FACTA UNIVERSITATIS
Series: **Physics, Chemistry and Technology** Vol. 14, N° 3, 2016, Special Issue, pp. 319 – 335
DOI: 10.2298/FUPCT1603319S

SELF-ADJOINTNESS, CONFINEMENT AND THE CASIMIR EFFECT †

UDC 535.14: 533.5

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Abstract. *An influence of a classical magnetic field on the vacuum of the quantized charged spinor matter field confined between two parallel material plates is studied. In the case of the uniform magnetic field transverse to the plates, the Casimir effect is shown to be repulsive, independently of a choice of boundary conditions and of a distance between the plates.*

Key words: *magnetic field, Casimir effect, vacuum, spinor matter*

1. Introduction

Almost seven decades ago, Casimir [1] predicted an attraction between grounded metal plates as a macroscopic effect of vacuum fluctuations in quantum field theory. Since then, his prediction has been confirmed experimentally with great precision, opening prospects for its application in modern nanotechnology, see review in [2].

The detected Casimir force (or pressure) between parallel plates separated by distance a,

Received March 3rd, 2016; accepted May 27th, 2016.
†**Acknowledgement:** The work Yu.A.S. was supported by the National Academy of Sciences of Ukraine (project No.0112U000054) and the ICTP – SEENET-MTP grant PRJ-09 "Strings and Cosmology". The work V.M.G. was supported by the Swiss National Science Foundation grant SCOPE IZ 7370-152581.
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$$F_{EM} = -\frac{\pi^2}{240a^4}, \tag{1}$$

is due to the vacuum fluctuations of the quantized electromagnetic field only [1]. As to the vacuum fluctuations of other quantized fields, their contribution to the Casimir effect was theoretically considered erstwhile, see, e.g., [2]. It suffices to note here that this contribution is of order of a^{-4} at $a \ll \lambda_C$ and of order of $a^{-4}(a/\lambda_C)^{\nu}$ $\exp(-2a/\lambda_C)$ at $a \gg \lambda_C$, where $\lambda_C = m^{-1}$ is the Compton wavelength of the matter field of mass m; the sign of this contribution, as well as exponent ν, depends on a boundary condition and the spin of the matter field. Usually, the Casimir effect is validated experimentally for the macroscopic separation of plates: $a > 10^{-8}\,\mathrm{m}$. So, even if one takes the lightest massive particle, electron ($\lambda_C = 3.86 \times 10^{-13}\,\mathrm{m}$), then it becomes clear that the case of $a \ll \lambda_C$ has no relation to physics reality. Whereas, in the realistic case of $a \gg \lambda_C$, the contribution of the vacuum fluctuations of quantized massive matter fields to the Casimir effect is vanishing.

However, quantized massive matter fields can be charged, and as those perceive an influence from external (classical) electromagnetic fields. We shall study an impact of a classical magnetic field on the vacuum of the quantized massive matter field; both the quantized and classical fields are confined to a bounded spatial region. A crucial point for our analysis is a choice of boundary conditions, and we adhere to the most general one. Namely, the principles of comprehensibility and mathematical consistency require that operators of physical observables in quantum mechanics be self-adjoint, see, e.g., [3]. To put it simply, a multiple action is well defined for a self-adjoint operator only, allowing for the construction of functions of the operator, such as resolvent, evolution, heat kernel and zeta-function operators, with further implications upon second quantization.

The mathematical demand for the self-adjointness of a differential operator acting on wave functions in a bounded spatial region is a somewhat more general than the physical demand for the confinement of appropriate quantized matter fields within this region. The concept of confined matter fields is quite familiar in the context of condensed matter physics: collective excitations (e.g., spin waves and phonons) exist only inside material objects and do not spread outside. Nonetheless, a quest for boundary conditions ensuring the confinement of the quantized matter was initiated in particle physics in the context of a model description of hadrons as composites of quarks and gluons [4, 5]. If an hadron is an extended object occupying spatial region Ω bounded by surface $\partial\Omega$, then the quark matter field, $\psi(\mathbf{r})$, is subject to the MIT bag boundary condition [6]:

$$[I + \mathrm{i}\beta(\mathbf{n} \cdot \boldsymbol{\alpha})]\psi(\mathbf{r})|_{\mathbf{r}\in\partial\Omega} = 0, \tag{2}$$

where α^1, α^2, α^3 and β are the generating elements of the Dirac-Clifford algebra. However, the point is that this boundary condition is not the only one. The most general set of boundary conditions in the case of a simply-connected boundary involves four arbitrary parameters [7, 8], and the explicit form for this set has been given [9]; the set is compatible with the self-adjointness of the Dirac hamiltonian operator, and its four parameters can be interpreted as the self-adjoint extension parameters.

Thus, let us consider in general the quantized spinor matter field that is confined to the three-dimensional spatial region Ω bounded by the two-dimensional surface $\partial\Omega$. To study a response of the vacuum to the classical magnetic field, we restrict ourselves to the case of a boundary consisting of two parallel planes; the classical magnetic field strength is assumed to be uniform and orthogonal to the planes. As was already explained, we start from the most general set of mathematically acceptable (i.e. compatible with the self-adjointness) boundary conditions. Further follow physical constraints that the spinor matter be confined within the planes and that the spectrum of the wave number vector in the direction which is orthogonal to the planes be unambiguously (although implicitly) determined. Employing these mathematical and physical restrictions, we consider the generalized Casimir effect which is due to vacuum fluctuations of the quantized spinor matter field in the presence of the classical uniform magnetic field; the pressure from the vacuum onto the bounding planes will be found.

