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[2] E. B. Saff, R. S. Varga, On incomplete polynomials II, Pacific J. Math. 92 (1981) 161–172.

[3] P. Erdős, On the distribution of the roots of orthogonal polynomials, in: G. Alexits, S. B. Steckhin (Eds.), Proceedings of a Conference on Constructive Theory of Functions, Akademiai Kiado, Budapest, 1972, pp. 145–150.

[4] D. Allen, Relations between the local and global structure of finite semigroups, Ph. D. Thesis, University of California, Berkeley, 1968.

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Original Scientific Paper

REFLEXIVITY OF LINEAR n-NORMED SPACE WITH RESPECT TO b-LINEAR FUNCTIONAL

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Abstract. In this paper, we discuss various consequences of Hahn-Banach theorem for bounded b-linear functional in linear n-normed space and describe the notion of reflexivity of linear n-normed space with respect to bounded b-linear functional. The concepts of strong convergence and weak convergence of a sequence of vectors with respect to bounded b-linear functionals in linear n-normed space have been introduced and some of their properties are being discussed.
Keywords: Hahn-Banach theorem, reflexivity of normed linear space, weak and strong convergence, linear n-normed space, n-Banach space.

1. Introduction

The dual space of a normed linear space is the set of all bounded linear functionals on the space. In some cases, the dual of the dual space, i. e., second dual space of a normed space, under a specific mapping-called the natural embedding, is isometrically isomorphic to the original space. Such normed spaces are known as reflexive spaces. This concept was introduced by H. Hahn in 1927 and called reflexivity by E. R Lorch in 1939. Hahn recognized the importance of reflexivity in his study of linear equations in normed spaces. Weak convergence of sequence of vectors in a normed space is a certain kind of interplay between a normed space and its dual

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space. This concept demonstrates a fundamental principle of functional analysis which in turn states that the investigation of normed spaces is generally linked with that of their dual spaces. Weak convergence has various applications in the calculus of variations, general theory of differential equations and in fact, plays an important role in many problems of analysis.

The notion of linear 2-normed space was introduced by S. Gahler [2]. A survey of the theory of linear 2-normed space can be found in [1]. The concept of 2-Banach space is briefly discussed in [8]. H. Gunawan and Mashadi [5] developed the generalization of a linear 2-normed space for $n \geq 2$. P. Ghosh and T. K. Samanta [3] developed Uniform Boundedness Principile and Hahn-Banach theorem for bounded b-linear functionals in linear n-normed space. They also studied slow convergence of sequences of b-linear functionals in linear n-normed space [4].

In this paper, some important consequences of the Hahn-Banach theorem for bounded b-linear functionals in case of linear n-normed spaces are discussed. We shall introduce the notion of b-relexivity of linear n-normed space and see that a closed subspce of a b-reflexive n-Banach space is also b-reflexive. Finally, b-weak convergence and b-strong convergence of a sequence of vectors in a linear n-normed space in terms of bounded b-linear functionals are introduced and characterized.

2. Preliminaries

Theorem 2.1. *[6] Let $\{T_k\}$ be a sequence of bounded linear operators $T_k : Y \to Z$ from a Banach space Y into a normed space Z such that $\{\|T_k(x)\|\}$ is bounded for every $x \in Y$. Then the sequence of the norms $\{\|T_k\|\}$ is bounded.*

Definition 2.1. [5] Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field with $dim\,X \geq n$, where n is a positive integer. A real valued function $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ is called an n-norm on X if

(N1) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,

(N2) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutations of $x_1, x_2, \cdots, x_n$,

(N3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \, \|x_1, x_2, \cdots, x_n\| \quad \forall \ \alpha \in \mathbb{K}$,

(N4) $\|x + y, x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

hold for all $x, y, x_1, x_2, \cdots, x_n \in X$. The pair $(X, \|\cdot, \cdots, \cdot\|)$ is then called a linear n-normed space. For particular value $n = 2$, the space X is said to be a linear 2-normed space [2].

Throughout this paper, X will denote linear n-normed space over the field $\mathbb{K}$ $(\mathbb{R} \ or \ \mathbb{C})$ associated with the n-norm $\|\cdot, \cdots, \cdot\|$.

Definition 2.2. [5] A sequence $\{x_k\} \subseteq X$ is said to converge to $x \in X$ if

$$\lim_{k \to \infty} \| x_k - x, e_2, \cdots, e_n \| = 0$$

for every $e_2, \cdots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \| x_l - x_k, e_2, \cdots, e_n \| = 0$$

for every $e_2, \cdots, e_n \in X$. The space X is said to be complete or n-Banach space if every Cauchy sequence in this space is convergent in X. 2-Banach space [8] is a particular case of n-Banach space for $n = 2$.

Definition 2.3. [7] We define the following open and closed ball in X:

$$B_{\{e_2, \cdots, e_n\}}(a, \delta) = \{x \in X : \| x - a, e_2, \cdots, e_n \| < \delta\} \text{ and}$$

$$B_{\{e_2, \cdots, e_n\}}[a, \delta] = \{x \in X : \| x - a, e_2, \cdots, e_n \| \leq \delta\},$$

where $a, e_2, \cdots, e_n \in X$ and δ be a positive number.

Definition 2.4. [7] A subset G of X is said to be open in X if for all $a \in G$, there exist $e_2, \cdots, e_n \in X$ and $\delta > 0$ such that $B_{\{e_2, \cdots, e_n\}}(a, \delta) \subseteq G$.

Definition 2.5. [7] Let $A \subseteq X$. Then the closure of A is defined as

$$\overline{A} = \left\{ x \in X \mid \exists \{x_k\} \in A \text{ with } \lim_{k \to \infty} x_k = x \right\}.$$

The set A is said to be closed if $A = \overline{A}$.

Definition 2.6. [3] Let W be a subspace of X and $b_2, b_3, \cdots, b_n$ be fixed elements in X and $\langle b_i \rangle$ denote the subspaces of X generated by b_i, for $i = 2, 3, \cdots, n$. Then a map $T : W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:

$$(I)\ T(x + y, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) + T(y, b_2, \cdots, b_n)$$

$$(II)\ T(kx, b_2, \cdots, b_n) = k\,T(x, b_2, \cdots, b_n).$$

A b-linear functional is said to be bounded if there exists a real number $M > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq M \| x, b_2, \cdots, b_n \| \quad \forall x \in W.$$

The norm of the bounded b-linear functional T is defined by

$$\| T \| = \inf\{M > 0 : |T(x, b_2, \cdots, b_n)| \leq M \| x, b_2, \cdots, b_n \| \ \forall x \in W\}.$$

The norm of T can be expressed by any one of the following equivalent formula:

$$(I)\ \|T\| = \sup\{\,|T(x, b_2, \cdots, b_n)| \,:\, \|x, b_2, \cdots, b_n\| \leq 1\,\}.$$

$$(II)\ \|T\| = \sup\{\,|T(x, b_2, \cdots, b_n)| \,:\, \|x, b_2, \cdots, b_n\| = 1\,\}.$$

$$(III)\ \|T\| = \sup\left\{\frac{|T(x, b_2, \cdots, b_n)|}{\|x, b_2, \cdots, b_n\|} \,:\, \|x, b_2, \cdots, b_n\| \neq 0\right\}.$$

Also, we have

$$|T(x, b_2, \cdots, b_n)| \leq \|T\|\,\|x, b_2, \cdots, b_n\| \;\; \forall\, x \in W.$$

Let X_F^* denote the Banach space of all bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ with respect to the above norm.

Definition 2.7. [3] A set $\mathcal{A}$ of bounded b-linear functionals defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ is said to be pointwise bounded if for each $x \in X$, the set $\{T(x, b_2, \cdots, b_n) : T \in \mathcal{A}\}$ is a bounded set in $\mathbb{K}$ and uniformly bounded if there exists $K > 0$ such that $\|T\| \leq K \;\; \forall\, T \in \mathcal{A}$.

Theorem 2.2. *[3] Let X be a n-Banach space over the field $\mathbb{K}$. If a set $\mathcal{A}$ of bounded b-linear functionals on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ is pointwise bounded, then it is uniformly bounded.*

Theorem 2.3. *[3] Let X be a linear n-normed space over the field $\mathbb{R}$ and W be a subspace of X. Then each bounded b-linear functional T_W defined on $W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ can be extended onto $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ with preservation of the norm. In other words, there exists a bounded b-linear functional T defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ such that*

$$T(x, b_2, \cdots, b_n) = T_W(x, b_2, \cdots, b_n) \;\forall\, x \in W$$

and $\|T_W\| = \|T\|$.

Theorem 2.4. *[3] Let X be a linear n-normed space over the field $\mathbb{R}$ and x_0 be an arbitrary non-zero element in X. Then there exists a bounded b-linear functional T defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ such that*

$$\|T\| = 1 \quad \text{and} \quad T(x_0, b_2, \cdots, b_n) = \|x_0, b_2, \cdots, b_n\|.$$

Theorem 2.5. *[3] Let X be a linear n-normed space over the field $\mathbb{R}$ and $x \in X$. Then*

$$\|x, b_2, \cdots, b_n\| = \sup\left\{\frac{|T(x, b_2, \cdots, b_n)|}{\|T\|} : T \in X_F^*,\ T \neq 0\right\}.$$

3. Consequences of Hahn-Banach theorem in linear n-normed space

In this section, we shall consider some immediate corollaries and important consequences of the Hahn-Banach extension theorem for bounded b-linear functional [3] in case of linear n-normed space.

Theorem 3.1. *Let X be a linear n-normed space over the field $\mathbb{R}$ and let x, y be two distinct points of X such that the set $\{x, b_2, \cdots, b_n\}$ or $\{y, b_2, \cdots, b_n\}$ are linearly independent. Then there exists $T \in X_F^*$ such that*

$$T(x, b_2, \cdots, b_n) \neq T(y, b_2, \cdots, b_n).$$

Proof. Consider, $z = x - y$. Then $\theta \neq z \in X$ and therefore by Theorem 2.4, there exists $T \in X_F^*$ such that

$$T(z, b_2, \cdots, b_n) = \| z, b_2, \cdots, b_n \|$$

and $\| T \| = 1$. Thus

$$\begin{aligned} & T(x - y, b_2, \cdots, b_n) = \| x - y, b_2, \cdots, b_n \| \neq 0 \\ \Rightarrow \quad & T(x, b_2, \cdots, b_n) - T(y, b_2, \cdots, b_n) \neq 0 \\ \Rightarrow \quad & T(x, b_2, \cdots, b_n) \neq T(y, b_2, \cdots, b_n). \end{aligned}$$

□

Corollary 3.1. *If $X \neq \{\theta\}$ is a linear n-normed space, then there are always non-trivial bounded b-linear functionals on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, i. e., $X \neq \{\theta\} \Rightarrow X_F^* \neq \{O\}$, O being a null operator.*

Proof. This is an immediate consequence of Theorem 2.4. □

Corollary 3.2. *Let X be a linear n-normed space. Then for all $T \in X_F^*$,*

$$T(x, b_2, \cdots, b_n) = 0 \Rightarrow x = \theta.$$

Proof. If possible let $x \neq \theta$. Then by Corollary 3.1, there exists $T \in X_F^*$ such that $T(x, b_2, \cdots, b_n) \neq 0$. This is a contradiction to the given hypothesis. Hence the results follows. □

We now proceed to present another implication of the Hahn-Banach theorem for bounded b-linear functional and establish that there are always sufficient bounded b-linear functionals on a linear n-normed space which separate points from proper subspaces.

Theorem 3.2. *Let X be a linear n-normed space over the field $\mathbb{R}$ and W be a subspace of X and let $x_0 \in X$ such that $x_0, b_2, \cdots, b_n$ are linearly independent and suppose $d = \inf_{x \in W} \| x_0 - x, b_2, \cdots, b_n \| > 0$. Then there exists $T \in X_F^*$ such that*

(*I*) $T\,(x_0, b_2, \cdots, b_n) = 1$,

(*II*) $T\,(x, b_2, \cdots, b_n) = 0 \;\; \forall\, x \in W \;\; and \;\; \|T\| = \frac{1}{d}$.

Proof. Let $W_0 = W + \langle x_0 \rangle$ be the space spannded by W and x_0. Since $d > 0$, we have $x_0 \notin W$. Therefore, each $x \in W_0$ can be expressed uniquely in the form $x = y + \alpha x_0$, $y \in W$ and $\alpha \in \mathbb{R}$. We define a functional as follows:

$$T_1 : W_0 \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{R},\; T_1\,(y + \alpha x_0, b_2, \cdots, b_n) = \alpha.$$

Then clearly T_1 is a b-linear functional on $W_0 \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ satisfying

$$T_1\,(x, b_2, \cdots, b_n) = 0 \;\; \forall\, x \in W \;\; and \;\; T_1\,(x_0, b_2, \cdots, b_n) = 1.$$

Also, for each $x \in W_0$, we have

$$\begin{aligned}
|T_1\,(x, b_2, \cdots, b_n)| &= |T_1\,(y + \alpha x_0, b_2, \cdots, b_n)| = |\alpha| \\
&= \frac{|\alpha|\,\|x, b_2, \cdots, b_n\|}{\|x, b_2, \cdots, b_n\|} = \frac{|\alpha|\,\|x, b_2, \cdots, b_n\|}{\|y + \alpha x_0, b_2, \cdots, b_n\|} \\
&= \frac{|\alpha|\,\|x, b_2, \cdots, b_n\|}{|\alpha|\,\left\|\frac{y}{\alpha} + x_0, b_2, \cdots, b_n\right\|} \\
&= \frac{\|x, b_2, \cdots, b_n\|}{\left\|x_0 - \left(-\frac{y}{\alpha}\right), b_2, \cdots, b_n\right\|} \\
&\leq \frac{\|x, b_2, \cdots, b_n\|}{d} \quad \left[\,since\; -\frac{y}{\alpha} \in W\right].
\end{aligned}$$

This shows that T_1 is a bounded b-linear functional with $\|T_1\| \leq \frac{1}{d}$. To prove $\|T_1\| \geq \frac{1}{d}$, we consider a sequence $\{x_k\}$, $x_k \in W$ such that

$$\lim_{k \to \infty} \|x_0 - x_k, b_2, \cdots, b_n\| = d.$$

Now,

$$\begin{aligned}
1 &= |T_1\,(x_0, b_2, \cdots, b_n) - T_1\,(x_k, b_2, \cdots, b_n)| \\
&= |T_1\,(x_0 - x_k, b_2, \cdots, b_n)| \\
&\leq \|T_1\|\,\|x_0 - x_k, b_2, \cdots, b_n\|. \\
&\leq \|T_1\| \lim_{k \to \infty} \|x_0 - x_k, b_2, \cdots, b_n\| \\
&= \|T_1\|\, d \Rightarrow \|T_1\| \geq \frac{1}{d}.
\end{aligned}$$

Thus, we have established that there exists a bounded b-linear functional T_1 on $W_0 \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ such that

$$T_1\,(x, b_2, \cdots, b_n) = 0 \;\; \forall\, x \in W,\;\; T_1\,(x_0, b_2, \cdots, b_n) = 1 \;\; and \;\; \|T_1\| = \frac{1}{d}.$$

Applying the Theorem 2.3, we obtain a b-linear functional $T \in X_F^*$ such that

$$T\,(x,\, b_2,\, \cdots,\, b_n) = T_1\,(x,\, b_2,\, \cdots,\, b_n) \;\; \forall\, x \in W_0 \;\; and \; \|\, T\,\| = \|\, T_1\,\| = \frac{1}{d}.$$

So,

$$T\,(x,\, b_2,\, \cdots,\, b_n) = T_1\,(x,\, b_2,\, \cdots,\, b_n) = 0 \;\; \forall\, x \in W \;\; and$$

$$T\,(x_0,\, b_2,\, \cdots,\, b_n) = T_1\,(x_0,\, b_2,\, \cdots,\, b_n) = 1.$$

Hence, the proof of the theorem is complete. □

Remark 3.1. Theorem 3.2 is a generalization of Theorem 2.4 and its derivation is as follows:

Consider $W = \{0\}$ and $d = \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|$, then by Theorem 3.2, there exists a bounded b-linear functional $T_0 \in X_F^*$ such that

$$\|\, T_0\,\| = \frac{1}{d} = \frac{1}{\|\, x_0,\, b_2,\, \cdots,\, b_n\,\|} \quad and \;\; T_0\,(x_0,\, b_2,\, \cdots,\, b_n) = 1.$$

Now, for all $x \in X$, we define

$$T\,(x,\, b_2,\, \cdots,\, b_n) = \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|\; T_0\,(x,\, b_2,\, \cdots,\, b_n).$$

Then

$$\begin{aligned} T\,(x_0,\, b_2,\, \cdots,\, b_n) &= \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|\; T_0\,(x_0,\, b_2,\, \cdots,\, b_n) \\ &= \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|. \end{aligned}$$

Also,

$$\begin{aligned} \|\, T\,\| &= \sup\left\{ \frac{|\, T\,(x,\, b_2,\, \cdots,\, b_n)\,|}{\|\, x,\, b_2,\, \cdots,\, b_n\,\|} : \|\, x,\, b_2,\, \cdots,\, b_n\,\| \neq 0 \right\} \\ &= \sup\left\{ \frac{|\, \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|\;\, T_0\,(x,\, b_2,\, \cdots,\, b_n)\,|}{\|\, x,\, b_2,\, \cdots,\, b_n\,\|} : \|\, x,\, b_2,\, \cdots,\, b_n\,\| \neq 0 \right\} \\ &= \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|\; \sup\left\{ \frac{|\, T_0\,(x,\, b_2,\, \cdots,\, b_n)\,|}{\|\, x,\, b_2,\, \cdots,\, b_n\,\|} : \|\, x,\, b_2,\, \cdots,\, b_n\,\| \neq 0 \right\} \\ &= \|\, x_0,\, b_2,\, \cdots,\, b_n\,\|\; \|\, T_0\,\| = 1. \end{aligned}$$

Corollary 3.3. *Let X be a linear n-normed space over the field $\mathbb{R}$ and W be a subspace of X and let $x_0 \in X$ such that $x_0,\, b_2,\, \cdots,\, b_n$ are linearly independent and suppose $d = \inf\limits_{x \in W} \|\, x_0 - x,\, b_2,\, \cdots,\, b_n\,\| > 0$. Then*

(I) $T\,(x_0,\, b_2,\, \cdots,\, b_n) = d$,

(II) $T\,(x,\, b_2,\, \cdots,\, b_n) = 0 \;\; \forall\, x \in W$ *and* $\|\, T\,\| = 1$, *for some* $T \in X_F^*$.

Proof. By Theorem 3.2, there exists $T_1 \in X_F^*$ such that

$$T_1\,(x_0, b_2, \cdots, b_n) = 1,\; T_1\,(x, b_2, \cdots, b_n) = 0 \quad \forall\, x \in W$$

and $\|T_1\| = \frac{1}{d}$. Define the bounded b-linear functional T on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ by $T = d\,T_1$. Then

$$\begin{aligned} T\,(x_0, b_2, \cdots, b_n) &= d\,T_1\,(x_0, b_2, \cdots, b_n) = d, \\ T\,(x, b_2, \cdots, b_n) &= d\,T_1\,(x, b_2, \cdots, b_n) = 0 \quad \forall\, x \in W \end{aligned}$$

with $\|T\| = d\,\|T_1\| = \frac{d}{d} = 1$. This completes the proof. □

Corollary 3.4. *Let X be a linear n-normed space over the field $\mathbb{R}$ and W be a closed linear subspace of X and let $x_0 \in X - W$ such that $x_0, b_2, \cdots, b_n$ are linearly independent and suppose $d = \inf\limits_{x \in W} \|x_0 - x, b_2, \cdots, b_n\|$. Then there exists $T \in X_F^*$ such that*

(*I*) $T\,(x_0, b_2, \cdots, b_n) = 1$,

(*II*) $T\,(x, b_2, \cdots, b_n) = 0 \quad \forall\, x \in W$ *and* $\|T\| = \frac{1}{d}$.

Proof. It can be easily verified that $\inf\limits_{x \in W} \|x_0 - x, b_2, \cdots, b_n\| = 0$ if and only if $x_0 \in \overline{W}$. But $W = \overline{W}$ and it follows that $x_0 \notin \overline{W}$. Hence

$$d = \inf_{x \in W} \|x_0 - x, b_2, \cdots, b_n\| > 0.$$

Now, the proof of this corollary follows from Theorem 3.2. □

Corollary 3.5. *Let X be a linear n-normed space over the field $\mathbb{R}$ and W be a closed linear subspace of X and let $x_0 \in X - W$ such that $x_0, b_2, \cdots, b_n$ are linearly independent. Then there exists $T \in X_F^*$ such that*

$$T\,(x_0, b_2, \cdots, b_n) \neq 0 \;\textit{ and }\; T\,(x, b_2, \cdots, b_n) = 0 \quad \forall\, x \in W.$$

Proof. Proof of this corollary directly follows from that of the corollary 3.4. □

The Hahn-Banach Theorem for bounded b-linear functional and its consequences can be used to revel much among the properties of linear n-normed space and its dual space. Next theorem relates separability of the dual space to the separability of its original space.

Theorem 3.3. *Let X be a linear n-normed space over the field $\mathbb{R}$ and X_F^* be the Banach space of all bounded b-linear functionals defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then the space X is separable if X_F^* is separable.*

Proof. Since X_F^* is separable, there exists a countable set $S = \{ T_k \in X_F^* : k \in \mathbb{N} \}$ such that S is dense in X_F^*, i.e., $\overline{S} = X_F^*$. For each $k \in \mathbb{N}$, choose $x_k \in X$ such that $\| x_k, b_2, \cdots, b_n \| = 1$ and $| T_k (x_k, b_2, \cdots, b_n) | \geq \frac{1}{2} \| T_k \|$. Let W be the closed subspace of X generated by the sequence $\{ x_k \}_{k=1}^{\infty}$, i.e., $W = \overline{span} \{ x_k \in X : k \in \mathbb{N} \}$. Suppose $W \neq X$. Let $x_0 \in X - W$ such that $x_0, b_2, \cdots, b_n$ are linearly independent. By Corollary 3.5, there exists $0 \neq T \in X_F^*$ such that

$$T (x_0, b_2, \cdots, b_n) \neq 0 \;\; \textit{and} \;\; T (x, b_2, \cdots, b_n) = 0 \;\; \forall \, x \in W.$$

Since

$$\{ x_k \}_{k=1}^{\infty} \subseteq W , \;\; T (x_k, b_2, \cdots, b_n) = 0, \; k \in \mathbb{N}.$$

Thus,

$$\begin{aligned}
\frac{1}{2} \| T_k \| &\leq | T_k (x_k, b_2, \cdots, b_n) | \\
&= | T_k (x_k, b_2, \cdots, b_n) - T (x_k, b_2, \cdots, b_n) | \\
&\leq \| T_k - T \| \, \| x_k, b_2, \cdots, b_n \| \\
&= \| T_k - T \| \; [\textit{ since } \| x_k, b_2, \cdots, b_n \| = 1].
\end{aligned}$$

Again, since $\overline{S} = X_F^*$, for each $T \in X_F^*$, there exists a sequence $\{ T_k \}$ in S such that $\lim_{k \to \infty} T_k = T$. Therefore,

$$\| T \| \leq \| T_k - T \| + \| T_k \| \leq 3 \, \| T_k - T \| \;\; \forall k \in \mathbb{N}.$$

Taking limit on both sides as $k \to \infty$, it follows that $T = 0$, which contradicts the assumption that $W \neq X$. Hence, $W = X$ and thus X is separable. $\square$

4. Reflexivity of linear n-normed space

Recall that given a linear n-normed space $X \neq \{ 0 \}$, the dual space X_F^* is a normed space with respect to the norm $\| \cdot \| : X_F^* \to \mathbb{R}$ defined by

$$\| T \| = \sup \{ | T (x, b_2, \cdots, b_n) | : x \in X, \| x, b_2, \cdots, b_n \| = 1 \}.$$

Furthermore, X_F^* is a Banach space. Also, by Corollary 3.1, $X_F^* \neq \{ O \}$ and, therefore, as a normed space X_F^* has its own dual space $(X_F^*)^*$, denoted by X_F^{**} and is called the second dual space of X, which is again a Banach space under the norm

$$\| \varphi \| = \sup \{ | \varphi (T) | \; : \; T \in X_F^*, \| T \| \leq 1 \}, \varphi \in X_F^{**}.$$

Theorem 4.1. *Let X be a real linear n-normed space. Given $x \in X$, let*

$$\varphi_{(x, F)} (T) = T (x, b_2, \cdots, b_n) \;\; \forall \, T \in X_F^*. \tag{4.1}$$

Then $\varphi_{(x, F)}$ is a bounded linear functional on X_F^*. Furthermore, the mapping $(x, b_2, \cdots, b_n) \to \varphi_{(x, F)}$ is an isometric isomorphism of $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ onto the subspace $\left\{ \varphi_{(x, F)} : (x, b_2, \cdots, b_n) \in X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \right\}$ of X_F^{**}.

Proof. Let $\alpha, \beta \in \mathbb{R}$. Then, for all $T_1, T_2 \in X_F^*$, we have

$$\begin{aligned} \varphi_{(x, F)}(\alpha T_1 + \beta T_2) &= (\alpha T_1 + \beta T_2)(x, b_2, \cdots, b_n) \\ &= \alpha T_1(x, b_2, \cdots, b_n) + \beta T_2(x, b_2, \cdots, b_n) \\ &= \alpha \varphi_{(x, F)}(T_1) + \beta \varphi_{(x, F)}(T_2). \end{aligned}$$

So, $\varphi_{(x, F)}$ is linear functional. Also, for all $T \in X_F^*$, we have

$$\left|\varphi_{(x, F)}(T)\right| = |T(x, b_2, \cdots, b_n)| \leq \|x, b_2, \cdots, b_n\| \, \|T\|.$$

Consequently, $\varphi_{(x, F)} \in X_F^{**}$ with $\left\|\varphi_{(x, F)}\right\| \leq \|x, b_2, \cdots, b_n\|$. Moreover, such $\varphi_{(x, F)}$ is unique. So, for every fixed $x \in X$ there corresponds a unique bounded linear functional $\varphi_{(x, F)} \in X_F^{**}$ given by (4.1). This defines a function $J : X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to X_F^{**}$ given by $J(x, b_2, \cdots, b_n) = \varphi_{(x, F)}$. We now verify that J is an isomorphism between $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ and the range of J, which is a subspace of X_F^{**}.

(*I*) Let $x, y \in X$ *and* $\alpha, \beta \in \mathbb{R}$. Then for all $T \in X_F^*$, we have

$$\begin{aligned} & [J(\alpha x + \beta y, b_2, \cdots, b_n)](T) = \varphi_{(\alpha x + \beta y, F)}(T) \\ = \;& T(\alpha x + \beta y, b_2, \cdots, b_n) \\ = \;& \alpha T(x, b_2, \cdots, b_n) + \beta T(y, b_2, \cdots, b_n) \\ = \;& \alpha \varphi_{(x, F)}(T) + \beta \varphi_{(y, F)}(T) = \left(\alpha \varphi_{(x, F)} + \beta \varphi_{(y, F)}\right)(T) \\ = \;& [\alpha J(x, b_2, \cdots, b_n) + \beta J(y, b_2, \cdots, b_n)](T). \\ \Rightarrow \;& J(\alpha x + \beta y, b_2, \cdots, b_n) = \alpha J(x, b_2, \cdots, b_n) + \beta J(y, b_2, \cdots, b_n). \end{aligned}$$

This shows that J is a b-linear operator.

(*II*) J preserves the norm:
For each $(x, b_2, \cdots, b_n) \in X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, we have

$$\begin{aligned} \|J(x, b_2, \cdots, b_n)\| &= \left\|\varphi_{(x, F)}\right\| \\ &= \sup\left\{\frac{\left|\varphi_{(x, F)}(T)\right|}{\|T\|} : T \in X_F^*, T \neq 0\right\} \\ &= \sup\left\{\frac{|T(x, b_2, \cdots, b_n)|}{\|T\|} : T \in X_F^*, T \neq 0\right\} \\ &= \|x, b_2, \cdots, b_n\| \quad [\textit{by Theorem 2.5}]. \end{aligned} \tag{4.2}$$

(*III*) J is injective:
Let $x, y \in X$ with $x \neq y$ such that $\{x, b_2, \cdots, b_n\}$ or $\{y, b_2, \cdots, b_n\}$ are linearly independent. Then by (4.2), we get

$$\begin{aligned} & \|x - y, b_2, \cdots, b_n\| \neq 0 \\ \Rightarrow \;& \|J(x - y, b_2, \cdots, b_n)\| \neq 0 \\ \Rightarrow \;& \|J(x, b_2, \cdots, b_n) - J(y, b_2, \cdots, b_n)\| \neq 0 \\ \Rightarrow \;& J(x, b_2, \cdots, b_n) \neq J(y, b_2, \cdots, b_n). \end{aligned}$$

We thus conclude that J is an isomeric isomorphism of $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ onto the subspace of X_F^{**}. This completes the proof.

□

Definition 4.1. Let X be a linear n-normed space over the field $\mathbb{R}$. The isometric isomorphism $J : X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to X_F^{**}$ defined by

$$J\,(x, b_2, \cdots, b_n) = \varphi_{(x, F)} \;\; \forall\, x \in X \;\; \textit{and} \;\; \varphi_{(x, F)} \in X_F^{**}$$

is called the b-natural embedding or the b-canonical mapping of $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ into the second dual space X_F^{**}.

Definition 4.2. A linear n-normed space X is said to be b-reflexive if the b-natural embedding J, maps the space $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ onto its second dual space X_F^{**}, i.e., $J\,(X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle) = X_F^{**}$.

Theorem 4.2. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in a linear n-normed space X. Suppose*

$$\sup_{1 \le k < \infty} |T\,(x_k, b_2, \cdots, b_n)| < \infty \;\; \forall\, T \in X_F^{*}. \textit{ Then}$$
$$\sup_{1 \le k < \infty} \|x_k, b_2, \cdots, b_n\| < \infty.$$

Proof. Consider the b-natural embedding

$$(x, b_2, \cdots, b_n) \to \varphi_{(x, F)},\; (x, b_2, \cdots, b_n) \in X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle.$$

Since $\{x_k\}_{k=1}^{\infty}$ is a sequence of vectors in X, $\left\{\varphi_{(x_k, F)}\right\}_{k=1}^{\infty}$ is a sequence of bounded linear functionals in X_F^{**}. Also,

$$\left|\varphi_{(x_k, F)}(T)\right| = |T\,(x_k, b_2, \cdots, b_n)| \le \sup_{1 \le k < \infty} |T\,(x_k, b_2, \cdots, b_n)|.$$

Therefore, $\left\{\varphi_{(x_k, F)}(T)\right\}_{k=1}^{\infty}$ is bounded for each $T \in X_F^{*}$. Applying the Principle of Uniform Boundedness (Theorem 2.1), to the family $\left\{\varphi_{(x_k, F)}\right\}_{k=1}^{\infty}$, we conclude that $\left\{\left\|\varphi_{(x_k, F)}\right\|\right\}_{k=1}^{\infty}$ is bounded and hence by (4.2), the sequence $\{\|x_k, b_2, \cdots, b_n\|\}_{k=1}^{\infty}$ is bounded. This proves the theorem. □

Theorem 4.3. *A closed subspace of a b-reflexive n-Banach space is b-reflexive.*

Proof. Let X be a b-reflexive n-Banach space and Y be a closed subspace of X. Let $T : X_F^{*} \to Y_F^{*}$ be an operator defined by

$$(T f)(y, b_2, \cdots, b_n) = f\,(y, b_2, \cdots, b_n) \;\; \forall\, y \in Y,\; f \in X_F^{*},$$

where Y_F^{*} denotes the Banach space of all bounded b-linear functionals defined on $Y \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Then for $f \in X_F^{*}$,

$$\|T f\| = \sup\left\{\frac{|f\,(y, b_2, \cdots, b_n)|}{\|y, b_2, \cdots, b_n\|} : \|y, b_2, \cdots, b_n\| \neq 0\right\} = \|f\|.$$

Let J_Y be the b-natural embedding of $Y \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ into Y_F^{**}. That is,

$$J_Y\,(y, b_2, \cdots, b_n) = \psi_{(y, F)} \;\forall\, y \in Y,\; \psi_{(y, F)} \in Y_F^{**}. \textit{ Define}$$

$T_1 : Y_F^{**} \to X_F^{**}$ by $\left(T_1\, \psi_{(y, F)}\right)(f) = \psi_{(y, F)}(T f)$, $f \in X_F^{*}$. We now verify that $T_1\, \psi_{(y, F)} \in X_F^{**}$.

(*I*) $T_1\, \psi_{(y, F)}$ is linear functional:
Let $\alpha, \beta \in \mathbb{R}$. Then for every $f, g \in X_F^{*}$ and $y \in Y$, we have

$$\begin{aligned}
& \left(T_1\, \psi_{(y, F)}\right)(\alpha f + \beta g)(y, b_2, \cdots, b_n) \\
= \;& \psi_{(y, F)}\,[T\,(\alpha f + \beta g)]\,(y, b_2, \cdots, b_n) \\
= \;& \psi_{(y, F)}\,[\alpha\, T\,(f\,(y, b_2, \cdots, b_n)) + \beta\, T\,(g\,(y, b_2, \cdots, b_n))] \\
= \;& \alpha \psi_{(y, F)}(T f)(y, b_2, \cdots, b_n) + \beta \psi_{(y, F)}(T g)\,(y, b_2, \cdots, b_n) \\
= \;& \left[\alpha\, \psi_{(y, F)}(T f) + \beta\, \psi_{(y, F)}(T g)\right](y, b_2, \cdots, b_n) \\
= \;& \left[\alpha\,\left(T_1\, \psi_{(y, F)}\right)(f) + \beta\,\left(T_1\, \psi_{(y, F)}\right)(g)\right](y, b_2, \cdots, b_n). \\
\Rightarrow \;& \left(T_1 \psi_{(y, F)}\right)(\alpha f + \beta g) = \alpha\left(T_1 \psi_{(y, F)}\right)(f) + \beta\left(T_1 \psi_{(y, F)}\right)(g).
\end{aligned}$$

(*II*) $T_1\, \psi_{(y, F)}$ is bounded:
Since $\psi_{(y, F)}$ preserves the norm,

$$\left\|\left(T_1\, \psi_{(y, F)}\right)(f)\right\| = \left\|\psi_{(y, F)}(T f)\right\| = \|T f\| = \|f\|.$$

So, $T_1\, \psi_{(y, F)} \in X_F^{**}$ and hence T_1 is well-defined. Since X is b-reflexive, the b-natural embedding $J_X : X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to X_F^{**}$ defined by

$$J_X\,(x, b_2, \cdots, b_n) = \varphi_{(x, F)}\,,\; \varphi_{(x, F)} \in X_F^{**}$$

is such that $J_X\,(X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle) = X_F^{**}$. Therefore, $T_1 \psi_{(y, F)} \in X_F^{**}$ implies that $J_X^{-1}\left(T_1\, \psi_{(y, F)}\right) \in X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Write $(x, b_2, \cdots, b_n) = J_X^{-1}\left(T_1\, \psi_{(y, F)}\right)$ so that $J_X\,(x, b_2, \cdots, b_n) = T_1\, \psi_{(y, F)}$. We need to prove that $x \in Y$. Let, if possible, $x \in X - Y$ such that $x, b_2, \cdots, b_n$ are linearly independent. Then by Corollary 3.5, there exists a bounded b-linear functional $f \in X_F^{*}$ such that $f\,(x, b_2, \cdots, b_n) \neq 0$ and $f\,(y, b_2, \cdots, b_n) = 0$ for all $y \in Y$. Consequently, $T f = 0$ and as such $\psi_{(y, F)}(T f) = 0$. This leads to $\varphi_{(x, F)}(f) = 0$ and hence $f\,(x, b_2, \cdots, b_n) = 0$, which is a contradiction. Thus, we conclude that $(x, b_2, \cdots, b_n) = J_X^{-1}\left(T_1\, \psi_{(y\, F)}\right) \in Y \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. This verifies that $J_X^{-1}\,(T_1\,(Y_F^{**})) \subset Y \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Now, let $\psi \in Y_F^{**}$. Set $(x_0, b_2, \cdots, b_n) = J_X^{-1}\,(T_1\, \psi)$ so that $(x_0, b_2, \cdots, b_n) \in Y \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. Let $g \in Y_F^{*}$. Then there exists a b-linear functional $f \in X_F^{*}$ such that

$$f\,(y, b_2, \cdots, b_n) = g\,(y, b_2, \cdots, b_n) \;\;\forall\, y \in Y \;\textit{ and }\; g = T f.$$

Therefore,

$$\begin{aligned}\psi(g) &= \psi(Tf) = (T_1\psi)(f) = [J_X(x_0, b_2, \cdots, b_n)](f) \\ &= \varphi_{(x_0, F)}(f) = f(x_0, b_2, \cdots, b_n) = g(x_0, b_2, \cdots, b_n).\end{aligned}$$

This proves that $J_Y(x_0, b_2, \cdots, b_n) = \psi_{(x_0, F)}$ and hence

$$J_Y(Y \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle) = Y_F^{**}.$$

This proves that Y is b-reflexive. □

5. b-weak convergence and b-strong convergence in linear n-normed space

In this section, we shall introduce b-weak convergence and b-strong convergence relative to bounded b-linear functionals in linear n-normed space and establish that these two types of convergence are equivalent in case of finite dimensional linear n-normed space.

Definition 5.1. A sequence $\{x_k\}$ in a linear n-normed space X is said to be b-weakly convergent if there exists an element $x \in X$ such that for every $T \in X_F^*$,

$$\lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n).$$

The vector x is called the b-weak limit of the sequence $\{x_k\}$ and we say that $\{x_k\}$ converges b-weakly to x. Note that, for each $T \in X_F^*$, $\{T(x_k, b_2, \cdots, b_n)\}$ is a sequence of scalars in $\mathbb{K}$. Therefore, b-weak convergence means convergence of the sequence of scalars $\{T(x_k, b_2, \cdots, b_n)\}$ for every $T \in X_F^*$.

Theorem 5.1. *Let $\{x_k\}$ be b-weakly convergent sequence in X. Then*

(I) the b-weak limit of $\{x_k\}$ is unique.

(II) $\{\|x_k, b_2, \cdots, b_n\|\}$ is bounded sequence in $\mathbb{K}$.

Proof. (I) Suppose that $\{x_k\}$ converges b-weakly to x as well as to y. Then for all $T \in X_F^*$, we get

$$\begin{aligned}T(x, b_2, \cdots, b_n) &= \lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) \\ &= T(y, b_2, \cdots, b_n).\end{aligned}$$

This shows that

$$\begin{aligned}&T(x, b_2, \cdots, b_n) - T(y, b_2, \cdots, b_n) = 0 \quad \forall\, T \in X_F^*. \\ \Rightarrow\quad &T(x - y, b_2, \cdots, b_n) = 0 \quad \forall\, T \in X_F^*.\end{aligned}$$

Hence, by Corollary 3.2, $x = y$.

Proof of (II) Since $\{x_k\}$ converges b-weakly to x, we have

$$\lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) \quad \forall\, T \in X_F^*.$$

Therefore, for each $T \in X_F^*$, $\{T(x_k, b_2, \cdots, b_n)\}$ is a convergent sequence in $\mathbb{K}$ and so the sequence $\{T(x_k, b_2, \cdots, b_n)\}$ is bounded. Consequently, there exists a constant K_T (depending on T) such that $|T(x_k, b_2, \cdots, b_n)| \leq K_T \;\forall\, k \in \mathbb{N}$. Let $(x, b_2, \cdots, b_n) \to \varphi_{(x, F)}$ be the b-natural embedding of $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ into X_F^{**}. Then for each $k \in \mathbb{N}$, by (4.2), we have

$$\left\| \varphi_{(x_k, F)} \right\| = \| x_k, b_2, \cdots, b_n \|$$

and

$$\left| \varphi_{(x_k, F)}(T) \right| = |T(x_k, b_2, \cdots, b_n)| \leq K_T \quad \forall\, k \in \mathbb{N}.$$

Thus, $\left\{ \varphi_{(x_k, F)}(T) \right\}$ is bounded for each $T \in X_F^*$. But the space X_F^* being a Banach space, by the Principle of Uniform Boundedness (Theorem 2.1), it follows that $\left\{ \left\| \varphi_{(x_k, F)} \right\| \right\}$ is bounded and hence $\{\| x_k, b_2, \cdots, b_n \|\}_{k=1}^{\infty}$ is bounded. □

Theorem 5.2. *Let $\{x_k\}$ and $\{y_k\}$ be two sequences in a linear n-normed space X. If $\{x_k\}$ and $\{y_k\}$ converges b-weakly to x and y, respectively then for any scalar α and β, $\{\alpha x_k + \beta y_k\}$ converges b-weakly to $\alpha x + \beta y$.*

Proof. Since $\{x_k\}$ and $\{y_k\}$ converges b-weakly to x and y, we have

$$\lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) \;\textit{and}$$
$$\lim_{k \to \infty} T(y_k, b_2, \cdots, b_n) = T(y, b_2, \cdots, b_n) \quad \forall\, T \in X_F^*.$$

Now, for all $T \in X_F^*$, we have

$$\begin{aligned}
& \lim_{k \to \infty} T(\alpha x_k + \beta y_k, b_2, \cdots, b_n) \\
= & \lim_{k \to \infty} [T(\alpha x_k, b_2, \cdots, b_n) + T(\beta y_k, b_2, \cdots, b_n)] \\
= & \lim_{k \to \infty} \alpha T(x_k, b_2, \cdots, b_n) + \lim_{k \to \infty} \beta T(y_k, b_2, \cdots, b_n) \\
= & \alpha T(x, b_2, \cdots, b_n) + \beta T(y, b_2, \cdots, b_n) \\
= & T(\alpha x + \beta y, b_2, \cdots, b_n).
\end{aligned}$$

This shows that $\{\alpha x_k + \beta y_k\}$ converges b-weakly to $\alpha x + \beta y$. □

Theorem 5.3. *A sequence $\{x_k\}$ in X converges b-weakly to $x \in X$ if and only if*

(I) the sequence $\{\| x_k, b_2, \cdots, b_n \|\}$ is bounded and

(II) $\lim\limits_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) \;\; \forall\, T \in M$, *where* M *is fundamental or total subset of* X_F^*.

Proof. In the case of b-weak convergence, (I) follows from the Theorem 5.1 and since $M \subset X_F^*$, (II) follows from the definition of b-weak convergence of $\{x_k\}$.

Conversely, suppose that (I) and (II) hold. By (I), there exists a constant L such that

$$\|x_k, b_2, \cdots, b_n\| \leq L \;\; \forall\, k \in \mathbb{N} \;\; \text{and} \;\; \|x, b_2, \cdots, b_n\| \leq L.$$

Since $\overline{span\, M} = X_F^*$, for each $T \in X_F^*$, there exists a sequence $\{T_m\}$ in $span\, M$ such that $\lim\limits_{m \to \infty} T_m = T$. Hence, for any given $\epsilon > 0$, there exists $T_m \in span\, M$ such that $\|T_m - T\| < \frac{\epsilon}{3L}$. Furthermore, by the hypothesis (II), there exists $K \in \mathbb{N}$ such that

$$|T_m(x_k, b_2, \cdots, b_n) - T_m(x, b_2, \cdots, b_n)| < \frac{\epsilon}{3} \;\; \forall\, m > K.$$

Now, for $m > K$, we have

$$\begin{aligned}
& |T(x_k, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \\
\leq\; & |T(x_k, b_2, \cdots, b_n) - T_m(x_k, b_2, \cdots, b_n)| + \\
+\; & |T_m(x_k, b_2, \cdots, b_n) - T_m(x, b_2, \cdots, b_n)| \\
+\; & |T_m(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \\
<\; & \|T_m - T\|\,\|x_k, b_2, \cdots, b_n\| + \frac{\epsilon}{3} + \|T_m - T\|\,\|x, b_2, \cdots, b_n\| \\
<\; & \frac{\epsilon}{3L} \cdot L + \frac{\epsilon}{3} + \frac{\epsilon}{3L} \cdot L = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \\
\Rightarrow\; & \lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) \;\; \forall\, T \in X_F^*.
\end{aligned}$$

Hence, $\{x_k\}$ converges b-weakly to $x \in X$. □

Definition 5.2. A sequence $\{x_k\}$ in X is said to be b-strongly convergent if there exists a vector $x \in X$ such that $\lim\limits_{k\to\infty} \|x_k - x, b_2, \cdots, b_n\| = 0$. The vector x is called b-strong limit and we say that $\{x_k\}$ converges b-strongly to x.

Theorem 5.4. *If a sequence* $\{x_k\}$ *in* X *converges b-strongly to* x, *then* $\{x_k\}$ *converges b-weakly to* x *in* X.

