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A MODIFIED FREQUENCY FORMULATION FOR NONLINEAR MECHANICAL VIBRATIONS

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Abstract. *The paper elucidates the characteristics of nonlinear oscillators that contain quasi-linear terms arising in mechanical engineering. Analytical solutions for this type of nonlinear oscillator are often difficult to obtain; they may involve singular and discontinuous terms, which can make them significantly more difficult to obtain. In this paper, a modified frequency formulation for this type of nonlinear oscillator is presented. While the mathematical proof is not provided, illustrative examples are included to demonstrate the remarkable simplicity and reliability of the approach. This paper challenges the conventional wisdom in analytical methods and offers a promising new direction for further investigation.*

Key words: *Nonlinear vibration, Non-perturbative approach, Quasi-linear oscillator, Discontinuous oscillator, Fractional potential, Zigzag oscillation*

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1. INTRODUCTION

In the ever-evolving landscape of contemporary science and technology, nonlinear vibration systems [1, 2] have risen to prominence as a foundational building block with far-reaching consequences across a diverse array of disciplines. Specifically, they have provided substantial support to the domain of mechanical engineering [3], particularly the field of micro-electromechanical systems (MEMS) [4]. It is imperative to comprehend the frequency-amplitude relationship to identify novel prospects and propel the advancement of related technologies. For instance, the capacity to discern the slightest tremors of starquakes, a capability of paramount importance in the realm of astronomy [5], exemplifies this phenomenon. These seismic events offer invaluable insights into the internal structure and composition of stars. Additionally, high-frequency vibrations have been observed in the coupling of excitons in π-conjugated molecules [6].

The nonlinearity exhibited by oscillators can have various origins. Cubic-quintic nonlinearity is a frequently employed phenomenon in the domain of mechanical engineering [7-9]. However, there are also nonlinearities of the form u^n), with $n < 1$. For instance, the integer n could be either 1/3 or -1. These nonlinear oscillators consist of both singular oscillators and discontinuous oscillators. The procurement of approximate solutions for these oscillatory phenomena poses a substantial challenge in the realms of mathematics and mechanical engineering. The complexity inherent to this phenomenon stems from the fact that singular oscillators frequently possess unique properties, thereby rendering traditional solution methods ineffective. Conversely, discontinuous oscillators pose significant challenges due to their sudden fluctuations in behavior. Confronting these challenges necessitates the development of innovative methodologies and a profound comprehension of both mathematical principles and mechanical systems.

A paucity of known periodic functions exists for the approximation of nonlinear oscillators. In the domain of nonlinear vibration theory, the cosine function and the sine function are the most frequently employed. While it is possible to theoretically model a zigzag periodic motion through an infinite series of cosine or sine functions, this approach is mathematically intricate. Engineers generally prioritize the key factors that influence the period or frequency of these oscillators. However, contemporary analytical methodologies predominantly center on the pursuit of an approximate cosine or sine solution for a complex nonlinear vibration system, with the objective of deriving an expression for the frequency. In order to make a more precise prediction of the impact of parameters on the frequency, it is necessary to undertake a cumbersome derivation process. However, this approach frequently falls short in providing highly accurate results due to the inherent limitations of using cosine and sine functions. Alternative methods, such as numerical simulations or advanced experiment techniques [10], have the potential to serve as an auxiliary means of vibration analysis. However, these approaches are costly and cannot provide more accurate predictions or overcome the complexity associated with traditional methods.

Another periodic function that must be considered is the Jacobi elliptic functions [7]. These functions are particularly well-suited for Duffing-like oscillators due to their specific mathematical properties, which can accurately capture the behavior of such oscillators. However, the differential operations of these mechanisms are intricate, posing a challenge for engineers to comprehend and utilize. The primary concern of an engineer is not the approximate solution per se; rather, the emphasis is placed on the precision of the frequency-amplitude relationship. This relationship is of paramount importance, as it constitutes the foundation for the optimal design of a nonlinear vibration system. A precise understanding of the frequency-amplitude relationship is essential for engineers to fine-tune the parameters of the system, ensuring its optimal performance and reliability. Absent a reliable estimate of this relationship, the design process becomes a speculative endeavor, with the potential to yield suboptimal results and inefficiencies.

