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# LI-HE'S MODIFIED HOMOTOPY PERTURBATION METHOD FOR DOUBLY-CLAMPED ELECTRICALLY ACTUATED MICROBEAMS-BASED MICROELECTROMECHANICAL SYSTEM

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**Abstract**. *This paper highlights Li-He's approach in which the enhanced perturbation method is linked with the parameter expansion technology in order to obtain frequency amplitude formulation of electrically actuated microbeams-based microelectromechanical system (MEMS). The governing equation is a second-order nonlinear ordinary differential equation. The obtained results are compared with the solution achieved numerically by the Runge-Kutta (RK) method that shows the effectiveness of this variation in the homotopy perturbation method (HPM).*

**Key Words**: *Microelectromechanical systems, Enhanced perturbation method, Parameter expansion method, Nonlinear oscillator, Amplitude-frequency relationship*

## 1. INTRODUCTION

The last two decades have witnessed rapid advancement in nonlinear sciences arising in oscillation theory and other fields of physics [1-5]. Several methods were developed to find periodic solutions of nonlinear oscillatory systems, for example, variation iteration method (VIM) [6-7], homotopy perturbation method [8-9], Hamiltonian approach (HA) [10], energy balance method (EBM) [11], spreading residue harmonic balance method

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(SRHBM) [12], iteration perturbation method (IPM) [13], and other methods [14-15]. HPM was proposed in the later 1990s [16-17] and now has been established into a mature phase for ordinary differential equations [18-19], partial differential equations [20-21], and differential equations of fractional order [22-24]. It is widely applied for nonlinear oscillation problems of conservative oscillators [19,25], attachment oscillator [26], Fangzhu oscillator [27], micro systems' oscillators [28-29], and fractional-order oscillators [30]. Generally, a single iteration of this technique leads to a high accuracy of the solution.

Many researchers devoted their efforts and time to scrutinize the applicability of HPM for nonlinear problems and used it with the parameter expansion technology [31-32], the supporting terms [33-35], and the Laplace transform [18,21]. Recently, an adjustment in the perturbation method is proposed by Filobello-Nino [36], named the enhanced perturbation method. This method is highly accurate and provides better results because it deals with the problems with both small and large values of the perturbation parameters. Li and He [37] adopted this modification to link the enhanced perturbation method with the parameter expansion technology [19,31], and highly accurate results can be achieved for nonlinear oscillators. Ji et. al., [38] employed hybridization of Li-He's technique with EBM to find an approximate solution of the nonlinear problem of a tangent packing system.

The microelectromechanical system (MEMS) refers to the high technology devices of small sizes; it has become a hot topic in both academic and industrial communities [39-40]. The MEMS are intelligent structures and their systems are commonly micron or nanometer. Microelectronic technology is the origin of these tiny devices used in vibrators, sensors, switches, and so on [41-42]. Spring-base structures [43-45], nanotubes [15,46], and microbeams [11-12,14] can be considered as some of the potential and very applicable nano/microstructures in various sensing and actuating devices. These structures are modeled by generally using Galerikin's method and represented by nonlinear mathematical models. Different types of forcing nonlinearities such as electrostatic force [11,43], electromagnetic force [41,44], and van der Waals force [12] make the solution process extremely difficult. Therefore, approximate solutions of these nonlinear models are important for predicting their dynamic behavior.

Recently Fu et al. [11] studied electrically excited microbeams-based MEMS oscillator by employing the EBM [11]. The electrostatic force was used for actuation while the solution was depicted as an amplitude-frequency relationship. In this paper, we link the enhanced perturbation method with the parameter expansion technology [19,31] and propose an amplitude-frequency formula based on Li-He's approach [37] in order to find an approximate solution of the aforementioned model. The nonlinear frequency obtained from the proposed technology is compared with the frequency achieved numerically using the Runge-Kutta method (RK) for verification. We also match the results of Li-He's technique with those attained from EBM [11] to ensure the effectiveness of the suggested approach over EBM.