Proof. Suppose $\{x_k\}$ converges b-strongly to x. Then for every $T \in X_F^*$, we have

$$\begin{aligned}
& |T(x_k, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \\
=\; & |T(x_k - x, b_2, \cdots, b_n)| \leq \|T\|\,\|x_k - x, b_2, \cdots, b_n\| \\
& \to 0 \;\; \text{as } k \to \infty \;\; [\,\text{since } \{x_k\} \text{ converges } b\text{-strongly to } x\,] \\
\Rightarrow\; & \lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) \;\; \forall\, T \in X_F^*.
\end{aligned}$$

Hence, $\{x_k\}$ converges b-weakly to x in X. □

Theorem 5.5. *In a finite dimensional linear n-normed space, b-weak convergence implies b-strong convergence.*

Proof. Let X be a linear n-normed space with $dim X = d \geq n$. Then there exists a basis $\{e_1, e_2, \cdots, e_d\}$ for X. Let $\{x_k\}$ be a sequence in X such that $\{x_k\}$ converges b-weakly to x. Now, we can write

$$\begin{aligned} x_k &= a_{k,1} e_1 + a_{k,2} e_2 + \cdots + a_{k,d} e_d, \ (k = 1, 2, \cdots), \\ x &= a_1 e_1 + a_2 e_2 + \cdots\cdots + a_d e_d, \end{aligned}$$

where $a_{k,1}, a_{k,2}, \cdots, a_{k,d}, a_1, a_2, \cdots, a_d \in \mathbb{R}$. Consider the b-linear functionals $\{T_1, T_2, \cdots, T_d\}$ in X_F^* such that $T_i(e_j, b_2, \cdots, b_n) = 1$ if $i = j$ and $T_i(e_j, b_2, \cdots, b_n) = 0$ if $i \neq j$, $1 \leq i, j \leq d$. Now, for $1 \leq i \leq d$, we have

$$\begin{aligned} T_i(x_k, b_2, \cdots, b_n) &= T_i\left(\sum_{j=1}^{d} a_{k,j} e_j, b_2, \cdots, b_n\right) \\ &= \sum_{j=1}^{d} a_{k,j} T_i(e_j, b_2, \cdots, b_n) = a_{k,i} \end{aligned}$$

and similarly, $T_i(x, b_2, \cdots, b_n) = a_i$, $(1 \leq i \leq d)$. Since

$$\lim_{k \to \infty} T(x_k, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) \quad \forall\, T \in X_F^*,$$

in particular, we have

$$\lim_{k \to \infty} T_i(x_k, b_2, \cdots, b_n) = T_i(x, b_2, \cdots, b_n), \ (1 \leq i \leq d).$$

Thus,

$$\lim_{k \to \infty} a_{k,i} = a_i, \ (1 \leq i \leq d). \tag{5.1}$$

Therefore,

$$\begin{aligned} \|x_k - x, b_2, \cdots, b_n\| &= \left\| \sum_{i=1}^{d} (a_{k,i} - a_i) e_i, b_2, \cdots, b_n \right\| \\ &\leq \sum_{i=1}^{d} |a_{k,i} - a_i| \, \|e_i, b_2, \cdots, b_n\| \\ &\to 0 \ \textit{as} \ k \to \infty \ [\ \textit{by}\ (5.1)\ ] \\ &\Rightarrow \lim_{k \to \infty} \|x_k - x, b_2, \cdots, b_n\| = 0 \end{aligned}$$

and hence $\{x_k\}$ converges b-strongly to x in X. □

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SOME RESULTS ON YAMABE SOLITONS ON NEARLY HYPERBOLIC SASAKIAN MANIFOLDS

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Abstract. We classify almost Yamabe on nearly hyperbolic Sasakian manifolds whose potential vector field is torse-forming admitting semi-symmetric metric connection and quarter symmetric non-metric connection. Certain results of such solitons on CR-sub-manifolds of nearly hyperbolic Sasakian manifolds with respect to such connection are obtained. Finally, a non-trivial example is given to validate some of our results.
Keywords: Sasakian manifolds, CR-sub-manifolds, Yamabe solitons.

1. Introduction

Much progress has been done in recent years in the study of soliton solutions of the Ricci flow, the mean curvature flow and the Yamabe flow. Soliton solutions correspond to self-similar solutions of the corresponding flow. The Yamabe flow,

$$\frac{\partial}{\partial t}g(t) = -R(t)g(t), \tag{1.1}$$

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where $R(t)$ is the scalar curvature of the metric $g(t)$, was introduced by Hamilton [14], as an approach to solve the Yamabe problem. In dimension $n(=2)$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n > 2$ the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric while the Ricci flow does not in general.

A Yamabe soliton on a Riemannian manifold (M, g) of dimension n is a special solution of the Yamabe flow. A triplet structure (g, κ, λ) satisfies

$$(1.2) \qquad \frac{1}{2}\mathfrak{L}_{\kappa}g(X,Y) = (\hat{\delta} - \lambda)g(X,Y)$$

for all X, Y on M is known as a Yamabe soliton, where $\mathfrak{L}_{\kappa}$ denotes the Lie derivative of the metric g along the vector field κ, $\hat{\delta}$ is the scalar curvature and λ is a constant. The beauty of such =soliton depends on the the flavor of λ. The soliton is said to be expanding, steady or shrinking, according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. If $\lambda \in C^{\infty}(M)$, then the metric satisfying (1.2) is called almost Yamabe soliton [2]. Thus the almost Yamabe solitons are the generalization of Yamabe solitons. Moreover, if κ is the gradient of some function $\widetilde{\phi}$ on M then it is known as gradient Yamabe soliton. In context of geometry, the Yamabe solitons are special solution of Yamabe flow under some regulation. There are several geometers that light up quite extensively on the beauty of Yamabe flow and Yamabe soliton (see,[9], [11], [12], [16]).

A vector field κ on a Riemannian manifold (M, g) is known as torse-forming vector field [21] if it satisfies

$$(1.3) \qquad \nabla_X \kappa = \psi X + \theta(X)\kappa, \;\; \forall\, X \in \chi(M),$$

where $\psi \in C^{\infty}(M)$ and θ is a 1-form. The beauty of such vector field is as follows:

i) It is concircular if the 1-form θ vanishes identically [20],
ii) For concurrent, $\psi = 1$ and $\theta = 0$ [22],
iii) It is recurrent if $\psi = 0$,
iv) For parallel if $\psi = \theta = 0$.

In 2017, Chen [8] initiated a new type vector field known as torqued vector field if the vector field κ satisfying (1.2) with $\theta(\kappa) = 0$, where ψ is called torqued function with the 1-form θ is the torqued form of κ.

Bejancu introduced the concept of CR-sub-manifolds of Kähler manifold as a generalization of invariant and anti-invariant sub-manifolds [3]. After that, CR-sub-manifolds of Sasakian manifold was studied by Hsu [15] and Kobayashi [17]. Yano and Kon [23] studied contact CR-sub-manifolds. As per this motivation, several geometers studied CR-sub-manifolds of almost contact manifolds (see, [1],[4],[5],[18]). The almost hyperbolic (f, ξ, η, g)-structure was defined and studied by Upadhyay and Dube [19]. CR-sub-manifolds of trans-hyperbolic Sasakian manifold studied by Bhatt and Dube [6]. Apart from that, Golab [13] introduced the idea of semi-symmetric and quarter symmetric connections. Lovejoy Das et al. [10] studied

CR-sub-manifolds of LP-Sasakian manifold with semi-symmetric non-metric connection. CR-sub-manifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by Siddiqi and Rizvi [1].

The sections of this paper are organized as follows. After introduction, Section 2 contains some definitions and basic results. In Section 3, we recall the notion of semi-symmetric metric connection and quarter symmetric non-metric connection on nearly hyperbolic Sasakian manifold. Section 4 is devoted to CR-sub-manifolds of nearly hyperbolic Sasakian manifolds with respect to semi-symmetric metric connection and quarter symmetric non-metric connection. In Section 5, we study Yamabe soliton whose potential vector field is torse-forming vector field on nearly hyperbolic Sasakian manifold with respect to such connection. Section 6 is concerned with the study of Yamabe soliton with a torse-forming vector field on CR-sub-manifolds of nearly hyperbolic Sasakian manifolds. Furthermore, we study almost Yamabe soliton with torse-forming vector field taking κ^t and κ^n as tangential and normal components of such vector field on CR-sub-manifolds of nearly hyperbolic Sasakian manifolds admitting such connection in Section 7.

2. Preliminaries

Let $\mathbb{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \tag{2.1}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y tangent to $\mathbb{M}$ [7]. As per this consequences

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.3}$$

where I is the identity of the tangent bundle $T\mathbb{M}$, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is Riemannian metric tensor of $\mathbb{M}$. An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\mathbb{M}$ is called hyperbolic Sasakian manifold if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

for all tangent vectors X, Y and a Riemannian metric g and Riemannian connection ∇ on $\mathbb{M}$. With reference to (2.4), an almost hyperbolic contact metric manifold $\mathbb{M}$ with (ϕ, ξ, η, g)-structure is called a nearly hyperbolic Sasakian manifold if

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X. \tag{2.6}$$

Let $\grave{M}$ be a submanifold immersed in $\mathbb{M}$, the Riemannian metric g induced on $\grave{M}$. Let $T\grave{M}$ and $T^{\perp}\grave{M}$ be the Lie algebra of vector fields tangential to $\grave{M}$ and normal to $\grave{M}$ respectively and $\grave{\nabla}$ be the induced Levi-Civita connection on $\grave{M}$, then the Gauss and Weingarten formulae are given respectively by

$$\nabla_X Y = \grave{\nabla}_X Y + h(X,Y), \ \forall \ X, Y \in T\grave{M}, \tag{2.7}$$

$$\nabla_X N = -A_N X + \nabla^{\perp} N, \ \forall \ N \in T^{\perp}\grave{M}, \tag{2.8}$$

where $\nabla_X Y$ and $\{h(X,Y), \nabla^{\perp}_X N\}$ belong to $T\grave{M}$ and $T^{\perp}\grave{M}$, respectively. The second fundamental form h and Weingarten map A_N associated with N as

$$g(h(X,Y),N) = g(A_N X, Y). \tag{2.9}$$

For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, we can write

$$X = PX + QX, \ \ PX \in \Gamma(D), \ QX \in \Gamma(D^{\perp}), \tag{2.10}$$

$$\phi N = BN + CN, \ \ BN \in \Gamma(D^{\perp}), \ CN \in \Gamma(\mu). \tag{2.11}$$

3. Semi-symmetric Metric Connection and Quarter symmetric non-metric connection

Firstly, we define a semi-symmetric metric connection [13]:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{3.1}$$

such that

$$(\widetilde{\nabla}_X g)(Y,Z) = 0. \tag{3.2}$$

With the help of (2.6) and (3.1), we get

$$(\widetilde{\nabla}_X \phi)Y + \phi(\widetilde{\nabla}_X Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y) - g(X, \phi Y)\xi. \tag{3.3}$$

On interchanging X and Y, equation (3.3) reduces to

$$(\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_Y X) = (\nabla_Y \phi)X + \phi(\nabla_Y X) - g(Y, \phi X)\xi, \tag{3.4}$$

Adding (3.3) and (3.4), we obtain

$$\begin{aligned} &(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_X Y - \nabla_X Y) + \phi(\widetilde{\nabla}_Y X - \nabla_Y X) \\ &= (\nabla_X \phi)Y + (\nabla)_Y \phi)X - g(X, \phi Y)\xi - g(Y, \phi X)\xi. \end{aligned} \tag{3.5}$$

Keeping in mind (2.1), (2.3), (2.6) and (3.1) above equation turn up

$$\begin{aligned} &(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X \\ &= 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X - \eta(X)\phi Y - \eta(Y)\phi X. \end{aligned} \tag{3.6}$$

Also from (2.5) and (3.1), we get

$$\widetilde{\nabla}_X \xi = \phi X - X - \eta(X). \tag{3.7}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure (ϕ, ξ, η, g) is called nearly hyperbolic Sasakian manifold with semi-symmetric metric connection if it bearing (3.5) and (3.6). With the help of (2.7), (2.8) and (3.1) the Gauss and Weingarten formulae on nearly hyperbolic Sasakian manifold with semi-symmetric metric connection as follows

$$\widetilde{\nabla}_X Y = \grave{\nabla}_X Y + h(X,Y), \ \forall \ X, Y \in T\grave{M}, \tag{3.8}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp} N, \ \forall \ N \in T^{\perp}\grave{M}, \tag{3.9}$$

Also we recall the notion of a quarter symmetric non-metric connection [13] by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X, \tag{3.10}$$

such that

$$(\widehat{\nabla}_X g)(Y, Z) = \eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y). \tag{3.11}$$

From (2.6) and (3.9), we have

$$\begin{aligned} &(\widehat{\nabla}_X \phi)Y + (\widehat{\nabla}_Y \phi)X \\ &= 2g(X,Y)\xi - \eta(X)Y - 2\eta(Y)X - 2\eta(X)\phi Y - 2\eta(X)\eta(Y)\xi. \end{aligned} \tag{3.12}$$

An almost hyperbolic contact manifold is called nearly hyperbolic Sasakian [7] manifold with quarter symmetric non-metric connection if it satisfies (3.11). Therefore from (2.5) and (3.9), we obtain

$$\widehat{\nabla}_X \xi = 2\phi X. \tag{3.13}$$

Therefore Gauss and Weingarten formulae on nearly hyperbolic Sasakian manifold bearing quarter symmetric non-metric connection are given respectively by

$$\widehat{\nabla}_X Y = \grave{\nabla}_X Y + h(X,Y), \ \forall \ X, Y \in T\grave{M}, \tag{3.14}$$

$$\widehat{\nabla}_X N = -A_N X + \nabla^{\perp} N, \ \forall \ N \in T^{\perp}\grave{M}, \tag{3.15}$$

4. CR-sub-manifolds of a Nearly hyperbolic Sasakian Manifold

Definition 4.1. [4] An m-dimensional Riemannian submanifold (M, g) of an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that

i) D is invariant under ϕ, i.e., $\phi D \subset D$.

ii) The orthogonal complement distribution $D^{\perp} : x \to D_x^{\perp} \subset T_x M$ of the distribution D on M is totally real, i.e., $\phi D^{\perp} \subset T^{\perp} M$.

If dim $D^{\perp}$=0 (resp., dim D=0), then the CR-submanifold is known as an invariant (resp., anti-invariant) submanifold.

Definition 4.2. [4] If the distribution D $(resp., D^{\perp})$ is horizontal (resp., vertical), then the pair $(D, D^{\perp})$ is called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ (resp., $\xi \in \Gamma(D^{\perp})$). The CR-submanifold is also called ξ-horizontal (resp., ξ-vertical) if $\xi \in \Gamma(D)$ $(resp., \xi \in \Gamma(D^{\perp}))$.

The orthogonal complement $\phi D^{\perp} \in T^{\perp} M$ is given by

$$TM = D \oplus D^{\perp}, \;\; T^{\perp} M = \phi D^{\perp} \oplus \mu, \tag{4.1}$$

where $\phi\mu$=μ.

Let $\grave{M}$ be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with semi-symmetric metric connection $\widetilde{\nabla}$. The Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are given, respectively,

$$\widetilde{\nabla}_X Y = \grave{\widetilde{\nabla}}_X Y + \widetilde{h}(X, Y), \tag{4.2}$$

$$\widetilde{\nabla}_X N = -\widetilde{A}_N X + \widetilde{\nabla}_X^{\perp} N \tag{4.3}$$

for any $X, Y \in \Gamma(T\mathbb{M})$, where $\widetilde{\nabla}_X Y$, $\widetilde{A}_N X \in \Gamma(TM)$. Here $\grave{\widetilde{\nabla}}$, $\widetilde{h}$ and $\widetilde{A}_N$ are called the induced connection on M, the second fundamental form and the Weingarten mapping with respect to $\widetilde{\nabla}$, respectively. In view of (3.7), (3.9) and (4.2), we get

$$\grave{\widetilde{\nabla}}_X Y + \widetilde{h}(X, Y) = \grave{\nabla}_X Y + h(X, Y) + \eta(Y) X - g(X, Y)\xi. \tag{4.4}$$

Using (2.10) and (2.11) in the equation (4.4) and comparing the tangential and normal components on both sides, we obtain

$$P\grave{\widetilde{\nabla}}_X Y = P\grave{\nabla}_X Y + \eta(Y) PX - \alpha g(X, Y) P\xi, \tag{4.5}$$

$$\widetilde{h}(X, Y) = h(X, Y) + \eta(Y)\phi Q X, \tag{4.6}$$

$$Q\grave{\widetilde{\nabla}}_X Y = Q\grave{\nabla}_X Y - g(X, Y) Q\xi, \tag{4.7}$$

for any $X, Y \in (T\mathbb{M})$.
Let $\grave{M}$ be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with quarter symmetric metric connection $\widehat{\nabla}$. Then Gauss and Weingarten formulas with respect to $\widehat{\nabla}$ as follows,

$$\widehat{\nabla}_X Y = \grave{\widehat{\nabla}}_X Y + \widehat{h}(X, Y), \tag{4.8}$$

$$\widehat{\nabla}_X N = -\widehat{A}_N X + \widehat{\nabla}^{\perp}_X N \tag{4.9}$$

for any $X, Y \in \Gamma(T\mathbb{M})$, where $\widehat{\nabla}_X Y$, $\widehat{A}_N X \in \Gamma(T\mathbb{M})$. Here $\grave{\widehat{\nabla}}$, $\widehat{h}$ and $\widehat{A}_N$ are called the induced connection on $\mathbb{M}$, the second fundamental form and the Weingarten mapping with respect to $\widehat{\nabla}$, respectively. In view of (3.9), (3.13) and (4.8), we get

$$\widehat{\nabla}_X Y + \widetilde{h}(X, Y) = \grave{\nabla}_X Y + h(X, Y) + \eta(Y)\phi X. \tag{4.10}$$

Using (2.10) and (2.11) in (4.10) and comparing the tangential and normal components on both sides, we obtain

$$P\widehat{\nabla}_X Y = P\grave{\nabla}_X Y + \eta(Y)P\phi X, \tag{4.11}$$

$$\widetilde{h}(X, Y) = h(X, Y), \tag{4.12}$$

$$Q\widehat{\nabla}_X Y = Q\grave{\nabla}_X Y + \eta(Y)Q\phi X, \tag{4.13}$$

for any $X, Y \in (T\mathbb{M})$.

In this sequel we state the following result.

Theorem 4.1. *Let $\grave{M}$ be a CR-Submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to semi-symmetric metric connection $\widetilde{\nabla}$ then we have*

i) If $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widetilde{\nabla}$ then induced connection $\grave{\widetilde{\nabla}}$ is also a semi-symmetric metric connection.

ii) If $\grave{M}$ ξ-vertical $\Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widetilde{\nabla}$ then induced connection $\grave{\widetilde{\nabla}}$ is also a semi-symmetric non-metric connection.

iii) The Gauss formula with respect to semi-symmetric metric connection is of the form

$$\widetilde{\nabla}_X Y = \grave{\widetilde{\nabla}}_X Y + h(X, Y) + \eta(Y)\phi Q X, \tag{4.14}$$

iv) The weingarten formula with respect to semi-symmetric metric connection is of the form

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N) X \tag{4.15}$$

Proof. With the help of (4.2) and (4.6) we get (iii). Also, from (2.8) and (3.1) we yield (iv). With reference to (4.5), if $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widetilde{\nabla}$ then result (i) is verifying. On the other hand, with the help of (4.7) if $\grave{M}$ is ξ-vertical, $X, Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widetilde{\nabla}$, we obtain our desired result(ii). This tells us that the proof is completed. □

Theorem 4.2. *Let $\grave{M}$ be a CR-Submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to quarter symmetric non-metric connection $\widehat{\nabla}$ then we have*

i) If $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widehat{\nabla}$ then induced connection $\grave{\widehat{\nabla}}$ is also a quarter symmetric non metric connection.
ii) If $\grave{M}$ ξ-vertical, $X, Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widehat{\nabla}$ then induced connection $\grave{\widehat{\nabla}}$ is also a quarter symmetric non-metric connection.
iii) The Gauss formula with respect to quarter symmetric non-metric connection is of the form

$$\widehat{\nabla}_X Y = \grave{\nabla}_X Y + h(X, Y), \tag{4.16}$$

iv) The weingarten formula with respect to quarter symmetric non-metric connection is of the form

$$\widehat{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)\phi X \tag{4.17}$$

Proof. With the help of (4.8) and (4.12) we get (iii). Also, from (2.8) and (3.9) we yield (iv). With reference to (4.11), if $\grave{M}$ ξ-horizontal, $X, Y \in \Gamma(D)$ and D is parallel with respect to $\widetilde{\nabla}$ then result (i) is verifying. On the other hand, with the help of (4.13) if $\grave{M}$ is ξ-vertical, $X, Y \in \Gamma(D^{\perp})$ and $D^{\perp}$ is parallel with respect to $\widetilde{\nabla}$, we obtain our desired result(ii). We completed the proof. □

5. Yamabe solitons with potential vector field is torse-forming

As per this consequence of our analysis in this section we have the following geometric characterization of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ admitting semi-symmetric metric connection $\widetilde{\nabla}$ and quarter symmetric non-metric connection $\widehat{\nabla}$. Thus, in view of my thought,we can state the following result.

Theorem 5.1. *A Yamabe soliton (g, κ, λ) on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to semi symmetric metric connection $\widetilde{\nabla}$ is invariant if and only if*

$$2\eta(\kappa)g(X, Y) = \{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}.$$

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a semi symmetric metric connection$\widehat{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widetilde{\mathfrak{L}}_\kappa g)(X, Y) = (\widetilde{\hat{\delta}} - \lambda)g(X, Y). \tag{5.1}$$

From the definition of Lie derivative, equations (2.3) and (3.1), we obtain

$$(\widetilde{\mathfrak{L}}_\kappa g)(X, Y) = g(\widetilde{\nabla}_X \kappa, Y) + g(X, \widetilde{\nabla}_Y \kappa) \tag{5.2}$$

$$= g(\nabla_X \kappa, Y) + g(X, \nabla_Y \kappa) + 2\eta(\kappa)g(X, Y) - \{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}$$
$$= (\mathfrak{L}_\kappa g)(X, Y) + 2\eta(\kappa)g(X, Y) - \{g(X, \kappa)\eta(Y) + g(Y, \kappa)\eta(X)\}$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.1) and (5.2), we get

$$\frac{1}{2}(\mathfrak{L}_\kappa g)(X,Y) + \eta(\kappa)g(X,Y) - \frac{1}{2}\{g(X,\kappa)\eta(Y) + g(Y,\kappa)\eta(X)\} = (\widetilde{\hat{\delta}} - \lambda)g(X,Y). \tag{5.3}$$

This indicate that proof is completed. □

Theorem 5.2. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to semi-symmetric metric connection. If κ is a torse-forming vector field, then the soliton (g, κ, λ) is expanding, steady and shrinking according as $\lambda = \widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$ <> =0, unless $\lambda = \widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}$ is constant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a semi-symmetric metric connection$\widetilde{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widetilde{\mathfrak{L}}_\kappa g)(X,Y) = (\widetilde{\hat{\delta}} - \lambda)g(X,Y). \tag{5.4}$$

From the definition of Lie derivative, equations (1.3) and (3.1), we obtain

$$\begin{aligned}(\widetilde{\mathfrak{L}}_\kappa g)(X,Y) &= g(\widetilde{\nabla}_X\kappa, Y) + g(X, \widetilde{\nabla}_Y\kappa)\\ &= 2\psi g(X,Y) + \{\theta(X)g(\kappa,Y) + \theta(Y)g(\kappa,X)\}\\ &\quad +2\eta(\kappa)g(X,Y) - \{\eta(X)g(\kappa,Y) + \eta(Y)g(\kappa,X)\}\end{aligned} \tag{5.5}$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.4) and (5.5), we get

$$\begin{aligned}(\psi - \widetilde{\hat{\delta}} + \lambda)g(X,Y) &= \frac{1}{2}\{\eta(Y)g(\kappa,X) + \eta(X)g(\kappa,Y)\}\\ &\quad -\frac{1}{2}\{\theta(X)g(\kappa,Y) + \theta(Y)g(\kappa,X)\} - \eta(\kappa)g(X,Y)\end{aligned} \tag{5.6}$$

On contracting (5.6), we have

$$\lambda = \widetilde{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}. \tag{5.7}$$

This leads to the Theorem 5.2 □

In this sequel, we write the following corollaries.

Corollary 5.1. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with respect to a semi-symmetric*

metric connection $\widetilde{\nabla}$. *Then following relations hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}{=}C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> {=}0$
concircular	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}{=}C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> {=}0$
concurrent	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}{=}C$	$1 - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> {=}0$
recurrent	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\}{=}C$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa) + (n-1)\eta(\kappa)\} <> {=}0$
parallel	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}{=}C$	$\widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> {=}0$
torqued	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\}{=}C$	$\psi - \widetilde{\hat{\delta}} - \frac{1}{n}\{(n-1)\eta(\kappa)\} <> {=}0$

Theorem 5.3. *A Yamabe soliton* (g, κ, λ) *on an n-dimensional nearly hyperbolic Sasakian manifold* $\mathbb{M}^n(\phi, \xi, \eta, g)$ *with respect to quarter symmetric metric connection* $\widehat{\nabla}$ *always invariant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a quarter symmetric metric connection$\widehat{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widehat{\mathfrak{L}}_{\kappa} g)(X, Y) = (\widehat{\hat{\delta}} - \lambda) g(X, Y). \tag{5.8}$$

From the definition of Lie derivative, equations (2.3) and (3.9), we obtain

$$\begin{aligned}(\widehat{\mathfrak{L}}_{\kappa} g)(X, Y) &= g(\widehat{\nabla}_X \kappa, Y) + g(X, \widehat{\nabla}_Y \kappa) \\ &= g(\nabla_X \kappa, Y) + g(X, \nabla_Y \kappa) + \eta(\kappa) g(\phi X, Y) + \eta(\kappa) g(X, \phi Y) \\ &= (\mathfrak{L}_{\kappa} g)(X, Y),\end{aligned} \tag{5.9}$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.8) and (5.9), we get

$$\frac{1}{2}(\mathfrak{L}_{\kappa} g)(X, Y) = (\widehat{\hat{\delta}} - \lambda) g(X, Y). \tag{5.10}$$

Proof is completed. □

Theorem 5.4. *Let* (g, κ, λ) *be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold* $\mathbb{M}^n(\phi, \xi, \eta, g)$ *with respect to quarter symmetric metric connection* $\widehat{\nabla}$. *If* κ *is a torse-forming vector field, then the soliton* (g, κ, λ) *is expanding, steady and shrinking according as* $\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <> {=}0$, *unless* $\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\}$ *is constant.*

Proof. Let (g, κ, λ) be a Yamabe soliton on $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to a quarter symmetric metric connection$\widehat{\nabla}$. So from (1.2), we have

$$\frac{1}{2}(\widehat{\mathfrak{L}}_{\kappa} g)(X, Y) = (\widehat{\hat{\delta}} - \lambda) g(X, Y). \tag{5.11}$$

From the definition of Lie derivative, equations (1.3) and (3.9), we obtain

$$
\begin{aligned}
(\widehat{\mathfrak{L}}_\kappa g)(X,Y) &= g(\widehat{\nabla}_X \kappa, Y) + g(X, \widehat{\nabla}_Y \kappa) \\
&= 2\psi g(X,Y) + \theta(X) g(\kappa, Y) + \theta(Y) g(\kappa, X)
\end{aligned}
\tag{5.12}
$$

for all $X, Y \in \chi(\mathbb{M})$. With the help of (5.11) and (5.12), we get

$$
(\psi - \widehat{\hat{\delta}} + \lambda) g(X,Y) = -\frac{1}{2}\{\theta(X) g(\kappa, Y) + \theta(Y) g(\kappa, X)\}
\tag{5.13}
$$

Taking contraction (5.13), we have

$$
\lambda = \widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\}.
\tag{5.14}
$$

This leads to the Theorem 5.4. □

In this sequel, we write the following corollaries.

Corollary 5.2. *Let (g, κ, λ) be a Yamabe soliton on an n-dimensional nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$ with respect to quarter symmetric metric connection $\widehat{\nabla}$. Then following relations hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} = C$	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <> =0$
concircular	$\widehat{\hat{\delta}} - \psi = C$	$\widehat{\hat{\delta}} - \psi <> =0$
concurrent	$\widehat{\hat{\delta}} - 1 = C$	$\widehat{\hat{\delta}} - 1 <> =0$
recurrent	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} = C$	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} <> =0$
parallel	$\widehat{\hat{\delta}} = C$	$\widehat{\hat{\delta}} <> =0$
torqued	$\widehat{\hat{\delta}} - \psi = C$	$\widehat{\hat{\delta}} - \psi <> =0$

6. Yamabe solitons whose potential vector field is torse-forming on CR-submanifold of nearly hyperbolic Sasakian manifold

In this section, we study Yamabe soliton whose potential vector field is a torse-forming on CR-sub-manifolds of nearly hyperbolic Sasakian manifold with respect to the induced connection $\grave{\grave{\nabla}}$ and $\grave{\nabla}$. We state the following theorem as:

Theorem 6.1. *Let $\grave{M}$ be a CR-submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, admitting semi-symmetric metric connection $\widetilde{\nabla}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\widetilde{\nabla}$. If (g, κ, λ) be a*

Yamabe soliton on M and κ is a torse-forming vector field, then (g,κ,λ) is expanding, steady and shrinking according as $\widetilde{\hat{\delta}}-\psi-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\} <> =0$, unless $\widetilde{\hat{\delta}}-\psi-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\}$ is constant.

Proof. If $\grave{M}$ is ξ-horizontal for all X, $Y \in \Gamma(D)$ and D is parallel with respect to $\grave{\grave{\nabla}}$, then in view of (4.5), we have

$$\grave{\grave{\nabla}}_X Y = \grave{\nabla}_X Y + \eta(Y)X - g(X,Y)\xi. \tag{6.1}$$

With the help of Theorem 5.2 and (3.1), we conclude that the induced connection $\grave{\nabla}$ is also semi-symmetric metric connection. This leads to the proof of the Theorem 6.1 □

In this sequel, we write the following corollaries.

Corollary 6.1. *Let $\grave{M}$ be a CR-submanifold nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi,\xi,\eta,g)$, $n>1$, admitting a semi-symmetric metric connection $\grave{\grave{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\grave{\grave{\nabla}}$. If (g,κ,λ) be a Yamabe soliton on M and κ is a torse-forming vector field, then the following results hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\}=C$	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\} <> =0$
concircular	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
concurrent	$1-\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$1-\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
recurrent	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\}=C$	$\widetilde{\hat{\delta}}-\frac{1}{n}\{\theta(\kappa)+(n-1)\eta(\kappa)\} <> =0$
parallel	$\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$
torqued	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\}=C$	$\psi-\widetilde{\hat{\delta}}-\frac{1}{n}\{(n-1)\eta(\kappa)\} <> =0$

Theorem 6.2. *Let $\grave{M}$ be a CR-submanifold of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi,\xi,\eta,g)$, $n>1$, admitting quarter symmetric non- metric connection $\widehat{\nabla}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\widehat{\nabla}$. If (g,κ,λ) be a Yamabe soliton on M and κ is a torse-forming vector field, then (g,κ,λ) is expanding, steady and shrinking according as $\lambda=\widehat{\hat{\delta}}-\psi-\frac{1}{n}\{\theta(\kappa)\} <> =0$, unless $\lambda=\widehat{\hat{\delta}}-\psi-\frac{1}{n}\{\theta(\kappa)\}$ is constant.*

Proof. If $\grave{M}$ is ξ-horizontal for all X, $Y \in \Gamma(D)$ and D is parallel with respect to $\grave{\hat{\nabla}}$, then in view of (4.11), we have

$$\grave{\hat{\nabla}}_X Y = \grave{\nabla}_X Y + \eta(Y)\phi X, \tag{6.2}$$

With the help of Theorem 5.5 and (3.9), we conclude that the induced connection $\grave{\hat{\nabla}}$ is also quarter symmetric non-metric connection. This leads to the statement of the Theorem 6.2. $\square$

In this sequel, we write the following corollaries.

Corollary 6.2. *Let $\grave{M}$ be a CR-submanifold nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, admitting induced quarter symmetric non-metric connection $\grave{\hat{\nabla}}$ is ξ-horizontal (resp. ξ-vertical) and D is parallel with respect to $\grave{\hat{\nabla}}$. If (g, κ, λ) be a Yamabe soliton on M and κ is a torse-forming vector field, then the following results hold*

κ	*condition of existence*	*condition of shrinking, steady and expanding*
torse-forming	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} = constant$	$\widehat{\hat{\delta}} - \psi - \frac{1}{n}\{\theta(\kappa)\} <> =0$
concircular	$\widehat{\hat{\delta}} - \psi = constant$	$\widehat{\hat{\delta}} - \psi <> =0$
concurrent	$\widehat{\hat{\delta}} - 1 = constant$	$\widehat{\hat{\delta}} - 1 <> =0$
recurrent	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} = constant$	$\widehat{\hat{\delta}} - \frac{1}{n}\{\theta(\kappa)\} <> =0$
parallel	$\widehat{\hat{\delta}} = constant$	$\widehat{\hat{\delta}} <> =0$
torqued	$\widehat{\hat{\delta}} - \psi = constant$	$\widehat{\hat{\delta}} - \psi <> =0$

7. Almost Yamabe solitons whose potential vector field is torse-forming on CR-submanifold of nearly hyperbolic Sasakian manifold

In this section, we classify almost Yamabe solitons whose potential field is torse-forming on CR-submanifold of nearly hyperbolic Sasakian manifold with respect to a semi-symmetric metric connection and quarter symmetric non-metric connection. At this stage, we denote κ^t and κ^n as tangential and normal component of such vector field. For almost Yamabe soliton we have the following.

Theorem 7.1. *An almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with a semi-symmetric metric connection of type $\widetilde{\nabla}$ satisfies*

$$
\begin{aligned}
(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) &= g(A_{\kappa^n}X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \\
&+ \frac{1}{2}\{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\} \qquad (7.1)
\end{aligned}
$$

for any vector fields X, Y on M.

Proof. In view of (1.3), (3.1), (4.14) and (4.15), we have

$$\begin{aligned}\psi X + \theta(P)\kappa = \widetilde{\nabla}_X \kappa = \widetilde{\nabla}_X(\kappa^t + \kappa^n) = \grave{\nabla}_X \kappa^t + h(X, \kappa^t) + \eta(\kappa^t)\phi Q X \\ -A_{\kappa^n} X + \nabla^{\perp}_X \kappa^n + \eta(\kappa^n) X - g(X, \kappa^n)\xi.\end{aligned} \tag{7.2}$$

On comparing tangential and normal component of (7.2), we obtain

$$\grave{\nabla}_X \kappa^t = \psi X + \theta(P)\kappa + A_{\kappa^n} X - \eta(\kappa^n) X + g(X, \kappa^n)\xi \tag{7.3}$$

and

$$h(X, \kappa^t) = -\nabla^{\perp}_X \kappa^n - \eta(\kappa^n)\phi Q X. \tag{7.4}$$

From the definition of Lie derivative and (7.3), we have

$$\begin{aligned}\mathfrak{L}_{\kappa^t} g(X, Y) = 2\psi g(X, Y) + 2g(A_{\kappa^n} X, Y) - 2\eta(\kappa^n) g(X, Y) + \{\theta(X) g(\kappa, Y) \\ +\theta(Y) g(X, \kappa)\} + \{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}.\end{aligned} \tag{7.5}$$

Using (7.5) in (1.2), we yield

$$\begin{aligned}(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X) g(\kappa, Y) + \theta(Y) g(X, \kappa)\} \\ +\frac{1}{2}\{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}.\end{aligned} \tag{7.6}$$

This proves our assertion. □

Corollary 7.1. *If an almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with semi-symmetric metric connection is minimal, then*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) n = \theta(\kappa). \tag{7.7}$$

Corollary 7.2. *Let (g, κ^t, λ) be an almost Yamabe soliton on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, and ξ-horizontal (resp.ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\nabla}$ satisfies*

$$\begin{aligned}(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) g(X, Y) = g(A_{\kappa^n} X, Y) + \frac{1}{2}\{\theta(X) g(\kappa, Y) + \theta(Y) g(X, \kappa)\} \\ +\frac{1}{2}\{g(\kappa^n, X)\eta(Y) + g(Y, \kappa^n)\eta(X)\}\end{aligned} \tag{7.8}$$

for any vector fields X, Y on M.

Corollary 7.3. *If an almost Yamabe soliton (g, κ^t, λ) on CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $(n > 1)$ and ξ-horizontal (resp. ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\nabla}$ is minimal, then*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n)) n = \theta(\kappa) \tag{7.9}$$

Theorem 7.2. *An almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with quarter symmetric non-metric connection $\widehat{\nabla}$ satisfies*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A_{\kappa^n} X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \tag{7.10}$$

for any vector fields X, Y on M.

Proof. In view of (1.3), (3.9), (4.16) and (4.17), we have

$$\begin{aligned} \psi X + \theta(P)\kappa = \widehat{\nabla}_X \kappa = \widehat{\nabla}_X(\kappa^t + \kappa^n) &= \grave{\nabla}_X \kappa^t + \widehat{h}(X, \kappa^t) - \widehat{A}_{\kappa^n} X + \widehat{\nabla}^{\perp}_X \kappa^n \\ &= \grave{\nabla}_X \kappa^t + h(X, \kappa^t) - \widehat{A}_{\kappa^n} X + \nabla^{\perp}_X \kappa^n + \eta(\kappa^n)\phi X. \end{aligned} \tag{7.11}$$

On comparing tangential and normal component of (7.11), we obtain

$$\grave{\nabla}_X \kappa^t = \psi X + \theta(X)\kappa + A_{\kappa^n} X - \eta(\kappa^n)\phi X, \tag{7.12}$$

and

$$h(X, \kappa^t) = -\nabla^{\perp}_X \kappa^n. \tag{7.13}$$

From the definition of Lie derivative and (7.12), we have

$$\mathfrak{L}_{\kappa^t} g(X, Y) = 2\psi g(X, Y) + 2g(A_{\kappa^n} X, Y) + \{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\}. \tag{7.14}$$

Using (7.14) in (1.2), we yield

$$(\hat{\delta} - \lambda - \psi)g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \tag{7.15}$$

This proves our assertion. □

Corollary 7.4. *If an almost Yamabe soliton (g, κ^t, λ) on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, with quarter symmetric non-metric connection is minimal, then*

$$(\hat{\delta} - \lambda - \psi)n = \theta(\kappa). \tag{7.16}$$

Corollary 7.5. *Let (g, κ^t, λ) be an almost Yamabe soliton on a CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $n > 1$, and ξ-horizontal (resp.ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\widehat{\nabla}}$ satisfies*

$$(\hat{\delta} - \lambda - \psi + \eta(\kappa^n))g(X, Y) = g(A^n_\kappa X, Y) + \frac{1}{2}\{\theta(X)g(\kappa, Y) + \theta(Y)g(X, \kappa)\} \tag{7.17}$$

for any vector fields X, Y on M.

Corollary 7.6. *If an almost Yamabe soliton (g, κ^t, λ) on CR-submanifold $\grave{M}$ of nearly hyperbolic Sasakian manifold $\mathbb{M}^n(\phi, \xi, \eta, g)$, $(n > 1)$ and ξ-horizontal (resp. ξ-vertical), $X, Y \in \Gamma(D)$, D is parallel with induced connection $\grave{\hat{\nabla}}$ is minimal, then*

$$(\hat{\delta} - \lambda - \psi)n = \theta(\kappa) \tag{7.18}$$

8. Example

Example 8.1. Let us consider on $\mathbb{R}^{2n+1}$ the following hyperbolic Sasakian structure (ϕ, ξ, η, g) given by

$$\eta = \frac{1}{2}\left(dz - \sum_{i=n}^{n} y^i dx_i\right), \quad \xi = \frac{\partial}{\partial z},$$

$$g = -\eta \otimes \eta - \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi \circ \left(coshx_i \frac{\partial}{\partial x_i} + sinhy_i \frac{\partial}{\partial y_i} + z\frac{\partial}{\partial z}\right)$$

$$= \sum_{i=1}^{n}\left(sinhy_i \frac{\partial}{\partial x_i} + coshx_i \frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{n} sinhy, y^i \frac{\partial}{\partial z},$$

where $\{x^i, y^i, z\}$,$i = 1, ...n$ are the denoting the Cartesian coordinates.

The equation $t(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, 0, x_5)$ define a CR-sub-manifolds in $\mathbb{R}^5$ with its hyperbolic Sasakian structure (ϕ, ξ, η, g). For this fact we take the orthogonal basis

$$E_1 = coshx_5 \frac{\partial}{\partial x_1} + sinhx_5 \frac{\partial}{\partial x_2}, \quad E_2 = sinhx_5 \frac{\partial}{\partial x_1} + coshx_5 \frac{\partial}{\partial x_2}$$

$$E_3 = coshx_5 \frac{\partial}{\partial x_3} + sinhx_5 \frac{\partial}{\partial x_4}, \quad E_4 = sinhx_5 \frac{\partial}{\partial x_3} + coshx_5 \frac{\partial}{\partial x_4}, \quad E_5 = \frac{\partial}{\partial x_5} = \xi,$$

and define $D = span\{E_1, E_2\}$ and $D^\perp = span\{E_3\}$. In this case it is clear that $TM = D \oplus D^\perp \oplus \langle\xi\rangle$.