2. Problem Formulation

In a previous publication, Ji-Huan He [11] considered the following nonlinear oscillator:

$$u'' + a_1 u + \sum_{n=1}^{N} a_{2n+1} u^{2n+1} = 0 \tag{1}$$

and proposed the following frequency formulation:

$$\omega^2 = a_1 + \sum_{n=1}^{N} \frac{3}{2n+2} a_{2n+1} A^{2n} \tag{2}$$

where a_{2n+1} (n=0~N) are constants, A is the amplitude. The frequency formulation is effective for nonlinear oscillators with high order nonlinearities, for example, the cubic-quintic-septic Duffing oscillator [12]:

$$\omega^2 = a_1 + \sum_{n=1}^{N} a_{2n+1} (\bar{u}_{2n+1})^{2n} \tag{3}$$

This is to choose a location point for each nonlinear term, u^{2n+1}:

$$\bar{u}_{2n+1} = \left(\frac{3}{2n+2}\right)^{1/2n} A \tag{4}$$

This frequency formulation holds immense importance in the realms of both mathematics and nonlinear vibration theory. It offers a swift and effective means of understanding the two main factors of oscillators and their intricate relationship. This knowledge is essential for delving deep into the vibration properties and enabling advanced applications. Take, for example, for the most energy harvesting devices or vibration dampers, their low-frequency property implies a large amplitude motion. As an illustration, we now consider the following nonlinear oscillator [13].

$$(1 + au^2)u'' + auu'^2 - u(1 - u^2) = 0 \tag{5}$$

where α is a constant. It can be re-written as

$$u'' - u + u^3 + auu'^2 + au^2 u'' = 0 \tag{6}$$

By the frequency formulation of Eq. (3), we have

$$\omega^2 = -1 + \bar{u}^2 + a\bar{u}'^2 + a\overline{uu''} \tag{7}$$

The location point is chosen as

$$\bar{u} = \frac{\sqrt{3}}{2} A \tag{8}$$

Its derivatives can be approximately calculated as

$$\bar{u}' = \omega\sqrt{A^2 - \bar{u}^2} = \frac{1}{2} A\omega \tag{9}$$

$$\bar{u}'' = -\omega^2 \bar{u} = -\frac{\sqrt{3}}{2} A\omega^2 \tag{10}$$

So we have

$$\omega^2 = -1 + \frac{3}{4} A^2 + a\frac{1}{4} A^2\omega^2 - a\frac{3}{4} A^2\omega^2 = -1 + \frac{3}{4} A^2 + \frac{1}{2} aA^2\omega^2 \tag{11}$$

After a simple calculation, we have

$$\omega = \sqrt{\frac{-1 + \frac{3}{4} A^2}{1 + \frac{1}{2} aA^2}} \tag{12}$$

which is exactly same as that in Ref. [14]. There are an abundant number of publications on the applications of the original frequency-amplitude formulation and its various modifications. He and Liu [15] gave a rigorous mathematical perspective for He's frequency formulation. Ismail et al. [16] and Hashemi [17] found that this formulation is extremely well-suited for strongly nonlinear oscillators. Their studies have demonstrated its effectiveness in analyzing and understanding the complex periodic properties. Kawser et al. conducted a comparison between the frequency formulation and numerical simulation for the analysis of the jet engine vibration system. They discovered that the former is not only simple to use but also highly effective [18]. Tsaltas [19] concluded that the one-step frequency formulation holds great promise as a new approach for nonlinear oscillators. Additionally, other applications of the frequency formulation are equally captivating [20,21]. These applications span different fields and highlight the versatility and importance of the frequency formulation in understanding and solving a wide range of problems related to nonlinear oscillators.

