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NONLOCAL MODIFIED GRAVITY AND ITS COSMOLOGICAL SOLUTIONS *

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Abstract. Besides great achievements and many nice properties, general relativity as theory of gravity is not a complete theory. There are many attempts to its modification. One of promising modern approaches towards more complete theory of gravity is its nonlocal modification. We present here a brief review of nonlocal gravity with some its cosmological solutions. In particular, we pay special attention to two attractive nonlocal models, in which nonlocality is expressed by an analytic function of the d'Alembert operator $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$. In these models, we are mainly interested in nonsingular bounce solutions for the cosmic scale factor.

1. Introduction

It is well known that General Relativity is the Einstein theory of gravity, which is usually presented as tensorial equation of motion for gravitational (metric) field $g_{\mu\nu}$: $\quad R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor, and speed of light is taken $c = 1$. This Einstein equation can be derived from the Einstein-Hilbert action $S = \frac{1}{16\pi G}\int\sqrt{-g}\,R\,d^4x + \int\sqrt{-g}\mathcal{L}_m\,d^4x$, where $g = \det(g_{\mu\nu})$ and $\mathcal{L}_m$ is Lagrangian of matter.

Motivations for modification of general relativity usually come from some problems of quantum gravity, string theory, astrophysics and cosmology (for a review, see [1, 2, 3]). Modifications with higher order derivatives can improve problem with UV divergences. Here, we are mainly interested in cosmological motives to modify the

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Einstein theory of gravity. If general relativity is gravity theory for the universe as a whole and the universe has Friedmann-Lemaître-Robertson-Walker (FLRW) metric, then there is in the universe about 68% of *dark energy*, 27% of *dark matter*, and only about 5% of *visible matter* [4]. The visible matter is described by the Standard model of particle physics. However, existence of this 95% of dark energy-matter content of the universe is rather hypothetical, because it has been not verified in the laboratory conditions and there is no non-gravitational indications of its presence at the cosmic scale. There is also problem of the Big Bang singularity. Namely, under rather general conditions, general relativity contains cosmological solutions for scale factor $a(t)$ with $a(0) = 0$, what means an infinite matter density at the beginning. When physical theory contains singularity, it is evident indication that in the vicinity of singularity such theory has to be appropriately modified.

In this paper, we briefly review nonlocal modification of general relativity in a way to point out cosmological solutions without Big Bang singularity. We consider two nonlocal models and present their nonsingular bounce cosmological solutions. To have more complete presentation of these models we also give some power-law singular solutions of the form $a(t) = a_0\, |t|^{\alpha}$.

In Section 2 we describe some general characteristics of nonlocal modified gravity, which are useful for understanding what follows in the sequel. Section 3 contains a review of both nonsingular bounce and singular cosmological solutions for two nonlocal gravity models without matter. Last section contains a discussion with some concluding remarks.

2. Nonlocal Modified Gravity

Any well founded modification of the Einstein theory of gravity has to contain general relativity and to be verified on the dynamics of the Solar system. In other words, it has to be a generalization of the general theory of relativity. Mathematically, it should be formulated within the pseudo-Riemannian geometry in terms of covariant quantities and take into account equivalence of the inertial and gravitational mass. Consequently, the Ricci scalar R in gravity Lagrangian $\mathcal{L}_g$ of the Einstein-Hilbert action has to be replaced by a function which, in general, may contain not only R but also any covariant construction which is possible in the Riemannian geometry. However there are infinitely many such possibilities. Unfortunately, so far there is no acceptable theoretical principle which could make definite choice. The Einstein-Hilbert action can be viewed as realization of the principle of simplicity in construction of $\mathcal{L}_g$.

We consider here nonlocal modified gravity. Usually a modified gravity model contains an infinite number of spacetime derivatives in the form of some power expansions of the d'Alembert operator $\Box = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$ or its inverse $\Box^{-1}$, or some combination of both. In this article, we are mainly interested in nonlocality expressed in the form of an analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, where coefficients f_n should be determined from various conditions. However, some models with

$\Box^{-1}R$, have been also considered (see, e.g. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein). For nonlocal gravity with $\Box^{-1}$ see also [15, 16]. Many aspects of nonlocal gravity models have been considered, see e.g. [17, 18, 19, 20, 21] and references therein.

Motivation to modify gravity in a nonlocal way comes mainly from string theory, in particular from string field theory and p-adic string theory. Namely, strings are one-dimensional extended objects and, consequently, their field theory description contains spacetime nonlocality. We will discuss some features in the framework of p-adic string theory in Section 4.

