**FACTA UNIVERSITATIS**
Series: **Working and Living Environmental Protection** Vol. 12, N° 1, 2015, pp. 39 - 47

# SELF EXCITED VIBRATION OF A LINE ELEMENT OF BUILDING LINES STRUCTURE

*UDC 531/534:624.131.52*

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**Abstract**. *Structure element of a building connected with the ground in a line is usually modeled as a beam with the Winkler type support. The elastic property of the support is assumed to be linear or with cubic nonlinearity. Unfortunately, the experiments do not prove such an assumption. It is evident that the nonlinearity is transformed into a real positive number which does not need to be an integer. In this paper, the generalization of the beam with Winkler support is done by introducing the nonlinearity of any non-integer order. The line structure, i.e. beam, has transversal vibrations. The mathematical description of these vibrations is a nonlinear partial differential equation. To solve the equation, we suggest an analytic procedure. The solution is assumed as a product of a time and a displacement function. After averaging, the problem is transformed into a second order nonlinear differential equation. The approximate solution has the form of a cosine (ca) Ateb function. Once the obtained results have been analyzed, the influence of support properties on the system behavior is considered. The attention is given to the influence of the Winkler-Pasternak foundation, too.*

**Key words**: *nonlinear vibration, Winkler type support, Pasternak foundation*

## 1. INTRODUCTION

The problem of beam vibration continually supported with elastic foundation is not a new one. Winkler was the first to introduce a continualy distributed linear support [1] which was assumed to model the connection between the beam and the foundation. The elastic property is supposed to be a linear one. The vibrations of the beam on the linear elastic foundation is modeled with a liner partial differential equation. Depending on the boundary conditions of the supproted beam, the solution of the equation are obtained. Further investigation in beam vibration required the improvement of the mathematical model due to improvement of foundation modelling. Thus, two types of models appear:

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Received November 12, 2014 / Accepted February 27, 2015
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one which includes the – Pasternak type of foundation [2] and the other which takes into consideration the nonlinear properties of the foundation. Usually, instead of linear elastic function, a weak nonlinear cubic order elastic force is considered [3]. The nonlinearity is assumed to be weak in comparison to the linear terms. A significant number of papers has been published showing developed various mathematical models for solving partial differential equations with small nonlinearity [4-7]. Recently, the equation with strong cubic nonlinearity has recently been considered. Results obtained by solving this equation are more appropriate than for the small cubic nonlinearity. Experimetnal investigation show that nonlinearity in foundation shoud not be of cubic type. Usually, the nonlinearity is a deflection function with the order which may be any positive rational number (integer or non-integer) not smaller than 1. In this paper the self-excited vibration of a simply supported uniform beam on such strong nonlinear foundation is considered.

## 2. MATHEMATICAL MODEL OF THE SYSTEM

Mathematical model of vibration of a uniform beam on a nonlinear foundation is as follows

$$EI(\frac{\partial^4 u}{\partial x^4}) + \rho A(\frac{\partial^2 u}{\partial t^2}) + c_\beta u|u|^{\beta-1} = 0, \tag{1}$$

where $EI$ is the rigidity of beam, $\rho A$ is the elementary mass, $c_\beta$ is the coefficient of rigidity of foundation and $\beta \geq 1$ is the order of nonlinearity. For the simply supported beam the boundary conditions are

$$(\frac{\partial^2 u}{\partial x^2})_l = (\frac{\partial^2 u}{\partial x^2})_0 = 0, \quad u(0) = u(l) = 0, \tag{2}$$

where $l$ is the length of the beam.

Let us assume the solution of (1) in the simple form [6]

$$u(x,t) = X(t)\sin(n\pi x/l) \tag{3}$$

where $X(t)$ is an unknown time variable function, $l$ is the length of the beam and $n$=1,2,... The solution (3) satisfies initial conditions (2). Substituting (3) into (1) it is

$$(\ddot{X}\rho A - XEI\left(\frac{n\pi}{l}\right)^4)\sin\left(\frac{n\pi x}{l}\right) + c_\beta X|X|^{\beta-1}(\sin\left(\frac{n\pi x}{l}\right))^\beta = 0. \tag{4}$$