Example 8.2. Let us consider the 5-dimensioanl manifold $M = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinated in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$e_1 = coshx_5 \frac{\partial}{\partial x_1} + sinhx_5 \frac{\partial}{\partial x_2}, \quad e_2 = sinhx_5 \frac{\partial}{\partial x_1} + coshx_5 \frac{\partial}{\partial x_2}$$

$$e_3 = coshx_5 \frac{\partial}{\partial x_3} + sinhx_5 \frac{\partial}{\partial x_4}, \quad e_4 = sinhx_5 \frac{\partial}{\partial x_3} + coshx_5 \frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial x_5} = \xi,$$

which are linearly independent at each point of M and hence form a basis tangent space T_pM.
Let g be the Riemannian metric on M define by

$$g(e_i, e_i) = -1, for 1 \leq i \leq 4 \quad and \quad g(e_5, e_5) = -1, \tag{8.1}$$

$$g(e_i, e_j) = 0, \quad for \quad 1 \neq j \quad and \quad 1 \leq i \leq 5 \quad 1 \leq j \leq 5. \tag{8.2}$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for all $X \in (M)$ and let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = -e_4, \quad \phi(e_4) = -e_3, \quad \phi(e_5) = 0.$$

Thus $e_5 = \xi$, the structure (ϕ, ξ, η, g) define an almost hyperbolic contact metric structure on M. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,$$

$$[e_1, e_5] = -e_2, [e_2, e_5] = -e_1, [e_3, e_5] = e_4, [e_4, e_5] = -e_3,$$

The Levi-Civita connection ∇ of the Riemannian metric g is given by,

$$2g(\nabla_X Y, Z) \tag{8.3}$$

$$= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

which is known as Koszul's formula. After using koszul's formula, we find

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -e_5, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = -e_2,$$

$$\nabla_{e} e_1 = -e_5, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = -e_1,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_4 = -e_5, \quad \nabla_{e_3} e_5 = -e_4,$$

$$\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = -e_5, \quad \nabla_{e_4} e_3 = -e_5, \quad \nabla_{e_4} e_4 = 0, \quad \nabla_{e_4} e_5 = -e_3,$$

$$\nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0,$$

By using the definition of semi-symmetric metric connection (3.1) and from above expressions we find

$$\widetilde{\nabla}_{e_1} e_1 = -e_5, \quad \widetilde{\nabla}_{e_1} e_2 = -e_5, \quad \widetilde{\nabla}_{e_1} e_3 = 0, \quad \widetilde{\nabla}_{e_1} e_4 = 0, \quad \widetilde{\nabla}_{e_1} e_5 = -e_1 - e_2,$$

$$\widetilde{\nabla}_{e_2} e_1 = -e_5, \quad \widetilde{\nabla}_{e_2} e_2 = -e_5, \quad \widetilde{\nabla}_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = -e_1 - e_2,$$

$$\widetilde{\nabla}_{e_3} e_1 = 0, \quad \widetilde{\nabla}_{e_3} e_2 = 0, \quad \widetilde{\nabla}_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = -e_5, \quad \nabla_{e_3} e_5 = e_3 - e_4,$$

$$\widetilde{\nabla}_{e_4} e_1 = 0, \quad \widetilde{\nabla}_{e_4} e_2 = -e_5, \quad \widetilde{\nabla}_{e_4} e_3 = -e_5, \quad \widetilde{\nabla}_{e_4} e_4 = -e_5, \quad \widetilde{\nabla}_{e_4} e_5 = -e_3 - e_4,$$

$$\widetilde{\nabla}_{e_5} e_1 = 0, \quad \widetilde{\nabla}_{e_5} e_2 = 0, \quad \widetilde{\nabla}_{e_5} e_3 = 0, \quad \widetilde{\nabla}_{e_5} e_4 = 0, \quad \widetilde{\nabla}_{e_5} e_5 = 0,$$

Therefore, the non-vanishing components of the Riemannian curvatures, the Ricci curvatures and the Scalar curvature with respect to the semi-symmetric metric connection as follows:

$$\widetilde{R}(e_1,e_2)e_1=0,\widetilde{R}(e_1,e_2)e_2=0,\widetilde{R}(e_1,e_3)e_1=-e_3-e_4,\widetilde{R}(e_1,e_3)e_3=e_1+e_2,$$

$$\widetilde{R}(e_1,e_2)e_1=e_2,\widetilde{R}(e_1,e_2)e_2=-e_1,\widetilde{R}(e_1,e_3)e_1=0,\widetilde{R}(e_1,e_3)e_3=0,$$

$$\widetilde{R}(e_1,e_4)e_1=-e_3-e_4,\widetilde{R}(e_1,e_4)e_4=e_1+e_2,\widetilde{R}(e_1,e_5)e_1=-e_5,$$

$$\widetilde{R}(e_1,e_5)e_5=-e_1-e_2,\widetilde{R}(e_2,e_3)e_2=-e_3-e_4,\widetilde{R}(e_2,e_3)e_3=-e_1-e_2,$$

$$\widetilde{R}(e_2,e_4)e_3=0,\widetilde{R}(e_3,e_4)e_4=0,\widetilde{R}(e_3,e_5)e_3=-e_5,$$

$$\widetilde{R}(e_3,e_5)e_5=-e_3-e_4,\widetilde{R}(e_4,e_5)e_4=-e_5,\widetilde{R}(e_4,e_5)e_5=-e_3-e_4,$$

From these Riemannian curvatures tensors, we calculate

$$\widetilde{S}(e_1,e_1)=\quad\widetilde{S}(e_2,e_2)=\quad\widetilde{S}(e_3,e_3)=\widetilde{R}(e_4,e_4)=\widetilde{S}(e_5,e_5)=-4$$

Since $\{e_1,e_2,e_3,e_4,e_5\}$ form a basis of a 5-dimensional almost hyperbolic contact metric structure. Thus any vector field $X,Y,Z\in\chi(M^5)$ can be written as

$$X=a_1e_1+b_1e_2+c_1e_3+d_1e_4+t_1e_5,$$

$$Y=a_2e_1+b_2e_2+c_2e_3+d_2e_4+t_2e_5,$$

$$Z=a_3e_1+b_3e_2+c_3e_3+d_3e_4+t_3e_5,$$

where $a_i,b_i,c_i,d_i,t_i\in\mathrm{Re}^+,\ i=1,2,3,4,5$ such that

$$\left\{\frac{(a_1a_2+b_1b_2+c_1c_2+d_1d_3)}{t_1}+t_1\left(\frac{b_2}{b_1}-\frac{a_2}{a_1}-\frac{c_2}{c_1}-1\right)\right\}\neq 0.$$

If we consider the 1-form θ by $\theta(X)$=-$g(X,e_5)$, for any $X\in\chi(M)$ and considering $\psi\in C^\infty(M)$ as

$$\psi=\left\{\frac{(a_1a_2+b_1b_2+c_1c_2+d_1d_3)}{t_1}+t_1\left(\frac{b_2}{b_1}-\frac{a_2}{a_1}-\frac{c_2}{c_1}-1\right)\right\}.$$

So the relation

(8.4) $$\nabla_XY=\psi X+\theta(X)Y,$$

holds. As per this consequences Y is a torse-forming vector field. Thus from (9.3), we get

$$(\mathfrak{L}_Y g)(X,Z) = g(\nabla_X Y, Z) + g(X, \nabla_Z Y) \\ = 2\psi g(X,Z) + \theta(X)g(Y,Z) + \theta(Z)g(Y,X). \tag{8.5}$$

Also, we calculate

$$\begin{cases} g(X,Z) = a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3 \\ g(Y,Z) = a_2a_3 + b_2b_3 + c_2c_3 + d_2d_3 - t_2t_3 \\ g(Y,X) = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 - t_1t_2 \end{cases}. \tag{8.6}$$

Also

$$\begin{cases} \theta(X) = t_1 \\ \theta(Y) = t_2 \\ \theta(Z) = t_3 \end{cases}. \tag{8.7}$$

With the help of above equation (9.2) can be reduced

$$\begin{aligned} \tfrac{1}{2}(\mathfrak{L}_Y g)(X,Z) = &\left\{ \tfrac{(a_1a_2+b_1b_2+c_1c_2+d_1d_3)}{t_1} + t_1\left(\tfrac{b_2}{b_1} - \tfrac{a_2}{a_1} - \tfrac{c_2}{c_1} - 1\right)\right\} \\ &\times \{a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3 \\ &-\tfrac{1}{2}\, t_1(a_2a_3 + b_2b_3 + c_2c_3 + d_2d_3 - t_2t_3) \\ &+t_3(a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3)\} \end{aligned} \tag{8.8}$$

Also,

$$(\widetilde{\hat{\delta}} - \lambda)g(X,Z) = (-16 - \lambda)\{a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3 - t_1t_3\} \tag{8.9}$$

We consider that $a_1a_3+b_1b_3+c_1c_3+d_1d_3-t_1t_3 \neq 0$ and $5t_1(a_2a_3+b_2b_3+c_2c_3+d_2d_3-t_2t_3)+5t_3(a_1a_3+b_1b_3+c_1c_3+d_1d_3-t_1t_3)+2t_2(a_1a_3+b_1b_3+c_1c_3+d_1d_3-t_1t_3)$=0.

we get (g, Y, λ) is a Yamabe soliton, i.e., $\frac{1}{2}\mathfrak{L}_Y g(X,Z)$=$(\widetilde{\hat{\delta}} - \lambda)g(X,Z)$ holds, unless

$$\begin{aligned} \lambda &= -16 - \left\{ \tfrac{(a_1a_2+b_1b_2+c_1c_2+d_1d_3)}{t_1} + t_1\left(\tfrac{b_2}{b_1} - \tfrac{a_2}{a_1} - \tfrac{c_2}{c_1} - 1\right)\right\} - \tfrac{1}{5}t_2 \\ &= \widetilde{r} - \psi - \tfrac{1}{5}\theta(Y) \\ &= constant \end{aligned}$$

So the existence of Yamabe soliton (g, Y, λ) on a 5-dimensional hyperbolic Sasakian manifold with semi symmetric metric connection $\widetilde{\nabla}$ with potential vector field Y as torse-forming thus the Theorem 5.2 is verified.

Example 8.3. In Example 8.2, we consider the hyperbolic Sasakian manifold $M\ (\phi, \eta, \xi, g)$ with quarter symmetric non-metric connection. Using the equation (3.9), we obtain:

$$\begin{array}{lllll} \widehat{\nabla}_{e_1}e_1 = 0, & \widehat{\nabla}_{e_1}e_2 = -e_5, & \widehat{\nabla}_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, & \widehat{\nabla}_{e_1}e_5 = -e_2, \\ \widehat{\nabla}_{e}e_1 = -e_5, & \widehat{\nabla}_{e_2}e_2 = 0, & \widehat{\nabla}_{e_2}e_3 = 0, & \widehat{\nabla}_{e_2}e_4 = 0, & \widehat{\nabla}_{e_2}e_5 = -e_1, \\ \widehat{\nabla}_{e_3}e_1 = 0, & \widehat{\nabla}_{e_3}e_2 = 0, & \widehat{\nabla}_{e_3}e_3 = 0, & \widehat{\nabla}_{e_3}e_4 = -e_5, & \widehat{\nabla}_{e_3}e_5 = -e_4, \\ \widehat{\nabla}_{e_4}e_1 = 0, & \widehat{\nabla}_{e_4}e_2 = -e_5, & \widehat{\nabla}_{e_4}e_3 = -e_5, & \widehat{\nabla}_{e_4}e_4 = 0, & \widehat{\nabla}_{e_4}e_5 = -e_3, \\ \widehat{\nabla}_{e_5}e_1 = 0, & \widehat{\nabla}_{e_5}e_2 = 0, & \widehat{\nabla}_{e_5}e_3 = 0, & \widehat{\nabla}_{e_5}e_4 = 0, & \widehat{\nabla}_{e_5}e_5 = 0, \end{array}$$

Therefore, the non-vanishing components of the Riemannian curvatures, the Ricci curvatures and the Scalar curvature with respect to the quarter-symmetric non-metric connection are as follows:

$$\widehat{R}(e_1,e_2)e_1 = e_2, \quad \widehat{R}(e_1,e_2)e_2 = -e_1, \quad \widehat{R}(e_1,e_3)e_1 = 0, \quad \widehat{R}(e_1,e_3)e_3 = 0,$$

$$\widehat{R}(e_1,e_4)e_1 = 0, \quad \widehat{R}(e_1,e_4)e_4 = 0, \quad \widehat{R}(e_1,e_5)e_1 = -e_5, \quad \widehat{R}(e_1,e_5)e_5 = -e_1,$$

$$\widehat{R}(e_2,e_3)e_2 = 0, \quad \widehat{R}(e_2,e_3)e_3 = 0, \quad \widehat{R}(e_2,e_4)e_3 = 0, \quad \widehat{R}(e_3,e_4)e_4 = 0,$$

$$\widehat{R}(e_2,e_5)e_2 = -e_5, \quad \widehat{R}(e_2,e_5)e_5 = -e_2, \quad \widehat{R}(e_3,e_4)e_3 = e_4, \quad \widehat{R}(e_3,e_4)e_4 = -e_3,$$

$$\widehat{R}(e_3,e_5)e_3 = -e_5, \quad \widehat{R}(e_3,e_5)e_5 = -e_3, \quad \widehat{R}(e_4,e_5)e_4 = -e_5, \quad \widehat{R}(e_4,e_5)e_5 = -e_4,$$

From these Riemannian curvatures tensors components with quarter semi-symmetric non-metric connection we calculate:

$$\widehat{S}(e_1,e_1) = \quad \widehat{S}(e_2,e_2) = \quad \widehat{S}(e_3,e_3) = \widehat{R}(e_4,e_4) = 0, \widehat{S}(e_5,e_5) = -4$$

$$\widehat{r} = -4.$$

Therefore, the constructed metric of the hyperbolic Sasakian manifold with quarter-symmetric non-metric connection is Yamabe solion. It is shown that the scalar curvature with respect to the quarter-symmetric non-metric connection $\widehat{r} = -4$ and $\lambda = -4 < 0$ i.e is admitting shrinking Yamabe soliton.

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ON WEAKLY SYMMETRIC AND SPECIAL WEAKLY RICCI SYMMETRIC LP-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC SEMI-METRIC CONNECTION

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Abstract. The aim of this paper is to study the geometric properties of LP-Sasakian manifolds with respect to Levi-Civita connection when they are weakly symmetric, weakly Ricci symmetric and special weakly symmetric with respect to semi-symmetric semi-metric connection. An illustration of three dimensional LP-Sasakian manifold is given.
Keywords: LP-Sasakian manifolds, Levi-Civita connection, weakly Ricci symmetric LP-Sasakian manifolds.

1. Introduction

The concept of an LP-Sasakian manifold was first developed in 1989 by K. Matsumoto [9]. The identical idea was then independently suggested by I. Mihai and R. Rosca [11], who produced multiple results on this manifold. Additionally, Venkatesha and C.S. Bagewadi [19], I. Mihai, A.A. Shaikh and U.C. De [12], A.A. Shaikh [18], C. Ozgur [14] and others have explored the LP-Sasakian manifold.

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Subsequently, numerous geometers have published various works in this field ([8], [4], [15], [16], [6]).

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly symmetric* if there exist 1-forms $\alpha, \beta, \gamma, \delta$ and σ such that

$$
\begin{aligned}
(1.1) \qquad (\nabla_X R)(Y,Z,V,U) &= \alpha(X)R(Y,Z,V,U) + \beta(Y)R(X,Z,V,U) \\
&+ \gamma(Z)R(Y,X,V,U) + \delta(V)R(Y,Z,X,U) \\
&+ \sigma(U)R(Y,Z,V,X),
\end{aligned}
$$

holds for all vector fields $X, Y, ..., V \in X(M)$, where R is the Riemannian curvature tensor of (M^n, g) of type $(0, 4)$ and ∇ is the covariant differentiation with respect to the Riemannian metric g. A weakly symmetric manifold is said to be *proper* if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case.

Let $\{e_i\}$, $(i = 1, 2, ..., n)$ be an orthonormal basis of the tangent space at point of the manifold. Then, putting $Y = U = e_i$ in (1.1) and taking summation for $1 \leq i \leq n$, we obtain

$$
\begin{aligned}
(1.2) \qquad (\nabla_X S)(Z,V) &= \alpha(X)S(Z,V) + \gamma(Z)S(X,V) + \delta(V)S(Z,X) \\
&+ \beta(R(X,Z)V) + \sigma(R(X,V)Z).
\end{aligned}
$$

A Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly Ricci-symmetric* if there exist 1-forms ρ, μ, ν such that the relation

$$
(1.3) \qquad (\nabla_X S)(Y,Z) = \rho(X)S(Y,Z) + \mu(Y)S(X,Z) + \nu(Z)S(X,Y),
$$

holds for any vector fields X, Y, Z, where S is the Ricci tensor of type $(0, 2)$ of the manifold M^n. A weakly Ricci-symmetric manifold is said to be proper if $\rho = \mu = \nu = 0$ is not the case.

An n-dimensional Riemannian manifold (M^n, g) is called a special weakly Ricci-symmetric $(SWRS)_n$ manifold if

$$
(1.4) \qquad (\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(X,Y),
$$

where α is a 1-form and is defined by

$$
(1.5) \qquad \alpha(X) = g(X, \rho),
$$

where ρ is the associated vector field.

We are the following known result.

Lemma 1.1. *[13] If $M : g = c$ is a surface in R^n, then the gradient vector field is a non-vanishing normal vector field on the entire surface M.*

2. LP-Sasakian manifold

A differentiable manifold of dimensional n(odd) is called LP-Sasakian manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy:

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

for all $X, Y \in TM$.

Also LP-Sasakian manifold M^n satisfies

$$(\nabla_X \phi)Y = \{g(X,Y)\xi + 2\eta(Y)\eta(X)\xi\}, \tag{2.3}$$

$$\nabla_X \xi = \phi X, \tag{2.4}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Example of LP-Sasakian manifold: Consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3; z \neq 0\}$, where (x,y,z) are the standard co-ordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M^n given by

$$E_1 = \frac{e^z}{x}\frac{\partial}{\partial x}, \quad E_2 = \frac{e^{z-ax}}{y}\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}. \tag{2.5}$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1,E_2) &= g(E_1,E_3) = g(E_2,E_3) = 0, \\ g(E_1,E_1) &= g(E_2,E_2) = 1 \text{ and } g(E_3,E_3) = -1. \end{aligned}$$

The (ϕ, ξ, η) is given by

$$\begin{aligned} &\eta = -dz, \quad \xi = E_3 = \frac{\partial}{\partial z}, \\ &\phi E_1 = -E_1, \quad \phi E_2 = -E_2, \quad \phi E_3 = 0. \end{aligned}$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= -1, \quad \phi^2 U = U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U,W) + \eta(U)\eta(W), \quad g(U,\xi) = \eta(U), \end{aligned}$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1,E_2] = -\frac{ae^z}{x}E_2, \quad [E_1,E_3] = -E_1, \quad [E_2,E_3] = -E_2. \tag{2.6}$$

The Levi-Civita connection with respect to above metric g is given by Koszula forumula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Then we have,

$$\begin{array}{lll} \nabla_{E_1}E_1 = -E_3, & \nabla_{E_1}E_2 = 0, & \nabla_{E_1}E_3 = -E_1, \\ \nabla_{E_2}E_1 = \dfrac{ae^z}{x}E_2, & \nabla_{E_2}E_2 = -\dfrac{ae^z}{x}E_1 - E_3, & \nabla_{E_2}E_3 = -E_2, \\ \nabla_{E_3}E_1 = 0, & \nabla_{E_3}E_2 = 0, & \nabla_{E_3}E_3 = 0. \end{array}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations $(2.1), (2.2), (2.3)$ and (2.4). Thus M^n is LP-Sasakian manifold.

Also, in LP-Sasakian manifold M^n the following relations hold:

$$\eta(R(X,Y)Z) = \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.7}$$

$$R(X,Y)\xi = \{\eta(Y)X - \eta(X)Y\}, \tag{2.8}$$

$$R(\xi,X)Y = \{g(X,Y)\xi - \eta(Y)X\}, \tag{2.9}$$

$$R(\xi,X)\xi = \{\eta(X)\xi + X\}, \tag{2.10}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.11}$$

$$Q\xi = (n-1)\xi, \tag{2.12}$$

for any vector fields X, Y, Z, where $R(X,Y)Z$ is the curvature tensor and S is the Ricci tensor.

3. Semi-symmetric semi-metric connection

A. Friedmann and J.A. Schouten [5] introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{3.1}$$

where η is a 1-form. Motivated by studies of author in [1], introduced the notion of semi-symmetric semi-metric connection $\widetilde{\nabla}$ on a contact metric manifold and it is defined as

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi, \tag{3.2}$$

where ∇ is Levi-Civita connection. A study on semi-symmetric connections and their properties can be found in [20, 3, 5, 7]. More recently, Mobin Ahmad and M. Danish Siddiqui [1] have studied a nearly Sasakian manifold with a semi-symmetric semi-metric connection, proving the results of integrability conditions of distribution of semi-invariant submanifolds of an approximately Sasakian manifold, inspired by research done by the author in [1]. Our focus is on LP-Sasakian manifolds that exhibit weakly symmetry.

A relation between the curvature tensor of M^n with respect to the semi-symmetric semi-metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ is given by

$$\begin{aligned}(3.3)\ \widetilde{R}(X,Y)Z &= R(X,Y)Z + 2[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\xi + [g(X,\phi Y) \\ &- g(Y,\phi X)]Z + [g(Y,Z)\phi X - g(X,Z)\phi Y],\end{aligned}$$

where $\widetilde{R}$ and R are the Riemannian curvatures of the connections $\widetilde{\nabla}$ and ∇ respectively. From (3.3), it follows that

$$(3.4)\qquad \widetilde{S}(Y,Z) = S(Y,Z) + 2\eta(Y)\eta(Z) + 2g(Y,Z) - g(Z,\phi Y) + Tg(Y,Z),$$

where $T = trace\phi = g(\phi e_i, e_i)$, $\widetilde{S}$ and S are the Ricci tensors of the connections $\widetilde{\nabla}$ and ∇ respectively.

Taking Z instead of ξ, the above expression becomes

$$(3.5)\qquad \widetilde{S}(Y,\xi) = [(n-1) + T]\eta(Y).$$

4. Weakly symmetric LP-Sasakian manifold admitting semi-symmetric semi-metric connection

Let $\widetilde{M^n}$ denote LP-Sasakian manifold admitting semi-symmetric semi-metric connection. Let $\widetilde{M^n}$ be weakly symmetric. Then equation (1.2) may be written as

$$\begin{aligned}(4.1)\qquad (\widetilde{\nabla}_X\widetilde{S})(Z,V) &= \alpha(X)\widetilde{S}(Z,V) + \gamma(Z)\widetilde{S}(X,V) + \delta(V)\widetilde{S}(Z,X) \\ &+ \beta(\widetilde{R}(X,Z)V) + \sigma(\widetilde{R}(X,V)Z).\end{aligned}$$

Taking covariant differentiation of the Ricci tensor $\widetilde{S}$ with respect to X, we have

$$(4.2)\qquad (\widetilde{\nabla}_X\widetilde{S})(Z,V) = \widetilde{\nabla}_X\widetilde{S}(Z,V) - \widetilde{S}(\widetilde{\nabla}_X Z, V) - \widetilde{S}(Z,\widetilde{\nabla}_X V).$$

Putting $V = \xi$ in (4.2) and by virtue of (2.1), (2.4), (2.11), (3.2), (3.4), we find

$$\begin{aligned}(4.3)\qquad (\widetilde{\nabla}_X\widetilde{S})(Z,\xi) &= (n-1)\eta(\nabla_X Z) - (n-1)\eta(X)\eta(Z) - (n-1)g(X,Z) \\ &+(n-1)g(Z,\phi X) + X(T)\eta(Z) + T\eta(\nabla_X Z) - T\eta(X)\eta(Z) \\ &-Tg(X,Z) + Tg(Z,\phi X) + (n-1)g(Z,\phi X) - S(Z,\phi X) \\ &-2g(Z,\phi X) + g(Z,X) + \eta(X)\eta(Z).\end{aligned}$$

On the other hand replacing V with ξ in (4.1) and use (2.1), (2.11), (3.3), (3.4), (3.5), we immediately obtain

$$(4.4)\ (\widetilde{\nabla}_X\widetilde{S})(Z,\xi) = [(n-1) + T]\alpha(X)\eta(Z) + [(n-1) + T]\gamma(Z)\eta(X)$$

$$
\begin{aligned}
&+ \quad \delta(\xi)S(Z,X) + 2\delta(\xi)\eta(Z)\eta(X) + 2\delta(\xi)g(Z,X) \\
&- \quad \delta(\xi)g(X,\phi Z) + T\delta(\xi)g(Z,X) + \eta(Z)\beta(X) \\
&- \quad \eta(X)\beta(Z) + g(X,\phi Z)\beta(\xi) - g(Z,\phi X)\beta(\xi) \\
&+ \quad \eta(Z)\beta(\phi X) - \eta(X)\beta(\phi Z) + \eta(Z)\sigma(X) - g(X,Z)\sigma(\xi) \\
&- \quad 2g(X,Z)\sigma(\xi) - 2\eta(X)\eta(Z)\sigma(\xi) + \eta(Z)\sigma(\phi X).
\end{aligned}
$$

Hence, comparing the right hand side of the equations (4.3) and (4.4), we get

$$
\begin{aligned}
(4.5)\qquad & (n-1)\eta(\nabla_X Z) - (n-1)\eta(X)\eta(Z) - (n-1)g(X,Z) + (n-1)g(Z,\phi X) \\
& +X(T)\eta(Z) + T\eta(\nabla_X Z) - T\eta(X)\eta(Z) - Tg(X,Z) + Tg(Z,\phi X) \\
& +(n-1)g(Z,\phi X) - S(Z,\phi X) - 2g(Z,\phi X) + g(Z,X) + \eta(X)\eta(Z) \\
& = [(n-1)+T]\alpha(X)\eta(Z) + [(n-1)+T]\gamma(Z)\eta(X) + \delta(\xi)S(Z,X) \\
& +2\delta(\xi)\eta(Z)\eta(X) + 2\delta(\xi)g(Z,X) - \delta(\xi)g(X,\phi Z) + T\delta(\xi)g(Z,X) \\
& +\eta(Z)\beta(X) - \eta(X)\beta(Z) + g(X,\phi Z)\beta(\xi) - g(Z,\phi X)\beta(\xi) \\
& +\eta(Z)\beta(\phi X) - \eta(X)\beta(\phi Z) + \eta(Z)\sigma(X) - g(X,Z)\sigma(\xi) \\
& -2g(X,Z)\sigma(\xi) - 2\eta(X)\eta(Z)\sigma(\xi) + \eta(Z)\sigma(\phi X).
\end{aligned}
$$

Plugging $Z = \xi$ in (4.5) and using these equations (2.1), (2.4), (2.11), we get the equation

$$
\begin{aligned}
(4.6)\qquad -X(T) &= -[(n-1)+T]\alpha(X) + [(n-1)+T]\gamma(\xi)\eta(X) \\
&+ [(n-1)+T]\delta(\xi)\eta(X) - \beta(X) - \eta(X)\beta(\xi) \\
&- \beta(\phi X) - \sigma(X) - \eta(X)\sigma(\xi) - \sigma(\phi X).
\end{aligned}
$$

At this stage we can't give any geometric meaning to this equation. If we take $X = \xi$, then

$$
\begin{aligned}
\xi(T) &= [(n-1)+T][\alpha(\xi)+\gamma(\xi)+\delta(\xi)], \\
(4.7)\qquad i.e,\quad gradT.\xi &= [(n-1)+T][\alpha(\xi)+\gamma(\xi)+\delta(\xi)].
\end{aligned}
$$

Since $[(n-1)+T] \neq 0$, we have $gradT$ is normal to ξ if and only if $[\alpha(\xi) + \gamma(\xi) + \delta(\xi)] = 0$.

Thus by Lemma 1.1 we can state the following:

Theorem 4.1. *Let $\widetilde{M}^n$ be weakly symmetric LP-Sasakian manifold with respect to semi-symmetric semi-metric connection. Then the sum of 1-forms α, γ and δ on vanish on the characteristic vector field ξ if and only if the gradient of trace of the endomorphism ϕ is normal to M^n along ξ.*

5. On special weakly Ricci-symmetric LP-Sasakian manifold admitting semi-symmetric semi-metric connection

Let $\widetilde{M}^n$ be special weakly Ricci-symmetric LP-Sasakian manifold. Then (1.4) may be written as

$$
(5.1)\qquad (\widetilde{\nabla}_X \widetilde{S})(Y,Z) = 2\alpha(X)\widetilde{S}(Y,Z) + \alpha(Y)\widetilde{S}(X,Z) + \alpha(Z)\widetilde{S}(X,Y).
$$

Taking cyclic sum of (5.1). This implies that

$$(\widetilde{\nabla}_X\widetilde{S})(Y,Z)+(\widetilde{\nabla}_Y\widetilde{S})(Z,X)+(\widetilde{\nabla}_Z\widetilde{S})(X,Y) = 4[\alpha(X)\widetilde{S}(Y,Z)+\alpha(Y)\widetilde{S}(Z,X)+\alpha(Z)\widetilde{S}(X,Y)]. \tag{5.2}$$

Let $\widetilde{M}^n$ admit a cyclic Ricci tensor. Then (5.2) reduces to

$$0=\alpha(X)\widetilde{S}(Y,Z)+\alpha(Y)\widetilde{S}(Z,X)+\alpha(Z)\widetilde{S}(X,Y). \tag{5.3}$$

Now setting $Z=\xi$ in (5.3) and yield (2.1), (3.4), (3.5), we get

$$\begin{aligned}0 &= [(n-1)+T]\alpha(X)\eta(Y)+[(n-1)+T]\alpha(Y)\eta(X)+\alpha(\xi)S(X,Y)\\ &+ 2\alpha(\xi)\eta(X)\eta(Y)+2\alpha(\xi)g(X,Y)-\alpha(\xi)g(Y,\phi X)+T\alpha(\xi)g(X,Y).\end{aligned} \tag{5.4}$$

Again setting $Y=\xi$ in (5.4) and employ (1.5) and (2.1), we obtain

$$2\eta(\rho)\eta(X)=\alpha(X). \tag{5.5}$$

Changing X to ξ in (5.5) and make use of (1.5) and (2.1), it follows that

$$\eta(\rho)=0. \tag{5.6}$$

By virtue of (5.6) in (5.5), we procure $\alpha(X)=0$, for all X.
This lead us to the following

Theorem 5.1. *Let $\widetilde{M}^n$ be special weakly Ricci-symmetric LP-Sasakian manifold M^n with respect to semi-symmetric semi-metric connection and admits a cyclic Ricci tensor. Then the 1-form α must vanish on M^n. However the converse holds trivially.*

Next setting $Z=\xi$ in (5.1), we have the following

$$(\widetilde{\nabla}_X\widetilde{S})(Y,\xi)=2\alpha(X)\widetilde{S}(Y,\xi)+\alpha(Y)\widetilde{S}(X,\xi)+\alpha(\xi)\widetilde{S}(X,Y). \tag{5.7}$$

The left hand side can be written in the form

$$(\widetilde{\nabla}_X\widetilde{S})(Y,\xi)=\widetilde{\nabla}_X\widetilde{S}(Y,\xi)-\widetilde{S}(\widetilde{\nabla}_XY,\xi)-\widetilde{S}(Y,\widetilde{\nabla}_X\xi). \tag{5.8}$$

By view of (1.5), (2.1), (2.11), (3.2), (3.4), (3.5), we infer that

$$\begin{aligned}&(n-1)\eta(\nabla_XY)-(n-1)\eta(X)\eta(Y)-(n-1)g(X,Y)+(n-1)g(Y,\phi X)\\ &+X(T)\eta(Y)+T\eta(\nabla_XY)-T\eta(X)\eta(Y)-Tg(X,Y)+Tg(Y,\phi X)\\ &+(n-1)g(Y,\phi X)-S(Y,\phi X)-2g(Y,\phi X)+g(Y,X)+\eta(X)\eta(Y)\\ &=2[(n-1)+T]\alpha(X)\eta(Y)+[(n-1)+T]\alpha(Y)\eta(X)\\ &+\eta(\rho)\{S(X,Y)+2\eta(X)\eta(Y)+2g(X,Y)-g(Y,\phi X)+Tg(X,Y)\}.\end{aligned} \tag{5.9}$$

Choosing $Y = \xi$ in (5.9) and utilize (1.5) and (2.1), (2.4), (2.11), gives

$$\begin{aligned} (5.10) \qquad -X(T) &= -2[(n-1)+T]\alpha(X) + 2[(n-1)+T]\eta(\rho)\eta(X), \\ (5.11) \qquad i.e, \quad X(T) &= 2[\eta(\rho)\eta(X) - \alpha(X)][(n-1)+T]. \end{aligned}$$

We know that $X(T) = gradT \cdot X$. Since $[(n-1)+T] \neq 0$. $gradT$ is normal to M^n if and only if $\eta(\rho)\eta(X) = \alpha(X)$.
Hence we state the following lemma 1.1.

Theorem 5.2. *Let $\widetilde{M}^n$ be special weakly Ricci-symmetric LP-Sasakian manifold M^n with respect to semi-symmetric semi-metric connection. Then the gradient of the trace of the endomorphism of T is normal to M^n if and only if $\eta(\rho)\eta(X) = \alpha(X)$.*

If we put $X = \xi$ in $\eta(\rho)\eta(X) = \alpha(X)$, then $\eta(\rho) = 0$. Thus $\alpha(X) = 0$.

Hence we can restate the Theorem 5.2 as follows:

Corollary 5.1. *Let $\widetilde{M}^n$ be special weakly symmetric LP-Sasakian manifold with respect to semi-symmetric semi-metric connection. Then the gradient of the trace of the endomorphism of T is normal to M^n along ξ if and only if 1-form vanish on the whole space M^n.*

We conclude from the above results:

Conclusion: If $\widetilde{M}^n$ is weakly symmetric LP-Sasakian manifold, then the sum of the 1-forms vanish along the characteristic vector field ξ if and only if the trace of endomorphism of ϕ is normal to M^n along ξ, whereas if $\widetilde{M}^n$ is special weakly Ricci-symmetric then the 1-form vanishes for every vector field if and only if trace of endomorphism ϕ is normal to M^n along ξ. If $\widetilde{M}^n$ admits cyclic Ricci tensor then the 1-form vanish the whole manifold M^n without any endomorphism.

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MANNHEIM PARTNER TRAJECTORIES RELATED TO PAFORS

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Abstract. In this study, we consider the concept of Mannheim partner trajectories related to the Positional Adapted Frame on Regular Surfaces (PAFORS) for the particles moving on the different regular surfaces in Euclidean 3-space. We give the relations between the PAFORS elements of these aforementioned trajectories. Also, we obtain the relations between Darboux basis vectors of these trajectories. Furthermore, some special cases of these trajectories are written.
Keywords: Mannheim partner trajectories, Positional Adapted Frame on Regular Surfaces, Darboux basis vectors.

1. Introduction

The surface theory is one of the most popular fundamental areas in differential geometry although its history is very long. The well-known moving frame Frenet-Serret frame has played an important role in the development of this theory. The steps which are performed by Frenet and Serret helped to adapt the moving frames to the curves on regular surfaces. This success was achieved by French mathematician Darboux [3]. He constructed a moving frame that is called today as Darboux frame for surface curves. Darboux frame is well-defined at every non-umbilic point

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of a surface. Therefore, it exists at every point of a regular surface curve [3, 15, 22]. Darboux frame has been used as a convenient tool for discussing many topics in the surface theory. Until today, a lot of researchers have performed many significant studies on the surface theory by means of Darboux frame. In [6, 11, 20, 26, 27], one can easily find some of these studies.

Another popular area in differential geometry is the curve theory. The concept of the special curves is an important part of this theory. In Euclidean 3-space E^3, curve pairs like Mannheim curve pairs are well-known examples of special curves. The topic of moving frames has an important place in the investigation of the local theory of these kinds of curve pairs. Developing new moving frames has always been an important effort for mathematicians. The groundbreaking discovery in this regard is the discovery of the Frenet-Serret frame, as everyone will agree. Most of the moving frames developed later include one of the basis vectors of the Frenet-Serret frame in common. Bishop frame [1], type 2-Bishop frame [29], type 3-Bishop frame [25], q-frame [5], Flc-frame [4], N-C-W frame [23], N-Bishop frame [10] can be given as examples to them. Similar to these moving frames, recently, Özen and Tosun have introduced a new moving frame on regular surfaces in Euclidean 3-space which is shortly called PAFORS by using the Darboux frame for the trajectories with non-vanishing angular momentum [17]. The authors have followed similar steps followed in the study [18] to construct this frame. The same authors also give some characterizations on asymptotic, slant helical, and geodesic trajectories with respect to PAFORS in the study [19]. Then, the idea of this new frame has been expanded to the Minkowski 3-space by Gürbüz in the study [8]. Gürbüz has taken into consideration the evolution of an electric field according to PAFORS in Minkowski 3-space in the aforementioned study.

Mannheim partner curves (according to Frenet-Serret frame) are interesting and popular special curves. The principal normal line of one of these partners matches up with the binormal line of the other partner at the corresponding points of them. Mannheim carried out the first study in 1878 on this topic [2, 13]. In the early 2000s, Mannheim partner curves were studied by Liu and Wang [12, 28]. In [12], the authors specified the necessary and sufficient conditions for a curve to possess a Mannheim partner curve in Euclidean 3-space and Minkowski 3-space. Then, Mannheim offsets of ruled surfaces were defined in [16]. On the other hand, dual Mannheim curves were discussed [7] and [21]. Another thing that can be of importance is that this topic was expanded to different frames such as Darboux frame and Bishop frame. Kazaz et al. [9] determined the Mannheim partner D-curves taking into consideration the Darboux frames of the curves on surfaces. Similar to this study, Masal and Azak investigated the Mannheim B-curves utilizing the Bishop frame [14].

In this paper, we investigate Mannheim partner trajectories related to PAFORS. Firstly, in Section 2, we mention the necessary information to understand the ensuing sections. In Section 3, Mannheim partner trajectories related to PAFORS are defined, and the relations between the PAFORS elements of these trajectories are given. Also, the relations between Darboux basis vectors of these trajectories are obtained. Moreover, some special cases of these trajectories are characterized according to PAFORS curvatures of these trajectories. Then, we give conclusions in Section 4.

2. Preliminaries

In this section, we remind some required terminology used throughout this paper.

In E^3, the standard inner product of any two vectors $\mathcal{W} = (w_1, w_2, w_3)$ and $\mathcal{X} = (x_1, x_2, x_3)$ are expressed as $\langle \mathcal{W}, \mathcal{X} \rangle = w_1x_1 + w_2x_2 + w_3x_3$. Based on this equality, the norm of the vector $\mathcal{W}$ is given by $\|\mathcal{W}\| = \sqrt{\langle \mathcal{W}, \mathcal{W} \rangle} = \sqrt{w_1^2 + w_2^2 + w_3^2}$. On the other hand, for a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$, if the condition $\|{}^{d\alpha}/_{ds}\| = 1$ for all $s \in I$ is satisfied, α is called a unit speed curve. In such a case, the parameter s is said to be an arc-length parameter of α. Also, if the derivative of a differentiable curve does not equal to zero everywhere along this curve, it is called a regular curve. Any regular curve always has a unit speed parameterization [24]. We must emphasize that the symbol prime $'$ will be used to show the differentiation with respect to the arc-length parameter s in the rest of this study.

The researchers generally make use of the Frenet-Serret frame to investigate many properties of regular curves. However, if these regular curves lie on regular surfaces, then using the Darboux frame offers more possibilities than the Frenet-Serret frame.

Let us suppose that a particle R moves on a regular surface M in the Euclidean 3-space along the trajectory $\alpha = \alpha(s)$ that is a unit speed curve. Thus, we can express α as $\alpha : I \subset \mathbb{R} \to M \subset \mathbb{E}^3$. The base vectors of the Darboux frame of the trajectory α are presented as $\{\mathbf{T}(s), \mathbf{Y}(s), \mathbf{U}(s)\}$ along α where $\mathbf{T}$ is called the unit tangent vector, $\mathbf{U}$ is called the unit normal vector. The remaining basis vector $\mathbf{Y}$ of the Darboux frame is found by means of the equality $\mathbf{Y} = \mathbf{U} \times \mathbf{T}$. It should be specified that the second-order derivatives of the curves, which we will consider in this article, are always non-zero (it means that $\alpha''(s)$ is zero nowhere). For Darboux frame, the derivative formulas are constructed as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{Y}'(s) \\ \mathbf{U}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & \tau_g(s) \\ -k_n(s) & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix},$$

where k_g is geodesic curvature, k_n is normal curvature and τ_g is geodesic torsion of the curve α [6, 15].

Assume that the angular momentum vector of the aforesaid particle R about the origin does not vanish during the motion. In that case, PAFORS $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ is well defined along the trajectory $\alpha = \alpha(s)$. The base vectors of PAFORS are given as follows:

$$\begin{cases} \mathbf{T}(s) = \mathbf{T}(s), \\ \mathbf{G}(s) = \dfrac{\langle \alpha(s), \mathbf{U}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{Y}(s) + \dfrac{\langle \alpha(s), \mathbf{Y}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{U}(s), \\ \mathbf{H}(s) = \dfrac{\langle -\alpha(s), \mathbf{Y}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{Y}(s) + \dfrac{\langle \alpha(s), \mathbf{U}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{U}(s). \end{cases}$$

The relation between the Darboux frame and PAFORS exists as follows:

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{H}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(s) & -\sin\varphi(s) \\ 0 & \sin\varphi(s) & \cos\varphi(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix}. \tag{2.1}$$

Here, $\varphi(s)$ is the angle between the vectors $\mathbf{Y}(s)$ and $\mathbf{G}(s)$ that is positively oriented from $\mathbf{Y}(s)$ to $\mathbf{G}(s)$ [17].
Furthermore, the derivative formulas of PAFORS are given as in the following [17]:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{G}'(s) \\ \mathbf{H}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{G}(s) \\ \mathbf{H}(s) \end{pmatrix}, \tag{2.2}$$

where

$$\begin{cases} k_1(s) = k_g(s)\cos\varphi(s) - k_n(s)\sin\varphi(s), \\ k_2(s) = k_g(s)\sin\varphi(s) + k_n(s)\cos\varphi(s), \\ k_3(s) = \tau_g(s) - \varphi'(s). \end{cases}$$

Additionally, the rotation angle $\varphi(s)$ is calculated by using the following equation [17]:

$$\varphi(s) = \begin{cases} \arctan\left(-\frac{\langle \alpha(s), \mathbf{Y}(s) \rangle}{\langle \alpha(s), \mathbf{U}(s) \rangle}\right) \quad if \quad \langle \alpha(s),\, \mathbf{U}(s) \rangle > 0, \\ \arctan\left(-\frac{\langle \alpha(s), \mathbf{Y}(s) \rangle}{\langle \alpha(s), \mathbf{U}(s) \rangle}\right) + \pi \quad if \quad \langle \alpha(s),\, \mathbf{U}(s) \rangle < 0, \\ -\frac{\pi}{2} \quad if \quad \langle \alpha(s), \mathbf{U}(s) \rangle = 0\ , \quad \langle \alpha(s),\, \mathbf{Y}(s) \rangle > 0, \\ \frac{\pi}{2} \quad if \quad \langle \alpha(s), \mathbf{U}(s) \rangle = 0\ , \quad \langle \alpha(s),\, \mathbf{Y}(s) \rangle < 0. \end{cases}$$

Also, the elements of the set $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s), k_1(s), k_2(s), k_3(s)\}$ are called as PAFORS apparatuses of the trajectory $\alpha = \alpha\,(s)$ [17].

In order to remind the asymptotic curve and geodesic curve, we can present the following conditions [15]:

1. $k_n = 0$ if and only if $\alpha = \alpha(s)$ is an asymptotic curve.
2. $k_g = 0$ if and only if $\alpha = \alpha(s)$ is a geodesic curve.

Theorem 2.1. [19] *Suppose that $\alpha = \alpha(s)$ is an asymptotic curve on the regular surface M with the condition $k_g \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ if and only if $k_2 = 0$.*

Theorem 2.2. [19] *Assume that $\alpha = \alpha(s)$ is an asymptotic curve on the regular surface M with the condition $k_g \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ if and only if $k_1 = 0$.*

Theorem 2.3. [19] *Suppose that $\alpha = \alpha(s)$ is a geodesic curve on the regular surface M with the condition $k_n \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ if and only if $k_1 = 0$.*

Theorem 2.4. [19] *Assume that $\alpha = \alpha(s)$ is a geodesic curve on the regular surface M with the condition $k_n \neq 0$. Then, $\alpha = \alpha(s)$ is a curve whose position vector lies on the corresponding plane $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ if and only if $k_2 = 0$.*

For more detailed and comprehensive information about PAFORS, see [8, 17, 19].

3. Mannheim Partner Trajectories Related to PAFORS Lying on Different Regular Surfaces

In this section of this study, we introduce the Mannheim partner trajectories related to PAFORS and obtain some characterizations and geometric interpretations of them.

Definition 3.1. Let R and $\widehat{R}$ be the moving point particles on regular surfaces M and $\widehat{M}$ in Euclidean 3-space E^3. Let us show the unit speed parametrization of the trajectories of R and $\widehat{R}$ with $\alpha = \alpha(s)$ and $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$, respectively. Let $\{\mathbf{T}, \mathbf{G}, \mathbf{H}, k_1, k_2, k_3\}$ and $\left\{\widehat{\mathbf{T}}, \widehat{\mathbf{G}}, \widehat{\mathbf{H}}, \widehat{k}_1, \widehat{k}_2, \widehat{k}_3\right\}$ represent the PAFORS apparatus of the trajectories α and $\widehat{\alpha}$, respectively. If the PAFORS base vector $\mathbf{G}$ coincides with the PAFORS base vector $\widehat{\mathbf{H}}$ at the corresponding points of the trajectories α and $\widehat{\alpha}$, $\widehat{\alpha}$ is said to be a Mannheim partner trajectory of α related to PAFORS. Additionally, the pair $\{\alpha, \widehat{\alpha}\}$ is called a Mannheim pair related to PAFORS.

With the help of the definition of Mannheim pair related to PAFORS, we can give the following equation:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ 0 & 0 & 1 \\ -\sin\psi & \cos\psi & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{T}} \\ \widehat{\mathbf{G}} \\ \widehat{\mathbf{H}} \end{pmatrix}, \tag{3.1}$$

where ψ is the angle between the tangent vectors $\mathbf{T}$ and $\widehat{\mathbf{T}}$.

