FACTA UNIVERSITATIS
Series: **Mechanical Engineering** Vol. 13, N° 2, 2015, pp. 81 - 90

Original scientific paper

APPLICATION OF THE METHOD OF DIMENSIONALITY REDUCTION TO CONTACTS UNDER NORMAL AND TORSIONAL LOADING

UDC 539.3

Emanuel Willert[1], Markus Hess[1], Valentin L. Popov[1,2]

[1]Berlin University of Technology, Germany
[2]National Research Tomsk State University, Russia

Abstract. *Recently the method of dimensionality reduction (MDR) has been introduced to solve axisymmetric contact problems easily and exactly. The list of tasks that this method can deal with comprises normal, tangential, adhesive and rolling contacts with simply connected contact areas between elastic or viscoelastic bodies. Due to its simplicity and easy applicability the MDR provides the possibility of fast and comprehensive studies of contact problems in technological or biological systems, for example bearings, artificial hip joints, wheel-rail systems or others. Within the complicated three-dimensional contact theory those studies, in most cases, cannot be done without a tremendous mathematical or numerical effort.*
In view of all this, the torsional contact problems have been disregarded until now, although it is known that torsion is a major reason of wear and possible failure of system components. Therefore, in the present paper, we extend the MDR to contacts of axisymmetric profiles under superimposed normal and torsional loading.

Key Words: *Contact Mechanics, Method of Dimensionality Reduction, Torsion, Friction, Stick, Slip*

1. INTRODUCTION

Pure torsional contacts or normal and torsional contacts coupled by friction were not given much attention in the past, although torsional loading is known to be a major reason of wear and fatigue.

In [1] Lubkin gave the shear stress distribution for the torsional contact between two elastic spheres with partial slip. Hetényi and McDonald Jr. calculated the stresses and displacements for the full-sliding contact between an elastic sphere and an elastic halfspace using Hankel transforms [2]. Also based on Hankel transforms, i.e. Bessel functions, Kartal

Received May 29, 2015 / Accepted July 1, 2015
Corresponding author: Emanuel Willert
Technische Universität Berlin, Institut für Mechanik, Str. des 17. Juni 135, 10623 Berlin, Germany
E-mail: e.willert@tu-berlin.de

et al. analyzed the torsional contact with partial slip between flat-ended elastic cylinders [3]. Jäger determined the stress distributions in the form of Abel transforms for the torsional contact with partial slip between axisymmetric bodies of arbitrary profile shape [4]. He thereby used a superposition of flat-punch-solutions to solve the contact problem of arbitrarily shaped bodies. In the experimental work [5] Trejo et al. investigated the friction between an elastomer and a randomly rough surface using a torsional contact configuration.

In a series of recent papers, Popov and collaborators have introduced the so-called method of dimensionality reduction (MDR), which allows solving normal contact problems of axisymmetric elastic and viscoelastic bodies as well as tangential contact problems with a constant coefficient of friction for arbitrary loading histories [6, 7]. In the monographs [8] and [9], the MDR has been summarized and many applications have been provided. Moreover, an introduction into its usage in the form of a user's handbook can be found in [10].

In the present paper, we are showing that the contact with torsion (rotation around the normal axis to the contact plane) can also be described with the MDR as long as there is no slip in the contact or the thickness of the slip annulus at the edge of the contact is small compared to the contact radius.

The paper is organized as follows: In the Section 2 we reproduce, for convenience of the reader, the derivation of the MDR equations for the normal contact following [9]. In the Section 3 the application of the MDR to contacts with torsion without slip is discussed. In the Section 4 the torsional contact with a narrow slip region is considered. Section 5 closes the paper.

2. Method of Dimensionality Reduction for the Normal Contact

In this section, the equations of the MDR for the normal contact of axisymmetric profiles $f(r)$ with a compact area of contact are deduced. Thereby we follow the idea of Jäger [11] to derive the solution of an axisymmetric contact problem by summation of differential flat punch solutions.

