FACTA UNIVERSITATIS
Series: **Physics, Chemistry and Technology** Vol. 14, N° 3, 2016, Special Issue, pp. 243 – 255
DOI: 10.2298/FUPCT1603243I

CAPACITARY ESTIMATE ON THE SPACE OF ENDS OF TREE BASED ON ORLICZ NORM †

UDC 517.987.1 + 519.216/.217 + 514.12 + 511.225 + 519.172.1

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Abstract. *In this article, we will focus on a significance of Ben Amor's result which reveals an important relationship between Orlicz norm and a capacitary estimate. We will derive a lower capacitary estimates from spectral analytic overviews based on the scheme and recent development of stochastic analytic schemes on the ends of a tree. In particular, as an application of our analytical approach, we will shed light on a capacitary estimate for singleton given as an end of the tree.*

Key words: *capacitary estimate, tree, Orlicz norm*

1. Introduction

A class of Markov processes on the field of p-adic numbers constructed by Albeverio and Karwowski associates the spectral theory for spectral analysis in [1] and their method was improved so that a more general class of Markov processes on ends of tree is covered in [2]. It is noteworthy to recall that their transition semi-groups are explicitly described. This is partly because capacitary estimates have been discussed in [11] and [10] based on the kernels determined by transition probability rooted in [8], where the use of probabilistic counterpart of the Bessel kernels is proposed.

Received April 17$^{\text{th}}$, 2016; accepted June 23$^{\text{rd}}$, 2016.
† **Acknowledgement:** This work was supported by Japan Society for the Promotion of Science JP26400150.
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It is widely accepted that the natural random walk on the binary tree gives a reinterpretation of a Markov process on the Cantor set equivalently on the ring of 2-adic integers by restricting our attention on the displacements of the random walk while the particle is traveling on the ends of tree, which are attached to the tree as geometric ideal boundary points. Historically, Baxter suggested in [3] that such relationship of random walk on tree and Markov process on the ends of tree can be discussed. Afterwards, in [14] a clearer potential theoretic relationship is suggested when those ends constitute a compact set, which covers how the harmonic extension into the tree is determined by the boundary values given on the ends. In [15], it is discuss ed that the complete orthonormal system in the family of the square integrable functions on the ends of tree plays an important role for a construction of Markov process on the ends of tree and the harmonic extension is taken without assuming compactness of the ends of tree.

Recently, capacitary estimate on the ends of tree is discussed based on the complete orthonormal system in [7]. However, any comparison of capacity with Radon measure has not been discussed persistently based on the complete orthonormal system. The main objective of this article is building a scheme on capacitary estimate based on the complete orthonormal system taken as in [15] and [12]. We will look also at probabilistic significance of Orlicz space theory pointed out in [16], which showed that an estimate on a reasonably given Orlicz norm implies a lower estimate on capacity of compact sets. In this article, we will derive a lower capacitary estimate for compact sets in space of ends of tree from a spectral analytical method by closely looking at a relationship between the Orlicz space and spectral decomposition of functions associated with nodewise given Dirichlet spaces in [12].

More specifically, along the scheme built by Ben Amor, a regular Dirichlet space on $(\mathcal{E}, \mathcal{F})$ on $L^2(\Sigma^+, \mu)$ and an Orlicz space $L^{(\Phi,m)}$ will be required in our discussion, where Σ^+ stands for the space consisting of ends of a tree, Φ for an N-function and μ, m for Radon measures on Σ^+, respectively. We will rely on Ben Amor's result which showed that the validity of the inequality

$$|u^2|_{L^{(\Phi,\nu)}} \leq M_1(\mathcal{E}(u,u) + (u,u)_{L^2(\Sigma^+,\mu)}), \qquad \text{for any } u \in \mathcal{F} \tag{1}$$

with some positive constant M_1 is equivalent to the validity of the capacitary estimate

$$m(K)\Psi^{-1}(1/m(K)) \leq M_2 \mathrm{Cap}(K), \qquad \text{for any compact set } K \text{ in } \Sigma^+, \tag{2}$$

with some positive constant M_2, where Cap stands for the capacity associated with the Dirichlet space $(\mathcal{E}, \mathcal{F})$.

in Section 2, we will look back at that the tree equipped with nodewise given Dirichlet spaces in [12] is so well-designed as to yield a space Σ^+ consisting of ends of tree and a Dirichlet space on Σ^+. In Section 3, we will briefly recapitulate the notions of Luxemburg norm and Orlicz space. We will pay attention to its aspect tightly related to the Dirichlet space theory. In Section 4, we will achieve our objective based on Ben Amor's result by showing an upper estimate on Orlicz norm on Σ^+. We will show the key inequality (1.1) so that the lower capacitary estimate on compact set on Σ^+ is obtained by assuming the so called ∇'-condition, In the final section, we will shift our attention to the scheme on a tree with a root, which will cover a fundamental capacitary estimate on Cantor-like subsets of Σ^+ associated with a tree with a constant branching number.