## 2. Problem Statement

Consider a doubly-clamped microbeam of length $L$, width $b$, thickness $h$ and density $\rho$ shown in Fig. 1 with coordinate system $OXYZ$. Equation of motion as deflection of microbeam can be expressed with a partial differential equation as

$$EI\frac{\partial^4 W}{\partial x^4}+\rho S\frac{\partial^2 W}{\partial \tau^2}-\left[\tilde{N}+\frac{ES}{2L}\int_0^L\left(\frac{\partial W}{\partial x}\right)dx\right]\frac{\partial^2 W}{\partial x^2}-F(x,\tau)=0 \tag{1}$$

where $W(x,\tau)$ is the function of location $x$ while time $\tau$ represents the deflection of microbeam, $E$ is the Young's modulus, $I=bh^3/12$ and $S=bh$ are moment of inertia about $Y$-axis and area of cross section, respectively, $\tilde{N}$ is the axial load between microbeam and its substrate and $F(x, \tau)$ is the actuation force resulting from electrostatic excitation [42].

$$F(x,\tau)=\frac{bv^2\varepsilon_v}{2}\left[\frac{1}{(d-W)^2}-\frac{1}{(d+W)^2}\right] \tag{2}$$

where $v$ denotes Poisson ratio, $\varepsilon_v$ is dielectric constant with usual value of 8.85 $\mathrm{PFm}^{-1}$ and $d$ is the initial gap between the substrate and the beam. As the whole study is performed for a doubly-clamped microbeam, the boundary conditions will be

$$W(0,\tau)=W(L,\tau)=0,\qquad \left.\frac{\partial W}{\partial x}\right|_{(0,\tau)}=\left.\frac{\partial W}{\partial x}\right|_{(L,\tau)}=0 \tag{3}$$

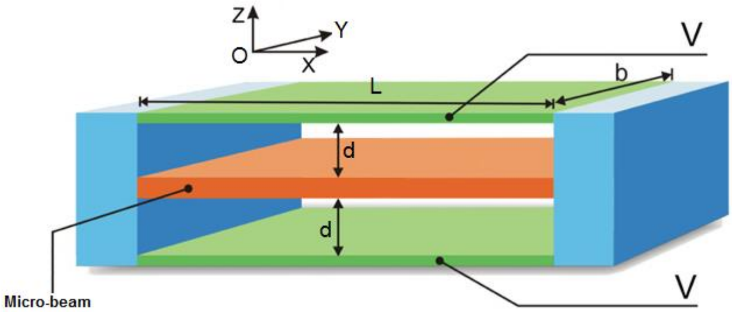


**Fig. 1** Model of doubly clamped electrically actuated microbeam-based MEMS

For simplicity, the variables of deflection of microbeam, location, and time in nondimensional form can be taken as

$$w=\frac{W}{d},\quad \eta=\frac{x}{L},\quad t=\frac{\tau}{\tilde{T}} \tag{4}$$

where
$$\tilde{T}=\sqrt{\frac{\rho hbL^4}{EI}} \tag{5}$$

Eq. (1) after substituting the nondimensional variables from Eq. (4) is given by

$$\frac{\partial^4 w}{\partial \eta^4}+\frac{\partial^2 w}{\partial t^2}-\left[\sigma+\beta\int_0^1\left(\frac{\partial w}{\partial \eta}\right)d\eta\right]\frac{\partial^2 w}{\partial \eta^2}-\frac{\kappa^2}{4}\left[\frac{1}{(1-w)^2}-\frac{1}{(1+w)^2}\right]=0 \tag{6}$$

where nondimensional parameters axial load $N$, aspect ratio $\alpha$ and parameter of electrostatic force $V$ in Eq. (4) are as follows

$$N = \frac{\tilde{N}L^2}{EI}, \quad \alpha = 6\left(\frac{d}{h}\right)^2, \quad V = \frac{24L^4 v^2 \varepsilon_v}{Ed^3h^3} \tag{7}$$

Also the boundary conditions in nondimensional form can be expressed as

$$w(0,t) = w(1,t) = 0, \qquad \left.\frac{\partial w}{\partial \eta}\right|_{(0,\tau)} = \left.\frac{\partial w}{\partial \eta}\right|_{(1,\tau)} = 0 \tag{8}$$

We apply method of separation of variables in order to find the solution of Eq. (6) subject to boundary conditions of Eq. (8). Hence defection function $w(\eta,t)$ can be written as the product of two functions.

