FACTA UNIVERSITATIS
Series: **Working and Living Environmental Protection** Vol. 21, N° 4, Special Issue, 2024, pp. 401 - 408
https://doi.org/10.22190/FUWLEP240924038H

Original scientific paper

NONLINEAR VIBRATION OF A BEAM SUBJECTED TO MECHANICAL IMPACT AND WINKLER-PASTERNAK FOUNDATION

UDC 624.072.2:519.957

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Abstract. *The simultaneous effects of mechanical impact and Winkler-Pasternak foundation on the dynamic response of an Euler-Bernoulli beam are studied. By means of the Galerkin-Bubnov procedure, the governing equation with partial derivatives is reduced to an ordinary differential equation. This nonlinear equation is solved by means of the Optimal Homotopy Asymptotic Method (OHAM).*

Key words: *nonlinear vibration, OHAM, mechanical impact, Winkler-Pasternak foundation*

1. INTRODUCTION

Vibration of a beam under mechanical impact and resting on a nonlinear Winkler-Pasternak foundation is interesting as the basic research on vibration problems considering practical bridges. Some of the previous works have been the study of Ansari et al. [1]. They found the existence of the attractors at modest oscillation levels during investigations with realistic parameters. Abiala [2] used the finite element method and Neimark's integration to obtain the dynamic response of beams under uniformly distributed moving loads. The fourth-order Runge-Kutta method is applied by Ding et al. [3] to three types of conventional boundary conditions. The geometrical nonlinearities are considered by Nbendjo and Woafo [4], showing that the single-mode dynamic of the beam can be described by a Φ^6 potential with various configurations. Poorjamishidian et al. [5] analyzed the nonlinear vibration for a simply supported beam with a constant velocity carrying a moving mass.

Received September 24, 2024 / Accepted October 10, 2024
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Pirmoradian and Karinpour [6] explored dynamic stability in the Hamiltonian formulation. The vibration of viscoelastic axially moving Rayleigh and Euler-Bernoulli beam is investigated by Shariati et al. [7]. Herisanu and Marinca [8] studied the nonlinear vibration of a beam under mechanical impact in the presence of an electromagnetic actuator.

The present study is devoted to the nonlinear forced vibration of a beam resting on a nonlinear Winkler-Pasternak elastic foundation subjected to a mechanical impact. The time response of the beam has been obtained using the Optimal Homotopy Asymptotic Method (OHAM). The results show that the result of the analytical procedure has a very good correspondence with numerical integration results.

2. THE GOVERNING EQUATION

The physical model of a simply supported beam of length L subjected to a mechanical impact by the force F and resting on the Winkler-Pasternak foundation with linear and nonlinear springs K_1 and K_3 respectively is presented in Fig. 1.

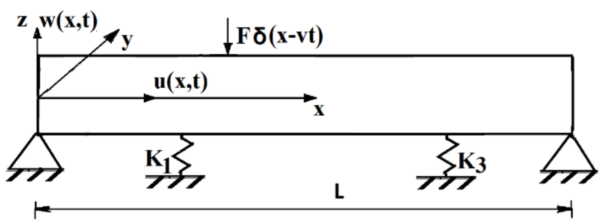

Fig. 1 Geometry of the beam under mechanical impact and elastic foundation

The Young's modulus E, a mass density ρ and the cross-sectional area A of the beam are supposed to be constants. The transverse and longitudinal displacements are w(x,t) and u(x,t), respectively.

The deflection of an element of length ds of the beam at rest is defined by

$$ds = \left[\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]^{1/2} dx \tag{1}$$

If the terms of the forms u_x^2, $u_x w_x^2$, u_x^3, $u_x^2 w_x^2$, u_x^4, where $u_x = \frac{\partial u}{\partial x}$, are neglected, from Eq.(1) one can get:

$$\frac{dx}{ds} = [(1+u_x)^2 + w_x^2]^{-\frac{1}{2}} \cong 1 - u_x - \frac{1}{2}w_x^2 + \frac{3}{8}w_x^4 \tag{2}$$

The unit vector parallel to the defined element (1) can be written in the form

$$\bar{k} = [(1+u_x)\bar{\imath} + w_k\bar{\jmath}]\frac{dx}{ds} \tag{3}$$

The tension in the beam is

$$T = -EAe \tag{4}$$

in which e is defined as

$$e = \frac{dx-ds}{dx} = 1 - \frac{ds}{dx} \cong u_x + \frac{1}{2}w_x^2 - \frac{3}{8}w_x^4 \tag{5}$$

The dynamics of the beam is defined by the equation [4]:

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = -\frac{\partial}{\partial x}\left(T\bar{k}\right)\bar{J} + F_{mi} + F_{wp} \tag{6}$$

where I is the moment of inertia of the beam cross-section, F_{mi} and F_{wp} are the impact force and nonlinear elastic foundation, respectively.

