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SIMPLIFIED HAMILTONIAN-BASED FREQUENCY-AMPLITUDE FORMULATION FOR NONLINEAR VIBRATION SYSTEMS

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Abstract. *The Hamiltonian-based frequency formulation has been hailed as an unprecedented success for it gives a straightforward insight into a complex nonlinear vibration system with simple calculation. This paper gives a systematical analysis of the formulation, and two simplified formulations are suggested. The cubic-quintic Duffing oscillator is used as an example to show extremely simple calculation and remarkable accuracy. It can be used as a paradigm for many other applications, and the one-step solving process has cleaned up the road of the nonlinear vibration theory.*

Key Words: *He's Frequency Formulation, Duffing Equation, Hamilton Principle, One-step Solving*

1. INTRODUCTION

Nonlinear vibration systems arise everywhere in mechanical engineering, as in thin-walled structures [1], rotor vibration [2,3], gyroscope systems [4,5], micro-electro-mechanical systems [6,7], packaging systems [8], building vibration [9], heavy machinery [10], to name but a few. The most famous nonlinear oscillators include the pendulum systems [11, 12], van der Pol oscillator and Duffing oscillator [13]. The spring-pendulum systems have many applications in satellites, submarines, aircraft, and energy harvesting device [14]. The recent research frontier in the nonlinear vibration theory is the fractal vibration; hence the description of a vibrating system in a fractal space [15-18].

Generally a nonlinear vibration system is complex and it is difficult to solve it exactly. In the last 200 years, engineers had little mathematical tools. The most available one was the perturbation method. However, a small parameter assumption involved in the perturbation method hinders its widely applications for accurate prediction of the periodic properties of a practical problem; hence it is valid only for weakly nonlinear oscillators.

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The homotopy perturbation method [19-21] and the variational iteration method [22-24] are much advanced ones for strongly nonlinear oscillators, but both methods require some special mathematics skills for problem-solving; especially the former asks for a suitable homotopy equation, and the latter requires a complex identification of the Lagrange multiplier involved in the iteration algorithm.

For engineering problems, as claimed by the Chinese mathematician Ji-Huan He, the simpler is better for a complex problem [18]. Now the condition is completely changed; the Hamiltonian-based frequency formulation [25] sheds a promising light on a fast but reliable insight into the physical properties of a nonlinear oscillator with simple calculation. The Hamiltonian-based frequency formulation is mathematically beautiful and physically penetrating since only a simple integration calculation is needed. It gives a vivid picture of the whole solution properties.

The 2021 Hamiltonian-based frequency formulation [25] is a modification of He's 2006 frequency formulation [26]. The formulation observes a nonlinear vibration system in an energy view by establishing a Hamilton principle, which can elucidate clearly the energy conservation of the vibrating process, that is the total energy of kinetic and potential energy keeps unchanged during oscillation. As the formulation is derived from an energy form of the nonlinear oscillator, its approximate solution is valid for the whole solution domain without limitations such as those in the traditional perturbation method.

The frequency formulation begins with two arbitrary guesses of the frequency, and two residual integrals have to be calculated to estimate an accurate frequency. It is amazing that the accurate result does not depend upon the initial choice of two arbitrary frequencies even for strong nonlinearity. Due to these unprecedented advantages, the formulation caught immediate attention after publication due to its extremely simplicity and remarkable reliability. It has been successfully applied to Kundu-Mukherjee-Naskar oscillator [27], nonlinear vibration of embedded carbon nanotube [28], fractal vibration systems [29,30], packaging systems [31] and MEMS oscillators [32]. It has been hailed as an unprecedented success. Though the achievement has contributed to the fast development of the nonlinear vibration theory, the solving process is relative complex, so a modification is needed.