2. Self-adjointness and boundary conditions

Defining a scalar product as $(\tilde{\chi}, \chi) = \int\limits_{\Omega} \mathrm{d}^3 r\, \tilde{\chi}^\dagger \chi$, we get, using integration by parts,

$$(\tilde{\chi}, H\chi) = (H^\dagger \tilde{\chi}, \chi) - \mathrm{i} \int\limits_{\partial\Omega} \mathrm{d}\mathbf{s} \cdot \tilde{\chi}^\dagger \boldsymbol{\alpha} \chi, \tag{3}$$

where

$$H = H^\dagger = -\mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m \tag{4}$$

is the formal expression for the Dirac hamiltonian operator and $\boldsymbol{\nabla}$ is the covariant derivative involving both the affine and bundle connections (natural units $\hbar = c = 1$ are used). Operator H is Hermitian (or symmetric in mathematical parlance),

$$(\tilde{\chi}, H\chi) = (H^\dagger \tilde{\chi}, \chi), \tag{5}$$

if

$$\int\limits_{\partial\Omega} \mathrm{d}\mathbf{s} \cdot \tilde{\chi}^\dagger \boldsymbol{\alpha} \chi = 0. \tag{6}$$

The latter condition can be satisfied in various ways by imposing different boundary conditions for χ and $\tilde{\chi}$. However, among the whole variety, there may exist a possibility that a boundary condition for $\tilde{\chi}$ is the same as that for χ; then the domain of definition of $H^\dagger$ (set of functions $\tilde{\chi}$) coincides with that of H (set of functions χ), and operator H is self-adjoint. The action of a self-adjoint operator results in functions belonging to its domain of definition only, and, therefore, a multiple action and functions of such an operator can be consistently defined.

Condition (6) is certainly fulfilled when the integrand in (6) vanishes, i.e.

$$\tilde{\chi}^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{7}$$

To fulfill the latter condition, we impose the same boundary condition for χ and $\tilde{\chi}$ in the form

$$\chi|_{\mathbf{r}\in\partial\Omega} = K\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = K\tilde{\chi}|_{\mathbf{r}\in\partial\Omega}, \tag{8}$$

where K is a matrix (element of the Dirac-Clifford algebra) which is determined by conditions

$$K^2 = I, \quad K^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})K = -\mathbf{n}\cdot\boldsymbol{\alpha}. \tag{9}$$

It should be noted that, in addition to (7), the following combination of χ and $\tilde{\chi}$ is also vanishing at the boundary:

$$\tilde{\chi}^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})K\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{\chi}^\dagger K^\dagger(\mathbf{n}\cdot\boldsymbol{\alpha})\chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{10}$$

Using the standard representation for the Dirac matrices, one can get matrix K in the off-diagonal form [9]

$$K = \frac{(1+u^2-v^2-\mathbf{t}^2)\beta + (1-u^2+v^2+\mathbf{t}^2)I}{2\mathrm{i}(u^2-v^2-\mathbf{t}^2)}(u\mathbf{n}\cdot\boldsymbol{\alpha} + v\beta\gamma^5 - \mathrm{i}\mathbf{t}\cdot\boldsymbol{\alpha}), \tag{11}$$

where $\mathbf{t} = (t^1, t^2)$ is a two-dimensional vector which is tangential to the boundary, $\mathbf{t}\cdot\mathbf{n} = 0$, and $\gamma^5 = \mathrm{i}\alpha^1\alpha^2\alpha^3$. Matrix K is Hermitian in two cases only when it takes forms

$$K_+ = -\mathrm{i}\beta(\mathbf{n}\cdot\boldsymbol{\alpha}) \quad (u=1, \quad v=0, \quad \mathbf{t}=0) \tag{12}$$

and

$$K_- = \mathrm{i}v\beta\gamma^5 + \mathbf{t}\cdot\boldsymbol{\alpha} \quad (u=0, \quad v^2+\mathbf{t}^2=1). \tag{13}$$

Matrix K_+ (12) corresponds to the choice of the standard MIT bag boundary condition [6], cf. (2),

$$(I-K_+)\chi|_{\mathbf{r}\in\partial\Omega} = (I-K_+)\tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = 0, \tag{14}$$

when relation (10) takes form

$$\tilde{\chi}^\dagger \beta \chi|_{\mathbf{r}\in\partial\Omega} = 0. \tag{15}$$

It is instructive to go over from off-diagonal matrix K (11) to Hermitian matrix $\tilde{K}$, presenting boundary condition (8) as

$$\chi|_{\mathbf{r}\in\partial\Omega} = \tilde{K}\chi|_{\mathbf{r}\in\partial\Omega}, \quad \tilde{\chi}|_{\mathbf{r}\in\partial\Omega} = \tilde{K}\tilde{\chi}|_{\mathbf{r}\in\partial\Omega}, \tag{16}$$

with $\tilde{K} = \tilde{K}^\dagger$ determined by conditions

$$\tilde{K}^2 = I, \quad [\tilde{K}, \mathbf{n}\cdot\boldsymbol{\alpha}]_+ = 0. \tag{17}$$

This transition is implemented with the use of the block-diagonal Hermitian matrix, N, obeying condition

$$(I - N)K = K^\dagger(I - N); \tag{18}$$

namely, the result is

$$\tilde{K} = (I - N)K + N. \tag{19}$$

Using parametrization

$$\begin{gathered}
u = -\frac{\sin\tilde{\varphi}}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \quad v = \frac{\sin\varphi\cos\theta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \\
t^1 = \frac{\sin\theta\cos\eta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \quad t^2 = \frac{\sin\theta\sin\eta}{\cos\varphi\cos\theta + \cos\tilde{\varphi}}, \\
-\pi/2 < \varphi \le \pi/2, \quad -\pi/2 \le \tilde{\varphi} < \pi/2, \quad 0 \le \theta < \pi, \quad 0 \le \eta < 2\pi,
\end{gathered} \tag{20}$$

one gets

$$\begin{gathered}
K = \mathrm{i}\frac{\beta\cos\varphi\cos\theta + I\cos\tilde{\varphi}}{\cos^2\varphi\cos^2\theta - \cos^2\tilde{\varphi}} \\
\times[\mathbf{n}\cdot\boldsymbol{\alpha}\sin\tilde{\varphi} - \beta\gamma^5\sin\varphi\cos\theta + \mathrm{i}(\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta)],
\end{gathered} \tag{21}$$

where

$$[\mathbf{n}\cdot\boldsymbol{\alpha},\, \alpha^1]_+ = [\mathbf{n}\cdot\boldsymbol{\alpha},\, \alpha^2]_+ = [\alpha^1, \alpha^2]_+ = 0. \tag{22}$$

