RETURN TIME STATISTICS OF EXTREME EVENTS IN DISCRETE NONLINEAR LATTICES [†]

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Abstract. *Time statistics of extreme events (EEs) in one-dimensional discrete Salerno lattices is investigated numerically. We show that the dependence of the mean return time of EEs on the amplitude threshold can be used as a criterion to differentiate between various dynamical regimes of the extreme events. Also, we found that dispersion of points on the time probability distribution curve can be an indicator of the appearance of EEs in the system, but it has to be complemented with other statistical measures. The results obtained here can be used to distinguish between different dynamical regimes and as identifiers of the EEs existence in the lattice system.*

Key words: *extreme events, nonlinear photonic lattice, time statistics*

1. INTRODUCTION

Extreme events (EE) – rogue or freak waves - can be seen as transient or persistent structures with huge heights (amplitudes), statistically not so significant in the system under observation. Originally, the term 'rogue wave' referred to isolated ocean waves of considerable height that 'appear out of nowhere and disappear without a trace' (Akhmediev et al., 2009) in relatively calm seas. Theoretical investigations of ocean freak waves usually use the nonlinear Schrödinger (NLS) equation and it has been shown that the probability of their appearance is not negligible (Onorato et al., 2001). The modulation instability (MI) is considered as the main origin of EEs in nonlinear systems (Akhmediev and Ankiewicz, 2011, Onorato et al., 2006). MI induces local exponential growth of an initially sinusoidal long-wavelength perturbation of a plane wave solution. Extreme events of this type have also been observed in nonlinear optical systems (Solli et al., 2007), ultra-cold matter (Bludov et al., 2009), microwave experiments (Höhmann et al., 2010), etc.

Regarding the criterion to define an event as the extreme one, usually the significant height is used (Kharif and Pelinovsky, 2003). Based on the observations with water waves, the significant wave height H_s is defined as the average height of one-third of the highest waves in the height distribution. An extreme event can be classified as the one with a height greater than $h_{th}=2.2\ H_s$.

In this paper, we focus on one-dimensional discrete nonlinear lattices and investigate EEs that appear as a result of discrete soliton or breather generation and their mutual interactions. We address the question of correlation of probability of EEs generation with the return time statistics of these events that we explore in detail. We use the Salerno model (SM) (Salerno, 1992) which interpolates between a fully integrable discrete lattice of the Ablowitz-Ladik (AL) type (Ablowitz and Ladik, 1976) and the nonintegrable discrete nonlinear Schrödinger (DNLS) type (Molina and Tsironis, 1993).

The paper is organized as follows. The Salerno model of wave propagation through the system is presented in Section 2. Details on time statistics together with results and discussion are given in Section 3. In Section 4, conclusions are briefly summarized.

2. THE SALERNO MODEL

We consider the wave propagation in one-dimensional nonlinear lattice described by the Salerno model equations,

$$i \frac{d\psi_n}{dt} = -(1+\mu|\psi_n|^2)(\psi_{n+1}+\psi_{n-1})-\gamma|\psi_n|^2\psi_n, \tag{1}$$

where μ and γ are two nonlinearity parameters corresponding to nonlocal and local nonlinearity, respectively. For $\mu=0$ the model becomes the DNLS equation with cubic nonlinearity, while for $\gamma=0$ it reduces to AL equation. Model (1) conserves the first moment (related with the norm) and the Hamiltonian (corresponding to the energy of the system). Equations (1) exhibit MI which may give rise to spatially localized solutions in the form of discrete solitons and breathers (DBs) (Flach and Willis, 1998) and it was proposed that they might serve as models for freak waves, i. e. extreme events.

For the purpose of numerical simulations, the variables Ψ_n in Eq.(1) are rescaled as $\psi_n = \xi_n / \sqrt{\mu}$, so that in terms of ξ_n dynamical equations read

$$i \frac{d\xi_n}{dt} = -(1+|\xi_n|^2)(\xi_{n+1}+\xi_{n-1})-\Gamma|\xi_n|^2\xi_n, \tag{2}$$

where $\Gamma=\gamma/\mu$. This way, the whole two-dimensional parameter space (γ, μ) is scaled by $\mu=1$, leaving γ as a free parameter. By letting Γ attain large values, we can go close to the DNLS limit. The exact DNLS limit ($\mu=0$) has to be calculated separately.

