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VARIATIONAL APPROACH TO MICRO-ELECTRO-MECHANICAL SYSTEMS

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Abstract. *This paper presents a novel approach for formulating a variational principle tailored to microelectromechanical systems (MEMS) through the utilization of the semi-inverse method. The resulting variational principle is that of least action, which is of considerable significance. The newly presented variational principle has the potential to be applied in a number of advantageous ways. One of the primary applications of this approach is the determination of the pull-in voltage. The application of this principle allows for a more accurate and efficient determination of the pull-in voltage. This is of paramount importance for the optimal functioning and optimization of MEMS devices. The enhanced accuracy in determining the pull-in voltage enables more precise design and greater reliability of MEMS-based systems. Additionally, the enhanced computational efficiency allows for the saving of valuable time and resources during the design process. Furthermore, the paper addresses the topic of fractal MEMS and puts forth a novel approach to fractional differentiation based on two-scale fractal differentiation, which is anticipated to facilitate the discovery of new insights and optimization strategies for MEMS devices.*

Key words: *Semi-inverse method, Pull-in instability, Fractal MEMS, Two-scale fractal derivative based fractional derivative, Wearable textiles*

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1. INTRODUCTION

Micro-Electro-Mechanical Systems (MEMS) have been instrumental in the modern technological landscape, primarily due to their ability to facilitate the development of miniature integrated devices that effectively combine mechanical and electrical functionalities [1]. Their applications are found in critical fields, such as wearable textiles [2]. MEMS sensors have been developed for the monitoring of physiological parameters, including heart rate and body temperature, with exceptional precision. Flexible actuators have been incorporated into adaptive clothing, thereby empowering it to respond dynamically to environmental stimuli. In the domain of biomedical engineering, MEMS devices play a pivotal role in facilitating targeted drug delivery and real-time diagnostics. These devices leverage their microscale size to interact seamlessly with biological systems, a property that has been demonstrated to support inner ear theragnostics [4]. Furthermore, MEMS oscillators function as essential components in communication systems, providing stable timing signals that are crucial for 6G and subsequent generations [5].

Roy et al. [6] developed a generative AI-assisted piezo-MEMS ultrasound device tailored for plant dehydration monitoring. This device merges piezo-MEMS technology with generative AI to enable efficient tracking of plant water status. Liang and Lin [7] introduced a MEMS electrothermal actuator (ETA) featuring highly linear actuation and stable characteristics. The actuator under consideration consists of a bilayer cantilever array of Au and Si_3N_4, and it is compatible with standard CMOS manufacturing processes. The device's surface roughness engineering prevents stiction, exhibits exceptional linearity between displacement and squared driving voltage, achieves an ultra-low power consumption of 7.5 mW, and maintains stable performance over 100 cycles. Potential applications include optical modulation and medical imaging, among others.

These diverse applications necessitate accurate models to predict dynamic behaviors, particularly pull-in instability – a phenomenon where electrostatic forces surpass mechanical restoring forces, leading to device failure and compromising operational reliability [8].

The variational principle has long served as a foundational tenet within the realm of theoretical physics, offering a unifying framework for the analysis of system dynamics through the minimization of energy. From Hamilton's principle, which governs classical mechanics, to its extensions in field theory and quantum mechanics, variational methods integrate conservation laws and dynamic constraints into a single formulation [9]. In the field of MEMS research, early applications of variational principles centered on linear oscillations and elementary geometries [10], drawing on techniques such as the variational iteration method [11] and finite element analysis [12]. However, these approaches encounter challenges due to nonlinear electrostatic-mechanical coupling, microscale effects (e.g., surface tension), and complex fractal geometries. These limitations hinder the accuracy of predictions of critical parameters like pull-in voltage [13, 14]. Concurrently, Niu has substantiated the robust minimum condition of a variational formulation applicable to MEMS systems [15].

Against this backdrop, the need for a tailored variational principle for MEMS becomes evident [16]. Traditional models fail to capture singularities in displacement fields and overlook fractal-induced dynamics, leading to design inefficiencies. This paper addresses these gaps by developing a novel variational formulation via the semi-inverse method, with applications to pull-in instability analysis and fractal MEMS modeling.

2. MEMS OSCILLATOR

Following the notable developments of the Duffing oscillator and van der Pol oscillator, the MEMS oscillator is identified as the subsequent most crucial component in the domain of nanotechnology. A MEMS oscillator represents a revolutionary timing device that has gained significant prominence within the field of electronics. It employs microelectromechanical systems technology to facilitate precise and reliable oscillatory behavior. The micromachining technology that emerged in the late 1980s enables the fabrication of sensors and actuators with a micron-scale resolution. Such micro transducers can be integrated with signal conditioning and processing circuitry to form micro-electromechanical-systems, which are capable of performing real-time distributed control. The system operates periodically in general, but the pseudo-pull-in stability is a subject of research study [17, 18].