Although significant achievements have been attained, there is still ample room for further improvement, particularly for nonlinear oscillators with a nonlinear term of the form u^n where $n<1$. This type of oscillator encompasses singular oscillators and zigzag oscillators, which are extremely challenging to solve analytically.

3. Frequency Formulation for Quasi-Linear Oscillator and Singular Oscillators

The aforementioned frequency formulation is applicable to $n>1$. It is evident that:

$$\bar{u}_{2n+1} > A, \text{for } n < 1/2\ . \tag{13}$$

However, this is incompatible with the characteristics of a conservative oscillator, which exhibits vibrations from $u=A$ to $u=-A$. There exists a large set of nonlinear oscillators with an irrational term [22] or a term of the form $u^{1/(2n+1)}$ [23]. Though there were some effective modifications of the frequency formulation [24, 25], here a more effective formulation will be recommended.

The following nonlinear oscillator is now considered.

$$u'' + \sum_{i=1}^{N} b_i u^{n_i} = 0, n_i < 2 \tag{14}$$

where b_i (i=1~N) are constants. For periodic solutions, it requires

$$\sum_{i=1}^{N} b_i u^{(n_i-1)} > 0 \tag{15}$$

The previous approaches to this type of nonlinear oscillators are some famous analytical methods, e.g., the variational iteration method [26] and the homotopy perturbation method [27, 28], all these methods require sophisticated technology with complex calculation, however, in the practical applications, the frequency-amplitude relationship is the most important factor. A simple yet reliable approach is always highly welcome. Now, this paper presents the following frequency formulation:

$$\omega^2 = \sum_{i=1}^{N} \frac{6}{n_i+5} b_i A^{(n_i-1)} \tag{16}$$

It can be also written in the form

$$\omega^2 = \sum_{i=1}^{N} b_i (\bar{u}_{n_i})^{(n_i-1)} \tag{17}$$

where the location point for each nonlinear term is given as

$$\bar{u}_{n_i} = (\frac{6}{n_i+5})^{1/(n_i-1)} A \tag{18}$$

The new frequency formulation is extremely simple. A more straightforward approach is always advantageous for engineers. The frequency formulation demonstrates considerable promise for a diverse array of prospective applications. This novel approach has the potential to transform analytical techniques within the domain of vibration theory. The method provides a rapid yet reliable comprehension of intricate vibration systems through a straightforward step that is more efficient than traditional analytical methods, which entail complex and detailed calculations. It can be utilized for the optimization of complex vibration systems without the necessity for onerous computations. The remarkable versatility of this new formulation represents a significant advancement in the field of nonlinear vibration theory. It is adaptable to a variety of situations and requirements, thereby facilitating the realization of novel possibilities and opportunities. As research and development continue, the potential applications of this approach are likely to expand, with the prospect of substantial benefits for the field of nonlinear science. The following examples demonstrate the efficacy and simplicity of this approach.

4. EXAMPLES

4.1. Example 1. Nonlinear Oscillator with Fractional Potential

Nonlinear oscillators with fractional potential are also called as quasi-linear oscillators, the nonlinear term is always in the form, $u^{1/(2n+1)}$, where n is a natural number. In this example, we consider the following oscillator.

$$u'' + \beta u^{1/3} = 0 \tag{19}$$

where β is a positive constant. Its variational formulation is

$$J(u) = \int \left\{ \frac{1}{2} u'^2 - \frac{3}{4} \beta u^{4/3} \right\} dt \tag{20}$$

The potential is represented by a fractional exponent, and thus Eq. (20) is designated as a nonlinear oscillator with fractional potential. This nonlinear oscillator has been the subject of extensive attention in the open literature, as evidenced by references [29, 30]. The nonlinear nature of the system presents a considerable challenge in attempting to find a general closed-form solution through analytical means. The nonlinear characteristics introduce intricate interactions and behaviors that render the derivation of an approximate solution an exceptionally challenging endeavor.

