**FACTA UNIVERSITATIS**
Series: **Mechanical Engineering** Vol. 22, Nº 2, 2024, pp. 199 - 216
https://doi.org/10.22190/FUME230605035E

**Original scientific paper**

# THE UP-GRATING RANK APPROACH TO SOLVE THE FORCED FRACTAL DUFFING OSCILLATOR BY NON- PERTURBATIVE TECHNIQUE

**Yusry O. El-Dib[1], Nasser S. Elgazery[1], Haifa A. Alyousef[2]**

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt
[2]Department of Physics, College of Science, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia

ORCID iDs: Yusry O. El-Dib https://orcid.org/0000-0001-6381-5918
Nasser S. Elgazery https://orcid.org/0000-0003-4691-2526
Haifa A. Alyousef https://orcid.org/0000-0003-1727-0958

**Abstract**. *The current research studies a fractal Duffing oscillator in the presence of periodic force. To find an analytic solution for this oscillator, the aspects explained in the following are considered. First, we obtain an alternative unforced fractal fourth-order equation and then convert it into a continuous space. Therefore, the non-perturbative (NP) approach is used to calculate the analytic solution for the alternate equation in the second-order form after reducing its rank. It is seen that the analytical and numerical solutions agree very well. The computations reveal that for every value of the fraction parameter, the approximation and numerical solutions are identical. The present study gives reliability in the technique of reducing the order of differential equations. Furthermore, the required periodic solution is also obtained by Galerkin's technique. In contrast to the traditional technique, which works to transform the variable and is valid only in the absence of external forces, if there is an external force, it leads to significant mathematical difficulties. The current technique works on the operator, which is simple and effective when investigating fractal oscillators with external forces, easy to obtain analytic solutions, and doesn't lead to any mathematical difficulties*.

**Key words**: *Forced fractal Duffing oscillator, Non-perturbative (NP) technique, Fractal space, Galerkin's approach*

Received: June 05, 2023 / Accepted October 07, 2023
**Corresponding author**: Nasser S. Elgazery
Department of Mathematics, Faculty of Education, Ain Shams University, El Makrizy Street, Roxy, Heliopolis: 11566, Cairo, Egypt
E-mail: nasersaleh@edu.asu.edu.eg

© 2024 by University of Niš, Serbia | Creative Commons License: CC BY-NC-ND

## 1. INTRODUCTION

Fractals have been a fascinating characteristic of many natural occurrences since the seventeenth century, and they have a significant impact on a wide range of applications [1,2]. Fractals have been widely applied in many fields, such as fluid mechanics, materials design, and petroleum engineering, as well as in other engineering problems to describe and express them clearly and effectively [3,4]. This also appears clearly in the application of the theory of fractals in the investigation within the branched networks to study thermal conductivity [5,6]. For the electrical and mechanical properties of composite materials, the fractal approach offers straightforward and effective tools [7]. Moreover, the fractal rheological models provide a fast physical understanding of the rheological characteristics of the fluids [8].

Due to its numerous large-scale applications for modeling a variety of phenomena that emerge in most domains, the idea of two-scale fractals theory has recently drawn significant attention from scientists [9]. Where this theory enables each problem to be observed on two different scales, one of which is related to a continuous problem and the other to a discrete problem. The hidden qualities of a polymer procedure that cannot be exposed by other models can be determined using the fractal two-phase flow model [10-12]. He et al. [13] used a novel fractal vibration architecture model using the two-scale fractals theory. Anjum et al. [14,15] also investigate two different fractional models with two-scale for population expansion and tsunami waves, respectively.

An interesting topic in mechanical engineering sciences and physics is nonlinear oscillations. One of the most well-liked nonlinear oscillators in engineering, physics, and biology was first introduced in 1918 by the German electrical engineer "Georg Duffing". Since then, Duffing oscillators have been widely applied in nature. Using a traditional perturbation method, Nayfeh and Mook derived a periodic solution for the Duffing oscillator, which simulates a nonlinear structural vibration. Moreover, Nayfeh used the multiple-scale methodology to produce a periodic analytical solution for a Duffing oscillator in the presence of a damping coefficient [16]. The homotopy perturbation technique was later developed by renowned Chinese scientist Ji-Huan He [17] and utilized in several engineering and physics investigations to determine analytical solutions for the Duffing oscillator and other mechanical vibrations. On the other hand, recent years have seen an increase in interest in mechanical engineering applications for fractal oscillators, as a subfield of vibration theory. Wang [18] used a novel frequency formulation to study a fractal nonlinear oscillator. The same author also found an analytical solution for a fractal nonlinear oscillator in [19] by employing the effective He's frequency formulation. Further, Wang and Wei [20] have presented a newfangled, straightforward, and effective frequency formula to deal with fractal nonlinear oscillators. Also, two approximate solutions for the fractal Yao-Cheng and Duffing oscillators have been introduced in [21-23], built on the harmonic balance and antiquated Chinese approaches.

Since the solution of systems related to the study of fractal oscillators plays an important and useful role in describing and explaining many phenomena that occur on irregular boundaries or in porous media. Essentially, it is usually impossible to find exact solutions for these nonlinear oscillators. As a result, mathematicians have been developing different analytical methods to solve this type of nonlinear fractal problem, such as using extended approximation approaches. The most prominent of these methods are perturbation and multiple-scale techniques [16]. Sheng et al. [24] have presented a fractal model for a shale matrix using a multi-scale porous structure. Miao et al. [25] investigated another fractal model for a fractured porous medium. Furthermore, other fractal models for describing the gas flow inside microfractures in layers of rocks in the depths of the oceans and underground have been mentioned in the literature [26-28]. On the other hand, Yang et al. [29] used the local fractional differential transform approach to get the precise solutions in a fractal heat transfer. Additionally, several efforts have been established to analyze these phenomena using the non-perturbative method. By using the non-perturbative method, El-Dib and Elgazery [30,31] established a practical method for solving fractal models with damping forces. On the other hand for more relevant works focusing on different methods and models in the literature, the semi-analytical technique is one of the methods that play an important role in finding solutions of physical problems [32,33]. Furthermore, for some numerical solutions see sinc-Bernoulli collocation procedure [34].

Fractal oscillators have recently attracted more attention in mechanical engineering and physics. Despite this, all previous studies depended on a classical technique that works to transform the variable. This traditional technique is limited only by the absence of external forces. Here it should be noted that there were some timid attempts to deal with fractal oscillators under external forces, such as the attempt of Elías-Zúñiga et al. [35], where they faced many mathematical difficulties. Regarding our new technique, it is a different and distinct technique from its predecessor, as it works on the operator. It is also simple and effective when investigating fractal oscillators with external forces, easy to obtain analytic solutions, and doesn't lead to any mathematical difficulties.

The aforementioned factors serve as an inspiration for this investigation goal, which is to look into the fractal Duffing oscillator in the presence of a periodic force via the non-perturbative technique. As opposed to earlier works, with direct handling of the operator, the present technique is simple to deal with forced fractal problems and easy to obtain analytic solutions.

## 2. PROBLEM STATEMENT

Indeed, Ji-Huan He's publications [36,37] aiming to formulate the oscillation frequency have considered the conservative nonlinear oscillator:

$$\ddot{u} + f(u) = 0;\; u(0) = A \text{ and } \dot{u}(0) = 0 \tag{1}$$

Here, $f(u)$ represents the nonlinear restoring force, $u$ is the displacement, and $A$ is a constant.