Then matrix N takes form

$$\begin{gathered}
N = \beta\cos\varphi\cos\tilde{\varphi}\cos\theta - \beta\gamma^5(\mathbf{n}\cdot\boldsymbol{\alpha})\sin\varphi\sin\tilde{\varphi}\cos\theta \\
+\mathrm{i}(\alpha^1\cos\eta + \alpha^2\sin\eta)(\mathbf{n}\cdot\boldsymbol{\alpha})\sin\tilde{\varphi}\sin\theta,
\end{gathered} \tag{23}$$

and one gets

$$\tilde{K} = [\beta e^{i\varphi\gamma^5}\cos\theta + (\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta]e^{i\tilde{\varphi}\mathbf{n}\cdot\boldsymbol{\alpha}}; \tag{24}$$

in particular,

$$K_+ = \tilde{K}|_{\varphi=0,\,\tilde{\varphi}=-\pi/2,\,\theta=0}, \quad K_- = \tilde{K}|_{\varphi=\pi/2,\,\tilde{\varphi}=0}. \tag{25}$$

Thus, the explicit form of the boundary condition ensuring the self-adjointness of operator H (4) is

$$\left\{I - [\beta e^{i\varphi\gamma^5}\cos\theta + (\alpha^1\cos\eta + \alpha^2\sin\eta)\sin\theta]e^{i\tilde{\varphi}\mathbf{n}\cdot\boldsymbol{\alpha}}\right\}\chi|_{\mathbf{r}\in\partial\Omega} = 0 \tag{26}$$

(the same condition is for $\tilde{\chi}$). Four parameters of boundary condition (26), $\varphi, \tilde{\varphi}, \theta$ and η, are interpreted as the self-adjoint extension parameters.

In the context of the Casimir effect, one usually considers spatial region Ω with a disconnected boundary consisting of two connected components, $\partial\Omega = \partial\Omega^{(+)}\bigcup\partial\Omega^{(-)}$. Choosing coordinates $\mathbf{r} = (x, y, z)$ in such a way that x and y are tangential to the boundary, while z is normal to it, we identify the position of $\partial\Omega^{(\pm)}$ with, say, $z = \pm a/2$. In general, there are 8 self-adjoint extension parameters: φ_+, $\tilde{\varphi}_+$, θ_+ and η_+ corresponding to $\partial\Omega^{(+)}$ and φ_-, $\tilde{\varphi}_-$, θ_- and η_- corresponding to $\partial\Omega^{(-)}$. However, if some symmetry is present, then the number of self-adjoint extension parameters is diminished. For instance, if the boundary consists of two parallel planes, then the cases differing by the values of η_+ or η_- are physically indistinguishable, since they are related by a rotation around a normal to the boundary. To avoid this unphysical degeneracy, one has to fix

$$\theta_+ = \theta_- = 0, \tag{27}$$

and there remains 4 self-adjoint extension parameters: φ_+, $\tilde{\varphi}_+$, φ_- and $\tilde{\varphi}_-$. Operator H (4) acting on functions which are defined in the region bounded by two parallel planes is self-adjoint, if the following condition holds:

$$\left\{I - \beta\exp[i(\varphi_\pm\gamma^5 \pm \tilde{\varphi}_\pm\alpha^z)]\right\}\chi|_{z=\pm a/2} = 0 \tag{28}$$

(the same condition holds for $\tilde{\chi}$). The latter ensures the fulfilment of constraints

$$\tilde{\chi}^\dagger\alpha^z\chi|_{z=\pm a/2} = 0 \tag{29}$$

and

$$\tilde{\chi}^\dagger\beta\exp\left\{i[\varphi_\pm\gamma^5 \pm (\tilde{\varphi}_\pm + \pi/2)\alpha^z]\right\}\chi|_{z=\pm a/2} = 0. \tag{30}$$

3. Induced vacuum energy in the bundle curvature background

Let us consider the quantized charged massive spinor field in the background of a static uniform magnetic field; then $\boldsymbol{\nabla} = \boldsymbol{\partial} - \mathrm{i}e\mathbf{A}$ and the connection can be chosen as $\mathbf{A} = (-yB, 0, 0)$, where B is the value of the magnetic field strength which is directed along the z-axis in Cartesian coordinates $\mathbf{r} = (x, y, z)$, $\mathbf{B} = (0, 0, B)$. The one-particle energy spectrum is

$$E_{nk} = \pm\omega_{nk}, \quad \omega_{nk} = \sqrt{2n|eB| + k^2 + m^2}, -\infty < k < \infty, n = 0, 1, 2, \ldots, \tag{31}$$

k is the value of the wave number vector along the z-axis, and n labels the Landau levels. Using the explicit form of the complete set of solutions to the Dirac equation, one can obtain the formal expression for the vacuum expectation value of the energy density

$$\varepsilon^{\infty} = -\frac{|eB|}{2\pi^2} \int\limits_{-\infty}^{\infty} \mathrm{d}k \sum_{n=0}^{\infty} \iota_n \omega_{nk}, \tag{32}$$

where the superscript on the left-hand side indicates that the magnetic field fills the whole (infinite) space; the appearance of factor $\iota_n = 1 - \frac{1}{2}\delta_{n0}$ on the right-hand side is due to the fact that there is one solution for the lowest Landau level, $\psi^{(0)}_{q0k}(\mathbf{r})$ (q is the value of the wave number vector along the x-axis, $-\infty < q < \infty$), and there are two solutions otherwise, $\psi^{(j)}_{qnk}(\mathbf{r})$ ($j = 1, 2$), $n \geq 1$. The integral and the sum in (32) are divergent and require regularization and renormalization. This problem has been solved long ago by Heisenberg and Euler [10] (see also [11]), and we just list here their result

$$\varepsilon^{\infty}_{\mathrm{ren}} = \frac{1}{8\pi^2} \int\limits_0^{\infty} \frac{\mathrm{d}\tau}{\tau} \mathrm{e}^{-\tau} \left[\frac{eBm^2}{\tau} \coth\left(\frac{eB\tau}{m^2}\right) - \frac{m^4}{\tau^2} - \frac{1}{3}e^2B^2 \right]; \tag{33}$$

note that the renormalization procedure involves subtraction at $B = 0$ and renormalization of the charge.