We consider the evolution of an initially uniform state (plane wave solution), slightly amplitude modulated by some initial noise to accelerate the development of MI. This uniform solution is chosen in the interval where linear stability analysis has shown that it was unstable (Maluckov et al., 2009). We numerically integrate the system of Eqs.(2) with periodic boundary conditions using a sixth order Runge-Kutta algorithm for different values of Γ. The size of the lattice is $N=101$ and the time step is fixed to 10^{-4}.

In the previous study on this matter (Maluckov et al., 2009), different dynamical regimes in terms of EEs have been reported. Three main regimes can be selected: the regime with dominant local nonlinearity, the regime with competing local and nonlocal nonlinearity, and

the regime with dominant nonlocal nonlinearity (i.e. vicinity of the AL limit). It was shown that the EEs are most probable in the near integrable (AL) limit. In this paper, we want to probe into the return time statistics of these events and to correlate time statistics with other statistical measures employed previously. The goal is to identify additional measures that can be used to differentiate between various dynamical regimes.

3. TIME STATISTICS – RESULTS AND DISCUSSION

We focus on the time statistics of extreme events. We investigate in detail the return time of EEs and the probability distribution of the return time. The return time r is defined as a time interval between the appearance at a given position of two successive events with amplitudes above certain predetermined height threshold q. The procedure regarding the calculation of the return time probability is given in (Maluckov et al., 2013, Mančić et al., 2017).

Firstly, we study the mean return time R of an EE. Since it is a threshold dependent quantity, we investigate its dependence on a threshold q for different dynamical regimes, i.e. for different values of Γ. The results are presented in Figure 1.
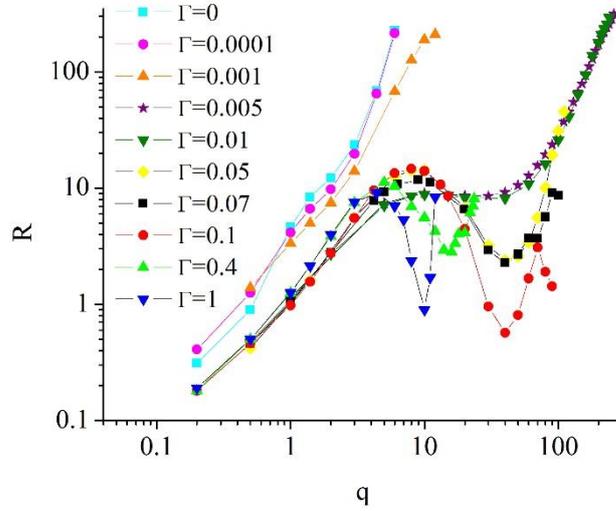

Fig. 1 The mean return time R of extreme events as a function of the threshold amplitude q for different values of Γ. The results are given in the log-log presentation.

By observing the curves $R(q)$ for different Γ, we can distinguish three different regimes. The first regime (i.e. the regime with dominant nonlocal nonlinearity) corresponds to very small values of Γ ($\Gamma < 0.005$) and is characterized by the increase of R with q. In the second regime with dominant local nonlinearity, corresponding to high values of Γ ($\Gamma > 0.01$), curves $R(q)$ have a complex behaviour and one can notice maximum and minimum points on the graphs. There is a third, transient regime between the above mentioned regimes.

Here, R grows with q up to a certain value where it saturates for the further increase of q up to a certain level, when it starts to grow with q again.