Theorem 3.1. *Suppose that $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ is any Mannheim pair related to PAFORS. Then, the distance between the corresponding points of α and $\widehat{\alpha}$ is constant.*

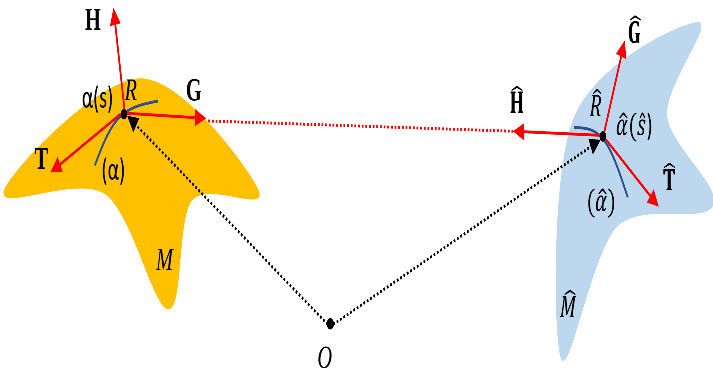

FIG. 3.1: Mannheim partner trajectories related to PAFORS

Proof. According to the definition of Mannheim trajectories related to PAFORS, the following equation can be given:

$$\alpha(s) = \widehat{\alpha}(\widehat{s}) + \eta(\widehat{s})\,\widehat{\mathbf{H}}(\widehat{s}), \tag{3.2}$$

where η is a real valued smooth function of $\widehat{s}$ (cf. Figure 3.1). Differentiating the equation (3.2) with respect to $\widehat{s}$ and using the equation (2.2), we have:

$$\mathbf{T}\frac{ds}{d\widehat{s}} = \left(1 - \eta\widehat{k}_2\right)\widehat{\mathbf{T}} - \eta\widehat{k}_3\widehat{\mathbf{G}} + \eta'\widehat{\mathbf{H}}. \tag{3.3}$$

Since $\mathbf{T}, \widehat{\mathbf{T}}$ and $\widehat{\mathbf{G}}$ are orthogonal to $\widehat{\mathbf{H}}$, we have $\eta' = 0$ with the help of the inner product. Thus, η is a non-zero constant and then we can rewrite the equation (3.3) as follows:

$$\mathbf{T}\frac{ds}{d\widehat{s}} = \left(1 - \eta\widehat{k}_2\right)\widehat{\mathbf{T}} - \eta\widehat{k}_3\widehat{\mathbf{G}}. \tag{3.4}$$

Hence, the distance between the corresponding points of α and $\widehat{\alpha}$ can be written as follows:

$$d\left(\alpha(s), \widehat{\alpha}(\widehat{s})\right) = \|\alpha(s) - \widehat{\alpha}(\widehat{s})\| = \left\|\eta\widehat{\mathbf{H}}\right\| = |\,\eta\,|\,.$$

Therefore, we obtain the distance between each corresponding points of α and $\widehat{\alpha}$ as non-zero constant. □

Theorem 3.2. *Let $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ be any Mannheim pair related to PAFORS. In that case, the following equation is satisfied.*

$$\frac{d}{ds}(\cos\psi) = k_2\left\langle \mathbf{H}, \widehat{\mathbf{T}}\right\rangle + \widehat{k}_1\frac{d\widehat{s}}{ds}\left\langle \mathbf{T}, \widehat{\mathbf{G}}\right\rangle$$

Proof. Since ψ is the angle between the tangent vectors $\mathbf{T}$ and $\widehat{\mathbf{T}}$, we can write $\left\langle \mathbf{T}, \widehat{\mathbf{T}} \right\rangle = \|\mathbf{T}\| \left\|\widehat{\mathbf{T}}\right\| \cos\psi = \cos\psi$. If this equation is differentiated with respect to the parameter s, we obtain:

$$\begin{aligned} \frac{d}{ds}(\cos\psi) &= \frac{d}{ds}\left\langle \mathbf{T}, \widehat{\mathbf{T}} \right\rangle \\ &= \left\langle k_1\mathbf{G} + k_2\mathbf{H},\, \widehat{\mathbf{T}} \right\rangle + \left\langle \mathbf{T}, (\widehat{k}_1\widehat{\mathbf{G}} + \widehat{k}_2\widehat{\mathbf{H}})\frac{d\widehat{s}}{ds} \right\rangle \cdot \end{aligned}$$

Then, the last equation yields the desired result. $\square$

Corollary 3.1. *The angles between the tangent vectors at the corresponding points of a Mannheim pair (related to PAFORS) are generally not constant.*

Theorem 3.3. *Let $\{\alpha = \alpha(s), \ \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair related to PAFORS. Then, the following equation is satisfied:*

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds} & -\eta\widehat{k}_3\frac{d\widehat{s}}{ds} & 0 \\ 0 & 0 & 1 \\ \eta\widehat{k}_3\frac{d\widehat{s}}{ds} & \left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{T}} \\ \widehat{\mathbf{G}} \\ \widehat{\mathbf{H}} \end{pmatrix} \cdot \tag{3.5}$$

Proof. Let $\{\alpha, \widehat{\alpha}\}$ be a Mannheim pair related to PAFORS. With the help of the equations (3.1) and (3.4), we get:

$$\cos\psi\frac{ds}{d\widehat{s}}\widehat{\mathbf{T}} + \sin\psi\frac{ds}{d\widehat{s}}\widehat{\mathbf{G}} = \left(1-\eta\widehat{k}_2\right)\widehat{\mathbf{T}} - \eta\widehat{k}_3\widehat{\mathbf{G}}.$$

From the previous equation, we can write:

$$\begin{cases} \cos\psi = \left(1-\eta\widehat{k}_2\right)\dfrac{d\widehat{s}}{ds}, \\ \sin\psi = -\eta\widehat{k}_3\dfrac{d\widehat{s}}{ds} \cdot \end{cases} \tag{3.6}$$

Substituting the equation (3.6) in the equation (3.1), we have the equation (3.5). $\square$

Corollary 3.2. *The tangent of the angle between the unit tangent vectors of the Mannheim partner trajectories (related to PAFORS) $\alpha = \alpha(s)$ and $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$ is given as follows:*

$$\tan\psi = \frac{-\eta\widehat{k}_3}{1-\eta\widehat{k}_2} \cdot \tag{3.7}$$

Corollary 3.3. *Let $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair (related to PAFORS). In that case, the following equation is satisfied*

$$\int \cos\psi ds + \eta\int \widehat{k}_2 d\widehat{s} = \widehat{s} + c_1,$$

where c_1 shows the integration constant.

Corollary 3.4. *Let $\{\alpha=\alpha(s),\widehat{\alpha}=\widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair (related to PAFORS). Then, the following equation is satisfied.*

$$\int \sin\psi ds+\eta\int \widehat{k_3}d\widehat{s}=0$$

Theorem 3.4. *Let $\{\alpha=\alpha(s),\widehat{\alpha}=\widehat{\alpha}(\widehat{s})\}$ be a Mannheim pair related to PAFORS and their Darboux frame be denoted by $\{\mathbf{T},\mathbf{Y},\mathbf{U}\}$ and $\left\{\widehat{\mathbf{T}},\widehat{\mathbf{Y}},\widehat{\mathbf{U}}\right\}$, respectively. In that case, the relations between the Darboux base vectors of this pair are given by*

$$\begin{aligned}
\widehat{\mathbf{T}}&=\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}\mathbf{T}-\eta\widehat{k}_3\sin\varphi\frac{d\widehat{s}}{ds}\mathbf{Y}-\eta\widehat{k}_3\cos\varphi\frac{d\widehat{s}}{ds}\mathbf{U},\\
\widehat{\mathbf{Y}}&=\eta\widehat{k}_3\sin\widehat{\varphi}\frac{d\widehat{s}}{ds}\mathbf{T}+\left(\cos\widehat{\varphi}\cos\varphi+\left(1-\eta\widehat{k}_2\right)\sin\widehat{\varphi}\sin\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{Y}\\
&\quad+\left(-\cos\widehat{\varphi}\sin\varphi+\left(1-\eta\widehat{k}_2\right)\sin\widehat{\varphi}\cos\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{U},\\
\widehat{\mathbf{U}}&=\eta\widehat{k}_3\cos\widehat{\varphi}\frac{d\widehat{s}}{ds}\mathbf{T}+\left(-\sin\widehat{\varphi}\cos\varphi+\left(1-\eta\widehat{k}_2\right)\cos\widehat{\varphi}\sin\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{Y}\\
&\quad+\left(\sin\widehat{\varphi}\sin\varphi+\left(1-\eta\widehat{k}_2\right)\cos\widehat{\varphi}\cos\varphi\frac{d\widehat{s}}{ds}\right)\mathbf{U},
\end{aligned}$$

where φ is the angle between the vectors $\mathbf{U}$ and $\mathbf{H}$ and also, $\widehat{\varphi}$ is the angle between the vectors $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{H}}$.

Proof. With the help of the equation (2.1), the following equations

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos\varphi&-\sin\varphi\\0&\sin\varphi&\cos\varphi\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{Y}\\ \mathbf{U}\end{pmatrix}\tag{3.8}$$

and

$$\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{Y}}\\ \widehat{\mathbf{U}}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos\widehat{\varphi}&\sin\widehat{\varphi}\\0&-\sin\widehat{\varphi}&\cos\widehat{\varphi}\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{G}}\\ \widehat{\mathbf{H}}\end{pmatrix}\tag{3.9}$$

can be seen easily. Also, we can write the following equation according to the equation (3.5):

$$\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{G}}\\ \widehat{\mathbf{H}}\end{pmatrix}=\begin{pmatrix}\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}&0&\eta\widehat{k}_3\frac{d\widehat{s}}{ds}\\-\eta\widehat{k}_3\frac{d\widehat{s}}{ds}&0&\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}\\0&1&0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}.\tag{3.10}$$

Substituting the equation (3.10) in the equation (3.9) gives us the following:

$$\begin{pmatrix}\widehat{\mathbf{T}}\\ \widehat{\mathbf{Y}}\\ \widehat{\mathbf{U}}\end{pmatrix}=\begin{pmatrix}\left(1-\eta\widehat{k}_2\right)\frac{d\widehat{s}}{ds}&0&\eta\widehat{k}_3\frac{d\widehat{s}}{ds}\\-\eta\widehat{k}_3\cos\varphi\frac{d\widehat{s}}{ds}&\sin\widehat{\varphi}&\left(1-\eta\widehat{k}_2\right)\cos\widehat{\varphi}\frac{d\widehat{s}}{ds}\\\eta\widehat{k}_3\sin\widehat{\varphi}\frac{d\widehat{s}}{ds}&\cos\widehat{\varphi}&-\left(1-\eta\widehat{k}_2\right)\sin\widehat{\varphi}\frac{d\widehat{s}}{ds}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}.\tag{3.11}$$

If the equation (3.8) is considered in the equation (3.11), the desired equations are found. □

Theorem 3.5. *Let* $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair related to PAFORS. In that case, the following relations can be given:*

1. $k_1 = \dfrac{\widehat{k}_2 - \eta \widehat{k}_2^{\,2} - \eta \widehat{k}_3^{\,2}}{1 - 2\eta \widehat{k}_2 + \eta^2 \left(\widehat{k}_2^{\,2} + \widehat{k}_3^{\,2} \right)}$

2. $\widehat{k}_2 = \dfrac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2 \left(k_1^2 + k_3^2 \right)}$

where ξ *is a constant satisfying* $|\,\xi\,| = |\,\eta\,|$.

Proof. 1. Assume that $\{\alpha, \widehat{\alpha}\}$ is a Mannheim pair related to PAFORS. With the help of the well-known identity $\cos^2\psi + \sin^2\psi = 1$, we get:

$$\left(\frac{d\widehat{s}}{ds} \right)^2 \left(\left(1 - \eta \widehat{k}_2 \right)^2 + \eta^2 \widehat{k}_3^{\,2} \right) = 1$$

using the equation (3.6). Then, we can write:

$$\left(\frac{ds}{d\widehat{s}} \right)^2 = 1 - 2\eta \widehat{k}_2 + \eta^2 \left(\widehat{k}_2^{\,2} + \widehat{k}_3^{\,2} \right). \tag{3.12}$$

By differentiating the equation (3.4) according to the parameter $\widehat{s}$ and by using the equation (2.2), we have:

$$\begin{aligned} \frac{d^2 s}{d\widehat{s}^2}\mathbf{T} + k_1 \left(\frac{ds}{d\widehat{s}} \right)^2 \mathbf{G} + k_2 \left(\frac{ds}{d\widehat{s}} \right)^2 \mathbf{H} = & \left(-\eta \left(\widehat{k}_2 \right)' + \eta \widehat{k}_1 \widehat{k}_3 \right) \widehat{\mathbf{T}} \\ & + \left(\widehat{k}_1 \left(1 - \eta \widehat{k}_2 \right) - \eta \widehat{k}_3^{\,\prime} \right) \widehat{\mathbf{G}} \\ & + \left(\widehat{k}_2 \left(1 - \eta \widehat{k}_2 \right) - \eta \widehat{k}_3^{\,2} \right) \widehat{\mathbf{H}}. \end{aligned} \tag{3.13}$$

The last equation yields:

$$k_1 \left(\frac{ds}{d\widehat{s}} \right)^2 = \left(1 - \eta \widehat{k}_2 \right) \widehat{k}_2 - \eta \widehat{k}_3^{\,2}. \tag{3.14}$$

If we substitute the equation (3.12) in the equation (3.14), we get the desired result.

2. We can easily see the equality:

$$\widehat{\alpha}(\widehat{s}) = \alpha(s) + \xi \mathbf{G}(s)$$

where ξ is a constant satisfying $\mid \eta \mid = \mid \xi \mid$ (cf. Figure 3.1). Derivating this equation according to the s twice, we get:

$$\widehat{\mathbf{T}}\frac{d\widehat{s}}{ds} = (1-\xi k_1)\,\mathbf{T} + \xi k_3\mathbf{H} \tag{3.15}$$

and

$$\begin{aligned}\frac{d^2\widehat{s}}{ds^2}\widehat{\mathbf{T}} + \widehat{k}_1\left(\frac{d\widehat{s}}{ds}\right)^2\widehat{\mathbf{G}} + \widehat{k}_2\left(\frac{d\widehat{s}}{ds}\right)^2\widehat{\mathbf{H}} = & \left(-\xi k_1' - \xi k_2 k_3\right)\mathbf{T} \\ & + \left(k_1\left(1-\xi k_1\right) - \xi k_3^2\right)\mathbf{G} \\ & + \left(k_2\left(1-\xi k_1\right) + \xi k_3'\right)\mathbf{H}.\end{aligned} \tag{3.16}$$

By the equation (3.1), it can be seen that $\widehat{\mathbf{T}} = \cos\psi\mathbf{T} - \sin\psi\mathbf{H}$. Thus, we get:

$$\frac{d\widehat{s}}{ds}\cos\psi\mathbf{T} - \frac{d\widehat{s}}{ds}\sin\psi\mathbf{H} = (1-\xi k_1)\,\mathbf{T} + \xi k_3\mathbf{H}$$

and also $\frac{d\widehat{s}}{ds}\cos\psi = 1-\xi k_1$, $-\frac{d\widehat{s}}{ds}\sin\psi = \xi k_3$. From here we can write:

$$\left(\frac{d\widehat{s}}{ds}\right)^2 = 1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right). \tag{3.17}$$

The inner product of the vectors at the right and left sides of the equation (3.16) with the vector $\mathbf{G}$ gives us the following:

$$\widehat{k}_2\left(\frac{d\widehat{s}}{ds}\right)^2 = k_1 - \xi k_1^2 - \xi k_3^2. \tag{3.18}$$

Consequently, by using the equation (3.17), we have:

$$\widehat{k}_2 = \frac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right)}$$

and the proof is completed.

□

With the help of the Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 3.5, we can give the following corollaries.

Corollary 3.5. *Let* $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair (related to PAFORS). If* $\widehat{k}_2 = \widehat{k}_3 = 0$, *then* $k_1 = 0$.

Corollary 3.6. *Let* $\{\alpha = \alpha(s), \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair (related to PAFORS). If* $k_1 = k_3 = 0$, *then* $\widehat{k}_2 = 0$.

Corollary 3.7. *Let* $\{\alpha = \alpha(s), \ \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair related to PAFORS. Then, the followings are satisfied:*

1. *Suppose that the geodesic curvature of* α *never equals to zero. Then,* $\alpha = \alpha(s)$ *is an asymptotic curve whose position vector lies on the corresponding plane* $Sp\{\mathbf{T}(\mathbf{s}), \mathbf{Y}(\mathbf{s})\}$ *if and only if* $\dfrac{\widehat{k}_2 - \eta\widehat{k}_2^{\,2} - \eta\widehat{k}_3^{\,2}}{1 - 2\eta\widehat{k}_2 + \eta^2\left(\widehat{k}_2^{\,2} + \widehat{k}_3^{\,2}\right)} = 0.$

2. *Assume that the geodesic curvature of* $\widehat{\alpha}$ *never equals to zero. Then,* $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$ *is an asymptotic curve whose position vector lies on the corresponding plane* $Sp\{\widehat{\mathbf{T}}(\widehat{\mathbf{s}}), \widehat{\mathbf{U}}(\widehat{\mathbf{s}})\}$ *if and only if* $\dfrac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right)} = 0.$

Corollary 3.8. *Let* $\{\alpha = \alpha(s), \ \widehat{\alpha} = \widehat{\alpha}(\widehat{s})\}$ *be a Mannheim pair related to PAFORS. Then, the followings are satisfied:*

1. *Suppose that the normal curvature of* α *never equals to zero. Then,* $\alpha = \alpha(s)$ *is a geodesic curve whose position vector lies on the corresponding plane* $Sp\{\mathbf{T}(\mathbf{s}), \mathbf{U}(\mathbf{s})\}$ *if and only if* $\dfrac{\widehat{k}_2 - \eta\widehat{k}_2^{\,2} - \eta\widehat{k}_3^{\,2}}{1 - 2\eta\widehat{k}_2 + \eta^2\left(\widehat{k}_2^{\,2} + \widehat{k}_3^{\,2}\right)} = 0.$

2. *Assume that the normal curvature of* $\widehat{\alpha}$ *never equals to zero. Then,* $\widehat{\alpha} = \widehat{\alpha}(\widehat{s})$ *is a geodesic curve whose position vector lies on the corresponding plane* $Sp\{\widehat{\mathbf{T}}(\widehat{\mathbf{s}}), \widehat{\mathbf{Y}}(\widehat{\mathbf{s}})\}$ *if and only if* $\dfrac{k_1 - \xi k_1^2 - \xi k_3^2}{1 - 2\xi k_1 + \xi^2\left(k_1^2 + k_3^2\right)} = 0.$

4. Conclusions

The main purpose of this study is to lead the studies investigating the special classes of regular surface curves (traced out by a moving particle) by means of the new and convenient moving frame PAFORS. In accordance with this purpose, we choose the Mannheim partner curves which are well-known and preferred widely. We think this choice makes the study more remarkable.

In this study, Mannheim partner trajectories related to PAFORS are defined for the particles moving along the different regular surfaces in Euclidean 3-space. Also, the relations are given between the PAFORS elements of these aforementioned trajectories. Moreover, the relations are obtained between Darboux basis vectors of these trajectories, and some special cases of these trajectories are characterized.

We state that we plan to discuss the Bertrand partner trajectories related to PAFORS in the future study.

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VARIATIONS OF SEPARABILITY AND SUPERTIGHTNESS OF HYPERSPACES

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Abstract. For a Hausdorff non-compact space X, relationships between closure-type properties of the hyperspace $(\Lambda, \tau_{\Delta}^{+})$ and covering properties of that of X have been studied. We then investigate selective separability and some variations of this property. Finally supertightness of $(\Lambda, \tau_{\Delta}^{+})$ has been studied.
Keywords: Hausdorff space, compactness, separability, supertightness.

1. Introduction

In this paper we consider some stronger versions of separability in hyperspaces. In [27], Marion Scheepers introduced a general notation for selection principles as follows:

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets of an infinite set X. Then,
• $S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{b_n : n \in \mathbb{N}\}$ such that for each $n, b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.
• $S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ of finite sets such that for each $n, B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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If $\mathcal{A}$ and $\mathcal{B}$ stand for the family of all dense subsets of X (where we denote the set of all dense subsets of X by $\mathcal{D}$), then $S_{fin}(\mathcal{D}, \mathcal{D})$ is called the selective separability of X. I. Juhász and S. Shelah in their paper [13] proved that a compact space X has countable π-weight whenever every dense subspace of X is separable. Selective separability of X follows from countable π-weight of X and implies that all dense subspaces of X are separable. Therefore, the above-mentioned theorem of Juhász and Shelah implies that, in compact spaces, selective separability coincides with countable π-weight.

In [3], spaces X satisfying $S_{fin}(\mathcal{D}, \mathcal{D})$ or $S_1(\mathcal{D}, \mathcal{D})$ are called M-separable and R-separable, respectively. Also, X is said to be H-separable if for every sequence $\{D_n : n \in \mathbb{N}\}$ of elements of $\mathcal{D}$, one can pick finite $F_n \subset D_n$ so that for every nonempty open subset O of X, the intersection $O \cap F_n$ is nonempty for all but finitely many n. Naturally, M-, R-, and H-, are motivated by analogy with well-known Menger, Rothberger, and Hurewicz properties. Recall that X is Menger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exist finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, so that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ covers X; X is Rothberger if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exist $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, so that $\{U_n : n \in \mathbb{N}\}$ covers X; X is Hurewicz if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exist finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, so that for every $x \in X$, $x \in \bigcup \mathcal{V}_n$, for all but finitely many n. Also a family $\mathcal{P}$ of open sets in X is called a π-base for X if every nonempty open set in X contains a nonempty element of $\mathcal{P}$; where $\pi w(X) = \min\{|\mathcal{P}| : \mathcal{P}$ is a π-base for $X\}$ is the π-weight of X. The following implications are obvious:

$$\begin{tikzcd}
& & \text{R-separable} \arrow[dl] & \\
\text{Separable} & \text{M-separable} \arrow[l] & & \text{Countable } \pi\text{-weight} \arrow[ul] \arrow[dl] \\
& & \text{H-separable} \arrow[ul] &
\end{tikzcd}$$

Let us now recall some backgrounds of hyperspace topology. Given a Hausdorff non-compact space X, we denote the family of nonempty closed subsets (resp., closed subsets, compact subsets) of a topological space X by $CL(X)$ (resp., 2^X, $\mathbb{K}(X)$). For a subset $U \subset X$ and a family $\mathcal{U}$ of subsets of X, we write:

$$U^- = \{A \in CL(X) : A \cap U \neq \phi\},$$

$$U^+ = \{A \in CL(X) : A \subset U\},$$

$$U^c = X \setminus U,$$

$$\mathcal{U}^c = \{U^c : U \in \mathcal{U}\}.$$

The most known and popular among the topologies on 2^X are Fell topology and Vietoris topology. J. M. G. Fell [11] introduced a topology τ_F on 2^X having a subbase consisting of all sets of the form V^{-}, where V is an open subset of X plus all sets of the form $(K^{c})^{+}$, where K is a compact subset of X. The Fell topology τ_{F} has a basic open subset of the form $(\bigcap_{i=1}^{n} V_{i}^{-}) \cap (K^{c})^{+}$, where $V_{1}, V_{2}, ..., V_{n}$ are open subsets of X and K is a compact subset of X.

If compact subsets in the definition above are replaced by closed sets, we obtain the stronger Vietoris topology τ_{V} [21]. A basic open subset of the Vietoris topology τ_{V} is of the form: $< U_{1}, U_{2}, ..., U_{n} >= \{A \in 2^{X} : A \subset \bigcup_{i=1}^{n} U_{i}, A \cap U_{i} \neq \phi$, for $1 \leq i \leq n\}$, where $U_{1}, U_{2},, U_{n}$ are open subsets of X, for $n \in \mathbb{N}$.

Let Δ be a subset of 2^{X} closed for finite unions and containing all singletons. The upper Δ-topology, denoted by Δ^{+}, is the topology whose subbase is the collection $\{(D^{c})^{+} : D \in \Delta\} \cup \{2^{X}\}$. If Δ is the family of all finite subsets of X (resp., the collection of compact subsets of X), the corresponding Δ^{+}-topology known as co-finite topology (resp., co-compact topology) will be denoted by $\mathbf{Z}^{+}$ (resp., $\mathbf{F}^{+}$).

We have the inclusions: $\mathbf{Z}^{+} \subseteq \mathbf{F}^{+} \subseteq \tau_{F} \subseteq \tau_{V}$.

Let $\Delta \subseteq CL(X)$ be a subfamily of $CL(X)$ closed under finite unions and containing all singletons. Then, the hit-and-miss topology on $CL(X)$ with respect to Δ (first studied in the abstract in [23] and then in [7]), denoted by τ_{Δ}^{+}, has as a base, the family

$$\{(\bigcap_{i=1}^{m} V_{i}^{-}) \cap (B^{c})^{+} : B \in \Delta \text{ and } V_{i} \in \tau \text{ for } i \in \{1, 2, ..., m\},\ m \in \mathbb{N}\}.$$

Following [32], the basic element $(\bigcap_{i=1}^{m} V_{i}^{-}) \cap (B^{c})^{+}$ will be denoted by $(V_{1}, ..., V_{m})_{B}^{+}$.

Two important cases of the hit-and-miss topology are the Vietoris topology, τ_{V}, when $\Delta = CL(X)$ ([31], [21]) and the Fell topology, τ_{F}, when $\Delta = \mathbb{K}(X)$ ([11]).

By a cover, we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \cup\,\mathcal{U}$ and $X \notin \mathcal{U}$. k-covers and ω-covers play important roles in selection principles [2], [14], [15]. Different Δ-covers exposed many dualities in hyperspace topologies such as Fell topology, Vietoris topology, $\mathbf{Z}^{+}$, $\mathbf{F}^{+}$ ([5], [15], [16], [19], [10], [9], [8], [22], [26]).

Throughout the paper all spaces are assumed to be Hausdorff, non-compact. Along this paper, unless we say the opposite, we will take a family $\Lambda \subseteq CL(X)$ that is closed under finite unions. Also we shall use $[X]^{<\omega}$ to denote all finite subsets of X.

2. Definitions and Results

Let us recall that an open cover $\mathcal{U}$ of a space X is called an ω-cover [12] (respectively, a k-cover [20]) if every finite (respectively, compact) subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$. An open cover $\mathcal{U}$ of X is called a γ-cover [12] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Notice that it is equivalent to the assertion: Each finite subset of X belongs to all but finitely many members of $\mathcal{U}$. Also Lj. D. R. Kočinac in his paper [16] introduced a stronger version of γ-cover as: an open cover $\mathcal{U}$ of a space X is called a γ_k-cover of X if each compact subset of X is contained in all but finitely many elements of $\mathcal{U}$ and X is not a member of the cover.

For a space (X, τ) and a point $x \in X$ we use

- $\mathcal{O}$: the collection of open covers of X;
- Ω : the collection of ω-covers of X;
- $\mathcal{K}$: the collection of k-covers of X;
- Γ : the collection of all γ-covers of X;
- Γ_k : the collection of all γ_k-covers of X;
- $\Omega_x = \{A \subset X : x \in ClA\}$;
- $\mathcal{D}_\tau$: the collection of all dense subsets of the space (X, τ).

As $\mathbf{F}^+$ and $\mathbf{Z}^+$ are miss type hyperspace topologies, they are dual to k-covers and ω-covers in selection principles. The Fell topology and the Vietoris topology are hit-and-miss topologies of types of subbasic open sets: those that hit a variable open subset plus those that miss a compact subset (in case of Fell topology) or a closed subset (in case of Vietoris topology). Z. Li in his paper [19] introduced the definitions of hit-and-miss type covers to study the selection principles in $CL(X)$ under τ_F and τ_V. The following definition of hit-and-miss type covers has been introduced in [6].

Definition 2.1. [6] Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-cover of X, if for any $D \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $D^c \cap V_i \neq \phi$, for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $D \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The family of all $c_\Delta(\Lambda)$-covers of X will be denoted by $\mathbb{C}_\Delta(\Lambda)$.

Next we recall the relative version of the above type of covers as follows.

Definition 2.2. [29] Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-cover of Y, if for any $D \in \Delta$ with $D \subseteq Y$ and open subsets $V_1, ..., V_m$ of X, with $Y^c \cap V_i \neq \phi$, for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $D \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. We denote by $\mathbb{C}^*_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of $Y \subseteq X$, with $Y \neq X$.

Lemma 2.1. *[29] Let Y be an open subset of a space X with $Y \neq X$ and $\mathcal{U} \subseteq \Lambda^c$ be a cover of Y. Then the following statements are equivalent:*

(i) $\mathcal{U}$ *is a* $c_\Delta(\Lambda)$*-cover of* Y*.*
(ii) $Y^c \in Cl_{\tau_\Delta^+}(\mathcal{U}^c)$*.*

Lemma 2.2. *For a space* X*,* $E \in \Lambda$ *and a collection* $\mathcal{A} \subset \Lambda$*,* $\mathcal{A} \in \Omega_E^{\tau_\Delta^+}$ *implies* $\{(A \cup E)^c : A \in \mathcal{A}\}$ *is a* $c_\Delta(\Lambda)$*-cover of* E^c*, where* $\Omega_E^{\tau_\Delta^+} = \{\mathcal{A} \subset CL(X) : E \in Cl_{\tau_\Delta^+}(\mathcal{A})\}$*.*

Proof. Let $D \in \Delta$ be such that $D \subset E^c$ and let $V_1, ..., V_m$ be open sets in X with $E \cap V_i \neq \phi$, for all $i = 1, ..., m$. Then $(V_1, ..., V_m)_D^+$ is a τ_Δ^+-neighbourhood of E. As $\mathcal{A} \in \Omega_E^{\tau_\Delta^+}$, there exists $A \in \mathcal{A}$ such that $A \in (V_1, ..., V_m)_D^+$. Now choose $x_i \in A \cap V_i$, for $1 \leq i \leq m$ and consider the set $F = \{x_i : 1 \leq i \leq m\}$. Then $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also $D \subset (A \cup E)^c$ and $(A \cup E)^c \cap F = \phi$. Hence $\{(A \cup E)^c : A \in \mathcal{A}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. □

We next recall the definition of $\Delta\gamma$-covers of a space as follows.

Definition 2.3. [29] Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $\Delta\gamma$-cover of X, if each $B \in \Delta$ belongs to all but finitely many elements of $\mathcal{U}$ and for any $B \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $B^c \cap V_i \neq \phi$ for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The set of all $\Delta\gamma$-covers of X is denoted by $\Delta\Gamma$.

Next recall the relative version of the above type of covers as follows.

Definition 2.4. [28] Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $\Delta\gamma$-cover of Y, if each $B \subseteq Y$ with $B \in \Delta$ belongs to all but finitely many elements of $\mathcal{U}$ and for any $B \subseteq Y$ with $B \in \Delta$ and open subsets $V_1, ..., V_m$ of X, with $Y^c \cap V_i \neq \phi$ for any $i \in \{1, ..., m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \phi$ and for each $i \in \{1, ..., m\}$, $F \cap V_i \neq \phi$. The set of all $\Delta\gamma$-covers of $Y \subseteq X$ is denoted by $\Delta\Gamma^*$.

Remark 2.1. If we consider $\Delta = \mathbb{K}(X)$ and $\Lambda = CL(X)$ (resp., $\Delta = \Lambda = CL(X)$) in Definitions 2.3 and 2.4 above, we get the definitions of γ_{k_F}-covers (resp., γ_{c_V}-covers) of X and also the definitions of γ_{k_F}-covers (resp., γ_{c_V}-covers) of a subset Y of X, with $Y \neq X$.

It is easy to observe that $\Delta\Gamma \subset \mathbb{C}_\Delta(\Lambda)$.

Lemma 2.3. *[28] Let* X *be a topological space,* Y *be an open subset of* X *and* $\mathcal{U} = \{U_n : n \in \mathbb{N}\} \subseteq \Lambda^c$ *be a cover of* Y*. Then the following statements are equivalent:*

(i) $\mathcal{U}$ *is a* $\Delta\gamma$*-cover of* Y*.*
(ii) $\{U_n^c : n \in \mathbb{N}\}$ *converges to* Y^c *in* (Λ, τ_Δ^+)*.*

Recall now that an open cover $\mathcal{U}$ of a space X is called
(i) ω-groupable [15], [17] (k-groupable [9]) if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite (compact) set $C \subset X$, for all but finitely many n there is an $U \in \mathcal{U}_n$ such that $C \subset U$,
(ii) weakly groupable [2] (k-weakly groupable [9]) if there is a partition of $\mathcal{U}$ into countably many finite, pairwise disjoint sets $\mathcal{U}_n$, for $n \in \mathbb{N}$, such that each finite (compact) subset of X is contained in $\bigcup \mathcal{U}_n$, for some n.

Also recall that a countable element D from $\mathcal{D}$ is said to be groupable [17], [18] if there is a partition $D = \bigcup_{n\in\mathbb{N}} D_n$ into finite pairwise disjoint sets such that each nonempty open set of the space intersects D_n, for all but finitely many n. Let $\mathcal{D}^{gp}$ denote the family of groupable elements of $\mathcal{D}$.

For a space X, we denote:

- Ω^{gp} - the family of ω-groupable covers of X;
- $\mathcal{K}^{gp}$ - the family of k-groupable covers of X;
- $\mathcal{O}^{wgp}$ the family of weakly groupable covers of X;
- $\mathcal{O}^{k-wgp}$ the family of k-weakly groupable covers of X;
- $(\Omega_E^{\tau_\Delta^+})^{gp}$ - the family of groupable elements of $\Omega_E^{\tau_\Delta^+}$.

Following Definitions 5.1 and 5.5 of [19], where the classes $\mathcal{K}_F^{gp}$ of k_F-groupable covers and $\mathcal{C}_V^{gp}$ of c_V-groupable covers are introduced, we define the general notion of a Δ-groupable $c_\Delta(\Lambda)$-cover as follows.

Definition 2.5. A $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of a space X is said to be Δ-groupable if it can be expressed as a union of infinitely many finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any subset B of X with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi$ $(1 \leq i \leq m)$, there exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and a finite set F_n with $F_n \cap V_i \neq \phi$ $(1 \leq i \leq m)$ such that $B \subset U_n$ and $F_n \cap U_n = \phi$. We denote the family of all Δ-groupable covers of X by $\mathbb{C}_\Delta(\Lambda)^{gp}$.

Definition 2.6. Let (X, τ) be a topological space and $Y \subseteq X$ with $Y \neq X$. A $c_\Delta(\Lambda)$-cover $\mathcal{U}$ of Y is said to be Δ-groupable if it can be expressed as a union of infinitely many finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any subset $B \subseteq Y$ with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap Y^c \neq \phi$ $(1 \leq i \leq m)$, there exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and a finite set F_n with $F_n \cap V_i \neq \phi$ $(1 \leq i \leq m)$ such that $B \subset U_n$ and $F_n \cap U_n = \phi$. We denote the family of all Δ-groupable covers of $Y \subseteq X$ with $Y \neq X$ by $\mathbb{C}^*_\Delta(\Lambda)^{gp}$.

Lemma 2.4. *For a space X, $E \in \Lambda$ and a collection $\mathcal{A} \subset \Lambda$, $\mathcal{A} \in (\Omega_E^{\tau_\Delta^+})^{gp}$ implies $\{(A \cup E)^c : A \in \mathcal{A}\}$ is a Δ-groupable cover of E^c.*

Proof. Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a partition of $\mathcal{A}$ into finite, pairwise disjoint sets such that each τ^+_Δ-neighbourhood of E meets $\mathcal{B}_n$ for all but finitely many n. Then by Lemma 2.2, $\mathcal{U} = \{(A \cup E)^c : A \in \mathcal{A}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. Write $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, where for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{(B \cup E)^c : B \in \mathcal{B}_n\}$. Let $D \in \Delta$ be such that $D \subset E^c$ and let $V_1, ..., V_m$ be open sets in X with $E \cap V_i \neq \phi$, for all $i = 1, ..., m$. Then $(V_1, ..., V_m)^+_D$ is a τ^+_Δ-neighbourhood of E. Hence there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, there exists $B_n \in \mathcal{B}_n$ such that $B_n \in (V_1, ..., V_m)^+_D$. Now choose $x_i \in B_n \cap V_i$, for $1 \leq i \leq m$ and consider the set $F = \{x_i : 1 \leq i \leq m\}$. Then $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also $D \subset (B_n \cup E)^c$ and $(B \cup E)^c \cap F = \phi$. Hence $\{(A \cup E)^c : A \in \mathcal{A}\}$ is a Δ-groupable cover of E^c. □

Definition 2.7. A cover $\mathcal{U}$ of a space X is weakly Δ-groupable if it can be expressed as a union of infinitely many finite, pairwise disjoint subfamilies $\mathcal{U}_n \subset \mathcal{U}$ such that for any subset B of X with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi\ (1 \leq i \leq m)$, there exist $\mathcal{U}_n$ and a finite set F with $F \cap V_i \neq \phi\ (1 \leq i \leq m)$ such that $B \subset \cup\, \mathcal{U}_n$ and $F \cap (\cup\, \mathcal{U}_n) = \phi$. We denote the family of all weakly Δ-groupable covers of X by $\mathbb{C}^{wgp}_\Delta$.

Lemma 2.5. *[6] A family $\mathcal{U} \subseteq \Lambda^c$ is a $c_\Delta(\Lambda)$-cover of X if and only if the family $\mathcal{U}^c$ is a dense subset of (Λ, τ^+_Δ).*

Lemma 2.6. *For a space X and a countable subset $\mathcal{A} \subset CL(X)$, the following statements are equivalent:*

(i) $\mathcal{A}$ is a groupable dense subset of $(CL(X), \tau^+_\Delta)$.
(ii) $\mathcal{A}^c$ is a Δ-groupable cover of X.

Proof. (i) ⇒ (ii): Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a partition into finite pairwise disjoint sets such that each open set of $(CL(X), \tau^+_\Delta)$ intersects $\mathcal{B}_n$ for all but finitely many n. We claim that $\mathcal{A}^c = \bigcup_{n\in\mathbb{N}} \mathcal{B}^c_n$ is a Δ-groupable cover of X. Indeed, let $K \in \Delta$ be a subset of X and $V_1, ..., V_m$ be open in X with $(X \setminus K) \cap V_i \neq \phi$, for $1 \leq i \leq m$. Then $(V_1, ..., V_m)^+_K$ is a τ^+_Δ-open set in $CL(X)$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exists $B_n \in \mathcal{B}_n$ such that $B_n \in (V_1, ..., V_m)^+_K$. Let $U_n = B^c_n$, for $n \geq n_0$. Then $U_n \in \mathcal{B}^c_n$. Choose $x^{(n)}_i \in V_i \cap B_n$, for $1 \leq i \leq m$ and consieder $F = \{x^{(n)}_i : 1 \leq i \leq m\}$. Then F is a finite subset of X with $F \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also $K \subset U_n$ and $F \cap U_n = \phi$. Hence $\mathcal{B}^c_n$ is a $c_\Delta(CL(X))$-cover of X.

(ii) ⇒ (i): Let $\mathcal{A}^c = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ be a partition of $\mathcal{A}^c$ that witnesses (ii). We claim that $\mathcal{A}$ is a groupable dense subset of $(CL(X), \tau^+_\Delta)$. Let $(V_1, ..., V_m)^+_D$ be a τ^+_Δ-open set in $(CL(X), \tau^{+}_{\Delta})$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exist $U_n \in \mathcal{U}_n$ and $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \phi$, for all $i = 1, ..., m$ such that $D \subseteq U_n$ and $U_n \cap F_n = \phi$. Hence $U_n^c \in (V_1, ..., V_m)^{+}_{D}$, for all $n \geq n_0$, so that $\mathcal{A}$ is a groupable dense subset of $(CL(X), \tau^{+}_{\Delta})$. □

3. Selective separability of the hyperspace $(\Lambda, \tau^{+}_{\Delta})$

In this section we first start with the relationships between closure-type properties of the hyperspace $(\Lambda, \tau^{+}_{\Delta})$ and covering properties of that of X. We then discuss about the selective separability and variations of separability in $(\Lambda, \tau^{+}_{\Delta})$.

Theorem 3.1. *Let $\star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(i) X satisfies $S_{\star}(\mathbb{C}_{\Delta}(\Lambda), \mathbb{C}_{\Delta}(\Lambda))$.
(ii) $(\Lambda, \tau^{+}_{\Delta})$ satisfies $S_{\star}(\mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}, \mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)})$.
(where $\mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}$ denotes the family of dense subsets of $(\Lambda, \tau^{+}_{\Delta})$).

Proof. We prove the theorem for $\star = fin$, the other part being similar.

(i) $\Rightarrow$ (ii): Let $\{D_i : i \in \mathbb{N}\}$ be a family of dense subsets of $(\Lambda, \tau^{+}_{\Delta})$ such that $D_i \in \mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}$, for each $i \in \mathbb{N}$. Then by Lemma 2.5, $\{D_i^c : i \in \mathbb{N}\}$ is a family of open covers of X such that $D_i^c \in \mathbb{C}_{\Delta}(\Lambda)$, for all $i \in \mathbb{N}$. As X satisfies $S_{fin}(\mathbb{C}_{\Delta}(\Lambda), \mathbb{C}_{\Delta}(\Lambda))$, there exists a sequence $\{A_i : i \in \mathbb{N}\}$ of finite sets such that $A_i \subseteq D_i^c$ and $\bigcup_{i\in\mathbb{N}} A_i \in \mathbb{C}_{\Delta}(\Lambda)$, for each $i \in \mathbb{N}$. Then $\bigcup_{i\in\mathbb{N}} A_i^c \in \mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}$.

(ii) $\Rightarrow$(i): Assume that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a family of open covers of X such that $\mathcal{U}_n \in \mathbb{C}_{\Delta}(\Lambda)$. Consider $\mathcal{A}_n = \mathcal{U}_n^c$, for each $n \in \mathbb{N}$. Then by Lemma 2.5, $\mathcal{A}_n$ is a dense subset of $(\Lambda, \tau^{+}_{\Delta})$ for each $n \in \mathbb{N}$ such that $\mathcal{A}_n \in \mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}$. As $(\Lambda, \tau^{+}_{\Delta})$ satisfies $S_{fin}(\mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}, \mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)})$, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of finite subsets such that $A_n \subseteq \mathcal{A}_n$, for each $n \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{D}_{\mathbb{C}_{\Delta}(\Lambda)}$. Then $U_n = A_n^c$, for $n \in \mathbb{N}$ is such that $\bigcup_{n\in\mathbb{N}} U_n$ is an open cover of X and $\bigcup_{n\in\mathbb{N}} U_n \in \mathbb{C}_{\Delta}(\Lambda)$. □

Corollary 3.1. *(Theorem 3.6 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$.
(ii) X satisfies $S_1(\mathbb{C}_V, \mathbb{C}_V)$.

Corollary 3.2. *(Theorem 3.4 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ satisfies $S_1(\mathcal{D}, \mathcal{D})$.
(ii) X satisfies $S_1(\mathbb{K}_F, \mathbb{K}_F)$.

Corollary 3.3. *(Theorem 4.4 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.
(ii) X satisfies $S_{fin}(\mathbb{C}_V, \mathbb{C}_V)$.

Corollary 3.4. *(Theorem 4.2 in [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{D})$.
(ii) X satisfies $S_{fin}(\mathbb{K}_F, \mathbb{K}_F)$.

Recall here that a space X is M-separable [3] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subspaces of X one can select finite $F_n \subset D_n$ so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in X. Thus we have the following theorem.

Theorem 3.2. *For a space X, (Λ, τ_Δ^+) is M-separable if and only if X satisfies $S_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.*

Again a space X is R-separable [3] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subspaces of X one can pick $x_n \in D_n$ so that $\{x_n : n \in \mathbb{N}\}$ is dense in X. Thus we have the following theorem.

Theorem 3.3. *For a space X, (Λ, τ_Δ^+) is R-separable if and only if X satisfies $S_1(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.*

Theorem 3.4. *Let $\Phi, \Psi \in \{\Delta\Gamma^*, \mathbb{C}_\Delta^*(\Lambda)\}, \star \in \{1, fin\}$. Then for a space X the following statements are equivalent:*

(i) Each open set $Y \subset X$ with $Y \in \Lambda^c$ has the property $S_\star(\Phi, \Psi)$.
(ii) Each $E \in (\Lambda, \tau_\Delta^+)$ satisfies $S_\star(\Phi_E, \Psi_E)$.
(where Φ_E denotes the Φ family of covers of E and Ψ_E denotes the Ψ family of covers of E).

Proof. We prove the theorem for $\star = 1$, the other parts being similar.

(i) $\Rightarrow$ (ii): Let $E \in \Lambda$ and let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence such that for each $n \in \mathbb{N}$, $\mathcal{A}_n \in \Phi_E$. Then $\{\mathcal{A}_n^c : n \in \mathbb{N}\}$ is a sequence of open covers of E^c such that for each $n \in \mathbb{N}$, $\mathcal{A}_n^c \in \Phi$. As E^c has the property $S_1(\Phi, \Psi)$, there exists a sequence $\{A_n^c : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $A_n^c \in \mathcal{A}_n^c$ and $\{A_n^c : n \in \mathbb{N}\}$ is an open cover of E^c such that $\{A_n^c : n \in \mathbb{N}\} \in \Psi$. Hence $\{A_n : n \in \mathbb{N}\} \in \Psi_E$.

(ii) $\Rightarrow$ (i): Let Y be an open subset of X with $Y \in \Lambda^c$ and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of open covers of Y such that $\mathcal{F}_n \in \Phi_Y$, for $n \in \mathbb{N}$. Let $E = X \setminus Y$. Put $\mathcal{A}_n = \mathcal{F}_n^c, n \in \mathbb{N}$. Then $\mathcal{A}_n \subset \Lambda$ and $\mathcal{A}_n \in \Phi_E$, for $n \in \mathbb{N}$. As E satisfies $S_1(\Phi_E, \Psi_E)$, there exists a sequence $\{A_n : n \in \mathbb{N}\}$ such that $A_n \in \mathcal{A}_n$, for each $n \in \mathbb{N}$ and $\{A_n : n \in \mathbb{N}\} \in \Psi_E$. Hence $\{F_n = A_n^c : n \in \mathbb{N}\} \in \Psi$. $\square$

Recall that a space X has countable fan tightness [1] if whenever $x \in ClA_n$ for all $n \in \mathbb{N}$, one can choose finite $F_n \subset A_n$ so that $x \in Cl(\cup\{F_n : n \in \mathbb{N}\})$ and X has countable strong fan tightness [25] if whenever $x \in ClA_n$ for $n \in \mathbb{N}$, there are $x_n \in A_n$ such that $x \in Cl(\{x_n : n \in \mathbb{N}\})$. In view of these definitions we can restate the above theorem as follows.

Theorem 3.5. *For a space X, $(\Lambda, \tau_{\Delta}^{+})$ has countable strong fan tightness if and only if each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ satisfies $S_1(\mathbb{C}_{\Delta}^{*}(\Lambda), \mathbb{C}_{\Delta}^{*}(\Lambda))$.*

Proof. First let $Y \subsetneq X$ be open in X with $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of Y. Then by Lemma 2.1, $Y^c \in Cl_{\tau_{\Delta}^{+}}(\mathcal{U}_n^c)$. As $(\Lambda, \tau_{\Delta}^{+})$ has countable strong fan tightness, there exists $U_n^c \in \mathcal{U}_n^c$, for $n \in \mathbb{N}$ such that $Y^c \in Cl_{\tau_{\Delta^+}}(\{U_n^c : n \in \mathbb{N}\})$. Hence $\{U_n : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of Y.

