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CANONICAL APPROACH TO THE CLOSED STRING NON-COMMUTATIVITY *

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Abstract. We consider the propagation of the closed bosonic string in the weakly curved background. We show that the closed string non-commutativity is essentially connected to the T-duality and nontrivial background. From the T-duality transformation laws, connecting the canonical variables of the original and T-dual theory, we find the structure of the Poisson brackets in the T-dual space corresponding to the fundamental Poisson brackets in the original theory. We find that the commutative original theory is equivalent to the non-commutative T-dual theory, in which Poisson brackets close on winding and momenta numbers and the coefficients are proportional to the background fluxes.

1. Introduction

Recently, in Refs. [1, 2] the non-commutativity of the closed string coordinates was found to exist in the presence of the nontrivial background fields fluxes. In these papers the different T-dual backgrounds of the three dimensional torus were considered. Refs. [1, 2] motivated us to investigate the closed string non-commutativity. Here we present a different approach, based on the canonical method and an analogy with the open string non-commutativity investigated in [3].

The open string non-commutativity has a source in a Kalb-Ramond field [4, 5]. In the constant background only the open string endpoints attached to a Dp-brane are non-commutative, with the non-commutativity parameter proportional to the Kalb-Ramond field.

For the open string described by the action

$$S(x) = \kappa \int_{\Sigma} (\frac{\eta^{\alpha\beta}}{2} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu}) \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{1.1}$$

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the minimal action principle in addition to the equation of motion produces the boundary conditions as well. Solving the boundary conditions, one obtains the expression for the initial coordinate x^μ, in terms of the effective coordinate q^μ and the effective momenta p_μ (which are even parts of the original coordinates and momenta)

$$x^\mu = q^\mu - 2\Theta^{\mu\nu} \int^\sigma d\sigma_1 p_\nu(\sigma_1)\,. \tag{1.2}$$

Here, the non-commutativity parameter $\Theta^{\mu\nu}$ is defined in terms of the effective metric $G^E_{\mu\nu}$ by

$$\Theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1} B G^{-1})^{\mu\nu}\,, \quad G^E_{\mu\nu} = (G - 4B^2)_{\mu\nu}\,. \tag{1.3}$$

Because the original coordinates x^μ are the linear combination of both effective coordinates and momenta, using the relation $\{q^\mu(\sigma), p_\nu(\bar\sigma)\} = 2\delta^\mu{}_\nu \delta_s(\sigma, \bar\sigma)$ one obtains

$$\{x^\mu(0), x^\nu(0)\} = -2\Theta^{\mu\nu}\,, \quad \{x^\mu(\pi), x^\nu(\pi)\} = 2\Theta^{\mu\nu}\,. \tag{1.4}$$

So, the ends of the open string attached to Dp-brane become non-commutative in a presence of the constant Kalb-Ramond field $B_{\mu\nu}$. If the action does not contain the Kalb-Ramond field then there is no non-commutativity, because the solution depends only on the effective coordinate q^μ.

The open string non-commutativity follows from the fact that original coordinate can be expressed in terms of the effective coordinates and effective momenta. This relation is a consequence of the boundary conditions at the open string endpoints. The closed string does not have endpoints. So, to follow the analogy with open string we are going to express coordinates of the closed string in terms of coordinates and momenta of some other theory. We are going to show that for some background fields we can take T-dual theory as that other theory.

We consider the weakly curved background which depends on all the coordinates. In paper [6] we proposed the T-dualization procedure for such a background, which is the generalization of the well known Buscher procedure [7]. T-dualising all the coordinates we find the T-dual theory, and the transformation laws between the original and T-dual coordinates. We express these laws in the canonical form, and use them to find the relation between Poisson brackets in the original and T-dual spaces. From the transformation laws we obtain that the original coordinates depend on both T-dual coordinates and T-dual momenta. Herefrom one obtains the closed string non-commutativity, and T-duality is a way to observe it. T-duals of all the Poisson brackets between coordinates close on the winding numbers and momenta of the T-dual background. The coefficients are fluxes introduced in [1, 2].