A flat punch of radius a, indenting an elastic half space with effective elastic modulus E^* by indentation depth d, produces displacement u_z

$$u_z = \begin{cases} d\ , & r < a \\ \dfrac{2}{\pi} d \cdot \arcsin(a/r), & r > a \end{cases} . \tag{1}$$

The resulting pressure distribution will be

$$p(r) = \begin{cases} \dfrac{1}{\pi} \dfrac{E^* d}{\sqrt{a^2 - r^2}}\ , & r < a \\ 0\ , & r > a \end{cases} \tag{2}$$

and the total normal force

$$F_N = 2E^* a d\ . \tag{3}$$

Hence, contact stiffness k_z is

$$k_z(a) = \frac{\mathrm{d}F_N}{\mathrm{d}d} = 2E^* a\ . \tag{4}$$

Note that this equation is valid for any profile shape, if a is understood as the current contact radius.

Let us assume a contact between a rigid indenter of shape $z = f(r)$ and an elastic half space. The indentation depth due to normal force F_N will be d and contact radius a. For any given profile shape any of those three parameters will unambiguously define the other two. Especially, the indentation depth is a definite function of the contact radius, which we will denote by

$$d = g(a) . \tag{5}$$

Firstly we show that the complete solution of the normal contact problem will be unambiguously determined by function $d = g(a)$.

Analyzing the complete process of indentation from its very first moment until the final indentation, the current values of the normal force, indentation depth and contact radius are given by $\tilde{F}_N$, $\tilde{d}$ and $\tilde{a}$. During the indentation process, the indentation depth changes from $\tilde{d} = 0$ to $\tilde{d} = d$, the contact radius accordingly from $\tilde{a} = 0$ to $\tilde{a} = a$ and the normal force from $\tilde{F}_N = 0$ to $\tilde{F}_N = F_N$. The final normal force can be written as

$$F_N = \int_0^{F_N} \mathrm{d}\tilde{F}_N = \int_0^a \frac{\mathrm{d}\tilde{F}_N}{\mathrm{d}\tilde{d}} \frac{\mathrm{d}\tilde{d}}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a} . \tag{6}$$

If we take into account that the differential contact stiffness of an area with radius $\tilde{a}$ is given by (4) and the indentation depth by (5), we get

$$F_N = 2E^* \int_0^a \tilde{a} \frac{\mathrm{d}g(\tilde{a})}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a} , \tag{7}$$

which gives after partial integration

$$F_N = 2E^* \left[a \cdot g(a) - \int_0^a g(\tilde{a}) \mathrm{d}\tilde{a} \right] = 2E^* \left[\int_0^a \big(d - g(\tilde{a}) \big) \mathrm{d}\tilde{a} \right] . \tag{8}$$

Let us calculate the pressure distribution within the contact area. An infinitesimal indentation $\mathrm{d}\tilde{d}$ of an area with radius $\tilde{a}$ will, due to (2), produce the pressure

$$\mathrm{d}p(r) = \frac{1}{\pi} \frac{E^*}{\sqrt{\tilde{a}^2 - r^2}} \mathrm{d}\tilde{d} . \tag{9}$$

The pressure distribution at the end of the indentation process is given by the sum of all infinitesimal pressure components,

$$p(r) = \int_{d(r)}^{d} \frac{1}{\pi} \frac{E^*}{\sqrt{\tilde{a}^2 - r^2}} \mathrm{d}\tilde{d} = \int_r^a \frac{1}{\pi} \frac{E^*}{\sqrt{\tilde{a}^2 - r^2}} \frac{\mathrm{d}g(\tilde{a})}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a} . \tag{10}$$

Hence, function $d = g(a)$ unambiguously defines the pressure distribution and therefore the total normal force as well. That is why the solution of the contact problem is reduced to the determination of this function.

