lie commutator of r_1 and r_2 and it satisfies the basic commutator identities: for each $r_1, r_2, r_3 \in R$, $[r_1 + r_2, r_3] = [r_1, r_3] + [r_2, r_3]$, $[r_1, r_2 + r_3] = [r_1, r_2] + [r_1, r_3]$, $[r_1 r_2, r_3] = r_1 [r_2, r_3] + [r_1, r_3] r_2$, $[r_1, r_2 r_3] = [r_1, r_2] r_3 + r_2 [r_1, r_3]$. We denote the

Received March 22, 2022, revised: July 11, 2023, accepted: September 04, 2023
Communicated by Dijana Mosić
Corresponding Author: Ayşe Engin. E-mail addresses: ayseayraan@gmail.com (A. Engin), neseta@comu.edu.tr (N. Aydın)
2010 *Mathematics Subject Classification.* Primary 16N60; Secondary 16W30

identity mapping of R by id_R; that is, the mapping $id_R : R \to R$ is defined as $id_R(r_1) = r_1$, for all $r_1 \in R$. For a non-empty subset A of R, $C_R(A)$ is defined as $C_R(A) = \{r \in R : [r, x] = 0$, for all $x \in A\}$.

Let α, β be any two mapping of R. An additive mapping $\delta : R \to R$ is called an (α, β)-derivation if $\delta(r_1r_2) = \delta(r_1)\alpha(r_2) + \beta(r_1)\delta(r_2)$ holds, for all $r_1, r_2 \in R$. An additive mapping $\varphi : R \to R$ is called a right generalized $(\alpha, \beta)-$derivation (a left generalized $(\alpha, \beta)-$derivation) of R associated with δ, if $\varphi(r_1r_2) = \delta(r_1)\alpha(r_2) + \beta(r_1)\varphi(r_2)$ $(\varphi(r_1r_2) = \varphi(r_1)\alpha(r_2) + \beta(r_1)\delta(r_2))$, for all $r_1, r_2 \in R$ and φ is said to be a generalized (α, β)-derivation of R with δ if it is both a right and a left generalized (α, β)-derivation of R associated with δ.

Many authors have investigated the relationship between the commutativity of a ring and the act of derivation ((α, β)-derivation, reverse derivation, (α, β)-reverse derivation, generalized reverse derivation, etc.) defined on the ring. Herstein (1957) was the first to introduce the concept of reverse derivation. An additive mapping $g : R \to R$ is a reverse derivation if $g(r_1r_2) = g(r_2)r_1 + r_2g(r_1)$, for all $r_1, r_2 \in R$. In [4], it is shown that if a prime ring R with a characteristic different from two admits non-zero reverse derivation g, then g is a derivation of R. An additive mapping $d : R \to R$ is an $(\alpha, \beta)-$reverse derivation if $d(r_1r_2) = d(r_2)\alpha(r_1) + \beta(r_2)d(r_1)$, for all $r_1, r_2 \in R$. In [8], Chaudhry and Thaheem shown that if a semiprime ring R admits non-zero (α, β)-reverse derivation d, then d is (α, β)-reverse derivation of R. Here, α and β are automorphism of R. An additive mapping $H : R \to R$ is called l-generalized reverse derivation (r-generalized reverse derivation) In [1], A. Aboubakr and S. Gonzalez (2015) introduced one-sided generalized reverse derivation. An additive mapping $H : R \to R$ is called an l-generalized reverse derivation (r-generalized reverse derivation) if there exists a reverse derivation $g : R \to R$ such that $H(r_1r_2) = H(r_2)r_1 + r_2g(r_1)$ $(H(r_1r_2) = g(r_2)r_1 + r_2H(r_1))$, for all $r_1, r_2 \in R$. In [1], they have indicated that if a semiprime ring R admits non-zero one-sided generalized reverse derivation H associated with reverse derivation g, then H is a one-sided generalized derivation with associated derivation g. Reverse derivation, generalized reverse derivation, (α, β)-reverse derivation, generalized (α, β)-reverse derivation, multiplicative reverse derivation, multiplicative generalized reverse derivation, multiplicative (α, β)-reverse derivation, and multiplicative generalized (α, β)-reverse derivation of prime or semiprime rings have been studied by a lot of scholars in the literature. (see [2],[3],[4], [9],[10],[12],[13],[14],[15],[16].)

This paper extends the notion of one-sided reverse derivation to one-sided generalized (α, β)-reverse derivation.

Definition 1.1. Let R be a ring, α, β be a mapping of R, and γ be an (α, β)-reverse derivation of R. An additive mapping $F : R \to R$ is said to be an r-generalized (α, β)-reverse derivation of R associated with γ if

$$F(r_1r_2) = \gamma(r_2)\alpha(r_1) + \beta(r_2)F(r_1)$$

for all $r_1, r_2 \in R$, F is said to be an $l-$generalized (α, β)-reverse derivation of R associated with γ if

$$F(r_1r_2) = F(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1)$$

for all $r_1, r_2 \in R$ and F said to be a generalized (α, β) $-$reverse derivation of R associated with γ if it is both an r-generalized and $l-$generalized (α, β)-reverse derivation of R associated with γ.

When $\alpha = \beta = id_R$, an r-generalized (l-generalized) (α, β)-reverse derivation is a r-generalized (l-generalized) reverse derivation. Thus, the one-sided generalized reverse derivation is a special case of one-sided generalized (α, β)-reverse derivation.

This study consists of 2 parts. In the first part, we show that If R is a 2-torsion free semiprime ring, α, β are automorphisms of R, and $\gamma : R \to R$ is a non-zero (α, β)-reverse derivation, then γ is an (α, β)-derivation on R. With this result, we will show that the concepts of (α, β)-reverse derivation and (α, β)-derivation overlap in 2 torsion-free semiprime rings in which α and β are automorphisms of the ring. In the second part, we give a generalization of [1, Theorem 3.1, Theorem 3.2, and Corollary 3.3], which is the main result of the article. In that case, one-sided generalized (α, β)-reverse derivation and one-sided generalized (β, α)-derivation overlap in a semiprime ring where only one of α and β is an epimorphism of the ring. Thus we will show that the intersection of the set of all generalized (α, β)-derivation and the set of all generalized (α, β)-reverse derivation is different from the empty set. At the end of the paper, we showed that in case α is a homomorphism of R and β is an epimorphism of R; there is no non-zero generalized (α, β)-reverse derivation associated with (α, β)-reverse derivation of noncommutative prime ring R.

From now on, R is an associative ring, Z is the center of R, and $\alpha, \beta : R \to R$ are homomorphisms.

2. Preliminary

In this section, we give some auxiliary results that will need later. We begin our discussion with several examples related to (α, β)-reverse derivation and one-sided generalized (α, β)-reverse derivation.

Lemma 2.1. [7, Lemma 3] *If the prime ring R contains a commutative non-zero right ideal I, then R is commutative.*

Lemma 2.2. [7, Lemma 4] *Let b and ab be in the center of a prime ring R. If b is not zero, then a is in Z, the center of R.*

Lemma 2.3. [11, Corollary 2.1] *Let R be a 2-torsion free semiprime ring, α, β be automorphisms of R and $L \nsubseteq Z(R)$ be a non-zero square-closed Lie ideal of R. If $\delta : R \to L$ satisfying*

$$(2.1) \qquad \left(a^2\right)^\delta = a^\delta \alpha(a) + \beta(a) a^\delta, \textit{ for all } a \in L$$

and $a^\delta, \beta(a) \in L$, then δ is a (α, β) $-$derivation on L.

Example 2.1. Consider the ring $R = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ the ring of integers. Let us define $\alpha : R \to R$, $\beta : R \to R$, and $d : R \to R$ as follows:

$$\alpha\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_{11} \end{bmatrix}$$

$$\beta\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{22} & 0 \\ 0 & 0 \end{bmatrix}$$

$$d\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & a_{11} - a_{22} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that d is both an (α, β)-reverse derivation and an (α, β)-derivation.

Example 2.2. Consider the ring $R = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ the ring of integers. Define the mappings $\alpha : R \to R$, $\beta : R \to R$, and $d : R \to R$ as follows:

$$\alpha\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_{11} \end{bmatrix}$$

$$\beta\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{22} & 0 \\ 0 & 0 \end{bmatrix}$$

$$d\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_{12} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that d is an (α, β)-reverse derivation. But d is not an (α, β)-derivation.

Example 2.3. Consider the ring $R = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ the ring of integers. Define the mappings $\alpha : R \to R$, $\beta : R \to R$, and $d : R \to R$ as follows:

$$\alpha\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix}$$

$$\beta\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$d\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that d is an (α, β)-derivation. But d is not an (α, β)-reverse derivation.

Example 2.4. Let $(R_1, +, *)$ be a commutative ring and $(R_2, \oplus, \circledast)$ be a noncommutative ring. Let's consider operation $\otimes : R_2 \times R_2 \to R_2$, $r \otimes s = s \circledast r$. With these operations $(R_2, \oplus, \otimes)$ called opposite ring and it is shown R_2^{op}. α, β are homomorphisms of R_2, $\delta : R_2 \to R_2^{op}$ is an $(\beta, \alpha)\,-$derivation, and $\varphi : R_2 \to R_2^{op}$ is a left generalized $(\beta, \alpha)\,-$derivation with δ. Define the mappings $\widetilde{\alpha}, \widetilde{\beta} : R_2 \times R_1 \to R_2 \times R_1$,and $\widetilde{\delta}, \widetilde{\varphi} : R_2 \times R_1 \to R_2^{op} \times R_1$ as follows:

$$\begin{aligned} \widetilde{\alpha}(r, s) &= (\alpha(r), s) \\ \widetilde{\beta}(r, s) &= (\beta(r), s) \\ \widetilde{\delta}\,(r, s) &= (\delta(r), s) \\ \widetilde{\varphi}(r, s) &= (\varphi(r), s)\,. \end{aligned}$$

Then it is straightforward to verify that $\widetilde{\varphi}$ is an $l-$generalized (α, β)-reverse derivation with (α, β) $-$reverse derivation $\widetilde{\delta}$ of $R_2 \times R_1$. But $\widetilde{\varphi}$ is not a generalized (α, β)-derivation with (α, β)-derivation $\widetilde{\delta}$ of $R_2 \times R_1$.

Example 2.5. Let $(R_1, +, *)$ and $(R_2, \oplus, \otimes)$ be a rings as defined in example 2.4. Let α, β be homomorphisms of R_2, $\delta : R_2 \to R_2^{op}$ be an (β, α) $-$derivation, and $\varphi : R_2 \to R_2^{op}$ be a right generalized (β, α) $-$derivation with δ. Define the mappings $\widetilde{\alpha}, \widetilde{\beta} : R_2 \times R_1 \to R_2 \times R_1$ and $\widetilde{\delta}, \widetilde{\varphi} : R_2 \times R_1 \to R_2^{op} \times R_1$ as follows:

$$\begin{aligned}
\widetilde{\alpha}(r, s) &= (\alpha(x), s) \\
\widetilde{\beta}(r, s) &= (\beta(x), s) \\
\widetilde{\delta}\,(r, s) &= (\delta(x), s) \\
\widetilde{\varphi}(r, s) &= (\varphi(x), s)\,.
\end{aligned}$$

Then it is straightforward to verify that $\widetilde{\varphi}$ is an $r-$generalized (α, β)-reverse derivation with (α, β)-reverse derivation $\widetilde{\delta}$ of $R_2 \times R_1$. But $\widetilde{\varphi}$ is not a generalized (α, β)-derivation with (α, β)-derivation $\widetilde{\delta}$ of $R_2 \times R_1$.

3. (α, β) $-$Reverse Derivation

Theorem 3.1. *Let R be a 2-torsion free semiprime ring, α, β be automorphisms of R. If $\gamma : R \to R$ is a non-zero (α, β)-reverse derivation, then γ is an (α, β)-derivation on R.*

Proof. Suppose that R is non-commutative ring. Let $r_1 \in R$. From the hypothesis, we get

$$\gamma(r_1^2) = \gamma(r_1)\alpha(r_1) + \beta(r_1)\gamma(r_1).$$

This equation ensures equality of (2.1). We know that the ring R is a square closed Lie ideal of R. So, we can think of R instead of L in Lemma 2.3. Thus, γ is an (α, β) $-$derivation on R because of Lemma 2.3. While R is a commutative ring, (α, β)-reverse derivation of R is (α, β)-derivation of R. So, the proof ends. □

Theorem 3.2. *Let R be a semiprime ring, ϱ is a non-zero two-sided ideal of R, α be an epimorphism of ϱ and β be a homomorphism of ϱ(or α be a homomorphism of ϱ and β be an epimorphism of ϱ). There exists $\gamma : \varrho \to R$ a non-zero (α, β)-reverse derivation iff $\gamma(\varrho) \subset C_R(\varrho)$ and γ is (β, α)-derivation on ϱ.*

Proof. We only prove case of no parenthesis. The another one has the same argument. Let $x_1, x_2, x_3 \in \varrho$. Since γ is $(\alpha, \beta)-$reverse derivation on ϱ, we have

(3.1)
$$\gamma(x_1x_2x_3) = \gamma(x_1(x_2x_3)) = \gamma(x_3)\alpha(x_2)\alpha(x_1) + \beta(x_3)\gamma(x_2)\alpha(x_1) + \beta(x_2)\beta(x_3)\gamma(x_1)$$

and

(3.2)
$$\gamma(x_1x_2x_3) = \gamma((x_1x_2)x_3) = \gamma(x_3)\alpha(x_1)\alpha(x_2) + \beta(x_3)\gamma(x_2)\alpha(x_1) + \beta(x_3)\beta(x_2)\gamma(x_1).$$

From (3.1) and (3.2),

$$\gamma(x_3)\,[\alpha(x_1),\alpha(x_2)] = [\beta(x_3),\beta(x_2)]\,\gamma(x_1). \tag{3.3}$$

Replacing x_3 by x_2 in (3.3),

$$\gamma(x_2)\,[\alpha(x_1),\alpha(x_2)] = 0$$

for all $x_1, x_2 \in \varrho$. Because α is an epimorphism of ϱ, for each $x_1, x_2 \in \varrho$, we get

$$\gamma(x_2)\,[x_1,\alpha(x_2)] = 0. \tag{3.4}$$

Take $r \in R$. Substituting x_1x_3r for x_1 in (3.4), we obtain $\gamma(x_2)x_1x_3\,[r,\alpha(x_2)] = 0$, for all $x_1, x_2, x_3 \in \varrho, r \in R$. So implies that

$$\gamma(x_2)\varrho\varrho\,[R,\alpha(x_2)] = (0) \tag{3.5}$$

for all $x_2 \in \varrho$. Because ϱ is a semiprime ring, it must contain a family ρ of prime ideals such that $\cap\rho = (0)$. Let ρ_φ be a typical member of this family and $x_2 \in \varrho$; by (3.5),

$$\gamma(x_2)\varrho \subset \rho_\varphi \text{ or } [R,\alpha(x_2)] \subset \rho_\varphi.$$

Let $M = \{x_2 \in \varrho : \gamma(x_2)\varrho \subset \rho_\varphi\}$ and $N = \{x_2 \in \varrho : [R,\alpha(x_2)] \subset \rho_\varphi\}$. Clearly, each group M and N is additive subgroup of ϱ such that $\varrho = M \cup N$. But a group cannot be a set union of two proper subgroups. Hence, $M = \varrho$ or $N = \varrho$. Since ρ_φ is ideal of ϱ, it holds that $\gamma(\varrho)\varrho\,[R,\alpha(\varrho)] \subset \rho_\varphi$. Thus $\gamma(\varrho)\varrho\,[R,\alpha(\varrho)] \subset \cap\rho = (0)$. Because α is an epimorphism of ϱ, it provides that $\gamma(\varrho)\varrho\,[R,\varrho] = 0$. Let $x_1, x_2, x_3 \in \varrho, r \in R$. Means that,

$$\gamma(x_1)x_2\,[r,x_3] = 0. \tag{3.6}$$

Let $x_4 \in \varrho$. In (3.6), replacing r by $x_4\gamma(x_1)$ and x_3 by x_2,we get

$$\gamma(x_1)x_2x_4\,[\gamma(x_1),x_2] = 0. \tag{3.7}$$

In (3.6), substituting x_2 by x_4, we get $\gamma(x_1)x_4\,[r,x_3] = 0$. In this equation replacing x_3 by x_2, r by $\gamma(x_1)$ and multiply from the left by x_2, it holds

$$x_2\gamma(x_1)x_4\,[\gamma(x_1),x_2] = 0. \tag{3.8}$$

From (3.7) and (3.8),

$$[\gamma(x_1),x_2]\,x_4\,[\gamma(x_1),x_2] = 0, \text{ for all } x_1,x_2,x_4 \in \varrho. \tag{3.9}$$

Since ϱ is a semiprime ring,

$$[\gamma(x_1),x_2] = 0, \text{ for all } x_1,x_2 \in \varrho.$$

That is $\gamma(\varrho) \subset C_R(\varrho)$. We get

$$\begin{aligned}\gamma(x_1x_2) &= \gamma(x_2)\alpha(x_1) + \beta(x_2)\gamma(x_1)\\ &= \gamma(x_1)\beta(x_2) + \alpha(x_1)\gamma(x_2)\end{aligned}$$

for all $x_1, x_2 \in \varrho$. This means that γ is (β,α)-derivation on ϱ. The converse is trivial. □

If consider R instead of ϱ in Theorem 3.2, we get

Corollary 3.1. *Let R be a semiprime ring, α be an epimorphism of R and β be a homomorphism of R(or α be a homomorphism of R and β be an epimorphism of R). There exists $\gamma : R \to R$ a non-zero (α, β)-reverse derivation iff central γ is (β, α)-derivation on R.*

Corollary 3.2. *Let R be a prime ring, α be an epimorphism of R and β be a homomorphism of R(or α be a homomorphism of R and β be an epimorphism of R). There exists $\gamma : R \to R$ a non-zero (α, β)-reverse derivation iff R is commutative and γ is an (α, β)-derivation of R.*

Proof. We only prove a case in which α is an epimorphism of R and β is a homomorphism of R. Another case has the similar argument. By Corollary 3.1, γ is a central (β, α) $-$derivation of R. Let $r_1, r_2 \in R$. It is clear that

$$[\gamma(r_1 r_2), \beta(r_2)] = 0.$$

Applying Lie commutator features, we get

$$\begin{aligned} [\gamma(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1), \beta(r_2)] &= [\gamma(r_2)\alpha(r_1), \beta(r_2)] + [\beta(r_2)\gamma(r_1), \beta(r_2)] \\ &= \gamma(r_2)\,[\alpha(r_1), \beta(r_2)] + [\gamma(r_2), \beta(r_2)]\,\alpha(r_1) \\ &\quad + \beta(r_2)\,[\gamma(r_1), \beta(r_2)] + [\beta(r_2), \beta(r_2)]\,\gamma(r_1) \end{aligned}$$

for all $r_1, r_2 \in R$. In the last equation, since $\gamma(r_1), \gamma(r_2) \in Z$, we have

$$\gamma(r_2)\,[\alpha(r_1), \beta(r_2)] = 0$$

for all $r_1, r_2 \in R$. Let $r_3 \in R$. Since α is an epimorphism of R, we get

$$\gamma(r_2) r_3\,[r_1, \beta(r_2)] = 0.$$

Thus, for each $r_2 \in R$, we write

$$\gamma(r_2) R\,[R, \beta(r_2)] = (0).$$

By the primeness of R, for each $r_2 \in R$, we get

$$\gamma(r_2) = 0 \text{ or } \beta(r_2) \in Z.$$

Let $M = \{r_2 \in R : \gamma(r_2) = 0\}$ and $N = \{r_2 \in R : \beta(r_2) \in Z\}$. Clearly, each group M and N is additive subgroup of R such that $R = M \cup N$. But a subgroup cannot be a set union of two proper subgroups. Hence, $M = R$ or $N = R$. Since γ is a non-zero $(\alpha, \beta)-$reverse derivation of R, it happens $\beta(R) \subset Z$. Since $\gamma(r_1 r_2) \in Z$ and Z is a subring of R, we have

$$\gamma(r_2)\alpha(r_1) \in Z, \text{ for all } r_1, r_2 \in R.$$

In view of Lemma 2.2, for each $r_1 \in R$, we have $\alpha(r_1) \in Z$. In addition, since α is an epimorphism of R, we have R is commutative. Therefore, we conclude that

$$\gamma(r_1 r_2) = \gamma(r_2 r_1) = \gamma(r_1)\alpha(r_2) + \beta(r_1)\gamma(r_2)$$

for all $r_1, r_2 \in R$. This implies γ is an (α, β)-derivation of R. $\square$

4. One-Sided Generalized (α,β) −Reverse Derivation

Theorem 4.1. *Let R be a semiprime ring, ϱ is a non-zero two-sided ideal of R, α be an epimorphism of ϱ, β be homomorphism of ϱ and $\gamma : \varrho \to R$ be a non-zero (α,β)-reverse derivation. There exists $F : \varrho \to R$, a l-generalized (α,β)-reverse derivation associated with γ iff $F(\varrho), \gamma(\varrho) \subset C_R(\varrho)$ and F is $r-$generalized (β,α)-derivation associated with (β,α)-derivation γ on ϱ.*

Proof. Let $x_1, x_2, x_3 \in \varrho$. Using the definition of l-generalized (α,β)-reverse derivation one can easily see that

$$F(x_1(x_2x_3)) = F(x_3)\alpha(x_2)\alpha(x_1) + \beta(x_3)\gamma(x_2)\alpha(x_1) + \beta(x_2)\beta(x_3)\gamma(x_1) \tag{4.1}$$

and

$$F((x_1x_2)x_3) = F(x_3)\alpha(x_1)\alpha(x_2) + \beta(x_3)\gamma(x_2)\alpha(x_1) + \beta(x_3)\beta(x_2)\gamma(x_1) \tag{4.2}$$

Combining (4.1) and (4.2),

$$F(x_3)\left[\alpha(x_2), \alpha(x_1)\right] = \left[\beta(x_3), \beta(x_2)\right]\gamma(x_1). \tag{4.3}$$

Substituting x_3 by x_2 in (4.3),

$$F(x_2)\left[\alpha(x_2), \alpha(x_1)\right] = 0$$

for all $x_1, x_2 \in \varrho$. Since α is an epimorphism of ϱ, for each $x_1, x_2 \in \varrho$, we have

$$F(x_2)\left[\alpha(x_2), x_1\right] = 0. \tag{4.4}$$

Taking $x_3 \in \varrho, r \in R$. Replacing x_1 by x_1x_3r in (4.4), $F(x_2)x_1x_3\left[\alpha(x_2), r\right] = 0$. For each $x_2 \in \varrho$, we have $F(x_2)\varrho\varrho\left[\alpha(x_2), R\right] = (0)$. Now, when similar steps are applied to the steps from equality (3.5) to equality (3.9), for each $x_1, x_2, x_3 \in \varrho$, we have $\left[F(x_1), x_2\right] x_3 \left[F(x_1), x_2\right] = 0$. Since ϱ is a semiprime ring,

$$\left[F(x_1), x_2\right] = 0$$

for all $x_1, x_2 \in \varrho$. That is $F(\varrho) \subset C_R(\varrho)$. Moreover, if γ is (α,β) −reverse derivation of R, then by Theorem 3.2, $\gamma(\varrho) \subset C_R(\varrho)$ and γ is an (β,α) −derivation on ϱ. Hence,

$$F(x_1x_2) = F(x_2)\alpha(x_1) + \beta(x_2)\gamma(x_1)$$

$$= \gamma(x_1)\beta(x_2) + \alpha(x_1)F(x_2)$$

for all $x_1, x_2 \in \varrho$ and F is a $r-$generalized (β,α) −derivation associated with $(\beta,\alpha)-$ derivation γ on ϱ. The converse is a trivial. □

Corollary 4.1. *Let R be a semiprime ring, α be an epimorphism of R, β be homomorphism of R and $\gamma : R \to R$ be a non-zero (α,β)-reverse derivation. There exists $F : R \to R$, a l-generalized (α,β)-reverse derivation associated with γ iff $F(I), \gamma(I) \subset Z$ and F is r-generalized (β,α)-derivation associated with (β,α)-derivation γ of R.*

Theorem 4.2. *Let R be a semiprime ring, ϱ is a non-zero two-sided ideal of R, α be a homomorphism of ϱ, β be an epimorphism of ϱ and $\gamma : \varrho \to R$ be a non-zero (α, β)-reverse derivation. There exists $F : \varrho \to R$, a r-generalized (α, β)-reverse derivation associated with γ iff $F(\varrho), \gamma(\varrho) \subset C_R(\varrho)$ and F is $l-$generalized (β, α)-derivation associated with (β, α)-derivation γ on ϱ.*

Proof. By a similar proof in Theorem 4.1, desired is achieved. □

Corollary 4.2. *Let R be a semiprime ring, α be an homomorphism of R, β be an epimorphism of R and $\gamma : R \to R$ be a non-zero (α, β)-reverse derivation. There exists $F : R \to R$, a r-generalized (α, β)-reverse derivation associated with γ iff $F(R), \gamma(R) \subset Z$ and F is $l-$generalized (β, α)-derivation associated with (β, α)-derivation γ of R.*

Theorem 4.3. *Let R be a semiprime ring, α and β be an epimorphisms of R and $\gamma : R \to R$ be a non-zero $(\alpha, \beta)-$reverse derivation. If there exists $F : R \to R$, a non-zero l-generalized (α, β)-reverse derivation (r-generalized (α, β)-reverse derivation) associated with γ then R contains a non-zero central ideal.*

Proof. Assume that $F : R \to R$ is a l-generalized (α, β)-reverse derivation associated with non-zero (α, β)-reverse derivation γ of R. From Corollary 4.1, it holds $\gamma(R), F(R) \subset Z$. For all $r_1, r_2 \in R$,

$$[F(r_1r_2), \beta(r_2)] = 0$$

is obtained. This means

$$F(r_2)\,[\alpha(r_1), \beta(r_2)] = 0$$

for all $r_1, r_2 \in R$. Because α is an epimorphism of R, for each $r_1, r_2 \in R$, we get $F(r_2)\,[r_1, \beta(r_2)] = 0$. Let r_3. Replacing r_1 by r_1r_3 in $F(r_2)\,[r_1, \beta(r_2)] = 0$, we get

$$F(r_2)r_1\,[r_3, \beta(r_2)] = 0.$$

Now, when similar steps are applied to the steps from equality (3.5) to equality (3.9), for each $r_1, r_2, r_3 \in R$, we have $F(r_1)\,[r_2, \beta(r_3)] = 0$. Because β is an epimorphism of R, we get $F(r_1)\,[r_2, r_3] = 0$. That is

$$[F(r_1)r_2, r_3] = 0$$

for all $r_1, r_2, r_3 \in R$. This means $F(R)R \subset Z$. Since F is non-zero $l-$generalized $(\alpha, \beta)-$reverse derivation and R is semiprime, $F(R)R \neq (0)$. $F(R)R$ is obviously central ideal of R. The proof has a similar argument if F is $r-$generalized $(\alpha, \beta)-$reverse derivation of R. □

Corollary 4.3. *Let R be a semiprime ring, α and β be an epimorphisms of R and $\gamma : R \to R$ be a non-zero (α, β)-reverse derivation. If there exists $F : R \to R$, a non-zero generalized (α, β)-reverse derivation associated with γ then R contains a non-zero central ideal.*

Corollary 4.4. *Let R be a prime ring, α and β be an epimorphisms of R and $\gamma : R \to R$ be a non-zero (α,β)-reverse derivation. If there exists $F : R \to R$, a non-zero generalized (α,β)-reverse derivation associated with d then R is commutative ring and F is a generalized (α,β)-derivation associated with an (α,β)-derivation γ of R.*

Theorem 4.4. *Let R be a noncommutative prime ring, α be a homomorphism of R and β be an epimorphism of R. If $F : R \to R$ is a generalized (α,β)-reverse derivation associated with non-zero (α,β)-reverse derivation γ of R then $F = \gamma$.*

Proof. Assume that $F : R \to R$ is a generalized (α,β)-reverse derivation associated with non-zero (α,β)-reverse derivation γ of R. Let $r_1, r_2 \in R$. Then,

$$F(r_1r_2) = F(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1) = \gamma(r_2)\alpha(r_1) + \beta(r_2)F(r_1).$$

That is,

$$(F-\gamma)(r_2)\alpha(r_1) - \beta(r_2)(F-\gamma)(r_1) = 0.$$

Let us introduce mapping $\varphi : R \to R$, $\varphi(r_1) = (F-\gamma)(r_1)$. Moreover, the last equation implies that

$$\varphi(r_2)\alpha(r_1) = \beta(r_2)\varphi(r_1). \tag{4.5}$$

Let $r_1, r_2 \in R$. Since F is an r-generalized (α,β)-reverse derivation (l- of generalized (α,β)-reverse derivation) R and γ is an (α,β)-reverse derivation of R, the mapping φ respectively ensures:

$$\begin{aligned}\varphi(r_1r_2) &= (F-\gamma)(r_1r_2) = \gamma(r_2)\alpha(r_1) + \beta(r_2)F(r_1) - \gamma(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1)\\ &= \beta(r_2)\varphi(r_1)\end{aligned}$$

and

$$\begin{aligned}\varphi(r_1r_2) &= (F-\gamma)(r_1r_2) = F(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1) - \gamma(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1)\\ &= \varphi(r_2)\alpha(r_1).\end{aligned}$$

That is,

$$\varphi(r_1r_2) = \beta(r_2)\varphi(r_1). \tag{4.6}$$

$$\varphi(r_1r_2) = \varphi(r_2)\alpha(r_1). \tag{4.7}$$

Let $r_3 \in R$. Writing r_2r_3 by r_2 in (4.5), we get

$$\varphi(r_2r_3)\alpha(r_1) - \beta(r_2r_3)\varphi(r_1) = 0.$$

In the last equality, using (4.6) and (4.7), we get

$$\beta([r_3, r_2])\varphi(r_1) = 0$$

for all $r_1, r_2, r_3 \in R$. Because β is an epimorphism, for each $r_1, r_2, r_3 \in R$, we have $[r_3, r_2]\,\varphi(r_1) = 0$. Given that R is a noncommutative prime ring, we get $\varphi = 0$. That is, $F = \gamma$. $\square$

Acknowledgment

This paper is derived from the first author's doctoral dissertation supervised by the second author.

REFERENCES

1. A. Aboubakr and S. Gonzalez: *$(\alpha, \beta)-$ reverse derivation on semiprime rings.* Siberian Mathematical Journal **56(2)** (2015), 199–205.
2. A. Ali and A. Bano: *Multiplicative (generalized) reverse derivations on semiprime ring.* European Journal of Pure and Applied Mathematics **11(3)** (2018), 717–729.
3. C. J. S. Reddy, N. Subbarayudu and C. V. S. R. Reddy: *A Note on multiplicative (generalized)-$(\alpha, \beta)-$reverse derivations on left Ideals in prime rings.* Global Journal of Pure and Applied Mathematics **18(2)** (2022), 425–431.
4. G. S. Sandhu and D. Kumar: *Annihilator conditions of multiplicative reverse derivations on prime rings.* International electronic journal of algebra **25(25)** (2019), 87–103.
5. I. N. Herstein: *Jordan derivations of prime rings.* Proc. Amer. Math. Soc. **8(6)** (1957), 1104–1110.
6. I. N. Herstein: *Rings with involution.* The University of Chicago Press, 1976.
7. J. H. Mayne: *Centralizing mappings of prime rings.* Canad. Math. Bull. **27(1)** (1984), 122–126.
8. M. A. Chaudhry and A. B. Thaheem: *On $(\alpha, \beta)-$derivations of semiprime rings II.* Demonstratıo Mathematica **37(4)** (2004), 793–802.
9. M. S. Samman and A. B. Thaheem: *Derivations on semiprime rings.* Int. Journal of Pure and Applied Math. **2(4)** (2003), 465–472.
10. M. S. Samman and N. Alyamani: *Derivations and reverse derivations in semiprime rings.* Int. Math. Forum **39(2)** (2007), 1895–1902.
11. N. U. Rehman and E. K. Söğütçü: *Lie ideals and jordan Triple $(\alpha, \beta)-$derivations in rings.* Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69(1)** (2020), 528–539.
12. S. K. Tiwari, R. K. Sharma and B. Dhara: *Some theorems of commutativity on semiprime rings with mappings.* Southeast Asian Bull. Math **42(2)** (2018), 279–292.
13. S. Huang: *Generalized reverse derivations and commutativity of prime rings.* Communications in Mathematics **27** (2019), 43–50.
14. Y. Ahmed and M. Aslam: *On diversity of generalized reverse derivations in rings.* Journal of Mechanics of Continua and Mathematical Sciences **15(6)** (2020), 118–126.
15. Z. Z. M. Alhaidary and A. H. Majeed: *Square closed Lie ideals and multiplicative (generalized) (α, β) reverse derivation of prime rings.* Journal of Discrete Mathematical Sciences & Cryptography **24(7)** (2021), 2037–2046.
16. Z. Z. M. Alhaidary and A. H. Majeed: *Commutativity results for multiplicative (generalized) (α, β) reverse derivations on prime rings.* Iraqi Journal of Science **62(9)** (2021), 3102–3113.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 33–49
https://doi.org/10.22190/FUMI220922003G
Original Scientific Paper

RANKS OF SUBMATRICES IN THE REFLEXIVE SOLUTIONS OF SOME MATRIX EQUATIONS

Sihem Guerarra and Radja Belkhiri

Faculty of Exact Sciences and Sciences of Nature and Life
Department of Mathematics and Informatics
University of Oum El Bouaghi, 04000, Algeria

ORCID IDs: Sihem Guerarra https://orcid.org/0000-0001-6888-5974
Radja Belkhiri N/A

Abstract. Maximal and minimal ranks of the two submatrices X_1 and X_2 in the (skew-) Hermitian reflexive solution $X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ of the matrix equation $AXA^* = C$, in the reflexive solution of the matrix equation $AXB = C$ are derived. Therefore, necessary and sufficient conditions for these reflexive solutions to have special forms, and the general expressions of these reflexive solutions are achieved.
Keywords: matrix equation, rank, reflexive solution.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices over $\mathbb{C}$ by $\mathbb{C}^{m\times n}$, the set of all $n \times n$ Hermitian matrices by $\mathbb{C}_H^{n\times n}$, the symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of a given matrix $A \in \mathbb{C}^{n\times m}$ respectively, I_n denotes the identity matrix of order n. The Moore-Penrose inverse of a matrix A, is defined to be the unique matrix A^+ satisfying:

$$AA^+A = A, A^+AA^+ = A^+, \left(AA^+\right)^* = AA^+, \left(A^+A\right)^* = A^+A.$$

Further, the symbols R_A and L_A stand for the two orthogonal projectors $L_A = I_n - A^+A$ and $R_A = I_m - AA^+$ induced by $A \in \mathbb{C}^{m\times n}$. For more informations and basic concepts about the Moore-Penrose generalized inverse see [1], [15].

Received September 22, 2022, revised: November 04, 2023, accepted: November 19, 2023
Communicated by Predrag Stanimirović
Corresponding Author: Sihem Guerarra. E-mail addresses: guerarra.siham@univ-oeb.dz (S. Guerarra), radja.belkhiri@univ-oeb.dz (R. Belkhiri)
2010 *Mathematics Subject Classification.* Primary 15A24; Secondary 15A03, 15A09, 15B57.

A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ and $P^2 = I$. Chen in [2] defined the following subspace of matrices:

$$\mathbb{C}_r^{n\times n}(P) = \left\{A \in \mathbb{C}^{n\times n}, A = PAP\right\}$$

where P is a generalized reflection matrix.

The matrix $A \in \mathbb{C}_r^{n\times n}(P)$ is said to be a generalized reflexive with respect to the generalized reflection matrix P. The generalized reflexive matrices have applications in system and control theory, in engineering, in scientific computations and various other fields (see [2], [3], [7]). In particular the reflexive solutions of the linear matrix equations

$$\begin{aligned} AXA^* &= C \\ AXB &= C \end{aligned}$$

where A, B, C are given matrices, and X is a variable matrix was widely studied by many authors (see [12], [13], [14]), also in [5] Deghan and Hajarian established new necessary and sufficient conditions for the existence of the reflexive (anti-reflexive) solutions to the linear matrix equation $AXB+CYD = E$ and derived representation of the general reflexive (anti-reflexive) solutions to this matrix equation, then in [6] they investigated the solvability of these matrix equations

$$\begin{aligned} A_1XB_1 &= D_1, \\ A_1X &= C_1, XB_2 = C_2, and \\ A_1X &= C_1, XB_2 = C_2, A_3X = C_3, XB_4 = C_4. \end{aligned}$$

over reflexive and anti reflexive matrices, in [4] Cvetković-Ilić studied the existence of a reflexive solution of the matrix equation $AXB = C$, with respect to the generalized reflection matrix P, Liu and Yuan [9] gave some conditions for the existence and the representations for the generalized reflexive and anti-reflexive solutions to matrix equation $AX = B$, In [10], Liu established some conditions for the existence and representations for the common generalized reflexive and anti-reflexive solutions of matrix equations $AX = B$ and $XC = D$, also Liu in [11] discussed the extremal ranks of the matrix expression $A - BXC$ where X is (anti-) reflexive matrix, and in [8] he established some conditions for the existence and the representations for the Hermitian reflexive and Hermitian anti-reflexive, and nonnegative definite reflexive solutions to the matrix equation $AX = B$ with respect to a generalized reflection matrix P by using the Moore-Penrose inverse.

This paper is organized as follows: In Section 2 we derive the extremal ranks of the (skew-) Hermitian reflexive solution of the matrix equation $AXA^* = C$, from these rank formulas we show some forms of the reflexive solution of $AXA^* = C$, also the general expressions of the solution is given. In Section 3, we consider the matrix equation $AXB = C$ over the general reflexive solution and give some forms for this solution.

First we begin by these lemmas to review some representations of the generalized reflection matrix P and the subspace $\mathbb{C}_r^{n\times n}(P)$ matrices.

Lemma 1.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix, so P can be expressed as*

$$P = U \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} U^*$$

where U is an unitary matrix.

Lemma 1.2. *The matrix $A \in \mathbb{C}_r^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} U^*$$

where $A_1 \in \mathbb{C}^{k\times k}$, $A_2 \in \mathbb{C}^{(n-k)\times(n-k)}$, U is an unitary matrix.

Definition 1.1. Given a generalized reflection matrix $P \in \mathbb{C}^{n\times n}$.
1. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a Hermitian reflexive matrix if $A = A^*$ and $A \in \mathbb{C}_r^{n\times n}(P)$.
2. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a skew-Hermitian reflexive matrix if $A = -A^*$ and $A \in \mathbb{C}_r^{n\times n}(P)$.

The Lemmas 1.3, 1.4 and 1.5 are found in [19] as [Theorem 2.5, Theorem 2.6 and Lemma 2.2] respectively.

Lemma 1.3. *[19] Let $H^{m\times n}$ be the set of all $m \times n$ matrices over the quaternion algebra. Suppose that the matrix equation*

$$AXA^* + BYB^* = C \tag{1.1}$$

where $A \in H^{m\times n}$, $B \in H^{m\times p}$, $C \in H^{m\times m}$, $C = C^*$, $X \in H^{n\times n}$, and $Y \in H^{p\times p}$, $G = \begin{bmatrix} A & B \end{bmatrix}$ has a Hermitian solution. Then,
The maximal and minimal ranks of the general Hermitian solution to (1.1) are given by

$$\max_{\substack{AXA^* + BYB^* = C \\ X = X^*}} r(X) = \min\left\{n, r\begin{bmatrix} B & C \end{bmatrix} + 2n - r(A) - r(G)\right\}$$

$$\min_{\substack{AXA^* + BYB^* = C \\ X = X^*}} r(X) = 2r[B, C] - r\begin{bmatrix} C & B \\ B^* & 0 \end{bmatrix}$$

.

$$\max_{\substack{AXA^* + BYB^* = C \\ Y = Y^*}} r(Y) = \min\left\{p, r\begin{bmatrix} A & C \end{bmatrix} + 2p - r(B) - r(G)\right\}$$

$$\min_{\substack{AXA^* + BYB^* = C \\ Y = Y^*}} r(Y) = 2r\begin{bmatrix} A & C \end{bmatrix} - r\begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}$$

.

Lemma 1.4. *[19] Let $H^{m\times n}$ be the set of all $m\times n$ matrices over the quaternion algebra. Suppose that the matrix equation (1.1), where $A\in H^{m\times n}$, $B\in H^{m\times p}$, $C\in H^{m\times m}$, $C=-C^*$, $X\in H^{n\times n}$, and $Y\in H^{p\times p}$, $G=\begin{bmatrix} A & B\end{bmatrix}$ has a skew Hermitian solution. Then,*
The maximal and minimal ranks of the general skew-Hermitian solution to (1.1) are given by

$$\max_{\substack{AXA^*+BYB^*=C\\ X=-X^*}} r(X)=\min\left\{n, r\begin{bmatrix} B & C\end{bmatrix}+2n-r(A)-r(G)\right\}$$

$$\min_{\substack{AXA^*+BYB^*=C\\ X=-X^*}} r(X)=2r[B,C]-r\begin{bmatrix} C & B\\ -B^* & 0\end{bmatrix}.$$

$$\max_{\substack{AXA^*+BYB^*=C\\ Y=-Y^*}} r(Y)=\min\left\{p, r\begin{bmatrix} A & C\end{bmatrix}+2p-r(B)-r(G)\right\}$$

$$\min_{\substack{AXA^*+BYB^*=C\\ Y=-Y^*}} r(Y)=2r\begin{bmatrix} A & C\end{bmatrix}-r\begin{bmatrix} C & A\\ -A^* & 0\end{bmatrix}.$$

Lemma 1.5. *[19] Consider the linear matrix equation (1.1), where $A\in H^{m\times n}$, $B\in H^{m\times p}$, $C\in H^{m\times m}$ are given, and $X\in H^{n\times n}$, $Y\in H^{p\times p}$ unknown.*
1) If $C=C^*$, and (1.1) has a Hermitian solution, then the general Hermitian solution to (1.1) can be expressed as

$$X=X_0+S_1L_GZL_GS_1^*+L_AV+V^*L_A \tag{1.2}$$

$$Y=Y_0-S_2L_GZL_GS_2^*+L_BW+W^*L_B \tag{1.3}$$

where X_0 and Y_0 are a special pair Hermitian solution of (1.1),

$$S_1=(I_n,0), S_2=(0,I_p), G=\begin{bmatrix} A & B\end{bmatrix} \tag{1.4}$$

Z is an arbitrary Hermitian quaternion matrix with consistent size, and V and W are arbitrary quaternion matrices with suitable sizes.
2) If $C=-C^*$, and (1.1) has a skew-Hermitian solution, then the general skew-Hermitian solution can be expressed as

$$X=X_0+S_1L_GZL_GS_1^*+L_AV-V^*L_A. \tag{1.5}$$

$$Y=Y_0-S_2L_GZL_GS_2^*+L_BW-W^*L_B. \tag{1.6}$$

where X_0 and Y_0 are a special pair skew-Hermitian solution of (1.1), and S_1, S_2, and G are the same as (1.4); Z is an arbitrary skew-Hermitian quaternion matrix with consistent size, and V and W are arbitrary quaternion matrices with suitable sizes.

2. Extremal ranks of submatrices in (skew-)Hermitian reflexive solution of $AXA^* = C$

In this section we will derive the extremal ranks of the (skew-) Hermitian reflexive solution of the matrix equation $AXA^* = C$, as consequences we will show some forms of the reflexive solution of $AXA^* = C$, and some applications on generelized inverses.

Consider the linear matrix equation

$$AXA^* = C \tag{2.1}$$

where A, C are given and X is unknown.

Theorem 2.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^{m\times m}$ be given, suppose that the matrix equation (2.1) has a Hermitian reflexive solution.$X = X^* \in \mathbb{C}_r^{n\times n}(P)$ Then,*
a) The maximal and minimal ranks of the two submatrices X_1 and X_2 in Hermitian reflexive solution $X = U\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ to the matrix equation (2.1) are given by

$$\max_{X_1=X_1^*} r(X_1) = \min\left\{k, r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} + 2k - r(A(I_n+P)) - r(A)\right\}. \tag{2.2}$$

$$\min_{X_1=X_1^*} r(X_1) = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix}. \tag{2.3}$$

(2.4)
$$\max_{X_2=X_2^*} r(X_2) = \min\left\{n-k, r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} + 2(n-k) - r(A(I_n-P)) - r(A)\right\}.$$

$$\min_{X_2=X_2^*} r(X_2) = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix}. \tag{2.5}$$

b) The general Hermitian reflexive solution to (2.1) can be expressed as

$$X = U\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^*$$

where

$$X_1 = X_{01} + S_1L_{AU}ZL_{AU}S_1^* + L_{\left(\frac{1}{2}A(I_n+P)U\right)}V + V^*L_{\left(\frac{1}{2}A(I_n+P)U\right)} \tag{2.6}$$
$$X_2 = X_{02} - S_2L_{AU}ZL_{AU}S_2^* + L_{\left(\frac{1}{2}A(I_n-P)U\right)}W + W^*L_{\left(\frac{1}{2}A(I_n-P)U\right)} \tag{2.7}$$

where $\begin{bmatrix} X_{01} & 0 \\ 0 & X_{02} \end{bmatrix}$ is a special Hermitian reflexive solution of (2.1), and

$$S_1 = (I_k, 0), S_2 = (0, I_{n-k})$$

V, W and Z are arbitrary matrices with suitable sizes.

Proof. a) From lemma 1.2 the Hermitian reflexive solution to $AXA^* = C$ can be written as

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$$

where $X_1 = X_1^* \in \mathbb{C}^{k\times k}$, $X_2 = X_2^* \in \mathbb{C}^{(n-k)\times(n-k)}$, and arbitrary unitary matrix $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, with $U_1 \in \mathbb{C}^{n\times k}$, $U_2 \in \mathbb{C}^{n\times(n-k)}$.
We denote $AU = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, where $A_1 \in \mathbb{C}^{m\times k}$, $A_2 \in \mathbb{C}^{m\times(n-k)}$, we have

$$\begin{aligned} AXA^* = C &\iff AU \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^* A^* = C \\ &\iff [A_1 \;\; A_2] \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \end{bmatrix} = C \\ &\iff A_1X_1A_1^* + A_2X_2A_2^* = C. \end{aligned} \tag{2.8}$$

Then, the two equations (2.1) and (2.8) are equivalent, so from Lemma 1.3 we have

$$\max_{\substack{A_1X_1A_1^*+A_2X_2A_2^*=C \\ X_1=X_1^*}} r(X_1) = \min\left\{k,\; r[A_2 \;\; C] + 2k - r(A_1) - r\begin{bmatrix} A_1 & A_2 \end{bmatrix}\right\} \tag{2.9}$$

$$\min_{\substack{A_1X_1A_1^*+A_2X_2A_2^*=C \\ X_1=X_1^*}} r(X_1) = 2r\begin{bmatrix} A_2 & C \end{bmatrix} - r\begin{bmatrix} C & A_2 \\ A_2^* & 0 \end{bmatrix}. \tag{2.10}$$

$$\max_{\substack{A_1X_1A_1^*+A_2X_2A_2^*=C \\ X_2=X_2^*}} r(X_2) = \min\left\{n-k,\; r\begin{bmatrix} A_1 & C \end{bmatrix} + 2(n-k) - r(A_2) - r\begin{bmatrix} A_1 & A_2 \end{bmatrix}\right\} \tag{2.11}$$

$$\min_{\substack{A_1X_1A_1^*+A_2X_2A_2^*=C \\ X_2=X_2^*}} r(X_2) = 2r\begin{bmatrix} A_1 & C \end{bmatrix} - r\begin{bmatrix} C & A_1 \\ A_1^* & 0 \end{bmatrix}. \tag{2.12}$$

From Lemmas 1.1 and 1.2 we can simplify:

$$\begin{aligned} r\begin{bmatrix} A_1 & C \end{bmatrix} &= r\begin{bmatrix} A_1 & 0 & C \end{bmatrix} \\ &= r\begin{bmatrix} \frac{1}{2}A(I_n+P)U & C \end{bmatrix} \\ &= r\begin{bmatrix} A(I_n+P) & C \end{bmatrix}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} r\begin{bmatrix} A_2 & C \end{bmatrix} &= r\begin{bmatrix} 0 & A_2 & C \end{bmatrix} \\ &= r\begin{bmatrix} \frac{1}{2}A(I_n-P)U & C \end{bmatrix} \\ &= r\begin{bmatrix} A(I_n-P) & C \end{bmatrix}, \end{aligned} \tag{2.14}$$

$$r\begin{bmatrix} C & A_1 \\ A_1^* & 0 \end{bmatrix} = r\begin{bmatrix} C & A_1 & 0 \\ A_1^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n+P)U \\ \frac{1}{2}U(I_n+P)A^* & 0 \end{bmatrix}$$

(2.15) $$= r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix},$$

$$r\begin{bmatrix} C & A_2 \\ A_2^* & 0 \end{bmatrix} = r\begin{bmatrix} C & 0 & A_2 \\ 0 & 0 & 0 \\ A_2^* & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n-P)U \\ \frac{1}{2}U(I_n-P)A^* & 0 \end{bmatrix}$$

(2.16) $$= r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix}.$$

Substituting (2.13)-(2.16) into (2.9)-(2.12) yields (2.2)-(2.5).
b) Necessary substitutions from (2.13)-(2.16) into (1.2)-(1.3) yields (2.6) and (2.7). □

Corollary 2.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^{m\times m}$ be given, suppose that the matrix equation (2.1) has a Hermitian reflexive solution.$X = X^* \in \mathbb{C}_r^{n\times n}(P)$ Then.*

a) Equation (2.1) has a Hermitian reflexive solution of the form $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ *if and only if*

$$r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix}.$$

b) All Hermitian reflexive solutions of equation (2.1) have the form $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ *if and only if*

$$r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} = r(A(I_n-P)) + r(A) - 2(n-k).$$

c) Equation (2.1) has a Hermitian reflexive solution of the form $X = U\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ *if and only if*

$$r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix}.$$

d) All Hermitian reflexive solutions of equation (2.1) have the form $X = U\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ *if and only if*

$$r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} = r(A(I_n+P)) + r(A) - 2k.$$

e) Equation (2.1) has a null solution if and only if

$$r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix},$$
$$r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix}.$$

f) All Hermitian reflexive solutions of equation (2.1) are nulls if and only if

$$r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} = r(A(I_n-P)) + r(A) - 2(n-k),$$
$$r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} = r(A(I_n+P)) + r(A) - 2k.$$

It is well known that, the generalized inverse A^- for a given matrix A is a solution of the matrix equation $AXA = A$, so we apply Corollary 2.1 to the equation $AXA = A$ we obtain this result.

Corollary 2.2. *Let $A \in \mathbb{C}^{n\times n}$, for some unitary matrix U. Then,*
a) A has a generalized inverse A^- of the form $A^- = U\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} A & A(I_n+P) \\ (I_n+P)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & A \end{bmatrix}.$$

b) A has a generalized inverse A^- of the form $A^- = U\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} A & A(I_n-P) \\ (I_n-P)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & A \end{bmatrix}.$$

A square complex matrix A is defined as EP (Equal-Range Projection) or (range-Hermitian) when both the matrix A and its conjugate transpose A^* have identical ranges. Tian in [18] compiled established characterizations for EP matrices and provided additional new characterizations for this class of matrices, hence if the two matrices N_1 and N_2 in Corollary (2.2) satisfy some conditions we have the result

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$, If N_1 and N_2 in Corollary (2.2) are nonsingular, for some unitary matrix U, we have:*
A is an EP matrix if and only if

$$r\begin{bmatrix} A & A(I_n+P) \\ (I_n+P)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & A \end{bmatrix}.$$

$$\begin{bmatrix} A & A(I_n-P) \\ (I_n-P)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & A \end{bmatrix}.$$

Proof. From ([18] Theorem 2.1) for a given matrix $A \in \mathbb{C}^{n\times n}$, the following statements are equivalent
i) A is EP
ii) A^{-} is EP
iii) There exists an unitary matrix U such that $UAU^{*} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where A_1 is nonsingular.
By applying i), ii) and iii) to a) and b) of Corollary (2.2) leads to result in Corollary (2.3). □

Theorem 2.2. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times m}$, $C = -C^{*}$, and assume that the matrix equation (2.1) has a skew-Hermitian reflexive solution $X = -X^{*} \in \mathbb{C}_r^{n\times n}(P)$ Then,*
(a) The maximal and minimal ranks of the two submatrices X_1 and X_2 in skew-Hermitian reflexive solution $X = U\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^{*}$ to the matrix equation (2.1) are given by

$$\max_{X_1=-X_1^{*}} r(X_1) = \min\left\{k, r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} + 2k - r(A(I_n+P)) - r(A)\right\}.$$

$$\min_{X_1=-X_1^{*}} r(X_1) = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n-P) \\ -(I_n-P)A^{*} & 0 \end{bmatrix}.$$

$$\max_{X_2=-X_2^{*}} r(X_2) = \min\left\{n-k, r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} + 2(n-k) - r(A(I_n-P)) - r(A)\right\}.$$

$$\min_{X_2=-X_2^{*}} r(X_2) = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n+P) \\ -(I_n+P)A^{*} & 0 \end{bmatrix}.$$

b) The general skew- Hermitian reflexive solution of (2.1) can be expressed as

$$X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^{*}$$

where

$$\begin{aligned} X_1 &= X_{01} + S_1 L_{AU} Z L_{AU} S_1^{*} + L_{\left(\frac{1}{2}A(I_n+P)U\right)} V - V^{*} L_{\left(\frac{1}{2}A(I_n+P)U\right)}, \\ X_2 &= X_{02} - S_2 L_{AU} Z L_{AU} S_2^{*} + L_{\left(\frac{1}{2}A(I_n-P)U\right)} W - W^{*} L_{\left(\frac{1}{2}A(I_n-P)U\right)} \end{aligned}$$

where $\begin{bmatrix} X_{01} & 0 \\ 0 & X_{02} \end{bmatrix}$ is a special skew-Hermitian reflexive solution of (2.1), and

$$S_1 = (I_k, 0), S_2 = (0, I_{n-k})$$

Z, V and W are arbitrary matrices with suitable sizes.

