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DESIGN OF COMPLEMENTARY RECURSIVE DIGITAL FILTERS BASED ON GROUP DELAY APPROXIMATION

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Abstract. *This paper describes a new procedure for design of complementary IIR digital filters based on group delay approximation. The filters are realized as parallel sum of two all-pass filters, a structure for which low complexity implementations exist. The problem with phase warping which is inevitable if filter design is made through phase approximation will be solved using a proposed method. Adequate initial solution is also proposed. Realized amplitude characteristics of complementary filters will be approximately equiripple. The design examples illustrate that the proposed algorithm is very efficient in terms of computation time and number of iterations.*

Key words: *all-pass filters, parallel connection, complementary filters, approximately linear phase, group delay approximation*

1. INTRODUCTION

Digital all-pass filter is efficient signal processing building block which is very useful in many applications as notch filtering, complementary filtering, multirate filtering,, etc. In many signal processing applications transfer function of the filter must be determined according to magnitude or phase response constraints which depend on the application [1].

In some signal processing applications digital filters linear phase characteristic is of vital interest. There are a lot of methods for design of digital filters with linear phase. One well known method is all-pass filter implementation for realization of selective digital filter with linear phase. Digital IIR filters designed by parallel connection of two all-pass filters exhibits small sensitivity of amplitude characteristic in the pass-band on coefficients quantization. In general these filters can provide arbitrary phase, but it is also possible to realize filter with approximately linear phase characteristic in both pass-band and stop-band [4], [5].

Because of these features and a fact that number of bits using for representation of digital filter coefficients depends on sensitivity of amplitude characteristic caused huge interest for

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these filter types in last several years [6],[7]. Realization of the filter through parallel connection of two all-pass filters allows that only with one additional adder we can realize a complementary filter [2],[3]. If the aim is to design only one filter, it is possible to independently control the value of the attenuation in the pass-band and stop-band by adequately choosing the number of poles located in the pass-band and stop-band. If the aim is realization of the complementary filters, input parameters are minimal attenuations in the stop-bands while obtained attenuations in the pass-bands are very small in that case.

Amplitude characteristics of these filters are functions of all-pass filters phases and design is usually carried out through all-pass filters phase approximation [12], [13], [14], [15], [16], [17]. However, during calculation of digital filters phase discontinuities can appear and this fact can cause difficulties during the calculation. For equiripple phase approximation case, if phase error is sufficiently small group delay characteristic will be approximately equiripple. Deviation from equiripple characteristic will be noticeable only near the pass-band edge i.e., only a few last extrema will be slightly higher comparing with other extrema in the approximation region. In this paper we used this fact in order to achieve design of complementary filters based on group delay approximation [10]. This method can be successfully used if prescribed minimal attenuation in the stop-bands is more than 40dB and this condition is fulfilled in almost all practical applications.

The main disadvantage of this approach lies in the fact that approximation regions for phase and group delay are not the same. The consequence will be that boundaries of the group delay approximation regions and pass-band and stop-band boundaries of resulting filters do not match. We proposed a way how this problem can be overcome.

At the end efficiency of the proposed method is illustrated with two examples.

2. APPROXIMATION

Transfer functions of complementary filters realized through parallel connection of two all-pass filters, displayed in Fig. 1, can be written in the next form:

$$H(z) = \frac{1}{2}(A_{N_1}(z) + A_{N_2}(z)) = \left|H(e^{j\omega})\right| e^{j\psi_H(\omega)} \tag{1}$$

$$G(z) = \frac{1}{2}(A_{N_1}(z) - A_{N_2}(z)) = \left|G(e^{j\omega})\right| e^{j\psi_G(\omega)} \tag{2}$$

where $A_{N1}(z)$ and $A_{N2}(z)$ are transfer functions of used all-pass filters.

