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MOMENT LYAPUNOV EXPONENTS AND STOCHASTIC STABILITY OF A THIN-WALLED BEAM SUBJECTED TO AXIAL LOADS AND END MOMENTS

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Abstract. *In this paper, the Lyapunov exponent and moment Lyapunov exponents of two degrees-of-freedom linear systems subjected to white noise parametric excitation are investigated. The method of regular perturbation is used to determine the explicit asymptotic expressions for these exponents in the presence of small intensity noises. The Lyapunov exponent and moment Lyapunov exponents are important characteristics for determining both the almost-sure and the moment stability of a stochastic dynamic system. As an example, we study the almost-sure and moment stability of a thin-walled beam subjected to stochastic axial load and stochastically fluctuating end moments. The validity of the approximate results for moment Lyapunov exponents is checked by numerical Monte Carlo simulation method for this stochastic system.*

Key words: *Eigenvalues, Perturbation, Stochastic stability, Thin-walled beam, Mechanics of solids and structures*

1. INTRODUCTION

In recent years there has been considerable interest in the study of the dynamic stability of non-gyroscopic conservative elastic systems whose parameters fluctuate in a stochastic manner. To have a complete picture of the dynamic stability of a dynamic system, it is important to study both the almost-sure and the moment stability and to determine both the maximal Lyapunov exponent and the pth moment Lyapunov exponent. The maximal Lyapunov exponent is defined by

$$\lambda_q = \lim_{t \to \infty} \frac{1}{t} \log \| \boldsymbol{q}(t; q_0) \| \tag{1}$$

where $q(t; q_0)$ is the solution process of a linear dynamic system. The almost-sure stability depends upon the sign of the maximal Lyapunov exponent which is an exponential growth rate of the solution of the randomly perturbed dynamic system. A negative sign of the maximal Lyapunov exponent implies the almost-sure stability whereas a non-negative value indicates instability. The exponential growth rate $E[\|q(t; q_0, \dot{q}_0\|^p]$ is provided by the moment Lyapunov exponent defined as

$$\Lambda_q(p) = \lim_{t \to \infty} \frac{1}{t} \log E[\|q(t; q_0)\|^p] \tag{2}$$

where $E[\ ]$ denotes the expectation. If $\Lambda_q(p) < 0$ then, by definition $E[\|q(t; q_0, \dot{q}_0\|^p] \to 0)$ as $t \to \infty$ and this is referred to as the pth moment stability. Although the moment Lyapunov exponents are important in the study of the dynamic stability of the stochastic systems, the actual evaluations of the moment Lyapunov exponents are very difficult.

Arnold et al. [1] constructed an approximation for the moment Lyapunov exponents, the asymptotic growth rate of the moments of the response of a two-dimensional linear system driven by real or white noise. A perturbation approach was used to obtain explicit expressions for these exponents in the presence of small intensity noises. Khasminskii and Moshchuk [2] obtained an asymptotic expansion of the moment Lyapunov exponents of a two-dimensional system under white noise parametric excitation in terms of the small fluctuation parameter ε, from which the stability index was obtained. Sri Namachchivaya et al. [3] used a perturbation approach to calculate the asymptotic growth rate of a stochastically coupled two-degrees-of-freedom system. The noise was assumed to be white and of small intensity in order to calculate the explicit asymptotic formulas for the maximum Lyapunov exponent. Sri Namachchivaya and Van Roessel [4] used a perturbation approach to obtain an approximation for the moment Lyapunov exponents of two coupled oscillators with commensurable frequencies driven by small intensity real noise with dissipation. The generator for the eigenvalue problem associated with the moment Lyapunov exponents was derived without any restriction on the size of pth moment. Kozić et al. [5] investigated the Lyapunov exponent and moment Lyapunov exponents of a dynamic system that could be described by Hill's equation with frequency and damping coefficient fluctuated by white noise. The procedure employed in Khasminskii and Moshchuk [2] was applied to obtain an asymptotic expansion of the Lyapunov exponent and moment Lyapunov exponents of an oscillatory system under two white-noise parametric excitations in terms of the small fluctuation parameter. These results were used to obtain explicit expressions of an asymptotic expansion of the moment and almost sure stability boundaries of the simply supported beam which was subjected to the axial compressions and varying damping which were two random processes. In [6, 7], Kozić et al. investigated the Lyapunov exponent and moment Lyapunov exponents of two degrees-of-freedom linear systems subjected to white noise parametric excitation. In [6], almost-sure and moment stability of the flexural-torsion stability of a thin elastic beam subjected to a stochastically fluctuating follower force were studied. In [7], moment Lyapunov exponents and stability boundary of the double-beam system under stochastic compressive axial loading were obtained. In [9], Pavlović et al. investigated the dynamic stability of thin-walled beams subjected to combined action of stochastic axial loads and stochastically fluctuating end moments. By using the direct Lyapunov method, the authors obtained the almost-sure stochastic boundary and uniform

stochastic stability boundary as the function of characteristics of stochastic process and geometric and physical parameters.

Deng et al. [12] investigated the Lyapunov exponent and moment Lyapunov exponents of flexural-torsional viscoelastic beam, under parametric excitation of white noise. The system of stochastic differential equations of motion is first decoupled by using the method of stochastic averaging for dynamic systems with small damping and weak excitations. The moment and almost-sure stability boundaries and critical excitation are obtained analytically which are confirmed by numerical simulation. Also, Deng in [13] studied the moment stochastic stability and almost-sure stochastic stability through the moment Lyapunov exponents and the largest Lyapunov exponent of flexural-torsional viscoelastic beam, under the parametric excitation of a real noise.

Stochastic stability of a viscoelastic plate in supersonic flow as well typical example of a coupled non-gyroscopic system through Lyapunov exponent and moment Lyapunov exponents and are investigated by Deng et al. [14]. The excitation is modelled as a bounded noise process. By using the method of stochastic averaging, the equations of motion are decoupled into Itô differential equations, from which moment Lyapunov exponents are readily obtained. The Lyapunov exponents are obtained from the relation with moment Lyapunov exponents.

The aim of this paper is to determine a weak noise expansion for the moment Lyapunov exponents of the four-dimensional stochastic system. The noise is assumed to be white noise of such small intensity that an asymptotic growth rate can be obtained. We apply the perturbation theoretical approach given in Khasminskii and Moshchuk [2] to obtain second-order weak noise expansions of the moment Lyapunov exponents. The Lyapunov exponent is then obtained using the relationship between the moment Lyapunov exponents and the Lyapunov exponent. These results are applied to study the pth moment stability and almost-sure stability of a thin-walled beam subjected to stochastic axial loads and stochastically fluctuating end moments. The motion of such an elastic system is governed by the partial differential equations in [9] by Pavlović et al. The approximate analytical results of the moment Lyapunov exponents are compared with the numerical values obtained by the Monte Carlo simulation approach for these exponents of a four-dimensional stochastic system.

2. THEORETICAL FORMULATION

Consider linear oscillatory systems described by equations of motion of the form

$$\ddot{q}_1 + \omega_1^2 q_1 + 2\varepsilon\beta_1\dot{q}_1 - \sqrt{\varepsilon}K_{11}\xi_1(t)q_1 - \sqrt{\varepsilon}K_{12}\xi_2(t)q_2 = 0,$$
$$\ddot{q}_2 + \omega_2^2 q_2 + 2\varepsilon\beta_2\dot{q}_2 - \sqrt{\varepsilon}K_{21}\xi_1(t)q_1 - \sqrt{\varepsilon}K_{22}\xi_2(t)q_2 = 0,$$
(3)

where q_1, q_2 are generalized coordinates, ω_1, ω_2 are natural frequencies and $2\varepsilon\beta_1$, $2\varepsilon\beta_2$ represent small viscous damping coefficients. The stochastic terms $\sqrt{\varepsilon}\xi_1(t)$ and $\sqrt{\varepsilon}\xi_2(t)$ are white-noise processes with small intensity with zero mean and autocorrelation functions

$$R_{\xi_1\xi_1}(t_1,t_2) = E[\xi_1(t_1)\xi_1(t_2)] = \sigma_1^2\delta(t_2 - t_1),$$
$$R_{\xi_2\xi_2}(t_1,t_2) = E[\xi_2(t_1)\xi_2(t_2)] = \sigma_2^2\delta(t_2 - t_1),$$
(4)

where σ_1, σ_2 are the intensity of the random process $\xi_1(t)$ and $\xi_2(t)$, and $\delta(\ )$ is the Dirac delta.

