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Original scientific paper

A CIRCULAR SECTOR VIBRATION SYSTEM IN A POROUS MEDIUM: A FRACTAL-FRACTIONAL MODEL AND HE'S FREQUENCY FORMULATION

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Abstract. *A circular sector is commonly used in a linkage mechanism, and its frequency property plays an important role in optimization of the linkage mechanism. Fast insight into its vibration property with simple calculation is very meaningful in scientific research. This paper studies the vibration of the circular sector in a porous medium (e.g. water), and a fractal-fractional oscillator is established using the two-scale fractal derivative. He's frequency formula and Ma's modification are used to elucidate the circular sector's periodic property in a porous medium, the results show that the fractal dimension of the porous medium plays an important role in vibration attenuation.*

Key words: *He's frequency formula, Ma's modification, Circular sector oscillator, Fractal variational theory, Two-scale transform*

1. INTRODUCTION

A circular sector is commonly used in a linkage mechanism, it might vibrate in water or oil, and its vibration property has caught much attention in mechanical engineering [1]. There are many engineering applications in literature, such as the base of structures, the wipers of cars, smoothing filters and many other vibration systems [2-4].

The traditional vibration theory cannot effectively elucidate the effect of the porous medium's geometry on the vibration property, now the condition is changed, because of the new born fractal vibration theory [5]. The fractal vibration theory can make up for the deficiency of the traditional vibration theory. Let's take the vibration of a sector in the air as an example. In all previous vibration theories, the influence of air on vibration characteristics is considered as an air drag; however, the performance of thin air in a microgravity environment is different from that of near surface atmosphere. The vibration

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model constructed by the fractal vibration theory can consider the influence of molecules' size and distribution on vibration motion, the acceleration in the fractal space is a combination of the damping force and the inertia force in the traditional vibration system [6-10], so the vibration system in a porous medium(e.g. air, water) can be modelled by a fractal-fractional model without considering the damping effect.

In general, the fractal vibration follows a fractal variational theory, that is, the vibration system in a porous medium obeys the fractal energy conservation law, the fractal kinetic energy and the potential energy are changed during the vibrating process, but their total energy keeps unchanged. Fractal nonlinear systems truly describe the dynamic problems of engineering science, and its academic research has greatly expanded the field of human cognition. Many phenomena can be explained by the fractal theory, which makes us realize that the world is totally discontinuous and nonlinear. The fractal nonlinear vibration can be closer to practical problems in both depth and breadth. It is very important to study the analytical or approximate solutions of nonlinear vibration equations that can provide deep insight into the essence of general properties of a fractal vibrating system. So, there are many analytical and numerical methods to find an approximate solution of a differential equation involving fractal-fractional derivatives, such as the variational iteration method [11-13], the homotopy perturbation [14,15], the Hamiltonian approach [16], and the Taylor series method [17].

The most important property of a nonlinear system is the frequency-amplitude relationship, so how to estimate quickly its relationship is an urgent problem in practical applications. Many researchers devoted their efforts to studying fractal calculus that provides a powerful tool to characterizing the periodic behavior of a nonlinear oscillator [18-20]. Chun-Hui He suggested a fractal nano/microelectromechanical (N/MEMS) system [21], Ji-Huan He gave a tutorial review on fractal space and fractional calculus [22,23]. He, et al. studied the fractal Duffing oscillator with arbitrary conditions [24].

Ji-Huan He suggested a frequency formula for a conservation nonlinear oscillator [25] and it was further improved to an extremely simple formula in Refs. [26,27], and it has been widely used to solve nonlinear oscillator problems, for examples, the N/MEMS oscillator [28], the attachment oscillator [29], the Tangent oscillator [30], the fractal undamped Duffing equation [31], the large amplitude vibration system [32] and the vibration system on a porous foundation [33]. Many modifications were appeared in literature to deal with more complex vibration systems [34-37].

The Hamiltonian-based frequency formula is a modification of He's frequency formula [38]. The frequency formula starts with two arbitrary guesses for the frequency, and two residual integrals must be computed to estimate a more accurate frequency. This paper applies He's frequency formula and Ma's modification to study the fractal vibration systems and uses the fractal circular sector oscillator as an example. The results show that the two methods are very effective for fractal nonlinear oscillators.

2. Fractal Circular Sector Oscillator

As the fractal variational theory is helpful in establishing a governing equation in a fractal space, it has become a significantly hot topic in both mathematics and mechanical engineering. Many fractal variational principles were appeared for the internal temperature response of a porous concrete [39], the fractal Benney-Lin equation [40], the fractal shallow water wave [41], the fractal Bogoyavlenskii system [42], the fractal Schrodinger system [43], the fractal solitary wave [44] and the fractal economics [45].