Let us turn now to the quantized charged massive spinor field in the background of a static uniform magnetic field in spatial region Ω bounded by two parallel planes $\partial\Omega^{(+)}$ and $\partial\Omega^{(-)}$; the position of $\partial\Omega^{(\pm)}$ is identified with $z = \pm a/2$, and the magnetic field is orthogonal to the planes. The solution to the Dirac equation in region Ω is chosen as a superposition of two plane waves propagating in opposite directions along the z-axis,

$$\psi^{(j)}_{qnl}(\mathbf{r}) = \psi^{(j)}_{qnk_l}(\mathbf{r}) + \psi^{(j)}_{qn-k_l}(\mathbf{r}), \quad j = 0, 1, 2, \tag{34}$$

where the values of wave number vector k_l ($l = 0, \pm 1, \pm 2, \ldots$) are determined from the boundary condition, see (28):

$$\left\{I-\beta\exp[\mathrm{i}(\varphi_\pm\gamma^5\pm\tilde{\varphi}_\pm\alpha^z)]\right\}\psi^{(j)}_{qnl}(\mathbf{r})|_{z=\pm a/2}=0,\quad (j=1,2)\quad n\geq 1 \tag{35}$$

and

$$\left[I+\frac{\beta}{2}\left(\pm\alpha^z\gamma^5-1\right)\mathrm{e}^{\mathrm{i}(\varphi_\pm-\tilde{\varphi}_\pm)\gamma^5}\Theta(\pm eB)\right. \tag{36}$$
$$\left.-\frac{\beta}{2}\left(\pm\alpha^z\gamma^5+1\right)\mathrm{e}^{\mathrm{i}(\varphi_\pm+\tilde{\varphi}_\pm)\gamma^5}\Theta(\mp eB)\right]\psi^{(0)}_{q0l}(\mathbf{r})|_{z=\pm a/2}=0;$$

the step function is introduced as $\Theta(u)=1$ at $u>0$ and $\Theta(u)=0$ at $u<0$. This boundary condition ensures that the normal component of current $\mathbf{J}_{qnlj}(\mathbf{r})=\psi^{(j)\dagger}_{qnl}(\mathbf{r})\boldsymbol{\alpha}\psi^{(j)}_{qnl}(\mathbf{r})$ $(j=0,1,2)$ vanishes at the planes, see (29),

$$J^z_{qnlj}(\mathbf{r})|_{z=\pm a/2}=0, \tag{37}$$

which signifies that the quantized matter is confined within the planes.

The boundary condition depends on four self-adjoint extension parameters, $\varphi_+,\tilde{\varphi}_+,\varphi_-$ and $\tilde{\varphi}_-$, in the case of $n\geq 1$, see (35), and on two self-adjoint extension parameters, $\varphi_+-\tilde{\varphi}_+$ and $\varphi_-+\tilde{\varphi}_-$ $(eB>0)$, or $\varphi_++\tilde{\varphi}_+$ and $\varphi_--\tilde{\varphi}_-$ $(eB<0)$, in the case of $n=0$, see (36). It should be noted that value $k_l=0$ is allowed for special cases only, when the following condition holds:

$$\sin\frac{\varphi_+-\varphi_-+\tilde{\varphi}_++\tilde{\varphi}_-}{2}\sin\frac{\varphi_+-\varphi_--\tilde{\varphi}_+-\tilde{\varphi}_-}{2}=0. \tag{38}$$

The spectrum of k_l is determined from a transcendental equation which in general possesses two branches and allows for complex values of k_l (details are published elsewhere). It is not clear which of the branches should be chosen, and, therefore, we restrict ourselves to boundary conditions corresponding to the case of a single branch. The latter is ensured by imposing constraint

$$\varphi_+=\varphi_-=\varphi,\quad \tilde{\varphi}_+=\tilde{\varphi}_-=\tilde{\varphi}. \tag{39}$$

Then relations (28) and (30) take forms

$$\left\{I-\beta\exp[\mathrm{i}(\varphi\gamma^5\pm\tilde{\varphi}\alpha^z)]\right\}\chi|_{z=\pm a/2}=0 \tag{40}$$

and

$$\tilde{\chi}^\dagger\beta\exp\left\{\mathrm{i}[\varphi\gamma^5\pm(\tilde{\varphi}+\pi/2)\alpha^z]\right\}\chi|_{z=\pm a/2}=0. \tag{41}$$

respectively, while the equation determining the spectrum of k_l takes form

$$\cos(k_l a) + \frac{\mathrm{sgn}(E_{nk_l})\,\omega_{nk_l}\cos\tilde{\varphi} - m\cos\varphi}{k_l\sin\tilde{\varphi}}\sin(k_l a) = 0, \qquad (42)$$

where $\mathrm{sgn}(u) = \Theta(u) - \Theta(-u)$ is the sign function; note that the spectrum is real, consisting of values of the same sign, say, $k_l > 0$ (values of the opposite sign ($k_l < 0$) should be excluded to avoid double counting). In the case of $\tilde{\varphi} = 0$, the spectrum of k_l is independent of the number of the Landau level, n, and of the sign of the one-particle energy, $\mathrm{sgn}(E_{nk_l})$; moreover, it is independent of φ, since the determining equation takes form

$$\sin(k_l a) = 0; \qquad (43)$$

note that value $k_l = 0$ is admissible in this case, see (38)-(39). In what follows, we shall consider the most general case of two self-adjoint extension parameters, φ and $\tilde{\varphi}$, when the k_l-spectrum depends on n and on $\mathrm{sgn}(E_{nk_l})$, see (42).

Using the explicit form of the complete set of wave functions $\psi^{(j)}_{qnl}(\mathbf{r})$ ($j = 0, 1, 2$), we obtain the following expression for the vacuum expectation value of the energy per unit area of the boundary surface

$$\frac{E}{S} \equiv \int\limits_{-a/2}^{a/2} \mathrm{d}z\,\varepsilon = -\frac{|eB|}{2\pi}\sum_{\mathrm{sgn}(E_{nk_l})}\sum_{l}\sum_{n=0}^{\infty}\iota_n\omega_{nk_l}. \qquad (44)$$