Proof. The poof is similar to that of Theorem 2.1. □

Corollary 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^{m\times m}$ be given and assume that the matrix equation (2.1) has a skew-Hermitian reflexive solution $X = -X^* \in \mathbb{C}_r^{n\times n}(P)$ Then.*
a) Equation (2.1) has a skew-Hermitian reflexive solution of the form
$X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} C & A(I_n + P) \\ -(I_n + P) A^* & 0 \end{bmatrix} = 2r \begin{bmatrix} A(I_n + P) & C \end{bmatrix}.$$

b) All skew-Hermitian reflexive solutions of equation (2.1) have the form $X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} A(I_n + P) & C \end{bmatrix} = r(A(I_n - P)) + r(A) - 2(n - k).$$

c) Equation (2.1) has a skew-Hermitian reflexive solution of the form
$X = U \begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} C & A(I_n - P) \\ -(I_n - P) A^* & 0 \end{bmatrix} = 2r \begin{bmatrix} A(I_n - P) & C \end{bmatrix}.$$

d) All skew-Hermitian reflexive solutions of equation (2.1) have the form $X = U \begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} A(I_n - P) & C \end{bmatrix} = r(A(I_n + P)) + r(A) - 2k.$$

e) Equation (2.1) has a null solution if and only if

$$r \begin{bmatrix} C & A(I_n + P) \\ -(I_n + P) A^* & 0 \end{bmatrix} = 2r \begin{bmatrix} A(I_n + P) & C \end{bmatrix},$$
$$r \begin{bmatrix} C & A(I_n - P) \\ -(I_n - P) A^* & 0 \end{bmatrix} = 2r \begin{bmatrix} A(I_n - P) & C \end{bmatrix}.$$

f) All skew-Hermitian reflexive solutions of equation (2.1) are null solutions if and only if

$$r \begin{bmatrix} A(I_n + P) & C \end{bmatrix} = r(A(I_n - P)) + r(A) - 2(n - k),$$
$$r \begin{bmatrix} A(I_n - P) & C \end{bmatrix} = r(A(I_n + P)) + r(A) - 2k.$$

3. Extremal ranks of submatrices in generalized reflexive solution of $AXB = C$

In this section we will review special forms of the reflexive solution of the equation $AXB = C$ with respect to the generalized reflexion matrix P.
Consider the linear matrix equation

$$AXB = C \tag{3.1}$$

where A, B and C are given, and X is unknown.

The following Lemma is the same that corollary 3.5 in [20], (also it is the same that Theorem 2.2 in [17]).

Lemma 3.1. *[20]We adopt the following notations:*

$$\begin{aligned} J_3 &= \{X_1 \in H^{p_1\times q_1} \mid A_3X_1B_1 + A_4X_2B_2 = C_3\} \\ J_4 &= \{X_2 \in H^{p_2\times q_2}|, A_3X_1B_1 + A_4X_2B_2 = C_3\}. \end{aligned}$$

Assume that $A_3 \in H^{s\times p_1}$, $A_4 \in H^{s\times p_2}$, $B_1 \in H^{q_1\times t}$, $B_2 \in H^{q_2\times t}$, $C_3 \in H^{s\times t}$, *and the matrix equation*

$$A_3X_1B_1 + A_4X_2B_2 = C_3. \tag{3.2}$$

is consistent. Then the extremal ranks of the solution to (3.2) are given by

$$\begin{aligned}
\max_{X_1\in J_3} r(X_1) &= \min\left\{ \begin{array}{c} p_1, q_1, p_1+q_1+r\begin{bmatrix} C_3 & A_4 \end{bmatrix} - r\begin{bmatrix} A_3 & A_4 \end{bmatrix} - r(B_1), \\ p_1+q_1+r\begin{bmatrix} B_2 \\ C_3 \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_3). \end{array} \right\}. \\
\min_{X_1\in J_3} r(X_1) &= r\begin{bmatrix} C_3 & A_4 \end{bmatrix} + r\begin{bmatrix} B_2 \\ C_3 \end{bmatrix} - r\begin{bmatrix} C_3 & A_4 \\ B_2 & 0 \end{bmatrix}. \\
\max_{X_2\in J_4} r(X_2) &= \min\left\{ \begin{array}{c} p_2, q_2, p_2+q_2+r\begin{bmatrix} C_3 & A_3 \end{bmatrix} - r\begin{bmatrix} A_3 & A_4 \end{bmatrix} - r(B_2), \\ p_2+q_2+r\begin{bmatrix} B_1 \\ C_3 \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_4). \end{array} \right\}. \\
\min_{X_2\in J_4} r(X_2) &= r\begin{bmatrix} C_3 & A_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ C_3 \end{bmatrix} - r\begin{bmatrix} C_3 & A_3 \\ B_1 & 0 \end{bmatrix}.
\end{aligned}$$

Lemma 3.2. *[16] Let* $A_1 \in \mathcal{F}^{m\times p}$, $B_1 \in \mathcal{F}^{q\times n}$, $A_2 \in \mathcal{F}^{m\times s}$, $B_2 \in \mathcal{F}^{t\times n}$ *and* $C \in \mathcal{F}^{m\times n}$ *be given over an arbitrary field* $\mathcal{F}$, *and suppose that the matrix equation*

$$A_1XB_1 + A_2YB_2 = C \tag{3.3}$$

is solvable. Then its general solutions for X *and* Y *can be expressed as:*

$$X = X_0 + S_1L_GUR_HT_1 + L_{A_1}V_1 + V_2R_{B_1}. \tag{3.4}$$

$$Y = Y_0 + S_2L_GUR_HT_2 + L_{A_2}W_1 + W_2R_{B_2}. \tag{3.5}$$

where $S_1 = [I_p, 0]$, $S_2 = [0, I_s]$, $T_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 \\ I_t \end{bmatrix}$, $G = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, $H = \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}$ *and* X_0, Y_0 *are a pair of particular solutions to Eq (3.3),* U, V_1, V_2, W_1 *and* W_2 *are arbitrary*

Theorem 3.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$ are given, suppose that the matrix equation (3.1) has a reflexive solution.$X \in \mathbb{C}_r^{n\times n}(P)$ Then*
(a) The maximal and minimal ranks of the two submatrices X_1 and X_2 in a reflexive solution $X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ to the matrix equation (3.1) are given by

$$\max_{X_1} r(X_1) = \min \left\{ \begin{array}{c} k, 2k + r\begin{bmatrix} C & A(I_n - P) \end{bmatrix} - r(A) - r((I_n + P)B), \\ 2k + r\begin{bmatrix} (I_n - P)B \\ C \end{bmatrix} - r(B) - r(A(I_n + P)). \end{array} \right\}. \tag{3.6}$$

$$\min_{X_1} r(X_1) = r\begin{bmatrix} C & A(I_n - P) \end{bmatrix} + r\begin{bmatrix} (I_n - P)B \\ C \end{bmatrix} - r\begin{bmatrix} C & A(I_n - P) \\ (I_n - P)B & 0 \end{bmatrix}. \tag{3.7}$$

$$\max_{X_2} r(X_2) = \min \left\{ \begin{array}{c} n - k, 2(n - r) + r\begin{bmatrix} C & A(I_n + P) \end{bmatrix} - r(A) - r((I_n - P)B), \\ 2(n - k) + r\begin{bmatrix} (I_n + P)B \\ C \end{bmatrix} - r(B) - r(A(I_n - P)). \end{array} \right\}. \tag{3.8}$$

$$\min_{X_2} r(X_2) = r\begin{bmatrix} C & A(I_n + P) \end{bmatrix} + r\begin{bmatrix} (I_n + P)B \\ C \end{bmatrix} - r\begin{bmatrix} C & A(I_n + P) \\ (I_n + P)B & 0 \end{bmatrix}. \tag{3.9}$$

b) The general reflexive solution to (3.1) can be expressed as

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$$

where

$$\begin{aligned} X_1 &= X_0 + S_1 L_{AU} Z R_{U^*B} T_1 + L_{\left(\frac{1}{2}A(I_n+P)U\right)} Z_1 + Z_2 R_{\left(\frac{1}{2}U^*(I_n+P)B\right)}, \\ X_2 &= Y_0 + S_2 L_{AU} Z R_{U^*B} T_2 + L_{\left(\frac{1}{2}A(I_n-P)U\right)} Z_3 + Z_4 R_{\left(\frac{1}{2}U^*(I_n-P)B\right)}. \end{aligned}$$

where $S_1 = [I_k,\ 0]$, $S_2 = [0, I_{n-k}]$, $T_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix}$, and $\begin{bmatrix} X_0 & 0 \\ 0 & Y_0 \end{bmatrix}$ is a particular reflexive solution to equation (3.1),
Z, Z_1, Z_2, Z_3 and Z_4 are arbitrary matrices with appropriate sizes.

Proof. a) From lemma 1.2 the reflexive solution to $AXB = C$ can be written as

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$$

for arbitrary unitary matrix $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, with $U_1 \in \mathbb{C}^{n\times k}$, $U_2 \in \mathbb{C}^{n\times(n-k)}$. We denote

$$AU = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, U^* B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $A_1 \in \mathbb{C}^{m\times k}$, $A_2 \in \mathbb{C}^{m\times(n-k)}$, $B_1 \in \mathbb{C}^{k\times l}$, $B_2 \in \mathbb{C}^{(n-k)\times l}$. So,

$$\begin{aligned} AXB = C \quad &\Longleftrightarrow \quad AU \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^* B = C \\ &\Longleftrightarrow \quad \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = C \\ &\Longleftrightarrow \quad A_1X_1B_1 + A_2X_2B_2 = C. \end{aligned} \tag{3.10}$$

Then, the two equations (3.1) and (3.10) are equivalent, now we adopt the following notations:

$$S_1 = \{X_1 \in \mathbb{C}^{k\times k} \mid A_1X_1B_1 + A_2X_2B_2 = C\},$$
$$S_2 = \{X_2 \in \mathbb{C}^{(n-k)\times(n-k)} \mid A_1X_1B_1 + A_2X_2B_2 = C\}.$$

From Lemma 3.1 we have

$$\max_{X_1\in S_1} r(X_1) = \min \left\{ \begin{array}{c} r, 2k + r\begin{bmatrix} C & A_2 \end{bmatrix} - r\begin{bmatrix} A_1 & A_2 \end{bmatrix} - r(B_1), \\ 2k + r\begin{bmatrix} B_2 \\ C \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_1). \end{array} \right\} \tag{3.11}$$

$$\min_{X_1\in S_1} r(X_1) = r\begin{bmatrix} C & A_2 \end{bmatrix} + r\begin{bmatrix} B_2 \\ C \end{bmatrix} - r\begin{bmatrix} C & A_2 \\ B_2 & 0 \end{bmatrix} \tag{3.12}$$

$$\max_{X_2\in S_2} r(X_2) \tag{3.13}$$
$$= \min \left\{ \begin{array}{c} n-k, 2(n-r) + r\begin{bmatrix} C & A_1 \end{bmatrix} - r\begin{bmatrix} A_1 & A_2 \end{bmatrix} - r(B_2), \\ 2(n-k) + r\begin{bmatrix} B_1 \\ C \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_2). \end{array} \right\}$$

$$\min_{X_2\in S_2} r(X_2) = r\begin{bmatrix} C & A_1 \end{bmatrix} + r\begin{bmatrix} B_1 \\ C \end{bmatrix} - r\begin{bmatrix} C & A_1 \\ B_1 & 0 \end{bmatrix}. \tag{3.14}$$

From Lemmas 1.1 and 1.2 we can simplify:

$$\begin{aligned} r\begin{bmatrix} C & A_2 \end{bmatrix} &= r\begin{bmatrix} C & 0 & A_2 \end{bmatrix} \\ &= r\begin{bmatrix} C & \tfrac{1}{2}A(I_n - P)U \end{bmatrix} \\ &= r\begin{bmatrix} C & A(I_n - P) \end{bmatrix}, \\ r\begin{bmatrix} C & A_1 \end{bmatrix} &= r\begin{bmatrix} C & A_1 & 0 \end{bmatrix} \end{aligned} \tag{3.15}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n+P)U \end{bmatrix}$$

$$= r\begin{bmatrix} C & A(I_n+P) \end{bmatrix}, \tag{3.16}$$

$$r\begin{bmatrix} B_2 \\ C \end{bmatrix} = r\begin{bmatrix} 0 \\ B_2 \\ C \end{bmatrix}$$

$$= r\begin{bmatrix} \frac{1}{2}U^*(I_n-P)B \\ C \end{bmatrix}$$

$$= r\begin{bmatrix} (I_n-P)B \\ C \end{bmatrix}, \tag{3.17}$$

$$r\begin{bmatrix} B_1 \\ C \end{bmatrix} = r\begin{bmatrix} B_1 \\ 0 \\ C \end{bmatrix}$$

$$= r\begin{bmatrix} \frac{1}{2}U^*(I_n+P)B \\ C \end{bmatrix}$$

$$= r\begin{bmatrix} (I_n+P)B \\ C \end{bmatrix}. \tag{3.18}$$

$$r\begin{bmatrix} C & A_2 \\ B_2 & 0 \end{bmatrix} = r\begin{bmatrix} C & 0 & A_2 \\ 0 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n-P)U \\ \frac{1}{2}U^*(I_n-P)B & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)B & 0 \end{bmatrix} \tag{3.19}$$

$$r\begin{bmatrix} C & A_1 \\ B_1 & 0 \end{bmatrix} = \begin{bmatrix} C & A_1 & 0 \\ B_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n+P)U \\ \frac{1}{2}U^*(I_n+P)B & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)B & 0 \end{bmatrix} \tag{3.20}$$

Substituting (3.15)-(3.20) into (3.11)-(3.14) yields results of Theorem 3.1.
b) Obvious from formulas (3.4)-(3.5) of Lemma (3.2) and necessary changes from (3.15)-(3.20). □

Corollary 3.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$ are given, we suppose that the matrix equation (3.1) has a reflexive solution. Then* *a) Equation (3.1) has a reflexive solution of the form* $X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} C & A(I_n + P) \\ (I_n + P)B & 0 \end{bmatrix} = r \begin{bmatrix} C & A(I_n + P) \end{bmatrix} + r \begin{bmatrix} (I_n + P)B \\ C \end{bmatrix}.$$

b) All reflexive solutions of equation (3.1) have the form $X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$\begin{aligned} r \begin{bmatrix} C & A(I_n + P) \end{bmatrix} &= r(A) + r((I_n - P)B) - 2(n-k), \\ r \begin{bmatrix} (I_n + P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n - P)) - 2(n-k). \end{aligned}$$

c) Equation (3.1) has a reflexive solution of the form $X = U \begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} C & A(I_n - P) \\ (I_n - P)B & 0 \end{bmatrix} = r \begin{bmatrix} C & A(I_n - P) \end{bmatrix} + r \begin{bmatrix} (I_n - P)B \\ C \end{bmatrix}.$$

d) All reflexive solutions of equation (3.1) have the form $X = U \begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$\begin{aligned} r \begin{bmatrix} C & A(I_n - P) \end{bmatrix} &= r(A) + r((I_n + P)B) - 2k, \\ r \begin{bmatrix} (I_n - P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n + P)) - 2k. \end{aligned}$$

e) Equation (3.1) has a null reflexive solution if and only if

$$\begin{aligned} r \begin{bmatrix} C & A(I_n + P) \\ (I_n + P)B & 0 \end{bmatrix} &= r \begin{bmatrix} C & A(I_n + P) \end{bmatrix} + r \begin{bmatrix} (I_n + P)B \\ C \end{bmatrix}, \\ \text{and } r \begin{bmatrix} C & A(I_n - P) \\ (I_n - P)B & 0 \end{bmatrix} &= r \begin{bmatrix} C & A(I_n - P) \end{bmatrix} + r \begin{bmatrix} (I_n - P)B \\ C \end{bmatrix}. \end{aligned}$$

f) All reflexive solutions of equation (3.1) are nulls if and only if

$$\begin{aligned} r \begin{bmatrix} C & A(I_n + P) \end{bmatrix} &= r(A) + r((I_n - P)B) - 2(n-k), \\ r \begin{bmatrix} (I_n + P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n - P)) - 2(n-k). \end{aligned}$$

and

$$\begin{aligned} r \begin{bmatrix} C & A(I_n - P) \end{bmatrix} &= r(A) + r((I_n + P)B) - 2k, \\ r \begin{bmatrix} (I_n - P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n + P)) - 2k. \end{aligned}$$

Acknowledgments

The authors are grateful for the detailed comments from the referees, which significantly improved the quality of the paper.

REFERENCES

1. A. Ben-Israel and T. N. E. Greville: *Generalized Inverses: Theory and Applications.* 2^{nd} ed, Springer, 2003.
2. H. Chen: *Generalized reflexive matrices: Special properties and applications.* SIAM J. Matrix Anal. Appl **19** (1998), 140-153.
3. C. Chen and A. Sameh: *Numerical linear algebra algorithms on the cedar system. Parallel computations and their impact on mechanics, AMD.* The American Society of Mechanical Engineers. **86** (1987), 101-125.
4. D. S. Cvetković-Ilić: *The reflexive solutions of the matrix equation $AXB = C$.* Comput. Math. Appl. **51** (2006), 897-902.
5. M. Dehghan and M. Hajarian: *On the reflexive solution of the matrix equation $AXB + CYD = E$.* Bull. Korean Math. Soc. **46 No 3** (2009), 511-519.
6. M. Dehghan and M. Hajarian: *The reflexive and anti reflexive solutions of a linear matrix equation and systems of matrix equations.*Rocky Mountain journal of mathematics. **40 No 3** (2010), 825-848.
7. F. Ding and T. Chen: *Hierarchical gradient-based identification of multivariable discrete-time systems.* Automatica J. IFAC. **41 No 2** (2005), 315-325.
8. X. Liu: *Hermitian and nonnegative definite reflexive and anti-reflexive solutions to $AX = B$.* Int. J. Comput. Math. **95 No 8** (2018), 1666-1671.
9. X. Liu and Y. Yuan: *Generalized reflexive and anti-reflexive solutions of $AX = B$.* Calcolo. **53** (2016), 59-66.
10. X. Liu: *The common (P, Q) generalized reflexive and anti-reflexive solutions to $AX = B$ and $XC = D$.* Calcolo. **53** (2016), 227-234.
11. X. Liu: *The Extremal Ranks and Inertias of Matrix Expressions with Respect to Generalized Reflexive and Anti-Reflexive Matrices with Applications.* Filomat. **32 No 19** (2018), 6653-6666.
12. Y. X. Peng, X. Y. Hu and L. Zhang: *An iteration method for the symmetric solutions and the optimal approximation solution of the matrix equation $AXB = C$.* Appl. Math. Comp. **160** (2005), 763-777.
13. Z. Y. Peng: *An iterative method for the least squares symmetric solution of the linear matrix equation $AXB = C$.* Appl. Math. Comp. **170** (2005), 711-723.
14. X. Peng, X. Hu and L. Zhang: *The reflexive and anti-reflexive solutions of the matrix equation $A^H XB = C$.* J. Comput. Appl. Math. **200** (2007), 749-760.
15. P. S. Stanimirović: *G-inverses and canonical forms.* Facta universitatis (Nis) Ser. Math. Inform. **15** (2000), 1-14.
16. Y. Tian: *The solvability of two linear matrix equations.* Linear Multilinear Algebra. **48** (2000), 123-147.
17. Y. tian: *Ranks and independence of solutions of the matrix equation $AXB+CYD = M$.* Acta Math. Univ. Comenianae. **1** (2006), 75-84.

18. Y. Tian and W. Hongxing: *Characterizations of EP matrices and weighted-EP matrices*. Linear Algebra Appl. **434 No 5** (2011), 1295-1318.

19. Q. W. Wang and J. Jiang: *Extreme ranks of (skew-) Hermitian solutions to a quaternion matrix equation*. Electron. J. Linear Algebra. **20** (2010), 552–573.

20. Q. W. Wang and C. K. Li: *Ranks and the least-norm of the general solution to a system of quaternion matrix equations*. Linear. Algebra. Appl. **430** (2009), 1626-1640.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 51–65
https://doi.org/10.22190/FUMI221122004U
Original Scientific Paper

A NEW GLANCE TO THE ASPECTS OF Q-HELICES

Yasin Ünlütürk[1], Cumali Ekici[2] and Doğan Ünal[3]

[1] **Faculty of Science, Kırklareli University**
Department of Mathematics, 39100 Kırklareli, Türkiye
[2] **Faculty of Science, Eskişehir Osmangazi University**
Department of Mathematics and Computer Sciences, 26040 Eskişehir, Türkiye
[3] **Faculty of Engineering, Kırklareli University**
Department of Software Engineering, 39100 Kırklareli, Türkiye

ORCID IDs: Yasin Ünlütürk https://orcid.org/0000-0002-6395-5272
Cumali Ekici https://orcid.org/0000-0002-3247-5727
Dogan Ünal https://orcid.org/0000-0001-5480-2998

Abstract. In this examination, we take q-helices into consideration. By q-helices, we mean curves due to the quasi-frame (abbv. q-frame) whose vector fields make constant angles with a non-zero fixed axis. One by one, all types of these q-helices we study in the work are therefore classified in three dimensional Euclidean space. Additionally, we study Darboux q-helices by using Darboux vector obtained with respect to q-frames fields of a curve. For a curve enclosed with q-frame as a general case, we reach some results for Darboux q-helices.
Keywords: q-frame, q-helices, the relations between q-helices, Darboux q-helices.

1. Introduction

A necessary and sufficient condition for a curve to be of constant slope is that the ratio of curvature to torsion be constant. This expression is a famous theorem characterizing helices which was proposed by M.A. Lancret in 1802, but its first proof was given by B. de Saint Venant in 1845 in his work published at Journal Ec. Polyt. 30, 1845, p. 26. [20].

Slant helices as more general forms of helices were conceptualized by Izumiya and Takeuchi [9]. Several authors introduced different types of helices and investigated their properties [10, 11, 12, 21].

Received November 22, 2022, accepted: January 21, 2023
Communicated by Mića Stanković
Corresponding Author: Yasin Ünlütürk. E-mail addresses: yasinunluturk@klu.edu.tr (Y. Ünlütürk), cekici@ogu.edu.tr (C. Ekici), doganunal@klu.edu.tr (D. Ünal)
2010 *Mathematics Subject Classification.* 53A04

Researches are increasing on $k-$type slant helices with their various aspects [13, 16, 17, 21]. The meaning of $k-$type slant helices is related to the class of curves having a property that the scalar product of frame's vector field and a fixed axis is constant [8]. For example, general helices are $0-$type helices and also 1-type slant helix is one whose normal vector field makes a constant angle with a non zero fixed axis. This subject "*$k-$type slant helices*" has been studied and developed in different types of spaces such as Euclidean, Galilean, and Lorentzian spaces [1, 13, 15, 18]. Another approach called as "k-type Darboux slant helices" is based on the idea that Darboux vector obtained by the frame fields in which curves' behaviour is taken into consideration makes a constant angle with a non-zero fixed axis is seen in the works [13, 16, 17, 23, 25].

The different suggestions to frame a curve such as parallel transport frame, Frenet frame, and etc. are prevalent approaches in differential geometry of curves [2, 3, 4, 5, 6, 22, 24]. The way to establish the quasi-frame was firstly paved with introducing the quasi normal vector of a space curve by Coquillart [3]. Then Shin et al. defined the quasi-normal vector for each point of the curve which lies in the plane perpendicular to the tangent of the curve at this point [19]. The local theory of space curves via q-frame was studied by Dede in [4].

In this research, q-helices by which we mean curves whose q-frame fields make a constant angle with a non-zero fixed axis. We give the necessary and sufficient conditions for curves due to the q-frame to be q-helices. Then we obtain some results of the relations between q-helices and Darboux q-helices. Also we classify Darboux q-helices as special ones whose Darboux vector makes a constant angle with a non-zero fixed axis by choosing the curve as one of the types of q-helices, and also the general case.

2. Preliminaries

The three dimensional Euclidean space $\mathbb{E}^3$ is a real vector space $\mathbb{R}^3$ equipped with

$$g = dx_1^2 + dx_2^2 + dx_3^3, \tag{2.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$.

Let $\gamma : I \longrightarrow \mathbb{E}^3$ be an arc-length parametrized curve which has at least four continuous derivatives, then the curve γ has a natural frame called as Frenet frame with the equations below:

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' &= -\tau\mathbf{N} \end{aligned} \tag{2.2}$$

where κ and τ are the curvature and the torsion functions of the curve γ, respectively. We designate unit tangent vector field with $\mathbf{T}$, unit principle normal vector field with $\mathbf{N}$ and the unit binormal vector field with $\mathbf{B}$. We exclude the condition $\mathbf{T}'(s) = 0$ for some $s \in I$ along with this paper [7].

The quasi-frame (abbv. q-frame) as an alternative frame to Frenet trihedron has been introduced as follows: Given a space curve $\gamma(t)$, the q-frame composes of three orthonormal vectors. These vectors are the unit tangent vector $\mathbf{T}$, the quasi-normal $\mathbf{N}_q$ and the quasi-binormal vector $\mathbf{B}_q$, respectively. The q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q, \mathbf{k}\}$ is given by

$$\mathbf{T} = \frac{\gamma'}{\|\gamma'\|}, \mathbf{N}_q = \frac{\mathbf{T}\wedge\mathbf{k}}{\|\mathbf{T}\wedge\mathbf{k}\|}, \mathbf{B}_q = \mathbf{T} \wedge \mathbf{N}_q \tag{2.3}$$

where $\mathbf{k}$ is the projection vector.

For clarity, the projection vector $\mathbf{k}$ has been chosen as $\mathbf{k} = (0, 0, 1)$ along with the paper. Nevertheless, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ become parallel. Hence, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel, the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

Let $\gamma(s)$ be a curve that is parameterized by arc length s. The variation equations of the q-frame is given ([4]) as

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}'_q \\ \mathbf{B}'_q \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix}, \tag{2.4}$$

where the q-curvatures are

$$k_1 = \frac{\langle \mathbf{T}', \mathbf{N}_q \rangle}{\|\gamma'\|}, \quad k_2 = \frac{\langle \mathbf{T}', \mathbf{B}_q \rangle}{\|\gamma'\|}, \quad k_3 = -\frac{\langle \mathbf{N}_q, \mathbf{B}'_q \rangle}{\|\gamma'\|}. \tag{2.5}$$

3. The q-helices

In this section, we study different types of q-helices which means $k-$type slant helices of curves via q-frame in Euclidean 3-space $\mathbb{E}^3$. By q-helices, we intend the curves whose q-frame vector fields make a constant angle with a non-zero fixed axis. These types of helices within the q-frame are enclosed as depending on a constant angle between the tangent vector field $\mathbf{T}$ and the fixed vector $\mathbf{U}$, the quasi-normal vector field $\mathbf{N}_q$ and the fixed vector $\mathbf{U}$, and the quasi-binormal vector field $\mathbf{B}_q$ and the fixed vector $\mathbf{U}$.

Definition 3.1. A curve γ in $\mathbb{E}^3$ given by the q-frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$ is called a slant helix of type-0, a slant helix of type-1 and a slant helix of type-2 if there exists a non zero fixed direction $\mathbf{U} \in \mathbb{E}^3$ such that satisfies, respectively,

$$\langle \mathbf{T}, \mathbf{U} \rangle = \cos\theta_1, \quad \langle \mathbf{N}_q, \mathbf{U} \rangle = \cos\theta_2, \quad \langle \mathbf{B}_q, \mathbf{U} \rangle = \cos\theta_3, \tag{3.1}$$

where θ_1, θ_2 and θ_3 are constant angles. The fixed direction $\mathbf{U}$ is called axis of the q-helices.

The vector $\mathbf{U}$ can be written as a combination of q-frame fields as subsequent

$$\mathbf{U} = \lambda_1 \mathbf{T} + \lambda_2 \mathbf{N}_q + \lambda_3 \mathbf{B}_q, \tag{3.2}$$

where

$$\lambda_1 = \langle \mathbf{T}, \mathbf{U} \rangle, \lambda_2 = \langle \mathbf{N}_q, \mathbf{U} \rangle, \lambda_3 = \langle \mathbf{B}_q, \mathbf{U} \rangle.$$

Since U is a fixed vector field, its differentiation vanishes, that is,

$$\begin{aligned}\mathbf{U}' &= (\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2)\mathbf{T} + (\lambda_2' + \lambda_1 k_1 - \lambda_3 k_3)\mathbf{N}_q + (\lambda_3' + \lambda_1 k_2 + \lambda_2 k_3)\mathbf{B}_q \\ &= 0.\end{aligned} \tag{3.3}$$

By (3.3), the following system is obtained as

$$\begin{aligned}\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 &= 0,\\ \lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 &= 0,\\ \lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 &= 0.\end{aligned} \tag{3.4}$$

In the following subsections, we study q-helices based on the system of differential equations (3.4).

3.1. The q-helices of type-0

Theorem 3.1. *Let γ be a curve due to the q-frame in $\mathbb{E}^3$. Then γ is a q-helix of type-0 if and only if*

$$\left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds\right) k_1 + \left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds\right) k_2 = 0. \tag{3.5}$$

Proof. A q-helix of type-0 satisfies the condition

$$\lambda_1 = \langle \mathbf{T}, \mathbf{U} \rangle = \cos\theta_1, \tag{3.6}$$

where θ_1 is a constant angle. Therefore, by substituting $\lambda_1 = \cos\theta_1$ into the system (3.4), it turns into

$$\begin{aligned}\lambda_2 k_1 + \lambda_3 k_2 &= 0,\\ \lambda_2' - \lambda_3 k_3 + \cos\theta_1 k_1 &= 0,\\ \lambda_3' + \lambda_2 k_3 + \cos\theta_1 k_2 &= 0.\end{aligned} \tag{3.7}$$

From $(3.7)_1$,

$$\lambda_3 = -\frac{k_1}{k_2}\lambda_2, \qquad \lambda_2 = -\frac{k_2}{k_1}\lambda_3. \tag{3.8}$$

By using (3.8) in the equations $(3.7)_1$, and $(3.7)_2$, we get the following linear differential equations of first order:

$$\lambda_2' + \frac{k_1 k_3}{k_2}\lambda_2 = -\cos\theta_1 k_1, \tag{3.9}$$

$$\lambda_3' - \frac{k_2 k_3}{k_1}\lambda_3 = -\cos\theta_1 k_2. \tag{3.10}$$

The solution of (3.9) is

$$\lambda_2 = -\cos\theta_1 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds, \tag{3.11}$$

and the solution of (3.10) is

$$\lambda_3 = -\cos\theta_1 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.12}$$

Substituting (3.11) and (3.12) into $(3.7)_1$ gives the condition to be q-helices of type-0 as follows:

$$\left(e^{-\int \frac{k_1k_3}{k_2}ds}\int k_1e^{\int \frac{k_1k_3}{k_2}ds}ds\right)k_1+\left(e^{\int \frac{k_2k_3}{k_1}ds}\int k_2e^{-\int \frac{k_2k_3}{k_1}ds}ds\right)k_2=0. \tag{3.13}$$

Conversely, suppose that the relation (3.5) holds, also the fixed vector filed $\mathbf{U}$ can be composed of

$$\begin{aligned}\mathbf{U}=\cos\theta_1\mathbf{T}&-\left(\cos\theta_1e^{-\int \frac{k_1k_3}{k_2}ds}\int k_1e^{\int \frac{k_1k_3}{k_2}ds}ds\right)\mathbf{N}_q\\&-\left(\cos\theta_1e^{\int \frac{k_2k_3}{k_1}ds}\int k_2e^{-\int \frac{k_2k_3}{k_1}ds}ds\right)\mathbf{B}_q.\end{aligned} \tag{3.14}$$

We obtain $\mathbf{U}'=\mathbf{0}$ by using (3.6). Hence γ is a q-helices of type-0. □

Corollary 3.1. *If γ is a q-helix of type-0, an axis of γ is*

$$\begin{aligned}\mathbf{D}_0=\cos\theta_1\mathbf{T}&+\left(-\cos\theta_1e^{-\int \frac{k_1k_3}{k_2}ds}\int k_1e^{\int \frac{k_1k_3}{k_2}ds}ds\right)\mathbf{N}_q\\&+\left(-\cos\theta_1e^{\int \frac{k_2k_3}{k_1}ds}\int k_2e^{-\int \frac{k_2k_3}{k_1}ds}ds\right)\mathbf{B}_q.\end{aligned} \tag{3.15}$$

Remark 3.1. If the tangent vector field $\mathbf{T}$ of the curve γ and the fixed axis $\mathbf{D}_0$ are orthogonal to each other, that is, $\cos\theta_1=\cos\frac{\pi}{2}=0$, then the q-helix of type-0 can not occur since the vanishing of the axis $\mathbf{D}_0$.

3.2. The q-helices of type-1

Theorem 3.2. *Let γ be a curve due to the q-frame in $\mathbb{E}^3$. Then γ is a q-helix of type-1 if and only if*

$$\left(e^{\int \frac{k_1k_2}{k_3}ds}\int k_1e^{-\int \frac{k_1k_2}{k_3}ds}ds\right)k_1+\left(e^{-\int \frac{k_2k_3}{k_1}ds}\int k_3e^{\int \frac{k_2k_3}{k_1}ds}ds\right)k_3=0. \tag{3.16}$$

Proof. A q-helix of type-1 satisfies the condition

$$\lambda_2=\langle\mathbf{N}_q,\mathbf{U}\rangle=\cos\theta_2, \tag{3.17}$$

where θ_2 is a constant angle. Therefore, by substituting $\lambda_2=\cos\theta_2$ into the system (3.4), it turns into

$$\begin{aligned}&\lambda_1'-\lambda_2k_1-\lambda_3k_2=0,\\&\lambda_1k_1-\lambda_3k_3=0,\\&\lambda_3'+\lambda_1k_2+\lambda_2k_3=0.\end{aligned} \tag{3.18}$$

From $(3.18)_2$,

$$\lambda_3=\frac{k_1}{k_3}\lambda_1,\qquad \lambda_1=\frac{k_3}{k_1}\lambda_3. \tag{3.19}$$

By using (3.19) in the equations $(3.18)_1$ and $(3.18)_3$, we get the following linear differential equations of first order:

$$\lambda_1'-\frac{k_1k_2}{k_3}\lambda_1=\cos\theta_2k_1, \tag{3.20}$$

$$\lambda_3' + \frac{k_2k_3}{k_1}\lambda_3 = -\cos\theta_2 k_3. \tag{3.21}$$

The solution of (3.20) is

$$\lambda_1 = \cos\theta_2 e^{\int \frac{k_1k_2}{k_3}ds} \int k_1 e^{-\int \frac{k_1k_2}{k_3}ds} ds, \tag{3.22}$$

and the solution of (3.21) is

$$\lambda_3 = -\cos\theta_2 e^{-\int \frac{k_2k_3}{k_1}ds} \int k_3 e^{\int \frac{k_2k_3}{k_1}ds} ds. \tag{3.23}$$

Substituting (3.22), and (3.23) into $(3.18)_2$ gives the condition to be q-helices of type-1 as follows:

$$\left(e^{\int \frac{k_1k_2}{k_3}ds} \int k_1 e^{-\int \frac{k_1k_2}{k_3}ds} ds\right) k_1 + \left(e^{-\int \frac{k_2k_3}{k_1}ds} \int k_3 e^{\int \frac{k_2k_3}{k_1}ds} ds\right) k_3 = 0. \tag{3.24}$$

Conversely, suppose that the relation (3.16) holds, also the fixed vector field $\mathbf{U}$ can be composed of

$$\begin{aligned} \mathbf{U} = &\left(\cos\theta_2 e^{\int \frac{k_1k_2}{k_3}ds} \int k_1 e^{-\int \frac{k_1k_2}{k_3}ds} ds\right) \mathbf{T} + \cos\theta_2 \mathbf{N}_q \\ &- \left(\cos\theta_2 e^{-\int \frac{k_2k_3}{k_1}ds} \int k_3 e^{\int \frac{k_2k_3}{k_1}ds} ds\right) \mathbf{B}_q. \end{aligned} \tag{3.25}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.17). Hence γ is a q-helix of type-1. □

Corollary 3.2. *If γ is a q-helix of type-1, an axis of γ is*

$$\begin{aligned} \mathbf{D}_1 = &\left(\cos\theta_2 e^{\int \frac{k_1k_2}{k_3}ds} \int k_1 e^{-\int \frac{k_1k_2}{k_3}ds} ds\right) \mathbf{T} + \cos\theta_2 \mathbf{N}_q \\ &+ \left(-\cos\theta_2 e^{-\int \frac{k_2k_3}{k_1}ds} \int k_3 e^{\int \frac{k_2k_3}{k_1}ds} ds\right) \mathbf{B}_q. \end{aligned} \tag{3.26}$$

Remark 3.2. If the tangent vector field $\mathbf{N}_q$ of the curve γ and the fixed axis $\mathbf{D}_1$ are orthogonal to each other, that is, $\cos\theta_2 = \cos\frac{\pi}{2} = 0$, then the q-helix of type-1 can not occur since the vanishing of the axis $\mathbf{D}_1$.

3.3. The q-helices of type-2

Theorem 3.3. *Let γ be a curve due to the q-frame in $\mathbb{E}^3$. Then γ is a q-helix of type-2 if and only if*

$$\left(e^{-\int \frac{k_1k_2}{k_3}ds} \int k_2 e^{\int \frac{k_1k_2}{k_3}ds} ds\right) k_2 + \left(e^{\int \frac{k_1k_3}{k_2}ds} \int k_3 e^{-\int \frac{k_1k_3}{k_2}ds} ds\right) k_3 = 0. \tag{3.27}$$

Proof. A q-helix of type-2 satisfies the condition

$$\lambda_3 = \langle \mathbf{B}_q, \mathbf{U} \rangle = \cos\theta_3, \tag{3.28}$$

where θ_3 is a constant angle. Therefore, by substituting $\lambda_3 = \cos\theta_3$ into the system (3.4), it turns into

$$\begin{aligned} \lambda_1 k_2 + \lambda_2 k_3 &= 0. \\ \lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 &= 0, \\ \lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 &= 0. \end{aligned} \tag{3.29}$$

From $(3.29)_1$,

$$\lambda_2 = -\tfrac{k_2}{k_3}\lambda_1, \qquad \lambda_1 = -\tfrac{k_3}{k_2}\lambda_2. \tag{3.30}$$

By using (3.30) in the equations $(3.29)_2$, and $(3.29)_3$, we get the following linear differential equations of first order:

$$\lambda_1' + \tfrac{k_1 k_2}{k_3}\lambda_1 = \cos\theta_3 k_2, \tag{3.31}$$

$$\lambda_2' - \tfrac{k_1 k_3}{k_2}\lambda_2 = \cos\theta_3 k_3. \tag{3.32}$$

The solutions of (3.31) and (3.32) are

$$\lambda_1 = \cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds, \tag{3.33}$$

$$\lambda_2 = \cos\theta_3 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds, \tag{3.34}$$

respectively.

Substituting (3.33) and (3.34) into $(3.29)_1$ gives the condition to be q-helices of type-2 as follows:

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) k_2 + \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) k_3 = 0. \tag{3.35}$$

Conversely, suppose that the relation (3.27) holds, also the fixed vector field $\mathbf{U}$ can be composed of

$$\begin{aligned} \mathbf{U} = &\left(\cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} \\ &+ \left(\cos\theta_3 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q + \cos\theta_3 \mathbf{B}_q. \end{aligned} \tag{3.36}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (3.27) and (3.28). Hence γ is a q-helix of type-2. □

Corollary 3.3. *If γ is a q-helix of type-2, an axis of γ is*

$$\begin{aligned} \mathbf{D}_2 = &\left(\cos\theta_3 e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) \mathbf{T} \\ &+ \left(\cos\theta_3 e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) \mathbf{N}_q + \cos\theta_3 \mathbf{B}_q. \end{aligned} \tag{3.37}$$

Remark 3.3. If the tangent vector field $\mathbf{B}_q$ of the curve γ and the fixed axis $\mathbf{D}_2$ are orthogonal to each other, that is, $\cos\theta_3 = \cos\frac{\pi}{2} = 0$, then the q-helix of type-2 can not occur since the vanishing of the axis $\mathbf{D}_2$.

3.4. The relations of q-helices to each other

In this part, we give the relations of q-helices to each other based on the consequences of Theorems 3.1, 3.2 and 3.3.

Corollary 3.4. *Let γ be a q-helix of type-0 in $\mathbb{E}^3$. Then γ is a q-helix of type-1 if and only if*

$$k_1 = 0 \quad or \quad k_2 = ck_3, \tag{3.38}$$

where c is a constant.

Proof. Using (3.14) at the condition to be a q-helix of type-1 as follows:

$$\langle \mathbf{N}_q, \mathbf{U} \rangle = -\cos\theta_1 e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds. \tag{3.39}$$

The expression in (3.39) becomes constant if the cases (3.38) are satisified. □

Corollary 3.5. *Let γ be a q-helix of type-0 in $\mathbb{E}^3$. Then γ is a q-helix of type-2 if and only if*

$$k_2 = 0 \quad or \quad k_1 = -ck_3 \tag{3.40}$$

where c is a constant.

Proof. Using (3.14) at the condition to be a q-helix of type-2 as follows:

$$\langle \mathbf{B}_q, \mathbf{U} \rangle = -\cos\theta_1 e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds. \tag{3.41}$$

The expression in (3.41) becomes constant if the cases (3.40) are satisified. □

Corollary 3.6. *Let γ be a q-helix of type-1 in $\mathbb{E}^3$. Then γ is a q-helix of type-0 if and only if*

$$k_1 = 0 \quad or \quad k_3 = -ck_2 \tag{3.42}$$

where c is a constant.

Proof. Using (3.25) at the condition to be a q-helix of type-0 as follows:

$$\langle \mathbf{T}, \mathbf{U} \rangle = \cos\theta_2 e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds. \tag{3.43}$$

The expression in (3.43) becomes constant if the cases (3.42) are satisified. □

Corollary 3.7. *Let γ be a q-helix of type-1 in $\mathbb{E}^3$. Then γ is a q-helix of type-2 if and only if*

$$k_3 = 0 \quad or \quad k_1 = -ck_2 \tag{3.44}$$

where c is a constant.

Proof. Using (3.25) at the condition to be a q-helix of type-2 as follows:

$$\langle \mathbf{B}_q, \mathbf{U}\rangle = -\cos\theta_2 e^{-\int \frac{k_2k_3}{k_1}ds} \int k_3 e^{\int \frac{k_2k_3}{k_1}ds} ds. \tag{3.45}$$

The expression in (3.45) becomes constant if the cases (3.44) are satisified. □

Corollary 3.8. *Let γ be a q-helix of type-2 in $\mathbb{E}^3$. Then γ is a q-helix of type-0 if and only if*

$$k_2 = 0 \quad or \quad k_3 = ck_1. \tag{3.46}$$

where c is a constant.

Proof. Using (3.36) at the condition to be a q-helix of type-0 as follows:

$$\langle \mathbf{T}, \mathbf{U}\rangle = \cos\theta_3 e^{-\int \frac{k_1k_2}{k_3}ds} \int k_2 e^{\int \frac{k_1k_2}{k_3}ds} ds. \tag{3.47}$$

The expression in (3.47) becomes constant if the cases (3.46) are satisified. □

Corollary 3.9. *Let γ be a q-helix of type-2 in $\mathbb{E}^3$. Then γ is a q-helix of type-1 if and only if*

$$k_3 = 0 \quad or \quad k_2 = -ck_1. \tag{3.48}$$

where c is a constant.

Proof. Using (3.36) at the condition to be a q-helix of type-1 as follows:

$$\langle \mathbf{N}_q, \mathbf{U}\rangle = \cos\theta_3 e^{\int \frac{k_1k_2}{k_3}ds} \int k_1 e^{-\int \frac{k_1k_2}{k_3}ds} ds. \tag{3.49}$$

The expression in (3.49) becomes constant if the cases (3.48) are satisified. □

The above results can be put together with the following corollary:

Corollary 3.10. *Let γ be a curve via q-frame in $\mathbb{E}^3$. Then*

(i) *The curve γ is both a q-helix of type-0 and type-1 provided that*

$$k_1 = 0 \quad or \quad k_2 = Ak_3,$$

where A is an arbitrary constant.

(ii) *The curve γ is both a q-helix of type-0 and type-2 provided that*

$$k_2 = 0 \quad or \quad k_1 = Bk_3,$$

where B is an arbitrary constant.

(iii) *The curve γ is both a q-helix of type-1 and type-2 provided that*

$$k_3 = 0 \quad or \quad k_2 = Ck_1,$$

where C is an arbitrary constant.

4. The Darboux q-helices

In this part of our research, we classify the Darboux q-helices. First we study the conditions of q-helices of type-0, type-1 and type-2 to be Darboux q-helices, respectively. Finally, we obtain the general case for q-helices to be Darboux helices.

From [4], the Darboux vector of a curve due to the q-frame is as follows:

$$\partial = k_3\mathbf{T} - k_2\mathbf{N}_q + k_1\mathbf{B}_q. \tag{4.1}$$

We have to give the description of Darboux q-helices as follows:

Definition 4.1. A unit speed curve γ framed by q-frame whose Darboux vector ∂ is said to be a Darboux helix provided that there exists a non-zero fixed direction $\mathbf{U} \in \mathbb{E}^3$ such that satisfies

$$\langle \partial, \mathbf{U} \rangle = \cos\varphi, \tag{4.2}$$

φ is a constant angle between the vectors ∂ and $\mathbf{U}$.

Differentiating (4.2) and rearranging the equation gives the system

$$\begin{aligned} \lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 &= 0, \\ \lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 &= 0, \\ \lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 &= 0. \end{aligned} \tag{4.3}$$

Based upon the system (4.3), we take the q-helices of type-0, type-1 and type-2, and a curve frame by a q-frame to be Darboux helices into consideration, respectively, in the subsequent four cases:

Case 1: Let γ be a q-helix of type-0. Hence the equation (3.6) holds. Then we have the equation

$$\langle \partial', \mathbf{U} \rangle = \lambda_1 k_3' - \lambda_2 k_2' + \lambda_3 k_1' = 0. \tag{4.4}$$

Using (3.6) and (4.4) in the system (4.3) results in the following system:

$$\begin{aligned} \cos\theta_1 k_3' - \lambda_2 k_2' + \lambda_3 k_1' &= 0, \\ \lambda_2 k_1 + \lambda_3 k_2 &= 0, \\ \lambda_2' + \cos\theta_1 k_1 - \lambda_3 k_3 &= 0, \\ \lambda_3' + \cos\theta_1 k_2 + \lambda_2 k_3 &= 0. \end{aligned} \tag{4.5}$$

Applying $(4.5)_2$ into the equations $(4.5)_3$ and $(4.5)_4$, the functions λ_2 and λ_3 are found as in (3.11) and (3.12). If the values obtained are substituted to the equation $(4.5)_1$, then it follows that

$$k_3' + k_2' e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds - k_1' e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds = 0. \tag{4.6}$$

Also from (3.13), we have

$$\left(e^{\int \frac{k_2 k_3}{k_1} ds} \int k_2 e^{-\int \frac{k_2 k_3}{k_1} ds} ds \right) = -\frac{k_1}{k_2} \left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds \right). \tag{4.7}$$

Substituting (4.7) into (4.6) gives

$$k_3' + \left(k_2' + \frac{k_1 k_1'}{k_2}\right)\left(e^{-\int \frac{k_1 k_3}{k_2} ds} \int k_1 e^{\int \frac{k_1 k_3}{k_2} ds} ds\right) = 0 \tag{4.8}$$

which is the condition for a q-helix of type-0 to be a Darboux helix.

Conversely, suppose that the relation (4.8) holds, it can be seen that the axis given in (3.14) is a fixed one.

Case 2: Let γ be a q-helix of type-1. Hence the equation (3.17) holds. Using (3.17), and (4.4) in the system (4.3), we find the system

$$\begin{aligned} &\lambda_1 k_3' - \cos\theta_2 k_2' + \lambda_3 k_1' = 0, \\ &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0, \\ &\lambda_1 k_1 - \lambda_3 k_3 = 0, \\ &\lambda_3' + \lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{4.9}$$

Applying $(4.9)_3$ into the equations $(4.9)_2$ and $(4.9)_4$, the functions λ_1 and λ_3 are found as in (3.22) and (3.23). If the values obtained are substituted to the equation $(4.9)_1$, then it follows that

$$k_3' e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds - k_2' - k_1' e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds = 0 \tag{4.10}$$

Also from (3.24), we obtain

$$\left(e^{-\int \frac{k_2 k_3}{k_1} ds} \int k_3 e^{\int \frac{k_2 k_3}{k_1} ds} ds\right) = -\frac{k_1}{k_3}\left(e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) \tag{4.11}$$

Substituting (4.11) into (4.10), we attain the equation

$$k_2' + \left(\frac{k_1 k_1'}{k_3} - k_3'\right)\left(e^{\int \frac{k_1 k_2}{k_3} ds} \int k_1 e^{-\int \frac{k_1 k_2}{k_3} ds} ds\right) = 0 \tag{4.12}$$

which is the condition for a q-helix of type-1 to be a Darboux helix.

Conversely, suppose that the relation (4.12) holds. It can be seen that the axis given in (3.25) is a fixed one.

Case 3: Let γ be a q-helix of type-2. So the equation (3.28) holds. Using (3.28) and (4.4) in the system (4.3), the system is as follows:

$$\begin{aligned} &\lambda_1 k_3' - \lambda_2 k_2' + \cos\theta_3 k_1' = 0, \\ &\lambda_1' - \lambda_2 k_1 - \lambda_3 k_2 = 0, \\ &\lambda_2' + \lambda_1 k_1 - \lambda_3 k_3 = 0, \\ &\lambda_1 k_2 + \lambda_2 k_3 = 0. \end{aligned} \tag{4.13}$$

Applying $(4.13)_4$ into the equations $(4.13)_2$ and $(4.13)_3$, the functions λ_1 and λ_2 are obtained as in (3.33) and (3.34). If the values obtained are put into the equation $(4.13)_1$, then it is followed that

$$k_3' e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds - k_2' e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds + k_1' = 0. \tag{4.14}$$

Also from (3.35), we obtain

$$\left(e^{-\int \frac{k_1 k_2}{k_3} ds} \int k_2 e^{\int \frac{k_1 k_2}{k_3} ds} ds\right) = -\frac{k_3}{k_2} \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right). \tag{4.15}$$

Replacing (4.15) into (4.14), we reach the result

$$k_1' - \left(k_2' + \frac{k_3 k_3'}{k_2}\right) \left(e^{\int \frac{k_1 k_3}{k_2} ds} \int k_3 e^{-\int \frac{k_1 k_3}{k_2} ds} ds\right) = 0 \tag{4.16}$$

which is the condition for a q-helix of type-2 to be a Darboux helix.

Conversely, suppose that the relation (4.16) holds. It can be seen that the axis given in (3.36) is a fixed one.

Case 4 (General Case): Let γ be a curve due to the q-frame in $\mathbb{E}^3$. From (4.2), we obtain

$$\lambda_1 k_3 - \lambda_2 k_2 + \lambda_3 k_1 = \cos\varphi. \tag{4.17}$$

Differentiating (4.17) and after some arrangements, we find

$$\lambda_1 k_3' - \lambda_2 k_2' + \lambda_3 k_1' = 0. \tag{4.18}$$

With the aid of (4.17) and (4.18), we arrive

$$\lambda_3 = \frac{\left(k_2' k_3 - k_2 k_3'\right)\lambda_2 - \cos\varphi k_3'}{k_1' k_3 - k_1 k_3'}, \tag{4.19}$$

and

$$\lambda_1 = \frac{\left(k_2' k_1 - k_2 k_1'\right)\lambda_2 - \cos\varphi k_1'}{k_3' k_1 - k_3 k_1'}, \tag{4.20}$$

respectively. Substituting (4.19) and (4.20) into $(4.3)_2$ delivers the linear DE as

$$\lambda_2' + \left(\frac{k_2' k_1^2 - k_2 k_1 k_1' + k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'}\right) \lambda_2 = \frac{\cos\varphi\left(k_1 k_1' + k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'}. \tag{4.21}$$

The solution of (4.21) is

$$\lambda_2 = \cos\varphi e^{\int \frac{k_2 k_1 k_1' + k_2 k_3 k_3' - k_2' k_1^2 - k_2' k_3^2}{k_3' k_1 - k_3 k_1'} ds} \int \frac{\left(k_1 k_1' + k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'} e^{\int \frac{k_2' k_1^2 - k_2 k_1 k_1' + k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'} ds} ds. \tag{4.22}$$

Using (4.17) and (4.18), we obtain

$$\lambda_1 = \frac{\left(k_1' k_2 - k_1 k_2'\right)\lambda_3 + \cos\varphi k_2'}{k_2' k_3 - k_2 k_3'}, \tag{4.23}$$

and

$$\lambda_2 = \frac{\left(k_1' k_3 - k_1 k_3'\right)\lambda_3 + \cos\varphi k_3'}{k_2' k_3 - k_2 k_3'}, \tag{4.24}$$

respectively. Replacing (4.23) and (4.24) into $(4.3)_3$, we have the following differential equation

$$\lambda_3' + \left(\frac{k_1' k_2^2 - k_1 k_2 k_2' + k_1' k_3^2 - k_1 k_3 k_3'}{k_2' k_3 - k_2 k_3'}\right)\lambda_3 = \frac{\cos\varphi k_2 k_2' + \cos\varphi k_3 k_3'}{k_2 k_3' - k_2' k_3}. \tag{4.25}$$

The solution of (4.25) is

$$\lambda_3 = \cos\varphi e^{\int \frac{k_1 k_2 k_2' + k_1 k_3 k_3' - k_1' k_3^2 - k_1' k_2^2}{k_2' k_3 - k_2 k_3'} ds} \int \frac{k_2 k_2' + k_3 k_3'}{k_2 k_3' - k_2' k_3} e^{\int \frac{k_1' k_2^2 - k_1 k_2 k_2' + k_1' k_3^2 - k_1 k_3 k_3'}{k_2' k_3 - k_2 k_3'} ds} ds. \tag{4.26}$$

From (4.17) and (4.18), we attain

$$\lambda_2 = \frac{\left(k_3' k_1 - k_3 k_1'\right)\lambda_1 + \cos\varphi k_1'}{k_2' k_1 - k_2 k_1'}, \tag{4.27}$$

and

$$\lambda_3 = \frac{\left(k_2' k_3 - k_2 k_3'\right)\lambda_1 - \cos\varphi k_2'}{\left(k_1' k_2 - k_1 k_2'\right)}, \tag{4.28}$$

respectively. Usage of the equations (4.27) and (4.28) at $(4.3)_1$ allows the equation

$$\lambda_1' + \left(\frac{k_3 k_1' - k_3' k_1 + k_2' k_3 - k_2 k_3'}{k_1 k_2' - k_1' k_2}\right)\lambda_1 = \frac{\cos\varphi k_1 k_1' + \cos\varphi k_2 k_2'}{k_1 k_2' - k_1' k_2}. \tag{4.29}$$

The solution of (4.29) is

$$\lambda_1 = \cos\varphi e^{\int \frac{k_3' k_1 - k_3 k_1' - k_2' k_3 + k_2 k_3'}{k_1 k_2' - k_1' k_2} ds} \int \frac{\cos\varphi k_1 k_1' + \cos\varphi k_2 k_2'}{k_1 k_2' - k_1' k_2} e^{\int \frac{k_3 k_1' - k_3' k_1 + k_2' k_3 - k_2 k_3'}{k_1 k_2' - k_1' k_2} ds} ds. \tag{4.30}$$

Substituting (4.22), (4.26) and (4.30) into (4.18) gives the condition for a curve to be a Darboux q-helix as follows:

$$\begin{aligned} &\left(e^{\int \frac{k_1 k_2 k_2' + k_1 k_3 k_3' - k_1' k_3^2 - k_1' k_2^2}{k_2' k_3 - k_2 k_3'} ds} \int \frac{k_2 k_2' + k_3 k_3'}{k_2 k_3' - k_2' k_3} e^{\int \frac{k_1' k_2^2 - k_1 k_2 k_2' + k_1' k_3^2 - k_1 k_3 k_3'}{k_2' k_3 - k_2 k_3'} ds} ds\right) k_1' \\ &+ \left(e^{\int \frac{k_3' k_1 - k_3 k_1' - k_2' k_3 + k_2 k_3'}{k_1 k_2' - k_1' k_2} ds} \int \frac{k_1 k_1' + k_2 k_2'}{k_1 k_2' - k_1' k_2} e^{\int \frac{k_3 k_1' - k_3' k_1 + k_2' k_3 - k_2 k_3'}{k_1 k_2' - k_1' k_2} ds} ds\right) k_3' \\ &= \left(e^{\int \frac{k_2 k_1 k_1' + k_2 k_3 k_3' - k_2' k_1^2 - k_2' k_3^2}{k_3' k_1 - k_3 k_1'} ds} \int \frac{\left(k_1 k_1' + k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'} e^{\int \frac{k_2' k_1^2 - k_2 k_1 k_1' + k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'} ds} ds\right) k_2'. \end{aligned} \tag{4.31}$$

Conversely, suppose that the relation (4.31) holds, the fixed vector filed $\mathbf{U}$ can also be composed of

$$\begin{aligned} \mathbf{U} = &\left(\cos\varphi e^{\int \frac{k_3' k_1 - k_3 k_1' - k_2' k_3 + k_2 k_3'}{k_1 k_2' - k_1' k_2} ds} \int \frac{\cos\varphi k_1 k_1' + \cos\varphi k_2 k_2'}{k_1 k_2' - k_1' k_2} e^{\int \frac{k_3 k_1' - k_3' k_1 + k_2' k_3 - k_2 k_3'}{k_1 k_2' - k_1' k_2} ds} ds\right) \mathbf{T} \\ &+ \left(\cos\varphi e^{\int \frac{k_2 k_1 k_1' + k_2 k_3 k_3' - k_2' k_1^2 - k_2' k_3^2}{k_3' k_1 - k_3 k_1'} ds} \int \frac{\left(k_1 k_1' + k_3 k_3'\right)}{k_3' k_1 - k_3 k_1'} e^{\int \frac{k_2' k_1^2 - k_2 k_1 k_1' + k_2' k_3^2 - k_2 k_3 k_3'}{k_3' k_1 - k_3 k_1'} ds} ds\right) \mathbf{N}_q \\ &+ \left(\cos\varphi e^{\int \frac{k_1 k_2 k_2' + k_1 k_3 k_3' - k_1' k_3^2 - k_1' k_2^2}{k_2' k_3 - k_2 k_3'} ds} \int \frac{k_2 k_2' + k_3 k_3'}{k_2 k_3' - k_2' k_3} e^{\int \frac{k_1' k_2^2 - k_1 k_2 k_2' + k_1' k_3^2 - k_1 k_3 k_3'}{k_2' k_3 - k_2 k_3'} ds} ds\right) \mathbf{B}_q \end{aligned} \tag{4.32}$$

We obtain $\mathbf{U}' = \mathbf{0}$ by using (4.17) and (4.31). Hence γ is a Darboux q-helix.

