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FROM WINKLER'S FOUNDATION TO POPOV'S FOUNDATION

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Abstract. *In recent years, the method of dimensionality reduction (MDR) has started to figure as a very convenient tool for dealing with a wide class of elastic contact problems. The MDR modeling framework introduces an equivalent punch profile and a one-dimensional Winkler-type elastic foundation, called henceforth Popov's foundation. While the former mainly accounts for the geometry of contact configuration, the Popov foundation inherits the main characteristics of both the contact interface (like friction and adhesion) and the contacting elastic bodies (e.g., anisotropy, viscoelasticity or inhomogeneity). The discussion is illustrated with an example of the Kendall-type adhesive contact for an isotropic elastic half-space.*

Key Words: *Elastic Contact, Winkler Foundation, Method of Dimensionality Reduction, Contact Stiffness, Adhesion Strength*

1. INTRODUCTION

Contact phenomena [1,2] can be encountered in diverse applications ranging from engineering (wheel/rail contact [3], tribological systems [4], etc.) to medicine (e.g., contact of articular cartilage layers [5]). Depending on the material's deformation response to external loading, contact geometry configuration, and accompanying surface effects such as friction or adhesion, a particular contact problem can be very complicated for analytical treatment. While a number of numerical techniques have been developed in the last few decades [6,7], still analytical and semi-analytical models [8,9] of contact interactions are preferred over computer simulations, especially if qualitative understanding of the contact problem is required.

For many years, the theory of local contact of elastic bodies, which was created by Heinrich Hertz [10], was one of the most difficult parts of the solid mechanics courses. In recent years, the method of dimensionality reduction (MDR) has been developed by

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Valentin L. Popov and his collaborators [11] into a mature and effective semi-analytical framework for analysis of the Hertzian type elastic contact.

A great advantage of the MDR is that it reduces the contact problem to a much simpler one for a one-dimensional Winkler foundation. However, the matter is not so simple, and a vulgar interpretation of the MDR methodology may lead to erroneous conclusions (see, e.g., the author's discussion [12]). In the present paper, we consider the adhesion aspect of elastic contact, how it is introduced into the core MDR for axisymmetric JKR (Johnson–Kendall–Roberts)-type contact [13], and how it could be generalized into the non-axisymmetric case.

2. Winkler's Foundation

In this section, we briefly outline the basics of the Winkler foundation model in the light of the comparison with an elastic half-space model.

2.1. Response equation of the Winkler foundation

Consider an elastic foundation bounded by a flat surface, which is assumed to be smooth (that is, contact is frictionless) and non-sticky (non-adhesive). Let an absolutely rigid body be pressed against the foundation surface by a normal force, F. The external load is equilibrated by the contact pressures, $p(x_1, x_2)$, distributed over a contact area. According to the hypothesis introduced by Emil Winkler (see, in particular, [14]), the contact pressures are determined solely by the local normal displacements, $u_3(x_1, x_2)$, of the surface points which come into contact, i.e.,

$$p(x_1, x_2) = ku_3(x_1, x_2)\,, \tag{1}$$

Constant k is called the coefficient of foundation.

Under the assumption that the contact is unilateral, so that the contact pressure density is not allowed to take negative values, the contact area will be determined by the displaced surface points.

2.2. Contact stiffness and incremental contact stiffness

In the case of a flat-ended rigid body, which touches the foundation surface over a certain domain, Ω, generally speaking, we will have

$$u_3(x_1, x_2) = \delta + \beta_1 x_2 - \beta_2 x_1\,, \tag{2}$$

where β_1 and β_2 are small angles of the rigid body's rotation.

In view of (1), the equilibrium equation implies that

$$F = k(\delta A + \beta_1 S_1 - \beta_2 S_2)\,, \tag{3}$$

where A is the area of Ω, S_1 and S_2 are the first moments of area Ω in the x_1 and x_2 directions, respectively, i.e.,

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix} = \iint \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} \mathrm{d}x_1 \mathrm{d}x_2\,.$$

Let (x_1^c, x_2^c) denote the center of mass of Ω, that is

$$\begin{Bmatrix} x_1^c \\ x_2^c \end{Bmatrix} = \frac{1}{A} \iint \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \mathrm{d}x_1 \mathrm{d}x_2 ,$$

where the integration is carried out over the domain Ω.

Then, in view of Eqs. (1) and (2), Eq. (3) can be rewritten as

$$F = kA\delta_c , \tag{4}$$

where $\delta_c = u_3(x_1^c, x_2^c)$ is the vertical displacement of the rigid body's flat base at the mass center of Ω.