Using the transformation

$$q_1 = x_1, \ \dot{q}_1 = \omega_1 x_2, \ q_2 = x_3, \ \dot{q}_2 = \omega_2 x_4 \tag{5}$$

and denoting

$$p_{ij} = K_{ij}\sigma_j, \ \text{(i, j=1,2),} \tag{6}$$

the above Eqs. (3) can be represented in the first-order form by a set of Stratonovich differential equations

$$d\mathbf{X} = \mathbf{A_0}\mathbf{X}dt + \varepsilon\mathbf{A}\mathbf{X}dt + \sqrt{\varepsilon}\ \mathbf{B_1}\mathbf{X} \circ dw_1(t) + \sqrt{\varepsilon}\ \mathbf{B_2}\mathbf{X} \circ dw_2(t), \tag{7}$$

where $X = (x_1 \ \ x_2 \ \ x_3 \ \ x_4)^T$ is the state vector of the system, $w_1(t)$ and $w_2(t)$ are the standard Weiner processes and A_0, A, B_1 and B_2 are constant 4×4 matrices given by

$$\mathbf{A_0} = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta_2 \end{bmatrix},$$

$$\mathbf{B_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_{22} & 0 \end{bmatrix}, \quad \mathbf{B_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & p_{12} & 0 \\ 0 & 0 & 0 & 0 \\ p_{21} & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Applying the transformation

$$x_1 = a\cos\varphi\cos\theta_1, \quad x_2 = -a\cos\varphi\sin\theta_1, \quad x_3 = a\sin\varphi\cos\theta_2, \quad x_4 = -a\sin\varphi\sin\theta_2,$$

$$P = \|a\|^p = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{p/2}, \tag{9}$$

$$0 \le \theta_1 \le 2\pi, \qquad 0 \le \theta_2 \le 2\pi, \qquad 0 \le \varphi \le \pi/2, \quad -\infty < p < \infty,$$

and employing Itô's differential rule, yields the following set of Itô equations for the pth power of the norm of the response and phase variables φ, θ_1, θ_2:

$$
\begin{aligned}
d\|a\|^p &= \varepsilon\alpha_1^* dt + \sqrt{\varepsilon}\gamma_{11}^* \circ dw_1(t) + \sqrt{\varepsilon}\gamma_{12}^* \circ dw_2(t), \\
d\varphi &= \varepsilon\alpha_2^* dt + \sqrt{\varepsilon}\gamma_{21}^* \circ dw_1(t) + \sqrt{\varepsilon}\gamma_{22}^* \circ dw_2(t), \\
d\theta_1 &= (\omega_1 + \varepsilon\alpha_3^*)dt + \sqrt{\varepsilon}\gamma_{31}^* \circ dw_1(t) + \sqrt{\varepsilon}\gamma_{32}^* \circ dw_2(t), \\
d\theta_2 &= (\omega_2 + \varepsilon\alpha_4^*)dt + \sqrt{\varepsilon}\gamma_{41}^* \circ dw_1(t) + \sqrt{\varepsilon}\gamma_{42}^* \circ dw_2(t).
\end{aligned}
\tag{10}
$$

In the previous transformations, a represents the norm of the response, θ_1 and θ_2 are the angles of the first and second oscillators, respectively, and φ describes the coupling or exchange of energy between the first and second oscillator

In the previous equation we have introduced the following markings

$$\alpha_1^* = -2pP(\beta_1 \sin^2\theta_1 \cos^2\varphi + \beta_2 \sin^2\theta_2 \sin^2\varphi), \quad \alpha_2^* = (\beta_1 \sin^2\theta_1 - \beta_2 \sin^2\theta_2)\sin 2\varphi$$

$$\alpha_3^* = -\beta_1 \sin 2\theta_1, \quad \alpha_4^* = -\beta_2 \sin 2\theta_2, \quad \gamma_{11}^* = -\frac{pP}{2}[p_{11}\sin 2\theta_1 \cos^2\varphi + p_{22}\sin 2\theta_2 \sin^2\varphi]$$

$$\gamma_{21}^* = \frac{1}{4}[p_{11}\sin 2\theta_1 - p_{22}\sin 2\theta_2]\sin 2\varphi, \quad \gamma_{31}^* = -p_{11}\cos^2\theta_1, \quad \gamma_{41}^* = -p_{22}\cos^2\theta_2, \quad (11)$$

$$\gamma_{12}^* = -\frac{pP}{2}[p_{12}\sin\theta_1 \cos\theta_2 + p_{21}\cos\theta_1 \sin\theta_2]\sin 2\varphi,$$

$$\gamma_{22}^* = \frac{1}{2}[p_{12}\sin\theta_1 \cos\theta_2 \sin^2\varphi - p_{21}\cos\theta_1 \sin\theta_2 \cos^2\varphi],$$

$$\gamma_{32}^* = -p_{12}\cos\theta_1 \cos\theta_2 \, tg\varphi \quad \gamma_{42}^* = -p_{21}\cos\theta_1 \cos\theta_2 \cot\varphi$$

The Itô version of Eqs.(10) have the following form

$$d\|a\|^p = \varepsilon\alpha_1 dt + \sqrt{\varepsilon}\gamma_{11}\, dw_1(t) + \sqrt{\varepsilon}\gamma_{12}\, dw_2(t),$$
$$d\phi = \varepsilon\alpha_2 dt + \sqrt{\varepsilon}\gamma_{21}\, dw_1(t) + \sqrt{\varepsilon}\gamma_{22}\, dw_2(t),$$
$$d\theta_1 = (\omega_1 + \varepsilon\alpha_3)dt + \sqrt{\varepsilon}\gamma_{31}\, dw_1(t) + \sqrt{\varepsilon}\gamma_{32}\, dw_2(t), \qquad (12)$$
$$d\theta_2 = (\omega_2 + \varepsilon\alpha_4)dt + \sqrt{\varepsilon}\gamma_{41}\, dw_1(t) + \sqrt{\varepsilon}\gamma_{42}\, dw_2(t),$$

where α_i are given in Appendix 1 and $\gamma_{ij} = \gamma_{ij}^*$, (i, j=1, 2, 3, 4).

Following Wedig [11], we perform the linear stochastic transformation

$$S = T(\varphi,\theta_1,\theta_2)P, \quad P = T^{-1}(\varphi,\theta_1,\theta_2)S, \qquad (13)$$

introducing the new norm process S by means of the scalar function $T(\varphi,\theta_1,\theta_2)$ which is defined on the stationary phase processes θ_1, θ_2 and φ

$$dS = P(\omega_1 T'_{\theta_1} + \omega_2 T'_{\theta_2})dt + \varepsilon P(\alpha_1 T + m_0 T'_\phi + m_1 T'_{\theta_1} + m_2 T'_{\theta_2} +$$
$$+ m_{00}T''_{\phi\phi} + m_{01}T''_{\phi\theta_1} + m_{02}T''_{\phi\theta_2} + m_{11}T''_{\theta_1\theta_1} + m_{12}T''_{\theta_1\theta_2} + m_{22}T''_{\theta_2\theta_2})\, dt +$$
$$+ \sqrt{\varepsilon}P(T\gamma_{11} + T'_\phi\gamma_{21} + T'_{\theta_1}\gamma_{31} + T'_{\theta_2}\gamma_{41})\, dw_1(t) + \qquad (14)$$
$$+ \sqrt{\varepsilon}P(T\gamma_{12} + T'_\phi\gamma_{22} + T'_{\theta_1}\gamma_{32} + T'_{\theta_2}\gamma_{42})\, dw_2(t),$$

where

$$m_0 = \alpha_2 + \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}, \quad m_1 = \alpha_3 + \gamma_{11}\gamma_{31} + \gamma_{12}\gamma_{32}, \quad m_2 = \alpha_4 + \gamma_{11}\gamma_{41} + \gamma_{12}\gamma_{42},$$

$$m_{00} = \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2), \quad m_{01} = \gamma_{21}\gamma_{31} + \gamma_{22}\gamma_{32}, \quad m_{02} = \gamma_{21}\gamma_{41} + \gamma_{22}\gamma_{42}, \qquad (15)$$

$$m_{11} = \frac{1}{2}(\gamma_{31}^2 + \gamma_{32}^2), \quad m_{12} = \gamma_{31}\gamma_{41} + \gamma_{32}\gamma_{42}, \quad m_{22} = \frac{1}{2}(\gamma_{41}^2 + \gamma_{42}^2).$$

If the transformation function $T(\varphi,\theta_1,\theta_2)$ is bounded and non-singular, both processes P and S possess the same stability behavior. Therefore, transformation function $T(\varphi,\theta_1,\theta_2)$ is chosen so that the drift term, of the Itô differential Eq. (15), does not depend on the phase processes θ_1, θ_2 and φ, so that

$$dS = \Lambda(p)\,S\,dt + S\,T^{-1}\left(T\gamma_{11} + T'_{\phi}\gamma_{21} + T'_{\theta_1}\gamma_{31} + T'_{\theta_2}\gamma_{41}\right)dw_1(t) + $$
$$+ S\,T^{-1}\left(T\gamma_{12} + T'_{\phi}\gamma_{22} + T'_{\theta_1}\gamma_{32} + T'_{\theta_2}\gamma_{42}\right)dw_2(t)\,. \tag{16}$$

By comparing Eqs. (14) and (16), it can be seen that such a transformation function $T(\varphi,\theta_1,\theta_2)$ is given by the following equation

$$[L_0 + \varepsilon L_1]T(\varphi,\theta_1,\theta_2) = \Lambda(p)T(\varphi,\theta_1,\theta_2). \tag{17}$$

In (17) L_0 and L_1 are the following first and second-order differential operators

$$L_0 = \omega_1\frac{\partial}{\partial\theta_1} + \omega_2\frac{\partial}{\partial\theta_2},$$

$$L_1 = a_1\frac{\partial^2}{\partial\varphi^2} + a_2\frac{\partial^2}{\partial\theta_1^2} + a_3\frac{\partial^2}{\partial\theta_2^2} + +a_4\frac{\partial^2}{\partial\varphi\partial\theta_1} + a_5\frac{\partial^2}{\partial\varphi\partial\theta_2} + a_6\frac{\partial^2}{\partial\theta_1\partial\theta_2} + \tag{18}$$

$$+ b_1\frac{\partial}{\partial\varphi} + b_2\frac{\partial}{\partial\theta_1} + b_3\frac{\partial}{\partial\theta_2} + c,$$

where a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3 and c are given in Appendix 2.