We can give the following theorem containing the cases above:

Theorem 4.1. *Let γ be a curve due to the q-frame in Euclidean 3-space $\mathbb{E}^3$. Then*

(i) γ is a Darboux q-helix satisfying the condition to be q-helix of type-0 if and only if the equation (4.8) is satisfied.

(ii) γ is a Darboux q-helix satisfying the condition to be q-helix of type-1 if and only if the equation (4.12) is satisfied.

(iii) γ is a Darboux q-helix satisfying the condition to be q-helix of type-2 if and only if the equation (4.16) is satisfied.

(iv) γ is a Darboux q-helix if and only if the equation (4.31) is satisfied, and the fixed axis is given as in (4.32).

5. Conclusion

Helices are very special curves by which many patterns can be modelled in nature. In the present examination, we considered these special curves from the point of view of frame fields which describe the behaviour of the curves. The original aspect of our research is to deal quasi-frame (abbv. q-frame) in Euclidean 3-space. For all vector fields of the mentioned frame, slant helices, which are recalled, in the context of the paper, as q-helices, have been worked out in Euclidean 3-space. Additionally, the Darboux q-helices are obtained by Darboux vector which has been formed by q-frame fields.

REFERENCES

1. A. Alİ, R. Lopez and M. Turgut: *$k-$type partially null and pseudo null slant helices in Minkowski 4-space.* Math. Commun. **17** (2012), 93–103.
2. R. L. Bishop: *There is more than one way to frame a curve.* Am. Math. Mon. **82** (1975), 246–251.
3. S. Coquillart: *Computing offsets of B-spline curves.* Computer-Aided Design **19** (1987), 305–309.
4. M. Dede, C. Ekİcİ and A. Görgülü: *Directional q-frame along a space curve.* IJARCSSE **5** (2015), 775–780.
5. M. Dede, C. Ekİcİ and İ. Güven: *Directional Bertrand Curves.* GU. J. Sci. **31** (2018), 202–211.
6. M. Dede, M. Ç. Aslan, and C. Ekİcİ: *On a variational problem due to the B-Darboux frame in Euclidean 3-spaces.* Math. Meth. Appl. Sci. **44** (2021), 12630–12639.
7. P. M. Do Carmo: *Differential Geometry of Curves and Surfaces.* Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1976.

8. M. Ergüt, H. B. Öztekin and S. Aykurt: *Non-null $k-$slant helices and their spherical indicatrices in Minkowski 3-space.* J. Adv. Res. Dyn. Control Syst. **2** (2021), 1–12.

9. S. Izumiya and N. Takeuchi: *New special curves and developable surfaces.* Turk. J. Math. **28** (2004), 531–537.

10. L. Kula and Y. Yayli: *On slant helix and its spherical indicatrix.* Appl. Math. Comput. **169** (2005), 600–607.

11. L. Kula, N. Ekmekçi, Y. Yayli and K. İlarslan: *Characterizations of slant helices in Euclidean 3-space.* Turk. J. Math. **34** (2010), 261–273.

12. R. S. Millman and G. D. Parker: *Elements of Differential Geometry.* Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1977.

13. E. Nesovic, U. Öztürk and E. B. Koç Öztürk: *On $k-$type pseudo null Darboux helices in Minkowski 3-space.* Math. Anal. Appl. **439** (2016), 690–700.

14. E. Nesovic, E. B. Koç Öztürk and U. Öztürk: *On $k-$type null Cartan slant helices in Minkowski 3-space.* Math. Meth. Appl. Sci. **41** (2018), 7583–7598.

15. E. Nesovic, U. Öztürk and E. B. Koç Öztürk: *On T-Slant, N-Slant and B-Slant helices in Galilean space G_3.* J. Dyn. Syst. Geom. Theor. **16** (2018), 187–199.

16. U. Öztürk and E. Nesovic: *On pseudo null and null Cartan Darboux helices in Minkowski 3-space.* Kuwait J. Sci. **43** (2016), 64-82.

17. U. Öztürk and Z. B. Alkan: *Darboux helices in three dimensional Lie groups.* AIMS Mathematics **5** (2020), 3169–3181.

18. J. Quian and Y. Ho Kim: *Null helix and k-type null slant helices in E_1^4.* Rev. Un. Mat. Argentina **57** (2016), 71–83.

19. H. Shin, S. K. Yoo, S. K. Cho and W. H. Chung: *Directional Offset of a Spatial Curve for Practical Engineering Design.* ICCSA **3** (2003), 711–720.

20. D. J. Struik: *Lectures on Classical Differential Geometry.* Dover, New York, 1988.

21. M. Turgut and S. Yilmaz: *Characterizations of some special helices in E^4.* Sci. Magna **4** (2008), 51–55.

22. G. Uğur Kaymanli, C. Ekici and M. Dede: *Directional evolution of the ruled surfaces via the evolution of their directrix using q-frame along a timelike space curve.* EJOSAT **20** (2020), 392–396.

23. Y. Ünlütürk and T. Körpinar: *On Darboux helices in Complex space C^3.* JOSA **4** (2019), 851–858.

24. S. Yilmaz, and M. Turgut: *A new version of Bishop frame and an application to spherical images.* J. Math. Anal. Appl. **371** (2010), 764–776.

25. E. Ziplar, A. Şenol and Y. Yayli: *On Darboux helices in Euclidean 3-space.* Glob. J. Sci. Front. Res. **12** (2012), 73–80.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 67–86
https://doi.org/10.22190/FUMI221202005D
Original Scientific Paper

2-RULED HYPERSURFACES IN A WALKER 4-MANIFOLD

Mohamed Ayatola Dramé[1], Ameth Ndiaye[2] and Abdoul Salam Diallo[3]

[1] Université Alioune Diop de Bambey, UFR SATIC
Département de Mathématiques, Équipe de Recherche en Analyse Non Linéaire et Géométrie (ER ANLG), B. P. 30, Bambey, Sénégal
[2] Université Cheik Anta Diop de Dakar, FASTEF
Département de Mathématiques, B. P. 880, Dakar, Sénégal

ORCID IDs: Mohamed Ayatola Dramé https://orcid.org/0000-0002-7953-7825
Ameth Ndiaye https://orcid.org/0000-0003-0055-1948
Abdoul Salam Diallo https://orcid.org/0000-0003-4254-5829

Abstract. The hypersurface is one of the most important objects in a space. Many authors studied diffrent geometric aspects of hypersurfaces in a space. In this paper, we define three types of 2-ruled hypersurfaces in a Walker 4-manifold. We obtain the Gaussian and mean curvatures of the 2-ruled hypersurfaces of type-1, type-2 and type-3. We give some characterizations about its minimality. We also deal with the first Laplace-Beltrami operators of these types of 2-ruled hypersurfaces in the considered Walker 4-manifold.
Keywords: 2-ruled hypersurface, Walker manifolds.

1. Introduction

The study of hypersurface of a given ambiant space M is an interesting problem which enriches our knowledge and understanding of the geometry of the space itself. The theory of ruled surfaces in $\mathbb{R}^3$ is a classical subject in diffrential geometry. The study of ruled surfaces of a given ambiant space M is also a natural and interesting problem. A surface Σ in M is said to be ruled if every point of Σ is on (a open geodesic segment) in M that lies in Σ (see [21]). Locally a ruled surface is made by a one parameter family of geodesic segments [7]. Ruled surfaces are one parameter set

Received December 02, 2022, revised: June 04, 2023, accepted: June 09, 2023
Communicated by Uday Chand De
Corresponding Author: Abdoul Salam Diallo. E-mail addresses: rabanyayatoulah@gmail.com (M. A. Dramé), ameth1.ndiaye@ucad.edu.sn (A. Ndiaye), abdoulsalam.diallo@uadb.edu.sn (A. S. Diallo)
2010 *Mathematics Subject Classification.* Primary 14J70, 53A07; Secondary 53A10, 53C50

of lines and they are one of the important topics of classifical differential geometry. A ruled surface is defined as

$$\varphi(s,t) = \alpha(s) + tX(s), s, t \in I \subset \mathbb{R},$$

where the curve $\alpha(s)$ is called base curve and $X(s)$ is called the ruling of the ruled surface. A lots of studies have been done about different characterizations of ruled surfaces in 3-dimensional Euclidean, Minkowskian, Galilean and pseudo-Galilean space (see [9, 10, 12, 13, 14, 18] and references therein). In [2], the authors define a quaternionic operator whose scalar part is a real parameter and vector part is a curve in three dimensional real vector space $\mathbb{R}^3$. They prove that quaternion product of this operator and a spherical curve represent a ruled surface in $\mathbb{R}^3$ if the vector part of the quaternionic operator is perpendicular to the position vector of the spherical curve. Also in [3], the authors show that the split quaternion product of a split quaternion operator and a curve, which lies on Lorentzian unit sphere or on hyperbolic unit sphere, parametrizes a ruled surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ if the vector part of the operator is perpendicular to the position vector of the spherical curve. Recently, in [19], the authors have constructed two special families of ruled surfaces in a three dimensional strict Walker manifold. They show that the local degeneracy (resp. non-degeneracy) to one of this family has a strong consequence on the geometry of the ambiant Walker manifold. Ruled hypersurfaces in higher dimensions have also been studied by many authors [4, 5]. In [6], the intrinsic classification of irreducible ruled hypersurfaces of $\mathbb{R}^4$ has been given. In [16], a new approach to investigating ruled real hypersurfaces in complex hyperbolic space $\mathbb{CH}^n$ is given. In the paper [15], the authors study ruled real hypersurfaces in the complex quadric.

A 2-ruled hypersurface in $\mathbb{R}^4$ is a one-parameter family of planes in $\mathbb{R}^4$. This is a generalization of ruled surfaces in $\mathbb{R}^3$. In [25], the author study singularities of 2-ruled hypersurfaces in Euclidean 4-space. After defining a non-degenerate 2-ruled hypersurface, he gives a necessary and sufficient condition for such a map germ to be right-left equivalent to the cross cap $\times$ interval. Also, the author in [25] discusses the behavior of a generic 2-ruled hypersurface map. In [1], the authors obtain the Gauss map (unit normal vector field) of a 2-ruled hypersurface in Euclidean 4-space with the aid of its general parametric equation. They also obtain Gaussian and mean curvatures of the 2-ruled hypersurface and they give some characterizations about its minimality. Finally, they deal with the first and second Laplace-Beltrami operators of 2-ruled hypersurfaces in $\mathbb{E}^4$. Recently, in [17] the authors have defined three types of 2-ruled hypersurfaces in the Minkowski 4-space $\mathbb{E}^4_1$. They obtain Gaussian and mean curvatures of the 2-ruled hypersurfaces of type-1 and type-2, and some characterizations about its minimality. They also deal with the first Laplace-Beltrami operators of these types of 2-ruled hypersurfaces in $\mathbb{E}^4_1$.

Motivated by the above two works, in this paper we study the 2-ruled hypersurfaces in a Walker 4-manifold. We define three types of 2-ruled hypersurfaces and we gives Gaussian and mean curvatures of the 2-ruled hypersurface and some characterizations about its minimality. Our paper is organized as follows: We introduce the topic in section 1., then we recall some basics notions on pseudo-Riemannian manifolds in section 2.. Finally, we study 2-ruled hypersurfaces on a Walker 4-manifold in section 3.

2. Preliminaries

In this section, we recall some basics notions on pseudo-Riemannian manifolds taken from the book [11]. We begin with some algebraic preliminaries on non-degenerate bilinear forms on an m-dimensional real vector space V.

Let $g : V \times V \to \mathbb{R}$ be a symmetric bilinear form. We say that g is non-degenerate if $g(u, v) = 0$ for each $v \in V$ implies $u = 0$, otherwise g is called degenerate. A non-degenerate symmetric bilinear form on V is called a pseudo-Euclidean metric on V. It may induce either a non-degenerate or a degenerate symmetric bilinear form on a subspace W of V; then W is said to be a non-degenerate or a degenerate subspace, respectively. We say that g is positive (negative) definite provided that $u \neq 0$ implies $g(u, u) > 0(< 0)$. If g is non-degenerate, there exists an ordered basis $(e_1, e_2, \ldots, e_m)$ of V such that:

$$\begin{aligned} g(e_i, e_i) &= -1, \quad 1 \leq i \leq q, \\ g(e_i, e_i) &= 1, \quad q+1 \leq i \leq m, \\ g(e_i, e_j) &= 0, \quad i \neq j, \end{aligned}$$

where q is uniquely determined and $(q, m-q)$ is the signature of g. Obviously, in the case $q = 0$ or $q = m$, the first or the second condition has to be dropped. The integer q is called the index of g on V and it is the largest dimension of a subspace $W \subset V$ on which the induced metric is negative definite.

A pseudo-Riemannian metric g on an m-dimensional manifold M is a symmetric tensor field of type $(0, 2)$ on M such that for any $p \in M$ the tensor g is a non-degenerate symmetric bilinear form on the tangent space T_pM of constant index. We call (M, g) a pseudo-Riemannian manifold. Frequently, we denote by M_q^m an m-dimensional pseudo-Riemannian manifold of index q. In the particular case $m > 2$ and $q = 1$, we call (M, g) a Lorentzian manifold. Obviously, if $q = 0, (M, g)$ is a Riemannian manifold.

Let N_s^n be a submanifold of a pseudo-Riemannian manifold M_q^m. If the pseudo-Riemannian metric tensor g_M of M_q^m induces a pseudo-Riemannian metric tensor, a Riemannian metric tensor or a degenerate metric tensor g_N on N_s^n, then N_s^n is called a pseudo-Riemannian submanifold, a Riemannian submanifold or a degenerate submanifold, respectively, of M_q^m. Let M_q^m be an m-dimensional pseudo-Riemannian manifold with pseudo- Riemannian metric tensor g_M of index q. Denoting by $\langle , \rangle$ the associated nondegenerate inner product on M_q^m, a tangent vector X to M_q^m is said to be spacelike if $\langle X, X \rangle > 0$ (or $X = 0$), timelike if $\langle X, X \rangle < 0$ or lightlike (null) if $\langle X, X \rangle = 0$ and $X \neq 0$. The set of null vectors of T_pM is called the null cone at $p \in M$.

Let $M_1^m(c)$ be an m-dimensional Lorentzian space form of constant curvature c, that is, $M_1^m(c)$ is the de Sitter space-time $\mathbb{S}_1^m(c)$, Minkowski space-time $\mathbb{R}_1^4(c)$ or the anti-de Sitter space-time $\mathbb{H}_1^m(c)$ according to $c > 0, c = 0$ or $c < 0$. For simplicity, we suppose that the constant curvature c of $M_1^m(c)$ is equal to $1, 0, -1$ according to whether $c > 0, c = 0, c < 0$.

Now, we describe some basic examples of pseudo-Riemannian manifolds. Let $\mathbb{R}_q^m$ be an m-dimensional pseudo-Euclidean space with metric tensor given by

$$g = -\sum_{i=1}^{q}(du_i)^2 + \sum_{i=q+1}^{m}(du_i)^2,$$

where $(u_1, \ldots, u_m)$ is a coordinate system of $\mathbb{R}_q^m$. So $(\mathbb{R}_q^m, g)$ is a flat pseudo-Riemannian manifold of index q. Putting:

$$\mathbb{S}_1^m(1) = \{u \in \mathbb{R}_1^{m+1}, \langle u, u \rangle = 1\},$$

one obtains an m-dimensional pseudo-Riemannian manifold of index q and of constant curvature $c = 1$. In the theory of general relativity, $\mathbb{S}_1^4(c)$ is called the de Sitter space-time. Putting:

$$\mathbb{H}_1^m(-1) = \{u \in \mathbb{R}_2^{m+1}, \langle u, u \rangle = -1\},$$

one obtains an m-dimensional pseudo-Riemannian manifold of index q and of constant curvature $c = -1$. $\mathbb{H}_1^m(-1)$ is called the anti-de Sitter space. We end this section by the following remark.

Remark 2.1. In contrast to the Riemannian case, there are topological obstructions to the existence of a Lorentz metric on a manifold M. Such a metric exists if either M is non-compact, or M is compact and has Euler number $\chi(M) = 0$.

3. 2-ruled hypersurfaces on a Walker 4-manifold

Hypersurfaces are one of the important objects in a space. Hypersurfaces in a manifold of constant curvature have been studied by many authors. Many ambiant spaces are not always of constant curvature. In this paper, we will studied 2-ruled hypersurfaces in a Walker 4-manifold.

A Walker 4-manifold noted M, is a pseudo-Riemannian manifold, which admits a field of parallel null 2-planes with signature $(++--)$. This class of manifold is locally isometric to (U, g_f) where U is an open of $\mathbb{R}^4$ and g_f is the metric given, respectively to the local coordinates basis by $\{\partial_i = \frac{\partial}{\partial_{u_1}}\}_{i=1,2,3,4}$ by

$$\begin{aligned} g_f(\partial_1, \partial_3) &= g_f(\partial_2, \partial_4) = 1, \\ g_f(\partial_i, \partial_j) &= g_{f_{ij}}(u_1, u_2, u_3, u_4) \quad \text{for} \quad i, j = 3, 4. \end{aligned}$$

The pseudo-Riemannian geometry of Walker metrics satisfying $g_{f_{34}} = 0$ has been studied by Chaichi et al. [8]. The purpose of this paper is to characterize some metrics propertiers of Walker satisfying : $g_{f_{33}} = g_{f_{44}} = 0$. More precesily, we will consider Walker metrics of the following form:

$$g_f = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & f \\ 0 & 1 & f & 0 \end{pmatrix}, \tag{3.1}$$

where $f = f(u_3, u_4)$ denotes a differentiable function defined U. We denote by $f_3 = \frac{\partial f(u_3,u_4)}{\partial u_3}$ and $f_4 = \frac{\partial f(u_3,u_4)}{\partial u_4}$ for any function $f(u_3, u_4)$. It follows after some straightforward calculations that the non zero christoffel symbols of a Walker metric (3.1) are:

$$\Gamma^2_{33} = f_3 \quad \text{and} \quad \Gamma^1_{44} = f_4.$$

We deduce that the Levita-Civita connection of a Walker metric is given by

$$\nabla_{\partial_3}\partial_3 = f_3\partial_2 \quad \text{and} \quad \nabla_{\partial_4}\partial_4 = f_4\partial_1.$$

Since we will deal with 2-ruled hypersurface in Walker 4-manifold, we now define the de Sitter 3-space, the anti-de Sitter space 3-space and the light cone at the origin, respectivily, by

$$\mathbb{S}^3_1 = \{x \in M, \|u\| = 1\}, \tag{3.2}$$

$$\mathbb{H}^3_+(-1) = \{u \in M, \|u\| = -1\}, \tag{3.3}$$

$$\mathcal{LC} = \{x \in M, \|u\| = 0\}, \tag{3.4}$$

where $\|u\| = \sqrt{g_f(u,u)}$.

If $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ are three vectors in M, then the vector product is defined by

$$\overrightarrow{u} \times_f \overrightarrow{v} \times_f \overrightarrow{w} = \begin{pmatrix} 0 & -f & 1 & 0 \\ -f & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \det \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}. \tag{3.5}$$

If

$$\begin{aligned} \varphi : I_1 \times I_2 \times I_3 &\to M \\ (u_1, u_2, u_3) &\mapsto \varphi(u_1, u_2, u_3), \end{aligned}$$

with

$$\varphi(u_1, u_2, u_3) = \big(\varphi_1, \varphi_2, \varphi_3, \varphi_4\big), \tag{3.6}$$

where $\varphi_i = \varphi_i(u_1, u_2, u_3), i = 1, 2, 3$, is a hypersurface in M. The Gauss map (i.e., the unit normal vector field), the matrix forms of the first and second fundamental forms are

$$G_f = \frac{\varphi_{u_1} \times_f \varphi_{u_2} \times_f \varphi_{u_2}}{\|\varphi_{u_1} \times_f \varphi_{u_2} \times \varphi_{u_3}\|}, \tag{3.7}$$

$$[g_{ij}] = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{3.8}$$

and

$$[h_{ij}] = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}, \tag{3.9}$$

respectively, where: $g_{ij} = g_f(\varphi_{u_i}, \varphi_{u_j}), h_{ij} = g_f(\varphi_{u_i u_j}, G_f)_{i,j\in\{1,2,3\}}$, with $\varphi_{uv} = \sum_{k=1}^{4}\Big\{\frac{\partial^2\varphi_k}{\partial v \partial u} + \sum_{ij} \Gamma^k_{ij} \frac{\partial \varphi_i}{\partial u}\frac{\partial \varphi_j}{\partial v}\Big\}\partial_k$. Also, the matrix of shape operator of the hypersurface φ (3.6) is

$$S_f = [s_{ij}] = [g^{ij}] \cdot [h_{ij}], \tag{3.10}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. With aid of (3.8)-(3.10), the Gaussian curvature and mean curvature of a hypersurface in M are given by

$$K_f = \frac{\det[h_{ij}]}{\det[g_{ij}]}, \tag{3.11}$$

and

$$3H_f = \mathrm{trace}(S_f), \tag{3.12}$$

respectively.

3.1. 2-ruled hypersurfaces of type-1 in M

In this subsection, we give the definition of 2-ruled hypersurfaces of type-1 and state some results on Gaussian and mean curvatures. By a 2-ruled hypersurface of type-1 in M, we mean a map $\varphi : I_1 \times I_2 \times I_3 \to M$ of the form

$$\varphi(u_1, u_2, u_3) = \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1), \tag{3.13}$$

where $\alpha : I_1 \to M, \beta : I_2 \to \mathbb{S}^3_1$ and $\gamma : I_3 \to \mathbb{S}^3_1$ are smooth maps, $\mathbb{S}^3_1$ is the de Sitter 3-space of M and I_1, I_2, I_3 are open intervals. We call α a base curve and two curves β and γ director curves. The planes $(u_2, u_3) \to \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1)$ are called rulings [25].

Putting:

$$\left\{\begin{array}{lcl}\alpha(u_1) & = & \big(\alpha_1(u_1),\alpha_2(u_1),\alpha_3(u_1),\alpha_4(u_1)\big)\\ \beta(u_1) & = & \big(\beta_1(u_1),\beta_2(u_1),\beta_3(u_1),\beta_4(u_1)\big)\\ \gamma(u_1) & = & \big(\gamma_1(u_1),\gamma_2(u_1),\gamma_3(u_1),\gamma_4(u_1)\big),\end{array}\right. \tag{3.14}$$

then, the equation (3.13) becomes:

$$\varphi(u_1,u_2,u_3)=\begin{pmatrix}\alpha_1(u_1)+u_2\beta_1(u_1)+u_3\gamma_1(u_1)\\ \alpha_2(u_1)+u_2\beta_2(u_1)+u_3\gamma_2(u_1)\\ \alpha_3(u_1)+u_2\beta_3(u_1)+u_3\gamma_3(u_1)\\ \alpha_4(u_1)+u_2\beta_4(u_1)+u_3\gamma_4(u_1)\end{pmatrix}. \tag{3.15}$$

We see that $\langle\beta_i,\beta_i\rangle=\langle\gamma_i,\gamma_i\rangle=1$ and we state: $\alpha_i=\alpha_i(u_1)$, $\beta_i=\beta_i(u_1)$, $\gamma_i=\gamma_i(u_1)$, $\varphi_i=\varphi_i(u_1,u_2,u_3)$, $f'=\frac{\partial f(u_1)}{\partial u_1}$, $f''=\frac{\partial^2 f(u_1)}{\partial u_1\partial u_1}$, $i\in\{1,2,3,4\}$ and $f\in\{\alpha,\beta,\gamma\}$. We denote by

$$E_{ij} = \gamma_i(\alpha'_j+u_2\beta'_j+u_3\gamma'_j) \tag{3.16}$$
$$F_{ij} = \beta_i(\alpha'_j+u_2\beta'_j+u_3\gamma'_j). \tag{3.17}$$

Now, let us prove the following theorem which contains the Gauss map of the 2-ruled hypersurface of type-1 defined in (3.15).

Theorem 3.1. *The Gauss map of the 2-ruled hypersurface of type-1 of the form (3.15) is given by*

$$\begin{aligned}G_f(u_1,u_2,u_3) &= \frac{G_1(u_1,u_2,u_3)\partial_1+G_2(u_1,u_2,u_3)\partial_2}{A}\\ &\quad+\frac{G_3(u_1,u_2,u_3)\partial_3+G_4(u_1,u_2,u_3)\partial_4}{A},\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}G_1(u_1,u_2,u_3) &= -f\Big(\beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})+\beta_4(E_{31}-E_{13})\Big)\\ &\quad+\beta_1(E_{24}-E_{42})+\beta_2(E_{41}-E_{14})+\beta_4(E_{12}-E_{21}),\\ G_2(u_1,u_2,u_3) &= -f\Big(\beta_2(E_{34}-E_{43})+\beta_3(E_{42}-E_{24})+\beta_4(E_{23}-E_{32}\Big)\\ &\quad+\beta_1(E_{32}-E_{23})+\beta_2(E_{13}-E_{31})+\beta_3(E_{21}-E_{12}),\\ G_3(u_1,u_2,u_3) &= \beta_2(E_{34}-E_{43})+\beta_3(E_{42}-E_{24})+\beta_4(E_{23}-E_{32}),\\ G_4(u_1,u_2,u_3) &= \beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})\\ &\quad+\beta_4(E_{31}-E_{13}),\end{aligned} \tag{3.19}$$

and

$$A=\sqrt{2G_1G_3+2G_2G_4+2fG_3G_4}, \tag{3.20}$$

with $G_1=G_1(u_1,u_2,u_3)$, $G_2=G_2(u_1,u_2,u_3)$, $G_3=G_3(u_1,u_2,u_3)$ *and* $G_4=G_4(u_1,u_2,u_3)$.

Proof. If we differentiate (3.15), we get:

$$\begin{cases} \varphi_{u_1}(u_1,u_2,u_3) &= \big(\alpha_1'+u_2\beta_1'+u_3\gamma_1', \alpha_2'+u_2\beta_2'+u_3\gamma_2', \\ & \quad \alpha_3'+u_2\beta_3'+u_3\gamma_3', \alpha_4'+u_2\beta_4'+u_3\gamma_4'\big) \\ \varphi_{u_2}(u_1,u_2,u_3) &= \big(\beta_1,\beta_2,\beta_3,\beta_4\big) \\ \varphi_{u_3}(u_1,u_2,u_3) &= \big(\gamma_1,\gamma_2,\gamma_3,\gamma_4\big). \end{cases}$$

By using, the vector product define in (3.5), we get:

$$\begin{aligned} \varphi_{u_1}\times_f\varphi_{u_2}\times_f\varphi_{u_3} = & \Big(-f\Big(\beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})+\beta_4(E_{31}-E_{13})\Big) \\ & +\Big(\beta_1(E_{24}-E_{42})+\beta_2(E_{41}-E_{14})+\beta_4(E_{12}-E_{21})\Big)\Big)\partial_1 \\ & +\Big(-f\Big(\beta_2(E_{34}-E_{43})+\beta_3(E_{42}-E_{24})+\beta_4(E_{23}-E_{32})\Big) \\ & +\Big(\beta_1(E_{32}-E_{23})+\beta_2(E_{13}-E_{31})+\beta_3(E_{21}-E_{12})\Big)\Big)\partial_2 \\ & +\Big(\beta_2(E_{43}-E_{34})+\beta_3(E_{24}-E_{42})+\beta_4(E_{32}-E_{23})\Big)\partial_3 \\ & +\Big(\beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})+\beta_4(E_{31}-E_{13})\Big)\partial_4. \end{aligned}$$

Now using the unit normal vector formula in (3.7), we get the result. □

From (3.8), we obtain the matrix of the first fundamental form:

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & 1 & e \\ c & e & 1 \end{bmatrix}, \tag{3.21}$$

where

$$\begin{aligned} a &= 2f(\alpha_3'+u_2\beta_3'+u_3\gamma_3')(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\ &\quad +2\sum_{i=1}^{2}(\alpha_i'+u_2\beta_i'+u_3\gamma_i')(\alpha_{i+2}'+u_2\beta_{i+2}'+u_3\gamma_{i+2}'), \\ b &= f(F_{34}-F_{43})+\sum_{i=1}^{2}(F_{i(i+2)}+F_{(i+2)i}), \\ c &= f(E_{34}-E_{43})+\sum_{i=1}^{2}(E_{i(i+2)}+E_{(i+2)i}), \\ e &= f(\beta_3\gamma_4+\beta_4\gamma_3)+\sum_{i=1}^{2}(\beta_i\gamma_{i+2}+\beta_{i+2}\gamma_i), \end{aligned} \tag{3.22}$$

and we obtain the inverse matrix $[g^{ij}]$ of $[g_{ij}]$ as:

$$[g^{ij}] = \frac{1}{\det[g_{ij}]}\begin{bmatrix} 1-e^2 & ce-b & be-c \\ ce-b & a-c^2 & bc-ae \\ be-c & bc-ae & a-b^2 \end{bmatrix}, \tag{3.23}$$

where

$$\det[g_{ij}] = -b^2 + 2cbe - c^2 - ae^2 + a = B. \tag{3.24}$$

Furthermore, from (3.9), the matrix form of the second fundamental from of the 2-ruled hypersurface (3.15) is obtained by

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{3.25}$$

where

$$\begin{aligned}
h_{11} &= \frac{f_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4')}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{\sum_{i=1}^{2} G_{i+2}(\alpha_{i''} + u_2\beta_{i''} + u_3\gamma_{i''})}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\
h_{12} = h_{21} &= \frac{f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' - u_3\gamma_4')}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{\sum_{i=1}^{2} G_{i+2}\beta_i'}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
h_{13} = h_{31} &= \frac{f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4')}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{\sum_{i=1}^{2} G_{i+2}\gamma_i'}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
h_{22} &= \frac{f_3\beta_3^2G_4 + f_4\beta_4^2G_3}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\
h_{33} &= \frac{f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\
h_{23} = h_{32} &= \frac{f_3\beta_3\gamma_3G_4}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{f_4\beta_4\gamma_4G_3}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}.
\end{aligned} \tag{3.26}$$

We can see easily that the $\det[h_{ij}] = h_{11}h_{22}h_{33} + 2h_{12}h_{13}h_{23} - h_{12}^2h_{33} - h_{13}^2h_{22} - h_{23}^2h_{11} \neq 0$.

Then we can give the following theorem by using (3.11)

Theorem 3.2. *The 2-ruled hypersurface of type-1 defined in (3.15) is no flat.*

Corollary 3.1. *The 2-ruled hypersurface of type-1 defined in (3.15) is flat if f is nonzero constant.*

Proof. From (3.9), the matrix of second fundamental form of the 2-ruled hypersurface (3.15) is obtained by

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & 0 & 0 \\ h_{31} & 0 & 0 \end{bmatrix}, \tag{3.27}$$

where

$$\begin{aligned} h_{11} &= \frac{\sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')}{\sqrt{\sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\ h_{12} &= h_{21} = \frac{\sum_{i=1}^{2} G_{i+2}\beta_i'}{\sqrt{\sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\ h_{13} &= h_{31}\frac{\sum_{i=2}^{2} G_{i+2}\gamma_i'}{\sqrt{\sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\ h_{22} &= h_{23} = h_{33} = 0. \end{aligned} \tag{3.28}$$

So we have $\det(h_{ij}) = 0$, hence $G_f = 0$. □

Now, we will prove the following theorem about the mean curvature.

Theorem 3.3. *The 2-ruled hypersurface of type-1 defined in (3.15) is minimal, if*

$$\begin{aligned} 0 = {} & (1-e^2)\Big[f_3G_4(\alpha_3' + u_2\beta_2' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\ & + 2(ce-b)\Big[f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\ & + 2(be-c)\Big[f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\ & + 2(bc-ae)\Big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\Big] + (a-c^2)\Big[f_3\beta_3^2G_4 + f_4\beta_4^2G_3\Big] \\ & + (a-b^2)\Big[f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3\Big]. \end{aligned} \tag{3.29}$$

Proof. By (3.10), the matrix of the shape operator is

$$S = \begin{bmatrix} 1-e^2 & ce-b & be-c \\ ce-b & a-c^2 & bc-ae \\ be-c & bc-ae & a-b^2 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix},$$

where $h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}$ are the same in (3.26). Then we get the coefficients of S by

$$\begin{aligned} S_{11} &= (1-e^2)h_{11} + (ce-b)h_{12} + (be-c)h_{13} \\ S_{22} &= (ce-b)h_{12} + (a-c^2)h_{22} + (bc-ae)h_{23} \\ S_{33} &= (be-c)h_{13} + (bc-ae)h_{23} + (a-b^2)h_{33}. \end{aligned}$$

And using (3.26) and (3.12), we see that the 2-ruled hypersurface is minimal if

$$S_{11} + S_{22} + S_{33} = 0.$$

Then, the proof is complete. □

Corollary 3.2. *If the curves β and γ are orthogonal, then the 2-ruled hypersurface of type-1 defined in (3.15) is minimal if*

$$\begin{aligned} 0 = \; & \Big[f_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\ & -2b\Big[f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_4') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' + \gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\ & -2c\Big[f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\ & +2bc\Big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\Big] + (a-c^2)\Big[f_3\beta_3^2G_4 + f_4\gamma_4^2G_3\Big] \\ & +(a-b^2)\Big[f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3\Big]. \end{aligned} \tag{3.30}$$

The Laplace-Beltrami operator of a smooth function $\varphi = \varphi(u_1, u_2, u_3)$ of class C^3 with respect to the first fundamental form of a hypersurface is defined as follows:

$$\Delta\varphi = \frac{1}{\sqrt{\det[g_{ij}]}} \sum_{i,j}^{3} \frac{\partial}{\partial u_i}\left(\sqrt{\det[g_{ij}]}\, g^{ij} \frac{\partial \varphi}{\partial u_j}\right). \tag{3.31}$$

Using (3.31), we get the Laplace-Beltrami operator of the 2-ruled hypersurface of type-1 (3.15) by

$$\Delta\varphi = (\Delta\varphi_1, \Delta\varphi_2, \Delta\varphi_3, \Delta\varphi_4),$$

where

$$\begin{aligned}\Delta\varphi_i \;=\; & \frac{1}{\sqrt{B}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{(1-e^2)\varphi_{iu_1} + (ce-b)\varphi_{iu_2} + (be-c)\varphi_{iu_3}}{\sqrt{\det[g_{ij}]}}\Big) \\ & + \frac{\partial}{\partial u_2}\Big(\frac{(ce-b)\varphi_{iu_1} + (a-c^2)\varphi_{iu_2} + (bc-ae)\varphi_{iu_3}}{\sqrt{\det[g_{ij}]}}\Big) \\ & + \frac{\partial}{\partial u_3}\Big(\frac{(be-c)\varphi_{iu_1} + (bc-ae)\varphi_{iu_2} + (a-b^2)\varphi_{iu_3}}{\sqrt{\det[g_{ij}]}}\Big)\Big]. \qquad (3.32)\end{aligned}$$

That is

$$\begin{aligned}\Delta\varphi_i \;=\; & \frac{1}{\sqrt{B}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{(1-e^2)(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (ce-b)\beta_i + (be-c)\gamma_i}{\sqrt{\det[g_{ij}]}}\Big) \\ & + \frac{\partial}{\partial u_2}\Big(\frac{(ce-b)(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (a-c^2)\beta_i + (bc-ae)\gamma_i}{\sqrt{\det[g_{ij}]}}\Big) \\ & + \frac{\partial}{\partial u_3}\Big(\frac{(be-c)(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (bc-ae)\beta_i + (a-b^2)\gamma_i}{\sqrt{\det[g_{ij}]}}\Big)\Big]. \qquad (3.33)\end{aligned}$$

If we suppose that β and γ are orthogonal, then the Laplace-Beltrami operator of the 2-ruled hypersuface of type-1 (3.15) is given by

$$\begin{aligned}\Delta\varphi_i \;=\; & \frac{1}{\sqrt{a-b^2-c^2}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{(\alpha_i' + u_2\beta_i' + u_3\gamma_i') - b\beta_i - c\gamma_i}{\sqrt{a-b^2-c^2}}\Big) \\ & + \frac{\partial}{\partial u_2}\Big(\frac{-b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (a-c^2)\beta_i + bc\gamma_i}{\sqrt{a-b^2-c^2}}\Big) \\ & + \frac{\partial}{\partial u_3}\Big(\frac{-c(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + bc\beta_i + (a-b^2)\gamma_i}{\sqrt{a-b^2-c^2}}\Big)\Big]. \qquad (3.34)\end{aligned}$$

Theorem 3.4. *The components of the Laplace-Beltrami operator of the* 2*-ruled hypersurface of type-*1 *defined in (3.15) are*

$$\begin{aligned}\Delta\varphi_i \;=\; & \frac{1}{W^{\frac{3}{2}}\sqrt{W}}\Big[(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'' - (b\beta_i)_{u_1} - (c\gamma_i)_{u_1})W \\ & - V_1(\alpha_i' + u_2\beta_i' + u_2\gamma_i' - b\beta_i - c\gamma_i) \\ & + (-b\beta_i' + ((a-c^2)\beta_i)_{u_2} + (bc\gamma_i)_{u_2})W - V_2(-b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\ & + (a-c^2)\beta_i + bc\gamma_i) \\ & + (-c\gamma_i' + (bc\beta_i)_{u_3} + ((a-b^2)\gamma_i)_{u_3})W - V_3(-c(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\ & + bc\beta_i + (a-b^2)\gamma_i)\Big], \qquad (3.35)\end{aligned}$$

where $i = 1,2,3,4$*;* β *and* γ *are orthogonal;* $W = a - b^2 - c^2$, $V_1 = a_{u_1} - 2bb_{u_1} - 2cc_{u_1}$, $V_2 = a_{u_2} - 2bb_{u_2} - 2cc_{u_2}$, $V_3 = a_{u_3} - 2bb_{u_3} - 2cc_{u_3}$.

3.2. 2-Ruled hypersurfaces of type-2 in M

A 2-ruled hypersurface of type-2 in M means (the image of) a map $\varphi : I_1 \times I_2 \times I_3 \to M$ of the form

$$\varphi(u_1, u_2, u_3) = \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1), \tag{3.36}$$

where $\alpha : I_1 \to M$, $\beta : I_2 \to \mathbb{H}^3_+(-1)$, $\gamma : I_3 \to \mathbb{H}^3_+(-1)$ are smooth maps, $\mathbb{H}^3_+(-1)$ is the anti-de Sitter space 3-space of M and I_1, I_2, I_3 are open intervals. We call α a base curve, β and γ director curves. The planes $(u_2, u_3) \mapsto \alpha(u_1)+u_2\beta(u_1)+u_3\gamma(u_1)$ are called rulings. So, if we take

$$\begin{cases} \alpha(u_1) &= \big(\alpha_1(u_1), \alpha_2(u_1), \alpha_3(u_1), \alpha_4(u_1)\big) \\ \beta(u_1) &= \big(\beta_1(u_1), \beta_2(u_1), \beta_3(u_1), \beta_4(u_1)\big) \\ \gamma(u_1) &= \big(\gamma_1(u_1), \gamma_2(u_1), \gamma_3(u_1), \gamma_4(u_1)\big) \end{cases} \tag{3.37}$$

in (3.36), then we can write the 2-ruled hypersurface of type-2 as

$$\varphi(u_1, u_2, u_3) = \begin{pmatrix} \alpha_1(u_1) + u_2\beta_1(u_1) + u_3\gamma_1(u_1) \\ \alpha_2(u_1) + u_2\beta_2(u_1) + u_3\gamma_2(u_1) \\ \alpha_3(u_1) + u_2\beta_3(u_1) + u_2\gamma_3(u_1) \\ \alpha_4(u_1) + u_2\beta_4(u_1) + u_3\gamma_4(u_1) \end{pmatrix}. \tag{3.38}$$

We see that $\langle\beta_i, \beta_i\rangle = \langle\gamma_i, \gamma_i\rangle = -1$ and we state $\alpha_i = \alpha_i(u_1)$, $\beta_i = \beta_i(u_1)$, $\gamma_i = \gamma_i(u_1)$, $\varphi_i = \varphi_i(u_1, u_2, u_3)$, $f' = \frac{\partial f(u_1)}{\partial u_1}$, $f'' = \frac{\partial^2 f(u_1)}{\partial u_1 \partial u_1}$, $i \in \{1, 2, 3, 4\}$ and $f \in \{\alpha, \beta, \gamma\}$.

From (3.8), we obtain the matrix of the first fundamental form

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & -1 & e \\ c & e & -1 \end{bmatrix}. \tag{3.39}$$

And we obtain the inverse matrix $[g^{ij}]$ of $[g_{ij}]$ as

$$[g^{ij}] = \frac{1}{\det[g_{ij}]} \begin{bmatrix} 1-e^2 & ce+b & be+c \\ ce+b & -a-c^2 & bc-ae \\ be+c & bc-ae & -a-b^2 \end{bmatrix}. \tag{3.40}$$

where a, b, c and e are the same in (3.22) and

$$\det[g_{ij}] = b^2 + 2cbe + c^2 - ae^2 + a = C. \tag{3.41}$$

Furthermore, from (3.9), the matrix form of the second fundamental from of the 2-ruled hypersurface (3.38) is the same given in (3.25) and (3.26). And we have the following theorem since the $\det[h_{ij}] \neq 0$.

Theorem 3.5. *The 2-ruled hypersurface of type-2 defined in (3.38) is not flat.*

Corollary 3.3. *The* 2*-ruled hypersurface of type-*2 *defined in (3.38) is flat if* f *is nonzero constant.*

For the mean curvature we have:

Theorem 3.6. *The* 2*-ruled hypersurface of type-*2 *defined in (3.38) is minimal in* M*, if*

$$\begin{aligned}
0 &= (1-e^2)\Big[f_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4G_3(\alpha_4'+u_2\gamma_4'+u_3\gamma_4') \\
&\quad +\sum_{i=1}^{2}G_{i+2}(\alpha_i''+u_2\beta_i''+u_3\gamma_i'')\Big] \\
&\quad +2(ce+b)\Big[f_3\beta_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4\beta_4G_3(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\
&\quad +\sum_{i=1}^{2}G_{i+2}\beta_i'\Big] \\
&\quad +2(be+c)\Big[f_3\gamma_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4\gamma_4G_3(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\
&\quad +\sum_{i=1}^{2}G_{i+2}\gamma_i'\Big] \\
&\quad +2(bc-ae)\Big[f_3\beta_3\gamma_3G_4+f_4\beta_4\gamma_4G_3\Big]+(-a-c^2)\Big[f_3\beta_3^2G_4+f_4\beta_4^2G_3\Big] \\
&\quad +(-a-b^2)\Big[f_3\gamma_3^2G_4+f_4\gamma_4^2G_3\Big]. \qquad (3.42)
\end{aligned}$$

Proof. By (3.10), the matrix of the shape operator is

$$S=\begin{bmatrix} 1-e^2 & ce+b & be+c \\ ce+b & -a-c^2 & bc-ae \\ be+c & bc-ae & -a-b^2 \end{bmatrix}\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix},$$

where $h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}$ are the same in (3.26). Then, we get the coefficients of S by

$$\begin{aligned}
S_{11} &= (1-e^2)h_{11}+(ce+b)h_{12}+(be+c)h_{13}, \\
S_{22} &= (ce+b)h_{12}+(-a-c^2)h_{22}+(bc-ae)h_{23}, \\
S_{33} &= (be+c)h_{13}+(bc-ae)h_{23}+(-a-b^2)h_{33}.
\end{aligned}$$

And using (3.26) and (3.12), we see that the 2-ruled hypersurface of type-2 defined in (3.38) is minimal if

$$S_{11}+S_{22}+S_{33}=0.$$

Then, the proof is complete. □

Corollary 3.4. *If the curves β and γ are orthogonal, then the 2-ruled hypersurface of type-2 defined in (3.38) is minimal if*

$$
\begin{aligned}
0 \;=\;& \Big[f_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\
&+\sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\
&+2b\Big[f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\
&+\sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\
&+2c\Big[f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_2' + u_3\gamma_4') \\
&+\sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\
&+2bc\Big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\Big] + (-a-c^2)\Big[f_3\beta_3^2G_4 + f_4\gamma_4^2G_3\Big] \\
&+(-a-b^2)\Big[f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3\Big]. \qquad (3.43)
\end{aligned}
$$

To end this subsection, we will give the operator of Laplace-Beltrami in the following theorem:

Theorem 3.7. *The components of the Laplace-Beltrami operator of the 2-ruled hypersurface of type-2 defined in (3.38) are*

$$
\begin{aligned}
\Delta\varphi_i \;=\;& \frac{1}{R^{\frac{3}{2}}\sqrt{T}}\Big[(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'') + (b\beta_i)_{u_1} + (c\gamma_i)_{u_1})T \\
&-R_1(\alpha_i' + u_2\beta_i' + u_3\gamma_i' + b\beta_i + c\gamma_i)\Big] \\
&+(b\beta_i' + ((-a-c^2)\beta_i)_{u_2} + (bc\gamma_i)_{u_2})T - R_2(b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\
&+(-a-c^2)\beta_i + bc\gamma_i) \\
&+(c\gamma_i' + (bc\beta_i)_{u_3} + ((-a-b^2)\gamma_i)_{u_3})T - R_3(c(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\
&+bc\beta_i + (-a-b^2)\gamma_i)\Big], \qquad (3.44)
\end{aligned}
$$

where $i = 1, 2, 3, 4$; β and γ are orthogonal; $T = a+b^2+c^2$, $R_1 = a_{u_1}+2bb_{u_2}+2cc_{u_1}$, $R_2 = a_{u_2} + 2bb_{u_2} + 2cc_{u_2}$, $R_3 = a_{u_3} + 2bb_{u_3} + 2cc_{u_3}$.

3.3. 2-Ruled hypersurfaces of type-3 in M

A 2-ruled hypersurface of type-3 in M means (the image of) a map $\varphi : I_1\times I_2\times I_3 \to M$ of the form

$$\varphi(u_1, u_2, u_3) = \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1), \qquad (3.45)$$

where $\alpha : I_1 \to M$, $\beta : I_2 \to \mathcal{LC}$, $\gamma : I_3 \to \mathcal{LC}$ are smooth maps, $\mathcal{LC}$ is the light cone of M and I_1, I_2, I_3 are open intervals. We call α a base curve, β and γ director curves. The planes $(u_2, u_3) \mapsto \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1)$ are called rulings. So, if we take

$$\left\{ \begin{array}{lcl} \alpha(u_1) & = & \big(\alpha_1(u_1), \alpha_2(u_1), \alpha_3(u_1), \alpha_4(u_1)\big), \\ \beta(u_1) & = & \big(\beta_1(u_1), \beta_2(u_1), \beta_3(u_1), \beta_4(u_1)\big), \\ \gamma(u_1) & = & \big(\gamma_1(u_1), \gamma_2(u_1), \gamma_3(u_1), \gamma_4(u_1)\big), \end{array} \right. \tag{3.46}$$

in (3.45), then we can write the 2-ruled hypersurface of type-3 as

$$\varphi(u_1, u_2, u_3) = \begin{pmatrix} \alpha_1(u_1) + u_2\beta_1(u_1) + u_3\gamma_1(u_1) \\ \alpha_2(u_1) + u_2\beta_2(u_1) + u_3\gamma_2(u_1) \\ \alpha_3(u_1) + u_2\beta_3(u_1) + u_2\gamma_3(u_1) \\ \alpha_4(u_1) + u_2\beta_4(u_1) + u_3\gamma_4(u_1) \end{pmatrix}. \tag{3.47}$$

We see that $\langle \beta_i, \beta_i \rangle = \langle \gamma_i, \gamma_i \rangle = -1$ and we state $\alpha_i = \alpha_i(u_1)$, $\beta_i = \beta_i(u_1)$, $\gamma_i = \gamma_i(u_1)$, $\varphi_i = \varphi_i(u_1, u_2, u_3)$, $f' = \frac{\partial f(u_1)}{\partial u_1}$, $f'' = \frac{\partial^2 f(u_1)}{\partial u_1 \partial u_1}$, $i \in \{1, 2, 3, 4\}$ and $f \in \{\alpha, \beta, \gamma\}$.

From (3.8), we obtain the matrix of the first fundamental form

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & 0 & e \\ c & e & 0 \end{bmatrix}. \tag{3.48}$$

And we obtain the inverse matrix $[g^{ij}]$ of $[g_{ij}]$ as

$$[g^{ij}] = \frac{1}{\det[g_{ij}]} \begin{bmatrix} -e^2 & ce & be \\ ce & -c^2 & bc - ae \\ be & bc - ae & -b^2 \end{bmatrix}, \tag{3.49}$$

where a, b, c and e are the same in (3.22) and

$$\det[g_{ij}] = 2bce - ae^2 = D. \tag{3.50}$$

Furthermore, from (3.9), the matrix form of the second fundamental from of the 2-ruled hypersurface (3.47) is the same given in (3.25) and (3.26). And we have the following theorem since the $\det[h_{ij}] \neq 0$.