The quantity

$$K = kA \tag{5}$$

is called the contact stiffness.

Now, let a rigid body be bounded by a surface

$$x_3 = -\Phi(x_1, x_2) , \tag{6}$$

such that $\Phi(x_1, x_2) \geq 0$ and $\Phi(0,0) = 0$. (This is not a restrictive assumption.)

If the rigid body is translationally displaced into the Winkler foundation to some depth, δ, then the surface displacements are

$$u_3(x_1, x_2) = \delta - \Phi(x_1, x_2) , \tag{7}$$

And contact domain Ω_δ will be determined by inequality $\delta - \Phi(x_1, x_2) > 0$.

In this case, the contact force is given by

$$F = k \iint_{\Omega_\delta} (\delta - \Phi(x_1, x_2))_+ \mathrm{d}x_1 \mathrm{d}x_2 , \tag{8}$$

where $(x)_+ = x + |x|$ is the positive part function.

The force-displacement relation (8) is nonlinear, but the incremental contact stiffness can be simply evaluated as follows (cf. Eq. (5)):

$$\frac{\mathrm{d}F}{\mathrm{d}\delta} = kA_\delta . \tag{9}$$

Here, A_δ is the area of the domain Ω_δ.

2.3. Comparison with the elasticity theory model

Let us compare the simple solution of the contact problem obtained in the framework of the Winkler foundation model with the corresponding results for an isotropic elastic half-space (with Young's modulus E and Poisson's ratio ν). In particular, what we are interested in is the contact stiffness for a flat-ended indenter, which can be represented as

$$K = 2E^* \alpha , \tag{10}$$

where $E^* = E/(1-\nu^2)$ is the reduced elastic modulus, and α is the so-called harmonic capacity radius of the current contact area (see, e.g. [15]).

The main difference between formulas (5) and (10) is that they predict different variations of the contact stiffness under a similarity scaling of the contact area, since A and α have dimensions of L^2 and L, respectively, where L is the dimension of length.

Explicit formulas for α (or for the harmonic capacity) are known only in a limited number of cases (for instance, for an annular contact area [9]). Also, some approximations for the α-related characteristic (such as contact compliance or constriction resistance) can be found in the literature (see, e.g., [16,17]). In particular, using the cross-property connection established by Sevostianov and Kachanov [18], the following approximation can be written out for the harmonic capacity radius of a circular cluster of N identical circular microcontacts:

$$\alpha = \left(\frac{1}{N a_i} + \frac{1}{R_{\mathrm{H}}} \right)^{-1} .$$

Here, a_i is the radius of identical microcontacts, R_{H} is Holm's radius [19], whose value was estimated in [20,21].

Finally, it is clear that for a circular contact area α coincides with the contact radius.

3. Popov's Foundation

In this section we recast the underlying concept of the MDR into a simple form. For a complete and detailed review of the MDR, the reader is referred to [22–26].

3.1. Equivalent profile

Returning back to the contact problem for a rigid indenter, which is bounded by the surface (6), let us now assume that it is pressed against an elastic half-space. Considering the indentation as a one-parametric process, that is the case for normal translational displacement of the indenter, both contact force F and indenter displacement δ can be regarded as one-valued functions of harmonic capacity radius α. In particular, let the latter function be designated as

$$\delta = g(\alpha) \tag{11}$$

The MDR reduces the elastic contact problem to a much simpler contact problem for a one-dimensional Winkler-type linearly deformable foundation, which will be called the Popov foundation, and a rigid punch of equivalent profile. The latter is described by the equation

$$z = -g(|x|) \ , \tag{12}$$

where x and z are horizontal and vertical coordinates, and function $g(x)$, $x \geq 0$, is defined according to Eq. (11). Note that, by definition, the equivalent punch is symmetric, and, therefore, the contact interval will be symmetric with respect to the z-axis as well.

3.2. Stiffness coefficient of the Popov foundation

The intuitive simplicity of solving the MDR equivalent contact problem is explained not only by the simple character of the deformation response of the Popov linearly deformable foundation, but by the exact correspondences (equalities) between the kinematic and force parameters, which are denoted by the same symbols δ and F. While the first (kinematic) correspondence is facilitated by Eqs. (11) and (12), the second (force) correspondence is achieved by tuning the Popov foundation coefficient, k_z. It is to note here that the MDR transformation rules for the exact mapping of any axisymmetric normal contact with and without adhesion have been first given by Heß [27].