Due to (4) it is evident that (3) is not an exact solution of (1). It represents only an approximate solution. Besides, the equation (4) depends on two variables $x$ and $t$. The aim is to eliminate from (4) the functions with variable $x$. Let us divide the relation (4) with $\sin(n\pi x/l)$. We have

$$(\ddot{X}\rho A + XEI\left(\frac{n\pi}{l}\right)^4) + c_\beta X|X|^{\beta-1}(\sin\left(\frac{n\pi x}{l}\right))^{\beta-1} = 0. \tag{5}$$

This is at this point where the averaging procedure is introduced. Integrating the sinus function over its period of $2\pi$, we obtain

$$\ddot{X}\rho A + XEI\left(\frac{n\pi}{l}\right)^4 + c_\beta C_\beta X\left|X\right|^{\beta-1} = 0, \tag{6}$$

where

$$\varphi = \frac{n\pi x}{l}, \quad C_\beta = \frac{1}{2\pi}\int_0^{2\pi}\left|\sin(\varphi)\right|^{\beta-1} d\varphi. \tag{7}$$

The equation (6) is a strongly non-linear. Namely, the linear terms are much more smaller than the last term in (6). It requires the equation to be treated as a strong nonlinear one, i.e.,

$$\ddot{X} + k_\beta^2 X\left|X\right|^{\beta-1} = -\varepsilon k_1^2 X, \tag{8}$$

where $\varepsilon$<<1 is a parameter of the linear term, while $k_\beta$ is the coefficient of the nonlinear term, i.e.,

$$k_\beta^2 = \frac{c_\beta C_\beta}{\rho A}, \quad \varepsilon k_1^2 = \frac{EI}{\rho A}\left(\frac{n\pi}{l}\right)^4. \tag{9}$$

Now, the main task is to obtain the frequency properties of the system. It requires to find the solution of the strong nonlinear differential equaiton (8). Unfortunately, it is not an easy task.

## 3. Solving of the Equation

Let us assume a procedure for obtaining approximate solution of (8). The suggested method is based on the exact solution of the pure nonlinear differential equation ($\varepsilon$=0). As is supposed that the equation (8) is the perturbed version of the pure nonlinera equation, the approximate solution is assumed in the form of the exact solution but with time variable parameters.

For $\varepsilon$=0, the equation (8) transforms into a pure nonlinear equation

$$\ddot{X} + k_\beta^2 X\left|X\right|^{\beta-1} = 0. \tag{10}$$

For this equation, the exact analytical solution has the form of an Ateb function [8]:

$$X = X_0 ca(\beta, 1, \psi), \tag{11}$$

where the phase angle of the cosine-Ateb function *ca* is

$$\psi = \Omega t + \theta, \tag{12}$$

with $\theta$=const., $X_0$ is the amplitude of the oscillatory function and $\Omega$ is the frequency of the function which depends on the amplitude of vibration $X_0$ as

$$\Omega = \sqrt{\frac{k_\beta^2\left|X_0\right|^{\beta-1}(\beta+1)}{2}}. \tag{13}$$

The first time derivative of (11) is

$$\dot{X} = -\frac{2X_0\Omega}{\beta+1} sa(1,\beta,\psi), \tag{14}$$

where *sa* is the sine-Ateb function [8].

### 3.1. Approximate solution

We assume the solution of (8) and its time derivative in the form (11) and (14) but with time variable parameters:

$$X = X_0(t) ca(\beta,1,\psi(t)), \tag{15}$$

$$\dot{X} = -\frac{2X_0(t)\Omega(X_0)}{\beta+1} sa(1,\beta,\psi(t)), \tag{16}$$

where

$$\dot{\psi}(t) = \Omega(X_0) + \dot{\theta}(t), \Omega(X_0) = \sqrt{\frac{k_\beta^2 |X_0(t)|^{\beta-1}(\beta+1)}{2}} \tag{17}$$

and $X_0(t)$ and $\theta(t)$ are unknown functions. Substituting the assumed solution into (8) it is

$$\dot{X}_0 sa + \frac{2X_0\dot{\theta}}{\beta+1} ca^\beta = \varepsilon k_1^2 X_0 |X_0|^{(1-\beta)/2} \sqrt{\frac{2}{(\beta+1)k_\beta^2}} ca. \tag{18}$$