Conversely, let $E \in \Lambda$ be such that $E \in Cl(\mathcal{U}_n)$. Then by Lemma 2.1, $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(\Lambda)$-covers of E^c. By the given condition, there exists $U_n^c \in \mathcal{U}_n^c$, for $n \in \mathbb{N}$ such that $\{U_n^c : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. Hence $E \in Cl_{\tau_{\Delta}^{+}}(\{U_n : n \in \mathbb{N}\})$, so that $(\Lambda, \tau_{\Delta}^{+})$ has countable strong fan tightness. $\square$

Theorem 3.6. *For a space X, $(\Lambda, \tau_{\Delta}^{+})$ has countable fan tightness if and only if each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ satisfies $S_{fin}(\mathbb{C}_{\Delta}^{*}(\Lambda), \mathbb{C}_{\Delta}^{*}(\Lambda))$.*

Proof. First let $Y \subsetneq X$ be open in X with $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of Y. Then by Lemma 2.1, $Y^c \in Cl_{\tau_{\Delta}^{+}}(\mathcal{U}_n^c)$. As $(\Lambda, \tau_{\Delta}^{+})$ has countable fan tightness, there exist finite $\mathcal{V}_n^c \subset \mathcal{U}_n^c$, for $n \in \mathbb{N}$, such that $Y^c \in Cl_{\tau_{\Delta^+}}(\bigcup\{\mathcal{V}_n^c : n \in \mathbb{N}\})$. Hence $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of Y.

Conversely, let $E \in \Lambda$ be such that $E \in Cl(\mathcal{U}_n)$. Then by Lemma 2.1, $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(\Lambda)$-covers of E^c. By the given condition, there exist finite $\mathcal{V}_n^c \subset \mathcal{U}_n^c$, for $n \in \mathbb{N}$, such that $\bigcup\{\mathcal{V}_n^c : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of E^c. Hence $E \in Cl_{\tau_{\Delta}^{+}}(\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\})$. $\square$

Corollary 3.5. *(Theorem 3.2 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ has countable strong fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_1(\mathbb{C}_V^{*}, \mathbb{C}_V^{*})$.

Corollary 3.6. *(Theorem 3.1 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ has countable strong fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_1(\mathbb{K}_F^{*}, \mathbb{K}_F^{*})$.

Corollary 3.7. *(Theorem 4.3 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_V)$ has countable fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_{fin}(\mathbb{C}_V^{*}, \mathbb{C}_V^{*})$.

Corollary 3.8. *(Theorem 4.1 of [19]) For a space X, the following are equivalent:*

(i) $(CL(X), \tau_F)$ has countable fan tightness.
(ii) Each open subset Y of X with $Y \subset X$ satisfies $S_{fin}(\mathbb{K}_F^*, \mathbb{K}_F^*)$.

Theorem 3.7. *For a space X, the following statements are equivalent:*

(i) X satisfies $S_1(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$.
(ii) $(CL(X), \tau_\Delta^+)$ satisfies $S_1(\mathcal{D}_{\tau_\Delta^+}, \mathcal{D}^{gp}_{\tau_\Delta^+})$.

Proof. (i) $\Rightarrow$ (ii): Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $(CL(X), \tau_\Delta^+)$. For each $n \in \mathbb{N}$, put $\mathcal{U}_n = \mathcal{D}_n^c$. Then $\mathcal{U}_n$ is a $c_\Delta(CL(X))$-cover of X, for each $n \in \mathbb{N}$. By (i) applied to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, there exists a sequence $\{D_n^c : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $D_n^c \in \mathcal{U}_n$ and $\{D_n^c : n \in \mathbb{N}\}$ is a Δ-groupable cover of X. Hence by Lemma 2.6, $\{D_n : n \in \mathbb{N}\}$ is a groupable dense subset of $(CL(X), \tau_\Delta^+)$.

(ii) $\Rightarrow$ (i): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(CL(X))$-covers of X. Put $\mathcal{A}_n = \mathcal{U}_n^c$, $n \in \mathbb{N}$. Then by Lemma 2.5 for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is a sequence of dense subsets of $(CL(X), \tau_\Delta^+)$. By (ii), there exists a sequence $\{A_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}_n$ and $\mathcal{B} = \{A_n : n \in \mathbb{N}\} \in \mathcal{D}^{gp}_{\tau_\Delta^+}$. Again by Lemma 2.6, $\mathcal{B}^c$ is a Δ-groupable cover of X. Hence $\{A_n^c : n \in \mathbb{N}\}$ guarantees for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ that X satisfies $S_1(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$. $\square$

Next recall that a space X is H-separable [3] if for every sequence $\{D_n : n \in \mathbb{N}\}$ of dense subspaces of X, one can pick finite $F_n \subset D_n$ so that for every nonempty open set $O \subset X$, the intersection $O \cap F_n$ is nonempty for all but finitely many n. Thus we have the following theorem.

Theorem 3.8. *For a space X, $(CL(X), \tau_\Delta^+)$ is H-separable if and only if X satisfies $S_{fin}(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$.*

Proof. First let, $(CL(X), \tau_\Delta^+)$ be H-separable and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(CL(X))$-covers of X. Then by Lemma 2.5, $\{\mathcal{U}_n^c : n \in \mathbb{N}\}$ is a sequence of dense subsets of $CL(X)$. By H-separability of $(CL(X), \tau_\Delta^+)$, there exist finite $\mathcal{V}_n^c \subset \mathcal{U}_n^c$, $n \in \mathbb{N}$, such that for every non-empty open set W of $CL(X)$, $W \cap \mathcal{V}_n^c \neq \phi$, for all but finitely many $n \in \mathbb{N}$. We claim that $\bigcup \mathcal{V}_n$ is a Δ-groupable cover of X. Indeed, Let $D \in \Delta$ and $V_1, ..., V_m$ be open in X with $D^c \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Then $(V_1, ..., V_m)_D^+$ is a τ_Δ^+-open set in $CL(X)$ and hence there exists $n_0 \in \mathbb{N}$ such that $(V_1, ..., V_m)_D^+ \cap \mathcal{V}_n^c \neq \phi$, for all $n \geq n_0$. Choose $V_n^c \in (V_1, ..., V_m)_D^+ \cap \mathcal{V}_n^c$, for all $n \geq n_0$. Next choose $x_i^{(n)} \in (V_1, ..., V_m)_D^+ \cap V_n^c$, for all $1 \leq i \leq m$ and consider the set $F_n = \{x_i^{(n)} : 1 \leq i \leq m\}$. Then $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \phi$, for all $1 \leq i \leq m$. Also, $D \subset V_n$ and $V_n \cap F_n = \phi$, for all $n \geq n_0$. Hence $\bigcup \mathcal{V}_n$ is a Δ-groupable cover of X.

Conversely, let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $CL(X)$. By Lemma 2.5, $\{\mathcal{D}_n^c : n \in \mathcal{N}\}$ is a sequence of $c_\Delta(CL(X))$-covers of X. As X satisfies $S_{fin}(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta(CL(X))^{gp})$, there exist finite $\mathcal{B}_n^c \subset \mathcal{D}_n^c$, $n \in \mathbb{N}$, such that $\bigcup \mathcal{B}_n^c$ is a Δ-groupable cover of X. Then every τ_Δ^+-open set intersects all but finitey many$\mathcal{B}_n$. Hence $(CL(X), \tau_\Delta^+)$ is H-separable. □

Corollary 3.9. *(Theorem 5.4 of [19]) For a space X, the following statements are equivalent:*

(i) $(CL(X), \tau_V))$ satisfies $S_1(\mathcal{D}, \mathcal{D}^{gp})$.
(ii) X satisfies $S_1(\mathbb{C}_V, \mathbb{C}_V^{gp})$.

Corollary 3.10. *(Theorem 5.2 of [19]) For a space X, the following statements are equivalent:*

(i) $(CL(X), \tau_F)$ satisfies $S_1(\mathcal{D}, \mathcal{D}^{gp})$.
(ii) X satisfies $S_1(\mathbb{K}_F, \mathbb{K}_F^{gp})$.

Theorem 3.9. *For a space X, the following statements are equivalent:*

(i) $(CL(X), \tau_\Delta^+)$ satisfies: for each sequence $\{\mathcal{D}_n : n \in \mathbb{N}\}$ of dense subsets of $(CL(X), \tau_\Delta^+)$ there is a finite $\mathcal{B}_n \subset \mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ can be partitioned into a union of finite sets $\mathcal{C}_n$, $n \in \mathbb{N}$, so that $\{\bigcap \mathcal{C}_n : n \in \mathbb{N}\}$ is dense in $(CL(X), \tau_\Delta^+)$.
(ii) X satisfies $S_{fin}(\mathbb{C}_\Delta(CL(X)), \mathbb{C}_\Delta^{wgp})$.

Proof. (i) ⇒ (ii): Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(CL(X))$-open covers of X. Then for each $n \in \mathbb{N}$, $\mathcal{A}_n = \mathcal{U}_n^c$ is a dense subset of $(CL(X), \tau_\Delta^+)$. By (i), there exist finite $\mathcal{B}_n \subset \mathcal{A}_n$, for each $n \in \mathbb{N}$, such that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a union of finite pairwise disjoint sets $\mathcal{C}_n$ and $\{\bigcap \mathcal{C}_n : n \in \mathbb{N}\}$ is dense in $(CL(X), \tau_\Delta^+)$. Let $\mathcal{V} = \mathcal{B}^c$ and $\mathcal{W}_n = \mathcal{C}_n^c$, for each $n \in \mathbb{N}$. We now claim that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a weakly Δ-groupable cover of X. Let $K \in \Delta$, $V_1, V_2, ..., V_m$ be open sets of X with $V_i \cap K^c \neq \phi$ $(1 \leq i \leq m)$. Then there exists a $n_0 \in \mathbb{N}$ such that $\bigcap \mathcal{C}_{n_0} \in (V_1, ..., V_m)_K^+$. Choose $x_i \in V_i \cap (\bigcap \mathcal{C}_{n_0})$, for $1 \leq i \leq m$. Now consider $F = \{x_i : 1 \leq i \leq m\}$. Hence $K \subset (\bigcap \mathcal{C}_{n_0})^c = \bigcup \mathcal{W}_{n_0}$ and $F \cap (\bigcup \mathcal{W}_{n_0}) = \phi$.

(ii) ⇒ (i): Let $\{\mathcal{D}_n : n \in \mathbb{N}\}$ be a sequence of dense subsets of $(CL(X), \tau_\Delta^+)$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \mathcal{D}_n^c$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(CL(X))$-covers of X. By (ii), for each $n \in \mathbb{N}$, there is a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a weakly Δ-groupable cover of X. Thus $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a union of countably many finite pairwise disjoint sets $\mathcal{W}_n$ satisfying: for each subset $K \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap K^c \neq \phi$ $(1 \leq i \leq m)$, there exist a n_0 and a finite set F with $F \cap V_i = \phi$, for $1 \leq i \leq m$ such that $K \subset \bigcup \mathcal{W}_{n_0}$ and $F \cap (\bigcup \mathcal{W}_{n_0}) = \phi$. Hence $\bigcap \mathcal{C}_{n_0} \in (V_1, ..., V_m)^+_K$. Let $\mathcal{B}_n = \mathcal{V}^c_n$ and $\mathcal{C}_n = \mathcal{W}^c_n$, for each $n \in \mathbb{N}$. Then $\mathcal{B}_n$ is finite set of $\mathcal{D}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ can be partitioned into a union $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ of finite sets $\mathcal{C}_n$, for $n \in \mathbb{N}$, such that $\{\bigcap \mathcal{C}_n : n \in \mathbb{N}\}$ is dense in $(CL(X), \tau^+_\Delta)$. $\square$

Recall that a space X is weakly Fréchet in the strict sense [24] if whenever $x \in Cl A_n$ for all $n \in \mathbb{N}$, there are finite $F_n \subset A_n$ such that every neighbourhood of x intersects all but finitely many F_n.

Theorem 3.10. *For a space X, (Λ, τ^+_Δ) is weakly Fréchet in the strict sense if and only if each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ has $S_{fin}(\mathbb{C}^*_\Delta(\Lambda), \mathbb{C}^*_\Delta(\Lambda)^{gp})$.*

Proof. First let $Y \subsetneq X$ be such that $Y^c \in \Lambda$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $c_\Delta(\Lambda)$-covers of Y. Then by Lemma 2.1, $\{\mathcal{U}^c_n : n \in \mathbb{N}\}$ is a sequence of subsets of (Λ, τ^+_Δ) such that $Y^c \in Cl_{\tau^+_\Delta} \mathcal{U}^c_n$, for each $n \in \mathbb{N}$. Since (Λ, τ^+_Δ) is weakly Fréchet in the strict sense, there exist finite $\mathcal{V}^c_n \subset \mathcal{U}^c_n$, $n \in \mathbb{N}$, such that each neighbourhood of Y^c intersects all but finitely many $\mathcal{V}^c_n$. We now show that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a Δ-groupable cover of Y. Let $B \subseteq Y$ with $B \in \Delta$ and $V_1, ..., V_m$ be open subsets of X with $Y^c \cap V_i \neq \phi$, for $1 \leq i \leq m$ so that $(V_1, ..., V_m)^+_B \cap \Lambda$ is a τ^+_Δ-neighbourhood of Y^c in the space (Λ, τ^+_Δ). Thus there exists $n_0 \in \mathbb{N}$ such that $(V_1, ..., V_m)^+_B \cap \mathcal{V}^c_n \cap \Lambda \neq \phi$, for all $n \geq n_0$. Let $V^c_n \in \mathcal{V}^c_n$ be such that $V^c_n \in (V_1, ..., V_m)^+_B \cap \Lambda$ and choose $x^{(n)}_i \in V^c_n \cap V_i$, for $1 \leq i \leq m$. Now form the set $F_n = \{x^{(n)}_1, ..., x^{(n)}_m\}$. Then $F_n \in [X]^{<\omega}$ with $F_n \cap V_i \neq \phi$, for $1 \leq i \leq m$, $F_n \cap V_n = \phi$ and $B \subseteq V_n$, for all $n \geq n_0$.

Conversely, let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of subsets of Λ and $E \in \Lambda$ be such that $E \in Cl_{\tau^+_\Delta}(\mathcal{A}_n)$, for $n \in \mathbb{N}$. Then $\{\mathcal{A}^c_n : n \in \mathbb{N}\}$ is a sequence of $c_\Delta(\Lambda)$-covers of E^c, for each $n \in \mathbb{N}$. Hence by the given condition there exist finite $\mathcal{B}^c_n \subset \mathcal{A}^c_n$, $n \in \mathbb{N}$, such that $\bigcup \mathcal{B}^c_n$ is a Δ-groupable cover of E^c. Hence (Λ, τ^+_Δ) is weakly Fréchet in the strict sense. $\square$

4. Supertightness of (Λ, τ^+_Δ)

In [29], the authors have posed an open problem as: "Is it possible to characterize the supertightness of the hyperspace Λ by means of $c_\Delta(\Lambda)$-covers of Y, for some open subset $Y \subseteq X$?" In this section we give an affirmative answer to the question. Let us first recall that a family $\mathcal{P}$ of nonempty subsets of a space X is said to be a π-network at p [30] if every neighbourhood of p contains some member of $\mathcal{P}$.

Definition 4.1. [30, 24] A space X is said to have countable supertightness if $p \in X$ and $\mathcal{P}$ is a π-network at p consisting of finite subsets of X, then there is a countable subfamily $\mathcal{F} \subset \mathcal{P}$ such that $\mathcal{F}$ is a π-network at p.

We now define the following.

Definition 4.2. Let Y be a subspace of X. A partitioned $c_\Delta(\Lambda)$-cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ (where $\mathcal{U} \subseteq \Lambda^c$) is called a finite p-$c_\Delta(\Lambda)$-cover of Y if each $\mathcal{U}_\alpha$ is finite and for any subset $B \subseteq Y$ with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi$ $(1 \leq i \leq m)$, there exists $\alpha \in A$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $i = 1, 2, ..., m$ such that $B \subset U$ and $F \cap U = \phi$, for each $U \in \mathcal{U}_\alpha$.

Theorem 4.1. *For a space X, the following are equivalent:*

(i) (Λ, τ_Δ^+) has countable supertightness.
(ii) For each open subset $Y \subsetneq X$ with $Y^c \in \Lambda$ and each finite p-$c_\Delta(\Lambda)$-cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ of Y, there exists a countable subset $A' \subset A$ such that $\bigcup_{\alpha \in A'} \mathcal{U}_\alpha$ is a finite p-$c_\Delta(\Lambda)$-cover of Y.

Proof. (i) $\Rightarrow$ (ii): Let $Y \subsetneq X$ be an open subset of X with $Y^c \in \Lambda$ and $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ be a finite p-$c_\Delta(\Lambda)$-cover of Y. Then $\{\mathcal{U}_\alpha^c : \alpha \in A\}$ is a π-network at Y^c. Indeed let $Y^c \in (V_1, ..., V_m)_D^+ \cap \Lambda$. Then there exists $\alpha \in A$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $i = 1, ..., m$ such that $D \subset U$ and $F \cap U = \phi$, for all $U \in \mathcal{U}_\alpha$. Then $U^c \in (V_1, ..., V_m)_D^+ \cap \Lambda$, for each $U \in \mathcal{U}_\alpha$. Hence $\{\mathcal{U}_\alpha^c : \alpha \in A\}$ is a π-network at Y^c consisting of finite subsets of Λ. As (Λ, τ_Δ^+) has countable supertightness, there exists a countable subset $A' \subset A$ such that $\{\mathcal{U}_\alpha^c : \alpha \in A'\}$ is a π-network at Y^c. Hence $\bigcup_{\alpha \in A'} \mathcal{U}_\alpha$ is a finite p-$c_\Delta(\Lambda)$-cover of Y.

(ii) $\Rightarrow$ (i): Let $E \in \Lambda$ and $\{\mathcal{A}_\alpha : \alpha \in A\}$ be a π-network at E, where each $\mathcal{A}_\alpha$ is a finite subset of A. Then for any neighbourhood $(V_1, ..., V_m)_D^+ \cap \Lambda$ of E, there exists $\alpha \in A$ such that $\mathcal{A}_\alpha \subset (V_1, ..., V_m)_D^+ \cap \Lambda$. Let

$$A' = \{\alpha \in A : E^c \cap F^c \neq \phi, \text{ for each } F \in \mathcal{A}_\alpha\}.$$

Then $A' \neq \phi$ and $\{\mathcal{A}_\alpha : \alpha \in A'\}$ is a π-network at A. Hence $\bigcup_{\alpha \in A'} \mathcal{A}_\alpha^c$ is a finite p-$c_\Delta(\Lambda)$-cover of E^c. By (ii), there exists a countable family $\{\mathcal{A}_{\alpha_n} : n \in \mathbb{N}\} \subset \{\mathcal{A}_\alpha : \alpha \in A'\}$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{A}_{\alpha_n}^c$ is a finite p-$c_\Delta(\Lambda)$-cover of E^c. Hence $\{\mathcal{A}_{\alpha_n} : n \in \mathbb{N}\}$ is a π-network at E, so that (Λ, τ_Δ^+) has countable supertightness. $\square$

Definition 4.3. [4] A space X is supertight at $p \in X$ if whenever $\mathcal{P}$ is a π-network at p consisting of countable subsets of X, there is a countable subfamily $\mathcal{F} \subset \mathcal{P}$ such that $\mathcal{F}$ is a π-network at p. A space is supertight if all its points are supertight.

Definition 4.4. Let Y be a subspace of X. A partitioned $c_{\Delta}(\Lambda)$-cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_{\alpha}$ (where $\mathcal{U} \subseteq \Lambda^c$) is called a countable p-$c_{\Delta}(\Lambda)$-cover of Y if each $\mathcal{U}_{\alpha}$ is countable and for any subset $B \subseteq Y$ with $B \in \Delta$, open sets $V_1, V_2, ..., V_m$ of X with $V_i \cap B^c \neq \phi (1 \leq i \leq m)$, there exists $\alpha \in A$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \phi$, for all $i = 1, 2, ..., m$ such that $B \subset U$ and $F \cap U = \phi$, for each $U \in \mathcal{U}_{\alpha}$.

Theorem 4.2. *For a space X, the following are equivalent:*

(i) $(\Lambda, \tau_{\Delta}^{+})$ is supertight.
(ii) For each open subset $Y \subseteq X$ with $Y \neq X$ and each countable p-$c_{\Delta}(\Lambda)$-groupable cover $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_{\alpha}$ of Y, there exists a countable subset $A' \subset A$ such that $\bigcup_{\alpha \in A'} \mathcal{U}_{\alpha}$ is a countable p-$c_{\Delta}(\Lambda)$-cover of Y.

Proof. Same as Theorem 4.1. □

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CHEMICALLY REACTIVE MHD FLOW THROUGH A SLENDERING STRETCHING SHEET SUBJECTED TO NON-LINEAR RADIATION FLOW OVER A LINEAR AND NON-LINEAR STRETCHING SHEET

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Abstract. In this analysis, the MHD flow and nth-order dispersion of chemically reactive species over a slendering stretching sheet are studied numerically. The partial slip boundary condition and non-linear form of thermal radiation are also considered in this research. To get non-linear ordinary differential equations from the system of partial differential equations governing the flow, energy, and concentration, similarity transformations are applied. Using the shooting technique and the Runge-Kutta scheme, the resultant equations are integrated numerically. The numerical results in terms of temperature, velocity, and concentration are represented graphically. Results from this research indicate that an increase in the wall thickness parameter reduces momentum and heat transfer effects when a magnetic field is present.
Keywords: Chemically reactive fluid, MHD slip flow, slendering stretching sheet, non-linear Rosseland thermal radiation.

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1. Introduction

The combined analysis of heat and momentum transport with a chemical reaction (CR) on a constantly moving sheet has a significant role in many processes due to which these problems obtained a lot of attention recently. These developments include surface evaporation of the water body, transfer of heat in a misty refrigerating tower, drying, and the stream within a desert cooler. After the innovative study of Sakiadis [29], who investigated BLF beyond a constant solid surface, many researchers studied this problem with various aspects. Crane [10] studied the flow past a stretching plate. In a numerical study, the characteristics of heat and mass transport with nth-order CR over a linearly SS were discussed by Ferdows and Al-Mdallal [14]. Makinde et al. [22] described the effects of BL flow with the transmission of convective temperature at the surface in the existence of thermal diffusion and MHD. Rashidi et al. [26] examined the heat and mass transport with free convection in magnetohydrodynamic liquid flow under the effects of buoyancy force and radiation past SS. Mabood et al. [20] studied the combined heat and mass transport impacts on magnetohydrodynamic fluid flow through SS under the impact of first-order CR. Babu and Sandeep [5, 4, 6] inspected the hydromagnetic flow past a slendering stretching sheet (SS) along with various presumptions. All the above studies discussed the fluid flow over a flat SS with different assumptions and physical geometries. In real-world applications, the SS not necessarily be flat, we may be confronted by sheets with variable thickness (VT). Plates having VT are commonly present in acoustical components, nuclear reactor technology, naval structures, and machine design and are also one of the essential characteristics in the investigation of orthotropic plate vibration. Initially, Lee [19] discussed the idea of needles by considering VT and solved the problem numerically. Later, Fang et al. [13] analyzed the boundary layer (BL) flow over SS with VT. Khader and Megahed [18] presented the numerical solution of Newtonian fluid flow through a non-linear SS with VT and velocity slip condition (SC). Subhashini et al. [31] investigated the two-fold solutions of two-dimensional laminar thermal diffusive flows past SS with VT. The ramifications of the magnetohydrodynamic nanofluid flow comprising Ag and TiO_2 nanoparticles through a slender SS with VT are analyzed by Acharya et al. [2]. Babu et al. [7] deliberated the dissipative hydromagnetic flow with the influence of temperature-dependent variable viscosity over a slender SS. The radiative effects on hydromagnetic fluid with heat and mass transport have several important practical applications i.e., in astrophysical power technology, planetary vehicle re-entry, electronic power manufacturing, removal of nuclear surplus and suspension of chemical impurities through water-saturated dust, and many more. Magyari and Pantokratoras [21] inspected the effect of thermal radiation (TR) on various BL flows using linearized Rosseland approximation. Mushtaq et al. [24] studied the impacts of nonlinear TR on the two-dimensional viscous flow of nanoliquids because of the presence of solar energy. Devi and Prakash [11] explored the influences of TR on hydromagnetic liquid flow past a slendering SS. Qayyum et al. [28] scrutinized the third-grade MHD nanofluid flow over a slendering SS under the effects of heat generation/absorption and TR heat. A radiative ferrofluid flow along with the impact of aligned magnetic field and frictional heating through a slendering

SS is examined by Reddy et al. [27]. Mousavi et al. [23] explored the dual solutions for water-based TiO_2-Cu nanofluid flow in the presence of TR over a continuously moving thin needle. Due to the significance of slip flow in many industrial thermal problems and manufacturing fluid dynamics, slip effects with various configurations have been analyzed in the literature. Wang [33] discussed the flow through a SS in the existence of partial slip. In another study, Wang [32] explored the viscous flow over a SS under the impacts of velocity SC and suction force. Fang et al. [12] analytically explained the MHD viscous flow problem with slip condition over SS. BL flow with fixed heat flux surface and velocity SC through a uniform plate was deliberated by Aziz [3]. For a BL flow, Hayat et al. [16] deliberated the hydromagnetic flow and heat transport characteristics over SS with velocity and thermal SCs. Bhattacharyya et al. [8] inspected the BL forced convective flow past a porous plate. Velocity and thermal SCs were also considered. Ibrahim and Shankar [17] examined the heat transport and BL flow of nano liquid past SS with solutal slip BCs. Hasnain et al. [15] deliberated the outcomes of velocity slip on dusty ferrofluid in a channel through spongy media. In the existing exploration, we analyze the impact of nth-order CR on the hydromagnetic viscous liquid past a continually moving sheet with VT. The non-linear TR and slip boundary conditions towards a sheet are also considered. A numerical technique is employed to get the approximate solution of obtained coupled non-linear PDEs. The influence of the Hartman number, the parameter of wall thickness, the radiation parameter, the Schmidt number, and the parameter of velocity power index on liquid velocity, temperature, and concentration profiles is examined through their graphic illustrations.

2. Problem development

The two-dimensional, laminar, and time-independent flow of Newtonian liquid under the effects of Lorentz force with constant density through an impermeable SS with BL and VT is considered. The sheet is situated in the xz-plane, the x-axis is towards the motion of SS however y-axis is considered vertically. The SS velocity is assumed as $U_w(x) = U_0(x+b)^m$. We further suppose that the thickness of the sheet is not fixed and is written as $y = A(x+b)^{(1-m)/2}$. To do away with the pressure gradient, a small enough value of A is chosen to make the sheet thin enough. The magnetic field $B(x) = B_0(x+b)^{(m-1)/2}$ is taken vertically upward to fluid flow. Because of the supposition of neglectable magnetic Reynolds number, the outer electric field is insignificant and there is no effect of an induced magnetic field. Figure 2.1 signifies the physical model of a slendering SS along with varying thickness. For this problem, we take $m \neq 1$, it is because the sheet becomes flat by considering $m = 1$. Moreover, non-linear TR is considered in the present numerical analysis. Under these physical considerations, the mathematical model for the proposed boundary layer flow is specified as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

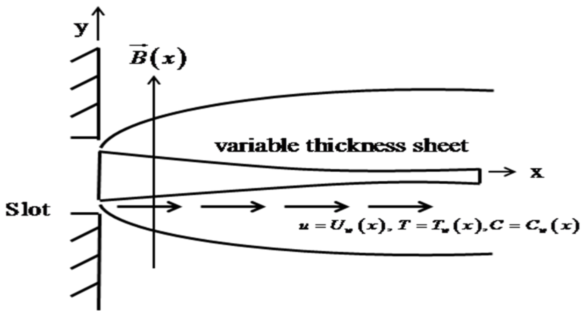

FIG. 2.1: Physical model of a slendering SS along with varying thickness

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B\left(x\right)^2 u}{\rho}, \tag{2.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho c_p}\frac{\partial q_r}{\partial y}, \tag{2.3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-k_n\left(x\right)\left(C-C_\infty\right)^n, \tag{2.4}$$

where $k_n(x)=k(b+x)^{(m-1)(n+1)/2}$ represents the change of n^{th}-order homogeneous CR.
The relevant BCs of heat, momentum, and concentration fields are:

$$\begin{gathered}
u\left(x,y\right)=U_w\left(x\right)+h_1^*\left(\frac{\partial u}{\partial y}\right),\\
v\left(x, A\left(x+b\right)^{\frac{1-m}{2}}\right)=0,\\
T\left(x,y\right)=T_w\left(x\right)+h_2^*\left(\frac{\partial T}{\partial y}\right),\\
C\left(x,y\right)=C_w\left(x\right)+h_3^*\left(\frac{\partial C}{\partial y}\right),\quad \text{at}\ \ y=A\left(x+b\right)^{\frac{1-m}{2}},\\
u\left(x,\infty\right)=0, T\left(x,\infty\right)=T_\infty, C\left(x,\infty\right)=C_\infty,\ \ (m\neq 1)
\end{gathered} \tag{2.5}$$

here

$$h_1^*=\left[\tfrac{2-f_1}{f_1}\right]\xi_1\left(x+b\right)^{\frac{1-m}{2}},\quad h_2^*=\left[\tfrac{2-a}{a}\right]\xi_2\left(x+b\right)^{\frac{1-m}{2}},\quad \xi_2=\left(\tfrac{2\gamma_1}{\gamma_1+1}\right)\tfrac{\xi_1}{\mathrm{Pr}},$$
$$h_3^*=\left[\tfrac{2-c}{c}\right]\xi_3\left(x+b\right)^{\frac{1-m}{2}},\quad \xi_3=\left(\tfrac{2\gamma_2}{\gamma_2+1}\right)\tfrac{\xi_1}{Sc}.$$

To obtain a similar solution we considered a special form of wall temperature and wall concentration defined as (Subhashini et al. [31])

$$T_w\left(x\right)=T_0\left(x+b\right)^{\frac{1-m}{2}}+T_\infty,\ \ C_w\left(x\right)=C_0\left(x+b\right)^{\frac{1-m}{2}}+C_\infty,\ \ \left(m\neq 1\right). \tag{2.6}$$

Applying Rosseland approximation for optically thick medium, the radiation heat flux is taken as (Raptis [25], Brewster [9], and Sparrow and Cess [30])

$$q_r = -\frac{4\sigma^*}{k^*}\frac{\partial T^4}{\partial y} = -\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}. \tag{2.7}$$

By using Eq. (2.7) in Eq. (2.3) , we get

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[\left(\alpha + \frac{16\sigma^* T^3}{3k^*\rho c_p}\right)\frac{\partial T}{\partial y}\right]. \tag{2.8}$$

Similarity transformations in the following form are considered to simplify the flow problem (see Khader and Megahed [18])

$$\eta = y\sqrt{\frac{m+1}{2}\frac{U_0\,(x+b)^{m-1}}{\upsilon}}, \qquad u = U_0\,(x+b)^m f'(\eta), \tag{2.9}$$

$$\begin{gathered} v = -\sqrt{\tfrac{m+1}{2}\upsilon\, U_0\,(x+b)^{m-1}}\left[f'(\eta)\,\eta\left(\tfrac{m-1}{m+1}\right) + f(\eta)\right], \quad (m \neq 1), \\ \theta = \tfrac{T-T_\infty}{T_w(x)-T_\infty} \quad \text{with} \quad T = T_\infty\,(1+(\theta_w-1)\,\theta), \\ \theta_w = \tfrac{T_w}{T_\infty}, \qquad \phi = \tfrac{C-C_\infty}{C_w(x)-C_\infty}, \end{gathered}$$

Using similarity transformations (2.9), the continuity Eq. (2.1) is inevitably fulfilled and Eqs. (2.2), (2.4) and (2.8) with BCs (2.5) take the form

$$f''' = \left(\frac{2m}{m+1}\right)(f')^2 - ff'' + M^2 f', \tag{2.10}$$

$$\left(1 + R_d\,(1+(\theta_w-1)\,\theta)^3\,\theta'\right)' = \Pr\left(\left(\frac{1-m}{m+1}\right)f'\theta - f\theta'\right), \tag{2.11}$$

$$\phi'' = Sc\left(\left(\frac{1-m}{m+1}\right)f'\phi - f\phi'\right) + Sc\gamma\phi^n. \tag{2.12}$$

with

$$\begin{gathered} f(\lambda) = \lambda\left(\tfrac{1-m}{m+1}\right)(1 + h_1 f''(\lambda)), \qquad f'(\lambda) = 1 + h_1 f''(\lambda), \\ \theta(\lambda) = 1 + h_2\,\theta'(0), \qquad \phi(\lambda) = 1 + h_3\,\phi'(0), \end{gathered} \tag{2.13}$$

$$f'(\infty) = 0, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0, (m \neq 1),$$

where

$$R_d = \frac{16\sigma^* T_\infty^3}{3kk^*}, \quad M^2 = \frac{2\sigma B_0^2}{(1+m)\,\rho U_0}, \quad \gamma = \frac{2\,k\,C_0^{n-1}}{(1+m)\,U_0}, \quad \Pr = \frac{\upsilon}{\alpha}, \quad Sc = \frac{\upsilon}{D}.$$

Moreover, $R_d = 0$ shows no TR effect, > 0 represents the destructive CR whereas < 0 represents the constructive CR and

$$\lambda = A\sqrt{\frac{U_0\,(m+1)}{2\nu}}, \qquad h_1 = \left[\frac{2-f_1}{f_1}\right]\xi_1\sqrt{\frac{U_0\,(m+1)}{2\nu}},$$

$$h_2 = \left[\frac{2-a}{a}\right]\xi_2\sqrt{\frac{U_0\,(m+1)}{2\nu}}\ , \quad h_3 = \left[\frac{2-c}{c}\right]\xi_3\sqrt{\frac{U_0\,(m+1)}{2\nu}}.$$

The domain of Eqs. (2.10)-(2.12) with BC's Eq. (2.13) is $[\boldsymbol{\lambda},\ \boldsymbol{\infty}]$. To accommodate the calculation we transform domain $[\boldsymbol{\lambda},\ \boldsymbol{\infty}]$ into $[\mathbf{0},\ \boldsymbol{\infty}]$, for this let $F(\boldsymbol{\xi})=F(\boldsymbol{\eta}$-$\boldsymbol{\lambda})=f(\boldsymbol{\eta})$. Using this transformation Eqs. (2.10)–(2.12) become

$$F''' = \left(\frac{2m}{m+1}\right)(F')^2 - FF'' + M^2F', \tag{2.14}$$

$$\left(1 + R_d\left(1 + (\theta_w - 1)\,\Theta\right)^3\Theta'\right)' = \Pr\left(\left(\frac{1-m}{m+1}\right)F'\Theta - F\Theta'\right), \tag{2.15}$$

$$\Phi'' = Sc\left(\left(\frac{1-m}{m+1}\right)F'\Phi - F\Phi'\right) + Sc\gamma\,\Phi^n, \tag{2.16}$$

and the BC's are

$$\begin{gathered} F(0) = \lambda\left(\tfrac{1-m}{m+1}\right)(1 + h_1F''(0)), \quad F'(0) = 1 + h_1F''(0), \\ \Theta(0) = 1 + h_2\,\Theta'(0), \quad \Phi(0) = 1 + h_3\,\Phi'(0), \\ F'(\infty) = 0, \quad \Theta(\infty) = 0, \quad \Phi(\infty) = 0, \quad (m \neq 1). \end{gathered} \tag{2.17}$$

The skin-drag parameter C_f, the local Nusselt number Nu_x and the local Sherwood number Sh_x are defined as

$$C_f = \frac{1}{\frac{1}{2}\rho U_w^2}\mu\left.\frac{\partial u}{\partial y}\right|_{y=A(x+b)^{\frac{1-m}{2}}} = 2\sqrt{\frac{m+1}{2}}\,(Re_x)^{-\frac{1}{2}}F''(0), \tag{2.18}$$

$$\begin{gathered} Nu_x = -\frac{(x+b)}{(T_w(x) - T_\infty)}\left.\frac{\partial T}{\partial y}\right|_{y=A(x+b)^{\frac{1-m}{2}}} + (q_r)_w = \\ -\sqrt{\frac{m+1}{2}}\left(1 + R_d\theta_w^3\right)(Re_x)^{\frac{1}{2}}\,\Theta'(0), \end{gathered} \tag{2.19}$$

$$Sh_x = -\frac{(x+b)}{(C_w(x) - C_\infty)}\left.\frac{\partial C}{\partial y}\right|_{y=A(x+b)^{\frac{1-m}{2}}} = -\sqrt{\frac{m+1}{2}}\,(Re_x)^{\frac{1}{2}}\,\Phi'(0), \tag{2.20}$$

where $\mathrm{Re}_x = U_wX/\upsilon$ and $X=(x+b)$ is the local Reynolds number.

3. Numerical scheme

Non-linear differential equations (2.14)-(2.16) with boundary conditions (2.17) are solved using the shooting technique together with the fourth-order Runge-Kutta method. Our system of equations must be transformed into a first-order initial value system for this technique by declaring:

$$y_1 = F, y_2 = F', y_3 = F'', y_3' = \left(\frac{2m}{m+1}\right) y_2^2 - y_1 y_3 + M^2 y_2, \tag{3.1}$$

$$\begin{aligned} y_4 = \Theta, y_5 = \Theta',\\ y_5' = \frac{1}{1 + R_d \left(1 + (\theta_w - 1) y_4\right)^3} \left(-3R_d \left(1 + (\theta_w - 1) y_4\right)^2 (\theta_w - 1) y_5^2\right)\\ -\frac{1}{1 + R_d \left(1 + (\theta_w - 1) y_4\right)^3} \left(\left(1 + R_d (\theta_w - 1) y_4\right)^3\right) y_5\\ +\frac{1}{1 + R_d \left(1 + (\theta_w - 1) y_4\right)^3} \left(Pr \left(\left(\frac{1-m}{m+1}\right) y_2 y_4 - y_1 y_5\right)\right), \end{aligned} \tag{3.2}$$

$$y_6 = \Phi,\ y_7 = \Phi',\ y_7' = Sc \left(\left(\frac{1-m}{m+1}\right) y_2 y_6 - y_1 y_7\right) + Sc\gamma\, (y_6)^n, \tag{3.3}$$

with boundary conditions

$$\begin{aligned} &y_1(0) = \lambda \left(\tfrac{1-m}{m+1}\right)(1 + h_1 u_1), \quad y_2(0) = 1 + h_1 u_1, \quad F''(0) = u_1,\\ &y_4(0) = 1 + h_2 u_2, \quad \Theta' = u_2, \quad y_7(0) = 1 + h_3 u_3, \quad \Phi' = u_3. \end{aligned}$$

4. Results and discussion

The solution of ODE's (2.14)–(2.16) with BC's (2.17) is numerically determined by using the shooting method together with the 4^{th}-order algorithm of Runge-Kutta. The influences of all involved constraints on the momentum, concentration, and temperature inside the BL are displayed in Figures 4.1-4.6.

The effect of Hartman number M on liquid velocity is seen in Figure 4.1a. Slip and no-slip velocity conditions are taken into consideration. It is evident from Figure 4.1a that both the liquid velocity and BL thickness decline with an increase in M for both slip and no-slip conditions. Lorentz force (a force manifesting owing to the combined action of magnetic and electric fields) is responsible for this attenuation since it works against transport phenomena more potently. Figure 4.1b represents the variation of wall thickness parameter $\boldsymbol{\lambda}$ and power index parameter m on liquid velocity. It is observed from this Figure that augmentation in m causes an upsurge in sheet slenderness which enables the fluid to flow more rapidly due to this flow velocity accelerates and ultimately boundary layer thickness becomes

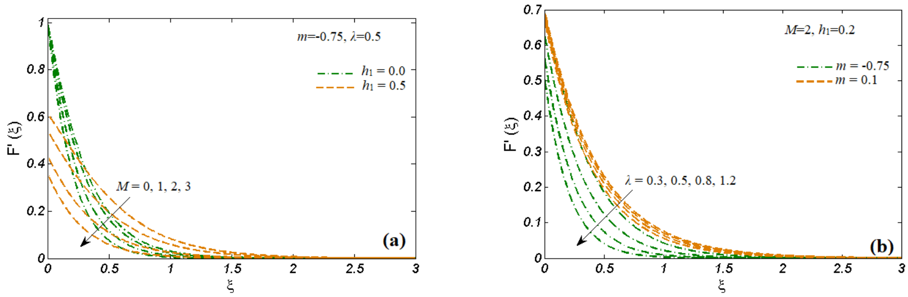

FIG. 4.1: Momentum transfer for distinct values of (a) M and h_1 (b) $\boldsymbol{\lambda}$ and m.

thicker. However, the parameter of the wall thickness $\boldsymbol{\lambda}$ creates retardation in the flow velocity and consequently, BL thickness reduces with a rise in wall thickness parameter $\boldsymbol{\lambda}$.

Figure 4.1a exhibits the influences of the M on dimensionless temperature. It is detected that the temperature profiles enhance when Hartman number M is increased, and results are the same when we consider velocity slip as well as non-slip velocity. Since Lorentz force acts as a resistive force for fluid movement thus heat is generated and therefore the thermal BL thickness rises when M escalates. Figure 4.1b displays the variation of the power index of velocity m and thickness of wall parameter $\boldsymbol{\lambda}$ on the temperature of the liquid. It is depicted that both the thickness of thermal BL and temperature is the increasing function of m whereas decreases with increasing wall thickness parameter $\boldsymbol{\lambda}$. Heat transfers faster through the thinner surface and in this case, an increase in m tends to reduce sheet thickness. As a result, a higher value for m leads to a hotter temperature profile.

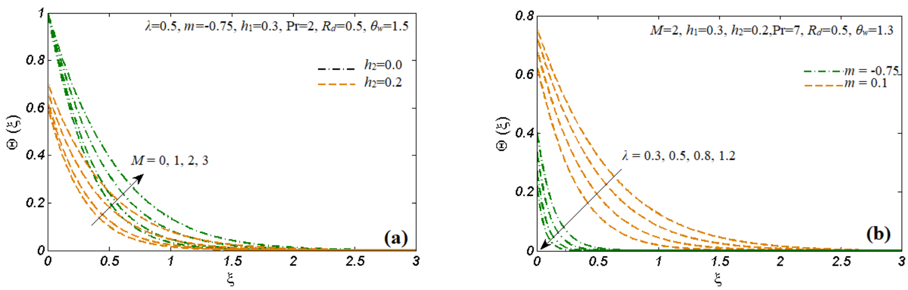

FIG. 4.2: Heat transfer for distinct values of (a) M and h_2 (b) m and λ

Figure 4.3a is illustrated to show the variation in the temperature profiles for Pr and Rd. It is noticed from this fig. that the temperature profiles along with thermal BL thickness decrease with high Pr. Physically, the thermal diffusivity falls when Pr increases therefore heat is diffused slowly far from the heated sheet. However, the temperature profiles and thickness of thermal BL augments with increments in radiation parameter Rd. Figure 4.3b is the graphical depiction of variation in $\boldsymbol{\theta}_w$ for temperature profiles. It is detected that heat travels effectively as thickness for thermal BL is found to grow with $\boldsymbol{\theta}_w$.

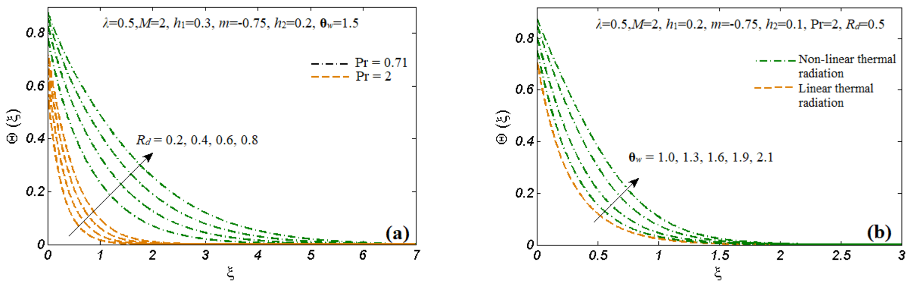

FIG. 4.3: Heat transport for distinct values of (a) R_d and Pr (b) $\boldsymbol{\theta}_w$

The influence of M on the concentration profile is demonstrated in Figure 4.4a. Both the concentration and thickness of its BL are found to increase with M, and this is true for both the slip and no-slip scenarios. The fluid experiences friction due to Lorentz force by accumulative friction among the layers, which is why species distribution increases. Figure 4.4b reveals the behavior of species concentration for different values of m and $\boldsymbol{\lambda}$. It shows that species concentration enhances when m is increased and falls with the augmentation in $\boldsymbol{\lambda}$. As the temperature of the liquid escalates with m, the species concentration also increases. Comparison of the

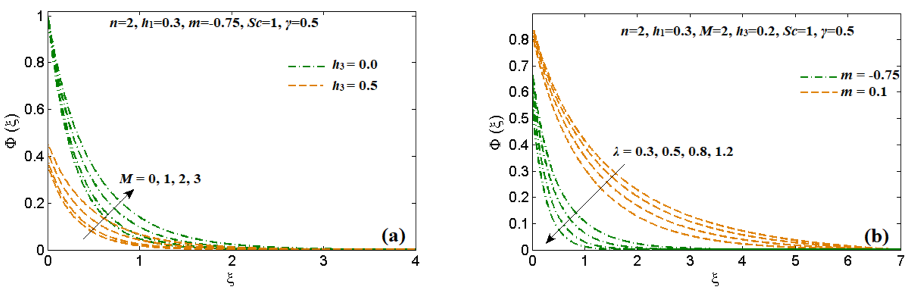

FIG. 4.4: Concentration profile for distinct values of (a) M and h (b) $\boldsymbol{\lambda}$ and m.

effects of no-slip velocity vs slip velocity on species concentration as a function of Sc are shown in Figure 4.5a. Schmidt number describes the ratio of the viscous BL thickness and thickness of the concentration BL so from this figure, we see that increasing Schmidt number Sc decreases the solute BL. Figure 4.5b displays the impacts of the rate of CR parameter on the species concentration for no-slip velocity and slip velocity conditions. For both cases, the liquid concentration decreases for destructive CR (γ>0) and increases for constructive CR (γ<0). Destructive CR behaves similarly to Schmidt number therefore, with destructive CR thickness of solute BL falls while it increases with constructive CR. Therefore, the reaction rate is important in adjusting the solute BL in the reactive concentration distribution.

