FACTA UNIVERSITATIS
Series: **Mechanical Engineering** Vol. 20, Nº 3, 2022, pp. 539 - 552
https://doi.org/10.22190/FUME220703035F

Original scientific paper

DYNAMIC CHARACTERISTICS OF MIXTURE UNIFIED GRADIENT ELASTIC NANOBEAMS

Seyed Ali Faghidian[1], Abdelouahed Tounsi[2,3]

[1]Department of Mechanical Engineering, Science and Research Branch,
Islamic Azad University, Tehran, Iran
[2]YFL (Yonsei Frontier Lab), Yonsei University, Seoul, South Korea
[3]Material and Hydrology Laboratory, University of Sidi Bel Abbes,
Faculty of Technology, Civil Engineering Department, Algeria

Abstract. *The mixture unified gradient theory of elasticity is invoked for the rigorous analysis of the dynamic characteristics of elastic nanobeams. A consistent variational framework is established and the boundary-value problem of dynamic equilibrium enriched with proper form of the extra non-standard boundary conditions is detected. As a well-established privilege of the stationary variational theorems, the constitutive laws of the resultant fields cast as differential relations. The wave dispersion response of elastic nano-sized beams is analytically addressed and the closed form solution of the phase velocity is determined. The free vibrations of the mixture unified gradient elastic beam is, furthermore, analytically studied. The dynamic characteristics of elastic nanobeams is numerically evaluated, graphically illustrated, and commented upon. The efficacy of the established augmented elasticity theory in realizing the softening and stiffening responses of nano-sized beams is evinced. New numerical benchmark is detected for dynamic analysis of elastic nanobeams. The established mixture unified gradient elasticity model provides a practical approach to tackle dynamics of nano-structures in pioneering MEMS/NEMS.*

Key words: *Stationary variational principle, Nanobeam, Free vibrations, Wave dispersion, Stress gradient elasticity, Strain gradient theory*

Received: July 03, 2022 / Accepted August 25, 2022
Corresponding author: Seyed Ali Faghidian
Department of Mechanical Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran
E-mail: faghidian@gmail.com; faghidian@srbiau.ac.ir

1. INTRODUCTION

Micro-/nano-sensors are one of the representative elements of micro-/nano-electromechanical systems (MEMS/NEMS) that found a variety of applications in pioneering engineering systems [1, 2]. Structural analysis of vibrating modules of MEMS/NEMS is a challenging issue that stimulated a great deal of interest [3-15].

In view of inadequacy of the classical elasticity theory in appropriate prediction of the peculiar behavior of nano-scale structures, several augmented elasticity theories are introduced in literature. The constitutive model associated with the augmented elasticity theory is, accordingly, enriched via incorporation of the gradient effects or of the nonlocality. Within the framework of the strain gradient theory, the material response is assumed to be a function of the classical kinematics along with the gradients of various order [16]. The strain gradient theory can efficiently predict the smaller-is-stiffer phenomenon at the ultra-small scale [17-21]. Alternatively, the material response can be considered to depend on the classical kinetic variables together with the gradients of various order which yields the stress gradient theory [22, 23]. The long-range interactions can be modeled within the context of the nonlocal elasticity theory wherein the constitutive model is modified employing the nonlocal kernel function [24]. The smaller-is-softer phenomenon is, therefore, realized via implementation of the nonlocal elasticity model. As the nonlocality concept is a topic of major interest in the size-dependent mechanics [25-35], severe concern exists in its application to nano-scale continua with finite domain [36].

Either the strain gradient theory of elasticity or the nonlocal elasticity model can solely predict the stiffening or the softening material behavior, and therefore, they cannot cover the wide spectrum of material characteristics at the ultra-small scale. To overcome this deficiency, dissimilar size-dependent elasticity models are integrated to introduce novel augmented theories, such as the nonlocal strain gradient model [37, 38], the nonlocal modified gradient theory [39, 40], the higher-order nonlocal gradient theory [41-43], the nonlocal surface elasticity [44-46], and the mixture stress gradient theory [47]. Utilization of the integrated size-dependent elasticity models for nanoscopic study of the field quantities, accordingly, received increasing interest in recent literature [48-51].

The present study provides significant insight and critical analysis of the dynamic characteristics of nano-scaled structures and offers necessary guidance for the reliable assessment of the dynamic response of nano-components in advanced MEMS/NEMS. The mixture unified gradient theory of elasticity, as a proficient alternative to the two-phase local/nonlocal gradient theory, is invoked to properly realize the size-effects. Both the softening and the stiffening material behaviors can be effectively captured in view of consistent unification of the stress gradient, the strain gradient, and the classical elasticity theory. The wave dispersion response of the flexural waves along with the free vibrations of nano-sized elastic beams is rigorously examined. The dynamic characteristics of mixture unified gradient elastic nanobeams are analytically addressed and numerically demonstrated.

The paper proceeds as follows; a stationary variational framework is conceived in Section 2 to suitably integrate all the governing equations into a single functional. The corresponding boundary-value problem of dynamic equilibrium is determined and equipped with suitable extra non-standard boundary conditions. Nanoscopic analysis of the flexural wave dispersion is performed in Section 3 wherein the analytical solution of the phase velocity is derived. Section 3 is, furthermore, enriched with numerical illustrations of the wave dispersion features in an elastic nanobeam. The free vibrations of nano-sized beams, associated with the framework of the mixture unified gradient theory of elasticity, is addressed in Section 4. Fundamental frequencies of elastic nano-scaled beams are analytically determined, numerically illustrated and commented upon. Section 5 summarizes the paper and draws the conclusion.