Selective amplitude characteristics are obtained by adequate signal phase difference in two parallel branches. Resulting amplitude characteristics are given by:

$$\left|H(e^{j\omega})\right| = \left|\cos\frac{\Phi_1(\omega) - \Phi_2(\omega)}{2}\right| \tag{3}$$

$$\left|G(e^{j\omega})\right| = \left|\sin\frac{\Phi_1(\omega) - \Phi_2(\omega)}{2}\right| \tag{4}$$

and corresponding phases are:

$$\psi_H(\omega) = \frac{\Phi_1(\omega) + \Phi_2(\omega)}{2} \tag{5}$$

$$\psi_G(\omega) = \frac{\Phi_1(\omega) + \Phi_2(\omega) - \pi}{2} \tag{6}$$

where $\Phi_1(\omega)$ and $\Phi_2(\omega)$ denote phase of all-pass filters. In regions where phase difference is approximately equal to zero pass-bands and in regions where phase difference is approximately equal to π rad, stop-bands are obtained.

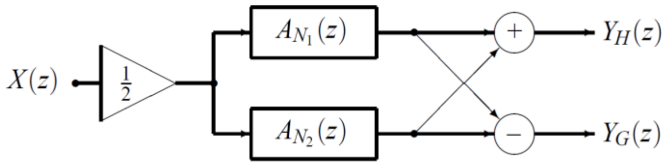

Fig. 1 Realization of IIR selective complementary digital filters using parallel connection of two all-pass filters

The dependence between prescribed minimal attenuation in the stop-band and maximal attenuation in the pass-band of complementary filter is displayed in Fig. 2. It is obvious that if prescribed minimal attenuation is more than 40dB, corresponding maximal attenuation in the pass-band will be very small (less than 10^{-3}dB).

If phase $\Phi_1(\omega)$ and $\Phi_2(\omega)$ of all-pass functions $A_{N1}(z)$ and $A_{N2}(z)$, from Fig. 1 fulfils condition:

$$\Phi_1(\omega) = \begin{cases} -k\omega, & \omega \in \left[0, \omega_p\right] \\ -\pi - k\omega, & \omega \in \left[\omega_s, \pi\right] \end{cases} \tag{7}$$

$$\Phi_2(\omega) = -k\omega, \qquad \omega \in \left[0, \pi\right] \tag{8}$$

then the relations (1) and (3) represent amplitude and phase characteristic of low-pass filter, while relations (2) and (4) represent amplitude and phase characteristic of complementary high-pass filter, where ω_p and ω_s are boundary frequencies of pass-band and stop-band respectively. Ideal phase characteristics (7) and (8) can be approximated in the filter design process.

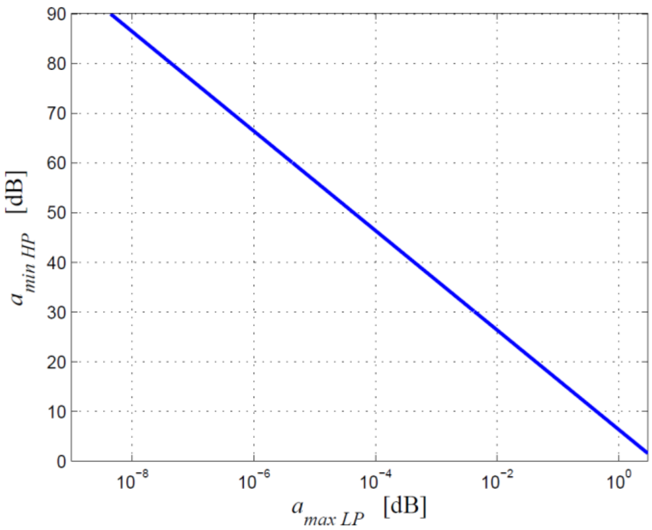

Fig. 2 Dependence between maximal attenuation in the pass-band and minimal attenuation in the stop-band of complementary filter

Amplitude characteristics of filters obtained using parallel connections of two all-pass networks depend on all-pass phases mutual relation. From this reason in existing literature design of all-pass parallel structure filters is usually done through all-pass phases approximation. However, during the calculation of digital filter phase discontinuities will appear making certain difficulties. This problem will be overcome if we use group delay approach. In this case group delay of all-pass filters $A_{N1}(z)$ and $A_{N2}(z)$ have to satisfy next condition:

$$\tau_1(\omega) = -\frac{d\Phi_1(\omega)}{d\omega} = \begin{cases} \tau_{id}, & \omega \in [0, \omega_p] \\ \tau_{id}, & \omega \in [\omega_s, \pi] \end{cases} \tag{9}$$

$$\tau_2(\omega) = -\frac{d\Phi_2(\omega)}{d\omega} = \tau_{id}, \quad \omega \in [\omega_s, \pi] \tag{10}$$

Every pole and every zero of all-pass transfer function contribute to phase with π/2rad, and conditions (7) and (8) will be satisfied if orders of all-pass transfer functions $A_{N1}(z)$ and $A_{N2}(z)$ differs for one. If order of all-pass transfer function $A_{N1}(z)$ is N, than order of all-pass transfer function $A_{N2}(z)$ is N-1.