We expect this differentiation of regimes to correlate with dynamical evolution of the system, i. e. regimes from the same group to have a similar dynamical evolution. Indeed, if we look at the dynamical evolution of a system for different Γ (Figure 2), these similarities are evident. For small values of Γ (first regime, closeness of AL limit), discrete breathers of relatively small amplitudes are mobile and almost noninteracting (Fig.2(a) upper row). As Γ increases (transient regime), there is an onset of interactions between localized modes, they merge and form discrete rogue waves (i.e. extreme events) as transient structures with very high amplitudes (Fig. 2(b) upper row) – this corresponds to the transient regime on the graph $R(q)$. With further increase of Γ, DNLS-type of behaviour dominates, i.e. the self-trapping is the main localization effect resulting in narrow high amplitude localized structures in the lattice, which are classified as persistent RW structures (Fig. 2(c) upper row). The corresponding $R(q)$ graphs have maxima and minima, the maximum shifts toward the lower values of q with increasing Γ. Differentiation between dynamical regimes is also evident if we look at the amplitude profiles for a certain moment of time given in the lower portion of Fig. 2. These dynamical regimes have been reported in (Maluckov et al., 2009).
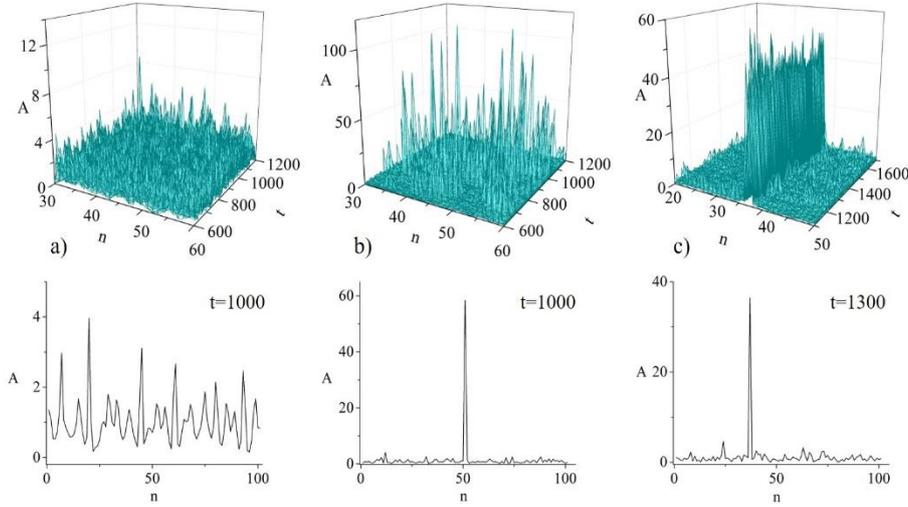

Fig. 2 Evolution of scaled amplitudes (upper row) and amplitude profile at moment t (lower row): (a) $\Gamma=0.0001$, t=1000 (first regime), (b) $\Gamma=0.005$, t=1000 (transient regime) and (c) $\Gamma=0.1$, t=1300 (second regime).

As a next step and before going deeper into the time statistics of EEs, we wanted to check if these findings, regarding the $R(q)$ behaviour, correlate with other statistical measures reported earlier, namely, with the height probability density (HPD) $P_h(h)$ (Maluckov et al., 2009). We note that the tails of HPDs are related to extreme events and the appearance of plateau on these curves indicates an increase in the EE probability. The

probability of EEs, P_{ee}, is obtained by integration of the (normalized) HPD from $h=h_{th}$ up to infinity. The normalized HPD curves for different values of Γ are presented in Figure 3.
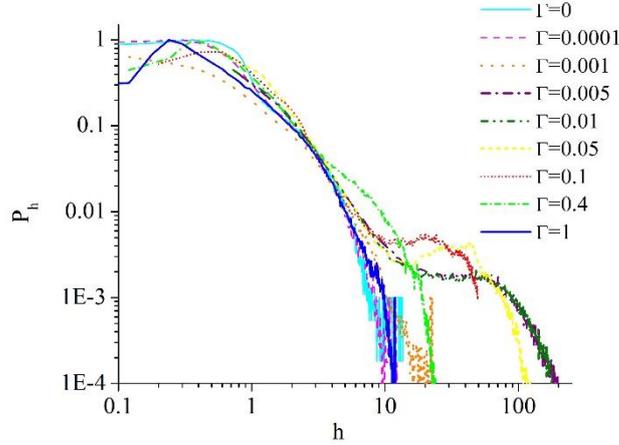

Fig. 3 The normalized height probability density $P_h(h)$ for different values of Γ.