The dimensionless MEMS oscillator can be written as [17]:

$$w'' + w - \frac{m}{1-w} = 0 \tag{1}$$

where w is the dimensionless displacement, $0<w<1$, and m is voltage-related parameter.

This equation arises from the dynamic balance between mechanical restoring forces and electrostatic forces in a parallel-plate MEMS structure. When m is small, Eq. (1) has periodic solution, however, when m is larger than a threshold value, m^*, the pull-in motion occurs, and m^* is often called as the pull-in voltage. This equation possesses distinctive characteristics, namely: 1) zero initial conditions, i.e., $w(0)=0$ and $w'(0)=0$; and 2) the singularity at $w=1$. Nevertheless, it can be effectively solved by the homotopy perturbation method [19, 20, 21], the variational iteration method [22, 23], and He-Liu's modified frequency formulation [24], albeit with the aid of certain special techniques. The variational principle in MEMS systems constitutes a fundamental concept that facilitates comprehension and analysis of the behavior of these complex systems (see Ref. [10]).

To explain Eq. (1) in a mechanical view, we consider the motion for a particle with unit mass in a potential field V:

$$V(w) = \frac{1}{2}w^2 + m\ln(1-w) \tag{2}$$

The variational principle of MEMS oscillator given in Eq. (1) can be written in the form

$$J(w) = \int \left\{ \frac{1}{2}w'^2 - V(w) \right\} dt = \int \left\{ \frac{1}{2}w'^2 - \frac{1}{2}w^2 - m\ln(1-w) \right\} dt \tag{3}$$

According to the variational formulation given in Eq. (3), the following identity

$$\frac{1}{2}w'^2 + \frac{1}{2}w^2 + m\ln(1-w) = H \tag{4}$$

where H is the Hamilton invariant, it can be calculated by the zero initial conditions, and it results in $H=0$, so we have

$$\frac{1}{2}w'^2 + \frac{1}{2}w^2 + m\ln(1-w) = 0 \tag{5}$$

We introduce a new variable, *K*, defined as:

$$K = \frac{1}{2} w'^2 . \tag{6}$$

The new variable is the kinetic energy, from Eq. (5), we have:

$$K = -\frac{1}{2} w^2 - m \ln(1-w) . \tag{7}$$

For small displacement, Eq. (7) can be approximately expressed as

$$K = -\frac{1}{2} w^2 - m(-w - \frac{1}{2} w^2) = -\frac{1}{2} w^2 + mw + \frac{1}{2} mw^2 , \tag{8}$$

or

$$w = \frac{2m + \sqrt{4m^2 - 8K(1-m)}}{2(1-m)} . \tag{9}$$

Eq. (9) implies that the displacement (*w*) is a function of *m* and *K*.

3. Criterion for Pull-In Instability

Pull-in instability in MEMS refers to a phenomenon where an electrostatically actuated MEMS device collapses or snaps to a new stable state. This occurs due to the interaction between electrostatic forces and mechanical restoring forces. As the applied voltage increases, the electrostatic force overcomes the mechanical restoring force, leading to instability. It can cause failure in MEMS devices and is an important consideration in their design. Understanding and predicting pull-in instability is crucial for ensuring the reliable operation of MEMS.

The criterion for pull-in instability in MEMS is a key aspect in understanding and designing these devices. As mentioned earlier, the criterion for pull-in instability is that when the kinetic energy and the accelerated speed become zero simultaneously [16]. This means that at this point, the device is in a state of unstable equilibrium and is likely to collapse or snap to a new stable state. This criterion provides a quantitative measure for engineers to determine when pull-in instability may occur. For example, in the design of a MEMS actuator, engineers can use this criterion to calculate the maximum allowable voltage that can be applied before pull-in instability occurs. By understanding this criterion, engineers can design MEMS devices that are more stable and reliable, and avoid potential failures due to pull-in instability.