Following meticulous and comprehensive calculations, a notable accomplishment is the derivation of the frequency-amplitude relationship for Eq. (20), as documented in [29, 30], which is:

$$\omega = 1.0768 A^{-1/3} . \tag{21}$$

This relationship, with a relative error of 0.59%, provides a crucial link between two fundamental properties of the oscillator and serves as a valuable resource for further studies and applications. This provides a basis for analyzing and predicting the behavior of the oscillator in different scenarios, thereby opening up new avenues for research and practical applications in a variety of fields.

Notwithstanding the significant advancements achieved through meticulous and intricate computations, engineers have been keenly awaiting a straightforward yet efficacious methodology to swiftly and dependably comprehend the vibration characteristics. The time has now come to seize this opportunity. By employing the frequency formulation of Equation (17), the desired results can be obtained with immediate effect.

$$\omega^2 = \frac{6}{\frac{1}{3} + 5} \beta A^{-2/3} = \frac{18}{16} \beta A^{-2/3} \tag{22}$$

or

$$\omega = 1.060660 \beta^{1/2} A^{-1/3} \tag{23}$$

It is noteworthy that it exhibits the same form as that of Eq. (22). Given its simplicity, the frequency formulation represents a significant accomplishment in the field.

The accuracy of this method is comparable to that attained by the variational iteration method or the homotopy perturbation method. The one-step approach is a highly attractive proposition. The simplicity and efficiency of this approach render it an attractive option, particularly in situations where rapid insights and the frequency-amplitude relationship are required without the intricacy of more elaborate methods.

The exact frequency is [29, 30]

$$\omega_{exact} = 1.070451\beta^{1/2}A^{-1/3} \tag{24}$$

The relative error is 0.91%, which is deemed to be within an acceptable range for engineering applications.

4.2. Example 2. Discontinuous Oscillator with Sign Function

The study of nonlinear oscillators with discontinuous terms represents a fascinating and rich area of inquiry within the broader fields of physics and mechanical engineering. Discontinuous terms may originate from impacts, switches, or sudden alterations in system parameters. The aforementioned discontinuities present considerable difficulties in the analysis and comprehension of the oscillator's behavior. For example, a nonlinear term with the sign function frequently plays a pivotal role in such systems, as it can represent abrupt changes in direction or state. The study of nonlinear oscillators with discontinuous terms represents a vibrant and multifaceted research domain, with implications that extend far beyond the immediate field of inquiry and have the potential to inform a vast array of practical applications. An investigation of the frequency-amplitude relationship may facilitate the discovery of new possibilities for engineering and scientific advancement. Researchers are engaged in ongoing efforts to develop new methods for the analysis and control of these oscillators. This includes the creation of novel theoretical models and experimental techniques, with the aim of gaining deeper insights into their behavior and the design of more efficient and reliable systems.

In this example, we consider a nonlinear oscillator subjected to a constant force, as illustrated in Fig. 1.

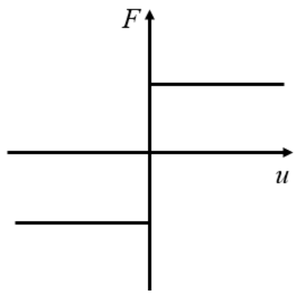

Fig. 1 Discontinuous oscillator with discontinuous restoring force

The restoring force follows: when u>0, $F(u)$=1; when u<0, $F(u)$=−1. The governing equation can be expressed as

$$u'' + \beta\,\mathrm{sgn}(u) = 0 \tag{25}$$

where $\beta=1/m$, m is the mass of the oscillator, and sgn(u) is defined as

$$\mathrm{sgn}(u) = \begin{cases} u^0, u > 0 \\ 0, u = 0 \\ -u^0, u < 0 \end{cases} = \begin{cases} 1, u > 0 \\ 0, u = 0 \\ -1, u < 0 \end{cases}. \tag{26}$$

Considerable effort has been dedicated to seeking an accurate analytical approximation for Equation (26), as referenced in [31-33]. The primary mathematical tool utilized is the homotopy perturbation method. In the application of this method to nonlinear vibration systems, the construction of the homotopy equation, the selection of the initial guess, and the elimination of secular terms during the lengthy calculation process represent the primary focal points [34]. These aspects are of paramount importance in guaranteeing the efficacy and precision of the method in analyzing and approximating nonlinear vibration phenomena.