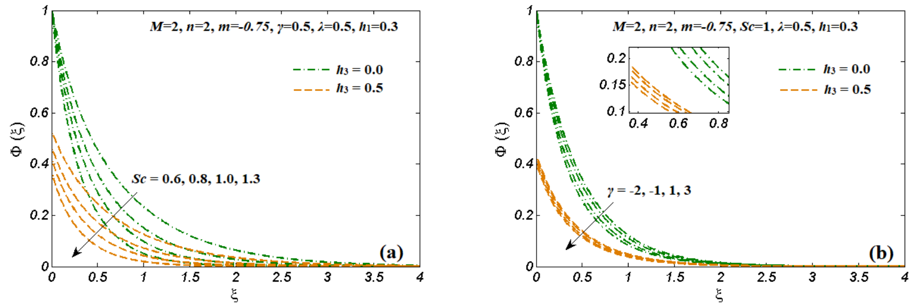

FIG. 4.5: Concentration behavior for distinct values of (a) Sc and h_3 (b) γ and h_3.

Figure 4.6a shows the influence of both parameters λ and velocity power index m on $F''(0)$. Figure 4.6b illustrates the upshot of $\Theta\prime(0)$ with λ for distinct values of Rd. $\Theta\prime(0)$ increases with λ, while diminishes with increasing values of Rd. Figure 4.6c depicts that $\Phi\prime(0)$ is increased with an increment in Sc and λ. It is also depicted from this figure that $\Phi\prime(0)$ falls with the higher values of reaction-order parameter n.

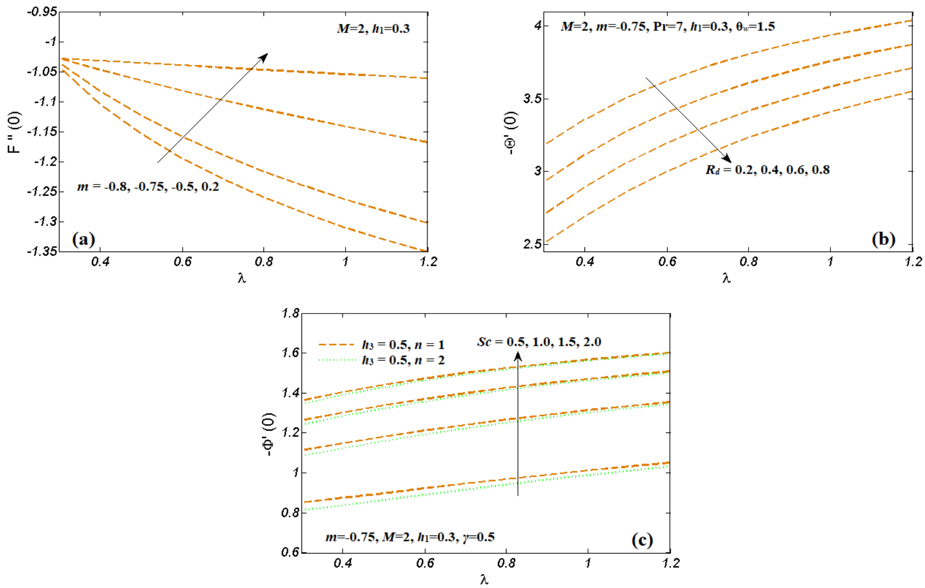

FIG. 4.6: Upshot of (a) $F''(0)$ for m (b) $\boldsymbol{\Theta'}(0)$ for R_d (c) $\boldsymbol{\Phi'}(0)$ for Sc versus $\boldsymbol{\lambda}$.

To ensure the accuracy of new results, we compared them to previous studies'

Table 4.1: Numerical comparative values of $F''(0)$ when $\boldsymbol{\lambda}$=0.5 and M=0

m	Fang et al. [13] (Numerical Method)	Subhashini et al. [31] (Numerical Method)	Present Results (Numerical Method)
-0.51	-1.1859	-1.1860	-1.1860
-0.55	-1.2807	-1.2821	-1.2808
-0.60	-1.4522	-1.4531	-1.4522
-0.65	-1.7095	-1.7103	-1.7095
-0.70	-2.0967	-2.0974	-2.0967
-0.75	-2.6882	-2.6891	-2.6882
-0.80	-3.6278	-3.6282	-3.6278
-0.85	-5.2477	-5.2481	-5.2477
-0.90	-8.5457	-8.5463	-8.5457
-0.95	-18.5194	-18.5209	-18.5194
-0.99	-98.5034	-98.5046	-98.4642

Table 4.2: Comparison with the numerical and analytical solution for $F''(0)$ when M=0

m	$\boldsymbol{\lambda}$	Fang et al. [13] (Shooting Method)	Abdel-wahed et al. [1] (Optimal homotopy asymptotic method)	Present Results
0.50	0.25	-0.93380	-0.92641	-0.93376
1.00		-1.00000	-1.00000	-1.00000
5.00		-1.11860	-1.12623	-1.11858
0.50	0.5	-0.97990	-0.96335	-0.97994
1.00		-1.00000	-1.00000	-1.00000
2.00		-1.02340	-1.03339	-1.02339

findings and discovered they were in good accord which is represented in Table 4.1. Table 4.2 compares the current results to both numerical and analytical approaches and shows that they are in good agreement.

5. CONCLUDING REMARKS

The present work of hydromagnetic flow and dispersion of CRS towards a slendering SS with slip condition has been studied. Non-linear Rosseland thermal radiation is also considered within heat transfer. A comparison with available literature is also carried out. The key effects of the existing study can be prescribed as below:

- Since the magnetic field creates a drag force, liquid velocity and thickness of BL reduce when Hartman number M for both slip and no-slip conditions is increased. Whereas, increasing values of the Hartman number M boosts the heat transfer and concentration field.

- The velocity, temperature, and CRS concentration profiles fall with increment in the thickness of wall parameter $\boldsymbol{\lambda}$ however, rise with a velocity power index m.
- Both radiation parameter R_d and $\boldsymbol{\theta}_w$ increase the temperature profiles.
- Prandtl and Schmidt's numbers decline the heat transfer and concentration field, respectively.
- Destructive CR (γ>0) reduces while constructive CR γ<0) enhances the species concentration with both slip and no-slip conditions.

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Original Scientific Paper

ON HOMOGENEOUS 2-DIMENSIONAL FINSLER MANIFOLDS WITH ISOTROPIC FLAG CURVATURES

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Abstract. We show that every Finsler surface with isotropic main scalar and isotropic flag curvature is Riemannian or relatively constant Landsberg metric. Using it, we prove that every homogeneous Finsler surface with isotropic flag curvature and isotropic main scalar is Riemannian or locally Minkowskian.
Keywords: Finsler surface, Landsberg metric, Riemannian surface.

1. Introduction

For a given Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(\Pi, y)$ is a function of tangent planes $\Pi = \text{span}\{y, v\} \subset T_xM$ and directions $y \in \Pi \setminus \{0\}$. If F is a Riemannian metric, then the flag curvature is independent of the direction and can be written as $\mathbf{K} = \mathbf{K}(\Pi)$. In this special case, $\mathbf{K}$ is called the sectional curvature of F. Also, F is said to be of scalar flag curvature if the flag curvature is a scalar function on the slit tangent space, namely $\mathbf{K} = \mathbf{K}(x, y)$. F is called of isotropic flag curvature if the flag curvature $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on the manifold M. A Riemannian metric is of scalar curvature if and only if $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on M, which is a constant in dimension $n > 2$ by the Schur lemma. One

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of the important problems in Finsler geometry is to study and characterize Finsler metrics of isotropic flag curvature.

In order to study the class of Finsler metrics of isotropic flag curvature, one may consider 2-dimensional Finsler metrics. In Finsler geometry, the behavior of 2-dimensional Finsler metrics is different and sometimes contradictory to the higher dimensions. For example, all 2-dimensional Finsler metrics are C-reducible, while they need not be of Randers or Kropina type. Also, Finsler surfaces are of scalar flag curvature, while these cases are not valid for higher dimensions. Due to the latter issue, Z. Shen constructed three families of Finslerian surfaces on $\mathbb{S}^2$ and $\mathbb{D}^2$ with constant flag curvature that are not projectively flat, and thus the Beltrami's famous theorem in Finsler geometry lost its validity in the world of Finslerian surfaces [14].

To study of Finsler surfaces separately, L. Berwald made a special frame for Finsler surfaces, namely Berwald's frame. In this frame, a function appears that depends of the tangent space of Finsler surface and distinguishes each metric from the other metrics. This function is known as the main scalar of the Finsler surface and denoted by $I = I(x, y)$. In [9], Matsumoto gave some geometrical meanings of the main scalar of Finsler surfaces. Very soon, Berwald discovered that the Finsler surfaces with constant main scalar are Berwald, Landsberg or Douglas surfaces [4]. Then, he characterized two-dimensional Finsler metrics with isotropic main scalar $I = I(x)$. Using this characterization, Berwald succeeded to find the classification of two-dimensional projectively flat Finsler metrics with isotropic main scalar [4]. These studies shows that the class of Finsler surfaces with isotropic main scalars has important position in Finsler geometry and deserves to more studies.

Among the class of two-dimensional Finsler metrics, homogeneous Finsler surfaces are interesting, and until now little study has been done on these spaces. Then, it is natural to study homogeneous Finsler manifolds. A Finsler manifold is called homogeneous if its group of isometries acts transitively on the manifold. In [5], Deng and Hou proved that the group of isometries $I(M, F)$ of a Finsler manifold (M, F) is a Lie transformation group of the underlying manifold which can be used to study homogeneous Finsler manifolds. In this case, M can be written as the quotient manifold $I(M, F)/H$, where H is the stabilizer subgroup at a point in M. Recently, the authors proved that there is not any unicorn among the homogeneous Finsler surfaces [18]. In this paper, we study homogeneous Finsler surfaces with isotropic main scalar $I = I(x)$ and isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$, and prove the following rigidity result.

Theorem 1.1. *Every homogeneous Finsler surface with isotropic main scalar and isotropic flag curvature is Riemannian or locally Minkowskian.*

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent space and $TM_0 := TM - \{0\}$ the slit tangent space of M. A Finsler structure on manifold M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on

TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$; (iii) The quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive-definite on T_xM

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \Big[F^2(y + su + tv) \Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then, the pair (M, F) is called a Finsler manifold.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \Big[\mathbf{g}_{y+tw}(u, v) \Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y) \mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1} \mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y) u^i$, where

$$I_i := g^{jk} C_{ijk}.$$

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. The distortion $\tau = \tau(x, y)$ on TM associated with the Busemann-Hausdorff volume form

$$dV_{BH} = \sigma(x) dx$$

is defined by

$$\tau(x, y) = \ln \frac{\sqrt{\det \big(g_{ij}(x, y) \big)}}{\sigma(x)}.$$

By definition, the distortion τ is homogeneous of degree 1 with respect to y, i.e., the following holds

$$\tau(\lambda y) = \lambda \tau(y), \qquad \lambda > 0, \; y \in T_xM_0.$$

The following holds.

Lemma 2.1. ([13]) *Let F be a positive-definite Finsler metric on a manifold M. Then the following conditions are equivalent*
(a) $\tau = constant$;
(b) $\mathbf{I} = 0$;
(c) $\mathbf{C} = 0$;
In any case, F reduces to a Riemannian metric.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M, F).

Define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \partial/\partial x^i|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2} \mathbf{g}_y \big(\mathbf{B}_y(u, v, w), y \big).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L_{ijk}(y) u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2} y_l B^l{}_{ijk}.$$

$\mathbf{L}$ is called the Landsberg curvature and F is called a Landsberg metric if $\mathbf{L} = 0$. Also, F is called of relatively isotropic Landsberg curvature if

$$L_{ijk} = cFC_{ijk},$$

where $c = c(x)$ is a scalar function on M.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y) u^i$, where

$$J_i := g^{jk} L_{ijk}.$$

The quantity $\mathbf{J}$ is called the mean Landsberg curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J} = 0$. By definition, every Landsberg metric is a weakly Landsberg metric. F is called of relatively isotropic mean Landsberg curvature if

$$J_i = cFI_i,$$

where $c = c(x)$ is a scalar function on M.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y) u^k \partial/\partial x^i$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature.

For a flag $P := \mathrm{span}\{y, u\} \subset T_xM$ with the flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{2.2}$$

The flag curvature $\mathbf{K} = \mathbf{K}(x, y, P)$ is a function of tangent planes $P = \mathrm{span}\{y, v\} \subset T_xM$. This quantity tells us how curved the space is at a point. A Finsler metric F is of scalar flag curvature, if $\mathbf{K}(x, y, P) = \mathbf{K}(x, y)$ is independent of P. In this case, the flag curvature is just a scalar function on the tangent space of M.

The pulled-back bundle π^*TM admits a unique linear connection, called the Berwald connection. Let (M, F) be an n-dimensional Finsler manifold. Let $\{e_j\}$ be a local frame for π^*TM, $\{\omega^i, \omega^{n+i}\}$ be the corresponding local coframe for $T^*(TM_0)$ and $\{\omega^i_j\}$ be the set of local Berwald connection forms with respect to $\{e_j\}$. In local coordinate system, the Berwald connection determined by following

$$d\omega^i = \omega^j \wedge \omega^i_j, \tag{2.3}$$

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{2.4}$$

where

$$\omega^i := dx^i,$$
$$\omega^{n+k} := dy^k + y^j \omega^k{}_j.$$

Thus

$$g_{ij|k} = -2L_{ijk}, \qquad g_{ij,k} = 2C_{ijk}.$$

For a tensor $\mathbf{T} = T_{i\cdots k}\omega^i \otimes \cdots \otimes \omega^k$, we have

$$T_{i\cdots k\ \cdot m} = \frac{\partial T_{i\cdots k}}{\partial y^m}.$$

For a non-zero vector $y \in T_xM$, the tensor $\mathbf{T}$ induces a multi-linear form

$$\mathbf{T}_y(u, \cdots, w) := T_{i\cdots k}(x, y)u^i \cdots w^k$$

on T_xM. Let $\sigma(t)$ denote the geodesic with $\dot\sigma(0)=y$. We have

$$\frac{d}{dt}\Big[\mathbf{T}_{\dot\sigma(t)}\Big(U(t),\cdots,W(t)\Big)\Big]=T_{i\cdots k|m}(\sigma(t),\dot\sigma(t))\dot\sigma^m(t)U^i(t)\cdots W^k(t),$$

where $U(t)=U^i(t)\partial/\partial x^i|_{\sigma(t)},\cdots,W(t)=W^k(t)\partial/\partial x^k|_{\sigma(t)}$ are linearly parallel vector fields along σ. Thus the Landsberg curvature is given by

$$L_{ijk}=C_{ijk|m}y^m. \tag{2.5}$$

3. Proof of Theorem 1.1

It is well known that for any Minkowskian plane $(V,\mathcal{F})$ and any vector $v\in V$ with $\mathcal{F}(v)\neq 0$, there is a non-zero vector $w\in V$ such that is orthogonal to v with respect to the fundamental tensor raised by Minkowski functional $\mathcal{F}$. The special and useful Berwald frame was founded and developed method by Berwald in order to study of two-dimensional Finsler spaces [4]. It works under the assumption that the fundamental tensor is positive-definite.

Let (M,F) be a two-dimensional Finsler manifold. It is easy to see that for every $\mathbf{y}\in T_xM$, $x\in M$, there is a vector $\mathbf{y}^\perp\in T_xM_0$ such that

$$\mathbf{g}(\mathbf{y},\mathbf{y}^\perp)=0,\qquad \mathbf{g}(\mathbf{y}^\perp,\mathbf{y}^\perp)=F(\mathbf{y}).$$

The pair $\{\mathbf{y},\mathbf{y}^\perp\}$ is called the Berwald frame at $\mathbf{y}$.

Based on the Berwald frame, the Cartan torsion can be determined by a scalar function on slit tangent bundle. Let us define

$$\mathcal{I}(\mathbf{y}):=\frac{\mathbf{C}_{\mathbf{y}}(\mathbf{y}^\perp,\mathbf{y}^\perp,\mathbf{y}^\perp)}{F(\mathbf{y})}=\mathcal{I}(\mathbf{y}^\perp).$$

One can see that $\mathcal{I}(\lambda\mathbf{y})=\mathcal{I}(\mathbf{y})$ holds for $\forall\lambda>0$ and $\forall\mathbf{y}\in T_xM_0$. We call $\mathcal{I}$ the main scalar of Finsler metric F.

In most of literature of Finsler geometry, the special notation (ℓ,m) was used instead of $\{\mathbf{y},\mathbf{y}^\perp\}$. By considering this notation, for a scalar $T=T(x,y)$, we define the horizontal scalar derivatives $(T_{|1},T_{|2})$ and vertical scalar derivatives $(T_{,1},T_{,2})$ as follows

$$T_{|i}:=T_{|1}\ell_i+T_{|2}m_i,\quad FT_{,i}:=T_{,1}\ell_i+T_{,2}m_i,$$

where

$$T_{|i}:=\frac{\partial T}{\partial x^i}-G^j{}_i\frac{\partial T}{\partial y^j},\qquad FT_{,i}:=F\frac{\partial T}{\partial y^i}$$

denote the horizontal and vertical derivations with respect to the Berwald connection of F and

$$G^j{}_i:=\frac{\partial G^i}{\partial y^j}.$$

In order to prove Theorem 1.1, we need to know the special form of Berwald curvature of Finsler surface. We remark that the following identity holds

$$B^p_{jkl} = g^{ip}\Big\{C_{ijl|k} + C_{ikl|j} - C_{jkl|i} + L_{ijk,l}\Big\}. \tag{3.1}$$

See (10.19) at page 145 in [13]. On the other hand, the Cartan torsion of a Finsler surface (M, F) has no components in the direction ℓ^i, i.e., $C_{ijk}y^i = 0$. Then it can be written in the Berwald frame (ℓ, m) as follows

$$FC_{ijk} = \mathcal{I}m_i m_j m_k. \tag{3.2}$$

Taking a horizontal derivation of (3.2) implies that

$$FC_{ijk|s} = \big(\mathcal{I}_{|1}\ell_s + \mathcal{I}_{|2}m_s\big)m_i m_j m_k. \tag{3.3}$$

Contracting (3.3) with y^s yields

$$FL_{ijk} = \mathcal{I}_{|1}m_i m_j m_k. \tag{3.4}$$

By putting (3.3) and (3.4) in (3.1), we get

$$F{B^i}_{jkl} = \Big\{-2\mathcal{I}_{|1}\ell^i + \big(\mathcal{I}_{|1,2} + \mathcal{I}_{|2}\big)m^i\Big\}m_j m_k m_l. \tag{3.5}$$

Let us put

$$\mathcal{I}_2 := \mathcal{I}_{|1,2} + \mathcal{I}_{|2}.$$

Thus the Berwald curvature of Finsler surfaces is given by

$${B^i}_{jkl} = \frac{1}{F}\Big(\mathcal{I}_2 m^i - 2\mathcal{I}_{|1}\ell^i\Big)m_j m_k m_l. \tag{3.6}$$

By (3.2) and (3.6), we have

$${B^i}_{jkl} = -\frac{2\mathcal{I}_{,1}}{\mathcal{I}}C_{jkl}\ell^i + \frac{\mathcal{I}_2}{3F}\Big\{h_{jk}h^i_l + h_{kl}h^i_j + h_{lj}h^i_k\Big\}, \tag{3.7}$$

where $\mathbf{h} = h_{ij}dx^i dx^j$ denotes the angular metric. Then for a Finsler surface, the Berwald curvature can be written as follows

$${B^i}_{jkl} = \mu C_{jkl}\ell^i + \lambda\Big(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}\Big), \tag{3.8}$$

where

$$\mu := -\frac{2}{\mathcal{I}}\mathcal{I}_{|1}, \qquad \lambda := \frac{1}{3}\mathcal{I}_2.$$

Proposition 3.1. *Every non-Riemannian Finsler surface with isotropic main scalar and isotropic flag curvature is a relatively constant Landsberg metric.*

Proof. A 2-dimensional Finsler metrics F is of scalar curvature $\mathbf{K}=\mathbf{K}(x,y)$. This is equivalent to the following identity:

$$R^i{}_k=\mathbf{K}F^2\, h^i_k. \tag{3.9}$$

The following hold

$$L_{ijk|m}y^m=-\frac{1}{3}F^2\Big\{\mathbf{K}_{.i}h_{jk}+\mathbf{K}_{.j}h_{ik}+\mathbf{K}_{.k}h_{ij}+3\mathbf{K}C_{ijk}\Big\} \tag{3.10}$$

and

$$J_{k|m}y^m=-F^2\Big\{\mathbf{K}_{.k}+\mathbf{K}I_k\Big\}. \tag{3.11}$$

Contracting (3.8) with y_i implies that

$$L_{jkl}+\frac{1}{2}\mu FC_{jkl}=0. \tag{3.12}$$

Taking a trace of (3.12) implies that

$$J_i=-\frac{1}{2}\mu FI_i. \tag{3.13}$$

By taking a horizontal derivation of (3.13) along the Finslerian geodesics yields

$$J_{i|s}y^s=-\frac{F}{4}\Big(2\mu_{x^k}y^k-\mu^2F\Big)I_i. \tag{3.14}$$

By (3.11), (3.14) and $I_k=\tau_{.k}$, we get

$$\mathbf{K}_{y^i}+\frac{1}{4}\Big(4\mathbf{K}+\mu^2(x)-\frac{2}{F}\mu_{x^k}y^k\Big)\tau_{y^i}=0. \tag{3.15}$$

Now, suppose that $\mathbf{K}=\mathbf{K}(x)$ is a scalar function on M. Then (3.15) simplifies to

$$\Big(4\mathbf{K}+\mu^2-\frac{2}{F}\mu_0\Big)\tau_{y^i}=0. \tag{3.16}$$

where $\mu_0:=\mu_{x^k}y^k$.

Now, we claim that $\mu(x)=c$ is a constant. If this is false, then there is an open subset $\mathcal{U}$ such that $d\mu(x)\neq 0$ for any $x\in\mathcal{U}$. Clearly, at any $x\in\mathcal{U}$,

$$\mathbf{K}(x)\neq\frac{1}{4}\Big(-\mu(x)^2+\frac{2\mu_0(x)}{F(x,y)}\Big)$$

for almost all $y\in T_xM$. By (3.16), $\tau_{.i}=I_i=0$. Thus F is Riemannian on $\mathcal{U}$ by Deicke's theorem. This contradicts with the assumption. Then $\mu=$ *constant*. □

Proposition 3.2. *Let (M, F) be a Finsler surface. Suppose that F has isotropic main scalar and isotropic flag curvature. Then for any geodesic $\gamma = \gamma(t)$ and any parallel vector field $X = X(t)$ along γ, the following function*

$$\mathbf{C}(t) = \mathbf{C}_{\dot\gamma}\big(X(t), X(t), X(t)\big), \tag{3.17}$$

satisfies the following equation

$$\mathbf{C}(t) = \exp\Big(-\frac{1}{2}\mu t\Big)\mathbf{C}(0). \tag{3.18}$$

Proof. By definition, we have

$$\mathbf{L}_y(u, v, w) + \frac{1}{2}\mu F \mathbf{C}_y(u, v, w) = 0. \tag{3.19}$$

where $\mu = constant$. Let us define

$$\mathbf{L}(t) = \mathbf{L}_{\dot\gamma}\big(X(t), X(t), X(t)\big). \tag{3.20}$$

From the definition of $\mathbf{L}_y$, we have

$$\mathbf{L}(t) = \mathbf{C}^{'}(t). \tag{3.21}$$

Then, (3.19) can be written as follows

$$\mathbf{C}^{'}(t) = -\frac{1}{2}\mu \mathbf{C}(t). \tag{3.22}$$

Integration (3.22) gives (3.18). □

Proof of Theorem 1.1: The proof has two main cases as follows:

Case 1: If $\mu = 0$, then F is a Landsberg metric. In [18], we proved that every homogeneous Landsberg surface is Riemannian or locally Mikowskian.

Case 2: If $\mu \neq 0$. In this case, we have (3.18). In [17], it is proved that every homogeneous Finsler manifold is complete. By definition, every two points of a homogeneous Finsler manifold (M, F) map to each other under an isometry. This causes the norm of an invariant tensor under the isometries of a homogeneous Finsler manifold is a constant function on M, and consequently, it has a bounded norm. Using this fact, we showed that for a homogeneous Finsler manifold (M, F), every invariant tensor under the isometries of F has a bounded norm with respect to it [16]. Then letting $t \to -\infty$ in (3.18) and using $||\mathbf{C}|| < \infty$ implies that $\mathbf{C}(0) = 0$ and then $\mathbf{C}(t) = 0$. Then F reduces to a Riemannian metric. □

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ON THE EXISTENCE AND EXAMPLES OF HOMOGENEOUS GEODESICS IN GENERALIZED m-KROPINA SPACE

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Abstract. In this paper, we find a necessary and sufficient condition for a non-zero vector to be a geodesic vector in homogeneous generalized m-Kropina space. Further, we prove the existence of at least one homogeneous geodesic. However, it is conjectured that the outcomes and proofs in the case of Finsler geometry are not ideal, since gen-eral Finsler metrics are non-reversible. In Finsler geometry, the trajectory of unique homogeneous geodesic should be regarded as two geodesics with initial vectors X and $-X$. Hence, we construct an $(n+1)$-dimensional and a 4-dimensional space to find homogeneous geodesics explicitly.

Keywords: generalized m-Kropina space, Finsler geometry, homogeneous geodesic.

1. Introduction

A geodesic can be thought of literally as a curve that reduces the distance between two places. Homogeneous geodesics have gained attention in both Riemannian and Finsler geometry recently. A geodesic $\gamma(t) : \mathbb{R} \to M$ in a Finsler manifold

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(M, F) is said to be homogeneous geodesic, if there exists one-parameter group of isometries $\phi : \mathbb{R} \times M \to M$ such that

$$\gamma(t) = \phi(t, \gamma(0)), t \in \mathbb{R}.$$

Geodesics are treated similar to relative equilibria in mechanics and physics. The qualitative description of the behavior of the related mechanical system with symmetries depends on the description of such relative equilibria. Geodesics have always been exciting to find and study, and this has been true since geometry's inception. Due to the numerous uses of geodesics and homogeneous geodesics in physics [22, 5, 6, 23] and other mathematics disciplines, there has been an interest in their study recently.

There is a lot of literature in mechanics devoted to the investigation of relative equilibria. In [1], author extended Euler's theory of rigid-motions while studying left invariant Riemannian metrics on Lie groups. In [20], the author discussed that in homogeneous space with an invariant metric, geodesic flow can be seen as framework of Smale's mechanical system with symmetries. T$\acute{o}$th [26] studied the paths that were orbits of one-parameter symmetry group G. In fact, he discovered the conditions for solutions of Euler-Lagrange or Hamiltonian equations to coincide with the orbit of a one-parameter subgroup of a symmetry group.

Kajzer has studied the existence of homogeneous geodesics in [14]. In this study, the authors showed that in Lie groups with left invariant metrics, at least one homogeneous geodesic element can travel through the identity element. Kowalski and Szenthe [17] also showed that every homogeneous Riemannian manifold has at least one homogeneous geodesic across each point.

Additionally, Kowalski and Vl$\acute{a}\check{s}$ek [18] established a few examples of homogeneous Riemannian manifolds of any dim $n \geq 4$ with precisely one homogeneous geodesic. Latiffi [21] proposed the term 'geodesic vector' in homogeneous Finsler space and proved that any vector in every connected Lie group with a bi-invariant Finsler metric is a geodesic vector.

Recently, the existence of homogeneous geodesic for infinite series metric and exponential metric have been discussed in [15]. Also, some important results related to homogeneous Finsler spaces have been established in [25]. In homogeneous Kropina spaces, the existence of homogeneous geodesic through any arbitrary point have been discussed in [13] and it is also proved that under some conditions result holds for any (α, β)-homogeneous space. In this paper, homogeneous geodesics of 3-dimensional non-unimodular real Lie groups equipped with a left invariant Randers metric of Douglas type are also discussed as an example. In [10], author has showed the examples of homogeneous Randers manifold admitting just two homogeneous geodesic. In [2], authors have extended the study of left-invariant (α, β)-metrics on 4-dimensional Lie groups.

2. Preliminaries

In this section, we discuss basic definitions and notations of Finsler geometry. For more elaborate concepts of Finsler geometry and homogeneous Finsler geometry, refer [3, 4, 7]. Let V be an n-dimensional real vector space endowed with smooth norm F on $V\backslash\{0\}$, which is non-negative i.e., $F(u) \geq 0 \ \forall \ u \ \in V$, positively homogeneous i.e., $F(\lambda u) = \lambda F(u) \ \forall \ \lambda > 0$, and strongly convex i.e., if $\{u_1, u_2, ..., u_n\}$ be the basis of V such that $y = y^1u_1 + y^2u_2 + ... + y^nu_n$, then the Hessian matrix $(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^iy^j}\right)$, is positive definite at every point of $V\backslash\{0\}$. The pair (V, F) is called Minkowski space and F is called Minkowski norm.
Let M be a connected (smooth) manifold. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which satisfies:

1. F is smooth on slit tangent bundle $TM\backslash\{0\}$,

2. The restriction of F to any $T_xM, x \in M$ is a Minkowski norm.

The space (M, F) is called Finsler space. Let $\gamma : [0, 1] \to M$ be a C^1-curve. Then Finsler length $L(\gamma)$ of γ is defined as

$$L(\gamma) = \int_0^1 F(\gamma(t), \gamma^{'}(t))dt.$$

Further, Finsler distance $d_F(p, q)$ between two points $p, q \in M$ is defined as

$$d_F(p, q) = inf_\gamma L(\gamma),$$

where infimum is taken over all piecewise C^1-curves joining p and q.

Definition 2.1. Let $F = \alpha\phi(s); s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold with $||\beta|| < b_0$. Then, F is Finsler metric if and only if following conditions are satisfied:

$$\phi(s) > 0, \quad \phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0 \ \ \forall \ |s| \leq b < b_0. \tag{2.1}$$

An (α, β)-metric is said to be singular Finsler metric, if either $\phi(0)$ is not defined or $\phi(s)$ does not satisfy 2.1. In this paper, we study generalized m-Kropina spaces, which form a special class of (α, β)-metric. Kropina metric is a type of non-regular (α, β)-metric where $\phi(s) = \frac{1}{s}$, i.e., $F = \frac{\alpha^2}{\beta}$. The concept is proposed by Russian physicist V. K. Kropina [19]. Despite having singularities ($\beta = 0$), it is useful in the Lagrangian function's representation of the general dynamic system. Hence, due to the physical and applied importance of Kropina metric, we here investigate geodesics for generalized m-Kropina metric. Generalized m-Kropina metric is an important class of (α, β)-metric defined as

$$F(\alpha, \beta) = \frac{\alpha^{m+1}(x, y)}{\beta^m(x, y)}, \ (m \neq 0, 1).$$

Consider the inner product $\langle , \rangle$ on tangent space $T_xM, \ x \in M$ defined as

$$\langle u, v \rangle = a_{ij} u^i v^j, \ u, v \in T_xM,$$

where a_{ij} is a Riemannian metric.
Using the above defined inner product we induce an inner product on the cotangent space, T_x^*M, of M at x,

$$\langle dx^i, dx^j \rangle = a^{ij}.$$

Using this inner product, a linear isomorphism can be defined between T_xM and T_x^*M [9]. Hence, 1-form β corresponds to smooth vector field X on M given by

$$X|_x = b^i \frac{\partial}{\partial x^i}, \ b^i = a^{ij} b_j,$$

which further implies

$$\langle X|_x, y \rangle = \langle b^i \frac{\partial}{\partial x^i}, y^j \frac{\partial}{\partial x^j} \rangle = b^i y^j a_{ij} = b_j y^j = \beta(y).$$

Also, $||\beta|| = \alpha(X|_x) < 1$. On the basis of above discussion, w can conclude the following Lemma:

Lemma 2.1. *Let (M, α) be a Riemannian space. Then the generalized m-Kropina space, (M, F) where $F = \frac{\alpha^{m+1}}{\beta^m}, (m \neq -1, 0, 1)$ $\beta = b_i y^i$, a 1-form with $||\beta|| = \sqrt{b_i b^i}$, consists of Riemannian metric α along with a smooth vector field X on M, which satisfies $\alpha(X|_x) < 1 \ \forall \ x \in M$, i.e.,*

$$F(x, y) = \frac{\alpha(x, y)^{m+1}}{\langle X|_x, y \rangle^m},$$

where $\langle , \rangle$ is the inner product on T_xM induced by the Riemannian metric α.

Let (M, F) be a Finsler space. A diffeomorphism of M onto itself is said to be isometry, if it preserves the Finsler function, i.e., $F(\phi(p), d\phi_p(X)) = F(p, X)$ for any $p \in M$ and $X \in T_pM$. Let G be a Lie group and M a smooth manifold. If G has smooth action on M, then G is called Lie transformation group of M. A connected Finsler space (M, F) is said to be homogeneous Finsler space, if action of group of isometries of (M, F), denoted by $I(M, F)$ is transitive on M.
Let $G \subset I(M, F)$ be a connected Lie group acting transitively on Finsler space (M, F), and at a fixed point $p \in M$, let H be its isotropy group. Then M can be written as coset space G/H, with a G-invariant Finsler metric F. It is evident to

see that H is comapct, since action of H leaves invariant unit sphere in T_pM. Hence, we obtain reductive decomposition of $\mathfrak{g}$, Lie algebra of G as

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively and $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $Ad(\mathfrak{h})(\mathfrak{m}) \subset \mathfrak{m}$, where Ad denotes Adjoint representation of G.

Remark 2.1. [7] A homogeneous Finsler manifold $M = G/H$ is reductive homogeneous space.

Next proposition shows that G-invariant Finsler metrics on G/H can be identified with Minkowski norm F as follows:

Proposition 2.1. *[8] Let G/H be a reductive homogeneous manifold satisfying*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

Then there exists a one-to-one correspondence between the G-invariant Finsler metrics on G/H and the Minkowski norms F on $\mathfrak{m}$ which satisfy

$$F(Ad(h)x) = F(x), \ \forall \ h \in H, x \in \mathfrak{m}.$$

A regular smooth curve γ with velocity vector $T = \dot{\gamma}$, is said to be Finslerian geodesic, if it satisfies

$$D_T\left(\frac{T}{F(T)}\right) = 0,$$

with reference vector T. Here, D is defined from Chern connection, which is torsion free and almost metric compatible. A geodesic $\gamma(t)$ passing through origin $eH \in M = G/H$ is said to be homogeneous if it is one-parameter subgroup of G, i.e., $\gamma(t) = exp(tZ)(eH)$, $t \in \mathbb{R}$ and Z is a non zero vector in Lie algbera of G. A non-zero vector $X \in \mathfrak{g}$ is said to be a geodesic vector, if the curve $exp(tX)(eH)$ is constant speed geodesic of (M, F). If all the geodesics of a Riemannian manifold M are homogeneous, then M is callled g.o.(geodesic orbit) space.
A Finsler space (M, F) is called a Finsler g.o. space, if every geodesic of (M, F) is the orbit of a one-parameter subgroup of $G = I(M, F)$, i.e., if $\phi : \mathbb{R} \to M$ is a geodesic, then there exists a non-zero vector $Z \in \mathfrak{g} = Lie(G)$ and $p \in M$ such that $\phi(t) = exp(tZ).p$.
More precisely, a Finsler space (M, F) is called Finsler g.o.(geodesic orbit) space, if and only if the projections of all the geodesic vectors cover the set $T_{eH}(G/H) - \{0\}$. A Finsler g.o. space has vanishing S-curvature for Busemann volume form [21, 7]. Further, every Finsler g.o. space is homogeneous [7].
The following result provides criterion to study geodesic vector in Lie algbera level and hence provide a useful tool to study homogeneous geodesic.

Lemma 2.2. *[21] Suppose $(G/H, F)$ is a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if it satisfies*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \ \forall Z \in \mathfrak{g},$$

where the subscript $\mathfrak{m}$ denotes the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.

3. Necessary and sufficient condition

In this section, we discuss homogeneous geodesic in homogeneous generalized m-Kropina space. We provide some necessary and sufficient condition for a non-zero vector to be geodesic vector in homogeneous generalized m-Kropina space.

Corollary 3.1. *Let $(G/H, F)$ be a homogeneous Finsler space equipped with generalized m- Kropina metric arising from an invariant Riemannian metric $\langle , \rangle$ and an invariant vector field $\tilde{X}$, such that $X = \tilde{X}(H)$. Then necessary and sufficient condition for a non-zero vector $Y \in \mathfrak{g}$ to be a geodesic vector is*

(3.1)
$$\frac{\langle Y_{\mathfrak{m}}, Y_{\mathfrak{m}}\rangle^m}{\langle X, Y_{\mathfrak{m}}\rangle^{2m+1}} \left[(m+1)\langle X, Y_{\mathfrak{m}}\rangle\langle Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}\rangle - m\langle Y_{\mathfrak{m}}, Y_{\mathfrak{m}}\rangle\langle X, [Y, Z]_{\mathfrak{m}}\rangle\right] = 0, \quad \forall Z \in \mathfrak{m}.$$

Proof. Using the formula (2.7) for (α, β)-metric from [24], we get the following corollary directly by taking $\phi(s) = \frac{1}{s^m}$. □

Further, we use Theorem 2.2 of [24] to get the following remark:

Remark 3.1. Let $(G/H, F)$ be a homogeneous generalized m-Kropina space with assumptions same as taken in Theorem 3.1. Then the vector X is a geodesic vector of $(G/H, \langle , \rangle)$ if and only if it is a geodesic vector of $(G/H, F)$. In other words, a non zero vector is a geodesic vector of generalized m-Kropina metric if and only if it is a geodesic of its base Riemannian metric.

Also, as direct consequence of Corollary 3.1, we can conclude the following corollary:

Corollary 3.2. *Let $(G/H, F)$ be a homogeneous generalized m-Kropina space with assumptions same as taken in Theorem 3.1. Let $Y \in \mathfrak{g}$ be a non-zero vector such that $\langle X, [Y, Z]_{\mathfrak{m}}\rangle = 0 \ \forall \ Z \in \mathfrak{m}$. Then Y is a geodesic vector of $(G/H, \langle , \rangle)$ if and only if it is a geodesic vector of $(G/H, F)$.*

4. Existence

In this section, we prove the existence of atleast one homogeneous geodesic on homogeneous generalized m-Kropina space passing through origin.

Proposition 4.1. *Let $(G/H, F)$ be homogeneous generalized m-Kropina space. Then there exists atleast one homogeneous geodesic arising from each origin.*

Proof. Suppose $G \subset I(M,F)$ be connected Lie group acting transitively on (M,F). Let H be isotropy group at $\{eH\} \in G/H$. Let $\mathcal{K}$ be killing form and $rad\mathcal{K}$ be its null space.
Firstly, let us suppose $rad\mathcal{K} = \mathfrak{m}$. In [17], it is proved that Lie algebra $\mathfrak{g}$ has reductive decomposition $\mathfrak{m} + \mathfrak{h}$ such that $\mathfrak{m}$-projection $[\mathfrak{g},\mathfrak{g}]$ is a proper subspace of $\mathfrak{m}$. Consider $Y \in [\mathfrak{g},\mathfrak{g}]_{\mathfrak{m}}$ be a non-zero vector which satisfies $\langle Y, Y\rangle = 1$. Let $W = X \in [\mathfrak{g},\mathfrak{g}]^{\perp}_{\mathfrak{m}}$. We use Theorem 3.1 to check that W is a geodesic vector. Since, equation 3.1 with respect to W can be written as:

$$\frac{\langle W_{\mathfrak{m}}, W_{\mathfrak{m}}\rangle^m}{\langle X, W_{\mathfrak{m}}\rangle^{2m+1}}\left[\langle X, W_{\mathfrak{m}}\rangle\langle (m+1)W_{\mathfrak{m}}, [W,Z]_{\mathfrak{m}}\rangle - \langle W_{\mathfrak{m}}, W_{\mathfrak{m}}\rangle\langle mX, [W,Z]_{\mathfrak{m}}\rangle\right] = 0,$$

which implies that

$$\frac{\langle W_{\mathfrak{m}}, W_{\mathfrak{m}}\rangle^m}{\langle X, W_{\mathfrak{m}}\rangle^{2m+1}}\left[\langle X, [W,Z]_{\mathfrak{m}}\rangle\right] = 0.$$

This proves the existence of atleast one geodesic through origin.
Secondly, we suppose $rad\mathcal{K} \subsetneq \mathfrak{m}$. If $rad\mathcal{K}$ is a proper subset of $\mathfrak{m}$, then from [27], it is proved that $\mathfrak{m}$ can be decomposed into eigenspaces as $\mathfrak{m} = V_0 + V_1 + ...V_r$ with respect to $\mathcal{K}$-symmetric endomorphism defined as $K(X,Y) = \langle\theta(X), Y\rangle$ which satisfies $V_0 = radK_0$. Consider $\{f_1, f_2, f_3, ..., f_r\}$ be an orthonormal basis of $V = V_0 + V_1 + ... + V_r$ and θ be an endomorphism $\theta(f_i) = \lambda_i f_i$ for $i = 1,2,...,r$. Suppose that $X = X_0 + \sum_{i=1}^r x_i f_i, Y = Y_0 + \sum_{i=1}^r y_i f_i, \quad X_0, Y_0 \in V_0, \quad x_i, y_i \in \mathbb{R}$. Using Theorem 3.1, $Y \in \mathfrak{g}$ is a geodesic vector if and only if equation 3.1 equals to zero.

Hence, let us consider

$$\begin{aligned}
&[(m+1)\langle X, Y_{\mathfrak{m}}\rangle\langle Y_{\mathfrak{m}}, [Y,Z]_{\mathfrak{m}}\rangle - m\langle Y_{\mathfrak{m}}, Y_{\mathfrak{m}}\rangle\langle X, [Y,Z]_{\mathfrak{m}}\rangle] \\
&= [(m+1)\langle X_0 + \sum_{i=1}^r x_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle Y_0 + \sum_{i=1}^r y_i f_i, [Y,Z]_{\mathfrak{m}}\rangle \\
&\quad - m\langle Y_0 + \sum_{i=1}^r y_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle X_0 + \sum_{i=1}^r x_i f_i, [Y,Z]_{\mathfrak{m}}\rangle] \\
&= [(m+1)\langle X_0 + \sum_{i=1}^r x_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle Y_0, [Y,Z]_{\mathfrak{m}}\rangle \\
&\quad - m\langle Y_0 + \sum_{i=1}^r y_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle\langle X_0, [Y,Z]_{\mathfrak{m}}\rangle] \\
&\quad + (m+1)\langle X_0 + \sum_{i=1}^r x_i f_i, Y_0 + \sum_{i=1}^r y_i f_i\rangle K\left([Y,Z]_{\mathfrak{m}}, \sum_{i=1}^r y_i \frac{f_i}{\lambda_i}\right) \\
&\quad - m\langle Y_0 + \sum_{i=1}^r y_i f_i, Y_0 + \sum_{i=1}^r y_i f_i K\left([Y,Z]_{\mathfrak{m}}, \sum_{i=1}^r x_i f_i\right)\rangle
\end{aligned} \tag{4.1}$$

(4.2)

$$= (m+1)\left[\langle X_0 + \sum_{i=1}^{r} x_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle\right]\left[\langle Y_0, [Y,Z]_{\mathfrak{m}}\rangle + K\left([Y,Z]_{\mathfrak{m}}, \sum_{i=1}^{r} y_i \frac{f_i}{\lambda_i}\right)\right]$$
$$- m\left[\langle Y_0 + \sum_{i=1}^{r} y_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle][\langle X_0, [Y,Z]_{\mathfrak{m}}\rangle + K\left([Y,Z]_{\mathfrak{m}}, \sum_{i=1}^{r} x_i \frac{f_i}{\lambda_i}\right)\right]$$
$$= (m+1)[\langle X_0 + \sum_{i=1}^{r} x_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle] K(Z, [Y,Y]_{\mathfrak{m}})$$
$$- m\langle Y_0 + \sum_{i=1}^{r} y_i f_i, Y_0 + \sum_{i=1}^{r} y_i f_i\rangle K(Z, X_0 + \sum_{i=1}^{r} x_i \lambda_i f_i, Y_0 + \sum_{i=1}^{r} y_i \lambda_i f_i).$$

The first term in last resultant of above equation 4.2 vanishes, which on plugging into equation 3.1, we get

$$m\frac{\langle Y,Y\rangle^{m+1}}{\langle X,Y\rangle^{2m+1}}\left[K(Z,[x_0 + \sum_{i=1}^{r} x_i\lambda_i y_i], y_0 + \sum_{i=1}^{r} y_i \lambda_i f_i)\right]. \tag{4.3}$$

Above equation vanishes, whenever we have a solution in the form $(Y_0, y_1, ..., y_r, t)$. It is obvious to check that $\{Y_0 = X_0, y_1 = t_0 x_1, ..., y_r = t_0 x_r, t = t_0\}$ is a solution to satisfy above equation. This completes the proof. $\square$

In fact, in [11] author has showed existence of two homogeneous geodesics in any arbitrary homogeneous Finsler spaces. Hence, in particular, above proposition can be extended to say that there exists two homogeneous geodesics in this space. With this motivation in the next section, we construct an $(n+1)$-dimensional and 4-dimensional example and find homogeneous geodesics.

