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KANTOWSKI-SACHS MINISUPERSPACE COSMOLOGICAL MODEL ON NONCOMMUTATIVE SPACE[†]

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Abstract. *A vacuum homogeneous and anisotropic Kantowski-Sachs minisuperspace cosmological model is considered. In a classical case, Lagrangian of the model is reduced by a suitable coordinate transformation to Lagrangian of two decoupled oscillators with the same frequencies and with zero energy in total (an oscillator-ghost-oscillator system). The model is formulated also on noncommutative space.*

Key words: *3+1 formalism, Kantowski-Sachs model, noncommutativity*

1. INTRODUCTION

The study of early stages of the universe evolution demands quantum approach. In the absence of a complete theory of quantum gravity and quantum cosmology, it is important to analyze particular classical cosmological models and their quantum versions most often within the approach that uses canonical quantization or path integral quantization. Both of these approaches demand determination of the wave function (of the universe) for a quantum cosmological model. By doing so in the former case (canonical quantization) the wave function is acquired through solving Wheeler-DeWitt equation (DeWitt, 1967), in the latter case (path integral quantization) it can be done by Feynman minisuperspace propagator with appropriate boundary condition (the best known are the Hartle-Hawking no-boundary proposal (Hartle and Hawking, 1983) and the Vilenkin tunneling proposal (Vilenkin, 1982).

On the other hand, the study of physics of the black holes takes an important place in the theory of quantum gravity. Namely, the general relativity, besides the initial

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singularity of the universe, predicts the existence of local singularities i.e. the existence of black holes. That enables us to come to a significant conclusion related to the early stages of the creation of the universe by studying the characteristics of these objects.

In this paper we consider the Schwarzschild black hole interior as a homogeneous and anisotropic Kantowski-Sachs minisuperspace cosmological model without a matter field and the cosmological constant. This approach is based on a diffeomorphism between the Schwarzschild solution of the Einstein field equation and the corresponding solution of this cosmological model. Namely, within the Schwarzschild sphere, the coordinate transformation $t \leftrightarrow r$, transforms Schwarzschild metric to anisotropic Kantowski-Sachs metric. Within the classical analytical approach, by a suitable coordinate transformation, Lagrangian of the model, becomes Lagrangian of an oscillator-ghost-oscillator system of two decoupled oscillators, with equal frequencies and with zero total energy.

Motivated by a possible non-Archimedean and/or noncommutative structure of the space-time at the Planck scale (Vladimirov et al., 1994), we may discuss a *p*-adic and/or noncommutative form of this model. The prediction of the space-time discreteness structure around Planck distances ($\leq 10^{-33}$ cm), is common to both of these approaches (*p*-adic and noncommutative). In *p*-adic case this discreetness is implicitly present (Djordjevic et al., 2002), in noncommutative case it is explicitly present through commutation relations of noncommutative coordinates and/or their conjugate momenta. In this paper noncommutative form of the model is considered. Noncommutative coordinates are used for the first time by Wigner (Wigner, 1932) and Snyder (Snyder, 1947). This idea, used by Connes (Connes, 1985) and Woronowicz (Woronowicz, 1987) in noncommutative geometry, enabled the development of the new formulation of quantum gravity through noncommutative differential calculus (Maceda et al., 2004; Várilly, 2004; Wess and Zumino, 1990).

After determination and decoupling of Langrangian for the classical anisotropic Kantowski-Sachs model, we determine a classical action in this classical (commutative) case. In the noncommutative case, classical action is determined from Lagrangian of noncommutative version of the model.

2. The Classical Model

In the canonical formulation of the general relativity the starting point is 3+1 decomposition of metric. The Einstein-Hilbert action, within this approach, has the following form:

$$S = \frac{1}{16\pi G}\int_M [\,^{(3)}R + K^{ik}K_{ik} - K^2 - 2\Lambda] N\sqrt{\zeta}\, dt\, d^3x + \int_M L_m N\sqrt{\zeta}\, dt\, d^3x \tag{1}$$

where L_m is the Lagrangian of the matter field, G is the gravitational constant, $^{(3)}R$ is the Ricci scalar of the intrinsic 3-geometry, ζ is the determinant of the intrinsic 3-metric tensor h_{ik} (or first fundamental form), K_{ik} is the extrinsic curvature tensor (or second fundamental form), Λ is the cosmological constant and $K = K^i_{\ i}$.

The metric form for a homogeneous and anisotropic Kantowski-Sachs minisuperspace cosmological model is:

$$ds^2 = -\frac{N^2(t)}{l(t)}dt^2 + l(t)dr^2 + h^2(t)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

where h and l are scale factors of this metric.