2. MIXTURE UNIFIED GRADIENT ELASTIC BEAMS

To establish the elasticity framework of the mixture unified gradient theory, an elastic homogenous beam of symmetric cross-section is considered. In the undeformed state, the beam is referred to orthogonal Cartesian co-ordinates (x,y,z) with the x axis coinciding with the beam longitudinal axis and the y and z axes, respectively, coinciding with the width and the thickness directions. The beam is constrained at the ends $x=0$ and $x=L$ impeding any rigid-body motion. The beam is subjected to a generalized transversal load per unit length $\overline{f}$ where the inertia force is assumed to be incorporated as the body force. The shear and warping effects are overlooked, and accordingly, the kinematics of the beam in accordance with the classical beam theory is taken in the form

$$u_1 = -z\partial_x w(x,t), \qquad u_2 = 0, \qquad u_3 = w(x,t) \tag{1}$$

where u_1, u_2, u_3 denote the components of the displacement field at a generic point of the beam along with w designating the transverse displacement of the beam centroid at time t. The strain state of the beam, consequent to the assumed kinematics, can be written as

$$\varepsilon = -z\partial_{xx} w(x,t) = -z\chi(x,t) \tag{2}$$

with $\chi = \partial_{xx} w$ denoting the curvature of the beam centroidal axis. The variational functional $\mathfrak{A}$ consistent with the mixture unified gradient theory of elasticity is introduced as

$$\begin{aligned}\mathfrak{A} = \int_0^L \Bigg(& M_0(x,t)\partial_{xx} w(x,t) + \frac{1}{2I_E}(M_0(x,t))^2 + \frac{\ell_c^2}{2I_E}(\partial_x M_0(x,t))^2 \\ & + M_1(x,t)\partial_{xxx} w(x,t) + \frac{1}{2I_E(\aleph\ell_c^2 + \ell_s^2)}(M_1(x,t))^2 \\ & + \frac{\ell_c^2}{2I_E(\aleph\ell_c^2 + \ell_s^2)}(\partial_x M_1(x,t))^2 + \overline{f}(x,t)w(x,t)\Bigg)dx\end{aligned} \tag{3}$$

where the flexural stiffness I_E is defined by the second moment of the elastic area, weighted with the scalar field of the elastic modulus E, about the flexural axis. The resultant moments M_0 and M_1 are, correspondingly, defined as the dual fields of the curvature χ and of its first-order derivative along the beam axis $\partial_x\chi$. As the mixture parameter is denoted by $\aleph$, the stress gradient characteristic length ℓ_c and the strain gradient length-scale parameter ℓ_s are introduced to address the significance of the corresponding gradient elasticity theory.

Within the context of the stationary variational formulation, the kinetic field variables are also treated as independent variables along with the kinematic field variables; all the field variables are, therefore, considered to be subject to variation [52]. Assuming the virtual kinetic test fields to have compact support on the domain, the first variation of the functional $\mathfrak{A}$, following the integration by parts, reads

$$\begin{aligned}\delta\mathfrak{A} = \int_0^L \Big(& \left(\partial_{xx} M_0(x,t) - \partial_{xxx} M_1(x,t) + \bar{f}(x,t)\right)\delta w(x,t) \\ & + \delta M_0(x,t)\left(\partial_{xx} w(x,t) + \frac{1}{I_E}\left(M_0(x,t) - \ell_c^2 \partial_{xx} M_0(x,t)\right)\right) \\ & + \delta M_1(x,t)\left(\partial_{xxx} w(x,t) + \frac{1}{I_E(\aleph \ell_c^2 + \ell_s^2)}\left(M_1(x,t) - \ell_c^2 \partial_{xx} M_1(x,t)\right)\right)\Big) dx \\ & - \left(\partial_x M_0(x,t) - \partial_{xx} M_1(x,t)\right)\delta w\Big|_{x=0}^{x=L} + \left(M_0(x,t) - \partial_x M_1(x,t)\right)\partial_x \delta w\Big|_{x=0}^{x=L} \\ & + M_1(x,t)\partial_{xx}\delta w\Big|_{x=0}^{x=L}\end{aligned} \tag{4}$$

where the extra gradient-induced boundary terms are released in consideration of the virtual kinetic test fields having compact support on the domain [53]. Prescribing the stationarity of the functional $\delta\mathfrak{A} = 0$, the differential and boundary conditions of dynamic equilibrium for the elastic nano-sized beam consistent with the mixture unified gradient theory cast in the form

$$\begin{aligned} & \partial_{xx} M_0(x,t) - \partial_{xxx} M_1(x,t) + f(x) = A_\rho \partial_{tt} w(x,t) - I_\rho \partial_{xxtt} w(x,t) \\ & \left(\partial_x M_0(x,t) - \partial_{xx} M_1(x,t)\right)\delta w\Big|_{x=0}^{x=L} = 0 \\ & \left(M_0(x,t) - \partial_x M_1(x,t)\right)\partial_x \delta w\Big|_{x=0}^{x=L} = 0 \\ & M_1(x,t)\Big|_{x=0}^{x=L} = 0 \end{aligned} \tag{5}$$

where the curvature field is considered to have arbitrary variations at the beam ends. Noteworthily, the dynamic form of the differential condition of equilibrium is developed by application of the d'Alembert principle. The cross-sectional mass A_ρ is defined by the cross-sectional area weighted with the scalar field of the material density ρ. Likewise, the rotatory inertia I_ρ is introduced as the second moment of area, weighted with the scalar field of the material density ρ, about the flexural axis.

To further simplify the boundary-value problem of dynamic equilibrium, the concept of the total flexural moment is utilized within the framework of the gradient elasticity theory, and is introduced as

$$M(x,t) = M_0(x,t) - \partial_x M_1(x,t) \tag{6}$$

The boundary-value problem of the dynamic equilibrium of the mixture unified gradient elastic beam is, therefore, modified as

$$\begin{aligned} & \partial_{xx} M(x,t) + f(x) = A_\rho \partial_{tt} w(x,t) - I_\rho \partial_{xxtt} w(x,t) \\ & \partial_x M(x,t)\delta w\Big|_{x=0}^{x=L} = M(x,t)\partial_x \delta w\Big|_{x=0}^{x=L} = 0 \\ & M_1(x,t)\Big|_{x=0}^{x=L} = 0 \end{aligned} \tag{7}$$

As a distinguished privilege of implementing the stationary variational principle, the constitutive model of the associated elasticity theory is, furthermore, detected. The constitutive laws of the resultant fields consistent with the framework of the mixture unified gradient theory, accordingly, are cast as the subsequent ordinary differential relations

$$\begin{aligned} &M_0(x,t)-\ell_c^2\partial_{xx}M_0(x,t)=-I_E\partial_{xx}w(x,t) \\ &M_1(x,t)-\ell_c^2\partial_{xx}M_1(x,t)=-I_E(\aleph\ell_c^2+\ell_s^2)\partial_{xxx}w(x,t) \\ &M(x,t)-\ell_c^2\partial_{xx}M(x,t)=-I_E\left(\partial_{xx}w(x,t)-(\aleph\ell_c^2+\ell_s^2)\partial_{xxxx}w(x,t)\right) \end{aligned} \tag{8}$$

where the constitutive laws of the flexural resultants M_0, M_1 are directly determined in view of the stationarity of the functional $\delta\mathfrak{A}=0$, and the constitutive relation of the flexural moment M is detected utilizing its definition as Eq. (6).