The term of the action with constant part of the Kalb-Ramond field $b_{\mu\nu}$ is topological and consequently it does not contribute to the equations of motion. In the open string case it contributes to the boundary conditions and it is a source of the open string non-commutativity. In the closed string case it is absent from boundary conditions as well. Classically we can gauge it away and Kalb-Ramond field becomes infinitesimally small. But, if $b_{\mu\nu} = 0$ one loses topological contributions. In order to investigate the global structure of the theory with holonomies of the world sheet gauge fields in quantum theory we should preserve this term.

2. T-duality

T-duality is the symmetry which exist only in string theories [8]. It is a consequence of the fact that the fundamental objects in these theories are extended objects and not the point particles. T-duality connects physically equivalent theories giving effectively different prescriptions of a string, described while moving in the different background fields and therefore different space-times.

If one coordinate in original theory is compactified on the circle of radius R and one coordinate in the T-dual theory is compactified on the circle on radius $\tilde{R}$, then the mass squared of any state

$$M^2 = \frac{n^2}{R^2} + m^2 \frac{R^2}{\alpha'^2} + \text{oscillators}\,, \tag{2.1}$$

is invariant under transformation

$$n \leftrightarrow m, \quad R \leftrightarrow \tilde{R} \equiv \alpha'/R\,. \tag{2.2}$$

Here the numbers n and m are integers which denotes momentum and winding modes. The complete spectrums of the T-dual theories are the same.

It can be shown that the T-dual action ${}^\star S$ has the same form as initial one but with different background fields. The T-dual metric and T-dual Kalb-Ramond field

$${}^\star G^{\mu\nu} \sim (G_E^{-1})^{\mu\nu}\,, \quad {}^\star B^{\mu\nu} \sim \Theta^{\mu\nu}\,, \tag{2.3}$$

are proportional to the effective metric and non-commutativity parameter from the open string case. One can compare the original and T-dual Hamiltonian and obtain the canonical T-dual transformation laws. When string moves in the constant background then these laws are of the following form

$$\pi_\mu \cong \kappa y'_\mu\,, \quad {}^\star\pi^\mu \cong \kappa x'^\mu\,. \tag{2.4}$$

The momenta are T-dual to the sigma derivatives of the dual coordinates and vice versa. Note that coordinates do not depend on dual coordinates but only on dual momenta. So, because momenta commute for constant background there is no closed string non-commutativity

$$\{\pi_\mu, \pi_\nu\} = 0 \Longrightarrow \{y_\mu, y_\nu\} = 0\,. \tag{2.5}$$

3. Weakly curved background

We have learned that T-duality is not enough to establish closed string non-commutativity. We should generalize the background fields. The weakly curved background is the simplest coordinate dependent solution of the space time equations for the background fields [9]. It consists of the constant metric $G_{\mu\nu} = const$ and the coordinate dependent Kalb-Ramond field

$$B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3} H_{\mu\nu\rho} x^\rho \equiv b_{\mu\nu} + h_{\mu\nu}(x), \tag{3.1}$$

where $b_{\mu\nu}$ and $H_{\mu\nu\rho}$ are constant and $H_{\mu\nu\rho}$ is infinitesimal.

The standard Buscher's T-dualization procedure [7] is not applicable for the coordinate dependent backgrounds which depend on all the space-time coordinates. So, we generalized it.

The generalized Buscher's construction [6] has three steps. First is to gauge the global symmetry $\delta x^\mu = \lambda^\mu$, which is a symmetry even if $B_{\mu\nu}$ is linear in coordinate, and substitute the ordinary derivative with the covariant one