The incremental indentation stiffness for the Popov foundation is equal to $2\alpha k_z$, since 2α is the length of the one-dimensional contact interval. On the other hand, the incremental contact stiffness of an elastic half-space is given by Eq. (10). By equating these two values, we readily arrive at the following relation [11]:

$$k_z = E^* \ . \tag{13}$$

Let us emphasize [15] that, while in the one-dimensional equivalent contact problem α is the half-width of the contact interval, in the original elastic contact problem α has the meaning of the harmonic capacity radius of the current contact area, which corresponds to the same value of kinematic parameter δ.

4. Adhesive Strength of Elastic Contacts

In this section, we touch on the adhesive aspect of unilateral frictionless contact.

4.1. Adhesive strength of the Winkler foundation

Consider first a flat-ended contact with a Winkler foundation. To be more specific, we assume that a flat-ended indenter is pressed against a very thin compressible elastic layer bonded to an absolutely rigid substrate. It was shown by Aleksandrov [28] that the normal deformation of such isotropic elastic layer coincides with that of a Winkler foundation with the foundation constant

$$k = \frac{E_A}{h} \ , \tag{14}$$

where h is the layer thickness, and $E_A = E(1-\nu)/[(1+\nu)(1-2\nu)]$ is the so-called aggregate elastic modulus.

Under certain conditions (when contour Γ of contact area Ω is smooth with a variable curvature radius much larger than the layer thickness), the JKR-type detachment criterion on the contact contour can be formulated as follows [29]:

$$p\big|_{\Gamma} = p_c, \quad p_c = -\sqrt{\frac{2E_A\Delta\gamma}{h}} \ , \tag{15}$$

where $\Delta\gamma$ denotes the work of adhesion, and negative sign denotes tensile stresses.

Observe that Eq. (15) was derived with the help of asymptotic modeling technique from the stress-intensity factor (SIF) of the boundary layer (we refer to [30,31] for more details).

Now, let the flat-ended indenter is pulled from the layer surface with its base maintaining a horizontal position. The pull-off force, F_c, thus, is given by

$$F_c = Ap_c \ . \tag{16}$$

This means that the Winkler foundation based model of adhesive contact, Eqs. (14)–(16), predicts that the adhesive strength is proportional to the contact area.

4.2. Adhesive strength of the Kendall type contact

In the case of an isotropic elastic half-space, the pull-off force of a circular cylindrical indenter of radius a was evaluated by Kendall as

$$F_c = -\sqrt{8\pi a^3 E^* \Delta\gamma} \ , \tag{17}$$

However, generalization of Kendall's formula (17) to the non-axisymmetric case is not trivial. Following Kendall [32], we consider an arbitrary flat-ended punch making perfect contact with an elastic half-space of reduced modulus E^*. When a pull-off force F is applied to the punch, its displacement is given by

$$\delta = \frac{F}{2E^*\alpha} \ ,$$

where α is the harmonic capacity radius of the contact area, and the elastic energy will be

$$U_E = \frac{F^2}{4E^*\alpha} + \text{const.}$$

The surface energy is

$$U_S = -\Delta\gamma A \ ,$$

where A is the area of contact.

Further, the potential energy of the load $U_P = -F\delta + \text{const}$ is

$$U_P = -\frac{F^2}{2E^*\alpha} + \text{const.}$$

Thus, collecting the above formulas, we evaluate the total energy $U_T = U_S + U_E + U_P$ as follows:

$$U_T = -\Delta\gamma A - \frac{F^2}{4E^*\alpha} + \text{const.} \tag{18}$$

Observe that A and α represent two different integral characteristics of the contact area. In the general case, the following inequality takes place [33]:

$$\sqrt{\frac{A}{\pi}} \le \alpha \ . \tag{19}$$

So, differentiating Eq. (18) with respect to A, we readily get

$$\frac{\partial U_T}{\partial A} = -\Delta\gamma + \frac{F^2}{4E^*\alpha^2}\frac{\partial \alpha}{\partial A} \ . \tag{20}$$

Now, making use of the monotonicity property, which holds for both geometric quantities A and α, and assuming that $\alpha = 0$ for $A = 0$, we derive from (19) by differentiation that

$$\frac{1}{2\sqrt{\pi A}} \le \frac{\partial \alpha}{\partial A} \ . \tag{21}$$

Following the argumentation of Kendall [29], we state that detachment is possible when

$$\frac{\partial U_T}{\partial A} \ge 0 \ . \tag{22}$$

Thus, from Eqs. (20)–(22), it follows that

$$F_c^2 \le 8\sqrt{\pi A}E^*\alpha^2\Delta\gamma \ . \tag{23}$$

or using the inequality (19) once again, we arrive at the following relation [34]:

$$F_c^2 \le 8\pi E^*\alpha^3\Delta\gamma \ . \tag{24}$$

Another estimate for the pull-off force can be derived from the approximate solution obtained by Fabrikant [16] for an arbitrary flat-ended indenter

$$p(r,\theta) = \frac{Fa(\theta)}{2A\sqrt{a(\theta)^2 - r^2}} \ , \tag{25}$$

where the equation $r = a(\theta)$ determines the boundary of the contact area in the polar coordinates (r, θ).