All-pass filter $A_{N2}(z)$ has to guarantee constant group delay over whole frequency band [0, π]. If the order of all-pass filter is M it can be done if transfer function pole angles are equidistant following next relation:

$$\theta_i = \begin{cases} \dfrac{2i-1-M}{M}\pi, & i = 1,2,\ldots,M \text{ for } M \text{ even} \\ \dfrac{2i-M}{M}\pi, & i = 1,2,\ldots,M \text{ for } M \text{ odd with real pole at } \pi \\ \dfrac{2i+1-M}{M}\pi, & i = 1,2,\ldots,M \text{ for } M \text{ odd with real pole at } 0 \end{cases} \tag{11}$$

where pole modulo ρ_i is less than 0.5. In this case group delay of all-pass function with high accuracy approximates constant group delay $\tau_{id}=N_2=N$-1 in whole frequency band [0, π] in equiripple manner. In most case of practical applications it is enough to take that ρ_i =0.3, i=1,2, ..., N-1. The other option is to choose delay line of order N-1.

Group delay for all-pass filter of the order N is given with:

$$\tau(\rho,\theta,\omega) = \sum_{i=1}^{N} \frac{1-\rho_i^2}{1+\rho_i^2 - 2\rho_i \cos(\omega-\theta_i)}. \tag{12}$$

From the other side all-pass transfer function $A_{N1}(z)$ has to approximate group delay (9). In the case when order of filter is even or odd with real pole at π group delay has to satisfy the next system of equations:

$$\begin{aligned} \tau(\rho,\theta,\omega_i) - \tau_{id} &= (-1)^i \varepsilon_1 \mu_i, & 0 \le \omega_i \le \omega_p \quad & i = 1,2,\ldots,m_1+1 \\ \tau(\rho,\theta,\omega_i) - \tau_{id} &= (-1)^{i+1} \varepsilon_2 \nu_{i-m_1-1}, & \omega_s \le \omega_i \le \pi \quad & i = m_1+1,\ldots,m_1+m_2+2 \end{aligned} \tag{13}$$

where m_1 and m_2 represents number of extrema of group delay error function in the pass-band and stop-band respectively. Low-pass filter $H(z)$ boundary frequencies of pass-band and stop-band are marked with ω_p and ω_s. Parameter ε_1 is maximal group delay error in the pass-band while ε_2 is maximal group delay error in the stop-band for low-pass filter. Coefficients μ_i and ν_i are group delay deviation weighting factors. If μ_i=1 for i=1:m_1+1 and if ν_i=1 for i=1:m_2+1 equiripple characteristic will be obtained. Weighting factors are included in order to increase group delay error only near the boundary frequencies ω_p and ω_s in order to get approximately equiripple phase error. If group delay error was small enough, we concluded that it was sufficient to correct only three extrema starting from the boundary frequencies.

Our aim is to achieve similar group delay error in the pass-band and stop-band in order to provide almost identical attenuation in the stop-band for complementary filters. Taking into account the fact that poles in the pass-band and stop-band are almost equidistant in order to get similar group delay error in both bands it is necessary to choose parameters m_1 and m_2 in such way that next condition is fulfilled:

$$\omega_p : m_1 \approx (\pi - \omega_s) : m_2 \tag{14}$$

System of the linear equations (13) can be written in the next form:

$$\mathbf{A} \cdot \mathbf{\Delta} = \mathbf{B} \tag{15}$$

where elements of matrix **A** can be obtained from the next relation:

$$\mathbf{A}(i,j) = \begin{cases} \dfrac{\partial \tau(\rho^*, \theta^*, \omega_i)}{\partial \rho_j} & \text{for } i = 1 : N+2,\ j = 1 : N/2 \\ \dfrac{\partial \tau(\rho^*, \theta^*, \omega_i)}{\partial \theta_j} & \text{for } i = 1 : N+2,\ j = N/2+1 : N \\ (-1)^{i+1} \mu_i & \text{for } i = 1 : m_1 + 1,\ j = N+1 \\ (-1)^i \nu_i & \text{for } i = m_1 + 2 : N+2,\ j = N+2 \\ 0 & \text{elsewhere} \end{cases} \tag{16}$$