4. Casimir energy and force

Expression (44) can be regarded as purely formal, since it is ill-defined due to the divergence of infinite sums over l and n. To tame the divergence, a factor containing a regularization parameter should be inserted in (44). A summation over values $k_l > 0$, which are determined by (42), can be performed with the use of the following version of the Abel-Plana formula:

$$\sum_{\mathrm{sgn}(E_{nk_l})}\sum_{k_l>0} f(k_l^2) = \frac{a}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}k\,f(k^2) + \frac{2\mathrm{i}a}{\pi}\int\limits_{0}^{\infty}\mathrm{d}\kappa\,\Lambda(\kappa)\{f[(-\mathrm{i}\kappa)^2] - f[(\mathrm{i}\kappa)^2]\} \qquad (45)$$

$$-f(0) - \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{d}k\,f(k^2)\frac{m\cos\varphi\sin\tilde{\varphi}[k^2 - \mu_n(\varphi,\tilde{\varphi})]}{[k^2 + \mu_n(\varphi,\tilde{\varphi})]^2 + 4k^2m^2\cos^2\varphi\sin^2\tilde{\varphi}},$$

where

$$\Lambda(\kappa) = \Bigg(-[\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi}) \qquad (46)$$

$$+\frac{\sin\tilde{\varphi}}{a}\left\{-\kappa^2 m\cos\varphi(\cos 2\tilde{\varphi}\mathrm{e}^{2\kappa a}-1)+[(2\kappa\sin\tilde{\varphi}-m\cos\varphi)\mathrm{e}^{2\kappa a}\right.$$

$$\left.\left.+m\cos\varphi]\mu_n(\varphi,\tilde{\varphi})\right\}[\kappa^2-2\kappa m\cos\varphi\sin\tilde{\varphi}-\mu_n(\varphi,\tilde{\varphi})]^{-1}\right)$$

$$\times\left\{[\kappa^2-2\kappa m\cos\varphi\sin\tilde{\varphi}-\mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{4\kappa a}\right.$$

$$\left.-2[\kappa^2\cos 2\tilde{\varphi}-\mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{2\kappa a}+\kappa^2+2\kappa m\cos\varphi\sin\tilde{\varphi}-\mu_n(\varphi,\tilde{\varphi})\right\}^{-1}$$

and

$$\mu_n(\varphi,\tilde{\varphi})=2n|eB|\cos^2\tilde{\varphi}+m^2\sin(\varphi+\tilde{\varphi})\sin(\varphi-\tilde{\varphi}). \tag{47}$$

In (45), $f(u^2)$ as a function of complex variable u is assumed to decrease sufficiently fast at large distances from the origin of the complex u-plane, and this decrease is due to the use of some kind of regularization for (44). However, the regularization in the second integral on the right-hand side of (45) can be removed; then

$$\mathrm{i}\{f[(-\mathrm{i}\kappa)^2]-f[(\mathrm{i}\kappa)^2]\}=-\frac{|eB|}{\pi}\sum_{n=0}^{\infty}\iota_n\sqrt{\kappa^2-\omega_{n0}^2} \tag{48}$$

with the range of κ restricted to $\kappa>\omega_{n0}$ for the corresponding terms; here, recalling (31), $\omega_{n0}=\sqrt{2n|eB|+m^2}$. As to the first integral on the right-hand side of (45), one immediately recognizes that it is equal to ε^∞ (32) multiplied by a. Hence, if one ignores for a moment the terms in the last line of (45), then the problem of regularization and removal of the divergency in expression (44) is the same as that in the case of no boundaries, when the magnetic field fills the whole space. Defining the Casimir energy as the vacuum energy per unit area of the boundary surface, which is renormalized in the same way as in the case of no boundaries, we obtain

$$\frac{E_{\mathrm{ren}}}{S}=a\varepsilon^\infty_{\mathrm{ren}}-\frac{2|eB|}{\pi^2}a\sum_{n=0}^{\infty}\iota_n\int\limits_{\omega_{n0}}^{\infty}\mathrm{d}\kappa\Lambda(\kappa)\sqrt{\kappa^2-\omega_{n0}^2}+\frac{|eB|}{2\pi}\sum_{n=0}^{\infty}\iota_n\omega_{n0} \tag{49}$$

$$+\frac{|eB|}{2\pi^2}\int\limits_{-\infty}^{\infty}\mathrm{d}k\sum_{n=0}^{\infty}\iota_n\sqrt{k^2+\omega_{n0}^2}\frac{m\cos\varphi\sin\tilde{\varphi}[k^2-\mu_n(\varphi,\tilde{\varphi})]}{[k^2+\mu_n(\varphi,\tilde{\varphi})]^2+4k^2m^2\cos^2\varphi\sin^2\tilde{\varphi}},$$

$\varepsilon^\infty_{\mathrm{ren}}$ is given by (33). The sums and the integral in the last line of (49) (which are due to the terms in the last line of (45) and which can be interpreted as describing the proper energies of the boundary planes containing the sources of the magnetic field) are divergent, but this divergency is of no concern for us, because it has no physical consequences. Rather than the Casimir energy, a physically relevant quantity is the Casimir force per unit area of the boundary surface, i.e. pressure, which is defined as

$$F = -\frac{\partial}{\partial a}\frac{E_{\mathrm{ren}}}{S}, \tag{50}$$

and which is free from divergencies. We obtain

$$F = -\varepsilon_{\mathrm{ren}}^{\infty} + \Delta_{\varphi,\tilde{\varphi}}(a), \tag{51}$$

where

$$\Delta_{\varphi,\tilde{\varphi}}(a) = -\frac{2|eB|}{\pi^2}\sum_{n=0}^{\infty}\iota_n\int\limits_{\omega_{n0}}^{\infty}\mathrm{d}\kappa\Upsilon(\kappa)\sqrt{\kappa^2-\omega_{n0}^2} \tag{52}$$

and

$$\begin{gathered}\Upsilon(\kappa) \equiv -\frac{\partial}{\partial a}a\Lambda(\kappa) = \left[\upsilon_1(\kappa)\mathrm{e}^{6\kappa a} + \upsilon_2(\kappa)\mathrm{e}^{4\kappa a} + \upsilon_3(\kappa)\mathrm{e}^{2\kappa a} + \upsilon_4(\kappa)\right] \\ \times[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{4\kappa a} \\ -2[\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]\mathrm{e}^{2\kappa a} + \kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})\big\}^{-2},\end{gathered} \tag{53}$$

$$\begin{gathered}\upsilon_1(\kappa) = -(2\kappa a - 1)[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})] \\ \times[\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})] - 2[\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} - (2\kappa\sin\tilde{\varphi} - m\cos\varphi)\mu_n(\varphi,\tilde{\varphi})]\kappa\sin\tilde{\varphi},\end{gathered} \tag{54}$$