As expected, the same differentiation of regimes found regarding the behaviour of the $R(q)$ curve is found when observing the tendencies in the $P_h(h)$ curves. For small values of Γ (first regime), the tails of the HPD curves decay, indicating negligible probability for the occurrence of EEs (P_{ee} is of the order 0.02). In the transient regime, a plateau appears on the P_h curves, meaning that EEs are more probable now ($P_{ee} \sim 0.07$). With a further increase of Γ (entering the second regime), this plateau is still present and for $\Gamma=0.1$, P_{ee} has a maximum value of 0.11. A consecutive increase of Γ leads to a disappearance of the plateau and thus to a decrease of the P_{ee} (e.g. $P_{ee}=0.06$ for $\Gamma=0.4$). On the $R(q)$ graph, the position of the maximum shifts to smaller q values. For $\Gamma \gg 0.1$, P_{ee} decreases to 0.02 and less.

In the end, we address the main question of return time probability P_r of the EEs for the regimes identified above. One expects to observe a similar P_r behaviour for the corresponding thresholds q within separate regimes. The q values are chosen from the $R(q)$ curves in such a way as to correlate between different Γ values within one regime, i.e. q values are taken at some characteristic positions (where possible). The P_r curves for different Γ values, taken for the same threshold q, should show a similar behaviour. We start by examining the first regime. The results are presented in Figure 4. In Fig. 4(a) various thresholds q for which P_r curves are calculated are presented, whereas in Fig. 4(b), (c) and (d), curves $P_r(r/R)$ for those thresholds and for different Γ are given. And indeed, for all (fixed) q values, the corresponding curves for different Γ show a similar behaviour. These curves can be fitted with one (or two) power-law functions.
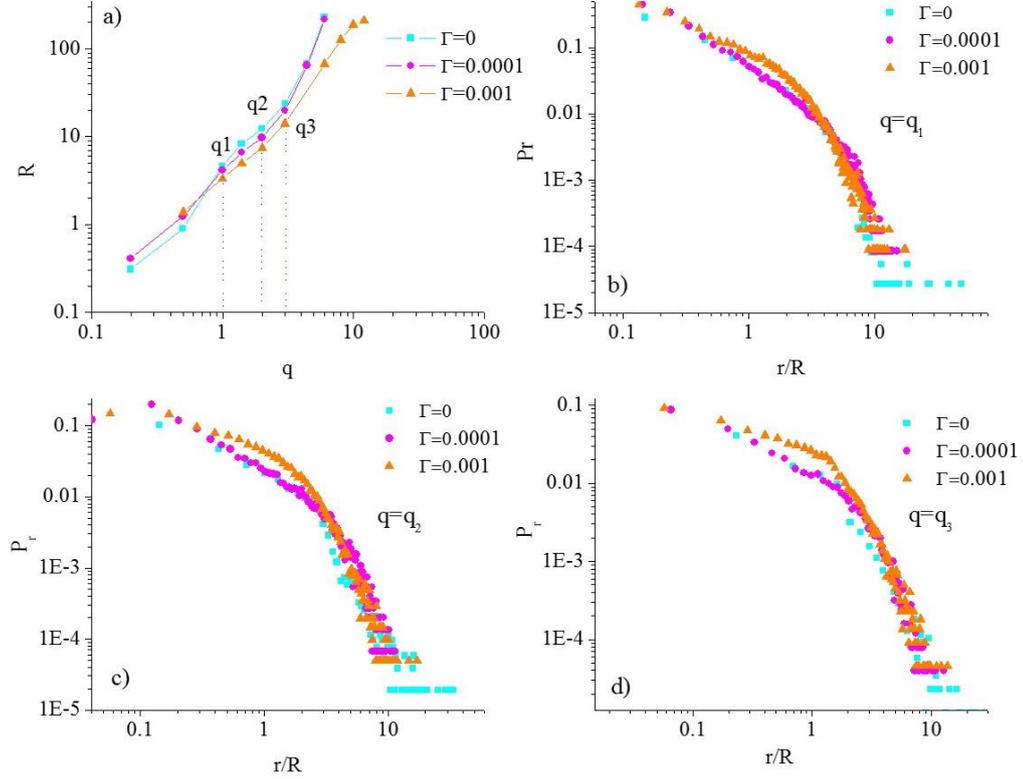
Fig. 4 (a) The mean return time R of extreme events as a function of the threshold
amplitude q for different values of Γ corresponding to the first regime (regime with
dominant nonlocal nonlinearity); Return time probability P_r (normalized so that
the surface below each curve is equal to 1) as a function of r/R for different Γ and
for $q=q_1$ (b), $q=q_2$ (c) and $q=q_3$ (d); ($q_3 \leq h_{th}$). The quantity r/R is the ratio between
the return time r, and the mean return time R, of an EE.