Eq. (17) defines an eigenvalue problem for a second-order differential operator of three independent variables, in which $\Lambda(p)$ is the eigenvalue and $T(\varphi,\theta_1,\theta_2)$ the associated eigenfunction. From Eq. (16), the eigenvalue $\Lambda(p)$ is seen to be the Lyapunov exponent of the pth moment of system (7), i. e., $\Lambda(p) = \Lambda_{x(t)}(p)$. This approach was first applied by Wedig [11] to derive the eigenvalue problem for the moment Lyapunov exponent of a two-dimensional linear Itô stochastic system. In the following section, the method of regular perturbation is applied to the eigenvalue problem (17) to obtain a weak noise expansion of the moment Lyapunov exponent of a four-dimensional stochastic linear system.

3. WEAK NOISE EXPANSION OF THE MOMENT LYAPUNOV EXPONENT

Applying the method of regular perturbation, both the moment Lyapunov exponent $\Lambda(p)$ and the eigenfunction $T(\varphi,\theta_1,\theta_2)$ are expanded in power series of ε as:

$$\Lambda(p) = \Lambda_0(p) + \varepsilon\Lambda_1(p) + \varepsilon^2\Lambda_2(p) + \cdots + \varepsilon^n\Lambda_n(p) + \cdots,$$
$$T(\varphi,\theta_1,\theta_2) = T_0(\varphi,\theta_1,\theta_2) + \varepsilon T_1(\varphi,\theta_1,\theta_2) + \varepsilon^2 T_2(\varphi,\theta_1,\theta_2) + \cdots + \varepsilon^n T_n(\varphi,\theta_1,\theta_2) + \cdots. \tag{19}$$

Substituting the perturbation series (19) into the eigenvalue problem (17) and equating terms of the equal powers of ε leads to the following equations

$$\varepsilon^0 \rightarrow L_0 T_0 = \Lambda_0(p) T_0 ,$$
$$\varepsilon^1 \rightarrow L_0 T_1 + L_1 T_0 = \Lambda_0(p) T_1 + \Lambda_1(p) T_0 ,$$
$$\varepsilon^2 \rightarrow L_0 T_2 + L_1 T_1 = \Lambda_0(p) T_2 + \Lambda_1(p) T_1 + \Lambda_2(p) T_0 ,$$
$$\varepsilon^3 \rightarrow L_0 T_3 + L_1 T_2 = \Lambda_0(p) T_3 + \Lambda_1(p) T_2 + \Lambda_2(p) T_1 + \Lambda_3(p) T_0 , \tag{20}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\varepsilon^n \rightarrow L_0 T_n + L_1 T_{n-1} = \Lambda_0(p) T_n + \Lambda_1(p) T_{n-1} + \Lambda_2(p) T_{n-2} + \cdots + \Lambda_{n-1}(p) T_1 + \Lambda_n(p) T_0 ,$$

where each function $T_i = T_i(\varphi, \theta_1, \theta_2)$, $i = 0, 1, 2, \ldots$ must be positive and periodic in the range $0 \leq \varphi \leq \pi/2$, $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_2 \leq 2\pi$.

3.1. Zeroth order perturbation

The zeroth order perturbation equation is $L_0 T_0 = \Lambda_0(p) T_0$ or

$$\omega_1 \frac{\partial T_0}{\partial \theta_1} + \omega_2 \frac{\partial T_0}{\partial \theta_2} = \Lambda_0(p) T_0 . \tag{21}$$

From the property of the moment Lyapunov exponent, it is known that

$$\Lambda(0) = \Lambda_0(0) + \varepsilon \Lambda_1(0) + \varepsilon^2 \Lambda_2(0) + \cdots + \varepsilon^n \Lambda_n(0) = 0 , \tag{22}$$

which results in $\Lambda_n(0) = 0$ for $n = 0$, 1, 2, 3,.... Since the eigenvalue problem (21) does not contain p, the eigenvalue $\Lambda_0(p)$ is independent of p. Hence, $\Lambda_0(0) = 0$ leads to

$$\Lambda_0(p) = 0 . \tag{23}$$

Now, partial differential Eqs. (21) have the form

$$\omega_1 \frac{\partial T_0}{\partial \theta_1} + \omega_2 \frac{\partial T_0}{\partial \theta_2} = 0 . \tag{24}$$

Solution of Eq.(24) may be taken as

$$T_0(\varphi, \theta_1, \theta_2) = \psi_0(\varphi) , \tag{25}$$

where $\psi_0(\varphi)$ is an unknown function of φ which has yet to be determined.

3.2. First order perturbation

The first order perturbation equation is

$$L_0 T_1 = \Lambda_1(p) T_0 - L_1 T_0 . \tag{26}$$

Since the homogeneous Eq. (24) has a non-trivial solution given by Eq. (25), for Eq. (26) to have a solution it is required, from the Fredholm alternative, that following is satisfied:

$$(L_0 T_1, T_0^*) = (\Lambda_1(p) T_0 - L_1 T_0, T_0^*) = 0 . \tag{27}$$

In the previous equation, $T_0^* = \Psi_0(\varphi)$ is an unknown solution of the associated adjoint differential equation of (24), and (f,g) denotes the inner product of functions $f(\varphi, \theta_1, \theta_2)$ and $g(\varphi, \theta_1, \theta_2)$ defined by

$$(f, g) = \int\limits_0^{\pi/2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} f(\varphi, \theta_1, \theta_2) g(\varphi, \theta_1, \theta_2) d\theta_1 d\theta_2 d\varphi \; . \tag{28}$$

Taking onto account (25), (26) and (28), the expression (27) has the form

$$\int\limits_0^{\pi/2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} (\Lambda_1(p)\psi_0 - L_1\psi_0)\Psi_0(\varphi) \, d\theta_1 d\theta_2 d\varphi = 0 \; , \tag{29}$$

and will be satisfied if and only if

$$\int\limits_0^{2\pi} \int\limits_0^{2\pi} (\Lambda_1(p)\psi_0 - L_1\psi_0) \, d\theta_1 d\theta_2 = 0 \; . \tag{30}$$

After the integration of the previous expression we have that

$$\bar{L}(\psi_0) = A_1(\phi)\frac{d^2\psi_0}{d\varphi^2} + B_1(\phi)\frac{d\psi_0}{d\varphi} + C_1(\varphi)\psi_0 - \Lambda_1(p)\psi_0 = 0 \; , \tag{31}$$

where

$$A_1(\varphi) = \int\limits_0^{2\pi} \int\limits_0^{2\pi} a_1(\varphi, \theta_1, \theta_2) \, d\theta_1 d\theta_2, \; B_1(\varphi) = \int\limits_0^{2\pi} \int\limits_0^{2\pi} b_1(\varphi, \theta_1, \theta_2) \, d\theta_1 d\theta_2 \; ,$$

$$C_1(\varphi) = \int\limits_0^{2\pi} \int\limits_0^{2\pi} c(\varphi, \theta_1, \theta_2) \, d\theta_1 d\theta_2. \tag{32}$$

Finally, A_1, B_1 and C_1 are

$$A_1(\varphi) = -\frac{1}{128}[p_{11}^2 + p_{22}^2 - 2(p_{12}^2 + p_{21}^2)]\cos 4\varphi -$$
$$-\frac{p_{12}^2 - p_{21}^2}{16}(\omega_1^2 - \omega_2^2)\cos 2\varphi +$$
$$+\frac{1}{128}[p_{11}^2 + p_{22}^2 + 6(p_{12}^2 + p_{21}^2)] \; ,$$

$$B_1(\varphi) = -\frac{1}{64}(p-1)[p_{11}^2 + p_{22}^2 - 2(p_{12}^2 + p_{21}^2)]\sin 4\varphi -$$
$$-\frac{1}{8}(p_{12}^2 \sin^2 \varphi \, tg\varphi - p_{21}^2 \cos^2 \varphi \cot \varphi) + \tag{33}$$
$$+\frac{1}{32}\left\{16\beta_1 - 16\beta_2 - [(p+2)(p_{11}^2 - p_{22}^2) + 2(p-1)(p_{12}^2 - p_{21}^2)]\right\}\sin 2\varphi \; ,$$

$$C_1(\varphi) = \frac{1}{128} p(p-2)[p_{11}^2 + p_{22}^2 - 2(p_{12}^2 + p_{21}^2)]\cos 4\varphi -$$
$$-\frac{1}{32}\left\{16\beta_1 - 16\beta_2 - [(p+2)(p_{11}^2 - p_{22}^2) - 4(p_{12}^2 - p_{21}^2)]\right\}\cos 2\varphi +$$
$$+\frac{1}{128} p\left\{-64\beta_1 - 64\beta_2 + [(10+3p)(p_{11}^2 + p_{22}^2) + 2(6+p)(p_{12}^2 + p_{21}^2)]\right\}$$

Since the coefficients (33) of the Eq.(31) are periodic functions of φ, a series expansion of the function $\psi_0(\varphi)$ may be taken in the form

$$\psi_0(\phi) = \sum_{k=0}^{N} K_k \cos 2k\varphi. \tag{34}$$

Substituting (34) in (31), multiplying the resulting equation by $\cos 2k\varphi$ ($k = 0$, 1, 2 ...) and integrating with respect φ from 0 to $\pi/2$ leads to a set of 2N+1 homogenuos linear equations for the unknown coefficients K_0, K_1, K_2...

$$\sum_{j=0}^{N} A_{jk} K_j = \Lambda_1(p) K_k , \tag{35}$$

where

$$A_{jk} = \int_0^{\pi/2} \overline{L}\big(\cos(2j\varphi)\big) \cos(2k\varphi) d\varphi, \quad k=0, 1, 2, 3, ...N. \tag{36}$$

When N tends to infinity, the solution (34) tends to the exact solution. The condition for system homogeneous linear equations (35) to have nontrivial solutions is that the determinant of system homogeneous linear equations (35) is equal to zero. The coefficients A_{jk} to order N=4 are presented in Appendix 3.

