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WEDGING OF FRICTIONAL ELASTIC SYSTEMS

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Abstract. *We consider discrete two-dimensional elastic systems with Coulomb friction contacts, and investigate the conditions that must be satisfied if these are to be capable of becoming 'wedged' --- i.e. of remaining with non-zero elastic deformations when all external loads have been removed. The condition for wedging is reduced to the requirement that a prescribed set of constraint vectors should fail to positively span the N-dimensional vector space of nodal displacements. We also show that the range of admissible wedged states increases monotonically with the coefficient of friction f and that there exists a unique critical coefficient* f_w *such that wedging is impossible for* $f < f_w$ *and possible for* $f > f_w$.

Key Words: *Wedging, Coulomb Friction, Positive Span, Contact Mechanics*

1. INTRODUCTION

If a system of contacting elastic bodies with frictional interfaces is subjected to time-varying loads, it can become *wedged*, meaning that it remains in a state of deformation with non-zero contact forces and tangential displacements, even when all external loads have been removed [1]. This concept is illustrated by the simple system of Fig. 1, comprising two rigid blocks, the upper block being supported by a spring of stiffness *k*. If the coefficient of friction between the blocks and/or between the lower block and the supporting plane surface is sufficiently high, the system will remain in the loaded configuration even when force *F* is removed.

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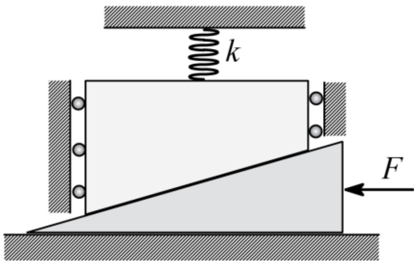

Fig. 1 A simple system susceptible to wedging

Wedging is important in many practical applications. For example, it may lead to incorrect configurations of assembled components in automated processes, such as pin-in-hole assembly [2,3]. But also, systems such as screwed fasteners [4] and hip replacements [5] depend crucially on the occurrence of wedging for their very functionality. Clearly we would like to be able to predict the frictional conditions under which wedging is possible for a given system.

If Coulomb friction is assumed with coefficient of friction f, the simple system of Fig. 1 leads us to expect that there should exist a critical value f_w, such that wedging is possible for $f > f_w$ and is not possible for $f < f_w$. One strategy for finding f_w is to postulate incipient slip throughout the contact area and explore the conditions under which a non-trivial solution exists with zero external loads. If such a solution can be found for a particular coefficient of friction f_0, and if the corresponding normal contact tractions are everywhere compressive, the same deformation pattern would define a wedged state for $f > f_0$ and hence f_0 would define an upper bound for f_w. This formulation leads to an eigenvalue problem for f_0 which was explored in the discrete formulation by Hassani and Hild [6, 7] and in the continuum formulation by Hild [8], principally from the perspective of the implied non-uniqueness of the quasi-static solution in frictional contact problems [9, 10]. A modified algorithm, tailored specifically to the wedging problem, was proposed by Barber and Hild [11], who used a finite element model to incrementally reduce the coefficient of friction from a wedged configuration in an attempt to discover the value of f at which the system finally relaxes back to the undeformed state.

In this paper, we shall explore the conditions for wedging in the restricted class of discrete two-dimensional problems with Coulomb friction that are `stationary' in the terminology of Dundurs and Stippes [12]. In other words, all contact nodes in the undeformed state remain in contact during deformation, and no additional contact nodes are established.

2. Problem Description

We first discretize the elastic contacting bodies [e.g. by the finite-element method] and use static reduction [13] to obtain the contact stiffness matrix

$$K = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} , \tag{1}$$

such that the normal and tangential nodal forces p_i, q_i and the corresponding relative nodal displacements w_i, v_i are related by an equation of the form

$$\begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{q}^w(t) \\ \boldsymbol{p}^w(t) \end{Bmatrix} + \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{Bmatrix} \quad . \tag{2}$$

We adopt the convention that the normal forces p_i are positive when compressive, and the normal displacements w_i are positive when the nodes separate. In Eq. (2), $p_i^w(t)$, $q_i^w(t)$ are the nodal forces that would be produced by the external loads if the nodes were all welded in contact at $\boldsymbol{v} = \boldsymbol{w} = \boldsymbol{0}$. The matrix $\boldsymbol{B}$ is a measure of the coupling between the normal and tangential contact problems [14] and plays a crucial role in the history-dependence of the frictional evolution problem.