This can be done as follows: Infinitesimal indentation $\mathrm{d}\tilde{d}$ mentioned above will, due to (1), produce surface displacement at $r = a > \tilde{a}$

$$\mathrm{d}u_z(a) = \frac{2}{\pi} \arcsin\left(\frac{\tilde{a}}{a} \right) \mathrm{d}\tilde{d} . \tag{11}$$

Again, the total displacement can be understood as a sum of all infinitesimal indentations:

$$u_z(a) = \frac{2}{\pi}\int_0^d \arcsin\left(\frac{\tilde{a}}{a}\right)\mathrm{d}\tilde{d} = \frac{2}{\pi}\int_0^a \arcsin\left(\frac{\tilde{a}}{a}\right)\frac{\mathrm{d}g(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a} \tag{12}$$

On the other hand, this displacement is given by $u_z(a) = d - f(a)$:

$$d - f(a) = \frac{2}{\pi}\int_0^a \arcsin\left(\frac{\tilde{a}}{a}\right)\frac{\mathrm{d}g(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}\ , \tag{13}$$

which gives after partial integration

$$f(a) = \frac{2}{\pi}\int_0^a \frac{g(\tilde{a})}{\sqrt{a^2-\tilde{a}^2}}\mathrm{d}\tilde{a}\ . \tag{14}$$

This is an Abel integration equation, which can be inverted [12]:

$$g(a) = a\int_0^a \frac{f'(\tilde{a})}{\sqrt{a^2-\tilde{a}^2}}\mathrm{d}\tilde{a}\ . \tag{15}$$

The MDR is mainly an interpretation of the equations (5), (8), (10) and (15), which, on the one hand, can be interpreted as just a mnemonic rule. However, in many ways it has a deeper physical meaning.

Let us assume an elastic foundation of independent equal springs, each at distance Δx from each other and with stiffness $\Delta k_z = E^* \Delta x$, as shown in Fig. 1.

Also, we define a one-dimensional profile $g(x)$ as a formal transformation of the three-dimensional axisymmetric profile $z = f(r)$ according to

$$g(x) = |x|\int_0^{|x|} \frac{f'(r)}{\sqrt{x^2-r^2}}\mathrm{d}r\ . \tag{16}$$

This transformation is illustrated in Fig. 2.

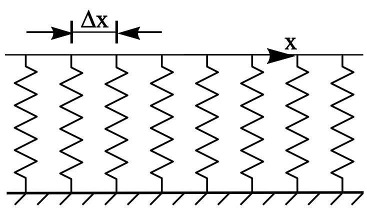

Fig. 1 Equivalent elastic foundation

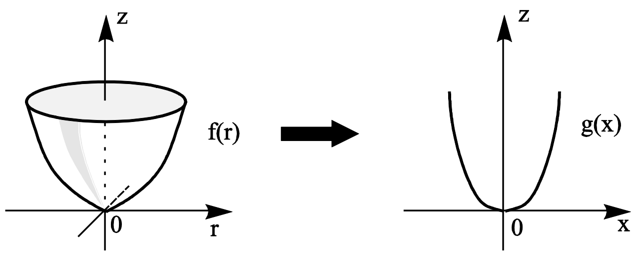

Fig. 2 Axisymmetric three-dimensional profile and one-dimensional analogue within the framework of the MDR (see Eq. (16))