Let us calculated the first time derivative of (15)

$$\dot{X} = \dot{X}_0 ca - \frac{2X_0\Omega}{\beta+1} sa - \frac{2X_0\dot{\theta}}{\beta+1} sa, \tag{19}$$

where $X_0 = X_0(t)$, $\Omega=\Omega(t)$, sa=sa(1,$\beta$,$\psi(t)$), ca=ca($\beta$,1,$\psi(t)$). Equating the relations (16) and (18), it is evident that the assumption (16) is regular only if the following relation is satisfied:

$$\dot{X}_0 ca - \frac{2X_0\dot{\theta}}{\beta+1} sa = 0. \tag{20}$$

Using (18) and (20), after some transformation it is

$$\dot{X}_0 = \varepsilon k_1^2 X_0 |X_0|^{(1-\beta)/2} \sqrt{\frac{2}{(\beta+1)k_\beta^2}} sa\, ca, \tag{21}$$

$$X_0\dot{\theta} = \varepsilon k_1^2 X_0 |X_0|^{(1-\beta)/2} \sqrt{\frac{\beta+1}{2k_\beta^2}} ca^2. \tag{22}$$

Relations (21) and (22) represent the two first order differential equations which correspond to the second order equation (8). Solving coupled equations (21) and (22) is very unconvenient, and the approximate procedure is introduced. Using the periodical property of the Ateb functions, they are averaged over the period

$$\Pi_\beta = B\left(\frac{1}{\beta+1}, \frac{1}{2}\right), \tag{23}$$

and B is the Beta function [8]. For

$$\langle sa\,ca \rangle = \frac{1}{2\Pi_\beta}\int_0^{2\Pi_\beta} sa\,ca\,d\psi = 0, \tag{24}$$

$$C = \langle ca^2 \rangle = \frac{1}{2\Pi_\beta}\int_0^{2\Pi_\beta} ca^2\,d\psi = \left(\frac{2}{3+\beta}\right)^{2/(\beta+1)}, \tag{25}$$

the averaged equations (21) and (22) are

$$\dot{X}_0 = 0, \quad X_0\dot{\theta} = \varepsilon k_1^2 C X_0 \left|X_0\right|^{(1-\beta)/2}\sqrt{\frac{\beta+1}{2k_\beta^2}}. \tag{26,27}$$

As the solution of (26) is $X_0$=const., we integrate (27) and obtain

$$\theta = \varepsilon k_1^2 C t \left|X_0\right|^{(1-\beta)/2}\sqrt{\frac{\beta+1}{2k_\beta^2}} + \theta_0, \tag{28}$$

where $\theta_0$ is a constant of integration. Substituting the result (28) into (17), the phase of the Ateb function is obtained

$$\psi = \theta_0 + t\sqrt{\frac{k_\beta^2\left|X_0\right|^{\beta-1}(\beta+1)}{2}}\left(1 + \frac{\varepsilon k_1^2 C}{k_\beta^2\left|X_0\right|^{\beta-1}}\right). \tag{29}$$

Based on (29) and the periodic property of the Ateb function, the frequency of (8) follows as

$$\Omega_1 = \frac{\pi}{\Pi_\beta}\sqrt{\frac{\beta+1}{2}}\left(\sqrt{k_\beta^2\left|X_0\right|^{\beta-1}} + \frac{\varepsilon k_1^2 C}{\sqrt{k_\beta^2\left|X_0\right|^{\beta-1}}}\right). \tag{30}$$

Substituting (29) into (11) and (17) it is

$$u = X_0 \sin\left(\frac{n\pi x}{l}\right) ca\left(\beta, 1, \theta_0 + t\sqrt{\frac{k_\beta^2\left|X_0\right|^{\beta-1}(\beta+1)}{2}}\left(1 + \frac{\varepsilon k_1^2 C}{k_\beta^2\left|X_0\right|^{\beta-1}}\right)\right).$$