In the mentioned publications, Ji-Huan He has used the function $f(u)$ to established the oscillation frequency $\boldsymbol{\omega}^2(\Omega)$, where $\Omega$ is the total frequency, in the form

$$\boldsymbol{\omega}^2(\Omega) = \lim_{u \to \frac{1}{2}A} \frac{df(u)}{du} \tag{2}$$

This formula has been well worked for the cubic polynomial of the function $f(u)$ provided that this function does not contain the velocity $\dot{u}$ or $\ddot{u}$ and the combination of them. For these shortcomings, the superior and most effective formula was developed by El-Dib [38], and it can be used to produce successive approximations to the solutions for non-linear oscillations [39]. It is better to elongate this method to equations having damping forces such that

$$\ddot{u} + g(u)\dot{u} + f(u,\dot{u},\ddot{u})u = 0 \tag{3}$$

If $\Omega$ is the total frequency to the above oscillator then it useful to write

$$\ddot{u} + \mu(\Omega)\dot{u} + \omega^2(\Omega)u = 0 \tag{4}$$

The following procedure is used to estimate $\mu(\Omega)$ and $\omega^2(\Omega)$:
The comparison of Eqs. (3) and (4) leads to the following error:

$$E(\mu,\omega^2) = \left|(f(u,\dot{u},\ddot{u}) - \omega^2)u(t) + (g(u) - \mu)\dot{u}(t)\right| \tag{5}$$

The mean square error is defined as

$$\bar{E}^2(\mu,\omega^2) = \int_0^T \left((f(u,\dot{u},\ddot{u}) - \omega^2)u(t) + (g(u) - \mu)\dot{u}(t)\right)^2 dt;\ T = \frac{\pi}{2\omega} \tag{6}$$

The minimum value requires that

$$\frac{d\bar{E}^2}{d(\omega^2)} = \frac{d}{d(\omega^2)}\int_0^T ((f(u,\dot{u},\ddot{u}) - \omega^2)^2 u^2(t) + 2(f(u,\dot{u},\ddot{u}) - \omega^2)(g(u) - \mu)u\dot{u})dt = 0 \tag{7}$$

$$\frac{d\bar{E}^2}{d\mu} = \frac{d}{d\mu}\int_0^T ((g(u) - \mu)^2 \dot{u}^2(t) + 2(f(u,\dot{u},\ddot{u}) - \omega^2)(g(u) - \mu)u\dot{u})dt = 0 \tag{8}$$

By suitable introducing trial solution in terms of the total frequency $\Omega$, the simplification leads to

$$\omega^2 \int_0^T u^2(t)dt - \int_0^T u^2(t) f(u,\dot{u},\ddot{u})dt = 0 \tag{9}$$

$$\mu\int_0^T \dot{u}^2(t)dt - \int_0^T \dot{u}^2(t) g(u)dt = 0 \tag{10}$$

Solving both Eq. (9) and Eq. (10) yields

$$\omega^2(\Omega) = \frac{\int_0^T u^2(t) f(u,\dot{u},\ddot{u})dt}{\int_0^T u^2(t)dt} \tag{11}$$

$$\mu(\Omega) = \frac{\int_0^T \dot{u}^2(t) g(u)dt}{\int_0^T \dot{u}^2(t)dt} \tag{12}$$

Performing the aforementioned integrals gives the equivalent damping coefficient and the equivalent conservative frequency.

## 3. The Fractal Forced Duffing Equation

Ineffective modelling of the system is a major obstacle for a nonlinear oscillation system in the fractal space. Particularly, the distribution of the periodic property's effect cannot be explained by differential equation models. In order to do this, the current work develops a fractal-differential model, and a forced fractal Duffing oscillator is used as an example to show the fundamental characteristics of the fractal oscillator subjected to harmonic force. Whereas a lot of research articles focused on a simple oscillation in fractal space [40-42], this work investigates a more generalized fractal vibration system that is modeled by the form:

$$\frac{du}{dt^{2\alpha}} + \omega_0^2 u + q u^3 = \delta \cos \Omega t; \;\; 0 < \alpha < 1 \tag{13}$$

Here, $\alpha$, $\omega_0$, $q$, $\Omega$, and $\delta$ are the non-zero coefficients, fractal factor, vibration natural frequency, Duffing coefficient, parametric force, and amplitude, respectively.

Assuming that the initial conditions are described as

$$u(0) = A, \; \frac{du(0)}{dt^{\alpha}} = 0 \tag{14}$$

The fractal theory has grown in importance in both mathematics and mechanical engineering because it is useful in developing a governing equation in a fractal environment. There are numerous definitions for the fractional derivatives. Most definitions often used ones include Riemann-Liouville, Caputo, Xiao-Jun Yang, and Jumarie for further justification see [43-45].

The He's fractal derivative is defined as [7, 46, 47]:

$$\frac{d^{\alpha}u}{dt^{\alpha}} = \Gamma(1+\alpha) \underset{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}}{Lim} \frac{u(t)-u(t_0)}{(t-t_0)^{\alpha}} \tag{15}$$

In this case, the fractal factor $\alpha$ runs along the $t$ dimension. If we see a motion on a vast size, it might alter continuously, but if we observe it on a tiny scale, it might discontinue. As a result, the fractal theory is an effective mathematical instrument for conducting more thorough global research. Moreover, this fractal derivative exhibits some of the following characteristics [48-50]:

$$\underset{\alpha \to 0}{Lim} \frac{du}{dt^{\alpha}} = u \;, \; \underset{\alpha \to 1}{Lim} \frac{du}{dt^{\alpha}} = \dot{u} \text{ and } \underset{\alpha \to 2}{Lim} \frac{du}{dt^{\alpha}} = \ddot{u} \tag{16}$$

Here the over-dot refers to the time classical derivative.

The fractal derivative given in Eq. (15) has widely been used to deal with porous or hierarchical structures [46-50] with great success. If we see a motion on a vast size, it might alter continuously, but if we observe it on a tiny scale, it might discontinue. As a result, the fractal theory is an effective mathematical instrument for conducting more thorough global research. The technique of creating mathematical models and the geometric physical interpretation of fractal derivatives were discussed in [51].

Defining the restoring force $f(u)$ in the current problem as

$$f(u) = \omega_0^2 u + q u^3 \tag{17}$$

Accordingly, Eq. (13) can be written as

$$\frac{du}{dt^{2\alpha}} + f(u) = \delta \cos \Omega t \tag{18}$$

Operating $\frac{d}{dt^{\alpha}}$ on Eq. (18) one time and $\frac{d}{dt^{2\alpha}}$ another time gives

$$\frac{du}{dt^{3\alpha}} + \frac{d}{dt^{\alpha}} f(u) = \delta \frac{d}{dt^{\alpha}} \cos \Omega t \tag{19}$$

$$\frac{du}{dt^{4\alpha}} + \frac{df(u)}{dt^{2\alpha}} = \delta \frac{d}{dt^{2\alpha}} \cos \Omega t \tag{20}$$

To estimate $\frac{d}{dt^{\alpha}} \cos \Omega t$ and $\frac{d}{dt^{2\alpha}} \cos \Omega t$ one may apply the following fractal derivative proposed by El-Dib and Elgazery [30,31]:

$$\frac{d..}{dt^{n\alpha}} = (S^{\alpha} \cos(\tfrac{1}{2}\pi\alpha) + S^{\alpha-1} \sin(\tfrac{1}{2}\pi\alpha) D)^{n} \; .. \tag{21}$$

