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KILLING FORMS ON TORIC SASAKI-EINSTEIN SPACES *

Mihai Visinescu

Abstract. We summarize recent results on the construction of Killing forms on Sasaki-Einstein manifolds. The complete set of special Killing forms of the Sasaki-Einstein spaces are presented. It is pointed out the existence of two additional Killing forms associated with the complex holomorphic volume form of Calabi-Yau cone manifold. In the case of toric Sasaki-Einstein manifolds the Killing forms are expressed in terms of toric data.

1. Introduction

In the last time Sasakian geometries, as an odd-dimensional analogue of Kähler geometries, have become of high interest. Sasakian structures in $2n-1$ dimensions are sandwiched between the Kähler cone of complex dimension n and the transverse Kähler structure of complex dimension $n-1$. In particular the Kähler cone is Ricci-flat, i.e. Calabi-Yau manifold, if and only if the corresponding Sasaki manifold is Einstein.

The interest in Sasaki-Einstein geometry has arisen due to its importance in AdS/CFT correspondence [9] which relates quantum gravity in certain background to ordinary quantum field theory without gravity. In a particular setting the AdS/CFT correspondence involves Sasaki-Einstein geometries in dimensions 5 and 7 in connection with superconformal field theories in dimensions 4 and 3 respectively.

The most general higher-dimensional metrics describing rotating black holes with NUT parameters in a asymptotically AdS spacetime were described in [2]. In certain scaling limits these geometries are related to Sasaki-Einstein manifolds. Recently nontrivial infinite families of toric Sasaki-Einstein manifolds were explicitly constructed [5, 6] and many new insights were obtained for AdS/CFT correspondence.

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The paper is organized as follows. In the next section we briefly describe various types of Killing tensors. In Section 3 we present the Sasakian geometry and its interrelation with Kähler geometry. The next two Sections are devoted to the symplectic and complex approaches of the metric cone. In Section 6 it is introduced the Delzant construction to obtain the toric description of Calabi-Yau spaces. In Section 7 the complete set of special Killing forms of the toric Sasaki-Einstein spaces are presented. The paper ends with conclusions in Section 8.

2. Killing forms

Killing vector fields represent a basic object of differential geometry connected with the infinitesimal isometries. The flow of a Killing vector field preserves a given metric and there exists a conserved quantity for the geodesic motions. A natural generalization of Killing vector fields is given by the conformal Killing vector fields with flows preserving a given conformal class of metrics. More generally, one can consider conformal Killing forms which are sometimes referred as twistor forms or conformal Killing-Yano tensors.

Definition 2.1. A conformal Killing-Yano tensor of rank p on a n dimensional Riemannian manifold (M, g) is a p-form ψ which satisfies

$$\nabla_X \psi = \frac{1}{p+1} X \lrcorner\, d\psi - \frac{1}{n-p+1} X^* \wedge d^* \psi, \tag{2.1}$$

for any vector field X on M.

Here we used the standard conventions: ∇ is the Levi-Civita connection with respect to the metric g, X^* is the 1-form dual to the vector field X, $\lrcorner$ is the operator dual to the wedge product and d^* is the adjoint of the exterior derivative d. If ψ is co-closed in (2.1), then we obtain the definition of a Killing-Yano tensor [19].

A particular class of Killing forms is represented by the special Killing forms:

Definition 2.2. A Killing form ψ is said to be a special Killing form if it satisfies for some constant c the additional equation

$$\nabla_X (d\psi) = c X^* \wedge \psi, \tag{2.2}$$

for any vector field X on M.

There is also a symmetric generalization of the Killing vectors:

Definition 2.3. A symmetric tensor $K_{i_1 \cdots i_r}$ of rank $r > 1$ satisfying the generalized Killing equation

$$K_{(i_1 \cdots i_r ; j)} = 0,$$

is called a Stäckel-Killing tensor.

Here a semicolon precedes an index of covariant differentiation associated with the Levi-Civita connection and a round bracket denotes a symmetrization over the indices within.