$$\begin{gathered}\upsilon_2(\kappa) = (4\kappa a - 3)\left\{[\kappa^2 - \mu_n(\varphi,\tilde{\varphi})]^2 - 4\kappa^2 m^2\cos^2\varphi\sin^2\tilde{\varphi}\right\} \\ +8\kappa^2[\kappa^2\cos^2\tilde{\varphi} - m^2\cos^2\varphi - \mu_n(\varphi,\tilde{\varphi})]\sin^2\tilde{\varphi} + 4[\kappa^2 + \mu_n(\varphi,\tilde{\varphi})]\kappa m\cos\varphi\sin\tilde{\varphi},\end{gathered} \tag{55}$$

$$\begin{gathered}\upsilon_3(\kappa) = -(2\kappa a - 3)[\kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})] \\ \times[\kappa^2\cos 2\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})] - 2[\kappa^2 m\cos\varphi\cos 2\tilde{\varphi} + (2\kappa\sin\tilde{\varphi} + m\cos\varphi)\mu_n(\varphi,\tilde{\varphi})]\kappa\sin\tilde{\varphi},\end{gathered} \tag{56}$$

$$\upsilon_4(\kappa) = -[\kappa^2 + 2\kappa m\cos\varphi\sin\tilde{\varphi} - \mu_n(\varphi,\tilde{\varphi})]^2. \tag{57}$$

5. Asymptotics at small and large separations of plates

In the case of a weak magnetic field, $|B| \ll m^2|e|^{-1}$, substituting the sum by integral $\int\limits_0^{\infty}\mathrm{d}n$ and changing the integration variable in (52), we get

$$\Delta_{\varphi,\tilde{\varphi}}(a) = -\frac{1}{\pi^2}\int\limits_m^{\infty}\mathrm{d}\kappa(\kappa^2-m^2)^{3/2}\int\limits_0^1\mathrm{d}v\sqrt{1-v}\,\tilde{\Upsilon}(\kappa,v), \quad |eB| \ll m^2, \tag{58}$$

where $\tilde{\Upsilon}(\kappa,v)$ is obtained from $\Upsilon(\kappa)$ (53) by substitution
$\mu_n(\varphi,\tilde{\varphi}) \to \tilde{\mu}_{v,\kappa^2}(\varphi,\tilde{\varphi})$ with

$$\tilde{\mu}_{v,\kappa^2}(\varphi,\tilde{\varphi}) = v(\kappa^2 - m^2)\cos^2\tilde{\varphi} + m^2\sin(\varphi+\tilde{\varphi})\sin(\varphi-\tilde{\varphi}). \tag{59}$$

In the limit of small distances between the plates, $ma \ll 1$, (58) becomes independent of the φ-parameter:

$$\Delta_{\varphi,\tilde{\varphi}}(a) = \frac{1}{4a^4}\left\{\frac{\pi^2}{30} - \int\limits_0^1 \mathrm{d}v\,\rho_{\tilde{\varphi}}(v)\left(1 - \frac{|\rho_{\tilde{\varphi}}(v)|}{\pi}\right)\left[\frac{3}{2}\sqrt{1-v}\,\rho_{\tilde{\varphi}}(v)\right.\right. \tag{60}$$
$$\left.\left.\times\left(1 - \frac{|\rho_{\tilde{\varphi}}(v)|}{\pi}\right) + \frac{v\sin 2\tilde{\varphi}}{1 - v\cos^2\tilde{\varphi}}\left(\frac{1}{2} - \frac{|\rho_{\tilde{\varphi}}(v)|}{\pi}\right)\right]\right\}, \quad \sqrt{|eB|}a \ll ma \ll 1,$$

where

$$\rho_{\tilde{\varphi}}(v) = \arcsin\left(\frac{\sin\tilde{\varphi}}{\sqrt{1 - v\cos^2\tilde{\varphi}}}\right). \tag{61}$$

Thus, $\Delta_{\varphi,\tilde{\varphi}}(a)$ in this case is power-dependent on the distance between the plates as a^{-4} with the dimensionless constant of proportionality, either positive or negative, depending on the value of the $\tilde{\varphi}$-parameter. In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{\pi^2}{120}\frac{1}{a^4}, \quad \sqrt{|eB|}a \ll ma \ll 1 \tag{62}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = -\frac{7}{8}\frac{\pi^2}{120}\frac{1}{a^4}, \quad \sqrt{|eB|}a \ll ma \ll 1. \tag{63}$$

In the limit of large distances between the plates, $ma \gg 1$, $\Delta_{\varphi,\tilde{\varphi}}(a)$ (58) takes form

$$\Delta_{\varphi,\tilde{\varphi}}(a) = \frac{2}{\pi^2}\int\limits_m^\infty \mathrm{d}\kappa\kappa(\kappa^2 - m^2)^{3/2}\mathrm{e}^{-2\kappa a}\int\limits_0^1 \mathrm{d}v\sqrt{1-v} \tag{64}$$
$$\times\left\{a\frac{\kappa^2\cos 2\tilde{\varphi} - \tilde{\mu}_{v,\kappa^2}(\varphi,\tilde{\varphi})}{\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \tilde{\mu}_{v,\kappa^2}(\varphi,\tilde{\varphi})}\right.$$
$$\left.-\frac{(2\kappa\sin\tilde{\varphi} - m\cos\varphi)\tilde{\mu}_{v,\kappa^2}(\varphi,\tilde{\varphi}) - \kappa^2 m\cos\varphi\cos 2\tilde{\varphi}}{[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde{\varphi} - \tilde{\mu}_{v,\kappa^2}(\varphi,\tilde{\varphi})]^2}\sin\tilde{\varphi}\right\},$$
$$|eB| \ll m^2, \quad ma \gg 1.$$

Clearly, (64) is suppressed as $\exp(-2ma)$. In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{1}{2\pi^{3/2}}\frac{m^{5/2}}{a^{3/2}}\mathrm{e}^{-2ma}\left[1 + O\left(\frac{1}{ma}\right)\right], \ |eB| \ll m^2, \ ma \gg 1 \tag{65}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = \left\{ \begin{array}{ll} -\frac{3}{16\pi^{3/2}}\frac{m^{3/2}}{a^{5/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \varphi = 0 \\ \\ -\frac{\tan^2(\varphi/2)}{2\pi^{3/2}}\frac{m^{5/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1+O(\frac{1}{ma})], & \varphi \neq 0 \end{array} \right\}, \quad (66)$$

$$|eB| \ll m^2, \quad ma \gg 1.$$

In the case of a strong magnetic field, $|B| \gg m^2|e|^{-1}$, one has

$$\Delta_{\varphi,\tilde{\varphi}}(a) = -\frac{|eB|}{\pi^2}\left[\int\limits_m^\infty \mathrm{d}\kappa\sqrt{\kappa^2-m^2}\Upsilon(\kappa)|_{n=0} \right. \quad (67)$$

$$\left. +2\sum_{n=1}^{\infty}\int\limits_{\sqrt{2n|eB|}}^{\infty}\mathrm{d}\kappa\sqrt{\kappa^2-2n|eB|}\Upsilon(\kappa)|_{m=0}\right], \quad |eB| \gg m^2.$$