Therefore, one gets

$$\frac{d \cos \Omega t}{dt^{\alpha}} = S^{\alpha} \cos(\tfrac{1}{2}\pi\alpha) \cos \Omega t - \Omega S^{\alpha-1} \sin(\tfrac{1}{2}\pi\alpha) \sin \Omega t \tag{22}$$

$$\frac{d \cos \Omega t}{dt^{2\alpha}} = b^{2}(\alpha)(a^{2}(\alpha) - \Omega^{2}) \cos \Omega t - 2\Omega a(\alpha) b^{2}(\alpha) \sin \Omega t \tag{23}$$

Here, $a(\alpha)$ and $b(\alpha)$ are described by

$$a(\alpha) = S \cot(\tfrac{1}{2}\pi\alpha) \text{ and } b(\alpha) = S^{\alpha-1} \sin(\tfrac{1}{2}\pi\alpha) \tag{24}$$

Employing Eqs. (22) and (23) into Eqs. (19) and (20) by using Eq. (18) yields

$$\frac{du}{dt^{3\alpha}} - S^{\alpha} \cos(\tfrac{1}{2}\pi\alpha) \frac{du}{dt^{2\alpha}} + \frac{d}{dt^{\alpha}} f(u) - S^{\alpha} \cos(\tfrac{1}{2}\pi\alpha) f(u) = -\delta\Omega S^{\alpha-1} \sin(\tfrac{1}{2}\pi\alpha) \sin \Omega t \tag{25}$$

$$\frac{du}{dt^{4\alpha}} + \frac{df(u)}{dt^{2\alpha}} - b^{2}(\alpha)\left(a^{2}(\alpha) - \Omega^{2}\right)\left(\frac{du}{dt^{2\alpha}} + f(u)\right) = -2\delta \Omega a(\alpha) b^{2}(\alpha) \sin \Omega t \tag{26}$$

Removing sin $\Omega t$ between Eqs. (25) and (26) yields

$$\frac{du}{dt^{4\alpha}} - 2S^{\alpha} \cos\left(\tfrac{1}{2}\pi\alpha\right)\left(\frac{du}{dt^{3\alpha}} + \frac{df}{dt^{\alpha}}\right) + b^{2}(\alpha)\left(a^{2}(\alpha) + \Omega^{2}\right)\left(\frac{du}{dt^{2\alpha}} + f(u)\right) + \frac{df}{dt^{2\alpha}} = 0 \tag{27}$$

This is the alternative form of the forced fractal Eq. (18) in the form of a fractal fourth-order equation. This equation is subject to the following initial conditions:

$$u(0^{\alpha}) = A, \frac{du(0^{\alpha})}{dt^{\alpha}} = 0^{\alpha}, \; \frac{du(0^{\alpha})}{dt^{2\alpha}} = \delta - f(A), \; \frac{du(0^{\alpha})}{dt^{3\alpha}} = -\delta S^{\alpha} \cos(\tfrac{1}{2}\pi\alpha) \tag{28}$$

To convert Eq. (27) and its corresponding initial conditions given in Eq. (28) into continuous space, the transformation given in Eq. (21) may be used. Therefore, with some simplifications, one gets

$$(a+D)^4 u - 2\,a(a+D)^3 u + b(a^2+\Omega^2)(a+D)^2 u + \frac{1}{b^2}(\Omega^2+D^2)\, f(u) = 0 \tag{29}$$

The operator $D$ represents the derivative corresponding to the variable $t$. Thus, Eq. (29) is a nonlinear fourth-order equation free of the variable coefficients

$$\begin{aligned} &u^{(4)} + 2\,a\,u^{(3)} + b(a^2+\Omega^2)\ddot{u} + 2\,a(b(a^2+\Omega^2) - a^2)\dot{u} + \frac{q}{b^2}(\Omega^2 u^3 + 6\dot{u}^2 u + 3u^2\ddot{u}) \\ &+ \frac{1}{b^2}(-a^4 b^2 + a^2 b^3 (a^2+\Omega^2) + \Omega^2 \boldsymbol{\omega}_0^2) u = 0 \end{aligned} \tag{30}$$

This is the alternative form of the forced fractal Duffing Eq. (13) in continuous space. Also, the alternative initial conditions in the continuous space are performed in the form

$$\left.\begin{aligned} &u(0) = A,\ \dot{u}(0) = -A\,a(\boldsymbol{\alpha}),\ \ddot{u}(0) = \frac{1}{b^2(\boldsymbol{\alpha})}(\boldsymbol{\delta} - f(A)) + a^2(\boldsymbol{\alpha})A, \\ &\dddot{u}(0) = -a^3(\boldsymbol{\alpha})A - \frac{a(\boldsymbol{\alpha})}{b^2(\boldsymbol{\alpha})}\big(4\boldsymbol{\delta} - 3\,f(A)\big) \end{aligned}\right\} \tag{31}$$

It is noteworthy, if the initial velocity is not zero such that $u(0) = A$ and $\frac{du(0)}{dt^\alpha} = B$, the present approach has a good applicable so the application of the formula given in Eq. (9) to the fractal initial condition has the form

$$\frac{du(0)}{dt^{\alpha}} = S^{\alpha}\cos(\tfrac{1}{2}\boldsymbol{\pi\alpha})u(0) + S^{\alpha-1}\sin(\tfrac{1}{2}\boldsymbol{\pi\alpha})\dot{u}(0) \tag{32}$$

Employing the non-zero initial conditions yields that $\dot{u}(0) = \frac{B}{S^{\alpha-1} sin\left(\frac{\pi\alpha}{2}\right)} - A\,S\,cot\left(\frac{\pi\alpha}{2}\right)$ and $u(0)=A$. Therefore, as $B \to 0$ the converted initial conditions given in Eq. (31) will arises.

In the transformation given in Eq. (21), the fractal parameter $S$ could be estimated by comparing between the linear frequency given in Eq. (30) with the linear one of the fractal Eq. (27). This comparison shows that

$$a^4 + \frac{b^3(\Omega^2 - b\,\boldsymbol{\omega}_0^2)}{b^2(b-1)}a^2 + \frac{\Omega^2\boldsymbol{\omega}_0^2(1-b^4)}{b^2(b-1)} = 0 \tag{33}$$

Here, $b(\boldsymbol{\alpha}) \neq 1$ is considered, unless $\boldsymbol{\alpha} = 1$. Employing the definition of $a(\boldsymbol{\alpha})$ given in Eq. (24) in the above equation yields

$$S^2(\Omega) = \frac{\tan^2(\frac{1}{2}\boldsymbol{\pi\alpha})}{2b(b-1)}\left[b^2(b\boldsymbol{\omega}_0^2 - \Omega^2) + \sqrt{b^4(\Omega^2 - b\boldsymbol{\omega}_0^2)^2 + 4(1-b)\Omega^2\boldsymbol{\omega}_0^2(1-b^4)}\right] \tag{34}$$

It is worthwhile to observe that the applied frequency $\Omega$ can be sought in terms of the fractal order $\alpha$. This can be read from Eq. (33) to have

$$\Omega^2(\alpha) = \frac{(b^2\omega_0^2 - (b-1)a^2)b^2a^2}{b^3a^2 + \omega_0^2(1-b^4)} \tag{35}$$

The trial solution for the nonlinear fourth-order Eq. (30) may be suggested in the form

$$u_0(t) = k_1\cos\omega t + k_2\sin\omega t + k_3\cos 3\omega t + k_4\sin 3\omega t \tag{36}$$