The criterion for pull-in instability is that when the kinetic energy and the accelerated speed become zero simultaneously. According to Eq. (1) and Eq. (7), we have:

$$w - \frac{m}{1-w} = 0 \tag{10}$$

$$-\frac{1}{2}w^2 - m\ln(1-w) = 0 \tag{11}$$

Solving Eqs. (10) and (11) simultaneously, we find the threshold values of the displacement and the voltage,

$$\begin{cases} w^* = 0.7153 \\ m^* = 0.2036 \end{cases} \tag{12}$$

That implies that in the event that the value of w is greater than w^* or the value of m is greater than m^*, the phenomenon of pull-in instability takes place, w^* is called as the pull-in displacement, and m^* is the pull-in voltage. This criterion for pull-in instability is highly accurate and extremely reliable. The accurate determination of the pull-in voltage provides a valuable benchmark for researchers and engineers working in the field of electrostatically actuated MEMS/NEMS. It allows for a more precise understanding and prediction of the occurrence of pull-in instability, enabling better design and optimization of these micro and nano devices. With such a highly accurate criterion, it becomes possible to make more informed decisions regarding the design parameters and operating conditions to avoid or manage pull-in instability effectively.

4. Semi-Inverse Method and a New Variational Formulation for MEMS Oscillator

The semi-inverse method [25] is a powerful approach for establishing a variational principle. It offers a systematic way to solve complex problems in various fields, including mechanics and engineering. Though the semi-inverse method, many new variational formulations were appeared in literature for incompressible fluids [26], lubrication problems [27], fractal solitary waves [28] and the simplified Navier-Stokes equations [29].

We re-write Eq. (1) in the following form:

$$w'' + w - w''w - w^2 - m = 0 \tag{13}$$

The semi-inverse method commences with the formulation of an informed hypothesis regarding the form of the variational formulation. Based on physical intuition or prior knowledge, a trial functional is selected that contains some unknown parameters. Subsequently, the variational problem is formulated by applying specific constraints and principles.

The fundamental concept is to partially invert the problem by selecting a trial functional that satisfies some of the essential properties of the solution. Subsequently, the functional is varied with respect to the unknown parameters, thereby obtaining the Euler-Lagrange equations.

The semi-inverse method is advantageous in that it reduces the complexity of the problem by narrowing down the search space. It offers a methodology for the acquisition of approximate solutions when the identification of exact solutions is challenging. Moreover, it is capable of handling nonlinear problems and can be extended to multi-dimensional and time-dependent systems.

In accordance with the semi-inverse method, the following assumption is made:

$$J(w,K)=\int\left\{\frac{1}{2}w'^2-\frac{1}{2}w^2+\frac{1}{4}w''w^2+\frac{1}{3}w^3+mw+F\right\}dt \tag{14}$$

where F is unknown yet.

Equation (14) represents a trial variational principle in our study. In this context, the independent variables play a crucial role in defining the problem. The kinetic energy (K) is considered an independent variable as it reflects the main dynamical property of the MEMS oscillator. When $K = 0$, the pull-in motion occurs. Additionally, the displacement (w) is another independent variable. When w=1, the pull-in instability occurs. Thus, K and w are the main parameters for studying the pull-in property.

The variational principle allows us to understand the significance of these variables and how they interact. A comprehensive understanding of these independent variables is essential for a detailed analysis of the variational principle and its applications in our study.

The Euler-Lagrange equation with respect to w is:

$$-w''-w+\frac{1}{2}w''w+\frac{1}{4}(w^2)''+w^2+m+\frac{\delta F}{\delta w}=0\ , \tag{15}$$

or

$$-w''-w+w''w+\frac{1}{2}w'^2+w^2+m+\frac{\delta F}{\delta w}=0\ , \tag{16}$$

where $\delta F/\delta w$ is the fractional derivative [30], defined as:

$$\frac{\delta F}{\delta w}=\frac{\partial F}{\partial w}-\frac{d}{dt}\left(\frac{\partial F}{\partial w'}\right)+\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial w''}\right)\ . \tag{17}$$

In view of Eqs. (13) and (6), we have:

$$\frac{\delta F}{\delta w}=w''+w-w''w-\frac{1}{2}w'^2-w^2-m=-\frac{1}{2}w'^2=-K\ . \tag{18}$$

From Eq. (18), F can be identified as:

$$F=-Kw+F_1\ , \tag{19}$$

where F_1 is an unknown function of K. So the trial-functional can be updated as:

$$J(w,K)=\int\left\{\frac{1}{2}w'^2-\frac{1}{2}w^2+\frac{1}{4}w''w^2+\frac{1}{3}w^3+mw-Kw+F_1\right\}dt \tag{20}$$

The Euler-Lagrange equation with respect to K is:

$$-w+\frac{\delta F_1}{\delta K}=0\ . \tag{21}$$

That is:

$$\frac{\delta F_1}{\delta K} = w = \frac{2m + \sqrt{4m^2 - 8K(1-m)}}{2(1-m)} . \tag{22}$$