The analogue of the conserved quantities associated with Killing vectors is given by the following proposition:

Proposition 2.1. *For any geodesic γ with tangent vector $\dot{\gamma}^i$*

$$Q_K = K_{i_1\cdots i_r}\dot{\gamma}^{i_1}\cdots\dot{\gamma}^{i_r},$$

is constant along γ.

Let us note that there is an important connection between these two generalizations of the Killing vectors. To wit, given two Killing-Yano tensors $\psi^{i_1,\dots,i_k}$ and $\sigma^{i_1,\dots,i_k}$ there is a Stäckel-Killing tensor of rank 2:

$$K_{ij}^{(\psi,\sigma)} = \psi_{ii_2\dots i_k}\sigma_j{}^{i_2\dots i_k} + \sigma_{ii_2\dots i_k}\psi_j{}^{i_2\dots i_k}.$$

This fact offers a method to generate higher order integrals of motion by identifying the complete set of Killing-Yano tensors.

3. Sasakian geometry

There are many equivalent definitions of the Sasakian structures. A simple and direct definition is the following:

Definition 3.1. A compact Riemannian manifold (Y,g) is Sasakian if and only if its metric cone $(X = C(Y) \cong \mathbb{R}_+ \times Y\,,\ \bar{g} = dr^2 + r^2\,g)$ is Kähler.

Here $r \in (0,\infty)$ may be considered as a coordinate on the positive real line $\mathbb{R}_+$. The Sasakian manifold (Y,g) is naturally isometrically embedded into the metric cone via the inclusion $Y = \{r=1\} = \{1\}\times Y \subset C(Y)$.

Let us denote by

$$\tilde{K} \equiv \mathcal{J}\Big(r\frac{\partial}{\partial r}\Big), \tag{3.1}$$

where $\mathcal{J}$ is the complex structure on the cone manifold. $\tilde{K} - i\mathcal{J}\tilde{K}$ is a holomorphic vector field on $C(Y)$ and the restriction K of $\tilde{K}$ to $Y\subset C(Y)$ is the Reeb vector field on Y. The Reeb vector field K is a Killing vector on (Y,g), has unit length and, in particular, is nowhere zero. Its integral curves are geodesics and the corresponding foliation $\mathcal{F}_K$ is called the Reeb foliation.

Let Y be a Sasaki-Einstein manifold of dimension $\dim_{\mathbb{R}} Y = 2n-1$ and its Kähler cone $X = C(Y)$ is of dimension $\dim_{\mathbb{R}} X = 2n$, $(\dim_{\mathbb{C}} X = n)$. Sasaki-Einstein geometry is naturally "sandwiched" between two Kähler-Einstein geometries as shown in the following proposition:

Proposition 3.1. *Let (Y, g) be a Sasaki manifold of dimension $2n-1$. Then the following are equivalent*

(1) (Y, g) is Sasaki-Einstein with $\mathrm{Ric}_g = 2(n-1)g$;

(2) The Kähler cone $(C(Y), \bar{g})$ is Ricci-flat, $\mathrm{Ric}_{\bar{g}} = 0$;

(3) The transverse Kähler structure to the Reeb foliation $\mathcal{F}_K$ is Kähler-Einstein with $\mathrm{Ric}^T = 2ng^T$.

The Kähler form ω is an exact 2-form and homogeneous degree 2 under the Euler angle $r\frac{\partial}{\partial r}$

$$\omega = -\frac{1}{2}d(r^2\eta) = -rdr\wedge\eta - \frac{1}{2}r^2 d\eta \quad , \quad \mathcal{L}_{r\frac{\partial}{\partial r}}\omega = 2\omega,$$

where η is the Sasakian 1-form of Y. It lifts to $C(Y)$ as

$$\eta = \mathcal{J}\Big(\frac{dr}{r}\Big) = i(\partial - \bar{\partial})\log r. \tag{3.2}$$

We use the same letter η by the abuse of notation. ¿From (3.1) and (3.2) it results that $\tilde{K}$ is dual to the 1-form $r^2\eta$. The Kähler form ω can be written as

$$\omega = \frac{1}{2}i\partial\bar{\partial}r^2,$$

which means that

$$F = \frac{r^2}{4}, \tag{3.3}$$

is the Kähler potential.