The transformed profile is now pressed into the elastic foundation described above. This is shown in Fig. 3. The surface displacement in normal direction at any point x will be given by the difference of indentation depth d and profile shape $g(x)$:

$$u_z^{1D}(x) = d - g(x) \ . \tag{17}$$

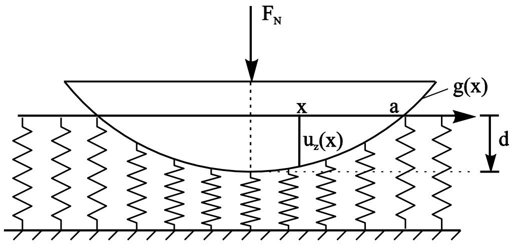

Fig. 3 MDR-model for the normal contact

For contacts without adhesion the displacement vanishes at the edge of the contact:

$$u_z^{1D}(a) = d - g(a) = 0 \ . \tag{18}$$

The normal force in a single spring is given by

$$\Delta F_N(x) = \Delta k_z(d - g(x)) = E^*(d - g(x))\Delta x \, , \tag{19}$$

from which the total normal force in the equilibrium state can be calculated by summation over all springs. In limiting case $\Delta x \to 0$ the sum will be the integral

$$F_N = E^* \int_{-a}^{a} u_z^{1D}(x)\mathrm{d}x = 2E^* \int_{0}^{a} (d - g(x))\mathrm{d}x \ . \tag{20}$$

It can be seen easily that the equations (18), (20) and (16) reproduce (5), (8) and (15). Hence, transformed profile $g(x)$ is the geometrical interpretation of dependence $d = g(a)$ for the given three-dimensional profile shape. By the equivalence of the equations presented above it is also shown that, instead of analyzing the three-dimensional contact problem, the described equivalent one-dimensional problem can be solved, obtaining the correct and exact results for the original contact.

In the next paragraph we are going to show how this can be done, by the same method, for torsional contact as well.

3. Description of the Torsional Contact with the Method of Dimensionality Reduction

Again, we start with the known solution for the torsional contact of a rigid flat cylinder. If a rigid flat-ended punch is pressed on an elastic half space and rotated around the axis of the cylinder by angle φ, the produced torsional moment, surface displacement and stress distribution will be given by equations [13]

$$M_z = \frac{16}{3} G a^3 \varphi \, , \tag{21}$$

$$u_\varphi(r) = \begin{cases} \varphi r\ , & r < a \\ \dfrac{2}{\pi}\varphi\left\{ r\arcsin\left(\dfrac{a}{r}\right) - a\sqrt{1-\dfrac{a^2}{r^2}} \right\}\ , & r > a \end{cases}\ , \tag{22}$$

$$\tau(r) = \begin{cases} \dfrac{4G\varphi}{\pi}\dfrac{r}{\sqrt{a^2-r^2}}\ , & r < a \\ 0\ , & r > a \end{cases}\ , \tag{23}$$

where G is the shear modulus. In the case of rotational symmetry and of no slip, the torsional contact problem completely decouples from the normal contact problem.

We analyze the torsional contact problem that is analogical to the normal contact problem described in the previous section, i.e. we imprint rotational surface displacement $u_\varphi(r) = r(\varphi - \psi(r))$ into an elastic half space and want to determine the shear stresses due to this displacement. $\psi(r)$ is the deviation of the torsional angle from the pure constant rotation with φ. This displacement is understood as a sum of infinitesimal torsional loadings of flat punches [4]. In analogy to (5) we introduce the function

$$\varphi = \Phi(a). \tag{24}$$

The complete torsional moment after the process of torque loading can be calculated as

$$M_z = \int_0^M \mathrm{d}\tilde{M}_z = \int_0^a \frac{\mathrm{d}\tilde{M}_z}{\mathrm{d}\tilde{\varphi}}\frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a} = \frac{16}{3}G\int_0^a \tilde{a}^3\frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}\ . \tag{25}$$

We introduce variable $u_y(x)$ in the following differential way:

$$\mathrm{d}u_y(x) = \begin{cases} x\mathrm{d}\varphi\ , & 0 < x < a \\ 0\ , & x > a \end{cases}. \tag{26}$$

At the end of the described process of infinitesimal torsional loadings this field will be

$$u_y(x) = \int_{\varphi(x)}^{\varphi} x\mathrm{d}\tilde{\varphi} = \int_x^a x\frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}\ . \tag{27}$$