In order to better understand nonlocal modified gravity itself, we investigate it without presence of matter. Models of nonlocal gravity which we mainly investigate are given by the action

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + R^q\mathcal{F}(\Box)R\Big), \quad q = +1, -1, \tag{2.1}$$

where Λ is cosmological constant, which is for the first time introduced by Einstein in 1917. Thus this nonlocality is given by the term $R^q\mathcal{F}(\Box)R$, where $q = \pm 1$ and $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$, i.e. we investigate two nonlocal gravity models: the first one with $q = +1$ and the second one with $q = -1$.

Before to proceed, it is worth mentioning that analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$, has to satisfy some conditions, in order to escape unphysical degrees of freedom like ghosts and tachyons, and to have good behavior in quantum sector (see discussion in [22, 23]).

3. Models, Their Equations of Motion and Cosmological Solutions

In the sequel we shall consider the above mentioned two nonlocal models (2.1) separately for $q = +1$ and $q = -1$.

We use the FLRW metric $ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\big)$ and investigate all three possibilities for curvature parameter $k = 0, \pm 1$. In the FLRW metric the Ricci scalar curvature is $R = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)$ and $\Box = -\partial_t^2 - 3H\partial_t$, where $H = \frac{\dot a}{a}$ is the Hubble parameter. Note that we use natural system of units in which speed of light $c = 1$.

3.1. Nonlocal Gravity Model Quadratic in R

Nonlocal gravity model which is quadratic in R is given by the action [24, 25]

$$S = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + R\mathcal{F}(\Box)R\Big). \tag{3.1}$$

This model is important because it is ghost free and has some nonsingular bounce solutions, which can be regarded as solution of the Big Bang cosmological singularity problem.

The corresponding equation of motion follows from the variation of the action (3.1) with respect to metric $g_{\mu\nu}$ and it is

$$\begin{aligned} &2R_{\mu\nu}\mathcal{F}(\Box)R - 2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)(\mathcal{F}(\Box)R) - \tfrac{1}{2}g_{\mu\nu}R\mathcal{F}(\Box)R \\ &+ \textstyle\sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\big(g_{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\Box^l R\partial_\beta\Box^{n-1-l}R + \Box^l R\Box^{n-l}R\right) \\ &-2\partial_\mu\Box^l R\partial_\nu\Box^{n-1-l}R\big) = \tfrac{-1}{8\pi G}(G_{\mu\nu} + \Lambda g_{\mu\nu}). \end{aligned} \tag{3.2}$$

When metric is of the FLRW form in (3.2) then there are only two independent equations. It is useful to use the trace and 00-component of (3.2), and respectively they are:

$$\begin{aligned} &6\Box(\mathcal{F}(\Box)R) + \textstyle\sum_{n=1}^{\infty} f_n\sum_{l=0}^{n-1}\left(\partial_\mu\Box^l R\partial^\mu\Box^{n-1-l}R + 2\Box^l R\Box^{n-l}R\right) \\ &= \tfrac{1}{8\pi G}R - \tfrac{\Lambda}{2\pi G}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &2R_{00}\mathcal{F}(\Box)R - 2(\nabla_0\nabla_0 - g_{00}\Box)(\mathcal{F}(\Box)R) - \tfrac{1}{2}g_{00}R\mathcal{F}(\Box)R \\ &+ \textstyle\sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\big(g_{00}\left(g^{\alpha\beta}\partial_\alpha\Box^l R\partial_\beta\Box^{n-1-l}R + \Box^l R\Box^{n-l}R\right) \\ &-2\partial_0\Box^l R\partial_0\Box^{n-1-l}R\big) = \tfrac{-1}{8\pi G}(G_{00} + \Lambda g_{00}). \end{aligned} \tag{3.4}$$

We are interested in cosmological solutions for the universe with FLRW metric and even in such simplified case it is rather difficult to find solutions of the above equations. To evaluate the above equations, the following Ansätze were used:

- Linear Ansatz: $\Box R = rR + s$, where r and s are constants.
- Quadratic Ansatz: $\Box R = qR^2$, where q is a constant.
- Qubic Ansatz: $\Box R = qR^3$, where q is a constant.
- Ansatz $\Box^n R = c_n R^{n+1}$, $n \geq 1$, where c_n are constants.

In fact these Ansätze make some constraints on possible solutions, but on the other hand they simplify formalism to find a particular solution.