Corresponding P_r curves for Γ values from the second regime and the choice of
thresholds q are shown in Figure 5. The principle behind the choice of thresholds q is
illustrated on the $R(q)$ curve for a single Γ value ($\Gamma=0.1$) for the reason of clarity
(Fig. 5(a)). As previously, for fixed q values, the corresponding P_r curves for different Γ
exhibit a similar behaviour (Fig. 4(b-f)). P_r curves in this regime and for $q=q_0$ to q_{2max} can
be fitted with two power law functions. For $q>q_{2max}$ dispersion of the points on P_r curves
is significant and the fitting procedure cannot be applied with satisfactory accuracy.
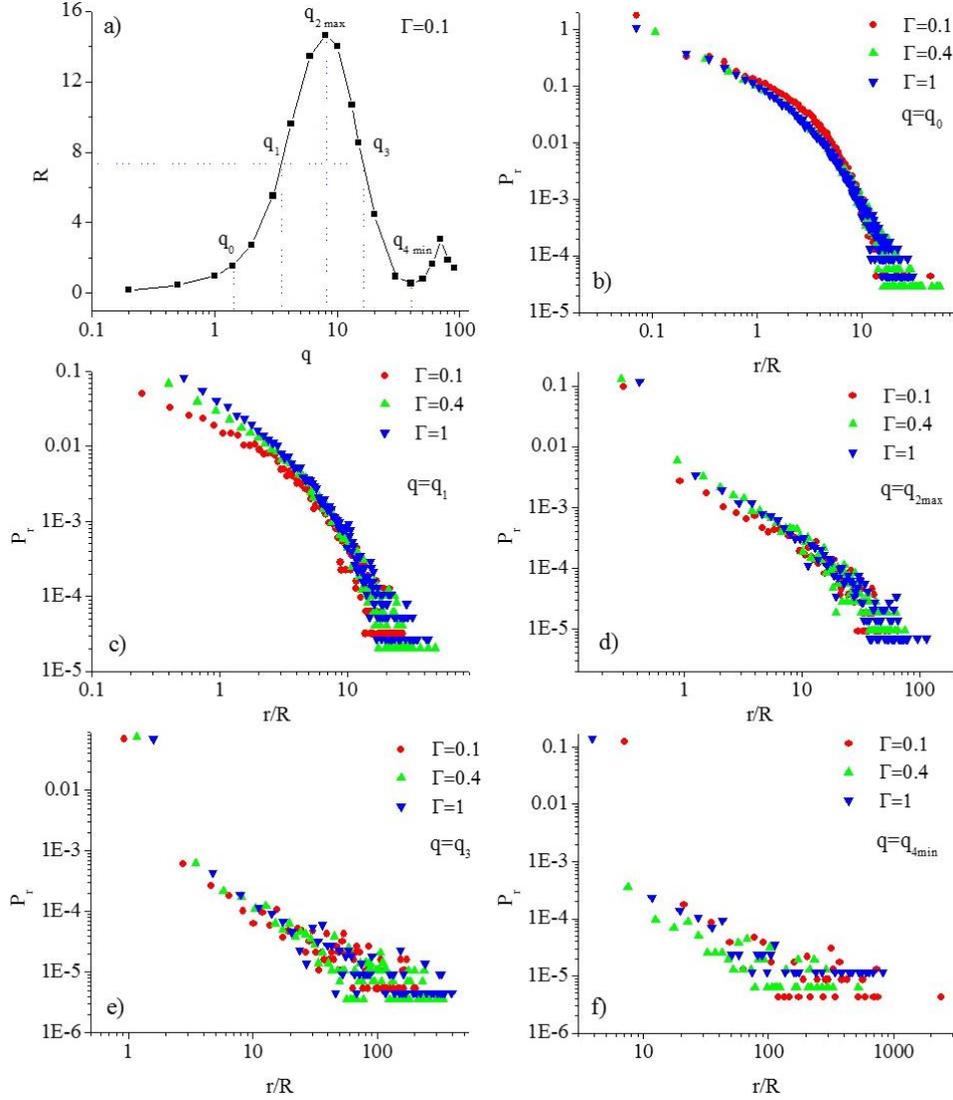
Fig. 5 (a) The mean return time R of extreme events as a function of the threshold amplitude q for different values of Γ corresponding to the second regime (regime with dominant local nonlinearity); Return time probability P_r (normalized so that the surface below each curve is equal to 1) as a function of r/R for different Γ and for $q=q_0$ (b), $q=q_1$ (c), $q=q_{2max}$ (d), $q=q_3$ (e) and $q=q_{4min}$ (f); ($q_{2max} \leq h_{th}$).