Theorem 3.8. *The 2-ruled hypersurface of type-3 defined in (3.47) is no flat.*

Corollary 3.5. *The 2-ruled hypersurface of type-3 is flat if f is non zero constant.*

For the mean curvature, we have:

Theorem 3.9. *The* 2*-ruled hypersurface of type*-3 *defined in (3.47) is minimal in* M*, if*

$$\begin{aligned} 0 \quad = \quad & -e^2\Big[f_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\gamma_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\ & +2ce\Big[f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\ & +2be\Big[f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\ & +2(bc-ae)\Big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\Big] - c^2\Big[f_3\beta_3^2G_4 + f_4\beta_4^2G_3\Big] \\ & -b^2\Big[h_3\gamma_3^2G_4 + h_4\gamma_4^2G_3\Big]. \end{aligned} \tag{3.51}$$

Proof. By (3.10) the matrix of the shape operator is

$$S = \begin{bmatrix} -e^2 & ce & be \\ ce & -c^2 & bc-ae \\ be & bc-ae & -b^2 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix},$$

where $h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}$ are the same in (3.26). Then we get the coefficients of S by

$$\begin{aligned} S_{11} &= -e^2h_{11} + ceh_{12} + beh_{13}, \\ S_{22} &= ceh_{12} - c^2h_{22} + (bc-ae)h_{23}, \\ S_{33} &= be + h_{13} + (bc-ae)h_{23} - b^2h_{33}. \end{aligned}$$

And using (3.26) and (3.12), we see that the 2-ruled hypersurface of type-3 defined in (3.47) is minimal if

$$S_{11} + S_{22} + S_{33} = 0.$$

Then, the proof is complete. □

Corollary 3.6. *If the curves* β *and* γ *are orthogonal, then the* 3*-ruled hypersurface of type*-3 *defined in (3.47) is minimal if*

$$\begin{aligned} 0 \quad = \quad & 2bc\big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\big] \\ & -c^2\big[f_3\beta_3^2G_4 + h_4\gamma_4^2G_3\big] \\ & -b^2\big[f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3\big]. \end{aligned} \tag{3.52}$$

To end this subsection, we will give the operator of Laplace-Beltrami in the following theorem:

Theorem 3.10. *The components of the Laplace-Beltrami operator of the* 2*-ruled hypersurface of type-*3 *defined in (3.47) are:*

$$\begin{aligned}
\Delta\varphi_i &= \frac{1}{L^{\frac{3}{2}}\sqrt{L}}\Big[(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'') + (b\beta_i)_{u_1} + (c\gamma_i)_{u_1})L \\
&\quad -J_1(\alpha_i' + u_2\beta_i' + u_3\gamma_i' + b\beta_i + c\gamma_i) \\
&\quad +(b\beta_i' + ((-a-c^2)\beta_i)_{u_2} + (bc\gamma_i)_{u_2})L - J_2(b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\
&\quad +(-a-c^2)\beta_i + bc\gamma_i) \\
&\quad +(c\gamma_i' + (bc\beta_i)_{u_3} + ((-a-b^2)\gamma_i)_{u_3})L - J_3(c(\alpha_i' + y\beta_i' + z\gamma_i') \\
&\quad +bc\beta_i + (-a-b^2)\gamma_i)\Big],
\end{aligned} \tag{3.53}$$

where $i = 1,2,3,4;\ ;\ L = 2bce - ae^2 2,\ J_1 = 2b_{u_1}ce + 2bc_{u_1}e + 2bce_{u_1} - 2aa_{u_1},$ $J_2 = 2b_{u_2}ce + 2bc_{u_2}e + 2bce_{u_2} - 2aa_{u_2},\ J_3 = 2b_{u_3}ce + 2bc_{u_3}e + 2bce_{u_3} - 2aa_{u_3}.$

Note that the hypersurfaces constructed in this paper are not flats. Unlike Euclidean and Minkowskian spaces, where the ruled hypersurfaces are flats.

4. Conclusion

We end this work by giving some applications of ruled surfaces and 2-ruled hypersurfaces as generalisations of the first one. Ruled surfaces have been applied in different areas such as CAD, electric discharge machining [1, 24]. The authors [23] present an elementary introduction to the theory of Betrand pairs of curves and ruled surfaces. Bertrand pairs of ruled surfaces are introduced as offsets in the context of line geometry. Also, the ruled surfaces has an important application area on kinematics [1, 22]. Additionally, ruled surfaces have an important application area in architecture [1]. For lightweight structures in the field of architecture and civil engineering, concrete shells with negative Gaussian curvature are frequently used. One class of such surfaces are the skew ruled surfaces [20].

Acknowledgments

The authors would like to thank the referee for his/her valuable suggestions and comments that helped them improve the paper.

REFERENCES

1. M. Altin, A. Kazan and D. W. Yoon: 2-*Ruled hypersurfaces in Euclidean* 4*-space.* J. Geom. Phys. **166** (2021), 104236, 13 pp.
2. S. Aslan, M. Bekar and Y. Yayli: *Ruled surfaces constructed by quaternions.* J. Geom. Phys. **161** (2021), 104048, 9pp.

3. S. Aslan, M. Bekar and Y. Yayli: *Ruled surfaces in Minkowski* 3*-spaces and split quaternion operators*. Adv. Appl. Clifford Algebras. **31** (2021), (5), 74, 14pp.
4. J. L. M. Barbosa and J. A. Delgado: *Ruled submanifolds of space forms with mean curvature of nonzero constant length*. Amer. J. Math. **106** (1984), 763-780.
5. J. L. M. Barbosa, L. P. M. Jorge and M. Dajczer: *Minimal ruled submanifolds of spaces of constant curvature*. Indiana Univ. Math. J. **33** (1984), 531-547.
6. E. Boeckx, O. Kowalski and L. Vanhecke: *Riemannian manifolds of conullity two*. World Scientific Publ., Singapore, 1996.
7. M. P. do Carmo: *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.
8. M. Chaichi, E. García-Río and Y. Matsushita: *Curvature properties of four-dimensional Walker metrics*. Classical Quantum Gravity **22** (2005), 559-577.
9. F. Dillen and W. Kühnel: *Ruled Weingarten surfaces in Minkowski* 3*-space*. Manuscripta Math. **98** (1999) 307-320.
10. B. Divjak and Z. Milin-Sipus: *Special curves on ruled surfaces in Galilean and presudo-Galilean spaces*. Acta Math. Hungar. **98** (2003), (3), 203-215.
11. M. Falcitelli, S. Ianus and A. M. Pastore: *Riemannian submersions and related topics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2004.
12. S. Flöry and H. Pottmann: *Ruled surfaces for rationalization and design in architecture*. In: ACADIA 10: LIFE Information, on Responsive Information and Variations in Architecture, 2010, pp. 103-109
13. E. Güler, H. H. Hacisalihoğlu and Y. H. Kim: *The Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-space*. Symmetry **10** (2018), (9), 1-11.
14. Y. H. Kim D. W. Yoon: *Classification of ruled surfaces in Minkowski* 3*-spaces*. J. Geom. Phys. **49** (2004), 89-100.
15. M. Kimura, H. Lee, J. de D. Pérez and Y. J. Suh: *Ruled real hypersurfaces in the complex quadric*. J. Geom. Anal. **31** (2021), (8), 7989-8012.
16. M. Kimura, S. Maeda and H. Tanabe: *New construction of ruled real hypersurfaces in a complex hyperbolic space and its applications*. Geom. Dedicata **207** (2020), 227-242.
17. A. Ndiaye and Z. Özdemir: 2*-ruled hypersurfaces in Minkowski* 4*-space and their constructions via octonions*. (2022), https://doi.org/10.48550/arXiv.2205.10808
18. A. Niang: *Surfaces minimales réglées dans l'espace de Minkowski ou Euclidien orienté de dimension* 3. Afrika Mat. **15** (2003), (3), 117-127.
19. A. Niang, A. Ndiaye and A. S. Diallo: *A classification of strict Walker* 3*-manifold*. Konuralp J. Math. **9** (2021), (1), 148-153.
20. K. Noack and D. Lordick: *Optimized ruled surfaces with an application to thin-walled concrete shells*. Cocchiarella, Luigi (ed.), ICGG 2018 - Proceedings of the 18th international conference on geometry and graphics. 40th anniversary - Milan, Italy, August 3-7, 2018. In 2 volumes. Cham: Springer; Milan: Politecnico de Milano (ISBN 978-3-319-95587-2/pbk; 978-3-319-95588-9/ebook). Advances in Intelligent Systems and Computing 809, (2019), pp. 338-349.
21. K. Nomizu and T. Sasaki: *Affine Differential Geometry. Geometry of Affine Immersions*. Cambridge University Press. Vol. 111. 1994.

22. H. Pottmann and J. Wallner: *Computational line geometry. Mathematics and Visualization*. Berlin: Springer. (2001).

23. B. Ravani and T. S. Ku: *Bertrand offsets of ruled and developable surfaces*. Comput. -Aided Des. **23** (1991), (2), 145-152 .

24. B. Ravani and J. W. Wan: *Computer aided eometric desin of line constructs*. Trans. ASME, J. Mec. Des. **113** (1991), (4), 363-371.

25. K. Saji, *Singularities of non-degenerate* 2*-ruled hypersurfaces in* 4*-space*, Hiroshima Math. J. **32** (2002), 309-323.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 87–98
https://doi.org/10.22190/FUMI230202006A
Original Scientific Paper

GENERALIZED η-RICCI SOLITONS ON TRANS-SASAKIAN MANIFOLDS ASSOCIATED TO THE SCHOUTEN-VAN KAMPEN CONNECTION

Shahroud Azami

Faculty of Science, Department of Pure Mathematics
Imam Khomeini International University, Qazvin, Iran

ORCID ID: Shahroud Azami https://orcid.org/0000-0002-8976-2014

Abstract. In this paper, we study generalized η-Ricci solitons with respect to the Schouten-van Kampen connection on trans-Sasakian manifolds. We give an example of generalized η-Ricci solitons on a trans-Sasakian manifold with respect to the Schouten-van Kampen connection to prove our results.
Keywords: manifolds, vector field, generalized Ricci solutions.

1. Introduction

The trans-Sasakian manifold was introduced by Oubina [37] as a class of almost contact metric manifolds. Later, Blair and Oubina [10] obtaned some properties of this manifolds. A trans-Sasakian manifold is usually denoted by $(M, \varphi, \xi, \eta, g, \sigma, \theta)$, where both σ and θ are smooth functions on M and (φ, ξ, η, g) is an almost contact metric structure. In this case, it is said to be of type (σ, θ). A trans-Sasakian manifold of type $(0, 0)$, $(0, \theta)$ and $(\sigma, 0)$ are cosymplectic, θ-Kenmotsu [1, 30, 36, 48] and σ-Sasakian [31], respectively. In [18, 19, 20, 21, 22, 23, 28, 34, 35, 49], the authors studied compact trans-Sasakian manifolds with some restrictions on the smooth functions σ, θ and the vector field ξ appearing in their definition for getting conditions under which a trans-Sasakian manifold is homothetic to a Sasakian manifold. In addition, in [43, 44, 49], interesting results on the geometry of trans-Sasakian manifolds are obtained.

Received February 02, 2023, accepted: April 01, 2023
Communicated by Mića Stanković
Corresponding Author: Shahroud Azami. E-mail: azami@sci.ikiu.ac.ir
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53D15, 53E20

Hamilton [25] introduced the concept of Ricci flow on a Riemannian manifold as follows:

$$\frac{\partial}{\partial t} g = -2S$$

where S is the Ricci tensor of a manifold. A self-similar solution to the Ricci flow is called a Ricci soliton which is a generalization of Einstein metric. A Ricci soliton [25] is a triplet (g, V, λ) on a pseudo-Riemannian manifold M such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\mathcal{L}_V$ is the Lie derivative in direction of the potential vector field V, S is the Ricci tensor, and λ is a real constant. Ricci solitons are important in physics and are often referred as quasi-Einstein [12, 13]. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively. If the vector field V is the gradient of a potential function ψ, that is, $V = \nabla\psi$, then g is called a gradient Ricci soliton. In 2016, Nurowski and Randall [33] introduced the concept of generalized Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat - 2\alpha S - 2\lambda g = 0, \tag{1.2}$$

where $V^\flat$ is the canonical 1-form associated to V. Also, as a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [16] which it is a 4-tuple (g, V, λ, ρ), where V is a vector field on M, λ and ρ are constants, and g is a pseudo-Riemannian metric satisfying the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. Many authors studied the η-Ricci solitons [5, 6, 7, 26, 29, 38, 42]. In particular, if $\rho = 0$, then the η-Ricci soliton equation becomes the Ricci soliton equation. Motivated by the above works M. D. Siddiqi [40] introduced the notion of generalized η-Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0. \tag{1.4}$$

Motivated by [2, 3, 11, 32] and the above works, we study generalized η-Ricci solitons on trans-Sasakian manifolds assoicated to the Schouten-van Kampen connection. We give an example of generalized η-Ricci soliton on a trans-Sasakian manifold with respect to the Schouten-van Kampen connection.

The paper is organized as follows. In Section 2, we recall some necessary and fundamental concepts and formulas on trans-Sasakian manifolds which be used throughout the paper. In Section 3, we give the main results and their proofs. In Section 4, we give an example of trans-Sasakian admitting the generalized η-Ricci solitons with respect to the Schouten-van Kampen connection.

2. Preliminaries

Let M be a $(2n + 1)$-dimensional manifold, φ be a $(1, 1)$-tensor field, ξ be a vector field, η be a 1-form, and g be a compatible Riemannian metric on M such that

$$\varphi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y. Then manifold (M, g) is called an almost contact metric manifold [8, 9] with an almost contact structure (φ, ξ, η, g). In this case, we have $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $g(X, \varphi Y) = -g(\varphi X, Y)$, and $\eta(X) = g(X, \xi)$. The fundamental 2-form Φ of M is given by

$$\Phi(X, Y) = g(X, \varphi Y),$$

for all vector fields X, Y. An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called trans-Sasakian manifold [37] if $(M \times \mathbb{R}, J, G)$ belong to the class W_4 [24], where J is the almost complex structure on $M \times \mathbb{R}$ given by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

for all vector field X on M, smooth function f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [10]

$$(\nabla_X \varphi)Y = \sigma\left(g(X, Y)\xi - \eta(Y)X\right) + \theta\left(g(\varphi X, Y)\xi - \eta(Y)\varphi X\right), \tag{2.3}$$

for all vector fields X, Y, for some smooth functions σ, θ on M. In this case, we say that the trans-Sasakian structure is of type (σ, θ). By virtue of (2.3), we have

$$\nabla_X \xi = -\sigma\varphi X + \theta(X - \eta(X)\xi), \tag{2.4}$$

$$(\nabla_X \eta)Y = -\sigma g(\varphi X, Y) + \theta g(\varphi X, \varphi Y), \tag{2.5}$$

for all vector fields X, Y. Using (2.3) and (2.4), we have

$$2\sigma\theta + \xi(\sigma) = 0, \tag{2.6}$$

$$\varphi(\nabla\sigma) = 2n\nabla\theta. \tag{2.7}$$

Further, we have the following relations [17]

$$\begin{aligned}
(2.8)\quad R(X, Y)\xi &= (\sigma^2 - \theta^2)(\eta(Y)X - \eta(X)Y) + 2\sigma\theta(\eta(Y)\varphi X - \eta(X)\varphi Y) \\
&\quad + (Y(\sigma))\varphi X - (X(\sigma))\varphi Y + (Y(\theta))\varphi^2 X - (X(\theta))\varphi^2 Y, \\
(2.9)\quad R(X, \xi)\xi &= (\sigma^2 - \theta^2 - \xi(\theta))\{X - \eta(X)\xi\}, \\
(2.10)\quad R(\xi, X)Y &= (\sigma^2 - \theta^2)\{g(X, Y)\xi - \eta(Y)X\} + (Y(\theta))\{X - \eta(X)\xi\} \\
&\quad + 2\sigma\theta\{g(\varphi Y, X)\xi + \eta(Y)\varphi X) + (Y(\sigma))\varphi X \\
&\quad + g(\varphi Y, X)\nabla\sigma - g(\varphi X, \varphi Y)\nabla\theta,
\end{aligned}$$

for all vector fields X, Y, where R is the Riemannian curvature tensor. From (2.8) and definition of the Ricci tensor S of a trans-Sasakian manifold M we also get

$$S(X, \xi) = \left(2n(\sigma^2 - \theta^2) - \xi(\theta)\right)\eta(X) - (2n - 1)X(\theta) - (\varphi X)\sigma, \tag{2.11}$$

for all vector field X.

Let M be an almost contact metric manifold and TM be the tangent bundle of M. We have two naturally defined distribution on tangent bundle TM as follows

$$H = \ker\eta, \qquad \hat{H} = \mathrm{span}\{\xi\}, \tag{2.12}$$

thus we get $TM = H \oplus \hat{H}$. Therefore, by this composition we can define the Schouten-van Kampen connection $\bar{\nabla}$ [4, 41] on M with respect to Levi-Civita connection ∇ as follows

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + ((\nabla_X \eta)(Y))\xi \tag{2.13}$$

for all vector fields X, Y. From [41] we have

$$\bar{\nabla}\xi = 0, \qquad \bar{\nabla}g = 0, \qquad \bar{\nabla}\eta = 0, \tag{2.14}$$

and the torsion $\bar{T}$ of $\bar{\nabla}$ is given by

$$\bar{T}(X, Y) = \eta(X)\nabla_X \xi - \eta(X)\nabla_Y \xi + 2d\eta(X, Y)\xi, \tag{2.15}$$

for all vector fields X, Y. Let $\bar{R}$ and $\bar{S}$ be the curvature tensors and the Ricci tensors of the connection $\bar{\nabla}$, respectively. From [27] on a trans-Sasakian we have

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma\{\eta(Y)\varphi X - g(\varphi X, Y)\xi\} - \theta\{\eta(Y)X - g(X, Y)\xi\} \tag{2.16}$$

and

$$\begin{aligned} \bar{S}(X, Y) &= S(X, Y) - (2n-2)\sigma\theta g(\varphi X, Y) + \{\xi(\theta) + 2n\theta^2\}g(X, Y) \\ &\quad -2\sigma^2\eta(X)\eta(Y) + \{(\varphi X)\sigma + (2n-1)(X\theta)\}\eta(Y), \end{aligned} \tag{2.17}$$

for all vector fields X, Y, where S denotes the Ricci tensor of the connection ∇. Hence,

$$\bar{S}(X, \xi) = 0, \tag{2.18}$$

$$\bar{S}(\xi, X) = (2n-1)\{(\xi\theta)\eta(X) - X\theta\} - (\phi X)\sigma. \tag{2.19}$$

From (2.16), we get

$$\begin{aligned} \overline{\mathcal{L}}_V g(X, Y) &= g(\bar{\nabla}_X V, Y) + g(X, \bar{\nabla}_Y V) \\ &= \mathcal{L}_V g(X, Y) - \sigma g(\varphi X, V)\eta(Y) - \sigma g(\varphi Y, V)\eta(X) \\ &\quad -2\theta\eta(V)g(X, Y) + \theta g(X, V)\eta(Y) + \theta g(Y, V)\eta(X), \end{aligned}$$

for all vector fields X, Y, V, where $\overline{\mathcal{L}}_V g$ denotes the Lie derivative of g along the vector field V with respect to $\bar{\nabla}$. Using (2.17), the Ricci operator $\bar{Q}$ of the connection $\bar{\nabla}$ is determined by

$$\bar{Q}X = QX - (2n-2)\sigma\theta\varphi X + \{\xi(\theta) + 2n\theta^2\}X - 2\sigma^2\eta(X)\xi + \{(\varphi X)\sigma + (2n-1)(X\theta)\}\xi, \tag{2.20}$$

for any vector field X. Let r and $\bar{r}$ be the scalar curvature of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\bar{\nabla}$. The equation (2.17) yields

$$\bar{r} = r + 2n\{2(\xi\theta) - \sigma^2 + (2n+1)\theta^2\}. \tag{2.21}$$

The generalized η-Ricci soliton associated to the Schouten-van Kampen connection is defined by

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta\otimes\eta + \lambda g = 0, \tag{2.22}$$

where $\bar{S}$ denotes the Ricci tensor of the connection $\bar{\nabla}$,

$$(\overline{\mathcal{L}}_V g)(Y,Z) := g(\bar{\nabla}_Y V, Z) + g(Y, \bar{\nabla}_Z V),$$

$V^\flat$ is the canonical 1-form associated to V that is $V^\flat(X) = g(V,X)$ for all vector field X, λ is a smooth function on M, and α, β, μ, ρ are real constants such that $(\alpha,\beta,\mu) \neq (0,0,0)$.

The generalized η-Ricci soliton equation reduces to

(1) the η-Ricci soliton equation when $\alpha = 1$ and $\mu = 0$,

(2) the Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$,

(3) the generalized Ricci soliton equation when $\rho = 0$.

3. Main results and their proofs

A trans-Sasakian manifold is said to η-Einstein if its Ricci tensor S is of the form

$$S = ag + b\eta\otimes\eta,$$

where a and b are smooth functions on manifold. Let M be a trans-Sasakian manifold. Now, we consider M satisfies the generalized η-Ricci soliton (2.22) associated to the Schouten-van Kampen connection and the potential vector field V is a pointwise collinear vector field with the structure vector field ξ, that is, $V = \gamma\xi$ for some function γ on M. Using (2.4) we get

$$\begin{aligned} \overline{\mathcal{L}}_{\gamma\xi} g(X,Y) &= \mathcal{L}_{\gamma\xi} g(X,Y) - 2\gamma\theta\,(g(X,Y) - \eta(X)\eta(Y)) \\ &= X(\gamma)\eta(Y) + Y(\gamma)\eta(X), \end{aligned} \tag{3.1}$$

for all vector fields X, Y. Also, we have

$$\xi^\flat \otimes \xi^\flat(X,Y) = \eta(X)\eta(Y), \tag{3.2}$$

for all vector fields X, Y. Applying $V = \gamma\xi$, (2.17), (3.1), and (3.2) in the equation (2.22) we infer

$$\alpha\bar{S}(X,Y) + \frac{\beta}{2}X(\gamma)\eta(Y) + \frac{\beta}{2}Y(\gamma)\eta(X) + (\mu\gamma^2 + \rho)\eta(X)\eta(Y) + \lambda g(X,Y) = 0, \tag{3.3}$$

for all vector fields X, Y. We plug $Y = \xi$ in the above equation and using (2.18) to yield

$$\frac{\beta}{2}X(\gamma) + \frac{\beta}{2}\xi(\gamma)\eta(X) + (\mu\gamma^2 + \rho + \lambda)\eta(X) = 0. \tag{3.4}$$

Taking $X = \xi$ in (3.4) gives

$$\beta\xi(\gamma) = -(\mu\gamma^2 + \rho + \lambda). \tag{3.5}$$

Inserting (3.5) in (3.4), we conclude

$$\beta X(\gamma) = -(\mu\gamma^2 + \rho + \lambda)\eta(X), \tag{3.6}$$

which yields

$$\beta d\gamma = -(\mu\gamma^2 + \rho + \lambda)\eta. \tag{3.7}$$

Applying (3.7) in (3.3) we obtain

$$\alpha\bar{S}(X, Y) = \lambda(-g(X, Y) + \eta(X)\eta(Y)), \tag{3.8}$$

which implies $\alpha\bar{r} = -2n\lambda$. We plug $Y = \xi$ in the equation (3.8) and using (2.19) to obtain

$$(2n-1)\{(\xi\theta)\eta(X) - X\theta\} - (\phi X)\sigma = 0, \tag{3.9}$$

for any vector field X. Using (2.7) we have $\xi\theta = 0$ then $(2n-1)X\theta = -(\phi X)\sigma$. This conclude that $\nabla\theta = 0$. Thus θ is a constant and $\varphi(\nabla\sigma) = 0$. Therefore, this leads to the following:

Theorem 3.1. *Let $(M, g, \varphi, \xi, \eta)$ be a trans-Sasakian and it admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $\alpha \neq 0$ and $V = \gamma\xi$ for some smooth function γ on M, then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection. Also, θ is a constant and $\varphi(\nabla\sigma) = 0$*

From (3.8) we also have the following:

Corollary 3.1. *Let $(M, g, \varphi, \xi, \eta)$ be a trans-Sasakian 3-dimensional manifold. If M admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $V = \gamma\xi$ for some smooth function γ on M, then $\alpha\bar{r} = -2n\lambda$.*

Now, let M be an η-Einstein quasi-Sasakian 3-dimensional manifold with respect to the Schouten-van Kampen connection and $V = \xi$. Then we get $\bar{S} = ag + b\eta \otimes \eta$ for some functions a and b on M. We have $\overline{\mathcal{L}}_\xi g = 0$, then

$$\begin{aligned}
&\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_\xi g + \mu\xi^\flat \otimes \xi^\flat + \rho\eta \otimes \eta + \lambda g \\
&= a\alpha g + b\alpha\eta \otimes \eta + \mu\eta \otimes \eta + \rho\eta \otimes \eta + \lambda g \\
&= (a\alpha + \lambda)g + (b\alpha + \mu + \rho)\eta \otimes \eta.
\end{aligned}$$

From the above equation M admits a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection if $\lambda = -a\alpha$ and $\rho = -b\alpha - \mu$.

Hence, we can state the following theorem:

Theorem 3.2. *Suppose that M is a η-Einstein trans-Sasakian manifold with respect to the Schouten-van Kampen connection, that is, $\bar{S} = ag + b\eta \otimes \eta$ for some constants a and b on M. Then manifold M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -b\alpha - \mu, -a\alpha)$ with respect to the Schouten-van Kampen connection.*

Definition 3.1. A vector field V is said to a conformal Killing vector field with respect to the Schouten-van Kampen connection if

$$(\overline{\mathcal{L}}_V g)(X, Y) = 2hg(X, Y), \tag{3.10}$$

for all vector fields X, Y, where h is some function on M. The conformal Killing vector field V is called

- proper when h is not constant,
- homothetic vector field when h is a constant,
- Killing vector field when $h = 0$.

Let vector field V is a conformal Killing vector field and satisfies in (3.10). By (3.10), (2.17), and (2.22) we have

$$\alpha\bar{S}(X, Y) + \beta hg(X, Y) + \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X, Y) = 0. \tag{3.11}$$

for all vector fields X, Y. By inserting $Y = \xi$ in the above equation we get

$$g(\beta h\xi + \mu\eta(V)V + \rho\xi + \lambda\xi, X) = 0. \tag{3.12}$$

Since X is arbitraray vector field we have the following theorem.

Theorem 3.3. *If the metric g of a trans-Sasakian manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection where V is conformally Killing vector field, that is $\mathcal{L}_V g = 2hg$ then*

$$(\beta h + \rho + \lambda)\xi + \mu\eta(V)V = 0. \tag{3.13}$$

Definition 3.2. A nonvanishing vector field V on pseudo-Riemannian manifold (M, g) is called torse-forming [46] if

$$\nabla_X V = fX + \omega(X)V, \tag{3.14}$$

for any vector field X, where ∇ is the Levi-Civita connection of g, f is a smooth function and ω is a 1-form. The vector field V is called

- concircular [15, 45] whenever in the equation (3.14) the 1-form ω vanishes identically,
- concurrent [39, 47] if in equation (3.14) the 1-form ω vanishes identically and $f = 1$,

- parallel vector field if in equation (3.14) $f = \omega = 0$,
- torqued vector field [14] if in equation (3.14) $\omega(V) = 0$.

Let $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ be a generalized η-Ricci soliton on a trans-Sasakian manifold with respect to the Schouten-van Kampen connection where V is a torse-forming vector filed with respect to the Schouten-van Kampen connection that is, $\bar{\nabla}_X V = fX + \omega(X)V$. Then

$$\alpha\bar{S}(X,Y) + (\overline{\mathcal{L}}_V g)(X,Y) + \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0, \tag{3.15}$$

for all vector fields X, Y. On the other hand,

$$(\overline{\mathcal{L}}_V g)(X,Y) = 2fg(X,Y) + \omega(X)g(V,Y) + \omega(Y)g(V,X), \tag{3.16}$$

for all vector fields X, Y. Applying (3.16) into (3.15) we arrive at

$$\begin{aligned}\alpha\bar{S}(X,Y) &+ [\beta f + \lambda]\, g(X,Y) + \rho\eta(X)\eta(Y) \\ &+ \frac{\beta}{2}\left[\omega(X)g(V,Y) + \omega(Y)g(V,X)\right] + \mu g(V,X)g(V,Y) = 0.\end{aligned} \tag{3.17}$$

We take contraction of the above equation over X and Y to obtain

$$\alpha\bar{r} + (2n+1)\left[\beta f + \lambda\right] + \rho + \beta\omega(V) + \mu|V|^2 = 0. \tag{3.18}$$

Therefore we have the following theorem.

Theorem 3.4. *If the metric g of a trans-Sasakian manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection where V torse-forming vector filed and satisfied in $\bar{\nabla}_X V = fX + \omega(X)V$, then*

$$\lambda = -\frac{1}{2n+1}\left[\alpha\left(r + 2n\{2(\xi\theta) - \sigma^2 + (2n+1)\theta^2\}\right) + 2n + \rho + \beta\omega(V) + \mu|V|^2\right] - \beta f. \tag{3.19}$$

4. Example

In this section, we give an example of trans-Sasakian manifold with respect to the Schouten-van Kampen connection.

Example 4.1. Let (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and $M = \{(x,y,z) \in \mathbb{R}^3 | z \neq 0\}$. We consider the linearly independent vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = z\frac{\partial}{\partial z}.$$

We define the metric g by $g(e_i, e_j) = 1$ if $i = j$ and $i, j \in \{1, 2, 3\}$ and otherwise $g(e_i, e_j) = 0$.

We define an almost contact structure (φ, ξ, η) on M by

$$\xi = e_3, \quad \eta(X) = g(X, e_3), \quad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all vector field X. Note the relations $\varphi^2(X) = -X + \eta(X)\xi$, $\eta(\xi) = 1$, and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ hold. Thus $(M, \varphi, \xi, \eta, g)$ defines an almost contact structure on M. We have

[,]	e_1	e_2	e_3
e_1	0	0	-e_1
e_2	0	0	-e_2
e_3	e_1	e_2	0

The Levi-Civita connection ∇ of M is determined by

$$\nabla_{e_i} e_j = \begin{pmatrix} e_3 & 0 & -e_1 \\ 0 & e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the structure (φ, ξ, η) is a trans-Sasakian structure with $\sigma = 0$ and $\theta = -1$. Now, using (2.16) we get the Schouten-van- Kampen connection on M as $\bar{\nabla}_{e_i} e_j = 0$ for $1 \leq i, j \leq 3$. Hence $\bar{S} = 0$ If we consider $V = \xi$ then $\overline{\mathcal{L}}_V g = 0$. Therefore $(g, \xi, \alpha, \beta, \mu, \rho = -\mu, \lambda = 0)$ is a generalized η-Ricci soliton on manifold M.

Declarations

- Funding
  This work does not receive any funding.

- Conflict of interest/Competing interests
  The authors declare that they have no competing interests.
- Availability of data and materials
  All data generated or analysed during this study are included in this published article.
- Authors' contributions
  The author read and approved the final manuscript.

REFERENCES

1. N. Aktan, M. Yildirim and C. Murathan: *Almost f-cosymplectic manifolds*, Mediterr. J. Math. **11**(2014), 775-787.
2. S. Azami: *Generalized Ricci solitons of three-dimensional Lorentzian Lie groups associated canonical connections and Kobayashi-Nomizu connections*, J. Nonlinear Math. Phys. **30**(1)(2023), 1-33.
3. S. Azami: *Generalized η-Ricci solitons on LP-Kenmotsu manifolds associated to the Schouten-Van Kampen connection*, U.P.B. Sci. Bull., Series A **85**(1)(2023), 53-64.

4. A. M. Blaga: *Cononical connections on para-Kenmotsu manifolds*, Novi Sad J. Math. **45**(2) (2015), 131-142.

5. A. M. Blaga: *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat **30**(2) (2016), 489-496.

6. A. M. Blaga: *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), 1-13.

7. A. M. Blaga and M. Crasmareanu: *Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry*, Filomat **31**(2) (2017), 499-504.

8. D. E. Blair: *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics Vol 509. Springer-Verlag, Berlin-New York (1976).

9. D. E. Blair: *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics Vol. 203. Birkhauser Boston Inc. (2002).

10. D. E. Blair and J. A. Oubina : *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Mathematiques **34** (1990), 199-207.

11. G. Calvaruso: *Three-dimensional homogeneous generalized Ricci solitons*, Mediterranean Journal of Mathematics, **14**(5)(2017), 1-21.

12. T. Chave and G. Valent: *Quasi-Einstein metrics and their renormalizability properties*, Helv. Phys. Acta., **69** (1996), 344-347.

13. T. Chave and G. Valent: *On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties*, Nuclear Phys. B., **478** (1996), 758-778.

14. B. Y. Chen: *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math., **41**(2) (2017), 239-250.

15. B. Y. Chen: *A simple characterization of generalized Robertson-Walker space-times*, Gen. Relativity Gravitation, **46**(12) (2014), Article ID 1833.

16. J. T. Cho and M. Kimura: *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. **61**(2) (2009), 205-212.

17. U. C. De and M. M. Tripathi: *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook Math. J. **2**(2003), 247-255.

18. U. C. De and A. Sarkar: *On three-Dimensional trans-Sasakian manifolds*, Extracta Math. **23**(2008), 265-277.

19. S. Deshmukh and M. M. Tripathi: *A note on trans-Sasakian manifolds*, Math. Slov. **63**(2013), 1361-1370.

20. S. Deshmukh, U. C. De and F. Al-Solamy: *Trans-Sasakian manifolds homothetic to Sasakian manifolds*, Publ. Math. Debr., **88**(2016), 439-448.

21. S. Deshmukh: *Trans-Sasakian manifolds homothetic to Sasakian manifolds*, Mediterr. J. Math. **13**(2016), 2951-2958.

22. S. Deshmukh and F. Alsolamy: *A note on compact Trans-Sasakian manifolds*, Mediterr. J. Math. **13**(2016), 2099-2104.

23. S. Deshmukh and U. C. De: *A note on Trans-Sasakian manifolds*, arXiv 2018, arxiv:1203.0860.

24. A. Gray and L. M. Hervalla: *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Math. Pura Appl. **123**(4)(1980), 35-58.

25. R. S. Hamilton: *The Ricci flow on surfaces, Mathematics and general relativity*, Contemp. Math. Santa Cruz, CA, 1986, 71, American Math. Soc., 1988, 237-262.

26. A. Haseeb, S. Pandey and R. Prasad: *Some results on η-Ricci solitons in quasi-Sasakian 3-manifolds*, Commun. Korean Math. Soc. **36**(2) (2021), 377-387.

27. A. Kazan and H. B. Karadag: *Trans-Sasakian manifolds with Schouten-van Kampen connection*, Ililias Journal of mathematics **7**(1)(2018), 1-12.

28. R. Ma and D. Pei: *Reeb flow invariant $\star$-Ricci operators on trans-Sasakian three-manifolds*, Math. Slov. **71**(2021), 749-756.

29. P. Majhi and U. C. De, D. Kar: *η-Ricci Solitons on Sasakian 3-Manifolds*, Anal. de Vest Timisoara LV(2) (2017), 143–156.

30. V. Mangione: *Harmonic maps and stability on f-Kenmotsu manifolds*, Int. J. Math. Math. Sci., 2008(2008).

31. J. C. Marrero: *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. **162**(1992), 77-86.

32. M. A. Mekki and A. M. Cherif: *Generalised Ricci solitons on Sasakian manifolds*, Kyungpook Math. J. **57**(2017), 677-682.

33. P. Nurowski and M. Randall: *Generalized Ricci solitons*, J. Geom. Anal. **26**(2016), 1280-1345.

34. M. Obata: *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Jpn. **14**(1962), 333-340.

35. M. Okumura: *Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures*, Tohoku Math. J. **16**(1964), 270-284.

36. Z. Olszak and R. Rosca: *Normal locally conformal almost cosymplectic manifolds*, Publ. Math. Debrecen **39**(1991), 315-323.

37. J. A. Oubina: *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), 187-193.

38. D. G. Prakasha and B. S. Hadimani: *η-Ricci solitons on para-Sasakian manifolds*, J. Geometry **108** (2017) 383-392.

39. J. A. Schouten: *Ricci Calculus*, Springer-Verlag (1954), Berlin.

40. M. D. Siddiqi: *Generalized η-Ricci solitons in trans Sasakian manifolds*, Eurasian bulltein of mathematics **1**(3)(2018), 107-116.

41. A. F. Solovév: *On the curvature of the connection induced on a hyperdistribution in a Riemannian space*, Geom. Sb. **19**(1978), 12-23.

42. M. Turana, C. Yetima and S. K. Chaubey: *On quasi-Sasakian 3-manifolds admitting η-Ricci solitons*, Filomat **33**(15) (2019), 4923-4930.

43. W. Wang and X. Liu: *Ricci tensors on trans-Sasakian 3-manifolds*, Filomat **32** (2018), 4365-4374.

44. Y. Wang and W. Wang: *A remark on trans-Sasakian 3-manifolds*, Rev. Union Mater. Argent. **60**(2019), 257-264.

45. K. Yano: *Concircular geometry I. Concircular tranformations*, Proc. Imp. Acad. Tokyo **16**(1940), 195-200.

46. K. Yano: *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 340-345.

47. K. Yano and B. Y. Chen: *On the concurrent vector fields of immersed manifolds*, Kodai Math. Sem. Rep., **23** (1971), 343-350.

48. A. Yildiz, U. C. De and M. Turan: *On 3-dimensional f-Kenmotsu manifolds and Ricci solitons*, Ukrainian Math. J. **65**(2013), 684-693.

49. Y. Zhao: *Some characterizations of property of trans-Sasakian 3-Manifolds*, Math. Slov. **71**(2021), 383-390.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 99–122
https://doi.org/10.22190/FUMI230203007P
Original Scientific Paper

A MIXTURE INTEGER-VALUED AUTOREGRESSIVE MODEL WITH A STRUCTURAL BREAK

Predrag M. Popović[1], Miroslav M. Ristić[2] and Milena S. Stojanović[2]

[1] **University of Niš, Faculty of Civil Engineering and Architecture**
Department of Mathematics, Physic and Informatics
Aleksandra Medvedeva 14, 18000 Niš, Serbia
[2] **University of Niš, Faculty of Sciences and Mathematics**
Department of Mathematics, Višegradska 33, 18000 Niš, Serbia

ORCID IDs: Predrag M. Popović https://orcid.org/0000-0003-2154-9038
Miroslav M. Ristić https://orcid.org/0000-0002-9067-2411
Milena S. Stojanović https://orcid.org/0000-0003-2566-3436

Abstract. In this manuscript we introduce a mixture integer-valued autoregressive model with a structural break. The introduced model is a mixture of an INAR(1) model with the binomial thinning operator and an INAR(1) model with the negative binomial thinning operator. Some properties of the introduced model are derived. The unknown parameters of the model are estimated by some methods and the performances of the obtained estimators are checked by simulations. At the end of the paper, two possible applications of the model are provided and discussed.
Keywords: Binomial thinning, Integer-valued autoregressive model, Mixture of INAR models, Structural break, Negative binomial thinning.

1. Introduction

In this manuscript we introduce a mixture integer-valued autoregressive model with a structural break motivated by the following real examples. We observe the number of infected people from a virus. At first, the activity of the virus is low and up to the moment τ, the number of infected people from the virus behaves like an INAR(1) model with the binomial thinning operator [13, 1]. After the moment τ, when a large number of reproductions are created, the activity of

Received February 03, 2023, revised: May 24, 2023, accepted: May 24, 2023
Communicated by Aleksandar Nastić
Corresponding Author: Predrag M. Popović. E-mail addresses: popovicpredrag@yahoo.com (P. Popović), miristic72@gmail.com (M. Ristić), milena_aleksic93@yahoo.com (M. Stojanović)
2010 *Mathematics Subject Classification.* Primary 62M10

the virus increases significantly and a greater number of people can be infected. Now, the number of infected people can be described as an INAR(1) model with the negative binomial thinning operator [14]. Also, we can observe the number of criminal offenses in some places. In the beginning, criminal groups are poorly organized and therefore, the number of criminal offenses can be described by an INAR(1) model with the binomial thinning operator. After some time criminal groups are better organized and their activities become more frequently. Therefore, after a moment τ, the number of crimes can be represented by an INAR(1) model with the negative binomial thinning operator.

Regarding these real examples, in this manuscript we introduce an integer-valued autoregressive model with one structural break τ based on two thinning operators, the binomial thinning operator " $\circ$ " and the negative binomial thinning operator " $*$ ". Exactly, we consider a time series model $\{X_t\}$, $t \in \mathcal{Z} \equiv \{0, \pm 1, \pm 2, \dots\}$, given by

$$X_t = \begin{cases} \alpha \circ X_{t-1} + \varepsilon_t, & t \leqslant \tau, \\ \beta * X_{t-1} + \varepsilon_t, & t > \tau, \end{cases} \tag{1.1}$$

where α and β belong to $(0,1)$, τ is an integer, all the counting series incorporated in $\alpha\circ$ and $\beta*$ are mutually independent sequences of independent and identically distributed (i.i.d.) random variables with Bernoulli(α), and Geometric($\beta/(1+\beta)$) distributions, respectively. Here, Bernoulli(α) indicates to a Bernoulli distributed random variable with success α and Geometric($\beta/(1+\beta)$) indicates to a geometric distributed random variable with mean β. We also suppose that $\{\varepsilon_t\}$ is a sequence of independent random variables, all the counting series are independent of X_t and ε_s for all t and s, and X_t and ε_s are independent for all $t < s$. The distribution of the random variable X_t can be different over the times $t \leqslant \tau$ and $t > \tau$, so the model given by (1.1) can be a non-stationary model. It is possible that the distribution of the random variable X_t is the same over all times $t \leqslant \tau$ and $t > \tau$ and in this case we have a process stationary in distribution. This type of process is also stationary in mean (a first-order stationary process) and it can be also a second-order stationary process only when the thinning parameters α and β are identical. In all other cases it is a non-stationary model, but stationary separately on each regime, i.e. it is stationary before and after a structural break.

The introduced model is an integer-valued autoregressive model with one structural break. Some other forms of these models have been widely investigated in the past. Also, special attentions have been given to development of the methods for detections of the breaks or the changes in these models. A comprehensive list of important references in detections of breaks and changes in time series can be found in [6] and [2]. Kashikar et al. [11] introduced an INAR model of the first- and higher-order with structural breaks. In their model, different binomial thinning operators have been used in each regime to generate values of the considered time series model. The authors have supposed that the innovations are independent Poisson distributed random variables with identical parameters inside each regime and with different parameters between different regimes. Hudecová [9] considered autoregressive binary time series with changes and has proposed method for detection of changes. Yu et al. [15] introduced an integer-valued moving average model with structural breaks and researched its properties. Hudecová et al. [10] developed procedures based on the probability generating function for detections of the changes in the integer-valued autoregressive model of the first order. Chen and Lee [4] introduced a zero-inflated generalized Poisson autoregressive models with structural breaks. Recently, Kim and Lee [12] introduced a residual-based CUSUM test for the PINAR(1) model which can be used as an alternative to classical CUSUM tests, while Cui and Wu [5] considered how to detect parameter changes in observation-driven models for count time series.

The manuscript is organized as follows. In Section 2, we introduce a first-order integer-valued autoregressive model as a mixture of two INAR(1) models based on two well-known thinning operators, the binomial and the negative binomial thinning operator. Some properties of the model with geometric marginals including conditional properties and correlation structure are derived. Section 3 covers some estimation issues. Here, we consider two methods of estimations, the conditional maximum likelihood method and the method of the conditional least squares. The performances of the obtained estimates are checked by simulations for different true values of the parameters, different sample sizes and different positions for structural break. In Section 4, we discuss possible applications of the introduced model on two real data sets about some criminal acts. The manuscript ends with some concluding remarks and discussion about further developments related to the introduced model and its generalization.

2. Construction and properties

In this section we derive some properties of the model introduced in the introduction. First, we start with the definition of the model in general case.

Definition 1. Suppose that α and β are real numbers from $(0, 1)$, τ is an integer, all the counting series incorporated in $\alpha\circ$ and $\beta*$ are mutually independent sequences of i.i.d. random variables with Bernoulli(α) and Geometric($\beta/(1+\beta)$) distributions. Also, suppose that $\{\varepsilon_t\}$ is a sequence of independent random variables, all the counting series are performed independently of X_t and ε_s for all t and s, and the random variables X_t and ε_s are independent for all $t < s$. A process $\{X_t\}$ is said to be a mixture integer-valued autoregressive model with a structural break if it satisfies equation (1.1) for all $t \in \mathcal{Z}$.

Now, we consider a case of the mixture integer-valued autoregressive model with a structural break under the assumption that the random variable X_t has the geometric distribution as follows. For $t \leqslant \tau$ we suppose that X_t has Geometric($\mu_1/(1+\mu_1)$) distribution and for $t > \tau$ we suppose that X_t has Geometric($\mu_2/(1+\mu_2)$) distribution. Both parameters μ_1 and μ_2 are positive real numbers. Thus, our model has 5 parameters: 2 thinning parameters α and β; 2 mean parameters μ_1 and μ_2, and 1 structural break parameter τ. The number of the unknown parameters can be reduced. First, it is possible that the random variable X_t has the same distribution over all times, so in this case we have that the parameters μ_1 and μ_2 are identical. Also, the thinning parameters α and β can be identical too. Thus, the number of unknown parameters can be reduced to 3 unknown parameters. As mentioned above, in the case when we have identical thinning parameters and identical mean parameters, our model is reduced to a second-order stationary process with same marginal distributions.

Our model is completely determined if the distribution of the random variable ε_t is known. Regarding this, the following theorem gives its distribution.

Theorem 1. *Let $\{X_t\}$, $t \in \mathcal{Z}$, be the mixture integer-valued autoregressive model with a structural break given by* (1) *and let us suppose that X_t has Geom$(\mu_1/(1+\mu_1))$ distribution for $t \le \tau$, and Geom$(\mu_2/(1+\mu_2))$ distribution for $t > \tau$. If $\alpha \in (0,1)$, $\beta \in \Big(0, \min\{\frac{\mu_2}{1+\mu_2}, \frac{\mu_2}{1+\mu_1}\}\Big]$, $\mu_1 > 0$, $\mu_2 > 0$, and $\tau \in \mathcal{Z}$, then the random variable ε_t is distributed as follows:*

$$\varepsilon_t \stackrel{d}{=} \begin{cases} 0, & w.p. \quad \alpha, \\ Geom\big(\frac{\mu_1}{1+\mu_1}\big), & w.p. \quad 1-\alpha, \end{cases} \qquad \text{if } t \le \tau, \tag{2.1}$$

$$\varepsilon_t \stackrel{d}{=} \begin{cases} Geom\big(\frac{\beta}{1+\beta}\big), & w.p. \quad \frac{\beta\mu_1}{\mu_2-\beta}, \\ Geom\big(\frac{\mu_2}{1+\mu_2}\big), & w.p. \quad 1-\frac{\beta\mu_1}{\mu_2-\beta}, \end{cases} \qquad \text{if } t = \tau+1, \tag{2.2}$$

$$\varepsilon_t \stackrel{d}{=} \begin{cases} Geom\big(\frac{\beta}{1+\beta}\big), & w.p. \quad \frac{\beta\mu_2}{\mu_2-\beta}, \\ Geom\big(\frac{\mu_2}{1+\mu_2}\big), & w.p. \quad 1-\frac{\beta\mu_2}{\mu_2-\beta}, \end{cases} \qquad \text{if } t \geqslant \tau+2. \tag{2.3}$$

Proof. Let Φ_{X_t}, $\Phi_{X_{t-1}}$ and Φ_{ε_t} be the probability generating functions of the random variables X_t, X_{t-1} and ε_t, respectively. Let us first consider the case $t \leqslant \tau$. In this case, we have that both random variables X_t and X_{t-1} have Geom$\big(\frac{\mu_1}{1+\mu_1}\big)$ distributions. Thus, from (1.1) we have that

$$\Phi_{X_t}(s) = \Phi_{X_{t-1}}(1-\alpha+\alpha s)\Phi_{\varepsilon_t}(s).$$

We obtain that

$$\Phi_{\varepsilon_t}(s) = \frac{\Phi_{X_t}(s)}{\Phi_{X_{t-1}}(1-\alpha+\alpha s)} = \frac{1+\alpha\mu_1-\alpha\mu_1 s}{1+\mu_1-\mu_1 s} = \alpha + (1-\alpha)\cdot\frac{1}{1+\mu_1-\mu_1 s}.$$

The function $\Phi_{\varepsilon_t}(s)$ is well defined for $\alpha \in (0,1)$. Thus, we obtain that the random variable ε_t is a mixture of 0 with probability α and the random variable which has geometric distribution Geom$\big(\frac{\mu_1}{1+\mu_1}\big)$ with probability $1-\alpha$.

If $t = \tau+1$, then X_t has a geometric distribution Geom$(\mu_2/(1+\mu_2))$ and X_{t-1} has a geometric distribution Geom$(\mu_1/(1+\mu_1))$. Thus,

$$\Phi_{X_t}(s) = \Phi_{X_{t-1}}\left(\frac{1}{1+\beta-\beta s}\right)\Phi_{\varepsilon_t}(s)$$

and we get

$$\Phi_{\varepsilon_t}(s) = \frac{\Phi_{X_t}(s)}{\Phi_{X_{t-1}}(\frac{1}{1+\beta-\beta s})} = \frac{1+\beta(1+\mu_1)-\beta(1+\mu_1)s}{(1+\beta-\beta s)(1+\mu_2-\mu_2 s)}$$
$$= \frac{\beta\mu_1}{\mu_2-\beta}\cdot\frac{1}{1+\beta-\beta s} + \left(1-\frac{\beta\mu_1}{\mu_2-\beta}\right)\cdot\frac{1}{1+\mu_2-\mu_2 s}.$$

The function $\Phi_{\varepsilon_t}(s)$ is well defined for $0 < \beta \leq \min\{1, \frac{\mu_2}{1+\mu_1}\}$. Thus, we obtain that the random variable ε_t is a mixture of two random variables which have geometric distribution $\text{Geom}\left(\frac{\beta}{1+\beta}\right)$ with probability $\frac{\beta\mu_1}{\mu_2-\beta}$ and geometric distribution $\text{Geom}\left(\frac{\mu_2}{1+\mu_2}\right)$ with probability $1-\frac{\beta\mu_1}{\mu_2-\beta}$.

The third case, $t \geqslant \tau_1 + 2$, can be considered in similar manner which implies that

$$\Phi_{\varepsilon_t}(s) = \frac{\beta\mu_2}{\mu_2-\beta}\cdot\frac{1}{1+\beta-\beta s} + \left(1-\frac{\beta\mu_2}{\mu_2-\beta}\right)\cdot\frac{1}{1+\mu_2-\mu_2 s},$$

where $0 < \beta \leq \frac{\mu_2}{1+\mu_2}$. Having in mind the above interval for parameter β, it follows that $\beta \in \left(0, \min\{\frac{\mu_2}{1+\mu_2}, \frac{\mu_2}{1+\mu_1}\}\right]$, which competes the proof of theorem. □

The introduced mixture integer-valued autoregressive model with a structural break is obviously a first-order Markov process, so the transition probabilities can be derived by considering the conditional probabilities $\pi(x_t|x_{t-1}) \equiv P(X_t = x_t|X_{t-1} = x_{t-1})$, where x_t and x_{t-1} are non-negative integers. These conditional probabilities are given by the following theorem and they will be used later for the derivation of the conditional log-likelihood function and for the conditional maximum likelihood estimation.

Theorem 2. *Let $\{X_t\}$, $t \in \mathcal{Z}$, be the model given by Definition 1 which satisfies the assumptions of Theorem 1. Let us define the functions $m(t) = \min(x_t, x_{t-1})$, $b(y,i,\theta) = \binom{y}{i}\theta^i(1-\theta)^{y-i}$, $g(y,\theta) = \frac{\theta^y}{(1+\theta)^{y+1}}$ and $h(y,i,\theta) = \binom{y+i-1}{i}\frac{\theta^i}{(1+\theta)^{y+i}}$. Let I_A be the indicator function of the event A. Then*

$$\pi(x_t|x_{t-1}) =$$
$$= \begin{cases} \sum\limits_{i=0}^{m(t)} b(x_{t-1},i,\alpha)\left[\alpha I_{\{x_t=i\}} + (1-\alpha)g(x_t-i,\mu_1)\right], & t\leqslant\tau \\ \sum\limits_{i=0}^{x_t} h(x_{t-1},i,\beta)\left[\frac{\beta\mu_1}{\mu_2-\beta}g(x_t-i,\beta) + \left(1-\frac{\beta\mu_1}{\mu_2-\beta}\right)g(x_t-i,\mu_2)\right], & t=\tau+1 \\ \sum\limits_{i=0}^{x_t} h(x_{t-1},i,\beta)\left[\frac{\beta\mu_2}{\mu_2-\beta}g(x_t-i,\beta) + \left(1-\frac{\beta\mu_2}{\mu_2-\beta}\right)g(x_t-i,\mu_2)\right], & t\geqslant\tau+2. \end{cases}$$

Proof. We will prove theorem only for the case $t \leqslant \tau$. All other cases can be proved similarly. Let us first suppose that $x_{t-1} > 0$. Since $t \leqslant \tau$, we have that

$$\pi(x_t|x_{t-1}) = P(\alpha \circ X_{t-1} + \varepsilon_t = x_t|X_{t-1} = x_{t-1}).$$

Now, the random variable $\alpha \circ X_{t-1}$ for given $X_{t-1} = x_{t-1}$ has the binomial distribution with parameters x_{t-1} and α. Let us denote this random variable as $Bin(x_{t-1}, \alpha)$. Then

$$\pi(x_t|x_{t-1}) = P(Bin(x_{t-1}, \alpha) + \varepsilon_t = x_t) = \sum_{i=0}^{m(t)} b(x_{t-1}, i, \alpha) P(\varepsilon_t = x_t - i).$$

According to distribution (2.1), we have that

$$P(\varepsilon_t = x_t - i) = \alpha I_{\{x_t = i\}} + (1 - \alpha) g(x_t - i, \mu_1).$$

Replacing this in the above equation we obtain the expression for the conditional probability when x_{t-1} is positive integer. When $x_{t-1} = 0$ we have that

$$\pi(x_t|0) = P(\varepsilon_t = x_t) = \alpha I_{\{x_t = 0\}} + (1 - \alpha) g(x_t, \mu_1).$$

□

As a next property we consider the covariance and the correlation structure of the introduced model. These properties will be used later for the estimation of the unknown parameters. This structure is given by the following theorem.

Theorem 3. *Let $\{X_t\}$, $t \in \mathcal{Z}$, be the model given by Definition 1 which satisfies the assumptions of Theorem 1. Then*

(*a*) *the covariance function of the random variables X_t and X_{t+k}, $k \geqslant 0$, is positive and it is given as*

$$\gamma_t(k) \equiv Cov(X_t, X_{t+k}) = \begin{cases} \alpha^k \mu_1 (1 + \mu_1), & \text{if } t + k \leqslant \tau, \\ \alpha^{\tau - t} \beta^{t+k-\tau} \mu_1 (1 + \mu_1), & \text{if } t \leqslant \tau < t + k, \\ \beta^k \mu_2 (1 + \mu_2), & \text{if } \tau < t, \end{cases}$$

(*b*) *the correlation function of the random variables X_t and X_{t+k}, $k \geq 0$, is positive, always less than 1 and given by*

$$\rho_t(k) \equiv Corr(X_t, X_{t+k}) = \begin{cases} \alpha^k, & \text{if } t + k \leqslant \tau, \\ \alpha^{\tau - t} \beta^{t+k-\tau} \sqrt{\frac{\mu_1(1+\mu_1)}{\mu_2(1+\mu_2)}}, & \text{if } t \leqslant \tau < t + k, \\ \beta^k, & \text{if } \tau < t. \end{cases}$$

Proof. (a) Let us first consider the covariance function between the random variables X_t and X_{t+k}. If $t + k \leq \tau$, then using the independency between X_t and the counting series incorporated in $\alpha \circ X_{t+k-1}$, and the independency between the random variables X_t and ε_{t+k}, we have that

$$\begin{aligned}\gamma_t(k) &= Cov(X_t, \alpha \circ X_{t+k-1} + \varepsilon_{t+k}) \\ &= \alpha Cov(X_t, X_{t+k-1}) \\ &= \alpha^k Var(X_t).\end{aligned}$$

Since $t \leqslant \tau$, then $Var(X_t) = \mu_1(1+\mu_1)$ and $\gamma_t(k) = \alpha^k \cdot \mu_1(1+\mu_1)$. Obviously, the covariance function is positive in this case.