Since the system is two-dimensional, each contact node i must, at any given time t, be in one of the four states

$$\begin{array}{llll} |q_i| \le fp_i & w_i = 0 & \dot{v}_i = 0 & \text{stick} \\ q_i = -fp_i & w_i = 0 & \dot{v}_i > 0 & \text{forward slip} \\ q_i = fp_i & w_i = 0 & \dot{v}_i < 0 & \text{backward slip} \\ q_i = 0 & p_i = 0 & w_i > 0 & \text{separation}, \end{array} \tag{3}$$

where the dot denotes a time derivative.

2.1. The eigenvalue problem

For a wedged state, there are no external loads and we are restricting attention to problems where all the nodes remain in contact ($\boldsymbol{w} = \boldsymbol{0}$), so Eq. (2) reduces to

$$\boldsymbol{q} = \boldsymbol{A}\boldsymbol{v} \;\; ; \;\; \boldsymbol{p} = \boldsymbol{B}\boldsymbol{v} . \tag{4}$$

For Hild's eigenvalue problem [7], each node must be in a state of stick, but with incipient slip $|q_i|=fp_i$. Clearly the direction of slip may be different at each node, but we can accommodate this by introducing a diagonal matrix $\boldsymbol{\Lambda}$ such that $\Lambda_i = \text{sgn}(q_i)$. In other words, $\Lambda_i = 1$ for incipient backward slip and -1 for incipient forward slip. We then have

$$\boldsymbol{q} = f\boldsymbol{\Lambda}\boldsymbol{p} \quad \text{or} \quad \boldsymbol{A}\boldsymbol{v} = f\boldsymbol{\Lambda}\boldsymbol{B}\boldsymbol{v} , \tag{5}$$

using Eq. (4). For any given $\boldsymbol{\Lambda}$, this defines a generalized linear eigenvalue problem for f, so the possibility of wedging can be explored by (i) solving the eigenvalue problem for all possible values of $\boldsymbol{\Lambda}$ [i.e. all possible diagonal matrices whose non-zero elements are all either +1 or −1] and then (ii) checking the resulting eigenfunctions to determine which permit a solution in which the normal nodal forces are all non-tensile.

2.2. The P-matrix condition

The perceptive reader will recognize a relation between the eigenvalue equation (5) and Klarbring's 'P-matrix' condition [15] for the frictional 'rate' problem [the statement of the Coulomb friction evolution problem in terms of time derivatives] to be well posed. For two-dimensional problems, Klarbring's condition is satisfied if and only if all matrices of the form $\boldsymbol{A} + f\boldsymbol{\Lambda}\boldsymbol{B}$ are P-matrices [16] — i.e. they have positive determinants as have all their principal minors. The matrix $\boldsymbol{A}$ is a stiffness matrix and hence is also a P-matrix, so Klarbring's condition is always satisfied in the absence of friction [$f = 0$]. Also, $\boldsymbol{A} + f\boldsymbol{\Lambda B}$ varies continuously as f is increased, so if the condition is to be violated, it must correspond to a condition where this matrix or one of its principal minors has a determinant equal to zero. Cases where the full matrix is singular correspond to eigenvalues of (5), but there appears to be no similar link between principal minors of the matrix and Hild's eigenvalue problem.

3. The Displacement Vector Space $\mathcal{V}$

Ahn *et al.* [17] introduced the idea of tracking the evolution of a frictional state in the N-dimensional vector space $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ and showed that the instantaneous state is represented by a point in this space that is 'pushed' by the frictional constraints during periods of slip. The inequalities $q_i \le fp_i$ and $-q_i \le fp_i$ governing stick at node i take the form

$$\sum_{j=1}^{N}(A_{ij} - fB_{ij})v_j \le fp_i^w(t) - q_i^w(t) \quad \text{and} \quad \sum_{j=1}^{N}(A_{ij} + fB_{ij})v_j \ge -fp_i^w(t) - q_i^w(t). \qquad (6)$$

For the simple 2-node case, the instantaneous state is defined by the point $P(v_1, v_2)$ and each inequality excludes the shaded region on one side of a straight line as shown in Fig. 2. Here, the lines I, II govern backward and forward slip respectively at node 1 and lines III, IV govern slip at node 2. If the external loads $p_i^w(t)$, $q_i^w(t)$ change, the constraint lines move whilst retaining the same slope. Fig. 2 illustrates the resulting motion of P if constraint IV first advances to the dotted line and then recedes, after which I advances.