3.1.1. Linear Ansatz and Nonsingular Bounce Cosmological Solutions

Using Ansatz $\Box R = rR + s$ a few nonsingular bounce solutions for the scale factor are found: $a(t) = a_0\cosh\left(\sqrt{\frac{\Lambda}{3}}t\right)$ (see [24, 25]), $a(t) = a_0 e^{\frac{1}{2}\sqrt{\frac{\Lambda}{3}}t^2}$ (see [27]) and $a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t})$ [29]. The first two consequences of this Ansatz are

$$\Box^n R = r^n(R + \frac{s}{r}),\ n \geq 1, \qquad \mathcal{F}(\Box)R = \mathcal{F}(r)R + \frac{s}{r}(\mathcal{F}(r) - f_0), \tag{3.5}$$

which considerably simplify nonlocal term.

Now we can search for a solution of the scale factor $a(t)$ in the form of a linear combination of $e^{\lambda t}$ and $e^{-\lambda t}$, i.e.

$$a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t}), \quad 0 < a_0, \lambda, \sigma, \tau \in \mathbb{R}. \tag{3.6}$$

Then the corresponding expressions for the Hubble parameter $H(t) = \frac{\dot{a}}{a}$, scalar curvature $R(t) = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k)$ and $\Box R$ are:

$$\begin{aligned} H(t) &= \frac{\lambda(\sigma e^{\lambda t} - \tau e^{-\lambda t})}{\sigma e^{\lambda t} + \tau e^{-\lambda t}}, \\ R(t) &= \frac{6\left(2a_0^2\lambda^2\left(\sigma^2 e^{4t\lambda} + \tau^2\right) + ke^{2t\lambda}\right)}{a_0^2\left(\sigma e^{2t\lambda} + \tau\right)^2}, \\ \Box R &= -\frac{12\lambda^2 e^{2t\lambda}\left(4a_0^2\lambda^2\sigma\tau - k\right)}{a_0^2\left(\sigma e^{2t\lambda} + \tau\right)^2}. \end{aligned} \tag{3.7}$$

We can rewrite $\Box R$ as

$$\Box R = 2\lambda^2 R - 24\lambda^4, \qquad r = 2\lambda^2,\ s = -24\lambda^4. \tag{3.8}$$

Substituting parameters r and s from (3.8) into (3.5) one obtains

$$\begin{aligned} \Box^n R &= (2\lambda^2)^n(R - 12\lambda^2),\ n \geq 1, \\ \mathcal{F}(\Box)R &= \mathcal{F}(2\lambda^2)R - 12\lambda^2(\mathcal{F}(2\lambda^2) - f_0). \end{aligned} \tag{3.9}$$

Using this in (3.3) and (3.4) we obtain

$$\begin{aligned} &36\lambda^2\mathcal{F}(2\lambda^2)(R - 12\lambda^2) + \mathcal{F}'(2\lambda^2)\left(4\lambda^2(R - 12\lambda^2)^2 - \dot{R}^2\right) \\ &-24\lambda^2 f_0(R - 12\lambda^2) = \tfrac{R - 4\Lambda}{8\pi G}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &(2R_{00} + \tfrac{1}{2}R)\left(\mathcal{F}(2\lambda^2)R - 12\lambda^2(\mathcal{F}(2\lambda^2) - f_0)\right) \\ &-\tfrac{1}{2}\mathcal{F}'(2\lambda^2)\left(\dot{R}^2 + 2\lambda^2(R - 12\lambda^2)^2\right) - 6\lambda^2(\mathcal{F}(2\lambda^2) - f_0)(R - 12\lambda^2) \\ &+6H\mathcal{F}(2\lambda^2)\dot{R} = -\tfrac{1}{8\pi G}(G_{00} - \Lambda). \end{aligned} \tag{3.11}$$

Substituting $a(t)$ from (3.6) into equations (3.10) and (3.11) one obtains two equations as polynomials in $e^{2\lambda t}$. Taking coefficients of these polynomials to be zero one obtains a system of equations and their solution determines parameters $a_0, \lambda, \sigma, \tau$ and yields some conditions for function $\mathcal{F}(2\lambda^2)$. For details see [29].

3.1.2. Quadratic Ansatz and Power-Law Cosmological Solutions

New Ansätze $\Box R = rR$, $\Box R = qR^2$ and $\Box^n R = c_n R^{n+1}$, were introduced in [28] and they contain solution for $R = 0$ which satisfies also equations of motion. When $k = 0$ there is only static solution $a = constant$, and for $k = -1$ solution is $a(t) = |t|$.