Dividing (27) by x and differentiating with respect to x we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_y(x)}{x}\right) = -\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} - \frac{u_y(a)}{a}\delta(x-a)\ . \tag{28}$$

Equation (25) can then be written in the following way:

$$M_z = -\frac{16}{3}G\int_0^a \tilde{a}^3\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{a}}\left(\frac{u_y(\tilde{a})}{\tilde{a}}\right) + \frac{u_y(a)}{a}\delta(\tilde{a}-a)\right]\mathrm{d}\tilde{a}\ , \tag{29}$$

which gives after partial integration

$$M_z = 16G\int_0^a \tilde{a}\cdot u_y(\tilde{a})\mathrm{d}\tilde{a}\ . \tag{30}$$

It is obvious that this equation can also be interpreted within a one-dimensional model.

Let us assume an elastic foundation with tangential stiffness

$$\Delta k_y = 8G\Delta x . \tag{31}$$

The force of a single spring is given by

$$\Delta F_y = \Delta k_y u_y(x) \tag{32}$$

and the resulting torsional moment by

$$\Delta M_z = 8Gxu_y(x)\Delta x . \tag{33}$$

The total moment can be calculated again by integration and will be

$$M_z = 8G\int_{-a}^{a} x\cdot u_y(x)\mathrm{d}x = 16G\int_{0}^{a} x\cdot u_y(x)\mathrm{d}x, \tag{34}$$

which coincides with (30).

To complete the solution of the described torsional contact problem, let us calculate function $\Phi(a)$ and the stress distribution. According to (23) the stress distribution can – analogically to the previous section – be calculated from function $\Phi(a)$:

$$\tau(r) = \frac{4G}{\pi}\int_{r}^{a}\frac{r}{\sqrt{\tilde{a}^2-r^2}}\frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}, \tag{35}$$

or with equation (28)

$$\tau(r) = -\frac{4G}{\pi}\int_{r}^{a}\frac{r}{\sqrt{\tilde{a}^2-r^2}}\left[\frac{\mathrm{d}}{\mathrm{d}\tilde{a}}\left(\frac{u_y(\tilde{a})}{\tilde{a}}\right)+\frac{u_y(a)}{a}\delta(\tilde{a}-a)\right]\mathrm{d}\tilde{a} , \tag{36}$$

which is equivalent to

$$\tau(r) = -\frac{4G}{\pi}\left[\int_{r}^{a}\frac{r}{\sqrt{\tilde{a}^2-r^2}}\frac{\mathrm{d}}{\mathrm{d}\tilde{a}}\left(\frac{u_y(\tilde{a})}{\tilde{a}}\right)\mathrm{d}\tilde{a}+\frac{u_y(a)}{a}\frac{r}{\sqrt{a^2-r^2}}\right]. \tag{37}$$

The displacement at the edge of the contact, i.e. at $r = a > \tilde{a}$, will be, according to (22),

$$u_\varphi(a) = a(\varphi-\psi(a)) = \frac{2}{\pi}\int_{0}^{a}\frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}}\left\{a\arcsin\left(\frac{\tilde{a}}{a}\right)-\tilde{a}\sqrt{1-\frac{\tilde{a}^2}{a^2}}\right\}\mathrm{d}\tilde{a} . \tag{38}$$

In the next section we will analyze the case of slip in the contact area. This will inevitably lead to requirement $\tau(a) = 0$. This given, (27) can be written as

$$u_y(x) = x(\varphi-\Phi(x)) \tag{39}$$

and partial integration of (38) will give

$$\psi(a) = \frac{4}{\pi a^2}\int_{0}^{a}\Phi(\tilde{a})\frac{\tilde{a}^2}{\sqrt{a^2-\tilde{a}^2}}\mathrm{d}\tilde{a} . \tag{40}$$