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu \,, \tag{3.2}$$

where v_α^μ are gauge fields. Second is to substitute the coordinate in the argument of the background fields with its invariant extension

$$x^\mu \to \Delta x^\mu_{inv} = \int_P d\xi^\alpha D_\alpha x^\mu,$$

which is the line integral of the covariant derivatives of the original coordinate. With this substitutions one finds the gauged action. In order to obtain equivalent theories we require that gauge fields should be nonphysical. So, third step is to add a new term to the lagrangian $y_\mu F^\mu$ where y_μ is Lagrange multiplier and F^μ is a field strength of the gauge fields v_α^μ. The T-dual action is obtained on the solution of the equations of motion for the gauge fields. It is defined on the doubled geometry with coordinates $(y_\mu, \tilde{y}_\mu)$, where $\dot{y}^\mu = \tilde{y}'_\mu$ and $\dot{\tilde{y}}_\mu = y'_\mu$. So, the coordinate from the original space is replaced by two coordinates $x^\mu \to (y_\mu, \tilde{y}_\mu)$. The T-dual background fields equal

$${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}(\Delta V) \,, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\Theta^{\mu\nu}(\Delta V), \tag{3.3}$$

and the T-dual action is of the form

$${}^\star S = \frac{\kappa^2}{2} \int d^2\xi \partial_+ y_\mu \Theta_-^{\mu\nu}(\Delta V) \partial_- y_\nu \,, \tag{3.4}$$

where

$$V^\mu = -\kappa \Theta_0^{\mu\nu} y_\nu + (g_E^{-1})^{\mu\nu} \tilde{y}_\nu \,, \tag{3.5}$$

and

$$\Theta^{\mu\nu}_{\pm}(x) = -\frac{2}{\kappa}\left(G_E^{-1}(x)\Pi_{\pm}(x)G^{-1}\right)^{\mu\nu}, \quad \Pi_{\pm\mu\nu} = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}\,. \tag{3.6}$$

T-dual transformation laws are obtained comparing the solutions of the equations of motions for the gauge fixed actions with respect to the Lagrange multipliers and gauge fields

$$\partial_{\pm}x^{\mu} = -\kappa\Theta^{\mu\nu}_{\pm}(\Delta V)\partial_{\pm}y_{\nu} \mp 2\kappa\Theta^{\mu\nu}_{0\pm}\beta^{\mp}_{\nu}(V)\,, \tag{3.7}$$

$$\partial_{\pm}y_{\mu} = -2\Pi_{\mp\mu\nu}(\Delta x)\partial_{\pm}x^{\nu} \mp \beta^{\mp}_{\mu}(x)\,, \tag{3.8}$$

where

$$\beta^{\pm}_{\mu}(x) = \mp\frac{1}{6}H_{\mu\rho\sigma}\partial_{\mp}x^{\rho}x^{\sigma}\,. \tag{3.9}$$

Expression for $\beta^{\pm}_{\mu}$ comes from the term

$$\int d^2\xi v^{\mu}_{+}B_{\mu\nu}(\delta V)v^{\nu}_{-} = \int d^2\xi\beta^{\alpha}_{\mu}(V)\delta v^{\mu}_{\alpha}\,. \tag{3.10}$$

The transformation laws can be presented in the canonical form as

$$x'^{\mu} = \frac{1}{\kappa}{}^{\star}\pi^{\mu} - \kappa\Theta^{\mu\nu}_{0}\beta^{0}_{\nu}(V) - (g_E^{-1})^{\mu\nu}\beta^{1}_{\nu}(V)\,, \tag{3.11}$$

$$y'_{\mu} = \frac{1}{\kappa}\pi_{\mu} - \beta^{0}_{\mu}(x)\,. \tag{3.12}$$

The infinitesimal quantities β^{α}_{μ} are an improvement in comparison to the flat space case. Also, as we will see, they are the source of a non-commutativity.

4. Non-commutativity of the closed string canonical variables

Finally, we are ready to consider closed string non-commutativity, as we introduced both necessary components, the T-duality and the weakly curved background. The canonical T-dual transformation law (3.12) express the dual coordinates y_{μ} in terms of the original coordinates x^{μ} and momenta π_{μ}. Let us take the following Poisson bracket in the original theory

$$\{x^{\mu}(\sigma),\pi_{\nu}(\bar\sigma)\} = \delta^{\mu}_{\nu}\delta(\sigma-\bar\sigma), \quad \{x^{\mu}(\sigma),x^{\nu}(\bar\sigma)\} = 0, \quad \{\pi_{\mu}(\sigma),\pi_{\nu}(\bar\sigma)\} = 0, \tag{4.1}$$

and find the corresponding Poisson brackets of the T-dual theory.