2. Tree and its ends

We take a set T consisting of countably infinite vertices and a map $\mathcal{A} : T \times T \to \{0, 1\}$. Each element in T is called a node as well and the pair $\{x, y\}$ of distinct nodes satisfying $\mathcal{A}(x, y) = 1$ will be called an edge. A sequence $(a_0, a_1, \ldots, a_n)$ of nodes in T is called a path, if $\mathcal{A}(x_i, x_{i+1}) = 1$ is satisfied for any $i = 0, \ldots, n-1$. If any pair of distinct elements y, z in T admits a path $(a_0, a_1, \ldots, a_n)$ with $a_0 = y$ and $a_n = z$, the pair $(T, \mathcal{A})$ is called a non-directed tree. If a sequence $(a_0, a_1, \ldots, a_n)$ satisfies $a_i \neq a_j$ for any distinct integers i, j, the sequence said to be simple. The the set $V(x)$ of nodes directly connected with $x \in T$ by an edge is given by $V(x) = \{y \mid \mathcal{A}(x, y) = 1\}$. Throughout the present article, we suppose that a non-directed tree $(T, \mathcal{A})$ satisfying the following properties is given as in [15] and [12]:

(i) the tree does not admit any path $(a_0, a_1, \ldots, a_n)$ satisfying $a_0 = a_n$ with distinct edges $\{a_0, a_1\}$,$\{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\}$,

(ii) $V(x)$ is a finite set and $\#(V(x)) \geq 3$ at any node x in T.

We introduce the notion of end of the tree in the next, similarly to [15] and [12]. An infinite sequence $(a_0, a_1, \ldots)$ of nodes is called a geodesic ray if any finite subsequence of $(a_0, a_1, \ldots, a_n, \ldots)$ is simple path. The set of geodesic rays is denoted by $\mathcal{R}$. We introduce the equivalence relation "$\sim$" on $\mathcal{R}$ defined by

$$(a_0, a_1, \ldots) \sim (b_0, b_1, \ldots) \quad \Leftrightarrow \quad \begin{array}{l} \text{there exists an integer} \\ k \text{ satisfying } a_{k+m} = b_m \text{ for any } m \geq 0. \end{array} \tag{3}$$

The quotient space $\mathcal{R}/\sim$ is denoted by Σ and each element in Σ is called an end. For establishing such hierarchical structure as the one associated with the field of p-adic numbers, we fix an element Δ in Σ and denote $\Sigma \setminus \{\Delta\}$ by Σ^+. Let us take a representative sequence $(\delta_0, \delta_1, \ldots)$ for Δ. Then any tree satisfying the these assumptions provides us with the situation where any node x outside $\{\delta_0, \delta_1, \ldots\}$ and any δ_i are connected by a unique simple path $(a_0, \ldots, a_n)$ as $a_0 = x$ and $a_n = \delta_i$. The length of simple path $(a_0, \ldots, a_n)$ is defined as n, i.e., the number of the nodes added to the initial node a_0 in the path. Our situation enables us to take a unique path with minimal length in the set $\{(a_0, \ldots, a_n) \mid a_0 = x, a_n = \delta_i,$ with some positive integer n for some $i\}$ of the simple paths. Consequently, we can focus only on the path with the minimal length. A path will be denoted by $(x, x', \ldots, \delta_i)$ and its length by $\ell(x)$.

Then, the map $\pi : T \to T$ is defined by

$$\pi(x) = \begin{cases} x' & \text{if } x \notin \{\delta_0, \delta_1, \ldots\}, \\ \delta_{i+1} & \text{if } x = \delta_i \text{ for some } i \geq 0. \end{cases} \tag{4}$$

T is represented as the disjoint union of its subsets $\{T_m \mid m \in \mathbb{Z}\}$ defined by $T_m = \{x \in T \mid \ell(x) - i_{(x)} = m\}$ for $m \in \mathbb{Z}$. Then, it turns out that $\pi(T_m) = T_{m-1}$ for any integer m.

Let us define S_x by $S_x = \{y \in T \mid \pi^k(y) = x \text{ for some non-negative integer } k\}$ for any $x \in T$ and $\Sigma_x^+ = \{\eta \in \Sigma^+ \mid \eta \text{ admits a geodesic ray } (a_0, a_1, \cdots) \text{ as a representative sequence of } \eta \text{ satsfying } a_0, a_1, \cdots \in S_x\}$. We can introduce a topology on Σ^+. As a matter of fact, the family $\{\Sigma_S^+ \mid S \subset T\}$ of subsets $\Sigma_S^+ = \cup_{x \in S} \Sigma_x^+$ determined by $S \subset T$ satisfies the axioms for open sets on Σ^+. We will regard Σ^+ as a topological space equipped with the family of open sets.