$$w(\eta,t) = \xi(\eta)\chi(t) \tag{9}$$

where $\chi(t)$ is the time function and $\xi(\eta)$ is the trail function satisfying all the boundary conditions mentioned in Eq. (8). In our study, we use trail function suggested as [11]

$$\varphi(\eta) = 16\eta^2(1-\eta)^2 \tag{10}$$

Substitute Eq. (9) into Eq. (6) and multiply the governing equation by $\varphi(\eta)(1-w^2)^2$ and then integrate over dimensionless domain in order to obtain

$$\begin{aligned}&\int_0^1 \varphi(1-\varphi^2\chi^2)^2\varphi'''' \, d\eta + \int_0^1 \varphi^2(1-\varphi^2\chi^2)^2 \ddot{\chi} \, d\eta - \\ &\quad \int_0^1 \varphi(1-\varphi^2\chi^2)^2 \left(N + \alpha\int_0^1 \left(\frac{\partial w}{\partial \eta}\right) d\eta\right)\chi\varphi'' \, d\eta - \int_0^1 V^2\varphi^2\chi d\eta = 0\end{aligned} \tag{11}$$

where over dot $(\dot{\bullet})$ represents differentiation with respect to time variable $t$ and prime $(\bullet')$ represents the partial differentiation with respect to coordinate variable $\eta$. Eq. (11) can be rewritten as

$$(c_0 + c_1\chi^2 + c_2\chi^4)\chi'' + c_3\chi + c_4\chi^3 + c_5\chi^5 + c_6\chi^7 = 0 \tag{12}$$

where coefficients $c_0$, $c_1$,…, $c_6$ can be determined as follows

$$c_0 = \int_0^1 \varphi^2 \, d\eta$$

$$c_1 = -2\int_0^1 \varphi^4 \, d\eta$$

$$c_2 = \int_0^1 \varphi^6 \, d\eta$$

$$c_3 = \int_0^1 (\varphi\varphi'''' - N\varphi\varphi'' - V^2\varphi^2) \, d\eta$$

$$c_4 = \int_0^1 \left( -2\varphi^3\varphi'''' + 2N\varphi^3\varphi'' - \alpha\varphi\varphi'' \int_0^1 \varphi'^2 d\eta \right) d\eta$$

$$c_5 = \int_0^1 \left( \varphi^5\varphi'''' - N\varphi^5\varphi'' + 2\alpha\varphi^3\varphi'' \int_0^1 \varphi'^2 d\eta \right) d\eta$$

$$c_6 = -\int_0^1 \left( \alpha\varphi^5\varphi'' \int_0^1 \varphi'^2 d\eta \right) d\eta$$

Eq. (12) is a second-order nonlinear ordinary differential equation and Li-He's approach will be employed for the solution under the following initial conditions

$$\chi(0) = A, \qquad \chi'(0) = 0 \tag{13}$$

where $A$ is the initial amplitude of the nonlinear oscillatory system.

## 3. BASIC IDEA OF LI-HE'S APPROACH

To understand the idea of Li-He's approach, consider the linear oscillator

$$x'' + \Omega^2 x = 0 \tag{14}$$

where $x$ is the function of time $t$ representing the general displacement and $\Omega$ is the angular frequency of the oscillator. Eq. (14) can be expressed in an operator form as

$$(D^2 + \Omega^2)x = 0 \tag{15}$$

where $D=d/dt$ is a differential operator. The enhanced perturbation method apply annihilator operator $D^2+\Omega^2$ to Eq. (15); we have

$$(D^2 + \Omega^2)(D^2 + \Omega^2)x = x'''' + 2\Omega^2 x'' + \Omega^4 x = 0 \tag{16}$$

This method can solve a wide class of non-linear problems. It is more effective in the case of nonlinear problems with the forced term but it can also apply to the problems without forced term [36]. After applying some suitable substitution, the Eq. (16) is a higher-order equation and can be rewritten into linear $\underline{L}$ and nonlinear $\underline{N}$ operator form as:

$$\underline{L}x + \underline{N}x = 0 \tag{17}$$

We can construct the homotopy equation for Eq. (17) as

$$H(\xi, q) = (1-q)[\underline{L}(\xi) - \underline{L}(\xi_0)] + q[\underline{L}(\xi) + \underline{N}(\xi)] = 0, \qquad q \in [0,1] \tag{18}$$