5. Examples of some homogeneous geodesic vectors

In this section, we visualize the homogeneous geodesics in an $(n+1)$-dimensional space and a 4-dimensional space. Let us consider a Lie algebra $\mathfrak{n}$ with orthonormal basis $\mathfrak{B} = \{e_1, e_2, ..., e_{n+1}\}$ generated by Lie brackets as follows:

$$[e_i, e_j] = 0, \quad \forall i, j \leq n$$
$$[e_{n+1}, e_i] = a_i e_i + e_{i+1}, \quad \forall i < n$$
$$[e_{n+1}, e_n] = a_n e_n$$

for arbitrary non-zero parameters $a_1, a_2, ..., a_n \in \mathbb{R}$. The family of Lie algebras $(\mathfrak{n}, \langle,\rangle)$ generates an (n-parameter) solvable Lie groups $\mathcal{N}$ with a set of invariant Riemannian metrics. In [18], authors showed that for generic choices of $\{a_i\}_{i=1}^{n}$ the corresponding group $\mathcal{N}$ acting by left translations is the maximal group of isometries. In [18] authors have assumed that $\mathcal{N}$ is diffeomorphic to $(n+1)$-dimensional

Euclidean space. We use a similar approach as in [10] to solve our further result. For the sake of simplicity, we shall consider metric F generated by the vector $X = ke_1, 0 < k < 1$. which are suitable for our purpose.

Example 5.1. Let (G, F) be an $(n+1)$-dimensional homogeneous generalized m-Kropina space, such that the parameters constructed above satisfies $min\{a_i\} > n$ and left-invariant metric F is determined by $X = ke_1$ and also $ka_1 < 1$. Then (G, F) admits exactly two geodesics whose initial vectors are $\tau_1 = c_1e_{n+1} + \frac{m}{m+1}kF(Y_{\mathfrak{m}})e_1$, and $\tau_2 = -c_1e_{n+1} + \frac{m}{m+1}kF(Y_{\mathfrak{m}})e_1$.

An arbitrary vector $Y \in \mathfrak{g}$ can be expressed with respect to the basis $\mathfrak{B} = \{e_1, e_2, ..., e_{n+1}\}$ as $Y = y_1e_1 + y_2e_2 + ... + y_{n+1}e_{n+1}$. The Lie brackets can be calculated as follows:

$$[Y, e_i] = y_{n+1}(a_ie_i + e_{i+1}), \quad 1 \leq i < n,$$

$$[Y, e_n] = y_{n+1}a_ne_n,$$

$$[Y, e_{n+1}] = -y_1a_1e_1 - \sum_{i=2}^{n}(y_{i-1} + y_ia_i)e_i.$$

Next, we plug the vector $Z \in \mathfrak{m}$ in equation 3.1 step by step for all elements of orthonormal basis $\mathfrak{B}$. Using Theorem 3.1 we get the $Y \in \mathfrak{g}$ is geodesic vector, if it satisfies the following homogeneous system of equations:

$$\begin{gathered}(m+1)[y_{n+1}(a_1y_1 + y_2) - mF(Y_{\mathfrak{m}})ka_1] = 0,\\ (m+1)[y_{n+1}(a_iy_i + y_{i+1})] = 0, \quad 1 < i < n\\ (m+1)y_{n+1}a_ny_n = 0,\\ (m+1)[-y_1^2a_1 - \sum_{i=2}^{n}(y_{i-1} + y_ia_i)y_i] - mF(Y_{\mathfrak{m}})ky_1a_1 = 0.\end{gathered}$$

In order to solve system of equations, first let us consider the case if $y_{n+1} \neq 0$. Due to homogeneity of equations, without loss of generality we may assume $y_{n+1} = \pm\, c$. Consequently, from all equations for $i = 1, ..., n$ we immediately get $y_n = y_{n-1} = ... = y_2 = 0$ and $y_1 = \left(\frac{m}{m+1}\right)kF(Y_{\mathfrak{m}})$. Hence, we obtain just two geodesics solutions for above system of equations.
Next, let us consider second case that $y_{n+1} = 0$, first n equations are satisfied immediately. For the last equation, we solve for polynomial $p(y_i) = 0$, where

$$p(y_i) = (m+1)y_1^2a_1 + (m+1)\sum_{i=2}^{n}y_iy_{i-1} + \sum_{i=2}^{n}y_i^2a_i + \sum_{i=2}^{n}y_i^2a_i + mka_1y_1F(Y_{\mathfrak{m}}).$$

On using the estimates $|y_iy_{i+1}| < 1$ and $\min a_i > n$, we get that $p(y_i) > 0$, which implies above system of equation doesn't have any other non trivial solution. This completes the proof.

Example 5.2. Consider a 4-dimensional (R^4, F) equipped with m-Kropina metric, which can be written as homogeneous space G/H where G is the 5-dimensional group of equiaffine transformations of a Euclidean space and H is group of rotations around origin. Also $\mathfrak{g}$

has reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, an orthonormal basis (e_1, e_2, e_3, e_4) of $\mathfrak{m}$ and generarator Λ of $\mathfrak{h}$. Using the multiplication table from [16], we have

$$[e_1, e_2] = 0, [e_1, e_3] = -e_1, [e_1, e_4] = e_1,$$

$$[e_2, e_3] = e_2, [e_2, e_4] = e_1, [e_3, e_4] = -2\Lambda,$$

$$[\Lambda, e_1] = -e_2, [\Lambda, e_2] = e_1, [\Lambda, e_3] = 2e_4, [\Lambda, e_4] = -2e_3.$$

Also, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Suppose $y \in \mathfrak{g}$ be geodesic vector,

$$y = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + q\Lambda$$

Using equation 3.1, we get the following set of equations:

$$\begin{aligned} &(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1(y_3 - y_4) - y_2q) \\ &- m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1(y_3 - y_4) - x_2q) = 0, \end{aligned} \tag{5.1}$$

$$\begin{aligned} &(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1(y_3 - y_4) - y_2q) \\ &- m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1(q - y_4) - x_2y_3) = 0, \end{aligned} \tag{5.2}$$

$$\begin{aligned} &(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(-y_1^2 + y_2^2 + 2qy_4) \\ &- m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(-x_1y_1 + x_2y_2 + 2qx_4) = 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1(y_1 + y_2) - 2qy_3) \\ &- m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1(y_1 + y_2) - 2qx_3) = 0. \end{aligned} \tag{5.4}$$

Using above equations, we also get

$$\begin{aligned} &(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_1 + y_2)(y_3 - q) \\ &- m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(x_1 + x_2)(y_3 - q) = 0, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &(m+1)(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)(y_2(y_1 + y_2) + 2q(y_4 - y_3)) \\ &- m(y_1^2 + y_2^2 + y_3^2 + y_4^2)(y_2(x_1 + x_2) + 2q(x_4 - x_3)) = 0. \end{aligned} \tag{5.6}$$

We consider some assumptions to see geodesic vectors explicitly:
A: $X = x_3e_3$
B: $X = x_4e_4$
C: $X = x_3(e_3 + e_4)$
D: $X = x_1(e_1 - e_2)$
A : For $X = x_3e_3$, equation 5.5 gives

$$(m+1)(x_3y_3)(y_1 + y_2)(y_3 - q) = 0.$$

(1) Let us suppose $y_1 = -y_2$, $y_3 \neq 0$ and $y_3 \neq q$. In this case equation 5.3, implies $2(m+1)(x_3y_3)qy_4 = 0$, which again gives two cases (a) and (b)	(a) $q = 0$ and $y_4 \neq 0$, this implies $y_1(y_3 - y_4) = 0$. If $y_1 = 0$, we also have $y_2 = 0$, which shows $y = y_3e_3 + y_4e_4$. If $y_3 - y_4 = 0$, implies $\boldsymbol{y = y_1(e_1 - e_2) + y_3(e_3 + e_4)}$.	(b) If $q \neq 0$ and $y_4 = 0$, then using equations 5.1 and 5.2, we get $y_1(y_1 + q) = 0$. And again here, if $y_1 = 0$, $y_3 \neq -q$ we have $y = y_3e_3 + q\Lambda$, otherwise for $y_1 \neq 0$ and $y_3 = -q$, we have $\boldsymbol{y = y_1(e_1 - e_2) + y_3(e_3 - \Lambda)}$.

(2) Next, we assume $y_1 \neq -y_2, y_3 \neq 0$, and $y_3 = q$. On plugging these into equation 5.3, we get $y_4 = \frac{y_1^2 - y_2^2}{2y_3}$, which gives geodesic vector $\boldsymbol{y = y_1e_1 + y_2e_2 + y_3e_3 + \frac{y_1^2 - y_2^2}{2y_3}e_4 + y_3\Lambda}$.

(3) At last we suppose $y_1 = -y_2, y_3 \neq 0$ and $y_3 = q$. On plugging into (5.3), we get $2(m+1)x_3q^2y_4 = 0$. This takes us to two cases:	(a) If $q = 0$ and $y_4 \neq 0$, this implies $\boldsymbol{y = y_4e_4}$ or $\boldsymbol{y = y_1(e_1 - e_2)}$.	(b) On taking $q \neq 0$, $y_4 = 0$ in (5.1), we get $2qy_1 = 0$ which vanishes y_1. So the geodesic vector is $\boldsymbol{y = y_3(e_3 + \Lambda)}$.

(4) Next we suppose, $y_1 \neq -y_2, y_3 \neq q, y_3 = 0$, from equation 5.4, we have $2mqx_3(y_1^2 + y_2^2 + y_4^2) = 0$. This gives that q vanishes and we get the geodesic vector as $\boldsymbol{y = y_1e_1 + y_2e_2 + y_4e_4}$.

Case (B) can be seen similar to the case(A). And it also coincides with the homogeneous geodesic in 4-dimensional Randers space example [12].

Case (C): On considering $X = x_3(e_3 + e_4)$, again from 5.5, we get

$$(m+1)x_3(y_3 + y_4)(y_1 + y_3)(y_3 - q) = 0.$$

This leads to different possiblities: (1) Let us Suppose $y_1 = -y_2, y_3 \neq q, y_3 \neq -y_4$,

(1) also from equation 5.6, we have $2q(m+1)x_3(y_3^2 - y_4^2) = 0$ which implies two cases, i.e., either $q = 0$ or $y_3 = y_4$	(a) If $q = 0$, and $y_3 \neq y_4$, from equation (5.1), we get $y_1(y_3 - y_4) = 0$, implies $y_1 = y_2 = 0$, which gives geodesic vector $\boldsymbol{y = y_3e_3 + y_4e_4}$.	If $q \neq 0$, $y_3 = y_4$, equation(5.1), gives $2qy_2(m+1)x_3(y_3+y_4) = 0$, which vanishes $y_3 = y_4 = 0$. Hence the geodesic vector takes the form $\boldsymbol{y = y_3(e_3 + e_4) + q\Lambda}$.

(2) In this case assume $y_1 \neq y_2, y_3 = q, y_3 \neq y_4$ using equation 5.6, we have

$$x_3(y_3 + y_4)[y_2(y_1 + y_2) + 2y_3y_4 - 2y_3^2] = 0.$$

Since, in this case $y_3 + y_4$ can't vanish. Hence, we get $2y_3^2 - 2y_3y_4 + y_2(y_3 + y_4) = 0$, which is quadratic in y_3. So the roots are $y_3 = \frac{y_4 \pm \sqrt{y_4^2 + 2(y_1 + y_2)y_2}}{2}$. So the geodesic vector y is written as

$$\boldsymbol{y_1e_1 + y_2e_2 + \frac{y_4 \pm \sqrt{y_4^2 + 2(y_1+y_2)y_2}}{2}(e_3 + \Lambda) + y_4e_4.}$$

(3) In third case, we assume $y_1 = -y_2, y_3 = q, y_3 \neq -y_4$, from using equation 5.6, we get $2x_3y_3(y_3+y_4)(y_3-y_4) = 0$, which leads to two cases:

(a) If $y_3 = 0$, the geodesic vector y takes form $\boldsymbol{y = y_1(e_1 - e_2) + y_4e_4.}$

(b) If $y_3 = y_4$, then $\boldsymbol{y = y_1(e_1 + e_2) + y_3(e_3 + e_4 + \Lambda).}$

(4) In this, let us assume $y_1 \neq y_2$, $y_3 \neq q$, $y_3 = y_4$, using the above assumptions in equation (5.4), we have $-4my_3^2qx_3 = 0$, which leads to two cases, i.e., either $y_3 = 0$ or $q = 0$	(a) $y_3 = 0$ implies geodesic vector takes the form $\boldsymbol{y = y_1e_1 + y_2e_2 + q\Lambda}$.	(b) If $q = 0$, we get $y = y_1e_1 + y_2e_2 + y_3(e_3 - e_4)$. If both $q = y_3 = 0$, y reduces to $\boldsymbol{y_1e_1 + y_2e_2}$.

(5) For this case, let us suppose $y_1 = -y_2, y_3 = -y_4, y_3 \neq q$. On plugging into equation 5.3, we get $-4mqy_3^2x_4 = 0$, which is similar to the case (4).

(6) For the last case, we take $y_1 \neq y_2, y_3 = q, y_3 = -y_4$. From equation 5.4, we have $-4my_3^3x_3 = 0$, which implies $y_3 = 0$ and this gives geodesic vector is $\boldsymbol{y = y_1e_1 + y_2e_2}$. For the last assumption $X = x_3(e_3 - e_4)$, we can retrace the steps of above to get the homogeneous geodesics.

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SOME FIXED POINT THEOREMS FOR (α, β)-ADMISSIBLE Z-CONTRACTION MAPPING IN METRIC-LIKE SPACES

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Abstract. The purpose of this paper is to establish some fixed point results in the setting of metric-like space by defining an (α, β)-admissible z-contraction mapping imbedded in simulation function. Our results generalize and extend several well known results in the literature of fixed point theory. A suitable example is also established to verify the validity of the results obtained.
Keywords: z-contraction mapping, fixed point, (α, β)-admissible mapping.

1. Introduction

As generalization of the standard metrics spaces, metric-like spaces were considered by Amini-Harandi [3] and proved some fixed point theorems. There after several authors have proved fixed and common fixed point theorem in metric-like space, for example see [1, 7, 5, 6, 9, 8, 10, 11, 21]. In 2012, Samet et al. [24] introduced the concept of α-contraction and α-admissible mappings and proved various fixed point theorems in complete metric spaces. Afterward, many authors obtain generalization of the result [24]. (For instance see [15, 17, 18, 19, 22]).

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Recently, Chandok[12] have introduced the notion of (α, β)-admissible mappings and obtained some fixed point results. Some of authors (For instance [13, 14]) obtained fixed point results by using the notion of (α, β)-admissible mappings and certain contractive conditions. On the other hand, Khojasteh et al [16] introduced a new class of mappings called simulation functions. In [16], they proved several fixed point theorems and shows that many results in the literature are simple consequences of their obtained results. In sequel, Argoubi et al.[4] modified the above said definition and proved some fixed point theorems with nonlinear contractions. There are many fixed point results in the setting of simulation function. (For instance [1, 14, 15, 20, 23]).

In this paper, we consider simulation functions to show the existence of fixed points of (α, β)-admissible z-contraction mapping in metric-like spaces. Our work generalizes and extends some previous results in the literature. We modify and generalize the results of Alsamir et al.[1], A. Dewangan et al.[14] and S. H, Cho[13]. Furthermore, we also give an examples to illustrate the main results.

2. Preliminaries

Let us recall some notations and definitions that we will need in the sequel. Throughout this paper we assume the symbols $\mathbb{R}$ and $\mathbb{N}$ as a set of real numbers and a set of natural numbers respectively.

Definition 2.1. [3] Let X be a non empty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a metric-like space (or a dislocated metric) on X if for any $x, y, z \in X$, the following conditions hold:

(σ_1) $\sigma(x, y) = 0 \Rightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is called metric-like space.

Then a metric-like on X satisfies all conditions of a metric except that $\sigma(x, x)$ may be positive for $x \in X$. Following [3], we have the following topological concepts.

Each metric-like σ on X generates a topology τ_σ on X, whose base is the family of open σ-balls, then for all $x \in X$ and $\epsilon > 0$

$$B_\sigma(X, \epsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \epsilon\}.$$

Now, let (X, σ) be a metric-like space. A sequence $\{x_n\}$ in the metric-like space (X, σ) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$.

Let (X, σ) be a metric-like space, and let $T : X \to X$ be a continuous mapping. Then $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} T(x_n) = T(x)$. A sequence $\{x_n\}$ is Cauchy in

(X,σ), iff $\lim_{n,m\to\infty}\sigma(x_m,x_n)$ exists and is finite. Moreover, the metric-like space (X,σ) is called complete, iff for every Cauchy sequence $\{x_n\}$ in X, there exists $x\in X$ such that

$$\lim_{n\to+\infty}\sigma(x_n,x)=\sigma(x,x)=\lim_{n,m\to\infty}\sigma(x_n,x_m).$$

It is clear that every metric space and partial metric space is a metric-like space but the converse is not true.

Example 2.1. Let $X=\{0,1\}$ and

$$\sigma(x,y)=\begin{cases}2, & \text{if x = y = 0;}\\ 1, & \text{if otherwise.}\end{cases}$$

Then (X,σ) is a metric-like space. It is neither a partial metric space ($\sigma(0,0)\nleq\sigma(0,1)$) nor a metric-like space ($\sigma(0,0)=2\neq 0$).

Remark 2.1. A subset A of a metric-like space (X,σ) is bounded if there is a point $b\in X$ and a positive constant k such that $\sigma(a,b)\le k$ for all $a\in A$.

Remark 2.2. [3] Let $X=\{0,1\}$ such that $\sigma(x,y)=1$ for each $x,y\in X$ and let $x_n=1$ for $n\in\mathbb{N}$. Then it is easy to see that $x_n\to 0$ and $x_n\to 1$ and so in metric-like space, the limit of convergence sequence is not necessarily unique.

The following lemma is known and useful for the rest of paper.

Lemma 2.1. *[3] Let (X,σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n\to x$, where $x\in X$ and $\sigma(x,y)=0$. Then for all $y\in X$ we have $\lim_{n\to\infty}\sigma(x_n,y)=\sigma(x,y)$.*

Definition 2.2. [12] Let X be a non-empty set, $T:X\to X$ and $\alpha,\beta:X\times X\to\mathbb{R}^+$. We say that T is an (α,β)-admissible mapping if $\alpha(x,y)\ge 1$ and $\beta(x,y)\ge 1$ imply that $\alpha(Tx,Ty)\ge 1$ and $\beta(Tx,Ty)\ge 1$ for all $x,y\in X$.

Khojasteh et al.[16] introduced a new class of mappings called simulation functions and proved several fixed point theorems and established that many results in the literature are simple consequences of their obtained results.

Definition 2.3. [16] A function $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called a simulation function if ζ satisfies the following conditions:

(ζ_1) $\zeta(0,0)=0$;

(ζ_2) $\zeta(t,s)<s-t$ for all $t,s>0$;

(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.

The following unique fixed point theorem is proved by Khojasteh et al.[16].

Theorem 2.1. *Let (X,d) be a metric space and $T : X \to X$ be a z-contraction with respect to a simulation function ζ, that is*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0$$

for all $x, y \in X$. Then T has a unique fixed point.

It is worth mentioning that the Banach contraction is an example of z-contraction by defining $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ via $\zeta(t,s) = \lambda s - t$ for all $s,t \in [0,\infty)$, where $\lambda \in [0,1)$.

Argoubi et al.[4] modified Definition (2.3) as follows.

Definition 2.4. [4] A simulation function is a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies the following conditions:

(*i*) $\zeta(t,s) < s - t$ for all $s,t > 0$;

(*ii*) If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.

It is clear that any simulation function in the sense of Khojasteh et al.[16](Definition (2.3)) is also a simulation function in the sense of Argoubi et al.[4] (Definition (2.4)). The converse is not true.

Example 2.2. [4] Define a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta(t,s) = \begin{cases} 1, & \text{if } \ (s,t) = (0,0); \\ \lambda s - t, & \text{otherwise.} \end{cases}$$

where $\lambda \in (0,1)$. Then ζ is a simulation function in the sense of Argoubi et al.[4].

Some other examples of simulation functions in the sense of Definition (2.3) (see [2, 16, 23])are as follows:

(*i*) $\zeta(t,s) = cs - t$ for all $t,s \in [0,\infty)$ where $c \in [0,1)$.

(*ii*) $\zeta(t,s) = s - \phi(s) - t$ for all $t,s \in [0,\infty)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi continuous function such that $\phi(t) = 0$ iff $t = 0$.

3. Main Results

Now, we are ready to prove our first result with the following definitions.

Definition 3.1. [1] Let (X,σ) be a metric-like space. Given $T : X \to X$ and $\alpha,\beta : X\times X \to \mathbb{R}^+$. Such T is said an (α,β)-admissible z-contraction with respect to ζ if

$$\zeta(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty),\sigma(x,y)) \geq 0 \tag{3.1}$$

for all $x,y \in X$, where ζ is a simulation function in the sense of Definition (2.3).

Now, we prove our first fixed point result.

Theorem 3.1. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be a (α,β)-admissible z-contraction mapping with respect to a ζ simulation function if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that*

$$\zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)),\psi(m(x,y))) \geq 0 \tag{3.2}$$

for all $x,y \in X$, where

$$m(x,y) = \max\left\{\sigma(x,y), \frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\right\}.$$

Assume that

1. *T is (α,β)-admissible;*
2. *there exists an element $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$;*
3. *T is σ-continuous.*

Then T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$.

Proof. By condition (2) of this theorem there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$. If $x_n = x_{n+1}$ for some n, $x_n = x_{n+1} = Tx_n$. So x_n is fixed point of T and the proof is completed. From now on assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. Since T is an (α,β)-admissible mapping, we derive

$$\alpha(x_0,Tx_0) = \alpha(x_0,x_1) \geq 1 \Rightarrow \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1.$$

Continuing this process, we get

$$\alpha(x_n,x_{n+1}) \geq 1, \quad \text{for} \quad \text{all} \quad n \geq 0. \tag{3.3}$$

Similarly,

$$\beta(x_n, x_{n+1}) \geq 1, \quad \text{for} \quad \text{all} \quad n \geq 0. \tag{3.4}$$

From (3.2)(3.3), and (3.4), we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(Tx_n, Tx_{n-1})), \psi(m(x_n, x_{n-1}))) \\ &= \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)), \psi(m(x_n, x_{n-1}))). \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned} m(x_n, x_{n-1}) &= \max\left\{\sigma(x_n, x_{n-1}), \frac{[1+\sigma(x_n, Tx_n)]\sigma(x_{n-1}, Tx_{n-1})}{1+\sigma(x_n, x_{n-1})}\right\} \\ &= \max\left\{\sigma(x_n, x_{n-1}), \frac{[1+\sigma(x_n, x_{n+1})]\sigma(x_{n-1}, x_n)}{1+\sigma(x_n, x_{n-1})}\right\} \\ &= \max\left\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\right\}. \end{aligned} \tag{3.6}$$

If follows from (3.5) and (3.6) that

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)), \\ &\quad \psi(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\})) \\ &< \psi(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\}) \\ &\quad -\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)). \end{aligned} \tag{3.7}$$

Consequently, we obtain that for all $n = 0, 1, 2, 3...$

$$\psi(\sigma(x_n, x_{n+1})) < \psi(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\}).$$

If $\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\} = \sigma(x_n, x_{n+1})$ for some n, then

$$\psi(\sigma(x_n, x_{n+1})) < \psi(\sigma(x_n, x_{n+1})),$$

which is a contradiction.

Hence $\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\} = \sigma(x_n, x_{n-1})$, for all $n \geq 0$ and hence from (3.7) we get,

$$0 < \psi(\sigma(x_n, x_{n-1})) - \psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n))$$

or

$$\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)) < \psi(\sigma(x_n, x_{n-1})). \tag{3.8}$$

By using the property of ψ, we get

$$\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n) < \sigma(x_n, x_{n-1}) \tag{3.9}$$

for all $n \geq 0$. The sequence $\{\sigma(x_n, x_{n-1})\}$ is nondecreasing, so there exists $r \geq 0$ such that $\lim_{n\to\infty} \sigma(x_n, x_{n-1}) = r$. We prove that

$$\lim_{n\to\infty} \sigma(x_n, x_{n-1}) = 0. \tag{3.10}$$

Suppose that $r > 0$. By the properties of ψ, (3.5), (3.8) and (3.9) and the condition (ζ_3)

$$0 \leq \lim_{n\to\infty} \sup \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)), \psi(\sigma(x_n, x_{n-1}))) < 0,$$

which is a contradiction. Therefore $r = 0$. This implies that $\lim_{n\to\infty} \sigma(x_n, x_{n-1}) = 0$.

Now we will show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can assume subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k,

$$\sigma(x_{n_k}, x_{m_k}) \geq \epsilon. \tag{3.11}$$

That is,

$$\sigma(x_{n_k}, x_{m_k-1}) < \epsilon. \tag{3.12}$$

By the triangular inequality and using (3.11) and (3.12), we get

$$\begin{aligned} \epsilon \leq \sigma(x_{n_k}, x_{m_k}) &\leq \sigma(x_{n_k}, x_{m_k-1}) + \sigma(x_{m_k-1}, x_{m_k}) \\ &< \epsilon + \sigma(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities and by using (3.10) and (3.11), we have

$$\lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.13}$$

Also, from the triangular inequality, we have

$$\sigma(x_{n_k}, x_{m_k}) \leq \sigma(x_{n_k}, x_{n_k+1}) + \sigma(x_{n_k+1}, x_{m_k}),$$

$$|\sigma(x_{n_k+1}, x_{m_k}) - \sigma(x_{n_k}, x_{m_k})| \leq \sigma(x_{n_k}, x_{n_k+1}).$$

On taking limit as $k \to \infty$ on both sides of above inequality and using (3.10) and (3.13), we get

$$\lim_{k\to\infty} \sigma(x_{n_k+1}, x_{m_k}) = \epsilon. \tag{3.14}$$

Similarly it is easy to show that

$$\lim_{k\to\infty} \sigma(x_{n_k+1}, x_{m_k+1}) = \lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k+1}) = \epsilon. \tag{3.15}$$

Moreover, since T is an (α, β)-admissible mapping, we have

$$\alpha(x_{n_k}, x_{m_k}) \geq 1 \quad \text{and} \quad \beta(x_{n_k}, x_{m_k}) \geq 1. \tag{3.16}$$

We deduce that

$$\begin{aligned} m(x_{n_k}, x_{m_k}) &= \max\left\{\sigma(x_{n_k}, x_{m_k}), \frac{[1+\sigma(x_{n_k}, Tx_{n_k})]\sigma(x_{m_k}, Tx_{m_k})}{1+\sigma(x_{n_k}, x_{m_k})}\right\} \\ &= \max\left\{\sigma(x_{n_k}, x_{m_k}), \frac{[1+\sigma(x_{n_k}, x_{n_k+1})]\sigma(x_{m_k}, x_{m_k+1})}{1+\sigma(x_{n_k}, x_{m_k})}\right\}. \end{aligned}$$

Taking $k \to \infty$ and using (3.10), (3.13) and (3.14), we obtain

$$\lim_{k\to\infty} \psi(m(x_{n_k}, x_{m_k})) = \epsilon. \tag{3.17}$$

By the fact T is an (α, β)-admissible z-contraction with respect to ζ, together with (3.13), (3.16) and (ζ_3), we get

$$\begin{aligned} 0 \leq \lim_{k\to\infty} \sup \zeta(\psi(\alpha(x_{n_k}, x_{m_k})\beta(x_{n_k}, x_{m_k})\sigma(x_{n_k+1}, x_{m_k+1})), \\ \psi(m(x_{n_k}, x_{m_k}))) < 0, \end{aligned}$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Owing to the fact that (X, σ) is a complete metric-like space, there exists some $u \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0 \tag{3.18}$$

Moreover, the continuity of T implies that

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \lim_{n\to\infty} \sigma(Tx_n, Tu) = \sigma(Tu, Tu) = 0. \tag{3.19}$$

Using Lemma 2.1 and (3.19), we have

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \sigma(u, Tu). \tag{3.20}$$

Continuing (3.19) and (3.20), we deduce that $\sigma(Tu, u) = \sigma(Tu, Tu)$. That is $Tu = u$. To prove the uniqueness of the fixed point, suppose that there exists $w \in X$ such that $Tw = w$ and $w \neq u$. Then

$$0 \leq \zeta(\psi(\alpha(u, w)\beta(u, w)\sigma(Tu, Tw)), \psi(m(u, w))) \tag{3.21}$$

where

$$m(u, w) = \max\left\{\sigma(u, w), \frac{[1+\sigma(u, Tu)]\sigma(w, Tw)}{1+\sigma(u, w)}\right\}$$

$$m(u,w) = \sigma(u,w) \tag{3.22}$$

from (3.21), (3.22) and (ζ_2) we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(u,w)\beta(u,w)\sigma(Tu,Tw)), \psi(\sigma(u,w))) \\ &< \psi(\sigma(u,w)) - \psi(\alpha(u,w)\beta(u,w)\sigma(Tu,Tw)). \end{aligned} \tag{3.23}$$

By using the property of ψ, we have

$$0 < \sigma(u,w) - \alpha(u,w)\beta(u,w)\sigma(Tu,Tw) \leq 0.$$

Which is a contradiction, so $u = w$. □

Theorem (3.1) remains true if we drop the continuity hypothesis by the following property:

(H): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all n, then there exists a subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k+1}) \geq 1$ and $\beta(x_{n_k}, x_{n_k+1}) \geq 1$ for all $k \in \mathbb{N}$ and $\alpha(x, Tx) \geq 1$ and $\beta(x, Tx) \geq 1$.

Theorem 3.2. *Let (X, σ) be a complete metric-like space and let T be a self-mapping on X satisfying the following conditions:*

(1) T is (α, β)-admissible;

(2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;

(3) (H) holds;

(4) T is an (α, β)-admissible z-contraction mapping with respect to a ζ simulation function if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that

$$\zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y))) \geq 0,$$

for all $x, y \in X$, where

$$m(x,y) = \max\Big\{\sigma(x,y), \frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\Big\}.$$

Then T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$.

Proof. Following the proof of Theorem (3.1), we construct a sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$, which converges to some $u \in X$. From definition of (α, β)-admissible mapping and (H), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k+1}) \geq 1$ and $\beta(x_{n_k}, x_{n_k+1}) \geq 1$ for all $k \in \mathbb{N}$. Thus applying condition (3.2) for all k, we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(Tx_{n_k}, Tu)), \psi(m(x_{n_k}, u))) \\ &= \zeta(\psi(\alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu)), \psi(m(x_{n_k}, u))) \\ &< \psi(m(x_{n_k}, u)) - \psi(\alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu)). \end{aligned} \tag{3.24}$$

By suing the property ψ, we have

$$0 < m(x_{n_k}, u) - \alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu). \tag{3.25}$$

Also from (3.22) and (3.25), we get

$$0 < \sigma(x_{n_k}, u) - \alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu) \tag{3.26}$$

which is equivalent to

$$\begin{aligned} \sigma(x_{n_k+1}, Tu) = \sigma(Tx_{n_k}, Tu) &\leq \alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(Tx_{n_k}, Tu) \\ &\leq \sigma(x_{n_k}, u). \end{aligned} \tag{3.27}$$

Letting $k \to \infty$ in the above, we have $\sigma(u, Tu) = 0$. Using similar arguments as above, we can show that u is a fixed point of T. The uniqueness of the fixed point of T is obtained by similar arguments as these given in the proof of Theorem (3.1)

Now, we apply Theorem (3.1) to obtain the following result which is known as Banach type. □

Corollary 3.1. *Let (x, σ) be a complete metric-like space and let T be a self-mapping on X satisfying the following conditions:*

(1) *T is (α, β)-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*

(3) *T is σ-continuous;*

(4) *$\psi(\alpha(x, y)\beta(x, y)\sigma(Tx, Ty)) \leq \lambda(\psi(m(x, y)))$, for all $x, y \in X$ and $\lambda \in [0, 1)$ and also $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) \leq t, \psi(0) = 0$.*

Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

Proof. Following the lines of Theorem (3.1), by taking as a σ-simulation function, $\zeta(t, s) = \lambda s - t$. □

Corollary 3.2. *Let (X, σ) be a complete metric-like space and T be a self-mapping on X satisfying the following conditions:*

(1) *T is (α, β)-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*

(3) *T is σ-continuous;*

(4) *there exists a lower semi continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma^{-1} = \{0\}$ such that*

$$\alpha(x, y)\beta(x, y)\sigma(Tx, Ty) \leq m(x, y) - \gamma(m(x, y))$$

for all $x, y \in X$. Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

Proof. Following the proof of Theorem (3.1), it sufficient to take $\zeta(t, s) = s - \gamma(s) - t$. □

If we consider in Theorem (3.1), $\alpha(x, y) = \beta(x, y) = 1$ for all $x, y \in X$, we have:

Corollary 3.3. *Let (X, σ) be a complete metric-like space and let T be a self-mapping on X. Suppose that there exists a σ-simulation function ζ such that*

$$\zeta(\psi(\sigma(Tx, Ty)), \psi(m(x, y))) \geq 0$$

for all $x, y \in X$. Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

We present the following illustrated example.

Example 3.1. Let $X = [0, \infty), \sigma(x, y) = x + y$ for all $x, y \in X$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{x}{4}, & \text{if } \ 0 \leq x \leq 1, \\ 4x, & \text{otherwise.} \end{cases}$$

consider $\zeta(t, s) = \lambda s - t$, where $0 \leq 1/4 < \lambda < 1$.

We define two mappings $\alpha, \beta : X \times X \to \mathbb{R}^+$ as

$$\alpha(x, y) = \begin{cases} \frac{5}{3}, & \text{if } \ 0 \leq x, y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta(x, y) = \begin{cases} \frac{3}{2}, & \text{if } \ 0 \leq x, y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\psi(t) = t$ for all $t \geq 0$. We shall prove that Corollary 3.1 can be applied. Clearly (X, σ) is a complete metric-like space. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$. Since $x, y \in [0,1]$ and so $Tx \in [0,1], Ty \in [0,1]$ and $\alpha(Tx,Ty) = 1$ and $\beta(Tx,Ty) = 1$. Hence T is (α,β)-admissible. Condition (2) is satisfied with $x_0 = 1$. Condition (3.2) is also satisfied with $x_n = T^n x_1 = 1/n$.

If $0 \leq x \leq 1$, then $\alpha(x,y) = 5/3$ and $\beta(x,y) = 3/2$.

Now

$$\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y)) = \alpha(x,y)\beta(x,y)\sigma(Tx,Ty), m(x,y)$$

where

$$\begin{aligned} m(x,y) &= \max\left\{x+y, \frac{[1+x+Tx](y+Ty)}{1+x+y}\right\} \\ &= \max\left\{x+y, \frac{[1+x+x/4](y+y/4)}{1+x+y}\right\} \\ &= \max\left\{x+y, \frac{[4+5x](5y)}{16(1+x+y)}\right\} = \{x+y\} \end{aligned}$$

$$\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y)) = \alpha(x,y)\beta(x,y)\sigma(Tx,Ty), x+y$$

$$\begin{aligned} \zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y))) &= \zeta(\alpha(x,y)\beta(x,y) \\ &\quad \sigma(Tx,Ty), x+y) \\ &= \lambda(x+y) - \\ &\quad \alpha(x,y)\beta(x,y)\sigma(Tx,Ty) \\ &= \frac{3}{4}(x+y) - \\ &\quad \left(\frac{5}{3}\right)\left(\frac{3}{2}\right)\left(\frac{x}{4}+\frac{y}{4}\right) \\ &= \frac{3}{4}(x+y) - \frac{5}{8}(x+y) \\ &= \left(\frac{3}{4}-\frac{5}{8}\right)(x+y) \\ &= \frac{1}{8}(x+y) \geq 0. \end{aligned}$$

If $0 \leq x \leq 1$ and $y > 1$, then $\zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y))) \geq 0$. Since $\alpha(x,y) = \beta(x,y) = 0$. Consequently, all assumptions of Corollary 3.1 are satisfied and hence T has a unique fixed point which is $u = 0$

4. Conclusion

In this attempt, we studied (α,β)-admissible z-contraction mappings imbedded in simulation function and proved some fixed point theorems in metric-like spaces. Our results are generalized and extended forms of recent results in the literature. Finally, we have illustrated an example in support of our obtained results.

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ON PROJECTIVELY FLAT GENERALIZED BERWALD (α, β)-METRICS

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Abstract. Every Berwald metric is a special generalized Berwald metric. In this paper, we study the class of projectively flat generalized Berwald (α, β)-metrics of isotropic S-curvature. We find some conditions under which this class of Finsler metrics reduces to the class of Berwald metrics.
Keywords: Berwald (α, β)-metric, Finsler metric, isotropic S-curvature.

1. Introduction

The geodesics curves of an arbitrary Finsler metric $F = F(x, y)$ on a manifold M are characterized by the following system of differential equations

$$\ddot{c}^i + 2G^i(\dot{c}) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients of F. Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and the other way around. In this case, there is a scalar function $P = P(x, y)$ defined on the slit tangent bundle $TM_0 = TM - \{0\}$ such that

$$G^i = \bar{G}^i + Py^i.$$

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Here, G^i and $\bar{G}^i$ denote the geodesic spray coefficients of F and $\bar{F}$, respectively [6]. The problem of projectively related Finsler metrics is quite old in geometry and its origin is formulated in Hilberts Fourth Problem: to determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines [2]. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's Fourth Problem. A Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called projectively flat if all geodesics are straight in U. In this case, F and the Euclidean metric on U are projectively related.

In order to find projectively flat Finsler metrics, one can search in the class of generalized Berwald metrics. A Finsler metric $F = F(x, y)$ on a manifold M is called a generalized Berwald metric if there exists a covariant derivative D on M such that the parallel translations induced by D preserve the Finsler function F [1][12]. In this case, F is called a generalized Berwald metric on M. If the covariant derivative D is also torsion-free, then F reduces to a Berwald metric. In this case, the spray coefficients of F is quadratic in direction y. By definition, the class of Berwald metrics belongs to the class of generalized Berwald metrics.

The class of generalized Berwald metrics is very large to search, and finding projectively flat Finsler metrics in this class is very complex. Thus, one can focus on a meaningful subclasses of these Finsler metrics, maybe the class of generalized Berwald (α, β)-metrics. An (α, β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M.

It is interesting to find some conditions under which a projectively flat generalized Berwald (α, β)-metric reduces to a Berwald metric. To find the mentioned condition, for an (α, β)-metric $F := \alpha\phi(s)$, let us put

$$Q := \frac{\phi'}{\phi - s\phi'}, \qquad \Psi := \frac{Q'}{2[1 + sQ + (b^2 - s^2)Q']}.$$

Define

$$\Lambda := b^ib^jb^kb^l\Big[\alpha\beta Q\Big]_{y^iy^jy^ky^l} \text{and} \quad \Upsilon := b^jb^jb^kb^lb^m\Big[\Psi\Big]_{y^iy^jy^ky^ly^m}. \tag{1.1}$$

Then, we will prove the following result.

Theorem 1.1. *Let $F = \alpha\phi(\beta/\alpha)$ be a projectively flat (α, β)-metric on a manifold M. Suppose that ϕ satisfies $\phi'(0) \neq 0$, $\Lambda \neq 0$ and $\Upsilon \neq 0$. Then F is a generalized Berwald metric of isotropic S-curvature if and only if it is a Berwald metric. In this case, F is a locally Minkowskian metric.*

We remark that the S-curvature is constructed by Zhongmin Shen for given comparison theorems on Finsler manifolds [11]. A natural problem is to study and characterize Finsler metrics of vanishing S-curvature. An n-dimensional Finsler metric is said to have isotropic S-curvature if $\mathbf{S} = (n + 1)cF$, for some scalar function $c = c(x)$ on M.

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent space and $TM_0 := TM - \{0\}$ the slit tangent space of M. A Finsler structure on manifold M is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$;
(iii) The quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is positive-definite on T_xM

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then, the pair (M, F) is called a Finsler manifold.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4} g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Big], \quad y \in T_xM. \tag{2.1}$$

$\mathbf{G}$ is called the spray associated to (M, F).

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \partial/\partial x^i|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature. Then F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$ [10].

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\big[(y^i) \in R^n \mid F\big(y^i \frac{\partial}{\partial x^i}|_x\big) < 1\big]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. Then for $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, the S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\big[\ln \sigma_F(x)\big].$$

This quantity was first introduced by Shen for a volume comparison theorem [10]. A Finsler metric F on an n-dimensional manifold M has isotropic S-curvature if

$$\mathbf{S} = (n+1)cF,$$

where $c = c(x)$ is a scalar function on M. Also, F has vanishing S-curvature if $\mathbf{S} = 0$.

It is known that a Finsler metric $F(x, y)$ on $\mathcal{U}$ is projective if and only if its geodesic coefficients G^i are in the form

$$G^i(x, y) = P(x, y)y^i,$$

where $P : T\mathcal{U} = \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$. We call $P(x, y)$ the *projective factor* of $F(x, y)$.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\partial/\partial x^i$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature.

For a flag $P := \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{2.2}$$

The flag curvature $\mathbf{K}(x, y, P)$ is a function of tangent planes $P = \mathrm{span}\{y, v\} \subset T_xM$. F is of scalar flag curvature if $\mathbf{K} = \mathbf{K}(x, y)$ is independent of flag P.

3. Proof of Theorem 1.1

An (α, β)-metric is a Finsler metric on a manifold M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For an (α, β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let us define

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad r_j := b^ir_{ij},$$
$$s_{i0} := s_{ij}y^j, \quad s_j := b^is_{ij}, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j.$$

Let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, put

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''.$$

In [4], Cheng-Shen characterized (α, β)-metrics with isotropic S-curvature on a manifold M of dimension $n \geq 3$. Soon, they found that their result holds for the class of (α, β)-metrics with constant length one-forms, only. Here, we modify their result as follows.

Lemma 3.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an non-Randers type (α, β)-metric on an manifold M of dimension $n \geq 3$. Suppose that β has constant length with respect to α. Then, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds*

(i) β satisfies

$$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{3.1}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3.2}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)k\epsilon F$.

(ii) β satisfies

$$r_{ij} = 0, \qquad s_j = 0. \tag{3.3}$$

In this case, $\boldsymbol{S} = 0$.

In [18], the following is proved.

Lemma 3.2. ([18]) *An (α, β)-metric satisfying $\phi'(0) \neq 0$ is a generalized Berwald manifold if and only if β has constant length with respect to α.*

By Lemmas 3.1 and 3.2, we get the following.

Lemma 3.3. *Let $F = \alpha\phi(\beta/\alpha)$ be an non-Randers type generalized Berwald (α, β)-metric on a manifold M of dimension $n \geq 3$ such that $\phi'(0) \neq 0$. Then, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds:*

(i) β satisfies

$$r_{ij} = \epsilon\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_j = 0, \tag{3.4}$$

where $\epsilon = \epsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3.5}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)k\epsilon F$.

(ii) β satisfies

$$r_{ij} = 0, \qquad s_j = 0. \tag{3.6}$$

In this case, $\boldsymbol{S} = 0$.

To prove Theorem 1.1, we need the following.