In particular, Ansatz $\Box R = qR^2$ is very interesting. The corresponding differential equation for the Hubble parameter, if $k = 0$, is

$$\dddot{H} + 4\dot{H}^2 + 7H\ddot{H} + 12H^2\dot{H} + 6q(\dot{H}^2 + 4H^2\dot{H} + 4H^4) = 0 \tag{3.12}$$

with solutions

$$H_\eta(t) = \frac{2\eta + 1}{3}\frac{1}{t + C_1}, \quad q_\eta = \frac{6(\eta - 1)}{(2\eta + 1)(4\eta - 1)}, \ \eta \in \mathbb{R} \tag{3.13}$$

and $H = \frac{1}{2}\frac{1}{t+C_1}$, what is equivalent to the Ansatz $\Box R = rR$ with $R = 0$.

The corresponding scalar curvature is given by

$$R_\eta = \frac{2}{3}\frac{(2\eta + 1)(4\eta - 1)}{(t + C_1)^2}, \ \eta \in \mathbb{R}. \tag{3.14}$$

By straightforward calculation one can show that $\Box^n R_n = 0$ when $n \in \mathbb{N}$. This simplifies the equations considerably. For this particular case of solutions operator $\mathcal{F}$ and trace equation (3.3) effectively become

$$\mathcal{F}(\Box) = \textstyle\sum_{k=0}^{n-1} f_k \Box^k, \tag{3.15}$$

$$\textstyle\sum_{k=1}^{n+1} f_k \sum_{l=0}^{k-1} (\partial_\mu \Box^l R \partial^\mu \Box^{k-1-l} R + 2\Box^l R \Box^{k-l} R) + 6\Box\mathcal{F}(\Box)R = \frac{R}{8\pi G}. \tag{3.16}$$

In particular case $n = 2$ the trace formula becomes

$$\begin{aligned} \tfrac{36}{35} f_0 R^2 + f_1(-\dot{R}^2 + \tfrac{12}{35}R^3) + f_2(-\tfrac{24}{35}R\dot{R}^2 + \tfrac{72}{1225}R^4) + f_3(-\tfrac{144}{1225}R^2\dot{R}^2) \\ = \tfrac{R}{8\pi G}. \end{aligned} \tag{3.17}$$

Some details on all the above three Ansätze can be found in [28].

3.2. Nonlocal Gravity Model with Term $R^{-1}\mathcal{F}(\Box)R$

This model was introduced recently [30] and its action may be written in the form

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R\Big), \tag{3.18}$$

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ and $f_0 = -\frac{\Lambda}{8\pi G}$ plays role of the cosmological constant. For example, $\mathcal{F}(\Box)$ can be of the form $\mathcal{F}(\Box) = -\frac{\Lambda}{8\pi G}e^{-\beta\Box}$.

The nonlocal term $R^{-1}\mathcal{F}(\Box)R$ in (3.18) is invariant under transformation $R \to CR$. It means that effect of nonlocality does not depend on the magnitude of scalar curvature R, but on its spacetime dependence, and in the FLRW case is sensitive only to dependence of R on time t. When $R = \textit{constant}$ there is no effect of nonlocality, but only of f_0 term, what corresponds to cosmological constant.

By variation of action (3.18) with respect to metric $g^{\mu\nu}$ one obtains the equations of motion for $g_{\mu\nu}$

$$\begin{aligned} &R_{\mu\nu}V - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)V - \tfrac{1}{2}g_{\mu\nu}R^{-1}\mathcal{F}(\Box)R \\ &+ \textstyle\sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\big(g_{\mu\nu}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + \Box^l(R^{-1})\Box^{n-l}R\right) \\ &-2\partial_\mu\Box^l(R^{-1})\partial_\nu\Box^{n-1-l}R\big) = -\tfrac{G_{\mu\nu}}{16\pi G}, \\ &V = \mathcal{F}(\Box)R^{-1} - R^{-2}\mathcal{F}(\Box)R, \end{aligned} \tag{3.19}$$

which derivation is rather complicate, see [31]. Note that operator $\Box$ acts not only on R but also on R^{-1}. There are only two independent equations when metric is of the FLRW type.