The noticeable advantage of the established stationary variational framework is to integrate all the governing equations, i.e. the differential condition of dynamic equilibrium, the classical and the extra non-standard boundary conditions along with the constitutive differential relations of the resultant fields, into the functional $\mathfrak{A}$. The constitutive laws of the resultant fields are of higher-order compared with the classical elasticity model, and therefore, to close the associated boundary-value problem of dynamic equilibrium on finite domains, extra non-standard boundary conditions should be imposed. Realizing the proper mathematical form of the non-standard flexural resultants is of supreme importance [54]; otherwise, it yields superfluous inference vis-à-vis the size-dependent response of ultra-small structures [21, 55]. The explicit constitutive relation of the resultant moment M_1 associated with the mixture unified gradient theory, subsequent to some straightforward mathematics, can be cast in the form

$$\begin{aligned} M_1(x,t)=&-\frac{(\aleph\ell_c^2+\ell_s^2)\ell_c^4}{\ell_c^2(1-\aleph)-\ell_s^2}\left(\partial_x q(x)-A_\rho\partial_{xtt}w(x,t)+I_\rho\partial_{xxxtt}w(x,t)\right) \\ &-I_E(\aleph\ell_c^2+\ell_s^2)\partial_{xxx}w(x,t)+I_E\frac{(\aleph\ell_c^2+\ell_s^2)^2\ell_c^2}{\ell_c^2(1-\aleph)-\ell_s^2}\partial_{xxxxx}w(x,t) \end{aligned} \tag{9}$$

Several types of the gradient elasticity theory are, noticeably, retrieved as special cases of the established mixture unified gradient theory under ad hoc assumptions on the gradient characteristic lengths. The classical elasticity model is obtained via either setting the gradient characteristic lengths to zero or via letting the mixture parameter approach unity in the absence of the strain gradient length-scale parameter. As the vanishing of the mixture parameter yields the unified gradient elasticity theory, the strain gradient and the stress gradient theory can be, likewise, retrieved via setting the pertinent gradient length-scale parameter to zero. Moreover, the mixture stress gradient theory can be recovered via vanishing the strain gradient length-scale parameter.

The peculiar size-dependent dynamic behavior of nano-scale beams can be competently captured within the framework of the mixture unified gradient theory as evinced via rigorous examination of the wave dispersion and the free vibrations of elastic inflected nanobeams.

3. WAVE DISPERSION CHARACTERISTICS

The size-dependent physical characteristics of nano-structures can be effectively realized by the analysis of the wave dispersion. Examination of dispersive waves can be, furthermore, exploited for inverse determination of the gradient characteristic parameters applying the inverse theory approach [56, 57].

To examine the flexural waves dispersing in the elastic nanobeam, the differential condition of dynamic equilibrium should be expressed in terms of the transverse displacement of the beam. The total flexural moment is, accordingly, obtained by imposing the dynamic equilibrium condition Eq. $(7)_1$ to the constitutive law of the flexural moment Eq. $(8)_3$, namely,

$$\begin{aligned} M(x,t) &= \ell_c^2 A_\rho \partial_{tt} w(x,t) - \ell_c^2 I_\rho \partial_{xxtt} w(x,t) \\ &\quad - I_E \partial_{xx} w(x,t) + (\aleph \ell_c^2 + \ell_s^2) I_E \partial_{xxxx} w(x,t) \end{aligned} \tag{10}$$

where the transverse body force is overlooked in the dynamic analysis of nano-sized beams. The governing equation on the dynamic response of the mixture unified gradient elastic beam is achieved via utilizing the detected result for the flexural moment in the differential condition of dynamic equilibrium as

$$\begin{aligned} &\ell_c^2 I_\rho \partial_{xxxxtt} w(x,t) + I_E \partial_{xxxx} w(x,t) \\ &\quad - (\aleph \ell_c^2 + \ell_s^2) I_E \partial_{xxxxxx} w(x,t) + A_\rho \partial_{tt} w(x,t) - (I_\rho + \ell_c^2 A_\rho) \partial_{xxtt} w(x,t) = 0 \end{aligned} \tag{11}$$

For the harmonic wave dispersion in infinitely extended homogeneous structures, vanishing the boundary conditions is tacitly fulfilled; the corresponding solution of the wave response, thus, takes the form

$$w(x,t) = W \exp(i\lambda(x - vt)) \tag{12}$$

with i being the unit imaginary number, λ and v, respectively, denote the wave number and the phase velocity along with W standing for the wave amplitude. By substituting the displacement solution Eq. (12) into the governing equation on the dynamics of the elastic nanobeam Eq. (11), based on the mixture unified gradient theory, the subsequent phase velocity is obtained

$$v = \lambda \frac{\sqrt{I_E + \lambda^2 I_E (\aleph \ell_c^2 + \ell_s^2)}}{\sqrt{1 + \lambda^2 \ell_c^2}\sqrt{A_\rho + \lambda^2 I_\rho}} \tag{13}$$

Noteworthily, the wave dispersion response of the nano-sized beam consistent with the unified gradient theory is recovered in the absence of the mixture parameter [40, 43].