Here, $\omega$ represents the oscillation frequency and the constant coefficients $k$'s can be estimated by using the initial conditions given in Eq. (31). Consequently,

$$\left.\begin{aligned} k_1 &= \frac{1}{8b^2\omega^2}(\delta + A(a^2 - f(A) + 9b^2\omega^2)),\\ k_2 &= -\frac{a}{8b^2\omega^3}(4\delta + A(a^2b^2 - 3f(A) + 9b^2\omega^2)),\\ k_3 &= -\frac{1}{8b^2\omega^2}(\delta + A(a^2 - f(A) + b^2\omega^2)),\\ k_4 &= \frac{a}{24b^2\omega^3}(4\delta + A(a^2b^2 - 3f(A) + b^2\omega^2)) \end{aligned}\right\} \tag{37}$$

Since the original system is a second-order derivative, the fourth-order derivative in Eq. (30) is artificial and should be reduced to the second-order one. In addition to reducing rank of Eq. (29) to the second-order form, our goal is to construct a linearizing form for it. The linearizing method can be used to achieve this goal. The following process may be suggested [52]:
Rearranged Eq. (30) in the form

$$g(u^{(4)}, u^{(3)}, \ddot{u})\ddot{u} + 2a(b(a^2+\Omega^2) - a^2)\dot{u} + F(u,\dot{u})u = 0 \tag{38}$$

The above two functions $g$ and $F$ are defined as

$$\left.\begin{aligned} g(u^{(4)}, u^{(3)}, \ddot{u}) &= \frac{1}{\ddot{u}}\left(u^{(4)} + 2a\,u^{(3)} + b(a^2+\Omega^2)\ddot{u} + \frac{3q}{b^2}u^2\ddot{u}\right),\\ F(u,\dot{u}) &= \frac{1}{b^2}(-a^4b^2 + a^2b^3(a^2+\Omega^2) + \Omega^2\omega_0^2) + \frac{q}{b^2}(\Omega^2u^2 + 6\dot{u}^2) \end{aligned}\right\} \tag{39}$$

Apply Caughey's linearized approach [53] to Eq. (38) with considered the suggested trial solution given in Eq. (36) yields

$$\sigma_0(\omega)\ddot{u} + 2a(b(a^2+\Omega^2) - a^2)\dot{u} + \sigma_1(\omega)u = 0 \tag{40}$$

Here, $\sigma_0(\omega)$ and $\sigma_1(\omega)$ are estimated using the following formula proposed by El-Dib [38]

$$\sigma_0(\omega) = \frac{\int_0^T g(u_0^{(4)}, u_0^{(3)}, \ddot{u}_0)\, \ddot{u}_0^2(t)\, dt}{\int_0^T \ddot{u}_0^2(t)\, dt}; \; T = \frac{2\pi}{\omega} \tag{41}$$

$$\sigma_1(\omega) = \frac{\int_0^T u_0^2(t)\, F(u_0, \dot{u}_0)\, dt}{\int_0^T u_0^2(t)\, dt} \tag{42}$$

The results of the above integrals are straightforward in their calculations and are available from the corresponding author.

It should be noticed that Eq. (40) can be expressed in the following manner:

$$\ddot{u} + 2\mu(\omega)\dot{u} + \varpi^2(\omega)u = 0 \tag{43}$$

Here, the coefficients $\mu$ ($\omega$) and $\varpi^2(\omega)$ are constants functions of the frequency $\omega$ given by

$$\mu(\omega) = \frac{a(b(a^2 + \Omega^2) - a^2)}{\sigma_0(\omega)} \tag{44}$$

$$\varpi^2(\omega) = \frac{\sigma_1(\omega)}{\sigma_0(\omega)} \tag{45}$$

Eq. (43) is a linear second-order equation having a damping force. The damping force is characterized by the coefficient $\mu$ ($\omega$). The solution of Eq. (43) has a damping behavior unless $\alpha$ is equal unity. More damping behavior is found as $\alpha$ having a greater decrease than unity [30,31,54]. The solution that covers this behavior is performed in the following form:

$$u(t) = e^{-\mu(\omega)t}(k_1 \cos\omega t + k_2 \sin\omega t + k_3 \cos 3\omega t + k_4 \sin 3\omega t) \tag{46}$$

The above solution is obtained due to applying the normal form technique. In this approach, the frequency $\omega$ has the form

$$\omega^2 = \varpi^2(\omega) - \mu^2(\omega) \tag{47}$$

## 4. Numerical Estimation

The numerical solution of the original forced fractal nonlinear oscillator given in Eq. (13) at $\alpha \to 1$ and analytical solution given in Eq. (46) were compared and shown in Fig. 1 for the system having $A = \omega_0 = 1$, $q = \delta = 0.01$, and $\Omega = 10$. The relative error is estimated in this calculation to be 0.1331 which reveals that the two solutions were in accord. Further, the comparison of the analytical solution given in Eq. (46) of the second-order Eq. (43) with the numerical solution of the fourth-order Eq. (30) has been displayed in Fig. 2 for the case of $\alpha$ is close to unity. The system that is used in these calculations has $A = \omega_0 = 1$, $q = \delta = 0.01$, and $\Omega = 10$. The relative error is estimated in this calculation to be 0.006445 which demonstrates that the analytical and numerical solutions are in good accord with one another. It is noted that the periodic solution is revealed in this calculation. This behavior is expected because $\alpha \to 1$ the fractal parameter $a(\alpha)$ should tend to zero. Consequently, the damping forces in both Eq. (30) and Eq. (43) will disappear. The damping behavior will appear due to the fractional order $\alpha$ decreasing away the unity as shown in Fig. 3. In the graph of Fig. 3 consequent values $\alpha$ are considered and collected together with the case of $\alpha \cong 1$. The agreement between the numerical solution and the analytical solution is still observed even when the value of $\alpha$ decreases from one to zero. The comparison shows that the damping behavior increases with decreasing in $\alpha$. In a word, the computations revealed that for every value of the fraction parameter, the approximation and numerical solutions are identical. From the above figures, the present study gives reliability to the technique of reducing the order of differential equations. This makes us think that the alternative equation can be handled, discussed, and then the solutions can be used as a reliable indicator.

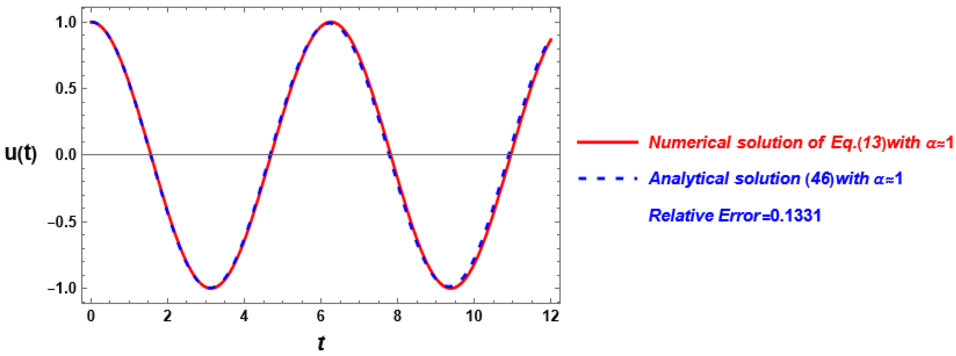


**Fig. 1** Comparison of the numerical solution of the original forced fractal nonlinear oscillator given in Eq. (13) with the analytical solution given in Eq. (46) at $\alpha$ is very close to unity for the system having $A = \omega_0 = 1$, $q = \delta = 0.01$, and $\Omega = 10$