2. PROBLEM STATEMENT

Consider the following conservative oscillator [25]:

$$w'' + p(w) = 0, w'(0) = 0 \quad w(0) = B \quad (1)$$

where $p(w)$ is an odd function of the displacement w and B is the displacement amplitude. Its Hamilton principle can be easily established by the semi-inverse method [33]

$$J(w) = \int \left\{ \frac{1}{2} w'^2 - P(w) \right\} dt \quad (2)$$

where $P(w)$ is the potential energy defined so that

$$\frac{d}{dw} P(w) = p(w) \quad (3)$$

According to Hamilton principle, we have the conservation law

$$\frac{1}{2}w'^2 + P(w) = H \quad (4)$$

where H is the Hamiltonian constant, which depends only upon the initial conditions, so the following first-order differential equation is obtained:

$$\frac{1}{2}w'^2 + P(w) - P(B) = 0 \quad (5)$$

The Hamiltonian-based frequency formulation is to find an approximate solution in the form

$$w(t) = B \cos \omega t \quad (6)$$

where ω is the frequency to be solved later. The Hamiltonian-based frequency formulation begins with two arbitrary tries ω_1 and ω_2 , and the following residual equations are obtained:

$$R_1(t) = \frac{1}{2}B^2\omega_1^2 \sin^2 \omega_1 t + P(B \cos \omega_1 t) - P(B) \quad (7)$$

$$R_2(t) = \frac{1}{2}B^2\omega_2^2 \sin^2 \omega_2 t + P(B \cos \omega_2 t) - P(B) \quad (8)$$

The residual integrals are defined as

$$\tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos \omega_1 t dt \quad (9)$$

$$\tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos \omega_2 t dt \quad (10)$$

where $T_1=2\pi/\omega_1$, $T_2=2\pi/\omega_2$.

The frequency formulation is [25]

$$\omega^2 = \frac{\tilde{R}_1 \omega_2^2 - \tilde{R}_2 \omega_1^2}{\tilde{R}_1 - \tilde{R}_2} \quad (11)$$

We can see that the frequency formulation requires only a simple calculation of two integrals, making it much attractive and promising in the nonlinear vibration theory.

3. MODIFIED HE'S FREQUENCY FORMULATIONS

The present author shows an interest in two arbitrary choices of ω_1 and ω_2 , e.g., $\omega_1=1$ and $\omega_2=2$, or $\omega_1=100$ and $\omega_2=0.1$, all cases leading to the same result. This two-trial assumption goes back to an ancient Chinese algorithm [26], which was developed to He's frequency formulation to solve nonlinear oscillators by Ji-Huan He [26], and Chun-Hui He's iteration algorithm for numerical simulation [34,35].

It is amazing that the Hamiltonian-based frequency formulation does not depend upon the choice of ω_1 and ω_2 . Here a strict proof is given.

From Eqs. (7) and (8), we have

$$\begin{aligned} R_1(t) &= \frac{B^2 \omega_1^2}{2} \sin^2 \omega_1 t + P(B \cos \omega_1 t) - P(B) \\ &= \frac{B^2 \omega_1^2 - 2P(B)}{2} - \frac{B^2 \omega_1^2}{2} \cos^2 \omega_1 t + P(B \cos \omega_1 t) \end{aligned} \quad (12)$$

$$\begin{aligned} R_2(t) &= \frac{B^2 \omega_2^2}{2} \sin^2 \omega_2 t + P(B \cos \omega_2 t) - P(B) \\ &= \frac{B^2 \omega_2^2 - 2P(B)}{2} - \frac{B^2 \omega_2^2}{2} \cos^2 \omega_2 t + P(B \cos \omega_2 t) \end{aligned} \quad (13)$$

The residual integrals become

$$\tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos \omega_1 t dt = \frac{B^2 \omega_1^2 - 6P(B)}{3\pi} + \frac{2}{\pi} \int_0^1 P(B\sqrt{1-x^2}) dx \quad (14)$$

$$\tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos \omega_2 t dt = \frac{B^2 \omega_2^2 - 6P(B)}{3\pi} + \frac{2}{\pi} \int_0^1 P(B\sqrt{1-x^2}) dx \quad (15)$$