where $q$ is embedding parameter and $\xi_0$ is initial solution of Eq. (17). It is clear from Eq. (8)

$$H(\xi, 0) = \underline{L}(\xi) - \underline{L}(\xi_0) = 0 \tag{19}$$

$$H(\xi, 1) = \underline{L}(\xi) + \underline{N}(\xi) = 0 \tag{20}$$

HPM uses embedding parameter $q$ as an expanding parameter [19], and basic assumption is that the solution of Eq. (17) can be specified as a power series in $q$:

$$\xi = \xi_0 + q\xi_1 + q^2\xi_2 + q^3\xi_3 + q^4\xi_4 + \cdots \tag{21}$$

Setting $q$=1 results in the approximate analytic solution of Eq. (17)

$$x = \lim_{q\to 1}\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4 + \cdots \tag{22}$$

## 4. SOLUTION OF MODEL PROBLEM

To apply Li-He's technique discussed in the above section, Eq. (12) can be expressed in the form

$$(1 + b_1\chi^2 + b_2\chi^4)\chi'' + b_3\chi + b_4\chi^3 + b_5\chi^5 + b_6\chi^7 = 0 \tag{23}$$

where $b_j = \dfrac{c_j}{c_0}$ $(j = 1, 2, \ldots, 6)$.

To reveal the solution process, consider Eq. (23) which is hard to be resolved analytically specially when $b_1$=−1, because the linear part has the form

$$\chi'' - \chi = 0$$

which has no periodic solution. We express Eq. (23) in an operator form as

$$[D^2(1 + b_1\chi + b_2\chi^3) + b_3 + b_4\chi^2 + b_5\chi^4 + b_6\chi^6]\chi = 0 \tag{24}$$

According to the enhanced perturbation method [36], we put on the annihilator operators $D^2$+1 to Eq. (14)

$$(D^2 + 1)\left[D^2(1 + b_1\chi + b_2\chi^3) + b_3 + b_4\chi^2 + b_5\chi^4 + b_6\chi^6\right]\chi = 0 \tag{25}$$

By applying the annihilator operators, a higher-order differential equation of Eq. (24) can be written as

$$\begin{aligned}&\chi'''' - b_3^2\chi - b_3b_4\chi^3 - b_3b_5\lambda\chi^5 - b_3b_6\chi^7 - b_1b_3\lambda\chi^2\chi'' - b_2b_3\lambda\chi^4\chi'' + b_4(6\chi\chi'^2 + \\ &\quad 3\chi^2\chi'') + b_5(20\chi^3\chi'^2 + 5\chi^4\chi'') + b_6(42\chi^5\chi'^2 + 7\chi^6\chi'') + b_1(2\chi'^2\chi'' + 2\chi\chi''^2 \\ &\quad + 4\chi\chi'\chi''' + \chi^2\chi'''') + b_2(12\chi^2\chi'^2\chi'' + 4\chi^3\chi''^2 + 8\chi^3\chi'\chi''' + \chi^4\chi'''') = 0\end{aligned} \tag{26}$$

The linear part becomes now

$$\chi'''' - b_3^2\chi = 0 \tag{27}$$

which represents a linear oscillator. For Eq. (26), the homotopy equation can be defined as

$$\begin{aligned}&\chi'''' - b_3^2\chi + p\Big[-b_3b_4\chi^3 - b_3b_5\lambda\chi^5 - b_3b_6\chi^7 - b_1b_3\lambda\chi^2\chi'' - b_2b_3\lambda\chi^4\chi'' + b_4(6\chi\chi'^2 \\ &\quad + 3\chi^2\chi'') + b_5(20\chi^3\chi'^2 + 5\chi^4\chi'') + b_6(42\chi^5\chi'^2 + 7\chi^6\chi'') + b_1(2\chi'^2\chi'' + 2\chi\chi''^2 \\ &\quad + 4\chi\chi'\chi''' + \chi^2\chi'''') + b_2(12\chi^2\chi'^2\chi'' + 4\chi^3\chi''^2 + 8\chi^3\chi'\chi''' + \chi^4\chi'''')\Big] = 0\end{aligned} \tag{28}$$