In the limit of extremely small distances between the plates, $ma \ll \sqrt{|eB|}a \ll 1$, the analysis is similar to that of the limit of $\sqrt{|eB|}a \ll ma \ll 1$, yielding the same results as (61)-(63). Otherwise, in the limit of $\sqrt{|eB|}a \gg 1$, only the first term in square brackets on the right-hand side of (67) matters. In the limit of small distances between the plates this term becomes φ-independent, yielding

$$\Delta_{\varphi,\tilde{\varphi}}(a) = \frac{|eB|}{4a^2}\left[\frac{1}{6} - \frac{|\tilde{\varphi}|}{\pi}\left(1-\frac{|\tilde{\varphi}|}{\pi}\right)\right], \quad \sqrt{|eB|}a \gg 1,\ ma \ll 1. \quad (68)$$

In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{|eB|}{24a^2}, \quad \sqrt{|eB|}a \gg 1,\ ma \ll 1, \quad (69)$$

$$\Delta_{\varphi,\pm\pi/4}(a) = -\frac{|eB|}{192a^2}, \quad \sqrt{|eB|}a \gg 1,\ ma \ll 1 \quad (70)$$

and

$$\Delta_{\varphi,-\pi/2}(a) = -\frac{|eB|}{48a^2}, \quad \sqrt{|eB|}a \gg 1,\ ma \ll 1. \quad (71)$$

In the limit of large distances between the plates, the first term in square brackets on the right-hand side of (67) yields

$$\Delta_{\varphi,\tilde{\varphi}}(a) = \frac{2|eB|}{\pi^2}\int\limits_m^\infty \mathrm{d}\kappa\kappa(\kappa^2-m^2)^{1/2}\mathrm{e}^{-2\kappa a} \quad (72)$$

$$\times\left\{a\frac{\kappa^2\cos 2\tilde\varphi - m^2\sin(\varphi+\tilde\varphi)\sin(\varphi-\tilde\varphi)}{\kappa^2 - 2\kappa m\cos\varphi\sin\tilde\varphi - m^2\sin(\varphi+\tilde\varphi)\sin(\varphi-\tilde\varphi)}\right.$$

$$\left.+\frac{\kappa^2 m\cos\varphi\cos 2\tilde\varphi - (2\kappa\sin\tilde\varphi - m\cos\varphi)m^2\sin(\varphi+\tilde\varphi)\sin(\varphi-\tilde\varphi)}{[\kappa^2 - 2\kappa m\cos\varphi\sin\tilde\varphi - m^2\sin(\varphi+\tilde\varphi)\sin(\varphi-\tilde\varphi)]^2}\sin\tilde\varphi\right\},$$

$$\sqrt{|eB|}a \gg ma \gg 1,$$

which is obviously suppressed as $\exp(-2ma)$. In particular, we get

$$\Delta_{\varphi,0}(a) = \frac{|eB|}{2\pi^{3/2}}\frac{m^{3/2}}{a^{1/2}}\mathrm{e}^{-2ma}\left[1 + O\left(\frac{1}{ma}\right)\right], \quad \sqrt{|eB|}a \gg ma \gg 1 \tag{73}$$

and

$$\Delta_{\varphi,-\pi/2}(a) = \left\{\begin{array}{ll} -\frac{|eB|}{16\pi^{3/2}}\frac{m^{1/2}}{a^{3/2}}\mathrm{e}^{-2ma}[1 + O(\frac{1}{ma})], & \varphi = 0 \\ \\ -\frac{|eB|\tan^2(\varphi/2)}{2\pi^{3/2}}\frac{m^{3/2}}{a^{1/2}}\mathrm{e}^{-2ma}[1 + O(\frac{1}{ma})], & \varphi \neq 0 \end{array}\right\}, \tag{74}$$

$$\sqrt{|eB|}a \gg ma \gg 1.$$

6. Summary and discussion

An influence of a background uniform magnetic field and boundary conditions on the vacuum of the quantized charged spinor matter field (of mass m) confined between two parallel plates has been comprehensively analyzed, and the Casimir force acting onto the plates is found to take the form of (51), where all dependence on the distance between the plates, a, and the choice of boundary conditions parametrized by φ and $\tilde\varphi$ is contained in the second term, $\Delta_{\varphi,\tilde\varphi}(a)$ (52), see (53)-(56). In the physically meaningful case, $ma \gg 1$, this term is exponentially damped as $\exp(-2ma)$, see (64)-(66), (72)-(74), and the Casimir force is given by the first term, $F = -\varepsilon^{\infty}_{\mathrm{ren}}$. This situation is to be contrasted with the case of hot dense matter in thermal equilibrium; the pressure in the latter case may become dependent on the distance between the plates and the choice of boundary conditions. We validate this statement in Appendix by considering, as an example, the axial current density in a strong (supercritical) magnetic field.

Returning to the vacuum effects, let us note that the Heisenberg-Euler vacuum energy density, $\varepsilon^{\infty}_{\mathrm{ren}}$, see (33), is negative (vanishing at $B = 0$ only), hence, the Casimir effect is repulsive, i.e. the pressure from the vacuum onto the plates is positive. Defining the critical value of the magnetic field as $B_{\mathrm{crit}} = m^2|e|^{-1}$, one can obtain the following expression for the Casimir force in the limit of a supercritical magnetic field, $|B| \gg B_{\mathrm{crit}}$, from (33):

$$F = \frac{1}{24\pi^2}\frac{1}{\lambda_C^4}\left(\frac{B}{B_{\rm crit}}\right)^2 \ln\frac{2|B|}{B_{\rm crit}} \tag{75}$$

(recall that $\lambda_C = m^{-1}$ is the Compton wavelength of the matter field). Note that the critical value is the lowest one, $B_{\rm crit} = 4.41 \times 10^{13}\,$G, for the case of quantized electron-positron matter, and supercritical magnetic fields with $|B| \gg 10^{13}\,$G may be attainable in some astrophysical objects, such as neutron stars and magnetars [12], and also gamma-ray bursters in scenarios involving protomagnetars [13]. A proper account for the influence of Casimir pressure (75) on physical processes in these objects should be taken.