The size-effects of the gradient length-scale parameters on the dispersive characteristics of waves are graphically illustrated and thoroughly discussed here. For the sake of consistency, the non-dimensional form of the gyration radius $\bar{\kappa}$, stress gradient characteristic parameter ζ, strain gradient characteristic parameters η, wave number $\bar{\lambda}$, and phase velocity $\bar{v}$ are defined as

$$\bar{\kappa} = \frac{1}{L}\sqrt{\frac{I_\rho}{A_\rho}}, \qquad \zeta = \frac{\ell_c}{L}, \qquad \eta = \frac{\ell_s}{L}, \qquad \bar{\lambda} = \lambda L, \qquad \bar{v} = vL\sqrt{\frac{A_\rho}{I_E}} \tag{14}$$

3D variation of the non-dimensional phase velocity in terms of the stress gradient characteristic parameter ζ and the strain gradient characteristic parameters η is, respectively, demonstrated in Figs. 1 and 2 where the logarithmic scaling of the non-dimensional wave number $\overline{\lambda}$ is utilized [58]. As the stress gradient characteristic parameter is assumed to range in the interval [0,1] in Fig. 1, two values of the mixture parameter $\aleph = 0,1/2$ are prescribed for a fixed strain gradient characteristic parameter $\eta = 1/3$. Likewise in Fig. 2, the strain gradient characteristic parameter is ranging in the interval [0,1] while two values of the mixture parameter $\aleph = 0,1/2$ are applied for a prescribed value of the stress gradient characteristic parameter $\zeta = 1/3$. In all the numerical illustrations of the wave dispersion, the (logarithm of) non-dimensional wave number $\overline{\lambda}$ is ranging in the interval $[10^{-1},10^{+1}]$. The non-dimensional gyration radius is, moreover, prescribed as $\overline{\kappa} = 1/20$.

As deducible from the numerical illustration in Fig. 1, a larger value of the stress gradient parameter ζ involves a smaller value of the phase velocity. The phase velocity of dispersive waves, therefore, reveals a softening response in terms of the stress gradient characteristic parameter ζ. Contrarily as demonstrated in Fig. 2, the phase velocity associated with waves disperse in the mixture unified gradient elastic beam increases by increasing the strain gradient parameter η, and accordingly, a stiffening behavior in terms of the strain gradient characteristic parameter η is observed. The effect of the stress gradient theory in the mixture unified gradient elasticity is continuously reinstated with the classical elasticity theory as the mixture parameter $\aleph$ varies from 0 to 1. A stiffening behavior in terms of the mixture parameter $\aleph$ is, thus, realized as the phase velocity of dispersive waves increases with increasing the mixture parameter $\aleph$.

The phase velocity detected within the framework of the mixture unified gradient elasticity is observed to remain unaffected for low wave numbers. This phenomenon is in complete agreement with the fact that the dispersive behavior of flexural waves is insensitive to the physical characteristics for large wavelengths. Accordingly, effects of the gradient characteristic parameters on the wave dispersion response are significantly enhanced at higher wave numbers. The classical dispersion relation of the phase velocity is, also, recovered as the gradient characteristic parameters tend to zero, or alternatively, as the mixture parameter approaches unity in the absence of the strain gradient characteristic parameter.

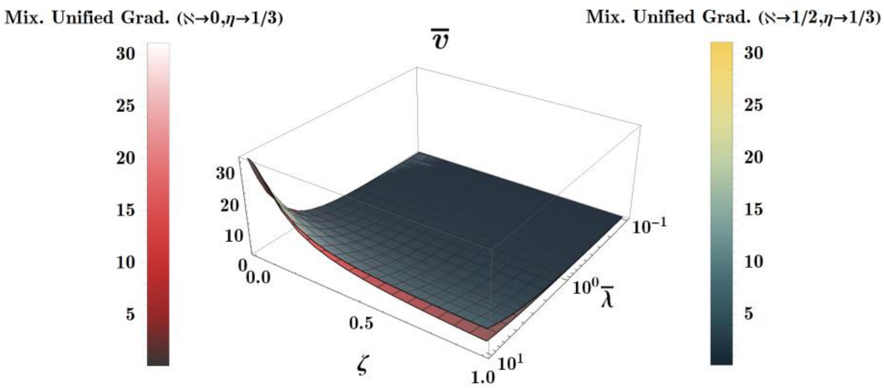

Fig. 1 Wave dispersion response by the mixture unified gradient theory: $\overline{v}$ vs. ζ for $\eta = 1/3$ and $\aleph = 0,1/2$

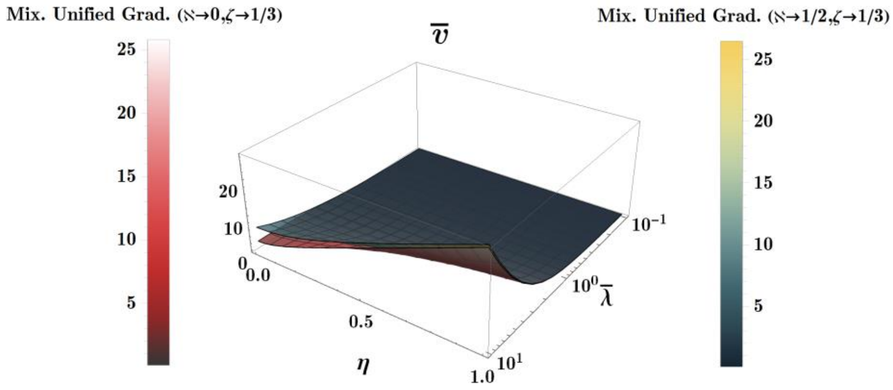

Fig. 2 Wave dispersion response by the mixture unified gradient theory: $\bar{v}$ vs. η for $\zeta = 1/3$ and $\aleph = 0,1/2$

4. FREE VIBRATIONS CHARACTERISTICS

The differential condition of dynamic equilibrium governing the free vibrations of a mixture unified gradient elastic beam is the same as differential equation addressed in Eq. (11) and can be detected employing literally the same approach. Nevertheless, the boundary-value problem of free vibrations of elastic nanobeams is defined on bounded structural domains, and accordingly, the governing differential equation of dynamic equilibrium is subject to the classical and the extra non-standard boundary conditions as expressed by Eq. $(7)_{2\text{-}3}$. The standard procedure of separating spatial and time variables is employed to analyze the free vibrations of elastic nanobeams, namely,

$$w(x,t) = \mathbb{W}(x)\exp(i\omega t) \tag{15}$$

with $\mathbb{W}$ and ω , respectively, denoting the spatial mode shape and the natural frequency of free vibrations. Prescribing the separation of variables Eq. (15) to the differential condition of dynamic equilibrium Eq. (11), the governing equation on the spatial mode shape $\mathbb{W}$ is detected as

$$\begin{aligned} -I_E(\aleph\ell_c^2+\ell_s^2)\partial_{xxxxxx}\mathbb{W}(x)+(I_E-\ell_c^2 I_\rho\omega^2)\partial_{xxxx}\mathbb{W}(x) \\ +(\ell_c^2 A_\rho\omega^2+I_\rho\omega^2)\partial_{xx}\mathbb{W}(x)-A_\rho\omega^2\mathbb{W}(x)=0 \end{aligned} \tag{16}$$