The solution and coefficient of the linear term can be expanding as

$$\chi = \chi_0 + p\chi_1 + p^2\chi_2 + \cdots\cdots \tag{29}$$

$$b_3^2 = \Omega^4 + p\Omega_1 + p^2\Omega_2 + \cdots\cdots \tag{30}$$

where $\Omega^4$ and $\Omega_i$ are constants and can be recognized by means of no secular term. Substituting Eq. (29) and Eq. (30) into Eq. (28) and continuing as that by the standard perturbation method, we have

$$\chi_0'''' - \Omega^4\chi_0 = 0, \qquad \chi_0(0) = A, \qquad \chi_0'(0) = 0 \tag{31}$$

$$\begin{aligned}&\chi_1'''' - \Omega^4\chi_1 - \Omega_1\chi_0 - b_3b_4\chi_0^3 - b_3b_5\chi_0^5 - b_3b_6\chi_0^7 - b_1b_3\chi_0^2\chi_0'' - b_2b_3\chi_0^4\chi_0'' + b_4(6\chi_0\chi_0'^2\\&+3\chi_0^2\chi_0'') + b_5(20\chi_0^3\chi_0'^2 + 5\chi_0^4\chi_0'') + b_6(42\chi_0^5\chi_0'^2 + 7\chi_0^6\chi_0'') + b_1(4\chi_0\chi_0'\chi_0''' +\\&2\chi_0'^2\chi_0'' + \chi_0^2\chi_0'''' + 2\chi_0\chi_0''^2) + b_2(12\chi_0^2\chi_0'^2\chi_0'' + 4\chi_0^3\chi_0''^2 + 8\chi_0^3\chi_0'\chi_0''' + \chi_0^4\chi_0'''') = 0\end{aligned} \tag{32}$$

By utilizing Eq. (12), we can take the initial approximate solution as

$$\chi_0 = A\cos\Omega t \tag{33}$$

Eq. (32) will get the form after employing the initial solution

$$\begin{aligned}&\chi_1'''' - \Omega^4\chi_1 - A\Omega_1\cos\Omega t - b_3b_4[A^3\cos^3\Omega t] - b_3b_5[A^5\cos^5\Omega t] - b_3b_6[A^7\cos^7\Omega t]\\&+b_1b_3[A^3\Omega^2\cos^3\Omega t] + b_2b_3[A^5\Omega^2\cos^5\Omega t] + b_4[6A^3\Omega^2\cos\Omega t - 9A^3\Omega^2\cos^3\Omega t]\\&+b_5[20A^5\Omega^2\cos^3\Omega t - 25A^5\Omega^2\cos^5\Omega t] + b_6[42A^7\Omega^2\cos^5\Omega t - 49A^7\Omega^2\cos^7\Omega t]\\&+b_1[-6A^3\Omega^4\cos\Omega t + 9A^3\Omega^4\cos^3\Omega t] + b_2[-20A^5\Omega^4\cos^3\Omega t + 25A^5\Omega^4\cos^5\Omega t] = 0\end{aligned} \tag{34}$$

After simple calculation, Eq. (34) can be written

$$\chi_1'''' - \Omega^4\chi_1 + \Gamma_1\cos\Omega t + \Gamma_2\cos 3\Omega t + \Gamma_3\cos 5\Omega t + \Gamma_4\cos 7\Omega t = 0 \tag{35}$$

where

$$\begin{aligned}\Gamma_1 = &-A\Omega_1 - \frac{3b_3b_4}{4}A^3 - \frac{5b_3b_5}{8}A^5 - \frac{35b_3b_6}{64}A^7 - \frac{3b_4}{4}A^3\Omega^2 - \frac{5b_5}{8}A^5\Omega^2\\&-\frac{35b_6}{64}A^7\Omega^2 + \frac{3b_1}{4}\lambda A^3\Omega^2 + \frac{5b_2}{8}\lambda A^5\Omega^2 + \frac{3b_1}{4}A^3\Omega^4 + \frac{5b_2}{8}A^5\Omega^4\end{aligned} \tag{36}$$