The variational formula of an oscillator in a fractal space is in the form

$$J(u)=\int\{\frac{1}{2}(\frac{du}{d\tau^{\theta}})^2-f(u)\}d\tau^{\theta} \tag{1}$$

where $(du/d\tau^{\theta})^2/2$ is the fractal kinetic energy and $f(u)$ is the potential energy.

When f=-cos$u$, it is the well-known pendulum oscillator [46,47]; when $f=u^2/2+au^4/4$, it is the well-known Duffing oscillator [48]. When $f=g\ln(3R^2/2-2Rk\cos u)/(2R)$, its variational formula is:

$$J(u)=\int\{\frac{1}{2}(\frac{du}{d\tau^{\theta}})^2-\frac{g}{2R}\ln(\frac{3}{2}R^2-2Rk\cos u)\}d\tau^{\theta} \tag{2}$$

It meets the following energy balance equation

$$\frac{1}{2}(\frac{du}{d\tau^{\theta}})^2+\frac{g}{2R}\ln(\frac{3}{2}R^2-2Rk\cos u)=H \tag{3}$$

where H is the Hamiltonian constant.

The fractal nonlinear oscillator can be obtained as follows

$$\frac{d}{d\tau^{\theta}}(\frac{du}{d\tau^{\theta}})+\frac{gk\sin u}{\frac{3}{2}R^2-2Rk\cos u}=0 \tag{4}$$

It is the fractal circular sector oscillator [49], where u is the angular displacement, R is the semicircular radius, g is the gravity acceleration, $\bar{R}$ is the height of mass center, $\bar{R}=2R\sin\alpha/(3\alpha)$, α indicates the semicircular angle, k is the dimensionless geometrical parameter, $k=\bar{R}/R=2\sin\alpha/(3\alpha)$, as illustrated in Fig. 1.

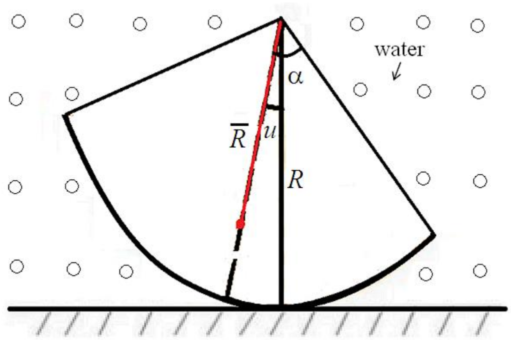

Fig. 1 Circular sector oscillator in water

By the two-scale transform [50], $t=\tau^{\theta}$, the fractal circular sector oscillator of Eq. (4) can be converted to an ordinary form as follows

$$u''+\frac{gk\sin u}{\frac{3}{2}R^2-2Rk\cos u}=0, u(0)=A, u'(0)=0 \tag{5}$$

where the derivative of the function u with respect to t.

3. HE'S FREQUENCY FORMULA

He's frequency formula [26,27] is an effective tool to learn the frequency-amplitude relationship of fractal oscillators in a timely and efficient way.

Consider the following nonlinear oscillator

$$u'' + h(u) = 0, u(0) = A, u'(0) = 0 \tag{6}$$

where $h(u)$ is a nonlinear function, and it satisfies $h(u)/u>0$.

He's frequency formula is：

$$\omega^2 = \left.\frac{dh(u)}{du}\right|_{u=0.5A} \tag{7}$$

or

$$\omega^2 = \frac{h(NA)}{NA} \tag{8}$$

where N is a constant, $0<N\leq 1$. N is recommended as √ 3/2 for nonlinear oscillators in [26, 27].

Convert Eq. (5) to the following approximate form

$$u'' + \omega^2 u = 0 \tag{9}$$

where the square of the frequency can be calculated by Eq.(8):

$$\omega^2 = \frac{gk\sin(\frac{\sqrt{3}}{2}A)}{\frac{\sqrt{3}}{2}A[\frac{3}{2}R^2 - 2Rk\cos(\frac{\sqrt{3}}{2}A)]} \tag{10}$$