After a complex calculation, Belendez et al. [30] obtained high-accuracy analytical approximations by the homotopy perturbation method, resulting in the following frequencies

$$\omega_1 = 1.128379 A^{-1/2}, \tag{27}$$

and

$$\omega_2 = 1.107452 A^{-1/2}. \tag{28}$$

The relative errors are 1.59% and 0.29%, respectively. Liu obtained the same result of Eq. (28) by the modified Lindstedt–Poincaré method [33]. Although the achievement has been attained, engineers are currently seeking a straightforward yet impactful methodology. By employing the frequency formulation, we have promptly derived the following outcome.

$$\omega^2 = \frac{6}{0+5}\beta A^{-1} = \frac{6}{5}\beta A^{-1}, \tag{29}$$

or

$$\omega = 1.0954\beta^{1/2} A^{-1/2}. \tag{30}$$

This result is comparable to those produced by the homotopy perturbation method [31] and the modified Lindstedt–Poincaré method [33]. The exact frequency is:

$$\omega_{ex} = 1.1107 A^{-1/2}. \tag{31}$$

The relative error is 1.37%, which is comparable to the results presented in Refs. [37, 39], where a length calculation and specialized skills are required. In contrast, the present approach is notable for its simplicity. In nonlinear theoretical analysis, a minimal calculation with one or two lines can have maximal influence in the academic community, as Ludwig Mies van der Rohe (1886–1969) said, "Less is more".

4.3. Example 3. Discontinuous Oscillator with Absolute Value Symbols

The study of discontinuous oscillators with absolute value symbols represents a fascinating area of inquiry within the field of nonlinear dynamics. The introduction of absolute value symbols can result in the emergence of intricate dynamics, which in turn presents significant analytical challenges.

Researchers employ a range of analytical techniques to examine these oscillators, undertaking significant efforts in this regard. Such techniques may include numerical simulations and analytical approximations. In this example, we consider a discontinuous oscillator with absolute value terms, as illustrated in Fig. 2.

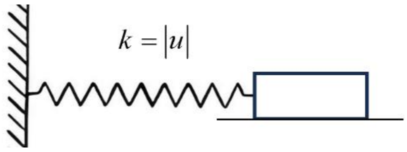

Fig. 2 Schematic of a discontinuous oscillator with absolute value terms

It is assumed that the equivalent elastic coefficient is proportional to the magnitude of displacement, and the oscillator can be expressed as:

$$u'' + |u|u = 0 . \tag{32}$$

By the frequency formulation, we have:

$$\omega^2 = \frac{6}{2+5}A = \frac{6}{7}A , \tag{33}$$

or

$$\omega = 0.92582\sqrt{A} , \tag{34}$$

while the exact one is [35]:

$$\omega_{exact} = 0.914681\sqrt{A} . \tag{35}$$

The relative error is 1.21%, which is an acceptable level of precision given the simplicity of the method employed.

Wang and He [36] achieved a notable result through a combination of sophisticated techniques and meticulous calculations, which is

$$\omega = \sqrt{\frac{8A}{3\pi}} = 0.9214\sqrt{A} . \tag{36}$$

This achievement was of tremendous significance because, prior to this, there was no analytical method that could effectively solve Eq. (33). Reference [36] made significant advancements in nonlinear science and nonlinear vibration theory. It provides an excellent illustration of the efficacy of meticulous analysis and innovative thinking in addressing complex problems and makes a valuable contribution to the expanding body of knowledge in the relevant discipline.