If $t \leqslant \tau < t+k$, then using the independency of the random variables considered in the first case, we have that

$$\begin{aligned}\gamma_t(k) &= Cov(X_t, \beta * X_{t+k-1} + \varepsilon_{t+k}) \\ &= \beta Cov(X_t, X_{t+k-1}) \\ &= \beta^{t+k-\tau} Cov(X_t, X_\tau).\end{aligned}$$

Since $Cov(X_t, X_\tau) = Cov(X_t, X_{t+(\tau-t)})$ and $\tau - t \geqslant 0$, we can apply the result of the first case which implies that $\gamma_t(k) = \beta^{t+k-\tau}\alpha^{\tau-t} Var(X_t)$. Finally, since $t \leqslant \tau$, we have that $\gamma_t(k) = \beta^{t+k-\tau}\alpha^{\tau-t}\mu_1(1+\mu_1)$. Obviously, the covariance function is positive in this case. The third case can be considered in similar way which implies the proof of the first part of theorem.

(b) The correlation function of the random variables X_t and X_{t+k} can be represented via the corresponding covariance function as

$$\rho_t(k) = \frac{\gamma_t(k)}{\sqrt{Var(X_t) \cdot Var(X_{t+k})}}.$$

If $t + k \leq \tau$, then we have that the random variables X_t and X_{t+k} have the same distributions which together with $\gamma_t(k) = \alpha^k Var(X_t)$ imply that $\rho_t(k) = \alpha^k$.

If $t \leqslant \tau < t+k$, then we have that the random variables X_t and X_{t+k} have geometric distributions with means μ_1 and μ_2, respectively. Then, using this and the result of the first part of theorem, we have that

$$\rho_t(k) = \frac{\alpha^{\tau-t}\beta^{t+k-\tau}\mu_1(1+\mu_1)}{\sqrt{\mu_1(1+\mu_1)\cdot\mu_2(1+\mu_2)}} = \alpha^{\tau-t}\beta^{t+k-\tau}\sqrt{\frac{\mu_1(1+\mu_1)}{\mu_2(1+\mu_2)}}.$$

The last case can be considered in similar way which implies the proof of the second part of theorem related to expression of the correlation function.

Obviously, the correlation function is positive in all cases. Let us now prove that the correlation function $\rho_t(k)$ is always less than 1. Since the parameters α and β belong to $(0,1)$, this conclusion is obviously in cases: $t + k \leq \tau$ and $t > \tau$. Let us consider the case $t \leqslant \tau < t+k$. Since $\beta \in \left(0, \min\{\frac{\mu_2}{1+\mu_2}, \frac{\mu_2}{1+\mu_1}\}\right)$, we have that $\beta < \mu_2/(1+\mu_1) < (1+\mu_2)/\mu_1$. From this we obtain that $\beta < \sqrt{\frac{\mu_2(1+\mu_2)}{\mu_1(1+\mu_1)}}$, which implies that $\rho_t(k)$ is less than 1. □

Remark 2.1. From the results of the previous theorem we can conclude that the correlation function of the random variables X_t and X_{t+k}, $k \geq 0$ can be written in the form $\rho_t(k) = ac^k$, where a is positive and $c \in \{\beta, \alpha\}$, which implies that $Corr(X_t, X_{t+k})$ converges to 0 when $k \longrightarrow \infty$.

In the following theorem we present some conditional properties of the introduced model which can be derived without using the complicated conditional probabilities.

Theorem 4. *Let $\{X_t\}$, $t \in \mathcal{Z}$, be the model given by Definition 1 which satisfies the assumptions of Theorem 1. Then:*

(a) The conditional expectation of the random variable X_{t+k}, $k \geqslant 0$, for given X_t is a linear function of X_t given by

$$E(X_{t+k}|X_t) = \begin{cases} \alpha^k(X_t - \mu_1) + \mu_1, & \text{if } t+k \leqslant \tau, \\ \alpha^{\tau-t}\beta^{t+k-\tau}(X_t - \mu_1) + \mu_2, & \text{if } t \leqslant \tau < t+k, \\ \beta^k(X_t - \mu_2) + \mu_2, & \text{if } \tau < t. \end{cases}$$

(b) The conditional variance of the random variable X_{t+k} for given X_t is of the form

$$V(X_{t+k}|X_t) = a_{t,k}X_t + b_{t,k}, \tag{2.4}$$

where $a_{t,k}$ and $b_{t,k}$ are given respectively as

$$a_{t,k} = \begin{cases} \alpha^k(1-\alpha^k), & t+k \leqslant \tau, \\ \frac{\alpha^{\tau-2t}\beta^{t+k-2\tau}}{1-\beta}(\alpha^t\beta^\tau + \alpha^t\beta^{\tau+1} - \alpha^\tau\beta^{t+k} - 2\alpha^t\beta^{t+k+1} & \\ +\alpha^\tau\beta^{t+k+1}), & t \leqslant \tau < t+k, \\ \dfrac{\beta^k(1+\beta)(1-\beta^k)}{1-\beta}, & \tau < t, \end{cases}$$

and

$$b_{t,k} = \begin{cases} \mu_1(1-\alpha^k)(1+\mu_1+\mu_1\alpha^k), & t+k \leqslant \tau, \\ \dfrac{\alpha^{-2t}\beta^{-2\tau}}{1-\beta}[\alpha^{2t}(1-\beta)\beta^{2\tau}\mu_2(1+\mu_2) - \alpha^{2\tau}(1-\beta)\beta^{2t+2k}\mu_1^2 & \\ +\alpha^{t+\tau}\beta^{t+k}(2\beta^{t+k+1} - \beta^\tau - \beta^{\tau+1})], & t \leqslant \tau < t+k, \\ \dfrac{(1-\beta^k)\mu_2[1-\beta-2\beta^{k+1}+(1-\beta)(1+\beta^k)\mu_2]}{1-\beta}, & \tau < t. \end{cases}$$

Proof. (a) Let us first consider the conditional expectation of the random variable X_{t+k}, $k \geqslant 0$, for given X_t. Let $t + k \leq \tau$. Then using the property of the binomial thinning operator $E(\alpha \circ X_{t+k-1}|X_{t+k-1}) = \alpha X_{t+k-1}$ and the first-order Markovian property, we have that $E(X_{t+k}|X_t) = \alpha E(X_{t+k-1}|X_t) + E(\varepsilon_{t+k})$. Applying the last equation $k-1$ more times, we have that $E(X_{t+k}|X_t) = \alpha^k X_t + \sum_{j=0}^{k-1} \alpha^j E(\varepsilon_{t+k-j})$. Since $t+k \leqslant \tau$, the random variables ε_{t+k}, ε_{t+k-1}, $\ldots$, ε_{t+1}, are identically distributed random variables. Also, from (2.1) we have that $E(\varepsilon_{t+k-j}) = (1-\alpha)\mu_1$ for $j \in \{0, 1, \ldots, k-1\}$. Thus,

$$E(X_{t+k}|X_t) = \alpha^k X_t + \mu_1(1-\alpha^k). \tag{2.5}$$

Let $t \leqslant \tau < t+k$. First, using the property of the negative binomial thinning operator $E(\beta * X_{t+k-1}|X_{t+k-1}) = \beta X_{t+k-1}$ and applying $t+k-\tau$ times the obtained equation, we have that

$$\begin{aligned} E(X_{t+k}|X_t) &= \beta E(X_{t+k-1}|X_t) + E(\varepsilon_{t+k}) \\ &= \beta^{t+k-\tau} E(X_\tau|X_t) + \sum_{j=0}^{t+k-\tau-2} \beta^j E(\varepsilon_{t+k-j}) + \beta^{t+k-\tau-1} E(\varepsilon_{\tau+1}). \end{aligned}$$

The random variables ε_{t+k}, ε_{t+k-1}, $\ldots$, $\varepsilon_{\tau+2}$, have the distribution (2.3) which implies that the expectation is $E(\varepsilon_{t+k-j}) = \mu_2(1-\beta)$ for $j \in \{0, 1, \ldots, t+k-\tau-2\}$. According to (2.2), we have that $E(\varepsilon_{\tau+1}) = \mu_2 - \beta\mu_1$. Replacing these results in expression of $E(X_{t+k}|X_t)$, we have that

$$E(X_{t+k}|X_t) = \beta^{t+k-\tau} E(X_\tau|X_t) - \beta^{t+k-\tau}\mu_1 + \mu_2.$$

Since $E(X_\tau|X_t) = E(X_{t+(\tau-t)}|X_t)$ and $\tau - t \geqslant 0$, using the result (2.5) for $k = \tau - t$, we obtain that

$$E(X_{t+k}|X_t) = \beta^{t+k-\tau}\alpha^{\tau-t}(X_t - \mu_1) + \mu_2.$$

The third case can be considered in similar way which implies the proof of the first part of theorem.

(b) The easiest way to calculate the conditional variance $Var(X_{t+k}|X_t)$ is to use the conditional probability generating function $g_{t,k}(s) = E(s^{X_{t+k}}|X_t)$ and the property

$$Var(X_{t+k}|X_t) = g''_{t,k}(1) + g'_{t,k}(1) - [g'_{t,k}(1)]^2. \tag{2.6}$$

We have three cases: $t+k \leq \tau$, $\tau < t$ and $t \leq \tau < t+k$. We will consider each case separately.

At first we suppose that $t+k \leq \tau$. Conditional probability generating function of the random variable X_{t+k}, $k \geqslant 0$, for given X_t is

$$\begin{aligned} g_{t,k}(s) &= E(s^{\alpha \circ X_{t+k-1} + \varepsilon_{t+k}} \mid X_t) \\ &= E[(1-\alpha+\alpha s)^{X_{t+k-1}}|X_t] E(s^{\varepsilon_{t+k}}) \\ &= g_{t,k-1}(1-\alpha+\alpha s)\varphi(s), \end{aligned} \tag{2.7}$$

where $\varphi(s) = \dfrac{1+\alpha\mu_1-\alpha\mu_1 s}{1+\mu_1-\mu_1 s}$. In fact, we have a recurrent formula $g_{t,k}(s)$ with an initial condition $g_{t,0}(s) = s^{X_t}$. Thus, we obtain that

$$g_{t,k}(s) = \frac{1+\mu_1\alpha^k-\mu_1\alpha^k s}{1+\mu_1-\mu_1 s}(1-\alpha+\alpha s)^{X_t}. \tag{2.8}$$

According to (2.6), we have

$$Var(X_{t+k}|X_t) = \alpha^k(1-\alpha^k)X_t + \mu_1(1-\alpha^k)(1+\mu_1+\mu_1\alpha^k). \tag{2.9}$$

Let us consider now the third case $\tau \leq t$. The second case will be considered at the end of the proof. Conditional probability generating function of the random variable X_{t+k}, $k \geqslant 0$, for given X_t is given by

$$\begin{aligned} g_{t,k}(s) &= E(s^{\beta * X_{t+k-1}+\varepsilon_{t+k}} \mid X_t) \\ &= E\Big[\Big(\frac{1}{1+\beta-\beta s}\Big)^{X_{t+k-1}} \mid X_t\Big]E(s^{\varepsilon_{t+k}}) \\ &= g_{t,k-1}\Big(\frac{1}{1+\beta-\beta s}\Big)\psi(s), \end{aligned} \tag{2.10}$$

where $\psi(s) = \frac{1+\beta(1+\mu_2)-\beta(1+\mu_2)s}{(1+\beta-\beta s)(1+\mu_2-\mu_2 s)}$. The function $g_{t,k}(s)$ can be written in the following form

$$g_{t,k}(s) = (K_k(s))^{X_t}B_k(s), \tag{2.11}$$

where

$$K_k(s) = \frac{1-\beta^k-\beta(1-\beta^{k-1})s}{1-\beta^{k+1}-\beta(1-\beta^k)s}$$

and

$$B_k(s) = \frac{1-\beta^{k+1}+(1-\beta)\beta^k\mu_2-\beta[1-\beta^k+(1-\beta)\beta^{k-1}\mu_2]s}{[1-\beta^{k+1}-\beta(1-\beta^k)s][1+\mu_2-\mu_2 s]}.$$

Then, by applying (2.6), we have that

$$\begin{aligned} Var(X_{t+k} \mid X_t) &= \frac{\beta^k(1+\beta)(1-\beta^k)}{1-\beta}X_t \\ &\quad + \frac{(1-\beta^k)\mu_2[1-\beta-2\beta^{k+1}+(1-\beta)(1+\beta^k)\mu_2]}{1-\beta}. \end{aligned}$$

At the end, we suppose that $t \leq \tau < t+k$. Since $t+k = \tau+1+(t+k-\tau-1)$, we have the following

$$\begin{aligned}
&E(s^{X_{t+k}} \mid X_t) = \\
&= E[(K_{t+k-\tau-1}(s))^{X_{\tau+1}} \mid X_t]B_{t+k-\tau-1}(s) \\
&= E[(K_{t+k-\tau-1}(s))^{\beta * X_\tau + \varepsilon_{\tau+1}} \mid X_t]B_{t+k-\tau-1}(s) \\
&= E\Big[\Big(\frac{1}{1+\beta-\beta K_{t+k-\tau-1}(s)}\Big)^{X_\tau} \mid X_t\Big] \\
&\times \frac{1+\beta(1+\mu_1)-\beta(1+\mu_1)K_{t+k-\tau-1}(s)}{(1+\beta-\beta K_{t+k-\tau-1}(s))(1+\mu_2-\mu_2 K_{t+k-\tau-1}(s))} B_{t+k-\tau-1}(s).
\end{aligned}$$

Let $L(s) = (1+\beta-\beta K_{t+k-\tau-1}(s))^{-1}$. Then, the next applies

$$\begin{aligned}
&E(s^{X_{t+k}} \mid X_t) = \\
&= E((L(s))^{X_\tau} \mid X_t) \frac{1+\beta(1+\mu_1)-\beta(1+\mu_1)K_{t+k-\tau-1}(s)}{(1+\beta-\beta K_{t+k-\tau-1}(s))(1+\mu_2-\mu_2 K_{t+k-\tau-1}(s))} B_{t+k-\tau-1}(s) \\
&= (1-\alpha^{\tau-t}+\alpha^{\tau-t}L(s))^{X_t} \frac{1+\mu_1\alpha^{\tau-t}-\mu_1\alpha^{\tau-t}L(s)}{1+\mu_1-\mu_1 L(s)} \\
&\times \frac{1+\beta(1+\mu_1)-\beta(1+\mu_1)K_{t+k-\tau-1}(s)}{(1+\beta-\beta K_{t+k-\tau-1}(s))(1+\mu_2-\mu_2 K_{t+k-\tau-1}(s))} B_{t+k-\tau-1}(s).
\end{aligned}$$

Finally, from (2.6), we have that

$$\begin{aligned}
&Var(X_{t+k} \mid X_t) = \\
&= \frac{\alpha^{\tau-2t}\beta^{t+k-2\tau}}{1-\beta}(\alpha^t\beta^\tau + \alpha^t\beta^{\tau+1} - \alpha^\tau\beta^{t+k} - 2\alpha^t\beta^{t+k+1} + \alpha^\tau\beta^{t+k+1})X_t \\
&+ \frac{\alpha^{-2t}\beta^{-2\tau}}{1-\beta}[\alpha^{2t}(1-\beta)\beta^{2\tau}\mu_2(1+\mu_2) - \alpha^{2\tau}(1-\beta)\beta^{2t+2k}\mu_1^2 \\
&+ \alpha^{t+\tau}\beta^{t+k}(2\beta^{t+k+1} - \beta^\tau - \beta^{\tau+1})]
\end{aligned}$$

and the theorem is proved by this. □

3. Estimation of the unknown parameters

In this section we consider estimation of the unknown parameters of our model. We suppose that we have a realization $(X_1, X_2, \dots, X_N)$ of size N of the mixture integer-valued autoregressive model with a structural break given by (1.1). We want to estimate the position of the structural break τ and to estimate the parameters α, β, μ_1 and μ_2. We consider two estimation methods: the conditional maximum likelihood method and the conditional least squares method. In the following two subsections we analyze both methods.

3.1. Conditional maximum likelihood estimation

The maximum likelihood method has been widely used for the estimation of the structural breaks and change points depending on the considered problem. The maximum likelihood method derives the estimators of the structural breaks or change points by maximizing the log likelihood function with respect to the structural breaks or given change points. For example, Hinkley and Hinkley [7] considered the maximum likelihood method for the estimation of one change point for the sequence of independent random variables with Bernoulli distributions, Horváth [8] used the maximum likelihood method to test changes in the parameters of independent normal distributed random variables, Avery and Anderson [3] used it for the estimation of two change points in DNA sequences etc.

Since our observations are dependent we consider the conditional maximum likelihood method which estimates the unknown structural break τ and the parameters α, β, μ_1 and μ_2 by maximizing the conditional log-likelihood function $L = \log P(X_i = x_i, 2 \leqslant i \leqslant N | X_1 = x_1)$. Using the fact that our model is a first-order Markov process and the results of Theorem 2, we obtain that the conditional log-likelihood function is given by

$$\begin{aligned} &L(\tau,\alpha,\beta,\mu_1,\mu_2) = \\ &= \sum_{t=2}^{\tau} \log\left\{ \sum_{i=0}^{m(t)} b(x_{t-1}, i, \alpha) \left[\alpha I_{\{x_t = i\}} + (1-\alpha) g(x_t - i, \mu_1) \right] \right\} \\ &+ \log\left\{ \sum_{i=0}^{x_{\tau+1}} h(x_\tau, i, \beta) \left[\frac{\beta\mu_1}{\mu_2 - \beta} g(x_{\tau+1} - i, \beta) + \left(1 - \frac{\beta\mu_1}{\mu_2 - \beta}\right) g(x_{\tau+1} - i, \mu_2) \right] \right\} \\ &+ \sum_{t=\tau+2}^{N} \log\left\{ \sum_{i=0}^{x_t} h(x_{t-1}, i, \beta) \left[\frac{\beta\mu_2}{\mu_2 - \beta} g(x_t - i, \beta) + \left(1 - \frac{\beta\mu_2}{\mu_2 - \beta}\right) g(x_t - i, \mu_2) \right] \right\}, \end{aligned}$$

where $m(t) = \min(x_t, x_{t-1})$.

We can consider two approaches based on the conditional maximum likelihood estimation method. Which approach will be used depends on the sample size. Thus, if the sample size is not too large, we can estimate the unknown parameters as follows. For each fixed and known τ we estimate the parameters α, β, μ_1 and μ_2 by maximizing the conditional log-likelihood function L. Let us denote these estimates by $\widehat{\alpha}_\tau$, $\widehat{\beta}_\tau$, $\widehat{\mu}_{1,\tau}$ and $\widehat{\mu}_{2,\tau}$. Then, the estimate of the structural break $\widehat{\tau}$ is obtained as the value τ which maximizes the function $L(\tau, \widehat{\alpha}_\tau, \widehat{\beta}_\tau, \widehat{\mu}_{1,\tau}, \widehat{\mu}_{2,\tau})$. This approach works very well and gives good estimates for any sample size, but can be slow when the sample size is greater than 2000. In this case, we can estimate all five parameters by maximizing the conditional log-likelihood function L with respect to all these parameters. The simulations show that the first approach gives better estimates, especially it estimates better the structural break.

3.2. Conditional least squares estimation

According to the results presented in the first part of Theorem 4, the conditional least squares estimators are obtained by minimizing the function Q given as

$$Q = \sum_{t=1}^{\tau-1}[X_{t+1} - \alpha X_t - (1-\alpha)\mu_1]^2 + (X_{\tau+1} - \beta X_\tau - \mu_2 + \beta\mu_1)^2 + \sum_{t=\tau+1}^{N-1}[X_{t+1} - \beta X_t - (1-\beta)\mu_2]^2$$

with respect to the unknown parameters τ, α, β, μ_1 and μ_2. If the parameter τ is known, then the estimators of the parameters α, β, μ_1 and μ_2 can be obtained as the solutions of the system of equations

$$\frac{\partial Q}{\partial \alpha} = \sum_{t=1}^{\tau-1}[X_{t+1} - \alpha X_t - (1-\alpha)\mu_1](-X_t + \mu_1) = 0,$$

$$\frac{\partial Q}{\partial \beta} = (X_{\tau+1} - \beta X_\tau - \mu_2 + \beta\mu_1)(-X_\tau + \mu_1) + \sum_{t=\tau+1}^{N-1}[X_{t+1} - \beta X_t - (1-\beta)\mu_2](-X_t + \mu_2) = 0,$$

$$\frac{\partial Q}{\partial \mu_1} = -(1-\alpha)\sum_{t=1}^{\tau-1}[X_{t+1} - \alpha X_t - (1-\alpha)\mu_1] + \beta(X_{\tau+1} - \beta X_\tau - \mu_2 + \beta\mu_1) = 0,$$

$$\frac{\partial Q}{\partial \mu_2} = (X_{\tau+1} - \beta X_\tau - \mu_2 + \beta\mu_1) + (1-\beta)\sum_{t=\tau+1}^{N-1}[X_{t+1} - \beta X_t - (1-\beta)\mu_2] = 0,$$

with respect to the parameters α, β, μ_1 and μ_2. Similarly as in the first approach discussed in the conditional maximum likelihood estimation, we can derive the conditional least squares estimates as follows. For each fixed and known true value of the parameter τ we estimate the remaining parameters α, β, μ_1 and μ_2 by minimizing the function Q. Let us denote again these estimates by $\widehat{\alpha}_\tau$, $\widehat{\beta}_\tau$, $\widehat{\mu}_{1,\tau}$ and $\widehat{\mu}_{2,\tau}$. Then, the estimate of the structural break $\widehat{\tau}$ is obtained as the value τ which minimizes the function $Q(\tau, \widehat{\alpha}_\tau, \widehat{\beta}_\tau, \widehat{\mu}_{1,\tau}, \widehat{\mu}_{2,\tau})$, i.e.

$$\hat{\tau} = \arg\min_{\tau} Q(\tau, \widehat{\alpha}_\tau, \widehat{\beta}_\tau, \widehat{\mu}_{1,\tau}, \widehat{\mu}_{2,\tau}),$$

where we consider the realized values of the function Q for all $\tau \in [1, N-1]$.

3.3. Simulations

To check the performances of the estimators obtained by the conditional maximum likelihood method and the conditional least squares method we simulated 1000 samples for the following cases of true values of the parameters. We considered two cases: (a) the true values are $\alpha = 0.2$, $\beta = 0.3$, $\mu_1 = 1$ and $\mu_2 = 2$; and (b) the true values are $\alpha = 0.4$, $\beta = 0.8$, $\mu_1 = 4$ and $\mu_2 = 10$. Thus, in the first case we supposed that the correlations between the observations are small and the mean parameters are moderate and similar. In the second case we supposed that the correlation and the mean in the case of the negative binomial thinning operator are significantly larger.

For both cases we consider samples of sizes $n = 100$, $n = 200$ and $n = 500$. For each different sample size we consider three different cases: $\tau = n/4$, $\tau = n/2$ and $\tau = 3n/4$.

The maximization and the minimization have been performed by using the `optim` function from R statistical software and using the method Nelder-Mead. Some parts of the code have been written in `Rcpp` to improve the speed of the estimation. For each case we considered the mean, median, lower (Q_1) and upper (Q_3) quartiles, and standard error (SE) of the obtained estimates. The flow of the estimation procedure can be divided into two steps. In the first step, for all $\tau \in [1, T-1]$ we estimate parameters α, β, μ_1 and μ_2. Consequently, we obtained values of functions L and Q discussed in the previous subsections. In the second step, from these calculated values, we pick the value of τ which maximise the value of L for CML and minimize the value of Q for the CLS method. All the results are presented in Tables 3.1–3.3.

For big enough and small enough sample size, we conclude that there is regularity. As the value of the structural parameter τ rises, as the value SE of $\widehat{\alpha}$ and $\widehat{\mu}_1$ decreases. For the value SE of $\widehat{\beta}$ and $\widehat{\mu}_2$, we can conclude the opposite. While the value of the structural parameter τ rises, the value SE of $\widehat{\beta}$ and $\widehat{\mu}_2$ also rises. This correctness is justified by the number of information we receive. When the value of the structural parameter τ increase, the number of information for $\widehat{\alpha}$ and $\widehat{\mu}_1$ grows and the number of information for $\widehat{\beta}$ and $\widehat{\mu}_2$ decreases. When we have more information the error is smaller and conversely.

We can notice that the estimates are quite close to the true values when the sample size is 500, while there are some deviations for samples of size 100. Actually, the CML method shows remarkable results even for samples of sizes 100 and 200, while the CLS method are not quite accurate in these cases, especially for parameters μ_1 and μ_2. The CML method provides consistent estimates for both set of parameters, while CLS gives slightly better results when the value of parameters are smaller, i.e. for the parameter set a). Especially interesting part is the estimation of the structural break (parameter τ). While the CML method is quite accurate for all tested cases, the CLS method shows some deviations from true value even for the samples of size 500.

After all we can conclude that the estimates converges to their true values with the increase of the sample size, while the convergence is faster with CML than with CLS method.

$n = 100,\ \tau = 25,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	40.3010	31	22	58	0.8389	$\widehat{\tau}$	56.178	53	32	83	0.8509
$\widehat{\tau}-\tau$	15.301	6	−3	33	0.8389	$\widehat{\tau}-\tau$	31.178	28	7	58	0.8509
$\widehat{\alpha}$	0.2113	0.1795	0.0240	0.3111	0.0065	$\widehat{\alpha}$	0.2539	0.2099	0.0663	0.3602	0.0074
$\widehat{\beta}$	0.2838	0.2836	0.1722	0.3761	0.0055	$\widehat{\beta}$	0.2392	0.2025	0.0300	0.3434	0.0073
$\widehat{\mu}_1$	0.9436	0.8422	0.5160	1.3039	0.0205	$\widehat{\mu}_1$	1.4065	1.2500	0.8447	1.6695	0.0335
$\widehat{\mu}_2$	2.1913	2.0686	1.7100	2.5004	0.0379	$\widehat{\mu}_2$	2.9528	2.3678	1.8130	3.1826	0.0679
$n = 100,\ \tau = 50,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	52.697	53	40.75	69.25	0.7504	$\widehat{\tau}$	64.957	65	51	82	0.6916
$\widehat{\tau}-\tau$	2.697	3	-9.25	19.25	0.7504	$\widehat{\tau}-\tau$	14.957	15	1	32	0.6916
$\widehat{\alpha}$	0.2059	0.1853	0.0728	0.2918	0.0058	$\widehat{\alpha}$	0.2113	0.1791	0.0674	0.3014	0.0060
$\widehat{\beta}$	0.2823	0.2667	0.1599	0.3919	0.0057	$\widehat{\beta}$	0.2386	0.1999	0.0305	0.3582	0.0073
$\widehat{\mu}_1$	0.9375	0.8892	0.6514	1.1087	0.0187	$\widehat{\mu}_1$	1.1521	1.0440	0.8236	1.2945	0.0225
$\widehat{\mu}_2$	2.2583	2.1321	1.6926	2.6426	0.0338	$\widehat{\mu}_2$	3.0559	2.4966	1.8853	3.4039	0.0664
$n = 100,\ \tau = 75,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	63.263	75	44	83	0.8781	$\widehat{\tau}$	73.353	80	70	89.25	0.7596
$\widehat{\tau}-\tau$	-11.737	0	-31	8	0.8781	$\widehat{\tau}-\tau$	-1.647	5	-5	14.25	0.7596
$\widehat{\alpha}$	0.2223	0.1937	0.1058	0.2901	0.0061	$\widehat{\alpha}$	0.1994	0.1661	0.0764	0.2721	0.0057
$\widehat{\beta}$	0.2820	0.2449	0.1072	0.4095	0.0071	$\widehat{\beta}$	0.2331	0.1606	0.0001	0.3600	0.0083
$\widehat{\mu}_1$	0.9271	0.8868	0.7090	1.0701	0.0163	$\widehat{\mu}_1$	1.1652	0.9633	0.8051	1.1488	0.0482
$\widehat{\mu}_2$	2.3613	2.1068	1.4056	2.9152	0.0464	$\widehat{\mu}_2$	3.1387	2.5917	1.8062	3.7004	0.0713
$n = 200,\ \tau = 50,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	68.376	54	45	75	1.4224	$\widehat{\tau}$	104.191	86	56	159	1.723
$\widehat{\tau}-\tau$	18.376	4	-5	25	1.4224	$\widehat{\tau}-\tau$	54.191	36	6	109	1.723
$\widehat{\alpha}$	0.1895	0.1748	0.0755	0.2741	0.0050	$\widehat{\alpha}$	0.2222	0.2080	0.1015	0.3206	0.0053
$\widehat{\beta}$	0.2980	0.2990	0.2356	0.3548	0.0039	$\widehat{\beta}$	0.2703	0.2590	0.1577	0.3518	0.0061
$\widehat{\mu}_1$	0.9489	0.8727	0.6749	1.1541	0.0162	$\widehat{\mu}_1$	1.2340	1.1950	0.8670	1.5363	0.0170
$\widehat{\mu}_2$	2.1757	2.0817	1.8718	2.3235	0.0259	$\widehat{\mu}_2$	2.8910	2.2823	1.9894	2.9201	0.0651
$n = 200,\ \tau = 100,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	101.519	102	91	117	1.1599	$\widehat{\tau}$	127.34	117	101.75	160	1.2865
$\widehat{\tau}-\tau$	1.519	2	-9	17	1.1599	$\widehat{\tau}-\tau$	27.34	17	1.75	60	1.2865
$\widehat{\alpha}$	0.1943	0.1854	0.1256	0.2527	0.0038	$\widehat{\alpha}$	0.2035	0.1960	0.1005	0.2842	0.0044
$\widehat{\beta}$	0.2964	0.2927	0.2223	0.3644	0.0040	$\widehat{\beta}$	0.2701	0.2594	0.1431	0.3541	0.0061
$\widehat{\mu}_1$	0.9363	0.9254	0.7842	1.0716	0.0097	$\widehat{\mu}_1$	1.1220	1.0431	0.8798	1.2406	0.0177
$\widehat{\mu}_2$	2.1895	2.1050	1.8272	2.4042	0.0276	$\widehat{\mu}_2$	2.9048	2.3379	1.9872	2.9028	0.0637
$n = 200,\ \tau = 150,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	139.624	151	137.75	162	1.3726	$\widehat{\tau}$	156.556	159	150	179	1.1203
$\widehat{\tau}-\tau$	-10.376	1	-12.25	12	1.3726	$\widehat{\tau}-\tau$	6.556	9	0	29	1.1203
$\widehat{\alpha}$	0.1982	0.1943	0.1395	0.2436	0.0035	$\widehat{\alpha}$	0.1928	0.1893	0.1221	0.2542	0.0034
$\widehat{\beta}$	0.3099	0.2914	0.1975	0.4010	0.0056	$\widehat{\beta}$	0.2770	0.2483	0.0962	0.4009	0.0074
$\widehat{\mu}_1$	0.9559	0.9457	0.8239	1.0627	0.0128	$\widehat{\mu}_1$	1.0598	0.9968	0.8846	1.1105	0.0170
$\widehat{\mu}_2$	2.3912	2.2345	1.7516	2.7357	0.0370	$\widehat{\mu}_2$	3.1465	2.5441	2.0297	3.4113	0.0668

Table 3.1: Conditional maximum likelihood and conditional least squares estimates for different cases of true values of the parameters

$n = 500,\ \tau = 125,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	135.523	128	119	142	1.7701	$\widehat{\tau}$	191.46	139	127	190.25	3.6558
$\widehat{\tau}-\tau$	10.523	3	-6	17	1.7701	$\widehat{\tau}-\tau$	66.46	14	2	65.25	3.6558
$\widehat{\alpha}$	0.1933	0.1882	0.1343	0.2454	0.0031	$\widehat{\alpha}$	0.1994	0.1896	0.1129	0.2687	0.0039
$\widehat{\beta}$	0.2969	0.2977	0.2655	0.3290	0.0017	$\widehat{\beta}$	0.2859	0.2859	0.2327	0.3328	0.0039
$\widehat{\mu}_1$	0.9482	0.9449	0.8151	1.0640	0.0075	$\widehat{\mu}_1$	1.0919	1.0107	0.8798	1.2274	0.0104
$\widehat{\mu}_2$	2.0448	2.0370	1.9024	2.1460	0.0118	$\widehat{\mu}_2$	2.4803	2.0982	1.9556	2.2773	0.0531
$n = 500,\ \tau = 250,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	255.23	253	246	265.25	1.4007	$\widehat{\tau}$	289.542	262	251	298.25	2.1692
$\widehat{\tau}-\tau$	5.23	3	-4	15.25	1.4007	$\widehat{\tau}-\tau$	39.542	12	1	48.25	2.1692
$\widehat{\alpha}$	0.1952	0.1935	0.1606	0.2279	0.0019	$\widehat{\alpha}$	0.1991	0.1945	0.1481	0.2446	0.0024
$\widehat{\beta}$	0.2954	0.2923	0.2539	0.3355	0.0020	$\widehat{\beta}$	0.2893	0.2838	0.2297	0.3405	0.0038
$\widehat{\mu}_1$	0.9742	0.9713	0.8955	1.0601	0.0042	$\widehat{\mu}_1$	1.0360	1.0127	0.9278	1.1172	0.0051
$\widehat{\mu}_2$	2.0509	2.0371	1.8962	2.1879	0.0079	$\widehat{\mu}_2$	2.3906	2.1054	1.9496	2.3229	0.0429
$n = 500,\ \tau = 375,\ \alpha = 0.2,\ \beta = 0.3,\ \mu_1 = 1,\ \mu_2 = 2$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	372.582	377	369	391	1.8381	$\widehat{\tau}$	394.592	386	375	419	1.8383
$\widehat{\tau}-\tau$	-2.418	2	-6	16	1.8381	$\widehat{\tau}-\tau$	19.592	11	0	44	1.8383
$\widehat{\alpha}$	0.1983	0.1997	0.1720	0.2291	0.0015	$\widehat{\alpha}$	0.2017	0.1963	0.1568	0.2406	0.0025
$\widehat{\beta}$	0.2971	0.2922	0.2341	0.3555	0.0032	$\widehat{\beta}$	0.2803	0.2754	0.1946	0.3551	0.0048
$\widehat{\mu}_1$	0.9715	0.9734	0.9106	1.0404	0.0041	$\widehat{\mu}_1$	1.0258	1.0023	0.9322	1.0779	0.0082
$\widehat{\mu}_2$	2.1009	2.0907	1.8623	2.3062	0.0158	$\widehat{\mu}_2$	2.5460	2.2123	1.9534	2.5673	0.0481
$n = 100,\ \tau = 25,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	27.315	25	22	31	0.374	$\widehat{\tau}$	52.778	47	25	86	0.9949
$\widehat{\tau}-\tau$	2.315	0	-3	6	0.374	$\widehat{\tau}-\tau$	27.778	22	0	61	0.9949
$\widehat{\alpha}$	0.3823	0.3939	0.3196	0.4604	0.0042	$\widehat{\alpha}$	0.6072	0.7011	0.3497	0.8357	0.0090
$\widehat{\beta}$	0.7953	0.8013	0.7595	0.8356	0.0018	$\widehat{\beta}$	0.5914	0.6536	0.4690	0.8017	0.0077
$\widehat{\mu}_1$	3.8424	3.6386	2.6067	4.9137	0.0555	$\widehat{\mu}_1$	4.8332	2.9441	1.5312	8.1632	0.1367
$\widehat{\mu}_2$	10.1301	9.6837	7.7273	11.9703	0.1113	$\widehat{\mu}_2$	6.5530	5.7130	4.1488	8.2864	0.1325
$n = 100,\ \tau = 50,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	49.606	50	46	56	0.4526	$\widehat{\tau}$	49.159	47	23	76	0.9257
$\widehat{\tau}-\tau$	-0.394	0	-4	6	0.4526	$\widehat{\tau}-\tau$	-0.841	-3	-27	26	0.9257
$\widehat{\alpha}$	0.3901	0.3899	0.3441	0.4414	0.0028	$\widehat{\alpha}$	0.4820	0.4096	0.2909	0.7275	0.0086
$\widehat{\beta}$	0.7776	0.7962	0.7471	0.8369	0.0036	$\widehat{\beta}$	0.6073	0.6687	0.5105	0.7896	0.0074
$\widehat{\mu}_1$	3.8695	3.8411	3.1233	4.6293	0.0396	$\widehat{\mu}_1$	4.1442	3.0642	1.9004	5.4229	0.1905
$\widehat{\mu}_2$	10.1451	9.3875	7.1875	12.2876	0.1524	$\widehat{\mu}_2$	6.8419	6.0001	4.1815	8.2232	0.1612
$n = 100,\ \tau = 75,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$											
CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\widehat{\tau}$	69.431	75	67	78	0.5738	$\widehat{\tau}$	49.801	49	24	75	0.9141
$\widehat{\tau}-\tau$	-5.569	0	-8	3	0.5738	$\widehat{\tau}-\tau$	-25.199	-26	-51	0	0.9141
$\widehat{\alpha}$	0.3986	0.4027	0.3614	0.4390	0.0027	$\widehat{\alpha}$	0.3847	0.3467	0.2244	0.5031	0.0076
$\widehat{\beta}$	0.7389	0.7898	0.7034	0.8460	0.0058	$\widehat{\beta}$	0.5889	0.6245	0.4577	0.7546	0.0068
$\widehat{\mu}_1$	3.7731	3.7484	3.2043	4.3642	0.0361	$\widehat{\mu}_1$	3.6433	3.3320	2.2435	4.5701	0.0699
$\widehat{\mu}_2$	10.0257	9.0245	5.4754	13.107	0.1956	$\widehat{\mu}_2$	6.2752	5.7130	4.1102	7.5401	0.1171

Table 3.2: Conditional maximum likelihood and conditional least squares estimates for different cases of true values of the parameters

$n = 200,\ \tau = 50,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$

CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\hat{\tau}$	52.079	50	47	56	0.4835	$\hat{\tau}$	99.802	77.5	39	179	2.1926
$\hat{\tau}-\tau$	2.079	0	-3	6	0.4835	$\hat{\tau}-\tau$	49.802	27.5	-11	129	2.1926
$\widehat{\alpha}$	0.3949	0.3962	0.3464	0.4409	0.0027	$\widehat{\alpha}$	0.6169	0.7021	0.3983	0.8215	0.0077
$\widehat{\beta}$	0.7955	0.7977	0.7721	0.8223	0.0012	$\widehat{\beta}$	0.6260	0.7097	0.5035	0.7978	0.0076
$\widehat{\mu}_1$	3.9902	3.8837	3.1168	4.7795	0.0385	$\widehat{\mu}_1$	4.8815	2.7730	1.8929	8.3710	0.1216
$\widehat{\mu}_2$	9.9912	9.7218	8.2559	11.3364	0.0797	$\widehat{\mu}_2$	7.5863	7.0549	4.9478	9.0451	0.1589

$n = 200,\ \tau = 100,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$

CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\hat{\tau}$	100.343	100	96	105	0.4853	$\hat{\tau}$	88.594	72	35	140	1.9663
$\hat{\tau}-\tau$	0.343	0	-4	5	0.4853	$\hat{\tau}-\tau$	-11.406	-28	-65	40	1.9663
$\widehat{\alpha}$	0.3977	0.3990	0.3668	0.4298	0.0016	$\widehat{\alpha}$	0.5060	0.4450	0.3367	0.7127	0.0074
$\widehat{\beta}$	0.7944	0.7996	0.7650	0.8298	0.0016	$\widehat{\beta}$	0.6658	0.7230	0.6100	0.7899	0.0062
$\widehat{\mu}_1$	3.9851	3.9772	3.5154	4.4516	0.0235	$\widehat{\mu}_1$	3.6213	2.5677	1.9437	4.6694	0.0807
$\widehat{\mu}_2$	10.0778	9.65	7.9898	11.8108	0.1068	$\widehat{\mu}_2$	6.9571	6.756	4.6707	8.2727	0.1127

$n = 200,\ \tau = 150,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$

CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\hat{\tau}$	147.603	150	144.75	155	0.6018	$\hat{\tau}$	105.155	108.5	51	157	1.8747
$\hat{\tau}-\tau$	-2.397	0	-5.25	5	0.601	$\hat{\tau}-\tau$	-44.845	-41.5	-99	7	1.8747
$\widehat{\alpha}$	0.3990	0.3994	0.3739	0.4246	0.0013	$\widehat{\alpha}$	0.4139	0.3896	0.2983	0.4885	0.0060
$\widehat{\beta}$	0.7810	0.7943	0.7449	0.8349	0.0029	$\widehat{\beta}$	0.6495	0.6914	0.5943	0.7713	0.0059
$\widehat{\mu}_1$	3.9524	3.9663	3.5614	4.3297	0.0193	$\widehat{\mu}_1$	3.5984	3.5848	2.3423	4.3866	0.0517
$\widehat{\mu}_2$	10.2504	9.6642	7.1598	12.4546	0.1494	$\widehat{\mu}_2$	6.4835	6.0226	4.4474	7.4225	0.1027

$n = 500,\ \tau = 125,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$

CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\hat{\tau}$	126.589	125	123	130	0.4317	$\hat{\tau}$	214.71	123	74	440.25	5.6351
$\hat{\tau}-\tau$	1.589	0	-2	5	0.4317	$\hat{\tau}-\tau$	89.71	-2	-51	315.25	5.6351
$\widehat{\alpha}$	0.3976	0.3995	0.3726	0.4241	0.0013	$\widehat{\alpha}$	0.6082	0.5840	0.444	0.7999	0.0063
$\widehat{\beta}$	0.7979	0.7995	0.7820	0.8150	0.0008	$\widehat{\beta}$	0.6796	0.7602	0.6771	0.8054	0.0067
$\widehat{\mu}_1$	4.0661	3.9800	3.5957	4.5168	0.0223	$\widehat{\mu}_1$	4.4798	2.3832	2.0444	7.9776	0.1049
$\widehat{\mu}_2$	9.9584	9.8093	8.8011	11.0295	0.0499	$\widehat{\mu}_2$	8.1966	8.2114	5.9552	9.8036	0.1216

$n = 500,\ \tau = 250,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$

CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\hat{\tau}$	251.822	250	248	256	0.4341	$\hat{\tau}$	183.334	124.5	67.75	282.25	4.7419
$\hat{\tau}-\tau$	1.822	0	2	6	0.4341	$\hat{\tau}-\tau$	-66.666	-125.5	-182.25	32.25	4.7419
$\widehat{\alpha}$	0.4008	0.4018	0.3831	0.4187	0.0008	$\widehat{\alpha}$	0.5115	0.4664	0.3784	0.6269	0.0057
$\widehat{\beta}$	0.7977	0.7993	0.7777	0.8186	0.0009	$\widehat{\beta}$	0.7231	0.7659	0.7187	0.7973	0.0051
$\widehat{\mu}_1$	3.9649	3.9336	3.6907	4.2383	0.0128	$\widehat{\mu}_1$	3.3701	2.2528	1.9854	4.0988	0.0659
$\widehat{\mu}_2$	10.0991	9.9783	8.7784	11.3061	0.0597	$\widehat{\mu}_2$	7.3444	7.2657	5.7130	8.3134	0.1048

$n = 500,\ \tau = 375,\ \alpha = 0.4,\ \beta = 0.8,\ \mu_1 = 4,\ \mu_2 = 10$

CML	Mean	Median	Q_1	Q_3	SE	CLS	Mean	Median	Q_1	Q_3	SE
$\hat{\tau}$	374.349	375	370	381	0.5138	$\hat{\tau}$	268.048	297	137	382	4.3605
$\hat{\tau}-\tau$	-0.651	0	-5	6	0.5138	$\hat{\tau}-\tau$	-106.952	-78	-238	7	4.3605
$\widehat{\alpha}$	0.4001	0.4011	0.3851	0.4153	7e-04	$\widehat{\alpha}$	0.4315	0.4174	0.3657	0.4779	0.0038
$\widehat{\beta}$	0.7938	0.7973	0.766	0.8229	0.0014	$\widehat{\beta}$	0.7346	0.7499	0.697	0.7938	0.0036
$\widehat{\mu}_1$	4.0037	4.0093	3.7538	4.2318	0.0111	$\widehat{\mu}_1$	3.4848	3.6406	2.6151	4.2147	0.0365
$\widehat{\mu}_2$	10.1226	9.709	8.1434	11.5507	0.1103	$\widehat{\mu}_2$	6.6011	6.0863	4.5934	7.1362	0.1267

Table 3.3: Conditional maximum likelihood and conditional least squares estimates for different cases of true values of the parameters

4. Illustrative examples

In this section we discuss possible applications of our introduced model. According to the motivations mentioned in the introductory section, we consider two real data sets about some criminal counts observed in Pittsburgh. We consider two real data sets from the forecasting principles site (http://www.forecastingprinciples.com). Each data set contains 144 observations which represents monthly observed corresponding criminal counts in the period between January 1990 and December 2001.

4.1. Computer Aided Dispatch (CAD) calls about drug dealing

The first considered real data set represents Computer Aided Dispatch (CAD) calls about drug dealing registered in Pittsburgh 1011th tract. The sample mean and the sample variance are 3.1944 and 13.3605, respectively, which indicate that we deal with overdispersed data. The sample autocorrelation is 0.6509, so the observations are strongly correlated. The sample path, ACF and PACF plots are presented in Figure 4.1.

From the PACF plot, we can conclude that the first order integer-valued time series model will be adequate for this real data set. Also, from the sample path plot we can conclude that there are low activities about the CAD drug calls before the first 60 months. After that, the CAD drug calls significantly increase. This indicates that there is a change in model after 60 months, so our model with structural break can be considered as competitive model in comparison with GINAR(1) and NGINAR(1) models. For each model we derive the maximum likelihood estimates, values of the Akaike (AIC) and Bayesian (BIC) criterions and the root mean square (RMS). Beside this, we estimate the structural break for our model. All results are presented in Table 4.1.

Model	ML estimates	AIC	BIC	RMS
GINAR(1)	$\widehat{\alpha} = 0.3901$, $\widehat{\mu} = 2.7882$	633.0897	639.0293	2.9385
NGINAR(1)	$\widehat{\beta} = 0.6588$, $\widehat{\mu} = 3.0756$	608.0767	614.0163	2.7664
Mixture	$\widehat{\tau} = 62$, $\widehat{\alpha} = 0.0736$, $\widehat{\beta} = 0.6309$, $\widehat{\mu}_1 = 0.868$, $\widehat{\mu}_2 = 4.2307$	593.0997	604.9790	2.6405

Table 4.1: ML estimates, AIC, BIC, RMS for the CAD drug calls in 1011th tract

From the obtained results, we can conclude that our model provides the smallest values for the AIC and BIC criterions, which leads to conclusion that our model has captured very well the change of models. The estimated structural break is $\widehat{\tau} = 62$ which corresponds to earlier conclusion obtained from the sample path. Also, we can see that the smallest RMS is obtained for our model which means that our model gives best fit among the considered models. Fits for all three models are given in Figure 4.2. The bolded lines represent the fit of each model. From this figure we can conclude that our model very well capture behavior of CAD drug calls.

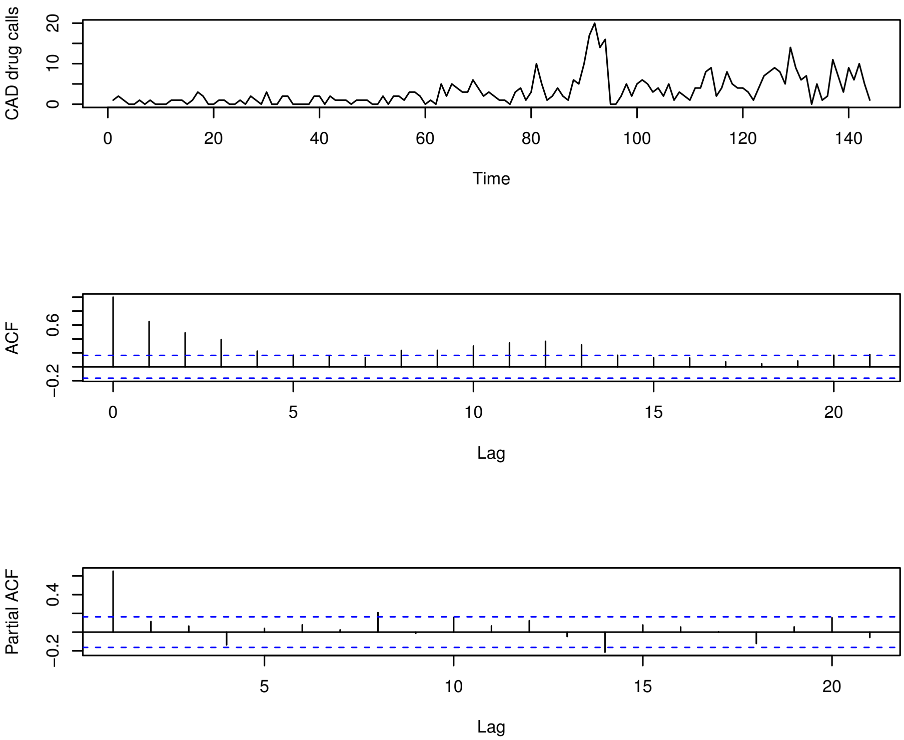

FIG. 4.1: The sample path, ACF and PACF for the CAD calls about drug dealing in 1011th tract

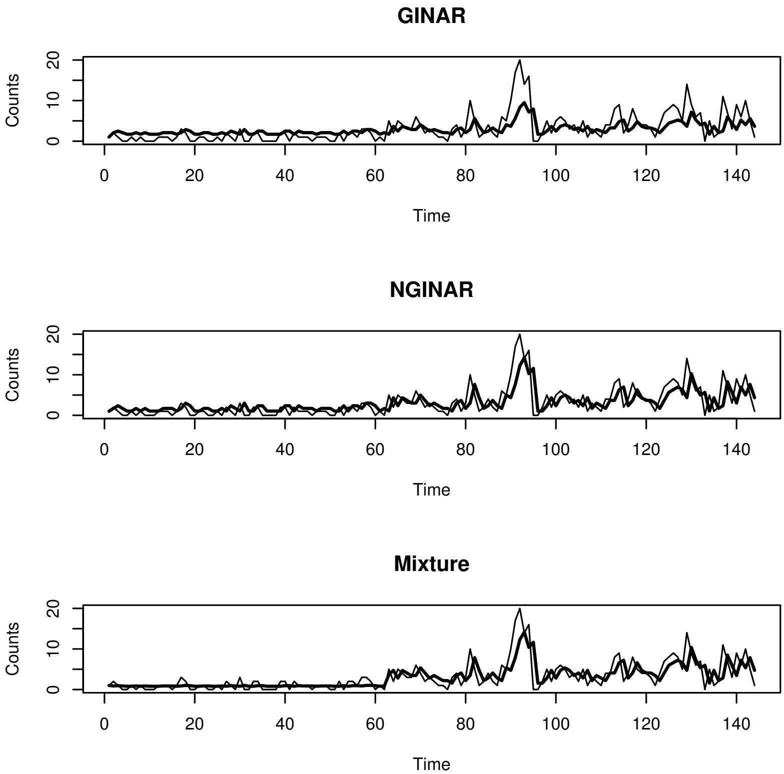

Fig. 4.2: The fits of all three models for the CAD drug calls data

4.2. Phone calls about registered shootings that can be reported also by civilians

The second considered actual dataset consists of phone calls about registered shootings that can be reported also by civilians registered in the 1017th Pittsburgh tract. The sample mean and the sample variance are 5.9722 and 32.3349, respectively. Again, we have overdispersed data. The sample autocorrelation is 0.4563 which indicates significant correlation between the observations. The Figure 4.3 represents the sample path, ACF and PACF plots.

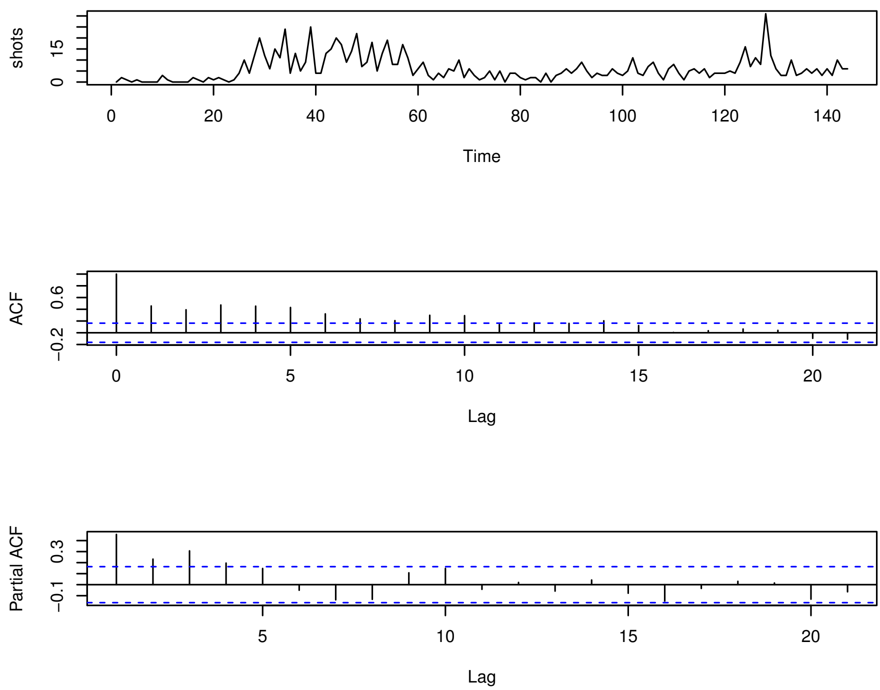

Fig. 4.3: The sample path, ACF and PACF for 1017 data

From the sample path we can observe that there is change in the behavior of the phone calls after around 20 months. This is justified by estimation in which we obtain that estimated structural break is 24 months. From the results presented in Table 4.2, we can conclude that our model very well fit the counts of phone calls.