So a modified frequency formulation is obtained, which is

$$\begin{aligned} \omega^2 &= \frac{\tilde{R}_1 \omega_2^2 - \tilde{R}_2 \omega_1^2}{\tilde{R}_1 - \tilde{R}_2} \\ &= \frac{\left(\frac{B^2 \omega_1^2 - 6P(B)}{3\pi} + \frac{2}{\pi} \int_0^1 P(B\sqrt{1-x^2}) dx \right) \omega_2^2 - \left(\frac{B^2 \omega_2^2 - 6P(B)}{3\pi} + \frac{2}{\pi} \int_0^1 P(B\sqrt{1-x^2}) dx \right) \omega_1^2}{\frac{B^2 \omega_1^2 - 6P(B)}{3\pi} - \frac{B^2 \omega_2^2 - 6P(B)}{3\pi}} \\ &= \frac{\frac{6P(B)}{3\pi} (\omega_1^2 - \omega_2^2) + \frac{2}{\pi} \int_0^1 P(B\sqrt{1-x^2}) dx (\omega_1^2 - \omega_2^2)}{\frac{B^2}{3\pi} (\omega_1^2 - \omega_2^2)} \\ &= \frac{6P(B) - 6 \int_0^1 P(B\sqrt{1-x^2}) dx}{B^2} \end{aligned} \quad (16)$$

This modification seems much simpler than the original one because no calculation of integrals is needed, so it is easily applicable.

Alternatively, we locate at [26]

$$\cos \omega_1 t = \cos \omega_2 t = k \quad (17)$$

Eqs. (7) and (8) become

$$\tilde{R}_1 = \frac{1}{2} B^2 \omega_1^2 (1 - k^2) + P(kB) - P(B) \quad (18)$$

$$\tilde{R}_2 = \frac{1}{2} B^2 \omega_2^2 (1 - k^2) + P(kB) - P(B) \quad (19)$$

Another modification of the Hamiltonian-based frequency formulation is

$$\begin{aligned} \omega^2 &= \frac{\omega_2^2 \tilde{R}_1 - \omega_1^2 \tilde{R}_2}{\tilde{R}_1 - \tilde{R}_2} \\ &= \frac{\omega_2^2 \left(\frac{1}{2} B^2 \omega_1^2 (1 - k^2) + P(kB) - P(B) \right) - \omega_1^2 \left(\frac{1}{2} B^2 \omega_2^2 (1 - k^2) + P(kB) - P(B) \right)}{\left(\frac{1}{2} B^2 \omega_1^2 (1 - k^2) + P(kB) - P(B) \right) - \left(\frac{1}{2} B^2 \omega_2^2 (1 - k^2) + P(kB) - P(B) \right)} \\ &= \frac{(\omega_2^2 - \omega_1^2) P(kB) - (\omega_2^2 - \omega_1^2) P(B)}{\frac{1}{2} (1 - k^2) B^2 (\omega_1^2 - \omega_2^2)} \\ &= \frac{P(B) - P(kB)}{\frac{1}{2} (1 - k^2) B^2} \end{aligned} \quad (20)$$

We recommend $k=1/\sqrt{2}$, the simplified Hamiltonian-based frequency formulation is

$$\omega^2 = \frac{4(P(B) - P(B/\sqrt{2}))}{B^2} \quad (21)$$

It is astonishing that all intermediate steps involved in the original Hamiltonian-based frequency formulation have disappeared, making the two modified formulations unprecedentedly simple.

4. EXAMPLES

Consider the following well-known Duffing equation [25]

$$w'' + w + \epsilon w^3 = 0, w'(0) = 0 \quad w(0) = B \quad (22)$$

For this example, we have

$$P(w) = \frac{1}{2} w^2 + \frac{1}{4} \epsilon w^4 \quad (23)$$

In order to compare the two modifications with the original one, we illustrate first the solving process of the original Hamiltonian-based frequency formulation, which begins with two arbitrary choices ω_1 and ω_2 , and the following residual equations are obtained:

$$\begin{aligned}
R_1(t) &= \frac{1}{2}B^2\omega_1^2 \sin^2 \omega_1 t + P(B \cos \omega_1 t) - P(B) \\
&= \frac{B^2\omega_1^2 - 2P(B)}{2} + \frac{B^2 - B^2\omega_1^2}{2} \cos^2 \omega_1 t + \frac{\epsilon B^4}{4} \cos^4 \omega_1 t
\end{aligned} \tag{24}$$

$$\begin{aligned}
R_2(t) &= \frac{1}{2}B^2\omega_2^2 \sin^2 \omega_2 t + P(B \cos \omega_2 t) - P(B) \\
&= \frac{B^2\omega_2^2 - 2P(B)}{2} + \frac{B^2 - B^2\omega_2^2}{2} \cos^2 \omega_2 t + \frac{\epsilon B^4}{4} \cos^4 \omega_2 t
\end{aligned} \tag{25}$$

The residual integrals are:

$$\tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos \omega_1 t dt = \frac{B^2}{3\pi} \omega_1^2 + \frac{4\epsilon B^4 + 10B^2 - 30P(B)}{15\pi} \tag{26}$$

$$\tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos \omega_2 t dt = \frac{B^2}{3\pi} \omega_2^2 + \frac{4\epsilon B^4 + 10B^2 - 30P(B)}{15\pi} \tag{27}$$

The Hamiltonian-based frequency formulation leads to the result

$$\omega^2 = \frac{\tilde{R}_1 \omega_2^2 - \tilde{R}_2 \omega_1^2}{\tilde{R}_1 - \tilde{R}_2} = 1 + \frac{7}{10} \epsilon B^2 \tag{28}$$

Now the modified formulation of Eq. (16) gives a one-step solving process:

$$\begin{aligned}
\omega^2 &= \frac{6P(B) - 6 \int_0^1 P(B\sqrt{1-x^2}) dx}{B^2} \\
&= \frac{6\left(\frac{1}{2}B^2 + \frac{1}{4}\epsilon B^4\right) - 6 \int_0^1 \left(\frac{1}{2}B^2(1-x^2) + \frac{1}{4}\epsilon B^4(x^4 - 2x^2 + 1)\right) dx}{B^2} \\
&= \frac{\left(3B^2 + \frac{3}{2}\epsilon B^4\right) - \left(2B^2 + \frac{4}{5}\epsilon B^4\right)}{B^2} \\
&= \frac{B^2 + \frac{7}{10}\epsilon B^4}{B^2} \\
&= 1 + \frac{7}{10}\epsilon B^2
\end{aligned} \tag{29}$$

which is the same as Eq. (28).

The modified formulation of Eq. (21) is also simple, which leads to the following result

$$\begin{aligned}\omega^2 &= \frac{4(P(B) - P(B/\sqrt{2}))}{B^2} \\ &= \frac{4\left(\frac{1}{2}B^2 + \frac{1}{4}\epsilon B^4 - \frac{1}{4}B^2 - \frac{1}{16}\epsilon B^4\right)}{B^2} \\ &= \frac{B^2 + \frac{3}{4}\epsilon B^4}{B^2} \\ &= 1 + \frac{3}{4}\epsilon B^2\end{aligned}\quad (30)$$

This result is the same as those obtained by the variational iteration method or the homotopy perturbation method [26].

Consider another example of the cubic-quintic Duffing oscillator:

$$w'' + w + aw^3 + bw^5 = 0, w'(0) = 0 \quad w(0) = B \quad (31)$$

The original Hamiltonian-based frequency formulation is to calculate the residuals:

$$\begin{aligned}R_1(t) &= \frac{1}{2}B^2\omega_1^2 \sin^2 \omega_1 t + P(B \cos \omega_1 t) - P(B) \\ &= \frac{B^2\omega_1^2 - 2P(B)}{2} + \frac{B^2 - B^2\omega_1^2}{2} \cos^2 \omega_1 t + \frac{aB^4}{4} \cos^4 \omega_1 t + \frac{bB^6}{6} \cos^6 \omega_1 t\end{aligned}\quad (32)$$