The current approach allows for a straightforward and effective solution, demonstrating the value of simplicity in achieving comparable outcomes. The new frequency formulation provides a novel avenue for future research in nonlinear vibration theory and nonlinear studies. The optimal approach is to employ a method that is both straightforward and efficacious.

4.4. Example 4. Singular Oscillator

A singular oscillator represents a distinctive category of dynamical systems that display a range of intriguing and frequently intricate behaviors. These oscillators possess singularities, which give rise to distinctive characteristics that are not observed in traditional oscillators.

The study of singular oscillators is of significant importance in a number of fields, including physics, engineering, and mathematics. It facilitates comprehension of complex systems and may yield novel insights and applications. Researchers are engaged in ongoing investigations into the properties and behaviors of singular oscillators, with the objective of unlocking their potential for innovative technologies and scientific discoveries [37]. The following example considers a singular oscillator:

$$u'' + \beta u^{-1} = 0 \tag{37}$$

According to the frequency formulation , we have:

$$\omega^2 = \frac{6}{-1+5}\beta A^{-2} = 1.5\beta A^{-2}, \tag{38}$$

or

$$\omega = 1.2247\beta^{1/2} A^{-1}. \tag{39}$$

The exact one is [37]:

$$\omega_{\text{exact}} = \frac{1.2533}{A}. \tag{40}$$

The relative error is 2.28 %. The singularity presents a challenge for both analytical and numerical methods. However, the one-step frequency formulation offers a solution that produces optimal results with minimal calculation.

4.5. Example 5. Nonlinear Oscillator with Fraction Term

The study of nonlinear oscillators with fraction terms represents a fascinating and promising area of mathematical inquiry. It presents a challenge to traditional analytical methods, as it may not conform to the standard mathematical models that are typically employed. Researchers employ sophisticated techniques to comprehend and anticipate the behavior of these oscillators.

Consider the following nonlinear oscillator with a fraction term:

$$u'' + \frac{u}{c_0 + \sum_{n=1}^{N} c_{2n} u^{2n}} = 0. \tag{41}$$

In order to utilize the aforementioned frequency formulation, we assume that A tends to infinity. Under this assumption, Eq. (41) becomes:

$$u'' + \sum_{n=1}^{N} c_{2n} u^{1-2n} = 0. \tag{42}$$

Its frequency-amplitude relationship is

$$\omega^2 = \sum_{n=1}^{N} \frac{6}{1-2n+5} b_{2n} A^{-2n} = \sum_{n=1}^{N} \frac{3}{3-n} b_{2n} A^{-2n} \tag{43}$$

For Eq. (42) the frequency formulation should be modified as:

$$\omega^2 = \frac{1}{c_0 + \sum_{n=1}^{N} \frac{3-n}{3} b_{2n} A^{2n}}. \tag{44}$$

In this modification, the case of $c_0=0$ and $N=1$ can be transformed into the form of Eq. (44). So it is suitable for large amplitude oscillation.
In order to show its effectiveness, we consider a simple case:

$$u'' + \frac{u}{1+u^2} = 0 . \tag{45}$$

By the frequency formulation of Eq. (45), the following result is obtained:

$$\omega = \frac{1}{\sqrt{1+\frac{2}{3}A^2}} . \tag{46}$$

The exact period can be calculated as:

$$T = 4\int_0^A \frac{1}{\sqrt{\ln(1+A^2)-\ln(1+u^2)}}\,du . \tag{47}$$

For the case A tends to infinity, we have:

$$\begin{aligned}\lim_{A\to\infty} T &= 4\lim_{A\to\infty}\int_0^A \frac{1}{\sqrt{\ln(1+A^2)-\ln(1+u^2)}}\,du \\ &= 4\int_0^A \frac{1}{\sqrt{\ln(A^2)-\ln(u^2)}}\,du = 2\sqrt{2\pi}A\end{aligned} \tag{48}$$