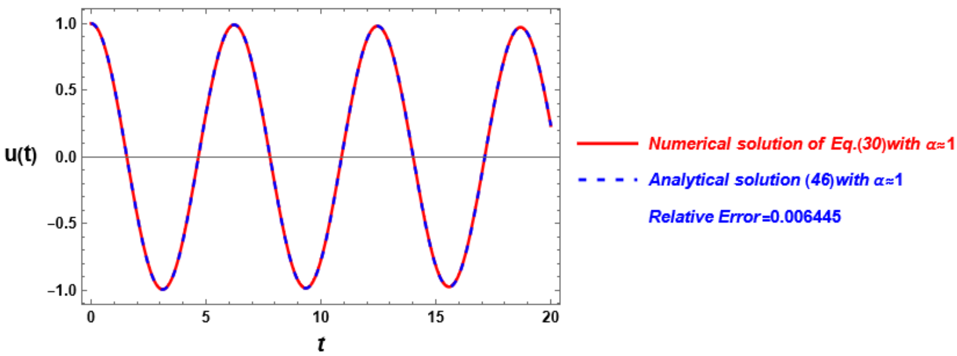


**Fig. 2** Comparison of the analytical solution given in Eq. (46) with the numerical one of Eq. (30) in the case of at $\alpha \to 1$ for the system having $A = \omega_0 = 1$, $q = \delta = 0.01$, and $\Omega = 10$

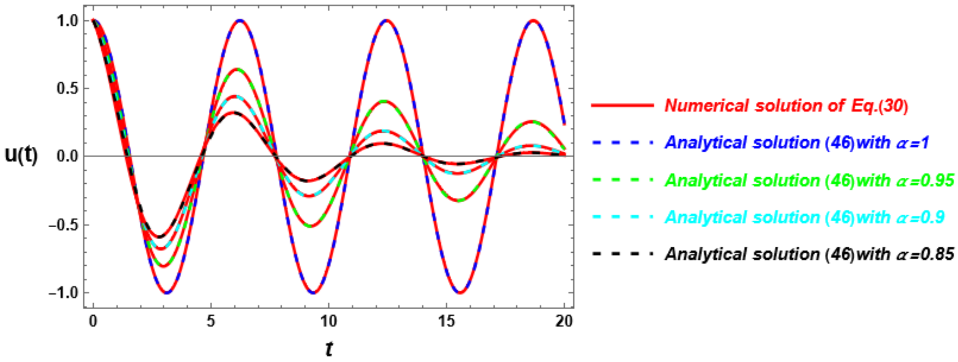


**Fig. 3** Represents to the influence of the variation in $\alpha$ on the comparison of the analytical solution given in Eq. (46) with the numerical one of Eq. (30) for $A = \omega_0 = 1$, $q = \delta = 0.01$, and $\Omega = 10$

In Figs. 4-6 the examinations are done for the amplitude and the frequency of the applied periodic force. For this purpose, the fractional order is held at $\alpha = 0.9$. The calculations are used in the same system as given in Fig. 2, with alternative values of the parameters $\delta$ and $\Omega$. It is noted that the relative error is 0.1189 when $\delta = 0.1$, see Fig. 4, but this error becomes 0.3281 as $\delta = 0.2$ shown in Fig. 5. Therefore, when comparing Fig. 4 with Fig. 5, it becomes clear that small delta values make the solution more accurate than larger values. When $\delta$ is held at $\delta = 0.1$ with changes in $\Omega$ from 7 to 10 as observed in Fig. 6, the relative error has increased for the case of large values of $\Omega$. Hence, the comparison between Figs. 4 and 6 means that the increase in $\Omega$ leads to improving the accuracy of the solution. It should be noted that one of the most important features of these calculations is that the fourth-order equation can be reduced to a second-order one via the non-perturbative approach.

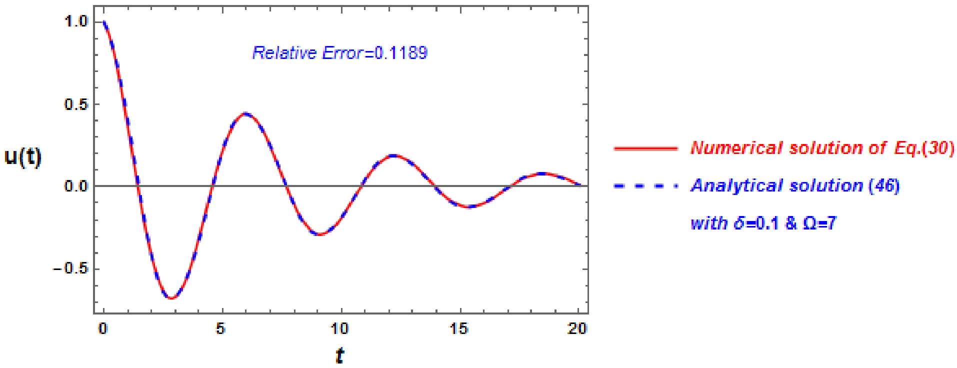


**Fig. 4** The relative error due to the comparison of the analytical solution given in Eq. (46) with the numerical one of Eq. (30) for $A = \omega_0 = 1$, $q = 0.01$, $\delta = 0.1$, $\alpha = 0.9$, and $\Omega = 7$

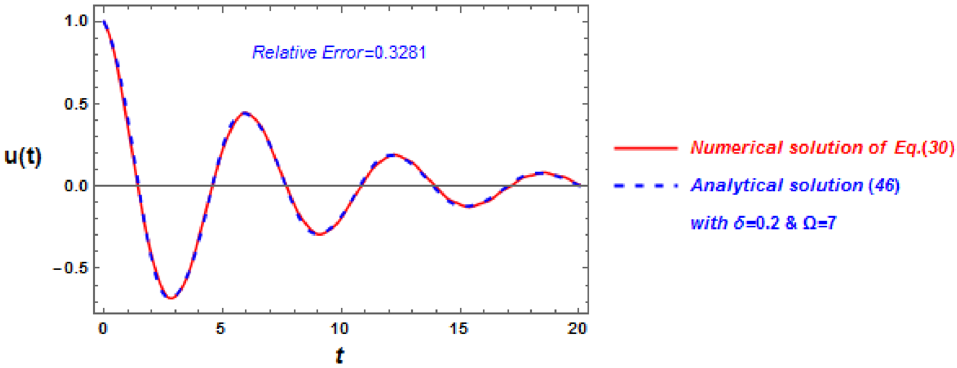


**Fig. 5** The relative error due to the comparison of the analytical solution given in Eq. (46) with the numerical one of Eq. (30) for $A = \omega_0 = 1$, $q = 0.01$, $\delta = 0.2$, $\alpha = 0.9$, and $\Omega = 7$

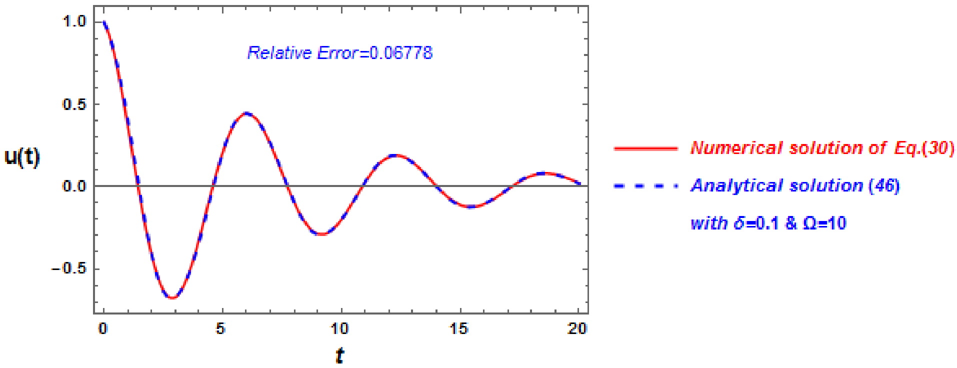


**Fig. 6** The relative error due to the comparison of the analytical solution given in Eq. (46) with the numerical one of Eq. (30) for $A = \omega_0 = 1$, $q = 0.01$, $\delta = 0.1$, $\alpha = 0.9$, and $\Omega = 10$