For initial conditions

$$u(0,x) = f(x), \quad \frac{\partial u(0,x)}{\partial t} = 0, \tag{31}$$

and the time derivative of (31)

$$\frac{\partial u}{\partial t} = X_0 \sqrt{\frac{k_\beta^2 |X_0|^{\beta-1}(\beta+1)}{2}} (1+\frac{\varepsilon k_1^2 C}{k_\beta^2 |X_0|^{\beta-1}}) \sin(\frac{n\pi x}{l})$$
$$sa(1,\beta,\theta_0 + t\sqrt{\frac{k_\beta^2 |X_0|^{\beta-1}(\beta+1)}{2}} (1+\frac{\varepsilon k_1^2 C}{k_\beta^2 |X_0|^{\beta-1}})). \tag{32}$$

we have

$$\theta_0 = 0, \quad f(x) = X_0 \sin\left(\frac{n\pi x}{l}\right). \tag{33}$$

Multiplying $(33)_2$ with $\sin(n\pi x/l)$ and integrating the equation over the period of this function the initial values for $X_{0n}$ follow as

$$\theta_0 = 0, \quad X_{0n} = \frac{n}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{34}$$

Finally, the approximate solution of (8) is

$$u = \sum_{n=1}^{\infty} X_{0n} \sin(\frac{n\pi x}{l}) ca(\beta,1,t\sqrt{\frac{k_\beta^2 |X_{0n}|^{\beta-1}(\beta+1)}{2}} (1+\frac{\varepsilon k_1^2 C}{k_\beta^2 |X_{0n}|^{\beta-1}})). \tag{35}$$

It can be concluded that in spite of the assumption that the form of vibration correspond to the linear oscillator the solution of (8) is quite complex. Nevertheless, the most important parameter of the system is its frequency of vibration.

### 3.2. Frequency of vibration

According to (30) and (34) the frequency of vibration of the n-th mode is

$$\Omega_{1n} = \frac{\pi}{\Pi_\beta} \sqrt{\frac{\beta+1}{2}} \left( \sqrt{k_\beta^2 |X_{0n}|^{\beta-1}} + \frac{\varepsilon k_1^2 C}{\sqrt{k_\beta^2 |X_{0n}|^{\beta-1}}} \right), \tag{36}$$

i.e.,

$$\Omega_{1n} = \frac{\pi}{B\left(\frac{1}{\beta+1}, \frac{1}{2}\right)} \sqrt{\frac{c_\beta C_\beta (\beta+1)}{2\rho A} |X_{0n}|^{\beta-1}} (1+\frac{EI\left(\frac{n\pi}{l}\right)^4 \left(\frac{2}{3+\beta}\right)^{2/(\beta+1)}}{c_\beta C_\beta |X_{0n}|^{\beta-1}}). \tag{37}$$

Analyzing the relation (37) it can be seen that the dependance of the frequency on the properties of foundation is very complex. For simplicity, let us rewritten (37) in the form

$$\Omega_{1n} \approx (\frac{c_\beta |X_{0n}|^{\beta-1}}{\rho A}\frac{\pi^2 C_\beta(\beta+1)}{2B^2} + \frac{EI\left(\frac{n\pi}{l}\right)^4}{\rho A}\frac{\pi^2(\beta+1)}{B^2}\left(\frac{2}{3+\beta}\right)^{2/(\beta+1)})^{1/2}. \quad (38)$$

From the second term of (38), it is seen that the rigidity coefficient of the Winkler foundation has no affects to the frequency of the free beam. The frequency of the free beam vibration is multiplied with parameters which depend on the order of nonlinearity β. It is the reason that the influence of the order of the foundation nonlinearity on the frequency of vibration is discussed.

To prove the correctness of the relation (38), the special case when the elasticity of foundation is linear will be considered. For that case the approximate frequency of vibration is approximately

$$\Omega_{1nl} = \sqrt{\frac{c_1}{\rho A} + \frac{EI\left(\frac{n\pi}{l}\right)^4}{\rho A}}). \quad (39)$$

The relation (39) corresponds to the well known one, where the quadratic value of the frequency is a sum of quadratic frequencies of the free beam and of a hamonic oscillator with a mass $\rho A$ and elasticity $c_1$.

If the nonlinearity of Winkler foundation is of cubic order, the relation (37) gives the approximate frequency as

$$\Omega_{1n3} \approx 0.91\sqrt{0.4330\frac{c_3}{\rho A}|X_{0n}|^2 + \frac{EI\left(\frac{n\pi}{l}\right)^4}{\rho A}}. \quad (40)$$

Comparing (40) with the value given in [3]

$$\Omega^*_{1n3} = \sqrt{0.5625\frac{c_3}{\rho A}|X_{0n}|^2 + \frac{EI}{\rho A}\left(\frac{n\pi}{l}\right)^4}, \quad (41)$$

where the nonlinearity of Winkler type is small, it is obvious that the form of the solutions is the same, but the coefficients differ. It is due to the fact that in (40) the linear term is considered as a small one.