On the other hand, the Schwarzschild metric is:

$$ds^2 = -(1-\frac{r_g}{r})dt^2 + (1-\frac{r_g}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3}$$

It diverges in case $r = 0$ (Schwarzschild or gravitational singularity) and on the so-called Schwarzschild sphere (the event horizon) for $r = r_g$ ("apparent" singularity[1]), where $r_g = 2Gm/c^2$ is the gravitational or Schwarzschild radius for an object of mass m.

For the metric form (3), we can see that when $r < r_g$, i.e. inside of the event horizon, Schwarzschild metric components $g_{00} = g_{tt}$ and $g_{11} = g_{rr}$ changed their signs. This suggests that space and time change roles in a way, once the event horizon is passed. Namely, the time coordinate, for an observer outside the event horizon, becomes spatial-like for another observer inside the Schwarzschild sphere. In this sense, for both observers, there is one possible direction of movement, for an outside observer in time and for an inside observer in space (right to singularity). Regarding the previous, we can consider a possibility that Schwarzschild sphere interior could be described by a square metric form acquired by substitution $t \leftrightarrow r$ in (2):

$$ds^2 = -(\frac{r_g}{t}-1)^{-1}dt^2 + (\frac{r_g}{t}-1)dr^2 + t^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

which metric components explicitly depend on time. This metric has the same form as the Kantowski-Sachs metric (2) with $l(t) = r_g / t - 1$, $h(t) = t$ and $N(t) = 1$.

From the Einstein-Hilbert action (1), without a matter field and the cosmological constant with the metric (2) one gets Lagrangian for the vacuum Kantowski-Sachs model in the form (Jalalzadeh and Vakili, 2012):

$$L = -\frac{V_0}{8\pi G}\left[\frac{1}{N}(h\dot{h}\dot{l} + \dot{h}^2 l) - N\right], \tag{5}$$

where V_0 has the meaning of a constant volume. By the rescaling of the lapse function $N = 2\sqrt{h^2 l}\tilde{N}$, with transformations (Brehme, 1977):

$$h = \frac{1}{2}(u-v)^2, \quad l = \left[\frac{u+v}{u-v}\right]^2, \tag{6}$$

Lagrangian (5) becomes decoupled:

$$L = -M_{pl}^2 V_0 \tilde{N}^{-1}(\dot{u}^2 - \dot{v}^2) + M_{pl}^2 V_0 \tilde{N}(u^2 - v^2) = -\frac{M_{pl}^2 V_0}{\tilde{N}}[(\dot{u}^2 - \tilde{N}^2 u^2) - (\dot{v}^2 - \tilde{N}^2 v^2)], \tag{7}$$

where $M_{pl} = \sqrt{\hbar c / 8\pi G} \approx 2.43\times 10^{18}$ GeV is the reduced Planck mass. Because of the possibility of a choice of the lapse function $\tilde{N}$ (gauge fixing), we can consider factor

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[1] This singularity can be removed by a suitable coordinate transformation (for example Kruskal-Szekeres transformation).

$-M_{pl}^2 V_0 / \tilde{N}$ in the formula (7) as a constant. Considering $\tilde{N} = \omega$, and bearing in mind the invariance of the system dynamics in relation to multiplication of Lagrangian with constant, instead of (7) we can consider:

$$L = (\dot{u}^2 - \omega^2 u^2) - (\dot{v}^2 - \omega^2 v^2), \tag{8}$$

as a Lagrangian which describes the model. This is the Lagrangian of an oscillator-ghost-oscillator system with zero energy. The corresponding Euler-Lagrangian equations are:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0. \tag{9}$$

The general solution of (9) is:

$$u = A_1 \cos(\omega t + A_2), \quad v = B_1 \cos(\omega t + B_2), \tag{10}$$

where A_1, A_2, B_1, B_2 are constants of integration. Then, particular solution for initial conditions $u(t') = u'$, $u(t'') = u''$, $v(t') = v'$ and $v(t'') = v''$, is:

$$u(t) = u' \frac{\sin(\omega(t'' - t))}{\sin(\omega(t'' - t'))} + u'' \frac{\sin(\omega(t - t'))}{\sin(\omega(t'' - t'))}; \; v(t) = v' \frac{\sin(\omega(t'' - t))}{\sin(\omega(t'' - t'))} + v'' \frac{\sin(\omega(t - t'))}{\sin(\omega(t'' - t'))}. \tag{11}$$

By substituting (11) into (8) the classical Lagrangian L^{cl} is obtained, and after its integration, in time, we get the classical action for this model:

$$S^{cl}(u'', v'', t''; u', v', t') = \int_{t'}^{t''} L^{cl}\, dt = \omega \left[\frac{u''^2 + u'^2}{\tan(\omega T)} - \frac{2u''u'}{\sin(\omega T)} \right] - \omega \left[\frac{v''^2 + v'^2}{\tan(\omega T)} - \frac{2v''v'}{\sin(\omega T)} \right], \tag{12}$$

where $T = t'' - t'$.