The analytical solution of the preceding differential equation for the spatial mode shape can be cast in the form

$$\begin{aligned} \mathbb{W}(x) = \mathbb{k}_1\exp(\Upsilon_1 x)+\mathbb{k}_2\exp(\Upsilon_2 x)+\mathbb{k}_3\exp(\Upsilon_3 x)+\mathbb{k}_4\exp(\Upsilon_4 x) \\ +\mathbb{k}_5\exp(\Upsilon_5 x)+\mathbb{k}_6\exp(\Upsilon_6 x) \end{aligned} \tag{17}$$

where unknown integration constants $\mathbb{k}_j$ (j=1..6) yet required to be determined via imposing the classical and the extra non-standard boundary conditions along with Υ_j(j=1..6) denoting the roots of the characteristic equation associated with the differential equation Eq. (16). Imposing the classical and the extra non-standard boundary conditions to the analytical solution addressed for the spatial mode shape yields a homogeneous sixth-order algebraic system in terms of the unknown integration constants $\mathbb{k}_j$ (j=1..6). The resulted system of algebraic equations should be singular to have the non-trivial solution, and as a result, the determinant of the coefficients of the homogeneous sixth-order algebraic system should vanish. The analytical solution approach yields a strongly nonlinear characteristic equation in terms of the fundamental frequency which requires to be numerically solved.

The influence of the gradient characteristic parameters along with the mixture parameter on the fundamental frequency of mixture unified gradient elastic beams is demonstrated in Figs. 3 and 4 for elastic nanobeams with kinematic constraints of interest in nano-mechanics, viz. fully-fixed and cantilever beams. The detected fundamental frequencies are, additionally, normalized utilizing the corresponding classical natural frequencies ω_0. The strain gradient and the stress gradient characteristic parameters are, respectively, ranging in the intervals [0.1,1] and [0,1], as two values of the mixture parameter $\aleph$ = 0,1/2 are implemented. The non-dimensional gyration radius of the elastic nanobeam is, also, set as $\bar{\kappa} = 1/20$.

Furthermore, the non-dimensional fundamental frequency $\bar{\omega}$ is defined for the sake of consistency of the illustrations as

$$\bar{\omega} = \omega L^2 \sqrt{\frac{A_\rho}{I_E}} \tag{18}$$

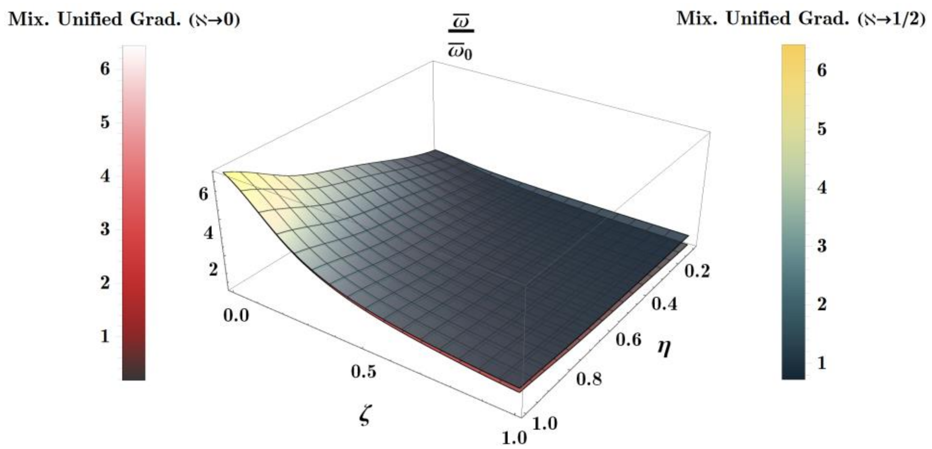

Fig. 3 Normalized fundamental frequency of mixture unified gradient elastic beams with fully-fixed ends

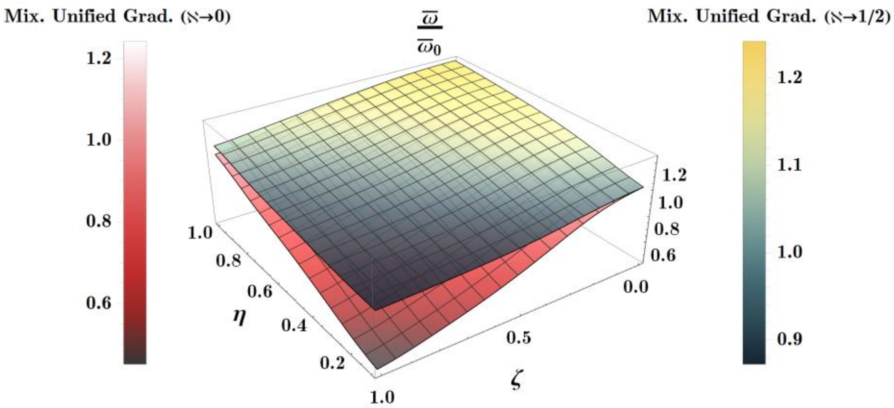

Fig. 4 Normalized fundamental frequency of mixture unified gradient elastic beams with cantilever ends

As noticeably observed from the numerical illustrations in Figs. 3 and 4, the stress gradient characteristic parameter ζ has the effect of decreasing the fundamental frequency, viz. a larger ζ involves a smaller natural frequency for given values of the characteristic parameters $\eta, \aleph$. The fundamental frequency of elastic nanobeams within the framework of the mixture unified gradient theory, therefore, reveals a softening response in terms of the stress gradient characteristic parameter. On the contrary, the fundamental frequency of free vibrations of a mixture unified gradient elastic beam increases by increasing either the strain gradient characteristic parameter η or the mixture parameter $\aleph$. A stiffening response in terms of the characteristic parameters $\eta, \aleph$ is, thus, confirmed for a given value of ζ.