Example 1 (A tree $T_{\mathbb{Q}_p}$ associated with the field $\mathbb{Q}_p$ of p-adic numbers). Let $T_{\mathbb{Q}_p}$ be the set consisting of all balls in $\mathbb{Q}_p$ and denote the radius of ball B by $\mathrm{r}(B)$. Then we define $\mathcal{A}_{\mathbb{Q}_p}(B, B')$ for $B, B' \in T_{\mathbb{Q}_p}$ by

$$\mathcal{A}_{\mathbb{Q}_p}(B, B') = \begin{cases} 1 & \text{if either } B \subset B',\ p\,\mathrm{r}(B) = \mathrm{r}(B') \text{ or } B' \subset B,\ p\,\mathrm{r}(B') = \mathrm{r}(B) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then it is not difficult to see the pair $(T_{\mathbb{Q}_p}, \mathcal{A}_{\mathbb{Q}_p})$ is a tree satisfying condition (i) and (ii). Take the ball Δ_0 centered at the origin 0 and with the radius 1 and a sequence $(\Delta_0, \Delta_1, \cdots)$ of elements in $T_{\mathbb{Q}_p}$ specified by $\Delta_i \subset \Delta_{i+1}$ and $\mathrm{r}(\Delta_i) = p^i$ for any $i = 0, 1, 2, \ldots$. In accordance with the choice of the end $\Delta_{\mathbb{Q}_p} \in \Sigma$ represented by the geodesic ray $(\Delta_0, \Delta_1, \cdots)$, the map π is defined by $\pi(B) = B'$ with the ball B' characterized by $B \subset B'$ and $p\,\mathrm{r}(B) = \mathrm{r}(B')$ and in addition a homeomorphism between Σ^+ and $\mathbb{Q}_p$ is obtained. In fact, any end $\eta \in \Sigma^+$ admits a geodesic ray $(B_0, B_1, \cdots)$ represented by a sequence of balls satisfying $B_0 \supsetneqq B_1 \supsetneqq \ldots$, which determines a singleton $\{a\} \subset \mathbb{Q}_p$ by $\{a\} = \cap_i B_i$. The map $\eta \mapsto a$ gives a bijection from Σ^+ to $\mathbb{Q}_p$ which is viewed as a homeomorphism.

The main assertions in [14] show that a Dirichlet form on the Cantor set is constructed so as to be a natural counterpart of the classical Douglas integral on the unit circle, where functions on the unit circle are replaced with ones on the Cantor set and the standard Brownian motion on the unit disk is replaced with a random walk on $T_{\mathbb{Z}_2}$ for accommodating the Dirichlet form to the generalization of the classical Douglas integral based on its probabilistic reinterpretation as in [5]. The results in [14] can be viewed as this sort of reconsideration natually arising from the case that the unit circle is replaced with $\mathbb{Z}_2$. A more general scheme founded on the same motives is built in [15]. A relationship between random walks on tree and a capacity on the ends of tree is discussed in [3].

A function taking constant on every Σ_y^+ for some disjoint open cover $\{\Sigma_y^+\}_{y\in S}$ of Σ^+ determined by some $S \subset T$ is said to be locally constant. The family of locally constant functions taking constant on every Σ_y^+ with $y \in T_{m+1}$ is denoted by $\mathcal{C}^m(\Sigma^+)$. The family of locally constant functions vanishing outside Σ_x^+ will be denoted by $\mathcal{C}(\Sigma_x^+)$ for every $x \in T$. The Stone-Weierstrass theorem shows that $\mathcal{C}(\Sigma_x^+)$ is contained densely in the family of continuous functions with support in Σ_x^+. In what follows, the intersection $\mathcal{C}(\Sigma_x^+) \cap \mathcal{C}^m(\Sigma^+)$ given by $x \in T_m$ will play an important role and be denoted by $\mathcal{C}_x$. In the present article, a node $x \in T$ will be called the confluent node for $y, z \in T$, if there exist positive integers m, ℓ such that $\pi^\ell(y) = \pi^m(z) = x$ and $\pi^{\ell-1}(y) \neq \pi^{m-1}(z)$. The confluent node for $y, z \in T$ will be denoted x by $[y, z]$.

$V(x)\setminus\{\pi(x)\}$ will be denoted by $S^+(x)$ and the positive integer $\#(S^+(x))-1 = \#(V(x)) - 2$ will be denoted by $n(x)$ for every $x \in T$. Throughout this article, we restrict our attention to the case that Σ^+ admits a Radon measure μ on Σ^+ with the support Σ^+ and a complete orthonormal system $\mathcal{V}$ of $L^2(\Sigma^+;\mu)$ is given so that it is divided into orthonormal systems $\{\mathcal{V}_x\}_{x\in T}$ each of which is assigned by $\mathcal{V}_x = \{\varphi \in \mathcal{V} \cap \mathcal{C}_x \mid (\varphi, 1_{\Sigma^+})_{L^2(\Sigma^+;\mu)} = 0\}$ and consists of $n(x)$ elements. The existence of such complete orthonormal system has been substantially ensured in [2] and turns to be an explicit assumption in [15]. Those authors dealt with the case that the complete orthonormal system coincides with the system of the eigenfunctions.