Using T-duality transformation law (3.12), we search for the corresponding Poisson structure in T-dual theory i.e. the expressions for the Poisson brackets between the T-dual string coordinates $y_{\mu}(\sigma)$, $\tilde y_{\mu}(\sigma)$ and momenta ${}^{\star}\pi^{\mu}(\sigma)$. This is done considering the brackets between

$$\Delta Y_{\mu}(\sigma,\sigma_0) = \int_{\sigma_0}^{\sigma} d\eta\, Y'_{\mu}(\eta) = Y_{\mu}(\sigma) - Y_{\mu}(\sigma_0), \tag{4.2}$$

$Y_\mu = (y_\mu, \tilde{y}_\mu)$ and calculating the equal time commutators. The fact that T-dual coordinates under T-duality transform to both coordinate and momenta dependent expressions, enables non-commutativity. The relation of the form

$$\{X'_\mu(\sigma), Y'_\nu(\bar{\sigma})\} \cong K'_{\mu\nu}(\sigma)\delta(\sigma-\bar{\sigma}) + L_{\mu\nu}(\sigma)\delta'(\sigma-\bar{\sigma}), \tag{4.3}$$

implies the following relation between coordinates

$$\{X_\mu(\tau,\sigma), Y_\nu(\tau,\bar{\sigma})\} \cong -\left[K_{\mu\nu}(\sigma) - K_{\mu\nu}(\bar{\sigma}) + L_{\mu\nu}(\bar{\sigma})\right]\theta(\sigma-\bar{\sigma}), \tag{4.4}$$

where $\theta(\sigma)$ is the periodic step function

$$\theta(\sigma) = \begin{cases} 0 & \text{if } \sigma = 0 \\ 1/2 & \text{if } 0 < \sigma < 2\pi, \\ 1 & \text{if } \sigma = 2\pi \end{cases} \quad \sigma \in [0, 2\pi]. \tag{4.5}$$

Using the transformation law (3.12), we can calculate Poisson brackets $\{y'_\mu, y'_\nu\}$, $\{y'_\mu(\sigma), \tilde{y}'_\nu(\bar{\sigma})\}$ and $\{\tilde{y}'_\mu(\sigma), \tilde{y}'_\nu(\bar{\sigma})\}$. We can re-express them in terms of fluxes: Christoffel connection corresponding to the effective metric $G^E_{\mu\nu}$

$$\Gamma^E_{\mu,\nu\rho} = \frac{1}{2}\Big(\partial_\nu G^E_{\mu\rho} + \partial_\rho G^E_{\mu\nu} - \partial_\mu G^E_{\nu\rho}\Big) = -\frac{4}{3}\Big(B_{\mu\sigma\nu}(G^{-1}b)^\sigma{}_\rho + B_{\mu\sigma\rho}(G^{-1}b)^\sigma{}_\nu\Big), \tag{4.6}$$

and the coefficient of the dual Kalb-Ramond field

$$Q^{\mu\nu}{}_\rho = -\frac{1}{3}\Big[(g^{-1})^{\mu\sigma}(g^{-1})^{\nu\tau} - \kappa^2\theta_0^{\mu\sigma}\theta_0^{\nu\tau}\Big]B_{\sigma\tau\rho}, \tag{4.7}$$

defined by the relation ${}^\star B^{\mu\nu}(\Delta V) = {}^\star b^{\mu\nu} + Q^{\mu\nu}{}_\rho \Delta V^\rho$. We have

1. $\{y'_\mu, y'_\nu\}$

$$K_{\mu\nu}[x] = \frac{3}{\kappa}h_{\mu\nu}[x] = \frac{1}{\kappa}B_{\mu\nu\rho}x^\rho, \quad L_{\mu\nu} = 0, \tag{4.8}$$