## 5. Periodic Solution

On the other side, the periodic solution of Eq. (43) can be established, for a non-zero coefficient $\mu(\omega)$. In this case, the frequency $\omega$ will recall $\hat{\omega}$ which is assumed to cover the periodic solution. To accomplish this periodic solution, Galerkin's technique may be applied to establish the frequency $\hat{\omega}$. To construct the residual function $R(\hat{\omega};t)$, one can insert the suggested solution given in Eq. (36) into Eq. (43), which gives

$$\begin{aligned} R(\hat{\omega};t) = &((\varpi^2 - \hat{\omega}^2)k_1 + 2\mu\hat{\omega}k_2)\cos\hat{\omega}t + ((\varpi^2 - \hat{\omega}^2)k_2 - 2\mu\hat{\omega}k_1)\sin\hat{\omega}t \\ &+ ((\varpi^2 - 9\hat{\omega}^2)k_3 + 6\mu\hat{\omega}k_4)\cos 3\hat{\omega}t + ((\varpi^2 - 9\hat{\omega}^2)k_4 - 6\mu\hat{\omega}k_3)\sin 3\hat{\omega}t \end{aligned} \tag{48}$$

To estimate the total frequency $\hat{\omega}$, one may apply the following Galerkin's formula:

$$\int_0^T R(\hat{\omega};t)\left(\cos\hat{\omega}t + \sin\hat{\omega}t + \cos 3\hat{\omega}t + \sin 3\hat{\omega}t\right)dt = 0 \tag{49}$$

Using the definitions given in Eq. (37) and the residual given in Eq. (48), the above integral leads to the following frequency equation:

$$c_4\hat{\omega}^4 + c_3\hat{\omega}^3 + c_2\hat{\omega}^2 + c_0 = 0 \tag{50}$$

Here, the coefficients $c$'s are listed below:

$$\left.\begin{aligned}
c_4 &= 9Ab^2(a - 2\mu(\hat{\omega}))\\
c_3 &= 12(\delta + a^2A - A^4q - A^2\omega_0^2 - 2Aab^2\mu(\hat{\omega}) + Ab^2\varpi^2(\hat{\omega}))\\
c_2 &= a(9A^4q + 9A^2\omega_0^2 - 12\delta - 13Ab^2)\varpi^2(\hat{\omega}) + 6(\delta + a^2A - A^4q - A^2\omega_0^2)\mu(\hat{\omega}) - 3Aa^3b^2\\
c_0 &= a(3A^4q + 3A^2\omega_0^2 - 4\delta - Aa^2b^2)\varpi^2(\hat{\omega})
\end{aligned}\right\} \tag{51}$$

Given the real roots of the frequency in Eq. (50), the periodic solution of Eq. (43) can be obtained in the form

$$u(t) = k_1\cos\hat{\omega}t + k_2\sin\hat{\omega}t + k_3\cos 3\hat{\omega}t + k_4\sin 3\hat{\omega}t \tag{52}$$

Numerical illustration for the current state is shown in the Figs. 7-12. The calculations are made for the periodic solution given in Eq. (52) where the frequency $\hat{\omega}$ is estimated from the frequency Eq. (50). It is noted that the coefficients $k$'s are depend on the frequency $\hat{\omega}$ as defined by Eq. (37). The numerical system that is considered in these calculations is $A = \omega_0 = 1$, $q = \delta = 0.1$, and $\Omega = 3$ with the variation of the fractal order $\alpha$. In Fig. 7, the smooth periodic curve is observed when the fractal order has the value $\alpha = 0.9$. The periodic curve due to $\alpha = 0.8$ that lies in Fig. 8 appears similar to the case of the first successive solution in [39]. This behavior of the periodic solution has been affected by decreasing in the fractal order as shown in Figs. 9-12.

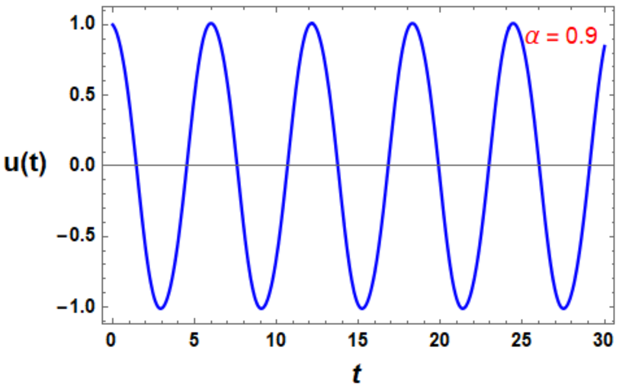


**Fig. 7** Representation of the periodic solution given in Eq. (52) for a system having $A = \omega_0 = 1$, $q = \delta = 0.1$, $\alpha = 0.9$, and $\Omega = 3$

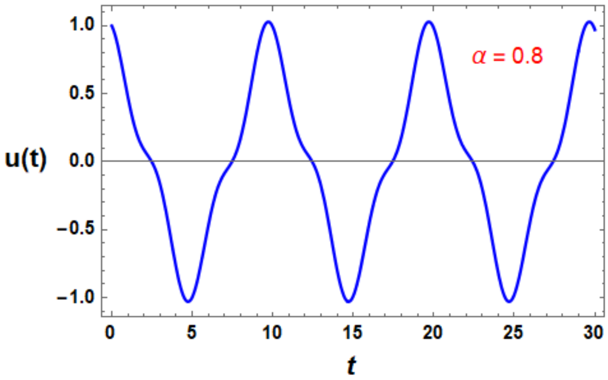


**Fig. 8** Representation of the periodic solution given in Eq. (52) for $A = \omega_0 = 1$, $q = \delta = 0.1$, $\alpha = 0.9$, and $\Omega = 3$

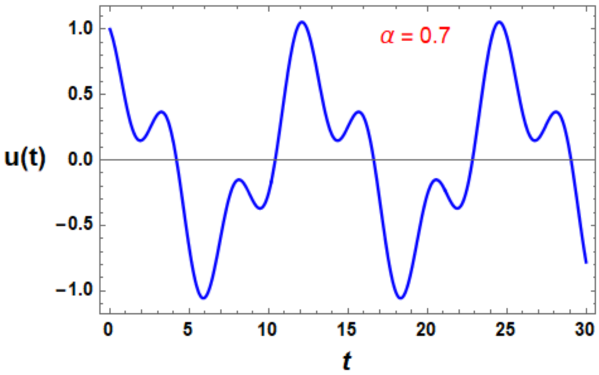


**Fig. 9** Representation of the periodic solution given in Eq. (52) for $A = \omega_0 = 1$, $q = \delta = 0.1$, $\alpha = 0.7$, and $\Omega = 3$