Table 1 Normalized fundamental frequencies of fully-fixed nanobeams: mixture unified gradient theory ($\aleph$ = 0)

	$\bar{\omega}/\bar{\omega}_0$				
ζ	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.5$	$\eta = 0.7$	$\eta = 1$
0^+	1.19212	2.16205	3.34212	4.57294	6.45084
0.2	0.72725	1.33248	2.06215	2.82252	3.98232
0.4	0.42807	0.78774	1.21969	1.66966	2.35591
0.6	0.29615	0.54564	0.84496	1.15673	1.63220
0.8	0.22516	0.41505	0.64276	0.87994	1.24164
1.0	0.18129	0.33426	0.51767	0.70869	1.00098

Table 2 Normalized fundamental frequencies of fully-fixed nanobeams: mixture unified gradient theory ($\aleph$ = 1/2)

	$\bar{\omega}/\bar{\omega}_0$				
ζ	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.5$	$\eta = 0.7$	$\eta = 1$
0^+	1.19212	2.16205	3.34212	4.57294	6.45084
0.2	0.91885	1.44405	2.13588	2.87683	4.02099
0.4	0.78774	1.02674	1.38602	1.79475	2.44615
0.6	0.74694	0.87520	1.08721	1.34389	1.76976
0.8	0.73043	0.80865	0.94587	1.12062	1.42238
1.0	0.72233	0.77449	0.86948	0.99511	1.21983

As expected, the mixture unified gradient theory of elasticity reveals a stiffening response in terms of the number of kinematic boundary constraints. In the numerical illustrations of free vibrations of elastic nanobeams within the context of the mixture unified gradient theory, the influence of the gradient characteristic parameters is more noticeable in elastic nanobeams with fully-fixed ends. The fundamental frequency of the classical elastic beam is, also, retrieved as either the gradient characteristic parameters tend to zero or as the mixture parameter approaches unity in the absence of the strain gradient characteristic parameter.

Numerical values of normalized fundamental frequencies of fully-fixed and cantilever nano-sized beams determined within the context of the mixture unified gradient theory are collected in Tables 1 through 4, correspondingly, for the mixture parameter $\aleph = 0$ and $\aleph = 1/2$.

Table 3 Normalized fundamental frequencies of cantilever nanobeams: mixture unified gradient theory ($\aleph = 0$)

	$\bar{\omega}/\bar{\omega}_0$				
ζ	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.5$	$\eta = 0.7$	$\eta = 1$
0^+	1.01961	1.10882	1.17691	1.21442	1.24183
0.2	0.94703	1.05644	1.14256	1.19185	1.22880
0.4	0.79510	0.93391	1.05452	1.13066	1.19190
0.6	0.64946	0.79839	0.94334	1.04607	1.13681
0.8	0.53583	0.67976	0.83312	0.95355	1.06970
1.0	0.45096	0.58396	0.73510	0.86370	1.00036

Table 4 Normalized fundamental frequencies of cantilever nanobeams: mixture unified gradient theory ($\aleph = 1/2$)

	$\bar{\omega}/\bar{\omega}_0$				
ζ	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.5$	$\eta = 0.7$	$\eta = 1$
0^+	1.01961	1.10882	1.17691	1.21442	1.24183
0.2	0.98474	1.07291	1.14878	1.19436	1.22960
0.4	0.93391	1.00737	1.08749	1.14557	1.19711
0.6	0.90107	0.95534	1.02689	1.08913	1.15380
0.8	0.88244	0.92161	0.98013	1.03862	1.10865
1.0	0.87161	0.90034	0.94687	0.99821	1.06713

5. Closing Remarks

A stationary variational framework is conceived to study the dynamic characteristics of elastic nanobeams within the context of the mixture unified gradient theory of elasticity. In view of consistent integration of the stress gradient theory, the strain gradient model, and the classical elasticity theory, the peculiar size-dependent response of the elastic nano-sized beams is efficiently captured. The differential condition of dynamic equilibrium, the classical and the extra non-standard boundary conditions, and the constitutive differential model of the elastic nanobeams are all incorporated to the introduced functional. To close the boundary-value problem associated with dynamics of elastic nanobeams, the appropriate mathematical form of the extra non-standard boundary conditions is determined and prescribed. Various augmented elasticity theories of gradient type are retrieved as special cases of the introduced mixture unified gradient theory via appropriate assumptions on the gradient characteristic lengths. The classical elasticity model is demonstrated to be recovered via either vanishing the gradient characteristic lengths or by approaching the mixture parameter to unity in the absence of the strain gradient length-scale parameter. The wave dispersion response of nano-sized beams is rigorously examined and the analytical solution of the phase velocity of dispersive waves is addressed. The dispersive response of waves within the context of the mixture unified gradient theory is graphically demonstrated and thoroughly discussed. The free vibrations of nano-sized beams consistent with the mixture unified gradient theory of elasticity is analytically examined. Numerical vales of the fundamental frequency of mixture unified gradient elastic beams with kinematic constraints of interest in nano-mechanics, nanobeams with fully-fixed and cantilever ends, are detected, graphically illustrated, and comprehensively commented up on. The expected peculiar size-dependent behaviors of elastic nanobeams, viz. the stiffening behavior in terms of the strain gradient characteristic parameter and the mixture parameter along with the softening response in terms of the stress gradient characteristic parameter, are effectually confirmed. A meticulous analysis of the dynamic characteristics of nano-scaled elastic beams is performed presenting the essential guide for the practical assessment of dynamics of beam-type nano-components of ground-breaking MEMS/NEMS.