and increment vector Δ is given with:

$$\mathbf{\Delta} = [\Delta\rho_1, \Delta\rho_2, \ldots, \Delta\rho_{N/2}, \Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_{N/2}, \Delta\varepsilon_1, \Delta\varepsilon_2]^T \tag{17}$$

while column vector **B** is given with

$$\mathbf{B}(i) = \begin{cases} (-1)^i \varepsilon_1 \mu_i + \tau_{id} - \tau(\rho^*, \theta^*, \omega_i) & \text{for } i = 1 : m_1 + 1 \\ (-1)^{i+1} \varepsilon_2 \nu_{i-m_1-1} + \tau_{id} - \tau(\rho^*, \theta^*, \omega_i) & \text{for } i = m_1 + 2 : m_1 + m_2 + 1 \end{cases} \tag{18}$$

where ρ^* and θ^* represents obtained pole modulo and pole phase from the previous iteration step.

Initial solution from equation (11) will guarantee that group delay is equiripple with mean value equal to the order of filter N. Now we are solving system of equations (15) where group delay will be decreased in step 0.2 from $\tau_{id}=N$-0.2 until $\tau_{id}=N$-1. During the calculation, for group delay approximation boundary frequencies values we shall use $\omega_p=0.9\omega_{pfinal}$ and $\omega_s=1.1\omega_{sfinal}$, where ω_{pfinal} and ω_{sfinal} are specified values for attenuation boundary frequencies. Using this, we shall get that attenuation at specified boundary frequency ω_{pfinal} will be for sure higher than attenuation in the pass-band. In order to get final solution we are solving again the same system of equations (15) where $\tau_{id}=N$-1 and using bisection method we are correcting boundary frequencies ω_p and ω_s until attenuation at frequency ω_{pfinal} decrease enough to be the same as attenuation at the first extremum of the pass-band.

Calculation stops when maximal element of increment vector Δ is less than specified small value.

2. EXAMPLES

The proposed method is illustrated with two examples. At the beginning of the first example all-pass filter is designed where group delay is approximated over regions [0:0.4π] and [0.6π: π]. The order of the delay line $A_{N2}(z)$ is 9 and the order of the all-pass filter $A_{N1}(z)$ is 10. Taking into account that there is symmetry between boundary frequencies, parameters m_1 and m_2 will be equal according to equation (14), ie. m_1=m_2=5. From the same reason weighting factors μ_i and ν_i will be symmetric, ie. μ_6=ν_1=2.5, μ_5=ν_2=1.57 and μ_4=ν_3=1.14.

As we mentioned earlier group delay approximation band is not the same as phase approximation band. In the last iteration next parameter values are used: ω_p=0.3892π and ω_s=0.6108π. Using weighting coefficients μ_i we shall get that attenuation minimum which is closest to the boundary frequency ω_p will be decreased and it will be almost identical to the other attenuation minimums. Similarly, ν_i coefficients do the same job with the first attenuation minimum of the low-pass filter which is closest to the boundary frequency ω_s. By this way we achieved not only approximately equiripple magnitude characteristic but minimal attenuation in both stop-bands is increased for a few decibels. Including pass-band and stop-band boundary frequencies in vector of extrema two new equations are formed. Added equations provide opportunity to modify maximal group delay errors ε_1 and ε_2. For given filter order optimal values will be obtained in the last iteration. In our example group delay errors are the same, ε_1=ε_2=0.0520 as consequence of the existing symmetry.

Poles of the obtained all-pass filter and weighting coefficients are given in Table 1.