This is again an Abel integration equation, which can be inverted [12]:

$$\Phi(a) = \frac{1}{2a}\int_{0}^{a}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\psi(r))\frac{\mathrm{d}r}{\sqrt{a^2-r^2}}. \tag{41}$$

4. Torsional Contact with a Narrow Slip Region

The boundary conditions at the surface of the half space in the presence of slip can generally be written in the form

$$\begin{aligned} \psi(r) &= 0\ , && \text{for } r < c, \\ \tau(r) &= \mu p(r)\ , && \text{for } c < r < a \end{aligned}\ , \tag{42}$$

with the radius of stick domain c and coefficient of friction μ. From (41) it is obvious that

$$\Phi(x) = 0\ , \qquad \text{for } x < c\,. \tag{43}$$

Hence, the shear stresses within the contact area will be

$$\tau(r) = \begin{cases} \dfrac{4G}{\pi} \displaystyle\int_c^a \frac{r}{\sqrt{\tilde{a}^2 - r^2}} \frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a}\ , & \text{for } \ r < c \\ \dfrac{4G}{\pi} \displaystyle\int_r^a \frac{r}{\sqrt{\tilde{a}^2 - r^2}} \frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a}\ , & \text{for } \ c < r < a \end{cases}\ . \tag{44}$$

Note, that (44) can always be written in the form

$$\tau(r) = \begin{cases} \mu\left(p_a(r) - p_c(r)\right)\ , & \text{for } \ r < c \\ \mu p_a(r)\ , & \text{for } \ c < r < a \end{cases}\ . \tag{45}$$

Here $p_a(r)$ and $p_c(r)$ denote known pressure distribution $p(r)$ with respective contact radii a and c. Thus the shear stress distribution in the whole contact area is known. The contact problem will be solved completely, if the integral equation

$$\frac{4G}{\pi} \int_r^a \frac{r}{\sqrt{\tilde{a}^2 - r^2}} \frac{\mathrm{d}\Phi(\tilde{a})}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a} = \mu p_a(r) = \frac{\mu E^*}{\pi} \int_r^a \frac{1}{\sqrt{\tilde{a}^2 - r^2}} \frac{\mathrm{d}g(\tilde{a})}{\mathrm{d}\tilde{a}} \mathrm{d}\tilde{a} \tag{46}$$

can be solved for function $\Phi(\tilde{a})$.

In case of a very small area of the slip domain, i.e. if $a - c \ll a$, this solution is elementary, because $r \approx a$ and therefore

$$\mathrm{d}\Phi = \lambda \mathrm{d}g\ , \qquad \text{for } c < x < a \tag{47}$$

with $\lambda = \dfrac{\mu E^*}{4Ga}$.

Geometrically, (47) together with (43) describes the following indentation process: First the indenter is pressed into the half space in a pure normal direction until the radius of stick domain c is reached. After that any infinitesimal indentation is a superposition of normal and torque loading, bound by (47). The solution of (47) with the boundary conditions (24) and (5) is given by

$$\Phi(x) = \varphi + \lambda(g(x) - d), \qquad \text{for } c < x < a\ . \tag{48}$$

Hence,

$$u_y(x) = x\begin{cases} \varphi\,, & \text{for} \quad x < c \\ \lambda u_z(x)\,, & \text{for} \quad c < x < a \end{cases}. \tag{49}$$

Again, as we assumed $a - c \ll a$, it is $x \approx a$ in the slip domain and therefore

$$u_y(x) = \begin{cases} x\varphi\,, & \text{for} \quad x < c \\ \dfrac{\mu E^*}{4G} u_z(x)\,, & \text{for} \quad c < x < a \end{cases} \tag{50}$$

If we choose $\mu^* = 2\mu$ as the equivalent coefficient of friction in the MDR-model for torsion, (50) can be written in the form

$$u_y(x) = \begin{cases} x\varphi\,, & \text{for} \quad x < c \\ \dfrac{\mu^* \Delta k_z}{\Delta k_y} u_z(x)\,, & \text{for} \quad c < x < a \end{cases}. \tag{51}$$

The radius of the stick domain is given by

$$8G\varphi c = 2\mu E^*(g(a) - g(c)) = \mu^* E^*(g(a) - g(c)) \ , \tag{52}$$

which agrees with the condition of no slip for the springs at the edge of the stick domain in the MDR-model. That is why this torsional contact problem with a finite coefficient of friction can be described by the MDR.
We emphasize again that the derivation starting with (47) is only valid for $a - c \ll a$.