2. $\{y'_\mu, \tilde{y}'_\nu\}$

$$\begin{aligned} K_{\mu\nu}[x,\tilde{x}] &= \frac{3}{\kappa}h_{\mu\nu}[\tilde{x}] - \frac{6}{\kappa}\Big[h[x]G^{-1}b + bG^{-1}h[x]\Big]_{\mu\nu} \\ &= \frac{1}{\kappa}B_{\mu\nu\rho}\tilde{x}^\rho - \frac{3}{2\kappa}\Gamma^E_{\rho,\mu\nu}x^\rho, \\ L_{\mu\nu}[x] &= \frac{1}{\kappa}g_{\mu\nu} - \frac{6}{\kappa}\Big[h[x]G^{-1}b + bG^{-1}h[x]\Big]_{\mu\nu} \\ &= \frac{1}{\kappa}g_{\mu\nu} - \frac{3}{2\kappa}\Gamma^E_{\rho,\mu\nu}x^\rho, \end{aligned} \tag{4.9}$$

with

$$\tilde{x}'^\mu = \frac{1}{\kappa}(G^{-1})^{\mu\nu}\pi_\nu + 2(G^{-1}B)^\mu{}_\nu x'^\nu. \tag{4.10}$$

3. $\{\tilde{y}'_\mu, \tilde{y}'_\nu\}$

$$\begin{aligned} K_{\mu\nu}[x] &= \frac{3}{\kappa}h_{\mu\nu}[x] + \frac{24}{\kappa}\Big[bh[x]b\Big]_{\mu\nu} + \frac{6}{\kappa}\Big[h[\tilde{x}]b - bh[\tilde{x}]\Big]_{\mu\nu} \\ &= -\frac{1}{\kappa}\Big[B_{\mu\nu\rho} - 6g_{\mu\alpha}Q^{\alpha\beta}{}_{\rho}g_{\beta\nu}\Big]x^\rho \\ &+ \Big[-\frac{3}{2\kappa}\Big(\Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho}\Big) + \frac{4}{\kappa}B_{\mu\nu\sigma}(G^{-1}b)^\sigma{}_\rho\Big]\tilde{x}^\rho. \end{aligned}$$

$$L_{\mu\nu} = 0\,. \tag{4.11}$$

For the above values of K and L, the relation (4.4) gives

$$\{y_\mu(\sigma), y_\nu(\bar{\sigma})\} \cong -\frac{1}{\kappa}B_{\mu\nu\rho}\big[x^\rho(\sigma) - x^\rho(\bar{\sigma})\big]\theta(\sigma - \bar{\sigma}), \tag{4.12}$$

$$\begin{aligned} \{y_\mu(\sigma), \tilde{y}_\nu(\bar{\sigma})\} \cong & -\Big\{\frac{1}{\kappa}B_{\mu\nu\rho}\big[\tilde{x}^\rho(\sigma) - \tilde{x}^\rho(\bar{\sigma})\big] - \frac{3}{2\kappa}\Gamma^E_{\rho,\mu\nu}\big[x^\rho(\sigma) - x^\rho(\bar{\sigma})\big] \\ & +\frac{1}{\kappa}g_{\mu\nu} - \frac{3}{2\kappa}\Gamma^E_{\rho,\mu\nu}\,x^\rho(\bar{\sigma})\Big\}\theta(\sigma - \bar{\sigma}), \end{aligned} \tag{4.13}$$

$$\begin{aligned} \{\tilde{y}_\mu(\sigma), \tilde{y}_\nu(\bar{\sigma})\} \cong & -\Big\{-\frac{1}{\kappa}\Big[B_{\mu\nu\rho} - 6g_{\mu\alpha}Q^{\alpha\beta}{}_\rho g_{\beta\nu}\Big]\big[x^\rho(\sigma) - x^\rho(\bar{\sigma})\big] \\ & +\Big[-\frac{3}{2\kappa}\Big(\Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho}\Big) + \frac{4}{\kappa}B_{\mu\nu\sigma}(G^{-1}b)^\sigma{}_\rho\Big]\big[\tilde{x}^\rho(\sigma) - \tilde{x}^\rho(\bar{\sigma})\big]\Big\}\cdot \\ & \cdot\theta(\sigma - \bar{\sigma})\,. \end{aligned} \tag{4.14}$$