3. The Model on Noncommutative Space

Space-time structure at the Planck scale is one of the most challenging questions in high energy physics. Two of the most justified approaches to this problem are: non-Archimedean (Djordjevic and Dragovich, 1997; Djordjevic et al., 2002, 2003, 2014) and a noncommutative approach which is presented in this part of the paper.

In the presence of noncommutativity of the spatial type i.e. (in Poisson brackets) $\{u, v\} = \theta \neq 0$, $\{u, \pi_u\} = \{v, \pi_v\} = 1$, $\{u, \pi_v\} = \{v, \pi_u\} = 0$ and $\{\pi_u, \pi_v\} = 0$, with transformations:

$$u \to u - \frac{\theta}{2}\pi_v = u + \theta \dot{v}; \qquad v \to v + \frac{\theta}{2}\pi_u = v + \theta \dot{u}, \tag{13}$$

two-oscillator Lagrangian (8) becomes:

$$L_\theta = [(1 + \omega^2\theta^2)\dot{u}^2 - \omega^2 u^2] - [(1 + \omega^2\theta^2)\dot{v}^2 - \omega^2 v^2] + 2\theta\omega^2[\dot{u}v - \dot{v}u]. \tag{14}$$

The corresponding Euler-Lagrangian equations are:

$$\ddot{u} + 2\theta\omega_\theta^2 \dot{v} + \omega_\theta^2 u = 0; \quad \ddot{v} + 2\theta\omega_\theta^2 \dot{u} + \omega_\theta^2 v = 0, \tag{15}$$

where $\omega_\theta = \omega / \sqrt{1 + \theta^2\omega^2}$. In the commutative regime ($\theta = 0$) Lagrangian (14) and equations (15) becomes (8) and (9), respectively.

The system of equations (15) has a solution of the following type (Sepangi et al., 2009):

$$u(t) = A_+ e^{\theta\omega^2 t}\sin(\omega_\theta t + \delta_+) + A_- e^{-\theta\omega^2 t}\sin(\omega_\theta t + \delta_-), \tag{16}$$

$$v(t) = -A_+ e^{\theta\omega^2 t}\sin(\omega_\theta t + \delta_+) + A_- e^{-\theta\omega^2 t}\sin(\omega_\theta t + \delta_-), \tag{17}$$

where $\omega_\theta = \omega\sqrt{1-\theta^2\omega^2}$. By substitution of the initial conditions $u(0) = u'$, $u(T) = u''$, $v(0) = v'$ and $v(T) = v''$ in (17) and (18) one gets constants of integration as follows:

$$A_\pm = \frac{[(u''\mp v'')^2 + (u'\mp v')^2 e^{\pm 2\theta\omega^2 T} - 2(u''\mp v'')(u'\mp v')e^{\pm\theta\omega^2 T}\cos(\omega_\theta T)]^{1/2}}{2e^{\pm\theta\omega^2 T}\sin(\omega_\theta T)};\ \delta_\pm = \arcsin\left(\frac{u'\mp v'}{A_\pm}\right). \tag{18}$$

By substituting (18) into (16) and (17) we obtain a classical solution of equations of motions whose replacement in (14) gives the classical Lagrangian of the model for the solution of Euler-Lagrange equations (15). Then, in the standard manner we obtain the corresponding classical action:

$$\begin{aligned}
&S_\theta^{cl}(u'',v'',T;u',v',0) = (1-\tfrac{1}{2}\omega^2\theta^2 - \omega^4\theta^4)\omega_\theta \times \{A_+^2[e^{2\theta\omega^2 T}\sin(2\omega_\theta T + 2\delta_+) - \sin(2\delta_+)\\
&+\frac{\theta\omega^2}{\omega_\theta}(e^{2\theta\omega^2 T}\cos(2\omega_\theta T + 2\delta_+) - \cos(2\delta_+))] + A_-^2[e^{-2\theta\omega^2 T}\sin(2\omega_\theta T + 2\delta_-) - \sin(2\delta_-)\\
&-\frac{\theta\omega^2}{\omega_\theta}(e^{-2\theta\omega^2 T}\cos(2\omega_\theta T + 2\delta_-) - \cos(2\delta_-))]\} + \theta\omega^2(1-\omega^4\theta^4)\\
&\times\{A_+^2[-e^{2\theta\omega^2 T}\cos(2\omega_\theta T + 2\delta_+) + \cos(2\delta_+)\\
&+\frac{\theta\omega^2}{\omega_\theta}(e^{2\theta\omega^2 T}\sin(2\omega_\theta T + 2\delta_+) - \sin(2\delta_+))] + A_-^2[e^{-2\theta\omega^2 T}\cos(2\omega_\theta T + 2\delta_-) - \cos(2\delta_-)\\
&+\frac{\theta\omega^2}{\omega_\theta}(e^{-2\theta\omega^2 T}\sin(2\omega_\theta T + 2\delta_-) - \sin(2\delta_-))]\}\\
&-\frac{2A_+A_-\theta^2\omega^4}{\omega_\theta}[\sin(\delta_+ + \delta_- + 2\omega_\theta T) - \sin(\delta_+ + \delta_-)]\\
&+\frac{\theta\omega^2}{2}[A_+^2(e^{2\theta\omega^2 T} - 1) - A_-^2(e^{-2\theta\omega^2 T} - 1)]\\
&+4A_+A_-\theta\omega^2 T[\theta\omega^2\cos(\delta_+ - \delta_-) + \omega_\theta\sin(\delta_- - \delta_+)].
\end{aligned} \tag{19}$$

4. CONCLUSION

In this paper we presented a homogeneous and anisotropic Kantowski-Sachs minisuperspace cosmological model in the classical and noncommutative case. In the classical case, we determined action and we saw that the dynamics of the model described the dynamics of the Schwarzschild black hole interior. In the noncommutative case, classical action was calculated, as well. The subject of further study may be the calculation of the Faynman propagators, which will lead to the formulation of quantum, commutative and noncommutative, Kantowski-Sachs model of the Schwarzschild black hole interior.

REFERENCES

Brehme, R.W., 1977. Am. J. Phys., 45, 423-428.
Connes, A., 1985. Inst. Hautes Etudes Sci. Publ. Math., 62, 257-360.
DeWitt, B.S., 1967. Phys. Rev., 160, 1113-1148.
Djordjevic, G.S., Dragovich, B., 1997. Mod. Phys. Lett. A, 12, 1455-1463.
Djordjevic, G.S., Dragovich B., Nesic, Lj., 2003. Quantum Probability and Related Topics, 6, 179-195.
Djordjevic, G.S., Dragovich, B., Nesic Lj., Volovich, I.V., 2002. Int. J. Mod. Phys. A, 17, 1413-1433.
Djordjevic, G.S., Nesic, Lj., Radovancevic, D., 2014. Int. J. Mod. Phys. A, 29, 1450155 (16 pages). doi:10.1142/S0217751X14501553
Hartle, J.B., Hawking, S.W., 1983. Phys. Rev. D, 28, 2960-2975.
Jalalzadeh S., Vakili, B., 2012. Int. J. Theor. Phys., 51, 263-275.
Maceda, M., Madore, J., Manousselis P., Zoupanos, G., 2004. Eur. Phys. J. C, 36, 529-534.
Sepangi, H.R., Shakerin B., Vakili, B., 2009. Class. Quantum Grav., 26, 065003 (20 pages). doi:10.1088/0264-9381/26/6/065003
Snyder, H.S., 1947. Phys. Rev., 71, 38-41.
Várilly, J.C., 2004. The Interface of noncommutative geometry and physics, in: Clifford Algebras, Volume 34 of the series Progress in Mathematical Physics. Birkhäuser, Boston, pp. 227-242.
Vilenkin, A., 1982. Phys. Lett., 117B, 25-28.
Vladimirov, V.S., Volovich I.V., Zelenov, E.I., 1994. *p*-Adic Analysis and Mathematical Physics, World Scientific, Singapore.
Wess, J., Zumino, B., 1990. Nucl. Phys. (Proc. Suppl.), 18B, 302-312.
Wigner, E., 1932. Phys. Rev., 40, 749-759.
Woronowicz, S.L., 1987. Pub. Res. Inst. Math. Sci., 23, 117-181.

KANTOVSKI-SAKS MINISUPERPROSTORNI KOSMOLOŠKI MODEL NA NEKOMUTATIVNOM PROSTORU

Razmatran je vakuumski homogeni i anizotropni Kantovski-Saks minisuperprostorni kosmološki model. U klasičnom slučaju, Lagranžijan modela je pogodnim koordinatnim transformacijama sveden na Lagranžijan dva dekuplovana oscilatora sa jednakim frekvencama i nultom ukupnom energijom ("oscillator-ghost-oscillator system"). Model je takođe formulisan na nekomutativnom prostoru.

Ključne reči: *3+1 formalizam, Kantovski-Saks model, nekomutativnost*