Substituting Eqs. (3)-(5) into Eq.(6), we obtain

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = -EA \frac{\partial}{\partial x}\left(e \frac{\partial w}{\partial x}\right) + F_{mi} + F_{wp} \tag{7}$$

It is known that the material term in Eq.(5) is small such that $\frac{d}{dx}(e) = 0$.

It follows that

$$u_x + \frac{1}{2} w_x^2 - \frac{3}{8} w_x^4 = e = C \tag{8}$$

where C is a constant which can be determined by integrating the last equation

$$u(x) = u(0) + Cx - \frac{1}{2}\int_0^x \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{3}{8}\int_0^x \left(\frac{\partial w}{\partial x}\right)^4 dx \tag{9}$$

Using the boundary conditions for the longitudinal displacement: u(L,t)=u(0,t)=0, from Eq.(9) we have that

$$e = C = \frac{1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx - \frac{3}{8L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^4 dx \tag{10}$$

From Eqs. (7) and (10) one can get

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = \frac{EA}{2L}\frac{\partial^2 w}{\partial x^2}\left[\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx - \frac{3}{4}\int_0^L \left(\frac{\partial w}{\partial x}\right)^4 dx\right] + F_{mi} + F_{wp} = 0 \tag{11}$$

The term F_{mi} represents the mechanical impact of load F:

$$F_{mi} = F\delta(x - vt) \tag{12}$$

where v is the speed of the load and δ is the Dirac-delta function.

The nonlinear elastic medium of the Winkler-Pasternak type is given by

$$F_{wp} = -K_1 w - K_3 w^3 \tag{13}$$

Inserting Eqs. (12) and (13) into Eq. (11) one retrieves:

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} - \frac{EA}{2L}\frac{\partial^2 w}{\partial x^2}\left[\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx - \frac{3}{4}\int_0^L \left(\frac{\partial w}{\partial x}\right)^4 dx\right] - $$

$$-K_1 w - K_3 w^3 = F\delta(x - vt) \tag{14}$$

with the simply supported beam, such that the boundary conditions are

$$w(0,t) = w(L,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \tag{15}$$

To express the governing Eqs. (14) and (15) in nondimensional form, the following parameters are defined:

$$\bar{w} = \frac{W}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad \bar{v} = vL\sqrt{\frac{\rho A}{EI}}$$

$$\bar{\alpha} = \frac{L^2 A}{2I}, \quad \bar{K}_1 = K_1 \frac{L^4}{EI}, \quad \bar{K}_3 = K_3 \frac{L^2}{EI} \quad \bar{f}_0 = \frac{FL^4}{EI} \tag{16}$$

Omitting the bar, Eq. (14) can be rewritten in nondimensional form as

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} - \alpha \frac{\partial^2 w}{\partial x^2} \left[\int_0^1 \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{3}{4} \int_0^L \left(\frac{\partial w}{\partial x} \right)^4 dx \right] -$$

$$-K_1 w - K_3 w^3 = f \delta(x - vt) \tag{17}$$

Using the Galerkin-Bubnov procedure, the solution of Eq.(17) can be assumed to be of the form

$$w(x,t) = X(x)T(t) \tag{18}$$

in which, taking into account the conditions (15), X(x) can be expressed as

$$X(x) = \sqrt{2}\sin\pi x \tag{19}$$

Inserting Eqs. (18) and (19) into Eq.(17) and multiplying this equation by X(x), and then integrating on the domain [0,1] it holds that

$$\ddot{T} + \omega^2 T + aT^3 + bT^5 = f sin\pi vt \tag{20}$$

where the dot denotes the derivative with respect to time and

$$\omega^2 = \int_0^1 \frac{d^2 X(x)}{dx^2} X(x)dx - K_1 \int_0^1 X^2(x)dx\,;$$

$$\text{a=-}\alpha \left[\int_0^1 \left(\frac{dX(x)}{dx} \right)^2 dx \right] \left[\int_0^1 \frac{d^2 X(x)}{dx^2} X(x)dx - K_3 \int_0^1 X^4(x)dx \right];$$

$$b = \frac{3}{4}\alpha \left[\int_0^1 \left(\frac{dX(x)}{dx} \right)^4 dx \right] \left[\int_0^1 \frac{d^2 X(x)}{dx^2} X(x)dx \right]; f = \sqrt{2} f_0 \tag{21}$$

for the nonlinear differential Eq. (20), the initial conditions are

$$T(0) = A,\, \dot{T}(0) = 0 \tag{22}$$

The Eqs. (20) and (22) are very difficult to be analytically solved to obtain exact solutions. In what follows, for Eqs. (20) and (22) we will apply the OHAM to obtain an analytical approximate solution.