$$\begin{aligned}R_2(t) &= \frac{1}{2}B^2\omega_2^2 \sin^2 \omega_2 t + P(B \cos \omega_2 t) - P(B) \\ &= \frac{B^2\omega_2^2 - 2P(B)}{2} + \frac{B^2 - B^2\omega_2^2}{2} \cos^2 \omega_2 t + \frac{aB^4}{4} \cos^4 \omega_2 t + \frac{bB^6}{6} \cos^6 \omega_2 t\end{aligned}\quad (33)$$

and the residual integrals:

$$\tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos \omega_1 t dt = \frac{B^2}{3\pi} \omega_1^2 + \frac{16bB^6 + 28aB^4 + 70B^2 - 210P(B)}{105\pi} \quad (34)$$

$$\tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos \omega_2 t dt = \frac{B^2}{3\pi} \omega_2^2 + \frac{16bB^6 + 28aB^4 + 70B^2 - 210P(B)}{105\pi} \quad (35)$$

The original Hamiltonian-based frequency formulation gives the following result

$$\omega^2 = \frac{\tilde{R}_1\omega_2^2 - \tilde{R}_2\omega_1^2}{\tilde{R}_1 - \tilde{R}_2} = 1 + \frac{7}{10}aB^2 + \frac{19}{35}bB^4 \quad (36)$$

Eq. (16) gives immediately the following results

$$P(w) = \frac{1}{2} w^2 + \frac{1}{4} a w^4 + \frac{1}{6} b w^6 \quad (37)$$

and

$$\begin{aligned} \omega^2 &= \frac{6P(B) - 6 \int_0^1 P(B\sqrt{1-x^2}) dx}{B^2} \\ &= \frac{6 \left(\frac{1}{2} B^2 + \frac{1}{4} a B^4 + \frac{1}{6} b B^6 \right) - 6 \int_0^1 \left(\frac{1}{2} B^2 (1-x^2) + \frac{1}{4} B^4 (1-x^2)^2 + \frac{1}{6} b B^6 (1-x^2)^3 \right) dx}{B^2} \\ &= \frac{\left(3B^2 + \frac{3}{2} a B^4 + b B^6 \right) - \left(2B^2 + \frac{4}{5} a B^4 + \frac{16}{35} b B^6 \right)}{B^2} \\ &= \frac{B^2 + \frac{7}{10} a B^4 + \frac{19}{35} b B^6}{B^2} \\ &= 1 + \frac{7}{10} a B^2 + \frac{19}{35} b B^4 \end{aligned} \quad (38)$$

This is the same with Eq. (36). The one-step solving process by Eq. (21) leads to the following result

$$\begin{aligned} \omega^2 &= \frac{4(P(B) - P(B/\sqrt{2}))}{B^2} \\ &= \frac{4 \left(\frac{1}{2} B^2 + \frac{1}{4} a B^4 + \frac{1}{6} b B^6 - \frac{1}{4} B^2 - \frac{1}{16} a B^4 - \frac{1}{48} b B^6 \right)}{B^2} \\ &= \frac{B^2 + \frac{3}{4} a B^4 + \frac{7}{12} b B^6}{B^2} \\ &= 1 + \frac{3}{4} a B^2 + \frac{7}{12} b B^4 \end{aligned} \quad (39)$$

Fig. 1 shows the comparison of approximate solution $w(t) = B \cos \omega t$, where the frequency is determined by Eq. (38) or Eq. (39), with the exact one when $a=b=1$ for various values of amplitudes.

4. CONCLUSIONS

This paper simplifies Hamiltonian-based frequency formulation for nonlinear oscillators. All the intermediate steps involved in the original formulation become totally unnecessary, making the formulation unprecedentedly simple. Both the simplified frequency formulations will play an important role in the nonlinear vibration theory for a fast and reliable insight into the periodic properties of a complex nonlinear oscillator, and it will open up a flood of opportunities for various applications. The one-step solving process given in this paper confirms again Ji-Huan He's famous saying: "The simpler is the better for the nonlinear vibration theory".