We take a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\Sigma^+;\mu)$ equipped with the inner product $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_{L^2(\Sigma^+;\mu)}$ for $u, v \in \mathcal{F}$. In accordance with the context in [2] and [12], we assume that $\mathcal{V} \subset \mathcal{F}$ and look into a relationship between the Dirichlet form $\mathcal{E}$ and the orthogonal projection P_x to the linear subspace $\mathcal{C}_{x,0}$ spanned by $\mathcal{V}_x$ at each node $x \in T$. In this article, we assume that there exists some positive number $\Lambda < \dfrac{1}{2}$ such that $|\mathcal{E}(P_x 1_{\Sigma_w^+} P_{\pi(x)} 1_{\Sigma_{w'}^+})| \leq \Lambda \mathcal{E}(P_{\pi(x)} 1_{\Sigma_w^+}, P_{\pi(x)} 1_{\Sigma_{w'}^+})$ for any $w, w' \in S^+(x)$ and $x \in T$ as in [13].

The results obtained in the final section of [2] give us the existence of $\mathcal{V}$ satisfying our assumptions and they yield that the conditions imposed on u in Lemma 1 in [13] is satisfied with $1_{\Sigma_x^+}$ in their framework. Accordingly, without losing the particular settings as in [13], we may assume that, for every $x \in T$, $1_{\Sigma_x^+}$ is described as the limit of the sequence $\{P^{\pi(x)}_{\pi^{k+1}(x)} 1_{\Sigma_x^+}\}_{k=1}^{\infty}$ with respect to the norm $\sqrt{\mathcal{E}_1(u, u)}$ on $\mathcal{F}$, where $P^{\pi(x)}_{\pi^{k+1}(x)}$ stands for the orthogonal projection to the linear subspace $\mathcal{C}^{\pi(x)}_{\pi^{k+1}(x),0}$ spanned by $\cup_{m=1}^{k+1} \mathcal{V}_{\pi^m(x)}$ in $L^2(\Sigma^+;\mu)$. This assumption clarifies the regularity of the Dirichlet space in terms of the complete orthonormal basis $\mathcal{V}$. In fact, the linear subspace spanned by $\mathcal{V}$ is contained densely in $\mathcal{F}$ with respect to the norm $\sqrt{\mathcal{E}_1(u, u)}$. Due to the Stone-Weierstrass theorem and the discussion as in Chapter VII in [18], it turns out that the family of continuous functions on Σ^+ with compact support contains the linear subspace spanned by finite linear combination of elements in $\{1_{\Sigma_x^+} \mid x \in T\}$ as a dense subset. We will admit a conventional notation π^0 to represent the identity map on T.

We suppose that a finite subset S of T with the property $y, z \in S, y \neq z \Rightarrow \Sigma_y^+ \cap \Sigma_z^+ = \emptyset$ and a symmetric bilinear form $\mathcal{E}$ on the real linear space $\mathcal{C}$ spanned by $\{1_{\Sigma_x^+} \mid x \in S\}$ are given. $(\mathcal{E}, \mathcal{C})$ is called a Dirichlet space, if it satisfies

(i) $\mathcal{E}(u, u) \geq 0$ for any $u \in \mathcal{C}$,

(ii) $\mathcal{E}(1_{\Sigma_y^+}, 1_{\Sigma_z^+}) \leq 0$ for any distinct $y, z \in S$,

(iii) $v = 1$ on $\cup_{x \in S} \Sigma_x^+$ implies $\mathcal{E}(u, v) = 0$ for any $u \in \mathcal{C}$.

When a Dirichlet space $(\mathcal{E}, \mathcal{C})$ is given, it admits the adjacent matrix A which is viewed as a linear operator in the Euclidean space $\mathcal{C}$ and specified by $\mathcal{E}(u, v) = -\sum_{y,z \in S} A_{y,z} u(y) v(z)$ for any $u, v \in \mathcal{C}$. It is easy to see the identities $A_{y,z} = A_{z,y}$ for any $y, z \in S$ and $\sum_{y \in S} A_{y,z} = 0$ for any $z \in S$. Another representation $\mathcal{E}(u, v) = \frac{1}{2} \sum_{y,z \in S} A_{y,z}(u(y) - u(z))(v(y) - v(z))$ for any $u, v \in \mathcal{C}$ of the bilinear form follows from these observations. If a symmetric bilinear form $\mathcal{E}$ with domain $\mathcal{C}$ satisfies (i), $\mathcal{E}$ said to be non-negative definite.