In the case when N=0, we assume a solution (34) in the form $\psi_0(\varphi) = K_0$. From conditions that $A_{00} = 0$, the moment Lyapunov exponent in the first perturbation is defined as

$$\Lambda_1(p) = -\frac{p}{2}(\beta_1 + \beta_2) + \frac{p(10+3p)}{128}(p_{11}^2 + p_{22}^2) + \frac{p(6+p)}{64}(p_{12}^2 + p_{21}^2). \tag{37}$$

In the case when N=1, the solution (34) has the form $\psi_0(\varphi) = K_0 + K_1 \cos 2\varphi$, then moment Lyapunov exponent in the first perturbation is the solution of the equation $\Lambda_1^2 + d_1^{(1)}\Lambda_1 + d_0^{(1)} = 0$ where coefficients $d_0^{(1)}$ and $d_1^{(1)}$ are presented in Appendix 4. In the case when N=2, the solution (34) has the form $\psi_0(\varphi) = K_0 + K_1 \cos 2\varphi + K_2 \cos 4\varphi$, the moment Lyapunov exponent in the first perturbation is the solution of the equation $\Lambda_1^3 + d_2^{(2)}\Lambda_1^2 + d_1^{(2)}\Lambda_1 + d_0^{(2)} = 0$ where coefficients $d_0^{(2)}$, $d_1^{(2)}$ and $d_2^{(2)}$ are presented in Appendix 5. However, for $N > 2$, it is impossible to obtain the explicit expressions of $\Lambda_1(p)$ and the numerical results must be given, for N = 3 and 4.

4. APPLICATION TO A THIN-WALLED BEAM SUBJECTED TO AXIAL LOADS AND END MOMENTS

The purpose of this section is to present the general results of the above sections in the context of real engineering applications and show how these results can be applied to physical problems. To this end, we consider the flexural-torsional vibration stability of a homogeneous, isotropic, thin walled beam with two planes of symmetry. The beam is assumed to be loaded in the plane of greater bending rigidity by two equal couples and stochastic axial loads and stochastically fluctuating end moments (Fig. 1).

The governing differential equations for the coupled flexural and torsional motion of the beam can be written as given by Pavlović et al. in [9]

$$\rho A \frac{\partial^2 U}{\partial T^2} + \alpha_u \frac{\partial U}{\partial t} + EI_y \frac{\partial^4 U}{\partial Z^4} + \overline{M}(T) \frac{\partial^2 \varphi}{\partial Z^2} + \overline{F}(T) \frac{\partial^2 U}{\partial Z^2} = 0,$$

$$\rho I_p \frac{\partial^2 \varphi}{\partial T^2} + \alpha_\varphi \frac{\partial \varphi}{\partial T} - \left(GJ - \overline{F}(t) \frac{I_p}{A} \right) \frac{\partial^2 \varphi}{\partial Z^2} + \overline{M}(T) \frac{\partial^2 U}{\partial Z^2} + EI_s \frac{\partial^4 \varphi}{\partial Z^4} = 0,$$ (38)

where U is the flexural displacement in the x-direction, ϕ is the torsional displacement, ρ is mass density, A is area of the cross-section of beam, I_y, I_p, I_S are axial, polar and sectorial moments of inertia, J is Saint–Venant torsional constant, E is Young modulus of elasticity, G is shear modulus, α_U, α_φ are viscous damping coefficients, T is time and Z is axial coordinate.
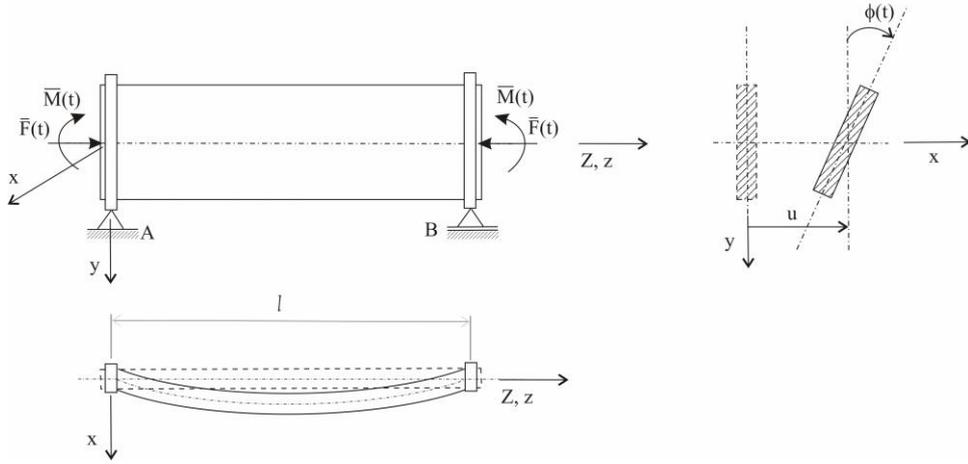
Fig. 1 Geometry of a thin-walled beam system

Using the following transformations

$$U = u\sqrt{\frac{I_p}{A}}, \ Z = zl, \ T = k_t t, \ \overline{F}(T) = F_{cr}F(t), \ \overline{M}(T) = M_{cr}M(t),$$

$$F_{cr} = \frac{\pi^2 EI_y}{l^2}, \ M_{cr} = \frac{\pi}{l}\sqrt{EI_y GJ}, \ k_t^2 = \frac{\rho Al^4}{EI_y}, \ e = \frac{AI_s}{I_y I_p},$$ (39)

$$\varepsilon\beta_1 = \frac{1}{2}\alpha_U \frac{l^2}{\sqrt{\rho AEI_y}}, \ \varepsilon\beta_2 = \frac{1}{2}\alpha_\varphi l^2 \sqrt{\frac{A}{\rho EI_y I_p^2}}, \ s = \frac{GJAl^2}{\pi^2 EI_y I_p},$$

where *l* is the length of the beam, F_{cr} is Euler critical force, M_{cr} is critical buckling moment for the simply supported narrow rectangular beam, S is slenderness parameter, β_1 and β_2 are reduced viscous damping coefficients, we get governing equations as

$$\frac{\partial^2 u}{\partial t^2} + 2\varepsilon\beta_1 \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial z^4} + \pi^2 \sqrt{s} M(t) \frac{\partial^2 \varphi}{\partial z^2} + \pi^2 F(T) \frac{\partial^2 u}{\partial z^2} = 0,$$

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\varepsilon\beta_2 \frac{\partial \varphi}{\partial T} - \pi^2 \left(s - F(t) \right) \frac{\partial^2 \varphi}{\partial z^2} + \pi^2 \sqrt{s} M(t) \frac{\partial^2 u}{\partial z^2} + e \frac{\partial^4 \varphi}{\partial z^4} = 0.$$ (40)

Taking free warping displacement and zero angular displacements into account, boundary conditions for the simply supported beam are

$$u(t,0) = u(t,1) = \left.\frac{\partial^2 u}{\partial z^2}\right|_{(t,0)} = \left.\frac{\partial^2 u}{\partial z^2}\right|_{(t,1)} = 0,$$

$$\varphi(t,0) = \varphi(t,1) = \left.\frac{\partial^2 \varphi}{\partial z^2}\right|_{(t,0)} = \left.\frac{\partial^2 \varphi}{\partial z^2}\right|_{(t,1)} = 0.$$ (41)

Consider the shape function $\sin(\pi z)$ which satisfies the boundary conditions for the first mode vibration, the displacement $u(t.z)$ and twist $\varphi(t,z)$ can be described by

$$u(t,z) = q_1(t)\sin \pi z, \quad \varphi(t,z) = q_2(t)\sin \pi z.$$ (42)

Substituting $u(t,z)$ and $\varphi(t,z)$ from (42) into the equations of motion (40) and employing Galerkin method unknown time functions can be expressed as

$$\ddot{q}_1 + \omega_1^2 q_1 + 2\beta_1\varepsilon \dot{q}_1 - K_{11}F(t)q_1 - K_{12}M(t)q_2 = 0,$$

$$\ddot{q}_2 + \omega_2^2 q_2 + 2\beta_2\varepsilon \dot{q}_2 - K_{21}M(t)q_1 - K_{22}F(t)q_2 = 0.$$ (43)

If we are defined the expressions

$$\omega_1^2 = \pi^4, \ \omega_2^2 = \pi^4(s+e), \ K_{11} = K_{22} = \pi^4, \ K_{12} = K_{21} = \pi^4\sqrt{s},$$ (44)

and assume that the compressive stochastic axial force and stochastically fluctuating end moment are white-noise processes (4) with small intensity

$$F(t) = \sqrt{\varepsilon}\xi_1(t), \quad M(t) = \sqrt{\varepsilon}\xi_2(t),$$ (45)

then Eq. (43) is reduced to Eq. (3).