Model	ML estimates	AIC	BIC	RMS
GINAR(1)	$\widehat{\alpha} = 0.3360$, $\widehat{\mu} = 5.2345$	787.0130	792.9526	5.1089
NGINAR(1)	$\widehat{\beta} = 0.4213$, $\widehat{\mu} = 5.4126$	782.5761	788.5157	5.0521
Mixture	$\widehat{\tau} = 24$, $\widehat{\alpha} = 0.1903$, $\widehat{\beta} = 0.3724$, $\widehat{\mu}_1 = 0.7662$, $\widehat{\mu}_2 = 5.9462$	768.2952	780.1744	4.8912

Table 4.2: ML estimates, AIC, BIC, RMS for the calls in 1017th tract

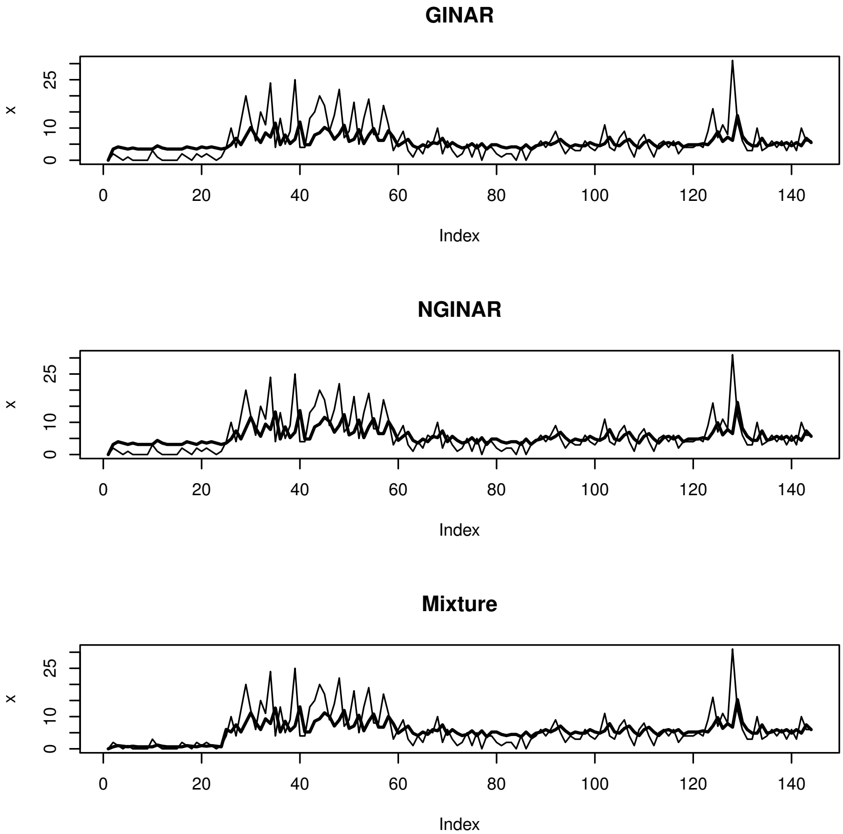

Fig. 4.4: The fits of all three models for the phone calls data

5. Concluding remarks

In this manuscript we have introduced an integer-valued autoregressive model of the first-order with a structural break as a mixture of two integer-valued autoregressive models with binomial and negative binomial thinning operators. The model has been constructed under motivations of the different behaviors of the considered objects before and after a break. Exactly, we have considered objects (virus, criminals etc.) which have low activities before a break leading to counts of small values and after a break have increasing activities leading to counts of large values. Because of that, we have used two different thinning operators, the binomial thinning for low activity and the negative binomial thinning for increasing activity. A model with different geometric marginals has been constructed and many of its properties are considered. Some of them are distribution of the innovations, conditional and unconditional properties, covariance and correlation structures. Two methods of estimations, conditional maximum likelihood and conditional least squares, are considered and the performances of their estimates have been checked by simulations. At the end, applicability of the model has been considered on two real data sets about criminal acts. It would be interesting to introduce some methods which can be used to detect the position of structural break. Standard CUSUM tests cannot be applied for our model because the change in thinning operators is not considered yet. Also, it would be interesting to consider more breaks and generalizes the model introduced in this paper.

REFERENCES

1. M.A. Al-Osh and A.A. Alzaid: *First-order integer-valued autoregressive (INAR(1)) process*, J. Time Ser. Anal. **8(3)** (1987), 261–275.
2. A. Aue and L. Horváth: *Structural breaks in time series*, J. Time Ser. Anal. **34(1)** (2013), 1–16.
3. P.J. Avery and D.A. Henderson: *Detecting a changed segment in DNA sequences*, J. Royal Stat. Soc.: Series C (Appl. Stat.) **48(4)** (1999), 489–503.
4. C.W.S. Chen and S. Lee: *Generalized Poisson autoregressive models for time series of counts*, Comput. Stat. Data Anal. **99** (2016), 51–67.
5. Y. Cui and R. Wu: *Test of parameter changes in a class of observation-driven models for count time series*, Commun. Stat. - Theory Methods (2019), 1–27.
6. J. Franke, C. Kirch and J.T. Kamgaing: *Changepoints in times series of counts*, J. Time Ser. Anal. **33(5)** (2012), 757–770.
7. D.V. Hinkley and E.A. Hinkley: *Inference about the change-point in a sequence of binomial variables*, Biometrika **57(3)** (1970), 477–488.
8. L. Horváth: *The maximum likelihood method for testing changes in the parameters of normal observations*, Ann. Stat. (1993), 671–680.
9. Š. Hudecová: *Structural changes in autoregressive models for binary time series*, J. Stat. Plan. Inference **143(10)** (2013), 1744–1752.
10. Š. Hudecová, M. Hušková and S. Meintanis: *Detection of changes in INAR models, Stochastic models, statistics and their applications* (2015), 11–18.

11. A.S. Kashikar, N. Rohan and T.V. Ramanathan: *Integer autoregressive models with structural breaks*, J. Appl. Stat. **40(12)** (2013), 2653–2669.

12. H. Kim and S. Lee: *On residual CUSUM statistic for PINAR(1) model in statistical design and diagnostic of control chart*, Commun. Stat. - Simul. Comput. (2019), 1–25.

13. E. McKenzie: *Some simple models for discrete variate time series*, JAWRA J. Amer. Water Res. Assoc. **21(4)** (1985), 645–650.

14. M.M. Ristić, H.S. Bakouch and A.S. Nastić: *A new geometric first-order integer-valued autoregressive (NGINAR(1)) process*, J. Stat. Plan. Inference **139(7)** (2009), 2218–2226.

15. K. Yu, H. Zou and D. Shi: *Integer-valued moving average models with structural changes*, Math. Probl. Eng. **2014** (2014).

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 123–139
https://doi.org/10.22190/FUMI230207008S
Original Scientific Paper

HYPERBOLIC KENMOTSU MANIFOLD ADMITTING A NEW TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

Abhishek Singh[1], Lovejoy S. Das[2], Pankaj[3] and Shraddha Patel[1]

[1] **Department of Mathematics and Statistics**
Dr. Rammanohar Lohia Avadh University, Ayodhya-224001 (U.P.), India
[2] **Department of Mathematics, Kent State University, Ohio, USA**
[3] **Department of Mathematics, Pranveer Singh Institute of Technology**
Kanpur-209305 (U.P.), India

ORCID IDs:		
	Abhishek Singh	https://orcid.org/0000-0002-8968-719X
	Lovejoy S. Das	https://orcid.org/0000-0002-2709-5113
	Pankaj	N/A
	Shraddha Patel	N/A

Abstract. In this work, we study a new type of semi-symmetric non-metric connection on hyperbolic Kenmotsu manifold. Some Riemannian curvature's characteristics on hyperbolic Kenmotsu manifold are investigated. The properties of semi-symmetric, locally φ-symmetric and Ricci semi-symmetric hyperbolic Kenmotsu manifold endowed with a new type of semi-symmetric non-metric connection are evaluated. A semi-symmetric and Ricci semi-symmetric hyperbolic Kenmotsu manifold with a semi-symmetric non-metric connection is also demonstrated, the Ricci soliton of data $(\mathfrak{g}_1, \xi^\flat, \lambda)$ is shrinking. Finally, we demonstrate our results with a 3-dimensional example.
Keywords: Semi-symmetric non-metric, hyperbolic Kenmotsu manifold, Ricci soliton, Einstein manifold, Ricci semi-symmetric.

1. Introduction

A. Friedmann and A. Schouten [16] first established the concept of a semi-symmetric linear connection on differentiable manifold in 1924. E. Bartolotti [6] gave a geometrical meaning to such a connection. Further, H. A. Hayden [17] introduced the concept of metric connection with non zero torsion tensor on a Rieman-

Received February 07, 2023, revised: May 27, 2023, accepted: June 04, 2023
Communicated by Uday Chand De
Corresponding Author: Shraddha Patel. E-mail addresses: abhi.rmlau@gmail.com (A. Singh), ldas@kent.edu (L. S. Das), pankaj.fellow@yahoo.co.in (Pankaj), shraddhapatelbbk@gmail.com (S. Patel)
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53D15, 53D10

nian manifold. Agashe and Chafle [2] define a semi-symmetric non-metric connection in Riemannian manifold. This was further studied by Agashe and Chafle [3], S. K. Chaubey and A. C. Pandey [11] and many other geometers like [8, 14, 18]. Sengupta, De and Binh [21], De and Sengupta [13] define new type of semi-symmetric non-metric connection on Riemannian manifold. Inline with this S. K. Chaubey and A. Yildiz [9] define another new type of semi-symmetric non-metric connection and studied different geometrical properties. On Riemannian manifold $(\Omega_{2n+1}, \mathfrak{g}_1)$, a linear connection $\widetilde{\nabla}$ is semi-symmetric if $\widetilde{\mathcal{T}}(\mathfrak{J}_1, \mathfrak{J}_2) = \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2$, $\forall\ \mathfrak{J}_1, \mathfrak{J}_2 \in \Gamma\Omega_{2n+1}$, where $\bar{\eta}$ is 1-form. Particularly, if $\mathfrak{J}_1 = \varphi\mathfrak{J}_1$ and $\mathfrak{J}_2 = \varphi\mathfrak{J}_2$, then the semi-symmetric connection reduces to the quarter-symmetric connection [15]. A semi-symmetric connection $\widetilde{\nabla}$ is metric if $\widetilde{\nabla}_{\mathfrak{g}_1} = 0$ & if $\widetilde{\nabla}_{\mathfrak{g}_1} \neq 0$, then it is non-metric. Since then, the properties of the semi-symmetric non-metric connection on different structures have been studied by many geometers [22, 12].

On the other hand, the almost hyperbolic (f, ξ, η, g)-structure was defined and studied by Upadhay and Dube [23]. Dube and Bhatt [7] studied CR-submanifold of trans-hyperbolic Sasakian manifold. Pankaj, S. K. Chaubey and Giilhanayar [20] studied Yamabe and gradient Yamabe soliton on 3-dimensional hyperbolic Kenmotsu manifolds. Mobin Ahmad and Kashif Ali [1] also studied CR-submanifold of a nearly hyperbolic Kenmotsu manifold admitting a quarter-symmetric non-metric connection. In the present article, it is initiated as follows: In section 2; contains some basic results of hyperbolic Kenmotsu manifolds. In section 3; we find some required results of the semi-symmetric non-metric connection. In section 4; we establish the relation between curvature tensor and semi-symmetric non-metric connection. The properties of semi-symmetric studied in section 5. Some results of locally φ-symmetric studied in section 6 and Ricci semi-symmetric hyperbolic Kenmatsu manifold equipped with semi-symmetric non-metric connection are investigated in section 7. We provided an example in section 8 and we also verified our results.

2. Hyperbolic Kenmotsu Manifold

Let $(\Omega_{2n+1}, \mathfrak{g}_1)$ be a contact manifold equipped with structure $(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$, where φ is a $(1,1)$-tensor field, $\xi^\flat$ is a vector field, $\bar{\eta}$ is 1-form and $\mathfrak{g}_1$ is a Riemannian metric [20] such that-

$$(2.1)\ \varphi^2\mathfrak{J}_1 = \mathfrak{J}_1 + \bar{\eta}(\mathfrak{J}_1)\xi^\flat, \qquad \bar{\eta}(\xi^\flat) = -1, \qquad \varphi\xi^\flat = 0, \qquad \bar{\eta}(\varphi\mathfrak{J}_1) = 0,$$

$$(2.2) \qquad \mathfrak{g}_1(\varphi\mathfrak{J}_1, \varphi\mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) - \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2),$$

$$(2.3) \qquad \mathfrak{g}_1(\varphi\mathfrak{J}_1, \mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \varphi\mathfrak{J}_2), \qquad \mathfrak{g}_1(\mathfrak{J}_1, \xi^\flat) = \bar{\eta}(\mathfrak{J}_1),$$

for all $\mathfrak{J}_1, \mathfrak{J}_2 \in \Gamma\Omega_{2n+1}$. A contact manifold Ω_{2n+1} is hyperbolic Kenmotsu manifold iff

$$(2.4) \qquad (\nabla_{\mathfrak{J}_1}\varphi)\mathfrak{J}_2 = \mathfrak{g}_1(\varphi\mathfrak{J}_1, \mathfrak{J}_2)\xi^\flat - \bar{\eta}(\mathfrak{J}_2)\varphi\mathfrak{J}_1,$$

where ∇ is Levi-Civita connection on Ω_{2n+1}. From (2.1), (2.2), (2.3) and (2.4), we find

$$d\bar{\eta} = 0, \qquad \nabla_{\mathfrak{J}_1}\xi^{\flat} = -\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\xi^{\flat}, \tag{2.5}$$

$$(\nabla_{\mathfrak{J}_1}\bar{\eta})\mathfrak{J}_2 = \mathfrak{g}_1(\varphi\mathfrak{J}_1, \varphi\mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) - \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2). \tag{2.6}$$

Also the hyperbolic Kenmotsu manifold hold the following relations:

$$\bar{\eta}(\mathcal{R}(\mathfrak{J}_1, \mathfrak{J}_2)\mathfrak{J}_3) = \mathfrak{g}_1(\mathfrak{J}_2, \mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_1) - \mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_2), \tag{2.7}$$

$$\mathcal{R}(\mathfrak{J}_1, \mathfrak{J}_2)\xi^{\flat} = \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2, \tag{2.8}$$

$$\mathcal{R}(\xi^{\flat}, \mathfrak{J}_1)\mathfrak{J}_2 = \mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2)\xi^{\flat} - \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1, \tag{2.9}$$

$$\mathcal{R}(\xi^{\flat}, \mathfrak{J}_1)\xi^{\flat} = -\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\xi^{\flat}, \tag{2.10}$$

$$\mathcal{S}^{\flat}(\mathfrak{J}_1, \xi^{\flat}) = 2n\bar{\eta}(\mathfrak{J}_1), \tag{2.11}$$

$$\mathcal{S}^{\flat}(\xi^{\flat}, \xi^{\flat}) = -2n, \tag{2.12}$$

$$\mathcal{Q}^{\flat}(\xi^{\flat}) = -2n\xi^{\flat}, \tag{2.13}$$

$\mathcal{S}^{\flat}$ and $\mathcal{Q}^{\flat}$ are related by

$$\mathcal{S}^{\flat}(\mathfrak{J}_1, \mathfrak{J}_2) = \mathfrak{g}_1(\mathcal{Q}^{\flat}\mathfrak{J}_1, \mathfrak{J}_2). \tag{2.14}$$

Definition 2.1. An almost contact manifold Ω_{2n+1} is an η-Einstein manifold (η-EM) if Ricci-tensor $\mathcal{S}^{\flat}$ is of the form

$$\mathcal{S}^{\flat}(\mathfrak{J}_1, \mathfrak{J}_2) = a_1\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) + a_2\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2), \tag{2.15}$$

where a_1 and a_2 are smooth functions on Ω_{2n+1}. If $a_2 = 0$, then manifold Ω_{2n+1} is an Einstein manifold (EM).

3. A new type of semi-symmetric non-metric connection

Let Ω_{2n+1} be hyperbolic Kenmotsu manifold. A linear connection $\widetilde{\nabla}$ on Ω_{2n+1} is given as

$$\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{J}_2 = \nabla_{\mathfrak{J}_1}\mathfrak{J}_2 + \frac{1}{2}\left[\bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2\right] \tag{3.1}$$

is known as a semi-symmetric non-metric connection $\widetilde{\nabla}$ if it satisfies

$$\widetilde{\mathcal{T}}(\mathfrak{J}_1, \mathfrak{J}_2) = \bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2 \tag{3.2}$$

and
(3.3)

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{g}_1)(\mathfrak{J}_2,\mathfrak{J}_3)=\frac{1}{2}\left[2\bar{\eta}(\mathfrak{J}_1)\,\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_2)\,\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_2)\right].$$

Now,

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\varphi)(\mathfrak{J}_2)=\frac{1}{2}\left[2\,(\nabla_{\mathfrak{J}_1}\varphi)\,\mathfrak{J}_2-\bar{\eta}(\mathfrak{J}_2)\,(\varphi\mathfrak{J}_1)\right],\tag{3.4}$$

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_2)=(\nabla_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_2),\tag{3.5}$$

$$(\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{g}_1)(\varphi\mathfrak{J}_2,\mathfrak{J}_3)=\frac{1}{2}\left[2\bar{\eta}(\mathfrak{J}_1)\,\mathfrak{g}_1(\varphi\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{g}_1(\mathfrak{J}_1,\varphi\mathfrak{J}_2)\right].\tag{3.6}$$

Changing $\mathfrak{J}_2$ by $\xi^\flat$ in (3.1), we have

$$\widetilde{\nabla}_{\mathfrak{J}_1}\xi^\flat=\nabla_{\mathfrak{J}_1}\xi^\flat-\frac{1}{2}\varphi^2\mathfrak{J}_1.\tag{3.7}$$

Replacing $\mathfrak{J}_1$ by $\xi^\flat$ in (3.3), we get

$$(\widetilde{\nabla}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_2,\mathfrak{J}_3)=\mathfrak{g}_1(\varphi\mathfrak{J}_2,\varphi\mathfrak{J}_3)=-\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_2)\,\bar{\eta}(\mathfrak{J}_3).\tag{3.8}$$

4. Curvature tensor of a hyperbolic Kenmotsu manifold endowed with semi-symmetric non-metric connection

The curvature tensor $\widetilde{\mathcal{R}}$ with $\widetilde{\nabla}$ defined as follows:

$$\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3=\widetilde{\nabla}_{\mathfrak{J}_1}\widetilde{\nabla}_{\mathfrak{J}_2}\mathfrak{J}_3-\widetilde{\nabla}_{\mathfrak{J}_2}\widetilde{\nabla}_{\mathfrak{J}_1}\mathfrak{J}_3-\widetilde{\nabla}_{[\mathfrak{J}_1,\mathfrak{J}_2]}\mathfrak{J}_3,\tag{4.1}$$

Using (3.1) in (4.1), we obtain

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3 &= \mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3+\frac{1}{2}[(\nabla_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_3)\mathfrak{J}_2-(\nabla_{\mathfrak{J}_1}\bar{\eta})(\mathfrak{J}_2)\mathfrak{J}_3\\ &\quad-(\nabla_{\mathfrak{J}_2}\bar{\eta})(\mathfrak{J}_3)\mathfrak{J}_1+(\nabla_{\mathfrak{J}_2}\bar{\eta})(\mathfrak{J}_1)\mathfrak{J}_3]\\ &\quad+\frac{1}{4}\left[\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_1-\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2\right],\end{aligned}\tag{4.2}$$

where,

$$\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3=\nabla_{\mathfrak{J}_1}\nabla_{\mathfrak{J}_2}\mathfrak{J}_3-\nabla_{\mathfrak{J}_2}\nabla_{\mathfrak{J}_1}\mathfrak{J}_3-\nabla_{[\mathfrak{J}_1,\mathfrak{J}_2]}\mathfrak{J}_3.\tag{4.3}$$

Now, using (2.6) in (4.2), we find

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3 &= \mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\,\mathfrak{J}_3+\frac{1}{2}\left[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\,\mathfrak{J}_1-\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\,\mathfrak{J}_2\right]\\ &\quad+\frac{3}{4}\left[\bar{\eta}(\mathfrak{J}_2)\,\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{J}_1-\bar{\eta}(\mathfrak{J}_1)\,\bar{\eta}(\mathfrak{J}_3)\,\mathfrak{J}_2\right].\end{aligned}\tag{4.4}$$

Contracting equation (4.4) along $\mathfrak{J}_1$, we get

$$\widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_2,\mathfrak{J}_3)=\mathcal{S}^{\flat}(\mathfrak{J}_2,\mathfrak{J}_3)+n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3). \tag{4.5}$$

By virtue of (2.14) and (4.5) gives

$$\widetilde{\mathcal{Q}}^{\flat}(\mathfrak{J}_2)=\mathcal{Q}^{\flat}(\mathfrak{J}_2)+n(\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\xi^{\flat}. \tag{4.6}$$

Again, contracting (4.5), we get

$$\widetilde{\tau}=\tau+n(2n-\frac{1}{2}). \tag{4.7}$$

Where $\widetilde{\mathcal{R}};\mathcal{R}$, $\widetilde{\mathcal{S}}^{\flat};\mathcal{S}^{\flat}$, $\widetilde{\mathcal{Q}}^{\flat}$; $\mathcal{Q}^{\flat}$ and $\widetilde{\tau}$; τ are curvature tensor, Ricci tensor, Ricci operators and scalar curvature respectively equipped with $\widetilde{\nabla}$ and Levi-Civita connection ∇.

Replacing $\mathfrak{J}_1=\xi^{\flat}$ in (4.4) and using (2.1), (2.3), we get

$$\begin{aligned}\widetilde{\mathcal{R}}(\xi^{\flat},\mathfrak{J}_2)\mathfrak{J}_3 &= \mathcal{R}(\xi^{\flat},\mathfrak{J}_2)\mathfrak{J}_3+\frac{1}{2}[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^{\flat}-\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2]\\ &\quad+\frac{3}{4}[\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\xi^{\flat}+\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2].\end{aligned} \tag{4.8}$$

Using (2.9) in above equation (4.8), we get

$$\widetilde{\mathcal{R}}(\xi^{\flat},\mathfrak{J}_2)\mathfrak{J}_3=\frac{3}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^{\flat}+\frac{3}{4}[\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\xi^{\flat}-\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2]. \tag{4.9}$$

Fix $\mathfrak{J}_3=\xi^{\flat}$ in (4.4) and using (2.1), (2.3), (2.8), we get

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\xi^{\flat} &= \frac{3}{4}\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\xi^{\flat}\\ &= \frac{3}{4}(\bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_1-\bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_2).\end{aligned} \tag{4.10}$$

Remark 4.1. Equation (4.10) shows that the manifold endowed with $\widetilde{\nabla}$ is regular.

In view of (2.3), (2.8), (4.4) and $\mathfrak{g}_1(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3),\mathfrak{J}_4)=-\mathfrak{g}_1(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_4),\mathfrak{J}_3)$, we have

$$\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3) = \frac{3}{2}[\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)]. \tag{4.11}$$

Contracting (4.10) with $\mathfrak{J}_1$, we find

$$\widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_2,\xi^{\flat})=\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2). \tag{4.12}$$

Taking $\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3=0$ in equation (4.4), we get

$$\begin{aligned}\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \frac{1}{2}\left[\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2-\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1\right]\\ &\quad+\frac{3}{4}\left[\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_1\right].\end{aligned}\tag{4.13}$$

In view of $'\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4)=\mathfrak{g}_1(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)$ and (4.13), we yields

$$\begin{aligned}'\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4) &= \frac{1}{2}\left[\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)-\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\right]\\ &\quad+\frac{3}{4}\left[\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\right].\end{aligned}\tag{4.14}$$

Contracting above equation along $\mathfrak{J}_1$ and $\mathfrak{J}_4$, we get

$$\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=-n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3).\tag{4.15}$$

Theorem 4.1. *A hyperbolic Kenmotsu manifold Ω_{2n+1} is an η-EM, if Riemannian curvature tensor endowed with $\widetilde{\nabla}$ is vanished.*

5. Semi-symmetric hyperbolic Kenmotsu manifold equipped with connection $\widetilde{\nabla}$

A contact manifold Ω_{2n+1} with connection $\widetilde{\nabla}$ is semi-symmetric if

$$(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{R}})(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5=0.$$

Then, we have

$$\begin{aligned}&\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5-\widetilde{\mathcal{R}}(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5\\ &-\widetilde{\mathcal{R}}(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4)\mathfrak{J}_5-\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_5=0.\end{aligned}\tag{5.1}$$

On changing $\mathfrak{J}_1=\xi^\flat$ in (5.1), we get

$$\begin{aligned}&\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5-\widetilde{\mathcal{R}}(\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5\\ &-\widetilde{\mathcal{R}}(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_4)\mathfrak{J}_5-\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_2)\mathfrak{J}_5=0.\end{aligned}\tag{5.2}$$

In view of (4.9), we obtain

$$\begin{aligned}2\mathfrak{g}_1(\mathfrak{J}_2,\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5) &= -\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5)-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\mathfrak{J}_5)\\ &\quad-2\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\bar{\eta}(\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_4)\mathfrak{J}_5)-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\widetilde{\mathcal{R}}(\xi^\flat,\mathfrak{J}_4)\mathfrak{J}_5)\\ &\quad+\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_2,\mathfrak{J}_4)\mathfrak{J}_5)-2\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\xi^\flat)\mathfrak{J}_5)\\ &\quad-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\xi^\flat)\mathfrak{J}_5)+\bar{\eta}(\mathfrak{J}_4)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_2)\mathfrak{J}_5)\\ &\quad-2\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_5)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\xi^\flat)-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_5)\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\xi^\flat)\end{aligned}$$

$$+\bar{\eta}(\mathfrak{J}_5)\,\bar{\eta}(\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\,\mathfrak{J}_2). \tag{5.3}$$

Using (2.1), (2.3), (4.9), (4.10) and (4.11) in (5.3), we get

$$\begin{aligned} {}'\widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4,\mathfrak{J}_5,\mathfrak{J}_2) &= \frac{3}{2}[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\,\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5) - \mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\,\mathfrak{g}_1(\mathfrak{J}_3,\mathfrak{J}_5)] \\ &+\frac{3}{4}[\bar{\eta}(\mathfrak{J}_4)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3) - \bar{\eta}(\mathfrak{J}_3)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)]. \end{aligned} \tag{5.4}$$

Hence, we have

$$\begin{aligned} \widetilde{\mathcal{R}}(\mathfrak{J}_3,\mathfrak{J}_4)\,\mathfrak{J}_5 &= \frac{3}{2}\,[\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5)\,\mathfrak{J}_3 - \mathfrak{g}_1(\mathfrak{J}_3,\mathfrak{J}_5)\,\mathfrak{J}_4] \\ &+\frac{3}{4}\,[\bar{\eta}(\mathfrak{J}_4)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{J}_3 - \bar{\eta}(\mathfrak{J}_3)\,\bar{\eta}(\mathfrak{J}_5)\,\mathfrak{J}_4]. \end{aligned} \tag{5.5}$$

Contracting (5.5) with $\mathfrak{J}_3$, we get

$$\widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_4,\mathfrak{J}_5) = 3n\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5) + \frac{3n}{2}\bar{\eta}(\mathfrak{J}_4)\,\bar{\eta}(\mathfrak{J}_5) \tag{5.6}$$

and

$$\widetilde{\mathcal{Q}}^{\flat}(\mathfrak{J}_4) = 3n\mathfrak{J}_4 + \frac{3n}{2}\bar{\eta}(\mathfrak{J}_4)\,\xi^{\flat}. \tag{5.7}$$

Again contracting (5.6), we have

$$\widetilde{\tau} = \frac{3n}{2}[4n+1]. \tag{5.8}$$

By virtue (2.15) and equation (5.6), we state:

Theorem 5.1. *A semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$, then Ω_{2n+1} is an η-EM.*

Now, using (4.5), (4.6), (4.7) in (5.6), (5.7) and (5.8), we obtain

$$\mathcal{S}^{\flat}(\mathfrak{J}_4,\mathfrak{J}_5) = 2n\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_5), \tag{5.9}$$

$$\mathcal{Q}^{\flat}\mathfrak{J}_4 = 2n(\mathfrak{J}_4) \tag{5.10}$$

and

$$\tau = 2n(2n+1). \tag{5.11}$$

Corollary 5.1. *A semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} endowed with $\widetilde{\nabla}$ is an EM with semi-symmetric non-metric connection $\widetilde{\nabla}$.*

The conformal curvature tensor $\widetilde{\mathcal{L}}^{\dagger}$ endowed with $\widetilde{\nabla}$ is defined as

$$\begin{aligned}\widetilde{\mathcal{L}}^{\dagger}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3-\frac{1}{2n-1}[\widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1-\widetilde{\mathcal{S}}^{\flat}(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2\\ &\quad+\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\widetilde{\mathcal{Q}}^{\flat}\mathfrak{J}_1-\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\widetilde{\mathcal{Q}}^{\flat}\mathfrak{J}_2]\\ &\quad+\frac{\widetilde{\tau}}{2n(2n-1)}[\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1-\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2].\end{aligned}\tag{5.12}$$

Using (5.5), (5.6), (5.7) and (5.8) in (5.12), we find

$$\begin{aligned}\widetilde{\mathcal{L}}^{\dagger}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \frac{3}{4(2n-1)}[\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2-\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1\\ &\quad+\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_2-\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{J}_1]\\ &\quad-\frac{3}{2(2n-1)}[\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^{\flat}-\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\xi^{\flat}].\end{aligned}\tag{5.13}$$

Taking $\mathfrak{J}_3=\xi^{\flat}$ in (5.13), we obtain

$$\widetilde{\mathcal{L}}^{\dagger}(\mathfrak{J}_1,\mathfrak{J}_2)\xi^{\flat} = 0.\tag{5.14}$$

Then, we have following result

Theorem 5.2. *A semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} admitting connection $\widetilde{\nabla}$ is $\xi^{\flat}$-conformally flat with $\widetilde{\nabla}$.*

6. Locally φ-symmetric hyperbolic Kenmotsu manifold admitting a connection $\widetilde{\nabla}$

Definition 6.1. A manifold Ω_{2n+1} endowed with $\widetilde{\nabla}$ is locally φ-symmetric [4] if

$$\varphi^2((\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)=0.$$

All vector fields $\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4$ orthogonal to $\xi^{\flat}$.

We know that

$$\begin{aligned}(\widetilde{\nabla}_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= \widetilde{\nabla}_{\mathfrak{J}_4}\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3-\mathcal{R}(\widetilde{\nabla}_{\mathfrak{J}_4}\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3\\ &\quad-\mathcal{R}(\mathfrak{J}_1,\widetilde{\nabla}_{\mathfrak{J}_4}\mathfrak{J}_2)\mathfrak{J}_3-\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)(\widetilde{\nabla}_{\mathfrak{J}_4}\mathfrak{J}_3).\end{aligned}\tag{6.1}$$

Using (3.1) and (2.7) in (6.1), we get

$$\begin{aligned}(\widetilde{\nabla}_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= (\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3+\frac{1}{2}[2\bar{\eta}(\mathfrak{J}_4)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3\\ &\quad-\bar{\eta}(\mathfrak{J}_1)\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3-\bar{\eta}(\mathfrak{J}_2)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3\\ &\quad-\bar{\eta}(\mathfrak{J}_3)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4+\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_4\end{aligned}$$

$$-\bar{\eta}(\mathfrak{J}_2)\,\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\,\mathfrak{J}_4]. \tag{6.2}$$

Covariant differentiation of (4.4) with respect to $\widetilde{\nabla}$ along $\mathfrak{J}_4$ and using (2.6), (3.5), (6.2), we obtain

$$\begin{aligned}(\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3 &= (\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3+\frac{1}{2}[2\bar{\eta}(\mathfrak{J}_4)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3\\ &\quad -\bar{\eta}(\mathfrak{J}_1)\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3-\bar{\eta}(\mathfrak{J}_2)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3\\ &\quad -\bar{\eta}(\mathfrak{J}_3)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4+\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_1)\mathfrak{J}_4\\ &\quad -\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_2)\mathfrak{J}_4+\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_1\\ &\quad -\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2+2\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_2\\ &\quad -2\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_1+2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_1)\mathfrak{J}_2\\ &\quad -2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_1-3\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_1\\ &\quad +3\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_2].\end{aligned} \tag{6.3}$$

Applying φ^2 on both side of equation (6.3) and using (2.1), (2.2), (2.3), we obtain

$$\begin{aligned}\varphi^2((\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3) &= \varphi^2((\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)+\frac{1}{2}[2\bar{\eta}(\mathfrak{J}_4)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3\\ &\quad +2\bar{\eta}(\mathfrak{J}_4)\bar{\eta}(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_1)\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3\\ &\quad -\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathcal{R}(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_2)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3\\ &\quad -\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_4)\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_3)\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4\\ &\quad -\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4)\xi^\flat+\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_4\\ &\quad +2\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\xi^\flat-\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_4\\ &\quad -2\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\xi^\flat+\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathfrak{J}_1\\ &\quad -\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathfrak{J}_2-2\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_1\\ &\quad -2\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_2)\xi^\flat-2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_2)\mathfrak{J}_1\\ &\quad +2\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_3)\mathfrak{J}_2+2\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_1)\mathfrak{J}_2\\ &\quad +2\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_4,\mathfrak{J}_1)\xi^\flat+3\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_2\\ &\quad -3\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\mathfrak{J}_1].\end{aligned} \tag{6.4}$$

Taking $\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3$ and $\mathfrak{J}_4$ orthogonal to $\xi^\flat$, then (6.4) yields

$$\varphi^2((\widetilde{\nabla}_{\mathfrak{J}_4}\widetilde{\mathcal{R}})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3)=\varphi^2((\nabla_{\mathfrak{J}_4}\mathcal{R})(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3). \tag{6.5}$$

Hence, the following theorem

Theorem 6.1. *The necessary and sufficient condition for manifold Ω_{2n+1} to be locally φ-symmetric equipped with ∇ is that it is also locally φ-symmetric endowed with $\widetilde{\nabla}$.*

7. Ricci semi-symmetric hyperbolic Kenmotsu manifold admitting a connection $\widetilde{\widetilde{\nabla}}$

A contact metric manifold Ω_{2n+1} endowed with $\widetilde{\widetilde{\nabla}}$ is Ricci semi-symmetric if $(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{S}}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4)=0$, then we have

$$\widetilde{\mathcal{S}}^\flat(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)+\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4)=0. \tag{7.1}$$

Replacing $\mathfrak{J}_1$ by $\xi^\flat$ and using (4.9), we have

$$\begin{aligned}\frac{3}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\xi^\flat,\mathfrak{J}_4)+\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\xi^\flat,\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)\\+\frac{3}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\xi^\flat)+\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\xi^\flat)\\-\frac{3}{4}\bar{\eta}(\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)=0.\end{aligned} \tag{7.2}$$

Equations (4.12) and (7.2) reduce to

$$\begin{aligned}\frac{9n}{4}\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{9n}{8}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_3)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)\\+\frac{9n}{4}\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)+\frac{9n}{8}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\bar{\eta}(\mathfrak{J}_4)\\-\frac{3}{4}\bar{\eta}(\mathfrak{J}_4)\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)=0.\end{aligned} \tag{7.3}$$

Taking $\mathfrak{J}_4=\xi^\flat$ and using (4.12), we have

$$\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=3n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3). \tag{7.4}$$

Using (4.5) in (7.4), we have

$$\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=2n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3). \tag{7.5}$$

Hence, we have the following theorem

Theorem 7.1. *A Ricci semi-symmetric hyperbolic Kenmotsu manifold Ω_{2n+1} equipped with $\widetilde{\widetilde{\nabla}}$ is an η-EM.*

Now, we have

$$\begin{aligned}(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{S}}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4)=-\widetilde{\mathcal{S}}^\flat(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_3,\mathfrak{J}_4)\\-\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_3,\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2)\mathfrak{J}_4).\end{aligned} \tag{7.6}$$

Using (4.4), (4.5) in (7.6), we have

$$(\widetilde{\mathcal{R}}(\mathfrak{J}_1,\mathfrak{J}_2).\widetilde{\mathcal{S}}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4) = (\mathcal{R}(\mathfrak{J}_1,\mathfrak{J}_2).\mathcal{S}^\flat)(\mathfrak{J}_3,\mathfrak{J}_4)-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)$$

$$
\begin{aligned}
&+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)\\
&+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)\\
&+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)\\
&+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\\
&+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\\
&-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3),
\end{aligned}
\tag{7.7}
$$

We suppose that $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}}^\flat=\mathcal{R}.\mathcal{S}^\flat$, then (7.7) can be expressed as

$$
\begin{aligned}
&-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_4)\\
&+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)+\frac{1}{2}\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)\\
&-\frac{3}{4}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)+\frac{3}{4}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_4)\\
&+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_3)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_4)-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)\\
&-\frac{3n}{2}\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_4)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)=0.
\end{aligned}
\tag{7.8}
$$

Replacing $\mathfrak{J}_4$ by $\xi^\flat$ in the (7.8) and using (2.1), (2.2), (2.3) and (2.11), we obtain

$$
\begin{aligned}
&\frac{n}{2}\bar{\eta}(\mathfrak{J}_1)\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)+\frac{5n}{2}\bar{\eta}(\mathfrak{J}_2)\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_3)+\frac{1}{4}\bar{\eta}(\mathfrak{J}_2)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_1)\\
&-\frac{1}{4}\bar{\eta}(\mathfrak{J}_1)\mathcal{S}^\flat(\mathfrak{J}_3,\mathfrak{J}_2)+\frac{3n}{2}\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)=0.
\end{aligned}
\tag{7.9}
$$

Putting $\mathfrak{J}_1=\xi^\flat$ in (7.9), we find

$$
\mathcal{S}^\flat(\mathfrak{J}_2,\mathfrak{J}_3)=2n\mathfrak{g}_1(\mathfrak{J}_2,\mathfrak{J}_3)-6n\bar{\eta}(\mathfrak{J}_2)\bar{\eta}(\mathfrak{J}_3)\Rightarrow r=4n(n+2).
\tag{7.10}
$$

Hence, we conclude the following theorem

Theorem 7.2. *A hyperbolic Kenmotsu manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$ satisfies $\widetilde{\mathcal{R}}.\widetilde{\mathcal{S}}^\flat-\mathcal{R}.\mathcal{S}^\flat=0$, then manifold Ω_{2n+1} is an η-EM.*

Definition 7.1. A Ricci soliton $(\mathfrak{g}_1, V_\flat, \lambda)$ on a Riemannian manifold is defined as

$$
(\pounds_{V_\flat}\mathfrak{g}_1)(\mathfrak{J}_1,\mathfrak{J}_2)+2\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)+2\lambda\mathfrak{g}_1(\mathfrak{J}_1,\mathfrak{J}_2)=0,
\tag{7.11}
$$

where $\pounds_{V_\flat}$ is a Lie-derivative along $V_\flat$ and λ is a constant. A triplet $(\mathfrak{g}_1, V_\flat, \lambda)$ is shrinking, steady and expanding according to $\lambda<0$, $\lambda=0$, $\lambda>0$ respectively [5].

We have two situations regarding the vector field $V_\flat$: $V_\flat \in Span\xi^\flat$ and $V_\flat \perp \xi^\flat$. We investigate only the case $V_\flat = \xi^\flat$. The Ricci soliton of data $(\mathfrak{g}_1, \xi^\flat, \lambda)$ on manifold Ω_{2n+1} equipped with $\widetilde{\nabla}$ can be defined by

$$(\widetilde{\mathcal{L}}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) + 2\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_1, \mathfrak{J}_2) + 2\lambda\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) = 0, \tag{7.12}$$

A straightforward calculation gives

$$(\widetilde{\mathcal{L}}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) = (\widetilde{\nabla}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) - \mathfrak{g}_1(\widetilde{\nabla}_{\mathfrak{J}_1}\xi^\flat, \mathfrak{J}_2) - \mathfrak{g}_1(\mathfrak{J}_1, \widetilde{\nabla}_{\mathfrak{J}_2}\xi^\flat). \tag{7.13}$$

Now using (2.1), (2.5), (3.7) and (3.8) in (7.13), we have

$$(\widetilde{\mathcal{L}}_{\xi^\flat}\mathfrak{g}_1)(\mathfrak{J}_1, \mathfrak{J}_2) = 2[\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) + \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2)]. \tag{7.14}$$

From (4.5), (5.9), (7.5) and (7.12), we yields

$$(1 + 3n + \lambda)\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) + (1 + \frac{3n}{2})\bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2) = 0. \tag{7.15}$$

Taking $\mathfrak{J}_1 = \mathfrak{J}_2 = \xi^\flat$ in (7.15), we get

$$\lambda = -\frac{3n}{2} < 0.$$

Thus, we state the following theorem

Theorem 7.3. *A triplet $(\mathfrak{g}_1, \xi^\flat, \lambda)$ on manifold Ω_{2n+1} endowed with $\widetilde{\nabla}$ is always shrinking.*

8. Example of hyperbolic Kenmotsu Manifold

Example 8.1. Let $\Omega_3 = (x, y, z) \in R^3 : z \neq 0$ be a 3-dimensional manifold with the standard coordinates (x, y, z) of R^3 [20]. Let $\varsigma_1 = e^z\frac{\partial}{\partial x}, \varsigma_2 = e^z\frac{\partial}{\partial y}$, $\varsigma_3 = \frac{\partial}{\partial z} = \xi^\flat$ be linear independent vector fields.

Suppose $\mathfrak{g}_1$ be the Ω_3 Riemannian metric specified by

$$\begin{aligned} &\mathfrak{g}_1(\varsigma_1, \varsigma_2) = \mathfrak{g}_1(\varsigma_2, \varsigma_3) = \mathfrak{g}_1(\varsigma_3, \varsigma_1) = 0, \\ &\mathfrak{g}_1(\varsigma_1, \varsigma_1) = 1, \quad \mathfrak{g}_1(\varsigma_2, \varsigma_2) = \mathfrak{g}_1(\varsigma_3, \varsigma_3) = -1, \end{aligned} \tag{8.1}$$

where

$$\mathfrak{g}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and φ is $(1,1)$-tensor field defined by

$$\varphi(\varsigma_1) = \varsigma_2, \varphi(\varsigma_2) = \varsigma_1, \varphi(\varsigma_3) = 0. \tag{8.2}$$

By using linearity of φ and $\mathfrak{g}_1$, we have

$$\bar{\eta}(\varsigma_3) = -1, \quad \varphi^2\mathfrak{J}_1 = \mathfrak{J}_1 + \bar{\eta}(\mathfrak{J}_1)\varsigma_3,$$
$$\mathfrak{g}_1(\varphi\mathfrak{J}_1, \varphi\mathfrak{J}_2) = -\mathfrak{g}_1(\mathfrak{J}_1, \mathfrak{J}_2) - \bar{\eta}(\mathfrak{J}_1)\bar{\eta}(\mathfrak{J}_2) \tag{8.3}$$

Here $\bar{\eta}(\mathfrak{J}_1) = \mathfrak{g}_1(\mathfrak{J}_1, \varsigma_3)$ defines a 1-form on Ω_3. Hence for $\xi^\flat = \varsigma_3$, the structure $(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$ defined on Ω_3. By applying definition $[\mathfrak{J}_1, \mathfrak{J}_2] = \mathfrak{J}_1(\mathfrak{J}_2 f) - \mathfrak{J}_2(\mathfrak{J}_1 f)$, the Lie bracket can be computed

$$\begin{aligned} &[\varsigma_1, \varsigma_1] = 0, \quad [\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = -\varsigma_1, \\ &[\varsigma_2, \varsigma_1] = 0, \quad [\varsigma_2, \varsigma_2] = 0, \quad [\varsigma_2, \varsigma_3] = -\varsigma_2, \\ &[\varsigma_3, \varsigma_1] = \varsigma_1, \quad [\varsigma_3, \varsigma_2] = \varsigma_2, \quad [\varsigma_3, \varsigma_3] = 0. \end{aligned} \tag{8.4}$$

Koszul's formula is given as

$$\begin{aligned} 2\mathfrak{g}_1\left(\nabla_{\mathfrak{J}_1}\mathfrak{J}_2, \mathfrak{J}_3\right) &= \mathfrak{J}_1\mathfrak{g}_1\left(\mathfrak{J}_2, \mathfrak{J}_3\right) + \mathfrak{J}_2\mathfrak{g}_1\left(\mathfrak{J}_3, \mathfrak{J}_1\right) - \mathfrak{J}_3\mathfrak{g}_1\left(\mathfrak{J}_1, \mathfrak{J}_2\right) \\ &\quad + \mathfrak{g}_1\left(\left[\mathfrak{J}_1, \mathfrak{J}_2\right], \mathfrak{J}_3\right) - \mathfrak{g}_1\left(\left[\mathfrak{J}_2, \mathfrak{J}_3\right], \mathfrak{J}_1\right) + \mathfrak{g}_1\left(\left[\mathfrak{J}_3, \mathfrak{J}_1\right], \mathfrak{J}_2\right). \end{aligned} \tag{8.5}$$

Now utilizing the above equation, we can compute

$$\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1 = -\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_2 = 0, \quad \nabla_{\varsigma_1}\varsigma_3 = -\varsigma_1, \\ &\nabla_{\varsigma_2}\varsigma_1 = 0, \quad \nabla_{\varsigma_2}\varsigma_2 = \varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = -\varsigma_2, \\ &\nabla_{\varsigma_3}\varsigma_1 = 0, \quad \nabla_{\varsigma_3}\varsigma_2 = 0, \quad \nabla_{\varsigma_3}\varsigma_3 = 0. \end{aligned} \tag{8.6}$$

Also $\mathfrak{J}_1 = \mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3$ and $\xi^\flat = \varsigma_3$, then we have

$$\begin{aligned} \nabla_{\mathfrak{J}_1}\xi^\flat &= \nabla_{\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3}\varsigma_3 \\ &= \mathfrak{J}_1^1\nabla_{\varsigma_1}\varsigma_3 + \mathfrak{J}_1^2\nabla_{\varsigma_2}\varsigma_3 + \mathfrak{J}_1^3\nabla_{\varsigma_3}\varsigma_3 \\ &= -\mathfrak{J}_1^1\varsigma_1 - \mathfrak{J}_1^2\varsigma_2 \end{aligned} \tag{8.7}$$

and

$$\begin{aligned} -\mathfrak{J}_1 - \bar{\eta}(\mathfrak{J}_1)\xi^\flat &= -\left(\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3\right) - \mathfrak{g}_1\left(\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3, \varsigma_3\right)\varsigma_3 \\ &= -\mathfrak{J}_1^1\varsigma_1 - \mathfrak{J}_1^2\varsigma_2 - \mathfrak{J}_1^3\varsigma_3 + \mathfrak{J}_1^3\varsigma_3 \\ &= -\mathfrak{J}_1^1\varsigma_1 - \mathfrak{J}_1^2\varsigma_2, \end{aligned} \tag{8.8}$$

where $\mathfrak{J}_1^1, \mathfrak{J}_1^2, \mathfrak{J}_1^3$ are scalars. From (8.7) and (8.8), the structure $(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$ is hyperbolic Kenmotsu structure. Therefore $\Omega_3(\varphi, \xi^\flat, \bar{\eta}, \mathfrak{g}_1)$ is hyperbolic Kenmotsu manifold. In reference of (2.1), (2.3), (3.1) and (8.6), we get

$$\widetilde{\nabla}_{\varsigma_1}\varsigma_1 = -\varsigma_3, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_2 = 0, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_3 = -\frac{3}{2}\varsigma_1,$$

$$\widetilde{\nabla}_{\varsigma_2}\varsigma_1 = 0, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_2 = \varsigma_3, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_3 = -\frac{3}{2}\varsigma_2, \tag{8.9}$$

$$\widetilde{\nabla}_{\varsigma_3}\varsigma_1 = \frac{1}{2}\varsigma_1, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_2 = \frac{1}{2}\varsigma_2, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_3 = 0.$$

From (3.2) and (3.3), we yields

$$\widetilde{\mathcal{T}}(\varsigma_1, \varsigma_3) = \bar{\eta}(\varsigma_3)\varsigma_1 - \bar{\eta}(\varsigma_1)\varsigma_3 = -\varsigma_1 \neq 0$$

and

$$\begin{aligned}(\widetilde{\nabla}_{\varsigma_1}\mathfrak{g}_1)(\varsigma_1, \varsigma_3) &= \frac{1}{2}[2\bar{\eta}(\varsigma_1)\mathfrak{g}_1(\varsigma_1, \varsigma_3) - \bar{\eta}(\varsigma_1)\mathfrak{g}_1(\varsigma_1, \varsigma_3) - \bar{\eta}(\varsigma_3)\mathfrak{g}_1(\varsigma_1, \varsigma_1)] \\ &= \frac{1}{2} \neq 0.\end{aligned}$$

Consequently, a new type of semi-symmetric non-metric connection defined in (3.1). Also,

$$\begin{aligned}\widetilde{\nabla}_{\mathfrak{J}_1}\xi^{\flat} &= \widetilde{\nabla}_{\mathfrak{J}_1^1\varsigma_1 + \mathfrak{J}_1^2\varsigma_2 + \mathfrak{J}_1^3\varsigma_3}\varsigma_3 \\ &= \mathfrak{J}_1^1\widetilde{\nabla}_{\varsigma_1}\varsigma_3 + \mathfrak{J}_1^2\widetilde{\nabla}_{\varsigma_2}\varsigma_3 + \mathfrak{J}_1^3\widetilde{\nabla}_{\varsigma_3}\varsigma_3 \\ &= -\frac{3}{2}\mathfrak{J}_1^1\varsigma_1 - \frac{3}{2}\mathfrak{J}_1^2\varsigma_2,\end{aligned} \tag{8.10}$$

Equation (3.7) can be verified by using (8.7) and (8.10).

The components of $\mathcal{R}$ with connection ∇ are given as

$$\mathcal{R}(\varsigma_1, \varsigma_2)\varsigma_1 = -\varsigma_2, \quad \mathcal{R}(\varsigma_1, \varsigma_3)\varsigma_1 = -\varsigma_3, \quad \mathcal{R}(\varsigma_2, \varsigma_3)\varsigma_1 = 0,$$

$$\mathcal{R}(\varsigma_1, \varsigma_2)\varsigma_2 = -\varsigma_1, \quad \mathcal{R}(\varsigma_1, \varsigma_3)\varsigma_2 = 0, \quad \mathcal{R}(\varsigma_2, \varsigma_3)\varsigma_2 = \varsigma_3, \tag{8.11}$$

$$\mathcal{R}(\varsigma_1, \varsigma_2)\varsigma_3 = 0, \quad \mathcal{R}(\varsigma_1, \varsigma_3)\varsigma_3 = -\varsigma_1, \quad \mathcal{R}(\varsigma_2, \varsigma_3)\varsigma_3 = -\varsigma_2,$$

also $\mathcal{R}(\varsigma_i, \varsigma_i)\varsigma_i = 0; i = 1, 2, 3$ from simple calculations. We can verify (2.7), (2.8), (2.9), (2.10) and (2.11).

Similarly, the component of $\widetilde{\mathcal{R}}$ endowed with connection $\widetilde{\nabla}$ are as under:

$$\widetilde{\mathcal{R}}(\varsigma_1,\varsigma_2)\varsigma_1=-\frac{3}{2}\varsigma_2,\quad \widetilde{\mathcal{R}}(\varsigma_1,\varsigma_3)\varsigma_1=-\frac{3}{2}\varsigma_3,\quad \widetilde{\mathcal{R}}(\varsigma_2,\varsigma_3)\varsigma_1=0,$$

$$\widetilde{\mathcal{R}}(\varsigma_1,\varsigma_2)\varsigma_2=-\frac{3}{2}\varsigma_1,\quad \widetilde{\mathcal{R}}(\varsigma_1,\varsigma_3)\varsigma_2=0,\qquad \widetilde{\mathcal{R}}(\varsigma_2,\varsigma_3)\varsigma_2=\frac{3}{2}\varsigma_3, \tag{8.12}$$

$$\widetilde{\mathcal{R}}(\varsigma_1,\varsigma_2)\varsigma_3=0,\qquad \widetilde{\mathcal{R}}(\varsigma_1,\varsigma_3)\varsigma_3=-\frac{3}{4}\varsigma_1,\quad \widetilde{\mathcal{R}}(\varsigma_2,\varsigma_3)\varsigma_3=-\frac{3}{4}\varsigma_2,$$

along with $\widetilde{\mathcal{R}}(\varsigma_i,\varsigma_i)\varsigma_i=0; i=1,2,3$. Thus, we can verify (4.4), (4.8), (4.9), (4.10) and (4.11).

The Ricci tensor $\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)$ of connection ∇ can be derived by using (8.11) in $\mathcal{S}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)=\sum_{i=1}^{3}\mathfrak{g}_1(\mathcal{R}(e_i,\mathfrak{J}_1)\mathfrak{J}_2,e_i)$. As follows:

$$\mathcal{S}^\flat(\varsigma_1,\varsigma_1)=2,\quad \mathcal{S}^\flat(\varsigma_2,\varsigma_2)=-2,\quad \mathcal{S}^\flat(\varsigma_3,\varsigma_3)=-2. \tag{8.13}$$

The Ricci tensor $\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)$ endowed with $\widetilde{\nabla}$ can be derived by using (8.12) in $\widetilde{\mathcal{S}}^\flat(\mathfrak{J}_1,\mathfrak{J}_2)=\sum_{i=1}^{3}\mathfrak{g}_1(\widetilde{\mathcal{R}}(e_i,\mathfrak{J}_1)\mathfrak{J}_2,e_i)$. It is as follows:

$$\widetilde{\mathcal{S}}^\flat(\varsigma_1,\varsigma_1)=3,\quad \widetilde{\mathcal{S}}^\flat(\varsigma_2,\varsigma_2)=-3,\quad \widetilde{\mathcal{S}}^\flat(\varsigma_3,\varsigma_3)=-\frac{3}{2}. \tag{8.14}$$

In view of (8.13) and (8.14), the scalar curvature can be calculated as under:

$$\tau=\sum_{i=1}^{3}\mathcal{S}^\flat(e_i,e_i)=\mathcal{S}^\flat(\varsigma_1,\varsigma_1)-\mathcal{S}^\flat(\varsigma_2,\varsigma_2)-\mathcal{S}^\flat(\varsigma_3,\varsigma_3)=6,$$

$$\widetilde{\tau}=\sum_{i=1}^{3}\widetilde{\mathcal{S}}^\flat(e_i,e_i)=\widetilde{\mathcal{S}}^\flat(\varsigma_1,\varsigma_1)-\widetilde{\mathcal{S}}^\flat(\varsigma_2,\varsigma_2)-\widetilde{\mathcal{S}}^\flat(\varsigma_3,\varsigma_3)=\frac{15}{2}.$$

Therefore, we can say that the example I provided completely correspond to our investigations.

REFERENCES

1. M. Ahmad and K. Ali: *CR-submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a quarter-symmetric non-metric connection*, J. Math. Comput. Sci. **3**(2013), no. 3, 905-917.
2. N. S. Agashe and M. R. Chafle: *A semi-symmetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. **23**(1992), 399-409.