The approximate solution of Eq. (4) can be formed as follows

$$u(\tau^\theta) = A\cos(\omega\tau^\theta) \tag{11}$$

4. MA'S MODIFICATION

Ma obtained a simplified form of Hamiltonian-based frequency formula as follows [38]:

$$\Omega^2 = \frac{H(A) - H(LA)}{0.5(1-L^2)A^2} \tag{12}$$

where $dH(u)/du=h(u)$, and L is recommend as $2^{-0.5}$ [38], the simplified Hamiltonian-based frequency formula converts to

$$\Omega^2 = \frac{4}{A^2}[H(A) - H(\frac{A}{\sqrt{2}})] \tag{13}$$

According to Eq.(5), we have

$$H(u) = \frac{g}{2R}\ln(\frac{3}{2}R^2 - 2Rk\cos u) \tag{14}$$

Ma's modification leads to the following result

$$\Omega^2 = \frac{2g}{RA^2}\ln\frac{\frac{3}{2}R - 2k\cos(A)}{\frac{3}{2}R - 2k\cos(A/\sqrt{2})} \tag{15}$$

The approximate solution of Eq. (4) can be formed as following

$$u(\tau^\theta) = A\cos(\Omega\tau^\theta) \tag{16}$$

The obtained approximate analytical solution must be compared with the numerical ones to demonstrate the validity of the above approaches. Therefore, Figs. 2-4 demonstrate the analytical solution from the simplified Hamiltonian-based frequency method (SHF), together with the numerical solution (NS) and the solution from He's frequency method (HF) for different values of A, R and α. The approximate periodic solutions agree well with exact numerical results.

The value of α changes from π/6 to π/3, the change amount is Δα=π/6, and the period increment is ΔT=4.5 in Fig. 2.

The value of α changes from π/2 to 2π/3, the change amount is Δα=π/6, and the period increment is ΔT=14 in Fig. 3. The larger the value of *α*, the more obvious the period change. It concludes that by increasing the semicircular angle *α* with constant semicircular radius *R*, the period of the oscillation will increase considerably.

In Fig. 4, the value of the amplitude A determines the maximum value of *u*. By comparing Figs. 3 and 4, it is found that by reducing the semicircular radius *R*, the period of the oscillation will decrease.

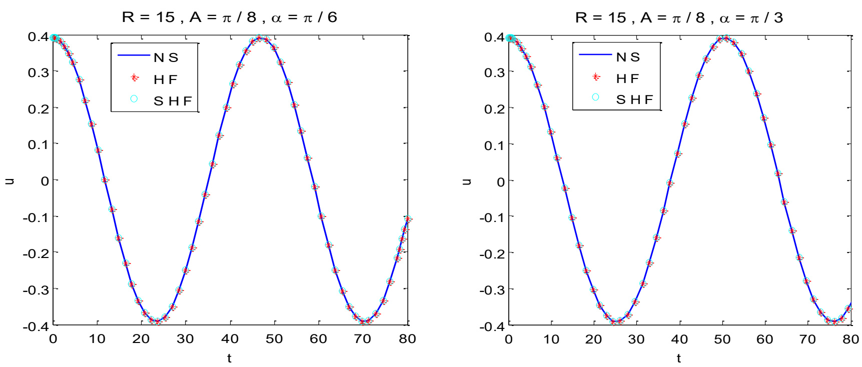

Fig. 2 Comparisons of the exact solutions (NS) with the approximate solutions based on Eqs. (11) (HF) and (16) (SHF) for α ranging from π/6 to π/3

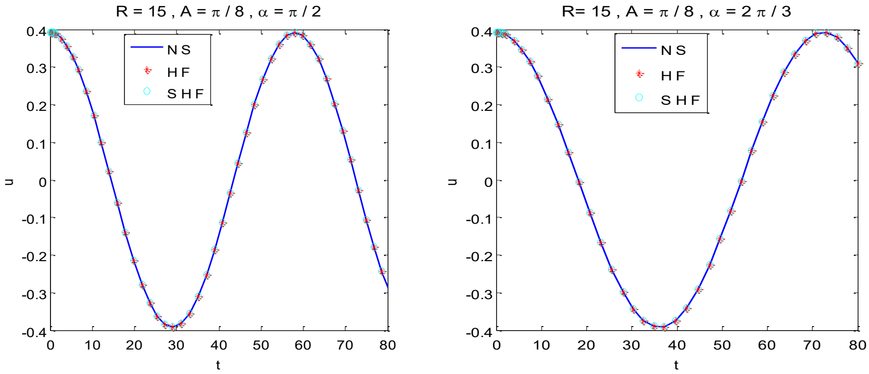

Fig. 3 Comparisons of the exact solutions (NS) with the approximate solutions based on Eqs. (11) (HF) and (16) (SHF) for α ranging from π/2 to 2π/3