Using the above result for the moment Lyapunov exponent in the first-order perturbation,

$$\Lambda(p) = \varepsilon\Lambda_1(p) + O(\varepsilon^2),$$ (46)

with the definition of the moment stability $\Lambda(p) < 0$, we determine analytically (the case where N = 0, $\Lambda_1(p)$ is shown with Eq.(37)) the pth moment stability boundary of the oscillatory system as

$$\beta_1 + \beta_2 > \pi^4 \frac{1+s+e}{s+e}\left(\frac{10+3p}{64}\sigma_1^2 + \frac{6+p}{32} s\sigma_2^2 \right).$$ (47)

It is known that the oscillatory system (40) is asymptotically stable only if the Lyapunov exponent $\lambda < 0$. Then expression

$$\lambda = \varepsilon\lambda_1 + O(\varepsilon^2), \tag{48}$$

is employed to determine the almost-sure stability boundary of the oscillatory system in the first-order perturbation

$$\beta_1 + \beta_2 > \pi^4 \frac{1+s+e}{s+e}\left(\frac{5}{32}\sigma_1^2 + \frac{3}{16}s\sigma_2^2\right). \tag{49}$$

In [9], Pavlović et al. by using the direct Lyapunov method, investigated the almost sure asymptotic stability boundary of an oscillatory system as the function of stochastic process, damping coefficient and geometric and physical parameters of the beam. According to the authors, the condition for almost sure stochastic stability may be expressed by the following expression

$$\pi^8(\sigma_1^2 + s\sigma_2^2)^2 - 2\pi^4(\sigma_1^2 + s\sigma_2^2)[\beta_1 + \beta_2(s+e)] + 4\beta_1\beta_2(s+e) > 0. \tag{50}$$

For the sake of simplicity in the comparison of results, in the following we assume that two viscous damping coefficients are equal

$$\beta_1 = \beta_2 = \beta, \tag{51}$$

For this case, we determine the almost-sure stability boundary as

$$\beta > \frac{3\pi^4}{32}\frac{1+s+e}{s+e}\left(\frac{5}{6}\sigma_1^2 + s\sigma_2^2\right), \tag{52}$$

and the pth moment stability boundary of the oscillatory system in the first-order perturbation as

$$\beta > \frac{\pi^4}{128}\frac{1+s+e}{s+e}[(10+3p)\sigma_1^2 + 2(6+p)s\sigma_2^2]. \tag{53}$$

Starting from Eq. (50), derived by Pavlović et al. [9], the almost sure stability boundary can be determined in the form

$$\beta > \frac{\pi^4}{2}(\sigma_1^2 + s\sigma_2^2). \tag{54}$$

With respect to standard I-section we can approximately take that ratios $h/b \approx 2$, $b/\delta_1 \approx 11$, $\delta/\delta_1 \approx 1.5$, where h is depth, b is width, δ is thickness of the flanges and δ_1 is thickness of the rib of I-section. These ratios give us $s \approx 0.01928(l/h)^2$ and $e \equiv 1.176$. For the narrow rectangular cross section, according to assumption $\delta/h < 0.1$, for thin-walled cross sections $s \approx 1.88(l/h)^2$ and $e \equiv 0$, which is obtained using the approximation $1 + (\delta/h)^2 \approx 1$.

a) I-section   **b)** Narrow rectangular cross section
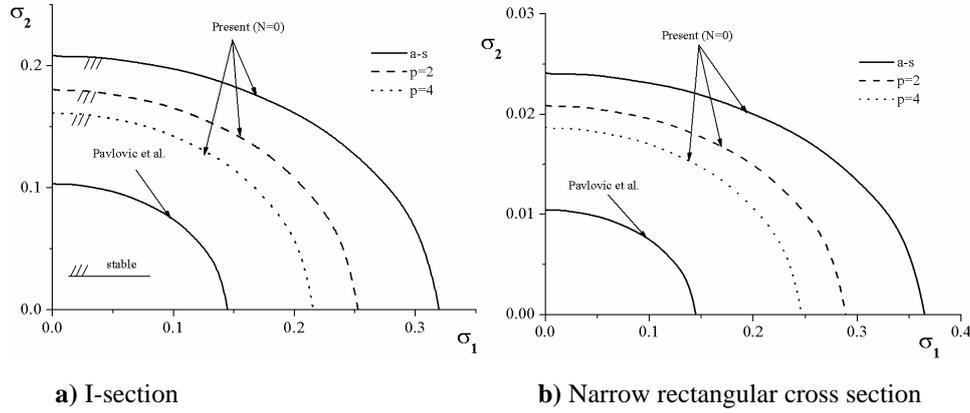
Fig. 2. Stability regions for almost-sure (a-s) and pth moment stability for $\varepsilon = 0.1$

Almost-sure stability boundary and *pth* moment stability boundary in the first-order perturbation for I-section are given in Fig. 2a, and for narrow rectangular cross section in Fig. 2b. It is evident that stability regions in the present study are higher compared to the results obtained by Pavlović et al. [9]. Also, the moment stability boundaries (53) are more conservative than the almost-sure boundary (52). It is evident that end moment variances are about ten times higher for I-section than for narrow rectangular section, when stochastic axial force vary only a little.

5. NUMERICAL DETERMINATION OF THE Pᵀᴴ MOMENT LYAPUNOV EXPONENT

Numerical determination of the pth moment Lyapunov exponent is important in assessing the validity and the ranges of applicability of the approximate analytical results. In many engineering applications, the amplitudes of noise excitations are not small so that the approximate analytical methods such as the method of perturbation or the method of stochastic averaging cannot be applied. Therefore, numerical approaches have to be employed to evaluate the moment Lyapunov exponents. The numerical approach is based on expanding the exact solution of the system of Itô stochastic differential equations in powers of the time increment h and the small parameter ε as proposed in Milstein and Tret'Yakov [8]. The state vector of the system (7) is to be rewritten as a system of Itô stochastic differential equations with small noise in the form

$$dx_1 = \omega_1 x_2 dt,$$
$$dx_2 = [-\omega_1 x_1 - \varepsilon 2\beta_1 x_2]dt + \sqrt{\varepsilon}\, p_{11} x_1 dw_1(t) + +\sqrt{\varepsilon}\, p_{12} x_3 dw_2(t),$$
$$dx_3 = \omega_2 x_4 dt,$$
$$dx_4 = [-\omega_2 x_3 - \varepsilon 2\beta_2 x_4]dt + \sqrt{\varepsilon}\, p_{22} x_3 dw_1(t) + +\sqrt{\varepsilon}\, p_{21} x_1 dw_2(t).$$

(55)

For the numerical solutions of the stochastic differential equations, the Runge-Kutta approximation may be applied, with error $R = O(h^4 + \varepsilon^4 h)$. The interval discretization is $[t_0, T]$: { t_k : k=0,1,2,3, ...M; $t_0 < t_1 < t_2 ....< t_M =$T} and the time increment is $h = t_{j+1} - t_j$.

The following Runge-Kutta method used to obtain the (k+1)th iteration of the state vector $X = (x_1, x_2, x_3, x_4)$

$$x_1^{(k+1)} = \left[1 - \frac{h^2\omega_1^2}{2} + \frac{h^4\omega_1^4}{24} + \sqrt{\varepsilon}\,\frac{p_{11}h^{3/2}\omega_1(\xi_1 + 2\eta_1)}{2} + \varepsilon\frac{\beta_1 h^3\omega_1^2}{3} \right] x_1^{(k)} +$$

$$+ \left[h\omega_1\left(1 - \frac{h^2\omega_1^2}{6}\right) + \sqrt{\varepsilon}\,\frac{p_{11}h^{5/2}\omega_1^2\xi_1}{6} + \varepsilon\beta_1 h^2\omega_1\left(\frac{h^2\omega_1^2}{9} - 1\right) \right] x_2^{(k)} +$$

$$+ \sqrt{\varepsilon}\,\frac{p_{12}h^{3/2}\omega_1(\xi_2 + 2\eta_2)}{2}\, x_3^{(k)} + \sqrt{\varepsilon}\,\frac{p_{12}h^{5/2}\omega_1\omega_2\xi_2}{6}\, x_4^{(k)}\ ,$$

$$x_2^{(k+1)} = \left[-h\omega_1\left(1 - \frac{h^2\omega_1^2}{6}\right) + \sqrt{\varepsilon}\,p_{11}h^{1/2}\xi_1\left(1 - \frac{h^2\omega_1^2}{3}\right) + \varepsilon\beta_1 h^2\omega_1\left(1 - \frac{h^2\omega_1^2}{6}\right) \right] x_1^{(k)} +$$

$$+ \left[1 - \frac{h^2\omega_1^2}{2} + \frac{h^4\omega_1^4}{24} + \sqrt{\varepsilon}\,\frac{p_{11}h^{3/2}\omega_1(\xi_1 - 2\eta_1)}{2} + \varepsilon 2\beta_1 h\left(-\frac{h^4\omega_1^4}{36} + \frac{h^2\omega_1^2}{3} - 1\right) \right] x_2^{(k)} +$$

$$+ \sqrt{\varepsilon}\,p_{12}h^{1/2}\xi_2\left(1 - \frac{h^2\omega_1^2}{6} - \frac{h^2\omega_2^2}{6}\right) x_3^{(k)} + \sqrt{\varepsilon}\,\frac{p_{12}h^{3/2}\omega_2(\xi_2 - 2\eta_2)}{2}\, x_4^{(k)}\ ,$$

$$x_3^{(k+1)} = \sqrt{\varepsilon}\,\frac{p_{21}h^{3/2}\omega_2(\xi_2 + 2\eta_2)}{2}\, x_1^{(k)} + \sqrt{\varepsilon}\,\frac{p_{21}h^{5/2}\omega_1\omega_2\xi_2}{6}\, x_2^{(k)} +$$

$$+ \left[1 - \frac{h^2\omega_2^2}{2} + \frac{h^4\omega_2^4}{24} + \sqrt{\varepsilon}\,\frac{p_{22}h^{3/2}\omega_2(\xi_1 + 2\eta_1)}{2} + \varepsilon\frac{\beta_2 h^3\omega_2^2}{3} \right] x_3^{(k)} +$$