Formula (25) implies the following approximation for the maximum SIF:

$$\max K_1(\theta) = -\frac{\sqrt{\pi}F}{2A}\sqrt{\max a(\theta)} \ . \tag{26}$$

Recall that we assume that $F < 0$, i.e, the pull-off force is assumed to be negative.

So, the substitution of (26) into the JKR detachment criterion (see [30,31])

$$\max K_1(\theta) = \sqrt{2E^*\Delta\gamma}$$

yields

$$F_c^2 \le \frac{16E^*A^2\Delta\gamma}{\pi D} \ , \tag{27}$$

where D is the diameter of the contact area.

Observe that formula (27) correctly predicts the onset of detachment for a flat-ended elliptical indenter.

4.3. On extension of the MDR to the Kendall type adhesive contact

In the case of a flat-ended cylindrical indenter, the shape function can be described as follows:

$$\Phi(x_1,x_2)=\begin{cases}0, & (x_1,x_2)\in\Omega,\\ \infty, & (x_1,x_2)\in\mathbb{R}^2\setminus\bar{\Omega}.\end{cases}$$

Here, Ω is the contact area, which does not change during the normal translational indentation.

Let α denote the harmonic capacity radius of Ω. Then, the shape function of the equivalent one-dimensional punch will simply be

$$g(x)=\begin{cases}0, & x<\alpha,\\ \infty, & x>\alpha.\end{cases}$$

Note that in the axisymmetric case, when α equals the contact radius a, the above definition coincides with that of [11]. This means that the pull-off force

$$F_c \le 2\alpha p_{1D}^c \tag{28}$$

must coincide with Kendall's result (17), when $\alpha = a$.

According to the logic of the rule of Heß for the adhesive contact between axially-symmetric bodies (see [11], section 4.2), the critical value of pressure $p_{1\mathrm{D}}^c$ may depend on α. The only question now is, what meaning should be attached to $p_{1\mathrm{D}}^c$? In view of the surprising difficulty of the evolution of the detachment process (see [35]), it is suggested to associate $p_{1\mathrm{D}}^c$ with critical force F_{c1} at which the detachment process starts (see [34] for details).

Observe that among the three upper estimates (23), (24), and (27), only the second one is a function of solely the harmonic capacity radius of the contact area. This fact limits the choice of approximations for $p_{1\mathrm{D}}^c$ to the following one:

$$p_{1D}^c = -\sqrt{2\pi\alpha E^*\Delta\gamma}\ . \tag{29}$$

It is interesting and significant that the substitution of Eq. (29) into Eq. (28) yields Kendall's formula, Eq. (17), in the axisymmetric case, when $\alpha = a$.

5. Discussion and Conclusion

The model of Winkler's foundation is based on two concepts: linear deformation of spring elements according to Hooke's law and non-interaction between the spring elements. There are known many generalizations of the Winkler model [36] and, in particular, its varied adaptations to the field of adhesion [37]. It is clear that Popov's foundation is a one-dimensional Winkler foundation. But it is more than that. The MDR (and correspondingly the Popov foundation) has been extended to cope with tangential [11] and torsional [38] contacts and to account for viscoelastic material's constitutive relationship [39] as well as for material grading [40,41].

Having been stemmed from a simple observation [22] that — as far as one is interested in the stiffness of the Hertzian type contact — the three-dimensional contact problem can be reduced to a one-dimensional problem for a Winkler foundation, the MDR has grown to become a comprehensive methodology for dealing with a wide class of elastic contacts.

Strictly speaking, a range of elastic contact problems solved by the MDR is characterized by a hierarchy of Popov foundations, or to be more precise, each type of contact interaction, which is covered by the MDR, requires its own Popov foundation. Since distinct types of contact can differ by diverse surface effects (like friction and adhesion), one can consider a generic case with a combination of the effects and forms of loading. In the MDR framework, this can be done in a straightforward way, assuming superposition of the effects. However, a care should be taken in this regard since the superposition of physical effects is not always valid.

To conclude, Popov’s foundation, which serves as the basis for the MDR, represents a major advance in developing a unified approach to effective dealing with elastic contacts.

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