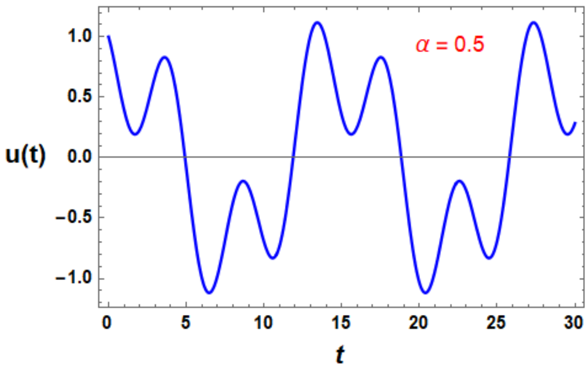


**Fig. 10** Representation of the periodic solution given in Eq. (52) for $A = \omega_0 = 1$, $q = \delta = 0.1$, $\alpha = 0.5$, and $\Omega = 3$

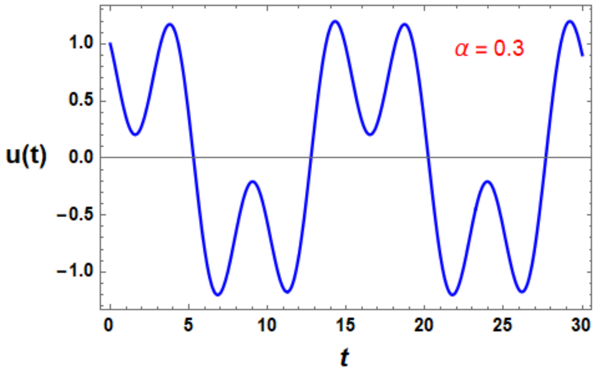


**Fig. 11** Representation of the periodic solution given in Eq. (52) for $A = \omega_0 = 1$, $q = \delta = 0.1$, $\alpha = 0.3$, and $\Omega = 3$

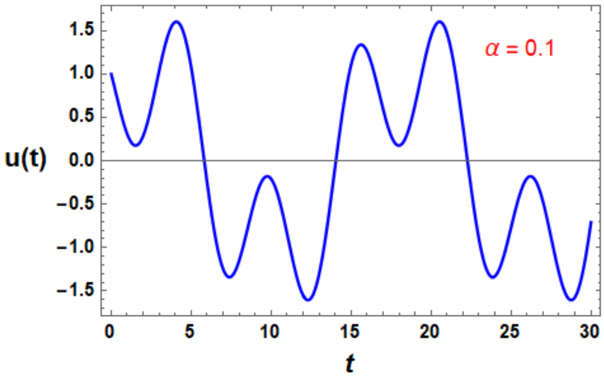


**Fig. 12** Representation of the periodic solution given in Eq. (52) for $A = \omega_0 = 1$, $q = \delta = 0.1$, $\alpha = 0.1$, and $\Omega = 3$

## 6. Conclusions

A fractal Duffing oscillator with periodic force was investigated in the current work. Ultimately, the present conclusion observations are summarized as follows:

- By using the rank upgrade technique, an alternative unforced fractal fourth-order equation was obtained and converted to its traditional derivative form in the continuous space.
- After reducing its rank, the non-perturbative approach was used to calculate the analytic solution for the alternate equation in the second-order form.
- It is seen that the analytical and numerical solutions agree quite well.
- The computations revealed that for every value of the fraction parameter $\alpha$, the approximation and numerical solutions are identical.
- $\frac{d}{dt^{\alpha}}\cos\Omega t$ and $\frac{d}{dt^{2\alpha}}\cos\Omega t$ was estimated, which is considered a new result that can be used in future works.

- The rank upgrade/reducing technique was introduced as a reliable approach to overcome difficulties.
- The present study gives reliability in the technique of reducing the order of differential equations.
- One of the most important features of these calculations lies in the fact that a fourth-order equation can be reduced to a second-order one via the non-perturbative approach.
- Galerkin's method was also used to find the required periodic solution.
- Decreasing the fractal order has an impact on the behaviour of periodic solution.
- In contrast to the traditional technique, which worked to transform the variable and was valid only in the absence of external forces, if there was an external force, it leads to great mathematical difficulties. The current technique works on the operator, which is simple and effective when fractal oscillators with external forces are investigated, it is easy to obtain analytic solutions, and the technique doesn't lead to any mathematical difficulties.

**Acknowledgement**: *The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R17), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.*

## REFERENCES

1. Zmeskal, O., Dzik, P., Vesely M., 2013, *Entropy of fractal systems*, Computers & Mathematics with Applications, 66(2), pp. 135-146.
2. Edgar , G. A., 2008, *Measure, Topology, and Fractal Geometry*, Edition Number 2, Springer New York, NY.
3. Chen, M., Yu, B., Xu, P., Chen, J., 2007, *A new deterministic complex network model with hierarchical structure*, Physica A: Statistical Mechanics and its Applications, 385(2), pp. 707-717.
4. Yang, S., Fu, H., Yu, B., 2017, *Fractal analysis of flow resistance in tree-like branching networks with roughened microchannels*, Fractals, 25(1), 1750008.
5. Miao, T., Chen, A., Xu, Y., Yang, S., Yu, B., 2016, *Optimal structure of damaged tree-like branching networks for the equivalent thermal conductivity*, International Journal of Thermal Sciences, 102, pp. 89–99.
6. Liang, M., Gao, Y., Yang, S., Xiao, B., Wang, Z., Li, Y., 2018, *An analytical model for two-phase relative permeability with Jamin effect in porous media*, Fractals, 26(3), 1850037.
7. He, J.-H., Kou, S.-J., He, C.-H., Zhang, Z.-W., Gepreel, K. A., 2021, *Fractal oscillation and its frequency-amplitude property*, Fractals, 29(4), 2150105.
8. Wang, K.-L., 2022, *Exact solitary wave solution for fractal shallow water wave model by He's variational method*, Modern Physics Letters B, 36(7), 2150602.
9. Ain, Q.T., He, J.-H., 2019, *On two-scale dimension and its applications*, Thermal Science, 23(3) Part B, pp. 1707-1712.
10. Li, X., Liu, Z., He, J.-H., 2020, *A fractal two-phase flow model for the fiber motion in a polymer filling process,* Fractals, 28(5), 2050093.
11. Zuo, Y., 2021, *A gecko-like fractal receptor of a three-dimensional printing technology: a fractal oscillator*, Journal of Mathematical Chemistry, 59(3), pp. 735-744.
12. Liu, C., 2021, *Periodic solution of fractal Phi-4 equation*, Thermal Science, 25(2) Part B, pp. 1345–1350.
13. He, C.-H., Liu, C., He, J.-H., Gepreel, K.A., 2021, *Low frequency property of a fractal vibration model for a concrete beam*, Fractals, 29(5), 2150117.
14. Anjum, N., He, C.-H., He, J.-H., 2021, *Two-scale fractal theory for the population dynamics*, Fractals, 29(7), 2150182.
15. Anjum, N., Ain, Q.T., Li, X.X., 2021, *Two-scale mathematical model for tsunami wave*, GEM - International Journal on Geomathematics, 12(1), 10.
16. Nayfeh, A.H., 2011, *Introduction to Perturbation Techniques*, John Wiley & Sons.