4. Symplectic approach

Let (y, ϕ) be the symplectic coordinates on X.

If (X, ω) is toric, the standard n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ acts effectively on X

$$\tau : \mathbb{T}^n \to \mathrm{Diff}(X, \omega), \tag{4.1}$$

preserving the Kähler form. $\partial/\partial\phi_i$ generate the $\mathbb{T}^n$ action, ϕ_i being the angular coordinates along the orbit of the torus action $\phi_i \sim \phi_i + 2\pi$. $\mathbb{T}^n$-invariant Kähler metric on X is

$$ds^2 = G_{ij}dy^i dy^j + G^{ij}d\phi_i d\phi_j, \tag{4.2}$$

where G_{ij} is the Hessian of the symplectic potential $G(y)$ in the y coordinates

$$G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j} \quad , \quad 1 \le i, j \le n,$$

and $G^{ij} = (G_{ij})^{-1}$.

The almost complex structure is

$$\mathcal{J} = \begin{pmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{pmatrix},$$

and the symplectic (Kähler) form is $\omega = dy^i \wedge d\phi_i$.

Associated to (X, ω, τ) there is a moment map $\mu : X \to \mathbb{R}^n$

$$\mu(y, \phi) = y, \tag{4.3}$$

i.e. the projection on the action coordinates:

$$y^i = -\frac{1}{2}\Big\langle r^2\eta, \frac{\partial}{\partial\phi_i}\Big\rangle.$$

The moment map exhibits the Kähler cone as a Lagrangian fibration over a strictly convex rational polyhedral cone $\mathcal{C} \subset \mathbb{R}^n$ by forgetting the angular coordinates ϕ_i [12]

$$\mathcal{C}\left\{y \in \mathbb{R}^n | l_a(y) \geqslant 0 \quad , \quad a = 1, \dots, d\right\},$$

with the linear function $l_a(y) = (y, v_a)$, where v_a are the inward pointing normal vectors to the d facets of the polyhedral cone. The set of vectors $\{v_a\}$

$$v_a = v_a^i \frac{\partial}{\partial\phi_i} \quad , \quad v_a^i \in \mathbb{Z},$$

is called a *toric data*.

5. Complex approach

The standard complex coordinates are w_i on $\mathbb{C}\backslash\{0\}$. Log complex coordinates are $z_i = \log w_i = x_i + i\phi_i$ and in these complex coordinates the metric is

$$ds^2 = F_{ij}dx_i dx_j + F_{ij}d\phi_i d\phi_j, \tag{5.1}$$

where F_{ij} is the Hessian of the Kähler potential (3.3). Note also that in the complex coordinates z_i the complex structures and the Kähler form are:

$$\mathcal{J} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad , \quad \omega = \begin{pmatrix} 0 & F_{ij} \\ -F_{ij} & 0 \end{pmatrix}. \tag{5.2}$$

The moment map of the $\mathbb{T}^n$-action with respect to ω is given by (4.3). The symplectic potential G and Kähler potential F are related by the Legendre transform

$$F(x) = \left(y^i\frac{\partial G}{\partial y^i} - G\right) \ (y = \partial F/\partial x).$$

Therefore F and G are Legendre dual to each other

$$F(x) + G(y) = \sum_j \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y^j} \text{ at } x_i = \frac{\partial G}{\partial y^i} \text{ or } y^i = \frac{\partial F}{\partial x_i}. \tag{5.3}$$

It follows from (4.2) and (5.1) that $F_{ij} = G^{ij}$ $(y = \partial F/\partial x)$.

6. Delzant construction

Let us note that not every polytope in $\mathbb{R}^n$ is the moment polytope of some triple (X, ω, τ). The image of X under the moment map μ (4.3) is a certain kind of convex rational polytope in $\mathbb{R}^n$ called a Delzant polytope [3, 1].