$$+ \left[h\omega_2\left(1 - \frac{h^2\omega_2^2}{6}\right) + \sqrt{\varepsilon}\,\frac{p_{22}h^{5/2}\omega_2^2\xi_1}{6} + \varepsilon\beta_2 h^2\omega_2\left(\frac{h^2\omega_2^2}{9} - 1\right) \right] x_4^{(k)},$$

$$\tag{56}$$

$$x_4^{(k+1)} = \sqrt{\varepsilon}\,p_{21}h^{1/2}\xi_2\left(1 - \frac{h^2\omega_1^2}{6} - \frac{h^2\omega_2^2}{6}\right) x_1^{(k)} + \sqrt{\varepsilon}\,\frac{p_{21}h^{3/2}\omega_2(\xi_2 - 2\eta_2)}{2}\, x_2^{(k)} +$$

$$+ \left[-h\omega_2\left(1 - \frac{h^2\omega_2^2}{6}\right) + \sqrt{\varepsilon}\,p_{22}h^{1/2}\xi_1\left(1 - \frac{h^2\omega_2^2}{3}\right) + \varepsilon\beta_2 h^2\omega_2\left(1 - \frac{h^2\omega_2^2}{6}\right) \right] x_3^{(k)} +$$

$$+ \left[1 - \frac{h^2\omega_2^2}{2} + \frac{h^4\omega_2^4}{24} + \sqrt{\varepsilon}\,\frac{p_{22}h^{3/2}\omega_1(\xi_1 - 2\eta_1)}{2} + \varepsilon 2\beta_2 h\left(-\frac{h^4\omega_2^4}{36} + \frac{h^2\omega_2^2}{3} - 1\right) \right] x_4^{(k)}\ .$$

Random variables ξ_i and η_i (i=1,2) are simulated as

$$P(\xi_i = -1) = P(\xi_i = 1) = \frac{1}{2}\ ,\quad P\left(\eta_i = \frac{-1}{\sqrt{12}}\right) = P\left(\eta_i = \frac{1}{\sqrt{12}}\right) = \frac{1}{2}\ . \tag{57}$$

Having obtained L samples of the solutions of the stochastic differential equations (56), the pth moment can be determined as follows

$$E\left[\left\|X(t_{k+1})\right\|^p\right] = \frac{1}{L}\sum_{j=1}^{L}\left\|X_j(t_{k+1})\right\|^p \ , \ \left\|X_j(t_{k+1})\right\| = \sqrt{[X_j(t_{k+1})]^T[X_j(t_{k+1})]} \ . \tag{58}$$

Using the Monte-Carlo technique by Xie [10], we numerically calculate the pth moment Lyapunov exponent for all values of p of interest as

$$\Lambda(p) = \frac{1}{T}\log E\left[\left\|X(T)\right\|^p\right]. \tag{59}$$

6. CONCLUSIONS

In this paper, the moment Lyapunov exponents of a thin-walled beam subjected to stochastic axial loads and stochastically fluctuating end moments under both white noises parametric excitations are studied. The method of regular perturbation is applied to obtain a weak noise expansion of the moment Lyapunov exponent in terms of the small fluctuation parameter. The weak noise expansion of the Lyapunov exponent is also obtained. The slope of the moment Lyapunov exponent curve at $p = 0$ is the Lyapunov exponent. When the Lyapunov exponent is negative, system (43) is stable with probability 1, otherwise it is unstable. For the purpose of illustration, in the numerical study we considered set system parameters $\beta_1 = \beta_2 = \beta = 1$, $\varepsilon = 0.1$, $L = 4000$, $h = 0.0005$, $M = 10000$ and $x_1(0) = x_2(0) = x_3(0) = 1/2$.

Typical results of the moment Lyapunov exponents $\Lambda(p)$ for system (43) given by Eq. (46) in the first perturbation are shown in Fig. 3 for I-section and the noise intensity $\sigma_1 = 0.1$ and $\sigma_2 = 0.15$. The accuracy of the approximate analytical results is validated and assessed by comparing them to the numerical results. The Monte Carlo simulation approach is usually more versatile, especially when the noise excitations cannot be described in such a form that can be treated easily using analytical tools. From the Central Limit Theorem, it is well known that the estimated pth moment Lyapunov exponent is a random number, with the mean being the true value of the pth moment Lyapunov exponent and standard deviation equal to $n_p / \sqrt{L}$, where n_p is the sample standard deviation determined from L samples. It is evident that the analytical result agrees very well with the numerical results, even for N = 0 when the function $\psi_0(\varphi)$ does not depend on φ and assumes the form $\psi_0(\varphi) = K_0$.

The moment Lyapunov exponents $\Lambda(p)$ in the first perturbation for narrow rectangular cross section and the noise intensity ($\sigma_1 = 0.15$ and $\sigma_2 = 0.01$ are shown in Fig. 4. Unlike the previous example, it is observed that the discrepancies between the approximate analytical and numerical results decrease for larger number N of series (34). Further increase of N number of members does not make sense, because the curves merge into one.
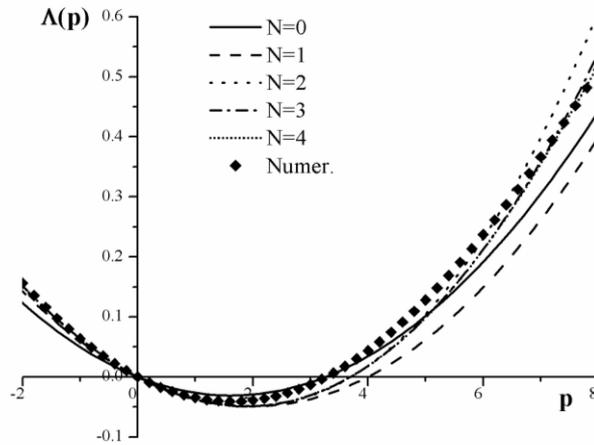
**Fig. 3** Moment Lyapunov exponent $\Lambda(p)$ for I-section ($\sigma_1 = 0.1$, $\sigma_2 = 0.15$)
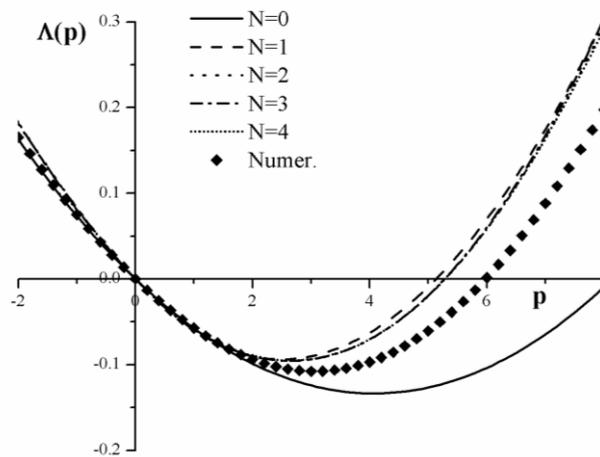
Fig. 4 Moment Lyapunov exponent $\Lambda(p)$ for narrow rectangular cross section
($\sigma_1 = 0.15$, $\sigma_2 = 0.01$)

If we consider the influence of cross-sectional area of stability boundary, generally speaking, the narrow rectangular cross section has smaller stability regions than the I-section. As for the influence of intensity of stochastic force, the end moment variances are about ten times higher for I-section than for narrow rectangular section, while the difference in axial force variances is small.

REFERENCES

1. Arnold, L., Doyle, M.N., Sri Namachchivaya, N., 1997, *Small noise expansion of moment Lyapunov exponents for two-dimensional systems*, Dynamics and Stability of Systems, 12(3), pp. 187-211.
2. Khasminskii, R., Moshchuk, N., 1998, *Moment Lyapunov exponent and stability index for linear conservative system with small random perturbation*, SIAM Journal of Applied Mathematics, 58(1), pp. 245-256.
3. Sri Namachchivaya, N., Van Roessel, H.J., Talwar, S., 1994, *Maximal Lyapunov exponent and almost-sure stability for coupled two-degree of freedom stochastic systems*, ASME Journal of Applied Mechanics, 61, pp. 446-452.
4. Sri Namachchivaya, N., Van Roessel, H.J., 2004, *Stochastic stability of coupled oscillators in resonance: A perturbation approach*, ASME Journal of Applied Mechanics, 71, pp. 759-767.
5. Kozić, P., Pavlović, R., Janevski, G., 2008, *Moment Lyapunov exponents of the stochastic parametrical Hill′s equation*, International Journal of Solids and Structures, 45(24), pp. 6056-6066.
6. Kozić, P., Janevski, G., Pavlović, R., 2009, *Moment Lyapunov exponents and stochastic stability for two coupled oscillators*, The Journal of Mechanics of Materials and Structures, 4(10), pp. 1689-1701.
7. Kozić, P., Janevski, G., Pavlović, R., 2010, *Moment Lyapunov exponents and stochastic stability of a double-beam system under compressive axial load*, International Journal of Solid and Structures, 47(10), pp. 1435-1442.
8. Milstein, N.G., Tret'Yakov, V.M., 1997, *Numerical methods in the weak sense for stochastic differential equations with small noise*, SIAM Journal on Numerical Analysis, 34(6), pp. 2142-2167.
9. Pavlović, R., Kozić, P., Rajković, P., Pavlović I., 2007, *Dynamic stability of a thin-walled beam subjected to axial loads and end moments*, Journal of Sound and Vibration, 301, pp. 690-700.
10. Xie, W.C., 2005, *Monte Carlo simulation of moment Lyapunov exponents*, ASME Journal of Applied Mechanics, 72, pp. 269-275.
11. Wedig, W., 1988, *Lyapunov exponent of stochastic systems and related bifurcation problems*, In: Ariaratnam, T.S., Schuëller, G.I., Elishakoff, I. (Eds.), *Stochastic Structural Dynamics–Progress in Theory and Applications*, Elsevier Applied Science, pp. 315 – 327.
12. Deng, J., Xie, W.C., Pandey M., 2014, *Moment Lyapunov exponents and stochastic stability of coupled viscoelastic systems driven by white noise*, Journal of mechanics of materials and structures, 9, pp. 27-50.
13. Deng, J., 2018, *Stochastic stability of coupled viscoelastic systems excited by real noise*, Mathematical problems in Engineering, Article ID 4725148.
14. Deng, J., Zhong, Z., Li, A., 2019, *Stochastic stability of viscoelastic plates under bounded noise excitation*, European Journal of Mechanics / A Solids, 78, Article ID 103849.