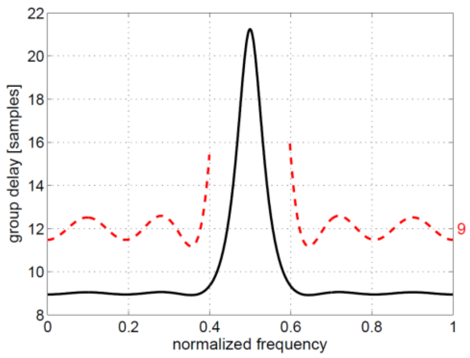

Fig. 3 Group delay of the all-pass filter of order N=10

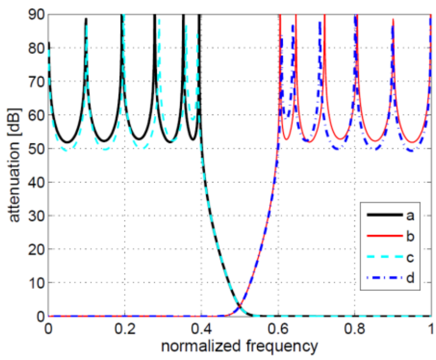

Fig. 4 Attenuation of the high-pass filter including (a) and without μ_i coefficients (c), and attenuation of the low-pass filter including (b) and without (d) ν_i coefficients, respectively

Table 1 Poles of the all-pass filter $A_{N1}(z)$

i	ρ_i	θ_i
	$m_1=m_2=5$ $\mu_6=\nu_1=2.5$ $\mu_5=\nu_2=1.57$ $\mu_4=\nu_3=1.14$	
1	0.555440768384734	±0.317690670860810
2	0.586948145572312	±0.946985650552696
3	0.874918332321571	±1.570796326794897
4	0.586948145572312	±2.194607003037097
5	0.555440768384734	±2.823901982728984

Group delay characteristic of the obtained all-pass filter is displayed in Fig. 3. Attenuation of the complementary filters is presented in Fig. 4. It is clear that using weighting coefficients minimums which are closest to the boundary frequencies become almost identical to the other minimums and at the same time minimal attenuation is increased from 49dB to 52dB.

Phase characteristic of the obtained all-pass filter and used delay line are displayed in Fig. 5 and corresponding all-pass filter approximation error is presented in Fig. 6.

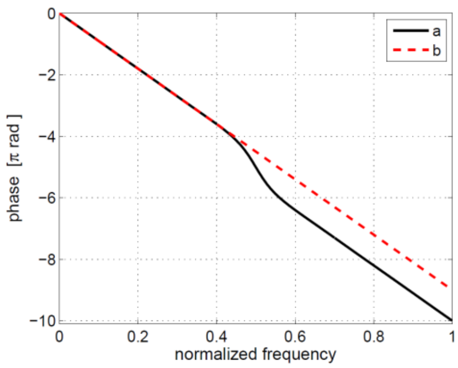

Fig. 5 Phase of the all-pass filter (a) and phase of the delay line (b)

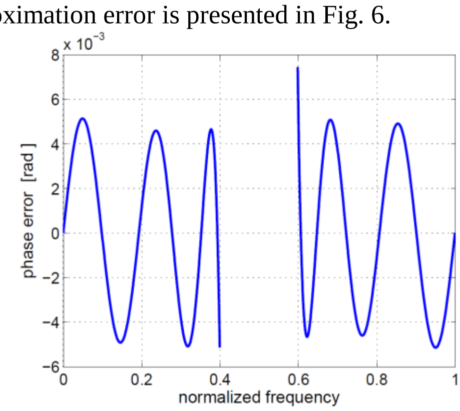

Fig. 6 Phase error of the all-pass filter

Table 2 Poles of the all-pass filter

i	ρ_i	θ_i
	$m_1=5$ $m_2=9$ $\mu_6=1.7$ $\mu_5=1.4$ $\mu_4=1.1$ $\nu_1=2.5$ $\nu_2=1.65$ $\nu_3=1.24$	
1	0.708980964894012	±0.227895868814686
2	0.728264847443748	±0.676998995789484
3	0.924326924585543	±1.099879576422855
4	0.726188578750450	±1.525174863602564
5	0.702338050763929	±1.975845566120250
6	0.696711004980318	±2.438398406539250
7	0.695084722741491	±2.906699180571980

In the second example we designed a filter with boundary frequencies ω_p=0.3π and ω_s=0.4π. The order of the delay line $A_{N2}(z)$ is 13 and the order of the all-pass filter $A_{N1}(z)$

is 14. In this example boundary frequencies are not symmetrical and according to equation (14) parameters m_1 and m_2 are 5 and 9 respectively. Weighting factors are displayed in Table 2 together with poles of the obtained all-pass filter.