In proving the following proposition, the Beurling-Deny formula which provides us with the representation

$$\mathcal{E}(u, v) = \frac{1}{2} \int \int_{\Sigma^+ \times \Sigma^+ \cap \{\eta \neq \zeta\}} (u(\eta) - u(\zeta))(v(\eta) - v(\zeta)) J(d\eta, d\zeta) \tag{6}$$

in the Dirichlet space theory (see [6] for the detail) will be crucially applied. In fact, it is not difficult to see that there is no Σ^+-valued continuous function defined on any intervals in the real line. Hereafter, we take the bilinear form $\mathcal{E}(P_x u, P_y v)$ and denote it by $\mathcal{E}_{x,y}(u, v)$. We will also take the linear subspace $\mathcal{C}_{\pi^k(x)}^{S^+(x)} = \oplus_{\pi(y)=x} \mathcal{C}_{y,0} \oplus \mathcal{C}_{x,0} \oplus \cdots \oplus \mathcal{C}_{\pi^{k-1}(x),0} \oplus \mathcal{C}_{\pi^k(x)}$ of $L^2(\Sigma^+; \mu)$ and the symmetric bilinear form

$$\begin{aligned}
&\sum_{\pi(y)=\pi(z)=x} \big(\mathcal{E}_{y,z}(u, v) + \mathcal{E}_{y,x}(u, v) + \mathcal{E}_{x,z}(u, v)\big) + \mathcal{E}_{x,x}(u, v) + \\
&\sum_{\ell=0}^{k-1} \big(\mathcal{E}_{\pi^\ell(x),\pi^{\ell+1}(x)}(u, v) + \mathcal{E}_{\pi^{\ell+1}(x),\pi^\ell(x)}(u, v) + \mathcal{E}_{\pi^{\ell+1}(x),\pi^{\ell+1}(x)}(u, v)\big)
\end{aligned} \tag{7}$$

for $u, v \in \mathcal{C}_{\pi^k(x)}$, which will be denoted by $\mathcal{E}_{\pi^k(x)}^{S^+(x)}(u, v)$. Similarly to [13], we can prove the following assertion:

Proposition 2.1. *If $\mathcal{E}(\varphi, \psi) = 0$ for any $\varphi \in \mathcal{C}_{y,0}, \psi \in \mathcal{C}_{z,0}$ with distinct $y, z \in T$ except $\pi(y) = \pi(z)$, $\pi(y) = z$ and $\pi(z) = y$, then bilinear form $\mathcal{E}_{y,z}$ defined on $\mathcal{C}_y \times \mathcal{C}_z$ associated with $y, z \in T$ with $\pi(y) = \pi(z)$, $\pi(y) = z$ or $\pi(z) = y$ and such bilinear forms constitute a family with the following properties:*

(i) $\mathcal{E}_{y,z}(u,v) = \mathcal{E}_{z,y}(v,u)$ *for any* $u \in \mathcal{C}_{y,0}, v \in \mathcal{C}_{z,0}$,

(ii) $\mathcal{E}_{y,z}(u, 1_{\Sigma_z^+}) = 0$ *for any* $u \in \mathcal{C}_{y,0}$ *and* $\mathcal{E}_{y,z}(1_{\Sigma_y^+}, v) = 0$ *for any* $v \in \mathcal{C}_{z,0}$,

(ii) *the symmetric bilinear form* $\mathcal{E}_{\pi^k(x)}^{S^+(x)}$ *determined by*

$$\mathcal{E}_{\pi^k(x)}^{S^+(x)}(u,v) = \sum_{\pi(y)=\pi(z)=x} \big(\mathcal{E}_{y,z}(u,v) + \mathcal{E}_{y,x}(u,v) + \mathcal{E}_{x,z}(u,v)\big) + \mathcal{E}_{x,x}(u,v)$$
$$+ \sum_{\ell=0}^{k-1} \big(\mathcal{E}_{\pi^\ell(x),\pi^{\ell+1}(x)}(u,v) + \mathcal{E}_{\pi^{\ell+1}(x),\pi^\ell(x)}(u,v) + \mathcal{E}_{\pi^{\ell+1}(x),\pi^{\ell+1}(x)}(u,v)\big)$$

with domain $\oplus_{\pi(y)=x}\mathcal{C}_{y,0} \oplus \mathcal{C}_{x,0} \oplus \cdots \oplus \mathcal{C}_{\pi^{k-1}(x),0} \oplus \mathcal{C}_{\pi^k(x)}$ *is a Dirichlet space for any* $x \in T$ *and non-negative integer* k.

3. N-function, Luxemburg norm and Orlicz space

Throughout the article, we take a finite Radon measure μ on Σ^+ with the support Σ^+. We introduce the Orlicz space based on the notion of Luxemburg norm given by an N-function according to the results stated in [17]. We introduce those notions on the space Σ^+.

Definition 3.1. *If a strictly increasing convex function* $\Phi : [0,\infty) \to [0,\infty)$ *with* $\Phi(0) = 0$ *satisfies*

$$\lim_{t\to 0+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0, \tag{8}$$

then Φ *is called an N-function. For any N-function* Φ*, the function defined by*

$$\Psi(y) = \sup\{x|y| - \Phi(x) \mid x \geq 0\}. \tag{9}$$

is called conjugate of Φ*.*

The following assertions are shown in [17]:

Theorem 3.2. *If* Φ *is an N-function,* Φ *admits the representation*

$$\Phi(x) = \int_0^x \varphi(t)dt \tag{10}$$

with a left-continuous function φ *which vanishes only at the origin and satisfies* $\lim_{t\to\infty}\varphi(t) = \infty$*. The conjugate* Ψ *of* Φ *defined by is represented as* $\Psi(y) = \int_0^y \varphi^{-1}(t)dt$*.*

Theorem 3.3. *Any N-function Φ satisfies*

$$a, b \geq 0 \Rightarrow \Phi(a) + \Phi(b) \leq \Phi(a+b) \tag{11}$$

and

$$a, b \geq 0 \Rightarrow \Phi^{-1}(a) + \Phi^{-1}(b) \geq \Phi^{-1}(a+b). \tag{12}$$

Definition 3.4. *If an N-function Φ satisfies*

$$x, y \geq 0 \Rightarrow \Phi(x)\Phi(y) \leq \Phi(xy), \tag{13}$$

then Φ is said to satisfy ∇'-condition.