Definition 6.1. A convex polytope P in $\mathbb{R}^n$ is Delzant if

(a) there are n edges meeting at each vertex p;

(b) the edges meeting at the vertex p are rational, i.e. each edge is of the form $1 + tu_i\,, 0 \leq t \leq \infty$ where $u_i \in \mathbb{Z}^n$;

(c) the $u_i, \ldots, u_n$ in (b) can be chosen to be a basis of $\mathbb{Z}^n$.

Delzant construction associates to every Delzant polytope $P \subset \mathbb{R}^n$ a closed connected symplectic manifold (M, ω) together with the Hamiltonian $\mathbb{T}^n$ action (4.1) and the moment map μ (4.3).

Let us write the Reeb vector (3.1) in the form:

$$\tilde{K} = b_i \frac{\partial}{\partial \phi_i}. \tag{6.1}$$

In the symplectic coordinates (y, ϕ) we have

$$r\frac{\partial}{\partial r} = 2y^i \frac{\partial}{\partial y^i},$$

and the components of the Reeb vector (6.1) are $b_i = 2G_{ij}y^j$.

Using the Delzant construction the general symplectic potential has the following form in terms of the toric data [7, 1]:

$$G = G^{can} + G_b + h, \tag{6.2}$$

where

$$G^{can} = \frac{1}{2}\sum_a l_a(y)\log l_a(y), \tag{6.3}$$

$$G_b = \frac{1}{2}\sum_a l_b(y)\log l_b(y) - \frac{1}{2} l_\infty(y)\log l_\infty(y), \tag{6.4}$$

with $l_b(y) = (b, y)\,,\ l_\infty(y) = \sum_a (v_a, y)$ and h is a homogeneous degree one function of variables y^i

$$h = \lambda_i y^i + t, \tag{6.5}$$

λ_i, t being some constants.

For a complete determination of the symplectic potential (6.2) it is necessary to find the Reeb vector $\tilde{K}$ (6.1) and the function h (6.5). There are known two different algebraic procedures to extract the components of the Reeb vector $\tilde{K}$ from the toric data. According to the AdS/CFT correspondence the volume of the Sasaki-Einstein space corresponds to the central charge of the dual conformal field theory. The first procedure is based on the maximization of the central charge (a-maximization) [8] used in connection with the computation of the Weyl anomaly in 4-dimensional field theory. The second one is known as volume minimization (or Z-minimization) [12].

Ricci form corresponding to $F(x)$ is given by

$$\rho = -i\partial\bar{\partial}\log\det(F_{ij}),$$

and the Ricci-flatness condition ($\rho = 0$) implies

$$\log\det(F_{ij}) = -2\gamma_i x_i + C,$$

where γ_i and C are constants.

Legendre transformation of this equation is the Monge-Ampère equation

$$\det(G_{ij}) = \exp\left(2\gamma_i \frac{\partial G}{\partial y^i} - C\right). \tag{6.6}$$

Using (6.3) and (6.4) we evaluate

$$\frac{\partial G^{can}}{\partial y^i} = \frac{1}{2}\sum_a (1 + \log l_a(y))\, v_i^a,$$

$$\frac{\partial G_b}{\partial y^i} = \frac{1}{2}(1 + \log l_b(y))\, b_i - \frac{1}{2}(1 + \log l_\infty(y))\sum_c v_i^c.$$

For a Calabi-Yau manifold X, by an appropriate $SL(n, \mathbb{Z})$ transformation, it is possible to bring the normal vectors of the polyhedral cone in the form

$$v_a = (1, w_a). \tag{6.7}$$

With this choice, from (6.6) we infer

$$-n = (b, \gamma), \tag{6.8}$$

and

$$\det(G_{ij}) = f(y)\prod_a [l_a(y)]^{-1},$$

where $f(y)$ is a smooth function on the polyhedral cone minus its apex [12]. Finally we get $(v_a, \gamma) = -1$ which can be solved using form (6.7)

$$\gamma = (-1, 0, 0, \ldots, 0). \tag{6.9}$$

Also, from (6.8) we obtain for the first component of the Reeb vector $b_1 = n$, a result which is valid for any toric data.