In this example group delay errors will not be the same, but ε_1=0.2286, ε_2=0.2052.

Group delay characteristic of the obtained all-pass filter is displayed in Fig. 7. Attenuation of the complementary filters is presented in Fig. 8. It is obvious that using weighting coefficients minimums which are closest to the boundary frequencies become almost identical to the other minimums and at the same time minimal attenuation is increased as in the previous example. If the choice of the parameters m_1 and m_2 is according to equation (14) the difference between minimum attenuations for low-pass and high-pass filter will be only a few decibels.

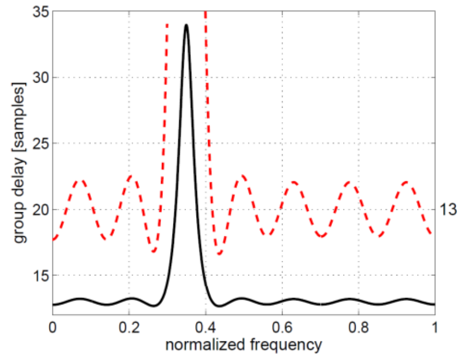

Fig. 7 Group delay of the all-pass filter of order N=14

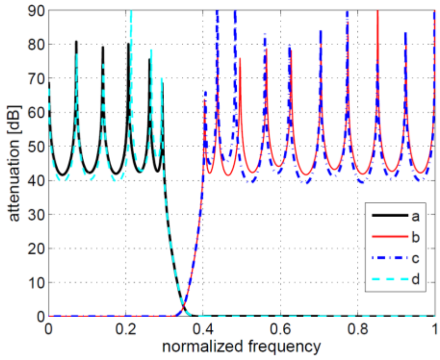

Fig. 8 Attenuation of the high-pass filter including (a) and without μ_i coefficients (c), and attenuation of the low-pass filter including (b) and without (d) ν_i coefficients, respectively

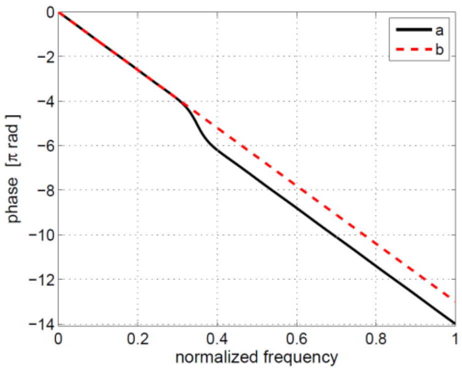

Fig. 9 Phase of the all-pass filter (a) and phase of the delay line (b)

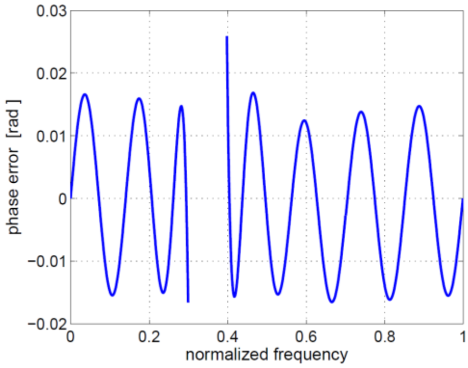

Fig. 10 Phase error of the all-pass filter

Phase characteristic of the obtained all-pass filter and used delay line are displayed in Figure 9 and corresponding all-pass filter approximation error is presented in Fig. 10.

3. CONCLUSION

In this paper we have outlined a procedure for design of complementary filters based on group delay approximation. The filters are realized as parallel sum of two all-pass filters. The well-known fact is that approximation regions for phase and group delay are not the same. Despite this fact using the proposed method, specified attenuation boundary frequencies will be reached. Solving system of linear equations poles of the filters can be derived easily. Finally, two examples are given in order to demonstrate the effectiveness of the proposed method.