3. THE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD

We will apply OHAM to the following nonlinear differential equation [9-11]:

$$L[T(t)] + N[T(t)] = 0 \tag{23}$$

whose boundary conditions are

$$B\left(T(t), \frac{dT(t)}{dt}\right) = 0 \tag{24}$$

In Eq.(23), L and N are linear operators and nonlinear operators, respectively. If $\bar{T}(t)$ is the unknown approximate solution of Eq. (23)-(24), then we can write that

$$\bar{T}(t) = T_0(t) + T_1(t) \tag{25}$$

where the initial approximation $T_0(t)$ can be evaluated from the linear equation

$$L[T_0(t)] = 0, \quad B\left(T_0(t), \frac{dT_0(t)}{dt}\right) = 0 \tag{26}$$

The first approximation $T_1(t)$ can be evaluated from the linear equation

$$L[T_1(t)] = H(t, C_1, C_2, \ldots, C_n) N[T_0(t)], \quad B\left(T_1(t), \frac{dT_1(t)}{dt}\right) = 0 \tag{27}$$

where $H(t,C_1,C_2,\ldots,C_n)$ is an arbitrary auxiliary function. This auxiliary function and $N[T_0(t)]$ should be of the same shape. The parameters C_1, C_2, …,C_n which appear on the first-order approximate solution obtained from Eq. (27) can be determined in many ways, using for example the least square method, the Galerkin method, the collocation method, the Ritz method or by minimizing the square residual error.

With these parameters known (namely the convergence-control parameters), the approximate solution (25) is well-determined.

4. Application of OHAM to the Nonlinear Vibration of the Beam

Making the transformations

$$\tau = \Omega t, T(t) = A\Psi(\tau) \tag{28}$$

in which Ω is the unknown natural frequency of the beam, Eq. (20) becomes

$$\Psi'' + \frac{\omega^2}{\Omega^2}\Psi + \frac{\alpha A^2}{\Omega^2}\Psi^3 + \frac{bA^4}{\Omega^2}\Psi^5 = \frac{f}{A\Omega^2} sin\frac{\pi v}{\Omega}\tau, \;\; \Psi(0) = 1, \Psi'(0) = 0 \tag{29}$$

where the prime denotes derivative with respect to τ.

The linear operator and nonlinear operator of Eq. (29) are respectively

$$L[\Psi(\tau)] = \Psi'' + \Psi\,; \;\; \mathrm{N}[\Psi(\tau)] = \left(\frac{\omega^2}{\Omega^2} - 1\right)\Psi + \frac{\alpha A^2}{\Omega^2}\Psi^3 + \frac{bA^4}{\Omega^2}\Psi^5 - \frac{f}{A\Omega^2} sin\frac{\pi v\tau}{\Omega} \tag{30}$$

The initial approximation $\psi_0(\tau)$ is determined from Eqs. (26) and (30):

$$\Psi_0'' + \Psi_0 = 0, \;\; \Psi_0(0) = 1, \Psi_0'(0) = 0 \tag{31}$$

and has the solution

$$\Psi_0(\tau) = cos\tau \tag{32}$$

Substituting Eq. (32) into the nonlinear operator (30) one gets:

$$\mathrm{N}[\Psi_0(\tau)] = M_1 cos\tau + M_3 cos3\tau + M_5 cos5\tau + Psin\frac{\pi v\tau}{\Omega} \tag{33}$$

where the constants M_i and P are given by

$$\mathrm{M}_1 = \frac{\omega^2}{\Omega^2} - 1 + \frac{3aA^2}{4\Omega^2} + \frac{5bA^4}{8\Omega^2}\,; \mathrm{M}_3 = \frac{aA^2}{4\Omega^2} + \frac{5bA^4}{16\Omega^2}; \mathrm{M}_5 = \frac{bA^4}{16\Omega^2}\,; P = -\frac{f}{A\Omega^2} \tag{34}$$

The auxiliary function $H(\tau,C_1,C_2,\ldots,C_n)$ from Eq. (27) is chosen such that the product $H(\tau,C_1,C_2,\ldots,C_n)N[\psi_0(\tau)]$ and $N[\psi_0(\tau)]$ be of the same form. The auxiliary functions $H(\tau,C_1,C_2,\ldots,C_n)$ and the natural number n are not unique. For example, we can alternatively choose these auxiliary functions in the following expressions:

$$\mathrm{H}_1(\tau, \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4) = \mathrm{C}_1 + 2\mathrm{C}_2 cos2\tau + 2\mathrm{C}_3 cos4\tau + 2\mathrm{C}_4 cos6\tau \tag{35}$$

$$\mathrm{H}_2(\tau, \mathrm{C}_1, \mathrm{C}_2) = \mathrm{C}_1 + 2\mathrm{C}_2 cos2\tau \tag{36}$$