To completely determine the Sasaki-Einstein metric one should solve the non-linear partial differential equation (6.6) for h (6.5). Some explicit solutions to the Monge-Ampère equation are presented in [13].

The $(n,0)$ holomorphic form of the Ricci-flat metric on the Calabi-Yau cone is

$$\Omega = e^{i\alpha}(\det F_{ij})^{1/2} dz_1 \wedge \cdots \wedge dz_n ,$$

with α a phase space which is fixed by requiring that Ω is a closed form. From (6.6) and (6.9) we have finally

$$\Omega = e^{x_1 + i\phi_1} dz_1 \wedge \cdots \wedge dz_n = dw_1 \wedge \cdots \wedge dw_n / (w_2 \dots w_n). \tag{6.10}$$

The Kähler potential F (3.3) is obtain by the Legendre transform (5.3). Using the evaluation of x_i

$$x_i = \frac{\partial G}{\partial y^i} = \frac{1}{2}\sum_a v_i^a \log l_a(y) + \frac{1}{2} b_i \left(1 + \log l_b(y)\right) - \frac{1}{2}\sum_c v_i^c \log l_\infty(y) + \lambda_i ,$$

we obtain finally

$$F(x) = \frac{r^2}{4} = \frac{1}{2}\sum_i b_i y^i - t ,$$

Detailed analysis shows that the constant t must be set to zero [4].

¿From (3.2) we have

$$\eta = \frac{2}{r^2}\frac{\partial F}{\partial x_j} d\phi_j = \frac{1}{r}\frac{\partial r}{\partial x_j} d\phi_j = \frac{2}{r^2} y^j d\phi_j .$$

Note that

$$\eta(\tilde{K}) = \frac{2}{r^2} y^j b_j = 1 ,$$

and

$$d\eta = \frac{2}{r^2}\left[G^{jk} - \frac{4}{r^2} y^j y^k\right] dx_k \wedge d\phi_j = \frac{2}{r^2}\left[G^{jk} - \frac{1}{r^2} G^{jm} G^{kn} b_m b_n\right] dx_k \wedge d\phi_j .$$

7. Hidden symmetries of the Sasaki-Einstein spaces

The Killing forms of the toric Sasaki-Einstein manifold Y are described by the special Killing forms (2.2) [14]

$$\Theta_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots , n-1 .$$

Besides these Killing forms, there are $n-1$ closed conformal Killing forms (also called $*$-Killing forms)

$$\Phi_k = (d\eta)^k \quad , \quad k = 1, \cdots, n-1.$$

Moreover in the case of the Calabi-Yau cone, the holonomy is $SU(n)$ and there are *two additional* Killing forms of degree n. In order to write explicitly these additional Killing forms we shall express the volume form of the metric cone in terms of the Kähler form (5.2)

$$d\mathcal{V} = \frac{1}{n!}\omega^n.$$

Here ω^n is the wedge product of ω with itself n times. The volume of a Kähler manifold can be also written as [14, 18]

$$d\mathcal{V} = \frac{i^n}{2^n}(-1)^{n(n-1)/2} dV \wedge \overline{dV},$$

where dV is the complex volume holomorphic $(n, 0)$ form (6.10) of $C(Y)$. The additional (real) Killing forms are given by the real respectively the imaginary part of the complex volume form.

Finally to extract the corresponding additional Killing forms of the Einstein-Sasaki spaces we make use of the fact that for any p-form ψ on the space Y we can define an associated $p+1$-form ψ^C on the cone $C(Y)$:

$$\psi^C := r^p dr \wedge \psi + \frac{r^{p+1}}{p+1} d\psi.$$

ψ^C is parallel if and only if ψ is a special Killing form (2.2) with constant $c = -(p+1)$ [14].