Finally, we present P_r curves corresponding to the transient regime (Fig. 6). The choice of q values is presented in Fig. 6(a) whereas P_r curves for fixed q and different Γ are given in Fig. 6 (b-d). The same conclusion as the one from previously explored regimes is valid here, too, P_r curves for fixed q and different Γ agree very well. P_r curves

for $q=q_0$ can be fitted with one or two power law functions, whereas curves corresponding to $q=q_1$, q_2 cannot be fitted due to a significant dispersion of data points.
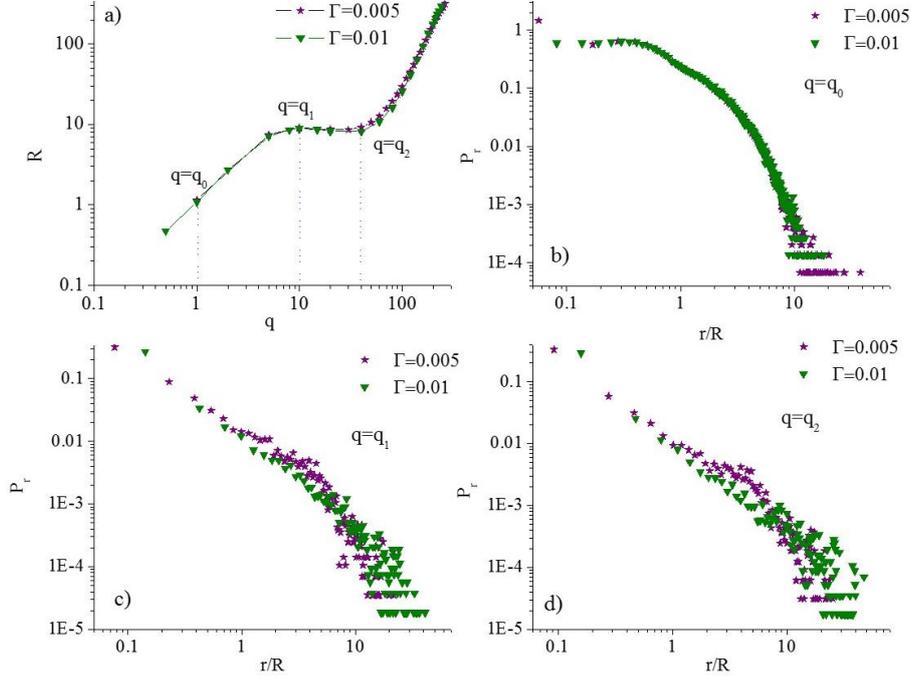

Fig. 6 (a) The mean return time R of extreme events as a function of the threshold amplitude q for different values of Γ corresponding to the transient regime; Return time probability P_r (normalized so that the surface below each curve is equal to 1) as a function of r/R for different Γ and for $q=q_0$ (b), $q=q_1$ (c) and $q=q_2$ (d); ($q_1 \leq h_{th}$).