APPENDIX 1

$$\alpha_1 = -2pP(\beta_1 \sin^2\theta_1 \cos^2\phi + \beta_2 \sin^2\theta_2 \sin^2\phi) +$$

$$+ \frac{pP}{2}\{\cos^2\theta_1 + [(p-1)\cos^2\phi + \sin^2\phi]\sin^2\theta_1\}(p_{11}^2\cos^2\theta_1\cos^2\phi + p_{12}^2\cos^2\theta_2\sin^2\phi) +$$

$$+ \frac{pP}{2}\{\cos^2\theta_2 + [(p-1)\sin^2\phi + \cos^2\phi]\sin^2\theta_2\}(p_{22}^2\cos^2\theta_2\sin^2\phi + p_{21}^2\cos^2\theta_1\cos^2\phi) +$$

$$+ \frac{p(p-2)P}{16}(p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1 \sin 2\theta_2 \sin^2 2\phi,$$

$$\alpha_2 = (\beta_1 \sin^2\theta_1 - \beta_2 \sin^2\theta_2)\sin 2\phi - \frac{1}{16}(p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1 \sin 2\theta_2 \sin 4\phi -$$

$$- \frac{1}{4}p_{11}^2\cos^2\theta_1\sin 2\phi(\cos 2\theta_1 - \cos 2\phi \sin^2\theta_1) + \frac{1}{4}p_{22}^2\cos^2\theta_2\sin 2\phi(\cos 2\theta_2 + \cos 2\phi \sin^2\theta_2) +$$

$$+ \frac{1}{2}p_{12}^2\cos^2\theta_2\sin^2\phi(\sin^2\theta_1 \sin 2\phi - \cos^2\theta_1 \ tg\phi) - \frac{1}{2}p_{21}^2\cos^2\theta_1\cos^2\phi(\sin^2\theta_2 \sin 2\phi - \cos^2\theta_2 \ cot\phi),$$

$$\alpha_3 = -\beta_1 \sin 2\theta_1 - \left[\frac{p_{11}^2}{2}\cos^2\theta_1 + \frac{p_{12}^2}{2}\cos^2\theta_2 tg^2\phi\right]\sin 2\theta_1$$

$$\alpha_4 = -\beta_2 \sin 2\theta_2 - \left[\frac{p_{22}^2}{2}\cos^2\theta_2 + \frac{p_{21}^2}{2}\cos^2\theta_1 \cot^2\phi\right]\sin 2\theta_2 \ .$$

APPENDIX 2

$$a_1 = \frac{1}{32}(p_{11}\sin 2\theta_1 - p_{22}\sin 2\theta_2)^2 \sin^2 2\phi + \frac{1}{2}(p_{12}\cos\theta_1\sin\theta_2\cos^2\phi - p_{21}\sin\theta_1\cos\theta_2\sin^2\phi)^2 \ ,$$

$$a_2 = \frac{p_{11}^2}{2}\cos^4\theta_1 + \frac{p_{12}^2}{2}\cos^2\theta_1\cos^2\theta_2 tg^2\phi \ , \quad a_3 = \frac{p_{22}^2}{2}\cos^4\theta_2 + \frac{p_{21}^2}{2}\cos^2\theta_1\cos^2\theta_2 \cot^2\phi \ ,$$

$$a_4 = -\frac{p_{11}}{4}(p_{11}\sin 2\theta_1 - p_{22}\sin 2\theta_2)\cos^2\theta_1\sin 2\phi - \frac{p_{12}}{4}(p_{12}\cos^2\theta_2\sin 2\theta_1\sin^2\phi tg\phi - p_{21}\cos^2\theta_1\sin 2\theta_2\sin 2\phi) \ ,$$

$$a_5 = -\frac{p_{22}}{4}(p_{11}\sin 2\theta_1 - p_{22}\sin 2\theta_2)\cos^2\theta_2\sin 2\phi - \frac{p_{21}}{4}(p_{12}\cos^2\theta_2\sin 2\theta_1\sin 2\phi - p_{21}\cos^2\theta_1\sin 2\theta_2\sin^2\phi\cot\phi) \ ,$$

$$a_6 = p_{11}p_{22}\cos^2\theta_1\cos^2\theta_2 \ ,$$

$$b_1 = (\beta_1\sin^2\theta_1 - \beta_2\sin^2\theta_2) - \beta_2\sin^2\theta_2\sin 2\phi + \frac{p-1}{16}(p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1\sin 2\theta_2\sin 4\phi -$$
$$-\frac{1}{4}p_{11}^2\cos^2\theta_1\sin 2\phi[\cos^2\theta_1 + 2(p-1)\cos^2\phi\sin^2\theta_1] + \frac{1}{4}p_{22}^2\cos^2\theta_2\sin 2\phi[\cos^2\theta_2 + 2(p-1)\sin^2\phi\sin^2\theta_2] -$$
$$-\frac{1}{2}p_{12}^2\cos^2\theta_2\sin^2\phi[(p-1)\sin^2\theta_1\sin 2\phi + \cos^2\theta_1 tg\phi] + \frac{1}{2}p_{21}^2\cos^2\theta_1\cos^2\phi[(p-1)\sin^2\theta_2\sin 2\phi + \cos^2\theta_2\cot\phi],$$

$$b_2 = -\beta_1\sin 2\theta_1 + p_{11}^2\sin\theta_1\cos^3\theta_1[(p-1)\cos^2\phi - \sin^2\phi] + \frac{p}{2}p_{11}p_{22}\cos^2\theta_1\sin 2\theta_2\sin^2\phi +$$
$$+\frac{1}{4}p_{12}^2\sin 2\theta_1\cos^2\theta_2(p-2+p\cos 2\phi)tg^2\phi + \frac{p}{2}p_{12}p_{21}\cos^2\theta_1\sin 2\theta_2\sin^2\phi \ ,$$

$$b_3 = -\beta_2\sin 2\theta_2 + p_{22}^2\sin\theta_2\cos^3\theta_2[(p-1)\sin^2\phi - \cos^2\phi] + \frac{p}{2}p_{11}p_{22}\sin 2\theta_1\cos^2\theta_2\cos^2\phi +$$
$$+\frac{1}{4}p_{21}^2\cos^2\theta_1\sin 2\theta_2(p-2-p\cos 2\phi)\cot^2\phi + \frac{p}{2}p_{12}p_{21}\sin 2\theta_1\cos^2\theta_2\cos^2\phi,$$

$$c = -2p(\beta_1\sin^2\theta_1\cos^2\phi + \beta_2\sin^2\theta_2\sin^2\phi) + \frac{p(p-2)}{16}(p_{11}p_{22} + p_{12}p_{21})\sin 2\theta_1\sin 2\theta_2\sin^2 2\phi +$$
$$+\frac{p}{2}(p_{11}^2\cos^2\theta_1\cos^2\phi + p_{12}^2\cos^2\theta_2\sin^2\phi)\{\cos^2\theta_1 + [(p-1)\cos^2\phi + \sin^2\phi]\sin^2\theta_1\} +$$
$$+\frac{p}{2}(p_{22}^2\cos^2\theta_2\sin^2\phi + p_{21}^2\cos^2\theta_1\cos^2\phi)\{\cos^2\theta_2 + [(p-1)\sin^2\phi + \cos^2\phi]\sin^2\theta_2\},$$