$$\begin{aligned}\Gamma_2 = &-\frac{b_3b_4}{4}A^3 - \frac{5b_3b_5}{16}A^5 - \frac{21b_3b_6}{64}A^7 - \frac{9b_4}{4}A^3\Omega^2 - \frac{45b_5}{16}A^5\Omega^2\\&-\frac{189b_6}{64}A^7\Omega^2 + \frac{b_1}{4}\lambda A^3\Omega^2 + \frac{5b_2}{16}\lambda A^5\Omega^2 + \frac{9b_1}{4}A^3\Omega^4 + \frac{45b_2}{16}A^5\Omega^4\end{aligned} \tag{37}$$

$$\Gamma_3 = -\frac{b_3b_5}{16}A^5 - \frac{7b_3b_6}{64}A^7 - \frac{25b_5}{16}A^5\Omega^2 - \frac{175b_6}{64}A^7\Omega^2 + \frac{b_2}{16}\lambda A^5\Omega^2 + \frac{25b_2}{16}A^5\Omega^4 \tag{38}$$

$$\Gamma_4 = -\frac{b_3b_6}{64}A^7 - \frac{49b_6}{64}A^7\Omega^2 \tag{39}$$

Requirement of no secular term needs

$$-A\Omega_1-\frac{3b_3b_4}{4}A^3-\frac{5b_3b_5}{8}A^5-\frac{35b_3b_6}{64}A^7-\frac{3b_4}{4}A^3\Omega^2-\frac{5b_5}{8}A^5\Omega^2-\frac{35b_6}{64}A^7\Omega^2+\frac{3b_1}{4}\lambda A^3\Omega^2+\frac{5b_2}{8}\lambda A^5\Omega^2+\frac{3b_1}{4}A^3\Omega^4+\frac{5b_2}{8}A^5\Omega^4=0 \tag{40}$$

If it is enough to obtain the first-order approximate solution, then from Eq. (30), we yield

$$\Omega_1=\lambda^2-\Omega^4 \tag{41}$$

Solving Ω from Eqs.(40) and (41) we have

$$\Omega=\sqrt{\frac{b_3+\frac{3}{4}b_4A^2+\frac{5}{8}b_5A^4+\frac{35}{64}b_6A^6}{1+\frac{3}{4}b_1A^2+\frac{5}{8}b_2A^4}} \tag{42}$$

and the corresponding approximate solution of Eq. (12) is

$$\chi(t)=A\cos\left(\sqrt{\frac{64c_3+48c_4A^2+40c_5A^4+35c_6A^6}{64c_0+48c_1A^2+40c_2A^4}}\;t\right) \tag{43}$$

which is different from the solution gained by EBM [11].

## 5. Results and Discussion

The analytic nonlinear frequency and the approximate solution for the vibration of microbeam can be calculated from Eqs. (42) and (43), respectively. A comparison between the frequencies obtained from Li-He's approach and those gained by RK is demonstrated in Table 1 for different parameters. It displays the high correctness of the proposed solution as the maximum percentage error is not greater than 1%. This table and the graphs of Figs. 2 and 3 specify that the analytic solution accomplished by Li-He's approach, Eqs. (42) and (43), can estimate the dynamic vibrational activities of the microbeams acceptably. This approves the validity of the proposed solution.

**Table 1** Comparison of analytic frequency obtained by Li-He and RK methods

| *A* | *N* | *A* | *V* | $\Omega_{RK}$ | $\Omega_{Li\text{-}He}$ | Error (%) |
|---|---|---|---|---|---|---|
| 0.15 | 12 | 20 | 15 | 22.9224 | 22.8896 | 0.1431 |
| 0.15 | 18 | 50 | 15 | 29.8489 | 29.8222 | 0.0269 |
| 0.3 | 6 | 35 | 10 | 28.8737 | 28.8882 | 0.0502 |
| 0.3 | 24 | 10 | 20 | 16.6002 | 16.6422 | 0.2530 |
| 0.45 | 12 | 20 | 5 | 28.5210 | 28.4952 | 0.0905 |
| 0.45 | 24 | 10 | 10 | 26.4555 | 26.5042 | 0.1841 |
| 0.6 | 6 | 25 | 5 | 29.0216 | 29.0262 | 0.0159 |
| 0.6 | 18 | 40 | 5 | 33.8716 | 34.0674 | 0.5781 |