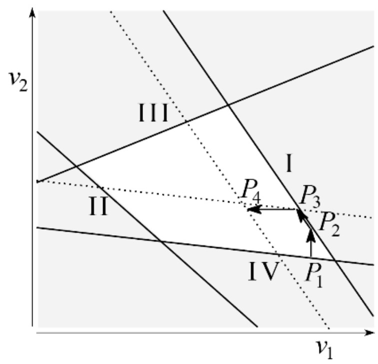

Fig. 2 Evolution of the frictional state for a two node system, as governed by the four constraints (inequalities) (6), here labeled I,II,III,IV, after Ahn *et al.* [17]

3.1. Constraint vectors

When there are no external loads, the inequalities (6) take the form

$$\sum_{j=1}^{N}(A_{ij} - fB_{ij})v_j \le 0 \quad \text{and} \quad \sum_{j=1}^{N}(A_{ij} + fB_{ij})v_j \ge 0 \qquad (7)$$

using Eq. (4). It is convenient to write these in the compact form

$$\mathcal{C}_k \cdot \boldsymbol{v} \le \mathbf{0}, \tag{8}$$

where $\mathcal{C}_k$, $k \in (1, 2N)$ comprise a set of unit *constraint vectors* defined by

$$\mathcal{C}_{2i\text{-}1} = \frac{(\boldsymbol{A} - f\boldsymbol{B})^T \boldsymbol{e}_i}{|(\boldsymbol{A} - f\boldsymbol{B})^T \boldsymbol{e}_i|}\,; \;\; \mathcal{C}_{2i} = -\frac{(\boldsymbol{A} + f\boldsymbol{B})^T \boldsymbol{e}_i}{|(\boldsymbol{A} + f\boldsymbol{B})^T \boldsymbol{e}_i|}\,, \tag{9}$$

and $\boldsymbol{e}_i$ is the unit vector in direction v_i. Each of these constraints excludes the region on one side of a hyperplane through the origin, and the corresponding constraint vector is defined so as to point perpendicularly *into* the excluded region.

3.2. A necessary and sufficient condition for wedging

For the purpose of this section, a *wedged state* is defined as one in which all nodes $i \in (1, N)$ are in contact [$w_i = 0$], and satisfy the unilateral inequality $p_i \ge 0$.

Theorem 1. *A wedged state is possible if and only if there exists a non-null displacement vector* **v** *that satisfies all of the inequalities (7). In other words, a necessary and sufficient condition for wedging is that there should exist at least one non-null N-vector* **U** *such that* $\mathcal{C}_k \cdot \boldsymbol{U} \le 0$ *for all* $k \in (1, N)$.

If this condition is satisfied by a given vector $\boldsymbol{U}$, it is clear that it will also be satisfied by $\lambda\boldsymbol{U}$, where λ is any positive scalar multiplier. Thus, the admissible wedging space, if it exists, will comprise a cone with vertex at the origin of $\mathcal{V}$. This case is illustrated for the 2-node case in Fig. 3(a), where wedging is possible for vectors $\boldsymbol{v}$ in the unshaded region between constraints II and III.

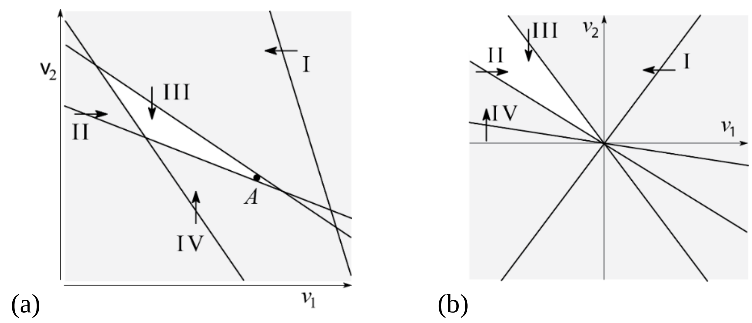

Fig. 3 (a) A two-node case where wedging is possible between constraints II and III. (b) A case where Klarbring's P-matrix condition is not satisfied, but where wedging is not possible

The arrows on the constraint boundaries in Fig. 3(a) indicate the direction of slip implied if the constraint were to move so as to exclude more space due to the imposition of external loads. In particular, if constraint II were to move so as to reduce the extent of the unshaded region, the point $P(v_1,v_2)$ would eventually become 'trapped' between II and III with no admissible slip motion being possible. This represents a special case where Klarbring's P-matrix criterion is violated and hence the rate problem is not well-posed. More general investigation of the two-node system shows that any set of constraints allowing a wedged region leads to a situation where the P-matrix criterion is violated.

However, the converse is not always true. Fig. 3(b) shows a case where *P* can become trapped at *A* between II and III, but where the slopes of the remaining constraints III, IV are such as to preclude wedging. We conclude that for the two-node system, violation of Klarbring's condition is a necessary but not sufficient condition for wedging. However, we are not aware of a proof of this result for the more general *N*- node case.