Definition 3.5. ([17]) For a Radon measure m on Σ^+ and N-function Φ, the subfamily

$$\{f \mid \sup\left\{ \int_{\Sigma+} |fg|\, dm \,\middle|\, \int_{\Sigma+} \Psi(|g|)\, dm \leq 1 \right\} < \infty\}$$

of all measurable function on Σ^+ is called Orlicz space and denoted by $L^{\Phi}(\Sigma^+, m)$. For each function f in $L^{\Phi}(\Sigma^+, m)$, its norm is defined by

$$|f|_{L^{(\Phi,m)}} = \sup\left\{ \int_{\Sigma+} |fg|\, dm \,\middle|\, \int_{\Sigma+} \Psi(|g|)\, dm \leq 1 \right\}.$$

One can propose another norm

$$|f|_{(\Phi,m)} = \inf\left\{ \lambda > 0 \,\middle|\, \int_{\Sigma+} \Phi(|f|/\lambda)\, dm \leq 1 \right\}$$

for any $f \in L^{(\Phi,m)}$. It is well known that

$$|f|_{(\Phi,m)} \leq |f|_{L^{(\Phi,m)}} \leq 2|f|_{(\Phi,m)}. \tag{14}$$

For any regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\Sigma^+; \mu)$ with a Radon measure μ on Σ^+, as pointed out in [16], we see that that the following conditions (i) and (ii) are equivalent:

(i) there exists a positive constant M_1 such that

$$|u^2|_{L^{(\Phi,m)}} \leq M_1(\mathcal{E}(u,u) + (u,u)_{L^2(\Sigma^+;\mu)}) \quad \text{for any } u \in \mathcal{F}, \tag{15}$$

(ii) there exists a positive constant M_2 such that

$$m(K)\Psi^{-1}(1/m(K)) \leq M_2 \mathrm{Cap}(K) \quad \text{for any compact set } K \text{ in } \Sigma^+, \tag{16}$$

where $\mathrm{Cap}(K)$ stands for the capacity of K associated with $(\mathcal{E}, \mathcal{F})$.

4. Estimate on Orlicz norm in Dirichlet space theory

In this section, we focus on the regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(\Sigma^+;\mu)$ satisfying the conditions in Propositon 2.1 and make an attempt on establishing estimate which bridges the Orlicz norm and capacitary estimate on compact sets in Σ^+. In our scheme already built in [12], the set $\{x\}\times\{1,\dots,n(x)\}$ is denoted by $N(x)$ and each element of the complete orthonormal system $\mathcal{V}$ of $L^2(\Sigma^+_o;\mu)$ is specified by the notation v_ν with some $\nu\in N(x)$ so that $\mathcal{V}_x=\{v_\nu \mid \nu\in N(x)\}$ is satisfied for any $x\in T$ consistently with Section 2.

For the purpose, we assume

(C.1) $\mu(\Sigma^+_x)\geq 1$ for any $x\in T_k$ with $k\leq 0$,

(C.2) there exists a sequence $\{g_k\}_{k=0}^\infty$ of measurable functions defined on Σ^+ satisfying $\int_{\Sigma^+}\Psi(|g_k(x)|)d\mu(x)\leq 1$ such that $|\mathcal{E}(u_x,u_{\pi(x)})|\leq \dfrac{1}{3}\displaystyle\int_{\Sigma^+_{\pi(x)}}|u_x u_{\pi(x)}|$ $\times\,|g_k(x)|d\mu(x)$ for any $x\in T_k$,

(C.3) $|\mathcal{E}(u_y,u_z)|\leq \dfrac{1}{12n(x)^2}(\mathcal{E}(u_y,u_y)+\mathcal{E}(u_z,u_z))$ for any $y,z\in T$ with $\pi(y)=\pi(y)=x$.