The trace of the equation (3.19) is

$$\begin{aligned} &RV + 3\Box V + \textstyle\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + 2\Box^l(R^{-1})\Box^{n-l}R\right) \\ &-2R^{-1}\mathcal{F}(\Box)R = \tfrac{R}{16\pi G}. \end{aligned} \tag{3.20}$$

The 00-component of (3.19) is

$$\begin{aligned} &R_{00}V - (\nabla_0\nabla_0 - g_{00}\Box)V - \tfrac{1}{2}g_{00}R^{-1}\mathcal{F}(\Box)R \\ &+ \textstyle\sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\big(g_{00}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + \Box^l(R^{-1})\Box^{n-l}R\right) \\ &-2\partial_0\Box^l(R^{-1})\partial_0\Box^{n-1-l}R\big) = -\tfrac{G_{00}}{16\pi G}. \end{aligned} \tag{3.21}$$

These trace and 00-component equations are equivalent for the FLRW universe in the equation of motion (20), but they are more suitable for usage.

3.2.1. Some Cosmological Solutions for Constant R

We are interested in some exact nonsingular cosmological solutions for the scale factor $a(t)$ in (3.19). The Ricci curvature R in the above equations of motion can be calculated by expression

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).$$

Case $k = 0$, $a(t) = a_0 e^{\lambda t}$.

We have $a(t) = a_0 e^{\lambda t}$, $\dot{a} = \lambda a$, $\ddot{a} = \lambda^2 a$, $H = \frac{\dot{a}}{a} = \lambda$ and $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 12\lambda^2$. Putting $a(t) = a_0 e^{\lambda t}$ in the above equations (3.20) and (3.21), they are satisfied with $\lambda = \pm\sqrt{\frac{\Lambda}{3}}$, where $\Lambda = -8\pi G f_0$ with $f_0 < 0$.

Case $k = +1$**,** $a(t) = \frac{1}{\lambda}\cosh\lambda t$**.**

Starting with $a(t) = a_0\cosh\lambda t$, we have $\dot a = \lambda a_0 \sinh\lambda t$, $\quad H = \frac{\dot a}{a} = \lambda\tanh\lambda t$ and $R = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{1}{a^2}\right) = 12\lambda^2$ if $a_0 = \frac{1}{\lambda}$. Hence equation (3.20) and (3.21) are satisfied for cosmic scale factor $a(t) = \frac{1}{\lambda}\cosh\lambda t$.

In a similar way, one can obtain another solution:

Case $k = -1$**,** $a(t) = \frac{1}{\lambda}|\sinh\lambda t|$**.**

Thus we have the following three cosmological solutions for $R = 12\lambda^2$:

1. $k = 0, \quad a(t) = a_0\, e^{\lambda t}$, nonsingular bounce solution.
2. $k = +1,\ a(t) = \frac{1}{\lambda}\cosh\lambda t$, nonsingular bounce solution.
3. $k = -1, \quad a(t) = \frac{1}{\lambda}\,|\sinh\lambda t|$, singular cosmic solution.

All of this solution have exponential behavior for large value of time t.

Note that in all the above three cases the following two tensors have also the same expressions:

$$R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}, \qquad G_{\mu\nu} = -\frac{1}{4}Rg_{\mu\nu}. \tag{3.22}$$

Minkowski background space follows from the de Sitter solution $k = 0, \quad a(t) = a_0e^{\lambda t}$. Namely, when $\lambda \to 0$ then $a(t) \to a_0$ and $H = R = 0$.

In all the above cases $\Box R = 0$ and thus coefficients $f_n,\ n \geq 1$ may be arbitrary. As a consequence, in these cases nonlocality does not play a role.

3.2.2. Some Power-Law Cosmological Solutions

Power-law solutions in the form $a(t) = a_0|t - t_0|^\alpha$, have been investigated by some Ansätze in [30] and without Ansätze [32]. The corresponding Ricci scalar and the Hubble parameter are:

$$R(t) = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{1}{a^2}\right) = 6\big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\big)$$

$$H(t) = \frac{\dot a}{a} = \frac{\alpha}{|t - t_0|}.$$

Now $\Box = -\partial_t^2 - \frac{3\alpha}{|t-t_0|}\partial_t$. An analysis has been performed for $\alpha \neq 0,\ \frac{1}{2}$, and also $\alpha \to 0, \quad \alpha \to \frac{1}{2}$ for $k = +1, -1, 0$. For details, the reader refers to [30, 32].