The large dispersion of data in the regimes with $\Gamma > 0.005$, i.e. the transient and the regime with significant local nonlinearity, can be associated with the leading role of self-trapping of light in the localization of energy. Let us be reminded that it is initiated by development of MI which is, by itself, a process dependent on the light intensity threshold. The result is the trapping of energy and, consequently, a huge increase in the light intensity at a few lattice sites. In our setup, the system is initially prepared so that the MI develops quickly. The largest part of injected energy is trapped in a few lattice sites and these sites can be treated as 'isolated' from the rest of the lattice due to the negligible energy exchange between them and the neighboring sites. This effect is more prompted if the local nonlinearity strength is higher ($\Gamma \geq 0.1$). The amplitude of events in 'isolated' sites is often above the threshold values derived with respect to the H_s. Being isolated, such events are persistent and the corresponding return time cannot be defined. Figuratively, it tends to infinity. This means that persistent huge amplitude structures are excluded from the estimation of P_r. Although the number of such events is not high, the energy that they carry is significant. On the other hand, the remaining energy is distributed among the rest of the

lattice sites. Some of the sites can temporary trap enough energy to be classified as EEs. These events can interact and move through the lattice, but due to their number, amplitude and calculation time we finally observe a significant dispersion of return times with respect to R.

According to the analysis, the existence of EEs can be related to the increase of dispersion on the P_r curve. However, a detailed study which would include other statistical measures is necessary to confirm the presence of EEs in the system.

4. CONCLUSION

In this paper, we have investigated time statistics of extreme events in one-dimensional discrete Salerno lattices which include two types of nonlinearity, local and nonlocal ones. We show that the dependence of the mean return time of EEs on the amplitude threshold, $R(q)$ can be used as a criterion to differentiate between various dynamical regimes of the extreme waves. The last were related to the interplay between the local and nonlocal nonlinearity in the sense that the local nonlinearity acts in the favor of creation of persistent EEs while the nonlocal one involves the interchange of energy between lattice sites and formation of transient EEs. Regarding the $R(q)$ behaviour, we identify three different regimes: the regime with dominant nonlocal nonlinearity, the regime with dominant local nonlinearity and the transient regime. Within each of these regimes, probability distributions of return time P_r for different values of nonlinearity parameter and for fixed amplitude threshold value exhibit a similar behaviour. These findings related to time statistics of EEs correlate with the results obtained in earlier studies where different statistical measures were used (Maluckov et al., 2009). Also, dispersion of the points on the P_r curve can be an indicator of the appearance of EEs in the system, but it has to be complemented with other statistical measures. Therefore, the measures derived from the return time statistics, such as the average return time, can be implemented to distinguish dynamical properties of the extreme events, as well as identifiers of the existence of extreme events in the lattice system.

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VREMENSKA STATISTIKA EKSTREMNIH DOGAĐAJA U DISKRETNIM NELINEARNIM REŠETKAMA

U ovom radu, numerički proučavamo vremensku statistiku ekstremnih događaja (ED) u diskretnim nelinearnim Salerno rešetkama. Pokazali smo da zavisnost srednjeg vremena povratka od vrednosti praga amplitude može da se koristi kao kriterijum za razlikovanje dinamičkih režima ekstremnih događaja. Takođe, disperzija tačaka na krivoj koja predstavlja raspodelu verovatnoće vremena povratka, može da bude indikator pojave ED u sistemu ali uz podatke dobijene iz dodatnih statističkih mera. Predstavljeni rezultati mogu se koristiti za razlikovanje dinamičkih režima kao i za nalaženje ED u nelinearnim diskretnim rešetkama.

Ključne reči: *ekstremni događaji, nelinearna fotonska rešetka, vremenska statistika*