Now F_1 can be identified as:

$$F_1 = \frac{2mK}{2(1-m)} - \frac{1}{24(1-m)^2}\left[4m^2 - 8K(1-m)\right]^{3/2} . \tag{23}$$

Finally, the following variational formulation is obtained:

$$\begin{aligned} J(w,K) &= \int \left\{ \frac{1}{2}w'^2 - \frac{1}{2}w^2 + \frac{1}{4}w''w^2 + \frac{1}{3}w^3 + mw - Kw \right\} dt \\ &+ \int \left\{ \frac{2mK}{2(1-m)} - \frac{1}{24(1-m)^2}\left[4m^2 - 8K(1-m)\right]^{3/2} \right\} dt \end{aligned} \tag{24}$$

The new obtained variational formulation given in Eq. (24) gives deeper insight into the physical property of the MEMS oscillator in an energy sense, but it is valid for the small displacement. When w is not small enough, the derivation of the variational principle is much more complex, and we obtain the following variational formulation

$$\begin{aligned} J(w,K) &= \int \left\{ \frac{1}{2}w'^2 - \frac{1}{2}w^2 + \frac{1}{4}w''w^2 + \frac{1}{3}w^3 + mw \right\} dt \\ &- \int \left[\frac{1}{6}w^3 + mw\ln(1-w) - mw - m\ln(1-w) \right] dt \\ &+ \int \left\{ a\left[K + \frac{1}{2}w^2 + m\ln(1-w) \right]^3 + b(K - \frac{1}{2}w'^2)^3 \right\} dt \end{aligned} \tag{25}$$

where a and b are nonzero constants.

Now we use the established variational formulation to investigate the pull-in voltage. Before pull-in inability, the kinetic energy and accelerated velocity tend to zero, under this extreme case, we have:

$$J(w) = \int \left\{ -\frac{1}{2}w^2 + \frac{1}{3}w^3 + mw \right\} dt . \tag{26}$$

The Euler-Lagrange equation becomes:

$$-w + w^2 + m = 0 \tag{27}$$

Solving m from Eq. (27), and submitting the result into Eq. (7), we have:

$$\frac{1}{2}w^2 + (w - w^2)\ln(1-w) = 0 \tag{28}$$

This equation can be solved by Newton iteration method, here introduces an ancient Chinese algorithm [31, 32]. We assume two trial solutions w_1=0.7 and w_2=0.75, Eq. (28)

produces the residuals R_1= -0.007834 and R_2=0.06249. The ancient Chinese algorithm results in the following approximate solution:

$$w = \frac{R_1 w_2 - R_2 w_1}{R_1 - R_2} = 0.71 . \tag{29}$$

We choose w_3=0.71, and the residual is R_3=-0.001737, which leads to:

$$w = \frac{R_1 w_3 - R_3 w_1}{R_1 - R_3} = 0.7128 . \tag{30}$$

The pull-in voltage can be calculated as:

$$m^* = w - w^2 = 0.2045 . \tag{31}$$

The precise value is 0.203632188, as demonstrated in Fig. 1, which illustrates the phase-space dynamics of the MEMS oscillator by plotting dimensionless velocity against displacement for varying voltage-related parameter.

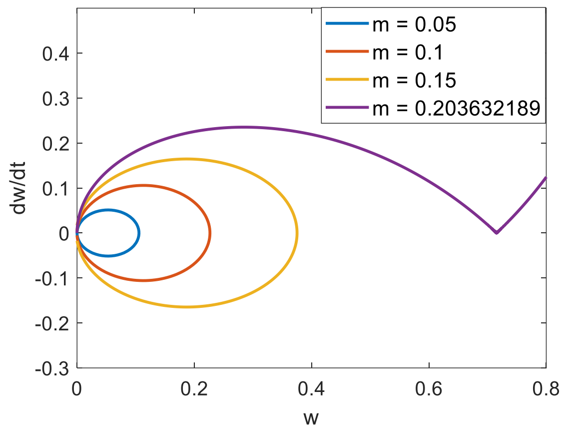

Fig. 1 Phase trajectories for m=0.05, 0.1, 0.15, and 0.203632189

Sub-critical voltage behavior manifests when $m < m^*$. For values of m ranging from 0.05 to 0.15, the trajectories exhibit closed, elliptical loops, thereby directly corresponding to periodic solutions where electrostatic and mechanical forces maintain stable equilibrium. These loops remain constrained within the range of values $0 < w < w^*$, thereby confirming that kinetic energy and acceleration never simultaneously reach zero—a finding that is consistent with the established pull-in criterion.