APPENDIX 3

$$A_{00} = -\Lambda_1(p) - \frac{p}{2}(\beta_1 + \beta_2) + \frac{p(10+3p)}{128}(p_{11}^2 + p_{22}^2) + \frac{p(6+p)}{64}(p_{12}^2 + p_{21}^2),$$

$$A_{10} = -\frac{p+2}{4}(\beta_1 - \beta_2) + \frac{1}{64}(p+2)^2(p_{11}^2 - p_{22}^2) + \frac{1}{4}(p_{12}^2 - p_{21}^2),$$

$$A_{20} = \frac{(p+2)(p+4)}{256}\left[p_{11}^2 + p_{22}^2 - 2(p_{12}^2 + p_{21}^2)\right] - \frac{17}{32}(p_{12}^2 + p_{21}^2),$$

$$A_{30} = \frac{3}{4}(p_{12}^2 - p_{21}^2), \qquad A_{40} = -(p_{12}^2 + p_{21}^2),$$

$$A_{01} = -\frac{p}{4}(\beta_1 - \beta_2) + \frac{p(p+2)}{64}(p_{11}^2 - p_{22}^2) - \frac{p}{16}(p_{12}^2 - p_{21}^2),$$

$$A_{11} = -\frac{1}{2}\Lambda_1(p) - \frac{p}{4}(\beta_1 + \beta_2) + \frac{7p^2 + 22p - 8}{512}(p_{11}^2 + p_{22}^2) + \frac{p^2 + 10p - 56}{256}(p_{12}^2 + p_{21}^2),$$

$$A_{21} = -\frac{p+4}{8}(\beta_1 - \beta_2) + \frac{p^2 + 6p + 8}{128}(p_{11}^2 - p_{22}^2) + \frac{p+20}{32}(p_{12}^2 - p_{21}^2),$$

$$A_{31} = \frac{p^2 + 10p + 24}{512}(p_{11}^2 + p_{22}^2) - \frac{p^2 + 10p + 216}{256}(p_{12}^2 + p_{21}^2), \qquad A_{41} = -(p_{12}^2 - p_{21}^2),$$

$$A_{02} = \frac{p(p-2)}{256}[(p_{11}^2 + p_{22}^2) - 2(p_{12}^2 + p_{21}^2)],$$

$$A_{12} = -\frac{p-2}{8}(\beta_1 - \beta_2) + \frac{(p+2)(p-2)}{128}(p_{11}^2 - p_{22}^2) + \frac{p-2}{16}(p_{12}^2 - p_{21}^2),$$

$$A_{22} = -\frac{1}{2}\Lambda_1(p) - \frac{p}{4}(\beta_1 + \beta_2) + \frac{3p^2 + 10p - 16}{256}(p_{11}^2 + p_{22}^2) + \frac{p^2 + 6p - 80}{128}(p_{12}^2 + p_{21}^2),$$

$$A_{32} = -\frac{p+6}{8}(\beta_1 - \beta_2) + \frac{7p^2 + 8p + 12}{512}(p_{11}^2 - p_{22}^2) + \frac{p+18}{16}(p_{12}^2 - p_{21}^2)$$

$$A_{42} = \frac{p^2 + 14p + 48}{512}(p_{11}^2 + p_{22}^2) - \frac{p^2 + 14p + 304}{256}(p_{12}^2 + p_{21}^2),$$

$$A_{03} = 0, \qquad A_{13} = \frac{p^2 - 6p + 8}{512}\left[(p_{11}^2 + p_{22}^2) - 2(p_{12}^2 + p_{21}^2)\right],$$

$$A_{23} = -\frac{p-4}{8}\left[(\beta_1 - \beta_2) - \frac{1}{16}(p+2)(p_{11}^2 - p_{22}^2) + \frac{3}{4}(p_{12}^2 - p_{21}^2)\right],$$

$$A_{33} = -\frac{1}{2}\Lambda_1(p) - \frac{p}{4}(\beta_1 + \beta_2) + \frac{3p^2 + 10p - 36}{256}(p_{11}^2 + p_{22}^2) + \frac{2p^2 + 12p - 312}{256}(p_{12}^2 + p_{21}^2),$$

$$A_{43} = -\frac{p+8}{8}(\beta_1 - \beta_2) + \frac{p^2 + 10p + 16}{128}(p_{11}^2 - p_{22}^2) + \frac{3p+56}{32}(p_{12}^2 - p_{21}^2),$$

$$A_{04} = 0, \ A_{14} = 0, \ A_{24} = \frac{p^2 - 10p + 24}{512}[(p_{11}^2 + p_{22}^2) - 2(p_{12}^2 + p_{21}^2)],$$

$$A_{34} = -\frac{p-6}{8}\left[(\beta_1 - \beta_2) - \frac{p+2}{16}(p_{11}^2 - p_{22}^2) + (p_{12}^2 - p_{21}^2)\right],$$

$$A_{44} = -\frac{1}{2}\Lambda_1(p) - \frac{p}{4}(\beta_1 + \beta_2) + \frac{3p^2 + 10p - 64}{256}(p_{11}^2 + p_{22}^2) + \frac{2p^2 + 12p - 512}{256}(p_{12}^2 + p_{21}^2),$$

APPENDIX 4

$$d_1^{(1)} = p(\beta_1 + \beta_2) + \left(\frac{1}{32} - \frac{21p}{28} - \frac{13p^2}{256}\right)(p_{11}^2 + p_{22}^2) + \left(\frac{7}{16} - \frac{11p}{64} - \frac{3p^2}{128}\right)(p_{12}^2 + p_{21}^2),$$

$$d_0^{(1)} = \frac{1}{8}p(p-2)\left(\beta_1^2 + \beta_2^2\right) + \frac{1}{4}p(2+3p)\beta_1\beta_2 +$$

$$+\left(-\frac{13p}{2048} + \frac{p^2}{8192} + \frac{5p^3}{4096} + \frac{5p^4}{32768}\right)(p_{11}^4 + p_{22}^4) + \left(-\frac{5p}{512} + \frac{p^2}{2048} + \frac{p^3}{512} + \frac{p^4}{8192}\right)(p_{12}^4 + p_{21}^4)$$

$$+\left(\frac{3p}{1024} + \frac{97p^2}{4096} + \frac{29p^3}{2048} + \frac{37p^4}{16384}\right)p_{11}^2 p_{22}^2 + \left(-\frac{37p}{256} + \frac{p^2}{1024} + \frac{p^3}{256} + \frac{p^4}{4096}\right)p_{12}^2 p_{21}^2 +$$

$$+\left(-\frac{23p}{512} + \frac{7p^2}{2048} + \frac{17p^3}{2048} + \frac{5p^4}{8192}\right)(p_{12}^4 + p_{21}^4) +$$

$$+\left(-\frac{15p}{512} + \frac{7p^2}{2048} + \frac{9p^3}{2048} + \frac{5p^4}{8192}\right)(p_{11}^2 p_{21}^2 + p_{22}^2 p_{12}^2) +$$

$$+\left(-\frac{3p}{64} - \frac{37p^2}{256} - \frac{21p^3}{512}\right)(\beta_1 p_{22}^2 + \beta_2 p_{11}^2) + \left(\frac{5p}{64} - \frac{5p^2}{256} - \frac{5p^3}{512}\right)(\beta_1 p_{11}^2 + \beta_2 p_{22}^2) +$$

$$+\left(\frac{5p}{32} - \frac{7p^2}{128} - \frac{3p^3}{256}\right)(\beta_1 p_{21}^2 + \beta_2 p_{12}^2) + \left(\frac{9p}{32} - \frac{15p^2}{128} - \frac{3p^3}{256}\right)(\beta_1 p_{12}^2 + \beta_2 p_{21}^2).$$

APPENDIX 5

$$d_2^{(2)} = \frac{3p}{2}(\beta_1 + \beta_2) + \left(\frac{5}{32} - \frac{31p}{128} - \frac{19p^2}{256}\right)(p_{11}^2 + p_{22}^2) + \left(\frac{27}{16} - \frac{17p}{64} - \frac{5p^2}{128}\right)(p_{12}^2 + p_{21}^2)$$

$$d_1^{(2)} = \left(\frac{1}{2} - \frac{3p}{8} - \frac{9p^2}{16}\right)\left(\beta_1^2 + \beta_2^2\right) - \left(1 - \frac{3p}{4} - \frac{15p^2}{8}\right)\beta_1\beta_2 +$$

$$+\left(\frac{3}{256} - \frac{47p}{2048} + \frac{41p^2}{32768} + \frac{61p^3}{8192} + \frac{35p^4}{32768}\right)(p_{11}^4 + p_{22}^4) + \left(-\frac{1}{128} - \frac{55p}{1024} + \frac{153p^2}{4096} + \frac{133p^3}{4096} + \frac{83p^4}{16384}\right)p_{11}^2 p_{22}^2 +$$

$$+\left(\frac{15}{64} - \frac{55p}{512} - \frac{39p^2}{2048} + \frac{13p^3}{2048} + \frac{3p^4}{8192}\right)(p_{12}^4 + p_{21}^4) + \left(\frac{55}{32} - \frac{231p}{256} - \frac{71p^2}{1024} + \frac{13p^3}{1024} + \frac{3p^4}{4096}\right)p_{12}^2 p_{21}^2 +$$

$$+\left(\frac{9}{64} - \frac{165p}{512} - \frac{109p^2}{2048} + \frac{47p^3}{2048} + \frac{17p^4}{8192}\right)(p_{11}^2 p_{12}^2 + p_{22}^2 p_{21}^2) + \left(\frac{3}{32} - \frac{153p}{256} - \frac{61p^2}{1024} + \frac{35p^3}{1024} + \frac{17p^4}{4096}\right)(p_{11}^2 p_{21}^2 + p_{22}^2 p_{12}^2) +$$

$$+\left(\frac{1}{8} + \frac{p}{8} - \frac{43p^2}{128} - \frac{25p^3}{256}\right)(\beta_1 p_{22}^2 + \beta_2 p_{11}^2) + \left(-\frac{1}{8} + \frac{3p}{16} - \frac{19p^2}{128} - \frac{13p^3}{256}\right)(\beta_1 p_{11}^2 + \beta_2 p_{22}^2) +$$

$$+\left(\frac{3}{8} + \frac{45p}{32} - \frac{7p^2}{32} - \frac{5p^3}{128}\right)(\beta_1 p_{21}^2 + \beta_2 p_{12}^2) + \left(-\frac{3}{8} + \frac{63p}{32} - \frac{5p^2}{16} - \frac{5p^3}{128}\right)(\beta_1 p_{12}^2 + \beta_2 p_{21}^2).$$