In what follows, $\min\{\mathcal{E}_{x,x}(u,u)\mid u\in\mathcal{F}$ satisfying $\|u\|_{L^2(\Sigma^+_x;\mu)}=1\}$ will be denoted by $\underline{\lambda}(x)$. First we show the following theorem aiming at the capacitary estimate:

Theorem 4.1. For any N-fucntion Φ satisfying ∇'-condition and Radon measure μ on Σ^+, if there exists some non-decreasing sequence $\{c_n\}$ satisfying $\inf_n c_n\geq 1$ and $\sum_{n=0}^\infty 1/c_n^2<\infty$ such that

$$\underline{\lambda}(x)+1\geq c_n^2\sup_{y\in S^+(x)}\frac{\Phi^{-1}(\mu(\Sigma^+_y))}{\mu(\Sigma^+_y)\Phi^{-1}(1)} \tag{17}$$

is satisfied with n determined by $x\in T_n$ for any $x\in T$, then there exists some positive conctant M_1 such that

$$|u^2|_{L^{(\Phi,\mu)}}\leq M_1(\mathcal{E}(u,u)+(u,u)_{L^2(\Sigma^+;\mu)})\quad \text{for any } u\in\mathcal{F}. \tag{18}$$

5. Tree with a root

As established in Section II in [7], we can start also with a tree T_o with its root o. Namely, we consider the case that there exists a node $o\in T_o$ such that any $x\in T_o$ is connected with o by a unique simple path $(o,\dots,x)$ and any element in the quotient space $\mathcal{R}/\sim$ admits representative element $(o,a_1,a_2,\dots)$. Such a unique vertex o will be called a root and $\mathcal{R}/\sim$ will be denoted by Σ^+_o. Then, the map $\pi: T_o\setminus\{o\}\to T$ is defined by

$$\pi(x) = \begin{cases} x' & \text{if } (o,\ldots,x',x) \text{ is a simple path connecting } o \text{ and } x, \\ o & \text{if } (o,x) \text{ is a simple path connecting } o \text{ and } x. \end{cases} \tag{19}$$

T_o is represented as the disjoint union of its subsets T_m with $m = 0, 1, 2 \ldots$ defined by $T_m = \{x \in T \mid \pi^m(x) = o\}$ for positive integer m and $T_0 = \{o\}$. It turns out that $\pi(T_m) = T_{m-1}$ for any positive integer m. Similarly to Section 2, for the the set $V(x)$ of nodes directly connected with $x \in T$, $V(x) \setminus \{\pi(x)\}$ will be denoted by $S^+(x)$ for any $x \in T_o$. We consider the case that the condition

$V(x)$ is a finite set and $\#V(x) \geq 3$ for any $x \in T_o$

is satisfied as Section 2.

We take a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(\Sigma_o^+; \mu)$ equipped with the inner product $\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_{L^2(\Sigma_o^+;\mu)}$ for $u, v \in \mathcal{F}$. As in the previous section, we assume that a complete orthonormal system $\mathcal{V} \subset \mathcal{F}$ of $L^2(\Sigma_o^+; \mu)$ is divided into orthonormal systems $\{\mathcal{V}_x\}_{x \in T_o}$ each of which is assigned by $\mathcal{V}_x = \{\varphi \in \mathcal{V} \cap \mathcal{C}_x \mid (\varphi, 1_{\Sigma_o^+})_{L^2(\Sigma_o^+;\mu)} = 0\}$ with $n(x) = \#S^+(x) - 1$ elements for every $x \in T_o \setminus \{o\}$ and by $\mathcal{V}_o = \mathcal{V} \cap \mathcal{C}_o$ with $n(o) = \#S^+(o)$ elements. Similarly again to Section 2, the orthogonal projection to the linear subspace $\mathcal{C}_{x,0}$ spanned by $\mathcal{V}_x$ is denoted by P_x at each node $x \in T_o$ and we assume that there exists some positive number $\Lambda < \dfrac{1}{2}$ such that $|\mathcal{E}(P_x 1_{\Sigma_w^+} P_{\pi(x)} 1_{\Sigma_{w'}^+})| \leq \Lambda \mathcal{E}(P_{\pi(x)} 1_{\Sigma_w^+}, P_{\pi(x)} 1_{\Sigma_{w'}^+})$ for any $w, w' \in S^+(x)$ and $x \in T_o \setminus \{o\}$. Due to this assumption, similarly to Section 2, we can prove the following fundamental assertion:

Proposition 5.1. *If $\mathcal{E}(\varphi, \psi) = 0$ for any $\varphi \in \mathcal{C}_{y,0}, \psi \in \mathcal{C}_{z,0}$ with distinct $y, z \in T_o$ except $\pi(y) = \pi(z)$, $\pi(y) = z$ and $\pi(z) = y$, then bilinear form $\mathcal{E}_{y,z}$ defined on $\mathcal{C}_y \times \mathcal{C}_z$ associated with $y, z \in T$ with $\pi(y) = \pi(z)$, $\pi(y) = z$ or $\pi(z) = y$ and such bilinear forms constitute a family with the following properties:*

(i) $\mathcal{E}_{y,z}(u, v) = \mathcal{E}_{z,y}(v, u)$ *for any* $u \in \mathcal{C}_{y,0}, v \in \mathcal{C}_{z,0}$,

(ii) $\mathcal{E}_{y,z}(u, 1_{\Sigma_z^+}) = 0$ *for any* $u \in \mathcal{C}_{y,0}$ *and* $\mathcal{E}_{y,z}(1_{\Sigma_y^+}, v) = 0$ *for any* $v \in \mathcal{C}_{z,0}$,