Proposition 3.1. *Let $F = \alpha\phi(\beta/\alpha)$ be a non-Randers type (α,β)-metric on a manifold M of dimension $n \geq 3$ such that $\Lambda \neq 0$. Then F is a generalized Berwald metric with vanishing S-curvature $\mathbf{S} = 0$ if and only if it is a Berwald metric.*

Proof. Let $G^i = G^i(x,y)$ and $G^i_\alpha = G^i_\alpha(x,y)$ denote the spray coefficients of F and α respectively in the same coordinate system. By (2.1), we have

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{3.7}$$

where

$$\begin{aligned} P &:= \alpha^{-1}\Theta(r_{00} - 2Q\alpha s_0), \\ Q^i &:= \alpha Q s^i{}_0 + \Psi(r_{00} - 2Q\alpha s_0)b^i. \end{aligned}$$

In [3], Cheng proved that every regular (α,β)-metric with isotropic S-curvature has vanishing S-curvature (see Theorem 2.4). In this case, by Lemma 3.3, we have $r_{00} = s_0 = 0$. Then (3.7) reduces to following

$$G^i = G^i_\alpha + \alpha Q s^i{}_0. \tag{3.8}$$

F is a projectively flat Finsler metric which is equal to following

$$G^i = Py^i, \tag{3.9}$$

where $P = P(x,y)$ is a local scalar function satisfying $P(x,\lambda y) = \lambda P(x,y)$. By (3.8) and (3.9), we have

$$Py^i = G^i_\alpha + \alpha Q s^i{}_0. \tag{3.10}$$

Multiplying (3.10) with b_i and y_i, respectively, imply that

$$P\beta = b_i G^i_\alpha, \tag{3.11}$$
$$P\alpha^2 = y_i G^i_\alpha. \tag{3.12}$$

Contracting (3.10) with β yields

$$P\beta y^i = \beta G^i_\alpha + \alpha\beta Q {s^i}_0. \tag{3.13}$$

By (3.11) and (3.13) it follows that

$$(b_r G^r_\alpha) y^i - \beta G^i_\alpha = \alpha\beta Q {s^i}_0. \tag{3.14}$$

The following holds

$$\big[(b_r G^r_\alpha) y^i - \beta G^i_\alpha\big]_{y^j y^k y^l y^m} = 0. \tag{3.15}$$

(3.14) and (3.15) give us

$$[\alpha\beta Q s^i_0]_{y^j y^k y^l y^m} = 0. \tag{3.16}$$

We have

$$\begin{aligned} \big[\alpha\beta Q s^i_0\big]_{y^j y^k y^l y^m} = {} & \big[\alpha\beta Q\big]_{y^j y^k y^l} {s^i}_m + \big[\alpha\beta Q\big]_{y^j y^k y^m} {s^i}_l + \big[\alpha\beta Q\big]_{y^j y^l y^m} {s^i}_k \\ & + \big[\alpha\beta Q\big]_{y^l y^k y^m} {s^i}_j + \big[\alpha\beta Q\big]_{y^j y^k y^l y^m} {s^i}_0 = 0 \end{aligned} \tag{3.17}$$

By part (b) of Lemma 3.3, we have $s^k = b^m s^k_m = 0$. Then multiplying (3.17) with $b^j b^k b^l b^m$ and considering (3.16) imply that

$$b^j b^k b^l b^m \big[\alpha\beta Q\big]_{y^j y^k y^l y^m} {s^i}_0 = 0 \tag{3.18}$$

By assumption, we get

$${s^i}_j = 0. \tag{3.19}$$

Putting (3.19) in (3.8) gives us $G^i = G^i_\alpha$. It implies that F is a Berwald metric. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1: The proof divided to three main cases as follows:

Case (i). F is not a Randers metric and $dim(M) \geq 3$**:** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a generalized Berwald non-Randers type (α, β)-metric on a manifold M of dimension $dim(M) \geq 3$. Suppose that F has isotropic S-curvature,

$\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M. In this case, by Lemma 3.3 we have

$$s_0 = 0, \tag{3.20}$$

$$r_{00} = c(b^2 - s^2)\alpha^2. \tag{3.21}$$

Since F is a projectively flat metric, then there exists a local scalar function $P = P(x,y)$ satisfies $P(x, \lambda y) = \lambda P(x,y)$. By (3.7) and (3.20), it follows that

$$Py^i = G^i_\alpha + \alpha Q s^i{}_0 + r_{00}\Big[\Theta\frac{y^i}{\alpha} + \Psi b^i\Big]. \tag{3.22}$$

Multiplying (3.22) with b_i and y_i, respectively, imply that

$$P\beta = b_i G^i_\alpha + r_{00}\Big[\Theta\frac{\beta}{\alpha} + \Psi b^2\Big], \tag{3.23}$$

$$P\alpha^2 = y_i G^i_\alpha + r_{00}\Big[\Theta\alpha + \Psi\beta\Big]. \tag{3.24}$$

(3.23) $\times\, \alpha^2$ − (3.24) $\times\, \beta$ yields

$$\Psi r_{00}(b^2\alpha^2 - \beta^2) = (y_i G^i_\alpha)\beta - (b_i G^i_\alpha)\alpha^2. \tag{3.25}$$

By (3.21) and (3.25), we get

$$c\Psi(b^2\alpha^2 - \beta^2)^2 = (y_i G^i_\alpha)\beta - (b_i G^i_\alpha)\alpha^2. \tag{3.26}$$

Since

$$\Big[(y_i G^i_\alpha)\beta - (b_i G^i_\alpha)\alpha^2\Big]_{y^j y^k y^l y^m y^p} = 0$$

then

$$\Big[c\Psi(b^2\alpha^2 - \beta^2)^2\Big]_{y^j y^k y^l y^m y^p} = 0.$$

It is easy to see that the following holds

$$b^t\Big[(b^2\alpha^2 - \beta^2)^2\Big]_{y^t} = 0.$$

Then

$$\begin{aligned} b^j b^k b^l b^m b^p \Big[c\Psi(b^2\alpha^2 - \beta^2)^2\Big]_{y^j y^k y^l y^m y^p} &= c b^j b^k b^l b^m b^p \Big[\Psi\Big]_{y^j y^k y^l y^m y^p}(b^2\alpha^2 - \beta^2)^2 \\ &= 0 \end{aligned} \tag{3.27}$$

According to the assumption, (3.27) implies that $c = 0$. Then $r_{00} = 0$ and by (3.20) we get $s_0 = 0$. By Lemma 3.3, F has vanishing S-curvature. Then by Proposition 3.1, we conclude that F is a Berwald metric. Since F is projectively flat metric then it is of scalar flag curvature $\mathbf{K} = \mathbf{K}(x,y)$. F is not Randers-type and then is not Riemannian. Then $\mathbf{K} = 0$, and F is a locally Minkowsian metric.

Case (ii). F is not a Randers metric and $dim(M) = 2$: Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a two-dimensional generalized Berwald non-Randers type (α,β)-metric on a manifold M. Suppose that F has isotropic S-curvature. By Theorem 2.4 of [3], every regular (α,β)-metric with isotropic S-curvature has vanishing S-curvature. In [13], it is proved that such metric reduces to a locally Minkowskian metric. This completes the proof.

Case (iii). F is a Randers metric: A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat and β is closed, i.e., $s_{ij} = 0$ (see [10]). On the other hand, in [18], it is proved that F is a generalized Berwald manifold if and only if β is of constant Riemannian length, namely $r_i + s_i = 0$. These imply

$$s_{ij} = 0, \quad r_i = 0. \tag{3.28}$$

In [4], it is proved that $F = \alpha + \beta$ has isotropic S-curvature $\mathbf{S} = (n+1)cF$ if and only if

$$e_{00} = 2c(\alpha^2 - \beta^2), \tag{3.29}$$

where $c = c(x)$ is a scalar function on M, $e_{00} = e_{ij}y^iy^j$ and $e_{ij} = r_{ij} + b_is_j + b_js_i$. By (3.28) and (3.29), we get

$$r_{ij} = 2c(a_{ij} - b_ib_j). \tag{3.30}$$

Multiplying (3.30) with b^i yields

$$r_j = 2c(1 - b^2)b_j. \tag{3.31}$$

Since $b < 1$ then by (3.28) and (3.31) we get $b_j = 0$ or $c = 0$. If $b_j = 0$ then F is Riemannian. If $c = 0$ then by (3.30) implies that $r_{ij} = 0$. By considering (3.28), β is parallel with respect to α and F reduces to a Berwald metric. This completes the proof. □

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$*$-CONFORMAL CURVATURE OF CONTACT METRIC MANIFOLDS

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Abstract. We introduce the $*$-conformal curvature tensor and ${}^*\eta$-Einstien manifolds in contact manifolds. We investigate this tensor in the three main classes of contact manifolds: Sasakian manifolds, Kenmotsu manifolds, and cosymplectic manifolds. We prove that a manifold is η-Einstienian if and only if be ${}^*\eta$-Einstienian manifold.
Keywords: $*$-conformal curvature, ${}^*\eta$-Einstien manifolds, Sasakian manifolds, Kenmotsu manifolds, Cosymplectic manifolds.

1. Introduction

There are many similar concepts in complex geometry and contact geometry. Tachibana introduces $*$-Ricci tensor within the framework of an almost Hermitian manifold in their work [23]. Afterward, Hamada introduces the $*$-Ricci tensor for the real hypersurfaces embedded in a non-flat complex space form [16]. This notion on an almost contact metric manifold $(M, g, \eta, \xi, \varphi)$ is defined as

$$(1.1)\qquad {}^*Ric(X_1, X_2) = \frac{1}{2}\, trace\{\mathbf{X_3} \ \to K(X_1, \varphi X_2)\varphi \mathbf{X_3}\},$$

for any vector field X_1, X_2. The $*$-Ricci operator ${}^*L$ is characterized by the relation $g({}^*LX_1, X_2) = {}^*Ric(X_1, X_2)$. With the help of the $*$-Ricci tensor, several authors have investigated $*$-Ricci soliton in contact geometry (see [14], [10], [25], [2]). In

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general, the equality ${}^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$ does not always hold.

In a Riemannian manifold (M^{2n+1}, g), the conformal curvature tensor C is expressed as

$$\begin{aligned} C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &- \quad \frac{1}{2n-1}\Big(Ric(X_2, X_3)X_1 - Ric(X_1, X_3)X_2 \\ &+ \quad g(X_2, X_3)\, LX_1 - g(X_1, X_3)\, LX_2\Big) \\ &+ \quad \frac{r}{2n(2n-1)}\Big(g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big), \end{aligned} \tag{1.2}$$

where K represents the curvature tensor of (1,3) type, Ric indicates the Ricci tensor, r is the scalar curvature and L is the Ricci operator of (M, g).

The paper is organized as follows: In Section 2, we express some preliminary definitions, then we proceed to investigate $*$-conformal curvature tensor of the contact manifolds. We examine some features of $*$-conformal curvature tensor.

In Section 3, we considered the Sasakian structure. Then, having the $*$-Ricci, we determined the relationship between η-Einstien and ${}^*\eta$-Einstien manifold.

Theorem 1.1. *Let M^{2n+1} be a manifold with a Sasakian structure (g, η, ξ, φ). The manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Then, we investigate the $*$-conformal curvature tensor of the Sasakian manifolds. In addition, we show that ξ-conformally flat and ξ-$*$conformally flat will not co-occur in Sasakian manifolds. By the condition ${}^*Ric(X_1, X_2)$ and ${}^*r$ for a $2n+1$-dimensional Sasakian manifold, we get the following $(0, 2)$-tensor

$${}^*T(X_1, X_2) = -\frac{{}^*Ric(X_1, X_2)}{2n-1} + \frac{{}^*r\ g(X_1, X_2)}{4n(2n-1)}.$$

We conclude that if $n > 1$, then $*$-conformal curvature tensor ans and ${}^*D(X_1, X_2)X_3$ do not vanish simultaneously.

In Section 4, we find some conditions for a Kenmotsu 3-manifold to have vanishing $*$-conformal curvature tensor. We show that for a special case, the $*$-conformal tensor of this manifold becomes zero as in the following Theorem.

Theorem 1.2. *If a Kenmotsu 3-manifold is of quasi-constant curvature of the form*

$$\begin{aligned} K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) \quad &- \quad \alpha\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\ &+ \quad g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big], \end{aligned}$$

where $\alpha = \frac{r}{2} + 2$, then $*$-conformal curvature tensor vanishes.

But in general, we show that on Kenmotsu manifolds, the $*$-conformal tensor cannot vanish identically. Similarly, the equivalence of η-Einstien and $^*\eta$-Einstien is also established in Kenmotsu manifolds. The same result about $*$-conformal curvature tensor and $^*D(X_1, X_2)X_3$ on the Sasakian manifold is obtained for the Kenmotsu manifold.

In the last section, we prove the $*$-conformal curvature tensor is identically zero on the 3-dimensional cosymplectic manifolds. We confirm a conformally flat cosymplectic manifold is an $^*\eta$-Einstien manifold. We prove the following theorem:

Theorem 1.3. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$. If M is a $*$-conformally flat manifold, then $^*D = 0$.*

2. Preliminaries

Definition 2.1. Consider a contact metric manifold $(M, g, \eta, \xi, \varphi)$ of dimension $2n+1$. The $*$-conformal curvature tensor for $(M, g, \eta, \xi, \varphi)$ is expressed as

$$\begin{aligned} {}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad & - \quad \frac{1}{2n-1}\Big({}^*Ric(X_2, X_3)X_1 - {}^*Ric(X_1, X_3)X_2 \\ & + \quad g(X_2, X_3)\,{}^*LX_1 - g(X_1, X_3)\,{}^*LX_2\Big) \\ & + \quad \frac{{}^*r}{2n(2n-1)}\Big(g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big), \end{aligned} \tag{2.1}$$

where $^*r$ represents the $*$-scalar curvature, which is the trace of the $*$-Ricci tensor.

Definition 2.2. A contact metric manifold is named $^*\eta$-Einstien if

$$ {}^*Ric(X_1, X_2) = c\, g(X_1, X_2) + d\, \eta(X_1)\eta(X_2), \qquad c, d \in C^\infty(M). \tag{2.2}$$

A differentiable manifold M^{2n+1} has an almost contact structure [2] if it admits a 1-form η, a characteristic vector field ξ, and a $(1,1)$-tensor field φ, which satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.3}$$

where I indicates the identity endomorphism. Then, by (2.3), can see that

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0. \tag{2.4}$$

If an almost contact manifold M^{2n+1} admits a Riemannian metric g with the property:

$$g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \qquad \forall X_1, X_2 \in \chi(M), \tag{2.5}$$

then $(M^{2n+1}, g, \eta, \xi, \varphi)$ is called an almost contact metric manifold. The 2-form $\Phi(X_1, X_2) = g(X_1, \varphi X_2)$ is called the fundamental 2-form on the almost contact metric manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$. An almost contact metric manifold is called normal if the (1,2)-type torsion tensor N_φ vanishes, where $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ is the Nijenhuis tensor of φ. A normal almost contact metric manifold is called a Sasakian manifold. A Sasakian manifold is also characterized by

$$\big(\nabla_{X_1}\varphi\big)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \qquad \forall X_1, X_2 \in \chi(M).$$

On a Sasakian manifold beside (2.3)-(2.5), we also have

$$\nabla_{X_1}\xi = -\varphi X_1, \qquad K(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{2.6}$$

where K denotes the curvature tensor of (1,3) type. The importance and application of Sasakian structures are in holomorphic statistical structures and are also related to string theory (see [1]).

If the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$, then the almost contact metric manifold is called almost Kenmotsu manifold. A normal almost Kenmutsu manifold is a Kenmutsu manifold, which is equivalent to:

$$(\nabla_{X_1}\varphi)X_2 = g(\varphi X_1, X_2)\xi - \eta(X_2)\varphi X_1, \qquad \forall X_1, X_2 \in \chi(M).$$

It is known that every Kenmotsu manifold is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a Kahler manifold, I is an open interval with coordinate t, and the warping function f defined by $f = ce^t$ for some positive constant c [19]. For a $(2n+1)$-dimensional Kenmotsu manifold, we have

$$\begin{aligned}
\nabla_{X_1}\xi &= X_1 - \eta(X_1)\xi, && (2.7)\\
K(X_1, X_2)\xi &= \eta(X_1)X_2 - \eta(X_2)X_1, && (2.8)\\
Ric(X_1, \xi) &= -2n\,\eta(X_1), && (2.9)\\
K(\xi, X_1)X_2 &= \eta(X_2)X_1 - g(X_1, X_2)\xi, && (2.10)\\
Ric(\phi X_1, \phi X_2) &= Ric(X_1, X_2) + 2n\,\eta(X_1)\eta(X_2). && (2.11)
\end{aligned}$$

An almost contact metric manifold is termed an almost cosymplectic manifold when both the 1-form η and 2-form Φ are closed. A normal almost cosymplectic manifold is called a cosymplectic manifold [3], [15]. Every cosymplectic manifold satisfies the following:

$$\nabla_{X_1}\xi = 0, \quad K(X_1, X_2)\xi = 0, \quad Ric(X_1, \xi) = 0. \tag{2.12}$$

The cosymplectic structure is a tool for time-dependent Hamiltonian mechanics. It has some applications in string theory, which shows the importance of cosymplectic manifolds.

Suppose that $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an almost contact metric manifold and ${}^*C$ is its $*$-conformal curvature tensor, which is defined by (2.1). A direct computation shows some symmetries of ${}^*C$.

Proposition 2.1. *In a contact metric manifold, the $*$-conformal curvature tensor obeys the following:*

1. $^*C(X_1, X_2)X_3 = -^*C(X_2, X_1)X_3$,
2. $$\begin{aligned} &^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 \\ &= -\tfrac{1}{2n-1}\{{}^*Ric(X_1, X_2)X_3 + {}^*Ric(X_2, X_3)X_1 + {}^*Ric(X_3, X_1)X_2 \\ &\quad - {}^*Ric(X_1, X_3)X_2 - {}^*Ric(X_2, X_1)X_3 - {}^*Ric(X_3, X_2)X_1\}. \end{aligned}$$

Definition 2.3. A contact metric manifold is called ξ-conformally flat and ξ-$*$conformally flat, respectively, if $C(X_1, X_2)\xi = 0$ and $^*C(X_1, X_2)\xi = 0$, respectively.

3. $*$-conformal curvature tensor in Sasakian manifolds

In [14], Ghash and Patra obtained the $*$-Ricci tensor in a $(2n + 1)$-dimensional Sasakian manifold as follows

$$^*Ric(X_1, X_2) = Ric(X_1, X_2) - (2n - 1)\, g(X_1, X_2) - \eta(X_1)\eta(X_2). \tag{3.1}$$

Equation (3.1) provides

$$^*LX_1 = LX_1 - (2n - 1)\, X_1 - \eta(X_1)\xi, \tag{3.2}$$

and

$$^*r = r - 4n^2. \tag{3.3}$$

Theorem 3.1. *Let M^{2n+1} be a manifold with a Sasakian structure (g, η, ξ, φ). The manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is a $^*\eta$-Einstien manifold.*

Proof. If $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold, then

$$\exists c, d \in C^\infty(M), \quad Ric(X_1, X_2) = c\, g(X_1, X_2) + d\, \eta(X_1)\eta(X_2). \tag{3.4}$$

From (3.1) and (3.4), we have

$$^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2), \tag{3.5}$$

where $\tilde{c} = c - (2n - 1)$ and $\tilde{d} = d - 1$. Thus, $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a $^*\eta$-Einstien manifold. In this case, there are smooth scalar functions $\tilde{c}$ and $\tilde{d}$

$$^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2). \tag{3.6}$$

By (3.6) and (3.1), we conclude that M is a η-Einstien manifold. $\square$

A Sasakian manifold is said to be a $\phi-$recurrent manifold if there exists a nonzero $1-$form A such that

$$\phi^2((\nabla_{X_1}K)(X_2,X_3)X_4) = A(X_1)\,K(X_2,X_3)X_4, \tag{3.7}$$

for arbitrary vector fields X_1, X_2, X_3, and X_4 on the manifold M [11]. As a result, a $\phi-$recurrent Sasakian manifold is an Einstein manifold. Thus, by Theorem 3.1, it follows that every $\phi-$recurrent Sasakian manifold is a $^*\eta$-Einstein manifold.

In 1968, Yano and Sawaki [27] defined quasi-conformal curvature tensor as follows:

$$\begin{aligned} W(X_1,X_2)X_3 &= [-(n-2)\,d]\,C(X_1,X_2)X_3 \\ &+ [c+(n-2)\,d]\,\tilde{C}(X_1,X_2)X_3, \end{aligned} \tag{3.8}$$

where c and d are arbitrary constants, C is the conformal curvature tensor, and $\tilde{C}$ given by

$$\begin{aligned} \tilde{C}(X_1,X_2)X_3 &= K(X_1,X_2)X_3 \\ &- \frac{r}{n(n-1)}\big[g(X_2,X_3)X_1 - g(X_1,X_3)X_2\big], \end{aligned} \tag{3.9}$$

where K is the Riemannian curvature tensor.

A quasi-conformally flat Sasakian manifold or a quasi-conformally semi-symmetric Sasakian manifold is an η-Einstein manifold [9]. Using Theorem 3.1, we infer every quasi-conformally flat or quasi-conformally semi-symmetric Sasakian manifold is a $^*\eta$-Einstein manifold.

By using (3.1), (3.2) and (3.3), from (2.1), we get

$$\begin{aligned} {}^*C(X_1,X_2)X_3 = C(X_1,X_2)X_3 &+ \frac{2n-2}{2n-1}\Big(g(X_2,X_3)X_1 - g(X_1,X_3)X_2\Big) \\ &+ \frac{1}{2n-1}\Big(\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\ &+ g(X_2,X_3)\eta(X_1)\xi - g(X_1,X_3)\eta(X_2)\xi\Big). \end{aligned} \tag{3.10}$$

In Sasakian manifolds, Proposition 2.1 reduces to Proposition 3.1.

Proposition 3.1. *In a Sasakian manifold, the $*$-conformal curvature tensor obeys the following:*

$$^*C(X_1,X_2)X_3 + {}^*C(X_2,X_3)X_1 + {}^*C(X_3,X_1)X_2 = 0.$$

In a 3-dimensional manifold, C vanishes identically, and hence, we have:

$$\begin{aligned} {}^*C(X_1,X_2)X_3 &= \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\ &+ g(X_2,X_3)\eta(X_1)\xi - g(X_1,X_3)\eta(X_2)\xi, \end{aligned} \tag{3.11}$$

In this case, (3.11) infers ${}^*C$ does not vanish identically. Indeed, for any non-zero vector filed $\tilde{X}$ in the kernel of η, we have

$$ {}^*C(2\tilde{X}+\xi, \tilde{X}+\xi)\xi = \tilde{X}. $$

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a Sasakian manifold. By putting $X_3 = \xi$ in (3.10), we have

$$ {}^*C(X_1, X_2)\xi = C(X_1, X_2)\xi + K(X_1, X_2)\xi. \tag{3.12} $$

Based on (3.12) and $K(X_1, X_2)\xi \neq 0$, we infer the Sasakian manifold does not become ξ-conformally flat and ξ-$*$conformally flat simultaneously.

Every Sasakian manifold is K-contact, but in general, every K-contact manifold is not Sasakian. For 3-dimensional manifolds, these are equivalent. In [28], the authors prove that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstien Sasakian manifold. From Theorem 3.1, we can say that a K-contact manifold is ξ-conformally flat if and only if it is a ${}^*\eta$-Einstien Sasakian manifold.

In [8], the authors defined the $(0,2)$-tensor field T on M^{2n+1} as follows:

$$ T(X_1, X_2) = -\frac{Ric(X_1, X_2)}{2n-1} + \frac{r\ g(X_1, X_2)}{4n(2n-1)}. \tag{3.13} $$

The conformal curvature tensor is given by

$$ \begin{aligned} C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad T(X_2, X_3)\cdot X_1 - T(X_1, X_3)\cdot X_2 \\ &+ \quad g(X_2, X_3)\ \hat{T}(X_1) - g(X_1, X_3)\ \hat{T}(X_2), \end{aligned} \tag{3.14} $$

where $T(X_1, X_2) = g(\hat{T}(X_1), X_2)$. For $n > 1$, If $C = 0$, then

$$ \nabla_{X_1} T(X_2, X_3) - \nabla_{X_2} T(X_1, X_3) = 0. \tag{3.15} $$

We put $D(X_1, X_2)X_3 := \nabla_{X_1} T(X_2, X_3) - \nabla_{X_2} T(X_1, X_3)$. Now, we define $(0,2)$-tensor field ${}^*T$ on a Sasakian manifold M^{2n+1} as follows:

$$ {}^*T(X_1, X_2) = -\frac{{}^*Ric(X_1, X_2)}{2n-1} + \frac{{}^*r\ g(X_1, X_2)}{4n(2n-1)}. \tag{3.16} $$

By (3.1) and (3.3), we can write (3.16) as follows

$$ {}^*T(X_1, X_2) = T(X_1, X_2) + \frac{n-1}{2n-1}\ g(X_1, X_2) + \frac{1}{2n-1}\ \eta(X_1)\eta(X_2). \tag{3.17} $$

Also, we define the conformal curvature tensor as follows:

$$ \begin{aligned} {}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad {}^*T(X_2, X_3)\cdot X_1 - {}^*T(X_1, X_3)\cdot X_2 \\ &+ \quad g(X_2, X_3){}^*\hat{T}(X_1) - g(X_1, X_3){}^*\hat{T}(X_2), \end{aligned} \tag{3.18} $$

where ${}^*T(X_1, X_2) = g({}^*\hat{T}(X_1), X_2)$. So $(0,1)$-tensor field ${}^*\hat{T}$ is given by

$$ {}^*\hat{T}(X_1) = \hat{T}(X_1) + \frac{n-1}{2n-1}\ X_1 + \frac{1}{2n-1}\ \eta(X_1)\xi. \tag{3.19} $$

By putting (3.17) and (3.19) in (3.18), we have

$$
\begin{aligned}
{}^*C(X_1,X_2)X_3 = C(X_1,X_2)X_3 \quad &+ \quad \frac{2(n-1)}{2n-1}\big[g(X_2,X_3)X_1 - g(X_1,X_3)X_2\big] \\
&+ \quad \frac{1}{2n-1}\big[g(X_2,X_3)\xi - \eta(X_3)X_2\big]\eta(X_1) \\
&- \quad \frac{1}{2n-1}\big[g(X_1,X_3)\xi - \eta(X_3)X_1\big]\eta(X_2).
\end{aligned}
\tag{3.20}
$$

We consider

$$
{}^*D(X_1,X_2)X_3 := \nabla_{X_1}{}^*T(X_2,X_3) - \nabla_{X_2}{}^*T(X_1,X_3).
\tag{3.21}
$$

A direct computation shows that

$$
\begin{aligned}
\nabla_{X_1}{}^*T(X_2,X_3) \quad &= \quad \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) + \lambda\ \nabla_{X_1}(\eta(X_2)\eta(X_3)) \\
&= \quad \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) \\
&+ \quad \lambda\ \big[\big(\nabla_{X_1}\eta(X_2)\big)\ \eta(X_3) + \eta(X_2)\ \big(\nabla_{X_1}\eta(X_3)\big)\big],
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
\nabla_{X_2}{}^*T(X_1,X_3) \quad &= \quad \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) + \lambda\ \nabla_{X_2}(\eta(X_1)\eta(X_3)) \\
&= \quad \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) \\
&+ \quad \lambda\ \big[\big(\nabla_{X_2}\eta(X_1)\big)\eta(X_3) + \eta(X_1)\ \big(\nabla_{X_2}\eta(X_3)\big)\big],
\end{aligned}
\tag{3.23}
$$

where $\mu = \frac{2n-2}{2n-1}$ and $\lambda = \frac{1}{2n-1}$. By putting (3.22) and (3.23) in (3.21), we have

$$
\begin{aligned}
{}^*D(X_1,X_2)X_3 = D(X_1,X_2)X_3 \quad &+ \quad \lambda\ \Big\{2g(X_1,\phi X_2)\eta(X_3) \\
&+ \quad (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\Big\}.
\end{aligned}
\tag{3.24}
$$

If M^{2n+1} is a conformally flat Sasakian manifold with $n > 1$, then

$$
{}^*D(X_1,X_2)X_3 = \lambda\ \Big\{2g(X_1,\phi X_2)\eta(X_3) + (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\Big\}.
\tag{3.25}
$$

From (3.24), it can be concluded that, if M^{2n+1} is a Sasakian manifold of dimension greater than 3, then $D(X_1,X_2)X_3 = 0$ and ${}^*D(X_1,X_2)X_3 = 0$ do not hold simultaneously, because otherwise, we have $d\eta = 0$, which is a contradiction with the Sasakian structure.

Example 3.1. We consider the Sasakian manifold $(\mathbb{R}^3, g, \eta, \xi, \varphi)$, where the 1-form η, vector field ξ, Riemannian metric g, and $(1,1)$-tensor field φ respectively as follows

$$
\eta = \frac{1}{2}(dz - ydx), \qquad \xi = 2\frac{\partial}{\partial z}, \qquad g = \eta\otimes\eta + \frac{1}{4}\Big((dx)^2 + (dy)^2\Big),
$$

and $\varphi == dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x} + ydz \otimes \frac{\partial}{\partial y}$. Also, the vector fields are given by

$$X_1 = 2\frac{\partial}{\partial y}, \qquad X_2 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \qquad X_3 = \xi.$$

So, we have

$$\varphi X_1 = X_2, \qquad \varphi X_2 = -X_1, \qquad \varphi\xi = 0.$$

We know that, $\mathbb{R}^3$ is a conformally flat manifold, then $C = 0$. By (3.10) and $C(X_1, X_2)X_3 = 0$, we have ${}^*C(X_1, X_2)X_3 = -y\, X_1$. Therefore, for this 3-dimensional Sasakian manifold, the tensor ${}^*C$ will not be zero. On the other hand, we know that since $C(X_1, X_2)X_3 = 0$, then $D(X_1, X_2)X_3 = 0$. Therefore, having (3.25), we calculate the tensor ${}^*D$ as follows:

$${}^*D(X_1, X_2)X_3 = -2.$$

4. $*$-conformal curvature tensor in Kenmotsu manifolds

In [25], the author proves that in a Kenmotsu 3-manifold the $*$-Ricci tensor is given by

$$\begin{aligned}
{}^*Ric(X_1, X_2) &= (\frac{r}{2} + 2)g(\varphi X_1, \varphi X_2), && (4.1)\\
{}^*r &= r + 4, && (4.2)\\
{}^*LX_1 &= (\frac{r}{2} + 2)\big[X_1 - \eta(X_1)\xi\big]. && (4.3)
\end{aligned}$$

By substituting (4.1), (4.2), and (4.3) into (2.1) yields

$$\begin{aligned}
{}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 &- (\frac{r}{2} + 2)\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big]\\
&+ (\frac{r}{2} + 2)\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\\
&+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big]. \qquad (4.4)
\end{aligned}$$

Definition 4.1. [18] If the curvature tensor K of an almost contact metric manifold obeys the subsequent condition, then is called quasi-constant curvature:

$$\begin{aligned}
K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) &+ \beta\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\\
&+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big], \qquad (4.5)
\end{aligned}$$

where $(X_1 \wedge X_2)(X_3) := g(X_2, X_3)X_1 - g(X_1, X_3)X_2$, α and β are smooth functions.

By some calculation, one concludes that the following holds.

Theorem 4.1. *If a Kenmotsu* 3*-manifold is of quasi-constant curvature of the form*

$$\begin{aligned}
K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) &- \alpha\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\\
&+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big], \qquad (4.6)
\end{aligned}$$

where $\alpha = \frac{r}{2} + 2$, then $*$-conformal curvature tensor vanishes.

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a Kenmostu manifold. By [21], we have

$$\begin{aligned} &(4.7) \quad {}^*Ric(X_1, X_2) &=& Ric(X_1, X_2) + (2n-1)g(X_1, X_2) + \eta(X_1)\eta(X_2),\\ &(4.8) \quad {}^*r &=& r + 4n^2,\\ &(4.9) \quad {}^*LX_1 &=& LX_1 + (2n-1)X_1 + \eta(X_1)\xi. \end{aligned}$$

By putting $X_2 = \xi$ in (4.7) and from (2.9), we have

$$ {}^*Ric(X_1, \xi) = 0, \tag{4.10}$$

from (2.11) and (4.7), we have

$$ {}^*Ric(\phi X_1, \phi X_2) = {}^*Ric(X_1, X_2). \tag{4.11}$$

Theorem 4.2. *Suppose M^{2n+1} is a manifold and (g, η, ξ, φ) is a Kenmotsu structure on M. The M is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. In [5], the contact metric structure is said to be η-Einstein if

$$ L = c\,I + d\,\eta \otimes \xi, \qquad c, d \in C^\infty(M). \tag{4.12}$$

Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a η-Einstein Kenmotsu manifold. By (4.9) and (4.12), we have

$$ {}^*L = \tilde{c}\,I + \tilde{d}\,\eta \otimes \xi, \tag{4.13}$$

where $\tilde{c} = c + (2n-1)$ and $\tilde{d} = c + 1$.

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a ${}^*\eta$-Einstein Kenmotsu manifold, then there are smooth functions $\tilde{c}$, and $\tilde{d}$ such that

$$ {}^*Ric(X_1, X_2) = \tilde{c}\,g(X_1, X_2) + \tilde{d}\,\eta(X_1)\eta(X_2). \tag{4.14}$$

By (4.14) and (4.7), we have

$$ Ric(X_1, X_2) = c\,g(X_1, X_2) + d\,\eta(X_1)\eta(X_2), \tag{4.15}$$

where $c = \tilde{c} - (2n-1)$ and $d = \tilde{d} - 1$. $\square$

By substituting (4.7), (4.8), and (4.9) into (2.1) yields

$$\begin{aligned} {}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3 &- \frac{2n-2}{2n-1}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big]\\ &- \frac{1}{2n-1}\big[g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\\ &+ \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\big]. \end{aligned} \tag{4.16}$$

By putting $X_3 = \xi$ in (4.16), we obtain

$$ {}^*C(X_1, X_2)\xi = C(X_1, X_2)\xi + K(X_1, X_2)\xi. \tag{4.17}$$

From (4.17), we conclude that if $C(X_1, X_2)\xi = 0$ then ${}^*C(X_1, X_2)\xi \neq 0$. In other words, the Kenmotsu manifold cannot be ξ-conformally flat and ξ-$*$conformally flat simultaneously.

In the Kenmotsu manifold, (2) results in ${}^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$. By Proposition 2.1 and ${}^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$, the $*$-conformal curvature tensor satisfies in Bianchi type identity, which leads to the next proposition.

Proposition 4.1. *In a Kenmotsu manifold, the $*$-conformal curvature tensor obeys the relation:*

$$ {}^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0. $$

Let us define

$$ C(X_1, X_2, X_3, X_4) := g\big(C(X_1, X_2)X_3, X_4\big), \qquad \forall X_1, X_2, X_3, X_4 \in \chi(M). $$

By substituting (4.7) into (2.1), we have

$$ \begin{aligned} {}^*C(X_1, X_2, X_3, X_4) &= C(X_1, X_2, X_3, X_4) \\ &- \frac{2n-2}{2n-1}\big[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\big] \\ &- \frac{1}{2n-1}\big[g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) \\ &+ g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3)\big]. \end{aligned} \tag{4.18} $$

Proposition 4.2. *For a Kenmotsu manifold, the $*$-conformal tensor cannot vanish identically.*

Proof. One can see that

$$ C(X_1, X_2, X_3, X_4) = -C(X_1, X_2, X_4, X_3). \tag{4.19} $$

Suppose that ${}^*C$ vanishes identically. Therefore, by (4.18) and (4.19), we have

$$ \begin{aligned} 2(2n-2) &\Big[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\Big] \\ &+ 2\big[g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) \\ &+ g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3)\big] = 0. \end{aligned} \tag{4.20} $$

Putting $X_3 = X_1 = \xi$ into (4.20) implies that

$$ (2n-1)\Big(g(X_2, X_4) - \eta(X_2)\eta(X_4)\Big) = 0. \tag{4.21} $$

Since $2n-1$ is an odd number, we have

$$ g(X_2, X_4) - \eta(X_2)\eta(X_4) = 0, \qquad \forall X_2, X_4 \in \chi(M), \tag{4.22} $$

which is impossible. $\square$

Using Propositions 4.1 and 4.2, one concludes that a Kenmotsu 3-manifold cannot be of quasi-constant curvature of the form (4.6).

Now, we consider $(0,2)$-tensor field ${}^{*}T$ on Kenmotsu manifold M^{2n+1} as follows:

$$ {}^{*}T(X_1,X_2) = -\frac{{}^{*}Ric(X_1,X_2)}{2n-1} + \frac{{}^{*}r\ g(X_1,X_2)}{4n(2n-1)}. \tag{4.23}$$

By (4.8) and (4.7), we can write (4.23) as follows:

$$ {}^{*}T(X_1,X_2) = T(X_1,X_2) + \frac{(1-n)}{(2n-1)}\ g(X_1,X_2) + \frac{-1}{2n-1}\ \eta(X_1)\eta(X_2). \tag{4.24}$$

Also, we define the conformal curvature tensor as follows:

$$\begin{aligned} {}^{*}C(X_1,X_2)X_3 = K(X_1,X_2)X_3 &+ {}^{*}T(X_2,X_3)\cdot X_1 - {}^{*}T(X_1,X_3)\cdot X_2 \\ &+ g(X_2,X_3){}^{*}\hat{T}(X_1) - g(X_1,X_3){}^{*}\hat{T}(X_2), \end{aligned} \tag{4.25}$$

where ${}^{*}T(X_1,X_2) = g({}^{*}\hat{T}(X_1),X_2)$. So ${}^{*}\hat{T}$ is given by

$$ {}^{*}\hat{T}(X_1) = \hat{T}(X_1) + \frac{(1-n)}{(2n-1)}\ X_1 + \frac{-1}{2n-1}\ \eta(X_1)\xi. \tag{4.26}$$

By putting (4.24) and (4.26) in (4.25), we have

$$\begin{aligned} {}^{*}C(X_1,X_2)X_3 = C(X_1,X_2)X_3 &+ \frac{2(1-n)}{(2n-1)}\big[g(X_2,X_3)X_1 - g(X_1,X_3)X_2\big] \\ &+ \Big(\frac{-1}{2n-1}\Big)\big[g(X_2,X_3)\xi - \eta(X_3)X_2\big]\eta(X_1) \\ &- \Big(\frac{-1}{2n-1}\Big)\big[g(X_1,X_3)\xi - \eta(X_3)X_1\big]\eta(X_2). \end{aligned} \tag{4.27}$$

We consider

$$ {}^{*}D(X_1,X_2)X_3 := \nabla_{X_1}{}^{*}T(X_2,X_3) - \nabla_{X_2}{}^{*}T(X_1,X_3). \tag{4.28}$$

Now, we consider can we conclude ${}^{*}D(X_1,X_2)X_3 = 0$ if ${}^{*}C(X_1,X_2)X_3 = 0$. So

$$\begin{aligned} \nabla_{X_1}{}^{*}T(X_2,X_3) &= \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) + \lambda\ \nabla_{X_1}(\eta(X_2)\eta(X_3)) \\ &= \nabla_{X_1}T(X_2,X_3) + \mu\ \nabla_{X_1}g(X_2,X_3) \\ &+ \lambda\ \big[\big(\nabla_{X_1}\eta(X_2)\big)\ \eta(X_3) + \eta(X_2)\ \big(\nabla_{X_1}\eta(X_3)\big)\big], \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \nabla_{X_2}{}^{*}T(X_1,X_3) &= \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) + \lambda\ \nabla_{X_2}(\eta(X_1)\eta(X_3)) \\ &= \nabla_{X_2}T(X_1,X_3) + \mu\ \nabla_{X_2}g(X_1,X_3) \\ &+ \lambda\ \big[\big(\nabla_{X_2}\eta(X_1)\big)\eta(X_3) + \eta(X_1)\ \big(\nabla_{X_2}\eta(X_3)\big)\big], \end{aligned} \tag{4.30}$$

where $\mu = \frac{2(1-n)}{(2n-1)}$ and $\lambda = \frac{-1}{2n-1}$. By putting (4.29) and (4.30) in (4.28), we have

$$\begin{aligned} {}^*D(X_1, X_2)X_3 &= D(X_1, X_2)X_3 \\ &+ \lambda \Big\{ (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1) \Big\}. \end{aligned} \tag{4.31}$$

Theorem 4.3. *Let M be a $2n+1$-dimension manifold with $n > 1$ and (g, η, ξ, φ) is a Kenmotsu structure on M. Then $D(X_1, X_2)X_3 = 0$ and ${}^*D(X_1, X_2)X_3 = 0$ do not hold at the same time.*

Proof. From (4.31), it is easily proved. □

Example 4.1. We consider the Kenmotsu manifold $(\mathbb{R}^3 - (0,0,0), g, \eta, \xi, \varphi)$, where the 1-form η, vector field ξ, Riemannian metric g, and $(1,1)$-tensor field φ respectively as follows

$$\eta = -\frac{1}{z}dz, \qquad \xi = -z\frac{\partial}{\partial z}, \qquad g = (dx)^2 + (dy)^2 + (dz)^2,$$

and $\varphi = dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$. Also, the vector fields are given by

$$X_1 = z\frac{\partial}{\partial x}, \qquad X_2 = z\frac{\partial}{\partial y}, \qquad X_3 = \xi.$$

So, we have

$$\varphi X_1 = -X_2, \qquad \varphi X_2 = X_1, \qquad \varphi\xi = 0.$$

By conformally flat manifold $\mathbb{R}^3$, we have $C = 0$. By (4.16) and $C = 0$, then ${}^*C(X_1, X_2)X_3 = 0$. We know that since $C(X_1, X_2)X_3 = 0$, then $D(X_1, X_2)X_3 = 0$. Therefore, having (4.31), ${}^*D(X_1, X_2)X_3 = 0$.

5. $*$-conformal curvature of the cosymplectic manifolds

Let (g, η, ξ, φ) be a cosymplectic structure on M^{2n+1}. In [17], it is proved that for a cosymplectic manifold

$${}^*Ric(X_1, X_2) = Ric(X_1, X_2), \tag{5.1}$$

and

$${}^*r = r. \tag{5.2}$$

Theorem 5.1. *Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a cosymplectic manifold. Then M is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. It is easy to conclude from (5.1) that for the cosymplectic manifold, the η-Einstien manifold and ${}^*\eta$-Einstien manifold are equivalent. □

Substituting (5.1) and (5.2) into (2.1) yields

$${}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3. \tag{5.3}$$

Proposition 5.1. *In a cosymplectic manifold, the $*$-conformal curvature tensor obeys the relation:*

$$^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0.$$

The following results are obtained from (5.3).

Corollary 5.1. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a cosymplectic manifold. Then M is a conformally flat if and only if it is a $*$-conformally flat.*

Corollary 5.2. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a cosymplectic manifold. Then M is a ξ-conformally flat if and only if it is a ξ-$*$conformally flat.*

The conformal curvature tensor is zero in dimension 3. Thus we have:

Proposition 5.2. *For a 3-dimensional cosymplectic manifold, $^*C$ is identically zero.*

We consider $(0, 2)$-tensor field $^*T$ on cosymplectic manifold M^{2n+1} as follows:

$$^*T(X_1, X_2) = -\frac{^*Ric(X_1, X_2)}{2n-1} + \frac{^*r\ g(X_1, X_2)}{4n(2n-1)}. \tag{5.4}$$

By (5.1) and (5.2), we can

$$^*T(X_1, X_2) = T(X_1, X_2). \tag{5.5}$$

Also, define the conformal curvature tensor as follows:

$$\begin{aligned} ^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \ &+ \ ^*T(X_2, X_3) \cdot X_1 - {}^*T(X_1, X_3) \cdot X_2 \\ &+ \ g(X_2, X_3)^*\hat{T}(X_1) - g(X_1, X_3)^*\hat{T}(X_2), \end{aligned} \tag{5.6}$$

where $^*T(X_1, X_2) = g(^*\hat{T}(X_1), X_2)$. So $(0, 1)$-tensor field $^*\hat{T}$ is given by

$$^*\hat{T}(X_1) = \hat{T}(X_1). \tag{5.7}$$

By putting (5.5) and (5.7) in (5.6), we have

$$^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3. \tag{5.8}$$

We consider

$$^*D(X_1, X_2)X_3 := \nabla_{X_1}{}^*T(X_2, X_3) - \nabla_{X_2}{}^*T(X_1, X_3). \tag{5.9}$$

On the other hand, we have

$$\nabla_{X_1}{}^*T(X_2, X_3) = \nabla_{X_1}T(X_2, X_3), \tag{5.10}$$

and

$$\nabla_{X_2}{}^{*}T(X_1, X_3) = \nabla_{X_2} T(X_1, X_3). \tag{5.11}$$

By putting (5.10) and (5.11) in (5.9), we have

$$^{*}D(X_1, X_2)X_3 = D(X_1, X_2)X_3. \tag{5.12}$$

We know that if $C = 0$ for a $2n + 1$-dimension cosymplectic manifold with $n \geqslant 1$, then $D = 0$. Now, if we assume $^{*}C = 0$, then according to (5.12), the following theorem is obtained.

Theorem 5.2. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$. If M is a $*$-conformally flat manifold, then $^{*}D = 0$.*

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