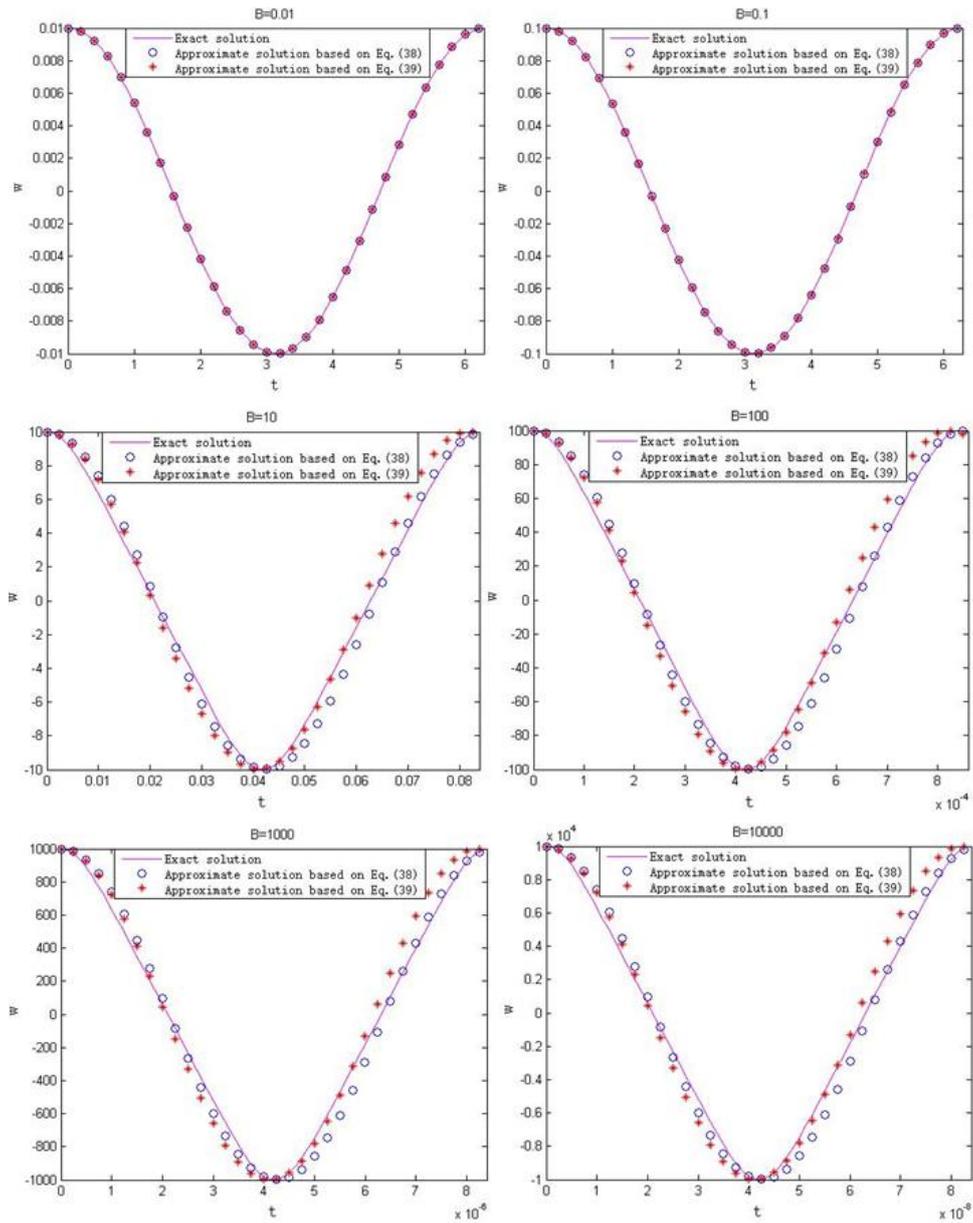


Fig. 1 Comparison of the exact solutions with the approximate solutions based on Eqs. (38) and (39) for amplitude B ranging from 0.01 to 10000

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