The critical voltage behavior emerges when the parameter m is equal to m^*. The innermost trajectory, collapsing to a point near $w = w^*$, visualizes the threshold at which the system teeters on instability. The loop contracts towards a singularity, thereby satisfying the pull-in condition: when $m=m^*$, the kinetic energy and acceleration simultaneously approach zero. This phase transition serves to substantiate the variational principle's forecast of a well-defined boundary between stable oscillation and pull-in motion.

Equation (31) demonstrates a notable level of precision. It is noteworthy that the ancient Chinese algorithm attains higher accuracy with a single iteration, underscoring its property of rapid convergence to a substantial level of precision. In addition, the variational principle facilitates precise voltage-controlled manipulation, thereby providing a robust framework for highly accurate voltage regulation. The synergy between the specified pull-in voltage value, the efficacy of the algorithm, and the precision of the variational principle underscores their collective potential for facilitating precise and regulated operations in the domains of MEMS design and optimization.

Fig. 2 demonstrates that when $m < m^*$ or $w < w^*$, the system exhibits periodic solutions. However, as the voltage parameter m increases, the displacement, w, approaches its threshold value, w^*.

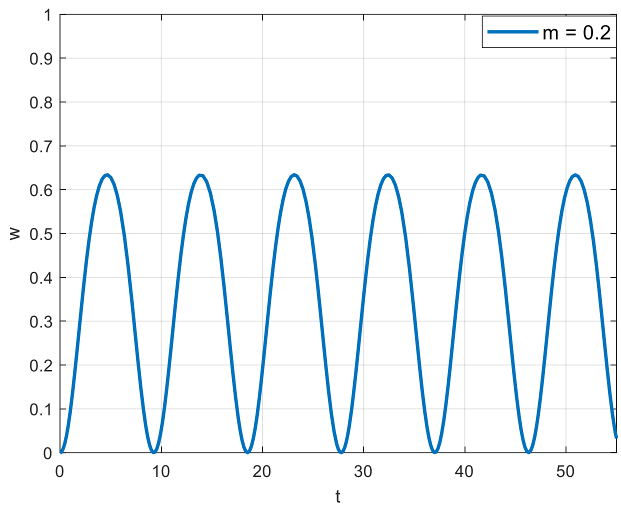

Fig. 2 Periodic solution for $m<m^*$

Once w exceeds w^*, pull-in motion occurs, as shown in Fig. 3. In contrast to the periodic behavior observed in Fig. 2, where the dimensionless displacement oscillates within the bounded range, $0<w<w^*$, under sub-critical voltages ($m<m^*$), the dynamics depicted in Fig. 3 reveal a striking shift when m exceeds the critical threshold, m^*. Here, w abandons all oscillatory characteristics and instead increases monotonically, advancing relentlessly toward the singularity at w=1. This unidirectional trajectory signifies an irreversible pull-in motion, a catastrophic transition driven by the breakdown of the delicate balance between electrostatic and mechanical forces that governs stable operation.

In the sub-critical regime (Fig. 2), mechanical restoring forces—rooted in the elastic properties of the MEMS structure—successfully counteract the electrostatic attraction between electrodes, limiting w to periodic oscillations. However, as m surpasses m^*, the electrostatic force undergoes a rapid amplification. This force exceeds the mechanical restoring capacity of the system, which weakens as the gap between electrodes narrows, as a result, the electrodes make contact, rendering the device inoperable.

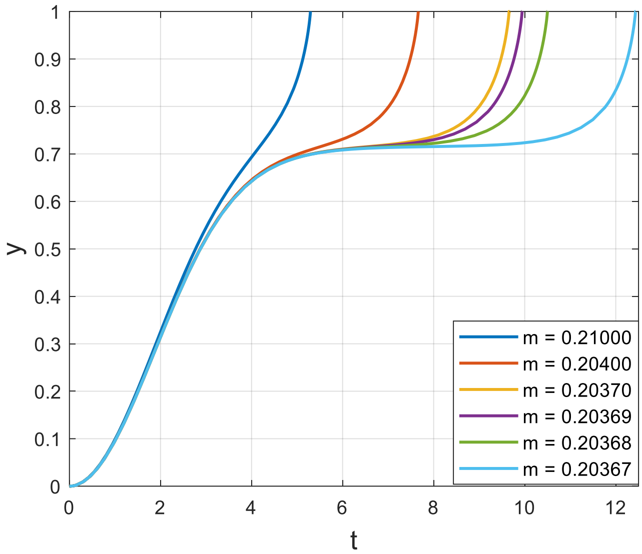

Fig. 3 Pull-in motion for $m>m^*$

Notably, Fig. 3 illustrates that the steepness of the pull-in curve intensifies with increasing m beyond m^*. This phenomenon arises because a higher m corresponds to a stronger initial electrostatic force, which accelerates the narrowing of the electrode gap. As the gap shrinks, the electrostatic force increases further, creating a positive feedback loop that steepens the trajectory. This behavior aligns with the predictions of the variational principle, which emphasizes that beyond m^*, the system's energy balance (governed by Eq. (5) is violated, with the potential energy dominated by electrostatic contributions that drive the irreversible collapse.