Explicit examples of the additional Killing forms are given in [15, 17] for $Y^{p,q}$ spaces [5] and in [16] for $L^{a,b,c}$ spaces [6, 10, 11].

8. Conclusions

In general it is a hard task to find the complete set of Killing forms trying to solve equation (2.1). In some cases it is possible to produce the complete set of Killing forms taking into account the geometrical features of the spaces. That is the case of Sasaki-Einstein spaces for which the explicit construction of the Killing forms is permitted.

Killing tensors play a fundamental role in the separability of field equations, pseudoclassical spinning models, the existence of quantum symmetry operators, supersymmetries, etc. The remarkable properties of Killing tensors offer new perspective in investigation of hidden symmetries of various spacetime structures.

REFERENCES

1. M. Abreu: *Kähler geometry of toric manifolds in symplectic coordinates*. Fields Institute Commun. **35** (2003), 1–24, AMS, Providence, RI, 2003.
2. W. Chen, H. Lü and C. Pope: *General Kerr-NUT-AdS metrics in all dimensions*. Class. Quantum Grav. **23** (2006), 5323–5340.
3. T. Delzant: *Hamiltoniens périodiques et images convexes de l'application moment*. Bull. Soc. Math. France **116** (1988), 315–339.
4. A. Futaki, H. Ono and G. Wang: *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*. J. Diff. Geom. **83** (2009), 585–636.
5. J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram: *Sasaki-Einstein metrics on* $S^2 \times S^3$. Adv. Theor. Math. Phys. **8** (2004), 711–734.
6. J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram: *A new infinite class of Sasaki-Einstein manifolds*. Adv. Theor. Math. Phys. **8** (2004), 987–1000.
7. V. Guillemin: *Kähler structures on toric varieties*. J. Diff. Geom. **40** (1994), 285–309.
8. K. Intriligator and B. Wecht: *The exact superconformal R-symmetry maximizes a*. Nucl. Phys. B **667** (2003), 183–200.
9. J. Maldacena: *The large N limit of superconformal field theory and supergravity*. Adv. Theor. Math. Phys. **2** (1998), 231–252.
10. D. Martelli and J. Sparks: *Toric Sasaki-Einstein metrics on* $S^2 \times S^3$. Phys. Lett. B **621** (2005), 208–212.
11. D. Martelli and J. Sparks: *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*. Commun. Math. Phys. **262** (2006), 51–89.
12. D. Martelli, J. Sparks and S.-T. Yau: *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*. Commun. Math. Phys. **268** (2006), 39–65.
13. K. Oota and Y. Yasui: *Toric Sasaki-Einstein manifolds and Heun equations*. Nucl. Phys. B **742** (2006), 275–294.
14. U. Semmelmann: *Conformal Killing forms on Riemannian manifolds*. Math. Z. **245** (2003), 503–527.
15. M. Visinescu: *Killing forms on the five-dimensional Einstein-Sasaki Y(p,q) spaces*. Mod. Phys. Lett. A **27** (2012), 1250217.
16. M. Visinescu: *Killing forms on 5D Sasaki-Einstein spaces*. Rom. J. Phys. **58** (2013), 675–682.
17. M. Visinescu: *Killing forms on Kerr-NUT-(A)dS spaces of Sasaki-Einstein type*. J. Geom. Symm. Phys. **30** (2013), 93–104.
18. M. Visinescu and G. Vîlcu: *Hidden symmetries of Euclideanised Kerr-NUT-(A)dS metrics in certain scaling limit*. SIGMA **8** (2012), 058.
19. K. Yano: *Some remarks on tensor fields and curvature*. Ann. of Math. **55** (1952), 328–347.

Mihai Visinescu
"Horia Hulubei" National Institute

for Physics and Nuclear Engineering
Department of Theoretical Physics
Reactorului 30, RO-077125, POB-MG6
Măgurele-Bucharest, Romania, EU
mvisin@theory.nipne.ro