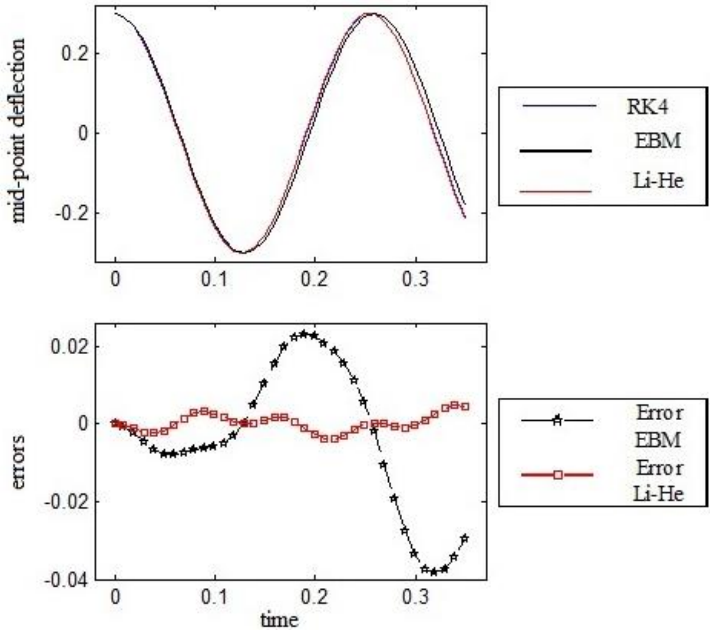


**Fig. 2** Comparison of solutions and errors of Li-He's approach with EBM for the parameters $A$=0.3, $N$=10, $\alpha$ =24, $V$=10

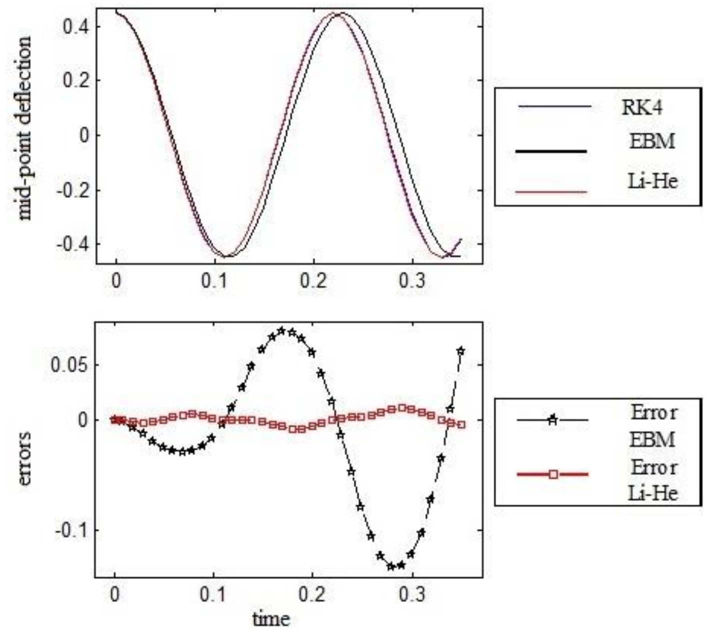


**Fig. 3** Comparison of solutions and errors of Li-He's approach with EBM for the parameters $A$=0.45, $N$=20, $\alpha$ =12, $V$=5

The top panels of Figs. 2 and 3 compare the solutions obtained from Li-He's technique (red line) expressed in Eq. (43), EBM (black line) depicted in Ref. [28] with the solution achieved by RK method (blue line). This comparison validates that the findings from the proposed method and those attained by the RK method match remarkably well. We also show the variation of errors for the said system in the bottom panels of Figs 2 and 3. Errors of EBM (black stars with solid line) and errors of Li-He's approach (red squares with solid line) against time confirm the supremacy of the proposed technique over the EBM.

## 6. CONCLUDING REMARKS

In this research study, we have applied hybridization of the enhanced perturbation method and the parameter expansion technology (collectively called Li-He's approach) to approximate the periodic behavior of electrically excited microbeams-based microelectromechanical system. The solution achieved from the proposed method has good agreement with the numerically ones gained by the Runge-Kutta technique. This method not only gives an alternative approximate solution to the oscillatory system by refining the order of the original differential equation but it also makes the solution process more accurate and reliable. This idea can also be implemented in other nonlinear oscillatory problems.

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