This monotonic, gap-closing motion captured in Fig. 3 thus serves as a critical experimental validation of the theoretical pull-in criterion: the simultaneous decay of kinetic energy and acceleration at $w = w^*$ marks the point of no return, beyond which the system is pulled inexorably toward the singularity—a hallmark of pull-in instability in electrostatically actuated MEMS devices.

5. Two-Scale Fractal Derive Based Fractional Derivative and its Application to MEMS System

The core variational principle, Eq. (24), is derived from the classical MEMS oscillator model, which presupposes idealized, smooth geometries. For fractional MEMS featuring fractal structures (e.g., rough surfaces or porous microstructures), the electrostatic-mechanical coupling is inherently modulated by multi-scale geometric irregularities. To address this disparity, the variational functional ought to be extended to incorporate fractal dimensions, thereby explicitly establishing a connection between the semi-inverse method and fractal dynamics.

The two-scale fractal, as a geometric concept, is tailored to describe porous media or unsmooth boundaries characterized by two distinct scales [33]. Within the domain of fractal MEMS systems, the two-scale fractal concept finds several pivotal applications:

1) Modeling Complex Geometries: Fractal MEMS devices often have complex geometries that cannot be accurately described by traditional geometric models. The two-scale fractal geometry provides a more realistic representation of these structures. For example, the surface of a fractal MEMS device may have irregularities and self - similar patterns at different scales, which can be modeled using two-scale fractals. This allows for a better understanding of how the geometry affects the device's physical properties such as capacitance, stiffness, and thermal conductivity.

2) Understanding Physical Processes: The two-scale fractal derivative helps in understanding the physical processes that occur within fractal MEMS devices. It can be used to analyze how mechanical and electrical properties change with respect to the fractal geometry. For instance, in the study of pull - in instability in MEMS, the two-scale fractal derivative can provide a more detailed analysis of how the instability is affected by the fractal structure of the device. This can lead to better design and optimization of MEMS devices to avoid or control the pull - in instability.

3) Novel Device Design: The concept of two-scale fractal can inspire novel designs for MEMS devices. By incorporating two-scale fractal geometries, it may be possible to achieve enhanced performance in terms of sensitivity, selectivity, or power consumption. For example, a fractal MEMS sensor with a two-scale fractal structure may have a higher sensitivity to certain physical quantities compared to a traditional MEMS sensor. This can open up new avenues for the development of advanced MEMS devices for various applications such as sensing, actuation, and signal processing.

The study of MEMS oscillators often requires a comprehensive understanding of their behavior under different geometric and physical conditions. In this regard, the generalization of the governing equations to account for more complex geometries, such as the two-scale fractal, is of great importance. This section focuses on detailing how the Eq. (1) for the MEMS oscillator can be extended to incorporate the two-scale fractal concept, providing a more accurate and detailed description of the oscillator's behavior.

Fractal MEMS systems [34, 35] hold the potential to be a truly revolutionary advancement in the extensive and ever-evolving field of microelectromechanical systems. These remarkable systems draw their inspiration from the captivating concepts such as the gecko effect [36], with its astonishing ability to allow geckos to adhere to various surfaces, and the lotus effect [37], which endows lotus leaves with their remarkable self-cleaning properties. The fractal surface at the nano scale is truly a wonder, as it possesses an extremely high surface energy, known as geometric potential [38, 39], which is widely used to design nano scale materials, demonstrating a creative and forward-thinking approach that could potentially reshape the landscape.

Tian and her colleagues, in their remarkable research efforts as documented in Ref. [40], put forward the highly innovative concept of fractal MEMS systems. This concept is aimed at overcoming the pull-in instability that often plagues such systems. By introducing this concept, they strive to make the systems significantly more reliable. The governing equation becomes:

$$\frac{d^{2D}w}{dt^{2D}}+w-\frac{m}{1-w}=0. \tag{32}$$

In this equation, *D* represents the two-fractal dimensions, which are essential in defining the fractal nature of the geometry. The two-scale fractal derivative is given as follows:

$$\frac{\partial^D w}{\partial t^D}(t_0) = \Gamma(1+D) \lim_{t-t_0 \to \Delta t} \frac{w(x,t) - w(x,t_0)}{(t-t_0)^D}. \tag{33}$$

This definition captures the essence of the fractal derivative by considering the limit of the difference quotient as the time interval approaches zero, weighted by the gamma function of the fractal dimension.