(iii) *for any non-negative integer k, the symmetric bilinear form $\mathcal{E}_{\pi^k(x)}^{S^+(x)}$ determined by*

$$\mathcal{E}_{\pi^k(x)}^{S^+(x)}(u, v) = \sum_{\pi(y)=\pi(z)=x} \big(\mathcal{E}_{y,z}(u, v) + \mathcal{E}_{y,x}(u, v) + \mathcal{E}_{x,z}(u, v)\big) + \mathcal{E}_{x,x}(u, v)$$
$$+ \sum_{\ell=0}^{k-1} \big(\mathcal{E}_{\pi^\ell(x),\pi^{\ell+1}(x)}(u, v) + \mathcal{E}_{\pi^{\ell+1}(x),\pi^\ell(x)}(u, v) + \mathcal{E}_{\pi^{\ell+1}(x),\pi^{\ell+1}(x)}(u, v)\big)$$

with domain $\oplus_{\pi(y)=x}\mathcal{C}_{y,0} \oplus \mathcal{C}_{x,0} \oplus \cdots \oplus \mathcal{C}_{\pi^{k-1}(x),0} \oplus \mathcal{C}_{\pi^k(x)}$ *is a Dirichlet space for any* $x \in T_o$*, where* $x \in \cup_{\ell=k}^{\infty} T_\ell$.

In this section, we do not assume (C.1)-(C.3). However, by taking a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ satisfying the conditions in Proposition 5.1, we can prove the following assertion similarly to the theorem in the previous section:

Proposition 5.2. Let Φ be an N-function satisfying ∇'-condition and μ, m be both Radon measures on Σ_o^+. If

$$\underline{\lambda}(x) + 1 \geq c_n^2 \sup_{y \in S^+(x)} \frac{\Phi^{-1}(\mu(\Sigma_y^+))}{m(\Sigma_y^+)\Phi^{-1}(1)} \tag{20}$$

is satisfied with n given by $x \in T_n$ at every $x \in T_o$, for some non-decreasing sequence $\{c_n\}$ satisfying $\inf c_n \geq 1$ and $\sum_{x \in T} 1/c_x^2 < \infty$, then there exists a positive constant M such that

$$m(K)\Psi^{-1}(1/m(K)) \leq M\mathrm{Cap}(K) \tag{21}$$

for any compact set $K \subset \Sigma_o^+$.

For any integer $q \geq 2$, let us denote the tree with a root satisfying $\#S^+(x) = q$ at each vertex x by $T^{(q)}$. We take trees $T^{(r)}$ and $T^{(s)}$ and Radon measures μ on $\Sigma^{+(r)}$ and m on $\Sigma^{+(s)}$ determined by $\mu(\Sigma_x^{+(r)}) = r^{-n}$ $(x \in T_n^{(r)})$ and $m(\Sigma_z^{+(s)}) = s^{-n}$ $(z \in T_n^{(s)})$ respectively. In the case $r > s$, we see that $T^{(s)}$ is naturally embedded in $T^{(r)}$ and $\Sigma^{+(s)}$ is in $\Sigma^{+(r)}$. By taking $p = (1/\log_{\frac{1}{r}} \frac{1}{s})$ and the N-function $\Phi(x) = x^p/p$, we see the following corollary:

Corollary 5.3. *Let* r, s *be a pair of integers satisfying* $r \geq 4$ *and* $r > s \geq 3$*. If there exists some non-decreasing sequence* $\{c_n\}$ *satisfying* $\inf c_n \geq 1$ *and* $\sum_{n=1}^{\infty} 1/c_n^2 < \infty$ *at every* $x \in T_n^{(r)}$ *such that* $\underline{\lambda}(x) + 1 \geq c_n^2 \sup_{y \in S^+(x)} \mu(\Sigma_y^{+(r)})^{\frac{1}{p}-1}$ *with* n *determined by* $x \in T_n$ *for any* $x \in T_o$*, then there exists some constant* M *such that*

$$m(\Sigma_z^{+(s)}) \leq M\mathrm{Cap}(\Sigma_x^{+(r)})$$

for any $x \in T^{(r)}, z \in T^{(s)}$ *satisfying* $x \in T_n^{(r)}, z \in T_n^{(s)}$ *with some* n.

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PROCENA KAPACITETA NA PROSTORU KRAJEVA DRVETA BAZIRANA NA ORLICZ NORMI

U ovom članku, fokusiraćemo se na značaj Ben Amorovog rezultata koji otkriva važnu relaciju izmedju Orlicz norme i procene kapaciteta. Izvešćemo donje procene kapaciteta pomoću spektralnih analitičkih pregleda na osnovu šeme i nedavnih razvoja stohastičkih analitičkih šema na krajevima nekog drveta. Kao primenu našeg analitičkog pristupa, posebno ćemo rasvetliti procenu kapaciteta za singleton dat kao kraj drveta.

Ključne reči: *procena kapaciteta, drvo, Orlicz norma*