REFERENCES

1. Dilena, M., Fedele Dell'Oste, M., Fernández-Sáez, J., Morassi, A., Zaera, R., 2020, *Hearing distributed mass in nanobeam resonators*, International Journal of Solids and Structures, 193–194, pp. 568-592.
2. Dilena, M., Fedele Dell'Oste, M., Morassi, A., Zaera, R., 2021, *The role of boundary conditions in resonator-based mass identification in nanorods,* Mechanics of Advanced Materials and Structures.
3. Jena, S.K., Chakraverty, S., Malikan, M., Mohammad-Sedighi, H., 2020, *Hygro-magnetic vibration of the single-walled carbon nanotube with nonlinear temperature distribution based on a modified beam theory and nonlocal strain gradient model*, International Journal of Applied Mechanics, 12, 2050054.
4. Sedighi, H.M., Malikan, M., Valipour, A., Żur, K.K., 2020, *Nonlocal vibration of carbon/boron-nitride nano-hetero-structure in thermal and magnetic fields by means of nonlinear finite element method*, Journal of Computational Design and Engineering,7, pp. 591-602.
5. Ghandourah, E.E., Ahmed, H.M., Eltaher, M.A., Attia, M.A., Abdraboh, A.M., 2021, Fr*ee vibration of porous FG nonlocal modified couple nanobeams via a modified porosity model*, Advances in Nano Research, 11, pp. 405-422.
6. Noroozi, M., Ghadiri, M., 2021, *Nonlinear vibration and stability analysis of a size-dependent viscoelastic cantilever nanobeam with axial excitation*, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 235, pp. 3624-3640.
7. Singhal, A., Mohammad Sedighi, H., Ebrahimi, F., Kuznetsova, I., 2021, *Comparative study of the flexoelectricity effect with a highly/weakly interface in distinct piezoelectric materials (PZT-2, PZT-4, PZT-5H, LiNbO3, BaTiO3)*, Waves in Random and Complex Media, 31, pp. 1780-1798.
8. Abouelregal, A.E., Mohammad Sedighi, H., 2021, *A new insight into the interaction of thermoelasticity with mass diffusion for a half-space in the context of Moore–Gibson–Thompson thermodiffusion theory*, Applied Physics A, 127, 582.
9. Jalaei, M.H., Thai, H.T., Civalek Ö., 2022, *On viscoelastic transient response of magnetically imperfect functionally graded nanobeams*, International Journal of Engineering Science 172, 103629.
10. Mohammad Sedighi, H., 2020, *Divergence and flutter instability of magneto-thermo-elastic C-BN hetero-nanotubes conveying fluid*, Acta Mechanica Sinica, 36, pp. 381-396.
11. Fang, J., Yin, B., Zhang, X., Yang, B., 2022, *Size-dependent vibration of functionally graded rotating nanobeams with different boundary conditions based on nonlocal elasticity theory*, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 236, pp. 2756-2774.

12. Faghidian, S.A., Żur, K.K., Reddy, J.N., Ferreira, A.J.M., 2022, *On the wave dispersion in functionally graded porous Timoshenko-Ehrenfest nanobeams based on the higher-order nonlocal gradient elasticity*, Composite Structures, 279, 114819.
13. Attia, A., Berrabah, A.T., Bousahla, A.A., Bourada, F., Tounsi, A., Mahmoud, S.R., 2021, *Free vibration analysis of FG plates under thermal environment via a simple 4-unknown HSDT*, Steel and Composite Structures, 41, pp. 899-910.
14. Hellal, H., Bourada, M., Hebali, H., Bourada, F., Tounsi, A., Bousahla, A.A., Mahmoud, S.R., 2021, *Dynamic and stability analysis of functionally graded material sandwich plates in hygro-thermal environment using a simple higher shear deformation theory*, Journal of Sandwich Structures & Materials, 23, pp. 814-851.
15. Liu, G., Wu, S., Shahsavari, D., Karami, B., Tounsi, A., 2022, *Dynamics of imperfect inhomogeneous nanoplate with exponentially-varying properties resting on viscoelastic foundation*, European Journal of Mechanics A/Solids, 95, 104649.
16. Mindlin, R.D., 1965, *Second gradient of strain and surface-tension in linear elasticity*, International Journal of Solids and Structures, 1, pp. 417–438.
17. Banerjee, J.R., Papkov, S.O., Vo, T.P., Elishakoff, I., 2021, *Dynamic stiffness formulation for a micro beam using Timoshenko–Ehrenfest and modified couple stress theories with applications*, Journal of Vibration and Control.
18. Ramezani, M., Rezaiee-Pajand, M., Tornabene, F., 2022, *Linear and nonlinear mechanical responses of FG-GPLRC plates using a novel strain-based formulation of modified FSDT theory*, International Journal of Non-Linear Mechanics, 140, 103923.
19. Jena, S.K., Chakraverty, S., Mahesh, V., Harursampath, D., 2022, *Application of Haar wavelet discretization and differential quadrature methods for free vibration of functionally graded micro-beam with porosity using modified couple stress theory*, Engineering Analysis with Boundary Elements, 140, pp. 167-185.
20. Jiang, Y., Li, L., Hu, Y., 2022, *Strain gradient elasticity theory of polymer networks*, Acta Mechanica.
21. Barretta, R., Faghidian, S.A., Marotti de Sciarra, F., 2019, *Aifantis versus Lam strain gradient models of Bishop elastic rods*, Acta Mechanica, 230, pp. 2799–2812.
22. Forest, S., Sab, K., 2012, *Stress gradient continuum theory*, Mechanics Research Communications, 40, pp. 16-25.
23. Polizzotto, C., 2014, *Stress gradient versus strain gradient constitutive models within elasticity*, International Journal of Solids and Structures, 51, pp. 1809-1818.
24. Eringen, A.C., 2002, *Nonlocal Continuum Field Theories*, Springer, New York.
25. Abouelregal, A.E., Mohammad-Sedighi, H., Faghidian, S.A., Shirazi, A.H., 2021, *Temperature-dependent physical characteristics of the rotating nonlocal nanobeams subject to a varying heat source and a dynamic load*, Facta Universitatis, Series: Mechanical Engineering, 19, pp. 633-56.
26. Pisano, A.A., Fuschi, P., Polizzotto, C., 2021, *Integral and differential approaches to Eringen's nonlocal elasticity models accounting for boundary effects with applications to beams in bending*, ZAMM Journal of Applied Mathematics and Mechanics, 101, 202000152.
27. Pisano, A.A., Fuschi, P., Polizzotto, C., 2020, *A strain-difference based nonlocal elasticity theory for small-scale shear deformable beams with parametric warping*, International Journal for Multiscale Computational Engineering, 18, pp. 83–102.
28. Akgöz, B., Civalek, Ö., 2017, *A size-dependent beam model for stability of axially loaded carbon nanotubes surrounded by Pasternak elastic foundation*, Composite Structures, 176, pp. 1028-1038.
29. Elishakoff, I., Ajenjo, A., Livshits, D., 2020, *Generalization of Eringen's result for random response of a beam on elastic foundation*, European Journal of Mechanics A/Solids, 81, 103931.
30. Civalek, Ö., Uzun, B., Yaylı, M.Ö., Akgöz, B., 2020, *Size-dependent transverse and longitudinal vibrations of embedded carbon and silica carbide nanotubes by nonlocal finite element method*, European Physical Journal Plus, 135, 381.
31. Hache, F., Challamel, N., Elishakoff, I., 2019, *Asymptotic derivation of nonlocal plate models from three-dimensional stress gradient elasticity*, Continuum Mechanics and Thermodynamics, 31, pp. 47–70.
32. Bendaida, M., Bousahla, A.A., Mouffoki, A., Heireche, H., Bourada, F., Tounsi, A., Benachour, A., Tounsi, A., Hussain, M., 2022, *Dynamic Properties of Nonlocal Temperature-Dependent FG Nanobeams under Various Thermal Environments*, Transport in Porous Media, 142, pp. 187–208.
33. Vinh, P.-V., Tounsi, A., 2021, *The role of spatial variation of the nonlocal parameter on the free vibration of functionally graded sandwich nanoplates*, Engineering with Computers.
34. Abouelregal, A.E., Ersoy, H., Civalek, Ö., 2021, *Solution of Moore–Gibson–Thompson Equation of an Unbounded Medium with a Cylindrical Hole*, Mathematics, 9, 1536.
35. Khorasani, M., Elahi, H., Eugeni, M., Lampani, L., Civalek, Ö., 2022, *Vibration of FG Porous Three-Layered Beams Equipped by Agglomerated Nanocomposite Patches Resting on Vlasov's Foundation*, Transport in Porous Media, 142, pp. 157-186.