Supercritical magnetic fields are not feasible in terrestrial laboratories where the maximal values of steady magnetic fields are of order of $10^5\,$G, see, e.g., [14]. In the case of a subcritical magnetic field, $|B|\ll B_{\rm crit}$, one obtains from (33):

$$F = \frac{1}{360\pi^2}\frac{1}{\lambda_C^4}\left(\frac{B}{B_{\rm crit}}\right)^4. \tag{76}$$

Let us compare this with the attractive Casimir force which is due to the quantized electromagnetic field, see F_{EM} (1), and define ratio

$$\frac{F}{F_{EM}} = -\frac{2}{3\pi^4}\left(\frac{a}{\lambda_C}\right)^4\left(\frac{B}{B_{\rm crit}}\right)^4. \tag{77}$$

At $a = 10^{-6}\,$m and $B = 10^5\,$G the attraction is prevailing over the repulsion by six orders of magnitude, $F_{EM}/F \approx -10^6$, and the Casimir force is $F_{EM} \approx -1.3\,$mPa. However, at $a = 10^{-5}\,$m and $B = 10^6\,$G the repulsion becomes dominant over the attraction by two orders of magnitude, $F/F_{EM} \approx -10^2$, and the Casimir force takes value $F \approx 0.009\,$mPa. Otherwise, the same value of the Casimir force is achieved at $a = 10^{-6}\,$m and $B = 10^7\,$G. Thus, an experimental observation of the influence of the background magnetic field on the Casimir pressure seems to be possible in some future in terrestrial laboratories.

Appendix

Let us consider hot dense ultrarelativistic spinor matter in the background of a static uniform magnetic field which is orthogonal to the bounding plates. Since field strength B, temperature T and chemical potential μ are assumed to be large,

$$|eB| >> m^2, \quad T >> m, \quad \mu >> m, \tag{78}$$

we employ a simplifying approximation neglecting the mass of the matter field and put $m = 0$ in the following. Then the equation determining the spectrum of the wave number vector in the direction of the magnetic field, see (42), takes form

$$\cos(k_l a) + k_l^{-1}\,\mathrm{sgn}(E_{nk_l})\,\omega_{nk_l}\cot\tilde{\varphi}\,\sin(k_l a) = 0, \tag{79}$$

whereas the boundary condition becomes dependent on one parameter, see (40). The z-component of the axial current density is defined as

$$J^{z5} = \sum_{\mathrm{sgn}(E_{nk_l})}\sum_{l}\sum_{n=0}^{\infty}\mathrm{sgn}(E_{nk_l})\psi_{qnl}^{(j)\dagger}(\mathbf{r})\alpha^z\gamma^5\psi_{qnl}^{(j)}(\mathbf{r}) \tag{80}$$
$$\times\left(\exp\left\{\left[\omega_{nk_l} - \mathrm{sgn}(E_{nk_l})\mu\right]/T\right\} + 1\right)^{-1}.$$

Only the lowest Landau level ($n = 0$) contributes to (80), thus the k_l-spectrum is

$$k_l = [l\pi - \mathrm{sgn}(E_{0k_l})\tilde{\varphi}]/a, \quad l \in \mathbb{Z}, \quad k_l > 0, \tag{81}$$

$\mathbb{Z}$ is the set of integer numbers. Then the calculation of the sums over l and $\mathrm{sgn}(E_{nk_l})$ yields

$$J^{z5} = -\frac{eB}{2\pi a}\mathrm{sgn}(\mu)F\left(|\mu|a + \mathrm{sgn}(\mu)\left[\tilde{\varphi} - \mathrm{sgn}(\tilde{\varphi})\frac{\pi}{2}\right];Ta\right), \tag{82}$$

where

$$F(s;t) = \frac{s}{\pi} - \frac{1}{\pi}\int_0^{\infty} dv\frac{\sin(2s)\sinh(\pi/t)}{[\cos(2s) + \cosh(2v)][\cosh(\pi/t) + \cos(v/t)]} \tag{83}$$
$$+\frac{\sinh\{[\arctan(\tan s)]/t\}}{\cosh[\pi/(2t)] + \cosh\{[\arctan(\tan s)]/t\}}.$$

In view of relation

$$\lim_{a\to\infty}\frac{1}{a}F(|\mu|a;\ Ta) = |\mu|/\pi, \tag{84}$$

the case of a magnetic field filling the whole (infinite) space is obtained from (82) as a limiting case:

$$\lim_{a\to\infty} J^{z5} = -\frac{eB}{2\pi^2}\mu. \tag{85}$$

Unlike this unrealistic case, the realistic case of a magnetic field confined to a region between the bounding plates is temperature dependent, see (82) and (83). In particular, we get

$$\lim_{T\to 0} J^{z5} = -\frac{eB}{2\pi a}\mathrm{sgn}(\mu)[\![[|\mu|a + \mathrm{sgn}(\mu)\tilde{\varphi}]/\pi + \Theta(-\mu\tilde{\varphi})]\!] \tag{86}$$

and

$$\lim_{T\to\infty} J^{z5} = -\frac{eB}{2\pi^2}\left\{\mu + [\tilde{\varphi} - \mathrm{sgn}(\tilde{\varphi})\pi/2]/a\right\}; \tag{87}$$

here $[\![u]\!]$ denotes the integer part of quantity u (i.e. the integer which is less than or equal to u).

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SAMOADJUNGOVANOST, KONFAJNMENT I KAZIMIROV EFEKAT

Izučavan je uticaj klasičnog magnetnog polja na vakuum kvantnog polja naelektrisane spinorske materije zatvorenog izmedju dve paralelne materijalne pločice. U slučaju uniformnog magnetnog polja tranzverzalnog na pločice pokazana je odbojnost Kazimirovog efekta, nezavisno od graničnih uslova i rastojanja izmedju pločica.

Ključne reči: *magnetno polje, Kazimirov efekat, vakuum, spinorska materija*