17. He, J.-H., 1999, *Homotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering, 178(3-4), pp. 257–262.
18. Wang, K.-L., 2021, *A new fractal transform frequency formulation for fractal nonlinear oscillators*, Fractals, 29(3), 2150062.
19. Wang, K.-L., 2021, *He's frequency formulation for fractal nonlinear oscillator arising in a microgravity space*, Numerical Methods for Partial Differential Equations, 37(2), pp. 1374-1384.
20. Wang, K.-L., Wei, C.-F., 2021, *A powerful and simple frequency formula to nonlinear fractal oscillators*, Journal of Low Frequency Noise, Vibration and Active Control, 40(3), pp. 1373-1379
21. Lu, J., Chen, L., 2022, *Numerical analysis of a fractal modification of Yao-Cheng oscillator*, Results in Physics, 105602.
22. Elías-Zúñiga, A., Martínez-Romero, O., Olvera-Trejo, D., Palacios-Pineda, L.M., 2021, *An efficient ancient Chinese algorithm to investigate the dynamics response of a fractal microgravity forced oscillator*, Fractals, 29(6), 2150144.
23. Elías-Zúñiga, A., Martínez-Romero, O., Olvera-Trejo, D., Palacios-Pineda, L.M., 2021, *Investigation of the steady-state solution of the fractal forced Duffing's oscillator using an ancient Chinese algorithm*, Fractals, 29(6), 2150133.
24. Sheng, M., Li, G., Tian, S., Huang, Z., Chen, L., 2016, *A fractal permeability model for shale matrix with multi-scale porous structure*, Fractals, 24(1), 1650002.
25. Miao, T., Chen, A., Xu, Y., Cheng, S., Yu, B., 2019, *A fractal permeability model for porous–fracture media with the transfer of fluids from porous matrix to fracture*, Fractals, 27(6), 1950121.
26. Xu, J., Wu, K., Li, R., Li, Z., Li, J., Xu, Q., Chen, Z., 2018, *Real gas transport in shale matrix with fractal structures*, Fuel, 219, pp. 353-363.
27. Wang, S., Wu, T., Cao, X., Zheng, Q., Ai, M., 2017, *A fractal model for gas apparent permeability in microfractures of tight/shale reservoirs*, Fractals, 25(3), 1750036.
28. Wu, T., Wang, S., 2020, *A fractal permeability model for real gas in shale reservoirs coupled with Knudsen diffusion and surface diffusion effects*, Fractals, 28(1), 2050017.
29. Yang, C.-Y., Zhang, Y.-D., Yang, X.-J., 2016, *Exact solutions for the differential equations in fractal heat transfer*, Thermal Science, 20(suppl. 3), pp. 747-750.
30. El-Dib, Y.O., Elgazery, N.S., 2022, *A novel pattern in a class of fractal models with the non-perturbative approach*, Chaos, Solitons & Fractals, 164, 112694.
31. El-Dib, Y.O., Elgazery, N.S., 2023, *An efficient approach to converting the damping fractal models to the traditional system*, Communications in Nonlinear Science and Numerical Simulation, 118, 107036.
32. Selivanov, M.F., Chornoivan, Y.O., 2018, *A semi-analytical solution method for problems of cohesive fracture and some of its applications*, International Journal of Fracture, 212, pp. 113-121.
33. Khater, M.M.A., Mousa, A.A., El-Shorbagy, M.A., Attia, R.A.M., 2021, *Analytical and semi-analytical solutions for Phi-four equation through three recent schemes*, Results in Physics, 22, 103954.
34. Moshtaghi, N., Saadatmandi, A., 2021, *Numerical solution of time fractional cable equation via the sinc-bernoulli collocation method*, Journal of Applied and Computational Mechanics, 7(4), pp. 1916-1924.
35. Elías-Zúñiga, A., Martínez-Romero, O., Olvera-Trejo, D., Palacios-Pineda, L.M., 2021, *Exact steady-state solution of fractals damped, and forced systems*, Results in Physics, 28, 104580.
36. He, J.-H., 2019, *The simplest approach to nonlinear oscillators*, Results in Physics, 15(2019), 102546.
37. He, J.-H., 2019, *The simpler, the better: Analytical methods for nonlinear oscillators and fractional oscillators*, Journal of Low Frequency Noise, Vibration and Active Control, 38(3-4), pp. 1252-1260.
38. El-Dib, Y.O., 2023, *Insightful and comprehensive formularization of frequency amplitude formula for strong or singular nonlinear oscillators*, Journal of Low Frequency Noise, Vibration and Active Control, 42(1), pp. 89-109.
39. El-Dib, Y.O., Alyousef, H.A., 2023, *Successive approximate solutions for nonlinear oscillation and improvement of the solution accuracy with efficient non-perturbative technique*, Journal of Low Frequency Noise, Vibration and Active Control, doi: 10.1177/14613484231161425.
40. Feng, G.-Q., 2021, *He's frequency formula to fractal undamped Duffing equation*, Journal of Low Frequency Noise, Vibration and Active Control, 40(4), pp. 1671-1676.
41. Elías-Zúñiga, A., Palacios-Pineda, L.M., Jimenez-Cedeno, I. H., Martínez-Romero, O., Olvera-Trejo, D., 2021, *Analytical solution of the fractal cubic–quintic Duffing equation*, Fractals, 29(4), 2150080.
42. Elías-Zúñiga, A., Palacios-Pineda, L.M., Martínez-Romero, O., Olvera-Trejo, D., 2021, *Dynamics response of the forced Fangzhu fractal device for water collection from air*, Fractals, 29(7), 2150186.
43. Podlubny, I., 1998, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier.
44. Abbas, S., Benchohra, M., N'Guérékata, G.M., 2012, *Topics in Fractional Differential Equations*, Springer Science & Business Media.

45. Wang, K.-L., Liu, S.-Y., 2016, *He's fractional derivative for non-linear fractional heat transfer equation*, Thermal Science, 20(3), pp. 793-796.
46. Wang, K.-L., Liu, S.-Y., 2017, *He's fractional derivative and its application for fractional Fornberg-Whitham equation*, Thermal Science, 21(5), pp. 2049-2055.
47. Wu, Y., Liu, Y.-P., 2021, *Residual calculation in He's frequency–amplitude formulation*, Journal of Low Frequency Noise, Vibration and Active Control, 40(2), pp. 1040-1047.
48. He, J.-H., Ain, Q.T., 2020, *New promises and future challenges of fractal calculus: From two-scale thermodynamics to fractal variational principle*, Thermal Science, 24(2), pp. 659-681.
49. El-Dib, Y.O., Elgazery, N.S., Alyousef, H.A., 2023, *Galerkin's method to solve a fractional time-delayed jerk oscillator*, Archive of Applied Mechanics, 93, pp. 3597-3607.
50. El-Dib, Y.O., Elgazery, N.S., Alyousef, H.A., 2023, *An instrumental insight for a periodic solution of a fractal Mathieu–Duffing equation*, Journal of Low Frequency Noise, Vibration and Active Control, doi: 10.1177/14613484231185907.
51. He, J.-H., 2018, *Fractal calculus and its geometrical explanation*, Results in Physics, 10, pp. 272-276.
52. El-Dib, Y.O., 2022, *An efficient approach to solving fractional Van der Pol–Duffing jerk oscillator*, Communications in Theoretical Physics., 74, 105006.
53. El-Dib, Y.O., 2022, *The simplest approach to solving the cubic nonlinear jerk oscillator with the non-perturbative method*, *Mathematical Methods* in the *Applied Sciences*, 45(9), pp. 5165-5183.
54. El-Dib, Y.O., Elgazery, N.S., Khattab, Y.M., Alyousef, H.A., 2023, *An innovative technique to solve a fractal damping Duffing-jerk oscillator*, Communications in Theoretical Physics, 75, 055001.