36. Romano, G., Diaco, M., 2021, *On formulation of nonlocal elasticity problems*, Meccanica, 56, pp. 1303-1328.
37. Aifantis, A.C., 2011, *On the gradient approach–relation to Eringen's nonlocal theory*, International Journal of Engineering Science, 49, pp. 1367–1377.
38. Polizzotto, C., 2015, *A unifying variational framework for stress gradient and strain gradient elasticity theories*, European Journal of Mechanics A/Solids, 49, pp. 430-440.
39. Faghidian, S.A., 2021, *Flexure mechanics of nonlocal modified gradient nanobeams*, Journal of Computational Design and Engineering, 8, pp. 949–959.
40. Faghidian, S.A., 2021, *Contribution of nonlocal integral elasticity to modified strain gradient theory*, European Physical Journal Plus, 136, 559.
41. Faghidian, S.A., Żur, K.K., Reddy, J.N., 2022, *A mixed variational framework for higher-order unified gradient elasticity*, International Journal of Engineering Science, 170, 103603.
42. Faghidian, S.A., 2020, *Two-phase local/nonlocal gradient mechanics of elastic torsion*, Mathematical Methods in the Applied Sciences.
43. Faghidian, S.A., *Higher-order mixture nonlocal gradient theory of wave propagation*, Mathematical Methods in the Applied Sciences.
44. Li, L., Lin, R., Ng, T.Y., 2020, *Contribution of nonlocality to surface elasticity*, International Journal of Engineering Science, 152, 103311.
45. Jiang, Y., Li, L., Hu, Y., 2022, *A nonlocal surface theory for surface–bulk interactions and its application to mechanics of nanobeams*, International Journal of Engineering Science, 172, 103624.
46. Zhu, X.W., Li, L., 2021, *Three-dimensionally nonlocal tensile nanobars incorporating surface effect: A self-consistent variational and well-posed model*, Science China Technological Sciences, 64, pp. 2495-2508.
47. Faghidian, S.A., Żur, K.K., Pan, E., Kim, J., 2022, *On the analytical and meshless numerical approaches to mixture stress gradient functionally graded nano-bar in tension*, Engineering Analysis with Boundary Element, 134, pp. 571-580.
48. Jena, S.K., Chakraverty, S., Mahesh, V., Harursampath, D., 2022, *Wavelet-based techniques for Hygro-Magneto-Thermo vibration of nonlocal strain gradient nanobeam resting on Winkler-Pasternak elastic foundation*, Engineering Analysis with Boundary Elements, 140, pp. 494-506.
49. Jena, S.K., Chakraverty, S., Malikan, M., 2020, *Stability analysis of nanobeams in hygrothermal environment based on a nonlocal strain gradient Timoshenko beam model under nonlinear thermal field*, Journal of Computational Design and Engineering, 7, pp. 685–699.
50. Monaco, G.T., Fantuzzi, N., Fabbrocino, F., Luciano, R., 2021, *Hygro-thermal vibrations and buckling of laminated nanoplates via nonlocal strain gradient theory*, Composite Structures, 262, 113337.
51. Monaco, G.T., Fantuzzi, N., Fabbrocino, F., Luciano, R., 2021, *Trigonometric solution for the bending analysis of magneto-electro-elastic strain gradient nonlocal nanoplates in hygro-thermal environment*, Mathematics, 9, 567.
52. Reddy, J.N., 2017, *Energy Principles and Variational Methods in Applied Mechanics*, 3rd ed., Wiley.
53. Yosida, K., 1980, *Functional Analysis*, 6th ed., Springer, New York.
54. Żur, K.K., Faghidian, S.A., 2021, *Analytical and meshless numerical approaches to unified gradient elasticity theory*, Engineering Analysis with Boundary Elements, 130, pp. 238-248.
55. Barretta, R., Faghidian, S.A., Marotti de Sciarra, F., Penna, R., Pinnola, F.P., 2020, *On torsion of nonlocal Lam strain gradient FG elastic beams*, Composite Structures, 233, 111550.
56. Faghidian, S.A., 2017, A*nalytical inverse solution of eigenstrains and residual fields in autofrettaged thick-walled tubes*, ASME Journal of Pressure Vessel Technology, 139, 031205.
57. Faghidian, S.A., 2017, *Analytical approach for inverse reconstruction of eigenstrains and residual stresses in autofrettaged spherical pressure vessels*, ASME Journal of Pressure Vessel Technology, 139, 041202.
58. Caprio, M.A., 2005, *LevelScheme: A level scheme drawing and scientific figure preparation system for Mathematica*, Computer Physics Communications, 171, pp. 107-118.