3.3. Positive span

Theorem 1 implies that wedging is *impossible* if and only if for every non-null *N*-vector $\boldsymbol{v}$, at least one of the 2*N* constraint inequalities (8) is violated — i.e. every point in $\mathcal{V}$ is excluded by at least one of the constraints. An alternative statement of this is condition is that *the set of* 2*N vectors* $\mathcal{C}_k$ *positively spans the N-dimensional vector space* $\mathcal{V}$. Algorithms for testing whether a given set of vectors spans a vector space are discussed by Regis [18].

If the coefficient of friction $f = 0$, the constraint vectors (9) for node *i* reduce to

$$\mathcal{C}_{2i\text{-}1} = \frac{\boldsymbol{A}^T \boldsymbol{e}_i}{|\boldsymbol{A}^T \boldsymbol{e}_i|} \; ; \;\; \mathcal{C}_{2i} = -\frac{\boldsymbol{A}^T \boldsymbol{e}_i}{|\boldsymbol{A}^T \boldsymbol{e}_i|} \quad , \tag{10}$$

and hence are equal and opposite. The same result is obtained for all values of *f*, for any node *i* for which

$$\boldsymbol{B}^T \boldsymbol{e}_i = \boldsymbol{0} \quad . \tag{11}$$

The pair of constraints (10) spans all vectors in $\mathcal{V}$ except those in the common orthogonal hyperplane. Thus, if (10) is satisfied at a subset of *M* nodes, the admissible region is reduced to the intersection of *M* such hyperplanes, which comprises a vector space $\mathcal{V}^*$ of dimension (*N-M*). In particular, if *M*=*N*, this reduces to a single point [the origin] and wedging is impossible. This arises (i) if $f = 0$, or (ii) if (11) is satisfied for all nodes $i \in (1, N)$, in which case $\boldsymbol{B}=\boldsymbol{0}$ and the system is 'uncoupled' [14].

At the other extreme, as $f \to \infty$, if Eq. (11) is not satisfied, we obtain

$$\mathcal{C}_{2i\text{-}1} \to -\frac{\boldsymbol{B}^T \boldsymbol{e}_i}{|\boldsymbol{B}^T \boldsymbol{e}_i|} \;\; ; \;\; \mathcal{C}_{2i} \to -\frac{\boldsymbol{B}^T \boldsymbol{e}_i}{|\boldsymbol{B}^T \boldsymbol{e}_i|} \quad , \tag{12}$$

and the two constraints for each node become identical. In this case, the complete set of 2*N* constraint vectors comprises only *N* independent vectors, but a minimum of *N*+1 independent vectors is needed to span an *N*-dimensional vector space [18, 19]. More generally, if (11) is satisfied at *M* (< *N*) nodes, there will be (*N-M*) constraints of the form (12), but these are insufficient to span the reduced vector space $\mathcal{V}^*$ of dimension (*N-M*) . We conclude that if $\boldsymbol{B} \neq \boldsymbol{0}$, the system must be capable of wedging at sufficiently large *f*.

As *f* is increased from $f = 0^+$, the admissible region due to the pair of vectors $\mathcal{C}_{2i-1}$, $\mathcal{C}_{2i}$ increases monotonically from half of the hyperplane orthogonal to $\boldsymbol{A}^T \boldsymbol{e}_i$ to the half-space bounded by the hyperplane $\boldsymbol{B}^T \boldsymbol{e}_i$. It follows that the admissible region due to the entire set of constraint vectors also increases monotonically [once it is non-null], so in general there must exist a unique critical coefficient of friction f_w such that wedging is possible for $f > f_w$ and impossible for $f < f_w$.

3.4. Tensile nodal forces and separation

Solutions of the system of equations (2) are physically meaningful only if all the normal nodal forces p_i are non-negative. However, if any $p_i < 0$ it follows that $|q_i| > fp_i$, so at least one of the two constraints $\mathcal{C}_{2i-1}\cdot\boldsymbol{v} \leq 0$ and $\mathcal{C}_{2i}\cdot\boldsymbol{v} \leq 0$ must be violated. Thus if a wedged state $\boldsymbol{v}$ for the system is identified by this criterion, it is not necessary to check the signs of the normal tractions, since these will necessarily be all positive.

4. CONCLUSIONS

The principal conclusion of the paper is that wedging for a discrete two-dimensional system with Coulomb friction is possible if and only if the set of $2N$ constraint vectors $\mathcal{C}_k$ defined by Eq. (9) fails to positively span the nodal displacement vector space $\mathcal{V}$. We also show that any states satisfying this condition automatically satisfy the condition that all nodes remain in contact with non-tensile nodal forces, and that for any given system, there exists a unique f_w such that wedging is possible for $f > f_w$ and not for $f < f_w$.

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