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REFERENCES

1. P.A. Regalia, S.K. Mitra and P.P. Vaidynathan, (1988): "The Digital All-Pass Filter: A Versatile Signal Processing Building Block", Proceedings of the IEEE, vol. 76, No. 1, January 1988, pp. 19-37.
2. P.P. Vaidyanathan, S.K. Mitra and Y. Neuvo, (1986): "A New Approach To the realization of Low Sensitivity IIR Digital Filters", IEEE Trans. on Acoust., Speech and Sig. Proc., vol. ASSP-34, pp. 350-361.
3. P.P. Vaidyanathan, P.A. Regalia and S. K. Mitra, (1987): "Design of Doubly Complementary IIR Digital Filters Using a Single Complex Allpass Filters with Multirate Applications", IEEE Trans. on Circuits and Systems, vol. CAS-34, No. 4, pp. 378-389.
4. B.M. Djurich and G.Z. Stancich, (2002): "Selective digital filters with linear phase", Int. J. Electron. Commun. (AEU), vol. 56, No. 2, pp. 131-134.
5. M. Lang, (1998): "Allpass filter design and applications", IEEE Trans. Sign. Process., vol. 46, pp. 2505-2514.
6. A. N. Willson and H. J. Orchard, (1995): "Insights into Digital Filters Made as the Sum of Two Allpass Functions", IEEE Trans. on Circuits and Systems, vol. 42, No.3, pp. 129-137.
7. Huang Jia-Sheu and S. Lowson, (1997): "Design of IIR digital filters using all-pass networks", ECCTD, pp. 529-532.
8. M. Lang, M. and T.I. Laakso, (1994): "Simple and robust method for the design of allpass filters using least--squares phase error criterion", IEEE Trans. Circ.\& Sys., vol. 41, pp. 40-48.
9. B.D. Raković and B.M. Djurić, (1972): "Chebyshev approximation of a constant group delay with constraints at the origin" IEEE Trans. on Circuit and Systems, vol. CT-19, pp. 466-475.
10. Ivan Selesnick, (1999): "Low-pass filters realizable as all-pass sums: design via a new flat delay filter", IEEE Trans. on Circuits and Systems - II: Analog and Digital Signal Processing, vol. 46, No. 1, pp. 40-50.
11. P.P. Vaidyanathan: "Some properties of IIR power-symetric filters", ICASSP 2007, pp. III 1449--III 1452.
12. Z. Jing, (1987): "A new method for digital allpass filter design", IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-35, Nov. 1987, pp. 1557-1564.
13. C.K. Chen and J.H. Lee, (1994): "Design of digital allpass filters using a weighted least squares approach", IEEE Trans. Circuits Syst. II, vol. 41, May 1994, pp. 346-350.
14. M. Ikehara, H. Tanaka and H. Huroda, (1994): "Design of IIR digital filters using allpass networks", IEEE Trans. Circuits Syst. II, vol. 41, March 1994, pp. 231-235.
15. X. Zhang and H. Iwakura: "Design of IIR digital filters based on eigenvalue problem", IEEE Trans. Signal Processing, vol. 44, June 1996, pp. 1325-1333.
16. X. Zhang and H. Iwakura, (1999): "Design of IIR Digital Allpass Filters Based on Eigenvalue Problem", IEEE Trans. Signal Processing, vol. 47, No. 2, Feb. 1999, pp. 554-559.
17. T. Nguyen, T. Laakso and R. Koilpillai, (1994): "Eigenfilter approach for the design of allpass filters approximating a given phase response", IEEE Trans. Signal Processing, vol. 42, No. 9, Sep. 1994, pp. 2257-2263.

PROJEKTOVANJE REKURZIVNIH DIGITALNIH FILTARA APROKSIMACIJOM GRUPNOG KAŠNJENJA

U radu je opisan nov postupak za projektovanje komplementarnih IIR digitalnih filtara baziran na aproksimaciji grupnog kašnjenja allpass filtara. Rezultujući selektivni filtar je realizovan paralelnom vezom dva allpass filtra. Ovakva struktura omogućava realizaciju komplementarnih filtara uz uštedu hardvera. Kod procedura za projektovanje filtara baziranih na aproksimaciji faze mogu da nastanu problemi sa razmotavanjem faze, koji se potpuno izbegavaju predloženim postupkom. Postupak određivanja početnih rešenja je takođe opisan u radu. Realizovane amplitudske karakteristike komplementarnih filtara su približno eliptičke. Priloženi primeri ilustruju efikasnost opisanog algoritma, koga karakteriše mali broj iteracija.

Ključne reči: *All-pass filtri, paralena veza, linearna faza, aproksimacija grupnog kašnjenja, konstantno grupno kašnjenje*