After two-dimensional reparametrizations, the σ dependent part takes the form

$$\big[X^\mu(f(\sigma)) - X^\mu(f(\bar{\sigma}))\big]\theta[f(\sigma) - f(\bar{\sigma})],$$

where $f(\sigma)$ is monotonically increasing function with properties $f(0) = 0$ and $f(2\pi) = 2\pi$. Therefore, Poisson bracket between different points is not reparametrization invariant. For fixed points, it can be fit to be arbitrary small, by the appropriate choice of the function $f(\sigma)$. So, only Poisson brackets at the same point are physically significant.

Taking $\sigma = \bar{\sigma}$ we obtain that all Poisson brackets vanish, and consequently, coordinates commute. But, taking $\sigma = \bar{\sigma} + 2\pi$, in the non-commutativity relation between the dual coordinates y's (4.12), we obtain the *closed string non-commutativity relation*

$$\{y_\mu(\sigma + 2\pi), y_\nu(\sigma)\} \cong -\frac{2\pi}{\kappa}B_{\mu\nu\rho}N^\rho, \tag{4.15}$$

where

$$N^\mu = \frac{1}{2\pi}\left[x^\mu(\sigma + 2\pi) - x^\mu(\sigma)\right], \tag{4.16}$$

is winding number for the initial coordinate x^μ. This result is in agreement with Ref.[2]. Similarly, from (4.13) and (4.14), we obtain

$$\begin{aligned} &\{y_\mu(\sigma+2\pi), \tilde{y}_\nu(\sigma)\} + \{y_\mu(\sigma), \tilde{y}_\nu(\sigma+2\pi)\} \cong \\ &-\frac{4\pi}{\kappa^2} B_{\mu\nu\rho} p^\rho + \frac{\pi}{\kappa}\left(3\Gamma^E_{\rho,\mu\nu} - 8B_{\mu\nu\lambda} b^\lambda{}_\rho\right) N^\rho, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \{\tilde{y}_\mu(\sigma+2\pi), \tilde{y}_\nu(\sigma)\} \cong\ & \frac{2\pi}{\kappa}\Big[-B_{\mu\nu\rho} - 6g_{\mu\alpha} Q^{\alpha\beta}{}_\rho g_{\beta\nu} + 2B_{\mu\nu}{}^\lambda g_{\lambda\rho} \\ +\ & 3\left(\Gamma^E_{\mu,\nu\lambda} - \Gamma^E_{\nu,\mu\lambda}\right) b^\lambda_\rho\Big] N^\rho \\ +\ & \frac{\pi}{\kappa^2}\left[3\left(\Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho}\right) p^\rho - 8B_{\mu\nu\lambda} b^\lambda{}_\rho\right] p^\rho\,. \end{aligned} \tag{4.18}$$

Using (4.10) and integrating from σ to $\sigma+2\pi$ we have

$$\frac{1}{2\pi}\left[\tilde{x}^\mu(\sigma+2\pi) - \tilde{x}^\mu(\sigma)\right] = \frac{1}{\kappa}(G^{-1})^{\mu\nu} p_\nu + 2(G^{-1})^{\mu\rho} b_{\rho\lambda} N^\lambda\,, \tag{4.19}$$

where

$$p_\mu = \frac{1}{2\pi}\int_\sigma^{\sigma+2\pi} d\eta\, \pi_\mu(\eta)\,. \tag{4.20}$$