In the case of general partial slip the solution for $\Phi(\tilde{a})$ in the slip domain is given by the inverse Abel transform of (46) [12]:

$$\Phi(a) - \Phi(x) = \frac{\mu}{2G}\int_x^a \frac{p(r)\mathrm{d}r}{\sqrt{r^2 - x^2}}, \qquad \text{for} \quad c < x < a \tag{53}$$

However, the resulting necessary spring displacement of the MDR-model

$$u_y(x) = x(\Phi(a) - \Phi(x)) = \frac{\mu x}{2G}\int_x^a \frac{p(r)\mathrm{d}r}{\sqrt{r^2 - x^2}}, \qquad \text{for} \quad c < x < a \tag{54}$$

cannot be interpreted as easily as in the case described above in (47) - (51).

5. Conclusion

In the present paper, we have extended the method of dimensionality reduction to contacts subjected to a superimposed normal and tangential loading. For the case of the simultaneous normal and torsional loading we have shown that the consideration of the original three-dimensional contact problem can be replaced by a contact with a one-dimensional elastic foundation with a properly defined coefficient of friction and normal and tangential stiffness if the slip annulus is small compared to the contact radius.

Acknowledgements: *This work was supported in parts by the German Research Society and the Tomsk State University Academic D.I. Mendeleev Fund Program.*

REFERENCES

1. Lubkin J.L., 1951, *The torsion of elastic spheres in contact*, Journal of Applied Mechanics, 18, pp. 183-187.
2. Hetényi M., McDonald Jr. P.H., 1958, *Contact Stresses Under Combined Pressure and Twist*, Journal of Applied Mechanics, 25, pp. 396-401.
3. Kartal M.E., Hills D.A., Nowell D., Barber J.R., 2010, *Torsional contact between elastically similar flat-ended cylinders*, International Journal of Solids and Structures, 47, pp. 1375-1380.
4. Jaeger J., 1995, *Axi-symmetric bodies of equal material in contact under torsion or shift*, Archive of Applied Mechanics, 65, pp. 478-487.
5. Trejo M., Fretigny C., Chateauminois A., 2013, *Friction of viscoelastic elastomers with rough surfaces under torsional contact conditions*, Physical Review, E88, 052401.
6. Popov V.L., Psakhie S.G., 2007, *Numerical simulation methods in tribology*, Tribology International, 40, pp. 916–923.
7. Hess M., 2011, *Über die exakte Abbildung ausgewählter dreidimensionaler Kontakte auf Systeme mit niedrigerer räumlicher Dimension*, Cullier Verlag, Berlin.
8. Popov V.L., Hess M., 2015, *Method of dimensionality reduction in contact mechanics and friction*, Springer.
9. Popov V.L., 2015, *Kontaktmechanik und Reibung*, 3. überarbeitete Auflage, Springer.
10. Popov V.L., Hess M., 2014, *Method of dimensionality reduction in contact mechanics and friction: A users handbook. I. Axially symmetric contacts*, Facta Universitatis, Mechanical Engineering, 12(1), pp. 1-14.
11. Jaeger J., 2005, *New Solutions in Contact Mechanics*, WIT Press.
12. Bracewell R., 1965, *The Fourier Transform and Its Applications*, McGraw-Hill Book Company.
13. Johnson K.L., 2001, *Contact mechanics*, Cambridge University Press.