4. Discussion and Concluding Remarks

To illustrate the form of the abave nonlocality (3.1) it is worth to start from exact effective Lagrangian at the tree level for p-adic closed and open scalar strings. This Lagrangian is as follows (see, e.g. [33]):

$$\begin{aligned} L_p = \quad & -\frac{m^D}{2g^2}\frac{p^2}{p-1}\varphi p^{-\frac{\Box}{2m^2}}\varphi - \frac{m^D}{2h^2}\frac{p^4}{p^2-1}\phi p^{-\frac{\Box}{4m^2}}\phi + \frac{m^D}{h^2}\frac{p^4}{p^4-1}\phi^{p^2+1} \\ & -\frac{m^D}{g^2}\frac{p^2}{p^2-1}\phi^{\frac{p(p-1)}{2}} + \frac{m^D}{g^2}\frac{p^2}{p^2-1}\varphi^{p+1}\phi^{\frac{p(p-1)}{2}}, \end{aligned} \tag{4.1}$$

where φ denotes open strings, D is spacetime dimensionality (in the sequel we shall take $D = 4$), and g and h are coupling constants for open and closed strings, respectively. Scalar field $\phi(x)$ corresponds to closed p-adic strings and could be related to gravity scalar curvature as $\phi = f(R)$, where f is an appropriate function. The corresponding equations of motion are:

$$p^{-\frac{\Box}{2m^2}}\varphi = \varphi^p\phi^{\frac{p(p-1)}{2}}, \quad p^{-\frac{\Box}{4m^2}}\phi = \phi^{p^2} + \frac{h^2}{2g^2}\frac{p-1}{p}\phi^{\frac{p(p-1)}{2}-1}\left(\varphi^{p+1}-1\right). \tag{4.2}$$

There are the following constant vacuum solutions: $(i)\,\varphi = \phi = 0$, $(ii)\,\varphi = \phi = 1$ and $(iii)\,\varphi = \phi^{-\frac{p}{2}} = constant$.

In the case that the open string field $\varphi = 0$, one obtains equation of motion only for closed string ϕ. One can now construct a toy nonlocal gravity model supposing that closed scalar string is related to the Ricci scalar curvature as $\phi = -\frac{1}{m^2}R = -\frac{4}{3g^2}(16\pi G)R$. Taking $p = 2$, we obtain the following Lagrangian for gravity sector:

$$\mathcal{L}_g = \frac{1}{16\pi G}R - \frac{8}{3}\frac{C^2}{h^2}R\,e^{-\frac{\ln 2\,\Box}{4m^2}}R - \frac{1024}{405g^6h^2}(16\pi G)^3R^5. \tag{4.3}$$

To compare third term to the first one in (4.3), let us note that $(16\pi G)^3R^5 = (16\pi GR)^4\frac{R}{16\pi G}$. It follows that $(GR)^4$ has to be dimensionless after rewriting it using constants c and $\hbar$. As Ricci scalar R has dimension $Time^{-2}$ it means that G has to be replaced by the Planck time as $t_P^2 = \frac{\hbar G}{c^5} \sim 10^{-88}s^2$. Hence $(GR)^4 \to (\frac{\hbar G}{c^5}R)^4 \sim 10^{-352}R^4$ and third term in (4.3) can be neglected with respect to the first one, except when $R \sim t_P^{-2}$. The nonlocal model with only first two terms corresponds to case considered above in this article. We shall consider this model including R^5 term elsewhere.

It is worth noting that the above two models with nonlocal terms $R\mathcal{F}(\Box)R$ and $R^{-1}\mathcal{F}(\Box)R$ are equivalent in the case when $R = constant$, because their equations of motion have the same solutions. These solutions do not depend on $\mathcal{F}(\Box) - f_0$. It would be useful to find cosmological solutions which have definite connection with the explicit form of nonlocal operator $\mathcal{F}(\Box)$.

Let us mention that many properties of (3.1) and its extended quadratic versions have been considered, see [22, 23, 26, 35, 36].

Nonlocal model (3.18) is a new one and was not considered before [30], it seems to be important and deserves further investigation. There are some gravity models modified by term R^{-1}, but they are neither nonlocal nor pass Solar system tests, see e.g. [34].

Note that nonlocal cosmology is related also to cosmological models in which matter sector contains nonlocality (see, e.g. [37, 38, 39, 40, 41, 42]). String field theory and p-adic string theory models have played significant role in motivation and construction of such models.

Nonsingular bounce cosmological solutions are very important (as reviews on bouncing cosmology, see e.g. [43, 44]) and their progress in nonlocal gravity may be a further step towards cosmology of the cyclic universe [45].

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