To complete the algebra we add the following relations

$$\begin{aligned} \{y_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \cong\ & \delta_\mu{}^\nu \delta(\sigma-\bar\sigma) + \kappa h_{\mu\rho}[x(\sigma)]\theta_0^{\rho\nu}\delta(\sigma-\bar\sigma) \\ +\ & \kappa h_{\mu\rho}[x'(\bar\sigma)]\theta_0^{\rho\nu}\theta(\sigma-\bar\sigma)\,, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \{\tilde{y}_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \cong\ & \Big[-2bG^{-1} - 3h[x(\sigma)]G^{-1} - 2\kappa b h[x(\sigma)]\theta_0\Big]_\mu^{\ \nu}\delta(\sigma-\bar\sigma) \\ -\ & \Big[3h[x'(\bar\sigma)]G^{-1} + 2\kappa b h[x'(\bar\sigma)]\theta_0\Big]_\mu^{\ \nu}\theta(\sigma-\bar\sigma), \end{aligned} \tag{4.22}$$

$$\{{}^\star\pi^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \cong 0. \tag{4.23}$$

Because the T-dual momenta ${}^\star\pi^\mu$ are bilinear in original coordinates, their Poisson bracket vanishes. The Poisson bracket between T-dual coordinates and momenta however gains the additional term linear in coordinates.

In doubled space the additional coordinate $\tilde{y}_\mu$ appears. It consists of the term linear in original momenta and the other terms bilinear in original coordinates. So, it produces nontrivial Poisson brackets with all variables $(y_\mu, \tilde{y}_\mu, {}^\star\pi^\mu)$, (4.13), (4.14) and (4.22).

5. Concluding remarks

We showed that we need two ingredients in order to have closed string non-commutativity: T-duality and nontrivial background. The T-dual transformation laws connect the world-sheet derivatives of the coordinates and momenta in the original and T-dual theory. The T-dual coordinates y_μ has two terms: one linear in original momenta and the other bilinear in original coordinates. This produces the nontrivial Poisson bracket $\{y_\mu, y_\nu\}$ (4.12) which is linear in coordinate. Consequently, the nontrivial infinitesimal expression β^0_μ, which exists only in the coordinate dependent backgrounds, is the source of the closed string non-commutativity. Note that in the case of open string moving in the flat background coordinate is linear function in both effective momenta and coordinates. So, the corresponding Poisson bracket is constant.

The general structure of the non-commutativity relations is

$$\{Y_\mu(\sigma), Y_\nu(\bar\sigma)\} = \{F_{\mu\nu\rho}\,[x^\rho(\sigma) - x^\rho(\bar\sigma)] + \tilde F_{\mu\nu\rho}\,[\tilde x^\rho(\sigma) - \tilde x^\rho(\bar\sigma)]\}\theta(\sigma-\bar\sigma)\,, \tag{5.1}$$

where $Y_\mu = (y_\mu, \tilde y_\nu)$ and $F_{\mu\nu\rho}$ and $\tilde F_{\mu\nu\rho}$ are constant and infinitesimal fluxes. At the same points, for $\sigma = \bar\sigma$, all Poisson brackets are zero. In the important particular case for $\sigma = \bar\sigma + 2\pi$ we get

$$\{Y_\mu(\sigma+2\pi), Y_\nu(\sigma)\} = 2\pi\left[(F_{\mu\nu\rho} + 2\tilde F_{\mu\nu\alpha} b^\alpha_\rho)N^\rho + \frac{1}{\kappa}\tilde F_{\mu\nu}{}^\rho p_\rho\right]\,, \tag{5.2}$$

where N^μ and p_μ are winding numbers and momenta of the original theory. We can rewrite it in the form

$$\{Y_\mu(\sigma+2\pi), Y_\nu(\sigma)\} = \oint_{C_\rho} F_{\mu\nu\rho} dx^\rho + \oint_{\tilde C_\rho} \tilde F_{\mu\nu\rho} d\tilde x^\rho\,, \tag{5.3}$$

where C_ρ and $\tilde C_\rho$ are cycles around which the closed string is wrapped. This generalizes the conjecture of Ref.[10] between closed string non-commutativity and fluxes.

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