## 4. EXAMPLE

Let us consider the first frequency mode of vibration. For that case the first approximate frequency of vibration is

$$\Omega_{11} \approx (\frac{c_\beta |X_{01}|^{\beta-1}}{\rho A}\frac{\pi^2 C_\beta(\beta+1)}{2B^2} + \frac{EI}{\rho A}\left(\frac{\pi}{l}\right)^4\frac{\pi^2(\beta+1)}{B^2}\left(\frac{2}{3+\beta}\right)^{2/(\beta+1)})^{1/2}. \quad (42)$$

To examine the influence of the order of nonlinearity of the Winkler foundation, we introduce the following numerical data into (42): $c_\beta=1$, $X_{01}=0.5$, $l=\pi$, $EI/\rho A=1$. According to (42) the frequency of vibration is calculated. The parameter $\beta$ is varied. The obtained values are shown in the Table 1.

**Table 1** Frequency – order of nonlinearity relation

| $\beta$ | $\Omega_{11}$ |
|---|---|
| 1 | 1.06066 |
| 5/3 | 0.84145 |
| 2 | 0.80609 |
| 3 | 0.84514 |
| 10/3 | 0.88590 |

Analysing the calculated values it is obvious that for the given values the frequancy of vibration is smaller for the nonlinear than for the linear Winkler foundation. Besides, for the order of nonlinearity in the interval [1,2] the frequency decreases with increasing the order, while in the interval [2,4] the frequency increases with $\beta$.

## 5. Conclusion

In this paper the influence of the order of the Winkler type nonlinearity on self-excited vibrations of a line structure (beam) has been investigated. A Bernouli-Euler beam is settled on the nonlinear foundation. Vibration is described by a partial differential equation. An approximate method for obtaining the solution has been developed. The motion is described in the form of two multipled functions: the first is the exact temporal function and the second is an approximate space function which satisfies the boundary conditions. It is shown that the beam on a linear Winkler foundation has higher frequencies than others on the nonlinear one. A general conclusion about tendency of frequency of a beam on nonlinear fundation can not be given a'priori, as the frequency - $\beta$ parameter function is implicit.

**Acknowledgement**: *This work was partially supported by the Secretariat for Science and Technological Development, Autonomous Province of Vojvodina (Proj. No 114-451-2094/2011 and Proj. ANKUV 2014) and Ministry of Education, Science and Technological Development of the Republic of Serbia (No.ON 174028 and IT 41007).*

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# SAMOPOBUDNE OSCILACIJE LINIJSKOG TEMELJNOG NOSAČA

*Linijski strukturni element zgrade koji je povezan sa podlogom obično se modelira kao greda sa Vinklerovim oslanjanjem. Pretpostavka je da su elastične osobine oslonca linearne ili sa kubnom nelinearnošću. Medjutim, eksperimentalna istraživanja ukazuju da ti parametric odstupaju od tih vrednosti i da je kod realnih sistema nelinearnost sa stepenom koji je pozitivan neceo realan broj. U ovom radu su korišćeni ti rezultati i izvršeno je upštavanje vezano za stepen nelinearnosti. Linijska struktura, odn., greda podložna je transverzalnim vibracijama i cilj rada je da se ispita kretanje za slučaj ma koje stepene nelinearnosti. Matematički model koji opisuje to kretanje je parcijalna nelinearna diferencijalna jednačina. U radu je dat približni analitički metod rešavanja ove jednačine. Rešenje se pretpostavi u obliku proizvoda vremenske funkcije i funkcije pomeranja. Nakon osrednjavanja, problem se transformiše u nelinearnu diferencijalnu jednačinu drugog reda koja ima približno rešenje oblika ca – Ateb funkcije. Analizom dobivenih rezultata zaključujemo o uticaju parametara podloge na ponašanje sistema. U radu je posebna pažnja posvećena i visko – elastičnom.*

Ključne reči: *nelinearne oscilacije, Vinklerov oslonac, Pasternakov oslonac.*