3. N. S. Agashe and M. R. Chafle: *On submanifolds of a Riemannian manifold with semi-symmetric non-metric connection*, Tensor N. S., **55(2)**(1994), 120-130.

4. A. Barman: *On P-Sasakian manifolds admitting semi-symmetric metric connection*, Publication de L' inst. Math., **95**(2014), 239-247.

5. C. S. Bagewadi and G. Ingalahalli: *Ricci Solitons in Lorentzian α-Sasakian manifolds*, Acta Math. Acad. P. N., **28**(2012), 59-68.

6. E. Bartolotti: *Sulla geometria della variata a connection affine*, Ann. diMat. **4(8)**(1930), 53-101.

7. L. Bhatt and K. K. Dube: *CR-submanifolds of trans-hyperbolic Sasakian manifold*, Acta Ciencialndica, **31**(2003), 91-96.

8. B. B. Chaturvedi and P. N. Pandey: *Semi-symmetric non-metric connection on a $\ddot{K}$ahler manifold*, Differential Geometry-Dynamical Systems, **10**(2008), 86-90.

9. S. K. Chaubey and A. Yildiz: *Riemannian manifold admitting a new type semi-symmetric non-metric connection*, Turkish Journal of Mathematics, **43**(2019), 1887-1904.

10. S. K. Chaubey: *On semi-symmetric non-metric connection*, Prog. of Math, **41-42**(2007), 11-20.

11. S. K. Chaubey and A. C. Pandey: *Some properties of a semi-symmetric non-metric connection on a Sasakian manifold*, Int. J. Contemp. Math. Sciences, **8**(2013), no. 16, 789-799.

12. U. C. De and D. Kamilya: *Hypersurfaces of a Riemannian manifold with semisymmetric non-metric connection*, J. Indian Inst. Sci., **75** (1995), 707-710.

13. U. C. De and J. Sengupta: *On a type of semi-symmetric non-metric connection*, Bull. Cal. Math. Soc., **92**(2000), no. 5, 375-384.

14. L. Das, M. Ahmad and A. Haseeb: *On semi-symmetric submanifolds of a nearly Sasakian manifold admitting a semi symmetric non-metric connection*, Journal of Applied Analysis, USA, **17**(2011), pp.1-12.

15. L. Das and M. Ahmad: *CR-submanifolds of a Lorenzian Para Sasakian Manifold enbowed with a quarter symmetric non-metric connection*, Math Science Research Journal, **13(7)**(2009), 161-169.

16. A. Friedmann and A. Schouten: *$\ddot{U}$ber die Geometric der halbsymmetrischen $\ddot{U}$bertragung*, Math. Zeitschr., **21**(1924), 211-223.

17. H. A. Hayden: *Subspaces of a space with torsion*, Proc. London Math. Soc., **34**(1932), 27-50.

18. L. K. Pandey and R. H. Ojha: *Semi-symmetric metric and non-metric connections in Lorentzian paracontact manifold*, Bull. Cal. Math. Soc., **93**(2001), no. 6, 497-504.

19. Pankaj, S. K. Chaubey and R. Prasad: *Sasakian manifolds with admitting a non-symmetric non-metric connection*, Palestine Journal of Mathematics, **9(2)**(2020), 698-710.

20. Pankaj, S. K. Chaubey and G. Ayar: *Yamabe and gradient Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manofolds*, Balkan Society of Geometers, **23**(2021), 175-188.

21. J. Sengupta, U. C. De and T. Q. Binh: *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math., **31**(2000), no.(1-2), 1659-1670.

22. M. M. Tripathi and N. Kakkar: *On a semi-symmetric non-metric connection in a Kenmotsu manifold*, Bull. Cal. Math. Soc., **16**(2001), no. 4, 323-330.

23. M. D. Upadhyay and K. K. Dube: *Almost contact hyperbolic* (ϕ, ξ, η, g)*-structure*, Acta. Math. Acad. Scient. Hung. Tomus, **28**(1976), 1-4.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 141–151
https://doi.org/10.22190/FUMI230310009A
Original Scientific Paper

A REPRESENTATION FORMULA FOR THE RESOLVENT OF CONFORMABLE FRACTIONAL STURM-LIOUVILLE OPERATOR

Bilender P. Allahverdiev[1,2], Hüseyin Tuna[3] and Yüksel Yalçınkaya[4]

[1] **Khazar University, AZ1096 Baku, Azerbaijan**
[2] **UNEC-Azerbaijan State University of Economics**
Department of Mathematics, Baku, Azerbaijan
[3] **Faculty of Art Sciences, Department of Mathematics**
Mehmet Akif Ersoy University, 15030 Burdur, Turkey
[4] **Faculty of Art Sciences, Department of Mathematics**
Süleyman Demirel University, 32260 Isparta, Turkey

ORCID IDs: Bilender P. Allahverdiev https://orcid.org/0000-0002-9315-4652
Hüseyin Tuna https://orcid.org/0000-0001-7240-8687
Yüksel Yalçınkaya https://orcid.org/0000-0002-1633-8343

Abstract. In this study, the resolvent of the conformable fractional Sturm–Liouville operator is considered. An integral representation for the resolvent of this operator is obtained.

Keywords: resolvent operator, partial differential equations, comforable fractional integral.

1. Introduction

Fractional analysis studies started with the correspondence between Leibniz and L'Hospital in 1695 and have continued until today. Euler made the first attempt in 1738 and tried to explain the fractional derivative of an x^a-shaped function with the help of the Gamma function. In 1820, Lacroix, in parallel with Euler's idea, introduced the half (1/2nd order) derivatives of x^a-shaped functions with a formula. The positive arbitrary derivative of a function was first defined by Fourier in 1822. In 1823, the problem known as the "Brachistochrone Problem" was formulated and

Received March 10, 2023, revised: December 28, 2023, accepted: December 29, 2023
Communicated by Dijana Mosić
Corresponding Author: Hüseyin Tuna. E-mail addresses: bilenderpasaoglu@gmail.com (B. P. Allahverdiev), hustuna@gmail.com (H. Tuna), matyuksel@hotmail.com (Y. Yalçınkaya)
2010 *Mathematics Subject Classification.* Primary 34L10; Secondary 34K08, 47A10

shown to be solved by Abel. Work on fractional calculus with Liouville in 1832; exponential showed the arbitrary order derivative of functions and any order integral of a function. Liouville's work was developed by Riemann in 1847 and the most basic definition, the Riemann–Liouville definition, was put forward ([6]). In recent years, Khalil et al. have defined the definition of conformable fractional derivative and conformable fractional integral using the classical derivative definition. Later in ([1]), Abdeljawad proved some properties of conformable fractional derivatives. Conformable fractional derivative aims to broaden the definition of classical derivative carrying the natural features of the classical derivative. The main difference of the conformable derivative from other fractional derivatives is that it has some of the properties of the classical derivative. For example, the rule of derivative of the product of two functions, the rule of derivative of the division of two functions, etc. In addition, with the help of the conformable differential equations obtained by the definition of derivative aims at a new look for differential equation theory ([8]).

On the other hand, resolvent operators play an important role in the spectral analysis of partial differential equations and in the theory of operators. In the classical Sturm–Liouville equation, the integral representation of the resolvent was first given by H. Weyl in 1910. Similar representations were obtained in [11, 10]. Examining the same problem under the conformable fractional calculus frame will yield interesting results. Firstly, we construct the resolvent operator of this equation. After, we will give a representation theorem for the resolvent operator.

Now, we will be given some definitions and properties related to conformable fractional calculus (see [9, 1, 2, 3, 4, 5, 7, 12]). Throughout this paper, we will fix $\alpha \in (0,1)$.

Definition 1.1. A function $f : [0,\infty) \longrightarrow \mathbb{R} =(-\infty,\infty)$ the conformable fractional derivative of order α of f at $\zeta > 0$ was defined by

$$T_\alpha f(\zeta) = \lim_{\xi\to\infty} \frac{f(\zeta+\xi\zeta^{1-\alpha}) - f(\zeta)}{\xi}, \qquad where \quad \zeta \in [0,\infty). \tag{1.1}$$

Definition 1.2. The conformable fractional integral starting from a of a function f of order $0 < \alpha \le 1$ is defined by

$$(I_\alpha^a f)(\zeta) = \int_a^\zeta f(\varsigma) d\alpha(\varsigma, a) = \int_a^\zeta (\varsigma - a)^{\alpha-1} f(\varsigma) d\varsigma.$$

Similarly, in the right case we have

$$({}^b I_\alpha f)(\zeta) = \int_\zeta^b f(\varsigma) d\alpha(b, \varsigma) = \int_\zeta^b (b-\varsigma)^{\alpha-1} f(\varsigma) d\varsigma.$$

Let us introduce the following space: $L_\alpha^2(0,b) := \left\{ f : \left(\int_0^b |f(\zeta)|^2 \, d_\alpha \zeta \right)^{1/2} < \infty \right\}$, where $0 < b \leq +\infty$. $L^2_\alpha(0,b)$ is a Hilbert space (see [9]) endowed with the inner product$(f,g) := \int_0^b f\overline{g} d_\alpha \zeta$.

2. Main Result

Let us consider the conformable fractional Sturm–Liouville equations

$$-T^2_\alpha y(\zeta) + V(\zeta)y(\zeta) = \lambda y(\zeta), \tag{2.1}$$

with the boundary conditions

$$y(0,\lambda)\cos\beta + T_\alpha y(0,\lambda)\sin\beta = 0, \tag{2.2}$$

$$y(b,\lambda)\cos\gamma + T_\alpha y\ b,\lambda)\sin\gamma = 0,\ \ \gamma,\beta \in \mathbb{R}, \tag{2.3}$$

where λ is a complex eigenvalue parameter, V is a real-valued function on $[0,\infty)$ and $V \in L^1_{\alpha,loc}(0,\infty)$.

We will denote by $\theta(\zeta,\lambda)$ and $\psi(\zeta,\lambda)$ the solutions of the equation (2.1) subject to the initial conditions

$$\begin{aligned} &\theta(0,\lambda) = \sin\beta,\ T_\alpha\theta(0,\lambda) = -\cos\beta, \\ &\psi(0,\lambda) = \cos\gamma,\ T_\alpha\theta(0,\lambda) = \sin\gamma. \end{aligned} \tag{2.4}$$

Let us define

$$Z_b(\zeta,\lambda) = \psi(\zeta,\lambda) + \ell(\lambda,b)\theta(\varsigma,\lambda) \in L^2_\alpha(0,b).$$

Lemma 2.1. *For each nonreal* λ,

$$Z_b(\zeta,\lambda) \quad\rightarrow\quad Z(\zeta,\lambda),$$

$$\int_0^b |Z_b(\zeta,\lambda)|^2\, d_\alpha t \quad\rightarrow\quad \int_0^\infty |Z(\zeta,\lambda)|^2\, d_\alpha\zeta,\ \ b\rightarrow\infty.$$

Proof. It is obvious that

$$Z_b(\zeta,\lambda) = Z(\zeta,\lambda) + [\ell(\lambda,b) - m(\lambda)]\theta(\zeta,\lambda)$$

where $Z(\zeta,\lambda) \in L^2_\alpha(0,\infty)$ and $m(\lambda)$ is the Titchmarsh–Weyl function ([5]). We know that $\ell(\lambda,b)$ varies on a circle with a finite radius r_b in the plane ([5]). In the limit-circle case, $\ell(\lambda,b) \longrightarrow m(\lambda)$([5]); therefore $Z_b(\zeta,\lambda) \rightarrow Z(\zeta,\lambda)$ and since $\theta(\zeta,\lambda) \in L^2_\alpha(0,\infty)$, we get$\int_0^b |Z_b(\zeta,\lambda)|^2 d_\alpha\zeta \rightarrow \int_0^\infty |Z(\zeta,\lambda)|^2 d_\alpha\zeta$.In the limit-point case ([5]), we have

$$|\ell(\lambda,b) - m(\lambda)| \leq r_b = (2\mathrm{Im}\lambda\int_0^b |\theta(\zeta,\lambda)|^2 d_\alpha\zeta)^{-1} \quad (\mathrm{Im}\lambda \neq 0).$$

As $r_b \to 0$, $Z_b(\zeta,\lambda) \to Z(\zeta,\lambda)$. Moreover,

$$\begin{aligned}\int_0^b |\{\ell(\lambda,b)-m(\lambda)\}\theta(\zeta,\lambda)|^2 d_\alpha\zeta &= |\ell(\lambda,b)-m(\lambda)|^2\int_0^b |\theta(\zeta,\lambda)|^2 d_\alpha\zeta \\ &\leq (4(\mathrm{Im}\lambda)^2\int_0^b |\theta(\zeta,\lambda)|^2 d_\alpha\zeta)^{-1}.\end{aligned}$$

Therefore,

$$\int_0^b |Z_b(\zeta,\lambda)|^2\, d_\alpha t \to \int_0^\infty |Z(\zeta,\lambda)|^2\, d_\alpha\zeta.$$

Let $f(.)\in L_\alpha^2(0,\infty)$. We put

$$G_b(\zeta,\varsigma,\lambda)=\begin{cases} Z_b(\zeta,\lambda)\theta(\zeta,\lambda), & \varsigma\leq\zeta \\ \theta(\zeta,\lambda)Z_b(\varsigma,\lambda), & \varsigma>\zeta,\end{cases}$$

$$(R_bf)(\zeta,\lambda)=\int_0^b G_b(\zeta,\varsigma,\lambda)f(\varsigma)d_\alpha\varsigma,\ \lambda\in\mathbb{C}. \tag{2.5}$$

Clearly, Eq. (2.5) satisfies the boundary value problem (2.1)-(2.3) and the problem (2.1) -(2.3) has a compact resolvent (see [4]). □

Let $\lambda_{m,b}$ and $\theta_{m,b}(\zeta):=\theta_{m,b}(\zeta,\lambda_{m,b})$ $(m\in\mathbb{N}:=\{1,2,3,...\})$ be the eigenvalues and eigenfunctions of the problem (2.1), (2.3), (2.5) and $\alpha_{m,b}^2=\int_0^b\theta_{m,b}^2(\zeta)d_\alpha\varsigma$.Then, we have [4]

$$\int_0^b |f(\zeta)|^2\, d_\alpha\zeta=\sum_{m=1}^\infty\frac{1}{\alpha_{m,b}^2}\left|\int_0^b f(\zeta)\varphi_{m,b}(\zeta)d_\alpha\zeta\right|^2. \tag{2.6}$$

Let

$$\varrho_b(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_{m,b}<0}\frac{1}{\alpha_{m,b}^2}, & for\quad \lambda\leq 0 \\ \sum_{0\leq\lambda_{m,b}<\lambda}\frac{1}{\alpha_{m,b}^2} & for\quad \lambda>0.\end{cases}$$

Thus the equality (2.6) can be written as

$$\int_0^b |f(\zeta)|^2\, d_\alpha t=\int_{-\infty}^\infty |F(\lambda)|^2\, d\varrho_b(\lambda), \tag{2.7}$$

where $F(\lambda)=\int_0^b f(\zeta)\varphi_{m,b}(\zeta)d_\alpha\zeta$.

Lemma 2.2. *For any positive S, there is a positive number $B=B(S)$ not depending on b so that*

$$V_{-S}^S\{\varrho_b(\lambda)\}=\sum_{-S\leq\lambda_{m,b}<S}\frac{1}{\alpha_{m,b}^2}=\varrho_b(S)-\varrho_b(-S)<B. \tag{2.8}$$

Proof. Let $\sin\beta \neq 0$. Since $\theta(\zeta,\lambda)$ is continuous in domain $-S \le \lambda \le S,\ 0 \le t \le b$,where a is an arbitrary fixed positive number and the condition $\theta(0,\lambda) = \sin\beta$, there exists a positive number h such that for $|\lambda| < S$,

$$\frac{1}{h}\left(\int_0^h \theta(\zeta,\lambda)d_\alpha\zeta\right)^2 > \frac{1}{2}\sin^2\beta. \tag{2.9}$$

Let

$$f_h(\zeta) = \begin{cases} \frac{1}{h}, & 0 \le t \le h \\ 0, & \zeta > h. \end{cases}$$

Then using (2.9), we get

$$\begin{aligned}
\int_0^h f_h^2(\zeta)d_\alpha\zeta &= \frac{1}{\alpha h^{2-\alpha}} \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{h}\int_0^h \theta(\zeta,\lambda)d_\alpha\zeta\right)^2 d\varrho_b(\lambda) \\
&\ge \int_{-S}^{S}\left(\frac{1}{h}\int_0^h \theta(\zeta,\lambda)d_\alpha\zeta\right)^2 d\varrho_\alpha(\lambda) \\
&> \frac{1}{2}\sin^2\beta\left\{\varrho_b(S) - \varrho_b(-S)\right\}.
\end{aligned}$$

If $\sin\beta = 0$, then we define $f_h(\zeta)$ by the formula $f_h(\zeta) = \begin{cases} \frac{1}{h^2}, & 0 \le \zeta \le h \\ 0, & \zeta > h. \end{cases}$ This proves the lemma. □

Now, we will give an expansion into a Fourier series of resolvent. After $\alpha-$integration by parts, we have

$$\begin{aligned}
&\int_0^b \left[-T_\alpha^2 y(\zeta,\lambda) + V(\zeta)y\zeta(\zeta,\lambda)\right]\theta_{m,b}(\zeta)d_\alpha\zeta \\
&= \int_0^b \left[-T_\alpha^2\varphi_{m,b}(\zeta) + V(\zeta)\theta_{m,b}(\zeta)\right]y(\zeta,\lambda)d_\alpha\zeta \\
&= -\lambda_{m,b}\int_0^b y(\zeta,\lambda)\theta_{m,b}(\zeta)d_\alpha\zeta = -\lambda_{m,b}\phi_m(\lambda) \qquad (m \in \mathbb{N}).
\end{aligned}$$

Set

$$y(\zeta,\lambda) = \sum_{m=1}^{\infty}\phi_m(\lambda)\psi_{m,b}(\zeta), \qquad a_m = \int_0^b f(\zeta)\psi_{m,b}(\zeta)d_\alpha\zeta\ (m \in \mathbb{N}).$$

Since $y(\zeta,\lambda)$ satisfies the equation

$$-T_\alpha^2 y(\zeta,\lambda)+(V(\zeta)-\lambda)y(\zeta,\lambda)=f(\zeta),$$

we obtain

$$\begin{aligned} a_m &= \int_0^b \left[-T_\alpha^2 y(\zeta,\lambda)+(V(\zeta)-\lambda)y(\zeta,\lambda)\right]\theta_{m,b}(\zeta)d_\alpha\zeta \\ &= -\lambda_{m,b}\phi_m(\lambda)+\lambda\phi_m(\lambda). \end{aligned}$$

Therefore, we have

$$\phi_m(\lambda)=\frac{a_m}{\lambda-\lambda_{m,b}} \qquad (m\in\mathbb{N})$$

and

$$\begin{aligned} y(\zeta,\lambda) &= \int_0^b G_b(\zeta,\varsigma,\lambda)f(\varsigma)d_\alpha\varsigma \\ &= \sum_{m=1}^{\infty}\frac{a_m\theta_{m,b}(\zeta)}{\lambda-\lambda_{m,b}}. \end{aligned}$$

Thus, we get the following expansion

$$(R_bf)(\zeta,z) = \sum_{m=1}^{\infty}\frac{\left\{\int_0^b f(\varsigma)\theta_{m,b}(\zeta)d_\alpha\varsigma\right\}\theta_{m,b}(\zeta)}{\alpha_{m,b}^2(z-\lambda_{m,b})} \tag{2.10}$$

$$= \int_{-\infty}^{\infty}\frac{\left\{\int_0^b f(\varsigma)\theta_{m,b}(\varsigma,\lambda)d_\alpha\varsigma\right\}\theta(\zeta,\lambda)}{z-\lambda}d\varrho_b(\lambda). \tag{2.11}$$

Lemma 2.3. *For each non real z and fixed ζ,*

$$\int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho_b(\lambda)<S. \tag{2.12}$$

Letting $f(\varsigma)=\frac{\theta_{m,b}(\varsigma)}{\alpha_{m,b}}$ in the (2.12), by virtue of the facts that the eigenfunctions $\theta_{m,b}(\zeta)$ are orthogonal, we obtain

$$\frac{1}{\alpha_{m,b}}\int_0^b G_b(\zeta,\varsigma,z)\theta_{m,b}(\varsigma)d_\alpha\varsigma=\frac{\theta_{m,b}(\zeta)}{\alpha_{m,b}(z-\lambda_{m,b})}. \tag{2.13}$$

From (2.13) and (2.6), we conclude that

$$\begin{aligned} \int_0^b \left|G_b(\zeta,\varsigma,z)\right|^2 d_\alpha\varsigma &= \sum_{m=1}^{\infty}\frac{\left|\theta_{m,b}(\zeta)\right|^2}{\alpha_{m,b}^2\left|z-\lambda_{m,b}\right|^2} \\ &= \int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho_b(\lambda). \end{aligned}$$

From Lemma 2.1, the last integral convergent.

By virtue of Lemma 2.2, the set$\{\varrho_b(\lambda)\}$ is bounded. Using a well-known theorem on passing to the limit inside a Stieltjes integral, we can find a sequence $\{b_k\}$ such that the function $\varrho_{b_k}(\lambda)$ converges to a monotone function $\varrho(\lambda)$ (as $b_k \to \infty$).

Lemma 2.4. *Let z be a non-real number and ζ be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\theta(\zeta,\lambda)}{z-\lambda} \right|^2 d\varrho(\lambda) \leq S. \tag{2.14}$$

The integral in (2.14) is given as generalized Riemann–Stieltjes integrals.

Proof. For arbitrary $\eta > 0$, it follows from (2.12) that$\int_{-\eta}^{\eta} \left| \frac{\varphi(\varsigma,\lambda)}{z-\lambda} \right|^2 d\varrho_b(\lambda) < S$.Letting $\eta \to \infty$, we get the desired result. □

Lemma 2.5. *For arbitrary $\eta > 0$, the inequalities*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty, \quad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty. \tag{2.15}$$

Proof. Let $\sin\beta \neq 0$. If we put $\zeta = 0$ in (2.14), we get$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty$.Let $\sin\beta = 0$. Then

$$\frac{1}{\alpha_{m,b}} \int_0^b T_{\alpha,\zeta} G_b(\zeta,\varsigma,z)\theta_{m,b}(\varsigma) d_\alpha\varsigma = \frac{T_{\alpha,\zeta}\theta_{m,b}(\zeta)}{\alpha_{m,b}(z-\lambda_{m,b})}.$$

From (2.7), we get

$$\int_0^b |T_{\alpha,\zeta} G_b(\zeta,\varsigma,z)|^2 d_\alpha\zeta = \int_{-\infty}^{\infty} \left| \frac{T_{\alpha,\zeta}\theta(\zeta,\lambda)}{z-\lambda} \right|^2 d\varrho_b(\lambda).$$

□

Lemma 2.6. *Let $f(.) \in L_\alpha^2(0,\infty)$, and let*

$$(Rf)(\zeta,z) = \int_0^\infty G(\zeta,\varsigma,z) f(\varsigma) d_\alpha\varsigma,$$

where

$$G(\zeta,\varsigma,z) = \begin{cases} Z(\zeta,z)\theta(\varsigma,z), & \varsigma \leq \zeta \\ \theta(\zeta,z)Z(\varsigma,z), & \varsigma > \zeta. \end{cases}$$

Then $\int_0^\infty |(Rf)(\zeta,z)|^2 d_\alpha t \leq \frac{1}{v^2}\int_0^\infty |f(t)|^2 d_\alpha\zeta$, *where* $z = u + iv$.

Proof. By (2.11) and (2.6), for each $b > 0$, we see that

$$\begin{aligned} \int_0^b |(R_b f)(\zeta, z)|^2 \, d_\alpha \zeta &= \sum_{m=1}^{\infty} \frac{\left| \int_0^b f(\zeta)\theta_{m,b}(\zeta) d_\alpha \varsigma \right|^2}{\alpha_{m,b}^2 |z - \lambda_{m,b}|^2} \\ &= \frac{1}{v^2} \int_0^b |f(\varsigma)|^2 \, d_\alpha \varsigma \end{aligned}$$

Let $\eta > 0$ be fixed. If $\eta < b$ then,

$$\begin{aligned} \int_0^\eta |(R_b f)(\zeta, z)|^2 \, d_\alpha t &\leq \int_0^b |(R_b f)(\zeta, z)|^2 \, d_\alpha \zeta \\ &\leq \frac{1}{v^2} \int_0^b |f(\varsigma)|^2 \, d_\alpha \varsigma \end{aligned}$$

Letting $b \longrightarrow \infty$, we have

$$\int_0^\eta |(Rf)(\zeta, z)|^2 \, d_\alpha t \leq \frac{1}{v^2} \int_0^\infty |f(\varsigma)|^2 \, d_\alpha \varsigma$$

□

Theorem 2.1. *(Integral Representation of the Resolvent). For every nonreal z and for each $f(.) \in L_\alpha^2(0, \infty)$, we obtain*

$$(Rf)(\zeta, z) = \int_{-\infty}^{\infty} \frac{\theta(\zeta, \lambda)}{z - \lambda} F(\lambda) d\varrho(\lambda), \tag{2.16}$$

where

$$F(\lambda) = \lim_{\sigma \to \infty} \int_0^\sigma f(\zeta)\theta(\zeta, \lambda) d_\alpha \zeta.$$

Proof. Suppose that $f(\zeta) = f_\sigma(\zeta)$ satisfies (2.2) and vanishes outside the interval $[0, \sigma]$, where $\sigma < b$. We put,$F_\sigma(\lambda) = \int_0^\sigma f_\sigma(\zeta)\theta(\zeta, \lambda) d_\alpha \zeta$.Let a arbitrary positive number. The right-hand side of (2.11) can then be rewritten in the form

$$\begin{aligned} (R_b f_\sigma)(\zeta, z) &= \int_{-\infty}^{\infty} \frac{\theta(\zeta, \lambda)}{z - \lambda} F_\sigma(\lambda) d\varrho_b(\lambda) \\ &= \int_{-\infty}^{-a} \frac{\theta(\zeta, \lambda)}{z - \lambda} F_\sigma(\lambda) d\varrho_b(\lambda) \\ &\quad + \int_{-a}^{a} \frac{\theta(\zeta, \lambda)}{z - \lambda} F_\sigma(\lambda) d\varrho_b(\lambda) \\ &\quad + \int_{a}^{\infty} \frac{\theta(\zeta, \lambda)}{z - \lambda} F_\sigma(\lambda) d\varrho_b(\lambda) \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{2.17}$$

Firstly, we will estimate I_1. By (2.11), we get

$$
\begin{aligned}
|I_1| &= \left| \int_{-\infty}^{-a} \frac{\theta(\zeta,\lambda)}{z-\lambda} F_\sigma(\lambda) d\varrho_b(\lambda) \right| \\
&= \left| \sum_{\lambda_{k,b}<-a} \frac{\theta_{k,b}(\zeta) \int_0^\sigma f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha \zeta}{\alpha_{k,b}^2 (z-\lambda_{k,b})} \right| \\
&\leq \Big(\sum_{\lambda_{k,b}<-a} \frac{\theta_{k,b}^2(\zeta)}{\alpha_{k,b}^2 \left|z-\lambda_{k,b}\right|^2} \Big)^{1/2} \\
&\quad \times \left(\sum_{\lambda_{k,b}<-a} \frac{1}{\alpha_{k,b}^2} \left| \int_0^\sigma f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha \zeta \right|^2 \right)^{1/2}. \qquad (2.18)
\end{aligned}
$$

Integrating twice by parts, we obtain

$$
\begin{aligned}
&\int_0^\sigma f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha \zeta \\
&= -\frac{1}{\lambda_{k,b}} \int_0^\sigma f_\sigma(\zeta) \left\{ -T_\alpha^2 \theta_{k,b}(\zeta) - v(\zeta)\theta_{k,b}(\zeta) \right\} d_\alpha \zeta \\
&= -\frac{1}{\lambda_{k,b}} \int_0^\sigma \left\{ -T_\alpha^2 f_\sigma(\zeta) - v(\zeta) f_\sigma(\zeta) \right\} \theta_{k,b}(\zeta) d_\alpha \zeta. \qquad (2.19)
\end{aligned}
$$

By Lemma 2.3, we have

$$
|I_1| \leq \frac{K^{1/2}}{a} \left(\sum_{\lambda_{k,b}<-a} \frac{1}{\alpha_{k,b}^2} \int_0^\sigma \left| -T_\alpha^2 f_\sigma(\zeta) + V(\zeta) f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha \zeta \right|^2 \right)^{1/2}.
$$

Using Bessel inequality, we get

$$
|I_1| \leq \frac{K^{1/2}}{a} \left[\int_0^\sigma \left| -T_\alpha^2 f_\sigma(\zeta) + V(\zeta) f_\sigma(\zeta) \right|^2 d_\alpha \zeta \right]^{1/2} = \frac{C}{a}.
$$

It is proved similarly that $|I_3| \leq \frac{C}{a}$. Then I_1 and I_3 tend to zero as $a \to \infty$, uniformly in b. Therefore we can use the generalization of the Helly selection theorem and obtain from the equality (2.17)

$$
(Rf_\sigma)(\zeta, z) = \int_{-\infty}^{\infty} \frac{\theta(\zeta,\lambda)}{z-\lambda} F_\sigma(\lambda) d\varrho(\lambda). \qquad (2.20)
$$

As is known, if $f(.) \in L_\alpha^2(0,\infty)$, then we find a sequence $\{f_\sigma(\varsigma)\}_{\sigma=1}^\infty$ that satisfies the previous conditions and tend to $f(\zeta)$ as $\sigma \to \infty$. From (2.6), the sequence of Fourier transform converges to the transform of $f(\zeta)$. Using Lemma 2.4 and Lemma 2.6, we can pass to the limit $\sigma \to \infty$ in (2.20). Thus, we get the desired result. $\square$

Remark 2.1. Using Theorem 2.1, we get

$$\int_0^\infty (Rf)(\varsigma, z)g(\varsigma)d_\alpha\varsigma = \int_{-\infty}^\infty \frac{F(\lambda)G(\lambda)}{z-\lambda}d\varrho(\lambda), \tag{2.21}$$

where

$$F(\lambda) = \lim_{\sigma\to\infty}\int_0^\sigma f(\zeta)\theta(\zeta,\lambda)d_\alpha\zeta,$$

$$G(\lambda) = \lim_{\sigma\to\infty}\int_0^\sigma g(\zeta)\theta(\zeta,\lambda)d_\alpha\zeta.$$

3. Conclusion

In this study, we consider a conformable fractional Sturm–Liouville operator. For this operator, a spectral function is constructed. Using this spectral function, a representation formula for the resolvent of conformable fractional Sturm–Liouville operator is obtained. The determination of whether the results obtained for the classical Sturm–Liouville problem are also valid for the conformable fractional Sturm–Liouville problem, is an explanation that will contribute to the literature.

REFERENCES

1. T. Abdeljawad: *On conformable fractional calculus.* J. Comput Appl. Math. **279 (2015)**, 57–66.
2. M. Abu Hammad and R. Khalil: *Abel's formula and Wronskian for conformable fractional differential equations.* Intern. J. Differ. Equ. Appl., **13 (3) (2014)**, 177–183.
3. M. Abu Hammad and R. Khalil: *Conformable fractional heat differential equations.* Intern. J. Pure. Appl. Math., **94 (2) (2014)**, 215–221.
4. B. P. Allahverdiev, H. Tuna and Y. Yalçinkaya: *Conformable fractional Sturm–Liouville equation.* Math. Meth. Appl. Sci., **42(10) (2019)**, 3508–3526.
5. D. Baleanu, F. Jarad and E. Uğurlu: *Singular conformable sequential differential equations with distributional potentials.* Quaest. (2018) Math., DOI:10.2989/16073606.2018.1445134.
6. P. L. Butzer and U. Westphal: *An Introduction to Fractional Calculus.* Applications of Fractional Calculus in Physics, World Scientic, New Jersey, USA, 2000.
7. T. Gulsen, E. Yilmaz and S. Goktas: *Conformable fractional Dirac system on time scales.* J Inequal Appl 2017, **161** (2017). https://doi.org/10.1186/s13660-017-1434-8.
8. R. Khalil and H. Abu-Shaab: *Solution of some conformable fractional differential equations.* Inter. J. Pure Appl. Math., **103(4)** (2015), 667–673.
9. R. Khalil, M. Al Horani, A. Yousef and M., Sababheh: *A new definition of fractional derivative.* J. Comput. Appl. Math., **264** (2014), 65–70.
10. B. M. Levitan and I. S. Sargsjan: *Sturm–Liouville and Dirac Operators.* Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1991 (translated from the Russian).

11. E. C. Titchmarsh: *Eigenfunction Expansions Associated with Second-order Differential Equations*. Part I. Second Edition Oxford, Great Britain: Clarendon Press, 1962.

12. E. Yilmaz, T. Gulsen and E. S. Panakhov: *Existence results for a conformable type Dirac system on time scales in quantum physics*. Appl. Comput. Math., **21(3)** (2022), 279–291.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 153–161
https://doi.org/10.22190/FUMI230525010S
Original Scientific Paper

ON INVARIANT CONTINUITY AND INVARIANT COMPACTNESS IN BANACH SPACES

Büşra Söylemez and Fatih Nuray

Faculty of Science, Department of Mathematics
Afyon Kocatepe University, Afyonkarahisar, Turkey

ORCID IDs: Büşra Söylemez N/A
Fatih Nuray https://orcid.org/0000-0003-0160-4001

Abstract. In this study, we have defined the concepts of invariant continuity, invariant compactness, invariant boundedness and invariant Cauchy sequence in normed linear spaces. In general, there is no relation between continuity and invariant continuity. We have proved that if f is a linear map, then continuity of f implies invariant continuity of f. Additionally, we have shown that continuity of f and invariant continuity of f is equal under a condition. Also, we have proved that every invariant convergent sequence is invariant Cauchy. Finally, we have proved that invariant continuous image of an invariant compact space is invariant compact.
Keywords: Invariant convergence, strongly invariant convergence, invariant continuity, invariant compactness, invariant Cauchy sequence.

1. Introduction

Let l_∞ denote the Banach space of all real bounded sequences with the usual norm $\|x\| = sup_k|x_k|$. Banach [1], recognized certain nonnegative linear functionals defined on l_∞ which remain invariant under shift operators. This extended functionals is known as the Banach limits. In 1948, Lorentz [5] defined a new type of convergence known as the almost convergence. Later, Kurtz [4] introduced the concept of almost convergent sequence in a normed linear space X as follows:

Received May 25, 2023, revised: October 26, 2023, accepted: December 26, 2023
Communicated by Dijana Mosić
Corresponding Author: Fatih Nuray. E-mail addresses: busra.soylemez@usr.aku.edu.tr (B. Söylemez), fnuray@uaku.edu.tr (F. Nuray)
2010 *Mathematics Subject Classification.* Primary 40A05 ; Secondary 26A15, 46B50

A sequence (x_k) in a normed linear space X is said to be *almost convergent* to $x \in X$ if

$$\lim_{n\to\infty} \left\| \frac{\sum_{i=1}^{n-1} x_{i+m}}{n} - x \right\| = 0, \text{uniformly in m.}$$

Raimi [12], defined the concept of invariant convergence (σ- convergence) which is generalization of almost convergence.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on l_∞ is said to be an *invariant mean or a σ-mean*, if and only if,

1. $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n (non-negative)
2. $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ (normal)
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in l_\infty$.

The mappings σ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$ denotes the mth iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

A sequence (x_k) is said to be *invariant convergent* to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} = L$$

uniformly in m [6]. In this case we write $(x_k) \longrightarrow L(V_\sigma)$ and L is called the σ-limit of (x_k).

Strongly invariant convergent sequence was defined by Mursaleen [7] as follows:

A sequence (x_k) is said to be *strongly invariant convergent* to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |x_{\sigma^j(m)} - L| = 0$$

uniformly in m. In this case we write $(x_k) \longrightarrow L[V_\sigma]$.

It is known that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$ (see [7]).

Several authors including Dündar et al.[3], Mursaleen [6], MA Mursaleen [8], Mursaleen and Edely [9], Pancaroğlu and Nuray [11], Raimi [12], Savaş and Nuray [13], Schaefer [14], Ulusu and Nuray [15] and others have studied invariant convergent sequences.

Continuity and compactness are related to convergence. In [10], Nanda, by using the definition of almost convergence sequence, defined the concepts of almost continuity function and almost compactness in any normed linear space.

In this study, we will introduce the concepts of invariant continuous function and invariant compactness in any normed linear space. Then we will give some relations between continuity and invariant continuity. We will also prove that the invariant continuous image of an invariant compact space is invariant compact.

2. Main Results

Now, we will define the concepts of invariant convergence and invariant continuity in any normed linear space.

Definition 2.1. Let X be a normed linear space. A sequence $(x_n) \in X$ is said to be invariant convergent to $x \in X$ if

$$\lim_{n\to\infty} \left\| \frac{\sum_{j=1}^{n} x_{\sigma^j(m)}}{n} - x \right\| = 0$$

uniformly in m.

$V_\sigma(X)$ will denote the set of all invariant convergent sequences in X, that is:

$$V_\sigma(X) = \left\{ (x_k) : \lim_{n\to\infty} \left\| \frac{\sum_{j=1}^{n} x_{\sigma^j(m)}}{n} - x \right\| = 0, \text{uniformly in m} \right\}.$$

Definition 2.2. Let X and Y be normed linear spaces and $f : X \longrightarrow Y$ be a mapping. f is said to be invariant continuous at a point $x \in X$ if

$$x_k \longrightarrow x(V_\sigma(X)) \quad \text{implies} \quad f(x_k) \longrightarrow f(x)(V_\sigma(X))$$

Remark 2.1. It is easy to prove that if f and g are invariant continuous then so is $f+g$. Also if k is real number and f is invariant continuous functions, then kf is invariant continuous. Thus, the set of all invariant continuous functions is a vector space.

We can introduce four types of continuity:

$$\lim_{n\to\infty} x_n = x \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) = f(x) \tag{2.1}$$

$$\lim_{n\to\infty} x_n = x \quad \text{implies} \quad \sigma - \lim_{n\to\infty} f(x_n) = f(x) \tag{2.2}$$

$$\sigma - \lim_{n\to\infty} x_n = x \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) = f(x) \tag{2.3}$$

$$\sigma - \lim_{n\to\infty} x_n = x \quad \text{implies} \quad \sigma - \lim_{n\to\infty} f(x_n) = f(x) \tag{2.4}$$

(2.1) is continuity and (2.4) is invariant continuity. We have

$$(2.3) \Rightarrow (2.1) \Rightarrow (2.2)$$

$$(2.3) \Rightarrow (2.4) \Rightarrow (2.2)$$

In general, there is no relation between continuity and invariant continuity. In [2], the following Lemma 2.1 and Theorem 2.1 were proved for almost continuity. We will prove similar lemmas for invariant continuity.

Lemma 2.1. *Let X and Y be normed linear spaces. If $f : X \longrightarrow Y$ is invariant continuous at $x_0 \in X$, then it is continuous at x_0.*

Proof. Firstly, we prove that the function f is bounded at x_0, i.e., there exists an $a > 0$ such that f is bounded on the interval $(x_0 - a, x_0 + a)$. To prove this it suffices to show that if $(x_n) \longrightarrow x_0$, then the sequence $f(x_n)_{n=1}^{\infty}$is bounded.

Let $(x_n) \longrightarrow x_0$. Then $(x_n) \longrightarrow x_0(V_\sigma(X))$ and by the assumption of Lemma, we have $f(x_n) \longrightarrow f(x_0)(V_\sigma(X))$. Hence $(f(x_n))_{n=1}^{\infty}$ as an invariant convergent sequence is bounded. Now, we can prove the continuity of the function f at the point x_0. Suppose that f is discontinuous at x_0. Since it is bounded on an interval $(x_0 - a, x_0 + a)$, there exists a sequence (y_n) of elements of $(x_0 - a, x_0 + a)$ such that $(y_n) \longrightarrow x_0$ and $f(y_n) \longrightarrow b \neq f(x_0)$. From this we get $f(y_n) \longrightarrow bV_\sigma(X))$. On the other hand from $(y_n) \longrightarrow x_0$ we have $(y_n) \longrightarrow x_0(V_\sigma(X))$ and so by the assumption of Lemma we get

$$f(y_n) \longrightarrow f(x_0) \neq b(V_\sigma(X)).$$

This contradicts $f(y_n) \longrightarrow b(V_\sigma(X))$. Hence, f is continuous and the proof is completed. □

Theorem 2.1. *Let X and Y be normed linear spaces. If $f : X \longrightarrow Y$ is invariant continuous at a point $x_0 \in X$, then f is a linear function.*

Proof. We will prove the theorem in two stages that special and general. First of all we will prove the following special case. Let $g : X \longrightarrow Y$ is invariant continuous at the point 0 and $g(0) = 0$. Let a, b, c be real numbers such that $a + b + c = 0$. Construct the sequence

$$(x_n)_{n=1}^{\infty} = a, b, c, a, b, c, ...$$

We show that this sequence is invariant convergent to 0. Let $\sigma(m) = m + 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} = 0$$

uniformly in m. Hence, (x_n) is invariant convergent to 0. According to the assumption of theorem $g(x_n) \longrightarrow g(0) = 0(V_\sigma(X))$, i.e.

$$g(x_n) = g(a), g(b), g(c), g(a), g(b), g(c), ...$$

is invariant convergent to 0. Additonally, a direct calculation shows that

$$\sigma - \lim g(x_n) = \frac{g(a) + g(b) + g(c)}{3}$$

Hence $g(a) + g(b) + g(c) = 0$. Since $c = -a - b$, we get $g(-a - b) = -g(a) - g(b)$. Putting $b = 0$ we have $g(-a) = -g(a)$.

Let x, y be arbitrary. Put $c = x + y$ $a = -x$ $b = -y$ then we get $g(x + y) = -g(-x) - g(-y) = g(x) + g(y)$. Hence the function g satisfies the Cauchy functional equation. It is continuous at 0 that Lemma 2.1. On the basis of well-known knowledge on Cauchy equation we get $g(x) = ax$ for $x \in X$, a being a constant.

Now we will prove the general case. Let $f : X \longrightarrow Y$ is invariant continuous at $x_0 \in X$. We introduce new coordinates $x' = x - x_0$, $y' = y - f(x_0)$. Put $g(x') = f(x) - f(x_0)$. Since g has the form $g(x') = ax'$, $f(x) - f(x_0) = a(x - x_0) = ax - ax_0$, $f(x) = ax + (f(x_0) - ax_0) = ax + b$ and f is linear. □

Remark 2.2. It follows from Theorem 2.1 that Lemma 2.1 cannot be conversed.

Theorem 2.2. *If f is a linear map, then continuity of f implies invariant continuity of f.*

Proof. For linear maps continuity implies invariant continuity. Let $(x_k) \longrightarrow x_0$ and f is continuous. So,

$$(x_k) \longrightarrow x_0 \quad \text{implies} \quad f(x_k) \longrightarrow f(x_0).$$

Since f is continuous and linear, f is bounded thus for $M > 0$ we can write

$$\|f(x_n) - f(x_0)\| = \|f(x_n - x_0)\| \leq M \|x_n - x_0\| .$$

Hence, we have

$$\left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) - f(x_0) \right\| = \left\| f\left(\frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} - x_0\right) \right\| \leq M \left\| \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} - x_0 \right\|$$

for each m. Thus f is invariant continuous and so continuity implies invariant continuity for linear maps.

But, the situation is changing for nonlinear map. For this, let us take the example of Teorem 1 in [10]. Let us consider the nonlinear map $f : L^2[0,1] \longrightarrow [0,1]$ defined by

$$[fx](s) = \int_0^1 x^2(t)\, \mathrm{d}t$$

Let (x_k) be a sequence which converges to x in $L^2[0,1]$. We have

$$\begin{aligned}
\|f(x_k) - f(x)\|^2 &= \int_0^1 \left(\int_0^1 (x_k^2(t) - x^2(t))\mathrm{d}t \right)^2 \mathrm{d}s \\
&\leq \int_0^1 \left(\int_0^1 (x_k(t) + x(t))^2 \mathrm{d}t \right) \left(\int_0^1 (x_k(t) - x(t))^2 \mathrm{d}t \right) \mathrm{d}s \\
&\leq N \|x_k - x\|^2
\end{aligned}$$

where $N = \sup_k \|x_k + x\|^2$. The continiuty of f follows from the above inequality.

Observe that if $x_k = sink\pi t$ and $\sigma(m) = m+1$ then,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}x_{\sigma^j(m)} = \lim_{n\to\infty}\frac{sin(m+1)\pi t + ... + sin(m+n)\pi t}{n} = 0$$

uniformly in m. So $(x_k) \longrightarrow 0(V_\sigma)$. But

$$\|f(x_k) - f(0)\| = \int_0^1 (sink\pi t)^2 \mathrm{d}t = \int_0^1 (\frac{1}{2} - \frac{cos2k\pi t}{2})\mathrm{d}t = \frac{1}{2}$$

for all k and so $f(x_k) \nrightarrow 0$. Thus f is not invariant continuous and this completes the proof. □

Theorem 2.3. *Let X and Y be normed linear spaces and $f : X \longrightarrow Y$. Continuity of f and invariant continuity of f are equivalent under the condition that*

$$\left\|\frac{f(x_{\sigma(m)}) + f(x_{\sigma^2(m)}) + ... + f(x_{\sigma^j(m)})}{n} - f(x_k)\right\| \to 0 \quad (n \longrightarrow \infty)$$

uniformly in m.

Proof. Let f be a continuous function as well as the condition holds. Let $(x_k) \to x$. Then

$$\|f(x_k) - f(x)\| \to 0$$

and

$$\left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x_k)\right\| \to 0 \quad (n \longrightarrow \infty)$$

unifromly in m. We can write,

$$\left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x)\right\| \le \left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x_k)\| + \|f(x_k) - f(x)\right\|$$

and

$$\left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x)\right\| \to 0 \quad (n \longrightarrow \infty)$$

unifromly in m. Hence f is invariant continuous.

Let $(x_k) \to x$ and f be invariant continuous at $x \in X$. Let the condition hold.Then

$$\left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x)\right\| \to 0 \quad (n \longrightarrow \infty)$$

unifromly in m. We have,

$$\|f(x_k) - f(x)\| \le \left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x_k)\right\| + \left\|\frac{\sum_{j=1}^{n} f(x_{\sigma^j(m)})}{n} - f(x)\right\|$$

and hence f is continuous. □

Definition 2.3. Let X and Y be normed linear spaces. A function $f : X \longrightarrow Y$ is said to be invariant bounded if there is a constant $M \geq 0$ such that for all n and m,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\| \leq M$$

Theorem 2.4. *Boundednes and invariant boundedness of functions are equivalent.*

Proof. Observe that

$$\|f(x_k)\| = \sup_{1,m} \left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\| \leq \sup_{n,m} \left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\|.$$

Also,

$$\sup_{n,m} \left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\| \leq \sup_{n} \frac{\sup_k \|f(x_k)\|}{n} \sum_{j=1}^{n} 1 = \sup_{k} \|f(x_k)\|$$

The result follows from the above two inequalities. □

Definition 2.4. A sequence (x_k) in X is said to be invariant Cauchy sequence if

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} \right\| \longrightarrow 0, \quad n, p \longrightarrow \infty$$

uniformly in m.

Theorem 2.5. *Let (x_k) be invariant Cauchy sequence. If*

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x_n \right\| \longrightarrow 0$$

uniformly in m, then it is Cauhy and vice-versa.

Proof. Let (x_k) be invariant Cauchy sequence and condition hold. We have

$$\|x_k - x_n\| \leq \left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x_n \right\|$$

$$+ \left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{k+1} \sum_{j=0}^{k} x_{\sigma^j(m)} \right\| + \left\| \frac{1}{k+1} \sum_{j=0}^{k} x_{\sigma^j(m)} - x_k \right\|$$

so (x_k) is Cauchy sequence.
Conversely, let (x_k) be Cauchy sequence and condition hold. Then

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} \right\| \leq$$

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x_n \right\| + \|x_n - x_p\| + \left\| \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} - x_p \right\|$$

so (x_k) is invariant Cauchy sequence. The proof is completed. □

Theorem 2.6. *Every invariant convergent sequence is invariant Cauchy sequence.*

Proof. Let the sequence (x_k) be invariant convergent to x. Then we can write

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} \right\| \leq$$

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x \right\| + \left\| \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} - x \right\|.$$

So (x_k) is invariant Cauchy sequence. □

Definition 2.5. A Banach space X is said to be invariant compact if every sequence in X has an invariant convergent subsequence.

Theorem 2.7. *Invariant continuous image of an invariant compact space is invariant compact.*

Proof. Let X and Y be normed linear spaces, K an invariant compact subspace of X and let $f : X \longrightarrow Y$ be invariant continuous. We have to show that $f(K) = \{f(x) : x \in K\}$ is also invariant compact.

Let $\{f(x_k)\}$, be a sequence in $f(K)$. Then (x_k) is a sequence in K. Since K is invariant compact, there is a subsequence (x_{k_n}) which is invariant convergent to $x \in X$. Observe that $\{f(x_{k_n})\}$ is a subsequence of $f(x_k)$. Since f is invariant continuous,

$$x_{k_n} \longrightarrow x(V_\sigma(X)) \quad \text{implies} \quad f(x_{k_n}) \longrightarrow f(x)(V_\sigma(X)).$$

Thus, $f(K)$ is invariant compact and the proof is completed. □

REFERENCES

1. S. Banach: *Theorie des operations linearies*. Warswaza, (1932).
2. J. Borsik and T. Salat: *On F-continuity of real functions*. Tatra Mt. Math. Publ. **2** (1993), 37-42.
3. E. Dündar, N. P. Akin and U. Ulusu: *Wijsman lacunary $\mathcal{I}$-invariant convergence of sequences of sets*. Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. **91(3)** (2021), 517-522.
4. J. C. Kurtz: *Almost convergence in Banach space*. Tohoku Math. J. **24(3)** (1972), 389-39.
5. G. G. Lorentz: *A contribution to the theory of divergent sequences*. Acta Math. **80** (1948), 167-190.
6. M. Mursaleen: *On finite matrices and invariant means*. Indian J. Pure Appl. Math. **10** (1979), 457-460.
7. M. Mursaleen: *Matrix transformation between some new sequence spaces*. Houston J. Math. **9** (1983), 505-509.
8. M. A. Mursaleen: *On σ-convergence by de la vallee poussin mean and matrix transformations*. Journal of Inequalities and Special Functions **8(3)** (2017), 119-124.
9. M. Mursaleen and O.H.H. Edely: *On the invariant mean and statistical convergence*. Applied Mathematics Letters **22** (2009), 1700-1704.
10. S. Nanda: *Almost continuity and almost compactness*. Journal of the Indian Math. Soc. Vol. **71**, Nos. 1-4 (2004), 227-232.
11. N. Pancaroğlu and F. Nuray: *On invariant statistically convergence and lacunary invariant statistical convergence of sequences of Sets*. Progress in Applied Mathematics **5(2)** (2013), 23-29.
12. R. A. Raimi: *Invariant means and invariant matrix methods of summability*. Duke. Math. J. **30** (1963), 81-94.
13. E. Savaş and F. Nuray: *On sigma-statistically convergence and lacunary sigma-statistically convergence*. Mathematica Slovaca **43(3)** (1993), 309-315.
14. P. Schaefer: *Infinite matrices and invariant means*. Proc. Amer. Math. Soc. **36** (1972), 104-110.
15. U. Ulusu and F. Nuray: *Lacunary $\mathcal{I}$-invariant convergence*. Cumhuriyet Sci. J. **41(3)** (2020), 617-624.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 163–175
https://doi.org/10.22190/FUMI230713011D
Original Scientific Paper

REPRODUCED PRINCIPAL IDEAL DOMAIN ON GENERAL HYPERRING $\mathbb{Z}_{p^n q^m}$

Roohallah Daneshpayeh[1] **and Sirus Jahanpanah**[2]

[1] **Faculty of Science, Department of Mathematics of Payame Noor University, P. O. Box 19395-4697, Tehran, Iran**
[2] **Faculty of Electronic Engineering, Department of Mathematics of Payame Noor University, P. O. Box 19395-4697, Tehran, Iran**

ORCID IDs: Roohallah Daneshpayeh https://orcid.org/0000-0003-2665-0872
Sirus Jahanpanah https://orcid.org/0000-0002-7268-9121

Abstract. Every classical algebra is a set equipped with binary operations that operate under certain axiom principles. The generalization of classical algebras to hyperalgebras has been created with the aim of generalizing operations to hyperoperations that apply to specific subject principles. This paper introduces the concept of reproduced general hyperrings as a generalization of rings and investigates and analyzes some of their essential properties. This study defines the notation of reproduced hyperideals in reproduced general hyperrings, consider the ideals of finite rings and obtain the finite and cyclic hyperideals. In the endl, we introduce and show that a principal Ideal domain finite reproduced general hyperring is Ideal-absorbing.
Keywords: hyperring, principal ideal domain, axioms.

1. Introduction

A ring is an algebraic structure that is equipped with two binary operations and in this regard it can connect two elements to only one element at the same time. From the practical point of view, connecting two elements to one element is a limitation because in practice we may need to connect a group of elements. Because algebraic structures are regular systems and their elements are related under specific subject principles, these structures can have many applications in the real world. Therefore, developing and removing the limitations of algebraic structures such as rings is

Received July 13, 2023, revised: September 05, 2023, accepted: September 18, 2023
Communicated by Amirhossein Nokhodkar and Predrag Stanimirović
Corresponding Author: Sirus Jahanpanah. E-mail addresses: rdaneshpayeh@pnu.ac.ir (R. Daneshpayeh), s.jahanpanah@pnu.ac.ir (S. Jahanpanah)
2010 *Mathematics Subject Classification.* Primary N2020; Secondary N2020

very important. A hyperring is just a ring, that is equipped with a hyperaddition, and hyperrings are considered in spaces of signs, also known as abstract real spectra and objects which arise naturally in the study of constructible sets in real geometry. Indeed, hyperrings as a generalization of rings are equipped with two hyperoperations and operation or two hyperoperations. Since hyperoperations are maps with a nonempty set range, can be helpful in the application of a group of elements. The theory of hyperrings as a generalization of rings can be considered as a type of elimination of the limitation of the connection of elements under the principles of a special subject. The concept of Krasner hyperring was introduced by Krasner [14], who used it as a tool for the approximation of valued fields or the second type of a hyperring as multiplicative hyperring (the multiplication is a hyperoperation, while the addition is an operation was introduced by R. Rota in 1982 [17]. Today, some researchers have investigated some works in hyperrings such as a study on special kinds of derivations in ordered hyperrings [16], the reducibility concept in general hyperrings [6], regular parameter elements and regular local hyperrings [4], hyperideals of (finite) general hyperrings [2], direct limit of Krasner (m, n)-hyperrings [1], a generalization of graded prime hyperideals over graded multiplicative hyperrings [11], extended centroid of hyperrings [18], weakly (k, n)-absorbing (primary) hyperideals of a Krasner (m, n)-hyperring [8] and contribution to study special kinds of hyperideals in ordered semihyperrings [15]. Fundamental relations are basic tools in algebraic hyperstructures theory and some researchers worked on fundamental relations of hyperrings such as Boolean rings based on hyperrings [3], commutative rings obtained from hyperrings (Hv-rings) with α^*–relations [9], Boolean rings obtained from hyperrings with $\eta^*_{1,m}$ relations [10], fundamental relation and automorphism group of very thin H_v–groups [12], height of prime hyperideals in Krasner hyperrings[5] and The fundamental Relations in Hyperrings [19]. Hamidi et al. constructed multigroups and hyperrings on every non–empty set, introduced and analyzed a special relation on hyperrings and extended it to the smallest strongly regular equivalence binary relation in such a way that the quotient of each given hyperring on this relation is a commutative Boolean ring with identity. They try to generalize the concept of rings to general hyperrings, to describe some of their properties and the differences between hyperrings and general hyperrings.