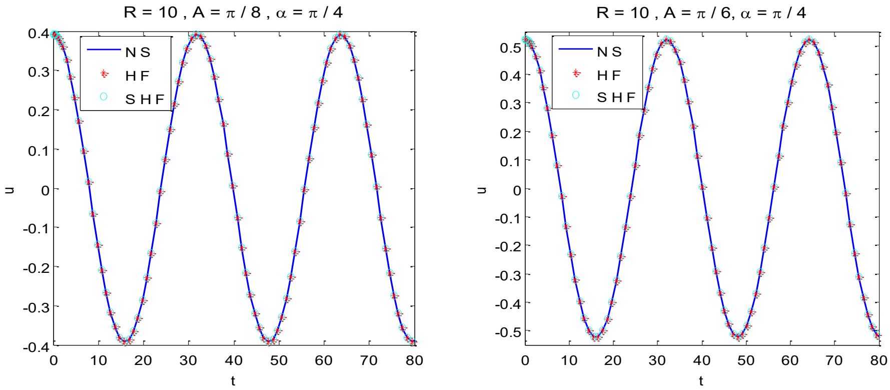

Fig. 4 Comparisons of the exact solutions (NS) with the approximate solutions based on Eqs. (11) (HF) and (16) (SHF) for amplitude A ranging from π/8 to π/6

Based on Eqs. (10) and (15), results are extracted from the analytical solution and are shown in Fig. 5. It visually shows the relationship between A, α and the frequency when the value of R is fixed. An increase in the semicircle radius R or semicircle α results in a decrease in frequency, which implies an increase in the oscillation period.

Five sequences of the fractal parameter θ are considered for Eq. (4) with the fixed parameters R=10, A=π/6, α=π/4 and shown together in Fig. 6. Numerical simulations indicate that the oscillation frequency becomes faster and the vibration attenuation occurs greater for increasing the values of the fractal exponent θ.

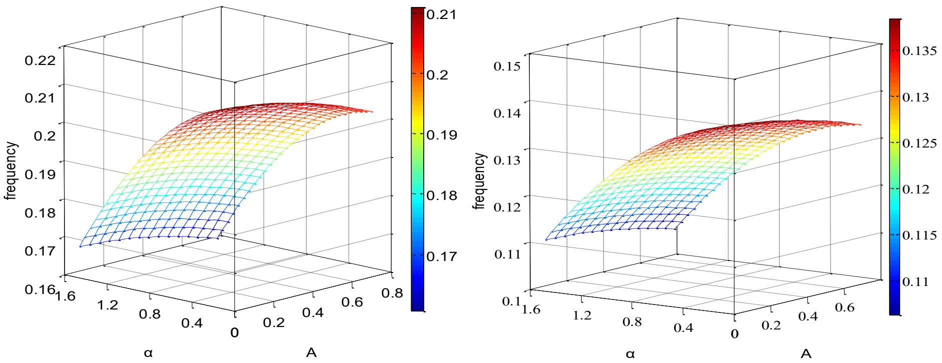

Fig. 5 Sensitivity analysis of frequency (0<A<π/4, 0<α<π/2) with R = 10 (left) and R = 15 (right)

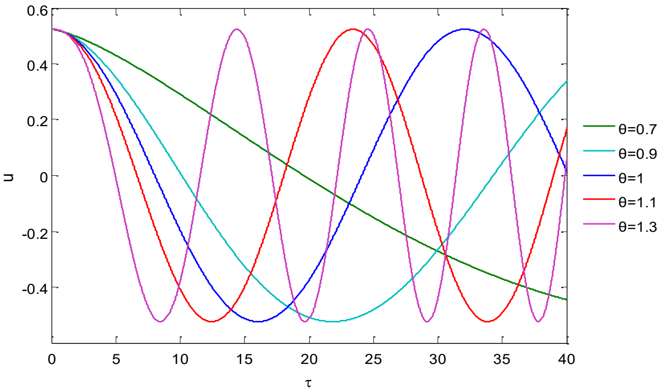

Fig. 6 Eq. (4) at different θ values for R=10, A=π/6, α=π/4

5. Conclusion

The most important property of a fractal nonlinear system is the relationship between frequency and amplitude, which plays a crucial role in construction engineering. This paper presents He's frequency formula and Ma's modification of Hamiltonian-based frequency formula and discusses how to quickly calculate the approximate frequency of the fractal circular sector in a porous medium. The result shows that two methods are accurate tools to quickly calculating the periodic properties of fractal nonlinear oscillators, and that the fractal dimension of a porous medium plays an important role in vibration attenuation. Both the frequency formulations will play an important role in the fractal nonlinear vibration theory and open up numerous opportunities to elucidate the effect of the porous medium's geometry on the vibration property, and also probably pave a new avenue for optimizing the circular sector in a nanofluid [51] where the nanoparticles' size and distribution can be effectively modelled by the fractal-fractional model.

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