So we have:

$$\lim_{A\to\infty} \omega_{exact} = \lim_{A\to\infty} \frac{2\pi}{T} = \sqrt{\frac{\pi}{2}}A^{-1} = 1.2533A^{-1} , \tag{49}$$

while from Eq. (28), we have:

$$\lim_{A\to\infty} \omega = \frac{1}{\sqrt{\frac{2}{3}A^2}} = 1.2247A^{-1} . \tag{50}$$

The relative error is 2.28%. It should be note that Eq. (48) is valid for large amplitude. For the small values of A, Eq. (46) can be approximated as:

$$u'' + u - u^3 = 0 . \tag{51}$$

By the frequency formulation given in Eq. (2), we have:

$$\omega = \sqrt{1-\frac{3}{2}A^2} = 1-\frac{3}{4}A^2 + O(A^2) . \tag{52}$$

Considering Eqs. (48) and (49), we recommend the following matching one:

$$\omega = \frac{1}{\sqrt{1+\frac{2m+3(1-m)}{3m+2(1-m)}A^2}} , \tag{53}$$

where m is the weighting factor, m=0 for A<1 and m=1 for A>1, and the value of m can be determined by matching Eq. (53) with the exact one when A=1.

Now this paper challenges the following nonlinear oscillators:

$$u'' + u + b_1 u^{1/3} + b_2 u^{1/5} + b_3 \operatorname{sgn}(u) + b_4 u^3 + b_5 u^5 = 0, \tag{54}$$

with arbitrary initial conditions:

$$u(0) = \alpha, u'(0) = \beta. \tag{55}$$

and this paper gives the following result without proof:

$$\begin{aligned}\omega^2 &= 1 + \frac{6}{\frac{1}{3}+5} b_1 A^{-2/3} + \frac{6}{\frac{1}{5}+5} b_2 u^{-4/5} + \frac{6}{0+5} b_3 A^{-1} + \frac{3}{2+2} b_4 A^2 + \frac{3}{2+4} b_5 A^4 \\ &= 1 + \frac{9}{8} b_1 A^{-2/3} + \frac{15}{13} b_2 u^{-4/5} + \frac{6}{5} b_3 A^{-1} + \frac{3}{4} b_4 A^2 + \frac{1}{2} b_5 A^4\end{aligned} \tag{56}$$

and

$$\alpha^2 + \frac{\beta^2}{\omega^2} = \mathrm{A}^2. \tag{57}$$

5. Discussion and Conclusion

Dozens of research groups around the world are engaged in the development of analytical methods for the study of the frequency-amplitude relationship of complex nonlinear oscillators [38].

The nonlinear vibration theory is cheaper and greener than other artificial methods, including sophisticated experiments, precise measurements, and time-consuming simulations. The nonlinear vibration theory has the potential to enhance the efficiency of energy harvester devices or to facilitate the conservation of mechanical vibrations, thereby reducing the emission of greenhouse gases. Furthermore, the frequency formulation can assist in the identification and search for optimal parameters for the devices currently under development.

To date, a number of analytical methods have been proposed for this purpose, with the homotopy perturbation method and the variational iteration method representing the most commonly used mathematical tools. The frequency formulation represents a particularly simple method.

This paper identifies the most fascinating aspect of the frequency formulation: its ability to link two primary factors that control periodic motion in a complex oscillator. The aforementioned examples illustrate the efficacy of the straightforward yet impactful frequency formulation in the context of scientific and technological advancement. The new frequency formulation presented in this paper represents a continuous process of improvement and expansion of its applications.

The discovery constituted a profound breakthrough that astonished the scientific community. While the effectiveness is illustrated by examples, it remains in its nascent stages, and there are numerous challenges and prospects for future research.

A rigorous mathematical proof of the frequency formulation is required to confirm its effectiveness. The formulation may encounter counter-examples in practical applications, which could lead to further technological developments. The innovation represents a significant advancement in technology and has the potential to transform existing practices.

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