$$\mathrm{H}_3(\tau, \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3) = \mathrm{C}_1 + 2\mathrm{C}_2 cos4\tau + 2\mathrm{C}_3 cos6\tau \tag{37}$$

$$\mathrm{H}_4(\tau, \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3) = \mathrm{C}_1 + 2\mathrm{C}_2 cos2\tau + 2\mathrm{C}_3 cos4\tau \tag{38}$$

and so on. Using only the expression (35), Eq. (27) can be written as follows:

$$\Psi_1'' + \Psi_1 = [M_1(C_1 + C_2) + M_3(C_2 + C_3) + M_5(C_3 + C_4)]cos\tau +$$
$$[M_3(C_1 + C_4) + M_1(C_2 + C_3) + M_5 C_2]cos3\tau +$$
$$[M_5 C_1 + M_3 C_2 + M_1(C_3 + C_4)]cos5\tau + [M_5 C_2 + M_3 C_3 + M_1 C_4]cos7\tau,$$
$$\Psi_1(0) = \Psi_1'(0) = 0 \tag{39}$$

No secular terms into Eq. (39) require that the coefficient of τ be zero. From this condition, one can put the natural frequency:

$$\Omega^2 = \omega^2 + \frac{3}{4}aA^2 + \frac{5bA^4}{8} + (\frac{aA^2}{4} + \frac{5bA^4}{16})\frac{C2+C3}{C1+C2} + \frac{bA^4}{16}\frac{C3+C4}{C1+C2} \tag{40}$$

The solution of Eq. (39) is

$$\Psi_1(\tau) = \frac{M_1(C_2+C_3)+M_3(C_1+C_4)+M_5C_2}{8}(cos\tau - cos3\tau) + \frac{M_1(C_3+C_4)+M_3C_2+M_5C_1}{24}(cos\tau - cos5\tau) + \frac{M_1C_1+M_3C_3+M_5C_2}{48}(cos\tau - cos7\tau) + \frac{M_3C_4+M_5C_3}{80}(cos\tau - cos9\tau) + \frac{M_3C_1}{120}(cos\tau - cos11\tau) \tag{41}$$

The approximate solution of Eqs. (20) and (22) are obtained from Eqs. (25), (28), (32) and (41) as:

$$\bar{T}(t) = Acos\Omega t + \frac{A[M_1(C_2+C_3)+M_3(C_1+C_4)+M_5C_2]}{8}(cos\Omega t - cos3\Omega t) + \frac{A[M_1(C_3+C_4)+M_3C_2+M_5C_1]}{24}(cos\Omega t - cos5\Omega t) + \frac{A[M_1C_1+M_3C_3+M_5C_2]}{48}(cos\Omega t - cos7\Omega t) + \frac{A[M_3C_4+M_5C_3]}{80}(cos\Omega t - cos9\Omega t) + \frac{AM_3C_1}{120}(cos\Omega t - cos11\Omega t) \tag{42}$$

in which Ω is evaluated from Eq. (40).

5. NUMERICAL RESULTS

To illustrate the accuracy of OHAM, we consider the parameters ω=1.3, A=0.5, a=0.24, b = 0.16, v = 0.414, f = 0.001.

Using a collocation approach, there are obtained the values C_1 = 0.092731330625, C_2= -1.2082598891606, C_3 =1.199106193799, C_4=-0.593752904778.

Figure 2 presents the solution (42) in comparison with the numerical integration results.

One can see that the error between the approximate solution and the numerical result is very small.

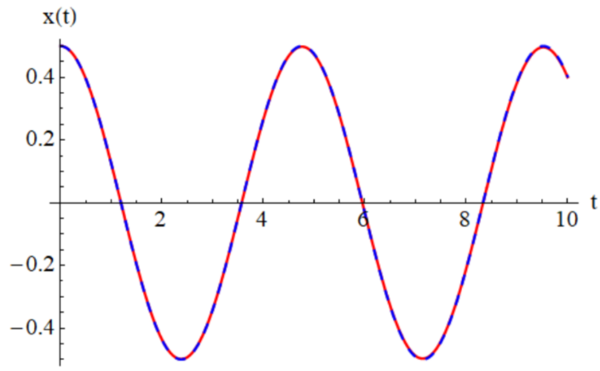

Fig. 2 Comparison between approximate solution (42) and numerical integration results: _ _ _ _ analytical; ______ numerical

6. CONCLUSIONS

In the present research, we propose the Optimal Homotopy Asymptotic Method (OHAM) to obtain an approximate analytical solution to the nonlinear differential equation of vibration of a beam subjected to mechanical impact and resting on the Winkler-Pasternak elastic nonlinear foundation. The validity of our procedure was demonstrated appropriately by choosing the linear operator and auxiliary function.

A numerical example is given and a very good agreement was found between the approximate analytical results and numerical simulation. Our proposed procedure is valid even if the nonlinear differential equation does not contain any small parameters.

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