The two-scale fractal derivative has found extensive application in modeling complex problems. As evidenced by numerous studies [41, 42], it has demonstrated its remarkable effectiveness in this regard. This derivative possesses the following properties:

$$\frac{\partial^D w}{\partial t^D}(t_0) \approx \Gamma(1+D)(\Delta t)^{1-D} \frac{\partial w}{\partial t} \tag{34}$$

$$\frac{\partial^{2D} w}{\partial t^{2D}} = \frac{\partial^D}{\partial t^D}\left(\frac{\partial^D w}{\partial t^D}\right) \tag{35}$$

$$\frac{\partial^D w}{\partial t^D} = \begin{cases} w, D = 0 \\ \dfrac{\partial w}{\partial t}, D = 1 \\ \dfrac{\partial^2 w}{\partial t^2}, D = 2 \end{cases} \tag{36}$$

The generalized equation has several important implications for the study of MEMS oscillators. Firstly, it provides a more accurate model for predicting the behavior of MEMS oscillators with fractal geometries. This is particularly useful in applications where the fractal nature of the MEMS structure plays a significant role. For example, in certain sensor and actuator designs, the fractal geometry can enhance the sensitivity and performance of the device. The generalized equation allows us to analyze how the fractal structure affects the oscillator's response to different inputs, such as voltage changes or external perturbations.

Secondly, the generalized equation allows for a deeper understanding of the physical processes involved in the operation of MEMS oscillators. By considering the fractal geometry, we can better analyze the effects of various factors on the oscillator's performance. For instance, the surface roughness of the MEMS device, which is often related to its fractal structure, can influence the dissipation of energy and the damping characteristics of the oscillator. Additionally, the distribution of material properties within the fractal structure can affect the mechanical and electrical properties of the oscillator, such as its stiffness and capacitance. The generalized equation enables us to study these effects in more detail, providing insights into how to optimize the design and performance of MEMS oscillators.

In conclusion, the generalization of Eq. (1) for the MEMS oscillator to the two-scale fractal case is a significant step forward in understanding the behavior of MEMS oscillators in the context of fractal geometry. This generalization not only allows for a more accurate description of the oscillator's behavior but also provides insights into the physical processes involved. Future research can further explore the applications and implications of this generalized equation for the development of more advanced MEMS devices. By continuing to study the behavior of MEMS oscillators under different geometric and physical conditions, we can improve the design and performance of these devices, leading to more efficient and reliable MEMS - based systems.

The fractional MEMS is indeed another highly captivating hot topic in the field of technology. In this domain, the fractional derivative, as demonstrated in references [43], plays an absolutely crucial role. Fractional MEMS systems truly represent a remarkable technological innovation. These systems are an enthralling combination of advanced engineering and profound mathematical concepts. Through the incorporation of fractional calculus principles, they manage to achieve not only enhanced performance but also expanded functionality. The utilization of fractional derivatives enables a more accurate description of the intricate behaviors and dynamic processes exhibited by these miniature mechanical and electrical systems. For instance, in the area of stochastic processes [44] and adaptive chaos control [45], the fractional derivative has shown great potential in providing a deeper understanding and better control of these complex phenomena. As research in this field continues to progress, fractional MEMS systems are expected to bring about even more significant technological breakthroughs and open up new avenues for applications in various industries.

Other definitions of the fractional derivatives can be modified in a similar way, for example, the Caputo-Fabrizio derivative and its generalization [46] can be updated as:

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \frac{1}{1-\alpha}\int_0^t \exp(-\frac{\alpha}{1-\alpha})\frac{\partial^D w}{\partial t^D} ds \tag{37}$$

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \frac{1}{1-\alpha}\int_0^t \frac{k(\alpha, s)}{k(\alpha, t)}\frac{\partial^D w}{\partial t^D} ds \tag{38}$$

We give a generalized definition of the two-scale fractal derivative based fractional derivative in the forms:

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \lambda(\alpha, D)\int_0^t \xi(t, s)\frac{\partial^D w(s)}{\partial s^D} ds \tag{39}$$