Motivation and advantage: Algebraic structures as one of the important branches of mathematics have many applications in the real world. These structures as an algebraic system equipped with several algebraic operations can be used as a mathematical model. In algebraic structures, under each operation, only two elements can be equalized to one element, and this limits the algebraic structures. Of course, this limitation can be overcome and covered by generalizing algebraic structures to algebraic superstructures. The advantage of algebraic superstructures is that, in addition to covering algebraic structures, they can relate both elements to a set of elements. This advantage allows us to connect a network of elements in the modeling of real-world problems. In this research, by developing rings into hyperrings within the context of a ring, we create a new achievement in hybrid substructures.

This paper introduces and works on the construction of reproduced general hyperrings and shows that this class of hyperstructures has some identity elements while having a unique zero element. It is natural to question as to what are the relationships between elements whence are considered in the same set concerning algebraic operations. Since any operation at most connects three elements, we need to extend more elements in defined axioms. It motivates us to introduce the concept of two algebraic hyperoperations in an underlying set. So the main motivation is to introduce some identity elements concerning algebraic hyperproducts and to consider the differences between other hyperstructures and structures. We obtained some theorems and corollaries that in special conditions are similar to corresponded theorems in (non-associative)rings, so we conclude that reproduced general hyperrings are a generalization of (non-associative)rings. Also, the concept of reproduced ideals is presented in this work and we analyze the hyperideals on principal ideal domain reproduced general hyperideals.

1.1. Preliminaries

In this section, we review some definitions and results from hyperstructures from [7, 13], which we need in what follows. Let R be a nonempty set, $\mathcal{P}^*(R) = \{S \mid \emptyset \neq\subseteq R\}$ and $\varrho = \{(x, X) \mid x \in R, X \in \mathcal{P}^*(R)\}$ be a map. Then ϱ is called a *hyperoperation* (*hypercomposition*), an *algebraic hypercompositional structure* (R, ϱ) is called a *hypergroupoid* and for all nonempty subsets S, T of R, $\varrho(S, T) = \bigcup\limits_{s\in S, t\in T} \varrho(s, t)$. An algebraic hypercompositional structure (R, ϱ), where ϱ is a binary hyperoperation, is called a *hypergroupoid* and a recall that a *hypergroupoid* (R, ϱ) is called a *semihypergroup*, if for all $x, y, z \in R$, $\varrho(\varrho(x, y), z) = \varrho(x, \varrho(y, z))$ and a semihypergroup (R, ϱ) is called a *hypergroup*, if for all $x \in R$, $\varrho(x, R)R = \varrho(R, x)$(*reproduction axiom*). A *general hyperring* is an algebraic hypercompositional structure (R, ϱ, ς), where (i) (R, ϱ) is a hypergroup, (ii) (R, ς) is a semihypergroup and (iii) for any $x, y, z \in R$: $\varsigma(x, \varsigma(y, z)) \subseteq \varrho(\varsigma(x, y), \varsigma(x, z))$ and $\varsigma(\varrho(x, y), z) \subseteq \varrho(\varsigma(x, z), \varsigma(y, z)$. A general hyperring $(R, \varrho, \iota, \varsigma)$ is called *commutative* (with unit element), if for all $x, y \in R$, $\varsigma(x, y) = \varsigma(y, x)$ (if there exists an element $\epsilon \in R$ such that for all $x \in R, \varsigma(\epsilon, x) = \varsigma(x, \epsilon) = \{x\}$). A nonempty subset I of R is called a (*right*)*left hyperideal*, if (1), (I, ϱ) is a hypergroup and (2), $(\varsigma(R, I) \subseteq I)(\varsigma(I, R) \subseteq I$. A hyperideal I is a both left and right hyperideal.

2. Hyperideals of general hyperrings

In this section, we apply the structure of rings and extend them to general hyperrings. Also the concept of reproduced ideals is introduced and investigated.

Definition 2.1. Let $(R, +, \cdot)$ be a ring. Then R is said to be a (ϱ, ς)-reproduced general hyperring, if there are hyperoperations "ϱ" and "ς", that (R, ϱ, ς) is a general hyperring and ϱ, ς are are dependent to $+$ and $\cdot$, respectively.

Theorem 2.1. *Assume $k \in \mathbb{N}$. Then $(\mathbb{Z}_{2k}, +, \cdot)$ is a (ϱ, ς)-reproduced general hyperring.*

Proof. Fix $\overline{0} \neq \overline{a} \in \mathbb{Z}_{2k}$, where $\overline{2a} = \overline{0}$. Clearly, $(\mathbb{Z}_{2k}, \varrho)$ is a hypergroup, where for any $\overline{x}, \overline{y} \in \mathbb{Z}_{2k}$, $\varrho(\overline{x}, \overline{y}) = \{\overline{x+y}, \overline{x+y+a}\}$. Now for any $\overline{x}, \overline{y} \in \mathbb{Z}_{2k}$, define $\varsigma(\overline{x}, \overline{y}) = \{\overline{xy}, \overline{xy+a}\}$. Simple computations show that $(\mathbb{Z}_{2k}, \varrho, \varsigma)$ is a general hyperring. □

Example 2.1. By Theorem 2.1, $(R = \{a, b, c, d, e, f\}, \varrho, \varsigma)$ is a (ϱ, ς)-reproduced general hyperring by the following hyperoperations:

ϱ	a	b	c	d	e	f
a	$\{a,d\}$	$\{b,e\}$	$\{c,f\}$	$\{d,a\}$	$\{e,b\}$	$\{f,c\}$
b	$\{b,e\}$	$\{c,f\}$	$\{d,a\}$	$\{e,b\}$	$\{f,c\}$	$\{a,d\}$
c	$\{c,f\}$	$\{d,a\}$	$\{e,b\}$	$\{f,c\}$	$\{a,d\}$	$\{b,e\}$
d	$\{d,a\}$	$\{e,b\}$	$\{f,c\}$	$\{a,d\}$	$\{b,e\}$	$\{c,f\}$
e	$\{e,b\}$	$\{f,c\}$	$\{a,d\}$	$\{b,e\}$	$\{c,f\}$	$\{d,a\}$
f	$\{f,c\}$	$\{a,d\}$	$\{b,e\}$	$\{c,f\}$	$\{d,a\}$	$\{b,e\}$

,

ς	a	b	c	d	e	f
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$
b	$\{a,d\}$	$\{b,e\}$	$\{c,f\}$	$\{d,a\}$	$\{e,b\}$	$\{f,c\}$
c	$\{a,d\}$	$\{c,f\}$	$\{e,b\}$	$\{a,d\}$	$\{c,f\}$	$\{e,b\}$
d	$\{d,a\}$	$\{a,d\}$	$\{d,a\}$	$\{a,d\}$	$\{d,a\}$	$\{a,d\}$
e	$\{a,d\}$	$\{e,b\}$	$\{c,f\}$	$\{a,d\}$	$\{e,b\}$	$\{c,f\}$
f	$\{a,d\}$	$\{f,c\}$	$\{e,b\}$	$\{d,a\}$	$\{c,f\}$	$\{b,e\}$

.

Theorem 2.2. *Let p be a prime and $k \in \mathbb{N}$. Then $\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$ is a $(\varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$-reproduced general hyperring.*

Proof. Let $x, y \in \mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$. Define

$$\varrho_{\sqrt{p}}(x,y) = \varrho_{\sqrt{p}}(y,x) = \begin{cases} \{0, \sqrt{p}\} & x = -y \text{ or } x = y = \sqrt{p}, \\ x + y & x, y \in \mathbb{Z}_{p^k}, x \neq -y, \\ y & x = 0 \text{ or } (\ x = \sqrt{p} \text{ and } y \notin \{0, \sqrt{p}\}) \end{cases}$$

and

$$\varsigma_{\sqrt{p}}(x,y) = \varsigma_{\sqrt{p}}(y,x) = \begin{cases} x.y & x, y \in \mathbb{Z}_{p^k}, \\ \sqrt{p} & x \in \mathbb{Z}_{p^k} \setminus \{mp\}, y = \sqrt{p}(m \in \mathbb{N}), \\ 0 & x = mp, y = \sqrt{p}(m \in \mathbb{N}), \\ \{0, \sqrt{p}\} & x = y = \sqrt{p}. \end{cases}$$

Computations show that $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$ is a general hyperring. □

Example 2.2. By Theorem 2.1, $(\mathbb{Z}_4 \cup \{\sqrt{2}\}, \varrho, \varsigma)$ is a (ϱ, ς)-reproduced general hyperring by the following hyperoperations:

ϱ	0	1	2	3	$\sqrt{2}$
0	0	1	2	3	$\sqrt{2}$
1	1	2	3	$\{0, \sqrt{2}\}$	1
2	2	3	$\{0, \sqrt{2}\}$	1	2
3	3	$\{0, \sqrt{2}\}$	1	2	3
$\sqrt{2}$	$\sqrt{2}$	1	2	3	$\{0, \sqrt{2}\}$

,

ς	0	1	2	3	$\sqrt{2}$
0	0	0	0	0	0
1	0	1	2	3	$\sqrt{2}$
2	0	2	0	2	0
3	0	3	2	1	$\sqrt{2}$
$\sqrt{2}$	0	$\sqrt{2}$	0	$\sqrt{2}$	$\{0, \sqrt{2}\}$

.

2.1. On hyperideals

Now, we present hyperideals of a general hyperring. In particular, we determine hyperideals of finite commutative general hyperrings.

Definition 2.2. Let (R, ϱ, ς) be a general hyperring and $\emptyset \neq I \subseteq R$. We say

(i) I is a general subhyperring of R, if (I, ϱ, ς) is a general hyperring;

(ii) I is a hyperideal of R, if $\varsigma(R, I) \cup \varsigma(I, R) \subseteq I$.

Theorem 2.3. *Suppose $(R, +, \cdot)$ is a general hyperring and $\emptyset \neq I \subseteq R$. Then I is a hyperideal of R if and only if the following hold:*

(i) *for any $x \in I$, $\varrho(x, I) = \varrho(I, x) = I$;*

(ii) *for any $r \in R$ and $x \in I$, we have $\varsigma(r, x) \cup \varsigma(x, r) \subseteq I$.*

Proof. Immediate by definition. □

Theorem 2.4. *Let (R, ϱ, ς) be a general hyperring and I be a hyperideal of R. Then*

(i) *$\forall\ r \in R, x \in I, n \in \mathbb{N}$, we have $\varrho\underbrace{(\varsigma(r,x), \varsigma(r,x), \ldots \varsigma(r,x))}_{n\ times} \subseteq I$;*

(iv) *if $x \in I$, then $\varsigma(x) \in I$.*

Proof. Immediate. □

Assume $(R, +, \cdot)$ is a general hyperring. We symbolize the set hyperideals of R by $\mathcal{I}(R)$. Clearly, $R \in \mathcal{I}(R) \neq \emptyset$ and will call R as a non-proper hyperideal of any general hyperring.

Example 2.3. Let $R = \{e, \iota, a, b\}$. Then (R, ϱ, ς) is a general hyperring as follows.

ϱ	e	ι	a	b
e	e	ι	a	b
ι	ι	R	$\{\iota, a\}$	$\{\iota, b\}$
a	a	$\{\iota, a\}$	R	$\{a, b\}$
b	b	$\{\iota, b\}$	$\{a, b\}$	R

and

ς	e	ι	a	b
e	e	e	e	e
ι	e	ι	a	b
a	e	a	a	a
b	e	b	a	$\{a, b\}$

.

Then $\mathcal{I}(R) = \{I = \{e\}, J = R\}$.

Theorem 2.5. *Assume* $(R, +, \cdot)$ *is a commutative general hyperring and* $I, I' \in \mathcal{I}(R)$*. Then*

(*i*) $\varrho(I, I') \in \mathcal{I}(R)$.

(*ii*) *if* $I \cap I' \neq \emptyset$*, then* $I \cap I' \in \mathcal{I}(R)$.

Proof. (*i*) Clearly, $\varrho(I, I') \neq \emptyset$. Let $a \in I$ and $a' \in I'$. Then for any $z \in \varrho(\varrho(a, a'), \varrho(I + I'))$, there is $b \in I, b' \in I'$, that $z \in \varrho(\varrho(a + a'), \varrho(b, b')) = \varrho(\varrho(a, b), \varrho(a', b')) \subseteq \varrho(I, I')$. If $z \in \varrho(I, I')$ be an arbitrary element in $\varrho(I, I')$, then there are $a, b, c \in I$, and $a', b', c' \in I'$, that $z \in \varrho(c, c') \subseteq \varrho(\varrho(a, b), \varrho(a', b')) = \varrho((a, a'), \varrho(b, b')) \subseteq \varrho(\varrho(a, a'), \varrho(I, I'))$. Hence $\varrho(\varrho(a, a'), \varrho(I, I')) = \varrho(I, I')$. Now, for any $r \in R, a \in I$ and $a' \in I'$, one obtains $\varsigma(r, \varrho(a, a')) \cup \varsigma(\varrho(a, a'), r) = \varrho(\varsigma(r, a), \varsigma(r, a')))\cup\varrho(\varsigma(a, r), \varsigma(a', r)) = \varrho(\varsigma(r, a), \varsigma(r, a')) \subseteq \varrho(I, I')$. Hence $\varrho(I, I') \in \mathcal{I}(R)$.

(*ii*) Since $I \cap I' \subseteq I$, we get that $I \cap I' \in \mathcal{I}(R)$. □

Theorem 2.6. *Assume* $(R, +, \cdot)$*,* $(S, +, \cdot)$ *are general hyperrings,* $f : R \to S$ *be a homomorphism, and* $I \in \mathcal{I}(R)$ *and* $J \in \mathcal{I}(S)$.

(*i*) *If* f *is an epimorphism, then* $f(I) \in \mathcal{I}(S)$.

(*ii*) $f^{-1}(J) \in \mathcal{I}(R)$.

Proof. (*i*) Since $\emptyset \neq I$, we have $f(I) \neq \emptyset$. Let $f(a) \in f(I)$. Then for every $f(b) \in f(I)$, there is $a' \in I$, that $b \in \varrho(a, a')$, and so $f(b) \in f(\varrho(a, a')) = \varrho(f(a), f(a')) \subseteq \varrho(f(a), I)$. Hence $f(I) \subseteq \varrho(f(a), f(I))$. If $c \in \varrho(f(a), f(I))$ is an arbitrary element, then there is $a' \in I$, that $c \in \varrho(f(a), f(a')) = f(\varrho(a, a')) \subseteq f(I)$. Hence, $\varrho(f(a), f(I)) = f(I)$. Now, for any $s \in S$ and $f(a) \in f(I)$, there is $r \in R$, that

$$\begin{aligned}\varsigma(s, f(a)) \cup \varsigma(f(a), s) &= \varsigma(f(r), f(a)) \cup \varsigma(f(a), f(r)) = (f(\varsigma(r, a))) \cup (f(\varsigma(a, r))) \\ &= f(\varsigma(r, a) \cup \varsigma(a, r)) \subseteq f(I).\end{aligned}$$

(*ii*) It is straightforward. □

2.2. Reproduced ideals in reproduced general hyperring $(\mathbb{Z}_n, \varrho, \varsigma)$

In this subsection, all reproduced ideals of finite reproduced general hyperring $R = (\mathbb{Z}_n, \varrho, \varsigma)$ are computed and it is proved that every reproduced ideal of the reproduced general hyperring $(\mathbb{Z}_n, \varrho, \varsigma)$ is characterized by the divisors of n.

Definition 2.3. Let $(R, +, \cdot)$ be a ring and $I \subseteq R$ be an ideal of R. We will call I as a reproduced ideal of (ϱ, ς)-reproduced general hyperring (R, ϱ, ς), if I is extended to a hyperideal of (R, ϱ, ς). We will denote the $\mathcal{RI}$ by the set of all reproduced ideals of (ϱ, ς)-reproduced general hyperring (R, ϱ, ς).

Theorem 2.7. *Let* $n, d \in \mathbb{N}$ *and* $\overline{x}, \overline{y} \in R$. *Then*

(*i*) $\langle \overline{x} \rangle \in \mathcal{I}(\mathbb{Z}_n, \varrho, \varsigma)$,

(*ii*) $\langle \overline{0} \rangle = \{\overline{0}\}$,

(*iii*) $\langle \overline{x} \rangle = \langle \overline{y} \rangle \Leftrightarrow \gcd(x, n) = \gcd(y, n) = d$.

Proof.

(*i*) Let $\overline{x} \in R$. By definition, we have $\langle \overline{x} \rangle = \bigcup_{k \in \mathbb{N}} \{\overline{kx}, \overline{0}\}$ and show that it is a hyperideal of R. Let $\overline{y} \in \langle \overline{x} \rangle$ and $\overline{z} \in \varrho(\overline{y}, \langle \overline{x} \rangle)$. Thus there is $k, k' \in \mathbb{N}$ and $\overline{w} \in \langle \overline{x} \rangle$ that $\overline{z} \in \varrho(\overline{y}, \overline{w})$ and so $\overline{z} \in \varrho(\overline{kx}, \overline{k'x}), \overline{z} \in \overline{kk'x}$ and $\overline{z} \in \langle \overline{0} \rangle$. There is $k'' \in \mathbb{N}$ that $\overline{z} \in \{\overline{k''x}, \overline{0}\} \subseteq \langle \overline{x} \rangle$. In a similar way, for any $\overline{r} \in \mathbb{Z}_n$ and $\overline{y} \in \langle \overline{x} \rangle$, we have $\varrho(\overline{r}, \langle \overline{x} \rangle) \subseteq \langle \overline{x} \rangle$. Hence $\langle \overline{x} \rangle \in \mathcal{I}(\mathbb{Z}_n, \varrho, \varsigma)$.

(*ii*) One can see that $\langle \overline{0} \rangle = \bigcup_{k \in \mathbb{N}} \{\overline{k0}, \overline{0}\} = \{\overline{0}\}$.

(*iii*) Let $\overline{z} \in \langle \overline{x} \rangle$. Then there is $k \in \mathbb{N}$, that $\overline{z} = \overline{kx}$ or $\overline{z} = \overline{0}$. Since $\gcd(x, n) = d$ and by item (*i*), there is $k' \in \mathbb{Z}$, that $x = k'd$. If $\overline{z} = \overline{kx}$, then $\overline{z} = \overline{kx} = \overline{kk'd} \in \langle \overline{x} \rangle$ and if $\overline{z} = \overline{0}$, then $\overline{z} = \overline{k0} = \overline{0} \in \langle \overline{x} \rangle$. Hence $\langle \overline{x} \rangle \subseteq \langle \overline{d} \rangle$. Let $\overline{z} \in \langle \overline{d} \rangle$. Then there is $k \in \mathbb{N}$, that $\overline{z} = \overline{kd}$ or $\overline{z} = \overline{0}$. Since $\gcd(x, n) = d$ and by item (*i*), there is $r, s \in \mathbb{Z}$ that $rx + ns = d$, and so $\overline{rkx} + \overline{nks} = \overline{kd}$. Applying Theorem 2.1, we get that $\overline{z} = \overline{krx}$ or $\overline{z} = \overline{0}$. Hence $\langle \overline{d} \rangle \subseteq \langle \overline{x} \rangle$. Also for $\gcd(y, n) = d$ the proof is similarly, then $\langle \overline{d} \rangle = \langle \overline{y} \rangle$, there for $\langle \overline{x} \rangle = \langle \overline{y} \rangle$. □

Example 2.4. Consider the general hyperring $(\mathbb{Z}_{100}, \varrho, \varsigma)$. By Theorem 2.7, we have the reproduced ideals of $R = (\mathbb{Z}_{100}, \varrho, \varsigma)$ as follows:

$$\begin{aligned}
&\mathcal{RI}(\mathbb{Z}_{100}, \varrho, \varsigma) = \{I_1 = \{\overline{0}\}, I_2 = \{\overline{0}, \overline{2}, \overline{4}, ..., \overline{98}\}, I_3 = \{\overline{0}, \overline{4}, \overline{8}, ..., \overline{96}\},\\
&I_4 = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \ldots, \overline{95}\}, I_5 = \{\overline{0}, \overline{25}, \overline{50}\}, I_6 = \{\overline{0}, \overline{10}, \overline{20}, \ldots, \overline{90}\},\\
&I_7 = \{\overline{0}, \overline{50}\}, I_8 = \{\overline{0}, \overline{20}, \ldots, \overline{80}\}, I_9 = \mathbb{Z}_{45}\}.
\end{aligned}$$

Theorem 2.8. *Let* $n \in \mathbb{N}$. *Then*

(*i*) $|\mathcal{RI}(\mathbb{Z}_n, \varrho, \varsigma)| = |Div(n)|$.

(*ii*) *for any* $\overline{x}, \overline{y} \in R, \langle \overline{x} \rangle \cap \langle \overline{y} \rangle = \langle \overline{lcm(x, y)} \rangle$.

Proof. (*i*) By Theorem 2.7, $I \in \mathcal{RI}$ if and only if there is $s\overline{x} \in R$, that $I = \langle \overline{x} \rangle$. Also for any $\overline{x} \in R, gcd(x, n) = d$ if and only if $\langle \overline{x} \rangle = \langle \overline{d} \rangle$. Thus $|\mathcal{RI}(\mathbb{Z}_n, \varrho, \varsigma)| = |Div(n)|$.

(*ii*) Let $\overline{x} \in R$. By definition, we have $\langle \overline{x} \rangle = \bigcup_{k \in \mathbb{N}} \{\overline{kx}, 0\}$. Clearly, there is $k_1, k_2 \in \mathbb{N}$ that $lcm(x, y) = k_1 x$ and $lcm(x, y) = k_2 y$. Hence $\overline{lcm(x, y)} \in \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$ and so $\langle \overline{lcm(x, y)} \rangle \subseteq \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$. Conversely. let $\overline{a} \in \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$. Then $\overline{a} = \overline{0}$ or there is $k_1, k_2 \in \mathbb{N}$ that $\overline{a} = \overline{k_1 x} = \overline{k_2 y}$. Thus $x \mid a$ and $y \mid a$ and so $lcm(x, y) \mid a$. Hence there is $k \in \mathbb{N}$ that $\overline{a} = \overline{k \times lcm(x, y)}$ and so $\langle \overline{a} \rangle \subseteq \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$. □

Corollary 2.1. *Let $n \in \mathbb{N}$. Then $\mathcal{RI}(\mathbb{Z}_n, \varrho, \varsigma) = \{\langle \overline{d} \rangle \mid d \in Div(n)\}$.*

Example 2.5. Let p, q, r be primes, $m, l, k \in \mathbb{N}$ and $n = p^m q^l r^k$. Then

$$\mathcal{I}(\mathbb{Z}_n, \varrho, \varsigma) = \{\langle \overline{p^{t_1} q^{t_2} r^{t_3}} \rangle \mid 0 \leq t_1 \leq m, 0 \leq t_2 \leq l, 0 \leq t_3 \leq k\}\rangle.$$

Corollary 2.2. *Assume $p_1, p_2, \ldots, p_k$ are primes, $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ and $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. Then $|\mathcal{RI}(\mathbb{Z}_n, \varrho, \varsigma)| = \sum_{i=1}^{k} (\alpha_i + 1)$.*

2.3. Reproduced ideals in $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}},)$

In this subsection, all reproduced ideals of finite reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}},)$ are computed and it is proved that every reproduced ideal of the reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}},)$ is characterized by the divisors of n.

Theorem 2.9. *Let p be a prime. Then in reproduced general hyperring $(\mathbb{Z}_p \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have $\mathcal{RI}(R) = \{R, \{\overline{0}\}, \{\overline{0}, \sqrt{p}\}\}$;*

Proof. Let $I \in \mathcal{RI}(R) \setminus \{\{\overline{0}\}, R\}$. Since $\emptyset \neq I$ is a hyperideal of R, there exists $a \in I$ and so $\{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\} \subseteq I$. In addition, $\forall\ r \neq \sqrt{p}$ we have $\upsilon_{\sqrt{p}}(r, \{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\}) \subseteq \{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\}$. Also for $r = \sqrt{p}$, we have $\upsilon_{\sqrt{p}}(r, \{a, 2a, 3a, \ldots, (p-1)a, \overline{0}\}) \subseteq \{\sqrt{p}, \overline{0}\}$. Thus $I = \{\sqrt{p}, \overline{0}\}$. □

Theorem 2.10. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have*

- (i) *if I is a nontrivial hyperideal of R, then $\sqrt{p} \in I$;*
- (ii) *$\forall\ 1 \leq m \leq p^{k-1}$, $I_p^{(m)} = \{m\overline{p}, 2m\overline{p}, \ldots, tm\overline{p}, \sqrt{p} \mid t \in \mathbb{N}$ is the smallest s.t $tm \equiv 0 \pmod{p^{k-1}}\}$ is a hyperideal of R.*

Proof. (i) Let $\overline{0} \neq x \in I$. Since I is a hyperideal of R and $\sqrt{p} \in R$, we get that $\upsilon_{\sqrt{p}}(\sqrt{p}, x) \subseteq I$. On other hand $\forall\ x \in I, \varsigma_{\sqrt{p}}(\sqrt{p}, x) = \overline{0}, \sqrt{p}$ or $\{\overline{0}, \sqrt{p}\}$. If $\upsilon_{\sqrt{p}}(\sqrt{p}, x) = \overline{0}$, then by definition there exists $m \in \mathbb{N}$ that $x = mp$. Hence there is $n \in \mathbb{N}$ that $\{\overline{0}, \sqrt{p}\} = \varrho_{\sqrt{p}} \underbrace{(x, x, \ldots x)}_{n\ times} \subseteq I$ and so in any case $\sqrt{p} \in I$.

(ii) Let $1 \leq m \leq p^{k-1}$ and $x, y \in I_p^{(m)} \setminus \{\sqrt{p}\}$. Then there exists $1 \leq k_1, k_2 \leq t \in \mathbb{N}$ that $x + y = (k_1 + k_2)(m\overline{p}) \subseteq I_p^{(m)}$, because of $\overline{0} \leq (k_1 + k_2)(m\overline{p}) \leq \overline{p^{k-1}}$. In addition $\forall\ \overline{x} \in I_p^{(m)}$, $\varrho_{\sqrt{p}}(\sqrt{p}, \overline{x}) = \{\overline{x}\} \subseteq I_p^{(m)}$ and $\varrho_{\sqrt{p}}(\sqrt{p}, \sqrt{p}) = \{\overline{0}, \sqrt{p}\} \subseteq I_p^{(m)}$, imply that $\forall\ x, y \in I_p^{(m)}, \varrho_{\sqrt{p}}(x, y) \subseteq I_p^{(m)}$. Also $\forall\ r \in R \setminus \{\sqrt{p}\}$ and $x \in I_p^{(m)} \setminus \{\sqrt{p}\}$ there exists $1 \leq k \leq t \in \mathbb{N}$ that $\upsilon_{\sqrt{p}}(r, x) = rk(m\overline{p}) \subseteq I_p^{(m)}$, because of $\overline{0} \leq (rkm)\overline{p} \leq \overline{p^{k-1}}$. On the other hand, $\upsilon_{\sqrt{p}}(\sqrt{p}, \overline{x}) \subseteq \{\overline{0}, \sqrt{p}\}$, implies that $\forall\ r \in R$ and $x \in I_p^{(m)}$, we have $\upsilon_{\sqrt{p}}(r, x) \subseteq I_p^{(m)}$. □

Theorem 2.11. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have*

(*i*) $\forall\ 1 \leq m \leq p^{k-1}$, *we have* $|I_p^{(m)}| = 1 + \dfrac{p^{k-1}}{\gcd(m, p^{k-1})}$;

(*ii*) $\forall\ 1 \leq m, m' \leq p^{k-1}$, $I_p^{(m)} = I_p^{(m')}$ *if and only if* $\gcd(p^{k-1}, m) = \gcd(p^{k-1}, m')$.

Proof. (*i*) Let $1 \leq m \leq p^{k-1}$. Using Theorem 2.10 (*i*), $\sqrt{p} \in I_p^m$, so $|I_p^m| = 1 + |\{t \in \mathbb{N} \mid t$ is the smallest s.t $tm \equiv 0 \pmod{p^{k-1}}\}| = q$. Suppose $t \in \mathbb{Z}$ is the smallest that $tm \equiv 0 \pmod{p^{k-1}}$. Thus $p^{k-1} \mid tm$. If $\gcd(p^{k-1}, m) = 1$, then $p^{k-1} \mid t$ and because t is the smallest, we obtain that $t = p^{k-1}$. But for $\gcd(p^{k-1}, m) = d \neq 1$, have $\dfrac{p^{k-1}}{d} \mid t$. Since $p^{k-1}m \equiv 0 \pmod{p^{k-1}}$ and $t \in \mathbb{N}$ is the smallest that $tm \equiv 0 \pmod{p^{k-1}}$, we get that $\dfrac{p^{k-1}}{\gcd(m, p^{k-1})} = t$.

(*ii*) Let $1 \leq m, m' \leq p^{k-1}$. Then by item (*ii*), $I_p^{(m)} = I_p^{(m')}$ if and only if $1 + \dfrac{p^{k-1}}{\gcd(m, p^{k-1})} = 1 + \dfrac{p^{k-1}}{\gcd(m', p^{k-1})} \Longleftrightarrow \gcd(p^{k-1}, m) = \gcd(p^{k-1}, m')$. □

Theorem 2.12. *Let p be a prime, $k \in \mathbb{N}$ and $1 \leq j \leq p^{k-1}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have*

(*i*) $I_p^{(p^i)} = I_p^{(rp^i)}$, *where* $rp^i \neq p^j$;

(*ii*) $\forall\ 1 \leq m \leq p^{k-1}$, $I_p^{(p^{k-1})} \subseteq I_p^{(m)}$;

(*iii*) $I_p^{(p^{k-1})} \subseteq I_p^{(p^{k-2})} \subseteq I_p^{(p^{k-3})} \subseteq I_p^{(p^{k-4})} \subseteq \ldots \subseteq I_p^{(p)}$.

Proof. The proof is similar to Theorem 2.11. □

Theorem 2.13. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have*

(*i*) $\mathcal{RI}(R) = \{R, \{\overline{0}\}, I_p^{(m)} \mid 1 \leq m \leq p^{k-1}\}$;

(*ii*) $|\mathcal{RI}(R)| = k$.

Proof. (*i*) Clearly $R, \{\overline{0}\} \in \mathcal{RI}(R)$. Let I be a nontrivial hyperideal of R, using Theorem 2.10 (*i*), $\overline{0}, \sqrt{p} \in I$. Suppose that $0 \neq a \in I$. If $\gcd(a, p^k) = 1$, then there exist $s, s' \in \mathbb{Z}$ that $1 = as + s'p^k$. It follows that $\overline{1} \in I$ and we get that $R = I$. But for $\gcd(a, p^k) = d \neq 1$, since p is a prime, there exist $1 \leq i \leq k$ in such a way that $d = p^i$, consequently $p^i \in I$.

(*ii*) It is immediate by (*i*). □

Definition 2.4. Let R be a reproduced general hyperring and $M \neq R$ be an arbitrary hyperideal of R.

(i) M is called a maximal hyperideal of R, if the only reproduced hyperideals containing M are M and R;

(ii) M is called a reproduced prime hyperideal of R, $\forall\ a, b \in R, \varsigma(a,b) \subseteq M$ implies that $a \in M$ or $b \in M$.

Theorem 2.14. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, $I_p^{(1)}$ is the reproduced maximal hyperideal of R.*

Proof. Applying Theorem 2.11, for any $I_p^{(m)}, I_p^{(m')}, \in \mathcal{HI}(R)$, we have $|I_p^{(m)}| \geq |I_p^{(m')}|$ if and only if $\dfrac{p^{k-1}}{\gcd(m, p^{k-1})} \geq \dfrac{p^{k-1}}{\gcd(m', p^{k-1})}$. In addition, for $|\dfrac{p^{k-1}}{\gcd(m, p^{k-1})}| = s$, s is maximum if and only if $\gcd(m, p^{k-1}) = 1$. Thus, $m = 1$, implies that $|I_p^{(m)}| \geq |I_p^{(m')}|$. □

Example 2.6. Consider the general hyperring $R = \mathbb{Z}_{125} \cup \{\sqrt{3}\}$. Computations show that

$$\begin{aligned} & I_5^{(1)} = I_5^{(2)} = I_5^{(3)} = I_5^{(4)} = I_5^{(6)} = I_5^{(7)} = I_5^{(8)} = I_5^{(9)} = I_5^{(11)} \\ & = I_5^{(12)} = I_5^{(13)} = I_5^{(14)} = I_5^{(16)} = I_5^{(17)} = I_5^{(18)} = I_5^{(19)} = I_5^{(21)} \\ & = I_5^{(22)} = I_5^{(23)} = I_5^{(24)} = \{\overline{5}, \overline{10}, \overline{15}, \overline{20}, \ldots, \overline{115}, \overline{120}, \overline{0}, \sqrt{5}\}, \\ & I_5^{(5)} = I_5^{(10)} = I_5^{(15)} = I_5^{(20)} = \{\overline{0}, \overline{25}, \overline{50}, \overline{75}, \overline{100}, \sqrt{5}\}, I_5^{(25)} = \{\overline{0}, \sqrt{5}\} \end{aligned}$$

and so $\mathcal{RI}(R) = \{I_5^{(1)}, I_5^{(5)}, I_5^{(25)}, \{\overline{0}\}, \mathbb{Z}_{125} \cup \{\sqrt{3}\}\}$.

Let (R, ϱ, ς) be a general hyperring. Then will denote $\mathcal{MRI}(R) = \{M \in \mathcal{RI}(R) \mid M \text{ is a maximal hyperideal}\}$ and $\mathcal{PRI}(R) = \{M \in \mathcal{RI}(R) \mid M \text{ is a prime hyperideal}\}$.

Theorem 2.15. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have*

(i) $\mathcal{MRI}(R) = \{I_p^{(m)} \mid gcd(m, p) = 1\}$.

(ii) $|\mathcal{MRI}(R)| = p^{k-2}(p-1)$.

Proof. (i) Let $I_p^{(m)} \in \mathcal{MRI}(R)$ and $m \neq 1$. Since $gcd(m, p) = 1$, we get that $I_p^{(m)} = I_p^{(1)}$ and so by Theorem 2.14.

(ii) By (i), $|\mathcal{MRI}(R)| = \varphi(p^{k-1})$, where φ is **Euler's phi** function. □

Theorem 2.16. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, $I_p^{(1)}$ is the reproduced prime hyperideal of R.*

Proof. Let p be a prime, $x, y \in \mathbb{Z}_{p^k} \cup \{\sqrt{p}\}$ and $\varsigma(x,y) \subseteq I_p^{(1)}$. Then $\varsigma(x,y) = \{0, \sqrt{p}\}, \varsigma(x,y) = \{\sqrt{p}\}, \varsigma(x,y) = \{0\}$ or there exists $1 \leq s \leq t$ such that $\varsigma(x,y) = \{sm\overline{p}\}$, where $tm \equiv 0 \pmod{p^{k-1}}$. If $\varsigma(x,y) = \{\sqrt{p}\}$, then $x \in \mathbb{Z}_{p^k} \setminus \{mp\}, y = \sqrt{p}(m \in \mathbb{N})$, and so $y \in I_p^{(1)}$. If $\varsigma(x,y) = \{0\}$, then $x = mp, y = \sqrt{p}(m \in \mathbb{N})$, and so $x, y \in I_p^{(1)}$. If $\varsigma(x,y) = \{0, \sqrt{p}\}$, then $x = y = \sqrt{p}$, and so $x, y \in I_p^{(1)}$. If there exists $1 \leq s \leq t$ such that $\varsigma(x,y) = \{s\overline{p} \neq \sqrt{p}\}$, where $tm \equiv 0 \pmod{p^{k-1}}$, then for $s = 1$, we have $\overline{p} \in I_p^{(1)}$. Thus $I_p^{(1)}$ is a reproduced prime hyperideal of R. □

Theorem 2.17. *Let p be a prime and $k \in \mathbb{N}$. Then in reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$, we have*

(*i*) $\mathcal{PRI}(R) = \{I_p^{(m)} \mid gcd(m,p) = 1\}$.

(*ii*) $|\mathcal{PRI}(R)| = p^{k-2}(p-1)$.

Proof. (*i*) Let $I_p^{(m)} \in \mathcal{PRI}(R)$ and $m \neq 1$. Since $gcd(m,p) = 1$, we get that $I_p^{(m)} = I_p^{(1)}$ and so by Theorem 2.16, $I_p^{(m)}$ is a reproduced prime hyperideal of R.

(*ii*) By (*i*), $|\mathcal{PRI}(R)| = \varphi(p^{k-1})$. □

Definition 2.5. Let (R, ϱ, ς) be a general hyperring. Then (R, ϱ, ς) is called an Ideal-absorbing, if the its set of all prime ideals and the its set of all maximal hyper ideals is equal.

Corollary 2.3. *Let p be a prime and $k \in \mathbb{N}$. Then the reproduced general hyperring $(\mathbb{Z}_{p^k} \cup \{\sqrt{p}\}, \varrho_{\sqrt{p}}, \varsigma_{\sqrt{p}})$ is an Ideal-absorbing.*

3. Conclusion and discussion

The current paper has defined the general hyperrings as a generalization of hyperrings and presented some properties in these hyperstructures. For each ring considered, it is possible to work only with the elements of the context set. This means that if we want to add another element to a ring, it is necessary to break all the principles of the axiom and it is possible that the new complex will not become a ring. But by adding an element to an arbitrary ring, a hyperring can be formed, and this is one of the limitations of rings that is solved by hyperrings. This advantage can be applied to all substructures including ideals and substructures Also,

(*i*) principal ideal domain reproduced general hyperrings are constructed,

(*ii*) the set of all prime ideals and the set of all maximal hyper ideals of principal ideal domain reproduced general hyperrings are computed, principal ideal domain reproduced general hyperrings are constructed.

(*iii*) the concept of Ideal-absorbing reproduced general hyperrings is defined and is proved that the principal ideal domain reproduced general hyperrings are Ideal-absorbing.

We hope that these results are helpful for further studies in general hyperring theory. In our future studies, we hope to obtain more results regarding fuzzy general hyperring, soft general hyperring, tropical general multifield and their applications.

REFERENCES

1. A. Asadi and R. Ameri: *Direct Limit of Krasner (m, n)-Hyperrings*, J. Sci. Islam. Repub. Iran. **31(1)** (2020), 75–83.
2. R. Ameri, M . Hamidi and H. Mohammadi: *Hyperideals of (Finite) General Hyperrings*, Mathematics Interdisciplinary Research. **6** (2021), 257–273.
3. R. Ameri, M. Hamidi and A. A. Tavakoli: *Boolean Rings Based on hyperrings*, J. Sci. Islam. Repub. Iran. **32(2)** (2021), 159–168.
4. H. Bordbar and I. Cristea: *Regular Parameter Elements and Regular Local Hyperrings*, Mathematics, **9(3)** (2021), 243.
5. H. Bordbara and I. Cristea: *Height of Prime Hyperideals in Krasner Hyperrings*, Filomat., **31:19** (2017), 6153-6163.
6. I. Cristea and M. Kankaras: *The Reducibility Concept in General Hyperrings*, Mathematics, **9(17)** (2021), 2037.
7. B. Davvaz and V. Leoreanu-Fotea: *Hyperring Theory and Applications*, International Academic Press. 2007.
8. B. Davvaz, G. Ulucak and U. Tekir: *Weakly (k, n)-absorbing (primary) hyperideals of a Krasner (m, n)-hyperring*, Hacet. J. Math. Stat. (2023).
9. B. Davvaz and T. Vougiouklis: *Commutative rings obtained from hyperrings (Hv-rings) with α^*–relations*, Commun. Algebra, **35** (2007), 3307–3320.
10. P. Ghiasvand, S. Mirvakili and B. Davvaz: *Boolean Rings Obtained from Hyperrings with $\eta^*_{1,m}$ Relations*, Iran. J. Sci. Technol Trans. Sci. **41(1)** (2017), 69–79.
11. P. Ghiasvand, M. Raeisi and S. Mirvakili: *A generalization of graded prime hyperideals over graded multiplicative hyperrings*, J. Algebra Appl. (2022).
12. M. Hamidi and A. R. Ashrafi: *Fundamental relation and automorphism group of very thin H_v–groups*, Comm. Algebra. **45(1)** (2017), 130–140.
13. M. Hamidi, F. Faraji, R. Ameri and K. Ahamadi Amoli: *Normal Injective Resolution Of General Krasner Hyprmodule*, J. Algebr. Syst. **10(1)** (2022), 121–145.
14. M. Krasner: *Approximation des corps values complet de caractristique $p > 0$ par ceux de caracteristique zero*, colloque, d'-Algebra superieure, (1956), Bruxells.
15. S. Omidi and B. Davvaz: *Contribution to study special kinds of hyperideals in ordered semihyperrings*, J. Taibah Univ. Sci. **11** (2017), 1083-1094.
16. Y. Rao, S. Kosari, A. Khan and N. Abbasizadeh: *A Study on Special Kinds of Derivations in Ordered Hyperrings*, Symmetry. **14(10)** (2022), 2205.
17. R. Rota: *Sugli iperanelli moltiplicativi*, Rend. Di Mat. **(4)2** (1982), 711-724.

18. H. Yazaril, B. Davvaz and D. Yilmaz: *Extended Centroid Of Hyperrings*, Gulf j. math. **8(1)** (2020), 6–15.

19. T. Vougiouklis: *The fundamental Relations in Hyperrings*, The general th International congress in Algebraic Hyperstructures hyperfield Proc.4 and its Applications (AHA 1990) World Scientific. 203–211.

FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 39, No 1 (2024), 177–181
https://doi.org/10.22190/FUMI230812012K
Original Scientific Paper

GENERATORS FOR THE ELLIPTIC CURVE $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$

Mehrdad Khazali[1], Hassan Daghigh[2] and Amir Alidadi[1]

[1] **Department of Mathematics, Higher Educational Complex of Bam, Iran**
[2] **Faculty of Mathematical Sciences, University of Kashan, Iran**

ORCID IDs: Mehrdad Khazali https://orcid.org/0000-0002-7460-6983
Hassan Daghigh https://orcid.org/0000-0002-4242-769X
Amir Alidadi https://orcid.org/0000-0001-9608-0663

Abstract. Let $\{E_{(p,q)}\}$ denote a family of elliptic curves over $\mathbb{Q}$ as defined by the Weierstrass equation $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$ where p and q are both prime numbers greater than 5. As evidence that this has two independent points, we already showed that at least the rank of $\{E_{(p,q)}\}$ is two. In this study, we show that the two independent points are part of a $\mathbb{Z}$-basis for the quotient of $E_{(p,q)}(\mathbb{Q})$ by its torsion subgroup.
Keywords: Independent points, Rank of an elliptic curve, Canonical Height.

1. Introduction

Let $\{E_{(1,m)}\}$ be a family of elliptic curves over $\mathbb{Q}$ as determined by the Weierstrass equation $E_{(1,m)} : y^2 = x^3 - x + m^2$ where m is an integer number greater than 1. Brown and Myers in [2] discovered that this family included two independent points. Fujita and Nara in [3] proved that the two independent points could be extended to form the basis for this family.

Let $\{E_{(n,1)}\}$ be a family of elliptic curves over $\mathbb{Q}$ as defined by the Weierstrass equation $E_{(n,1)} : y^2 = x^3 - n^2x + 1$ where n is an integer number greater than 1. In [1], Antoniewicz provided evidence that this family contained two independent points. Fujita and Nara in [3] showed that the two independent points could be extended to form the basis for this family.

The family of elliptic curves over $\mathbb{Q}$, as described by the Weierstrass equation $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$, where p and q are both prime numbers greater than 5,

Received August 12, 2023, revised: October 25, 2023, accepted: October 25, 2023
Communicated by Mojtaba Bahramian Mail, Amirhossein Nokhodkar Mail and Predrag Stanimirović
Corresponding Author: Mehrdad Khazali. E-mail addresses: mehrdad@bam.ac.ir (M. Khazali), hassan@kashanu.ac.ir (H. Daghigh), alidadi@bam.ac.ir (A. Alidadi)
2010 *Mathematics Subject Classification.* Primary 11G05; Secondary 14G05

is represented by the $\{E_{(p,q)}\}$. We recently proved that the points $P_1 = (0, q)$ and $P_2 = (-p, q)$ are independent points. In this essay, we describe how the two points P_1 and P_2 might be extended and expanded to serve as the basis for this family under particular circumstances. Theorem 1.1 demonstrates the most potent single assertion.

Theorem 1.1. *[Main Theorem]. Let $\{E_{(p,q)}\}$ denote a family of elliptic curves over $\mathbb{Q}$ as defined by the Weierstrass equation $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$ where p and q are both prime numbers greater than 5. If $p > 2\sqrt[4]{2}q$, then $P_1 = (0, q)$ and $P_2 = (-p, q)$ are part of a $\mathbb{Z}$-basis for the quotient of $E_{(p,q)}(\mathbb{Q})$ by its torsion subgroup.*

2. Upper and Lower bound

We continue exploring the idea of canonical height in this section because it is crucial for elliptic curve arithmetic. Point P's canonical height, expressed as

$$\begin{aligned} \hat{h} : E(\mathbb{Q}) &\longrightarrow [0, \infty) \\ P &\longmapsto \begin{cases} \lim\limits_{n\to\infty} \dfrac{h(2^nP)}{4^n} & P \neq \mathcal{O} \\ 0 & P = \mathcal{O} \end{cases} \end{aligned}$$

dose is not suitable for computation. The alternative definition of canonical height offered here with [6] is Tate's height. Therefore, we have

$$\hat{h}(P) = \hat{\lambda}_\infty(P) + \sum_{r|\Delta} \hat{\lambda}_r(P).$$

In fact, the canonical height is the sum of the archimedean local height and the local height, assuming that r is a prime number such that $r \mid \Delta$. We also note that the discriminant of $E_{(p,q)}$ is $\Delta = 16(4p^6 - 27q^4) = 16\Delta'$. We have previously shown that 3 and $5 \nmid \Delta'$. In this article, Δ' is assumed to be square-free. At the moment, we claim that the equation $y^2 = x^3 - p^2x + q^2$ is the global minimum.

Proposition 2.1. *The Weierstrass equation $y^2 = x^3 - p^2x + q^2$ is the global minimum.*

Proof. In view of Lemma 3.1 of [3]. □

Now, we compute $c_4 = 48p^2$, $c_6 = -864q^2$, $b_2 = 0$, $b_4 = -2p^2$, $b_6 = 4q^2$ and $b_8 = -p^4$. The upper and lower bounds of the canonical heights for P_1 and P_2 are established by the following theorems:

Theorem 2.1. *Let $\{E_{(p,q)}\}$ represent a family of elliptic curves over $\mathbb{Q}$ as defined by the Weierstrass equation $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$, where p and q are both prime* *numbers greater than 5. we consider* $P_1 = (0, q) \in E_{(p,q)}(\mathbb{Q})$ *and* $P_2 = (-p, q) \in E_{(p,q)}(\mathbb{Q})$. *If* $p > 2\sqrt[4]{2}q$, *then*

$$\hat{h}(P_1) \leqslant \frac{1}{2}log(p) + \frac{1}{24}log(2^{11}p^4), \quad \hat{h}(P_2) \leqslant \frac{1}{2}log(p) + \frac{1}{6}log(2^{11}p^4).$$

Proof. According to (4.1) of [6], we have

$$H = Max\{4, 2p^2, 8q^2, p^4\}.$$

The theorem's assumption leads to the conclusion that $H = p^4$. To compute the upper bound for canonical height for point P_1 based on Theorem (2.2) of [6], we must apply Equation 2.1.

$$\hat{\lambda}_\infty(P) = \frac{1}{8}\log(|(x^2 + p^2)^2 - 8q^2x|) + \frac{1}{8}\sum_{n=1}^{\infty} 4^{-n}\log(|z(2^nP)|). \tag{2.1}$$

Hence, we have

$$\hat{\lambda}_\infty(P_1) \leqslant \frac{1}{2}log(p) + \frac{1}{24}log(2^{11}p^4) = UB1,$$

and so for point P_2. According to Theorem (2.2) of [6], we must apply Equation 2.2.

$$\hat{\lambda}_\infty(P) = \frac{1}{2}\log(|x|) + \frac{1}{8}\sum_{n=0}^{\infty} 4^{-n}\log(|z(2^nP)|). \tag{2.2}$$

Hence, we have

$$\hat{\lambda}_\infty(P_2) \leqslant \frac{1}{2}log(p) + \frac{1}{6}log(2^{11}p^4) = UB2.$$

□

Theorem 2.2. *Let* $\{E_{(p,q)}\}$ *represent a family of elliptic curves over* $\mathbb{Q}$ *as defined by the Weierstrass equation* $E_{(p,q)} : y^2 = x^3 - p^2x + q^2$ *where* p *and* q *are both prime numbers greater than 5. Let* $P \in E_{(p,q)}(\mathbb{Q})$ *be a rational point on* $E_{(p,q)}$. *If* $p > 2\sqrt[4]{2}q$, *then*

$$\hat{h}(P) > \frac{1}{8}log(\frac{p^4}{2}) - \frac{1}{3}log(2) = LB.$$

Proof. We have two scenarios for computing the local height based on Proposition 2.1 and Theorem [6]. The condition $\lambda_2(P) = 0$ occurs if P reduces to a nonsingular point in module 2. Otherwise, P becomes a singular point modulo 2. According to (c) of Theorem (5.2) of [6], we have $\lambda_2(P) = -\frac{1}{3}log(2)$. Next, we show that

$$\hat{\lambda}_\infty(P) \geqslant \frac{1}{8}\log(|(x^2 + p^2)^2 - 8q^2x|) \geqslant \frac{1}{8}\log(|p^4 - 16q^4|) > \frac{1}{8}log(\frac{p^4}{2}),$$

therefore

$$\hat{h}(P) > \frac{1}{8}log(\frac{p^4}{2}) - \frac{1}{3}log(2).$$

□

3. Proof of Theorem 1.1

An important theorem applied to prove Theorem 3.1 is Theorem (3.1) of [5].

Theorem 3.1. *Let E be an elliptic curve with a rank of $r \geqslant 2$ over $\mathbb{Q}$. Let P_1' and P_2' be independent points in the $E(\mathbb{Q})$ modulo $E(\mathbb{Q})_{tors}$. Choose a basis $\{Q_1, Q_2, \dots, Q_r\}$ for $E(\mathbb{Q})$ modulo $E(\mathbb{Q})_{tors}$ according to the condition $P_1', P_2' \in \langle Q_1\rangle + \langle Q_2\rangle$. Assume that $E(\mathbb{Q})$ contains no infinite-order point Q with $\hat{h}(Q) \leqslant \lambda$ where λ is a positive real number. Then, index v of the span of P_1' and P_2' in $\langle Q_1\rangle + \langle Q_2\rangle$ satisfies*

$$v \leqslant \frac{2}{\sqrt{3}} \frac{\sqrt{R(P_1', P_2')}}{\lambda}$$

where

$$R(P_1', P_2') = \hat{h}(P_1')\hat{h}(P_2') - \frac{1}{4}(\hat{h}(P_1' + P_2') - \hat{h}(P_1') - \hat{h}(P_2'))^2 < \hat{h}(P_1')\hat{h}(P_2'),$$

thus

$$v \leqslant \frac{2}{\sqrt{3}} \frac{\sqrt{\hat{h}(P_1')\hat{h}(P_2')}}{\lambda}.$$

This has enabled us to demonstrate Theorem 1.1.

Proof. In addition to the fact that $2 \nmid v$ holds true, we support our claim with three theorems: 2.1, 2.2 and 3.1.

The right-hand side of the equation is now established as follows:

$$v \leqslant \frac{2}{\sqrt{3}} . \frac{\sqrt{UB1.UB2}}{LB}.$$

The calculation yields the value $v < 3$ for all prime numbers $p \geqslant 41$. The evidence is therefore persuasive. □

Acknowledgment

We would like to thank the referees for their careful reading and valuable comments.

REFERENCES

1. A. Antoniewicz: *On a family of elliptic curves*, Univ. Iagel. Acta Math. **43** (2005), 21–32.
2. E. Brown and T.M. Bruce: *Elliptic curves from Mordell to Diophantus and back*, The American mathematical monthly **109(7)** (2002), 639-649.
3. Y. Fujita and N. Tadahisa: *The Mordell-Weil bases for the elliptic curve of the form* $y^2 = x^3 - m^2x + n^2$, Publ. Math. Debrecen **92/1-2** (2018), 79-99.

4. M. Khazali and D. Hassan: *Family Of Elliptic Curves* $E(p, q) : y^2 = x^3 - p^2x + q^2$, Facta Universitatis, Series: Mathematics and Informatics **34(4)** (2019), 805-813.

5. S. Siksek: *Infinite descent on elliptic curves*, Rocky Mountain J. Math. **25(4)** (1995), 1501–1538. MR1371352.

6. J.H. Silverman: *Computing heights on elliptic curves*, Math. Comp. **51(183)** (1988), 339-358.

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51+004

FACTA Universitatis. Series, Mathematics and informatics / editors-in-chief Mića S. Stanković, Marko D. Petković. - 1986, no. 1- . - Niš : University of Niš, 1986- (Niš: Atlantis). - 24 cm

Pet puta godišnje. - Drugo izdanje na drugom medijumu: Facta Universitatis. Series: Mathematics and Informatics (Online) = ISSN 2406-047X
ISSN 0352-9665 = Facta Universitatis. Series: Mathematics and informatics
COBISS.SR-ID 5881090