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = \lambda(\alpha, D)\frac{\partial^D}{\partial t^D}\int_0^t \xi(t, s) w(s) ds \tag{40}$$

where λ is the normalization function and ξ is the kern function. By choosing suitably the normalization function and the kern function, the above definitions can be converted to a known fractional derivative, e.g., ABC fractional derivative [47, 48]. For example. if we choose:

$$\lambda(\alpha, D) = \frac{\Xi(\omega)}{1-\omega(t)}, \tag{41}$$

$$\xi(t,s) = E_{\omega(t)}\left\{\frac{\omega(t)}{\omega(t)-1}(t-s)\right\}, \tag{42}$$

where $\Xi(\omega)$ is the normalization function, satisfying $\Xi(0)=\Xi(1)=1$, $E_{\omega(t)}$ is the ML function, we can convert Eq. (39) and Eq. (40) to a modified ABC fractional derivative [47, 48].

Here, we introduce the two-scale fractal derivative based fractional derivative. In this context, within the known fractional derivative, the traditional derivatives are replaced by the two-scale fractal derivative. For example, the Caputo fractional derivative and Riemann-Liouville fractional derivative are updated respectively in the following manner:

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \frac{1}{\Gamma(D-\alpha)}\int_0^t (t-s)^{D-\alpha-1}\frac{\partial^D w(s)}{\partial s^D}ds, \tag{43}$$

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \frac{1}{\Gamma(D-\alpha)}\frac{\partial^D}{\partial t^D}\int_0^t (t-s)^{D-\alpha-1} w(s)ds. \tag{44}$$

When D=1, Eq. (43) and Eq. (44) turn back to Caputo fractional derivative and Riemann-Liouville fractional derivative, respectively.

6. CONCLUSIONS

In conclusion, this study has yielded several significant new findings, with particular emphasis on demonstrating the practical utility of the proposed variational principle through a specific MEMS system.

Firstly, the application of the semi-inverse method to establish a new variational principle for MEMS systems has proven to be a groundbreaking advancement. It surpasses certain limitations of traditional variational approaches by offering a more precise portrayal of the physical behavior of MEMS devices, facilitating enhanced prediction of their performance, and presenting improved computational efficiency, which is vital for design and optimization procedures demanding multiple simulations. This underscores the semi-inverse method as a potent tool for handling the intricate nature of MEMS systems.

To illustrate concretely the applicability of this new variational principle, we focused on a specific case: MEMS pressure sensors integrated into wearable textiles [49]. These sensors are critical for real-time physiological monitoring (e.g., heart rate, respiratory rate, and body temperature) due to their miniaturization, flexibility, and low power consumption. However, their reliability is heavily constrained by pull-in instability, which can lead to sudden performance degradation or failure under dynamic physiological conditions.

The proposed variational principle directly addresses this challenge. For the MEMS pressure sensor, we applied the new formulation to model the electrostatic-mechanical coupling behavior, where the sensor's diaphragm displacement, w, and voltage-related parameter, m, are the core variables. Using the derived criterion for pull-in instability (simultaneous zero kinetic energy and accelerated speed), we computed the pull-in voltage with higher precision compared to traditional methods. Specifically, the variational principle enabled us to incorporate microscale effects (e.g., surface tension and fractal surface roughness) that are critical for wearable applications but often overlooked in conventional models.

Furthermore, the two-scale fractal derivative-based fractional modeling, integrated with the variational principle, enhanced the accuracy of predicting the sensor's dynamic response under varying fractal surface geometries, which is common in wearable textiles due to fabric texture. This integration allowed us to quantify how surface roughness (characterized by fractal dimension D) influences pull-in behavior.

Secondly, this specific demonstration validates the potential of the new variational principle for future research. It serves as a foundation for further refinement, potentially leading to more inventive and dependable MEMS designs, particularly for wearable technologies where robustness and miniaturization are paramount. It also opens avenues for deeper investigations into the relationship between variational principles and microscale electromechanical performance.

Finally, the application of MEMS in wearable textiles is reaffirmed as highly promising. The specific MEMS pressure sensor case demonstrates that our variational approach not only improves design efficiency, but also enhances the reliability of wearable systems.

In summary, the proposed variational principle, validated through the MEMS pressure sensor in wearable textiles, advances both theoretical modeling and practical design of MEMS, paving the way for more robust and versatile microscale devices in emerging applications. Specifically, the modeling framework and analytical approach presented herein can be further extended to the study of MEMS graphene resonators [50], where precise characterization of dynamic behaviors and instability thresholds is equally critical for optimizing performance in high-frequency communication and sensing scenarios. This cross-application potential underscores the generalizability of our variational principle in addressing the core challenges of diverse MEMS configurations.

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